Sharp Interface Limits for Diffuse Interface Models with Contact Angle

Dissertation zur Erlangung des Doktorgrades
der Naturwissenschaften (Dr. rer. nat.)
der Fakultät Mathematik
der Universität Regensburg

vorgelegt von

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im Jahr 2020

Promotionsgesuch eingereicht am: 13.07.2020

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Abstract

We consider the sharp interface limit for the Allen-Cahn equation and some variants in a bounded smooth domain in the case of boundary contact. The Allen-Cahn equation is a diffuse interface model since (after a short generation time) solutions typically develop so-called diffuse interfaces, where the solution stays smooth but experiences steep gradients. Moreover, the equation contains a small parameter $\varepsilon > 0$ that corresponds to the thickness of the diffuse interfaces. The limit $\varepsilon \to 0$ is called "sharp interface limit" because - at least heuristically - the solutions should converge to step functions with the jump set evolving in time according to some sharp interface problem. We show the rigorous sharp interface limit, i.e. that solutions to the diffuse interface and the sharp interface model are related rigorously. The results are local in time and applicable as long as a smooth solution to the limit problem exists.

We consider the following cases:

- Convergence of the Allen-Cahn equation with Neumann boundary condition to the mean curvature flow with $90°$-contact angle in any dimension $N \geq 2$.
- Convergence of the vector-valued Allen-Cahn equation involving different choices for the potential and with Neumann boundary condition to the mean curvature flow with $90°$-contact angle in any dimension $N \geq 2$, but without the triple junction situation. For this case we expect that a similar strategy works. We give some comments in this direction.
- Convergence of an Allen-Cahn equation with a non-linear Robin boundary condition to the mean curvature flow with an α-contact angle in 2D for α close to $90°$.

For the convergence proofs we use the method of de Mottoni, Schatzman [deMS], i.e. we

1. Rigorously construct an approximate solution for the diffuse interface model with asymptotic expansions.
2. Estimate the difference of the exact and approximate solution to the diffuse interface model with a spectral estimate for a linear operator associated to the model.

The major novelty in the thesis is the consideration of boundary contact for the diffuse interfaces within the method of [deMS]. Therefore we construct suitable curvilinear coordinates. Based on the latter we rigorously set up the asymptotic expansions. In this process new parameter-dependent elliptic problems on the half space in $\mathbb{R}^2$ appear. For the $90°$-case these problems are solved with a splitting method in exponentially weighted Sobolev spaces. The latter seems not possible for angles $\alpha \neq 90°$ and we use the Implicit Function Theorem with respect to α in this case. Moreover, for the spectral estimate for the Allen-Cahn operator in every case (which is obtained by linearization at the approximate solution) we use a new idea: we construct an approximate first eigenfunction using asymptotic expansions. Then we split the space of H^1-functions over the domain into a "small" explicit space formally approximating the first eigenfunctions and the complementing space. Finally, we analyze the associated bilinear form on every part.

Zusammenfassung

Wir betrachten den scharfen Grenzschicht-Limes für die Allen-Cahn Gleichung und einige ihrer Varianten in einem beschränkten, glatten Gebiet im Fall von Randkontakt. Die Allen-Cahn Gleichung ist ein diffuses Grenzschicht-Modell, denn Lösungen der Gleichung bilden üblicherweise nach kurzer Zeit sogenannte diffuse Grenzschichten aus, in denen die Lösungen glatt bleiben, aber sich stark verändern. Hierbei enthält die Gleichung einen kleinen Parameter $\varepsilon > 0$, der proportional zur typischen Dicke der erzeugten diffusen Grenzschichten ist. Der Limes $\varepsilon \to 0$ heißt „scharfer Grenzschicht-Limes“, da - zumindest heuristisch - die Lösungen gegen Treppenfunktionen konvergieren sollten, deren Sprung gemäß eines scharfen Grenzschicht-Modells in der Zeit evolviert. Wir zeigen den rigorosen scharfen Grenzschicht-Limes, d.h. dass Lösungen des diffusen und des scharfen Grenzschicht-Problems rigoros in Beziehung gesetzt werden. Die Resultate gelten lokal in der Zeit und sind anwendbar solange eine glatte Lösung für das scharfe Grenzproblem existiert.

Wir betrachten die folgenden Fälle:

- Konvergenz der Allen-Cahn Gleichung mit Neumann-Randbedingung gegen den mittleren Krümmungsflüss mit $90°$-Kontaktwinkel für alle Dimensionen $N \geq 2$.
- Konvergenz der vektor-wertigen Allen-Cahn Gleichung mit mehreren Wahlen für das Potential und Neumann-Randbedingung gegen den mittleren Krümmungsflüss mit $90°$-Kontaktwinkel für alle Dimensionen $N \geq 2$, jedoch ohne den Fall von Tripel-Punkten. In diesem Fall erwarten wir, dass eine ähnliche Strategie funktioniert und wir bemerken einige Ideen hierzu.
- Konvergenz der Allen-Cahn Gleichung mit nichtlinearer Robin-Randbedingung gegen den mittleren Krümmungsfluss mit α-Kontaktwinkel in 2D für α nahe $90°$.

Für die Konvergenzresultate nutzen wir die Methode von de Mottoni, Schatzman [deMS], d.h. wir

1. Konstruieren rigoros eine Approximationslösung für das diffuse Grenzschicht-Modell mit Hilfe von asymptotischen Entwicklungen.
2. Schätzen die Differenz der exakten Lösung und der Approximationslösung zum diffusen Grenzschicht-Modell ab, indem wir eine Spektralabschätzung für einen linearen Operator nutzen, der in natürlicher Weise zum Modell gehört.

Die zentrale Neuheit in der Arbeit ist das Betrachten von Randkontakt für die diffuse Grenzschicht im Kontext der Methode von de Mottoni, Schatzman [deMS]. Zu diesem Zweck konstruieren wir geeignete krummlinige Koordinaten und bauen darauf die rigorosen asymptotischen Entwicklungen auf. Dabei kommen neue parameterabhängige elliptische Probleme auf dem Halbraum im $\mathbb{R}^2$ vor. Im $90°$-Fall lösen wir diese mit einer Aufspaltungsmethode in exponentiell gewichteten Sobolev-Räumen. Letzteres scheint nicht möglich für $\alpha \neq 90°$, weshalb wir in diesem Fall den Satz von der Impliziten Funktion bezüglich α verwenden. Außerdem nutzen wir für die Spektralabschätzung des Allen-Cahn-Operators in jedem Fall (diesen erhält man durch Linearisierung an der Approximationslösung) eine neue Idee: wir konstruieren eine approximative erste Eigenfunktion mittels asymptotischer Entwicklungen. Dann teilen wir den Raum der H^1-Funktionen über dem Gebiet auf in einen „kleineren“ expliziten Raum, der formal die ersten Eigenfunktionen approximiert, und dessen Komplementärraum. Dann analysieren wir die zugehörige Bilinearform auf den jeweiligen Teilräumen.

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1 Introduction

Interfaces and moving boundaries arise in manifold varieties in the natural sciences and their applications. Some prominent examples are the melting of ice, the motion of an oil droplet in water, crystal growth, biological membranes, tumour evolution, spinodal decomposition of polymers and antiphase boundaries in iron alloys. Naturally, the modelling and analysis of interfaces is a plentiful research area and of paramount importance. There are (among others; see e.g. Anderson, McFadden, Wheeler [AFW]) two important model categories: sharp interface models and diffuse interface models. These model types can be related by so-called sharp interface limits. The subject of this thesis belongs to this area. In the following we briefly describe and compare the previously mentioned model categories. Then we explain and motivate sharp interface limits. Based on an overview of existing results we delve deeper until arriving at the topic of this thesis.

Sharp Interface Models. The interface is represented as a surface of thickness zero, i.e. a hypersurface or more complicated objects (e.g. with junctions and clusters). Quantities (e.g. physical variables) are allowed to be discontinuous across the interface. These models typically involve an evolution law for hypersurfaces, possibly coupled with equations in the bulk domains (separated by the sharp interface) and relations on the interface. Often solutions develop singularities in finite time, in particular when the interface changes its topology (e.g. the breakup or coalescence of droplets). Examples for sharp interface models are the Stefan problem and the two-phase Navier-Stokes problem which can be used to model e.g. the melting of ice or an oil droplet in water, respectively, see for instance Prüss, Simonett [PS] and references therein. Usually the interface is unknown and part of the problem. Then such models are called "free boundary value problems". If the evolution is focussed on the motion of the surface itself and involves only its geometric quantities (normal velocity, curvature, contact angle etc.) one speaks of "geometric evolution equations". Well-known examples are the mean curvature flow, the surface diffusion flow and the Willmore flow, see e.g. Prüss, Simonett [PS].

Diffuse Interface Models. Such models involve a (typically smooth) order parameter (e.g. the density, the composition of two materials or an artificial variable) that distinguishes the bulk domains and experiences steep gradients in small transition zones ("diffuse interfaces") between them. In applications the diffuse interface can be viewed as a microscopically small mixing region of globally immiscible materials or phases occupying the bulk domains. The sharp interface can in principle be recovered as a level set of the order parameter. Quantities that are in the sharp interface models localized to the sharp interface usually have a diffuse analogue that is distributed throughout the diffuse interface. An interesting example is the correspondence of surface tension and capillary stress tensor in fluid mechanics, see [AFW]. Diffuse interface models may be more appropriate than sharp interface models to describe phenomena acting on length scales comparable to the interface thickness, e.g. interface thicking phenomena, complicated contact angle behaviour and topology changes, cf. [AFW], p.141. Moreover, the change of topology of the interface does typically not impose analytical or numerical difficulties in contrast to sharp interface models. Famous examples for diffuse interface models are the Cahn-Hilliard equation and the Allen-Cahn equation that can model e.g. spinodal decomposition of polymers or the motion of antiphase boundaries in iron alloys, respectively. See the overview article by Novick-Coen [N] on the Cahn-Hilliard equation and Allen, Cahn [AC].

Sharp Interface Limits. Typically diffuse interface models contain a small parameter that is proportional to the thickness of the diffuse interface. Heuristically, when this parameter is sent to zero, one obtains sharp interfaces evolving in time and thus a sharp interface model. Therefore such limits are called “sharp interface limits”.

It is an important task to connect diffuse interface models and sharp interface models via their sharp interface limits for the following reasons (see also the partly universal introduction in Caginalp, Chen [CC] and general comments in Caginalp, Chen, Eck [CCE]):

- Modelling and Analysis: both types of models can usually be derived or motivated with physical principles, phenomenological observations or geometrical arguments etc., but one always incorporates some constitutive assumptions. Often the derivation for the sharp interface models is more transparent and these models appear simpler and more qualitative. On the other hand, diffuse interface models are usually advantageous in more complicated situations and solutions typically have better analytical properties (cf. above). By identifying the sharp interface limit one confirms that the assumptions in the derivations are appropriate as well as that the models are compatible with each other and can be used to describe the same situation. Another motivation is the concept of using the diffuse interface model to extend solutions of the corresponding sharp interface model past singularities.

- Numerics: diffuse interface models are often simpler to solve numerically. By considering the sharp interface limit one justifies that the numerical solution to the diffuse interface model can be used to approximate the solution to the sharp interface model.

Concerning results for sharp interface limits: in general there are formal results and rigorous proofs for convergence.

Formal Sharp Interface Limits. The formal sharp interface limits are typically based on formal asymptotic expansions (see comments below) or numerical experiments (see e.g. Lee, Kim [LK]). However, see also [AFW], p.156ff for a “pillbox argument”, i.e. reasoning with a small test volume.

Asymptotic Expansions. Since we will use (rigorous) asymptotic expansions later, let us roughly explain the idea here, see also Eck, Garcke, Knabner [EGK], Sections 1.5-1.7 and references therein. The typical situation is a singularly perturbed equation in terms of a small parameter ε (for instance), i.e. the equation changes its type fundamentally (e.g. a change in differentiation order or a transition from parabolic to elliptic) when ε is evaluated at zero. The goal of an asymptotic expansion is to get an in depth understanding of the qualitative behaviour of solutions when ε is close to zero. Typically, in some portions of the domain of definition (in space and/or time) the qualitative behaviour of the solutions does not change much, but in different regions (often denoted by inner and outer regions) the qualitative properties are distinct and overlap in some transition regions. Typically, when using asymptotic expansions, in the inner and outer regions one makes a different ansatz for the solution with suitable ε-series and expands the equations into ε-series, usually with some Taylor-expansions. The expansions valid in the different regions should be compatible in the transition region, therefore “matching conditions” are imposed. Ideally, one can subsequently derive conditions for the coefficients in the ε-series, obtain an algorithm for the construction of the coefficients in the series and get an approximation for the exact solution. If one is just interested in deriving the leading order qualitative behaviour, only few terms in the expansion are needed (also called “formally matched asymptotics” then).

The expansion is "rigorous", if the construction is rigorous and the approximation error can be estimated. Typically, this involves more terms in the expansion, tedious calculations and remainder estimates.

For the above situation of interface problems, the diffuse interface model can be viewed as singularly perturbed in terms of the typical small parameter in the equation. The diffuse interface will be the "inner region" and the bulk domains the "outer region". See Sections 1.1-1.3 below for some references on formal sharp interface limits with asymptotic expansions in the case of some variants of the Allen-Cahn equation. Asymptotic expansions also appear e.g. in the context of singularly perturbed ODEs or the derivation of Prandtl's boundary layer equations, see Eck, Garcke, Knabner [EGK], Sections 1.5 and 6.6.

Rigorous Sharp Interface Limits. Regarding rigorous sharp interface limits, one can basically group such results into two types:

- Global time results using some kind of weak notion for the sharp interface system, e.g. viscosity solutions for mean curvature flow, varifold solutions, distributional solutions, etc.

- Local in time results that are applicable before singularities appear, i.e. as long as the interface does not develop singularities and stays smooth.

The subject of the thesis belongs to the rigorous, local in time results for sharp interface limits and uses the most-used method for these types of results, namely the so-called "method of de Mottoni and Schatzman" described below. De Mottoni and Schatzman [deMS] were the first to apply this method for the convergence of the Allen-Cahn equation to the mean curvature flow in $\mathbb{R}^N$, $N \in \mathbb{N}$. Here mean curvature flow for evolving hypersurfaces means that the normal velocity equals mean curvature. It is not convenient to attempt a complete list of references on sharp interface limits. However, we mention below other results that use the method of de Mottoni and Schatzman. Moreover, in Sections 1.1-1.3 below we cite existing results (obtained by distinct methods) that are related to the variants of the Allen-Cahn equation considered in this thesis.

The Method of de Mottoni and Schatzman. The idea is to carry out a rigorous asymptotic expansion. One assumes that there exists a local smooth solution to the limit sharp interface problem. This can usually be shown for small times. Then

1. One *rigorously* constructs an approximate solution to the diffuse interface model using asymptotic expansions based on the evolving surface that is (part of) the solution to the limit problem. In this process one has to solve model problems for the series coefficients.

2. Then one estimates the difference between exact and approximate solutions. This typically involves a spectral estimate for a linear operator associated to the diffuse interface equation.

This method also yields the typical profile of the solution which is usually not the case in other approaches, cf. the references in Section 1.1-1.3. Moreover, comparison principles are not needed in contrast to most of the other methods.

The method by de Mottoni and Schatzman was applied to other diffuse interface models as well. These results are based on general spectrum estimates in Chen [C2] for Allen-Cahn, Cahn-Hilliard and phase-field-type operators. There are results for the Cahn-Hilliard equation by Alikakos, Bates, Chen [ABC], the phase-field equations by Caginalp, Chen [CC], the mass-conserving Allen-Cahn equation by Chen, Hilhorst, Logak [CHL], the Cahn-Larché system by

Abels, Schaubeck [AS] and a Stokes/Allen-Cahn system by Abels, Liu [AL]. See also Schaubeck [Sb] for a result on a convective Cahn-Hilliard equation. Finally, Marquardt [Ma] studied the sharp interface limit for a Stokes/Cahn-Hilliard system. For the subtle variations in the rigorous asymptotic expansions and the spectral estimates see the inceptions to Sections 5-6.

Results of the Thesis and Content Overview. To the authors knowlegde, so far all the results[1] obtained with the method of de Mottoni and Schatzmann [deMS] have in common that the sharp interface is closed (i.e. compact, without boundary) and strictly contained in the domain of definition. This strongly motivates to apply the method in the case of boundary contact. The following results are obtained:

- Convergence of (solutions to) the Allen-Cahn equation with Neumann boundary condition to the mean curvature flow with $90°$-contact angle in any dimension $N \geq 2$. See Section 1.1 below. For the case $N = 2$ see also Abels, Moser [AM].
- Convergence of (solutions to) the vector-valued Allen-Cahn equation involving different choices for the potential and with Neumann boundary condition to the mean curvature flow with $90°$-contact angle in any dimension $N \geq 2$. See Section 1.2 below.
- Convergence of (solutions to) an Allen-Cahn equation with a non-linear Robin boundary condition to the mean curvature flow with an α-contact angle in the two-dimensional case for α close to $90°$. See Section 1.3 below.

In Section 2 we fix some notation and introduce function spaces. The major novel idea presented in the thesis is to construct and use *curvilinear coordinates* that are adapted to the problem, opposed to the well-known tubular neighbourhood coordinate system. This is done in Section 3. For every case considered in the thesis, in Section 4 we solve model problems appearing in the asymptotic expansions in Section 5. The spectral estimates are implemented in Section 6. The difference estimates and the proofs of the convergence theorems are carried out in Section 7.

Mean Curvature Flow (MCF) with Contact Angle. "Mean curvature flow" for evolving hypersurfaces means that the normal velocity equals mean curvature, where in this thesis "mean curvature" is for convenience defined as the sum of the principal curvatures. We often abbreviate "mean curvature flow" by "MCF".

For the convergence results in Sections 1.1-1.3 we will assume that the corresponding sharp interface models have a smooth solution on a time interval $[0, T_0]$. This is a prerequisite for the method of de Mottoni and Schatzman [deMS]. Recall that in this thesis mean curvature flow with $\frac{\pi}{2}$-contact angle in dimension $N \geq 2$ and α-contact angle, $\alpha \in (0, \pi)$, in 2D is considered.

The local well-posedness and existence of a smooth solution for small time for the considered sharp interface models starting from suitable initial sharp interfaces is basically well-known. At this point let us give some references in this direction. In Katsoulakis, Kossioris, Reitich [KKR], Section 2, a parametric approach is used to show local existence and uniqueness of classical solutions for MCF in arbitrary dimension and with fixed contact angle. In principle, it is also possible to reduce the evolution to a parabolic PDE by writing it over a reference hypersurface via suitable coordinates. For the typical procedure in the case of a closed interface see Prüss, Simonett [PS]. For curvilinear coordinates in the situation of boundary contact see Vogel [V] and

[1] The result by Abels, Moser [AM] is contained in the thesis.

Section 3 below. Moreover, note that in Huisken [Hu] the special case of MCF with $\frac{\pi}{2}$-contact angle in the graph case for cylindrical domains is considered and global existence and uniqueness of smooth solutions as well as convergence to a constant graph is obtained.

Notation for Sections 1.1-1.3. We need some notation in the context of the curvilinear coordinates in order to formulate the main theorems in Sections 1.1-1.3. For convenience let us introduce the expressions simultaneously at this point. The reader is encouraged to skip this remark first and jump back occasionally when studying Sections 1.1-1.3.

Remark 1.1 (**Domain, Sharp Interface and Coordinates**)**.** For the details see Section 3. For a sketch of the situation see Figure 1 below.

1. *Domain.* Let $N \in \mathbb{N}$, $N \geq 2$ and $\Omega \subset \mathbb{R}^N$ be a bounded, smooth domain (i.e. nonempty, open and connected[2]) with outer unit normal $N_{\partial\Omega}$. For $T > 0$ we set $Q_T := \Omega \times (0,T)$ and $\partial Q_T := \partial\Omega \times [0,T]$.

2. *Sharp Interface.* Consider $T_0 > 0$ and an evolving hypersurface $\Gamma = (\Gamma_t)_{t\in[0,T_0]}$ (with boundary, smooth, oriented, compact, connected[2]) suitably parametrized over a reference hypersurface Σ and such that $\partial\Gamma$ meets $\partial\Omega$ at contact angle $\frac{\pi}{2}$ if $N \geq 2$ or angle $\alpha \in (0,\pi)$ if $N = 2$. For Γ one can define the normal velocity V_{Γ_t} and mean curvature H_{Γ_t} at time $t \in [0,T_0]$ with respect to a unit normal $\vec{n}$ of Γ in the classical sense. MCF means $V_{\Gamma_t} = H_{\Gamma_t}$ for $t \in [0,T]$. See Section 3.1 for the concise assumptions and definitions.

3. *Coordinates.* We construct appropriate curvilinear coordinates (r,s) with values in $[-2\delta, 2\delta] \times \Sigma$ for some $\delta > 0$ describing a neighbourhood of Γ in $\overline{\Omega} \times [0,T_0]$. For the exact statements see in particular Theorem 3.3 and Theorem 3.7, with 2δ instead of δ there. Here r has the role of a signed distance function and s works like a tangential projection. The set $\overline{Q_{T_0}} = \overline{\Omega} \times [0,T_0]$ is split by Γ into two connected sets (Γ excluded) according to the sign of r. We denote them with $Q^\pm_{T_0}$. Then we have the disjoint union
$$\overline{Q_{T_0}} = \Gamma \cup Q^-_{T_0} \cup Q^+_{T_0}.$$
Finally, we introduce the tubular neighbourhoods $\Gamma(\eta) := r^{-1}((-\eta,\eta))$ for all $\eta \in (0,2\delta]$ and define a suitable normal derivative ∂_n and tangential gradient ∇_τ on $\Gamma(\eta)$. See Remark 3.4 and Remark 3.8.

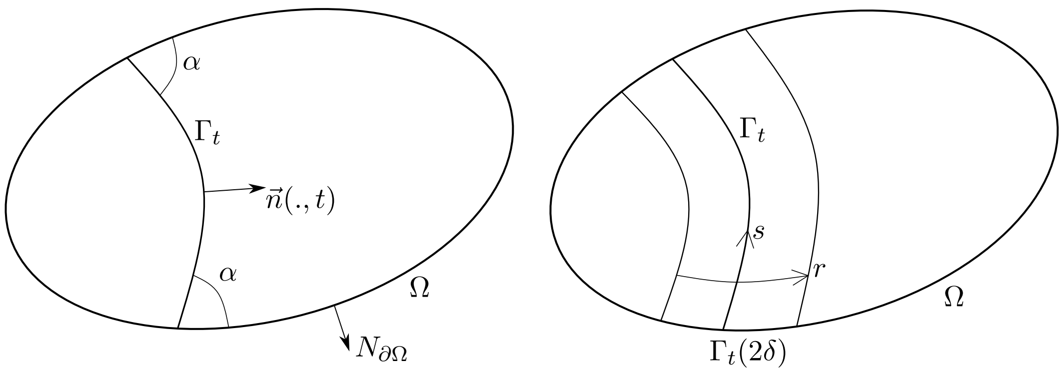

Figure 1: Sharp interface with α-contact angle and curvilinear coordinates.

[2] For convenience. The considerations can be adapted for the case of finitely many connected components.

1.1 Scalar-valued Allen-Cahn Equation with Neumann Boundary Condition, (AC)

Let $N \in \mathbb{N}$, $N \geq 2$ and Ω, $N_{\partial\Omega}$, Q_{T_0}, ∂Q_{T_0} for some $T_0 > 0$ be as in Remark 1.1, 1. Moreover, let $\varepsilon > 0$ small. For $u_\varepsilon : \overline{\Omega} \times [0, T_0] \to \mathbb{R}$ we consider the Allen-Cahn equation with homogeneous Neumann boundary condition, (AC)

$$\partial_t u_\varepsilon - \Delta u_\varepsilon + \frac{1}{\varepsilon^2} f'(u_\varepsilon) = 0 \qquad \text{in } Q_{T_0}, \tag{AC1}$$

$$\partial_{N_{\partial\Omega}} u_\varepsilon = 0 \qquad \text{on } \partial Q_{T_0}, \tag{AC2}$$

$$u_\varepsilon|_{t=0} = u_{0,\varepsilon} \qquad \text{in } \Omega, \tag{AC3}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a suitable smooth double well potential with wells of equal depth. A typical example is $f(u) = \frac{1}{2}(1 - u^2)^2$, see Figure 2. The precise conditions are

$$f \in C^\infty(\mathbb{R}), \quad f'(\pm 1) = 0, \quad f''(\pm 1) > 0, \quad \int_{-1}^{u} f' = \int_{1}^{u} f' > 0 \quad \forall u \in (-1, 1) \tag{1.1}$$

and we assume

$$u f'(u) \geq 0 \quad \text{ for all } |u| \geq R_0 \text{ and some } R_0 \geq 1. \tag{1.2}$$

Note that (1.2) is just a requirement for the sign of f'. The condition (1.2) is used to obtain uniform a priori bounds for solutions u_ε, see Section 7.1.1 below.

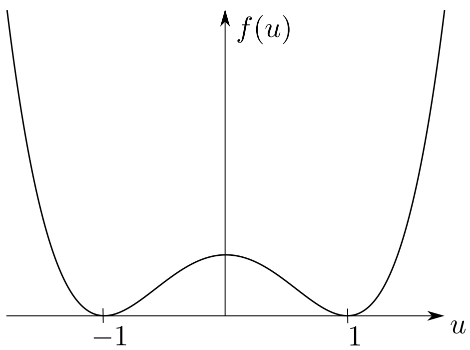

Figure 2: Typical form of the double-well potential, $f(u) = \frac{1}{2}(1 - u^2)^2$.

Motivation of (AC). The Allen-Cahn equation (similar to (AC1)) was originally introduced by Allen and Cahn [AC] to describe the evolution of antiphase boundaries in certain polycrystalline materials. For a summary on further motivations we refer to the introduction in Bronsard, Reitich [BR]. One can directly verify that equation (AC1)-(AC3) is the (by in time $\frac{1}{\varepsilon}$-accelerated) L^2-gradient flow to the (scalar) Ginzburg-Landau energy

$$E_\varepsilon(u) := \int_\Omega \frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} f(u) \, dx. \tag{1.3}$$

It will turn out that the time-scale is the right one for the sharp interface limit, cf. also Rubinstein, Sternberg, Keller [RSK] and the comments on “formal sharp interface limits” below.

For the sake of completeness, note that the limit $\varepsilon \to 0$ in energies of the form (1.3) (with similar potentials) has been considered in the context of Γ-convergence[3], see Modica [Mo1] (with mass constraint) and Sternberg [St] (with and without mass constraint). The Γ-limits are perimeter functionals which (at least formally) induce MCF with 90°-contact angle via the L^2-gradient flow. This also motivates to study the dynamical problem (AC1)-(AC3) associated to the energy (1.3) and its relation to MCF with 90°-contact angle in the limit $\varepsilon \to 0$.

Well-Posedness of (AC). This is in principle well-known, see the references in Bellettini [Be], Remark 15.1. An approach with weak solutions (obtained via time-discretization) can also be found in Bartels [Bar], Chapter 6.1. Moreover, equation (AC1)-(AC3) fits in the general framework of Lunardi [Lu], Section 7.3.1, where a semigroup approach and a Hölder-setting is used. Together with a priori boundedness of classical solutions (see Section 7.1.1 below) that can be obtained with maximum principle arguments, one can show global well-posedness for regular, bounded initial data. See e.g. Lunardi [Lu], Proposition 7.3.2. Higher regularity then follows using linear theory, cf. Lunardi, Sinestrari, von Wahl [LSW]. Finally, note that well-posedness for (AC1) on $\mathbb{R}^N$ is shown in [deMS] for bounded initial data with estimates for the heat semigroup.

Generation of Interfaces. Typically after a short time Ω is partitioned into subdomains where the solution u_ε of (AC1)-(AC3) is close to ± 1 and transition zones (diffuse interfaces; roughly $u_\varepsilon^{-1}([-1+\mu, 1-\mu])$ for $\mu > 0$ small) develop where $|\nabla u_\varepsilon|$ is large. See Figure 3 below for a typical situation. For a rigorous result in this direction ("generation of interfaces") see Chen [C1]. Formally, one can see this in the equation since the "reaction term" $f'(u_\varepsilon)/\varepsilon^2$ should be large for small times compared to the diffusion term Δu_ε. One also speaks of fast reaction/slow diffusion, see Rubinstein, Sternberg, Keller [RSK]. Neglecting Δu_ε, (AC1) becomes an ODE in time for each space point. For this ODE the stationary points are $0, \pm 1$, where 0 is unstable and ± 1 is stable. Moreover, one can also have a look at the energy (1.3). It is well-known that solutions to the corresponding gradient flow behave in such a way that the energy is non-increasing (and decreases in some optimal sense) in time. In this perspective values of u_ε away from ± 1 are penalized strongly and values of $|\nabla u_\varepsilon|$ away from 0 are penalized weakly.

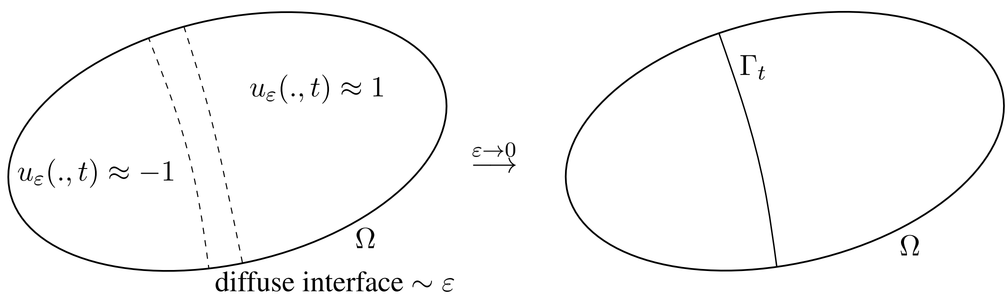

Figure 3: Diffuse interface and sharp interface limit.

Formal Sharp Interface Limit for (AC). Heuristically (or in sufficiently smooth cases) one can argue that the thickness of the diffuse interfaces is proportional to ε. Hence for $\varepsilon \to 0$ one should

[3] Of course this "Γ" has a different meaning than the Γ in Remark 1.1, 2.

obtain sharp interfaces evolving in time, cf. Figure 3. Formal asymptotic analysis by Rubinstein, Sternberg, Keller [RSK] yields that the limit sharp interface should evolve according to MCF and, if there is boundary contact, there should be a 90°-contact angle. Note that in [RSK] also the case with unequal wells is considered. In this situation, the limit interfaces formally move in the time scale $\tau = t/\varepsilon$ with speed proportional to the potential difference $f(-1) - f(1)$ in the direction of the region, where u_ε is close to 1. Moreover, the numerical experiments in Lee, Kim [LK] give another confirmation on a formal level for the sharp interface limit of (AC) to MCF with 90°-contact angle. Finally, note that the 1D-case is not interesting for finite time in the t-scale because patterns persist for and evolve in exponentially slow time scales $\tau = e^{-c/\varepsilon}t$, cf. Carr, Pego [CP]. More precisely, the sharp interface is a point and does not move in the time-scale t. This is consistent with MCF when the mean curvature of a point is defined as zero. Note that the method of de Mottoni, Schatzman [deMS] also works in this case with considerably simplified computations.

Rigorous Sharp Interface Limit Results for (AC). There are several rigorous results on the sharp interface limit for the Allen-Cahn equation ((AC1) on $\mathbb{R}^N$ or (AC)) to MCF (in the case of (AC) with $\frac{\pi}{2}$-contact angle). Via a comparison principle and the construction of sub- and supersolutions, Chen [C1] proves local in time convergence as long as the interface stays smooth. Moreover, de Mottoni and Schatzman [deMS] consider the $\mathbb{R}^N$-case and show convergence with strong norms for times when a smooth solution to MCF exists. This also works for (AC) when the interface is closed and strictly contained in Ω. Note that the papers by Chen, Hilhorst, Logak [CHL] and Abels, Liu [AL] also yield results for (AC) with strictly contained interface by simple adjustments. The resulting proofs and results are more optimized compared to [deMS]. For the subtle modifications and inspired ideas see the comments in the beginning of Sections 5-6 below.

Moreover, note that there is a paper by Sáez [Sa1], but unfortunately there is a severe gap in the proof of the main theorem, cf. [AM] for details.

For global in time results one has to use some weak formulation of MCF. There is the notion of viscosity solutions used by Evans, Soner, Souganidis [EvSS] for $\Omega = \mathbb{R}^N$ and by Katsoulakis, Kossioris, Reitich [KKR] in the case of a convex, bounded domain. In the latter the maximum principle is used and sub- and supersolutions to the Allen-Cahn equation are constructed using the distance function from the level set of a viscosity solution for MCF. Moreover, varifold solutions to MCF are used by Ilmanen [I] in the $\mathbb{R}^N$-case, by Mizuno, Tonegawa [MiT] for smooth, strictly convex, bounded domains and by Kagaya [Ka] without the convexity assumption. For varifold solutions only convergence of a subsequence is achieved. Finally, there is the conditional result by Laux, Simon [LS] where convergence of the (vector-valued; scalar case contained) Allen-Cahn equation to (multiphase) mean curvature flow in a BV-setting is obtained.

The New Rigorous Sharp Interface Limit Result for (AC). Finally, we state the convergence result for (AC) obtained in the thesis.

Theorem 1.2 (**Convergence of (AC) to MCF with** 90°**-Contact Angle**). *Let* $N \geq 2$, Ω, $N_{\partial\Omega}$, Q_T *and* ∂Q_T *for* $T > 0$ *be as in Remark 1.1, 1. Moreover, let* $\Gamma = (\Gamma_t)_{t\in[0,T_0]}$ *for some* $T_0 > 0$ *be a smooth evolving hypersurface with* $\frac{\pi}{2}$*-contact angle condition as in Remark 1.1, 2. and let* Γ *satisfy MCF. Let* $\delta > 0$ *small and the notation for* $Q^\pm_{T_0}$, $\Gamma(\delta)$, ∇_τ, ∂_n *be as in Remark 1.1, 3. Moreover, let* f *satisfy* (1.1)-(1.2). *Let* $M \in \mathbb{N}$ *with* $M \geq k(N) := \max\{2, \frac{N}{2}\}$.

Then there are $\varepsilon_0 > 0$ *and* $u^A_\varepsilon : \overline{\Omega} \times [0, T_0] \to \mathbb{R}$ *smooth for* $\varepsilon \in (0, \varepsilon_0]$ *with* $\lim_{\varepsilon\to 0} u^A_\varepsilon = \pm 1$ *uniformly on compact subsets of* $Q^\pm_{T_0}$ *and such that the following assertions hold:*

1. *If $M > k(N)$, then let $u_{0,\varepsilon} \in C^2(\overline{\Omega})$ with $\partial_{N_{\partial\Omega}} u_{0,\varepsilon} = 0$ on $\partial\Omega$ for $\varepsilon \in (0, \varepsilon_0]$ and*

$$\sup_{\varepsilon\in(0,\varepsilon_0]} \|u_{0,\varepsilon}\|_{L^\infty(\Omega)} < \infty \quad \text{and} \quad \|u_{0,\varepsilon} - u_\varepsilon^A|_{t=0}\|_{L^2(\Omega)} \leq R\varepsilon^{M+\frac{1}{2}} \tag{1.4}$$

for some $R > 0$ and all $\varepsilon \in (0, \varepsilon_0]$. Then for any set of solutions $u_\varepsilon \in C^2(\overline{Q_{T_0}})$ of (AC1)-(AC3) *for $\varepsilon \in (0, \varepsilon_0]$ with initial values $u_{0,\varepsilon}$ there are $\varepsilon_1 \in (0, \varepsilon_0]$, $C > 0$ such that*

$$\begin{aligned} \sup_{t\in[0,T]} \|(u_\varepsilon - u_\varepsilon^A)(t)\|_{L^2(\Omega)} + \|\nabla(u_\varepsilon - u_\varepsilon^A)\|_{L^2(Q_T\setminus\Gamma(\delta))} &\leq C\varepsilon^{M+\frac{1}{2}}, \\ \|\nabla_\tau(u_\varepsilon - u_\varepsilon^A)\|_{L^2(Q_T\cap\Gamma(\delta))} + \varepsilon\|\partial_n(u_\varepsilon - u_\varepsilon^A)\|_{L^2(Q_T\cap\Gamma(\delta))} &\leq C\varepsilon^{M+\frac{1}{2}} \end{aligned} \tag{1.5}$$

for all $\varepsilon \in (0, \varepsilon_1]$ and $T \in (0, T_0]$.

2. *If $k(N) \in \mathbb{N}$ and $M \geq k(N) + 1$, then there is a $\tilde{R} > 0$ small such that the assertion in 1. holds, when R, M in* (1.4)-(1.5) *are replaced by $\tilde{R}, k(N)$.*

3. *If $N \in \{2, 3\}$ and $M = 2(= k(N))$, then there is $T_1 \in (0, T_0]$ such that the assertion in 1. is valid but only such that* (1.5) *holds for all $\varepsilon \in (0, \varepsilon_1]$ and $T \in (0, T_1]$.*

Remark 1.3. 1. *Interpretation of Theorem 1.2.* One can interpret u_ε^A in the theorem as representation of a diffuse interface moving with Γ since u_ε^A is smooth but converges for $\varepsilon \to 0$ to a step function whose jump set is the solution to MCF with $\frac{\pi}{2}$-contact angle starting from Γ_0. The assumption on the initial values $u_{0,\varepsilon}$ in Theorem 1.2 essentially means that a diffuse interface already has developed and "sits" at the initial sharp interface Γ_0 at time $t = 0$, i.e. the generation of diffuse interfaces in the evolution is skipped. One also speaks of "well-prepared initial data", cf. [AL]. Hence Theorem 1.2 basically shows that the qualitative behaviour of diffuse interfaces with boundary contact, generated by (AC), is that of MCF with 90°-contact angle, at least as long as the evolution of the latter stays smooth.

2. *Layout of the Proof.* Required model problems, some ODEs on $\mathbb{R}$ and a linear elliptic equation on $\mathbb{R} \times (0, \infty)$ are considered in Section 4.1 and Section 4.2.1 below, respectively. The asymptotic expansions are carried out in Section 5.1 for $N = 2$ and Section 5.2 for $N \geq 2$. The approximate solution u_ε^A is defined in Section 5.1.3 for $N = 2$ and Section 5.2.3 for $N \geq 2$. Note that M corresponds to the number of terms in the expansion. The spectral estimate is proven in Section 6.2 for $N = 2$ and Section 6.3. The difference estimate is done in Section 7.2.1. Finally, Theorem 1.2 is obtained in Section 7.2.2. Note that for some parts the case $N = 2$ is considered separately for better readability.

3. The approximate solution u_ε^A obtained from the explicit construction equals ± 1 in the set $Q_{T_0}^\pm \setminus \Gamma(2\delta)$ and has a smooth, increasingly steep transition with a known "optimal profile" inbetween. Therefore Theorem 1.2 also yields the typical profile of solutions to (AC) across diffuse interfaces.

4. The level sets $\{u_\varepsilon^A = 0\}$, $\{u_\varepsilon = 0\}$ can be viewed as approximations for Γ. Note that in the explicit construction of u_ε^A in Sections 5.1-5.2 below the error from $\{u_\varepsilon^A = 0\}$ to Γ is of order ε and, in the case that f is even, of order ε^2, see Remark 5.8 and Remark 5.20. If one uses numerical computations for (AC) in order to approximate solutions to MCF with 90°-contact angle this is of importance, cf. also Caginalp, Chen, Eck [CCE].

5. In principle also estimates of better norms are possible in the situation of Theorem 1.2. The basic idea is to interpolate the already controlled norms with higher norms that can be estimated for exact solutions by some negative ε-orders. Cf. with Alikakos, Bates, Chen [ABC], Theorem 2.3 for a similar idea. However, note that this does not improve the approximation of Γ in the sense of 4.

6. Theorem 1.2 and the above comments hold analogously for closed Γ moving by MCF and compactly contained in Ω. The proof is basically contained since the constructions are localizable.

1.2 Vector-valued Allen-Cahn Equation with Neumann Boundary Condition, (vAC)

Let $N \in \mathbb{N}$, $N \geq 2$ and Ω, $N_{\partial\Omega}$, Q_{T_0}, ∂Q_{T_0} for some $T_0 > 0$ be as in Remark 1.1. Moreover, let $m \in \mathbb{N}$ and $W : \mathbb{R}^m \to \mathbb{R}$ be a suitable potential which will be specified below. Let $\varepsilon > 0$ be a small parameter. Then for $\vec{u}_\varepsilon : \overline{\Omega} \times [0, T] \to \mathbb{R}^m$ we consider the vector-valued Allen-Cahn equation with Neumann boundary condition, (vAC)

$$\partial_t \vec{u}_\varepsilon - \Delta \vec{u}_\varepsilon + \frac{1}{\varepsilon^2} \nabla W(\vec{u}_\varepsilon) = 0 \qquad \text{in } Q_T, \tag{vAC1}$$

$$\partial_{N_{\partial\Omega}} \vec{u}_\varepsilon = 0 \qquad \text{on } \partial Q_T, \tag{vAC2}$$

$$\vec{u}_\varepsilon|_{t=0} = \vec{u}_{0,\varepsilon} \qquad \text{in } \Omega. \tag{vAC3}$$

Motivation of (vAC). For a summary of motivations for vector-valued Allen-Cahn equations see Bronsard, Reitich [BR]. Another motivation is to approximate multiphase mean curvature flow in the sharp interface limit, where e.g. triple junctions of hypersurfaces are possible. One can compute directly that equation (vAC1)-(vAC3) is the (by in time $\frac{1}{\varepsilon}$-accelerated) L^2-gradient flow to the vector-Ginzburg-Landau energy

$$\check{E}_\varepsilon(\vec{u}) := \int_\Omega \frac{\varepsilon}{2} |\nabla \vec{u}|^2 + \frac{1}{\varepsilon} \nabla W(\vec{u}) \, dx. \tag{1.6}$$

For results in the direction of Γ-convergence with respect to $\varepsilon \to 0$ for energies like (1.6) (for several types of potentials and usually with mass constraint) see Baldo [Bal] and the references therein. The Γ-limits are (multiphase) perimeter functionals which (at least formally) induce (multiphase) MCF via the L^2-gradient flow. This gives another motivation to study the dynamical problem (vAC1)-(vAC3) and the connection to (multiphase) MCF in the limit $\varepsilon \to 0$.

The Potential W. Here we allow two types of potentials W. On one hand, we consider W with exactly two distinct minima and symmetry with respect to the hyperplane in the middle of these. On the other hand, we consider $m = 2$ and triple-well potentials W with symmetry. The first type is basically only interesting from a technical point of view, where with the second type one can describe e.g. three distinct phases in a polycristalline material, cf. Bronsard, Reitich [BR]. The precise requirements are as follows:

Definition 1.4. Let $W : \mathbb{R}^m \to \mathbb{R}$ be smooth and one of the two assumptions hold:

1. W has exactly two global minima $\vec{a}, \vec{b}$ with $W(\vec{a}) = W(\vec{b}) = 0$ in which D^2W is positive definite and W is symmetric with respect to the reflection $R_{\vec{a},\vec{b}} : \mathbb{R}^m \to \mathbb{R}^m$ at the hyperplane $\frac{1}{2}(\vec{a} + \vec{b}) + \mathrm{span}\{\vec{a} - \vec{b}\}^\perp$.

2. W is a symmetric triple well-potential for $m = 2$, i.e. W has exactly three global minima $\vec{x}_i$, $i = 1, 3, 5$ with $W(\vec{x}_i) = 0$ for $i = 1, 3, 5$ in which D^2W is positive definite and W is symmetric with respect to the symmetry group G of the equilateral triangle, cf. Kusche [Ku], Section 3.2 for the precise definition of G.

Moreover, in both cases we require $\vec{u} \cdot \nabla W(\vec{u}) \geq 0$ for all $\vec{u} \in \mathbb{R}^m$, $|\vec{u}| \geq \check{R}_0$ for some $\check{R}_0 > 0$. Furthermore, we assume that the kernel to a certain linear operator associated to W is one-dimensional. For the precise condition see Remark 4.28 below.

Remark 1.5. 1. An example for a typical triple-well potential that fulfils the conditions in Definition 1.4 can be found in Kusche [Ku], Section 3.4. See Figure 4 below for a sketch.

2. Compared to the scalar case, we always require symmetry properties for W. The assumption is used for example in Section 4.3.1 below. In order to relax this condition, one would have to find an appropriate substitute for the "equal-well"-assumption (1.1) for f in the scalar case.

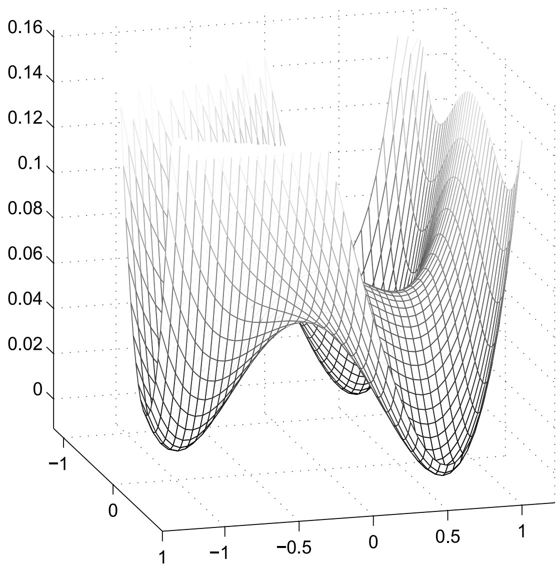

Figure 4: Typical triple-well potential W. The image is taken from Kusche [Ku].

Well-Posedness of (vAC). In general the analysis of systems is more challenging than that of single equations. However, the derivatives in (vAC1)-(vAC3) are decoupled and hence some methods from the scalar case can also be used for (vAC), e.g. regularity theory. Equation (vAC1)-(vAC3) matches the general setting of Lunardi [Lu], Section 7.3.1, where a semigroup method and Hölder-spaces are used. Moreover, by reduction to a scalar equation and maximum principle arguments, one can obtain a priori boundedness of classical solutions, see Section 7.1.2 below. Hence global well-posedness for regular, bounded initial data follows. Higher regularity can be obtained with linear theory for scalar equations, cf. Lunardi, Sinestrari, von Wahl [LSW].

Generation of Interfaces for (vAC). Analogously to the scalar case (AC) one can argue with formal arguments (system of fast reaction and slow diffusion; gradient flow to the energy (1.6)) that diffuse interfaces for solutions of (vAC) should develop after short time. Note that the transition between minima of W runs in $\mathbb{R}^m$. Moreover, in the case of a triple-well potential W also three-fold diffuse interfaces between the three minima of W are possible.

Formal Sharp Interface Limit for (vAC). Formal asymptotic calculations in [BR] yield that for a triple-well potential W in the sharp interface limit $\varepsilon \to 0$ one should obtain MCF together with:

- A $90°$-contact angle if a transition of two phases meets the boundary.
- A $120°$-triple junction if the three phases meet at an interior point.

See also Figure 5 below.

Rigorous Sharp Interface Limit Results for (vAC). We mention the conditional result by Laux, Simon [LS] on the convergence of the vector-valued Allen-Cahn equation to multiphase mean curvature flow in a BV-setting. Note that there is a work by Sáez [Sa2], but unfortunately there is a severe gap in the proof, cf. [AM] for details.

The New Rigorous Sharp Interface Limit Result for (vAC). Note that we only prove a result in the case of a two-fold transition, i.e. we do not consider the case of triple junctions. The latter is work in progress, see Section 1.2.1 below for some ideas.

Theorem 1.6 (Convergence of (vAC) to MCF with $90°$-Contact Angle). *Let $N \geq 2$, Ω, $N_{\partial\Omega}$, Q_T and ∂Q_T be as in Remark 1.1, 1. Moreover, let $\Gamma = (\Gamma_t)_{t\in[0,T_0]}$ for some $T_0 > 0$ be a smooth evolving hypersurface with $\frac{\pi}{2}$-contact angle condition as in Remark 1.1, 2. and let Γ satisfy MCF. Let $\delta > 0$ small and the notation for $Q_{T_0}^\pm$, $\Gamma(\delta)$, ∇_τ, ∂_n be as in Remark 1.1, 3. Moreover, let $W : \mathbb{R}^m \to \mathbb{R}$ be as in Definition 1.4 and $\vec{u}_\pm$ be any distinct pair of minimizers of W. Finally, let $M \in \mathbb{N}$ with $M \geq k(N) := \max\{2, \frac{N}{2}\}$.*

Then there are $\check{\varepsilon}_0 > 0$ and $\vec{u}_\varepsilon^A : \overline{\Omega}\times[0,T_0] \to \mathbb{R}^m$ smooth for $\varepsilon \in (0,\check{\varepsilon}_0]$ with $\lim_{\varepsilon\to 0} \vec{u}_\varepsilon^A = \vec{u}_\pm$ uniformly on compact subsets of $Q_{T_0}^\pm$ and such that the following assertions hold:

1. *If $M > k(N)$, then let $\vec{u}_{0,\varepsilon} \in C^2(\overline{\Omega})^m$ with $\partial_{N_{\partial\Omega}}\vec{u}_{0,\varepsilon} = 0$ on $\partial\Omega$ for $\varepsilon \in (0,\check{\varepsilon}_0]$ and*

$$\sup_{\varepsilon\in(0,\check{\varepsilon}_0]} \|\vec{u}_{0,\varepsilon}\|_{L^\infty(\Omega)^m} < \infty \quad \textit{and} \quad \|\vec{u}_{0,\varepsilon} - \vec{u}_\varepsilon^A|_{t=0}\|_{L^2(\Omega)^m} \leq R\varepsilon^{M+\frac{1}{2}} \tag{1.7}$$

for some $R > 0$ and all $\varepsilon \in (0,\check{\varepsilon}_0]$. Then for any set of solutions $\vec{u}_\varepsilon \in C^2(\overline{Q_{T_0}})^m$ of (vAC1)-(vAC3) *for $\varepsilon \in (0,\check{\varepsilon}_0]$ with initial values $\vec{u}_{0,\varepsilon}$ there are $\check{\varepsilon}_1 \in (0,\check{\varepsilon}_0]$, $C > 0$ with*

$$\begin{aligned} \sup_{t\in[0,T]} \|(\vec{u}_\varepsilon - \vec{u}_\varepsilon^A)(t)\|_{L^2(\Omega)^m} + \|\nabla(\vec{u}_\varepsilon - \vec{u}_\varepsilon^A)\|_{L^2(Q_T\setminus\Gamma(\delta))^{N\times m}} &\leq C\varepsilon^{M+\frac{1}{2}}, \\ \|\nabla_\tau(\vec{u}_\varepsilon - \vec{u}_\varepsilon^A)\|_{L^2(Q_T\cap\Gamma(\delta))^{N\times m}} + \varepsilon\|\partial_n(\vec{u}_\varepsilon - \vec{u}_\varepsilon^A)\|_{L^2(Q_T\cap\Gamma(\delta))^m} &\leq C\varepsilon^{M+\frac{1}{2}} \end{aligned} \tag{1.8}$$

for all $\varepsilon \in (0,\check{\varepsilon}_1]$ and $T \in (0,T_0]$.

