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Absolute Continuity of Solutions to Reaction-Diffusion Equations with Multiplicative Noise

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Abstract

We prove absolute continuity of the law of the solution, evaluated at fixed points in time and space, to a parabolic dissipative stochastic PDE on $L^2(G)$, where G is an open bounded domain in $\mathbb{R}^d$ with smooth boundary. The equation is driven by a multiplicative Wiener noise and the nonlinear drift term is the superposition operator associated to a real function that is assumed to be monotone, locally Lipschitz continuous, and growing not faster than a polynomial. The proof, which uses arguments of the Malliavin calculus, crucially relies on the well-posedness theory in the mild sense for stochastic evolution equations in Banach spaces.

Keywords Stochastic PDEs · Reaction-diffusion equations · Malliavin calculus

Mathematics Subject Classification (2010) 60H07 · 60H15

1 Introduction

Let G be a bounded domain of $\mathbb{R}^d$, $d > 1$, with smooth boundary. Consider a semilinear stochastic equation of the type

$$du(t) + Au(t)\,dt = f(u(t))\,dt + \sigma(u(t))B\,dW(t), \qquad u(0) = u_0, \tag{1}$$

where A is the negative generator of an analytic semigroup on $L^q(G)$, $q \geq 2$, $f : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz continuous decreasing function with polynomial growth, $\sigma : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function, B is a γ-Radonifying operator from $L^2(G)$ to $L^q(G)$, and W is a cylindrical Wiener process on $L^2(G)$ (precise assumptions on the data of the problem are provided in Section 2 below). Then (1) admits a unique mild solution which is continuous in space and time. Our aim is to prove that the law of the random variable $u(t, x)$

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is absolutely continuous with respect to Lebesgue measure for every fixed $(t, x) \in \mathbb{R}_+ \times G$. It seems that, somewhat surprisingly, this natural question has not been addressed in the literature. In fact, all results of which we are aware about existence (and regularity) of the density of solutions to SPDEs with multiplicative noise deal with the case where G is the whole space, $-A$ is the Laplacian, and the drift coefficient f is (globally) Lipschitz continuous (see, e.g., [13, 15, 16, 19] and references therein). Our results do not rely on any one of these assumptions. In particular, we essentially just assume that the semigroup generated by $-A$ is self-adjoint and given by a family of kernel operators, so that, for instance, very large classes of elliptic second-order operators are allowed, and the function f can be of polynomial type. Another major difference with respect to the above-mentioned works is that we rely almost exclusively on the interpretation of (1) as an equation for an $L^q(G)$-valued process, and that we view the pointwise Malliavin derivative of its solution as a process taking values in $L^q(G; H)$, where H is a suitably chosen Hilbert space. This point of view, which allows us to rely on powerful techniques of the functional-analytic approach to stochastic evolution equations on UMD Banach spaces, is probably the most interesting aspect of this work. The more common random field interpretation of (1), that seems the only one used in previous work, at least in connection with techniques of the Malliavin calculus, is used here very sparingly, essentially only to take the pointwise Malliavin derivative of the solution to (1).

Existence and regularity of the density of solutions to semilinear heat equations with additive noise, i.e. for the easier case where $\sigma = 1$ and $-A$ is the Laplacian, were obtained in [9]. Those results, however, depend heavily on the noise being additive, and cannot be extended to the general setting considered here. In fact, if the noise is additive, then the Malliavin derivative of the solution satisfies a deterministic equation with random coefficients, which yields quite strong estimates using pathwise arguments. On the other hand, if the noise is multiplicative, then the Malliavin derivative is only expected to satisfy a further stochastic evolution equation with quite singular initial condition, which is much more difficult to handle than the deterministic PDE arising in the case of additive noise. As a consequence, while in [9] we obtained existence as well as regularity of the density, here we can only show existence. As it is natural to expect, regularity could be obtained also in the case of multiplicative noise and Lipschitz continuous drift. However, we concentrate here only on the existence issue, and we shall deal with the regularity problem somewhere else, hopefully also in the general case where f is monotone and polynomially bounded.

Let us briefly describe the main content of the paper. We first show existence and uniqueness of a unique mild solution u to (1) which is continuous in space and time. This follows by relatively recent results on well-posedness in the mild sense for stochastic evolution equations in Banach spaces (see Section 2). Assuming that the semigroup generated by $-A$ is a family of kernel operators, the mild solution can be interpreted also in the sense of random fields. Considering first the case where f is Lipschitz continuous, so that the mild solution is the unique fixed point of an operator Φ, this reformulation allows to compute the Malliavin derivative of Φ applied to a class of sufficiently regular processes. Using estimates for stochastic convolutions in Banach spaces, we show that the fixed-point operator Φ leaves invariant a subspace of Malliavin differentiable processes with finite moment. This yields, by closability properties of the Malliavin derivative, that the unique mild solution to (1) is pointwise Malliavin differentiable. As a second step, we provide sufficient conditions ensuring that the Malliavin derivative is non-degenerate, adapting a method used in [16, Theorem 5.2] for equations on $\mathbb{R}^d$ (see Section 3). This yields, as is well known, the pointwise absolute continuity of the law of the solution. As mentioned above, the results should be interesting in their own right, as equations in domains (in dimension higher than one) do not appear to have been considered in the literature. Finally, in the general case of equations of reaction-diffusion type, the pointwise absolute continuity of the law of the solution is treated by localization techniques, i.e. by means of the Bouleau-Hirsch criterion (see Section 4), and by convergence results for stochastic evolution equations with locally Lipschitz continuous coefficients in spaces of continuous functions.

2 Well-Posedness in the Space of Continuous Functions

We are going to establish well-posedness in the mild sense for the stochastic equation (1) in a space of continuous functions, using general well-posedness results for stochastic evolution equations in UMD Banach spaces (see [6, 21]). Assuming that the semigroup generated by $-A$ is a family of integral operators, we shall also show that the solution thus obtained can be viewed as a solution in the sense of random field (cf. [3, 22]).

2.1 Preliminaries

Let us consider the following stochastic evolution equation, posed on a general Banach space X:

$$du(t) + Au(t)\,dt = f(u(t))\,dt + B(u(t))\,dW(t), \qquad u(0) = u_0, \tag{2}$$

where W is a cylindrical Wiener process on a Hilbert space U, and all other coefficients are specified below. The following well-posedness result is a slightly simplified version of [6, Theorem 4.9]. The space of γ-Radonifying operators from a Hilbert space K to a Banach space E will be denoted by $\gamma(K, E)$.

Theorem 2.1 *Let E be a UMD Banach space with type 2, such that X is densely and continuously embedded in E, and A be a sectorial, accretive operator on E such that the semigroup S on E generated by $-A$ restricts to a C_0-semigroup of contractions on X. Assume that $f : X \to X$ is locally Lipschitz continuous and there exists $m > 0$ such that*

$$\langle f(x+y) - f(y), x^* \rangle \lesssim 1 + \|y\|^m - \|x\|^m,$$

$$\|f(y)\| \lesssim 1 + \|y\|^m$$

for all $x, y \in X$ and $x^* \in \partial\|x\|$. Let $p > 2$ and assume that there exists a number $\eta \in \mathbb{R}_+$, with

$$\eta < \frac{1}{2} - \frac{1}{p},$$

such that $E_\eta := \mathsf{D}((I + A)^\eta)$ is densely and continuously embedded in X. If $B : X \to \gamma(U, E)$ is locally Lipschitz continuous with linear growth, and $u_0 \in L^p(\Omega; X)$, then there exists a unique X-valued mild solution to (2), which satisfies

$$\mathbb{E} \sup_{t \le T} \|u(t)\|_X^p \lesssim 1 + \mathbb{E}\,\|u_0\|_X^p.$$

Here $\partial\|x\|$ stands for the subdifferential at x, in the sense of convex analysis, of the convex function $\|\cdot\|$, that is, denoting the dual of X by X',

$$\partial\|x\| = \big\{x^* \in X' : \|x^*\| = 1,\ \langle x^*, x\rangle = 1\big\}.$$

Moreover, the notation $a \lesssim b$ means that there exists a constant N such that $a \leq Nb$. To emphasize the dependence of N on parameters $p_1, \ldots, p_n$, we shall write $a \lesssim_{p_1,\ldots,p_n} b$.

