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Frege, Dedekind, and the Modern Epistemology of Arithmetic

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Abstract In early analytic philosophy, one of the most central questions concerned the status of arithmetical objects. Frege argued against the popular conception that we arrive at natural numbers with a psychological process of abstraction. Instead, he wanted to show that arithmetical truths can be derived from the truths of logic, thus eliminating all psychological components. Meanwhile, Dedekind and Peano developed axiomatic systems of arithmetic. The differences between the logicist and axiomatic approaches turned out to be philosophical as well as mathematical. In this paper, I will argue that Dedekind's approach can be seen as a precursor to modern structuralism and as such, it enjoys many advantages over Frege's logicism. I also show that from a modern perspective, Frege's criticism of abstraction and psychologism is one-sided and fails against the psychological processes that modern research suggests to be at the heart of numerical cognition. The approach here is twofold. First, through historical analysis, I will try to build a clear image of what Frege's and Dedekind's views on arithmetic were. Then, I will consider those views from the perspective of modern philosophy of mathematics, and in particular, the empirical study of arithmetical cognition. I aim to show that there is nothing to suggest that the axiomatic Dedekind approach could not provide a perfectly adequate basis for philosophy of arithmetic.

1 Introduction

Frege's contribution to mathematics is best known for the logicist ideal of deriving arithmetic from the laws of logic. This established a paradigm in the study of foundations of mathematics that has retained considerable popularity to this day. During the same period, however, Dedekind and Peano were developing an equally influential paradigm for arithmetic, in which numbers were taken as something fixed by a direct axiomatization. In the Dedekind-Peano approach, numbers do not have an intrinsic

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character: their properties are given exhaustively by the axioms. Consequently, any structure that fulfils those axioms can be thought of as a structure of natural numbers. For Frege, this was not acceptable. In the logicist approach, numbers had to be somehow distinguished from other objects that could form a structure satisfying the axioms of arithmetic.[1]

In this paper, I focus on the differences of the two approaches with a twofold philosophical strategy. First, I will study the original writings of Frege and Dedekind and thus clarify the philosophical motivations behind their theories of arithmetic. This is not a straightforward matter since Frege and Dedekind are often seen to come from considerably different intellectual backgrounds. In *Die Grundlagen der Arithmetik*, where he presents the logicist program, Frege is commonly understood to write mainly in the role of a philosopher.[2] He formulates precise mathematical ideas, but the motivation for them seems to be largely philosophical in character. Dedekind, on the other hand, was primarily a mathematician. In his writings, philosophical issues always seem to have a minor role and the main focus is on developing the mathematical side. Nevertheless, there is important philosophical overlap in the subjects that the two address. Here, I want to clarify Dedekind's philosophical positions and compare them to Frege's. What we will find out is interesting: both had clear ideas that resonate strongly in modern philosophy of mathematics. Frege's logicist program, although failing in its original form, has been established in new forms in *neo-logicism* (or neo-Fregeanism). Meanwhile, Dedekind's ideas were precursors to *structuralism*, which is one of the most prominent theories in modern philosophy of mathematics.[3]

Second, aside from this historical perspective, I will give a systematic analysis of Frege's and Dedekind's approaches based on some new developments in the epistemology of mathematics. One of the main problems Frege saw in the contemporary philosophy of arithmetic was the idea that natural numbers come to us through a psychological process of *abstraction*, in which we abstract away qualities of collections of objects until we are left with the number of the objects. Some of Frege's criticism was very powerful and has remained so until modern times. Recently, however, there have emerged empirical data which suggest that abstraction may not be the conscious process that Frege criticizes. Rather, we could have a natural tendency to categorize observations in terms of quantities. If such a tendency can be conclusively established, it is likely to have important consequences in the epistemology of mathematics. In this paper, I argue that the Dedekind approach could be philosophically better suited for such results than the Frege one. If the laws of arithmetic can be established to conform to natural processes of observation, there is no need to justify them in terms of logical laws. At the very least, I will argue, Frege's criticism of psychologism loses much of its power against such naturalistic theories of abstraction.

[1] There is a danger of equivocation in the talk about late 19th century logicism, because Dedekind also described his position as deriving from logic. However, I believe that only Frege's brand of logicism corresponds to the way the term is usually understood in modern literature.

[2] This is the stand Kitcher (1992), among others, takes and it has been contested by, e.g., Wilson (2010) and Tappenden (1995). As will be seen, I believe that *Grundlagen* is fundamentally a mathematical work. Nevertheless, there seems to be little doubt that Frege generally comes from a more philosophical background than Dedekind.

[3] This has been contested by Corry (1996), but I believe that Sieg and Schlimm (2005) convincingly argue that Dedekind is indeed best understood as an early structuralist. This matter will be considered in more detail later on.

In Sect. 2, I briefly present the key mathematical and philosophical points of Frege's logicist programme, which I will then compare to Dedekind's approach in Sect. 3. In that section, I will also argue that Dedekind is best understood philosophically as a forefather of modern structuralism. In Sect. 4, I deal with the interpretations of Frege's and Dedekind's mathematical ideas, in particular some important problems in Frege's arguments against the Dedekind approach. Section 5 is about the epistemology of arithmetic, in particular the concepts of psychologism and abstraction that Frege heavily criticized. One particularly important target for Frege was Mill, and this criticism will also be examined. In addition, I will present the modern theory of psychological abstraction of quantities based on some recent empirical data. Finally, in Sect. 6, I will compare the approaches of Frege and Dedekind from a modern perspective, concluding that many of the problems that Frege saw with the Dedekind approach have proven to be less serious than originally thought.

One word about the sources is in place. In this paper, I have conformed to the established (e.g., Gillies 1982) methodology that when it comes to the philosophy of arithmetic, the important philosophical ideas of Frege and Dedekind can be found in *Grundlagen der Arithmetik* and *Was sind und was sollen die Zahlen?*, respectively. Their other works have also been used as sources when seen relevant. At times, I use the writings of other logicists, such as Russell, to elucidate the logicist perspective of Frege. Likewise, I use some arguments from modern structuralism to clarify the position of Dedekind. I have tried to be careful not to mix the original ideas with modern ones, but for the purpose of evaluating Frege and Dedekind from a modern perspective, I saw it necessary to include arguments also from their followers.

2 Frege's Logicism and the Natural Number

Let us begin by evaluating Frege's take on the contemporary philosophy of arithmetic. While there is no doubt that Frege made an important contribution to the foundations of arithmetic, as Tait (1997) points out, his work in mathematics was carried out in curious indifference to (or unawareness of) the important advances made around him. Whereas, mathematics developed in the late 19th century in huge leaps into a highly creative discipline, Frege—despite all his innovations in the field of logic—remained something of a conservative. One of his main concerns was the concept of number, yet he took little interest in such central developments as the analysis of the concept of infinite, the distinction between cardinal and ordinal numbers, and constructive and non-constructive mathematics.[4]

Even though his work on the subject was in this way limited, the concept of number was what Frege was most interested in his philosophy of mathematics. His main concern even in the early *Begriffsschrift* was deriving the laws of arithmetic from the laws of logic (Frege (1879, p. 5)):

[4] It should be noted that Frege (*Grundlagen* §85) does mention both Cantor's infinity and ordinal numbers and applauds (§86) his general work on the subject. But, this is the extent of the discussion of those ideas in *Grundlagen*.

> My initial step was to attempt to reduce the concept of ordering in a sequence to that of *logical* consequence, so as to proceed from there to the concept of number.[5]

The following *Die Grundlagen der Arithmetik* (1884) and the two-part *Die Grundgesetze der Arithmetik* (Frege 1893, 1903) were continuations of this project. *Grundlagen* is often seen as a philosophical sketch for the technical work of *Grundgesetze*, laying out the logicist aspirations and ideals which were to be filled out in the later work. While that is true to some degree, I concur with Boolos (1990) that the *Grundlagen* is fundamentally a mathematical work. It provides a definition of natural number based on an explicit definition of equicardinality. Such ideas are not mere sketches: they are at the very heart of Frege's system of arithmetic. That is the first reason I believe it is best to study Frege's philosophy of arithmetic with a focus on the *Grundlagen*, even though the crucial logicist program is left undeveloped in that book (it was to receive a full treatment in the *Grundgesetze*). The second reason is that in the *Grundlagen*, Frege takes part in many interesting philosophical discussions concerning arithmetic, including the central contemporary issues of abstraction and psychologism.