2. *If $k(N) \in \mathbb{N}$ and $M \geq k(N)+1$, then there is a $\check{R} > 0$ small such that the assertion in 1. holds, when R, M in* (1.7)-(1.8) *are replaced by $\check{R}, k(N)$.*

3. *If* $N \in \{2,3\}$ *and* $M = 2 (= k(N))$, *then there is* $\check{T}_1 \in (0, T_0]$ *such that the assertion in 1. is valid but only such that* (1.8) *holds for all* $\varepsilon \in (0, \check{\varepsilon}_1]$ *and* $T \in (0, \check{T}_1]$.

Remark 1.7.

1. The interpretation of Theorem 1.6 is analogous to the one of Theorem 1.2, where convergence of (AC) to MCF with 90°-contact angle is obtained, see Remark 1.3, 1.

2. *Layout of the Proof.* The new model problems, some vector-valued ODEs on $\mathbb{R}$ and a vector-valued linear elliptic equation on $\mathbb{R} \times (0, \infty)$ are solved in Section 4.3 and Section 4.4 below, respectively. The asymptotic expansions are done in Section 5.3 and the approximate solution $\vec{u}_\varepsilon^A$ is defined in Section 5.3.3. Note that M corresponds to the number of terms in the expansion. The spectral estimate is shown in Section 6.2 and the difference estimate is proven in Section 7.3.1. Finally, Theorem 1.6 is obtained in Section 7.3.2.

3. The comments for Theorem 1.2 in Remark 1.3, 3.-6. hold analogously. Here $\vec{u}_\pm$ has the role of ± 1 and the order of the approximation of Γ in the spirit of Remark 1.3, 4. is ε^2.

1.2.1 Notes on the Triple Junction Case

Let W be a triple-well potential as in Definition 1.4. The situation we have in mind is depicted in Figure 5. At the boundary contact points the angles are 90° and at the triple point 120°.

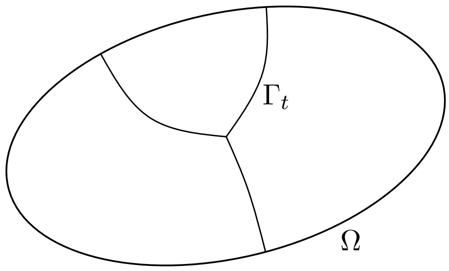

Figure 5: Triple junction.

Theorem 1.6 does not cover the situation of three-fold phase transitions in (vAC) and triple junctions for MCF, respectively. Nevertheless, because the arguments are localizable, due to Theorem 1.6 and its proof it is only left to consider the neighbourhood of a triple junction. This is work in progress. The plan roughly is as follows:

1. The construction is based on a smooth solution for MCF with triple junction. Local well-posedness is shown in Bronsard, Reitich [BR] with a parametric approach.

2. *Coordinates.* Show the existence of suitable curvilinear coordinates around the triple junction describing a neighbourhood in the surrounding space. The asymptotic expansion imposes conditions for the coordinates on the triple rod.

3. *Asymptotic Expansion.* Set up an appropriate asymptotic expansion at the triple point. This is challenging especially because in a straight-forward ansatz terms appear that blow up polynomially. This arises basically due to the asymptotics of the ansatz functions which have to match the two-phase transitions in the three interface directions. However, the situation is similar for the Allen-Cahn equation with nonlinear Robin-boundary condition,

where the sharp interface limit is MCF with an α-contact angle, see Section 1.3 below. In this case we solved the problem with suitable ε-scaled cut-off functions, see Section 5.4 below. This should work in the triple junction situation, too.

4. *Model Problems.* Solve the appearing model problems in the asymptotic expansion. The nonlinear equation appearing in the lowest order is considered in Bronsard, Gui, Schatzman [BGS]. It is left to solve the linearized equation appropriately. This should work similar to the approach in Section 4.3.3 below, where a linearized vector-valued ODE is solved. However, for this strategy one has to control the kernel of the associated linear operator. The condition on the potential W is similar to the one in Remark 4.28, but dimension equal two is natural due to the three-fold symmetry of W in $\mathbb{R}^2$. One has to justify this assumption by proving it for a typical triple-well potential.

5. *Spectral Estimate.* This should work similar to the 1D-type estimates in Chen [C2], cf. also the abstracted procedure for 1D-spectral estimates in Section 6.1 below. A fundamental difference to the spectral estimates in the boundary contact situations considered in Sections 6.2-6.5 is that there is no degeneracy in the spectrum due to the geometry. It should be possible to adapt many results on the spectrum and decay estimates in Kusche [Ku], Section 3, for the case without symmetry of the functions (but W is as above). This generalization is needed because symmetry of the approximate solution or the exact solution cannot be expected in case of curved geometries. In particular, uniform exponential decay estimates on large domains (in [Ku] equilateral triangles with smoothed edges) approximating $\mathbb{R}^2$ should help to prove the spectral estimate at the triple junction.

1.3 Scalar-valued Allen-Cahn Equation with Nonlinear Robin Boundary Condition, (AC$_\alpha$)

Let $N = 2$ and Ω, $N_{\partial\Omega}$, Q_{T_0}, ∂Q_{T_0} for some $T_0 > 0$ be as in Remark 1.1, 1. Let f be as in (1.1) such that (1.2) holds. We consider a fixed $\alpha \in (0, \pi)$. The parameter α will correspond to a fixed static contact angle of the limit interface in the sharp interface limit with boundary contact. Let $\sigma_\alpha : \mathbb{R} \to \mathbb{R}$ be smooth with $\operatorname{supp} \sigma'_\alpha \subset (-1, 1)$ and

$$\cos\alpha = \frac{\sigma_\alpha(-1) - \sigma_\alpha(1)}{\int_{-1}^{1} \sqrt{2(f(r) - f(-1))}\, dr}. \tag{1.9}$$

For the typical shape of σ_α see Figure 6 below. In order to fulfil the compatibility condition (1.9) and to have smoothness of σ_α with respect to α we choose σ_α for simplicity as follows:

Definition 1.8. Let $\hat{\sigma} : \mathbb{R} \to \mathbb{R}$ smooth with $\operatorname{supp} \hat{\sigma}' \subset (-1, 1)$ and such that

$$\hat{\sigma}(-1) - \hat{\sigma}(1) = \int_{-1}^{1} \sqrt{2(f(r) - f(-1))}\, dr.$$

Then we define $\sigma_\alpha := \cos\alpha\, \hat{\sigma}$.

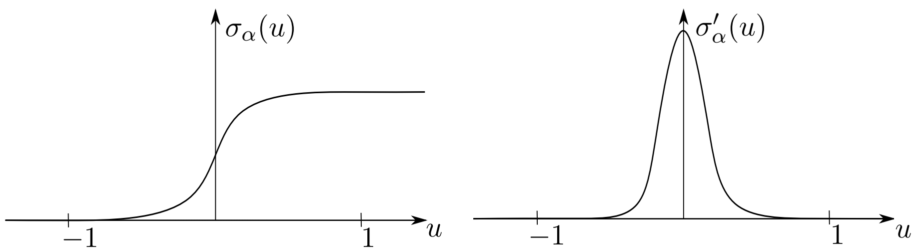

Figure 6: Typical shape of σ_α and σ'_α.

Let $\varepsilon > 0$ be small. For $u_{\varepsilon,\alpha} : \overline{\Omega} \times [0,T] \to \mathbb{R}$ we consider the Allen-Cahn equation with nonlinear Robin boundary condition, (AC$_\alpha$)

$$\partial_t u_{\varepsilon,\alpha} - \Delta u_{\varepsilon,\alpha} + \frac{1}{\varepsilon^2} f'(u_{\varepsilon,\alpha}) = 0 \qquad \text{in } Q_T, \tag{AC$_\alpha$1}$$

$$\partial_{N_{\partial\Omega}} u_{\varepsilon,\alpha} + \frac{1}{\varepsilon}\sigma'_\alpha(u_{\varepsilon,\alpha}) = 0 \qquad \text{on } \partial Q_T, \tag{AC$_\alpha$2}$$

$$u_{\varepsilon,\alpha}|_{t=0} = u_{0,\varepsilon,\alpha} \qquad \text{in } \Omega. \tag{AC$_\alpha$3}$$

Motivation of (AC$_\alpha$). One can directly verify that equation (AC$_\alpha$1)-(AC$_\alpha$3) is the (by in time $\frac{1}{\varepsilon}$-accelerated) L^2-gradient flow to the energy

$$E_{\varepsilon,\alpha}(u) := \int_\Omega \frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon} f(u)\, dx + \int_{\partial\Omega} \sigma_\alpha(u)\, d\mathcal{H}^{N-1}. \tag{1.10}$$

The first part is the usual (scalar) Ginzburg-Landau energy, cf. (1.3). The new term in the energy is a boundary contact energy. The idea is to adjust the gradient flow in the sense that distinct values of u are penalized differently when attained at the boundary. In Owen, Sternberg [OS] similar energies are considered but the possible boundary contact energy densities were different and rather physically motivated. The σ_α we use is motivated by the goal to obtain MCF with a static contact angle α in the sharp interface limit. Moreover, σ_α is chosen as simple as possible in order to shorten the proofs. It will turn out that f and σ_α are balanced suitably through (1.9). Note that (1.9) is reminiscent of the well-known Young's Equation that can be used to compute the contact angle of three adjacent media through surface tension relations.

Finally, note that Modica [Mo2] studied the Γ-convergence with respect to $\varepsilon \to 0$ for energies of the form (1.10) with mass and nonnegativity constraint. The Γ-limits are perimeter functionals, where additionally the perimeter of some part of the boundary is added but weighted with a constant corresponding to the potential and the boundary contact energy. This also motivates to study the dynamical problem (AC$_\alpha$1)-(AC$_\alpha$3) associated to (1.10) and the relation to MCF with contact angle distinct from $90°$ in the limit $\varepsilon \to 0$.

Well-Posedness of (AC$_\alpha$). The nonlinear boundary condition makes the analysis more difficult compared to (AC). Nevertheless, σ'_α is zero outside $(-1,1)$ and one can still obtain a priori boundedness of classical solutions, see Section 7.1.1 below. One possibility to show well-posedness is to construct weak solutions via time-discretization. Another way is to apply semigroup methods. Equation (AC$_\alpha$1)-(AC$_\alpha$3) fits for example in the abstract setting of Lunardi [Lu], Section 8.5.3. There local well-posedness in a Hölder-setting is obtained by linearization at the initial value and a fixed point argument. Then one can extend the solution to a maximal time interval. Higher regularity and smoothness can be obtained with linear theory, cf. Lunardi, Sinestrari, von Wahl [LSW]. In order to show global existence of solutions, one has to prove boundedness of the solution in $C^{2,\beta}(\overline{\Omega})$ with respect to time for some $\beta > 0$ small. Then the existence interval in the local well-posedness result can be taken uniformly. Here $C^{2,\beta}(\overline{\Omega})$ is the usual Hölder-space, see Definition 2.1 below. It should be possible to prove this with the a priori uniform boundedness and a "bootstrap argument", i.e. subsequently improving regularity or uniform estimates by using the equations and embeddings. More precisely, one could use extension operators for $C^k(\partial\Omega)$-Neumann boundary values ($k \geq 0$; see [LSW], Theorem 6.2.) and then L^p-maximal regularity for linear parabolic boundary value problems with zero Neumann boundary condition, see e.g. Prüss, Simonett [PS], Section 6.3. This way uniform estimates for regularity in time are not obtained, but one can use higher spatial regularity in the base space, see [PS], Section 6.3.5.

Formal Sharp Interface Limit for (AC$_\alpha$). In Owen, Sternberg [OS] formal asymptotic analysis is used to compute the contact angle in the sharp interface limit for some boundary contact energy densities. The calculations can be adapted for the σ_α in Definition 1.8, see also Remark 4.13 below. This yields that in the sharp interface limit $\varepsilon \to 0$ we should obtain MCF with an α-contact angle.

Rigorous Sharp Interface Limit Results for (AC$_\alpha$). As far as the author knows, there is no rigorous result so far in the literature. However, note that on the energy level there is a preparatory result in a varifold setting, see Kagaya, Tonegawa [KaT], in particular the remarks in [KaT], Section 5.3.

The Rigorous Sharp Interface Limit Result for (AC$_\alpha$). We obtain the following result:

Theorem 1.9 (Convergence of (AC$_\alpha$) to MCF with α-Contact Angle). *There is an $\overline{\alpha}_0 > 0$ small such that the following holds. Let $N = 2$, Ω, $N_{\partial\Omega}$, Q_T and ∂Q_T for $T > 0$ be as in Remark 1.1, 1. Moreover, let $\Gamma = (\Gamma_t)_{t\in[0,T_0]}$ for some $T_0 > 0$ be a smooth evolving hypersurface with α-contact angle as in Remark 1.1, 2. for fixed $\alpha \in \frac{\pi}{2} + [-\overline{\alpha}_0, \overline{\alpha}_0]$ and let Γ satisfy MCF. Let $\delta > 0$ small and the notation for $Q^\pm_{T_0}$, $\Gamma(\delta)$, ∇_τ, ∂_n be as in Remark 1.1, 3. Moreover, let f satisfy* (1.1)-(1.2) *and σ_α be as in Definition 1.8. Let $M \in \mathbb{N}$ with $M \geq 3$.*

Then there are $\delta_0 \in (0, \delta]$, $\varepsilon_0 > 0$ and $u^A_{\varepsilon,\alpha} : \overline{\Omega} \times [0, T_0] \to \mathbb{R}$ smooth for $\varepsilon \in (0, \varepsilon_0]$ such that $\lim_{\varepsilon\to 0} u^A_{\varepsilon,\alpha} = \pm 1$ uniformly on compact subsets of $Q^\pm_{T_0}$ and following assertions hold:

1. *If $M \geq 4$, then let $u_{0,\varepsilon,\alpha} \in C^2(\overline{\Omega})$ with $\partial_{N_{\partial\Omega}} u_{0,\varepsilon,\alpha} + \frac{1}{\varepsilon}\sigma'_\alpha(u_{0,\varepsilon,\alpha}) = 0$ on $\partial\Omega$ for $\varepsilon \in (0, \varepsilon_0]$ and such that for some $R > 0$ and all $\varepsilon \in (0, \varepsilon_0]$ it holds*
$$\sup_{\varepsilon\in(0,\varepsilon_0]} \|u_{0,\varepsilon,\alpha}\|_{L^\infty(\Omega)} < \infty \quad \text{and} \quad \|u_{0,\varepsilon,\alpha} - u^A_{\varepsilon,\alpha}|_{t=0}\|_{L^2(\Omega)} \leq R\varepsilon^M. \tag{1.11}$$
Then for any set of solutions $u_{\varepsilon,\alpha} \in C^2(\overline{Q_{T_0}})$ of (AC$_\alpha$1)-(AC$_\alpha$3) *for $\varepsilon \in (0, \varepsilon_0]$ with initial values $u_{0,\varepsilon,\alpha}$ there are $\varepsilon_1 \in (0, \varepsilon_0]$, $C > 0$ such that*
$$\begin{aligned}\sup_{t\in[0,T]} \|(u_{\varepsilon,\alpha} - u^A_{\varepsilon,\alpha})(t)\|_{L^2(\Omega)} + \|\nabla(u_{\varepsilon,\alpha} - u^A_{\varepsilon,\alpha})\|_{L^2(Q_T\setminus\Gamma(\delta_0))} &\leq C\varepsilon^M,\\ \sqrt{\varepsilon}\|\nabla_\tau(u_{\varepsilon,\alpha} - u^A_{\varepsilon,\alpha})\|_{L^2(Q_T\cap\Gamma(\delta_0))} + \varepsilon\|\partial_n(u_{\varepsilon,\alpha} - u^A_{\varepsilon,\alpha})\|_{L^2(Q_T\cap\Gamma(\delta_0))} &\leq C\varepsilon^M\end{aligned} \tag{1.12}$$
for all $\varepsilon \in (0, \varepsilon_1]$ and $T \in (0, T_0]$.

2. *If $M \geq 4$, then there is a $\tilde{R} > 0$ small such that the assertion in 1. holds, when R, M in* (1.11)-(1.12) *are replaced by $\tilde{R}, 3$.*

3. *If $M = 3$, then there is $T_1 \in (0, T_0]$ such that the assertion in 1. is valid but only such that* (1.12) *holds for all $\varepsilon \in (0, \varepsilon_1]$ and $T \in (0, T_1]$.*

Remark 1.10. 1. The interpretation of Theorem 1.9 is analogous to the one of Theorem 1.2, where convergence of (AC) to MCF with 90°-contact angle is obtained, see Remark 1.3, 1.

2. *Layout of the Proof.* The new model problems, a nonlinear elliptic problem on $\mathbb{R} \times (0, \infty)$ and the linearized problem are considered in Section 4.2.2 below. Note that interestingly, condition (1.9) turns out to be a necessary (and at least for α close to $\frac{\pi}{2}$ sufficient) condition for the solvability of the nonlinear equation, see Remark 4.13. The asymptotic expansions are carried out in Section 5.4 and the approximate solution $u^A_{\varepsilon,\alpha}$ is defined in Section 5.4.3. Note that M corresponds to the number of terms in the expansion. The spectral estimate is done in Section 6.5 and the difference estimate is shown in Section 7.4.1. Finally, Theorem 1.9 is proven in Section 7.4.2.

3. *Origin of* $\overline{\alpha}_0$. Theorem 1.9 is only shown for a small but uniform $\overline{\alpha}_0 > 0$. Let us comment at this point, where this restriction comes from. First, note that there is no restraint arising from the construction of the curvilinear coordinates in Section 3.2. The first restriction enters when we use the elliptic problem on $\mathbb{R} \times (0, \infty)$ from the 90°-case and the Implicit Function Theorem with respect to α to solve the model problems in Section 4.2.2, see also Section 4.2.2.3. The second restriction origins from the proof of the spectral estimate in Section 6.5. The reason is that we adapt the proof from the $\frac{\pi}{2}$-case in Section 6.2 and choose $\overline{\alpha}_0 > 0$ small such that similar arguments work, see also Remark 6.52, 1. The precise restriction on $\overline{\alpha}_0$ is manifested in Remark 5.33 and Theorem 6.51.

4. The comments for Theorem 1.2 in Remark 1.3, 3.-6. hold analogously, but the order of the approximation of Γ in the sense of Remark 1.3, 4. is ε in general. Cf. Remark 5.36.

Acknowledgements

First and foremost, I would like to thank my supervisor Prof. Dr. Helmut Abels for providing me a both very interesting and challenging topic for the thesis. I am deeply grateful for the continuous support and patience throughout the years, for various excellent lectures and many interesting discussions. Moreover, I gratefully acknowledge support through DFG, GRK 1692 “Curvature, Cycles and Cohomology”. Next, I would like to thank all my colleagues for the nice atmosphere and the enjoyable time at university. In particular, I am grateful to my office colleagues Dr. Simone Murro, Dr. Andreas Marquardt and Tobias Ameismeier for entertaining days at office and exciting discussions on mathematics. Especially I greatly thank Dr. Andreas Marquardt for sharing interesting insights within the theory of sharp interface limits. Finally, I am thankful to my family, especially to my mother, for the steady encouragement and support.

2 Notation and Function Spaces

Let $\mathbb{N}$ be the natural numbers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The symbol $\mathbb{K}$ stands for an element of $\{\mathbb{R}, \mathbb{C}\}$. Moreover, the Euclidean norm in $\mathbb{R}^m$, $m \in \mathbb{N}$ and the Frobenius norm in $\mathbb{R}^{m \times n}$, $m, n \in \mathbb{N}$ are for convenience denoted by $|.|$. The symbol " $\vec{}$ " indicates a vector or a vector-valued function. Moreover, objects (e.g. vectors, operators and constants) that are associated to a vector-valued setting often get the addition " $\check{}$ ". Furthermore, a subset $\Omega \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$ is called "domain", if Ω is open, nonempty and connected. Additionally, restrictions or evaluations of functions are often indicated by "$|_.$". The differential operators ∇, div and D^2 are defined to act just on spatial variables. Let X be a set and Y a normed space. Then $B(X, Y) := \{f : X \to Y \text{ bounded}\}$. Let X, Y be normed spaces over $\mathbb{K}$. Then $\mathcal{L}(X, Y)$ denotes the set of bounded linear operators $T : X \to Y$. Finally, note that we use the usual constant convention.

2.1 Unweighted Continuous and Continuously Differentiable Functions

Definition 2.1. Let $n, k \in \mathbb{N}$ and $\Omega \subseteq \mathbb{R}^n$ open and nonempty. Moreover, let B be a Banach space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Then

1. $C^0(\Omega, B) := \{f : \Omega \to B \text{ continuous}\}$ and analogously we define $C^0(\overline{\Omega}, B)$. Moreover,

$$C^k(\Omega, B) := \{f \in C^0(\Omega, B) : f \text{ is } k\text{-times countinuously Fréchet-differentiable}\},$$
$$C^k(\overline{\Omega}, B) := \{f \in C^k(\Omega, B) : f \text{ and derivatives up to order } k \text{ have } C^0\text{-extension to } \overline{\Omega}\}.$$

2. The spaces including boundedness for the function and all appearing derivatives are denoted with $C^0_b(\Omega, B)$, $C^0_b(\overline{\Omega}, B)$, $C^k_b(\Omega, B)$, $C^k_b(\overline{\Omega}, B)$ and equipped with the natural norms.

3. Above spaces with "∞" instead of "k" are defined via the intersection over all $k \in \mathbb{N}$.

4. $C^{k,\gamma}(\overline{\Omega}, B) := \{f \in C^k_b(\overline{\Omega}, B) : \|f\|_{C^{k,\gamma}(\overline{\Omega}, B)} < \infty\}$ are the Hölder-spaces, where

$$\|f\|_{C^{0,\gamma}(\overline{\Omega},B)} := \|f\|_{C^0_b(\overline{\Omega},B)} + \sup_{x,y \in \overline{\Omega}, x \neq y} \frac{\|f(x) - f(y)\|_B}{|x - y|^\gamma},$$
$$\|f\|_{C^{k,\gamma}(\overline{\Omega},B)} := \|f\|_{C^k_b(\overline{\Omega},B)} + \sup_{\beta \in \mathbb{N}_0^n, |\beta| = k} \|\partial_x^\beta f\|_{C^{0,\gamma}(\overline{\Omega},B)}.$$

5. Let U be an open subset of the interior $M^\circ := M \setminus \partial M$ of a smooth compact manifold M with (or without) boundary, where ∂M is defined via charts. Then $C^k(U, B)$, $C^k(\overline{U}, B)$, $C^k_b(U, B)$ and $C^k_b(\overline{U}, B)$ for every $k \in \mathbb{N}_0 \cup \{\infty\}$ are defined via local coordinates and the respective spaces on domains.

6. If $B = \mathbb{K}$ and it is clear from the context if $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, then we omit B in the notation.

7. $C^\infty_0(\Omega)$ is the set of $f \in C^\infty(\Omega, \mathbb{R})$ with compact support $\operatorname{supp} f \subseteq \Omega$. Moreover, $C^\infty_0(\overline{\Omega})$ denotes $\{f|_{\overline{\Omega}} : f \in C^\infty_0(\mathbb{R}^n)\}$.

Lemma 2.2. *Let U be an open subset of M°, where M is a smooth compact manifold M with (or without) boundary and dimension $l \in \mathbb{N}$. Let $x_j : U_j \to V_j \subseteq [0,\infty) \times \mathbb{R}^{l-1}$ for $j = 1, ..., L$ be charts of M and W_j open in $[0,\infty) \times \mathbb{R}^{l-1}$ with $\overline{W_j} \subset V_j$ compact for $j = 1, ..., L$ such that $\bigcup_{j=1}^{L} x_j^{-1}(W_j) = M$. Then $C_b^k(U, B)$ for $k \in \mathbb{N}_0$ is a Banach space with*

$$\|f\|_{C_b^k(U,B)} := \sum_{j=1}^{L} \|f \circ x_j^{-1}|_{x_j(U\cap U_j)\cap W_j}\|_{C_b^k(x_j(U\cap U_j)\cap W_j,B)} \quad \text{for all } f \in C_b^k(U,B).$$

Different choices of x_j, W_j yield equivalent norms. The analogous assertion holds for $C_b^k(\overline{U}, B)$.

Proof. The Banach space property follows directly. Moreover, the assumptions ensure that for different choices of x_j, W_j the relevant chart transformations have C^k-extensions to the closure of their domain. These induce bounded linear transformations of the associated C_b^k-spaces. □

2.2 Unweighted Lebesgue- and Sobolev-Spaces

2.2.1 Lebesgue-Spaces

Let $(M, \mathcal{A}, \mu)$ be a σ-finite, complete measure space and B be a Banach space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Then one can define the notions of (μ- or strongly-) measurable and (Bochner-)integrable functions $f : M \to B$ and the *Bochner(-Lebesgue)-Integral*, see Amann, Escher [AE], Chapter X for the definitions and properties. In particular the *Lebesgue-spaces* $L^p(M, B)$ for $1 \leq p \leq \infty$ are defined. If $B = \mathbb{K}$ and it is clear from the context if $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, then we omit B in the notation. We also use the Fubini Theorem for scalar-valued functions on σ-finite measure spaces, see Elstrodt [El], Satz V.2.4.

Later we need the notion of the support of a measurable function:

Remark 2.3. Let $\Omega \in \mathbb{R}^n$, $n \in \mathbb{N}$ be open and $f : \Omega \to B$ measurable. Then the support of f is

$$\operatorname{supp} f := \left[\bigcup_{U\subset\Omega \text{ open} : f=0 \text{ a.e. in } U} U \right]^c.$$

With topological properties of $\mathbb{R}^n$ one can show that $\operatorname{supp} f$ is closed and $f = 0$ a.e. on $(\operatorname{supp} f)^c$. Moreover, $\operatorname{supp}(f + g) \subseteq \operatorname{supp} f \cup \operatorname{supp} g$ for all $f, g : \Omega \to B$ measurable. Finally, for continuous f it holds $\operatorname{supp} f = \overline{\{x \in \Omega : f(x) \neq 0\}}$.

Moreover, we need the following transformation theorem:

Theorem 2.4 (Substitution Rule). *Let $U, V \subseteq \mathbb{R}^n$ be open, nonempty and $\Phi : U \to V$ be a C^1-diffeomorphism. Moreover, let B be a Banach space and $f : V \to B$. Then $f \in L^1(V, B)$ if and only if $(f \circ \Phi)|\det\, D\Phi| \in L^1(U, B)$. In this case it holds*

$$\int_V f\,dy = \int_U (f \circ \Phi)|\det\, D\Phi|\,dx.$$

This is [AE], Theorem X.8.14. Note that the corresponding assertion for measurable functions in general only holds if additionally f is almost separable-valued, cf. [AE], Theorem X.1.4.

Remark 2.5. Let (M, g) be a C^1-Riemannian submanifold of $\mathbb{R}^n$ with (or without) boundary. We denote with ∂M the boundary (defined via charts) and $M^\circ := M \setminus \partial M$ is the interior.

1. Let $\mathcal{L}_M$ be the *Lebesgue σ-Algebra of M* and λ_M the *Riemann-Lebesgue Volume Measure of M*. See [AE], Chapter XI and Chapter XII.1 for the definitions and properties. In particular $(M, \mathcal{L}_M, \lambda_M)$ is a σ-finite complete measure space (and satisfies many more properties). Therefore the Bochner-Integral and the Lebesgue spaces are defined.

2. For simplicity let g be the Euclidean Metric and let M have dimension m. Then one can show that λ_M coincides with the (properly scaled) m-dimensional Hausdorff-measure $\mathcal{H}^m$ on M. This should be even true for Lipschitz-submanifolds M, cf. Evans, Gariepy [EG], Chapter 3.3.4. For the definitions and properties of Hausdorff-measures, in particular the connections to Lebesgue measure cf. Evans, Gariepy [EG], Chapter 2. Therefore in the application later we write $\mathcal{H}^m$ instead of λ_M for convenience.

Finally, we show a transformation theorem for Riemannian submanifolds. Note that later we will only need the Euclidean metric, but the proof for the general case is the same.

Theorem 2.6 (**Substitution Rule for Riemannian Submanifolds of $\mathbb{R}^n$**). *Let (M, g) and (N, h) be C^1-Riemannian submanifolds of $\mathbb{R}^n$ with (or without) boundary and dimension m. Moreover, let $U \subset M^\circ$, $V \subset N^\circ$ be open and $\Phi : U \to V$ be a C^1-diffeomorphism. Then*

1. *Define $|\det d\Phi| : U \to \mathbb{R} : p \mapsto |\det d_p\Phi|$, where the latter is defined as the modulus of the determinant of the representation matrix of $d_p\Phi$ with respect to arbitrary orthonormal bases of T_pM and $T_{\Phi(p)}N$ for all $p \in U$. Then $|\det d\Phi|$ is well-defined and in $C^1(U, \mathbb{R})$.*

2. *Let B be a Banach space and consider $f : N \to B$. Then $f \in L^1(V, B)$ if and only if $(f \circ \Phi)|\det\, d\Phi| \in L^1(U, B)$. In this case it holds*
$$\int_V f\, d\lambda_N = \int_U (f \circ \Phi)|\det\, d\Phi|\, d\lambda_M.$$

Proof. Ad 1. The definition is independent of the choice of the orthonormal bases since the representation matrix corresponding to the change of basis on each of the tangent spaces has determinant ± 1. Via local representations one can prove that $|\det d\Phi| \in C^1(U, \mathbb{R})$. $\square_1$.

Ad 2. The assertion is compatible with restrictions on U and V. Therefore we can assume w.l.o.g. $U = M$ and $V = N$. Moreover, it is enough to prove one direction. Let $f \in L^1(V, B)$ and let (ψ, W) be a chart of M. Then $(\psi \circ \Phi^{-1}, \Phi(W))$ is a chart of N. Let $(g_{ij})_{i,j=1}^m$, $(h_{ij})_{i,j=1}^m$ be the local representations of g and h corresponding to (ψ, W) and $(\psi \circ \Phi^{-1}, \Phi(W))$, respectively. Furthermore, we set $G := \det[(g_{ij})_{i,j=1}^m]$ and $H := \det[(h_{ij})_{i,j=1}^m]$. The latter are viewed as maps from $\psi(W)$ to $\mathbb{R}$. Then Amann, Escher [AE], Theorem XII.1.10 yields $(f \circ (\psi \circ \Phi^{-1})^{-1})\sqrt{H} \in L^1(\psi(W), B)$. Choosing orthonormal bases for the related tangent spaces, one can show with the chain rule that $\sqrt{H} = |\det d\Phi| \circ \psi^{-1}\sqrt{G}$. Therefore we obtain
$$[(f \circ \Phi)|\det d\Phi|] \circ \psi^{-1}\sqrt{G} \in L^1(\psi(W), B)$$
and [AE], Theorem X.1.10 yields $(f \circ \Psi)|\det d\Phi| \in L^1(W, B)$ as well as
$$\int_{\Phi(W)} f\, d\lambda_N = \int_W (f \circ \Psi)|\det d\Phi|\, d\lambda_M.$$
In particular $(f \circ \Psi)|\det d\Phi| : M \to B$ is λ_M-measurable, cf. e.g. [AE], Proposition XII.1.8 and Theorem X.1.14. Finally, via Φ one can push forward a countable atlas for M and a corresponding C^1-partition of unity. Hence analogous computations as above and [AE], Proposition XII.1.11 yield the claim. $\square_2$.

2.2.2 Sobolev-Spaces on Domains in $\mathbb{R}^n$

Definition 2.7. 1. Let $\Omega \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$ be open and nonempty. Moreover, let $k \in \mathbb{N}_0$, $1 \leq p \leq \infty$ and B be a Banach space. Then $W^{k,p}(\Omega, B)$ are the usual *Sobolev-spaces*, where $W^{0,p}(\Omega, B) := L^p(\Omega, B)$. We also write $H^k(\Omega, B)$ instead of $W^{k,2}(\Omega, B)$. If $B = \mathbb{K}$ and it is clear from the context if $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, then we omit B in the notation.

2. Let $n \in \mathbb{N}$. Then $H^\beta(\mathbb{R}^n)$ for $\beta > 0$ are the well-known L^2-*Bessel-Potential spaces* and $W^{k+\mu,p}(\mathbb{R}^n)$ for $k \in \mathbb{N}_0$, $\mu \in (0,1)$ and $1 \leq p < \infty$ the *Sobolev-Slobodeckij spaces*.

For the definitions and properties of scalar-valued function spaces, in particular embeddings, interpolation results and trace theorems see Adams, Fournier [AF], Alt [Al], Leoni [Le] and Triebel [T1], [T2]. Many properties can be generalized to vector-valued function spaces over domains, see e.g. Kreuter [Kr] and the references therein. In particular:

Lemma 2.8. *Let $\Omega \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$ be open and nonempty, $k \in \mathbb{N}_0$, $1 \leq p < \infty$ and B be a Banach space. Then $C^\infty(\Omega, B) \cap W^{k,p}(\Omega, B)$ is dense in $W^{k,p}(\Omega, B)$.*

Proof. This follows via convolution analogously to the scalar case, cf. [Kr], Chapter 4.2. □

For transformations of Sobolev spaces we use

Theorem 2.9. *Let $\Omega_1, \Omega_2 \subseteq \mathbb{R}^n$ be open, nonempty and bounded. Moreover, let $l \in \mathbb{N}$, $l \geq 1$, $1 \leq p < \infty$ and B be a Banach space. Let $\Phi : \Omega_1 \to \Omega_2$ be a C^l-diffeomorphism with $\Phi \in C^l(\overline{\Omega_1})^n$ and $\Phi^{-1} \in C^l(\overline{\Omega_2})^n$ such that*

$$|\det D(\Phi^{-1})| \leq R_1 \quad \text{and} \quad \|\Phi\|_{C_b^l(\overline{\Omega_1})^n} \leq R_2.$$

Then $T : W^{k,p}(\Omega_2, B) \to W^{k,p}(\Omega_1, B) : f \mapsto f \circ \Phi$ is well-defined, continuous and linear for all $k \in \mathbb{N}_0$ with $0 \leq k \leq l$ and the operator norm is bounded by some $C(R_1) > 0$ if $k = 0$ and bounded by $C(R_1, R_2, p, l) > 0$ if k is arbitrary.

Proof. For $B = \mathbb{K}$ this follows from the proof of Adams, Fournier [AF], Theorem 3.41. One only needs density of $C^\infty(\Omega_2) \cap W_p^k(\Omega_2)$ in $W_p^k(\Omega_2)$, the substitution rule and the chain rule. Due to Lemma 2.8 and Theorem 2.4 analogous arguments apply for general B. □

For simplicity we only consider scalar-valued functions in the remainder of this section. In the following we need to know how Lebesgue and Sobolev spaces behave on product sets.

Lemma 2.10. *Let $\Omega_1 \subseteq \mathbb{R}^m, \Omega_2 \subseteq \mathbb{R}^n$ for $m, n \in \mathbb{N}$ be measurable. Then*

1. *Let $1 \leq p < \infty$ and $f \in L^p(\Omega_1 \times \Omega_2)$. Then $f(x_1, .) \in L^p(\Omega_2)$ for a.e. $x_1 \in \Omega_1$ and $Tf : \Omega_1 \to L^p(\Omega_2) : x_1 \mapsto f(x_1, .)$ is an element of $L^p(\Omega_1, L^p(\Omega_2))$. Moreover, the map $T : L^p(\Omega_1 \times \Omega_2) \to L^p(\Omega_1, L^p(\Omega_2))$ is an isometric isomorphism.*

2. *Let Ω_1, Ω_2 be open and $1 < p < \infty$. Then by restriction of T from 1. it holds*

$$W^{1,p}(\Omega_1 \times \Omega_2) \cong L^p(\Omega_1, W^{1,p}(\Omega_2)) \cap W^{1,p}(\Omega_1, L^p(\Omega_2))$$

and derivatives are compatible via T. Analogous assertions hold for higher orders.

3. *Let Ω_1, Ω_2 open, $k \in \mathbb{N}_0$ and $1 \leq p \leq \infty$. Then for $f \in W^{k,p}(\Omega_1)$ and $g \in W^k_p(\Omega_2)$ the product $(f \otimes g)(x_1, x_2) := f(x_1)g(x_2)$ is well-defined for a.e. $(x_1, x_2) \in \Omega_1 \times \Omega_2$. Moreover, it holds $f \otimes g \in W^{k,p}(\Omega_1 \times \Omega_2)$, the derivatives are natural and*

$$\|f \otimes g\|_{W^{k,p}(\Omega_1\times\Omega_2)} \leq C_{k,p}\|f\|_{W^{k,p}(\Omega_1)}\|g\|_{W^{k,p}(\Omega_2)}.$$

For $\Omega_1 = \mathbb{R}$, $\Omega_2 = \mathbb{R}_+$, $k = 1$ and $1 \leq p < \infty$ the trace is given by $\mathrm{tr}_{\partial\mathbb{R}^2_+}(f \otimes g) = g(0)f$.

Note that we restricted p in the second assertion due to a duality argument.

Proof. The first two assertions follow with ideas from Růžička [R], Chapter 2.1.1 and the Paragraph “Zusammenhang mit elementaren Definitionen” in Schweizer [Sw], p.188f. The first part of the third claim can be proven directly with the definitions. The trace assertion follows with a density argument. □

Moreover, we need the notion of domains with Lipschitz-boundary. See Alt [Al], Section A8.2 for the precise definition of a bounded Lipschitz-domain. We generalize this definition for parts of the boundary as a preparation for the next section.

Definition 2.11 (Lipschitz Condition). Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$ be open and nonempty. Then

1. Let $x \in \partial\Omega$. We say that Ω satisfies the *local Lipschitz condition* in x if $\partial\Omega$ is locally at x the graph of a Lipschitz function in a suitable orthogonal coordinate system such that Ω lies on one side of the graph. Cf. Alt [Al], Section A8.2 for more details.

2. We say Ω has *Lipschitz-boundary* if the local Lipschitz condition holds in x for all $x \in \partial\Omega$.

3. We call Ω a *Lipschitz-domain* if Ω is a domain and has Lipschitz-boundary.

Remark 2.12 (Integral on the Boundary of Bounded Lipschitz Domains in $\mathbb{R}^n$). Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$ be open and nonempty. Moreover, let Σ be open in $\partial\Omega$ (for example $\Sigma = \partial\Omega$) and assume that Ω satisfies the local Lipschitz condition in every point in the compact set $\overline{\Sigma}$. For simplicity we only consider scalar-valued functions.

1. Due to the Rademacher-Theorem, cf. Evans, Gariepy [EG], Chapter 3.1, one can define the notions of measurable and integrable functions $f : \Sigma \to \mathbb{R}$ and the integral over Σ in a natural way, cf. e.g. Alt [Al], Section A8.5. Then for $1 \leq p \leq \infty$ we denote the usual Lebesgue spaces by $L^p(\Sigma)$. The definitions are the same as the ones via $\mathcal{H}^{n-1}$ on Σ, cf. [EG], Chapter 3.3.4 and also the proof of [Al], A8.5(2).

2. The outer unit normal $N_{\partial\Omega}$ to Ω can be defined a.e. on Σ, cf. [Al], A8.5(3).

3. Let additionally $\Sigma = M \cup Z$ with a C^1-hypersurface M of $\mathbb{R}^n$ and a null-set Z with respect to $\mathcal{H}^{n-1}$ on $\mathbb{R}^n$. Then the notions in 1. are equivalent to the ones coming from the measure space $(M, \mathcal{L}_M, \lambda_M)$ introduced in Remark 2.5, 1. One can prove this by going into the constructions or via identification with $\mathcal{H}^{n-1}$ on Σ, cf. 1. and Remark 2.5, 2.

We need some properties of Sobolev spaces on domains in $\mathbb{R}^n$, where parts of the boundary satisfy the Lipschitz condition:

Theorem 2.13. *Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$ be open, bounded and let Σ be open in $\partial\Omega$ (e.g. $\Sigma = \partial\Omega$) such that Ω satisfies the local Lipschitz condition in every point in $\overline{\Sigma}$. Let $1 \leq p < \infty$. Then*

1. *$\{\psi \in C_0^\infty(\overline{\Omega}) : \operatorname{supp}\psi \subset \Omega \cup \Sigma\}$ is dense in $\{f \in W^{k,p}(\Omega) : \operatorname{supp} f \subset \Omega \cup \Sigma\}$ for all $k \in \mathbb{N}_0$, where $\operatorname{supp} f$ for measurable $f : \Omega \to \mathbb{R}$ is defined in Remark 2.3.*
2. *$C^0(\Omega \cup \overline{\Sigma}) \cap W^{1,p}(\Omega)$ is dense in $W^{1,p}(\Omega)$.*
3. *There is a unique bounded linear operator $\operatorname{tr} : W^{1,p}(\Omega) \to L^p(\Sigma)$ such that $\operatorname{tr} u = u|_\Sigma$ for all $u \in C^0(\Omega \cup \overline{\Sigma}) \cap W^{1,p}(\Omega)$.*
4. *Let $\Sigma = \partial\Omega$, i.e. Ω has Lipschitz-boundary. Then the Gauß Theorem holds for $W^{1,1}$-functions in weak form.*

Proof. Ad 1. This can be shown via localization with a suitable partition of unity and convolution similar to the proof of Alt [Al], Lemma A8.7. $\square_1$.

Ad 2. Note that due to compactness of $\overline{\Sigma}$, there is another set $\tilde{\Sigma}$ open in $\partial\Omega$ such that $\overline{\Sigma} \subset \tilde{\Sigma}$ and $\tilde{\Sigma}$ satisfies analogous properties as Σ. Therefore one can combine 1. and Lemma 2.8 with a partition of unity to show density of $C^0(\Omega \cup \overline{\Sigma}) \cap W^{1,p}(\Omega)$ in $W^{1,p}(\Omega)$. $\square_2$.

Ad 3. The proof of [Al], Theorem A8.6 can be directly adapted. $\square_3$.

Ad 4. See Alt [Al], Theorem A8.8. $\square_4$.

2.2.3 Sobolev Spaces on Domains in Compact Submanifolds of $\mathbb{R}^n$

In this section let (M, g) be a m-dimensional compact Riemannian submanifold of $\mathbb{R}^n$ with (or without) boundary and class C^l, where $l \in \mathbb{N} \cup \{\infty\}$, $l \geq 1$. Let $U \subset M^\circ$ be open and nonempty. Moreover, let B be a Banach space. Then $L^p(U, B)$ is defined due to Remark 2.5, 1.

Definition 2.14. Let $x_j : U_j \to V_j \subseteq [0, \infty) \times \mathbb{R}^{m-1}$ for $j = 1, ..., N$ be charts of M and W_j open in $[0, \infty) \times \mathbb{R}^{m-1}$ with $\overline{W_j} \subset V_j$ compact for $j = 1, ..., N$ and $\bigcup_{j=1}^N x_j^{-1}(W_j) = M$.

1. Then for $k \in \mathbb{N}$, $k \leq l$ and $1 \leq p < \infty$ we define the *Sobolev spaces*

   $$W^{k,p}(U, B) := \{f \in L^p(U, B) : f \circ x_j^{-1}|_{x_j(U \cap U_j) \cap W_j} \in W^{k,p}(x_j(U \cap U_j) \cap W_j, B) \, \forall j\}.$$

2. For $f \in W^{1,p}(U, B)$, $1 \leq p < \infty$ we define (in analogy to the scalar case) the *surface gradient*

   $$[\nabla_U f] \circ x_j^{-1}|_{x_j(U \cap U_j) \cap W_j} := \sum_{r,s=1}^m g^{rs} \circ x_j^{-1} \partial_{y_r}[f \circ x_j^{-1}|_{x_j(U \cap U_j) \cap W_j}] \partial_{y_s}(x_j^{-1})$$

   for all $j = 1, ..., N$, where $(g^{rs})_{r,s=1}^m$ is the inverse of the representation matrix of g with respect to x_j and the product of $\partial_{y_r}[f \circ x_j^{-1}|_{x_j(U \cap U_j) \cap W_j}] \in L^p(x_j(U \cap U_j) \cap W_j, B)$ with $\partial_{y_s}(x_j^{-1}) \in C^l(\overline{x_j(U \cap U_j) \cap W_j}, \mathbb{R}^n)$ is understood component-wise.

Lemma 2.15. *Consider the situation of Definition 2.14. Let $k \in \mathbb{N}$, $k \leq l$ and $1 \leq p < \infty$. Then*

1. *$W_p^k(U, B)$ is a Banach space with norm*

   $$\|f\|_{W^{k,p}(U,B)}^* := \sum_{j=1}^N \|f \circ x_j^{-1}|_{x_j(U \cap U_j) \cap W_j}\|_{W_p^k(x_j(U \cap U_j) \cap W_j, B)} \quad \textit{for all } f \in W_p^k(U, B).$$

   Different choices of (x_j, W_j) yield the same spaces with equivalent norms.

2. *$C^l(U,B) \cap W^k_p(U,B)$ is dense in $W^k_p(U,B)$.*

3. *$\nabla_U f$ is well-defined for all $f \in W^{1,p}(U,B)$ and independent of the choice of (x_j, W_j). Moreover, $\nabla_U f \in L^p(U,B^n)$ and*

$$\|f\|_{W^{1,p}(U,B)} := \|f\|_{L^p(U,B)} + \|\nabla_U f\|_{L^p(U,B^n)}$$

defines an equivalent norm on $W^{1,p}(U,B)$.

Later on $W^{1,p}(U,B)$ we always take the norm in Lemma 2.15, 3. Note that for higher orders one can also define coordinate independent norms, cf. with Hebey [He], Chapter 2 in the scalar case. Nevertheless, later we only need $W^{1,p}(U,B)$.

Proof. Ad 1. First, one can directly prove that $W^{k,p}(U,B)$ is a normed space with $\|.\|^*_{W^{k,p}(U,B)}$. Moreover, let $(f_i)_{i\in\mathbb{N}}$ be a Cauchy sequence in $W^{k,p}(U,B)$. Then because $L^p(U,B)$ and $W^{k,p}(x_j(U \cap U_j) \cap W_j, B)$ for $j = 1,...,N$ are Banach spaces, there are $f \in L^p(U,B)$ and $h_j \in W^{k,p}(x_j(U \cap U_j) \cap W_j, B)$ such that for all $j = 1,...,N$

$$f_i \overset{i\to\infty}{\longrightarrow} f \text{ in } L^p(U,B) \quad \text{and} \quad f_i \circ x_j^{-1}|_{x_j(U\cap U_j)\cap W_j} \overset{i\to\infty}{\longrightarrow} h_j \text{ in } W^{k,p}(x_j(U \cap U_j) \cap W_j, B).$$

Therefore $f \circ x_j^{-1}|_{x_j(U\cap U_j)\cap W_j} = h_j$ for all $j = 1,...,N$ and $f \in W^{k,p}(U,B)$ with $f_i \overset{i\to\infty}{\longrightarrow} f$ in $W^{k,p}(U,B)$. Hence $W^{k,p}(U,B)$ is a Banach space.

Now let $(\tilde{x}_j, \tilde{W}_j)$ for $j = 1,...,\tilde{N}$ be another combination of coordinates and sets as in Definition 2.14. We denote with $\tilde{W}^{k,p}(U,B)$ and $\|.\|^*_{\tilde{W}^{k,p}(U,B)}$ the corresponding space and norm. We have to show $W^{k,p}(U,B) = \tilde{W}^{k,p}(U,B)$ and that the norms are equivalent. It is enough to prove one direction. Let $f \in W^{k,p}(U,B)$ and fix $i \in \{1,...,\tilde{N}\}$. It holds

$$\tilde{x}_i(U \cap \tilde{U}_i) \cap \tilde{W}_i = \bigcup_{j=1}^{N} \tilde{x}_i(U \cap \tilde{U}_i \cap x_j^{-1}(W_j)) \cap \tilde{W}_i.$$

To obtain a suitable partition of unity for this note that $K_i := \overline{\tilde{x}_i(U \cap \tilde{U}_i) \cap \tilde{W}_i}$ is compact and $Y_{ij} := \tilde{x}_i(x_j^{-1}(W_j) \cap \overline{U} \cap U_i) \cap \overline{\tilde{W}_i}$ is open in K_i with $K_i \subseteq \bigcup_{j=1}^N Y_{ij} \subset \bigcup_{j=1}^N Y_{ij} \cup K_i^c$. Hence there are $\eta_{ij} \in C_0^\infty(\mathbb{R}^m)$, $j = 1,...,N$ such that $0 \le \eta_{ij} \le 1$, $\operatorname{supp} \eta_{ij} \subset Y_{ij} \cup K_i^c$ and $\sum_{j=1}^N \eta_{ij} \equiv 1$ on K_i. Therefore

$$f \circ \tilde{x}_i^{-1}|_{\tilde{x}_i(U\cap\tilde{U}_i)\cap\tilde{W}_j} = \sum_{j=1}^{N} \eta_{ij}[f \circ x_j^{-1}|_{x_j(U\cap U_j\cap \tilde{x}_i^{-1}(\tilde{W}_i))\cap W_j}] \circ (x_j \circ \tilde{x}_i^{-1})|_{\tilde{x}_i(U\cap\tilde{U}_i\cap x_j^{-1}(W_j))\cap\tilde{W}_i}.$$

Finally, due to Theorem 2.9 and since multiplication with smooth functions induce bounded linear operators on Sobolev spaces, we obtain $f \circ \tilde{x}_i^{-1}|_{\tilde{x}_i(U\cap\tilde{U}_i)\cap\tilde{W}_j} \in W^{k,p}(\tilde{x}_i(U \cap \tilde{U}_i) \cap \tilde{W}_j)$ and

$$\|f \circ \tilde{x}_i^{-1}|_{\tilde{x}_i(U\cap\tilde{U}_i)\cap\tilde{W}_j}\|_{W^{k,p}(\tilde{x}_i(U\cap\tilde{U}_i)\cap\tilde{W}_j)} \le C\|f\|^*_{W^{k,p}(U,B)},$$

where C does not depend on f. Since $i \in \{1,...,\tilde{N}\}$ was arbitrary, it follows that $f \in \tilde{W}^{k,p}(U,B)$ with $\|f\|^*_{\tilde{W}^{k,p}(U,B)} \le \tilde{C}\|f\|^*_{W^{k,p}(U,B)}$ with $\tilde{C}$ independent of f. This yields the claim. $\Box_1$.