Remark 2.2 In [6] the authors also require that

$$\langle -Ax + f(x+y), x^* \rangle \lesssim 1 + \|y\|^m + \|x\|$$

for every $x \in \mathsf{D}(A|_X)$ and $x, y \in X$. Since we are assuming that A is accretive in X, it follows that $\langle -Ax, x^* \rangle \leq 0$. Moreover,

$$\begin{aligned}\langle f(x+y), x^* \rangle &= \langle f(x+y) - f(y), x^* \rangle + \langle f(y), x^* \rangle \\ &\lesssim 1 + \|y\|^m + |\langle f(y), x^* \rangle| \lesssim 1 + \|y\|^m,\end{aligned}$$

hence their condition, under our assumptions, is automatically satisfied.

Remark 2.3 Further well-posedness results in L^q spaces for semilinear parabolic SPDEs of accretive type, with more natural assumptions on the nonlinear drift term f, can be found in [7, 8, 10–12]. See also [2] for related results in spaces of continuous functions.

We shall also need some basic facts on interpolation. The real and the complex interpolation functors are denoted by $(\cdot, \cdot)$ and $[\cdot, \cdot]$, respectively. Moreover, we shall write $X \hookrightarrow Y$ to mean that X is continuously embedded in Y.

Lemma 2.4 *Let X and Y be two Banach spaces forming an interpolation pair, A a positive operator on X, and $\theta, \theta' \in]0, 1[$, $q, q' \in [1, \infty]$ be constants. The following statements hold true:*

(a) *if $X \subset Y$ and $\theta < \theta'$, then $(X, Y)_{\theta,q} \hookrightarrow (X, Y)_{\theta',q'}$;*
(b) *$(X, Y)_{\theta,1} \hookrightarrow (X, Y)_{\theta,\infty}$;*
(c) *$(X, Y)_{\theta,1} \hookrightarrow [X, Y]_\theta \hookrightarrow (X, Y)_{\theta,\infty}$;*
(d) *$\big(X, \mathsf{D}(A)\big)_{\theta,1} \hookrightarrow \mathsf{D}(A^\theta) \hookrightarrow \big(X, \mathsf{D}(A)\big)_{\theta,\infty}$.*

Proof All statements can be found in [20]. Specific references are provided for each result: (a) and (b) are parts of Theorem 1.3.3, p. 25; (c) is a consequence of Theorem 1, p. 64, taking into account Definition 1.10.1, p. 61; (d) is part of Theorem 1.15.2, p. 101. □

2.2 Existence of a Unique Mild Solution

Let us now turn to equation (1), about which the following standing assumptions are assumed from now on.

Hypothesis 1 (a) The operator A is the realization on $L^q(G)$, $q \geq 2$, of a second-order strongly elliptic operator with C^∞ coefficients, with Dirichlet boundary conditions. (b) The function $f : \mathbb{R} \to \mathbb{R}$ is an odd polynomial of degree $m > 0$ with negative leading coefficient. (c) W is a cylindrical Wiener process on $L^2(G)$ defined on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$, with $T \in \mathbb{R}_+$, where $(\mathscr{F}_t)_{t\in[0,T]}$ is the completion of the filtration generated by W.

It follows by (b) that $|f(x)| \lesssim 1 + |x|^m$ for all $x \in \mathbb{R}$.

Proposition 2.5 *Assume that*

$$\frac{d}{2q} < \frac{1}{2} - \frac{1}{p},$$

$\sigma : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous with linear growth, and $B \in \gamma(L^2(G), L^q(G))$. If $u_0 \in L^p(\Omega; C(\overline{G}))$, then (1) admits a unique $C(\overline{G})$-valued mild solution u, which satisfies the estimate

$$\mathbb{E} \sup_{t \le T} \|u(t)\|^p_{C(\overline{G})} \lesssim 1 + \mathbb{E}\|u_0\|^p_{C(\overline{G})}.$$

Here $C(\overline{G})$ denotes the space of continuous functions on $\overline{G}$, the closure of G.

Proof We are going to verify that the assumptions of Theorem 2.1 are satisfied. It follows from Hypothesis 1 that, for any $q \ge 2$, A is a sectorial, accretive operator on $L^q(G)$, and that the semigroup S generated by $-A$ restricts to a C_0-semigroup on $C(\overline{G})$ (see, e.g., [17, Theorem 3.5, pp. 213-214 and Theorem 3.7, p. 217]). Moreover, denoting the evaluation operator on $C(\overline{G})$ associated to f by the same symbol, it is not difficult to see that f satisfies the assumptions of Theorem 2.1 (detail can be found in [6, Examples 4.2 and 4.5]). Moreover, one easily verifies that $u \mapsto \sigma(u)B$ is locally Lipschitz continuous and has linear growth as a map from $C(\overline{G})$ to $\gamma(U, L^q(G))$.

Let $\theta' < \theta$ be such that

$$\frac{d}{2q} < \theta' < \theta < \frac{1}{2} - \frac{1}{p}.$$

Setting $E := L^q := L^q(G)$, let us show that $E_\theta \hookrightarrow C(\overline{G})$ densely: recall that, by Lemma 2.4,

$$E_\theta \hookrightarrow \big(L^q, \mathsf{D}(A)\big)_{\theta,\infty} \hookrightarrow \big(L^q, \mathsf{D}(A)\big)_{\theta',1} \hookrightarrow \big[L^q, \mathsf{D}(A)\big]_{\theta'},$$

where, by the characterization of $\mathsf{D}(A)$ in [20, Theorem 4.9.1, p. 334],

$$\mathsf{D}(A) = H^2_{q,D}(G) := \big\{\phi \in H^2_q(G) : \ \phi|_{\partial G} = 0\big\}.$$

Moreover, thanks to [20, Theorem 3.3.4, p. 321], one has

$$\big[L^q, H^2_{q,D}\big]_{\theta'} = H^{2\theta'}_{q,D}$$

if $2\theta' \neq 1/q$. Since $d > 1$ and $2\theta' > d$ by hypothesis, the latter condition is obviously satisfied, hence $E_\theta \hookrightarrow H^{2\theta'}_{q,D} \subset H^{2\theta'}_q$. Finally, the Sobolev embedding theorem (cf. [20, Theorem 4.6.1, p. 328]) yields $H^{2\theta'}_q \hookrightarrow C(\overline{G})$, assuming that $2\theta' > d/q$, which is satisfied by hypothesis. We have thus shown that all assumptions of Theorem 2.1 are met, hence the claim is proved. □

Note that $p > 2$ imply that, for q large enough, the hypothesis $d/(2q) < 1/2 - 1/p$ is always satisfied.

Remark 2.6 Instead of assuming that f is an odd polynomial with negative leading coefficient, one could also assume that $f : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous, polynomially bounded, and quasi-monotone, i.e. that there exists $\lambda > 0$ such that $x \mapsto \lambda x - f(x)$ is increasing. In fact, assume that there exists $m > 0$ such that $|f(x)| \lesssim 1 + |x|^m$. By dissipativity of $f - \lambda I$,

$$\langle f(x+y) - \lambda(x+y) - (f(y) - \lambda y), x^* \rangle \le 0,$$

hence

$$\langle f(x+y) - f(y), x^* \rangle \le \lambda \langle x, x^* \rangle \le \lambda,$$

and

$$\langle f(x+y), x^* \rangle \leq \lambda + |\langle f(y)x^* \rangle| \lesssim \lambda + 1 + \|y\|^m.$$

2.3 Mild Solution as Random Field

We assume from now on, in addition to Hypothesis 1, the following condition on the semigroup S generated by $-A$.