It is of course well known that Russell found a paradox in the system of *Grundgesetze*, and the whole logicist project of deriving mathematics from logic turned out to be deeply problematic. Nevertheless, the ideas Frege presented in the *Grundlagen* are not necessarily tied to the system of *Grundgesetze*. In fact, as we will see, they have retained considerable popularity and power even though the original logicist program ultimately failed.

Let us set the failure—as well as the many successes—of *Grundgesetze* aside for a while and focus on the mathematical content of *Grundlagen*. In §62, Frege argued that we should use the principle of *equicardinality* to define the concept of natural number. This law of equicardinality is often called *Hume's Principle*[6] in the literature and it means simply the notion that numerical equality (or identity) should be understood in terms of a one-to-one correlation between the numbers. In modern terms, this can be stated as follows:

> The number of *F*s is equal to the number of *G*s if and only if there exists a bijection (one-to-one correspondence) between *F*s and *G*s.

With this definition of identity, Frege could give his famous definition of number (§68):

> the Number which belongs to the concept *F* is the extension of the concept "equal to the concept *F*."

This definition is one of the most celebrated in the philosophy of mathematics. Dummett (1991, 111), for example, calls the §62 of *Grundlagen,* where Hume's principle is stated as the basis of natural number, "arguably the most pregnant philosophical paragraph ever written."

[5] As in all quotations in this paper, the emphasis is in the original text.
[6] Coined by Boolos (1998), although Frege already makes it explicit in *Grundlagen* that the principle comes from Hume.

But as Frege pointed out (§63), defining identity of numbers in terms of a one-to-one correlation was in fact nothing out of the ordinary. In addition to Hume, contemporary mathematicians such as Schröder (1873) and in particular Cantor (1878) held the same view.[7] In this way, Dummett's hyperbole is a bit difficult to understand. Indeed, it could be said that much of the important work of Frege concerned *proving* Hume's principle. In order to do this (in the *Grundgesetze*), Frege introduced his infamous "Basic Law V," which—in one special form—states that the extensions of two concepts are equal if and only if the exact same objects fall under the two concepts. But, as what is now well-known, if we allow unrestricted comprehension—i.e., if in the system for every predicate there is a corresponding set—we can derive a contradiction in Frege's system. This way, Frege's logicist project failed, as he did not manage to derive the laws of arithmetic from a consistent system of logic.

But, let us put that aside for still a while and return to Frege's definition of number. So far, we have seen his general definition of what a number is, based on his definition of identity, i.e., Hume's principle. That, however, completes only half of the task at hand. We now have a characterization of what numbers in general are, but we also need a definition for individual numbers. After all, there must be something that distinguishes, say, the number three from the number four. In doing this, Frege follows the well-established (by, e.g., Dedekind 1887 and Cantor 1878) method of defining natural number recursively in terms of a successor relation. Starting by defining the number 0 as the "Number which belongs to the concept 'not identical with itself'" (§74), Frege moves to defining 1 as the "Number which belongs to the concept 'identical with 0'" (§77) and to the general definition of successor of a number *m* as the number *n* which follows *m* directly after *m* (§78).

Frege uses his construction "identical with" to move from 0 to 1, but essentially the same construction can be carried out with the help of sets, as Von Neumann (1923) and Zermelo (1908) later did. If we define 0 as the empty set Ø, we define 1 as the set formed of the empty set, that is, {Ø}. The number 2, in turn, can be defined as the set {{Ø}} and so on.[8] The operation of forming a set guarantees us that 1 follows directly after 0 and 2 directly after 1. Frege does something similar: since only one number, the number 0, is identical with 0, the number which belongs to the concept "identical with 0" must be 1. For Frege, the rest of the task, i.e., showing that for each *m* its successor *n* is a number, is more complicated than in the set-theoretic construction, but the general idea is similar: also, Frege defines his concept of successor in terms of how the number 1 is defined with the number 0.

In hindsight, we know that Frege's construction cannot be carried out from the laws of logic, but the situation is not as bleak as it appeared to Frege when he at the last second added the famous note to *Grundgesetze*, acknowledging his failure to deal with Russell's paradox in the book. Frege's logicist program can be revived in a new incarnation, taking Hume's principle as an *axiom* instead of trying to derive it as a theorem of logic. With Hume's principle, we have in second-order logic (which was Frege's logic in both *Grundlagen* and *Grundgesetze*) an interpretation of the usual Dedekind-Peano axioms of arithmetic. In fact, this is what Frege informally does in *Grundlagen*, although it was only proven in 1984 by Burgess. This so-called *Frege's Theorem* (coined in Boolos 1990) is an important result. Frege himself used an

[7] For Cantor, this of course led to famous results concerning the cardinalities of infinite sets.
[8] Here, Von Neumann and Zermelo differ, as will be seen. The approach given here is Zermelo's.

inconsistent system to derive Hume's principle, but he did (informally) show that we can derive arithmetic from a single consistent principle, and one which looks quite trivial.[9] It is nowadays often accepted that this is as close to the original ideal of logicism as we can get. The so-called neo-logicists (or neo-Fregeans) Hale and Wright (2001) have revived Frege's program in this second-order quasi-logicist incarnation in one of the most interesting developments in modern philosophy of mathematics. It is not what Frege wanted, however, and taking Hume's principle as a non-logical axiom makes the supposed logicist nature of neo-logicism questionable.[10]

3 Dedekind and Structuralism

Above, I have presented a brief description of what Frege's logicist program was about. It was ultimately a failure, but deriving mathematical truths as far as possible from the laws of logic has remained an important paradigm in the foundational study of mathematics. However, another tradition of formulating arithmetic emerged simultaneously with Frege's, one which also enjoys wide popularity both among mathematicians and philosophers. This approach was not foundational in the sense of Frege. Rather, it took the Euclidean form of fixing a system with intuitive axioms concerning natural numbers. It was influenced by the logic of Boole and Schröder, and although its most famous work is the Peano (1889) axiomatization, it is most comprehensively personified in Dedekind.[11]

The origins of the differences between the Frege and Dedekind approaches can be found deeper—we will return to them soon—but for the modern reader, the most important result of Dedekind (1887, §§71–73, §134) is that from his axiomatization, it follows that all "simply infinite systems" (nowadays we say *countably* infinite) are equivalent, up to isomorphism, with the system of natural numbers N. This in itself is a general theorem (of second-order arithmetic) and as such open to various philosophical interpretations. But, the crucial point here is that for Dedekind, the matter ends there. In his system, numbers do not have any other content than their position in the simply infinite system. This was radically different from Frege's approach which, while agreeing on how the natural numbers are formed with the successor function, was concerned with what numbers fundamentally *are*. Russell encapsulated the logicist opposition to Dedekind's approach well in his *Principles of Mathematics*:

> It is impossible that the ordinals should be, as Dedekind suggests, nothing but the terms of such relations as constitute a progression. If they are anything at all, they

[9] See Burgess (1984) or Boolos (1990).

[10] Not to mention the potentially problematic aspects of employing second-order logic, which does not enjoy the conceptual simplicity (e.g., completeness) first-order logic does. Although in Frege's time this would not have been considered problematic, in the modern discussion, the difference between first- and second-order logic is a crucial issue.