Ad 2. Due to 1. and Lemma 2.8 there are

$$(f_i^j)_{i\in\mathbb{N}} \subset C^\infty(x_j(U\cap U_j)\cap W_j, B)\cap W^{k,p}(x_j(U\cap U_j)\cap W_j, B)$$

such that $f_i^j \overset{i\to\infty}{\longrightarrow} f\circ x_j^{-1}|_{x_j(U\cap U_j)\cap W_j}$ in $W^{k,p}(x_j(U\cap U_j)\cap W_j, B)$. In order to get a suitable partition of unity note that $\overline{U} = \bigcup_{j=1}^N \overline{U}\cap x_j^{-1}(W_j)$ and $\overline{U}\cap x_j^{-1}(W_j)$ is open in $\overline{U}$. Hence there are $\chi_j \in C^l(M)$, $j=1,...,N$ such that $0\leq \chi_j\leq 1$, $\operatorname{supp}\chi_j \subset (\overline{U}\cap x_j^{-1}(W_j))\cup \overline{U}^c$ and $\sum_{j=1}^N \chi_j \equiv 1$ on M. By construction and Theorem 2.9 it follows that

$$f_i := \sum_{j=1}^N \chi_j(f_i^j\circ x_j) \in C^l(U,B)\cap W^{k,p}(U,B)$$

for all $i\in\mathbb{N}$ and $f_i \overset{i\to\infty}{\longrightarrow} f$ in $W^{k,p}(U,B)$. $\square_2$.

Ad 3. First let $f\in C^1(U,B)\cap W^{1,p}(U,B)$. Then the Hahn-Banach Theorem and the scalar case yield that $\nabla_U f$ is well-defined and independent of the choice of x_j, W_j. Therefore it holds $\nabla_U f\in C^0(U,B^n)$. Moreover, $\|[\nabla_U f]\circ x_j^{-1}|_{x_j(U\cap U_j)\cap W_j}\|_{B^n}$ and $\|\nabla[f\circ x_j^{-1}|_{x_j(U\cap U_j)\cap W_j}]\|_{B^n}$ satisfy uniform equivalence estimates on $x_j(U\cap U_j)\cap W_j$ for all $j=1,...,N$ with constants independent of f due to compactness. Hence the claim follows via density from 2. $\square_3$.

Moreover, we need a transformation theorem.

Theorem 2.16. *Let (M,g) and $(\tilde{M},\tilde{g})$ be m-dimensional compact Riemannian submanifolds of $\mathbb{R}^n$ with (or without) boundary and class C^l, where $l\in\mathbb{N}\cup\{\infty\}$, $l\geq 1$. Moreover, let $k\in\mathbb{N}_0$, $0\leq k\leq l$ and $1\leq p<\infty$ as well as B be a Banach space. Let $U\subset M^\circ$, $V\subset \tilde{M}^\circ$ be open and $\Phi: U\to V$ be a C^l-diffeomorphism such that $\Phi\in C^l(\overline{U})^m$ and $\Phi^{-1}\in C^l(\overline{V})^m$. Then $T: W^{k,p}(V,B)\to W^{k,p}(U,B): f\mapsto f\circ\Phi$ is a well-defined bounded linear operator.*

Note that for convenience we did not attempt to obtain a uniform estimate for the operator norm. In order to get such estimates for $k=1$ in the application later, we use Theorem 2.6 and uniform equivalence estimates for the surface gradient.

Proof. The case $k=0$ directly follows from Theorem 2.6. Now let (x_i, W_i) for $i=1,...,N$ and $(\tilde{x}_j,\tilde{W}_j)$ for $j=1,...,\tilde{N}$ be as in Definition 2.14 for M and $\tilde{M}$ respectively. W.l.o.g. we can assume $\Phi(U\cap x_i^{-1}(W_i))\subseteq \tilde{x}_j^{-1}(\tilde{W}_j)$ for some $j=j(i)\in\{1,...,\tilde{N}\}$ and all $i=1,...,N$. Otherwise one can simply refine the W_i. Then Theorem 2.9 yields the claim. $\square$

From now on let $B=\mathbb{K}$ for convenience. We need a product lemma analogous to Lemma 2.10, 1.-2. provided that one of the sets equals some U as in the beginning of the section. Note that the product of U with some open bounded set $\Omega\subseteq\mathbb{R}^q$ is again of the same type. Therefore the definitions and assertions in this section can also be applied for $\Omega\times U$ and $U\times\Omega$ instead of U.

Lemma 2.17. *Let (M,g) be a m-dimensional compact Riemannian submanifold of $\mathbb{R}^n$ of class C^1 and $U\subset M^\circ$ open. Moreover, let $\Omega\subset\mathbb{R}^q$, $q\in\mathbb{N}$ be open and bounded. Then*

1. *Let $1\leq p<\infty$ and $f\in L^p(U\times\Omega)$. Then $f(u,.)\in L^p(\Omega)$ for λ_U-a.e. $u\in U$ and $Tf: U\to L^p(\Omega): u\mapsto f(u,.)$ is an element of $L^p(U,L^p(\Omega))$. Moreover, the map $T: L^p(U\times\Omega)\to L^p(U,L^p(\Omega))$ is an isometric isomorphism.*

2. *Let* $1 < p < \infty$. *Then by restriction of* T *from 1. it holds*
$$W^{1,p}(U \times \Omega) \cong L^p(U, W^{1,p}(\Omega)) \cap W^{1,p}(U, L^p(\Omega))$$
and the derivatives ∇_U *as well as* $\nabla_\Omega = \nabla$ *are compatible in both spaces via* T. *Here* $\nabla_{U\times\Omega} = (\nabla_U, \nabla_\Omega)$ *canonically on* $W^{1,p}(U \times \Omega)$.

3. *Both assertions 1. and 2. also hold when we exchange* U *and* Ω.

Proof. For the proof let $x_j : U_j \to V_j$ and W_j for $j = 1, ..., N$ be as in Definition 2.14 for M. Moreover, we need a partition of unity as in the proof of Lemma 2.15, 3., i.e. let $\chi_j \in C^1(M)$, $j = 1, ..., N$ such that $0 \leq \chi_j \leq 1$, $\operatorname{supp}\chi_j \subset (\overline{U} \cap x_j^{-1}(W_j)) \cup \overline{U}^c$ and $\sum_{j=1}^N \chi_j \equiv 1$ on M. Furthermore, in the following we often denote restrictions to some set by "|" without the set if there is no ambiguity. Finally, note that we often use the notation u, y, z for points in U, V_j, Ω, respectively. This convention also clarifies how some derivatives are understood.

Ad 1. Let $f \in L^p(U\times\Omega)$. Then $f\circ(x_j^{-1}, \mathrm{id})| \in L^p([x_j(U\cap U_j)\cap W_j]\times\Omega)$ for all $j = 1, ..., N$ due to Theorem 2.6. Lemma 2.10 yields $f(x_j^{-1}(y), .) \in L^p(\Omega)$ for a.e. $y \in x_j(U \cap U_j) \cap W_j$ and the mapping $x_j(U \cap U_j) \cap W_j \to L^p(\Omega) : y \mapsto f(x_j^{-1}(y), .)$ is strongly measurable and in L^p for all $j = 1, ..., N$. Therefore $f(u, .) \in L^p(\Omega)$ for λ_U-a.e. $u \in U$ and with the well-known characterization for measurability, see Amann, Escher [AE], Theorem X.1.4 we obtain that $Tf : U \to L^p(\Omega) : u \mapsto f(u, .)$ is strongly measurable. Moreover, the Fubini Theorem implies that $u \mapsto \|f(u, .)\|^p_{L^p(\Omega)}$ is an element of $L^1(U)$ and
$$\int_{U\times\Omega} |f|^p \, d\lambda_{U\times\Omega}(u, z) = \int_U \|f(u, .)\|^p_{L^p(\Omega)} \, d\lambda_U(u).$$
Hence Tf is Bochner-integrable due to the Bochner Theorem, see Růžička [R], Satz 1.12. Therefore Tf is contained in $L^p(U, L^p(\Omega))$ with norm equal $\|f\|_{L^p(U\times\Omega)}$. In particular the map $T : L^p(U \times \Omega) \to L^p(U, L^p(\Omega))$ is well-defined and isometric.

It remains to prove that T is surjective. To this end consider $h \in L^p(U, L^p(\Omega))$. Then Theorem 2.6 yields $h \circ x_j^{-1}| \in L^p([x_j(U \cap U_j) \cap W_j], L^p(\Omega))$ for all $j = 1, ..., N$. Due to Lemma 2.10 there exist $h_j \in L^p([x_j(U \cap U_j) \cap W_j] \times \Omega)$ such that $[y \mapsto h_j(y, .)] = h \circ x_j^{-1}|_{x_j(U\cap U_j)\cap W_j}$ for all $j = 1, ..., N$. Then $h_j \circ (x_j, \mathrm{id}) \in L^p([U \cap x_j^{-1}(W_j)] \times \Omega)$ due to Theorem 2.6 and
$$f_g := \sum_{j=1}^N \chi_j[h_j \circ (x_j, \mathrm{id})] \in L^p(U \times \Omega). \tag{2.1}$$
By construction it holds $f_g(u, .) = h(u)$ for λ_U-a.e. $u \in U$, i.e. $Tf_h = h$. $\Box_{1.}$

Ad 2. Let $1 < p < \infty$ and $f \in W^{1,p}(U \times \Omega)$. We build up on the proof of 1. By definition $f \circ (x_j^{-1}, \mathrm{id})| \in W^{1,p}([x_j(U \cap U_j) \cap W_j] \times \Omega)$ for all $j = 1, ..., N$. Hence Lemma 2.10 yields
$$[y \mapsto f(x_j^{-1}(y), .)] \in W^{1,p}([x_j(U \cap U_j) \cap W_j], L^p(\Omega)) \cap L^p([x_j(U \cap U_j) \cap W_j], W^{1,p}(\Omega)),$$
$\partial_{y_i}[f(x_j^{-1}, \mathrm{id})|](y, .) = \partial_{y_i}[y \mapsto f(x_j^{-1}(y), .)]|_y$ and $\partial_{z_k}[f(x_j^{-1}, \mathrm{id})|](y, .) = \partial_{z_k}[f(x_j^{-1}(y), .)]$ for all $i = 1, ..., m$, $k = 1, ..., q$ and a.e. $y \in x_j(U \cap U_j) \cap W_j$, $j = 1, ..., N$. With 1., Definition 2.14 and Lemma 2.15 we obtain $Tf \in W^{1,p}(U, L^p(\Omega)) \cap L^p(U, W^{1,p}(\Omega))$ and since
$$[\nabla_{U\times\Omega} f]|_{(x_j^{-1},\mathrm{id})} = [\nabla_U f, \nabla_\Omega f]|_{(x_j^{-1},\mathrm{id})} = \left(\sum_{r,s=1}^m g^{rs} \partial_{y_r}(f|_{(x_j^{-1},\mathrm{id})}) \partial_{y_s}(x_j^{-1}), \nabla_z(f|_{(x_j^{-1},\mathrm{id})}) \right)$$

it follows that $[\nabla_U f](u,.) = \nabla_U[Tf]|_u$ and $[\nabla_\Omega f](u,.) = \nabla_z[Tf]|_u$ for λ_U-a.e. $u \in U$. Therefore the derivatives are compatible under T and Lemma 2.15, 3. yields the norm equivalence.

It is left to show that T on $W^{1,p}(U \times \Omega)$ with values in $W^{1,p}(U, L^p(\Omega)) \cap L^p(U, W^{1,p}(\Omega))$ is surjective. Therefore let h be in the target space and h_j for $j = 1, ..., N$ be as in 1. for h. Then due to Theorem 2.16 and Lemma 2.10 it holds $h_j \in W^{1,p}([x_j(U \cap U_j) \cap W_j] \times \Omega)$. Therefore Lemma 2.15 and Theorem 2.16 yield that f_h defined in (2.1) is an element of $W^{1,p}(U \times \Omega)$. $\square_2$.

Ad 3. Now we exchange the order of U and Ω. The proof is divided into four parts in accordance with the proofs of 1.-2. For the proof we denote the corresponding map in 1. with $\tilde{T}$.

Let $1 \leq p < \infty$ and $\tilde{f} \in L^p(\Omega \times U)$. Then $\tilde{f} \circ (\mathrm{id}, x_j^{-1})| \in L^p(\Omega \times [x_j(U \cap U_j) \cap W_j])$ for all $j = 1, ..., N$ due to Theorem 2.6. Lemma 2.10 yields $\tilde{f}(z, x_j^{-1}|) \in L^p(x_j(U \cap U_j) \cap W_j)$ for a.e. $z \in \Omega$ and the map $\Omega \to L^p(x_j(U \cap U_j) \cap W_j) : z \mapsto \tilde{f}(z, x_j^{-1}|)$ is strongly measurable and contained in L^p for all $j = 1, ..., N$. Therefore because of Theorem 2.6

$$\tilde{T}\tilde{f} : \Omega \to L^p(U) : z \mapsto \tilde{f}(z,.) = \sum_{j=1}^{N} \chi_j[\tilde{f}(z, x_j^{-1})] \circ x_j|_{U \cap x_j^{-1}(W_j)}$$

is strongly measurable. Now analogously to the proof of 1. we obtain with the Fubini Theorem and the Bochner Theorem that $\tilde{T}\tilde{f}$ is Bochner-integrable and $\tilde{T}\tilde{f} \in L^p(\Omega, L^p(U))$ with norm equal $\|\tilde{f}\|_{L^p(\Omega\times U)}$. In particular $\tilde{T} : L^p(\Omega \times U) \to L^p(\Omega, L^p(U))$ is well-defined and isometric.

Next we prove that $\tilde{T}$ is surjective. Therefore let $\tilde{h} \in L^p(\Omega, L^p(U))$. First note that due to Theorem 2.6 the map $L^p(U) \to L^p(x_j(U \cap U_j) \cap W_j) : \phi \mapsto \phi \circ x_j^{-1}|$ is bounded and linear for all $j = 1, ..., N$. Hence because of Lemma 2.10 there are $\tilde{h}_j \in L^p(\Omega \times [x_j(U \cap U_j) \cap W_j])$ such that $\tilde{h}_j(z,.) = \tilde{h}(z) \circ x_j^{-1}|$ for a.e. $z \in \Omega$ and $j = 1, ..., N$. Therefore Theorem 2.6 yields

$$\tilde{f}_{\tilde{h}} := \sum_{j=1}^{N} \chi_j[\tilde{h}_j \circ (\mathrm{id}, x_j)] \in L^p(\Omega \times U) \tag{2.2}$$

and by construction $\tilde{f}_{\tilde{h}}(z,.) = \tilde{h}(z)$ for a.e. $z \in \Omega$, i.e. $\tilde{T}\tilde{f}_{\tilde{h}} = \tilde{h}$. Hence $\tilde{T}$ is an isomorphism.

Now let $1 < p < \infty$ and $\tilde{f} \in W^{1,p}(\Omega \times U)$. Then $\tilde{f} \circ (\mathrm{id}, x_j^{-1})| \in W^{1,p}(\Omega \times [x_j(U \cap U_j)] \cap W_j)$ for $j = 1, ..., N$ by definition. Therefore Lemma 2.10 yields

$$[z \mapsto \tilde{f}(z, x_j^{-1}|)] \in W^{1,p}(\Omega, L^p([x_j(U \cap U_j) \cap W_j])) \cap L^p(\Omega, W^{1,p}([x_j(U \cap U_j) \cap W_j])),$$

$\partial_{y_i}[\tilde{f}(\mathrm{id}, x_j^{-1})|](z,.) = \partial_{y_i}[\tilde{f}(z, x_j^{-1}|)]$ and $\partial_{z_k}[\tilde{f}(\mathrm{id}, x_j^{-1})|](z,.) = \partial_{z_k}[z \mapsto \tilde{f}(z, x_j^{-1}|)]|_z$ for all $i = 1, ..., m$, $k = 1, ..., q$ and a.e. $z \in \Omega$, $j = 1, ..., N$. Using the isomorphism property of $\tilde{T}$ on L^p-spaces, Lemma 2.15 and Theorem 2.16 it follows that

$$T\tilde{f} = \sum_{j=1}^{N} \chi_j[z \mapsto \tilde{f}(z, x_j^{-1}|)] \circ x_j|_{U \cap x_j^{-1}(W_j)} \in W^{1,p}(\Omega, L^p(U)) \cap L^p(\Omega, W^{1,p}(U)),$$

$[\nabla_\Omega f](z,.) = \nabla_z[\tilde{T}\tilde{f}]|_z$ and $[\nabla_U f](z,.) = \nabla_U[\tilde{T}\tilde{f}]|_z$ for a.e. $z \in \Omega$. Hence the derivatives are compatible under $\tilde{T}$ and Lemma 2.15, 3. yields the norm equivalence.

Finally, we prove that $\tilde{T}$ on $W^{1,p}(\Omega \times U)$ with values in $W^{1,p}(\Omega, L^p(U)) \cap L^p(\Omega, W^{1,p}(U))$ is surjective. To this end let $\tilde{h}$ be in the target space and $\tilde{h}_j$ for $j = 1, ..., N$ be as above for $\tilde{h}$. Then due to Theorem 2.16 and Lemma 2.10 it holds $h_j \in W^{1,p}(\Omega \times [x_j(U \cap U_j) \cap W_j])$ for all $j = 1, ..., N$. Therefore Lemma 2.15 and Theorem 2.16 imply that $\tilde{f}_{\tilde{h}}$ defined in (2.2) is contained in $W^{1,p}(\Omega \times U)$. $\square_3$.

Finally, we need the notion of domains with Lipschitz-boundary in compact Riemannian submanifolds of $\mathbb{R}^n$ and the analogue of Theorem 2.13 in the case $\Sigma = \partial\Omega$.

Definition 2.18. Let (M, g) be a compact m-dimensional Riemannian submanifold of $\mathbb{R}^n$ with (or without) boundary and class C^1. Let $U \subset M^\circ$ be open and nonempty. Then

1. Let $u \in \partial U$. Then we say that U satisfies the *local Lipschitz condition* in u if this holds in local coordinates, i.e. for any chart $x : \tilde{U} \to V \subseteq [0, \infty) \times \mathbb{R}^{m-1}$ with $u \in \tilde{U}$ it follows that the domain $x(U \cap \tilde{U})$ satisfies the local Lipschitz condition in $x(u)$.

2. We say U has *Lipschitz-boundary* if the local Lipschitz-condition holds in u for all $u \in U$.

3. We call U a *Lipschitz-domain* in M if U is connected and has Lipschitz-boundary.

By definition M° has Lipschitz-boundary.

Remark 2.19. The local Lipschitz condition from Definition 2.11 for domains in $\mathbb{R}^n$ is invariant under C^1-diffeomorphisms (between open subsets of $\mathbb{R}^n$) defined on an open neighbourhood of the closure of the domain, cf. Hofmann, Mitrea, Taylor [HMT], Theorem 4.1. Therefore

1. The invariance under C^1-diffeomorphisms carries over to Definition 2.18.

2. It is enough to prove the condition in Definition 2.18, 1. for one admissible chart.

3. Definition 2.18 is consistent with Definition 2.11 in the case $m = n$.

Theorem 2.20. *Let (M, g) be a compact m-dimensional Riemannian submanifold of $\mathbb{R}^n$ with (or without) boundary and class C^l, $l \in \mathbb{N}_0 \cup \{\infty\}$. Let $U \subset M^\circ$ be a Lipschitz-domain and $1 \leq p < \infty$, $k \in \mathbb{N}_0$. Then $C^l(\overline{U})$ is dense in $W^{k,p}(U)$.*

Let (for convenience) additionally $\partial U = \Sigma \cup Z$ with an $(m-1)$-dimensional C^1-submanifold Σ of M and a null set Z with respect to $\mathcal{H}^{m-1}$. Then $L^p(\partial U) := L^p(\Sigma)$ is defined in Remark 2.5 and there is a unique bounded linear operator $\mathrm{tr} : W^{1,p}(U) \to L^p(\partial U)$ *such that* $\mathrm{tr}\, u = u|_{\partial U}$ *for all $u \in C^0(\overline{U}) \cap W^{1,p}(U)$.*

Proof. This follows from Theorem 2.13 via localization and a suitable partition of unity. □

2.3 Exponentially Weighted Spaces

We define all used spaces with exponential weight.

Definition 2.21. Let $1 \leq p \leq \infty, k \in \mathbb{N}_0, \mu \in (0, 1)$ and $\beta, \beta_1, \beta_2 \geq 0$.

1. Then we introduce with canonical norms

$$\begin{aligned} L^p_{(\beta_1,\beta_2)}(\mathbb{R}^2_+) &:= \{u \in L^1_{\mathrm{loc}}(\mathbb{R}^2_+) : e^{\beta_1|R|+\beta_2 H} u \in L^p(\mathbb{R}^2_+)\}, \\ W^{k,p}_{(\beta_1,\beta_2)}(\mathbb{R}^2_+) &:= \{u \in L^1_{\mathrm{loc}}(\mathbb{R}^2_+) : D^\gamma u \in L^p_{(\beta_1,\beta_2)}(\mathbb{R}^2_+)\}. \end{aligned}$$

   We also write H^k instead of $W^{k,2}$. Moreover, $C^k_{(\beta,\gamma)}(\overline{\mathbb{R}^2_+}) := C^k_b(\overline{\mathbb{R}^2_+}) \cap W^{k,\infty}_{(\beta,\gamma)}(\mathbb{R}^2_+)$.

2. In a similar way we define $L^p_{(\beta)}(\mathbb{R})$, $L^p_{(\beta)}(\mathbb{R}_+)$, $W^{k,p}_{(\beta)}(\mathbb{R})$, $W^{k,p}_{(\beta)}(\mathbb{R}_+)$, $C^k_{(\beta)}(\mathbb{R})$, $C^k_{(\beta)}(\overline{\mathbb{R}_+})$.

3. Let $L^p_{[\beta]}(\mathbb{R}^2_+) := L^p_{(\beta,\beta)}(\mathbb{R}^2_+)$, $W^{k,p}_{[\beta]}(\mathbb{R}^2_+) := W^{k,p}_{(\beta,\beta)}(\mathbb{R}^2_+)$ and $H^k_{[\beta]}(\mathbb{R}^2_+) := H^k_{(\beta,\beta)}(\mathbb{R}^2_+)$.

4. Let $\eta : \mathbb{R} \to \mathbb{R}$ be smooth with $\eta(R) = |R|$ for all $|R| \geq \overline{R}$ and some $\overline{R} > 0$. Then we define
$$W^{k+\mu,p}_{(\beta)}(\mathbb{R}) := \{u \in L^1_{\text{loc}}(\mathbb{R}) : e^{\beta\eta(R)}u \in W^{k+\mu,p}(\mathbb{R})\}$$
for $1 \leq p < \infty$ with natural norm.

The following lemma summarizes all the needed properties for these spaces:

Lemma 2.22. *1. The spaces in Definition 2.21 are Banach spaces.*

2. Equivalent norms: *Let $\eta : \mathbb{R} \to \mathbb{R}$ be as in Definition 2.21, 4., $k \in \mathbb{N}_0$, $1 \leq p \leq \infty$, $\beta_1, \beta_2 \geq 0$. Then $W^{k,p}_{(\beta_1,\beta_2)}(\mathbb{R}^2_+) = \{u \in L^1_{loc}(\mathbb{R}^2_+) : e^{\beta_1\eta(R)+\beta_2 H}u \in W^{k,p}(\mathbb{R}^2_+)\}$ and*
$$\sum_{\gamma\in\mathbb{N}_0^2, |\gamma|\leq k} \|e^{\beta_1|R|+\beta_2 H} D^\gamma u\|_{L^p(\mathbb{R}^2_+)}, \qquad \sum_{\gamma\in\mathbb{N}_0^2, |\gamma|\leq k} \|e^{\beta_1\eta(R)+\beta_2 H} D^\gamma u\|_{L^p(\mathbb{R}^2_+)}$$
$$\text{and} \quad \|e^{\beta_1\eta(R)+\beta_2 H}u\|_{W^{k,p}(\mathbb{R}^2_+)}$$
are equivalent for all $u \in W^{k,p}_{(\beta_1,\beta_2)}(\mathbb{R}^2_+)$. For $B > 0$ fixed the constants in the estimates can be taken uniformly in $\beta_1, \beta_2 \in [0, B]$. Analogous assertions hold for $\mathbb{R}$ instead of $\mathbb{R}^2_+$.

3. Density of smooth functions with compact support: *For all $k \in \mathbb{N}_0$, $1 \leq p < \infty$ and $\beta_1, \beta_2 \geq 0$ it holds: $C_0^\infty(\overline{\mathbb{R}^2_+})$ is dense in $W^{k,p}_{(\beta_1,\beta_2)}(\mathbb{R}^2_+)$, $C_0^\infty(\overline{\mathbb{R}_+})$ is dense in $W^{k,p}_{(\beta_2)}(\mathbb{R}_+)$ and $C_0^\infty(\mathbb{R})$ is dense in $W^{k,p}_{(\beta_1)}(\mathbb{R})$.*

4. Embeddings: *It holds $W^{k,p}_{(\beta_1,\beta_2)}(\mathbb{R}^2_+) \hookrightarrow W^{k,p}_{(\gamma_1,\gamma_2)}(\mathbb{R}^2_+)$ for all $k \in \mathbb{N}_0$, $1 \leq p \leq \infty$ and $0 \leq \gamma_1 \leq \beta_1$, $0 \leq \gamma_2 \leq \beta_2$, as well as*
$$L^p_{[\beta]}(\mathbb{R}^2_+) \hookrightarrow L^q_{[\beta-\varepsilon]}(\mathbb{R}^2_+) \quad \text{for all } \beta > \varepsilon > 0, 1 \leq q \leq p.$$
Analogous embeddings hold for spaces on $\mathbb{R}$ and $\mathbb{R}_+$.

5. Traces of weighted functions on $\mathbb{R}^2_+$: *For all $k \in \mathbb{N}$, $1 \leq p < \infty$ and $\beta \geq 0$ the trace operator*
$$\text{tr} : W^{k,p}_{(\beta,0)}(\mathbb{R}^2_+) \subset W^{k,p}(\mathbb{R}^2_+) \to W^{k-\frac{1}{p},p}_{(\beta)}(\mathbb{R})$$
is well-defined and bounded. Moreover, there is a coretract operator R_β (independent of k, p), i.e. R_β is a bounded operator from $W^{k-\frac{1}{p},p}_{(\beta)}(\mathbb{R})$ to $W^{k,p}_{(\beta,0)}(\mathbb{R}^2_+)$ with $\text{tr} \circ R_\beta = \text{id}$. Finally, all operator norms for fixed k, p are bounded uniformly in $\beta \geq 0$ if we take the third norm in Lemma 2.22, 2.

6. L^2-Poincaré Inequality for weighted functions on $\mathbb{R}_+$: *For $\beta > 0$ and all $u \in H^1_{(\beta)}(\mathbb{R}_+)$ it holds $\|u\|_{L^2_{(\beta)}(\mathbb{R}_+)} \leq \frac{1}{\beta}\|\partial_H u\|_{L^2_{(\beta)}(\mathbb{R}_+)}$.*

7. Reverse Fundamental Theorem for weighted L^2-functions on $\mathbb{R}_+$: *For $\beta > 0$ and v in $L^2_{(\beta)}(\mathbb{R}_+)$ it holds $-\int_.^\infty v\, ds =: w \in H^1_{(\beta)}(\mathbb{R}_+)$ with $\partial_H w = v$. In particular 6. is applicable.*

Remark 2.23. 1. Note that the choice $\int_0^{\cdot} v\,ds$ in Lemma 2.22, 7. would not be appropriate for integrability on $\mathbb{R}_+$.

2. From now on we will always use the third norm in Lemma 2.22, 2. for the weighted spaces.

Proof of Lemma 2.22. Ad 1. That all spaces are normed ones directly follows from the unweighted case. It is left to prove the completeness. For $L^p_{(\beta_1,\beta_2)}(\mathbb{R}^2_+)$ let $(u_l)_{l\in\mathbb{N}}$ be a Cauchy sequence. Then $(u_l)_{l\in\mathbb{N}}$ and $(e^{\beta_1|R|+\beta_2 H}u_l)_{l\in\mathbb{N}}$ are Cauchy sequences in $L^p(\mathbb{R}^2_+)$ and therefore converge to some u and v, respectively, in $L^p(\mathbb{R}^2_+)$. Since one finds a.e. convergent subsequences it follows that $e^{\beta_1|R|+\beta_2 H}u = v$ and hence $u_l \to u$ in $L^p_{(\beta_1,\beta_2)}(\mathbb{R}^2_+)$ for $l \to \infty$. For $W^{k,p}_{(\beta_1,\beta_2)}(\mathbb{R}^2_+)$ one shows the completeness with the case $k = 0$ and embeddings into the unweighted spaces. For the fractional Sobolev spaces, the completeness follows directly with the definition and the unweighted case. $\square_1$.

Ad 2. For $k = 0$ this directly follows from $c_\eta e^{|R|} \leq e^{\eta(R)} \leq C_\eta e^{|R|}$ for all $R \in \mathbb{R}$. In the case $k \geq 1$ one uses the product rule for distributions and smooth functions. $\square_2$.

Ad 3. The density properties directly carry over from the unweighted case since smooth functions with compact support stay in this class when multiplied with a smooth function. $\square_3$.

Ad 4. The first embedding is clear. For the second one we use Hölder's inequality. $\square_4$.

Ad 5. The trace operator tr is a bounded operator from $W^{k,p}(\mathbb{R}^2_+)$ onto $W^{k-\frac{1}{p},p}(\mathbb{R})$ if $k \in \mathbb{N}_0$, $1 \leq p < \infty$ and there is a coretract operator R independent of k, p, cf. Triebel [T2], Theorem 2.7.2 and the construction therein. For $u \in W^{k,p}_{(\beta,0)}(\mathbb{R}^2_+)$ we write

$$u = e^{-\beta\eta(.)} \cdot e^{\beta\eta(.)}u \in C_b^\infty(\overline{\mathbb{R}^2_+}) \cdot W^{k,p}(\mathbb{R}^2_+).$$

Then $W^{k-\frac{1}{p},p}(\mathbb{R}) \ni \operatorname{tr} u = e^{-\beta\eta(.)}\operatorname{tr}(e^{\beta\eta(.)}u) \in W^{k-\frac{1}{p},p}_{(\beta)}(\mathbb{R})$. Moreover, we have the estimate

$$\|\operatorname{tr} u\|_{W^{k-\frac{1}{p},p}_{(\beta)}(\mathbb{R})} = \|\operatorname{tr}(e^{\beta\eta(.)}u)\|_{W^{k-\frac{1}{p},p}(\mathbb{R})} \leq C_{k,p}\|e^{\beta\eta(.)}u\|_{W^{k,p}(\mathbb{R}^2_+)} = C_{k,p}\|u\|_{W^{k,p}_{(\beta,0)}(\mathbb{R}^2_+)}$$

for all $u \in W^{k,p}_{(\beta,0)}(\mathbb{R}^2_+)$. The coretract operator can be taken as $R_\beta v := e^{-\beta\eta(.)}R(e^{\beta\eta(.)}v)$ for all $v \in W^{k-\frac{1}{p},p}_{(\beta)}(\mathbb{R})$. One can directly verify the claimed properties. $\square_5$.

Ad 6. By density it is enough to prove the estimate for $u \in C_0^\infty(\overline{\mathbb{R}_+})$. With the Fundamental Theorem of Calculus, Fubini's Theorem and the Hölder inequality we obtain

$$\begin{aligned}\|u\|^2_{L^2_{(\beta)}(\mathbb{R}_+)} &= \int_0^\infty e^{2\beta H}u^2(H)\,dH \leq 2\int_0^\infty e^{2\beta H}\int_H^\infty |u\partial_s u|\,ds\,dH \leq \\ &\leq 2\int_0^\infty \int_0^s e^{2\beta H}\,dH|u\partial_s u|\,ds \leq \frac{1}{\beta}\int_0^\infty e^{2\beta s}|u\partial_s u|\,ds \leq \frac{1}{\beta}\|\partial_H u\|_{L^2_{(\beta)}(\mathbb{R}_+)}\|u\|_{L^2_{(\beta)}(\mathbb{R}_+)},\end{aligned}$$

where we used $\int_0^s e^{2\beta H}\,dH = \frac{1}{2\beta}[e^{2\beta s}-1] \leq \frac{1}{2\beta}e^{2\beta s}$. This shows the estimate. $\square_6$.

Ad 7. Let $v \in L^2_{(\beta)}(\mathbb{R}_+)$ for a $\beta > 0$ and $v_l \in C_0^\infty(\overline{\mathbb{R}_+})$ with $v_l \to v$ for $l \to \infty$. Then $u_l := -\int_\cdot^\infty v_l(s)\,ds \in C_0^\infty(\overline{\mathbb{R}_+})$ with $\frac{d}{dH}u_l = v_l$. From 5. we obtain that $(u_l)_{l\in\mathbb{N}}$ is a Cauchy sequence in $H^1_{(\beta)}(\mathbb{R}_+)$, hence by 1. there is a limit u in $H^1_{(\beta)}(\mathbb{R}_+)$ and $\frac{d}{dH}u = v$. Because of $u_l = -\int_\cdot^\infty v_l(s)\,ds \to -\int_\cdot^\infty v(s)\,ds$ for $l \to \infty$ pointwise, we get $u = -\int_\cdot^\infty v(s)\,ds$. $\square_7$.

3 Curvilinear Coordinates

Let $N \geq 2$ and $\Omega \subseteq \mathbb{R}^N$ be a bounded, smooth domain with outer unit normal $N_{\partial\Omega}$. In this section we show the existence of a curvilinear coordinate system describing a neighbourhood of a suitable evolving surface[4] in $\overline{\Omega}$ that meets the boundary $\partial\Omega$ at a given angle. More precisely, we consider the following situations (see also Figure 1):

1. Case $(\alpha, 2)$: contact angle $\alpha \in (0, \pi)$ and $N = 2$,
2. Case $(\frac{\pi}{2}, N)$: contact angle $\alpha = \frac{\pi}{2}$ and $N \geq 2$.

Some ideas in this chapter are motivated by Vogel [V], Proposition 3.1.

3.1 Requirements for the Evolving Surface

Let $\Sigma \subset \mathbb{R}^N$ be a smooth, orientable, compact and connected[5] hypersurface with boundary $\partial\Sigma$ and let $X_0 : \Sigma \times [0, T] \to \overline{\Omega}$ be smooth such that $X_0(., t)$ is an injective immersion for all $t \in [0, T]$. For technical reasons, assume that there is a smooth, orientable and connected hypersurface $\Sigma_0 \subset \mathbb{R}^N$ without boundary such that $\Sigma \subsetneq \Sigma_0$ and a smooth extension of X_0 to $\tilde{X}_0 : \Sigma_0 \times (-\tau_0, T + \tau_0) \to \mathbb{R}^N$ for some $\tau_0 > 0$ such that $\tilde{X}_0(., t)$ is an injective immersion for all $t \in (-\tau_0, T + \tau_0)$. Finally, we choose a smooth, orientable, compact and connected hypersurface $\tilde{\Sigma}$ with boundary such that $\Sigma \subsetneq \tilde{\Sigma}^\circ$ and $\tilde{\Sigma} \subsetneq \Sigma_0$.

Remark 3.1. Such $\Sigma_0, \tau_0, \tilde{X}_0$ should exist for any Σ, X_0 as above. For $N = 2$ this is clear, but for $N \geq 3$ this is more difficult to show. First, it should be possible to extend any Σ as above to a smooth orientable hypersurface $\hat{\Sigma} \subset \mathbb{R}^N$ without boundary by merging together local extensions in a suitable way. However, this is quite technical since one has to deal with fraying. Then X_0 can be extended to a smooth immersion $\hat{X}$ on an open neighbourhood of $\Sigma \times [0, T]$ in $\hat{\Sigma} \times \mathbb{R}$. Because immersions are locally injective (cf. O'Neill [O'N], Lemma 1.33), one can prove injectivity of $\hat{X}$ on a possibly smaller open neighbourhood of $\Sigma \times [0, T]$ in $\hat{\Sigma} \times \mathbb{R}$ with a contradiction argument and compactness, cf. also the proof of Theorem 3.3 below for a similar contradiction argument.

Since continuous bijections of compact into Hausdorff topological spaces are homeomorphisms, we know that $X_0(., t)$ is an embedding and $\Gamma_t := X_0(\Sigma, t) \subset \mathbb{R}^N$ is a smooth, orientable, compact and connected hypersurface with boundary for all $t \in [0, T]$. Moreover,

$$\Gamma := \bigcup_{t \in [0,T]} \Gamma_t \times \{t\}$$

is a smooth evolving hypersurface and

$$\overline{X}_0 := (X_0, \mathrm{pr}_t) : \Sigma \times [0, T] \to \Gamma : (s, t) \mapsto (X_0(s, t), t)$$

is a homeomorphism. We choose a smooth normal field $\vec{n} : \Sigma \times [0, T] \to \mathbb{R}^N$ meaning that $\vec{n}$ is smooth and $\vec{n}(., t)$ describes a normal field on Γ_t. Due to Depner [D], Lemma 2.40 the corresponding normal velocity is given by

$$V(s, t) := V_{\Gamma_t}(s) := \vec{n}(s, t) \cdot \partial_t X_0(s, t) \quad \text{for} \quad (s, t) \in \Sigma \times [0, T].$$

[4] For the definition of an evolving hypersurface cf. Depner [D], Definition 2.31.
[5] For simplicity. The considerations can be adjusted for the case of finitely many connected components.

Moreover, let $H(s,t) := H_{\Gamma_t}(s)$ for $(s,t) \in \Sigma\times[0,T]$ be the mean curvature which we choose to be the sum of the principal curvatures. The above definitions applied to $\tilde{X}_0$ on $\tilde{\Sigma} \times [-\frac{\tau_0}{2}, T + \frac{\tau_0}{2}]$ yield suitable extensions of $\Gamma_t, \Gamma, \vec{n}, V$ and H. For convenience, we use the same notation for $\vec{n}$.

Additionally, we require $(\Gamma_t)^\circ \subseteq \Omega$ and $\partial\Gamma_t \subseteq \partial\Omega$. Then the *contact angle* of Γ_t with $\partial\Omega$ in any boundary point $X_0(s,t)$, $(s,t) \in \partial\Sigma \times [0,T]$ with respect to $\vec{n}(s,t)$ is defined by

$$|\measuredangle(N_{\partial\Omega}|_{X_0(s,t)}, \vec{n}(s,t))| \in (0,\pi),$$

where $\measuredangle(N_{\partial\Omega}|_{X_0(s,t)}, \vec{n}(s,t))$ is taken in $(-\pi,\pi)$.

3.2 Coordinates: the Case $(\alpha, 2)$

Let the assumptions in the last Section 3.1 hold for dimension $N = 2$ and constant contact angle $\alpha \in (0,\pi)$ for times $t \in [0,T]$. By reparametrization we can choose w.l.o.g. $\Sigma := I := [-1,1]$ and we can choose $\tilde{\Sigma}$ and Σ_0 as intervals. We set $\tilde{I} := \tilde{\Sigma}$, $I_0 := \Sigma_0$ and define smooth tangent and normal fields on the curve Γ_t by

$$\vec{\tau}(s,t) := \frac{\partial_s X_0(s,t)}{|\partial_s X_0(s,t)|} \quad \text{and} \quad \vec{n}(s,t) := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \vec{\tau}(s,t) \quad \text{for all } (s,t) \in I \times [0,T].$$

The natural extensions to $\tilde{I} \times (-\tau_0, T+\tau_0)$ are denoted with the same symbols. Moreover, the contact points are $p^\pm(t) := X_0(\pm 1, t)$ and we set $\overline{p}^\pm(t) := (p^\pm(t), t)$ for $t \in [0,T]$.

Remark 3.2. We assume $|\partial_s X_0(s,t)| = 1$ for all $s \in I \setminus [-\frac{1}{2}, \frac{1}{2}]$ and $t \in [0,T]$. This can be achieved by reparametrization. More precisely, consider

$$B : I \times [0,T] \to I : (s,t) \mapsto \frac{2}{L(t)} \int_{-1}^{s} |\partial_s X_0(\sigma,t)|\, d\sigma - 1, \quad L(t) := \int_{-1}^{1} |\partial_s X_0(\sigma,t)|\, d\sigma.$$

Then B is smooth and $\partial_s B > 0$. Hence $B(.,t)$ is invertible for all $t \in [0,T]$ and the Inverse Mapping Theorem applied to a smooth extension of (B, pr_t) on $I \times [0,T]$ yields the smoothness of the inverse in (s,t). Hence $\tilde{X}_0(s,t) := X_0(B(.,t)^{-1}|_s, t)$ is a parametrization with $|\partial_s \tilde{X}_0(s,t)| \equiv L(t)/2$. Then another simple transformation yields the desired reparametrization.

The above condition on $\partial_s X_0$ is only needed for the case $\alpha \neq \frac{\pi}{2}$. More precisely, we use $|\partial_s X_0(\pm 1, t)| = 1$ in this Section 3.2 and for the asymptotic expansion of (AC$_\alpha$) at the contact points, see Section 5.4.2.1.1. Finally, the above condition on $\partial_s X_0$ is used for the proof of the spectral estimate for (AC$_\alpha$), see Section 6.5.

For the coordinates we choose a domain of definition that takes into account the contact angle structure. More precisely, for $\delta > 0$ consider the trapeze

$$S_{\delta,\alpha} := \left\{ (r,s) \in \mathbb{R}^2 : r \in (-\delta,\delta), s \in [-1 + \frac{\cos\alpha}{\sin\alpha} r, 1 - \frac{\cos\alpha}{\sin\alpha} r] \right\} \tag{3.1}$$

with upper and lower boundary

$$S^\pm_{\delta,\alpha} := \left\{ (r, s^\pm(r)) : r \in (-\delta,\delta) \right\}, \quad \text{where} \quad s^\pm(r) := \pm 1 \mp \frac{\cos\alpha}{\sin\alpha} r. \tag{3.2}$$

For $\alpha = \frac{\pi}{2}$ we have $S_{\delta,\frac{\pi}{2}} = (-\delta,\delta) \times I$ and $S^\pm_{\delta,\frac{\pi}{2}} = (-\delta,\delta) \times \{\pm 1\}$. For a sketch see Figure 7.

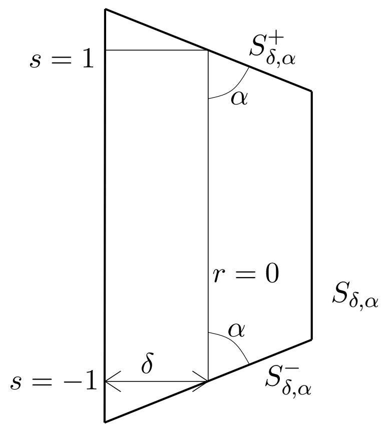

Figure 7: $S_{\delta,\alpha}$ and $S^\pm_{\delta,\alpha}$.

Theorem 3.3 (Coordinates, Case $(\alpha, 2)$). *Let the above assumptions hold. Then there is a $\delta > 0$ and a smooth map $\overline{S_{\delta,\alpha}} \times [0,T] \ni (r,s,t) \mapsto X(r,s,t) \in \overline{\Omega}$ with the following properties:*

1. *$\overline{X} := (X, \mathrm{pr}_t)$ is a homeomorphism onto a neighbourhood of Γ in $\overline{\Omega} \times [0,T]$. Moreover, $\overline{X}$ can be extended to a smooth diffeomorphism defined on an open neighbourhood of $\overline{S_{\delta,\alpha}} \times [0,T]$ in $\mathbb{R}^3$ mapping onto an open set in $\mathbb{R}^3$. The set*
$$\Gamma(\tilde{\delta}) := \overline{X}(S_{\tilde{\delta},\alpha} \times [0,T])$$
is an open neighbourhood of Γ in $\overline{\Omega} \times [0,T]$ for $\tilde{\delta} \in (0,\delta]$.

2. *$X|_{r=0} = X_0$ and X coincides with the usual tubular neighbourhood coordinate system for $s \in [-1+\mu_0, 1-\mu_0]$ for some $\mu_0 \in (0, \frac{1}{2})$ small. Furthermore, for $(r,s,t) \in \overline{S_{\delta,\alpha}} \times [0,T]$ it holds $X(r,s,t) \in \partial\Omega$ if and only if $(r,s) \in \overline{S^+_{\delta,\alpha} \cup S^-_{\delta,\alpha}}$.*

3. *Denote the inverse of $\overline{X}$ with (r, s, pr_t). Then*
$$|\nabla r|^2|_\Gamma = 1, \quad \partial_r(|\nabla r|^2 \circ \overline{X})|_{r=0} = 0 \quad \text{and} \quad \nabla r \cdot \nabla s|_\Gamma = 0.$$
Finally, we can choose $\nabla s \circ \overline{X}_0 = \partial_s X_0/|\partial_s X_0|^2$ and $\nabla r \circ \overline{X}_0 = \vec{n}$. Then it holds $V = -\partial_t r \circ \overline{X}_0$ and $H = -\Delta r \circ \overline{X}_0$.

Remark 3.4. 1. Let $Q_T := \Omega \times (0,T)$. There are unique connected $Q^\pm_T \subseteq \overline{Q_T} = \overline{\Omega} \times [0,T]$ such that $\overline{Q_T} = Q^-_T \cup Q^+_T \cup \Gamma$ (disjoint) and $\operatorname{sign} r = \pm 1$ on $Q^\pm_T \cap \Gamma(\delta)$. Moreover, we set
$$\begin{aligned}\Gamma^\pm(\tilde{\delta}, \mu) &:= \overline{X}((S^\circ_{\tilde{\delta},\alpha} \cap \{\pm s > 1-\mu\}) \times [0,T]),\\ \Gamma(\tilde{\delta}, \mu) &:= \Gamma(\delta) \setminus \overline{[\Gamma^-(\tilde{\delta},\mu) \cup \Gamma^+(\tilde{\delta},\mu)]} = \overline{X}((-\tilde{\delta},\tilde{\delta}) \times (-1+\mu, 1-\mu) \times [0,T])\end{aligned}$$
for $\tilde{\delta} \in (0,\delta]$ and $\mu \in (0,1]$. For $t \in [0,T]$ fixed let $\Gamma_t(\tilde{\delta}), \Gamma^\pm_t(\tilde{\delta},\mu)$ and $\Gamma_t(\tilde{\delta},\mu)$ be the respective sets intersected with $\mathbb{R}^2 \times \{t\}$ and then projected to $\mathbb{R}^2$. Here $\Gamma(\tilde{\delta})$ is as in Theorem 3.3.

2. Let $\tilde{\delta} \in (0, \delta]$. For a sufficiently smooth $\psi : \Gamma(\tilde{\delta}) \to \mathbb{R}$ we define the *tangential* and *normal derivative* by

$$\nabla_\tau \psi := \nabla s[\partial_s(\psi \circ \overline{X}) \circ \overline{X}^{-1}] \quad \text{and} \quad \partial_n \psi := \partial_r(\psi \circ \overline{X}) \circ \overline{X}^{-1},$$

respectively. In the part of $\Gamma(\delta)$ where the coordinate system is the orthogonal one, these definitions coincide with the ones in Abels, Liu [AL]:

$$\nabla_\tau \psi = \frac{\nabla s}{|\nabla s|}\frac{\nabla s}{|\nabla s|} \cdot \nabla \psi \quad \text{and} \quad \partial_n \psi = \nabla r \cdot \nabla \psi \quad \text{on } \Gamma(\tilde{\delta}, \mu_0).$$

This follows from $\nabla \psi|_{\overline{X}} = \nabla r|_{\overline{X}} \partial_r(\psi \circ \overline{X}) + \nabla s|_{\overline{X}} \partial_s(\psi \circ \overline{X})$. For $t \in [0, T]$ fixed and $\psi : \Gamma_t(\tilde{\delta}) \to \mathbb{R}$ smooth enough, we define $\nabla_\tau \psi$ and $\partial_n \psi$ analogously. In the orthogonal region similar identities as above hold. The same notation is used when ψ is only defined on open subsets of $\Gamma(\tilde{\delta})$ or $\Gamma_t(\tilde{\delta})$, $t \in [0, T]$. The same properties as before are valid in the orthogonal parts of the coordinate system.

3. For transformation arguments later we set

$$J(r, s, t) := J_t(r, s) := |\det D_{(r,s)} X(r, s, t)| \quad \text{for} \quad (r, s, t) \in \overline{S_{\delta,\alpha}} \times [0, T].$$

J is smooth and $c \leq J \leq C$ for some $c, C > 0$. Moreover, from the proof of Theorem 3.3 it follows that

$$J_t(r, s)^{-2} = \left[|\nabla r|^2 |\nabla s|^2 - (\nabla r \cdot \nabla s)^2\right]|_{\overline{X}(r,s,t)},$$

in particular $J_t(0, s) = |\partial_s X_0(s, t)|$ for all $(s, t) \in I \times [0, T]$.

As a starting point for the proof of Theorem 3.3 we show in the following lemma that there are graph descriptions of $\partial\Omega$ viewed from the tangential lines to $\partial\Omega$ at the contact points $p^\pm(t)$ for $t \in [0, T]$ in uniform neighbourhoods. See Figure 8 for a sketch of the situation.