Hypothesis 2 The semigroup $S = (S(t))_{t\geq 0}$ is sub-Markovian (i.e. $S(t)$ is positive and contracting in $L^\infty(G)$ for all $t \geq 0$) and admits a kernel, in the sense that there exists a function $K : \mathbb{R}_+ \times G^2 \to \mathbb{R}_+$ such that

$$\big[S(t)\phi\big](x) = \int_G K_t(x, y)\phi(y)\,dy$$

for every $\phi \in L^q(G)$, $q \geq 1$.

Let $Q := BB^*$, which is a symmetric and non-negative definite bounded operator. Recall that a cylindrical Q-Wiener process on $L^2 := L^2(G)$ is a Gaussian family of random variables $\mathcal{W} := \{W_h(t),\ h \in L^2, t \geq 0\}$ such that, for all $s, t \geq 0$ and $h, g \in L^2$, $\mathbb{E}(W_h(t)) = 0$ and

$$\mathbb{E}(W_h(t)W_g(s)) = (t \wedge s)\langle Qh, g\rangle_{L^2}$$

(in spite of the slight abuse of notation, no confusion should arise with the cylindrical Wiener process W). Let L^2_Q be the Hilbert space defined as the completion of L^2 with respect to the scalar product $\langle h, g\rangle_{L^2_Q} := \langle Qh, g\rangle_{L^2}$. Note that, denoting the pseudoinverse of $Q^{1/2}$ by $Q^{-1/2}$, if $(e^k)_{k\in\mathbb{N}}$ is a basis of L^2, then $(\bar{e}^k) := (Q^{-1/2}e^k)$ is a basis of L^2_Q. One can define stochastic integrals with respect to $\mathcal{W}$ as follows (see, e.g., [4, Sec. 2]): let $\{X(t,x) :\ (t,x) \in [0,T] \times G\}$ be a predictable process in $L^2(\Omega \times [0,T]; L^2_Q)$. Then

$$\int_0^T \int_G X(t,x)\,\mathcal{W}(dt,dx) := \sum_{k=1}^\infty \int_0^T \langle X(t,\cdot), \bar{e}^k\rangle_{L^2_Q}\,dW_{\bar{e}^k}(t), \tag{3}$$

and the isometry property reads

$$\mathbb{E}\left|\int_0^T \int_G X(t,x)\,\mathcal{W}(dt,dx)\right|^2 = \mathbb{E}\int_0^T \|X(t,\cdot)\|^2_{L^2_Q}\,dt.$$

In order to prove that the Malliavin derivative of the solution u of (1) satisfies a stochastic equation, we need to verify that u can be interpreted as a mild solution to (1) in the sense of random fields (see, e.g., [3, 4, 22]). This is indeed the case (cf. the analogous result for equations with additive noise in [9]).

Proposition 2.7 *Let the assumptions of Proposition 2.5 be satisfied. For any $(t,x) \in [0,T] \times G$, set $u(t,x) := [u(t)](x)$, where u is the unique $C(G)$-valued mild solution to (1). Then, for any $(t,x) \in]0,T] \times G$,*

$$\begin{aligned} u(t,x) = &\int_G K_t(x,y)u_0(y)\,dy + \int_0^t \int_G K_{t-s}(x,y) f(u(s,y))\,dy\,ds \\ &+ \int_0^t \int_G K_{t-s}(x,y)\sigma(u(s,y))\,\mathcal{W}(ds,dy). \end{aligned} \tag{4}$$

Proof As in the proof of [9, Proposition 3.1], it suffices to show that, for every $t \in]0, T]$ and for almost every $x \in G$, the process

$$(s, y) \mapsto K_{t-s}(x, y)\sigma(u(s, y))$$

belongs to $L^2(\Omega \times [0, T]; L^2_Q)$ and that

$$\int_0^t S(t-s)\sigma(u(s))B\, dW(s) = \int_0^t \int_G K_{t-s}(\cdot, y)\sigma(u(s, y))\mathcal{W}(ds, dy) \tag{5}$$

as an equality in L^2. Recalling that $(\bar{e}^k) = (Q^{-1/2}e^k)$, is a basis of the Hilbert space L^2_Q, one easily verifies that

$$\|K_{t-s}(x, \cdot)\sigma(u(s, \cdot))\|^2_{L^2_Q} = \sum_{k=1}^{\infty} \big([S(t-s)\sigma(u(s))](\tilde{e}^k)(x)\big)^2,$$

where $(\tilde{e}^k) := (Q^{1/2}e^k)$ is a basis of $Q^{1/2}(L^2)$. Note that

$$\mathbb{E}\int_0^t \sum_{k=1}^{\infty} \|[S(t-s)\sigma(u(s))](\tilde{e}^k)\|^2_{L^2} ds < \infty,$$

because the stochastic integral on the left-hand side of (5) is well defined. Thus, for almost all $x \in G$,

$$\mathbb{E}\int_0^t \|K_{t-s}(x, \cdot)\sigma(u(s, \cdot))\|^2_{L^2_Q} < \infty,$$

so the stochastic integral on the right-hand side of (5) is well defined. Using the standard formal expansion of the cylindrical Wiener process W as

$$W(t) = \sum_{k=1}^{\infty} e^k w_k(t),$$

where $w_k := W_{\bar{e}^k}$, $k \geq 1$, form a family of independent standard one-dimensional Wiener processes, one has

$$\int_0^t S(t-s)\sigma(u(s))B\, dW(s) = \sum_{k=1}^{\infty} \int_0^t \int_G K_{t-s}(\cdot, y)[\sigma(u(s))Be^k](y)\, dy\, dw_k(s).$$

Then (5) follows taking into account the definition (3) and that $BB^* = Q$. □

3 Equations with Lipschitz Continuous Coefficients

We assume throughout this section that the coefficients f and σ in equation (1) are Lipschitz continuous. We are going to prove that, for any fixed $(t, x) \in (0, T] \times G$, the law of the solution $u(t, x)$ to (1) is absolutely continuous with respect to the Lebesgue measure. For this, note that the Gaussian space where we will make use of the Malliavin calculus is determined by the isonormal Gaussian process on the Hilbert space $H := L^2(0, T; L^2_Q)$ that can be naturally associated to the cylindrical Q-Wiener process $\mathcal{W}$ defined in the previous section (see [14]).

We will first deal with the Malliavin differentiability of the solution, and then we shall provide sufficient conditions implying that the pointwise Malliavin derivative is non-degenerate.

We need further assumptions, that will be assumed to hold from now on.

Hypothesis 3 One has

$$\frac{d}{2q} < \frac{1}{2} - \frac{1}{p}.$$

Moreover, $B \in \gamma(L^2, L^q)$ and $u_0 \in C(\overline{G})$.

Hypothesis 4 The semigroup S is self-adjoint and Markovian.

Recall also that we assume that Hypotheses 1 and 2 are in force throughout. By Proposition 2.5, it follows that (1) admits a unique $C(\overline{G})$-valued mild solution u, and that (1) can also be written as an equality of random fields.

3.1 Pointwise Malliavin Differentiability of the Solution

The main result of this section is the following.

Theorem 3.1 *Let $u \in L^p(\Omega; C([0,T]; C(\overline{G})))$ be the unique mild solution to (1). Then*

$$u \in L^\infty([0,T] \times G; \mathbb{D}^{1,p})$$

and the family of Malliavin derivatives $\{Du(t,x)\}_{(t,x)\in[0,T]\times G}$ satisfies the following linear equation in H:

$$\begin{aligned} Du(t,x) \;=\; & v_0(t,x) + \int_0^t \int_G K_{t-s}(x,y) F(s,y) Du(s,y)\, dy\, ds \\ & + \int_0^t \int_G K_{t-s}(x,y) \Sigma(s,y) Du(s,y)\, \mathcal{W}(ds,dy) \end{aligned} \tag{6}$$

where

$$v_0(t,x) := (\tau, z) \mapsto K_{t-\tau}(x,z)\sigma(u(\tau,z))\, 1_{[0,t]}(\tau),$$

and $F, \Sigma \colon \Omega \times [0,T] \times G \to \mathbb{R}$ are adapted bounded random fields.