[11] As is the case with many interpretations of Dedekind, this is not universally accepted. Ferreirós (1999), for example, does not see Dedekind as a proponent of axiomatization. But, again, the research of Sieg and Schlimm (2005) seems to convincingly establish that Dedekind is indeed a key figure in the axiomatic tradition.

must be intrinsically something: they must differ from other entities as points from instants, or colours from sounds. (Russell 1903, p. 249)

Russell writes about ordinals (first, second, third…), but the same point applies to cardinals (one, two, three…) in Dedekind's account. If we take numbers to be merely terms in a progression (places in a structure, in modern parlance), then up to the least infinite ordinal ω, the cardinals are isomorphic to the ordinals.[12] But, the important thing is that numbers according to Russell must have some characteristic distinguishing them from other entities that form an isomorphic progression. No doubt we can attribute Russell's position also to Frege, and the view is intuitively an understandable one: after all, if we put a thousand apples in a row, we have a progression of ordinals isomorphic to the well-ordering of numbers {1, 2, 3…1000}. But, as Frege makes clear, we would hardly be ready to say that apples *are* numbers.

Naturally, Dedekind was aware of such possible counter-arguments, as it was a key issue in the philosophy of arithmetic at the time. The concept of *abstraction* was a very popular way of defining numbers in the 1880s, proposed in addition to Dedekind by such notable mathematicians as Cantor (1878) and Schröder (1873), as well as philosophers such as Husserl (1891). In essence, the idea was that when we take a collection of objects (like apples) and abstract away everything we possibly can, we arrive at the number of the objects. Thus, a plate of five apples, when we get rid of color, shape, etc. will give us the cardinality of the set, that is, the natural number five. This was the cause of much debate, and in the *Grundlagen* (§§21–24), Frege vehemently objected to such "psychologist" conceptions of number as bringing subjectivism to mathematics. It is not my purpose here to enter that general debate, even though it was certainly an interesting one. Instead, I want to focus on the Dedekind-Peano solution to it, which—depending on the point of view—either completely begs the question or provides an ingenious solution.

For Dedekind and Peano, the simple philosophical solution was that when we axiomatize natural numbers, *whatever* that fulfils the axioms is a system of natural numbers.[13] In

[12] For finite numbers, the isomorphism is trivial and the least infinite ordinal ω is identified with the smallest infinite cardinal N_o. For larger transfinite ordinals, the matter becomes trickier, as between the cardinal of countable infinity N_o and the cardinal of the smallest uncountable infinity N_1, there are (according to the continuum hypothesis) no infinite cardinals, yet there are uncountably many infinite *ordinals*.

[13] The Dedekind-Peano axioms can be presented in various ways, but in Peano's (1889) original work, the content of the nine axioms were as follows:

1. 1 is a number.
2. For all numbers a, $a = a$.
3. For all numbers a and b, $a = b$ if and only if $b = a$.
4. For all numbers a, b, and c: if $a = b$ and $b = c$, then $a = c$.
5. For all numbers a and b: if $a = b$, then b is a number.
6. For all numbers a, $a + 1$ is a number.
7. For all numbers a and b, $a = b$ if and only if $a + 1 = b + 1$.
8. For all numbers a, $a + 1 = 1$ is false.
9. If a set K is such that 0 belongs to K and for every number n: if n belongs to K, then $n + 1$ belongs to K, then every number belongs to K.

It should be noted that '$a + 1$' does not mean addition, but the *successor* of 1, usually notated as $S(a)$. The last axiom is the axiom of induction, and as we notice, in Peano's original axiomatization it is a second-order sentence. It can also be presented as a first-order axiom schema.

Dedekind's (1887) terms, all systems that are isomorphic with the simply infinite system are systems of natural numbers.[14] In modern parlance, Dedekind and Peano would be dubbed as *structuralists* over arithmetic: according to them, natural numbers are merely places in the structure determined by the axioms. As we have seen, Frege and Russell strongly disagreed with that position. For them, numbers were objects, and as such, there had to be something characteristic to numbers. That something was of course that numbers can be derived from logic, i.e., the laws of numbers must be in some way dependent on the laws of *thought*. But, differ as they did with Dedekind on the nature of numbers, Frege and Russell did not disagree with him about the *properties* of numbers. This tradition has endured largely unchanged to modern times. As Tait (1996, p. 239) puts it, in the post-Fregean philosophy, the question is not so much "What are the numbers?" as "What *besides* numbers are the numbers?"

Modern structuralists such as Resnik (1981) and Shapiro (1997) follow the Dedekind approach in answering that numbers are nothing besides their places in the natural number structure. Whether or not one is ultimately ready to accept that, there is a lot to like about their position. Frege's and Russell's goal was to build arithmetic on logic, but another popular project has been deriving natural numbers in set theory.[15] Defining natural numbers in terms of sets, however, can be done in various ways. In Von Neumann's (1923) account, identifying the empty set with the number zero, we get number one by forming the set out of it {Ø}, the number two by forming the set out of zero and one, that is, {Ø, {Ø}}, and so on. In Zermelo's (1908) account, we have similar constructions for the numbers zero and one, but two is the set {{Ø}}, etc. The two accounts are arithmetically equivalent, that is, they give us the same properties for natural numbers. However, set-theoretically, they are not equivalent, since in Von Neumann's approach, it holds that $1 = \{\emptyset\} \in \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} = 3$ while in Zermelo's approach, we have $1 = \{\emptyset\} \notin \{\{\{\emptyset\}\}\} = 3$. It would seem like a perfectly valid set-theoretical question to ask whether $1 \in 3$, yet arithmetically equivalent set-theoretical constructions give different answers. Moreover, it is not easy to see why one answer should be preferred over the other. This problem is of course not restricted to the set-theoretical approach: if we build set theory on logic, we face the same questions. For the structuralist, however, such a problem does not exist. For her, the numbers 1 and 3 are defined solely by their position in the structure of natural numbers, and in arithmetic, it does not make sense to ask whether $1 \in 3$. If numbers are not considered to be anything out of the context of their structure, many such problems vanish.

Above, I have described Dedekind as a kind of proto-structuralist, but this needs some clarification. At this point, we should have a brief historical interlude and examine the question how the views of Dedekind and Peano should be understood in terms of the modern discussion on structuralism. The standard image is that Dedekind

[14] For Dedekind, one important problem was the existence of infinite systems in general. His "proof" for this in *Was sind und was sollen die Zahlen?* is infamous in its reliance on the infinity of the number of thoughts. However, in modern philosophy of mathematics, such proofs of actual infinity are often not considered crucial for the position, as seen in, e.g., Hellman (1989). There are other ways of distinguishing the simply infinite system from other types of systems, ranging from the philosophical considerations of *potential* infinity to the Zermelo (1908) approach of taking the existence of infinite sets (or systems) as an axiom. This latter approach is, however, no doubt the kind of thing Russell (1919, p. 71) criticized as having the advantage of "theft over honest toil"—a criticism which was, incidentally, targeted against Dedekind's construction of real numbers.

[15] It should be noted that Dedekind's influence was crucial in this development. Indeed, we will see that Dedekind can also be interpreted as a proponent of the set theoretical approach.

developed the axiomatization of arithmetic, which Peano then took as the basis of his presentation.[16] In this view, the differences between the two approaches are not considered to be fundamental. Gillies (1982), however, holds the view that only Peano can be counted to have the above structuralist point of view, and Dedekind is actually better understood as a proponent of a set theoretic view of numbers. It is indeed clear that Dedekind (1887) defines numbers in terms of sets while Peano defines them by stating their properties as axioms, so there is an important methodological difference between them. Gillies (pp. 68–69) holds this to be a central difference between the two and rejects the standard image that Peano's and Dedekind's approaches are comparable. Gillies argues against this on the basis that Dedekind's approach leads to axiomatic set theory while Peano's approach leads to formal arithmetic in the sense of Hilbert. Whether we accept that or not, it was not what Peano (1891) thought at the time; for him, the important thing was that the two approaches agree in their characterization of the number.

These points are certainly not moot, but neither are they central when we consider the philosophical question of what a number is. After presenting his set theory, Dedekind (1887, §73) defines numbers as follows.