Lemma 3.5. *There is an $\eta > 0$ and $w_0^\pm : (-\eta, \eta) \times [0, T] \to \mathbb{R}$ smooth such that*

$$(-\eta, \eta) = B_\eta(0) \ni r \mapsto p^\pm(t) + r\vec{n}|_{(\pm 1,t)} + w_0^\pm(r, t)\vec{\tau}|_{(\pm 1,t)}$$

describes $\partial\Omega$ in the rectangle

$$R_\eta^\pm(t) := p^\pm(t) + B_\eta(0)\vec{n}|_{(\pm 1,t)} + B_{\eta c_\alpha}(0)\vec{\tau}|_{(\pm 1,t)}$$

with the stretching factor $c_\alpha := 1 + |\frac{\cos\alpha}{\sin\alpha}|$ and it holds

$$w_0^\pm(0, t) = 0, \quad \partial_r w_0^\pm(0, t) = \mp\frac{\cos\alpha}{\sin\alpha} \quad \textit{for all } t \in [0, T].$$

Note that the scaling in the rectangles is chosen such that the graph property is compatible with shrinking η for small η. This follows from the contact angle assumption and the Taylor Theorem.

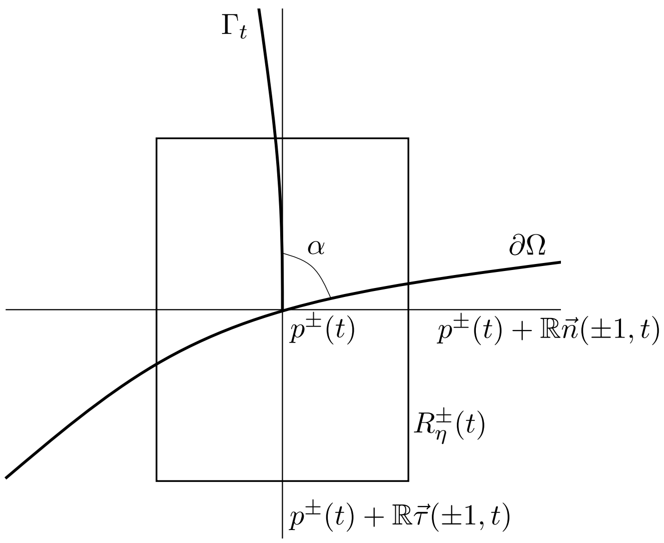

Figure 8: Construction of curvilinear coordinates 1.

Proof. Let us fix $t_0 \in [0, T]$. Then there is a graph parametrization of $\partial\Omega$ in a neighbourhood of $p^{\pm}(t_0)$ with a $\gamma : B_{\eta_0}(0) \subseteq \mathbb{R} \to \mathbb{R}$ in a rectangle as in the lemma in the $(\vec{n}, \vec{\tau})(\pm 1, t_0)$-coordinate system based at $p^{\pm}(t_0)$ for some $\eta_0 > 0$. The boundary points will stay nearby for small time variations: If $\varepsilon > 0$ is small, then

$$x^{\pm}(t) := (p^{\pm}(t) - p^{\pm}(t_0)) \cdot \vec{n}(\pm 1, t_0) \in B_{\eta_0/2}(0) \quad \text{for all } t \in \overline{B_\varepsilon(t_0)} \cap [0, T].$$

The idea is to invert the projection of $\partial\Omega$ to the space $p^{\pm}(t) + \mathbb{R}\vec{n}(\pm 1, t)$. Therefore we consider the smooth mapping $F^{\pm} : \overline{B_{\eta_0/2}(0)} \times (\overline{B_\varepsilon(t_0)} \cap [0, T]) \to \mathbb{R}$ defined by

$$F^{\pm}(x, t) := \left[p^{\pm}(t_0) + (x + x^{\pm}(t))\vec{n}(\pm 1, t_0) + \gamma(x + x^{\pm}(t))\vec{\tau}(\pm 1, t_0) - p^{\pm}(t)\right] \cdot \vec{n}(\pm 1, t).$$

Since $\partial_x F_{\pm}(0, t) \neq 0$ for all $t \in \overline{B_\varepsilon(t_0)} \cap [0, T]$, the inverse mapping theorem applied to a smooth extension of $(F^{\pm}, \mathrm{pr}_t)$ and a compactness argument for the domain of the inverse yields that there is an $\eta > 0$ such that for all $t \in \overline{B_\varepsilon(t_0)} \cap [0, T]$ there is an open neighbourhood V_t of 0 such that $F^{\pm}(., t) : V_t \to B_\eta(0)$ is a smooth diffeomorphism and

$$B_\eta(0) \times (\overline{B_\varepsilon(t_0)} \cap [0, T]) \to \mathbb{R} : (r, t) \mapsto F_{\pm}(., t)^{-1}|_r$$

is smooth. By construction $w_{\pm} : B_\eta(0) \times (\overline{B_\varepsilon(t_0)} \cap [0, T]) \to \mathbb{R} : (r, t) \mapsto$

$$\left[p^{\pm}(t_0) + (. + x^{\pm}(t))\vec{n}(\pm 1, t_0) + \gamma(. + x^{\pm}(t))\vec{\tau}(\pm 1, t_0) - p^{\pm}(t)\right]|_{F_{\pm}(.,t)^{-1}|_r} \cdot \vec{\tau}(\pm 1, t)$$

has the claimed properties for $t \in \overline{B_\varepsilon(t_0)} \cap [0, T]$ after possibly shrinking η. Finally, compactness of $[0, T]$ implies the lemma. $\square$

To use this for the definition of a curvilinear coordinate system, we have to introduce a suitable reparametrization over the upper and lower boundary of the trapeze. For $\alpha = \frac{\pi}{2}$ this is trivial.

Lemma 3.6. *Consider the situation in Lemma 3.5. Then there are $\delta > 0$ and*

$$(y^\pm, w^\pm) : [-\delta, \delta] \times [0, T] \to [-\frac{\eta}{2}, \frac{\eta}{2}] \times \mathbb{R}$$

smooth with $y^\pm(0,t) = w^\pm(0,t) = \partial_r w^\pm(0,t) = 0$ and $\partial_r y^\pm(r,t) > 0$ such that

$$\begin{aligned} &X_0|_{(\pm1,t)} + y^\pm(r,t)\,\vec{n}|_{(\pm1,t)} + w_0^\pm|_{(y^\pm(r,t),t)}\,\vec{\tau}|_{(\pm1,t)} \\ &= \tilde{X}_0|_{(s^\pm(r),t)} + r\,\vec{n}|_{(s^\pm(r),t)} + w^\pm|_{(r,t)}\,\vec{\tau}|_{(s^\pm(r),t)} \end{aligned}$$

for all $(r,t) \in [-\delta, \delta] \times [0, T]$, where $s^\pm(r) = \pm 1 \mp \frac{\cos\alpha}{\sin\alpha} r$ parametrizes $S^\pm_{\delta,\alpha}$. In particular $y^\pm(.,t)$ is invertible for all $t \in [0,T]$.

Proof. We show this with the Implicit Function Theorem. Consider for some $\mu > 0$ small

$$\begin{aligned} F^\pm : U := (-\mu, \mu) \times (-\mu, T+\mu) \times (\eta, \eta) \times \mathbb{R} \subseteq \mathbb{R}^4 \to \mathbb{R}^2 : (r,t,y,w) \mapsto \\ X_0|_{(\pm1,t)} + y\vec{n}_{(\pm1,t)} + w_0^\pm|_{(y,t)}\vec{\tau}|_{(\pm1,t)} - \tilde{X}_0|_{(s^\pm(r),t)} - r\vec{n}|_{(s^\pm(r),t)} - w\vec{\tau}|_{(s^\pm(r),t)}. \end{aligned}$$

Then $F^\pm$ is smooth and well-defined for small $\mu > 0$, $F^\pm(0,t,0,0) = 0$ and

$$D_{(y,w)}F^\pm|_{(0,t,0,0)} = \left(\vec{n}|_{(\pm1,t)} \mp \tfrac{\cos\alpha}{\sin\alpha}\vec{\tau}|_{(\pm1,t)} \quad -\vec{\tau}|_{(\pm1,t)}\right)$$

is invertible for all $t \in [0,T]$ with inverse

$$\left(D_{(y,w)}F^\pm|_{(0,t,0,0)}\right)^{-1} = \begin{pmatrix} (\vec{n}|_{(\pm1,t)})^\top \\ -(\vec{\tau}|_{(\pm1,t)})^\top \mp \frac{\cos\alpha}{\sin\alpha}(\vec{n}|_{(\pm1,t)})^\top \end{pmatrix}. \tag{3.3}$$

Hence the Implicit Function Theorem together with a compactness argument in the time variable yields the existence of a $\delta > 0$ and $(y^\pm, w^\pm) : [-\delta,\delta] \times [0,T] \to [-\frac{\eta}{2}, \frac{\eta}{2}] \times \mathbb{R}$ smooth such that

$$F^\pm(r,t,y^\pm(r,t),w^\pm(r,t)) = 0 \quad \text{and} \quad (y^\pm, w^\pm)|_{(0,t)} = 0 \quad \text{for all } (r,t) \in [-\delta,\delta] \times [0,T].$$

It is left to prove the explicit identities for the derivatives of $(y^\pm, w^\pm)$. We differentiate $F^\pm(r,t,y^\pm(r,t),w^\pm(r,t)) = 0$ with respect to r. This implies

$$0 = \partial_r F^\pm|_{(0,t,0,0)} + D_{(y,w)}F^\pm|_{(0,t,0,0)}\partial_r(y^\pm, w^\pm)|_{(0,t)}^\top.$$

Here $\partial_r F^\pm|_{(0,t,0,0)} = -\partial_s X_0|_{(\pm1,t)}(\mp\frac{\cos\alpha}{\sin\alpha}) - \vec{n}|_{(\pm1,t)}$. Because of $\partial_s X_0|_{(\pm1,t)} = \vec{\tau}|_{(\pm1,t)}$ due to Remark 3.2 and (3.3) it follows that $\partial_r(y^\pm, w^\pm)|_{(0,t)} = (1,0)$. □

Proof of Theorem 3.3. The idea for the definition of X is to extend the mapping in Lemma 3.6 such that it coincides with the usual tubular neighbourhood coordinate system outside a neighbourhood of the boundary and such that all the claimed properties are satisfied.

Therefore we first consider the construction of the standard tubular neighbourhood coordinate system in Hildebrandt [Hi], Chapter 4.6. Let I_1 be a closed interval such that $I \subset I_1^\circ$ and $I_1 \subset \tilde{I}^\circ$. Then similar ideas as in the proof of Lemma 3.5 above yield local graph parametrizations for $\tilde{\Gamma}_t := \tilde{X}_0(\tilde{I}, t)$ as in Lemma 1 in Hildebrandt [Hi], Chapter 4.6, viewed from the tangent lines in squares of uniform width around every point in $\tilde{X}_0(I_1, t)$ for all $t \in [0,T]$. Therefore the construction in [Hi], Satz 1 in Chapter 4.6, yields that for all $\delta \in (0, \delta_0]$, where $\delta_0 > 0$ is small but independent of t, it holds that

$$(-\delta, \delta) \times I_1 \ni (r, s) \mapsto \tilde{X}_0(s, t) + r\vec{n}(s, t) \in \mathbb{R}^2$$

is a diffeomorphism onto its image $U_\delta(t)$ and $U_\delta(t) \cap \overline{\Omega} = B_\delta(\tilde{\Gamma}_t) \cap \overline{\Omega}$ for all $t \in [0, T]$. Now we fix $\delta_0 > 0$ small enough.

We choose $\eta > 0$ small such that $R_\eta^\pm(t)$ is contained in $U_{\delta_0}(t)$, the assertions of Lemma 3.5 are fulfilled and such that the angles between the tangent vectors of $R_\eta^\pm(t) \cap \partial\Omega$ and $R_\eta^\pm(t) \cap \tilde{\Gamma}_t$, respectively, are smaller than a fixed $\beta > 0$ (which will be chosen later).

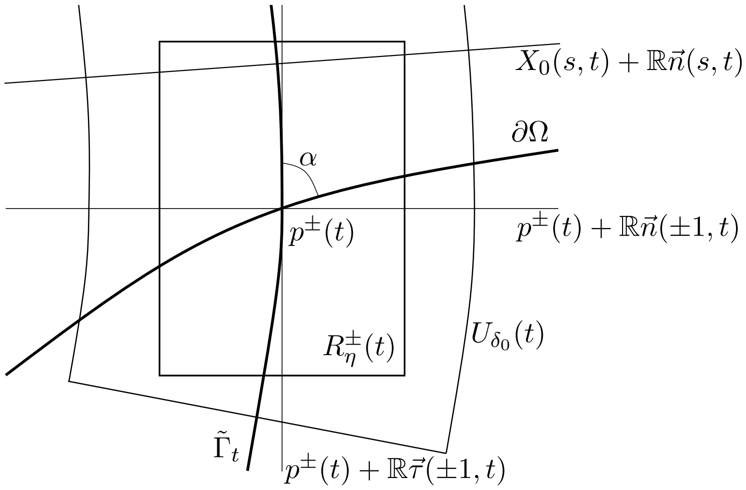

Figure 9: Construction of curvilinear coordinates 2.

Now we define X. Because of uniform continuity we can choose an $\varepsilon > 0$ such that for all s with $|s \mp 1| \leq \varepsilon$ and $t \in [0, T]$ it holds $s \in I_1$ and $\tilde{X}_0(s, t) \in R_{\eta/2}^\pm(t)$. Let $\chi : \mathbb{R} \to [0, 1]$ be a smooth cutoff-function with $\chi = 1$ for $|s \mp 1| \leq \frac{\varepsilon}{2}$ and $\chi = 0$ for $|s| \leq 1 - \varepsilon$, $|s| \geq 1 + \varepsilon$. Then we define $\vec{T} := \chi\vec{\tau}$ and for $\delta > 0$ small

$$X(r, s, t) := \tilde{X}_0(s, t) + r\vec{n}(s, t) + w(r, s, t)\vec{T}(s, t) \quad \text{ for } (r, s, t) \in [-\delta, \delta] \times \tilde{I} \times [0, T],$$

where $w(r, s, t) := w^{\operatorname{sign}(s)}(r, t)$ with $w^\pm$ from Lemma 3.6. In the following we show that the properties in the theorem are satisfied if $\delta > 0$ is small and $\beta > 0$ above was chosen properly.

Ad 1.-2. X is smooth and we compute

$$\begin{aligned}
\partial_r X(r, s, t) &= \vec{n}(s, t) + \partial_r w(r, s, t)\vec{T}(s, t), \\
\partial_s X(r, s, t) &= \partial_s X_0(s, t) + r\partial_s\vec{n}(s, t) + w(r, s, t)\partial_s\vec{T}(s, t)
\end{aligned} \tag{3.4}$$

for $(r, s, t) \in [-\delta, \delta] \times \tilde{I} \times [0, T]$. For $\delta > 0$ small $D_{(r,s)}X(r, s, t)$ is invertible because of $w|_{r=0} = 0$ and $\partial_r w|_{r=0} = 0$. Hence $\overline{X} := (X, \mathrm{pr}_t)$ is locally injective. Since $\overline{X}$ is injective on $\{0\} \times \tilde{I} \times [0, T]$, we obtain by contradiction with a compactness argument[6] that $\overline{X}$ is injective on $[-\delta, \delta] \times \tilde{I} \times [0, T]$ for $\delta > 0$ small.

[6] One could also use a geometric argument using angles and the Fundamental Theorem of Calculus. Nevertheless, this argument appears to be more complicated and it is difficult to generalize to higher dimensions.

More precisely, assume that there is no $\delta > 0$ such that $\overline{X}$ is injective on such a set. Then there are two distinctive sequences $(r_j^n, s_j^n, t_j^n) \in [-\frac{1}{n}, \frac{1}{n}] \times \tilde{I} \times [0,T]$ for $j = 1,2$ such that $\overline{X}(r_1^n, s_1^n, t_1^n) = \overline{X}(r_2^n, s_2^n, t_2^n)$ for all $n \in \mathbb{N}$. By compactness there are subsequences $(s_j^{n_k}, t_j^{n_k}) \longrightarrow (s_j, t_j)$ for $k \to \infty$ and $j = 1,2$. Since $\overline{X}$ is continuous, it follows that $\overline{X}(0, s_1, t_1) = \overline{X}(0, s_2, t_2)$. Hence $(s_1, t_1) = (s_2, t_2)$ and $(r_j^{n_k}, s_j^{n_k}, t_j^{n_k}) \longrightarrow (0, s_1, t_1)$ for $k \to \infty$ and $j = 1,2$. Local injectivity of $\overline{X}$ yields a contradiction. Altogether $\overline{X}$ restricted to $[-\delta, \delta] \times \tilde{I} \times [0,T]$ for $\delta > 0$ small is injective.

Moreover, due to the Inverse Function Theorem, $\overline{X}$ can locally be extended to a smooth diffeomorphism. Therefore with a similar argument as above one can show that $\overline{X}$ can be extended to a smooth diffeomorphism defined on an open neighbourhood of $[-\delta, \delta] \times \tilde{I} \times [0,T]$ in $\mathbb{R}^3$ mapping onto an open set in $\mathbb{R}^3$.

In the following we show $X(\overline{S_{\delta,\alpha}} \times [0,T]) \subset \overline{\Omega}$ and related properties for $\delta > 0$ small. If $|s| \leq 1 - \varepsilon$, then X coincides with the usual tubular neighbourhood coordinate system, i.e.

$$X(r,s,t) = X_0(s,t) + r\vec{n}(s,t) \quad \text{ for all } (r,s,t) \in [-\delta, \delta] \times [-1+\varepsilon, 1-\varepsilon] \times [0,T].$$

By compactness, $X_0([-1+\varepsilon, 1-\varepsilon], t)$ has a uniform positive distance from $\partial\Omega$. Therefore, by uniform continuity $X(r,s,t)$ stays in Ω for $|s| \leq 1-\varepsilon$, $r \in [-\delta, \delta], t \in [0,T]$ for $\delta > 0$ small. Moreover, because of $w = \partial_r w = 0$ for $r = 0$, the terms $|\partial_r X(r,s,t) - \vec{n}(s,t)|$ and $|\partial_s X(r,s,t) - \partial_s X_0(s,t)|$ for all $r \in [-\delta, \delta]$, $|s \mp 1| \leq \varepsilon$ and $t \in [0,T]$ are estimated by an arbitrary small constant $c_0 > 0$ for $\delta > 0$ small. If c_0 is small enough (depending on β), then

$$\measuredangle(\partial_r X(r,s,t), \vec{n}(\pm 1, t)) \leq 2\beta \quad \text{ and } \quad \measuredangle(\partial_s X(r,s,t), \vec{\tau}(\pm 1, t)) \leq 2\beta$$

for all $r \in [-\delta, \delta]$, $|s \mp 1| \leq \varepsilon$ and $t \in [0,T]$. Therefore because of Lemma 3.5, Lemma 3.6 and the Fundamental Theorem of Calculus it follows that for $\delta > 0$ small X maps $\overline{S_{\delta,\alpha}} \times [0,T]$ to $\overline{\Omega}$, X maps

$$([-\delta, \delta] \times [-1-\varepsilon, 1+\varepsilon] \times [0,T]) \backslash (\overline{S_{\delta,\alpha}} \times [0,T])$$

outside of $\overline{\Omega}$ and $X(r,s,t) \in \partial\Omega$ if and only if $(r,s) \in \overline{S_{\delta,\alpha}^+ \cup S_{\delta,\alpha}^-}$ provided that $\beta > 0$ was chosen sufficiently small before. Note that β can be chosen independently first, then $\eta > 0$ is chosen, then $\varepsilon > 0$ and finally $\delta > 0$. The assertion that $\Gamma(\tilde{\delta})$ for $\tilde{\delta} \in (0, \delta]$ is an open neighbourhood of Γ in $\overline{\Omega} \times [0,T]$ now follows from the extension property of $\overline{X}$ and the mapping properties of X above. $\square_{1.-2.}$

Ad 3. It remains to prove the explicit identities in Theorem 3.3. We have

$$((D_{(r,s)}X)^\top D_{(r,s)}X)^{-1} = \begin{pmatrix} |\nabla r|^2 & \nabla r \cdot \nabla s \\ \nabla r \cdot \nabla s & |\nabla s|^2 \end{pmatrix}\Bigg|_{\overline{X}}, \quad \begin{pmatrix} a & b \\ b & c \end{pmatrix} := (D_{(r,s)}X)^\top D_{(r,s)}X.$$

Using (3.4) and that $\vec{T}, \partial_s X_0, \partial_s \vec{n}$ are tangential, we obtain $a = 1 + (\partial_r w)^2 |\vec{T}|^2$,

$$b = w\partial_s \vec{T} \cdot \vec{n} + \partial_r w \partial_s X_0 \cdot \vec{T} + r\partial_r w \partial_s \vec{n} \cdot \vec{T} + w\partial_r w \partial_s \vec{T} \cdot \vec{T},$$
$$c = |\partial_s X_0|^2 + r^2 |\partial_s \vec{n}|^2 + w^2 |\partial_s \vec{T}|^2 + 2(r\partial_s X_0 \cdot \partial_s \vec{n} + w\partial_s X_0 \cdot \partial_s \vec{T} + rw\partial_s \vec{n} \cdot \partial_s \vec{T}).$$

The inverse can be computed explicitly. Since $w = \partial_r w = 0$ for $r = 0$, it follows that $a|_{(0,s,t)} = 1$, $b|_{(0,s,t)} = 0$ and $c|_{(0,s,t)} = |\partial_s X_0(s,t)|^2$ for $(s,t) \in I \times [0,T]$. Therefore

$$|\nabla r|^2|_\Gamma = 1, \quad \nabla r \cdot \nabla s|_\Gamma = 0 \quad \text{ and } \quad |\nabla s| \circ \overline{X}_0 = 1/|\partial_s X_0|.$$

Moreover, with $d := ac - b^2$ we obtain $\partial_r(|\nabla r|^2 \circ \overline{X}) = (d\partial_r c - c\partial_r d)/d^2$,

$$\partial_r((\nabla r \cdot \nabla s) \circ \overline{X}) = -\frac{d\partial_r b - b\partial_r d}{d^2} \quad \text{and} \quad \partial_r(|\nabla s|^2 \circ \overline{X}) = \frac{d\partial_r a - a\partial_r d}{d^2}.$$

We have $\partial_r a|_{(0,s,t)} = 0$, $\partial_r b|_{(0,s,t)} = \partial_r^2 w|_{(0,s,t)} \partial_s X_0 \cdot \vec{T}|_{(s,t)}$ and $\partial_r c|_{(0,s,t)} = 2\partial_s X_0 \cdot \partial_s \vec{n}|_{(s,t)}$ for $(s,t) \in I \times [0,T]$. Hence $\partial_r^k d|_{(0,s,t)} = \partial_r^k c|_{(0,s,t)}, k = 0, 1$ and $\partial_r(|\nabla r|^2 \circ \overline{X})|_{(0,s,t)} = 0$,

$$\partial_r((\nabla r \cdot \nabla s) \circ \overline{X})|_{(0,s,t)} = \frac{-\partial_r^2 w|_{(0,s,t)} \partial_s X_0 \cdot \vec{T}|_{(s,t)}}{|\partial_s X_0|^2|_{(s,t)}}$$

and $\partial_r(|\nabla s|^2 \circ \overline{X})|_{(0,s,t)} = -2\partial_s X_0 \cdot \partial_s \vec{n}|_{(s,t)}/|\partial_s X_0|^4|_{(s,t)}$ for all $(s,t) \in I \times [0,T]$.

Finally, we show that for the coordinate system constructed above the additional properties are satisfied. Because of (3.4) we have for all $(s,t) \in I \times [0,T]$

$$D_{(r,s)}X|_{(0,s,t)} = \left(\vec{n}, \partial_s X_0\right)|_{(s,t)}, \quad (D_{(r,s)}X|_{(0,s,t)})^{-1} = \begin{pmatrix} \vec{n}^\top \\ \partial_s X_0^\top/|\partial_s X_0|^2 \end{pmatrix}|_{(s,t)}.$$

This shows $\nabla r \circ \overline{X}_0 = \vec{n}$ and $\nabla s \circ \overline{X}_0 = \partial_s X_0/|\partial_s X_0|^2$. Furthermore, the chain rule applied to $r = r(X(r,s,t),t)$ yields

$$V(s,t) = \partial_t X|_{(0,s,t)} \cdot \vec{n}(s,t) = -\partial_t r|_{(X(0,s,t),t)} \quad \text{for } (s,t) \in I \times [0,T].$$

Well-known formulas for the mean curvature, cf. e.g. Depner [D], Chapter 2.1, imply

$$H_{\Gamma_t}(s) = -\mathrm{div}\left(\frac{\nabla r}{|\nabla r|}\right)|_{(X_0(s,t),t)} = -\left(\frac{\Delta r}{|\nabla r|} - \frac{1}{|\nabla r|^3}\nabla r^\top D^2 r \nabla r\right)|_{(X_0(s,t),t)}$$

for $(s,t) \in I \times [0,T]$. The second term vanishes because of $D^2 r \nabla r = \frac{1}{2}\nabla(|\nabla r|^2)$ and

$$\partial_r(|\nabla r|^2 \circ \overline{X})|_{(0,s,t)} = \partial_s(|\nabla r|^2 \circ \overline{X})|_{(0,s,t)} = 0 \quad \text{for } (s,t) \in I \times [0,T].$$

With $|\nabla r|^2|_\Gamma = 1$ we get $H(s,t) = -\Delta r|_{(X_0(s,t),t)}$ for $(s,t) \in I \times [0,T]$. $\square_{3.}$

3.3 Coordinates: the Case $(\frac{\pi}{2}, N)$

Let the assumptions in Section 3.1 hold for dimension $N \geq 2$ and constant contact angle $\frac{\pi}{2}$ for times $t \in [0,T]$. We adapt the ideas from the 2-dimensional case in the last Section 3.2 to the N-dimensional case. To this end we use the *outer unit conormal* $\vec{n}_{\partial\Sigma} : \partial\Sigma \to \mathbb{R}^N$, cf. Depner [D], Definition 2.28 on p.22. Moreover, we introduce the outer unit conormal for the evolving hypersurface Γ, $\vec{n}_{\partial\Gamma} : \partial\Sigma \times [0,T] \to \mathbb{R}^N$, where $\vec{n}_{\partial\Gamma}(\sigma,t) := \vec{n}_{\partial\Gamma_t}(\sigma)$ is the outer unit conormal with respect to $\partial\Gamma_t$ at $X_0(\sigma,t)$ for all $(\sigma,t) \in \partial\Sigma \times [0,T]$. One can show smoothness and

$$\vec{n}_{\partial\Gamma_t}(\sigma) = N_{\partial\Omega}|_{X_0(\sigma,t)} \quad \text{for all } (\sigma,t) \in \partial\Sigma \times [0,T]$$

with the considerations in [D]. Furthermore, we use the tubular neighbourhood coordinate system of $\partial\Sigma$ in $\tilde{\Sigma}$: for $\mu_1 > 0$ small there is a smooth diffeomorphism

$$\tilde{Y} : \partial\Sigma \times [-2\mu_1, 2\mu_1] \to R(\tilde{Y}) \subset \tilde{\Sigma}, \quad (\sigma, b) \mapsto \tilde{Y}(\sigma, b)$$

onto a neighbourhood $R(\tilde{Y})$ of $\partial\Sigma$ in $\tilde{\Sigma}$ such that $\tilde{Y}|_{b=0} = \mathrm{id}_{\partial\Sigma}$ and $Y := \tilde{Y}|_{\partial\Sigma\times[0,2\mu_1]}$ is a diffeomorphism onto a neighbourhood $R(Y)$ of $\partial\Sigma$ in Σ. We use the notation $(\tilde{\sigma},\tilde{b}) := \tilde{Y}^{-1}$. We define $\tilde{Y}$ via the exponential map on the normal bundle of $\partial\Sigma$ in $\tilde{\Sigma}$, cf. Proposition 7.26 in O'Neill [O'N]. Then

$$\partial_b Y(\sigma,0) = -\vec{n}_{\partial\Sigma}(\sigma) \quad \text{for all } \sigma \in \partial\Sigma. \tag{3.5}$$

Theorem 3.7 (Coordinates, Case $(\frac{\pi}{2},N)$). *Let the above assumptions hold. There exist $\delta > 0$ and a smooth map $[-\delta,\delta]\times\Sigma\times[0,T] \ni (r,s,t) \mapsto X(r,s,t) \in \overline{\Omega}$ with the following properties:*

1. *$\overline{X} := (X, \mathrm{pr}_t)$ is a homeomorphism onto a neighbourhood of Γ in $\overline{\Omega}\times[0,T]$. Moreover, $\overline{X}$ can be extended to a smooth diffeomorphism defined on an open neighbourhood of $[-\delta,\delta]\times\Sigma\times[0,T]$ in $\mathbb{R}\times\tilde{\Sigma}\times\mathbb{R}$ mapping onto an open set in $\mathbb{R}^{N+1}$. The set*

$$\Gamma(\tilde{\delta}) := \overline{X}((-\tilde{\delta},\tilde{\delta})\times\Sigma\times[0,T])$$

   is an open neighbourhood of Γ in $\overline{\Omega}\times[0,T]$ for $\tilde{\delta}\in(0,\delta]$.

2. *$X|_{r=0} = X_0$ and X coincides with the well-known tubular neighbourhood coordinate system for $s \in \Sigma\backslash Y(\partial\Sigma\times[0,\mu_0])$ for some $\mu_0 \in (0,\mu_1]$ small. Additionally, for points $(r,s,t)\in[-\delta,\delta]\times\Sigma\times[0,T]$ it holds $X(r,s,t)\in\partial\Omega$ if and only if $s\in\partial\Sigma$.*

3. *Let (r,s,pr_t) be the inverse of $\overline{X}$. Then $(\partial_{x_j}s|_{(x,t)})_{j=1}^N$ generate the tangent space $T_{s(x,t)}\Sigma$, $|\nabla r|_{(x,t)}| \geq c > 0$ for some $c>0$ independent of (x,t) and $D_x s(D_x s)^\top|_{(x,t)}$ is uniformly positive definite as a linear map in $\mathcal{L}(T_{s(x,t)}\Sigma)$ for all $(x,t)\in\overline{\Gamma(\delta)}$. Furthermore, we have*

$$|\nabla r|^2|_\Gamma = 1, \quad \partial_r(|\nabla r|^2\circ\overline{X})|_{r=0} = 0 \quad \text{and} \quad D_x s\nabla r|_\Gamma = 0$$

   and for all $(r,s,t)\in[-\delta,\delta]\times[\Sigma\backslash Y(\partial\Sigma\times[0,\mu_0])]\times[0,T]$ it holds

$$\nabla r|_{\overline{X}(r,s,t)} = \vec{n}(s,t) \quad \text{and} \quad D_x s|_{\overline{X}(r,s,t)}\vec{n}(s,t) = 0.$$

   Moreover, we can choose $\nabla r\circ\overline{X}_0 = \vec{n}$. Then it holds $V = -\partial_t r\circ\overline{X}_0$ and $H = -\Delta r\circ\overline{X}_0$.

4. *Let $(\sigma,b) := Y^{-1}\circ s : \overline{X}([-\delta,\delta]\times R(Y)\times[0,T]) \to \partial\Sigma\times[0,2\mu_1]$. Then*

$$N_{\partial\Omega}\cdot\nabla b|_{\overline{X}_0(\sigma,t)} = -D_x s N_{\partial\Omega}|_{\overline{X}_0(\sigma,t)}\cdot\vec{n}_{\partial\Sigma}|_\sigma, \quad |N_{\partial\Omega}\cdot\nabla b|_{\overline{X}_0(\sigma,t)}| \geq c > 0$$

   and $\nabla b\cdot\nabla r|_{\overline{X}_0(\sigma,t)} = 0$, $|\nabla b|_{\overline{X}_0(\sigma,t)}| \geq c > 0$ for all $(\sigma,t)\in\partial\Sigma\times[0,T]$.

Remark 3.8. 1. Let $Q_T := \Omega\times(0,T)$. There are unique connected $Q_T^\pm \subseteq \overline{Q_T} = \overline{\Omega}\times[0,T]$ such that $\overline{Q_T} = Q_T^-\cup Q_T^+\cup\Gamma$ (disjoint) and $\operatorname{sign} r = \pm1$ on $Q_T^\pm\cap\Gamma(\delta)$. Moreover, we set

$$\Gamma^C(\tilde{\delta},\mu) := \overline{X}((-\tilde{\delta},\tilde{\delta})\times Y(\partial\Sigma\times(0,\mu))\times[0,T]), \quad \Gamma(\tilde{\delta},\mu) := \Gamma(\tilde{\delta})\backslash\overline{\Gamma^C(\tilde{\delta},\mu)}$$

for $\tilde{\delta}\in(0,\delta]$ and $\mu\in(0,2\mu_1]$. For $t\in[0,T]$ fixed let $\Gamma_t(\tilde{\delta}), \Gamma_t^C(\tilde{\delta},\mu)$ and $\Gamma_t(\tilde{\delta},\mu)$ be the respective sets intersected with $\mathbb{R}^N\times\{t\}$ and then projected to $\mathbb{R}^N$. Here $\Gamma(\tilde{\delta})$ is defined in Theorem 3.7.

2. Let $\tilde{\delta} \in (0, \delta]$. For a sufficiently smooth $\psi : \Gamma(\tilde{\delta}) \to \mathbb{R}$ we define the *tangential* and *normal derivative* by

$$\nabla_\tau \psi := (D_x s)^\top [\nabla_\Sigma (\psi \circ \overline{X}) \circ \overline{X}^{-1}] \quad \text{and} \quad \partial_n \psi := \partial_r (\psi \circ \overline{X}) \circ \overline{X}^{-1},$$

respectively. For $t \in [0, T]$ fixed and $\psi : \Gamma_t(\tilde{\delta}) \to \mathbb{R}$ smooth enough, we define $\nabla_\tau \psi$ and $\partial_n \psi$ analogously. The same notation applies if ψ is only defined on an open subset of $\Gamma(\tilde{\delta})$ or $\Gamma_t(\tilde{\delta})$ for some $\mu \in (0, 2\mu_1]$ and $t \in [0, T]$. Note that in the case $N = 2$ and $\Sigma = [-1, 1]$ the definitions here coincide with the ones in Remark 3.4, 2. Important properties of ∇_τ and ∂_n will be shown in Corollary 3.10.

3. For transformation arguments we define

$$J(r, s, t) := J_t(r, s) := |\det d_{(r,s)} X(r, s, t)| \quad \text{for } (r, s, t) \in [-\delta, \delta] \times \Sigma \times [0, T],$$

where the determinant is taken with respect to an arbitrary orthonormal base of $T_s\Sigma$. The latter is well-defined, cf. Theorem 2.6, 1. Via local coordinates it follows that J is smooth and with a compactness argument we obtain that $0 < c \leq J \leq C$ for some $c, C > 0$.

The first step for the proof of Theorem 3.7 is to show an analogue of Lemma 3.5.

Lemma 3.9. *There is an $\eta > 0$ such that $\partial\Omega \cap R_\eta(\sigma, t)$ admits a graph parametrization over $X_0(\sigma, t) + [B_\eta(0) \cap T_{X_0(\sigma,t)} \partial\Omega]$ for all $(\sigma, t) \in \partial\Sigma \times [0, T]$, where*

$$R_\eta(\sigma, t) := X_0(\sigma, t) + (-\eta, \eta)\vec{n}_{\partial\Gamma}(\sigma, t) + [B_\eta(0) \cap T_{X_0(\sigma,t)} \partial\Omega].$$

Moreover, for $\eta > 0$ small there exists $w : (-\eta, \eta) \times \partial\Sigma \times [0, T] \to \mathbb{R}$ smooth such that $w|_{r=0} = \partial_r w|_{r=0} = 0$ and

$$(-\eta, \eta) \ni r \mapsto X_0(\sigma, t) + r\vec{n}(\sigma, t) + w(r, \sigma, t)\vec{n}_{\partial\Gamma}(\sigma, t)$$

describes $\partial\Omega$ in $X_0(\sigma, t) + (-\eta, \eta)\vec{n}(\sigma, t) + (-\eta, \eta)\vec{n}_{\partial\Gamma}(\sigma, t)$ for all $(\sigma, t) \in \partial\Sigma \times [0, T]$.

Again the assertions are compatible with shrinking η for small $\eta > 0$ which follows from the contact angle assumption and Taylor's Theorem.

Proof. Let $(\sigma_0, t_0) \in \partial\Sigma \times [0, T]$ be arbitrary. We choose a basis $\vec{v}_1, ..., \vec{v}_{N-2}$ of $T_{X_0(\sigma_0,t_0)} \partial\Gamma_{t_0}$ and extend it to smooth tangential vector fields $\vec{\tau}_1, ..., \vec{\tau}_{N-2}$ on $T\partial\Gamma$ such that locally in $\partial\Gamma$ around $X_0(\sigma_0, t_0)$ these are again bases in the corresponding tangential spaces over $\partial\Gamma_t$ for all $t \in B_\varepsilon(t_0) \cap [0, T]$, $\varepsilon > 0$ small. In coordinates one can apply similar arguments as in the proof of Lemma 3.5 to obtain smooth graph parametrizations of $\partial\Omega \cap R_\eta(\sigma, t)$ over $X_0(\sigma, t) + [B_\eta(0) \cap T_{X_0(\sigma,t)} \partial\Omega]$ for some $\eta > 0$ and $(\sigma, t) \in \partial\Sigma \times [0, T]$ close to (σ_0, t_0). More precisely, there is an $\eta > 0$ and a smooth

$$w_0 : (-\eta, \eta) \times B_\eta(0) \times U \times V \subseteq \mathbb{R} \times \mathbb{R}^{N-2} \times \Sigma \times [0, T] \to \mathbb{R},$$

where U, V are open neighbourhoods of σ_0, t_0 in $\partial\Sigma$, $[0, T]$, respectively, such that

$$\begin{aligned}(-\eta, \eta) \times B_\eta(0) \ni (r, r_1, ..., r_{N-2}) \mapsto & X_0(\sigma, t) + r\vec{n}(\sigma, t) + r_1\vec{\tau}_1(\sigma, t) + ... + r_{N-2}\vec{\tau}_{N-2}(\sigma, t) \\ & + w_0(r, r_2, ..., r_{N-2}, \sigma, t)\vec{n}_{\partial\Gamma}(\sigma, t)\end{aligned}$$

describes $\partial\Omega \cap R_\eta(\sigma,t)$ in $R_\eta(\sigma,t)$. Moreover, $w_0|_{r=0} = \partial_r w_0|_{r=0} = 0$. We set

$$w : (-\eta,\eta) \times U \times V \to \mathbb{R} : (r,\sigma,t) \mapsto w_0(r,0,...,0,\sigma,t)$$

and we observe that this definition is independent of the choice of $\vec{v}_1,...,\vec{v}_{N-2}$ and $\vec{\tau}_1,...,\vec{\tau}_{N-2}$ as well as (σ_0,t_0). By compactness $\eta > 0$ can be taken uniformly in $(\sigma_0,t_0) \in \partial\Sigma \times [0,T]$. $\square$

Proof of Theorem 3.7. Let Σ_1 be a compact hypersurface with boundary such that $\Sigma \subsetneq \Sigma_1^\circ$ and $\Sigma_1 \subsetneq \tilde{\Sigma}^\circ$. Similar as in the proof of Theorem 3.3 there is a $\delta_0 > 0$ such that for all $\delta \in (0,\delta_0]$ it holds that

$$(-\delta,\delta) \times \Sigma_1 \ni (r,s) \mapsto \tilde{X}_0(s,t) + r\vec{n}(s,t) \in \mathbb{R}^N$$

is a diffeomorphism onto its image $U_\delta(t)$ and $U_\delta(t) \cap \overline{\Omega} = B_\delta(\tilde{\Gamma}_t) \cap \overline{\Omega}$ for all $t \in [0,T]$, where we have set $\tilde{\Gamma}_t := \tilde{X}_0(\tilde{\Sigma},t)$.

We choose $\eta > 0$ small such that $R_\eta(\partial\Sigma,t)$ is contained in $U_{\delta_0}(t)$, the assertions of Lemma 3.9 are fulfilled and such that the angles between the tangent planes of $R_\eta(\sigma,t) \cap \partial\Omega$ are smaller than a fixed $\beta > 0$ (which will be chosen later).

Now we define X. Let $\vec{\tau} : \tilde{\Sigma} \times [0,T] \to \mathbb{R}^N$ be a smooth vector field with the property that $\vec{\tau}(s,t) \in T_{\tilde{X}_0(s,t)}\tilde{\Gamma}_t$ for all $(s,t) \in \tilde{\Sigma} \times [0,T]$ and $\vec{\tau}|_{\partial\Sigma\times[0,T]} = \vec{n}_{\partial\Gamma}$. Existence of such a $\vec{\tau}$ follows via local extensions of $\vec{n}_{\partial\Gamma}$ in submanifold charts of $\partial\Sigma$ with respect to $\tilde{\Sigma}$ and compactness arguments. Moreover, by uniform continuity there is an $\varepsilon \in (0,\mu_1]$ such that $\tilde{X}_0(\tilde{Y}(\sigma,\tilde{b}),t) \in R_{\frac{\eta}{2}}(\sigma,t)$ for all $(\sigma,\tilde{b},t) \in \partial\Sigma \times [-\varepsilon,\varepsilon] \times [0,T]$ as well as

$$|\partial_b\tilde{Y}(\sigma,\tilde{b}) + \vec{n}_{\partial\Sigma}(\sigma)| + \left|\frac{d}{d\tilde{b}}\tilde{X}_0(\tilde{Y}(\sigma,\tilde{b})) - \frac{d}{d\tilde{b}}\Big|_{\tilde{b}=0}\tilde{X}_0(\tilde{Y}(\sigma,.))\right| \leq c_0 \tag{3.6}$$

for a fixed $c_0 > 0$ small (to be determined later), where we used (3.5). Let $\chi : \mathbb{R} \to [0,1]$ be a smooth cutoff-function with $\chi = 1$ for $|b| \leq \frac{\varepsilon}{2}$ and $\chi = 0$ for $|b| \geq \varepsilon$. We define $\vec{T}(s,t) := \chi(\tilde{b}(s))\vec{\tau}(s,t)$ for $(s,t) \in \Sigma \times [0,T]$ and

$$X(r,s,t) := \tilde{X}_0(s,t) + r\vec{n}(s,t) + w(r,\tilde{\sigma}(s),t)\vec{T}(s,t) \in \mathbb{R}^N \quad \text{for } (r,s,t) \in [-\delta,\delta]\times\tilde{\Sigma}\times[0,T]$$

and $\delta > 0$ small. In the following we show that the properties in the theorem are satisfied if $\delta > 0$ is small and $\beta > 0$ as well as $c_0 > 0$ above were chosen properly.

Ad 1.-2. First of all, X is well-defined due to the cutoff-function. Moreover, X is smooth and

$$\partial_r X(r,s,t) = \vec{n}(s,t) + \partial_r w(r,\tilde{\sigma}(s),t)\vec{T}(s,t) \in \mathbb{R}^N, \tag{3.7}$$

$$d_s[X(r,.,t)] = d_s[\tilde{X}_0(.,t)] + r d_s[\vec{n}(.,t)] + d_s[w(r,\tilde{\sigma}(.),t)]\vec{T} + w|_{(r,\tilde{\sigma}(s),t)} d_s[\vec{T}(.,t)] \tag{3.8}$$

for all $(r,s,t) \in [-\delta,\delta] \times \tilde{\Sigma} \times [0,T]$, where $d_s[X(r,.,t)]$, $d_s[\tilde{X}_0(.,t)]$, $d_s[\vec{n}(.,t)]$ and $d_s[\vec{T}(.,t)]$ map from $T_s\tilde{\Sigma}$ to $\mathbb{R}^N$. Hence

$$d_{(r,s)}[X(.,t)] : \mathbb{R} \times T_s\tilde{\Sigma} \to \mathbb{R}^N, (v_1,v_2) \mapsto \partial_r X|_{(r,s,t)}v_1 + d_s[X(r,.,t)](v_2), \tag{3.9}$$

$$d_{(0,s)}[X(.,t)] : \mathbb{R} \times T_s\tilde{\Sigma} \to \mathbb{R}^N, (v_1,v_2) \mapsto \vec{n}(s,t)v_1 + d_s[\tilde{X}_0(.,t)](v_2), \tag{3.10}$$

where we used $w|_{r=0} = \partial_r w|_{r=0} = 0$. Since $d_s[\tilde{X}_0(.,t)] : T_s\tilde{\Sigma} \to T_{\tilde{X}_0(s,t)}\tilde{\Gamma}_t$ is an isomorphism and $\mathbb{R}^N = T_{\tilde{X}_0(s,t)}\tilde{\Gamma}_t \oplus N_{\tilde{X}_0(s,t)}\tilde{\Gamma}_t$, we obtain that $d_{(0,s)}[X(.,t)] : \mathbb{R} \times T_s\tilde{\Sigma} \to \mathbb{R}^N$ is invertible for every $(s,t) \in \tilde{\Sigma} \times [0,T]$. By compactness this is also valid for $d_{(r,s)}[X(.,t)]$ for every $(r,s,t) \in [-\delta,\delta] \times \tilde{\Sigma} \times [0,T]$ if $\delta > 0$ is small, cf. the similar argument in the proof of Theorem 3.3 above. The Inverse Function Theorem yields that $\overline{X}$ is locally injective and together with injectivity on $\{0\} \times \tilde{\Sigma} \times [0,T]$ we get similarly as in the 2-dimensional case by contradiction and compactness that $\overline{X}$ is injective on $[-\delta,\delta] \times \tilde{\Sigma} \times [0,T]$ for $\delta > 0$ small. Moreover, due to the Inverse Function Theorem, $\overline{X}$ can locally in $\mathbb{R} \times \Sigma_0 \times \mathbb{R}$ be extended to a smooth diffeomorphism. With a similar contradiction and compactness argument as before, it follows that $\overline{X}$ can be extended to a smooth diffeomorphism on an open neighbourhood of $[-\delta,\delta] \times \Sigma \times [0,T]$ in $\mathbb{R} \times \Sigma_0 \times \mathbb{R}$ mapping onto an open set in $\mathbb{R}^{N+1}$.

Next we prove that $X([-\delta,\delta] \times \Sigma \times [0,T]) \subset \overline{\Omega}$ if $\delta > 0$ is small and related properties. First, note that the set $\Gamma \backslash \overline{X}_0(Y(\partial\Sigma \times [0,\varepsilon] \times [0,T]))$ has a positive distance to $\partial\Omega \times [0,T]$ by compactness. Moreover,

$$X(r,s,t) = X_0(s,t) + r\vec{n}(s,t) \quad \text{for } (r,s,t) \in [-\delta,\delta] \times [\Sigma \backslash Y(\partial\Sigma \times [0,\varepsilon])] \times [0,T].$$

Therefore $X(r,s,t)$ stays in Ω for such (r,s,t) if $\delta > 0$ is small. For the remaining points we use geometric arguments with angles and Lemma 3.9. For $s \in Y(\partial\Sigma \times [0,\varepsilon])$ we observe that $Y(\tilde{\sigma}(s),.) : [0,\tilde{b}(s)] \to \Sigma$ is a curve from $\tilde{\sigma}(s)$ to s. Hence $X(r,Y(\tilde{\sigma}(s),.),t) : [0,b(s)] \to \mathbb{R}^N$ is a curve from $X(r,\tilde{\sigma}(s),t)$ to $X(r,s,t)$, where

$$\frac{d}{db}[X(r,Y(\tilde{\sigma}(s),b),t)] = d_{Y(\tilde{\sigma}(s),b)}[X(r,.,t)](\partial_b Y(\tilde{\sigma}(s),b)).$$

Because of (3.8) and $w|_{r=0} = \partial_r w|_{r=0} = 0$ we have

$$\left.\frac{d}{db}\right|_{b=0} [X(0,Y(\sigma,.),t)] = d_\sigma[X_0(.,t)](-\vec{n}_{\partial\Sigma}(\sigma)) \quad \text{for all } (\sigma,t) \in \partial\Sigma \times [0,T].$$

Here $d_\sigma[X_0(.,t)]$ is invertible from $T_\sigma\Sigma$ to $T_{X_0(\sigma,t)}\Gamma_t$ as well as from $T_\sigma\partial\Sigma$ to $T_{X_0(\sigma,t)}\partial\Gamma_t$. Therefore $d_\sigma[X_0(.,t)](\vec{n}_{\partial\Sigma}(\sigma)) \cdot \vec{n}_{\partial\Gamma}(\sigma,t) > 0$ and by compactness

$$d_\sigma[X_0(.,t)](\vec{n}_{\partial\Sigma}(\sigma)) \cdot \vec{n}_{\partial\Gamma}(\sigma,t) \geq c_1 > 0 \quad \text{for all } (\sigma,t) \in \partial\Sigma \times [0,T].$$

Due to (3.6) and (3.8) we obtain for all $(r,b,s,t) \in [-\delta,\delta] \times [0,\varepsilon] \times Y(\partial\Sigma \times [0,\varepsilon]) \times [0,T]$ that

$$-\vec{n}_{\partial\Gamma}|_{(\tilde{\sigma}(s),t)} \cdot \frac{d}{db}[X(r,Y(\tilde{\sigma}(s),b),t)] \geq \frac{c_1}{2} > 0 \tag{3.11}$$

provided that $\delta > 0$ is small and $c_0 > 0$ was chosen sufficiently small before. Moreover, it holds $X_0(Y(\sigma,b),t) \in R_{\frac{\eta}{2}}(\sigma,t)$ for $(\sigma,b,t) \in \partial\Sigma \times [0,\varepsilon] \times [0,T]$ by the choice of ε. This yields

$$X(r,Y(\sigma,b),t) \in R_{\frac{3\eta}{4}}(\sigma,t) \quad \text{for all } (r,\sigma,b,t) \in [-\delta,\delta] \times \partial\Sigma \times [0,\varepsilon] \times [0,T] \tag{3.12}$$

if $\delta > 0$ is small. Altogether we can determine the location of $X(r,Y(\sigma,b),t)$ geometrically: By (3.12) we know that $X(r,Y(\sigma,b),t)$ is contained in a cylinder where we have a suitable graph parametrization of $\partial\Omega$ due to Lemma 3.9. Moreover, (3.11) and the Fundamental Theorem of Calculus yield that $X(r,Y(\sigma,b),t)$ lies in a cone (where c_1 determines how close it can be to a half space) viewed from $X(r,Y(\sigma,0),t)$. Therefore if $\beta > 0$ in the beginning of the proof was chosen sufficiently small, the cone without the tip lies inside of Ω. Note that c_1 above is independent of $\beta, c_0, \eta, \varepsilon, \delta$. Therefore we can choose $\beta, c_0 > 0$ small first (both only depending on c_1), then $\eta > 0$, then $\varepsilon > 0$ and finally $\delta > 0$. This proves $X([-\delta, \delta] \times \Sigma \times [0,T]) \subset \overline{\Omega}$ and $X(r,s,t) \in \partial\Omega$ if and only if $s \in \partial\Sigma$. Moreover, with an analogous argument it follows that X maps

$$[-\delta, \delta] \times \tilde{Y}(\partial\Sigma \times [-\varepsilon, 0)) \times [0,T]$$

outside $\overline{\Omega}$ for possibly smaller $\beta, c_0, \eta, \varepsilon, \delta$. Finally, the extension property of $\overline{X}$ yields that $\Gamma(\tilde{\delta})$ is an open neighbourhood of Γ in $\overline{\Omega} \times [0,T]$ for $\tilde{\delta} \in (0, \delta]$ if $\delta > 0$ is small. $\square_{1.\text{-}2.}$

Ad 3. Consider $(r, s, \mathrm{pr}_t) := \overline{X}^{-1} : \overline{\Gamma(\delta)} \to [-\delta, \delta] \times \Sigma \times [0,T]$. Then

$$d_x[X^{-1}(.,t)] : \mathbb{R}^N \to \mathbb{R} \times T_{s(x,t)}\Sigma : \vec{v} \mapsto (d_x[r(.,t)], d_x[s(.,t)])\vec{v} = (\nabla r|_{(x,t)} \cdot \vec{v}, D_x s|_{(x,t)}\vec{v})$$

is invertible for all $(x,t) \in \overline{\Gamma(\delta)}$. Therefore $|\nabla r|_{(x,t)}| > 0$ and by compactness $|\nabla r|_{(x,t)}| \geq c > 0$ for all $(x,t) \in \overline{\Gamma(\delta)}$. Moreover, $(\partial_{x_j} s|_{(x,t)})_{j=1}^N$ generate $T_{s(x,t)}\Sigma$ for all such (x,t). In particular

$$D_x s (D_x s)^\top|_{(x,t)} = (\nabla s_i \cdot \nabla s_j|_{(x,t)})_{i,j=1}^N = \sum_{q=1}^N \partial_q s (\partial_q s)^\top|_{(x,t)}$$

is injective as a linear map in $\mathcal{L}(T_{s(x,t)}\Sigma)$ for all $(x,t) \in \overline{\Gamma(\delta)}$. The latter follows directly since $D_x s (D_x s)^\top|_{(x,t)}\vec{v} = 0$ for some $\vec{v} \in T_{s(x,t)}\Sigma$ implies $\sum_{q=1}^N |(\partial_q s|_{(x,t)})^\top \vec{v}|^2 = 0$ and hence $\vec{v} = 0$. Therefore $D_x s (D_x s)^\top|_{(x,t)}$ is positive definite on $T_{s(x,t)}\Sigma$ for all $(x,t) \in \overline{\Gamma(\delta)}$. In local coordinates the latter transforms to a linear map on $\mathbb{R}^{N-1}$. Note that by scaling it is equivalent to consider vectors in the sphere in $\mathbb{R}^{N-1}$ in order to verify the definition of positive definiteness. Therefore by compactness we obtain that $D_x s (D_x s)^\top|_{(x,t)}$ is uniformly positive definite as a linear map in $\mathcal{L}(T_{s(x,t)}\Sigma)$ for all $(x,t) \in \overline{\Gamma(\delta)}$.