The stochastic integral in (6) must be interpreted as an H-valued integral with respect to the cylindrical Q-Wiener process $\mathcal{W}$ (see, e.g., [16, Section 3]).

The following estimate plays an important role in the proof Theorem 3.1 as well as in several further developments. We shall write E^q_η, for any $q \geq 1$ and $\eta > 0$, to denote $(I + A)^{-\eta} L^q$.

Lemma 3.2 *Let $v \in L^p(\Omega; C([0,T]; C(\overline{G})))$ be adapted and $w : \Omega \times [0,T] \times G \to H$ be the process defined as*

$$w(t,x) := (\tau, z) \longmapsto K_{t-\tau}(x,z)\sigma(v(\tau,z))\, 1_{[0,t]}(\tau).$$

For any $\eta \in\,]d/(2q), 1/2 - 1/p[$ one has

$$\sup_{x\in G} \|w(t,x)\|_H^2 \lesssim \int_0^t \|S(t-s)\sigma(v(s))B\|^2_{\gamma(L^2, E^q_\eta)}\, ds.$$

Proof Since $H = L^2(0, T; L^2_Q)$ and $\langle Qh, h\rangle = \|B^*h\|^2_{L^2}$ for every $h \in L^2_Q$, denoting a complete orthonormal basis of L^2 by $(e^k)_{k\in\mathbb{N}}$, it follows by Plancherel's theorem that

$$\begin{aligned}
\|w(t,x)\|^2_H &= \int_0^t \|B^* K_{t-\tau}(x,\cdot)\sigma(v(\tau,\cdot))\|^2_{L^2}\,d\tau \\
&= \int_0^t \sum_{k\in\mathbb{N}} \langle K_{t-\tau}(x,\cdot)\sigma(v(\tau,\cdot)), Be^k\rangle^2\,d\tau \\
&= \int_0^t \sum_{k\in\mathbb{N}} \biggl(\int_G K_{t-\tau}(x,z)\sigma(v(\tau,z))[Be^k](z)\,dz\biggr)^2 d\tau \\
&= \int_0^t \sum_{k\in\mathbb{N}} \bigl[S(t-\tau)\sigma(v(\tau))Be^k\bigr](x)^2\,d\tau,
\end{aligned}$$

where we have used the integral representation of the semigroup S in the last step. Let $(\gamma_k)_{k\in\mathbb{N}}$ be a sequence of independent standard Gaussian random variable on an auxiliary probability space Ω'. Then

$$\|w(t,x)\|^2_H = \int_0^t \mathbb{E}'\biggl|\sum_{k\in\mathbb{N}} \gamma_k\bigl[S(t-\tau)\sigma(v(\tau))Be^k\bigr](x)\biggr|^2\,d\tau,$$

hence also, by Minkowski's inequality and the embedding $E^q_\eta \hookrightarrow L^\infty$,

$$\begin{aligned}
\sup_{x\in G}\|w(t,x)\|^2_H &\lesssim \int_0^t \mathbb{E}'\biggl\|\sum_{k\in\mathbb{N}} \gamma_k\bigl[S(t-\tau)\sigma(v(\tau))Be^k\bigr]\biggr\|^2_{E^q_\eta}\,d\tau \\
&= \int_0^t \|S(t-\tau)\sigma(v(\tau))B\|^2_{\gamma(L^2,E^q_\eta)}\,d\tau.
\end{aligned}$$

□

The proof of Theorem 3.1 uses a maximal inequality for stochastic convolutions that is a special (simpler) case of [21, Proposition 4.2]. We shall use the notation $R \diamond F$ to denote the process

$$R \diamond F\colon t \mapsto \int_0^t R(t-s)F(s)\,dW(s),$$

where R is an analytic semigroup of contractions on a UMD Banach space E and $F\colon \Omega\times\mathbb{R}_+ \to \mathscr{L}(L^2, E)$ is an L^2-strongly measurable and adapted process. Denoting the generator of R by $-C$, we shall write E_η, for any $\eta > 0$, to denote $\mathsf{D}((I+C)^\eta)$.

Proposition 3.3 *Let* $\alpha \in \,]0, 1/2[$, $p > 2$, $\theta \geq 0$ *be such that*

$$\theta < \alpha - \frac{1}{p},$$

and $T > 0$. *There exists* $\varepsilon > 0$ *such that*

$$\mathbb{E}\|R \diamond F\|^p_{C([0,T];E_\eta)} \lesssim T^{p\varepsilon}\int_0^T \mathbb{E}\|s \mapsto (t-s)^{-\alpha}F(s)\|^p_{\gamma(L^2(0,t;L^2),E)}.$$

We shall also need a deep result by Pisier (see [18, Theorem 1.2 and Remark 1.8] as well as [23, p. 5730]) on vector-valued extensions of analytic semigroup, according to which Hpothesis 4 implies that $(S(t)\otimes I_H)_{t\geq 0}$, where I_H denotes the identity of H, admits a (unique) extension from $L^q \otimes H$ to $L^q(H)$, denoted by S_H, which is again analytic. Let A_H denote the negative generator of S_H and $(\lambda + A_H)^{-1}_{\lambda>0}$ its resolvent. The Laplace transform identity

$$(\lambda + A_H)^{-1} = \int_0^\infty e^{-\lambda t} S_H(t)\, dt$$

implies that $(\lambda + A_H)^{-1}$ coincides with the unique continuous linear extension of $(\lambda + A)^{-1} \otimes I_H$ to $L^q(H)$. By Hypothesis 3 there exists $\eta \in]d/(2q), 1/2 - 1/p[$ such that $\mathsf{D}(A^\eta) \hookrightarrow L^\infty$, hence $(I + A)^{-\eta} \in \mathscr{L}(L^q, L^\infty)$. Since $(I + A)^{-\eta}$ is positivity preserving by Hypothesis 2, $(I + A)^{-\eta}$ admits a unique extension to a continuous linear operator from $L^q(H) := L^q(G; H)$ to $L^\infty(H) := L^\infty(G; H)$, with the same norm (see, e.g., [5, Theorem 12.2]). By the above, recalling well-known expressions for fractional powers of closed operators (see, e.g., [17, §2.6]), this extension coincides with $(I + A_H)^{-\eta}$. Therefore, setting $E^q_\eta(H) := (I + A)^{-\eta} L^q(H)$, we have $E^q_\eta(H) \hookrightarrow L^\infty(H)$.

Proof of Theorem 3.1 Let Φ be the fixed-point operator associated to equation (1), i.e.

$$\Phi : v \longmapsto S(t)u_0 + \int_0^t S(t-s) f(v(s))\, ds + \int_0^t S(t-s)\sigma(v(s)) B\, dW(s).$$

It follows by the (the proof) of Theorem 2.1 that the operator Φ, or a suitable power of it, is a contractive endomorphism of $L^p(\Omega; C([0, T]; C(\overline{G})))$. We are going to show that, for any $p > 2$, there exists $T_0 > 0$, a positive constants $c < 1$ depending on T_0, and a positive constant N depending on the $L^p(\Omega; C([0, T]; C(\overline{G})))$ norm of v, such that

$$\|D\Phi(v)\|_{L^\infty([0,T_0]\times G; L^p(\Omega; H))} \le N + c\|Dv\|_{L^\infty(0,T_0; L^p(\Omega; L^\infty(G; H)))}. \tag{7}$$

Let $v \in L^p(\Omega; C([0, T]; C(\overline{G})))$ be such that $Dv \in L^\infty(0, T; L^p(\Omega; L^\infty(G; H)))$. Writing

$$\begin{aligned}\big[\Phi(v)\big](t, x) &= \int_G K_t(x, y)u_0(y)\, dy + \int_0^t \int_G K_{t-s}(x, y) f(v(s, y))\, dy\, ds \\ &\quad + \int_0^t \int_G K_{t-s}(x, y)\sigma(v(s, y))\, \mathcal{W}(dy, ds),\end{aligned}$$

well-known criteria of Malliavin calculus imply that the Malliavin derivatives of all terms on the right-hand side exist, so that $D\big[\Phi(v)\big](t, x)$ can be written as the right-hand side of (6) with u replaced by v. The proof of (7) will be split in several steps, where each term appearing in the expression of $D\Phi(v)$ is estimated.