> If in the consideration of a simply infinite system *N* set in order by a transformation φ [*the successor function - Author*] we entirely neglect the special character of the elements; simply retaining their distinguishability and taking into account only the relations to one another in which they are placed by the order-setting transformation φ, then are these elements called *natural numbers* or *ordinal numbers* or simply *numbers*, and the base-element 1 is called the base-number of the number-series N. With reference to this freeing the elements from every other content (abstraction) we are justified in calling numbers a free creation of the human mind. The relations or laws which are derived entirely from the conditions $\alpha, \beta, \gamma, \delta$ [*Dedekind's versions of the Peano axioms - Author*] and therefore are always the same in all ordered simply infinite systems, whatever names may happen to be given to the individual elements, form the first object of the *science of numbers* or *arithmetic.*

I consider this to be the most important matter in Dedekind's approach. Essentially, Dedekind is saying that *however* we end up with the ordered simply infinite system, the system is the object of arithmetic.[17] In other words, we can choose any of the simply infinite systems and say that this is the system of natural numbers.[18] As I see it, this is stating the fundamental idea of structuralism remarkably clearly. Gillies is correct in that Dedekind's influence had more influence in the set-theoretical development while Peano's approach found its home in more formalistic circles, in particular with Hilbert. But, this is not the philosophical issue that we should be concerned with. In *Was sind und was sollen die Zahlen?*, Dedekind showed one way of defining natural numbers in terms of sets,

[16] See, e.g., Wang (1957).

[17] Of course, he (§133) also proves the crucial result that any two simply infinite systems *are* isomorphic.

[18] Here, for the modern reader, it is perhaps easiest to understand Dedekind as stating that we can choose any of the *models* of arithmetic. Because Dedekind's arithmetic is second-order, there are no differences between the models. The existence of non-standard models in first-order arithmetic makes this approach problematic in first-order approaches.

but—crucially to the matter at hand—he also made it clear that any other way of defining them will do, as long as it gives us a simply infinite system satisfying the conditions of the successor function. This includes the Peano axiomatization, so it is not easy to see why the approaches of Dedekind and Peano should be considered philosophically different, even though they have the methodological differences that Gillies reveals. [19]

Now the question is: what advantage does Frege's approach have over Dedekind's? There is no doubt that Hume's principle is a very intuitive and useful explanation of the concept of number. But, why should we prefer it to taking a direct axiomatization of the Dedekind-Peano type as our theory of arithmetic? There are many technical considerations in the matter, ones that we cannot get into here. But, one of the most important philosophical (as well as mathematical) questions is obviously the logicist basis of Frege's program. While we have seen that logicism as Frege originally conceived it is doomed to fail, there are still merits in the logicist approach. Second-order logic added with Hume's principle gives us the Dedekind-Peano axioms, which—while not the sort of result Frege and Russell hoped for—can still be interpreted as something of a success for logicism in general.

This way, it is not easy to see any immediate philosophical advantage between Frege's and Dedekind's approaches. The ontological questions concerning natural numbers seem to be distinct from the question whether we use logic and Hume's principle in the manner of Frege and the neo-logicists, or a direct axiomatization following Dedekind and Peano. However, that does not mean that certain philosophical theories are not better suited for one of the approaches.

4 Frege, Dedekind, and the Philosophy of Arithmetic

One thing we obviously gain if we take Dedekind's structuralist approach to the natural numbers is conceptual simplicity, in the sense that questions about logical or set theoretical foundations of arithmetic can be avoided. This has been seen both in the development of arithmetic and the philosophy of arithmetic, where the standard approach now is to take the first-order Peano axiomatization of arithmetic as the template. Now, the big Fregean question is: what do we *lose* by taking natural numbers to be mere places in the natural number structure?

As we have seen, in the *Grundlagen* and the following work by Frege and Russell, there were two main lines of criticism of philosophical theories of arithmetic. First, we should not think of numbers only as places in a structure. Second, we should not think of numbers in a psychologist way as following from a process of abstraction. Curiously, for, e.g., Dummett (1991), both of these positions are personified in Dedekind.[20] There

[19] As well as Dedekind's structuralism, there is some debate as to what *kind* of structuralist we should understand Dedekind to be, as seen in Reck (2003). In Reck's analysis, we should take Dedekind's notion of "free creation" seriously and thus advocate an interpretation that Dedekind is a logical structuralist rather than an *ante rem* one. While Reck's arguments are in many parts persuasive, I still believe that Dedekind is best understood as not taking a stand on the metaphysical issue. Thus, I do not read Dedekind here as an *ante rem* structuralist who believes that there exist a universal platonic structure of numbers, but neither do I want to strictly deny that possibility.

[20] This part of Dummett's interpretation of Frege is rather controversial. Angelelli (1994), for example, criticizes Dummett's extreme view of Frege as an enemy of abstraction. Although Angelelli's criticism is justified, there is little doubt that Frege had a different—and more reluctant—attitude toward abstraction than Dedekind.

is little doubt that the structuralist position can be attributed to Dedekind, but with the psychologist position this is not at all clear. Dummett (1991, p. 296) writes

> For Dedekind, however, the process of creation involved the operation of psychological abstraction, which needed a non-abstract system from which to begin; so it was for him a necessity, for the foundation of the mathematical theory, that there be such systems. That is why he included in his foundation of arithmetic a proof of the existence of a simply infinite system, which had, of necessity, to be a non-mathematical one.

But, we should note Dedekind's (1890, p. 101) remark:

> Does [the arithmetical] system *exist* at all in the realm of our ideas? Without a logical proof of existence it would always remain doubtful whether the notion of such a system might not perhaps contain internal contradictions. Hence the need for such proofs.

This is clearly related to Hilbert's (1925) formalist view that consistency and completeness of a mathematical theory imply existence of the objects in the theory, although Dedekind goes to a different direction. For both of them, lack of contradictions and existence are tied together. But, this formalism goes very badly together with the psychologism Dummett attributes to Dedekind. If completeness and consistency are tied to mathematical existence, where do we need the "non-abstract" systems? In fact, we do not, since as we remember, Dedekind's (1887, §73) process of abstraction is distinctly different from psychological abstraction:

> With reference to this freeing the elements from every other content (abstraction) we are justified in calling numbers a free creation of the human mind. The relations or laws which are derived entirely from the conditions $\alpha, \beta, \gamma, \delta$ and therefore are always the same in all ordered simply infinite systems, whatever names may happen to be given to the individual elements, form the first object of the *science of numbers* or *arithmetic*.

Granted, this is describing a process of abstraction, but is it in any way similar to the psychological abstraction that Frege criticizes? As I understand it, psychological abstraction must be something happening in the mathematician's mind from observations into concepts. But surely, this cannot be what Dedekind is saying. In terms of the analysis above, what he is describing is the mathematical abstraction of what natural numbers *are*, i.e., his structuralist conception of arithmetic. Dedekind is talking about *creating* natural numbers, not deriving them psychologically from non-mathematical systems. We will return to this potentially problematic concept of free creation later, but it should be obvious that Dedekind is writing here as a mathematician, describing the process of axiomatizing a theory of mathematics. In the 1880s, this was not the established method it currently is, and it needed some clarification. Dedekind may not have been perfectly clear with his philosophical concepts, but the overall spirit of his project is much better understood as mathematical abstraction of the laws of numbers, not a psychological explanation of how we arrive at the concept of a natural number.

Since we must thus reject Dummett's attribution of psychologism to Dedekind, the primary quarrel Frege had with Dedekind's conception of arithmetic was with the structuralist idea that numbers are defined solely by their place in the progression of natural numbers. Frege saw two main difficulties with Dedekind's approach. First, for Frege, numbers must be distinguished by something other than their position in the progression of natural numbers. Frege's (*Grundlagen* §42) argument for this is simple. After first rejecting numbers as being spatial or temporal, Frege claims:

> [we can invoke] a more generalized concept of series, but this too fails of its object; for their positions in the series cannot be the basis on which we distinguish the objects, since they must already have been distinguished somehow or other, for us to have been able to arrange them in a series. Any such arrangement always presupposes relations between the objects, whether spatial or temporal or logical.