Let $(r,s,t) \in [-\delta, \delta] \times \Sigma \times [0,T]$. Then $d_{(r,s)}[X(.,t)] \circ d_{X(r,s,t)}[X^{-1}(.,t)] = \mathrm{Id}_{\mathbb{R}^N}$. Hence

$$(d_{(r,s)}[X(.,t)])^{-1} = (d_{X(r,s,t)}[r(.,t)], d_{X(r,s,t)}[s(.,t)]) : \mathbb{R}^N \to \mathbb{R} \times T_s\Sigma.$$

On the other hand (3.10) implies

$$d_{X(0,s,t)}[r(.,t)] = \vec{n}(s,t)^\top \quad \text{and} \quad d_{X(0,s,t)}[s(.,t)] = d_s[X_0(.,t)]^{-1} P_{T_{X_0(s,t)}\Gamma_t}. \tag{3.13}$$

We can also write

$$d_{X(r,s,t)}[r(.,t)](\vec{v}) = D_x r|_{\overline{X}(r,s,t)}\vec{v} \quad \text{and} \quad d_{X(r,s,t)}[s(.,t)](\vec{v}) = D_x s|_{\overline{X}(r,s,t)}\vec{v} \tag{3.14}$$

for all $\vec{v} \in \mathbb{R}^N$. Altogether this yields $\nabla r|_{\overline{X}_0(s,t)} = \vec{n}(s,t)$ and $D_x s|_{\overline{X}_0(s,t)}\vec{n}(s,t) = 0$ for all $(s,t) \in \Sigma \times [0,T]$. With similar arguments we obtain

$$\nabla r|_{\overline{X}(r,s,t)} = \vec{n}(s,t) \quad \text{and} \quad D_x s|_{\overline{X}(r,s,t)}\vec{n}(s,t) = 0$$

for all $(r,s,t) \in [-\delta, \delta] \times [\Sigma \backslash Y(\partial\Sigma \times [0, \mu_0])] \times [0,T]$. Moreover, it holds

$$\partial_r(|\nabla r|^2 \circ \overline{X})|_{(0,s,t)} = 2\frac{d}{dr}[D_x r|_{X(.,s,t)}]|_{r=0} \cdot \nabla r|_{\overline{X}_0(s,t)}.$$

In order to use (3.14) we compute

$$\frac{d}{dr}[(d_{(r,s)}[X(.,t)])^{-1}]|_{r=0} = -(d_{(0,s)}X(.,t))^{-1} \circ \frac{d}{dr}[d_{(r,s)}X(.,t)]|_{r=0} \circ (d_{(0,s)}X(.,t))^{-1}$$

with the formula for the Fréchet derivative of the inverse of a differentiable family of invertible, linear operators. Here $(d_{(0,s)}X(.,t))^{-1} : \mathbb{R}^N \to \mathbb{R} \times T_s\Sigma$ is explicitly determined by (3.13). Furthermore, differentiating (3.7)-(3.8) with respect to r we obtain for all $(v_1, v_2) \in \mathbb{R} \times T_s\Sigma$

$$\frac{d}{dr}[d_{(r,s)}X(.,t)]|_{r=0}(v_1, v_2) = \partial_r^2 w(0, \sigma(s), t)\vec{T}(s,t)v_1 + d_s[\vec{n}(.,t)](v_2) \in \mathbb{R}^N.$$

Therefore it holds

$$\begin{aligned}
&\frac{d}{dr}[(d_{(r,s)}[X(.,t)])^{-1}]|_{r=0}(\vec{n}(s,t)) = -(d_{(0,s)}X(.,t))^{-1}\left(\frac{d}{dr}[d_{(r,s)}X(.,t)]|_{r=0}(1,0)\right)\\
&= -(d_{(0,s)}X(.,t))^{-1}\left(\partial_r^2 w(0,\sigma(s),t)\vec{T}(s,t)\right) = -(0, \partial_r^2 w(0,\sigma(s),t)d_s[X_0(.,t)]^{-1}\vec{T}(s,t)),
\end{aligned}$$

where $\partial_r(|\nabla r|^2 \circ \overline{X})|_{(0,s,t)}$ equals twice the first component, thus equals zero. Moreover, one can prove the identities for the normal velocity V and mean curvature H in an analogous way as in the case $N = 2$, cf. the proof of Theorem 3.3. $\square_3.$

Ad 4. Finally, we show the properties of b. Let $(\overline{\sigma}, \overline{b}) := Y^{-1} : R(Y) \to \partial\Sigma \times [0, 2\mu_1]$. Then $b = \overline{b} \circ s$ on $\overline{X}([-\delta,\delta] \times R(Y) \times [0,T])$ and by chain rule

$$d_x[b(.,t)] = d_{s(x,t)}\overline{b} \circ d_x[s(.,t)] : \mathbb{R}^N \to \mathbb{R},$$

where we are interested in boundary points $x = X_0(\sigma, t)$ for any $(\sigma, t) \in \partial\Sigma \times [0,T]$. For such x it holds $s(x,t) = \sigma$. Because of $Y(.,0) = \mathrm{id}_{\partial\Sigma}$ and (3.5) we have

$$d_{(\sigma,0)}Y : T_\sigma\partial\Sigma \times \mathbb{R} \to T_\sigma\Sigma : (v_1, v_2) \mapsto v_1 - \vec{n}_{\partial\Sigma}(\sigma)v_2.$$

Therefore $(d_\sigma\overline{\sigma}, d_\sigma\overline{b}) = (d_{(\sigma,0)}Y)^{-1} = (\mathrm{pr}_{T_\sigma\partial\Sigma}, -\vec{n}_{\partial\Sigma}(\sigma)^\top)$. Together with (3.13) we obtain

$$\nabla b|_{(x,t)} \cdot v = d_x[b(.,t)](v) = -\vec{n}_{\partial\Sigma}(\sigma) \cdot ((d_\sigma[X_0(.,t)])^{-1} \circ P_{T_x\Gamma_t}(v)) \quad \text{ for all } v \in \mathbb{R}^N.$$

Hence $\nabla b|_{\overline{X}_0(\sigma,t)} = \nabla b|_{(x,t)} \in T_{X_0(\sigma,t)}\Gamma_t$, in particular $\nabla b \cdot \nabla r|_{\overline{X}_0(\sigma,t)} = 0$, and

$$\nabla b|_{\overline{X}_0(\sigma,b)} \cdot N_{\partial\Omega}|_{X_0(\sigma,b)} = -\vec{n}_{\partial\Sigma}(\sigma) \cdot (d_\sigma[X_0(.,t)])^{-1}\vec{n}_{\partial\Gamma}(\sigma,t) \quad \text{ for all } (\sigma,t) \in \partial\Sigma \times [0,T].$$

Note that due to (3.13)-(3.14) it holds

$$(d_\sigma[X_0(.,t)])^{-1}\vec{n}_{\partial\Gamma}(\sigma,t) = D_x s N_{\partial\Omega}|_{\overline{X}_0(\sigma,t)}$$

for all $(\sigma,t) \in \partial\Sigma \times [0,T]$. Hence we obtain the identity in the theorem. Moreover, we know that $d_\sigma[X_0(.,t)]$ is invertible from $T_\sigma\Sigma$ to $T_{X_0(\sigma,t)}\Gamma_t$ as well as from $T_\sigma\partial\Sigma$ to $T_{X_0(\sigma,t)}\partial\Gamma_t$. This yields that $|\nabla b|_{\overline{X}_0(\sigma,t)} \cdot N_{\partial\Omega}|_{X_0(\sigma,t)}| > 0$ for all $(\sigma,t) \in \partial\Sigma \times [0,T]$ and by smoothness and compactness the latter is bounded from below by a uniform positive constant. Because of the Cauchy-Schwarz-Inequality, this estimate carries over to $|\nabla b|_{\overline{X}_0(\sigma,t)}|$. $\square_4.$

Finally, we show relations of ∂_n, ∇_τ defined in Remark 3.8, 2. to $\nabla, \nabla_\Sigma, \nabla_{\partial\Sigma}, \partial_b$.

Corollary 3.10. *Let $\psi : \Gamma(\tilde{\delta}) \to \mathbb{R}$ for some $\tilde{\delta} \in (0, \delta]$ be sufficiently smooth. Then*

1. $\nabla\psi = \partial_n\psi\nabla r + \nabla_\tau\psi$ *on* $\Gamma(\tilde{\delta})$ *and there are* $c, C > 0$ *independent of* ψ *and* $\tilde{\delta}$ *such that*
$$\begin{aligned} c(|\partial_n\psi| + |\nabla_\tau\psi|) \leq |\nabla\psi| \leq C(|\partial_n\psi| + |\nabla_\tau\psi|) \qquad &\text{on } \Gamma(\tilde{\delta}),\\ c|\partial_n\psi| \leq |\nabla r \partial_r(\psi \circ \overline{X}) \circ \overline{X}^{-1}| \leq C|\partial_n\psi| \quad &\text{on } \Gamma(\tilde{\delta}),\\ c|\nabla_\tau\psi| \leq |\nabla_\Sigma(\psi \circ \overline{X}) \circ \overline{X}^{-1}| \leq C|\nabla_\tau\psi| \qquad &\text{on } \Gamma(\tilde{\delta}). \end{aligned}$$
2. *It holds* $|\nabla\psi|^2 = |\partial_n\psi|^2 + |\nabla_\tau\psi|^2$ *on* $\Gamma(\tilde{\delta}, \mu_0)$.
3. *Set* $\overline{Y} : [-\delta, \delta] \times \partial\Sigma \times [0, 2\mu_1] \times [0, T] \to [-\delta, \delta] \times \Sigma \times [0, T] : (r, \sigma, b, t) \mapsto (r, Y(\sigma, b), t)$ *and* $\overline{\psi} := \psi \circ \overline{X} \circ \overline{Y}|_{(-\tilde{\delta},\tilde{\delta})\times\partial\Sigma\times[0,2\mu_1]\times[0,T]}$. *Then*
$$\tilde{c}(|\nabla_{\partial\Sigma}\overline{\psi}| + |\partial_b\overline{\psi}|) \leq |\nabla_\Sigma[\psi \circ \overline{X}] \circ \overline{Y}| \leq \tilde{C}(|\nabla_{\partial\Sigma}\overline{\psi}| + |\partial_b\overline{\psi}|)$$
on $(-\tilde{\delta}, \tilde{\delta}) \times \partial\Sigma \times [0, 2\mu_1] \times [0, T]$ *for some* $\tilde{c}, \tilde{C} > 0$ *independent of* $\psi, \tilde{\delta}$.

Analogous assertions hold for ψ *defined on* $\Gamma_t(\tilde{\delta})$, $t \in [0, T]$ *and for* ψ *defined on open subsets of* $\Gamma(\tilde{\delta})$ *or* $\Gamma_t(\tilde{\delta})$, $t \in [0, T]$ *with natural adjustments and uniform constants (w.r.t.* ψ, t *and the sets).*

Proof. We only consider $\psi : \Gamma(\tilde{\delta}) \to \mathbb{R}$. The case of sufficiently smooth $\psi : \Gamma_t(\tilde{\delta}) \to \mathbb{R}$ for any $t \in [0, T]$ and the case of other open sets can be shown with analogous arguments.

Ad 1. The second equivalence estimate is evident since $0 < \tilde{c} \leq |\nabla r| \leq \tilde{C}$ due to Theorem 3.7. Moreover, it holds $\psi = (\psi \circ \overline{X}) \circ \overline{X}^{-1}|_{(-\tilde{\delta},\tilde{\delta})\times\Sigma\times[0,T]}$. The chain rule yields
$$\nabla\psi|_{(x,t)} \cdot . = d_x[\psi(., t)] = d_{(r,s)}[\psi \circ \overline{X}(., t)] \circ d_x[X(., t)^{-1}] : \mathbb{R}^N \to \mathbb{R}$$
for all $(x, t) = \overline{X}(r, s, t) \in \Gamma(\tilde{\delta})$. Here
$$\begin{aligned} d_{(r,s)}[\psi \circ \overline{X}(., t)] : \mathbb{R} \times T_s\Sigma \to \mathbb{R} : (w, \vec{v}) &\mapsto d_r[\psi \circ \overline{X}|_{(.,s,t)}](w) + d_s[\psi \circ \overline{X}|_{(r,.,t)}](\vec{v})\\ &= \partial_{\tilde{r}}[\psi \circ \overline{X}|_{(.,s,t)}]|_r\, w + \nabla_\Sigma[\psi \circ \overline{X}|_{(r,.,t)}]|_s \cdot \vec{v}. \end{aligned}$$
Furthermore, $d_x[X(., t)^{-1}] : \mathbb{R}^N \to \mathbb{R} \times T_s\Sigma : \vec{u} \mapsto (\nabla r \cdot \vec{u}, D_x s\, \vec{u})$ is invertible for all $(x, t) = \overline{X}(r, s, t) \in \overline{\Gamma(\delta)}$ and the operator norm and the one of the inverse is uniformly bounded due to compactness. This yields $\nabla\psi = \partial_n\psi\nabla r + \nabla_\tau\psi$ on $\Gamma(\tilde{\delta})$ and
$$c(|\partial_n\psi| + |\nabla_\Sigma(\psi \circ \overline{X}) \circ \overline{X}^{-1}|) \leq |\nabla\psi| \leq C(|\partial_n\psi| + |\nabla_\Sigma(\psi \circ \overline{X}) \circ \overline{X}^{-1}|) \quad \text{on } \Gamma(\tilde{\delta})$$
with $c, C > 0$ independent of $\psi, \tilde{\delta}$. In order to show 1. it remains to prove the last equivalence estimate in the claim. The latter is valid since $D_x s$ is uniformly bounded and $D_x s(D_x s)^\top$ is uniformly positive definite on $T_{s(x,t)}\Sigma$ for all $(x, t) \in \overline{\Gamma(\delta)}$ due to Theorem 3.7. $\square_1$.

Ad 2. Consequence of 1. and $|\nabla r| = 1$, $D_x s\nabla r = 0$ on $\Gamma(\tilde{\delta}, \mu_0)$ due to Theorem 3.7, 3. $\square_2$.

Ad 3. The chain rule yields $d_{(\sigma,b)}[\overline{\psi}(r, ., t)] = d_{Y(\sigma,b)}[\psi \circ \overline{X}(r, ., t)] \circ d_{(\sigma,b)}Y : T_\sigma\partial\Sigma \times \mathbb{R} \to \mathbb{R}$ for all $(r, \sigma, b, t) \in (-\tilde{\delta}, \tilde{\delta}) \times \partial\Sigma \times [0, 2\mu_1] \times [0, T]$. Here $d_{(\sigma,b)}Y : T_\sigma\partial\Sigma \times \mathbb{R} \to T_{Y(\sigma,b)}\Sigma$ is invertible for all $(\sigma, b) \in \partial\Sigma \times [0, 2\mu_1]$ and the operator norm and the one of the inverse is uniformly bounded by compactness. Moreover, it holds
$$d_{(\sigma,b)}[\overline{\psi}(r, ., t)](\vec{v}, w) = \nabla_{\partial\Sigma}[\overline{\psi}(r, ., b, t)]|_\sigma \cdot \vec{v} + \partial_b[\overline{\psi}(r, \sigma, ., t)]|_b\, w$$
for all $(\vec{v}, w) \in T_\sigma\partial\Sigma \times \mathbb{R}$ and $d_{Y(\sigma,b)}[\psi \circ \overline{X}(r, ., t)](\vec{u}) = \nabla_\Sigma[\psi \circ \overline{X}(r, ., t)]|_{Y(\sigma,b)} \cdot \vec{u}$ for all $\vec{u} \in T_{Y(\sigma,b)}\Sigma$. This proves the claim. $\square_3$.

4 Model Problems

Unless otherwise stated we use real-valued function spaces in this section.

4.1 Some Scalar-valued ODE Problems on $\mathbb{R}$

In this section we prove existence and regularity results needed for ODEs appearing in the inner asymptotic expansion for (AC$_\alpha$), where $\alpha \in (0, \pi)$. Moreover, we show properties of a linear operator corresponding to a linearized ODE which will be important to solve the model problems on the half space in the next section. For the potential $f : \mathbb{R} \to \mathbb{R}$ in this section we assume (1.1).

4.1.1 The ODE for the Optimal Profile

The ODE system for the lowest order is

$$-w'' + f'(w) = 0, \quad w(0) = 0, \quad \lim_{z \to \pm\infty} w(z) = \pm 1. \tag{4.1}$$

Theorem 4.1. *Let f be as in* (1.1). *Then* (4.1) *has a unique solution $\theta_0 \in C^2(\mathbb{R})$. Moreover, θ_0 is smooth, $\theta_0' = \sqrt{2(f(\theta_0) - f(-1))} > 0$ and*

$$D_z^k(\theta_0 \mp 1)(z) = \mathcal{O}(e^{-\beta|z|}) \quad \textit{for } z \to \pm\infty \textit{ and all } k \in \mathbb{N}_0, \beta \in \left(0, \sqrt{\min\{f''(\pm 1)\}}\right).$$

Proof. This follows from Schaubeck [Sb], Lemma 2.6.1 and its proof. The idea is to solve the equivalent first order ODE

$$w' = \sqrt{\int_{-1}^{w} 2f'(s)\, ds}, \quad w(0) = 0.$$

Note that only ODE-methods and elementary arguments are used. $\square$

We call θ_0 the *optimal profile*. A rescaled version will be the typical profile of the solutions for the scalar-valued Allen-Cahn equation with Neumann boundary condition (AC1)-(AC3) from Section 1.1 across the interface. If f is even, then θ_0 is even, θ_0' is odd and θ_0'' even etc. For the typical double-well potential $f(u) = \frac{1}{2}(1 - u^2)^2$ shown in Figure 2 one can directly compute that the optimal profile is $\theta_0 = \tanh$, cf. Figure 10.

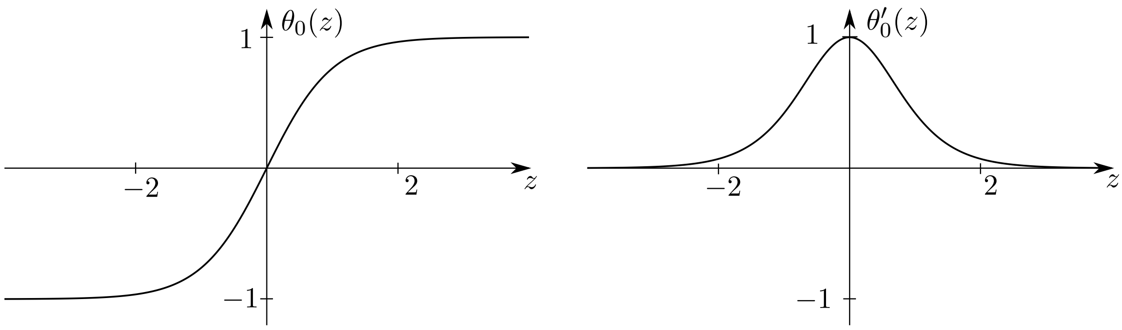

Figure 10: Typical optimal profile $\theta_0 = \tanh$ and the derivative θ_0'.

4.1.2 The Linearized Operator

The linearization at θ_0, i.e. $\mathcal{L}_0 := -\frac{d^2}{dz^2} + f''(\theta_0)$ will appear in the asymptotic expansion, too. In the next lemma we recall some properties of $\mathcal{L}_0$ viewed as an unbounded operator on $L^2(\mathbb{R}, \mathbb{K})$. The discrete spectrum $\sigma_d(L_0)$ is the set of discrete eigenvalues with finite algebraic multiplicity. Moreover, $\sigma_e(L_0) := \sigma(L_0) \setminus \sigma_d(L_0)$ is the essential spectrum.

Lemma 4.2. *The operator $L_0 : H^2(\mathbb{R}, \mathbb{K}) \subseteq L^2(\mathbb{R}, \mathbb{K}) \to L^2(\mathbb{R}, \mathbb{K}) : u \mapsto \mathcal{L}_0 u$ is self-adjoint, non-negative and $\sigma(L_0) \cap (-\infty, \min\{f''(\pm 1)\}) \subset \sigma_d(L_0)$. The lowest eigenvalue is 0 and $\ker L_0 = \operatorname{span}\{\theta_0'\}$. Moreover, with $(\ker L_0)^\perp := \{w \in L^2(\mathbb{R}, \mathbb{C}) : (w, \theta_0')_{L^2} = 0\}$ it holds*

$$
\begin{aligned}
0 < \nu_0 &:= \inf_{w \in H^2(\mathbb{R},\mathbb{C}) \cap (\ker L_0)^\perp, \|w\|_{L^2}=1} (L_0 w, w)_{L^2(\mathbb{R},\mathbb{C})} \\
&= \inf_{w \in H^1(\mathbb{R},\mathbb{C}) \cap (\ker L_0)^\perp, \|w\|_{L^2}=1} \int_{\mathbb{R}} |w'|^2 + f''(\theta_0)|w|^2 \, dz.
\end{aligned}
$$

Proof. It is enough to consider $\mathbb{K} = \mathbb{C}$ since $f''(\theta_0)$ is real-valued and the assertions directly carry over to the case $\mathbb{K} = \mathbb{R}$. Because $f''(\theta_0)$ is bounded from below, the Lax-Milgram Theorem and regularity theory imply that $\sigma(L_0)$ is bounded from below, in particular $\rho(L_0) \cap \mathbb{R} \neq \emptyset$. Since L_0 is densely defined and symmetric, it follows that L_0 is self-adjoint and it holds $\sigma(L_0) \subset \mathbb{R}$.

In order to show $\sigma(L_0) \cap (-\infty, \min\{f''(\pm 1)\}) \subset \sigma_d(L_0)$, we use Persson's Theorem, see Hislop, Sigal [HS], Theorem 14.11. The latter yields

$$
\inf \sigma_e(L_0) = \sup_{K \subset \mathbb{R} \text{ compact}} \inf\{(\phi, L_0\phi)_{L^2(\mathbb{R})} : \phi \in C_0^\infty(\mathbb{R} \setminus K, \mathbb{R}), \|\phi\|_{L^2} = 1\}.
$$

With the properties of $f''(\theta_0)$ we directly obtain $\min\{f''(\pm 1)\} \leq \inf \sigma_e(L_0)$. Then the definitions imply the subset-relation above.

Because of Theorem 4.1 it holds $\theta_0' \in H^2(\mathbb{R})$, $L_0\theta_0' = 0$ and $\theta_0' > 0$. In particular, since $0 < \min\{f''(\pm 1)\}$, it follows that $\lambda_0 := \inf \sigma(L_0)$ is an isolated eigenvalue with finite algebraic multiplicity. Weidmann [W], Satz 17.14 (cf. also Reed, Simon [RS], Theorem XIII, 48 and Faris, Simon [FS] for an english reference) yields that λ_0 is simple and corresponding eigenfunctions have a sign. Since eigenfunctions to distinct eigenvalues are orthogonal, by contradiction with $\theta_0' > 0$ it follows that $\lambda_0 = 0$ and the eigenspace $\ker L_0$ is spanned by θ_0'. This also yields that L_0 is non-negative due to Hislop, Sigal [HS], Proposition 5.12.

Finally, we prove the gap property. One can directly show that

$$
L_0^\perp := L_0|_{(\ker L_0)^\perp} : H^2(\mathbb{R}, \mathbb{C}) \cap (\ker L_0)^\perp \to (\ker L_0)^\perp
$$

is well-defined, self-adjoint and $\sigma(L_0) = \sigma(L_0^\perp) \cup \{0\}$. Assume $0 \in \sigma(L_0^\perp)$. Then 0 would be an eigenvalue of $L_0^\perp$ as an isolated point of the spectrum, see Hislop, Sigal [HS], Proposition 6.4. This is a contradiction to $\dim \ker L_0 = 1$. Therefore we obtain $\sigma(L_0^\perp) = \sigma(L_0) \setminus \{0\}$ and hence $\nu_0 > 0$ with [HS], Proposition 5.12. The last identity for ν_0 follows with a density argument and integration by parts. □

Remark 4.3.

1. It holds $\sigma_e(L_0) = [\min\{f''(\pm 1)\}, \infty)$. This follows from Lemma 4.2 if one proves $[\min\{f''(\pm 1)\}, \infty) \subseteq \sigma_e(L_0)$. The latter can be shown using Weyl sequences similar to the proof of Kusche [Ku], Proposition 2.1, where the vector-valued case is considered.

2. Note that the results in Section 6.1.3.1 for the corresponding operators on finite large intervals are obtained independently of Lemma 4.2. Therefore one could also use Lemma 6.6 together with a contradiction argument to show that 0 is simple and the lowest eigenvalue.

3. There is another way to prove that L_0 is non-negative. This is a natural conclusion from an energetic approach to construct θ_0. Such methods are used in the vector-valued case, cf. Theorem 4.26 below, but they can also be applied in the scalar case. See also Bellettini [Be], Chapter 15.

4. In order to show that 0 is a simple eigenvalue one can alternatively use Theorem 4.4, 1. below, where the linearized ODE is considered, with $A = 0$ and a contradiction argument.

4.1.3 The Linearized ODE

The following theorem is concerned with the solvability of the equation

$$-w'' + f''(\theta_0)w = A \quad \text{in } \mathbb{R}, \quad w(0) = 0, \tag{4.2}$$

which is obtained by linearization of (4.1) at θ_0'.

Theorem 4.4. *1. Let $A \in C_b^0(\mathbb{R})$. Then* (4.2) *has a solution $w \in C^2(\mathbb{R}) \cap C_b^0(\mathbb{R})$ if and only if $\int_{\mathbb{R}} A\theta_0' \, dz = 0$. In that case w is unique. Moreover, if $A(z) - A^\pm = \mathcal{O}(e^{-\beta|z|})$ for $z \to \pm\infty$ for some $\beta \in (0, \sqrt{\min\{f''(\pm 1)\}})$, then*

$$D_z^l \left[w - \frac{A^\pm}{f''(\pm 1)} \right] = \mathcal{O}(e^{-\beta|z|}) \quad \textit{for } z \to \pm\infty, \quad l = 0, 1, 2.$$

2. Let $U \subseteq \mathbb{R}^d$ (any set U is allowed, e.g. a point) and $A : \mathbb{R} \times U \to \mathbb{R}$, $A^\pm : U \to \mathbb{R}$ be smooth (i.e. locally smooth extendible) and the following hold uniformly in U:

$$D_x^k D_z^l \left[A(z,.) - A^\pm \right] = \mathcal{O}(e^{-\beta|z|}) \quad \textit{for } z \to \pm\infty, \quad k = 0, ..., K, l = 0, ..., L,$$

for some $\beta \in (0, \sqrt{\min\{f''(\pm 1)\}})$ and $K, L \in \mathbb{N}_0$. Then $w : \mathbb{R} \times U \to \mathbb{R}$, where $w(., x)$ is the solution of (4.2) *for $A(., x)$ for all $x \in U$, is also smooth and uniformly in U it holds*

$$D_x^k D_z^l \left[w(z,.) - \frac{A^\pm}{f''(\pm 1)} \right] = \mathcal{O}(e^{-\beta|z|}) \quad \textit{for } z \to \pm\infty, m = 0, ..., K, l = 0, ..., L + 2.$$

For our purpose $A^\pm = 0$ will be enough.

Proof. The result follows from the proof of Schaubeck [Sb], Lemma 2.6.2. The idea is to reduce to a first order ODE for the derivative of w/θ_0'. In order to show boundedness of the $w \in C^2(\mathbb{R})$ in [Sb] for $A \in C_b^0(\mathbb{R})$ provided $\int_{\mathbb{R}} A\theta_0' \, dz = 0$, one can use $\theta_0' > 0$, estimate A roughly in the formula for w in [Sb] and apply the convergence proof in [Sb] for the case of constant A there. Note that only ODE-methods and elementary arguments are used. □

Remark 4.5. One could also obtain solution operators in exponentially weighted Sobolev spaces using Lemma 4.2 and an argument as in the vector-valued case, cf. Theorem 4.31.

4.2 Some Scalar-valued Elliptic Problems on $\mathbb{R}^2_+$

4.2.1 An Elliptic Problem on $\mathbb{R}^2_+$ with Neumann Boundary Condition

Let $f : \mathbb{R} \to \mathbb{R}$ be as (1.1) and θ_0 be as in Theorem 4.1. For the contact point expansion for (AC) in any dimension $N \geq 2$ we have to solve the following model problem on $\mathbb{R}^2_+$: For data $G : \overline{\mathbb{R}^2_+} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ with suitable regularity and exponential decay find a solution $u : \overline{\mathbb{R}^2_+} \to \mathbb{R}$ with similar decay to

$$\left[-\Delta + f''(\theta_0(R))\right] u(R,H) = G(R,H) \quad \text{for } (R,H) \in \mathbb{R}^2_+, \tag{4.3}$$

$$-\partial_H u|_{H=0}(R) = g(R) \quad \text{for } R \in \mathbb{R}. \tag{4.4}$$

In Section 4.2.1.1 we show existence and uniqueness of weak solutions under suitable conditions on the data. The Lax-Milgram Theorem cannot be applied directly since coercivity fails. Therefore we split $G \in L^2(\mathbb{R}^2_+) = L^2(\mathbb{R}_+, L^2(\mathbb{R}))$ and $g \in L^2(\mathbb{R})$ orthogonally with respect to θ_0' in $L^2(\mathbb{R})$. To solve for the orthogonal parts we use the Lax-Milgram Theorem. For the parallel parts it turns out that for suitable G, g satisfying the compatibility condition

$$\int_{\mathbb{R}^2_+} G(R,H)\theta_0'(R)\, d(R,H) + \int_{\mathbb{R}} g(R)\theta_0'(R)\, dR = 0$$

there is an explicit solution formula. To obtain higher regularity one can apply standard theory.

In order to show suitable exponential decay one could proceed as follows, cf. Abels, Moser [AM], Section 2.4.2: One considers the functions $H \mapsto \|u(.,H)\|_{L^2(\mathbb{R})}$ and $R \mapsto \|u(R,.)\|_{L^2(\mathbb{R}_+)}$ and derives ordinary differential inequalities on appropriate sets. Then, if G, g are suitable, by contradiction one can show estimates of the type

$$\|u(.,H)\|_{L^2(\mathbb{R})} \leq C_u e^{-\nu H} \quad \text{for a.e. } H \in \mathbb{R}_+,$$

$$\|u(R,.)\|_{L^2(\mathbb{R}_+)} \leq C_u e^{-\nu |R|} \quad \text{for a.e. } R \in \mathbb{R},$$

where $\nu \in (0, \sqrt{\nu_0})$ and ν_0 is as in Lemma 4.2. Then by differentiating and rearranging the equations and by interpolation one gets similar estimates for the derivatives if the data are appropriate. With embeddings one also obtains pointwise estimates.

Here we proceed differently in terms of the exponential decay estimates: The splitting method introduced above seems not to work for the model problems in Section 4.2.2 arising in the contact point expansion for (AC$_\alpha$), cf. Paragraph 4.2.2.3 below. Therefore we introduce a functional analytic setting with exponentially weighted Sobolev spaces (defined in Section 2.3) in order to have isomorphisms between the solutions and the data for (4.3)-(4.4). The latter will be done in Paragraph 4.2.1.2 for several types of weighted Sobolev spaces. The rough idea is always to multiply the equation with the weights, use the product rule and known isomorphisms. This framework will then be used to solve the problems in Section 4.2.2 below for α close to $\frac{\pi}{2}$.

4.2.1.1 Weak Solutions and Regularity Let us start with the definition of a weak solution:

Definition 4.6. Let $G \in L^2(\mathbb{R}^2_+)$ and $g \in L^2(\mathbb{R})$. Then $u \in H^1(\mathbb{R}^2_+)$ is called *weak solution* of (4.3)-(4.4) if for all $\varphi \in H^1(\mathbb{R}^2_+)$ it holds that

$$a(u,\varphi) := \int_{\mathbb{R}^2_+} \nabla u \cdot \nabla \varphi + f''(\theta_0(R)) u\varphi\, d(R,H) = \int_{\mathbb{R}^2_+} G\varphi\, d(R,H) + \int_{\mathbb{R}} g(R)\varphi|_{H=0}(R)\, dR.$$

Regarding weak solutions we have the following theorem:

Theorem 4.7. *Let $G \in L^2(\mathbb{R}^2_+)$ and $g \in L^2(\mathbb{R})$. Then it holds:*

1. *$a : H^1(\mathbb{R}^2_+) \times H^1(\mathbb{R}^2_+) \to \mathbb{R}$ is not coercive.*
2. *If $G(.,H), g \perp \theta_0'$ for a.e. $H > 0$ in $L^2(\mathbb{R})$, then there is a weak solution u such that $u(.,H) \perp \theta_0'$ for a.e. $H > 0$ and it holds $\|u\|_{H^1(\mathbb{R}^2_+)} \leq C(\|G\|_{L^2(\mathbb{R}^2_+)} + \|g\|_{L^2(\mathbb{R})})$.*
3. *Weak solutions are unique.*
4. *If $G\theta_0' \in L^1(\mathbb{R}^2_+)$ and u is a weak solution with $\partial_H u\, \theta_0' \in L^1(\mathbb{R}^2_+)$, then the following compatibility condition holds:*
$$\int_{\mathbb{R}^2_+} G(R,H)\theta_0'(R)\, d(R,H) + \int_{\mathbb{R}} g(R)\theta_0'(R)\, dR = 0. \tag{4.5}$$
5. *If $G\theta_0' \in L^1(\mathbb{R}^2_+)$, then $\tilde{G}(H) := (G(.,H), \theta_0')_{L^2(\mathbb{R})}$ is well-defined for a.e. $H > 0$ and $\tilde{G} \in L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$. Moreover, we have the decomposition*
$$G = \tilde{G}(H)\frac{\theta_0'(R)}{\|\theta_0'\|^2_{L^2(\mathbb{R})}} + G^\perp(R,H), \quad g = (g, \theta_0')_{L^2(\mathbb{R})}\frac{\theta_0'(R)}{\|\theta_0'\|^2_{L^2(\mathbb{R})}} + g^\perp(R) \tag{4.6}$$
for some $G^\perp \in L^2(\mathbb{R}^2_+), g^\perp \in L^2(\mathbb{R})$ with $G^\perp(.,H), g \perp \theta_0'$ in $L^2(\mathbb{R})$ for a.e. $H > 0$.
6. *If $\|G(.,H)\|_{L^2(\mathbb{R})} \leq Ce^{-\nu H}$ for a.e. $H > 0$ and a constant $\nu > 0$, then $G\theta_0' \in L^1(\mathbb{R}^2_+)$. Let $\tilde{G}$ be defined as in 4. and the compatibility condition* (4.5) *hold. Then*
$$u_1(R,H) := -\int_H^\infty \int_{\tilde{H}}^\infty \tilde{G}(\hat{H})\, d\hat{H}\, d\tilde{H}\, \frac{\theta_0'(R)}{\|\theta_0'\|^2_{L^2(\mathbb{R})}} \tag{4.7}$$
is well-defined for a.e. $(R,H) \in \mathbb{R}^2_+$, $u_1 \in W^2_1(\mathbb{R}^2_+) \cap H^2(\mathbb{R}^2_+)$ and u_1 is a weak solution of (4.3)-(4.4) *for $G - G^\perp, g - g^\perp$ in* (4.6) *instead of G, g.*

In Theorem 4.7, 6. weaker conditions on G are enough, cf. Paragraph 4.2.1.2. The point is included for aesthetic reasons. Altogether we obtain an existence theorem for weak solutions:

Corollary 4.8. 1. *Let $g \in L^2(\mathbb{R})$, $G \in L^2(\mathbb{R}^2_+)$ with $\|G(.,H)\|_{L^2(\mathbb{R})} \leq Ce^{-\nu H}$ f.a.e. $H > 0$ and some $\nu > 0$. Let* (4.5) *hold. Then there is a unique weak solution of* (4.3)-(4.4).

2. *Let $k \in \mathbb{N}_0$ and $u \in H^1(\mathbb{R}^2_+)$ be a weak solution of* (4.3)-(4.4) *for $G \in H^k(\mathbb{R}^2_+)$ and $g \in H^{k+\frac{1}{2}}(\mathbb{R})$. Then $u \in H^{k+2}(\mathbb{R}^2_+) \hookrightarrow C^{k,\gamma}(\overline{\mathbb{R}^2_+})$ for all $\gamma \in (0,1)$ and it holds*
$$\|u\|_{H^{k+2}(\mathbb{R}^2_+)} \leq C_k(\|G\|_{H^k(\mathbb{R}^2_+)} + \|g\|_{H^{k+\frac{1}{2}}(\mathbb{R})} + \|u\|_{H^1(\mathbb{R}^2_+)}).$$

Proof. The first part directly follows from Theorem 4.7. For the second assertion, we apply Triebel [T2], Theorem 2.7.2 to obtain a $\overline{g} \in H^{k+2}(\mathbb{R}^2_+)$ that satisfies $(-\partial_H \overline{g})|_{H=0} = g$ and the estimate $\|\overline{g}\|_{H^{k+2}(\mathbb{R}^2_+)} \leq C\|g\|_{H^{k+\frac{1}{2}}(\mathbb{R})}$. Subtracting $\overline{g}$ from u and using standard regularity theory, we get $u \in H^{k+2}(\mathbb{R}^2_+)$ and the estimate. $\square$

Proof of Theorem 4.7. Ad 1. We consider smooth cut-off functions $\chi_n : \mathbb{R}_+ \to [0,1]$ for $n \in \mathbb{N}$ with $|\chi_n'| \leq C$, $\chi_n = 1$ for $H \leq n$ and $\chi_n = 0$ for $H \geq n+1$. If a would be coercive on $H^1(\mathbb{R}^2_+)$, then for some $c > 0$

$$\begin{aligned} a(\theta_0'(R)\chi_n(H), \theta_0'(R)\chi_n(H)) &= \int_{\mathbb{R}^2_+} (\theta_0'\chi_n')^2 + \left[(\theta_0'')^2 + f''(\theta_0)\theta_0'^2\right] \chi_n^2 \, d(R,H) \\ &= \int_{\mathbb{R}} (\theta_0')^2 \, dR \int_{\mathbb{R}_+} (\chi_n')^2 \, dH \geq c \int_{\mathbb{R}} (\theta_0')^2 \, dR \int_{\mathbb{R}_+} \chi_n^2 \, dH, \end{aligned}$$

where we used integration by parts with respect to $R \in \mathbb{R}$ and $\theta_0''' = f''(\theta_0)\theta_0'$ in the second term. This is a contradiction for $n \to \infty$. $\Box_{1.}$

Ad 2. Let $G \in L^2(\mathbb{R}^2_+)$ and $g \in L^2(\mathbb{R})$ with $G(.,H), g \perp \theta_0'$ in $L^2(\mathbb{R})$ for a.e. $H > 0$. In order to show the existence of a weak solution we apply the Lax-Milgram Theorem to the space $V := \{u \in H^1(\mathbb{R}^2_+) : u(.,H) \perp \theta_0' \text{ in } L^2(\mathbb{R}) \text{ for a.e. } H > 0\}$, the bilinear form

$$a : V \times V \to \mathbb{R} : (u,v) \mapsto \int_{\mathbb{R}^2_+} \nabla u \cdot \nabla v + f''(\theta_0(R))uv \, d(R,H)$$

and $x' \in V'$ defined by $x'(v) := \int_{\mathbb{R}^2_+} Gv + \int_{\mathbb{R}} g(R) v|_{H=0}(R) \, dR$ for all $v \in V$. First of all, V is a Hilbert space as a closed subspace of $H^1(\mathbb{R}^2_+)$. Here closedness follows from Lemma 2.10, 1. and linearity of $(., \theta_0')_{L^2(\mathbb{R})} : L^2(\mathbb{R}) \to \mathbb{R}$. Boundedness of a can be shown directly and coercivity on V follows from

$$\begin{aligned} a(v,v) &= \|\partial_H v\|^2_{L^2(\mathbb{R}^2_+)} + \int_0^\infty \int_{\mathbb{R}} (\partial_R v)^2 + f''(\theta_0(R))v^2 \, dR \, dH \\ &\geq \|\partial_H v\|^2_{L^2(\mathbb{R}^2_+)} + \delta(\|\partial_R v\|^2_{L^2(\mathbb{R}^2_+)} - \sup_{\mathbb{R}} |f''(\theta_0)| \|v\|^2_{L^2(\mathbb{R}^2_+)}) + (1-\delta)\nu_0 \|v\|^2_{L^2(\mathbb{R}^2_+)} \\ &\geq c\|v\|^2_{H^1(\mathbb{R}^2_+)} \end{aligned}$$

for all $v \in V$ and a $c > 0$ if $\delta > 0$ is sufficiently small, where we used Fubini's Theorem and Lemma 4.2. Therefore the Lax-Milgram Theorem implies that there is a unique $u \in V$ such that $a(u,\varphi) = x'(\varphi)$ for all $\varphi \in V$ and that the estimate holds. Hence u satisfies the definition of weak solution for all $\varphi \in V$. For $\varphi \in H^1(\mathbb{R}^2_+)$ let $\tilde{\varphi}(H) := (\varphi(.,H), \theta_0')_{L^2(\mathbb{R})}$ for a.e. $H > 0$. By Lemma 2.10 we have that $\tilde{\varphi} \in H^1(\mathbb{R}_+)$ and

$$\varphi = \tilde{\varphi}(H) \frac{\theta_0'(R)}{\|\theta_0'\|^2_{L^2(\mathbb{R})}} + \varphi^\perp \quad \text{with } \varphi^\perp \in H^1(\mathbb{R}^2_+) \text{ such that } \varphi^\perp(.,H) \perp \theta_0' \text{ for a.e. } H > 0.$$

Since the definition of weak solution is linear in φ, we only have to verify it for the parallel part, i.e. we need to show

$$\begin{aligned} &\int_{\mathbb{R}^2_+} \partial_H u \, \partial_H \tilde{\varphi}(H) \theta_0'(R) + \partial_R u \, \tilde{\varphi}(H) \theta_0''(R) + f''(\theta_0(R)) u \, \tilde{\varphi}(H) \theta_0'(R) \, d(R,H) \\ &= \int_{\mathbb{R}^2_+} G \tilde{\varphi}(H) \theta_0'(R) \, d(R,H) + \int_{\mathbb{R}} g(R) \tilde{\varphi}(0) \theta_0'(R) \, dR. \end{aligned}$$

The right hand side is zero because of the orthogonality condition for G, g. The second and the last term on the left hand side cancel since we can apply integration by parts in R for the second part and use $\theta_0''' = f''(\theta_0)\theta_0'$. Moreover, Lemma 2.10 and linearity of $(.,\theta_0')_{L^2(\mathbb{R})} : L^2(\mathbb{R}) \to \mathbb{R}$ imply

$$0 = \frac{d}{dH}(u(.,H),\theta_0')_{L^2(\mathbb{R})} = (\partial_H u(.,H),\theta_0')_{L^2(\mathbb{R})} \quad \text{for a.e. } H > 0.$$

Hence by the Fubini Theorem the first term above vanishes and u is a weak solution. $\square_2.$

Ad 3. Due to linearity it is enough to prove uniqueness for weak solutions u to the data $G, g = 0$. Given such a u, let us insert $\varphi = u$ in the Definition 4.6 of weak solution. This implies

$$\|\partial_H u\|^2_{L^2(\mathbb{R}^2_+)} + \int_{\mathbb{R}^2_+} (\partial_R u)^2 + f''(\theta_0(R))u^2 \, dR \, dH = 0.$$

Because of Fubini's Theorem, Lemma 4.2 and Lemma 2.10 the second integral is non-negative. This yields $\partial_H u = 0$ and from Lemma 2.10 we obtain that $u(.,H) \in L^2(\mathbb{R})$ is constant in $H > 0$. Thus $u = 0$, otherwise we get a contradiction to $u \in L^2(\mathbb{R}_+, L^2(\mathbb{R}))$. $\square_3.$

Ad 4. Let $G\theta_0' \in L^1(\mathbb{R}^2_+)$ and u be a weak solution such that $\partial_H u \, \theta_0' \in L^1(\mathbb{R}^2_+)$. Moreover, let $\chi_n : \mathbb{R}_+ \to [0,1]$ be as in the proof of 1. Then by inserting $\varphi = \chi_n(H)\theta_0'(R)$ in the definition of weak solution we obtain

$$\int_{\mathbb{R}^2_+} (\partial_R u \theta_0'' + f''(\theta_0) u \theta_0')\chi_n + \partial_H u \chi_n' \theta_0' \, d(R,H) = \int_{\mathbb{R}^2_+} G\theta_0' \chi_n \, d(R,H) + \int_{\mathbb{R}} g\theta_0' \, dR.$$

The first term on the left hand side vanishes since we can apply integration by parts in R and the second term converges to 0 for $n \to \infty$ because of the Dominated Convergence Theorem since $\partial_H u \theta_0' \in L^1(\mathbb{R}^2_+)$. Because of $G\theta_0' \in L^1(\mathbb{R}^2_+)$ the latter theorem applied to the first integral on the right hand side yields the compatibility condition (4.5). $\square_4.$

Ad 5. Let $G \in L^2(\mathbb{R}^2_+)$ with $G\theta_0' \in L^1(\mathbb{R}^2_+)$, $g \in L^2(\mathbb{R})$ and $\tilde{G}(H) := (G(.,H),\theta_0')_{L^2(\mathbb{R})}$ for $H > 0$. By Fubini's Theorem $\tilde{G}$ is well-defined a.e. on $\mathbb{R}_+$ and belongs to $L^1(\mathbb{R}_+)$. $\tilde{G} \in L^2(\mathbb{R}_+)$ follows from Lemma 2.10, 1. and linearity of $(.,\theta_0')_{L^2(\mathbb{R})} : L^2(\mathbb{R}) \to \mathbb{R}$. We define $G^\perp$ and $g^\perp$ according to equations (4.6). The claimed properties can be directly verified. $\square_5.$

Ad 6. Let $G \in L^2(\mathbb{R}^2_+)$ with $\|G(.,H)\|_{L^2(\mathbb{R})} \leq Ce^{-\nu H}$ for $C, \nu > 0$ and $g \in L^2(\mathbb{R})$. First, we show $G\theta_0' \in L^1(\mathbb{R}^2_+)$. Because of Lemma 2.10, 1. and since multiplication with θ_0' gives a bounded, linear operator from $L^2(\mathbb{R})$ to $L^1(\mathbb{R})$, we know that $\mathbb{R}_+ \ni H \mapsto G(.,H)\theta_0' \in L^1(\mathbb{R})$ is strongly measurable. The estimate for G ensures $G\theta_0' \in L^1(\mathbb{R}_+, L^1(\mathbb{R})) \cong L^1(\mathbb{R}^2_+)$.