STEP 1. Let us set, for every $(t, x), (\tau, z) \in [0, T] \times G$,

$$w_0(t, x) := (\tau, z) \mapsto K_{t-\tau}(x, z)\sigma(v(\tau, z))\, 1_{[0,t]}(\tau).$$

Let $\eta \in]d/(2q), 1/2 - 1/p[$. Lemma 3.2 yields

$$\|w_0(t, \cdot)\|^2_{L^\infty(H)} \lesssim \int_0^t \|S(t-\tau)\sigma(v(\tau))B\|^2_{\gamma(L^2, E^q_\eta)}\, d\tau,$$

where

$$\|S(t-\tau)\sigma(v(\tau))B\|_{\gamma(L^2, E^q_\eta)} \lesssim (t-\tau)^{-\eta}\, \|\sigma(v)\|_{C([0,T]; C(\overline{G}))}\, \|B\|_{\gamma(L^2, L^q)}.$$

This implies

$$\begin{aligned}\mathbb{E}\|w_0\|^p_{L^\infty([0,T]\times G;H)} &\lesssim \Big(1+\mathbb{E}\|v\|^p_{C([0,T];C(\overline{G}))}\Big)\|B\|^p_{\gamma(L^2,L^q)}\sup_{t\le T}\Big(\int_0^t (t-\tau)^{-2\eta}\,d\tau\Big)^{p/2}\\ &\lesssim \Big(1+\mathbb{E}\|v\|^p_{C([0,T];C(\overline{G}))}\Big)\|B\|^p_{\gamma(L^2,L^q)}\,T^{p(1-2\eta)/2},\end{aligned}$$

where the last term on the right-hand side is finite by assumption.

STEP 2. Let $\alpha < 1/2$ be such that $\eta < \alpha - 1/p$. Recalling that $E^q_\eta(H) \hookrightarrow L^\infty(H)$, Minkowski's and Jensen's inequalities yield

$$\begin{aligned}\Big\|\int_0^t S(t-s)F(s)Dv(s)\,ds\Big\|^2_{L^\infty(H)} &\lesssim_T \int_0^t \|S(t-s)F(s)Dv(s)\|^2_{E^q_\eta(H)}\,ds\\ &\lesssim \int_0^t (t-s)^{-2\eta}\|Dv(s)\|^2_{L^q(H)}\,ds.\end{aligned}$$

Since $\eta < \alpha - 1/p$ by assumption, we have $-2\eta > -2\alpha + 2/p$, hence $-2\eta = -2\alpha + 2/p + \varepsilon$, with $\varepsilon > 0$. Then

$$\begin{aligned}\int_0^t (t-s)^{-2\eta}\|Dv(s)\|^2_{L^q(H)}\,ds &= \int_0^t (t-s)^{-2\alpha}(t-s)^{2/p+\varepsilon}\|Dv(s)\|^2_{L^q(H)}\,ds\\ &\le t^{2/p+\varepsilon}\int_0^t (t-s)^{-2\alpha}\|Dv(s)\|^2_{L^q(H)}\,ds\\ &\lesssim_T \int_0^t s^{-2\alpha}\|Dv(t-s)\|^2_{L^q(H)}\,ds.\end{aligned}$$

As the measure μ on $[0,t]$ defined as

$$\mu(ds) := \frac{1-2\alpha}{t^{1-2\alpha}}s^{-2\alpha}\,ds$$

is a probability measure, it follows by Jensen's inequality that

$$\begin{aligned}\Big(\int_0^t s^{-2\alpha}\|Dv(t-s)\|^2_{L^q(H)}\,ds\Big)^{p/2} &= \Big(\frac{t^{1-2\alpha}}{1-2\alpha}\int_0^t \|Dv(t-s)\|^2_{L^q(H)}\,\mu(ds)\Big)^{p/2}\\ &\lesssim t^{(1-2\alpha)p/2}\int_0^t \|Dv(t-s)\|^p_{L^q(H)}\,\mu(ds)\\ &\lesssim t^{(1-2\alpha)(p/2-1)}\int_0^t s^{-2\alpha}\|Dv(t-s)\|^p_{L^q(H)}\,ds\\ &\lesssim_T \int_0^t (t-s)^{-2\alpha}\|Dv(s)\|^p_{L^q(H)}\,ds.\end{aligned}$$

Therefore

$$\mathbb{E}\Big\|\int_0^t S(t-s)F(s)Dv(s)\,ds\Big\|^p_{L^\infty(H)} \lesssim_T \int_0^t (t-s)^{-2\alpha}\mathbb{E}\|Dv(s)\|^p_{L^\infty(H)}\,ds.$$

STEP 3. Using again the continuous embedding $E^q_\eta(H) \hookrightarrow L^\infty(H)$, we have

$$\begin{aligned}
\mathbb{E}\|S \diamond (\Sigma D v B)\|^p_{C([0,t];L^\infty(H))} &\lesssim \mathbb{E}\|S \diamond (\Sigma D v B)\|^p_{C([0,t];E^q_\eta(H))} \\
&\lesssim_T \mathbb{E}\int_0^t \|(\tau - \cdot)^{-\alpha}\Sigma\, Dv\, B\|^p_{\gamma(L^2(0,\tau;L^2),L^q(H))}\,d\tau \\
&\lesssim \int_0^t \mathbb{E}\|(\tau - \cdot)^{-\alpha}\Sigma\, Dv\, B\|^p_{L^2(0,\tau;\gamma(L^2,L^q(H)))}\,d\tau,
\end{aligned}$$

where the third inequality follows by Proposition 3.3, as $L^q(H)$ is a UMD Banach space and $\eta < \alpha - 1/p$, and the fourth estimate follows by Fubini's theorem and the embedding

$$L^2(0,\tau;\gamma(L^2,L^q(H))) \hookrightarrow \gamma(L^2(0,\tau;L^2),L^q(H)),$$

which holds because $L^q(H)$ has type 2. Since $Dv(s) \in L^\infty(H)$ by assumption and $\Sigma \in L^\infty([0,T]\times G)$ by the Lipschitz continuity of σ, it follows that

$$\begin{aligned}
&\|(\tau - s)^{-\alpha}\Sigma(s)Dv(s)B\|_{\gamma(L^2,L^q(H))} \\
&\qquad \le (\tau - s)^{-\alpha}\|\Sigma\|_{L^\infty([0,T]\times G)}\|Dv(s)\|_{L^\infty(H)}\|B\|_{\gamma(L^2,L^q)},
\end{aligned}$$

hence

$$\begin{aligned}
&\|(\tau - \cdot)^{-\alpha}\Sigma\, Dv\, B\|^2_{L^2(0,\tau;\gamma(L^2,L^q(H)))} \\
&\qquad \le \|B\|^2_{\gamma(L^2,L^q)}\|\Sigma\|^2_{L^\infty([0,T]\times G)}\int_0^\tau (\tau - s)^{-2\alpha}\|Dv(s)\|^2_{L^\infty(H)}\,ds.
\end{aligned}$$