But here, Frege demands too much out of the concept of series.[21] If we take identical pebbles and put them in a row, can we not distinguish the pebbles solely by their position in the series? It seems strange to require some prior reasons for putting the pebbles in that particular order, and it seems even stranger to say that the order does not help us distinguish the pebbles from each other.[22] As Tait (1997, p. 23) observes, Frege seems to be confusing the notions of series and a linearly ordered set (also called a total order). We can have a series (n, n, n,…, n) where all members are identical, but in a linearly ordered set (0, 1, 2, 3,…, n), each number x, for which $x > y$, must be distinct from y. What Dedekind is doing is taking the *order* of the simply infinite series as what the natural numbers are. In the simply infinite series (n, n, n,…) we can by order alone stipulate that the first n corresponds to the natural number 0, the second n to the number 1, and so on: as long as the series is equipollent to the series of natural numbers (0, 1, 2,…), Dedekind's approach has no problems. All n can be the same *unit*, but by their different positions in the series, they are associated with different natural numbers.

Of course, the units can also be *distinct* from each other, thus answering another one of Frege's criticisms (§43):

> [W]e are, I imagine, fully entitled to speak of 45 million Germans without having first to have thought or put an average German 45 million times, which might be somewhat tedious.

But, we do not need to do anything like that, either. We can simply take the set of natural numbers from 1 to 45,000,000 and then check out whether the set of Germans is equipollent to that set. We can map the citizen Frege to 1 and the citizen Dedekind to 2, etc.: there is absolutely no need for them to be identical.

In his later writings, Frege retained this position. In his posthumous writings on the subject, this time targeted against Weierstrass (Frege (1914, p. 220)):

[21] Instead of a series, which refers to a sum in modern mathematics, we would now talk of a *sequence*. But, in this paper, I will use the term "series" in its meaning for Frege and Dedekind.
[22] This is meant to be as an analogy: of course, we are not putting the *same* pebble in the row many times, so there are other distinguishing characteristics in the pebbles.

> [According to Weierstrass] A numerical magnitude consists of several elements, and yet of only one unit, because each element is the unit. How is this to be imagined? Well, we take a railway wagon [..]. We posit this repeatedly and construct a goods train out of it. The goods train consists of several elements, namely goods wagons, but of only one unit.

Frege (p. 221) then points out the difficulty of this approach that we saw above and prepares for a counter-argument

> The layman will say 'But with a train the question of ordering comes in'. Not at all! We have only a single wagon which occurs repeatedly. In such a case there can be no talk of an ordering. Ordering comes in only when we have different things, not when we have a single thing which occurs repeatedly.

Frege's train analogy is rather unfortunate, since it seems intuitively obvious that talking about identical carriages can make sense only if they can be distinguished by their order on the train. In any case, the counter-argument is hard to follow. Of course, strictly speaking, he is not talking about identical carriages, but rather the same carriage occurring repeatedly. But, if it is the problematic analogy from physical objects to mathematical units that Frege's argument rests on, it is not particularly strong. It is hard to understand why ordering should be limited to different things. As we have seen, it seems quite natural to the modern reader that we can reach the linear ordering (1, 2, 3,…) from the series of units (*n, n, n,…*) by the simple process of taking each natural number as the ordinal denoting a place in the series.

5 Frege and the Epistemology of Arithmetic

In addition to the criticism of structuralism, in *Grundlagen*, Frege objected in particular to two understandings of the natural number. The first of these was the empiricism of Mill (1843) which Frege criticized for throwing away everything that is precious in mathematical knowledge:

> Often it is only after immense intellectual effort, which may have continued over centuries, that humanity at last succeeds in achieving knowledge of a concept in its pure form, in stripping off the irrelevant accretions which veil it from the eyes of the mind. What, then, are we to say of those who, instead of advancing this work where it is not yet completed, despise it, and betake themselves to the nursery, or bury themselves in the remotest conceivable periods of human evolution, there to discover, like John Stuart Mill, some gingerbread or pebble arithmetic. […] A procedure like this is surely the very reverse of rational, and as unmathematical, at any rate, as it could well be. (Frege 1884, p. vii)

While Mill was definitely not the simpleton Frege makes him out to be, his philosophy of arithmetic was admittedly something of a mess. The confusion is best seen in the problem Mill (1843, p. 170) saw with the law of identity $x = x$:

> How can we know that a forty-horse power is always equal to itself, unless we assume that all horses are of equal strength?

Since horses quite obviously are not of equal strength, Mill seems to argue that we cannot know that a forty-horse power is always equal to itself. But, this sounds like mere bad logic: however, much the strength of individual horses may differ, the strength of given forty horses is always the same as the strength of those same forty horses.[23] That the horse power may be a flawed concept (which it indeed would be, had it not been tied to the unit of the watt) because of the inequality of strengths of horses is a whole other question. It is not the job of arithmeticians to ensure that all people everywhere use numbers correctly for quantities. But, if it is even *possible* to do so, arithmetical knowledge would survive unscathed from Mill's challenge, since arithmetic seems to give us necessary knowledge of the world.

Frege was convinced that any attempt to found arithmetic on empirical origins is bound to take us away from the true essence of arithmetical truths—their inner nature—and move to the irrelevant details concerning the historical or personal discoveries of the truths.

However, what Frege saw as irrelevant seems irrelevant because of his prior philosophical conviction about the inner nature of mathematical propositions. In this, he seems to be conflating mathematics with the philosophy of mathematics. Clearly in mathematics, we do not want to bring in the empirical and psychological dimensions that may be involved. But, in order to dismiss those as irrelevant to philosophy, we already have had to reject the position that mathematical knowledge could be somehow based on an empirical foundation.

This is not so easy to accept nowadays. Mill's empirical account may have been crude, but in Kitcher (1983), we have seen a more sophisticated empirical approach to mathematics. His footsteps have been followed by Lakoff and Núñez (2000) who have provided the first more detailed empirical account of mathematics. Here, I cannot go into the merits or weaknesses of those projects, but even if they were deemed failures, the connection between the empirical and the mathematical is not something we can simply write off. There is a lot of promising empirical research concerning the way we handle basic mathematical (or proto-mathematical) concepts like numerosities. In the philosophy of mathematics, we should be open to such findings.[24]

We will return to this matter and especially its relevance to the Dedekind-Frege differences, but let us first try to build a better understanding of Frege's philosophy of arithmetic. The other target of Frege's criticism in *Grundlagen* was psychologism, which is naturally closely related to empiricism. In this, he particularly targets the thinking of Schröder (1873), to whom he attributes the view that we arrive at numbers by abstracting away every other property—say, color and shape—and thus only leaving the "frequency of units." This way, when we see a bag of oranges, we can by abstracting away everything that distinguishes the objects as oranges arrive at the *number* of the oranges. Frege (*Grundlagen* §§22-24) points out serious difficulties with this approach: often, there are various ways of "extracting" a number out of objects. If we have, say, a plateful of grapes, do we mean the number of individual

[23] Frege (*Grundlagen* §9) makes a similar point.
[24] For an overview of some of these results, see Dehaene (2011) and Dehaene and Brannon (2011).

grapes, the number of bunches of grapes, or perhaps something else? For Frege, a number cannot be a property of the objects because our choice of numbering is arbitrary in a way that the color of the grapes, for example, clearly is not.

However, as a criticism of the psychologist approach to arithmetic, Frege's analysis seems limited from a modern perspective. What he (§§21–28) focuses on are the difficulties of deriving a satisfactory mathematical account from the psychological conception of a number. Such difficulties, perhaps most importantly the problematic notion of inductive mathematical knowledge, are well-known and need no repeating here. But, a modern psychologist account of mathematics does not need to fall into such trappings. We can think of mathematical knowledge *emerging* psychologically from an empirical background and thus retaining its special mathematical character while having origins common to other modes of knowledge. I do not want to claim that such a satisfactory empirical account exists yet, but given the famous epistemological problems of Platonism—in particular Benacerraf's (1973) epistemological problem of physical subjects gaining knowledge of abstract non-physical objects—the line of thinking is not without its appeal.