Therefore we can define $\tilde{G}, G^\perp$ and $g^\perp$ as in 4. We show that u_1 defined via (4.7) is well-defined. Since $\tilde{G} \in L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ and $|\tilde{G}(H)| \leq Ce^{-\nu H}$ for a.e. $H > 0$, we obtain

$$\int_H^\infty \tilde{G}(\hat{H}) \, d\hat{H} = \int_0^\infty \tilde{G}(\hat{H}) \, d\hat{H} - \int_0^H \tilde{G}(\hat{H}) \, d\hat{H} \in W^1_1(\mathbb{R}_+) \cap H^1(\mathbb{R}_+) \quad \text{w.r.t. } H$$

with derivative $-\tilde{G}$. Analogously,

$$\int_H^\infty \int_{\tilde{H}}^\infty \tilde{G}(\hat{H}) \, d\hat{H} \, d\tilde{H} \in W^2_1(\mathbb{R}_+) \cap H^2(\mathbb{R}_+) \quad \text{w.r.t. } H$$

and the derivative is given by $-\int_H^\infty \tilde{G}(\hat{H})\,d\hat{H}$. Since $\theta_0' \in W_1^2(\mathbb{R}) \cap H^2(\mathbb{R})$, Lemma 2.10, 3. yields $u_1 \in W_1^2(\mathbb{R}^2_+) \cap H^2(\mathbb{R}^2_+)$ and we can explicitly calculate the derivatives. One can directly verify that

$$[-\Delta + f''(\theta_0)]\,u_1 = \frac{\theta_0'}{\|\theta_0'\|^2_{L^2(\mathbb{R})}}\tilde{G}.$$

Moreover, let the compatibility condition (4.5) hold. Then

$$-\partial_H u_1|_{H=0} = \frac{-\theta_0'}{\|\theta_0'\|^2_{L^2(\mathbb{R})}}\int_0^\infty \tilde{G}(\hat{H})\,d\hat{H} = \frac{-\theta_0'}{\|\theta_0'\|^2_{L^2(\mathbb{R})}}\int_{\mathbb{R}^2_+} G\theta_0'\,d(R,H) = \frac{\theta_0'(g,\theta_0')_{L^2(\mathbb{R})}}{\|\theta_0'\|^2_{L^2(\mathbb{R})}}.$$

Therefore u_1 is a strong solution of (4.3)-(4.4) for $G - G^\perp, g - g^\perp$ instead of G, g and thus a weak solution because of integration by parts. $\square_6$.

4.2.1.2 Solution Operators in Exponentially Weighted Spaces In the following the superscript "$\perp$" always means $u(.,H) \perp \theta_0'$ in $L^2(\mathbb{R})$ for a.e. $H \in \mathbb{R}_+$, if $u \in L^2(\mathbb{R}^2_+)$, and $u \perp \theta_0'$ if $u \in L^2(\mathbb{R})$. The symbol "$\|$" has the same meaning with "$\perp$" replaced by "$\|$".

Theorem 4.9 (Solution Operators for Decay in H). *1. For $\gamma \geq 0$ small the operator*

$$L_{\frac{\pi}{2}} := (-\Delta + f''(\theta_0(R)), -\partial_H|_{H=0}) : H^{2,\perp}_{(0,\gamma)}(\mathbb{R}^2_+) \to L^{2,\perp}_{(0,\gamma)}(\mathbb{R}^2_+) \times H^{\frac{1}{2},\perp}(\mathbb{R})$$

is well-defined and invertible. Moreover, for small $\gamma \geq 0$ the operator norm of $L_{\frac{\pi}{2}}^{-1}$ is uniformly bounded in the corresponding spaces.

2. For all $\gamma \in (0, \overline{\gamma}]$ and any $\overline{\gamma} > 0$ the operator

$$L_{\frac{\pi}{2}} : H^{2,\|}_{(0,\gamma)}(\mathbb{R}^2_+) \to \left\{(G,g) \in L^{2,\|}_{(0,\gamma)}(\mathbb{R}^2_+) \times H^{\frac{1}{2},\|}(\mathbb{R}) : \int_{\mathbb{R}^2_+} G\theta_0' + \int_{\mathbb{R}} g\theta_0' = 0\right\}$$

is well-defined, invertible and the norm of $L_{\frac{\pi}{2}}^{-1}$ is bounded by $C_{\overline{\gamma}}(1+\frac{1}{\gamma^2})$ for all $\gamma \in (0,\overline{\gamma}]$.

3. For $\gamma > 0$ small

$$L_{\frac{\pi}{2}} : H^2_{(0,\gamma)}(\mathbb{R}^2_+) \to \left\{(G,g) \in L^2_{(0,\gamma)}(\mathbb{R}^2_+) \times H^{\frac{1}{2}}(\mathbb{R}) : \int_{\mathbb{R}^2_+} G\theta_0' + \int_{\mathbb{R}} g\theta_0' = 0\right\}$$

and the operator norm of the inverse is bounded by $C(1+\frac{1}{\gamma^2})$ for small $\gamma > 0$.

Proof. Ad 1. $L_{\frac{\pi}{2}}$ is well-defined in the spaces because of Lemma 2.22, 2., 5. and since the orthogonality property can be shown via integration by parts as well as by differentiating the orthogonality condition for u with respect to H. In the case $\gamma = 0$ invertibility follows from Theorem 4.7, 2.-3. and Corollary 4.8, 2. Now let $\gamma > 0$. In order to solve $L_{\frac{\pi}{2}}u = (G,g)$ we make the ansatz $u = e^{-\gamma H}v$ with $v \in H^{2,\perp}(\mathbb{R}^2_+)$. By computing derivatives of u we obtain equations we want to solve for v. Note that the exponential factor does not destroy the orthogonality property. It holds

$$\partial_H u = -\gamma e^{-\gamma H}v + e^{-\gamma H}\partial_H v \quad \text{and} \quad \partial_H^2 u = \gamma^2 e^{-\gamma H}v - 2\gamma e^{-\gamma H}\partial_H v + e^{-\gamma H}\partial_H^2 v.$$

Therefore we consider

$$L_{\frac{\pi}{2}}v + N_\gamma v = (Ge^{\gamma H}, g), \quad \text{where } N_\gamma v := (-\gamma^2 v + 2\gamma\partial_H v, \gamma v|_{H=0}). \tag{4.8}$$

Here N_γ is a bounded linear operator from $H^{2,\perp}(\mathbb{R}^2_+)$ to $L^{2,\perp}(\mathbb{R}^2_+) \times H^{\frac{1}{2},\perp}(\mathbb{R})$ and the operator norm is estimated by $C(\gamma + \gamma^2)$. Hence a Neumann series argument yields that $L_{\frac{\pi}{2}} + N_\gamma$ is invertible in those spaces for small γ and the norm of the inverse is bounded uniformly. Let (G, g) be in $L^{2,\perp}_{(0,\gamma)}(\mathbb{R}^2_+) \times H^{\frac{1}{2}}(\mathbb{R})$. Then we obtain for small $\gamma > 0$ a unique $v \in H^{2,\perp}(\mathbb{R}^2_+)$ that solves (4.8). The above computations yield that $u := e^{-\gamma H}v \in H^{2,\perp}_{(0,\gamma)}(\mathbb{R}^2_+)$ is a solution of $L_{\frac{\pi}{2}}u = (G, g)$ and that it is unique. Finally, we have the estimate

$$\|u\|_{H^{2,\perp}_{(0,\gamma)}(\mathbb{R}^2_+)} = \|v\|_{H^{2,\perp}(\mathbb{R}^2_+)} \leq C\|(Ge^{\gamma H}, g)\|_{L^{2,\perp}(\mathbb{R}^2_+)\times H^{\frac{1}{2}}(\mathbb{R})} = C\|(G, g)\|_{L^{2,\perp}_{(0,\gamma)}(\mathbb{R}^2_+)\times H^{\frac{1}{2}}(\mathbb{R})},$$

where $C > 0$ is independent of $\gamma > 0$ small. $\square_1$.

Ad 2. Let $\gamma > 0$. Then $u \in H^{2,\|}_{(0,\gamma)}(\mathbb{R}^2_+)$ if and only if

$$e^{\gamma H}u \in H^2(\mathbb{R}^2_+) \quad \text{and} \quad u(R, H) = (u(., H), \theta_0')_{L^2(\mathbb{R})}\frac{\theta_0'(R)}{\|\theta_0'\|^2_{L^2(\mathbb{R})}} \text{ almost everywhere.}$$

Since $H^2(\mathbb{R}^2_+) \hookrightarrow H^2(\mathbb{R}_+, L^2(\mathbb{R}))$ by Lemma 2.10 and because multiplication with θ_0' is a bounded linear operator from $L^2(\mathbb{R})$ to $\mathbb{R}$, it follows that $u \in H^{2,\|}_{(0,\gamma)}(\mathbb{R}^2_+)$ is equivalent to $u(R, H) = \tilde{u}(H)\theta_0'(R)$ f.a.a. $(R, H) \in \mathbb{R}^2_+$ for some $\tilde{u} \in H^2_{(\gamma)}(\mathbb{R}_+)$. The operator $L_{\frac{\pi}{2}}$ acts as

$$L_{\frac{\pi}{2}}u = (-\partial_H^2\tilde{u}(H)\theta_0'(R), -\partial_H\tilde{u}(0)\theta_0'(R)) \in L^{2,\|}_{(0,\gamma)}(\mathbb{R}^2_+) \times H^{\frac{1}{2},\|}(\mathbb{R}).$$

Additionally, the compatibility condition (4.5) holds because of Theorem 4.7, 4. The latter could also be directly computed here. Altogether, $L_{\frac{\pi}{2}}$ is well-defined in the spaces. On the other hand, let (G, g) be in the space $L^{2,\|}_{(0,\gamma)}(\mathbb{R}^2_+) \times H^{\frac{1}{2},\|}(\mathbb{R})$ and let the compatibility condition (4.5) hold. Then $G = \tilde{G}(H)\theta_0'(R)$, $g = \tilde{g}\theta_0'$ for some $\tilde{G} \in L^2_{(\gamma)}(\mathbb{R}_+)$ and $\tilde{g} \in \mathbb{R}$. By Lemma 2.22, 6.-7. it holds

$$\tilde{u} := -\int_{.}^{\infty}\int_{\hat{H}}^{\infty}\tilde{G}(\overline{H})\,d\overline{H}\,d\hat{H} \in H^2_{(\gamma)}(\mathbb{R}_+)$$

with $\partial_H\tilde{u} = \int_{.}^{\infty}\tilde{G}(\hat{H})\,d\hat{H}$ and $\partial_H^2\tilde{u} = -\tilde{G}$ as well as $\|\tilde{u}\|_{H^2_{(\gamma)}(\mathbb{R}_+)} \leq c_{\overline{\gamma}}(1 + \frac{1}{\gamma^2})$ for all $\gamma \in (0, \overline{\gamma}]$, where $\overline{\gamma} > 0$ is arbitrary but fixed. Therefore $u_1 := \tilde{u}(H)\theta_0'(R) \in H^2_{(0,\gamma)}(\mathbb{R}^2_+)$ solves

$$\begin{aligned}
[-\Delta + f''(\theta_0)]u_1 &= -\partial_H^2\tilde{u}\theta_0' = G,\\
-\partial_H u_1|_{H=0} &= -\int_0^{\infty}\tilde{G}(\hat{H})\,d\hat{H}\theta_0' = \tilde{g}\theta_0' = g,
\end{aligned}$$

where the last equality follows from the compatibility condition (4.5). Hence u is a solution of $L_{\frac{\pi}{2}} = (G, g)$ and it is unique because of Theorem 4.7, 3. Moreover, we have

$$\|u_1\|_{H^2_{(0,\gamma)}(\mathbb{R}^2_+)} \leq C\|\tilde{u}\|_{H^2_{(\gamma)}(\mathbb{R}_+)} \leq Cc_{\overline{\gamma}}(1 + \frac{1}{\gamma^2})\|\tilde{G}\|_{L^2_{(\gamma)}(\mathbb{R}_+)} \leq C_{\overline{\gamma}}(1 + \frac{1}{\gamma^2})\|G\|_{L^2_{(0,\gamma)}(\mathbb{R}^2_+)}.$$

Altogether this proves the claim. $\square_2$.

Ad 3. Via (4.6) we have isomorphic splitting operators from $H^k_{(0,\gamma)}(\mathbb{R}^2_+)$ onto the direct sum $H^{k,\perp}_{(0,\gamma)}(\mathbb{R}^2_+) \oplus H^{k,\|}_{(0,\gamma)}(\mathbb{R}^2_+)$ for all $k \in \mathbb{N}_0$ (at this point only $k = 2$ needed) and the operator norms for fixed k are estimated by a constant independent of $\gamma \in (0, \overline{\gamma}]$. Therefore the claim follows from 1. and 2. $\square_{3.}$

Theorem 4.10 (Solution Operators for Decay in (R, H)). *Let $\overline{\gamma} > 0$ be such that Theorem 4.9, 3. holds for $\gamma \in (0, \overline{\gamma}]$. Then there is a non-decreasing $\overline{\beta} : (0, \overline{\gamma}] \to (0, \infty)$ such that*

$$L_{\frac{\pi}{2}} : H^2_{(\beta,\gamma)}(\mathbb{R}^2_+) \to Y_{(\beta,\gamma)} := \left\{ (G, g) \in L^2_{(\beta,\gamma)}(\mathbb{R}^2_+) \times H^{\frac{1}{2}}_{(\beta)}(\mathbb{R}) : \int_{\mathbb{R}^2_+} G\theta_0' + \int_{\mathbb{R}} g\theta_0' = 0 \right\}$$

is an isomorphism for all $\beta \in [0, \overline{\beta}(\gamma)]$ and the operator norm of the inverse is bounded by $\tilde{C}(1 + \frac{1}{\gamma^2})$ with $\tilde{C}$ independent of (β, γ).

Proof. The idea is similar as in the proof of Theorem 4.9, 1. $L_{\frac{\pi}{2}}$ is well-defined in the spaces due to Theorem 4.7, 4. In order to solve $L_{\frac{\pi}{2}} u = (G, g) \in Y_{(\beta,\gamma)}$, we make the ansatz $u = e^{-\beta\eta(R)} v$ for $v \in H^2_{(0,\gamma)}(\mathbb{R}^2_+)$, where $\eta : \mathbb{R} \to \mathbb{R}$ is as in Definition 2.21, 4. We compute

$$\begin{aligned} \partial_R u &= e^{-\beta\eta(R)}[\partial_R v - \beta\eta'(R) v], \\ \partial_R^2 u &= e^{-\beta\eta(R)}[\partial_R^2 v - 2\beta\eta'(R)\partial_R v + v(\beta^2\eta'(R)^2 - \beta\eta''(R))]. \end{aligned}$$

Therefore for $v \in H^2_{(0,\gamma)}(\mathbb{R}^2_+)$ we consider the equation

$$L_{\frac{\pi}{2}} v + (N_{(\beta,\gamma)} v, 0) = e^{\beta\eta(R)}(G, g), \quad N_{(\beta,\gamma)} v := 2\beta\eta'\partial_R v - v(\beta^2(\eta')^2 - \beta\eta''). \tag{4.9}$$

There is a problem with the compatibility condition (4.5) here. For $L_{\frac{\pi}{2}} v$ the latter is valid by Theorem 4.7, 4., but for $(N_{(\beta,\gamma)} v, 0)$ and $e^{\beta\eta(R)}(G, g)$ it does not hold necessarily. Therefore we enforce the condition on both sides artificially and look at the adjusted equation

$$\begin{aligned} &L_{\frac{\pi}{2}} v + \tilde{N}_{(\beta,\gamma)} v = (\overline{G}, \overline{g}), \qquad\qquad (4.10) \\ &\quad \tilde{N}_{(\beta,\gamma)} v := \left(N_{(\beta,\gamma)} v, - \left[\int_{\mathbb{R}^2_+} N_{(\beta,\gamma)} v \theta_0' \right] \frac{\theta_0'}{\|\theta_0'\|^2_{L^2(\mathbb{R})}} \right), \\ &\quad (\overline{G}, \overline{g}) := e^{\beta\eta(R)}(G, g) - \left(0, \left[\int_{\mathbb{R}^2_+} e^{\beta\eta(R)} G\theta_0' + \int_{\mathbb{R}} e^{\beta\eta(R)} g\theta_0' \right] \frac{\theta_0'}{\|\theta_0'\|^2_{L^2(\mathbb{R})}} \right). \end{aligned}$$

In order to solve (4.10), we observe that $\tilde{N}_{(\beta,\gamma)}$ is a bounded linear operator from $H^2_{(0,\gamma)}(\mathbb{R}^2_+)$ to $Y_{(0,\gamma)}$ with norm estimated by $C(\beta + \beta^2)$ for all $\beta \geq 0$ and $\gamma \in (0, \overline{\gamma}]$. Moreover, Theorem 4.9, 3. yields that $L_{\frac{\pi}{2}}$ is an isomorphism in these spaces and that the inverse is bounded by $\overline{C}(1 + \frac{1}{\gamma^2})$ for all $\gamma \in (0, \overline{\gamma}]$. We choose $\overline{\beta} = \overline{\beta}(\gamma)$ such that $C(\overline{\beta} + \overline{\beta}^2) \leq 1/[2\overline{C}(1 + \frac{1}{\gamma^2})]$ and such that $\overline{\beta} : (0, \overline{\gamma}] \to (0, \infty)$ is non-decreasing. Then a Neumann series argument yields that $L_{\frac{\pi}{2}} + \tilde{N}_{(\beta,\gamma)}$ is invertible from $H^2_{(0,\gamma)}(\mathbb{R}^2_+)$ onto $Y_{(0,\gamma)}$ for all $\beta \in [0, \overline{\beta}(\gamma)]$, $\gamma \in (0, \overline{\gamma}]$ and the norm of the inverse is bounded by $2\overline{C}(1 + \frac{1}{\gamma^2})$.

Now let $\beta \in [0, \overline{\beta}(\gamma)]$, $\gamma \in (0, \overline{\gamma}]$ and $v \in H^2_{(0,\gamma)}(\mathbb{R}^2_+)$ solve (4.10) for $(G, g) \in Y_{(\beta,\gamma)}$. Then $u := e^{-\beta\eta(R)}v \in H^2_{(\beta,\gamma)}(\mathbb{R}^2_+)$ is a solution of

$$L_{\frac{\pi}{2}} u = (G, g) + \left(0, \frac{e^{-\beta\eta(R)}\theta_0'}{\|\theta_0'\|^2_{L^2(\mathbb{R})}} \left[\int_{\mathbb{R}^2_+} (N_{(\beta,\gamma)}v + e^{\beta\eta(R)}G)\theta_0' + \int_{\mathbb{R}} e^{\beta\eta(R)} g\theta_0'\right]\right).$$

The compatibility condition (4.5) holds for $L_{\frac{\pi}{2}} u$ and (G, g). Since $\int_{\mathbb{R}} e^{-\beta\eta(R)}\theta_0'(R)^2\, dR$ is positive, it follows that the second term is zero for the solution, i.e. u is a solution of $L_{\frac{\pi}{2}} u = (G, g)$. By construction or alternatively by Theorem 4.7, 3. the solution is unique and we have the estimate

$$\|u\|_{H^2_{(\beta,\gamma)}(\mathbb{R}^2_+)} = \|v\|_{H^2_{(0,\gamma)}(\mathbb{R}^2_+)} \leq 2\overline{C}(1 + \frac{1}{\gamma^2})\|(\overline{G}, \overline{g})\|_{Y_{(0,\gamma)}} \leq \tilde{C}(1 + \frac{1}{\gamma^2})\|(G, g)\|_{Y_{(\beta,\gamma)}}$$

with $\tilde{C} > 0$ independent of β, γ and the functions. This proves the theorem. □

Theorem 4.11 (Solution Operators for Higher Regularity). *Let $\overline{\beta} : (0, \overline{\gamma}] \to (0, \infty)$ and $\overline{\gamma} > 0$ be as in Theorem 4.10. Then for all $k \in \mathbb{N}_0$, $\gamma \in (0, \overline{\gamma}]$ and $\beta \in [0, \overline{\beta}(\gamma)]$ it follows that*

$$L_{\frac{\pi}{2}} : H^{k+2}_{(\beta,\gamma)}(\mathbb{R}^2_+) \to Y^k_{(\beta,\gamma)} := \left\{(G, g) \in H^k_{(\beta,\gamma)}(\mathbb{R}^2_+) \times H^{k+\frac{1}{2}}_{(\beta)}(\mathbb{R}) : \int_{\mathbb{R}^2_+} G\theta_0' + \int_{\mathbb{R}} g\theta_0' = 0\right\}$$

is invertible and the operator norm of the inverse is bounded by $C(k)(1 + \frac{1}{\gamma^2})^{k+1}$.

Proof. $L_{\frac{\pi}{2}}$ is a well-defined, bounded linear operator in the above spaces. Let $(G, g) \in Y^k_{(\beta,\gamma)}$. Then by Theorem 4.10 there is a unique $u \in H^2_{(\beta,\gamma)}(\mathbb{R}^2_+)$ that solves $L_{\frac{\pi}{2}} u = (G, g)$. By regularity theory, cf. Corollary 4.8, 2., it follows that $u \in H^{k+2}(\mathbb{R}^2_+)$.

Now we show $\partial_R^l u \in H^2_{(\beta,\gamma)}(\mathbb{R}^2_+)$ for all $l = 1, ..., k$ and suitable estimates. To this end we apply ∂_R^l to the equation and get

$$L_{\frac{\pi}{2}} \partial_R^l u = \left(\partial_R^l G - \sum_{j=1}^{l} \binom{l}{j} \partial_R^j (f''(\theta_0))\partial_R^{l-j} u, \partial_R^l g\right) =: (G_l, g_l).$$

First we consider $l = 1$. It holds $G_1 = \partial_R G - \partial_R(f''(\theta_0))u \in L^2_{(\beta,\gamma)}(\mathbb{R}^2_+)$ and $g_1 \in H^{\frac{1}{2}}_{(\beta)}(\mathbb{R})$. Moreover, due to Theorem 4.7, 4. the compatibility condition (4.5) holds for (G_1, g_1). Therefore Theorem 4.10 and the uniqueness of solutions in Theorem 4.7, 3. implies $\partial_R u \in H^2_{(\beta,\gamma)}(\mathbb{R}^2_+)$ and

$$\begin{aligned}\|\partial_R u\|_{H^2_{(\beta,\gamma)}(\mathbb{R}^2_+)} &\leq \tilde{C}(1 + \frac{1}{\gamma^2})\|(G_1, g_1)\|_{Y_{(\beta,\gamma)}} \\ &\leq C(1 + \frac{1}{\gamma^2})(\|G\|_{H^1_{(\beta,\gamma)}(\mathbb{R}^2_+)} + \|g\|_{H^{\frac{3}{2}}_{(\beta)}(\mathbb{R})} + \|u\|_{L^2_{(\beta,\gamma)}(\mathbb{R}^2_+)}),\end{aligned}$$

where $\|u\|_{L^2_{(\beta,\gamma)}(\mathbb{R}^2_+)} \leq \tilde{C}(1 + \frac{1}{\gamma^2})\|(G, g)\|_{Y_{(\beta,\gamma)}}$ and we used the product rule to rewrite and estimate $e^{\beta\eta(R)}\partial_R g$. This shows the case $l = 1$. By mathematical induction on l it follows that $\partial_R^l u \in H^2_{(\beta,\gamma)}(\mathbb{R}^2_+)$ for $l = 1, ..., k$ and $\|\partial_R^l u\|_{H^2_{(\beta,\gamma)}(\mathbb{R}^2_+)} \leq C(k)(1 + \frac{1}{\gamma^2})^{k+1}$.

The remaining assertions and estimates will be shown by differentiating and rearranging the first equation in $L_{\frac{\pi}{2}}u = (G, g)$. For $l = 0, ..., k$ we have

$$\partial_R^l \partial_H^2 u = -\partial_R^l[-\partial_R^2 u + f''(\theta_0(R))u - G].$$

For $k = 1$ and $l = 1$ this implies $\partial_H^2 u \in H^1_{(\beta,\gamma)}(\mathbb{R}^2_+)$ together with a suitable estimate. Hence in the case $k = 1$ we are done. Now let $k \geq 2$. Then we obtain $\partial_R^l \partial_H^2 u \in H^2_{(\beta,\gamma)}(\mathbb{R}^2_+)$ for all $l = 0, ..., k-2$ with appropriate estimates. This also shows the case $k = 2$. Now let $k \geq 3$. Applying ∂_H^2 to the above equation and similar arguments as before yield $\partial_H^4 u \in H^1_{(\beta,\gamma)}(\mathbb{R}^2_+)$ in the case $k = 3$ and $\partial_R^l \partial_H^4 u \in H^2_{(\beta,\gamma)}(\mathbb{R}^2_+)$ for all $k \geq 4$ and $l = 0, ..., k-4$. Additionally, one also obtains suitable estimates. One can complete the argument with an induction proof. $\square$

Remark 4.12 (Dependence on Parameters). When the right hand sides (G, g) depend on independent variables, e.g. time $t \in [0, T]$, one directly obtains a solution u with the same regularity with respect to those variables because we have linear, bounded solution operators in Theorems 4.9-4.11. E.g. if for (β, γ) as in Theorem 4.11 and $n, k \in \mathbb{N}_0$ we have

$$(G, g) \in C^n([0, T], H^k_{(\beta,\gamma)}(\mathbb{R}^2_+) \times H^{k+\frac{1}{2}}_{(\beta)}(\mathbb{R})) \quad \text{with} \int_{\mathbb{R}^2_+} G\theta_0' + \int_{\mathbb{R}} g\theta_0' = 0,$$

then there is exactly one solution $u \in C^n([0, T], H^{k+2}_{(\beta,\gamma)}(\mathbb{R}^2_+))$ of $L_{\frac{\pi}{2}}u = (G, g)$. By embeddings, this can e.g. be applied for sufficiently smooth right hand sides with pointwise exponential decay for the functions and enough derivatives.

4.2.2 A Nonlinear Elliptic Problem on $\mathbb{R}^2_+$ and the Linearized Problem

Let $f : \mathbb{R} \to \mathbb{R}$ be as (1.1), $\alpha \in (0, \pi)$, $\hat{\sigma} : \mathbb{R} \to \mathbb{R}$ and $\sigma_\alpha = \cos\alpha\, \hat{\sigma}$ be as in Definition 1.8. In the contact point expansion for (AC$_\alpha$) we have to solve the following model problems:

4.2.2.1 The Nonlinear Elliptic Problem on $\mathbb{R}^2_+$ Find a smooth $v_\alpha : \overline{\mathbb{R}^2_+} \to \mathbb{R}$ such that with $A_\alpha := \begin{pmatrix} 1 & -\cos\alpha \\ -\cos\alpha & 1 \end{pmatrix}$ and θ_0 as in Theorem 4.1 it holds

$$-\mathrm{div} A_\alpha \nabla v_\alpha + f'(v_\alpha) = 0 \qquad \text{for } (R, H) \in \mathbb{R}^2_+, \tag{4.11}$$

$$N_{\partial\mathbb{R}^2_+} \cdot A_\alpha \nabla v_\alpha|_{H=0} + \sigma_\alpha'(v_\alpha)|_{H=0} = 0 \qquad \text{for } R \in \mathbb{R}, \tag{4.12}$$

$$\partial_R^k \partial_H^l [v_\alpha(R, H) - \theta_0(R)] = \mathcal{O}(e^{-c_{k,l}(|R|+H)}) \qquad \text{for all } k, l \in \mathbb{N}_0. \tag{4.13}$$

Here $N_{\partial\mathbb{R}^2_+} = (0, -1)^\top$. We choose $v_{\frac{\pi}{2}}(R, H) = \theta_0(R)$ for all $(R, H) \in \overline{\mathbb{R}^2_+}$.

Remark 4.13 (Compatibility Condition for σ_α). The condition (1.9) on α, σ_α, f can be derived as a necessary condition for the existence of a smooth solution v_α of (4.11)-(4.13).

This can be seen as follows: Let $\sigma_\alpha : \mathbb{R} \to \mathbb{R}$ be smooth with[7] $\operatorname{supp} \sigma_\alpha' \subset (-1, 1)$ and v_α sufficiently smooth solve (4.11)-(4.13), where $v_{\frac{\pi}{2}} := \theta_0$. We multiply (4.11) with $\partial_R v_\alpha$ and get

$$\frac{d}{dR}\left[\frac{1}{2}(\partial_R v_\alpha)^2 - \frac{1}{2}(\partial_H v_\alpha)^2 - f(v_\alpha)\right] + \frac{d}{dH}\left[\partial_R v_\alpha \partial_H v_\alpha - \cos\alpha(\partial_R v_\alpha)^2\right] = 0.$$

[7] In this remark the special form in Definition 1.8 is not needed.

Integrating with respect to R over $\mathbb{R}$ as well as H over $(0, H_0)$ for an arbitrary $H_0 > 0$ implies

$$0 = \int_{\mathbb{R}} [\partial_R v_\alpha (\partial_H v_\alpha - \cos\alpha \partial_R v_\alpha)]_{H=0}^{H_0} \, dR.$$

By the boundary condition (4.12) it holds $\partial_H v_\alpha - \cos\alpha \partial_R v_\alpha|_{H=0} = \sigma'_\alpha(v_\alpha)$. Therefore

$$0 = \int_{\mathbb{R}} [\partial_R v_\alpha (\partial_H v_\alpha - \cos\alpha \partial_R v_\alpha)]|_{H_0} \, dR - \int_{\mathbb{R}} \frac{d}{dR} [\sigma_\alpha(v_\alpha|_{H=0})] \, dR.$$

Using the asymptotics in (4.13) we obtain $0 = -\cos\alpha \int_{\mathbb{R}} (\theta'_0)^2 \, dR - [\sigma_\alpha(1) - \sigma_\alpha(-1)]$ by sending $H_0 \to \infty$. Hence σ_α has to fulfil

$$\cos\alpha = \frac{\sigma_\alpha(-1) - \sigma_\alpha(1)}{\int_{\mathbb{R}} (\theta'_0)^2 \, dR}. \tag{4.14}$$

Because of Theorem 4.1 it holds

$$\int_{\mathbb{R}} (\theta'_0)^2 = \int_{\mathbb{R}} \theta'_0 \sqrt{2(f(\theta_0) - f(-1))} = \int_{-1}^{1} \sqrt{2(f(r) - f(-1))} \, dr$$

and therefore (4.14) is equivalent to (1.9).

4.2.2.2 The Linearized Elliptic Problem on $\mathbb{R}^2_+$ Let A_α be as in the last Section 4.2.2.1 and v_α be a sufficiently smooth solution to (4.11)-(4.13), where $v_{\frac{\pi}{2}} = \theta_0$. The linearized problem reads as follows: For $G : \overline{\mathbb{R}^2_+} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ with suitable regularity and exponential decay find a solution $u : \overline{\mathbb{R}^2_+} \to \mathbb{R}$ with similar decay to

$$-\mathrm{div} A_\alpha \nabla u + f''(v_\alpha) u = G \quad \text{for } (R, H) \in \mathbb{R}^2_+, \tag{4.15}$$

$$N_{\partial\mathbb{R}^2_+} \cdot A_\alpha \nabla u|_{H=0} + \sigma''_\alpha(v_\alpha) u|_{H=0} = g \quad \text{for } R \in \mathbb{R}. \tag{4.16}$$

Remark 4.14 (**Compatibility Condition for the Data**). Let $\sigma_\alpha : \mathbb{R} \to \mathbb{R}$ be smooth with[8] $\operatorname{supp} \sigma'_\alpha \subset (-1, 1)$ and v_α sufficiently smooth solve (4.11)-(4.13), where $v_{\frac{\pi}{2}} := \theta_0$. It turns out that there is a necessary condition on the data G, g for a solution u of (4.15)-(4.16) to exist:

$$\int_{\mathbb{R}^2_+} G \partial_R v_\alpha + \int_{\mathbb{R}} g \partial_R v_\alpha|_{H=0} = 0. \tag{4.17}$$

Note that this is similar to the compatibility condition (4.5) and for $\alpha = \frac{\pi}{2}$ it is the same.

The condition (4.17) can be derived as follows: We test (4.15) with $\partial_R v_\alpha$ and obtain

$$\int_{\mathbb{R}^2_+} -\mathrm{div} A_\alpha \nabla u \partial_R v_\alpha + f''(v_\alpha) \partial_R v_\alpha u = \int_{\mathbb{R}^2_+} G \partial_R v_\alpha.$$

For the first term on the left hand side we apply integration by parts twice. This yields

$$-\int_{\mathbb{R}^2_+} \mathrm{div} A_\alpha \nabla u \partial_R v_\alpha = \int_{\mathbb{R}^2_+} \nabla v_\alpha \cdot A_\alpha \nabla u - \int_{\mathbb{R}} N_{\partial\mathbb{R}^2_+} \cdot A_\alpha \nabla u \partial_R v_\alpha|_{H=0},$$

$$\int_{\mathbb{R}^2_+} \nabla v_\alpha \cdot A_\alpha \nabla u = -\int_{\mathbb{R}^2_+} \mathrm{div} A_\alpha \nabla \partial_R v_\alpha u + \int_{\mathbb{R}} N_{\partial\mathbb{R}^2_+} \cdot A_\alpha \nabla \partial_R v_\alpha u|_{H=0}.$$

[8] In this remark the special form in Definition 1.8 is not needed.

Due to (4.16) it holds that $N_{\partial\mathbb{R}^2_+} \cdot A_\alpha \nabla u|_{H=0} = -\sigma''_\alpha(v_\alpha)u|_{H=0} + g$. Hence altogether we have

$$\int_{\mathbb{R}^2_+} \partial_R[-\mathrm{div} A_\alpha \nabla v_\alpha + f'(v_\alpha)]u + \int_{\mathbb{R}} \partial_R[N_{\partial\mathbb{R}^2_+} \cdot A_\alpha \nabla v_\alpha + \sigma'_\alpha(v_\alpha)]u - g\partial_R v_\alpha|_{H=0} = \int_{\mathbb{R}^2_+} G\partial_R v_\alpha.$$

Because of the equations (4.11)-(4.12) for v_α we obtain the compatibility condition (4.17).

4.2.2.3 Ideas and Solution Strategy for both Problems First some ideas are summarized that did not quite work out.

Concerning the nonlinear problem (4.11)-(4.13), one could consider shifted equations for $v_\alpha - \theta_0$ and try to solve them via energy methods. But since this is a problem on $\mathbb{R}^2_+$, it is not clear how to get coercivity or a minimizer. Another possibility would be to introduce approximate problems on finite and subsequently larger domains where energy methods work. Then one tries to get uniform estimates, e.g. in Hölder spaces with Schauder estimates, and tries to apply compactness arguments. The latter was basically the idea in Bronsard, Gui, Schatzman [BGS], where a symmetric solution (equivariant with respect to the equilateral triangle) to an elliptic problem on $\mathbb{R}^2$ with symmetric triple junction potential was constructed. There the constant solutions were ruled out by symmetry. The latter is not possible here. Moreover, introducing suitable boundary conditions[9] for the problems on finite domains is not an easy task because there is a nonlinearity. The construction and the estimates become technical and tedious.

For the linear problem (4.15)-(4.16) one could try to split functions in a similar way as in Section 4.2.1, i.e. with respect to $\partial_R v_\alpha(.,H)$ instead of θ'_0. One would like to solve for the orthogonal parts via the Lax-Milgram Theorem, but it is not clear how to get coercivity. In fact, it is a compact perturbation plus a small pertubation of a coercive problem, but this just gives a Fredholm property. However, for the parallel parts one can still obtain a solution formula up to some orthogonal error. Let us include this for the sake of completeness. We consider

$$G := \tilde{G}(H)\frac{\partial_R v_\alpha(R,H)}{\|\partial_R v_\alpha(.,H)\|^2_{L^2(\mathbb{R})}} \quad \text{and} \quad g := \tilde{g}\frac{\partial_R v_\alpha(R,0)}{\|\partial_R v_\alpha(.,0)\|^2_{L^2(\mathbb{R})}}.$$

Then the ansatz $u = \tilde{u}(H)\partial_R v_\alpha(R,H)$ leads to

$$\tilde{u}(H) := -\int_H^\infty \int_{\overline{H}}^\infty \frac{\tilde{G}(\hat{H})}{\|\partial_R v_\alpha(.,\hat{H})\|^2_{L^2(\mathbb{R})}}\, d\hat{H}\, d\overline{H}$$

such that u is a solution up to the error

$$2\tilde{u}'(H)\left[\partial_R\partial_H v_\alpha - (\partial_R\partial_H v_\alpha(.,H), \partial_R v_\alpha(.,H))_{L^2(\mathbb{R})}\frac{\partial_R v_\alpha(.,H)}{\|\partial_R v_\alpha(.,H)\|^2_{L^2(\mathbb{R})}} - 2\cos\alpha\,\partial_R^2 v_\alpha\right]$$

which is orthogonal to $\partial_R v_\alpha(.,H)$ in $L^2(\mathbb{R})$ f.a.a. $H > 0$.

Altogether this is not quite enough to solve the problems directly. Therefore we treat α as a parameter in the equations and use the functional analytic setting with exponentially weighted Sobolev spaces in Section 2.3. The latter allows for isomorphisms between the solution and the data for the linear problem in the case $\alpha = \frac{\pi}{2}$, cf. Section 4.2.1.2. The idea is to solve

[9] Considering strips $\mathbb{R} \times (0, H_0)$ for large H_0 is better for this. But this strategy becomes also technical.

the nonlinear problem (4.11)-(4.13) with the Implicit Function Theorem and the linear problem (4.15)-(4.16) with a Neumann series argument, both for α close to $\frac{\pi}{2}$. Here a problem to overcome is the compatibility condition (4.5). This will be dealt with in a similar way as in the proof of Theorem 4.10, i.e. first we subtract suitable terms in the boundary parts such that (4.5) is fulfilled, but in the end we show that those terms have to be zero for the solution. The latter involves similar computations as in the derivations of the compatibility conditions in Remarks 4.13-4.14. Furthermore, one also has to spend some thoughts on the regularity $m \in \mathbb{N}_0$ that one uses for the spaces in Theorem 4.11. More precisely, one can only apply the Implicit Function Theorem and the Neumann series argument in such a setting for finitely many m since otherwise the possible angles α depend on m. Moreover, m should be taken as low as possible to reduce the computations. We will solve the linear problem for $m = 0$ and $m = 1$ in order to have a "regularity theory" in exponentially weighted spaces due to uniqueness. It turns out that $m = 1$ for the nonlinear problem is enough to subsequently use this "regularity theory" for derivatives of v_α and to rigorously carry out computations as in Remarks 4.13-4.14. With induction arguments we obtain a smooth solution v_α to the nonlinear problem (4.11)-(4.13) and solution operators for the linear problem (4.15)-(4.16) for α close to $\frac{\pi}{2}$ in weighted Sobolev spaces with arbitrary integer regularity similar to Theorem 4.11, where the compatibility condition (4.17) is included in the data space. One could try to extend this to arbitrary $\alpha \in (0, \pi)$ via continuity arguments, e.g. the Leray-Schauder degree, but we did not persue this.

4.2.2.4 Solution of the Problems for α close to $\frac{\pi}{2}$

4.2.2.4.1 The Nonlinear Problem We rewrite the equations in a suitable way. Therefore let $\hat{v}_\alpha := v_\alpha - \theta_0$. Then the nonlinear equations (4.11)-(4.12) for v_α are equivalent to

$$L_{\frac{\pi}{2}} \hat{v}_\alpha = (G_\alpha, g_\alpha)(\hat{v}_\alpha), \tag{4.18}$$

where $L_{\frac{\pi}{2}}$ is as in Theorem 4.9 and

$$\begin{aligned} G_\alpha(v) &:= -2\cos\alpha\partial_R\partial_H v - [f'(\theta_0 + v) - f'(\theta_0) - f''(\theta_0)v], \\ g_\alpha(v) &:= -\cos\alpha[\partial_R v|_{H=0} + \theta_0'] - \sigma_\alpha'(\theta_0 + v)|_{H=0}. \end{aligned}$$

In order to achieve the compatibility condition (4.5) for the right hand sides, we define

$$\tilde{g}_\alpha(v) := g_\alpha(v) - \theta_0' c_N(\alpha, v), \quad c_N(\alpha, v) := \frac{1}{\|\theta_0'\|^2_{L^2(\mathbb{R})}} \left[\int_{\mathbb{R}^2_+} G_\alpha(v)\theta_0' + \int_{\mathbb{R}} g_\alpha(v)\theta_0' \right]$$

and consider the adjusted equation

$$L_{\frac{\pi}{2}} \hat{v}_\alpha = (G_\alpha, \tilde{g}_\alpha)(\hat{v}_\alpha). \tag{4.19}$$

In order to solve this with the Implicit Function Theorem we need

Lemma 4.15. *Let $\gamma \geq 0$ and $0 \leq \beta < \min\{\sqrt{f''(\pm 1)}\}$. Then the mappings*

$$(0, \pi) \times H^3_{(\beta,\gamma)}(\mathbb{R}^2_+) \to H^1_{(\beta,\gamma)}(\mathbb{R}^2_+) \times H^{\frac{3}{2}}_{(\beta)}(\mathbb{R}) : (\alpha, v) \mapsto (G_\alpha(v), g_\alpha(v))$$

are well-defined and continuously Fréchet-differentiable with

$$\frac{\partial}{\partial v}(G_{\cdot}, g_{\cdot})\Big|_{(\alpha,v)} = \left[\tilde{v} \mapsto -\begin{pmatrix} 2\cos\alpha\partial_R\partial_H\tilde{v} + [f''(\theta_0+v) - f''(\theta_0)]\tilde{v} \\ [\cos\alpha\partial_R\tilde{v} + \sigma''_\alpha(\theta_0+v)\tilde{v}]|_{H=0} \end{pmatrix}^\top\right],$$

$$\frac{\partial}{\partial \alpha}(G_{\cdot}, g_{\cdot})\Big|_{(\alpha,v)} = \left[\tilde{\alpha} \mapsto \tilde{\alpha}\sin\alpha \begin{pmatrix} 2\partial_R\partial_H v \\ [\partial_R v + \theta_0' + \hat{\sigma}'(\theta_0+v)]|_{H=0} \end{pmatrix}^\top\right].$$

We prove the following auxiliary Lemma:

Lemma 4.16. *Let $\beta, \gamma \geq 0$. Then the mapping*

$$\tilde{f} : H^3_{(\beta,\gamma)}(\mathbb{R}^2_+) \to H^2_{(\beta,\gamma)}(\mathbb{R}^2_+) : v \mapsto f'(\theta_0 + v) - f'(\theta_0)$$

is well-defined and C^1 with $[D\tilde{f}(v)](h) = [f''(\theta_0+v)]h$. The same is true for any $f' \in C^5(\mathbb{R})$ and $\theta_0 \in C^2_b(\mathbb{R})$.

Remark 4.17. As a byproduct of the proof we obtain that

$$\overline{f} : H^3_{(\beta,\gamma)}(\mathbb{R}^2_+) \to W^{2,4}_{(\beta,\gamma)}(\mathbb{R}^2_+) : v \mapsto f''(\theta_0 + v) - f''(\theta_0)$$

is well-defined and Lipschitz-continuous. The same holds for any $f'' \in C^5(\mathbb{R})$ and $\theta_0 \in C^2_b(\mathbb{R})$.

Proof of Lemma 4.16. First we show that $\tilde{f}$ is well-defined. Therefore let $v \in H^3_{(\beta,\gamma)}(\mathbb{R}^2_+)$. Then $\tilde{f}(v)$ is measurable. Moreover, $v \in C^1_b(\overline{\mathbb{R}^2_+})$ due to embeddings and hence $|\tilde{f}(v)| \leq C_{\|v\|_\infty}|v|$ because of $\tilde{f}(0) = 0$. In particular it holds

$$\tilde{f}(v) \in L^2_{(\beta,\gamma)}(\mathbb{R}^2_+) \quad \text{and} \quad \|\tilde{f}(v)\|_{L^2_{(\beta,\gamma)}(\mathbb{R}^2_+)} \leq C_{\|v\|_\infty}\|v\|_{L^2_{(\beta,\gamma)}(\mathbb{R}^2_+)}.$$

The first derivatives can be computed in the classical sense:

$$\nabla(\tilde{f}(v)) = [f''(\theta_0+v) - f''(\theta_0)]\begin{pmatrix}\theta_0' \\ 0\end{pmatrix} + f''(\theta_0+v)\nabla v.$$

As above we get $|f''(\theta_0+v) - f''(\theta_0)| \leq C_{\|v\|_\infty}|v|$ and $f''(\theta_0+v) \in C^1_b(\overline{\mathbb{R}^2_+})$. Therefore it holds $\tilde{f}(v) \in H^1_{(\beta,\gamma)}(\mathbb{R}^2_+)$. The above computations for f' instead of f yield that $f''(\theta_0+v) - f''(\theta_0)$ is contained in $H^1_{(\beta,\gamma)}(\mathbb{R}^2_+)$ and we have a formula for the first derivatives. Since $f''(\theta_0+v)$ is $C^1_b(\overline{\mathbb{R}^2_+})$, we can compute the derivatives of $\tilde{f}(v)$ to second order with the product rule:

$$\begin{aligned}
\partial_H\nabla(\tilde{f}(v)) &= f'''(\theta_0+v)\partial_H v\left[\begin{pmatrix}\theta_0' \\ 0\end{pmatrix} + \nabla v\right] + f''(\theta_0+v)\partial_H\nabla v, \\
\partial_R\nabla(\tilde{f}(v)) &= [(f'''(\theta_0+v) - f'''(\theta_0))\theta_0' + f'''(\theta_0+v)\partial_R v]\begin{pmatrix}\theta_0' \\ 0\end{pmatrix} \\
&\quad + [f''(\theta_0+v) - f''(\theta_0)]\begin{pmatrix}\theta_0'' \\ 0\end{pmatrix} + f'''(\theta_0+v)[\theta_0' + \partial_R v]\nabla v + f''(\theta_0+v)\partial_R\nabla v.
\end{aligned}$$

The assertions before for f'' instead of f and the product estimate

$$\|\partial_H v \nabla v\|_{L^2_{(\beta,\gamma)}(\mathbb{R}^2_+)} \leq C\|v\|^2_{W^{1,4}_{(\beta,\gamma)}(\mathbb{R}^2_+)} \leq C\|v\|^2_{H^2_{(\beta,\gamma)}(\mathbb{R}^2_+)}$$

as well as for $\partial_R v \nabla v$ instead of $\partial_H v \nabla v$ imply $\tilde{f}(v) \in H^2_{(\beta,\gamma)}(\mathbb{R}^2_+)$.

In the following we show that the candidate

$$\tilde{F} : H^3_{(\beta,\gamma)}(\mathbb{R}^2_+) \to \mathcal{L}(H^3_{(\beta,\gamma)}(\mathbb{R}^2_+), H^2_{(\beta,\gamma)}(\mathbb{R}^2_+)) : v \mapsto [h \mapsto [f''(\theta_0 + v)]h]$$

for the Fréchet derivative is well-defined and continuous. Since multiplication by $f''(\theta_0)$ defines a bounded linear operator on $H^2_{(\beta,\gamma)}(\mathbb{R}^2_+)$ and due to $H^3_{(\beta,\gamma)}(\mathbb{R}^2_+) \hookrightarrow W^{2,4}_{(\beta,\gamma)}(\mathbb{R}^2_+)$ as well as product estimates, it is enough to prove that

$$\overline{f} : H^3_{(\beta,\gamma)}(\mathbb{R}^2_+) \to W^{2,4}_{(\beta,\gamma)}(\mathbb{R}^2_+) : v \mapsto f''(\theta_0 + v) - f''(\theta_0)$$

is well-defined and Lipschitz-continuous. In an analogous way as before one can show that $\overline{f}(v)$ is an element of $W^{2,4}_{(\beta,\gamma)}(\mathbb{R}^2_+)$ for all $v \in H^3_{(\beta,\gamma)}(\mathbb{R}^2_+)$ and that one has the same formulas for the derivatives as for $\tilde{f}(v)$ with f replaced by f'. For the continuity let $v_j \in H^3_{(\beta,\gamma)}(\mathbb{R}^2_+)$ and $K > 0$ such that $\|v_j\|_\infty \leq K$ for $j = 1, 2$. Then we have the estimate

$$|\overline{f}(v_1) - \overline{f}(v_2)| = |f''(\theta_0 + v_1) - f''(\theta_0 + v_2)| \leq C_K|v_1 - v_2|.$$

Moreover, one can directly write down the derivatives for $\overline{f}(v_1) - \overline{f}(v_2)$. Here terms without an argument v_j cancel and one can estimate everything via null additions, embeddings and product estimates. Altogether we obtain Lipschitz continuity of $\overline{f}$ and hence the same is true for $\tilde{F}$.