Proceeding as in the previous step, we obtain

$$\mathbb{E}\|(\tau - \cdot)^{-\alpha}\Sigma\, Dv\, B\|^p_{L^2(0,\tau;\gamma(L^2,L^q(H)))} \lesssim_T \int_0^\tau (\tau - s)^{-2\alpha}\mathbb{E}\|Dv(s)\|^p_{L^\infty(H)}\,ds,$$

therefore, by Tonelli's theorem,

$$\begin{aligned}
\mathbb{E}\|S \diamond (\Sigma D v B)\|^p_{C([0,t];L^\infty(H))} &\lesssim_T \int_0^t\int_0^\tau (\tau - s)^{-2\alpha}\mathbb{E}\|Dv(s)\|^p_{L^\infty(H)}\,ds\,d\tau \\
&= \int_0^t \mathbb{E}\|Dv(s)\|^p_{L^\infty(H)}\int_s^t (\tau - s)^{-2\alpha}\,d\tau\,ds,
\end{aligned}$$

where

$$\int_s^t (\tau - s)^{-2\alpha}\,d\tau = \int_0^{t-s}\tau^{-2\alpha}\,d\tau = \frac{1}{1-2\alpha}(t-s)^{1-2\alpha},$$

hence

$$\mathbb{E}\|S \diamond (\Sigma D v B)\|^p_{C([0,t];L^\infty(H))} \lesssim_T \int_0^t (t-s)^{-2\alpha}\mathbb{E}\|Dv(s)\|^p_{L^\infty(H)}\,ds.$$

STEP 4. Setting

$$\begin{gathered}
\phi(t) := \mathbb{E}\|Dv(t)\|^p_{L^\infty(H)}, \qquad \psi(t) := \mathbb{E}\|D\Phi(v)(t)\|^p_{L^\infty(H)}, \\
N := 1 + \mathbb{E}\|v\|^p_{C([0,T];C(\overline{G}))},
\end{gathered}$$

the estimates in the previous steps can be written as

$$\psi(t) \lesssim_T N + \int_0^t (t-s)^{-2\alpha}\phi(s)\,ds,$$

hence, using the notation $h^*(s) := \sup_{r \leq s} |h(r)|$ for any function $h\colon \mathbb{R} \to \mathbb{R}$ for which it makes sense,

$$\psi(t) \lesssim_T N + \phi^*(t) \int_0^t (t-s)^{-2\alpha}\, ds = N + \frac{1}{1-2\alpha} t^{1-2\alpha} \phi^*(t),$$

thus also

$$\psi^*(t) \lesssim_T N + \frac{1}{1-2\alpha} t^{1-2\alpha} \phi^*(t),$$

from which (7) follows.

Let u_0 be identified with the process equal to u_0 for all $t \in [0,T]$, which clearly belongs to $L^p(\Omega; C([0,T]; C(\overline{G})))$ and is such that $Du_0 \in L^\infty(0,T; L^p(\Omega; L^\infty(G;H)))$, and introduce the sequence of processes (u_n), $u_n := \Phi(u_{n-1})$. Then u_n converges to u in $L^p(\Omega; C([0,T]; C(\overline{G})))$, possibly along a subsequence of the type (kn), with constant k (if Φ is not a contraction, but Φ^k is). In particular, (u_n) is bounded in $L^p(\Omega; C([0,T]; C(\overline{G})))$. This in turn implies, thanks to (7), that (Du_n) is bounded in $L^\infty([0,T_0] \times G; L^p(\Omega; H))$. Let us show that this actually implies that (Du_n) is bounded in $L^\infty([0,T] \times G; L^p(\Omega; H))$. In fact, setting

$$\phi_n(s) := \mathbb{E}\|Du_n(s)\|^p_{L^\infty(H)}, \qquad \phi_0 := 1 + \sup_{n\in\mathbb{N}} \mathbb{E}\|u_n\|^p_{C([0,T];C(\overline{G}))} < \infty,$$

we have already shown that

$$\phi_{n+1}(t) \lesssim_T \phi_0 + \int_0^t (t-s)^{-2\alpha} \phi_n(s)\, ds \qquad \forall t \in [0,T],$$

and that $(\phi_n^*(T_0))_n$ is bounded. We now proceed by induction: assuming that $(\phi_n^*(jT_0))_n$ is bounded, let us show that $(\phi_n^*((j+1)T_0))_n$ is also bounded. Let $jT_0 < t \leq (j+1)T_0$. We have

$$\begin{aligned}\phi_{n+1}(t) &\lesssim_T \phi_0 + \int_0^t (t-s)^{-2\alpha} \phi_n(s)\, ds \\ &= \phi_0 + \int_0^{jT_0} (t-s)^{-2\alpha} \phi_n(s)\, ds + \int_{jT_0}^t (t-s)^{-2\alpha} \phi_n(s)\, ds,\end{aligned}$$

where $t > jT_0$ implies $t - s > jT_0 - s$ and $(t-s)^{-2\alpha} < (jT_0 - s)^{-2\alpha}$, hence

$$\int_0^{jT_0} (t-s)^{-2\alpha} \phi_n(s)\, ds < \int_0^{jT_0} (jT_0 - s)^{-2\alpha} \phi_n(s)\, ds \leq \frac{(jT_0)^{1-2\alpha}}{1-2\alpha} \phi_n^*(T_0),$$

so that

$$\begin{aligned}\phi_{n+1}(t) &\lesssim_T \phi_0 + \frac{(jT_0)^{1-2\alpha}}{1-2\alpha} \phi_n^*(jT_0) + \int_{jT_0}^t (t-s)^{-2\alpha} \phi_n(s)\, ds \\ &\lesssim_T \phi_0 + \frac{(jT_0)^{1-2\alpha}}{1-2\alpha} \phi_n^*(jT_0) + \phi_n^*((j+1)T_0) \int_{jT_0}^t (t-s)^{-2\alpha}\, ds,\end{aligned}$$

where

$$\int_{jT_0}^t (t-s)^{-2\alpha}\, ds = \int_0^{t-jT_0} s^{-2\alpha}\, ds \leq \int_0^{T_0} s^{-2\alpha}\, ds = \frac{T_0^{1-2\alpha}}{1-2\alpha}.$$

This in turn implies, taking the supremum over $[0,(j+1)T_0]$,

$$\phi_{n+1}^*((j+1)T_0) \lesssim_T \phi_0 + \frac{(jT_0)^{1-2\alpha}}{1-2\alpha} \phi_n^*(jT_0) + \frac{T_0^{1-2\alpha}}{1-2\alpha} \phi_n^*((j+1)T_0).$$

Since $\phi_n^*(jT_0)$ is bounded uniformly with respect to n by the inductive assumption, we deduce that $\phi_n^*((j+1)T_0)$ is bounded uniformly over n as well, thus completing the inductive argument. This implies, by a standard argument based on the closure of the Malliavin derivative, that $u \in L^\infty([0,T] \times G; \mathbb{D}^{1,p})$.

Finally, the equation for Du follows immediately by differentiating equation (4), upon applying the chain rule for theMalliavin derivative (see, e.g., [14, Proposition 1.2.4]). □

3.2 Non-Degeneracy of the Malliavin Derivative

This section is devoted to study, for any fixed $(t,x) \in]0,T] \times G$, the norm of the Malliavin derivative of $u(t,x)$. Together with the results of the previous section, we will deduce the existence of the density for the law of the random variable $u(t,x)$. Recall that throughout the section we are assuming that f and σ are globally Lipschitz continuous functions.