Of course the idea here is not new, since psychologism has always been more concerned with the *discovery* of arithmetical facts. In modern philosophy of science, distinguishing between the contexts of discovery and *justification* is one of the cornerstones of argumentation. In the philosophy of mathematics, this distinction is fundamental. In the beginning of *Begriffsschrift* (1879), Frege presents the distinction clearly:

> we can inquire, on the one hand, how we have gradually arrived at a given proposition and, on the other, how it is finally to be most securely grounded. The first question may have to be answered differently for different persons; the second is more definite, and the answer to it is connected with the inner nature of the proposition considered.

In *Grundlagen* (§17), he restated the difference in terms of discovery and proof:

> we are concerned here not with the way in which [laws of number] are discovered but with the kind of ground on which their proof rests.

If we were ignorant of Frege's purpose, this would seem like a curiously cavalier attitude. The pebble arithmetic of Mill that Frege ridicules may be unfitting as ground for mathematical proofs, but it is certainly not irrelevant when considering the development of arithmetical thinking in an individual. Of course, it was the introduction of psychologism into the context of justification that Frege did not accept, but that cannot hide the limitations of his criticism of psychologism. Frege wanted to stay firmly in the realm of logic and the a priori, but this choice of framework cannot work as an *argument* against psychological concepts such as abstraction. For Frege's purposes in *Begriffschrift* and *Grundlagen*, abstraction may be irrelevant. But philosophically, the "inner nature" of arithmetical propositions may involve more than the logical structure.

Although the idea of separating the origins of laws of arithmetic from their justification has become to be associated with Frege, the idea was already present in Kantian philosophy. Indeed, when Mill was presenting his empiricist philosophy of mathematics, it was formulated as a criticism of the Kantian tradition, in particular, Whewell.

What Mill (1843, pp. 152–57) argued for was essentially the position that the contexts of discovery and justification cannot be separated in mathematics.[25] When it comes to geometry, Mill actually makes a convincing case. Ever since Plato, it has been common to think that the ideal circles described by Euclidean axioms are imperfectly replicated by the circles in the physical world. But, Mill turned this the other way around (p. 169):

> The proposition, 'A circle is a figure bounded by a line which has all its points equally distant from a point within it,' is called the definition of a circle; but the proposition from which so many consequences follow, and which is really a first principle in geometry, is, that figures answering to this description exist.

It hardly needs to be added that Mill in his empiricism takes the definition to answer (approximately) to the circles in the physical world, not some Platonic entity. When it comes to the context of discovery, it is safe to say that everybody agrees with Mill. It is hard to think how geometry could have developed without there being something in the physical world to answer roughly to the description of the circle. But, Mill goes further and holds that we must not forget the empirical aspect in the context of justification. This way, the theorems of geometry must always be empirically verified. They cannot be thought to be *necessary* truths.

In hindsight, it is possible to argue that Mill's case has actually become stronger, since non-Euclidean geometries and their applications in physics have indeed shown that that the choice of geometry for a theory of physics can be empirically justified. However, even if we were ready to accept that, it is important to resist the conclusion that since Euclidean geometrical truths were not necessary in the way the Kantians thought, the same must apply to the truths of *arithmetic.* We already knew (by, say, drawing lines on balloons) that there are different geometries, i.e., ones that satisfy a different set of axioms. But, it is hard to imagine potential theories of arithmetic that differ from our currently used axiomatizations in a similar manner.

Nevertheless, I claim that we can learn something from Mill's position also in the philosophy of arithmetic. Even if it were the case that arithmetical truths are necessary, we can think of them as at least partly empirically justified. Take a simple logical truth like "either there is a rhinoceros in this room or there is not." If we are inclined to believe in the correctness of two-valued logic, we are likely to think of this as a necessary truth. But, it is not absurd to say that our justification for believing the sentence to be necessarily true is in part empirical. Children quickly learn to observe the world in terms of congruent objects, which in most cases are clearly either perceived or not. Hence, the idea of excluded middle can be included in some form in children's mental processing much before they can understand its formal presentation. Similarly, it is undeniable that children learn facts about numbers empirically, by studying collections of physical objects. The truth of, say, $2+3=5$ is something that is initially reached by an empirical process such as counting pebbles, just like Mill argued.

Now the big question is: how can we deny that the context of justification does not depend at all on the context of discovery when it comes to arithmetic? To answer this, we must be clearer about what the two contexts consist of. Frege's criticism of psychologism focused on the concept of abstraction, but the form of abstraction he is concerned

[25] For more on the Mill-Whewell debate, see Gillies (1982, pp. 20–26).

with in the *Grundlagen* happens to be the one most prone to criticism. The account Frege criticizes can be called *conscious* abstraction, something in which we as active agents abstract away qualities until only the number remains. However, there is another type of process that can also be called abstraction in a sense relevant to Frege's criticism. In psychological and animal experiments, it has been detected time and again that infants and animals have a natural tendency to process observations in terms of numerosities.[26]

In the famous—and many times replicated—experiment of Wynn (1992), for example, it was found that infants react to unnatural numerosities in experiments. The infants first saw a doll and a screen put to cover it. Then, they saw another doll being put behind the screen. For half the test subjects, the other doll was removed before the screen was removed. The result was that the children spent longer looking at the unnatural setting corresponding to $1 + 1 = 1$.[27] In other versions of the experiment, the other variables have been removed, including initially removing a doll instead of adding one, and changing the size, shape, and location of the dolls behind the screen. The unnatural numerosities still remained as the single most surprising factor for the infants.[28]

Similar experiments have also been carried out with animals such as rats and small fish.[29] Rats can distinguish the quantity of tones from the total duration of them. Fish can recognize the numerosity of objects even when the total surface area illumination is the same. The experimental data we now have strongly support such a form of psychologism as the best explanation of at least the basic proto-arithmetical processes. Developed mathematical thinking and formal proofs are of course something vastly different, but in Frege's case, we are interested in those very basic processes behind our knowledge of quantities. When the empirical data points to primitive modes of proto-arithmetical thinking in mathematically undeveloped subjects, we can no longer dismiss the psychological origins as something philosophically irrelevant. If the disposition to process observations in terms of quantities can actually be more prominent than that of shapes, sizes, and locations, we are likely to be dealing with something fundamental to our capacity to observe the world.[30]

Of course, as a form of psychologism, the above account seems to be considerably different from the one that Frege objects to. First of all, it does not make sense to speak of conscious abstraction when we draw evidence from infants and animals. Rather, we should think of quantities as a natural, automatic, way of categorizing observations. Let us consider an experiment in which, instead of focusing on the length of a series of sounds, rats acted based on the number of sounds.[31] Does that mean the rat abstracted away the other properties—length, tone, pitch, etc.—and was left with the number of

[26] At the stage of proto-arithmetical cognition, it is better to speak of "numerosities," rather than natural numbers, in order to distinguish its primitive nature from developed arithmetical thinking.

[27] This is a standard method in the study of infants. To put it simply, infants get bored when they see something that they expect. When they see something surprising, they look at it longer.

[28] See Dehaene (2011), pp. 41–44. Although Wynn's result seems valid, we should be careful about postulating needlessly developed cognitive capacities for the subjects. Her paper, for example, was called "Addition and subtraction by human infants." But it seems quite problematic to assume that the infants are doing additions or subtractions in the process. It seems more likely that they are keeping one numerosity in their working memories. The infant and animal ability only applies to small quantities—usually no larger than four—and gets increasingly inaccurate as quantities become larger.

[29] See Butterworth (1999), Nieder (2011), and Agrillo et al (2009) for examples.

[30] See Pantsar (2014) for a more detailed account of the empirical data and its relevance to the epistemology of arithmetic.

[31] See Dehaene (2011), pp. 10–11.

sounds? This seems silly in just the way that Frege would ridicule. It is much more likely that it was simply the case that the rat had a natural way of categorizing the observations in terms of numerosities—one presumably developed for the evolutionary advantage of being familiar with the quantities of one's offspring and predators, locating one's nest (e.g., the third hole on the left) or other such reasons.