Finally, we have to verify the definition of the Fréchet derivative. Therefore, let v, h in $H^3_{(\beta,\gamma)}(\mathbb{R}^2_+)$ be arbitrary with norm bounded by $K > 0$. Then from Taylor's Theorem it follows

$$|\tilde{f}(v + h) - \tilde{f}(v) - \tilde{F}(v)h| = |f'(\theta_0 + v + h) - f'(\theta_0 + v) - f''(\theta_0 + v)h| \leq C_K|h|^2.$$

Therefore we obtain $\|\tilde{f}(v + h) - \tilde{f}(v) - \tilde{F}(v)h\|_{L^2_{(\beta,\gamma)}(\mathbb{R}^2_+)} \leq C_K\|h\|^2_{L^4_{(\beta,\gamma)}(\mathbb{R}^2_+)}$. Moreover,

$$\begin{aligned}\nabla[\tilde{f}(v + h) - \tilde{f}(v) - \tilde{F}(v)h] = {} & [f''(\theta_0 + v + h) - f''(\theta_0 + v)]\nabla h \\ & +[f''(\theta_0 + v + h) - f''(\theta_0 + v) - f'''(\theta_0 + v)h]\left[\begin{pmatrix}\theta_0' \\ 0\end{pmatrix} + \nabla v\right]\end{aligned}$$

and hence $|\nabla[\tilde{f}(v + h) - \tilde{f}(v) - \tilde{F}(v)h]| \leq C_K(|h|^2 + |h||\nabla h|)$. Furthermore,

$$\begin{aligned}\partial_H \nabla[\tilde{f}(v + h) - \tilde{f}(v) - \tilde{F}(v)h] = {} & [f''(\theta_0 + v + h) - f''(\theta_0 + v)]\partial_H \nabla h \\ & + f'''(\theta_0 + v + h)\partial_H h \nabla h + [f'''(\theta_0 + v + h) - f'''(\theta_0 + v)]\partial_H v \nabla h \\ & + [f''(\theta_0 + v + h) - f''(\theta_0 + v) - f'''(\theta_0 + v)h]\partial_H \nabla v \\ & + [f'''(\theta_0 + v + h) - f'''(\theta_0 + v) - f^{(4)}(\theta_0 + v)h]\partial_H v\left[\begin{pmatrix}\theta_0' \\ 0\end{pmatrix} + \nabla v\right] \\ & + [f'''(\theta_0 + v + h) - f'''(\theta_0 + v)]\partial_H h\left[\begin{pmatrix}\theta_0' \\ 0\end{pmatrix} + \nabla v\right].\end{aligned}$$

Hence we obtain

$$|\partial_H \nabla[\tilde{f}(v+h) - \tilde{f}(v) - \tilde{F}(v)h]| \le C_K[|h||\partial_H \nabla h| + |\partial_H h||\nabla h| + |h||\partial_H v||\nabla h|] + C_K[|h|^2|\partial_H \nabla v| + (|h|^2|\partial_H v| + |h||\partial_H h|)(1+|\nabla v|)].$$

Finally, analogous computations yield

$$|\partial_R \nabla[\tilde{f}(v+h) - \tilde{f}(v) - \tilde{F}(v)h]| \le C_K[|h||\partial_R \nabla h| + |\partial_R h||\nabla h| + |h|(1+|\partial_R v|)|\nabla h|] + C_K[|h|^2(1+|\partial_R \nabla v|) + (|h|^2|\partial_R v| + |h||\partial_R h|)(1+|\nabla v|)].$$

Altogether, with embeddings and product estimates we obtain

$$\|\tilde{f}(v+h) - \tilde{f}(v) - \tilde{F}(v)h\|_{H^2_{(\beta,\gamma)}(\mathbb{R}^2_+)} \le C_K \|h\|^2_{H^3_{(\beta,\gamma)}(\mathbb{R}^2_+)}.$$

This shows that $\tilde{f}$ is continuously Fréchet-differentiable and $D\tilde{f} = \tilde{F}$. □

Proof of Lemma 4.15. Let $\gamma \ge 0$ and $0 \le \beta < \min\{\sqrt{f''(\pm 1)}\}$. Then it holds $\theta_0' \in H^{\frac{3}{2}}_{(\beta)}(\mathbb{R})$ due to Theorem 4.1. Moreover, we write $\hat{\sigma}'(\theta_0 + .) = \hat{\sigma}'(\theta_0 + .) - \hat{\sigma}'(\theta_0) + \hat{\sigma}'(\theta_0)$, where $\hat{\sigma}'(\theta_0)$ is an element of $C_0^\infty(\mathbb{R})$. Therefore Lemma 4.16 applied to f and $\hat{\sigma}'$ as well as Lemma 2.22 yield that $(G_., g_.)$ are well-defined. By Zeidler [Z], Proposition 4.14 it is enough to prove continuous partial differentiability. For the derivatives in α this is clear since α only appears as a multiplication with $\cos\alpha$. Finally, the linear parts in v are no problem and the rest follows from Lemma 4.16. □

Now we can solve the nonlinear equations (4.11)-(4.12) for α close to $\frac{\pi}{2}$.

Theorem 4.18. *Let $\overline{\gamma}$, $\overline{\beta}(.)$ be as in Theorem 4.11, $\beta \in (0, \min\{\overline{\beta}(\gamma), \min\{\sqrt{f''(\pm 1)}\}\})$ and $\gamma \in (0, \overline{\gamma}]$. Then there is an $\overline{\alpha} = \overline{\alpha}(\beta, \gamma) > 0$ such that the following holds:* (4.18) *has a solution $\hat{v}_\alpha \in H^3_{(\beta,\gamma)}(\mathbb{R}^2_+)$ for $\alpha \in \frac{\pi}{2} + [-\overline{\alpha}, \overline{\alpha}]$ which is C^1 in α, $\hat{v}_{\frac{\pi}{2}} = 0$ and $v_\alpha := \theta_0 + \hat{v}_\alpha$ solves* (4.11)-(4.12) *and it holds $\int_{\mathbb{R}} \theta_0' \partial_R v_\alpha|_{H=0}\, dR > 0$.*

Proof. The mapping

$$F : (0,\pi) \times H^3_{(\beta,\gamma)}(\mathbb{R}^2_+) \to H^3_{(\beta,\gamma)}(\mathbb{R}^2_+) : (\alpha, v) \mapsto v - (L_{\frac{\pi}{2}})^{-1}(G_\alpha(v), \tilde{g}_\alpha(v)).$$

is well-defined and C^1 by Theorem 4.11 and Lemma 4.15. Moreover, it holds $F(\frac{\pi}{2}, 0) = 0$ and

$$\left.\frac{\partial F}{\partial v}\right|_{(\frac{\pi}{2},0)} (\tilde{v}) = \tilde{v} - (L_{\frac{\pi}{2}})^{-1}(0,0) = \tilde{v}.$$

Hence by the Implicit Function Theorem there is an $\overline{\alpha} > 0$ and $\hat{v}_. : \frac{\pi}{2} + [-\overline{\alpha}, \overline{\alpha}] \to H^3_{(\beta,\gamma)}(\mathbb{R}^2_+)$ continuously differentiable such that $\hat{v}_{\frac{\pi}{2}} = 0$ and $F(\alpha, \hat{v}_\alpha) = 0$, or equivalently (4.19), holds for all α in $\frac{\pi}{2} + [-\overline{\alpha}, \overline{\alpha}]$. The latter is equivalent to (4.11)-(4.12) for $v_\alpha = \theta_0 + \hat{v}_\alpha$ with 0 replaced by $\theta_0' c_N(\alpha, \hat{v}_\alpha)$ on the right hand side in (4.12), where c_N was defined before (4.19). One can carry out the same computations as in Remark 4.13 using $v_\alpha \in C^1_b(\overline{\mathbb{R}^2_+})$ for the nonlinear terms. Due to the compatibility condition (4.14) on $\hat{\sigma}$ and σ_α, respectively, this yields

$$0 = c_N(\alpha, \hat{v}_\alpha) \int_{\mathbb{R}} \theta_0' \partial_R v_\alpha|_{H=0}\, dR.$$

For $\overline{\alpha} > 0$ small it holds $\int_{\mathbb{R}} \theta_0' \partial_R v_\alpha|_{H=0}\, dR > 0$ because of $\int_{\mathbb{R}} \theta_0'^2 > 0$ and continuity of $\hat{v}_\alpha$ in α. Therefore we obtain $c_N(\alpha, \hat{v}_\alpha) = 0$ for all $\alpha \in \frac{\pi}{2} + [-\overline{\alpha}, \overline{\alpha}]$ and v_α is a solution of (4.11)-(4.12). This shows Theorem 4.18. □

Remark 4.19. From now on we fix $\gamma_0 \in (0,\overline{\gamma}]$ and $\beta_0 \in (0,\min\{\overline{\beta}(\frac{\gamma_0}{2}),\min\{\sqrt{f''(\pm 1)}\}\})$. Then Theorem 4.11 is valid for all $\beta \in [0,\beta_0]$ and $\gamma \in [\frac{\gamma_0}{2},\gamma_0]$. Moreover, we denote by $\overline{\alpha}$ and $\hat{v}_.$ the constant and the solution, respectively, obtained in Theorem 4.18 for β_0, γ_0. Due to the Lipschitz-continuity of $\hat{v}_. : \frac{\pi}{2}+[-\overline{\alpha},\overline{\alpha}] \rightarrow H^3_{(\beta_0,\gamma_0)}(\mathbb{R}^2_+)$, the identity $v_\alpha := \theta_0 + \hat{v}_\alpha$ and the embedding $H^3_{(\beta_0,\gamma_0)}(\mathbb{R}^2_+) \hookrightarrow C^1_{(\beta_0,\gamma_0)}(\overline{\mathbb{R}^2_+})$, by possibly shrinking $\overline{\alpha}$ we can assume that

$$\int_{\mathbb{R}} (\partial_\rho v_\alpha)^2|_Z \, d\rho \in [\frac{3}{4},\frac{5}{4}]\|\theta_0'\|^2_{L^2(\mathbb{R})} \text{ for } Z \geq 0 \quad \text{and} \quad \int_{\mathbb{R}^2_+} \partial_\rho \partial_Z v_\alpha \partial_\rho v_\alpha \, d(\rho,Z) \leq \frac{1}{4}\|\theta_0'\|^2_{L^2(\mathbb{R})}.$$

The latter estimates are not needed in this Section 4.2.2, but they will be important for asymptotic expansions and spectral estimates later, see the end of Section 5.4.2.2.2 and Section 6.5.

In order to get higher regularity we consider the linearized problem first:

4.2.2.4.2 The Linear Problem The linear equations (4.15)-(4.16) are equivalent to

$$\begin{aligned} L_\alpha u &= (G,g), \quad \text{where } L_\alpha := L_{\frac{\pi}{2}} + M_\alpha, \\ M_\alpha u &:= (2\cos\alpha \partial_R \partial_H u + [f''(v_\alpha) - f''(\theta_0)]u, [\cos\alpha \partial_R u + \sigma''_\alpha(v_\alpha)u]|_{H=0}). \end{aligned} \tag{4.20}$$

Again, we consider another equation where the compatibility condition (4.5) is enforced:

$$L_{\frac{\pi}{2}} u + \tilde{M}_\alpha u = (G, g - c_D(G,g)\theta_0'), \tag{4.21}$$

where $\tilde{M}_\alpha u := M_\alpha u - (0, c_M(\alpha,u)\theta_0')$ and

$$c_D(G,g) := \frac{1}{\|\theta_0'\|^2_{L^2(\mathbb{R})}} \left[\int_{\mathbb{R}^2_+} G\theta_0' + \int_{\mathbb{R}} g\theta_0' \right], \quad c_M(\alpha,u) := c_D(M^1_\alpha u, M^2_\alpha u).$$

In order to solve this with a Neumann series argument we show

Lemma 4.20. *Let $\hat{v}_.$ be as in Remark 4.19, $\beta,\gamma \geq 0$ and $k=0,1$. Then*

$$M_\alpha \in \mathcal{L}(H^{k+2}_{(\beta,\gamma)}(\mathbb{R}^2_+), H^k_{(\beta,\gamma)}(\mathbb{R}^2_+) \times H^{k+\frac{1}{2}}_{(\beta)}(\mathbb{R}))$$

is well-defined and for the operator norm $\|.\|$ it holds

$$\|M_\alpha - M_{\tilde{\alpha}}\| \leq C|\alpha - \tilde{\alpha}|$$

for all $\alpha,\tilde{\alpha} \in \frac{\pi}{2}+[-\overline{\alpha},\overline{\alpha}]$, where $C>0$ can be taken independent of β,γ if $\beta,\gamma \leq B$ for a fixed $B>0$. In particular, since $M_{\frac{\pi}{2}}=0$, we obtain $\|M_\alpha\| \leq C|\alpha - \frac{\pi}{2}|$ for all $\alpha \in \frac{\pi}{2}+[-\overline{\alpha},\overline{\alpha}]$.

Proof. For the terms involving derivatives the assertions are evident with Lemma 2.22. For the difference in f'' we use $\hat{v}_\alpha \in C^1_b(\overline{\mathbb{R}^2_+})$ by embeddings and Lipschitz-continuity in α due to the Fundamental Theorem of Calculus. This yields $f''(v_\alpha) - f''(\theta_0) \in C^1_b(\overline{\mathbb{R}^2_+})$ and

$$\|f''(v_\alpha) - f''(v_{\tilde{\alpha}})\|_{C^1_b(\overline{\mathbb{R}^2_+})} \leq c\|\hat{v}_\alpha - \hat{v}_{\tilde{\alpha}}\|_{C^1_b(\overline{\mathbb{R}^2_+})} \leq C|\alpha - \tilde{\alpha}|.$$

Since multiplication with functions in $C^1_b(\overline{\mathbb{R}^2_+})$ defines a product on $H^k_{(\beta,\gamma)}(\mathbb{R}^2_+)$ for $k=0,1$, the first component of M_α fulfils the assertions. Furthermore, it holds $\sigma_\alpha = \cos\alpha\,\hat{\sigma}$ and

$\hat{\sigma}''(\theta_0) \in C_0^\infty(\mathbb{R})$. From Remark 4.17 we obtain that $\hat{\sigma}''(v_\alpha) - \hat{\sigma}''(\theta_0) \in W^{2,4}(\mathbb{R}^2_+)$ with norm bounded independent of $\alpha \in \frac{\pi}{2} + [-\overline{\alpha}, \overline{\alpha}]$. Moreover, it holds

$$\|\hat{\sigma}''(v_\alpha) - \hat{\sigma}''(v_{\tilde{\alpha}})\|_{W^{2,4}(\mathbb{R}^2_+)} \leq C|\alpha - \tilde{\alpha}|$$

for all $\alpha, \tilde{\alpha} \in \frac{\pi}{2} + [-\overline{\alpha}, \overline{\alpha}]$. Since multiplication with functions in $W^{2,4}(\mathbb{R}^2_+)$ defines a product on $W^{k+1,4}_{(\beta,\gamma)}(\mathbb{R}^2_+)$ for $k = 0, 1$, the claim follows with Lemma 2.22. $\square$

This enables us to prove an existence theorem for the linear equations (4.15)-(4.16).

Theorem 4.21. *Let β_0, γ_0 and $\overline{\alpha}, \hat{v}_.$ be as in Remark 4.19. Then there is an $\alpha_0 \in (0, \overline{\alpha}]$ such that for all $\alpha \in \frac{\pi}{2} + [-\alpha_0, \alpha_0]$, $\beta \in [0, \beta_0]$, $\gamma \in [\frac{\gamma_0}{2}, \gamma_0]$ and $k = 0, 1$ it holds that*

$$\tilde{L}_\alpha := L_{\frac{\pi}{2}} + \tilde{M}_\alpha \in \mathcal{L}(H^{k+2}_{(\beta,\gamma)}(\mathbb{R}^2_+), Y^k_{(\beta,\gamma)})$$

is an isomorphism and $\|\tilde{L}_\alpha^{-1} - \tilde{L}_{\tilde{\alpha}}^{-1}\| \leq C|\alpha - \tilde{\alpha}|$ for all $\alpha, \tilde{\alpha} \in \frac{\pi}{2} + [-\alpha_0, \alpha_0]$ and β, γ, k as above. Here $Y^k_{(\beta,\gamma)}$ is as in Theorem 4.11. Furthermore, for $L_\alpha := L_{\frac{\pi}{2}} + M_\alpha$ we obtain that

$$L_\alpha : H^{k+2}_{(\beta,\gamma)}(\mathbb{R}^2_+) \to \left\{ (G, g) \in H^k_{(\beta,\gamma)}(\mathbb{R}^2_+) \times H^{k+\frac{1}{2}}_{(\beta)}(\mathbb{R}) : \int_{\mathbb{R}^2_+} G \partial_R v_\alpha + \int_{\mathbb{R}} g \partial_R v_\alpha|_{H=0} = 0 \right\}$$

is a well-defined isomorphism and the norm of the inverse is bounded independent of α, β, γ, k.

Finally, for $(G, g) \in H^k_{(\beta,\gamma)}(\mathbb{R}^2_+) \times H^{k+\frac{1}{2}}_{(\beta)}(\mathbb{R})$ with (4.17) and $u \in H^k_{(\beta,\gamma)}(\mathbb{R}^2_+)$

$$L_\alpha u = (G, g) \quad \textit{is equivalent to} \quad \tilde{L}_\alpha u = (G, g - c_D(G, g)\theta_0').$$

Remark 4.22 (Regularity Theorem). Theorem 4.21 implies a simple regularity theory. If $u \in H^2_{(\beta,\gamma)}(\mathbb{R}^2_+)$ solves $L_\alpha u = (G, g)$ for $(G, g) \in H^1_{(\beta,\gamma)}(\mathbb{R}^2_+) \times H^{\frac{3}{2}}_{(\beta)}(\mathbb{R})$ with the compatibility condition (4.17), then by uniqueness $u \in H^3_{(\beta,\gamma)}(\mathbb{R}^2_+)$ together with an estimate.

Proof of Theorem 4.21. First of all, we show that L_α is well-defined in the spaces. The regularity properties follow from Theorem 4.11 and Lemma 4.20. To prove the compatibility condition (4.17) one can carry out the same computations as in Remark 4.14. Note that therefore one applies ∂_R to the equations (4.11)-(4.12) for v_α and the regularity for $\hat{v}_\alpha$ obtained in Theorem 4.18 is enough.

Theorem 4.11 yields that $L_{\frac{\pi}{2}} : H^{k+2}_{(\beta,\gamma)}(\mathbb{R}^2_+) \to Y^k_{(\beta,\gamma)}$ is an isomorphism and the norm of the inverse is bounded by a uniform constant $C(\gamma_0)$ for $\alpha \in \frac{\pi}{2} + [-\overline{\alpha}, \overline{\alpha}]$ and β, γ, k as in the assertion. Moreover, Lemma 4.20 implies $\tilde{M}_\alpha \in \mathcal{L}(H^{k+2}_{(\beta,\gamma)}(\mathbb{R}^2_+), Y^k_{(\beta,\gamma)})$ and the norm is estimated by $C|\alpha - \frac{\pi}{2}|$ for a $C > 0$ independent of α, β, γ, k as above. Therefore, by a Neumann series argument we obtain for $\alpha_0 > 0$ with $C\alpha_0 \leq 1/C(\gamma_0)$ that $\tilde{L}_\alpha \in \mathcal{L}(H^{k+2}_{(\beta,\gamma)}(\mathbb{R}^2_+), Y^k_{(\beta,\gamma)})$ is an isomorphism and the inverse is uniformly bounded for $\alpha \in \frac{\pi}{2} + [-\alpha_0, \alpha_0]$ and β, γ, k as in the theorem. Furthermore, because of $L_\alpha - L_{\tilde{\alpha}} = M_\alpha - M_{\tilde{\alpha}}$ and Lemma 4.20, again with a Neumann series argument it follows that $\|\tilde{L}_\alpha^{-1} - \tilde{L}_{\tilde{\alpha}}^{-1}\| \leq C|\alpha - \tilde{\alpha}|$ for all $\alpha, \tilde{\alpha} \in \frac{\pi}{2} + [-\alpha_0, \alpha_0]$ and β, γ, k as above if $\alpha_0 > 0$ is small.

Now let (G, g) be in $H^k_{(\beta,\gamma)}(\mathbb{R}^2_+) \times H^{k+\frac{1}{2}}_{(\beta)}(\mathbb{R})$ such that (4.17) holds. The considerations before yield that there is exactly one solution $u \in H^{k+2}_{(\beta,\gamma)}(\mathbb{R}^2_+)$ of (4.21). The latter is equivalent to

$$L_\alpha u = (G, g) - (0, [c_D(G, g) + c_M(\alpha, u)]\theta_0').$$

Due to the compatibility condition (4.17) for $L_\alpha u$ and (G,g) we obtain

$$[c_D(G,g)+c_M(\alpha,u)]\int_{\mathbb{R}}\theta_0'\partial_R v_\alpha|_{H=0}\,dR=0$$

and therefore $c_D(G,g)+c_M(\alpha,u)=0$ because of Theorem 4.18. That means $L_\alpha u=(G,g)$. On the other hand, if there is another solution $\tilde{u}\in H^{k+2}_{(\beta,\gamma)}(\mathbb{R}^2_+)$ of $L_\alpha\tilde{u}=(G,g)$, Theorem 4.11 yields that the compatibility condition (4.5) holds for $(G,g)-M_\alpha u$. Hence we obtain (4.21). Finally,

$$\|u\|_{H^{k+2}_{(\beta,\gamma)}(\mathbb{R}^2_+)}\leq c\|(G,g-c_D(G,g)\theta_0')\|_{Y^k_{(\beta,\gamma)}}\leq C\|(G,g)\|_{Y^k_{(\beta,\gamma)}}$$

with a uniform constant $C>0$. This proves Theorem 4.21. $\square$

4.2.2.4.3 Higher Regularity for both Problems With Remark 4.22 (Regularity Theorem) we can now prove smoothness and the decay (4.13) for the v_α obtained in Remark 4.19.

Theorem 4.23. *Let $\beta_0,\gamma_0,\hat{v}_.$ be as in Remark 4.19 and α_0 be as in Theorem 4.21. Then $\hat{v}_.:\frac{\pi}{2}+[-\alpha_0,\alpha_0]\to H^k_{(\beta_0,\gamma_0)}(\mathbb{R}^2_+)$ is well-defined and Lipschitz-continuous for all $k\in\mathbb{N}_0$.*

For the proof we need the following auxiliary Lemma:

Lemma 4.24. *Let $\beta,\gamma\geq 0$, $k\in\mathbb{N}$ and $k\geq 3$, $q\in[2,\infty)$. Then*

$$H^k_{(\beta,\gamma)}(\mathbb{R}^2_+)\to W^{k-1,q}_{(\beta,\gamma)}(\mathbb{R}^2_+):v\mapsto f''(v+\theta_0)-f''(\theta_0)$$

is well-defined and Lipschitz-continuous. The same is true for any $f''\in C^\infty(\mathbb{R}),\theta_0\in C_b^\infty(\mathbb{R})$.

For the following $q=2$ will be enough. Arbitrary $q\in[2,\infty)$ are just included for induction.

Proof. The case $k=3$ can be proven analogously to the proof of Lemma 4.16 by using suitable embeddings, cf. also Remark 4.17. For arbitrary $k\geq 4$ we show the assertion via induction. Therefore let v be in $H^k_{(\beta,\gamma)}(\mathbb{R}^2_+)$ and let the claim hold for $3,...,k-1$ instead of k. Then

$$\nabla(f''(v+\theta_0)-f''(\theta_0))=[f^{(3)}(\theta_0+v)-f^{(3)}(\theta_0)]\begin{pmatrix}\theta_0'\\0\end{pmatrix}+f^{(3)}(\theta_0+v)\nabla v.$$

By the induction hypothesis

$$H^{k-1}_{(\beta,\gamma)}(\mathbb{R}^2_+)\to W^{k-2,q}_{(\beta,\gamma)}(\mathbb{R}^2_+)\cap W^{k-2,2q}_{(\beta,\gamma)}(\mathbb{R}^2_+):v\mapsto f^{(3)}(\theta_0+v)-f^{(3)}(\theta_0)$$

is well-defined and Lipschitz-continuous. Since $\theta_0',f^{(3)}(\theta_0)\in C_b^\infty(\mathbb{R})$ and because of the embedding $H^{k-1}_{(\beta,\gamma)}(\mathbb{R}^2_+)\hookrightarrow W^{k-2,2q}_{(\beta,\gamma)}(\mathbb{R}^2_+)$ for ∇v, we are done. $\square$

Proof of Theorem 4.23. Because $\hat{v}_\alpha$ is as in Remark 4.19 and solves (4.18), we can differentiate the latter with respect to R. This yields with L_α from (4.20) that

$$\begin{aligned}L_\alpha\partial_R\hat{v}_\alpha&=(G_1(\alpha),g_1(\alpha)),\\(G_1(\alpha),g_1(\alpha)):&=-([f''(\theta_0+\hat{v}_\alpha)-f''(\theta_0)]\theta_0',\cos\alpha\theta_0''+\sigma_\alpha''(v_\alpha)|_{H=0}\theta_0').\end{aligned}\tag{4.22}$$

Note that $\theta_0'' \in H^{\frac{3}{2}}_{(\beta_0)}(\mathbb{R})$ due to Theorem 4.1 and the choice of β_0 in Remark 4.19. Therefore Lemma 4.24 applied to f and $\hat{\sigma}$, the trace-assertion in Lemma 2.22 and Theorem 4.21 yield

$$\partial_R \hat{v}_\alpha = \tilde{L}_\alpha^{-1}[(G_1(\alpha), g_1(\alpha) - c_D(G_1(\alpha), g_1(\alpha))\theta_0')] \in H^3_{(\beta_0,\gamma_0)}(\mathbb{R}^2_+)$$

and Lipschitz-continuity in α. Furthermore, rearranging the first equation in (4.18) implies

$$\partial_H^2 \hat{v}_\alpha = -\partial_R^2 \hat{v}_\alpha + 2\cos\alpha \partial_R \partial_H \hat{v}_\alpha + f'(\hat{v}_\alpha + \theta_0) - f'(\theta_0). \tag{4.23}$$

With the properties of $\partial_R \hat{v}_\alpha$ and Lemma 4.24 we obtain $\partial_H^2 \hat{v}_\alpha \in H^2_{(\beta_0,\gamma_0)}(\mathbb{R}^2_+)$ and Lipschitz-continuity in α. Therefore $\hat{v}_. : \frac{\pi}{2} + [-\alpha_0, \alpha_0] \to H^4_{(\beta_0,\gamma_0)}(\mathbb{R}^2_+)$ is well-defined and Lipschitz-continuous in α. To abbreviate in the following, we do not explicitly state the Lipschitz-continuity in α. But this always holds.

Finally, we show the claim by induction. Therefore let the assertion hold for some $k \in \mathbb{N}$, $k \geq 4$. Then by applying ∂_R^{k-2} to (4.22) and using similar arguments as before we obtain $\partial_R^{k-2}\hat{v}_\alpha \in H^3_{(\beta_0,\gamma_0)}(\mathbb{R}^2_+)$. Hence $\partial_R^{k-2}\partial_H \hat{v}_\alpha \in H^2_{(\beta_0,\gamma_0)}(\mathbb{R}^2_+)$. Therefore after applying ∂_R^{k-3} to (4.23) we get $\partial_R^{k-3}\partial_H^2 \hat{v}_\alpha \in H^2_{(\beta_0,\gamma_0)}(\mathbb{R}^2_+)$. This yields $\partial_R^{k-3}\hat{v}_\alpha \in H^4_{(\beta_0,\gamma_0)}(\mathbb{R}^2_+)$. Now one can apply similar arguments inductively by considering the action of ∂_R^{k-l} on (4.23) for $l = 4, ..., k$. This implies $\partial_R^{k-l}\hat{v}_\alpha \in H^{l+1}_{(\beta_0,\gamma_0)}(\mathbb{R}^2_+)$ for $l = 4, ..., k$, in particular $\hat{v}_\alpha \in H^{k+1}_{(\beta_0,\gamma_0)}(\mathbb{R}^2_+)$. $\square$

Finally, we consider the operator L_α from Theorem 4.21 in spaces of higher regularity.

Theorem 4.25. *Let $\beta_0, \gamma_0, \hat{v}_.$ be as in Remark 4.19 and α_0 be as in Theorem 4.21. Then for all $\alpha \in \frac{\pi}{2} + [-\alpha_0, \alpha_0]$, $\beta \in [0, \beta_0]$, $\gamma \in [\frac{\gamma_0}{2}, \gamma_0]$ and $k \in \mathbb{N}_0$ it holds that*

$$L_\alpha : H^{k+2}_{(\beta,\gamma)}(\mathbb{R}^2_+) \to \left\{ (G, g) \in H^k_{(\beta,\gamma)}(\mathbb{R}^2_+) \times H^{k+\frac{1}{2}}_{(\beta)}(\mathbb{R}) : \int_{\mathbb{R}^2_+} G \partial_R v_\alpha + \int_{\mathbb{R}} g \partial_R v_\alpha|_{H=0} = 0 \right\}$$

is an isomorphism and the norm of the inverse is bounded independent of α, β, γ for fixed k.

Proof. For $k = 0, 1$ this follows from Theorem 4.21. For arbitrary $k \in \mathbb{N}$ the operator L_α is well-defined in the spaces. Now we use an induction proof. Let the assertion hold for $k - 1$ and $k - 2$, where $k \in \mathbb{N}$, $k \geq 2$. Moreover, let $(G, g) \in H^k_{(\beta,\gamma)}(\mathbb{R}^2_+) \times H^{k+\frac{1}{2}}_{(\beta)}(\mathbb{R})$ be such that the compatibility condition (4.17) holds. Then by the induction hypothesis there is exactly one solution $u \in H^{k+1}_{(\beta,\gamma)}(\mathbb{R}^2_+)$ of $L_\alpha u = (G, g)$ and we have the estimate

$$\|u\|_{H^{k+1}_{(\beta,\gamma)}(\mathbb{R}^2_+)} \leq C_{k-1}\|(G, g)\|_{H^{k-1}_{(\beta,\gamma)}(\mathbb{R}^2_+)\times H^{k-\frac{1}{2}}_{(\beta)}(\mathbb{R})}.$$

Since $k + 1 \geq 3$, we can apply ∂_R to the equation. This yields

$$L_\alpha \partial_R u = (\partial_R G - f^{(3)}(v_\alpha)\partial_R v_\alpha u, \partial_R g - \cos\alpha\, \hat{\sigma}^{(3)}(v_\alpha)\partial_R v_\alpha|_{H=0} u|_{H=0}).$$

Here it holds $v_\alpha \in C^k_b(\overline{\mathbb{R}^2_+})$ with uniform estimates due to Theorem 4.23. Therefore the right hand side is an element of $H^{k-1}_{(\beta,\gamma)}(\mathbb{R}^2_+) \times H^{k-\frac{1}{2}}_{(\beta)}(\mathbb{R})$. Hence the assertion for $k-1$ and $k-2$ (the latter for uniqueness) implies $\partial_R u \in H^{k+1}_{(\beta,\gamma)}(\mathbb{R}^2_+)$ and

$$\|\partial_R u\|_{H^{k+1}_{(\beta,\gamma)}(\mathbb{R}^2_+)} \leq C_{k-1}[\|(\partial_R G, \partial_R g)\|_{H^{k-1}_{(\beta,\gamma)}(\mathbb{R}^2_+)\times H^{k-\frac{1}{2}}_{(\beta)}(\mathbb{R})} + c_k\|u\|_{H^k_{(\beta,\gamma)}(\mathbb{R}^2_+)}].$$

Using $\partial_H^2 u = -\partial_R^2 u + 2\cos\alpha \partial_R \partial_H u + f''(v_\alpha)u - G$, we obtain $\partial_H^2 u \in H^k_{(\beta,\gamma)}(\mathbb{R}^2_+)$ and an estimate. Altogether this shows Theorem 4.25. $\square$

4.3 Some Vector-valued ODE Problems on $\mathbb{R}$

The structure of this section is similar to Section 4.1 which is the analogue in the scalar case. We consider vector-valued ODEs appearing in the inner asymptotic expansion of (vAC) and also the linear operator belonging to a linearized ODE. A crucial assumption to solve the linearized ODE will be that the kernel of the linearization is 1-dimensional. The latter is fulfilled for a typical potential, cf. the example in Remark 4.28 below.

Let $W : \mathbb{R}^m \to \mathbb{R}$ be as in Definition 1.4 and $\vec{u}_\pm$ be any distinct pair in $\{\vec{a}, \vec{b}\}$ or $\{\vec{x}_1, \vec{x}_3, \vec{x}_5\}$, respectively. From now on, we fix $\vec{u}_\pm$.

4.3.1 The Nonlinear ODE

The nonlinear ODE problem in the lowest order is the following: Find $\vec{u} : \mathbb{R} \to \mathbb{R}^m$ smooth with suitable decay such that

$$-\vec{u}'' + \nabla W(\vec{u}) = 0 \quad \text{on } \mathbb{R}, \quad \lim_{z\to\pm\infty} \vec{u}(z) = \vec{u}_\pm. \tag{4.24}$$

Theorem 4.26. *Let m, W, $\vec{u}_\pm$ be as above and let $\lambda > 0$ be such that $D^2W(\vec{u}_\pm) \geq \lambda I$. Then there is a smooth solution $\vec{u} : \mathbb{R} \to \mathbb{R}^m$ to* (4.24) *such that*

$$\partial_z^k[\vec{u} - \vec{u}_\pm](z) = \mathcal{O}(e^{-\beta|z|}) \quad \textit{for } z \to \pm\infty \textit{ and all } k \in \mathbb{N}_0, \beta \in \left(0, \sqrt{\lambda/2}\right).$$

Moreover, $\vec{u}$ can be chosen $R_{\vec{u}_-,\vec{u}_+}$-odd, i.e. $\vec{u}(-.) = R_{\vec{u}_-,\vec{u}_+}\vec{u}$ with $R_{\vec{u}_-,\vec{u}_+}$ as in Definition 1.4. In this case it holds $R_{\vec{u}_-,\vec{u}_+}\vec{u}'|_{z=0} \neq \vec{u}'|_{z=0}$.

Remark 4.27.

1. From now on, we fix a $R_{\vec{u}_-,\vec{u}_+}$-odd solution and simply denote it by $\vec{\theta}_0$.
2. The proof relies on minimizing an energy over an approporiate set (see below). Similar to Bronsard, Gui, Schatzman [BGS], Section 2, where the triple-well case is considered, it should be possible to determine the qualitative behaviour of the set of minimizers for both types of W in Definition 1.4 precisely. E.g. in the triple-well case the minimizers are trapped in the smaller sector between $\vec{u}_-$ and $\vec{u}_+$. But this is not needed here.

Proof of Theorem 4.26. Let $\xi : \mathbb{R} \to \mathbb{R}$ be smooth and odd such that $\xi(z) = \operatorname{sign}(z)$ for $|z| \geq 1$. Moreover, we define

$$\Xi(\vec{u}_-, \vec{u}_+) := \frac{1}{2}(\vec{u}_- + \vec{u}_+) + \xi\frac{1}{2}(\vec{u}_+ - \vec{u}_-). \tag{4.25}$$

Then Kusche [Ku], Section 2.1 for potentials W as in Definition 1.4, 1. and Bronsard, Gui, Schatzman [BGS], Section 2 for triple-well potentials W in Definition 1.4, 2., respectively, yield that

$$E : \Xi(\vec{u}_-, \vec{u}_+) + H^1(\mathbb{R})^m \to \mathbb{R} : \vec{u} \mapsto \int_{\mathbb{R}} \frac{1}{2}|\vec{u}'|^2 + W(\vec{u})\, dz$$

admits a global minimizer $\vec{u}$ that satisfies $\vec{u} \in C^3(\mathbb{R})^m \cap (H^2(\mathbb{R})^m + \Xi(\vec{u}_-, \vec{u}_+))$ and is $R_{\vec{u}_-,\vec{u}_+}$-odd. Moreover, $\vec{u}$ satisfies (4.24) and

$$\partial_z^k[\vec{u} - \vec{u}_\pm](z) = \mathcal{O}(e^{-\beta|z|}) \quad \text{for } z \to \pm\infty \text{ and } k = 0, 1, 2, 3, \beta \in (0, \sqrt{\lambda/2}).$$

Furthermore, one obtains $\vec{u} \in C^{k+1}(\mathbb{R})^m \cap (H^k(\mathbb{R})^m + \Xi(\vec{u}_-, \vec{u}_+))$ for all $k \in \mathbb{N}$, $k \geq 2$ and the decay properties by induction and differentiating the equation.

Finally, we show $R_{\vec{u}_-,\vec{u}_+}\vec{u}'|_{z=0} \neq \vec{u}'|_{z=0}$ for any smooth $\vec{u}: \mathbb{R} \to \mathbb{R}^m$ that solves (4.24) and is $R_{\vec{u}_-,\vec{u}_+}$-odd. This will be shown by contradiction with uniqueness for the initial value ODE problem for the part of $\vec{u}$ orthogonal to the hypersurface inbetween $\vec{u}_-$ and $\vec{u}_+$. Therefore let

$$\vec{v}^\perp := \frac{1}{2} P_{\text{span}\{\vec{u}_+ - \vec{u}_-\}}[\vec{v} - R_{\vec{u}_-,\vec{u}_+}\vec{v}] \quad \text{for every } \vec{v} \in \mathbb{R}^m.$$

Then $(\vec{u}')^\perp : \mathbb{R} \to \mathbb{R}$ is smooth and solves $-[(\vec{u}')^\perp]'' = [D^2W(\vec{u})\vec{u}']^\perp$. Since $\vec{u}$ is $R_{\vec{u}_-,\vec{u}_+}$-odd, this also holds for $\vec{u}''$ and hence $[(\vec{u}')^\perp]'(0) = 0$. Due to the boundary condition in (4.24) we obtain $(\vec{u}')^\perp \not\equiv 0$. Therefore $(\vec{u}')^\perp(0) \neq 0$, otherwise we get a contradiction to $(\vec{u}')^\perp \equiv 0$ due to ODE-theory. This proves $R_{\vec{u}_-,\vec{u}_+}\vec{u}'|_{z=0} \neq \vec{u}'|_{z=0}$. □

4.3.2 The Linearized Operator

We look at the operator obtained by linearization of the left hand side of the ODE (4.24) at $\vec{\theta}_0$, i.e.

$$\check{L}_0 : H^2(\mathbb{R}, \mathbb{K})^m \subseteq L^2(\mathbb{R}, \mathbb{K})^m \to L^2(\mathbb{R}, \mathbb{K})^m : \vec{u} \mapsto \check{\mathcal{L}}_0\vec{u} := \left[-\frac{d^2}{dz^2} + D^2W(\vec{\theta}_0) \right] \vec{u}. \quad (4.26)$$

Remark 4.28 (**Assumption** $\dim \ker \check{L}_0 = 1$)**.** $\vec{\theta}_0'$ is an element of $\ker \check{L}_0$ and $\vec{\theta}_0' \not\equiv 0$ due to Theorem 4.26. In order to have a spectral gap property that is needed for solving the vector-valued linearized ODE and the vector-valued $\mathbb{R}^2_+$-model problem in the next sections, we *assume* $\dim \ker \check{L}_0 = 1$ (this is independent of $\mathbb{K}$ since $D^2W(\vec{\theta}_0)$ is real-valued). This is reasonable, cf. Kusche [Ku], Section 3.4 for a typical triple-well potential that fulfils this. Note that the assumption should be stable under suitable "small" perturbations of the potential W due to the upper continuity of the nullity index for (semi-)Fredholm operators, cf. Kato [K], Theorem 5.22.

Lemma 4.29. *Let $\check{L}_0$ be as above and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Then $\check{L}_0$ is self-adjoint, $\check{L}_0 \geq 0$ and*

$$\sigma_e(\check{L}_0) = [\min\{\sigma(D^2W(\vec{u}_\pm))\}, \infty).$$

In particular $\sigma_d(\check{L}_0) \subset [0, \min\{\sigma(D^2W(\vec{u}_\pm))\})$. Moreover, if $\dim \ker \check{L}_0 = 1$, then with $(\ker \check{L}_0)^\perp := \{\vec{w} \in L^2(\mathbb{R}, \mathbb{C})^m : (\vec{w}, \vec{\theta}_0')_{L^2} = 0\}$ it holds

$$\begin{aligned} 0 < \check{\nu}_0 &:= \inf_{\vec{w} \in H^2(\mathbb{R},\mathbb{C})^m \cap \text{span}\{\vec{\theta}_0'\}^\perp, \|\vec{w}\|_{L^2}=1} (\check{L}_0\vec{w}, \vec{w})_{L^2(\mathbb{R},\mathbb{C})^m} \\ &= \inf_{\vec{w} \in H^1(\mathbb{R},\mathbb{C})^m \cap \text{span}\{\vec{\theta}_0'\}^\perp, \|\vec{w}\|_{L^2}=1} \int_{\mathbb{R}} |\vec{w}'|^2 + (D^2W(\vec{\theta}_0)\vec{w}, \vec{w})_{\mathbb{C}^m}\, dz. \end{aligned}$$

Proof. That $\check{L}_0$ is self-adjoint e.g. follows from Kusche [Ku], Proposition 1.1 or with a typical argument as in the proof of Lemma 4.2. The property $\check{L}_0 \geq 0$ is an outcome of the energetic approach in the proof of Theorem 4.26 (first for $\mathbb{K} = \mathbb{R}$, then it follows for $\mathbb{K} = \mathbb{C}$). Moreover, one can use Persson's Theorem and Weyl sequences to show the identity for $\sigma_e(\check{L}_0)$, cf. the proof of Proposition 2.1 in [Ku]. The remaining assertions can be deduced in the analogous way as in the proof of Lemma 4.2. □

4.3.3 The Linearized ODE

We have to consider the ODE that arises from the linearization of (4.24) at $\vec{\theta}_0$. More precisely, for $\vec{A}:\mathbb{R}\rightarrow\mathbb{R}^m$ with suitable regularity and decay we seek a function $\vec{u}:\mathbb{R}\rightarrow\mathbb{R}^m$ such that

$$\check{L}_0\vec{u}=\vec{A} \quad \text{and} \quad \check{B}\vec{u}=0, \tag{4.27}$$

where $\check{B}\in\mathcal{L}(H^l(\mathbb{R})^m,\mathbb{R})$ for some $l\in\{0,1,2\}$ with $\check{B}\vec{\theta}_0'\neq 0$. As before we make the assumption $\dim\ker\check{L}_0=1$, cf. Remark 4.28.

Remark 4.30. The additional condition with $\check{B}$ is imposed in order to get uniqueness below. The natural choice from a functional analytic point of view is $\check{B}:=(.,\vec{\theta}_0')_{L^2(\mathbb{R})^m}:L^2(\mathbb{R})^m\rightarrow\mathbb{R}$. However, the canonical choice for the application later is

$$\check{B}:=(\vec{u}_--\vec{u}_+)^\top[R_{\vec{u}_-,\vec{u}_+}-I](.)|_{z=0}:H^1(\mathbb{R})^m\rightarrow\mathbb{R},$$

where $R_{\vec{u}_-,\vec{u}_+}$ is defined analogously to Definition 1.4. The latter fulfils $\check{B}\vec{\theta}_0'\neq 0$ due to Theorem 4.26 and heuristically the condition $\check{B}\vec{u}=0$ means that $\vec{u}|_{z=0}$ is precisely in the middle of the two phases. E.g. for $\vec{u}_-=(-1,1)^\top$ and $\vec{u}_+=(1,1)^\top$ the latter reduces to $\vec{u}(0)_1=0$.

Theorem 4.31. *Let* $\dim\ker\check{L}_0=1$, *cf. Remark 4.28. Then it holds*

1. *Let $\vec{A}\in L^2(\mathbb{R})^m$. Then there is a $\vec{u}\in H^2(\mathbb{R})^m$ such that $\check{L}_0\vec{u}=\vec{A}$ if and only if $\int_{\mathbb{R}}\vec{A}\cdot\vec{\theta}_0'=0$. In this case $\vec{u}$ is unique up to multiples of $\vec{\theta}_0'$. In particular,* (4.27) *admits a unique solution $\vec{u}\in H^2(\mathbb{R})^m$ if and only if $\int_{\mathbb{R}}\vec{A}\cdot\vec{\theta}_0'=0$. Moreover, for all $k\in\mathbb{N}_0$*

$$\check{L}_0:\left\{\vec{u}\in H^{k+2}(\mathbb{R})^m:\check{B}\vec{u}=0\right\}\rightarrow\left\{\vec{A}\in H^k(\mathbb{R})^m:\int_{\mathbb{R}}\vec{A}\cdot\vec{\theta}_0'=0\right\}$$

   is an isomorphism and the inverse is bounded by some $c(\check{B},k)>0$.

2. *There is a $\check{\beta}_0>0$ small such that for all $\beta\in(0,\check{\beta}_0)$ and $k\in\mathbb{N}_0$*

$$\check{L}_0:\left\{\vec{u}\in H^{k+2}_{(\beta)}(\mathbb{R})^m:\check{B}\vec{u}=0\right\}\rightarrow\left\{\vec{A}\in H^k_{(\beta)}(\mathbb{R})^m:\int_{\mathbb{R}}\vec{A}\cdot\vec{\theta}_0'=0\right\}$$

   is an isomorphism and the norm of the inverse is bounded by a constant $C(\check{B},k)>0$.

Remark 4.32.

1. *Dependence on parameteres.* In Theorem 4.31 we obtained linear solution operators in suitable spaces with exponential decay. Therefore, if the right hand side in (4.27) depends on additional parameters and satisfies such exponential decay estimates, the regularity and decay carries over to the solution.

2. Using Theorem 4.31 one can directly obtain a result for right hand sides $\vec{A}$ that converge with appropriate rate to non-zero vectors $\vec{A}^\pm\in\mathbb{R}^m$ at $\pm\infty$. The idea is as follows: set

$$\vec{U}_\pm:=[D^2W(\vec{u}_\pm)]^{-1}(\vec{A}_\pm) \quad \text{and} \quad \vec{U}:=\Xi(\vec{U}_-,\vec{U}_+),$$

   where the latter is analogous to (4.25). Then formally it holds $\check{\mathcal{L}}_0\vec{u}=\vec{A}$ if and only if $\check{\mathcal{L}}_0(\vec{u}-\vec{U})=\vec{A}-\vec{A}_0$, where $\vec{A}_0:=\check{L}_0\vec{U}$. To this equation one can apply the results in Theorem 4.31 for suitable $\vec{A}$. Note that the compatibility condition is the same as before since $\int_{\mathbb{R}}\vec{A}_0\cdot\vec{\theta}_0'=0$ due to integration by parts. The case $\vec{A}^\pm\neq 0$ is not needed here but may be interesting for more sophisticated equations, e.g. a vector-valued Cahn-Hilliard equation. Moreover, the idea to reduce to $\vec{A}_\pm=0$ could also be helpful for triple junction cases. Finally, note the analogy in the limits at infinity to the ones in Theorem 4.4.

3. Kusche [Ku], Proposition 1.6 and Corollary 1.2 yield pointwise exponential decay estimates. The latter would be enough for our purpose. But the downside is that the exponent shrinks. Moreover, the proof of Theorem 4.31 is self-contained and simpler. However, in [Ku] there are also uniform estimates for finite large intervals which are important for the spectral estimates, cf. Section 6.1.4 below.

Proof of Theorem 4.31. Ad 1. Consider $\mathbb{K} = \mathbb{R}$ in Lemma 4.29. With the latter one can show that

$$\check{L}_0^\perp := \check{L}_0|_{(\ker \check{L}_0)^\perp} : H^2(\mathbb{R})^m \cap (\ker \check{L}_0)^\perp \to (\ker \check{L}_0)^\perp$$

is well-defined, self-adjoint and $\sigma(\check{L}_0) = \sigma(\check{L}_0^\perp) \cup \{0\}$. As in the proof of Lemma 4.2 it follows that $\sigma(\check{L}_0^\perp) = \sigma(\check{L}_0)\backslash\{0\}$ and hence $0 \in \rho(\check{L}_0^\perp)$. Moreover, the graph norm is equivalent to the $H^2(\mathbb{R})^m$-norm. This can e.g. be seen via direct estimates.