We will need an estimate for the norm of $Du(t,x)$ in

$$H(a,b) := L^2(a,b;L^2_Q), \qquad 0 \le a < b \le T.$$

Proposition 3.4 *Let $0 \le a < b \le T$, $p > 2$, and $\eta \in]d/(2q), 1/2 - 1/p[$. There exists a positive constant N, independent of a and b, such that*

$$\sup_{(t,x)\in[a,b]\times G} \mathbb{E}\|Du(t,x)\|^p_{H(a,b)} \le N(b-a)^{p(1/2-\eta)}.$$

Proof Repeating the proof of Theorem 3.1 with H replaced by $H(a,b)$, we get

$$\sup_{(t,x)\in[0,T]\times G} \mathbb{E}\|Du(t,x)\|^p_{H(a,b)} \le N \sup_{(t,x)\in[0,T]\times G} \mathbb{E}\|v_0(t,x)\|^p_{H(a,b)},$$

and, by Lemma 3.2,

$$\sup_{x\in G} \|v_0(t,x)\|^2_{H(a,b)} \lesssim \int_a^{t\wedge b} \|S(t-s)\sigma(u(s))B\|^2_{\gamma(L^2,E_\eta)}\,ds,$$

where

$$\|S(t-s)\sigma(u(s))B\|_{\gamma(L^2,E_\eta)} \lesssim (t-s)^{-\eta}\big(1 + \|u\|_{C([0,T\times\overline{G})}\big)\|B\|_{\gamma(L^2,L^q)}.$$

Therefore

$$\begin{aligned}\sup_{(t,x)\in[0,T]\times G} \mathbb{E}\|Du(t,x)\|^p_{H(a,b)} &\lesssim \big(1 + \|u\|^p_{L^p(\Omega;C([0,T\times\overline{G}))}\big) \sup_{t\le b}\left(\int_a^t (t-s)^{-2\eta}\,ds\right)^{p/2}\\ &\lesssim (b-a)^{(1-2\eta)p/2}.\end{aligned}$$

□

In the next result we establish sufficient conditions under which the norm of the Malliavin derivative of $u(t,x)$ does not vanish, almost surely.

Proposition 3.5 *Assume that there exists a constant $c > 0$ such that $|\sigma(z)| \ge c$ for all $z \in \mathbb{R}$ and that Q is positivity preserving. Let $(t,x) \in]0,T] \times G$, $\alpha \in]0,1/2[$, and $\eta \in]d/(2q), \alpha - 1/p[$. If there exist $\beta \in]0, 1-\alpha-\eta]$ such that*

$$\lim_{\delta\to 0} \frac{\delta^\beta}{\|K(x,\cdot)\|_{H(0,\delta)}} = 0, \tag{8}$$

then $\|Du(t,x)\|_H > 0$ almost surely.

Proof We are going to estimate $\mathbb{P}(\|Du(t,x)\|_H \leq 1/n)$ for $n \in \mathbb{N}$ and pass to the limit as $n \to \infty$. Let $\delta \in]0,1[$, and set, for compactness of notation, $H_\delta := H(t-\delta, t)$. The obvious inequality $\|a+b\| \geq \|a\| - \|b\|$ applied to the expression of Du given by Theorem 3.1 yields

$$\|Du(t,x)\|_H \geq \|v_0(t,x)\|_{H_\delta} - \|S * (FDu)(t,x) + S \diamond (\Sigma DuB)(t,x)\|_{H_\delta}.$$

Hence, simplifying the notation a bit and denoting the second term within the norm on the right-hand side by Y,

$$\mathbb{P}\big(\|Du(t,x)\|_H \leq 1/n\big) \leq \mathbb{P}\big(\|v_0(t,x)\|_{H_\delta} - \|Y\|_{H_\delta} \leq 1/n\big) = \mathbb{P}\big(\|Y\|_{H_\delta} \geq \|v_0(t,x)\|_{H_\delta} - 1/n\big).$$

Since Q as well as the semigroup S is positivity preserving, hence K is positive, and $\sigma : \mathbb{R} \to \mathbb{R}$ is continuous, we have

$$\begin{aligned}
\|v_0(t,x)\|^2_{H_\delta} &= \int_{t-\delta}^t \|K_{t-s}(x,\cdot)\sigma(u(s,\cdot))\|^2_{L^2_Q}\,ds \\
&= \int_{t-\delta}^t \int_G K_{t-s}(x,y)\sigma(u(s,y))\, Q[K_{t-s}(x,\cdot)\sigma(u(s,\cdot))](y)\,dy\,ds \\
&= \int_{t-\delta}^t \int_G K_{t-s}(x,y)|\sigma(u(s,y))|\, Q[K_{t-s}(x,\cdot)|\sigma(u(s,\cdot))|](y)\,dy\,ds \\
&\geq c^2 \int_{t-\delta}^t \int_G K_{t-s}(x,y) Q[K_{t-s}(x,\cdot)](y)\,dy\,ds \\
&= c^2 \int_0^\delta \|K_s(x,\cdot)\|^2_{L^2_Q}\,ds = c^2 \|K_\cdot(x,\cdot)\|^2_{H(0,\delta)}.
\end{aligned}$$

This implies that we can use Chebyshev's inequality to write, for n sufficiently large,

$$\begin{aligned}
\mathbb{P}\big(\|Du(t,x)\|_H \leq 1/n\big) &\leq \mathbb{P}\big(\|Y\|_{H_\delta} \geq c\|K_\cdot(x,\cdot)\|_{H(0,\delta)} - 1/n\big) \\
&\leq \frac{\mathbb{E}\|Y\|^p_{H_\delta}}{\big(c\|K_\cdot(x,\cdot)\|_{H(0,\delta)} - 1/n\big)^p},
\end{aligned}$$

where, thanks to Theorem 3.1 and Proposition 3.4,

$$\begin{aligned}
\mathbb{E}\|Y\|^p_{H_\delta} &= \mathbb{E}\|S * (FDu)(t,x) + S \diamond (\Sigma DuB)(t,x)\|^p_{H_\delta} \\
&\lesssim \delta^{p(1/2-\alpha)} \|Du\|^p_{L^\infty([0,T]\times G; L^p(\Omega; H_\delta))} \\
&\lesssim \delta^{p(1-\alpha-\eta)}.
\end{aligned}$$

Taking the limit as $n \to \infty$, we are left with

$$\mathbb{P}\big(\|Du(t,x)\|_H = 0\big) \lesssim \left(\frac{\delta^{1-\alpha-\eta}}{\|K_\cdot(x,\cdot)\|_{H(0,\delta)}}\right)^p$$

Since this inequality holds for every $\delta \in]0,1[$, and the limit of the right-hand side as $\delta \to 0$ is zero by assumption, it follows that $\mathbb{P}\big(\|Du(t,x)\|_H = 0\big) = 0$. □

As an immediate consequence of the above result and of Theorem 3.1 we obtain sufficient conditions for the pointwise absolute continuity of the law of the mild solution to (1), thanks to well-known criteria of the Malliavin calculus (see, e.g., [14, Theorem 2.1.3]).

Theorem 3.6 *Let $u \in L^p(\Omega; C([0,T]; C(\overline{G})))$ be the unique mild solution to equation (1), with f and σ Lipschitz continuous and $u_0 \in C(\overline{G})$. Assume that there exists $c > 0$ such that $|\sigma(z)| \geq c > 0$ for all $z \in \mathbb{R}$ and $Q = BB^*$ is positivity preserving. Let $(t,x) \in]0,T] \times G$,*

$\alpha \in]0, 1/2[$, and $\eta \in]d/(2q), \alpha - 1/p[$. If there exist $\beta \in]0, 1-\alpha-\eta]$ such that (8) is fulfilled, then the law of the random variable $u(t,x)$ is absolutely continuous with respect to Lebesgue measure.