So, can we call such automatic categorizing "abstraction?" I believe, here, we must distinguish between two types of abstraction. First, we have the psychological processes responsible for giving us concepts such as numbers. Frege actually (*Grundlagen* § 48) defends this type of abstraction, but only *after* the concepts are formed. After realizing that it is concepts, not objects, that have number:

> We now see also why there is a temptation to suggest that we get the number by abstraction from the things. What we do actually get by such means is the concept, and in this we then discover the number. Thus abstraction does genuinely often precede the formation of a judgment of number.

So, abstraction is really something that happens, but the abstraction does not happen directly from things to number, but via concepts. To exclude this middle step would, in another one of his lively similes (ibid.):

> […] would be an analogous confusion to maintain that the way to acquire the concept of fire risk is to build a frame house with timber gables, thatched roof and leaky chimneys.

Clearly, Frege makes a valid point in that we do not need to see a group of 15 things in order to acquire the number 15. But at the same time, his fire risk analogy is not too apt when we consider his reluctance to bring in any psychological process for the *origin* of numbers. After all, is the concept of fire risk not tied tightly to actual flammable things? And, if the same holds for numbers, should we not be interested in how we came around the concept of number in the first place? If so, the new empirical findings that in our cognitive architecture there may be a natural, automatic, way of "abstracting" things into numbers would go directly against Frege's claim. Frege is of course talking about a different process of abstraction, one that is conscious and in character philosophical. Against that form of abstraction, much of his criticism is valid. But, against the notion of primitive abstraction—an ability that we already have as infants and share with many animals—it is much less so. Here, I claim that in discussing the psychologism of arithmetic, this primitive abstraction is the notion we should apply.

That kind of primitive abstraction, however, cannot be the only concept of abstraction we need. After all, the kind of abstraction we employ in mathematics goes much further than some primitive conception of small quantities. It could become quite speculative to derive advanced mathematics from such theories.[32] Hence, in addition to the primitive psychological abstraction, we need a concept for *mathematical* abstraction. That was essentially also the concept that Frege had, and defended it against the conscious psychological abstraction of Schröder, as well as against Mill's empirical generalizations. In doing that,

[32] As seen in the work of Lakoff and Núñez (2000). What they suggest is no doubt a *possible* story, in fact quite plausible, but we can hardly say with much confidence that this is how mathematics actually developed.

I believe Frege was perfectly justified. Besides the primitive proto-mathematical origins, mathematics clearly also involves a peculiarly mathematical abstraction, and it is essentially different from empirical sciences. But, that does not mean that our mathematical abstractions are not determined at least partly by the primitive abstractions. It may not make sense to say that an axiomatization of arithmetic is an abstraction of the numerical ability that infants, rats, and fish have. But, it is highly plausible that there are fundamental similarities between the two. At the primitive level, Frege's criticism of psychologism and abstraction has limited power. We know that numbers are something more than abstracting the number 15 from seeing 15 things. But ultimately, numbers may not be anything more than a generalization of the primitive and automatic process of observing the world in terms of quantities. Even if they were, such a psychological process seems to be our first access to them. If Frege hits a target with his criticism of psychologism, this primitive process does not seem to have included it. Without that natural tendency toward quantities, there might not be any concept of number to analyze. In that case, strictly separating the contexts of discovery and justification no longer makes sense. The best modern empirical research suggests that this is a scenario we should take seriously in philosophy.

6 Conclusion: Frege, Dedekind, and Modern Philosophy of Mathematics

In addition to the requirement that numbers cannot be defined by their position in the progression of natural numbers alone, the other important point for Frege was that the laws of numbers must follow from the laws of thought. This could be seen also in the forms that the groundbreaking works of the two took, Frege's *Grundlagen der Arithmetik* and Dedekind's *Was sind und was sollen die Zahlen?*. In Dummett's (1991, p. 48) characterization:

> Dedekind's approach to the question […] differs utterly from Frege's. Dedekind tackled it more specifically in the spirit of a mathematician, Frege more in that of a philosopher; Dedekind's treatment was that of a pure mathematician, whereas Frege was concerned with applications. Dedekind's central concern was to characterise the abstract structure of the system of natural numbers; what those numbers are used for was for him a secondary matter.

There is no doubt that such differences exist between the two books, although it is not immediately obvious that Frege was concerned with what numbers are used for, rather than how we can analyze and characterize numbers. In any case, while for both the subject was the foundations of arithmetic, Frege clearly took the direction associated with much of the modern study of foundations, i.e., analyzing natural numbers in terms of logic. But, it is not as clear that Dedekind's approach is any less suited to be a philosophical foundation of arithmetic. Frege wanted to derive the laws of number from the laws of thought. But, what if we take primitive laws of numbers to *be* laws of thought? I propose this in the rough sense described earlier: that we have a natural tendency to characterize observations in terms of quantities.[33]

[33] Obviously, I do not want to claim that *all* laws of thought are about quantities. Indeed, Frege's laws of logic can retain their status in everything else, but in the Dedekindian approach suggested here, there is simply no need to derive numbers in terms of them.

I believe this is an interesting idea with many advantages. As we have seen, mathematically, the approach is widely accepted. But, it also seems to have considerable philosophical strength. First, it would retain the apparent objectivity of arithmetic. If there is a universal disposition toward thinking in terms of numerosities, we would not be in danger of bringing in a subjectivist or conventionalist philosophy where numbers are merely arbitrary constructions. Second, it would not require a problematic Platonist epistemology or ontology in order to achieve that. Third, it would help explain the applicability of arithmetic. If we cannot help thinking in terms of quantities, it is no surprise that our applications make use of numbers. This is of course only a partial explanation at best, since it leaves open the question how the applications are successful. It is impossible here to provide more than a rough sketch of an argument, but it is undeniable that the ability to process observations in terms of numbers has many immediate advantages for human beings, as well as animals. While this is still speculative in the current state of research, there is no denying the usefulness of quantities in many real-world applications. And just as obviously, our sophisticated applications of technology are built on physics, which is based on quantifying distance and time.

Finally, fourth—and curiously in philosophy by far least often acknowledged—this is what the best current empirical data tells us is the case. In the philosophy of mathematics, we traditionally have had precious little empirical support. Can we afford to ignore it now that it seems more and more likely that we have an inborn tendency to categorize observations in terms of numerosities?

In this paper, I have argued for a Dedekindian approach over a Fregean one in the philosophy of arithmetic on two counts. First, I tried to show that there are no foundational reasons for preferring a logicist approach to a structuralist one. Second, I have argued that when it comes to Frege's criticism of abstraction and psychologism, modern versions of those ideas could be immune to his arguments. I also hope to have provided reasons for thinking that Dedekind is best understood as a forefather of modern structuralism. Now, the final question is whether the above idea of primitive psychological abstraction is compatible with Dedekind's account. Did he not specifically state that numbers are "free creations of the human mind?" I believe that this statement must be understood in its proper context. In fact, the idea that laws of numbers should be understood as laws of thought can be found in Dedekind's initial introduction to *Was sind und was sollen die Zahlen?* (1963, p. 31):

> In speaking of arithmetic […] as a part of logic I mean to imply that I consider the number-concept entirely independent of the notions or intuitions of space and time, that I consider it an immediate result from the laws of thought. My answer to the problems propounded in the title of this paper is, then, briefly this: numbers are free creations of the human mind […]

Earlier, I argued that Dedekind meant mathematical abstraction when he stated that numbers are free creations. But although anachronistic, it is not at all impossible to include the kind of psychologism I have proposed above in Dedekind's position. If our laws of thought include ones involving numerosities, they are likely to be present in the psychological processes that make us categorize observations in terms of quantities. However, they are also likely to be mirrored in the developed abstract mathematical thinking. I am not claiming that Dedekind held such views, of course. Nevertheless, the structuralist view of numbers as places in the natural number structure and nothing else, added to the position that we have primitive laws of thought about quantities, fits remarkably well with the modern psychologist picture I have presented above. I hope to have shown that Dedekind can be seen as a forefather of this development.