Example 3.7 Assume that A has compact resolvent in L^2. Since A is accretive and self-adjoint, there exist an orthonormal basis $(e^k)_{k\in\mathbb{N}}$ of L^2 and a sequence $(\lambda_k)_{k\in\mathbb{N}} \geq 0$ such that $e^k \in \mathsf{D}(A)$, $Ae^k = \lambda_k e^k$ and $\lim_{k\to\infty} \lambda_k = +\infty$. Moreover, let $B = (I+A)^{-m}$, with $m \in \mathbb{N}$, and fix $(t,x) \in]0,T] \times G$. Since $Q = (I+A)^{-2m}$, one has, for any $\delta \in]0,1[$,

$$\begin{aligned}
\|K_{\cdot}(x,\cdot)\|^2_{H(0,\delta)} &= \int_0^\delta \int_G K_s(x,y)[QK_s(x,\cdot)](y)\,dy\,ds \\
&= \int_0^\delta \sum_{k\geq 0} (1+\lambda_k)^{-2m} \langle K_s(x,\cdot), e^k\rangle^2_{L^2}\,ds \\
&= \int_0^\delta \sum_{k\geq 1} (1+\lambda_k)^{-2m}\, e^{-2s\lambda_k} |e^k(x)|^2\,ds \\
&= \frac{1}{2}\sum_{k\geq 1} (1+\lambda_k)^{-2m}\, \lambda_k^{-1}\, (1-e^{-2\delta\lambda_k})|e^k(x)|^2.
\end{aligned}$$

Moreover, we have that

$$1 - e^{-2\delta\lambda_k} \geq \frac{2\delta\lambda_k}{1+2\delta\lambda_k} \geq \frac{2\delta\lambda_k}{1+2\lambda_k}.$$

Hence

$$\|K_{\cdot}(x,\cdot)\|^2_{H(0,\delta)} \geq \delta \sum_{k\geq 1} (1+\lambda_k)^{-2m}\,(1+2\lambda_k)^{-1}\,|e^k(x)|^2.$$

Assuming that $x \in G$ is such that there exists $k \in \mathbb{N}$ for which $e^k(x) \neq 0$, the quantity

$$C_x := \sum_{k\geq 1} (1+\lambda_k)^{-2m}\,(1+2\lambda_k)^{-1}\,|e^k(x)|^2$$

is strictly positive. Therefore we have $\|K_{\cdot}(x,\cdot)\|^2_{H(0,\delta)} \geq C_x\delta$, i.e.

$$\frac{\delta^{1/2}}{\|K_{\cdot}(x,\cdot)\|_{H(0,\delta)}} \leq C_x^{-1/2},$$

which implies that condition (8), hence also the assumptions of Theorem 3.6, are satisfied if we can find α and η such that $1-\alpha-\eta > 1/2$. This is possible if m is sufficiently large, so that $B \in \gamma(L^2, L^q)$ with q large and $d/(2q)$ is smaller than, say, $1/4$.

4 Reaction-Diffusion Equations

Let us now consider equation (1) in the general case, i.e. assuming that $f : \mathbb{R} \to \mathbb{R}$ is an odd polynomial with negative leading coefficient. As already observed, we could also assume that $x \mapsto f(x) - \lambda x$ is decreasing for some $\lambda \geq 0$, locally Lipschitz continuous, and with polynomial growth.

Let $u_0 \in C(\overline{G})$, and $u \in L^p(\Omega; C([0, T]; C(\overline{G})))$ be the unique mild solution to equation (1), the existence of which is guaranteed by Proposition 2.5. For every $n \in \mathbb{N}$, consider the function $f_n : \mathbb{R} \to \mathbb{R}$ defined as

$$f_n(x) = \begin{cases} f(x), & |x| \leq n, \\ f(nx/|x|), & |x| > n. \end{cases}$$

Then f_n is Lipschitz continuous, and the equation

$$du_n(t) + Au_n(t)\,dt = f_n(u_n(t))\,dt + \sigma(u_n(t))B\,dW(t), \qquad u(0) = u_0,$$

admits a unique mild solution $u_n \in L^p(\Omega; C([0, T]; C(\overline{G})))$. Moreover, by construction of u (see [6]), u_n coincides with u on the stochastic interval $[\![0, T_n]\!]$, where the stopping time T_n is defined as

$$T_n := \inf\bigl\{t \geq 0 : \ \|u_n(t)\|_{C(\overline{G})} \geq n\bigr\} \wedge T,$$

and $\lim_{n\to\infty} T_n = T$ almost surely. In particular, $u_n \to u$ in $L^r(\Omega; C([0, T]; C(\overline{G})))$ for all $r \in [1, p[$. Let $t \in]0, T]$ be arbitrary but fixed and set, for every $n \in \mathbb{N}$,

$$\Omega_n := \bigl\{\omega \in \Omega : \ t \leq T_n(\omega)\bigr\}.$$

Since (T_n) is a sequence of stopping times monotonically increasing to T as $n \to \infty$, (Ω_n) is a sequence in $\mathscr{F}$ monotonically increasing to Ω as $n \to \infty$. Clearly $\{t\} \times \Omega_n \subset [\![0, T_n]\!]$, hence $u(t) = u_n(t)$ on Ω_n, as an identity in $C(\overline{G})$. This implies that $u(t, x) = u_n(t, x)$ on Ω_n for every $x \in G$. Moreover, as f_n is Lipschitz continuous, Theorem 3.1 implies that $u_n(t, x) \in \mathbb{D}^{1,p}$ for every $x \in G$, for all $p \geq 1$. We have thus shown that $u(t, x) \in \mathbb{D}^{1,p}_{\text{loc}}$, with localizing sequence $(\Omega_n, u_n(t, x))$ (cf. [1, §III] or [14, §1.3.5]). This implies that $u(t, x)$ is Malliavin differentiable, i.e. that there exists a random variable $Du(t, x)$, independent of the chosen localizing sequence, such that $Du(t, x) = Du_n(t, x)$ on Ω_n.

We are now in the position to state and prove the main result of the paper.

Theorem 4.1 *Let $u \in L^p(\Omega; C([0, T]; C(\overline{G})))$ be the unique mild solution to (1) with initial datum $u_0 \in C(\overline{G})$. Assume that $Q = BB^*$ is positivity preserving and that there exists $c > 0$ such that $|\sigma(z)| \geq c > 0$ for all $z \in \mathbb{R}$. Let $(t, x) \in]0, T] \times G$, $\alpha \in]0, 1/2[$, and $\eta \in]d/(2q), \alpha - 1/p[$. If there exist $\beta \in]0, 1 - \alpha - \eta]$ such that*

$$\lim_{\delta \to 0} \frac{\delta^\beta}{\|K(x, \cdot)\|_{H(0,\delta)}} = 0,$$

then the law of the random variable $u(t, x)$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$.

Proof Let $(t, x) \in G_T$ be arbitrary but fixed. Then, by the Bouleau-Hirsch criterion (see [1, Proposition 7.1.4]), it suffices to prove that $\|Du(t, x)\|_H > 0$ almost surely. Since f_n is Lipschitz continuous for all $n \in \mathbb{N}$, $\|Du(t, x)\|_H > 0$ on Ω_n for all $n \in \mathbb{N}$. This readily implies that $\|Du(t, x)\|_H > 0$ almost surely: assume by contradiction that there exists $\Omega' \subset \Omega$ with strictly positive probability such that $\|Du(t, x)\|_H = 0$ on Ω'. Since Ω_n increases monotonically to Ω, there exists $n_0 \in \mathbb{N}$ such that $\mathbb{P}\left(\Omega''\right) > 0$, where $\Omega'' := \Omega_{n_0} \cap \Omega'$. In particular, by definition of Ω_{n_0}, one has $\|Du(t, x)\|_H > 0$ on Ω'' because $\Omega'' \subset \Omega_{n_0}$. This is clearly a contradiction, because $\Omega'' \subset \Omega'$. The claim is thus proved. $\square$

Remark 4.2 Very minor adjustments allow to consider the case where $\sigma : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous with linear growth. In fact, the construction of a unique global solution is obtained again by *récollement* of local solutions (see [6]), and the above reasoning can be repeated almost verbatim.

Remark 4.3 The setting of Example 3.7 obviously satisfies the assumptions of Theorem 4.1.

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