There is one final problem that we must look into. We must remember Frege's initial main concern: with psychologist theories of arithmetic, do we not lose the precision and universality—the whole necessary a priori character—that makes mathematics special and distinct from other disciplines? Are we stuck with Mill's pebble arithmetic? Unlike Frege, I do not think a proper psychologist philosophy has got anything to do with that. Mathematical theories can be based on psychological processes and then later *develop* an essentially a priori character. Once we are dealing with axiomatic systems of arithmetic, there is no further need to bring in psychological justifications for the laws of natural numbers. But, that does not mean such justifications are not possible—or indeed that they have not been *responsible* for the early development of arithmetical thinking. Mathematics can be essentially a priori, but in a context forced upon us by our cognitive architecture.[34]

From a modern perspective, with all the epistemological problems of Platonism, it would seem quite strange if we could draw no connection at all between *some* empirical origins and the developed mathematical theories. And, if there is a natural tendency to process observations in terms of quantities, the position that our laws of arithmetic are fundamentally laws of thought becomes quite appealing. However, when we ask what the natural numbers actually are, under this interpretation, there is no need to invoke concepts *beyond* those of arithmetic. This way, the position I have attributed to Dedekind in this paper—that the numbers do not need to be anything besides their places in the structure of natural numbers—does not seem to have any disadvantages that the Fregean logicist program avoids. Indeed, combined with an updated version of psychologism, the structuralist account that Dedekind helped develop may well prove to be the epistemologically most plausible platform for philosophy of arithmetic.

References

Agrillo, C., Dadda, M., Serena, G., & Bisazza, A. (2009). Use of number by fish. *PloS One, 4*(3), e4786.

Angelelli, I. (1994). Abstraction and number in Michael Dummett's Frege: philosophy of mathematics. *Modern Logic, 4*, 308–318.

Benacerraf, P. (1973). Mathematical truth. *The Journal of Philosophy, 70*, 661–679.

Boolos, G. (1990). The Standard of Equality of Numbers. In G. Boolos (ed.): *Meaning and Method: Essays in Honor of Hilary Putnam* , Cambridge: Cambridge University Press, pp. 261–277

Boolos, G. (1998). *Logic, logic and logic*. Cambridge: Harvard University Press, pp. 202–219

Cantor, G. (1878). Ein Beitrag zur Mannigfaltigkeitslehre. *Journal für reine und angewandte Mathematik, 84*, 242–258.

Corry, L. (1996). *Modern algebra and the rise of mathematical structures*. Basel: Birkhäuser.

Dedekind, R. (1887). *Was sind und was sollen die Zahlen?*, translated by W.W. Berman. In R. Dedekind (Ed.), *Essays in the Theory of Numbers*. New York: Dover.

Dedekind, R. (1890). Letter to Keferstein. In J. van Heijenoort (Ed.), *From Frege to Gödel: A source book in mathematical logic*, 1879–1937. New York: Harvard University Press, pp. 98–103.

Dehaene, S. (2011). *The number sense—revised & expanded edition*. New York: Oxford University Press.

Dehaene, S., & Brannon, E. (Eds.). (2011). *Space, time and number in the brain*. London: Academic Press.

[34] See Pantsar (2014) for more on such *contextual a priori* characterization of arithmetical knowledge.

Dummett, M. (1991). *Frege: philosophy of mathematics*. Cambridge: Harvard University Press.
Ferreirós, J. (1999). *Labyrinth of thought—a history of set theory and its role in modern mathematics*. Basel: Birkhäuser.
Frege, G. (1879). *Begriffsschift.* In van J. Heijenoort, (Ed.), *From Frege to Gödel: A source book in mathematical logic, 1879–1937*. New York: Harvard University Press, pp. 5–82.
Frege, G. (1884). *The foundations of arithmetic*. Oxford: Basil Blackwell.
Frege, G. (1893). *Grundgesetze der Arithmetik. Band I*. Jena: H. Pohle.
Frege, G. (1903). *Grundgesetze der Arithmetik. Band II*. Jena: H. Pohle.
Frege, G. (1914). Logic in mathematics. In G. Frege (Ed.), Posthumous Writings, Oxford: Wiley-Blackwell, pp. 203–250.
Gillies, D. A. (1982). *Frege, Dedekind, and Peano on the foundations of arithmetic*. Assen, Netherlands: Van Gorcum.
Hale, B., & Wright, C. (2001). *The reason's proper study*. Oxford: Clarendon Press.
Hellman, G. (1989). *Mathematics without numbers*. Oxford: Oxford University Press.
Hilbert, D. (1925). On the Infinite. In P. Becacerraf & H. Putnam (Eds.), *Philosophy of mathematics* (2nd ed.). Cambridge: Cambridge University Press.
Husserl, E. (1891). *Philosophie der Arithmetik*. Halle: Pfeffer.
Kitcher, P. (1983). *The nature of mathematical knowledge*. New York: Oxford University Press.
Kitcher, P. (1992). Frege's epistemology, *Philosophical Review* 88, pp. 235–262.
Lakoff, G., & Núñez, R. (2000). *Where mathematics comes from*. New York: Basic Books.
Mill, J. S. (1843). A system of logic. In J. M. Robson (Ed.), *Collected works of John Stuart Mill, vols. 7 & 8*. Toronto: University of Toronto Press.
Nieder, A. (2011). The neural code for number. In Dehaene & Brannon (Eds.), *Space time and number in the brain* (pp. 107–122). London: Academic Press.
Pantsar, M. (2014). An empirically feasible approach to the epistemology of arithmetic. *Synthese, 191*, 4201–4229.
Peano, G. (1889). The principles of arithmetic, presented by a new method. In J. van Heijenoort, (Ed.), *From Frege to Gödel: A source book in mathematical logic, 1879–1937*. New York: Harvard University Press, pp. 85–97.
Peano, G. (1891). *Sul concetto di numero*. In G. Peano (Ed.), *Opere Scelte Vol. III.* Rome: Cremonese, pp. 80–109.
Reck, E. (2003). Dedekind's sructuralism: an interpretation and partial defense. *Synthese, 137*, 369–419.
Resnik, M. (1981). Mathematics as a science of patterns: ontology and reference. *Noûs, 15*, 529–550.
Russell, B. (1903). *Principles of mathematics*. New York: Norton.
Russell, B. (1919). *Introduction to mathematical philosophy*. New York: Allen & Unwin.
Schröder, E. (1873). *Lehrbuch der Arithmetik und Algebra für Lehrer und Studirende I*. Leipzig: Teubner.
Sieg, W., & Schlimm, D. (2005). Dedekind's analysis of number: systems and axioms. *Synthese, 147*, 121–170.
Shapiro, S. (1997). *Philosophy of Mathematics: Structure and Ontology*. New York: Oxford University Press.
Tait, W. (1997). Frege versus cantor and Dedekind: on the concept of number. In W. Tait (Ed.), *Early analytic philosophy: Frege* (pp. 213–248). Russell, Wittgenstein, Chicago: Open Court.
Tappenden, J. (1995). Extending knowledge and "fruitful concepts", *Noûs* 29, pp. 427–467.
von Neumann, J. (1923). An axiomatization of set theory. In J. van Heijenoort (Ed.), *From Frege to Gödel: A source book in mathematical logic, 1879–1937*. New York: Harvard University Press, pp. 346–354.
Wang, H. (1957). The axiomatization of arithmetic. *Journal of Symbolic Logic, 22*, 145–158.
Wilson, M. (2010). Frege's mathematical setting. In T. Ricketts & M. Potter (Eds.), *The Cambridge companion to Frege*. Cambridge: University Press.
Wynn, K. (1992). Addition and subtraction by human infants. *Nature, 358*, 749–751.
Zermelo, E. (1908). Investigations in the foundations of set theory. In J. van Heijenoort (Ed.), *From Frege to Gödel: A source book in mathematical logic, 1879–1937*. New York: Harvard University Press, pp. 199–215.