Now let $\vec{v} \in H^2(\mathbb{R})^m$ solve $\check{L}_0\vec{v} = \vec{A}$ for some $\vec{A} \in L^2(\mathbb{R})^m$. Then due to integration by parts it holds $(\vec{A}, \vec{\theta}_0')_{L^2(\mathbb{R})^m} = (\vec{v}, \check{L}_0\vec{\theta}_0')_{L^2(\mathbb{R})^m} = 0$. On the other hand, let $\vec{A} \in L^2(\mathbb{R})^m$ be such that $\int_{\mathbb{R}} \vec{A} \cdot \vec{\theta}_0' = 0$. The latter is equivalent to $\vec{A} \in (\ker \check{L}_0)^\perp \cap L^2(\mathbb{R})^m$. The above considerations yield a unique solution $\vec{v} \in H^2(\mathbb{R})^m \cap (\ker \check{L}_0)^\perp$ to $\check{L}_0^\perp \vec{v} = \vec{A}$ and $\|\vec{v}\|_{H^2(\mathbb{R})^m} \leq C\|\vec{A}\|_{L^2(\mathbb{R})^m}$. Because of the assumption $\dim \ker \check{L}_0 = 1$ it holds $\ker \check{L}_0 = \operatorname{span}\{\vec{\theta}_0'\}$. This implies the uniqueness in $H^2(\mathbb{R})^m$ up to multiples of $\vec{\theta}_0'$. Due to $\check{B} \in \mathcal{L}(H^l(\mathbb{R})^2, \mathbb{R})$ for some $l \in \{0, 1, 2\}$ and $\check{B}\vec{\theta}_0' \neq 0$, we obtain that

$$\vec{u} := \vec{v} - \frac{\check{B}\vec{v}}{\check{B}\vec{\theta}_0'}\vec{\theta}_0' \in H^2(\mathbb{R})^m$$

is well-defined, the unique solution to (4.27) and that the estimate $\|\vec{u}\|_{H^2(\mathbb{R})^m} \leq C(\check{B})\|\vec{A}\|_{L^2(\mathbb{R})^m}$ holds. In particular the claim follows for $k = 0$. For arbitrary $k \in \mathbb{N}$ let $\vec{A} \in H^k(\mathbb{R})^m$ and $\vec{u} \in H^2(\mathbb{R})^m$ solve $\check{L}_0\vec{u} = \vec{A}$. Then it follows iteratively from the equation that $\vec{u} \in H^{k+2}(\mathbb{R})^m$. In particular $\check{L}_0$ is an isomorphism with respect to the spaces in the theorem for all $k \in \mathbb{N}$. The estimate for the inverse can also be shown iteratively using the equation. $\square_1$

Ad 2. We prove this with similar ideas as in Section 4.2.1.2, i.e. for $\vec{A} \in L^2_{(\beta)}(\mathbb{R})^m$ with $\int_{\mathbb{R}} \vec{A} \cdot \vec{\theta}_0' = 0$ we make the ansatz $\vec{u} = e^{-\beta\eta}\vec{v}$ with $\vec{v} \in H^2(\mathbb{R})^m$, where $\eta : \mathbb{R} \to \mathbb{R}$ is as in Definition 2.21, 4. It holds

$$\vec{u}'' = e^{-\beta\eta}[\vec{v}'' - 2\beta\eta'\vec{v}' + \vec{v}(\beta^2(\eta')^2 - \beta\eta'')].$$

Therefore we consider the equation

$$\check{L}_0\vec{v} + \check{N}_\beta\vec{v} = e^{\beta\eta}\vec{A}, \quad \check{N}_\beta\vec{v} := 2\beta\eta'\vec{v}' - \vec{v}(\beta^2(\eta')^2 - \beta\eta'').$$

In order to solve this, we want to use the spaces in 1. One problem is the compatibility condition for $\check{N}_\beta\vec{v}$ and the right hand side. Therefore we solve a different equation, where suitable terms are subtracted that enforce the compatibility condition. Moreover, $\vec{v}$ should satisfy $\check{B}_{(\beta)}\vec{v} = 0$, where $\check{B}_{(\beta)} := \check{B}(e^{-\beta\eta}.)$. This is a problem since then the space would depend on β. Therefore note that it is enough to show the assertion e.g. for $\check{B} = \check{B}_0 := (.)_1|_{z=0}$ since in the end one can subtract $\vec{\theta}_0'\check{B}\vec{u}/\check{B}\vec{\theta}_0'$ to get a solution of (4.27). If $\check{\beta}_0 > 0$ is small enough, in particular $\check{\beta}_0 \leq \sqrt{\lambda/2}$ with λ as in Theorem 4.26, the assertion carries over to general $\check{B}$. The advantage of $\check{B}_0$ is that $\check{B}_0(e^{-\beta\eta}\vec{v}) = 0$ if and only if $\check{B}_0\vec{v} = 0$. Therefore the space for $\vec{v}$ can be chosen uniformly in β. Hence we solve $\check{B}_0\vec{v} = 0$ together with

$$\check{L}_0\vec{v} + \check{M}_\beta\vec{v} = e^{\beta\eta}\vec{A} - \left[\int_{\mathbb{R}} e^{\beta\eta}\vec{A}\cdot\vec{\theta}_0'\right]\frac{\vec{\theta}_0'}{\|\vec{\theta}_0'\|_{L^2(\mathbb{R})^m}}, \tag{4.28}$$
$$\check{M}_\beta\vec{v} := \check{N}_\beta\vec{v} - \left[\int_{\mathbb{R}} \check{N}_\beta\vec{v}\cdot\vec{\theta}_0'\right]\frac{\vec{\theta}_0'}{\|\vec{\theta}_0'\|_{L^2(\mathbb{R})^m}}.$$

Using the properties of exponentially weighted Sobolev spaces in Lemma 2.22 one can directly show that

$$\check{N}_\beta, \check{M}_\beta \in \mathcal{L}(H^2_{(\beta)}(\mathbb{R})^m, L^2_{(\beta)}(\mathbb{R})^m)$$

with norm bounded by $C\beta$ for all $\beta \in [0, \check{\beta}_0)$ and any fixed $\check{\beta}_0 > 0$. Therefore the first part of the theorem and a Neumann series argument imply that for all $\beta \in [0, \check{\beta}_0)$ and $\check{\beta}_0 > 0$ small

$$\check{L}_0 + \check{M}_\beta : \left\{\vec{v} \in H^2(\mathbb{R})^m : \check{B}_0\vec{v} = 0\right\} \to \left\{\vec{A} \in L^2(\mathbb{R})^m : \int_{\mathbb{R}} \vec{A}\cdot\vec{\theta}_0' = 0\right\}$$

is an isomorphism and the norm of the inverse is bounded by a constant independent of β. Hence (4.28) admits a unique solution $\vec{v} \in H^2(\mathbb{R})^m$ with $\check{B}_0\vec{v} = 0$ for all $\beta \in [0, \check{\beta}_0)$. The computations above yield that $\vec{u} := e^{-\beta\eta}\vec{v} \in H^2_{(\beta)}(\mathbb{R})^m$ solves

$$\check{L}_0\vec{u} = \vec{A} + \left[-\int_{\mathbb{R}} e^{\beta\eta}\vec{A}\cdot\vec{\theta}_0' + \int_{\mathbb{R}} \check{N}_\beta\vec{v}\cdot\vec{\theta}_0'\right]\frac{e^{-\beta\eta}\vec{\theta}_0'}{\|\vec{\theta}_0'\|^2_{L^2(\mathbb{R})^m}}.$$

By assumption it holds $\int_{\mathbb{R}} \vec{A}\cdot\vec{\theta}_0' = 0$ and because of the first part of the theorem we have $\int_{\mathbb{R}} \check{L}_0\vec{u}\cdot\vec{\theta}_0' = 0$. Due to Theorem 4.26 it holds $\vec{\theta}_0' \neq 0$ and hence $\int_{\mathbb{R}} e^{-\beta\eta}|\vec{\theta}_0'|^2 > 0$. Therefore

$$\check{L}_0\vec{u} = \vec{A} \quad \text{and} \quad \check{B}_0\vec{u} = 0.$$

Finally, the first part in the theorem yields uniqueness and the estimate follows with the above considerations and Lemma 2.22. $\square_2$.

4.4 A Vector-valued Elliptic Problem on $\mathbb{R}^2_+$ with Neumann Boundary Condition

This section is analogous to Section 4.2.1, where the scalar case was done. Let $W : \mathbb{R}^m \to \mathbb{R}$ be as in Definition 1.4 and $\vec{\theta}_0$ be as in Remark 4.27, 1. In the contact point expansion for (vAC) in any dimension $N \geq 2$ the following model problem appears: For suitable data $\vec{G} : \overline{\mathbb{R}^2_+} \to \mathbb{R}^m$, $\vec{g} : \mathbb{R} \to \mathbb{R}^m$ find a solution $\vec{u} : \overline{\mathbb{R}^2_+} \to \mathbb{R}^m$ to the system

$$\left[-\Delta + D^2W(\vec{\theta}_0(R))\right]\vec{u}(R,H) = \vec{G}(R,H) \quad \text{for } (R,H) \in \mathbb{R}^2_+, \tag{4.29}$$
$$-\partial_H\vec{u}|_{H=0}(R) = \vec{g}(R) \quad \text{for } R \in \mathbb{R}. \tag{4.30}$$

As often in the last Section 4.3 we make the *assumption* that $\dim\ker\check{L}_0 = 1$, where $\check{L}_0$ is defined in (4.26), cf. also Remark 4.28. This implies a useful estimate for functions orthogonal to $\vec{\theta}_0'$, cf. Lemma 4.29. The solution strategy for (4.29)-(4.30) is completely analogous to Section 4.2.1. First of all, we show some assertions for weak solutions of the problem in Section 4.4.1. Then in Section 4.4.2 we obtain solution operators in exponentially weighted Sobolev spaces.

4.4.1 Weak Solutions and Regularity

In the vector-valued case weak solutions are defined as follows:

Definition 4.33. Let $\vec{G} \in L^2(\mathbb{R}^2_+)^m$ and $\vec{g} \in L^2(\mathbb{R})^m$. Then $\vec{u} \in H^1(\mathbb{R}^2_+)^m$ is called *weak solution* of (4.29)-(4.30) if for all $\vec{\varphi} \in H^1(\mathbb{R}^2_+)^m$ it holds that

$$\begin{aligned}\check{a}(\vec{u},\vec{\varphi}) &:= \int_{\mathbb{R}^2_+} \nabla\vec{u} : \nabla\vec{\varphi} + (D^2W(\vec{\theta}_0(R))\vec{u},\vec{\varphi})_{\mathbb{R}^m}\, d(R,H)\\ &= \int_{\mathbb{R}^2_+} \vec{G}\cdot\vec{\varphi}\, d(R,H) + \int_{\mathbb{R}} \vec{g}(R)\cdot\vec{\varphi}|_{H=0}(R)\, dR.\end{aligned}$$

We obtain the analogue of Theorem 4.7:

Theorem 4.34. *Let* $\dim\ker\check{L}_0 = 1$ *and consider* $\vec{G} \in L^2(\mathbb{R}^2_+)^m$ *and* $\vec{g} \in L^2(\mathbb{R})^m$. *Then*

1. $\check{a} : H^1(\mathbb{R}^2_+)^m \times H^1(\mathbb{R}^2_+)^m \to \mathbb{R}$ *is not coercive.*
2. *If* $\vec{G}(.,H), \vec{g} \perp \vec{\theta}_0'$ *for a.e.* $H > 0$ *in* $L^2(\mathbb{R})^m$, *then there exists a weak solution* $\vec{u}$ *with* $\vec{u}(.,H) \perp \vec{\theta}_0'$ *for a.e.* $H > 0$ *and it holds* $\|\vec{u}\|_{H^1(\mathbb{R}^2_+)^m} \leq C(\|\vec{G}\|_{L^2(\mathbb{R}^2_+)^m} + \|\vec{g}\|_{L^2(\mathbb{R})^m})$.
3. *Weak solutions are unique.*
4. *If* $\vec{G}\cdot\vec{\theta}_0' \in L^1(\mathbb{R}^2_+)$ *and* $\vec{u}$ *is a weak solution with* $\partial_H\vec{u}\cdot\vec{\theta}_0' \in L^1(\mathbb{R}^2_+)$, *then the following compatibility condition holds:*
$$\int_{\mathbb{R}^2_+} \vec{G}(R,H)\cdot\vec{\theta}_0'(R)\, d(R,H) + \int_{\mathbb{R}} \vec{g}(R)\cdot\vec{\theta}_0'(R)\, dR = 0. \tag{4.31}$$
5. *If* $\vec{G}\cdot\vec{\theta}_0' \in L^1(\mathbb{R}^2_+)$, *then* $\tilde{G}(H) := (\vec{G}(.,H),\vec{\theta}_0')_{L^2(\mathbb{R})^m}$ *is well-defined for a.e.* $H > 0$ *and* $\tilde{G} \in L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$. *Moreover, we have the decomposition*
$$\vec{G} = \tilde{G}(H)\frac{\vec{\theta}_0'(R)}{\|\vec{\theta}_0'\|^2_{L^2(\mathbb{R})^m}} + \vec{G}^\perp(R,H),\ \vec{g} = (\vec{g},\vec{\theta}_0')_{L^2(\mathbb{R})^m}\frac{\vec{\theta}_0'(R)}{\|\vec{\theta}_0'\|^2_{L^2(\mathbb{R})^m}} + \vec{g}^\perp(R), \tag{4.32}$$
where $\vec{G}^\perp \in L^2(\mathbb{R}^2_+)^m, \vec{g}^\perp \in L^2(\mathbb{R})^m$ *with* $\vec{G}^\perp(.,H), \vec{g}^\perp \perp \vec{\theta}_0'$ *in* $L^2(\mathbb{R})^m$ *f.a.e.* $H > 0$.
6. *If* $\|\vec{G}(.,H)\|_{L^2(\mathbb{R})^m} \leq Ce^{-\nu H}$ *for a.e.* $H > 0$ *and a constant* $\nu > 0$, *then* $\vec{G}\cdot\vec{\theta}_0' \in L^1(\mathbb{R}^2_+)$. *Let* $\tilde{G}$ *be defined as in 4. and the compatibility condition* (4.31) *hold. Then*
$$\vec{u}_1(R,H) := -\int_H^\infty \int_{\tilde{H}}^\infty \tilde{G}(\hat{H})\, d\hat{H}\, d\tilde{H}\, \frac{\vec{\theta}_0'(R)}{\|\vec{\theta}_0'\|^2_{L^2(\mathbb{R})^m}} \tag{4.33}$$
is well-defined for a.e. $(R,H) \in \mathbb{R}^2_+$, $\vec{u}_1 \in W^2_1(\mathbb{R}^2_+)^m \cap H^2(\mathbb{R}^2_+)^m$ *and* $\vec{u}_1$ *is a weak solution of* (4.3)-(4.4) *for* $\vec{G} - \vec{G}^\perp, \vec{g} - \vec{g}^\perp$ *in* (4.6) *instead of* $\vec{G}, \vec{g}$.

Proof. One can essentially copy the proof of Theorem 4.7. All multiplications of functions that are now vector-valued have to be interpreted as scalar products and $f''(\theta_0)$ has to be replaced by $D^2W(\vec{\theta}_0)$. Moreover, all spaces except for products and for $\tilde{G}$ change to vector-valued ones. Finally, one uses Lemma 4.29 instead of Lemma 4.2 to get coercivity of $\check{a}$ on the orthogonal parts and uniqueness of weak solutions. $\square$

Again this yields an existence theorem for weak solutions analogously to Corollary 4.8:

Corollary 4.35. *Let* $\dim\ker \check{L}_0 = 1$. *Then it holds*

1. *Let* $\vec{g} \in L^2(\mathbb{R})^m$, $\vec{G} \in L^2(\mathbb{R}^2_+)^m$ *with* $\|\vec{G}(.,H)\|_{L^2(\mathbb{R})^m} \leq Ce^{-\nu H}$ *for a.e.* $H > 0$ *and some* $\nu > 0$. *Let* (4.31) *hold. Then there is a unique weak solution of* (4.29)-(4.30).

2. *Let* $k \in \mathbb{N}_0$ *and* $\vec{u} \in H^1(\mathbb{R}^2_+)^m$ *be a weak solution of* (4.29)-(4.30) *for* $\vec{G} \in H^k(\mathbb{R}^2_+)^m$ *and* $\vec{g} \in H^{k+\frac{1}{2}}(\mathbb{R})^m$. *Then* $\vec{u} \in H^{k+2}(\mathbb{R}^2_+)^m \hookrightarrow C^{k,\gamma}(\overline{\mathbb{R}^2_+})^m$ *for all* $\gamma \in (0,1)$ *and*

$$\|\vec{u}\|_{H^{k+2}(\mathbb{R}^2_+)^m} \leq C_k(\|\vec{G}\|_{H^k(\mathbb{R}^2_+)^m} + \|\vec{g}\|_{H^{k+\frac{1}{2}}(\mathbb{R})^m} + \|\vec{u}\|_{H^1(\mathbb{R}^2_+)^m}).$$

Proof. The first part is a direct consequence of Theorem 4.34. The second assertion can be shown similar to the proof of Corollary 4.8, 2. using iteratively *scalar* regularity theory for every component of the elliptic equation, where $D^2W(\vec{\theta}_0)\vec{u}$ is viewed as part of the right hand side. $\square$

4.4.2 Solution Operators in Exponentially Weighted Spaces

With analogous adjustments as above one obtains solution operators in exponentially weighted Sobolev spaces similar to Section 4.2.1.2, cf. Theorems 4.9-4.11. We will just need the analogue of Theorem 4.11, hence we only formulate the latter in the vector-valued setting:

Theorem 4.36 (Solution Operators for the Vector-valued Case). *Let* $\dim\ker \check{L}_0 = 1$. *There exist* $\check{\gamma} > 0$ *and* $\check{\beta} : (0,\check{\gamma}] \to (0,\infty)$ *non-decreasing such that*

$$\check{L}_{\frac{\pi}{2}} := (-\Delta + D^2W(\vec{\theta}_0(R)), -\partial_H|_{H=0}) : H^{k+2}_{(\beta,\gamma)}(\mathbb{R}^2_+)^m \to \check{Y}^k_{(\beta,\gamma)},$$

$$\check{Y}^k_{(\beta,\gamma)} := \left\{ (\vec{G},\vec{g}) \in H^k_{(\beta,\gamma)}(\mathbb{R}^2_+)^m \times H^{k+\frac{1}{2}}_{(\beta)}(\mathbb{R})^m : \int_{\mathbb{R}^2_+} \vec{G}\cdot\vec{\theta}_0' + \int_{\mathbb{R}} \vec{g}\cdot\vec{\theta}_0' = 0 \right\}$$

is invertible for all $k \in \mathbb{N}_0$, $\gamma \in (0,\check{\gamma}]$ *and* $\beta \in [0,\check{\beta}(\gamma)]$ *and the operator norm of the inverse is bounded by* $\check{C}(k)(1+\frac{1}{\gamma^2})^{k+1}$.

Remark 4.37 (Dependence on Parameters). For data that depend on other indepenent variables the regularity directly carries over to the solution since we have linear and bounded solution operators. This is analogous to the scalar case, cf. Remark 4.12.

5 Asymptotic Expansions

In this section we carry out the rigorous asymptotic expansions for (AC), (vAC) and (AC$_\alpha$) in the situations mentioned in the introduction, see Sections 1.1-1.3. The expansions are based on the curvilinear coordinates from Section 3 and use the solutions for the model problems in Section 4.

More precisely, in Section 5.1 we start with the simplest case of the scalar-valued Allen-Cahn equation with Neumann boundary condition, (AC1)-(AC3), in two dimensions. Here we need the model problems in Section 4.1 and 4.2.1. Then we consider the N-dimensional case in Section 5.2 which is more technical but uses the same model problems. For $N = 2$ these computations are in principle the same as in Section 5.1 if some symbols are interpreted adequately. However, we decided to treat the case $N = 2$ separately, since there the fundamental ideas can be seen more clearly. In Section 5.3 we construct an approximate solution for the vector-valued Allen-Cahn equation, (vAC1)-(vAC3). Since we do not treat the triple junction case this is basically the same construction as in Section 5.2, just with vector-valued functions and the corresponding model problems in Sections 4.3-4.4. Finally, we consider the scalar Allen-Cahn equation with a nonlinear Robin boundary condition, (AC$_\alpha$1)-(AC$_\alpha$3) in Section 5.4. Here besides the model problems in Section 4.1 the ones in Section 4.2.2 appear.

Review of Rigorous Asymptotic Expansions used within the Method of [deMS]. At this point let us review the rigorous asymptotic expansions used in the literature for sharp interface limit results using the method of de Mottoni and Schatzman [deMS] and roughly compare with our expansions. For the latter see the end of this paragraph. We could simply start with our asymptotic expansions but then it is not apparent to the reader why an ansatz is used, what the difficulties are in general and how to adapt to other situations. Nevertheless the review is not needed in order to grasp the expansions in this thesis. In the following reviewed results, the sharp interface is always closed and strictly contained in the domain.

We start with de Mottoni, Schatzman [deMS]. The latter is the first rigorous sharp interface limit result via asymptotic expansions in the case of arbitrary curved geometries. As mentioned before, the Allen-Cahn equation on $\mathbb{R}^N$ is considered, i.e. (up to a scaling in time) equation (AC1) for $\Omega = \mathbb{R}^N$. The fundamental idea is that the zero-level set Γ_ε of the solution u_ε should approximate the sharp interface Γ for small ε. Moreover, the solution should change rapidly across the diffuse interface whose thickness is proportional to ε. For smooth hypersurfaces one can describe the neighbourhood in the surrounding space with the well-known tubular neighbourhood coordinate system via the signed distance and the tangential variable. Hence the idea was to make an ansatz for the solution u_ε close to the sharp interface (inner expansion) with ansatz functions depending on an ε-expansion of the signed distance divided by ε ("rescaled") and an ε-expansion of the tangential coordinate. Note that the ansatz functions for the expansion of the signed distance and tangential coordinate are assumed to depend on space and time. The zero-th order in the distance function and tangential variable is the one of the hypersurface that evolves by MCF. For the outer expansion in this situation, one can simply use the constant functions with the values where the potential takes its minima. The construction in [deMS] becomes very tedious, especially because of the expansion of the tangential coordinate. However, in each ε-order one can basically compute the next order of the distance function first and determine the tangential coordinate afterwards. The latter already gives a hint that there is a more efficient ansatz. We will come to this later.

The next result in line is the one of Alikakos, Bates, Chen [ABC], who consider the more complicated Cahn-Hilliard equation. The situation is considerably more difficult because the equation is of fourth order, a proper outer expansion is needed and also a boundary layer expansion close to the boundary of the domain is necessary. Moreover, blow up for ansatz functions in the traditional matched asymptotic expansions is a problem, cf. [ABC]. Therefore in the work [ABC] a new type of ansatz is invented. Let us call it the "[ABC]-technique". The idea is to use ansatz functions depending on an ε-expansion of the signed distance but, instead of an expansion of the tangential coordinate, the ansatz functions additionally depend on the space variable. This induces some "redundancy" in the ansatz, but also gives great flexibility. In particular it is possible to impose strong matching conditions on a whole transition region and to carry out the expansion rigorously with minimal differential geometric tools. The [ABC]-technique was then used by Caginalp, Chen [CC] for the phase-field equations with several parameter choices which containes the Cahn-Hilliard and the Allen-Cahn equation.

It turns out that the ansatz in [ABC] can be simplified for the phase-field equations for some choices of parameters, including the Allen-Cahn equation. This is done in Caginalp, Chen, Eck [CCE]. The main motivation was to simplify the ansatz such that the first order in the expansion of the distance function vanishes. This can be interpreted in the sense that the "approximate sharp interface" Γ_ε obtained from the level set of the solution to the diffuse interface model is $\mathcal{O}(\varepsilon^2)$-close to the sharp interface Γ. This is particularly interesting for numerical simulations, when the parameter ε cannot be taken arbitrarily small. Compared to the ansatz in [ABC] the higher orders in the expansion of the distance function are assumed to depend only on the tangential variable and time (opposed to space and time; cf. also (5.1) below) and the ansatz functions themselves depend additionally only on the tangential variable (instead of the space coordinate; cf. also (5.2) below). This ansatz requires some differential geometric calculations but is simpler compared to [deMS], [ABC]. It works rigorously and provides the "$\mathcal{O}(\varepsilon^2)$-approximation".

From that point on, the rigorous results in the literature use the [ABC]-technique and simpler variants like that introduced in [CCE]. In particular one can use a reduced ansatz for the distance function as in [CCE] but still allow a secondary dependence on the space variable instead of the tangential coordinate. Since the latter appears several times, let us denote this type of ansatz as the "[ABC]-[CCE]-ansatz" for the following. Which ansatz one should take is not clear a priori and presently relies rather on heuristics and experience than on a systematic treatment.

In Chen, Hilhorst, Logak [CHL] the mass-conserving Allen-Cahn equation is considered and an ansatz in the spirit of [CCE] is used. Schaubeck [Sb] uses the full [ABC]-technique for the results on a Cahn-Larché system and a convective Cahn-Hilliard equation. Abels, Liu [AL] consider a Stokes/Allen-Cahn system. They use an ansatz as in [CCE] for the Allen-Cahn part and an [ABC]-[CCE]-ansatz for the Stokes part. Finally, Marquardt [Ma] studied the sharp interface limit for a Cahn-Hilliard/Stokes system. He uses the [ABC]-[CCE]-ansatz for all parts. The latter indicates that the [ABC]-[CCE]-ansatz should also work for the Cahn-Hilliard equation.

Despite the similarity in the ansatz types, there is unfortunately no unified systematic calculus at the present. For each new equation type one has to separately compute the asymptotic expansions. At least often some of the model problems for the ansatz functions appearing in the expansions are similar, e.g. parameter-dependent ODEs on $\mathbb{R}$ as in Section 4.1. However, especially in cases with couplings very complicated model problems can appear (cf. [AL], Theorem 2.12) and the comments in [Ma], p.11. In [Ma] the problem was circumvented with the idea to use fractional ε-order terms, but this also requires tedious estimates.

Finally, let us roughly compare the above expansions to the ones presented in Sections 5.1-5.4. The major difference is that in the results reviewed above the sharp interface is closed and strictly contained in the domain, where in this thesis we consider interfaces with boundary contact. This requires asymptotic expansions at the contact points. In all the above mentioned expansions in the literature the usual tubular neighbourhood coordinate system is used which is the canonical choice for a closed surface. This might also work in the contact angle case, when one uses a smooth extension of the sharp interface. But the latter coordinate system does not perceive the curvilinear boundary of the domain. Hence setting up an asymptotic expansion in this way is uncomfortable and very tedious. Therefore we build up the asymptotic expansions on the curvilinear coordinates obtained in Section 3 which turns out to be fairly efficient. For the inner expansions (i.e. valid close to the interface but away from the contact points) we use a relatively simple ansatz in the spirit of [CCE]. The calculations for the inner expansions are analogous to (parts of) [CCE], [CHL], [AL], but new terms appear due to the curvilinear coordinates. For the contact point expansions we combine this ansatz with a dependence for the ansatz functions on the ε-divided coordinate variable that runs in the normal direction of the boundary of the domain. For the outer expansions we can simply take constant functions with the values where the potential takes its minima. Note that in a straightforward ansatz and construction at the contact points, it seems not possible to control the remainder terms that appear due to the errors in the rigorous Taylor expansions and the non-trivial asymptotic behaviour of the ansatz functions. The latter arises because the contact point expansion has to be matched appropriately with the inner expansion in order to glue them together in the end. Instead, we use an efficient ansatz for the contact point expansion that also simplifies the matching procedure. More precisely, we simply add the inner expansion terms suitably in the contact point expansion. For (AC) and (vAC) the latter are just summed up, for (AC$_\alpha$) we cut off with suitable ε-scaled cut-off functions. It turns out that this way the remaining higher order ansatz functions can be enforced to be exponentially decaying and that the remainder estimates work rigorously. This solves the problems with the diverging terms in a direct ansatz and the matching is simpler. Finally, note that the explicit identities for the coordinates obtained in Theorem 3.3 and Theorem 3.7 are crucial for the construction to work.

5.1 Asymptotic Expansion of (AC) in 2D

Let $N = 2$, $\Omega \subseteq \mathbb{R}^N$ be as in Remark 1.1, 1. and $\Gamma := (\Gamma_t)_{t\in[0,T]}$ be as in Section 3.1 with contact angle $\alpha = \frac{\pi}{2}$. In the following we use the same notation as in Sections 3.1-3.2. Moreover, let $\delta > 0$ be such that the assertions of Theorem 3.3 hold for 2δ instead of δ. In particular $(r, s) : \overline{\Gamma(2\delta)} \to [-2\delta, 2\delta] \times I$, where $I = [-1, 1]$, are curvilinear coordinates that describe a neighbourhood $\overline{\Gamma(2\delta)}$ of Γ in $\overline{\Omega} \times [0, T]$. Here r has the role of a signed distance function and s is like a tangential projection. Finally, we assume that Γ evolves according to MCF. Based on Γ we construct a smooth approximate solution u^A_ε to (AC1)-(AC3) with $u^A_\varepsilon = \pm 1$ in $Q^\pm_T \setminus \Gamma(2\delta)$, increasingly steep "transition" from -1 to 1 for $\varepsilon \to 0$ and such that $\{u^A_\varepsilon = 0\}$ "converges" to Γ for $\varepsilon \to 0$. Actually the latter will be valid in the sense that the maximal distance goes to zero.

Therefore let $M \in \mathbb{N}$ with $M \geq 2$. For $j = 1, ..., M$ we introduce height functions

$$h_j : I \times [0, T] \to \mathbb{R} \quad \text{and} \quad h_\varepsilon := \sum_{j=1}^{M} \varepsilon^{j-1} h_j.$$

Furthermore, we define a "stretched variable" (in the spirit of Chen, Caginalp, Eck [CCE]):

$$\rho_\varepsilon(x, t) := \frac{r(x, t)}{\varepsilon} - h_\varepsilon(s(x, t), t) \quad \text{for } (x, t) \in \overline{\Gamma(2\delta)}. \tag{5.1}$$

The idea is that $\{\rho_\varepsilon = 0\}$ should be close to $\{u_\varepsilon^A = 0\}$ and it should approximate the zero-level set Γ_ε of the exact solution u_ε. Here $\varepsilon \max|h_\varepsilon|$ is approximately the error from $\{\rho_\varepsilon = 0\}$ to $\Gamma = \{r = 0\}$ in normal direction.

In Section 5.1.1 we construct the inner expansion which is supposed to hold close to Γ but away from $\partial\Gamma$ and in Section 5.1.2 we consider the contact point expansion that is expected to be valid close to $\partial\Gamma$. It will turn out that our construction yields a suitable approximate solution u_ε^A to (AC1)-(AC3), see Section 5.1.3 below. Here the parameter M will correspond to the order of the approximation error.

5.1.1 Inner Expansion of (AC) in 2D

For the inner expansion we consider the following ansatz: Let $\varepsilon > 0$ be small and

$$u_\varepsilon^I := \sum_{j=0}^{M+1} \varepsilon^j u_j^I, \quad u_j^I(x,t) := \hat{u}_j^I(\rho_\varepsilon(x,t), s(x,t), t) \quad \text{for } (x,t) \in \overline{\Gamma(2\delta)}, \tag{5.2}$$

where

$$\hat{u}_j^I : \mathbb{R} \times I \times [0,T] \to \mathbb{R} : (\rho, s, t) \mapsto \hat{u}_j^I(\rho, s, t)$$

for $j = 0, ..., M+1$ and we set $\hat{u}_\varepsilon^I := \sum_{j=0}^{M+1} \varepsilon^j \hat{u}_j^I$. We will substitute u_ε^I into the Allen-Cahn equation (AC1) while ignoring the Neumann boundary condition (AC2) and expand it into ε-series with coefficients in (ρ_ε, s, t) up to $\mathcal{O}(\varepsilon^{M-1})$ in order to fulfil the equation up to some $\mathcal{O}(\varepsilon^M)$-error. Subsequently, the requirement that each coefficient vanishes will yield (s,t)-dependent ODEs in ρ for the $\hat{u}_j^I$, $j = 0, ..., M+1$. At the lowest order we will obtain the ODE (4.1) and in higher orders we get ODEs of type (4.2). Therefore due to Theorem 4.1 we will take the optimal profile θ_0 for the lowest order $\hat{u}_0^I$. Moreover, the solvability condition from Theorem 4.4 for the ODEs appearing in the higher orders will yield that MCF for Γ is necessary for our construction, and that the height functions h_j should satisfy non-autonomous linear parabolic PDEs. The boundary conditions for the h_j will be derived from the contact point expansion in the next Section 5.1.2. The initial values for the h_j will be chosen in a compatible way.

In order to carry out the expansions we need to know how the differential operators act on u_ε^I:

Lemma 5.1. *Let $\varepsilon > 0$, $\hat{w} : \mathbb{R} \times I \times [0,T] \to \mathbb{R}$ be sufficiently smooth and $w : \overline{\Gamma(2\delta)} \to \mathbb{R}$ be defined by $w(x,t) := \hat{w}(\rho_\varepsilon(x,t), s(x,t), t)$ for all $(x,t) \in \overline{\Gamma(2\delta)}$. Then*

$$\begin{aligned}
\partial_t w &= \partial_\rho \hat{w} \left[\frac{\partial_t r}{\varepsilon} - (\partial_t h_\varepsilon + \partial_t s \partial_s h_\varepsilon)\right] + \partial_s \hat{w}\, \partial_t s + \partial_t \hat{w}, \\
\nabla w &= \partial_\rho \hat{w} \left[\frac{\nabla r}{\varepsilon} - \nabla s \partial_s h_\varepsilon\right] + \partial_s \hat{w}\, \nabla s, \\
\Delta w &= \partial_\rho \hat{w} \left[\frac{\Delta r}{\varepsilon} - (\Delta s \partial_s h_\varepsilon + |\nabla s|^2 \partial_s^2 h_\varepsilon)\right] + \partial_s \hat{w}\, \Delta s + \partial_s^2 \hat{w}\, |\nabla s|^2 \\
&\quad + 2\partial_\rho \partial_s \hat{w}\, \nabla s \cdot \left[\frac{\nabla r}{\varepsilon} - \nabla s \partial_s h_\varepsilon\right] + \partial_\rho^2 \hat{w} \left|\frac{\nabla r}{\varepsilon} - \nabla s \partial_s h_\varepsilon\right|^2,
\end{aligned}$$

where the w-terms on the left hand side and derivatives of r or s are evaluated at $(x,t) \in \overline{\Gamma(2\delta)}$, the h_ε-terms at $(s(x,t),t)$ and the $\hat{w}$-terms at $(\rho_\varepsilon(x,t), s(x,t), t)$.

Proof. This follows directly from the chain rule. □

To expand the Allen-Cahn equation $\partial_t u^I_\varepsilon - \Delta u^I_\varepsilon + \frac{1}{\varepsilon^2} f'(u^I_\varepsilon) = 0$ into ε-series, we use Taylor expansions. First let us consider the f'-part. If the u^I_j are bounded, then

$$f'(u^I_\varepsilon) = f'(u^I_0) + \sum_{k=1}^{M+2} \frac{f^{(k+1)}(u^I_0)}{k!} \left[\sum_{j=1}^{M+1} u^I_j \varepsilon^j \right]^k + \mathcal{O}(\varepsilon^{M+3}) \quad \text{on } \overline{\Gamma(2\delta)}. \tag{5.3}$$

The terms in the ε-expansion that are needed explicitly are

$$\begin{aligned} \mathcal{O}(1): &\quad f'(u^I_0), \\ \mathcal{O}(\varepsilon): &\quad f''(u^I_0)u^I_1, \\ \mathcal{O}(\varepsilon^2): &\quad f''(u^I_0)u^I_2 + \frac{f^{(3)}(u^I_0)}{2!}(u^I_1)^2. \end{aligned}$$

For $k = 3, ..., M+1$ the order $\mathcal{O}(\varepsilon^k)$ is given by

$$\begin{aligned} \mathcal{O}(\varepsilon^k): \quad f''(u^I_0)u^I_k + &\quad [\text{some polynomial in } (u^I_1, ..., u^I_{k-1}) \text{ of order } \le k, \text{ where the} \\ &\quad \text{coefficients are multiples of } f^{(3)}(u^I_0), ..., f^{(k+1)}(u^I_0) \\ &\quad \text{and every term contains a } u^I_j\text{-factor}]. \end{aligned}$$

Let $u^I_{M+2} := 0$. Then the latter also holds for $k = M+2$. The other explicit terms in (5.3) are of order $\mathcal{O}(\varepsilon^{M+3})$.

The derivatives of r and s are functions of $(x,t) \in \overline{\Gamma(2\delta)}$ and we will expand them into ε-series with a Taylor expansion via $r(x,t) = \varepsilon(\rho_\varepsilon(x,t) + h_\varepsilon(s(x,t),t))$ for $(x,t) \in \overline{\Gamma(2\delta)}$. Then we replace ρ_ε by an arbitrary $\rho \in \mathbb{R}$ in order to get ODEs on $\mathbb{R}$. But later we just use the expansion rigorously for $r = \varepsilon(\rho_\varepsilon + h_\varepsilon) \in [-2\delta, 2\delta]$. Therefore let $g : \overline{\Gamma(2\delta)} \to \mathbb{R}$ be a smooth function. Then the Taylor expansion yields for $r \in [-2\delta, 2\delta]$ uniformly in (s,t):

$$\tilde{g}(r,s,t) := g(\overline{X}(r,s,t)) = \sum_{k=0}^{M+2} \frac{\partial_r^k \tilde{g}|_{(0,s,t)}}{k!} r^k + \mathcal{O}(|r|^{M+3}). \tag{5.4}$$

Only the first few terms in the ε-expansion are needed explicitly. These are

$$\begin{aligned} \mathcal{O}(1): &\quad g|_{\overline{X}_0(s,t)}, \\ \mathcal{O}(\varepsilon): &\quad (\rho + h_1(s,t))\partial_r \tilde{g}|_{(0,s,t)}, \\ \mathcal{O}(\varepsilon^2): &\quad h_2(s,t)\partial_r \tilde{g}|_{(0,s,t)} + (\rho + h_1(s,t))^2 \frac{\partial_r^2 \tilde{g}|_{(0,s,t)}}{2}. \end{aligned}$$

For $k = 3, ..., M$ the order $\mathcal{O}(\varepsilon^k)$ is

$$\begin{aligned} \mathcal{O}(\varepsilon^k): \quad h_k \partial_r \tilde{g}|_{(0,s,t)} &+ \frac{\partial_r^2 \tilde{g}|_{(0,s,t)}}{2} 2(\rho + h_1(s,t))h_{k-1}(s,t) \\ &+ [\text{some polynomial in } (\rho, h_1(s,t), ..., h_{k-2}(s,t)) \text{ of order } \le k, \\ &\quad \text{where the coefficients are multiples of } \partial_r^2 \tilde{g}|_{(0,s,t)}, ..., \partial_r^k \tilde{g}|_{(0,s,t)}]. \end{aligned}$$

Let $h_{M+1} := h_{M+2} := 0$. Then the latter also holds for $k = M+1, M+2$. The other explicit terms in (5.4) are bounded by ε^{M+3} times some polynomial in $|\rho|$ if the h_j are bounded. Later, these terms and the $\mathcal{O}(|r|^{M+3})$-term in (5.4) for each choice of g will be multiplied with terms that decay exponentially in $|\rho|$. Then these remainder terms will become $\mathcal{O}(\varepsilon^{M+3})$.

For the higher orders in the expansion the following definition is useful:

Definition 5.2 (Notation for Inner Expansion of (AC) in 2D). 1. We call (θ_0, u_1^I) the *zero-th inner order* and (h_j, u_{j+1}^I) the *j-th inner order* for $j = 1, ..., M$.

2. Let $k \in \{-1, ..., M+2\}$. We denote with P_k^I the set of polynomials in ρ with smooth coefficients in $(s,t) \in I \times [0,T]$ depending only on the h_j for $1 \leq j \leq \min\{k, M\}$.

3. Let $k \in \{-1, ..., M+2\}$ and $\beta > 0$. We denote with $R_{k,(\beta)}^I$ the set of smooth functions $R : \mathbb{R} \times I \times [0,T] \to \mathbb{R}$ that depend only on the j-th inner orders for $0 \leq j \leq \min\{k, M\}$ and satisfy uniformly in (ρ, s, t):

$$|\partial_\rho^i \partial_s^l \partial_t^n R(\rho, s, t)| = \mathcal{O}(e^{-\beta|\rho|}) \quad \text{for all } i, l, n \in \mathbb{N}_0.$$

Finally, $\hat{R}_{k,(\beta)}^I$ is defined analogously with functions $R : \mathbb{R} \times [0,T] \to \mathbb{R}$.[10]

Now we expand the Allen-Cahn equation (AC1) for $u_\varepsilon = u_\varepsilon^I$ into ε-series. If we write down an equation or assertion for (ρ, s, t), it is meant to hold for all $(\rho, s, t) \in \mathbb{R} \times I \times [0,T]$. Moreover, we often omit the argument (s,t) in the h_j-terms.

5.1.1.1 Inner Expansion: $\mathcal{O}(\varepsilon^{-2})$ We obtain that the $\mathcal{O}(\frac{1}{\varepsilon^2})$-order is zero if

$$-|\nabla r|^2|_{\overline{X}_0(s,t)} \partial_\rho^2 \hat{u}_0^I(\rho, s, t) + f'(\hat{u}_0^I(\rho, s, t)) = 0.$$

Because of Theorem 3.3 we have $|\nabla r|^2|_{\overline{X}_0(s,t)} = 1$. Moreover, $\{\rho_\varepsilon = 0\}$ should approximate the zero level set of u_ε^I. Hence we require $\hat{u}_0^I(0, s, t) = 0$. Finally, $\hat{u}_0^I$ should connect the values ± 1, i.e. $\lim_{\rho \to \pm\infty} \hat{u}_0^I(\rho, s, t) = \pm 1$. Altogether due to Theorem 4.1 we have to define

$$\hat{u}_0^I(\rho, s, t) := \theta_0(\rho).$$

5.1.1.2 Inner Expansion: $\mathcal{O}(\varepsilon^{-1})$ From the $\partial_t u$-part we get $\frac{1}{\varepsilon}\partial_t r|_{\overline{X}_0(s,t)} \theta_0'(\rho)$ and from Δu:

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\left[\partial_r(|\nabla r|^2 \circ \overline{X})|_{(0,s,t)} \varepsilon(\rho + h_1(s,t))\theta_0''(\rho) + |\nabla r|^2|_{\overline{X}_0(s,t)} \varepsilon \partial_\rho^2 \hat{u}_1^I(\rho, s, t)\right] \\
&+ \frac{1}{\varepsilon}\left[\theta_0'(\rho)\Delta r|_{\overline{X}_0(s,t)} + 2(\nabla r \cdot \nabla s)|_{\overline{X}_0(s,t)}(\partial_s \theta_0'(\rho) - \partial_s h_1(s,t)\theta_0''(\rho))\right] \\
&= \frac{1}{\varepsilon}\left[\partial_\rho^2 \hat{u}_1^I(\rho, s, t) + \theta_0'(\rho)\Delta r|_{\overline{X}_0(s,t)}\right],
\end{aligned}$$

where we used Theorem 3.3. Therefore the $\mathcal{O}(\frac{1}{\varepsilon})$-order cancels if

$$\mathcal{L}_0 \hat{u}_1^I(\rho, s, t) + \theta_0'(\rho)(\partial_t r - \Delta r)|_{\overline{X}_0(s,t)} = 0, \quad \text{where } \mathcal{L}_0 := -\partial_\rho^2 + f''(\theta_0).$$

Because of Theorem 4.4 this parameter-dependent ODE together with $\hat{u}_1^I(0, s, t) = 0$ and boundedness in ρ has a (unique) solution $\hat{u}_1^I$ if and only if $(\partial_t r - \Delta r)|_{\overline{X}_0(s,t)} = 0$. The latter is fulfilled since it is equivalent to MCF for Γ by Theorem 3.3. Thus we define $\hat{u}_1^I := 0$.

[10] Note that this set only appears in the contact point expansion later.

5.1.1.3 Inner Expansion: $\mathcal{O}(\varepsilon^0)$ From $\partial_t u$ we obtain

$$\begin{aligned}
&\frac{1}{\varepsilon}\left[\partial_t r|_{\overline{X}_0(s,t)}\varepsilon\partial_\rho \hat{u}_1^I + \partial_r(\partial_t r\circ\overline{X})|_{(0,s,t)}\varepsilon(\rho+h_1(s,t))\theta_0'(\rho)\right]\\
&+\theta_0'(\rho)\left[-\partial_t h_1(s,t) - \partial_t s|_{\overline{X}_0(s,t)}\partial_s h_1(s,t)\right] + \partial_s\theta_0\partial_t s|_{\overline{X}_0(s,t)} + \partial_t\theta_0(\rho)\\
&=\theta_0'(\rho)\left[(\rho+h_1(s,t))\partial_r(\partial_t r\circ\overline{X})|_{(0,s,t)} - \partial_t h_1(s,t) - \partial_t s|_{\overline{X}_0(s,t)}\partial_s h_1(s,t)\right],
\end{aligned}$$

and from Δu:

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\theta_0''(\rho)\left[\varepsilon^2\frac{1}{2}(\rho+h_1)^2\partial_r^2(|\nabla r|^2\circ\overline{X})|_{(0,s,t)} + \varepsilon^2 h_2\partial_r(|\nabla r|^2\circ\overline{X})|_{(0,s,t)}\right]\\
&+\frac{1}{\varepsilon^2}\partial_\rho^2\hat{u}_1^I\varepsilon^2(\rho+h_1)\partial_r(|\nabla r|^2\circ\overline{X})|_{(0,s,t)} + \frac{1}{\varepsilon^2}|\nabla r|^2|_{\overline{X}_0(s,t)}\varepsilon^2\partial_\rho^2\hat{u}_2^I\\
&+\frac{1}{\varepsilon}\left[\theta_0'(\rho)\varepsilon(\rho+h_1)\partial_r(\Delta r\circ\overline{X})|_{(0,s,t)} + \varepsilon\partial_\rho\hat{u}_1^I\Delta r|_{\overline{X}_0(s,t)}\right]\\
&+\frac{1}{\varepsilon}2\nabla r\cdot\nabla s|_{\overline{X}_0(s,t)}\left[\partial_s\partial_\rho\hat{u}_1^I\varepsilon - \partial_s h_1\varepsilon\partial_\rho^2\hat{u}_1^I - \varepsilon\partial_s h_2\theta_0''(\rho)\right]\\
&+\frac{1}{\varepsilon}2\partial_r((\nabla r\cdot\nabla s)\circ\overline{X})|_{(0,s,t)}\varepsilon(\rho+h_1)\left[\partial_s\theta_0'(\rho) - \partial_s h_1\theta_0''(\rho)\right]\\
&+\Delta s|_{\overline{X}_0(s,t)}\partial_s\theta_0(\rho) + |\nabla s|^2|_{\overline{X}_0(s,t)}\partial_s^2\theta_0(\rho) - 2|\nabla s|^2|_{\overline{X}_0(s,t)}\partial_s h_1\partial_s\theta_0'(\rho)\\
&-\theta_0'(\rho)\left[\Delta s|_{\overline{X}_0(s,t)}\partial_s h_1 + |\nabla s|^2|_{\overline{X}_0(s,t)}\partial_s^2 h_1\right] + |\nabla s|^2|_{\overline{X}_0(s,t)}(\partial_s h_1)^2\theta_0''(\rho),
\end{aligned}$$

which equals due to Theorem 3.3:

$$\begin{aligned}
&\theta_0''(\rho)\left[\frac{1}{2}(\rho+h_1)^2\partial_r^2(|\nabla r|^2\circ\overline{X})|_{(0,s,t)} + (\partial_s h_1)^2|\nabla s|^2|_{\overline{X}_0(s,t)}\right] + \partial_\rho^2\hat{u}_2^I\\
&+\theta_0''(\rho)2\partial_r((\nabla r\cdot\nabla s)\circ\overline{X})|_{(0,s,t)}(\rho+h_1)(-\partial_s h_1)\\
&-\theta_0'(\rho)\left[\Delta s|_{\overline{X}_0(s,t)}\partial_s h_1 + |\nabla s|^2|_{\overline{X}_0(s,t)}\partial_s^2 h_1\right] + \theta_0'(\rho)(\rho+h_1)\partial_r(\Delta r\circ\overline{X})|_{(0,s,t)}.
\end{aligned}$$

Since $\hat{u}_1^I = 0$, the contribution from the f'-part is $f''(\theta_0)\hat{u}_2^I$. Therefore for the cancellation of the $\mathcal{O}(1)$-term in the expansion for the Allen-Cahn equation we require

$$\begin{aligned}
-\mathcal{L}_0\hat{u}_2^I(\rho,s,t) &= R_1(\rho,s,t), \qquad (5.5)\\
R_1(\rho,s,t) &:= \theta_0'(\rho)\left[(\rho+h_1)\partial_r((\partial_t r-\Delta r)\circ\overline{X})|_{(0,s,t)} - \partial_t h_1 + |\nabla s|^2|_{\overline{X}_0(s,t)}\partial_s^2 h_1\right.\\
&\qquad\left. -(\partial_t s-\Delta s)|_{\overline{X}_0(s,t)}\partial_s h_1\right]\\
&\quad+\theta_0''(\rho)\left[-\frac{1}{2}(\rho+h_1)^2\partial_r^2(|\nabla r|^2\circ\overline{X})|_{(0,s,t)}\right.\\
&\qquad\left.+2(\rho+h_1)\partial_s h_1\partial_r((\nabla r\cdot\nabla s)\circ\overline{X})|_{(0,s,t)} - |\nabla s|^2|_{\overline{X}_0(s,t)}(\partial_s h_1)^2\right].
\end{aligned}$$

If h_1 is smooth, then R_1 is smooth and together with all derivatives decays exponentially in $|\rho|$ uniformly in (s,t) with rate β for any $\beta\in(0,\min\{\sqrt{f''(\pm 1)}\})$ due to Theorem 4.1. Hence Theorem 4.4 yields that there is a unique bounded solution $\hat{u}_2^I$ to (5.5) together with $\hat{u}_2^I(0,s,t)=0$ if and only if $\int_{\mathbb{R}} R_1(\rho,s,t)\theta_0'(\rho)\,d\rho = 0$. Since $\int_{\mathbb{R}}\theta_0'(\rho)\theta_0''(\rho)\,d\rho = 0$ due to integration by parts, the nonlinearities in h_1 drop out and we obtain a linear non-autonomous parabolic equation for h_1 with principal part $\partial_t - |\nabla s|^2|_{\overline{X}_0(s,t)}\partial_s^2$:

$$\partial_t h_1 - |\nabla s|^2|_{\overline{X}_0(s,t)}\partial_s^2 h_1 + a_1\partial_s h_1 + a_0 h_1 = f_0 \quad \text{on } I\times[0,T]. \tag{5.6}$$

Here with

$$\begin{aligned} d_1 &:= \int_{\mathbb{R}} \theta_0'(\rho)^2\,d\rho, & d_2 &:= \int_{\mathbb{R}} \theta_0'(\rho)^2\rho\,d\rho, & d_3 &:= \int_{\mathbb{R}} \theta_0'(\rho)^2\rho^2\,d\rho,\\ d_4 &:= \int_{\mathbb{R}} \theta_0'(\rho)\theta_0''(\rho)\rho\,d\rho, & d_5 &:= \int_{\mathbb{R}} \theta_0'(\rho)\theta_0''(\rho)\rho^2\,d\rho, & d_6 &:= \int_{\mathbb{R}} \theta_0'(\rho)\theta_0''(\rho)\rho^3\,d\rho, \end{aligned}$$

we have set for all $(s,t)\in I\times[0,T]$

$$a_1(s,t) := (\partial_t s - \Delta s)|_{\overline{X}_0(s,t)} - 2\frac{d_4}{d_1}\partial_r((\nabla r\cdot\nabla s)\circ\overline{X})|_{(0,s,t)}, \tag{5.7}$$

$$a_0(s,t) := -\partial_r((\partial_t r - \Delta r)\circ\overline{X})|_{(0,s,t)} + \frac{d_4}{d_1}\partial_r^2(|\nabla r|^2\circ\overline{X})|_{(0,s,t)}, \tag{5.8}$$

$$f_0(s,t) := \frac{d_2}{d_1}\partial_r((\partial_t r - \Delta r)\circ\overline{X})|_{(0,s,t)} - \frac{d_5}{2d_1}\partial_r^2(|\nabla r|^2\circ\overline{X})|_{(0,s,t)}. \tag{5.9}$$

If h_1 is smooth and solves (5.6), then Theorem 4.4 yields a solution $\hat{u}_2^I$ to (5.5) with the decay $\hat{u}_2^I \in R_{1,(\beta)}^I$ for any $\beta\in(0,\min\{\sqrt{f''(\pm1)}\})$.

Remark 5.3. If additionally f is even, then θ_0' is even and θ_0'' is odd. Hence $d_2 = d_5 = 0$ and $f_0 = 0$. Therefore the equation (5.6) for h_1 is homogeneous in this case.

5.1.1.4 Inner Expansion: $\mathcal{O}(\varepsilon^k)$ For $k = 1,...,M-1$ we compute the order $\mathcal{O}(\varepsilon^k)$ in (AC1) for $u_\varepsilon = u_\varepsilon^I$ and derive equations for the $(k+1)$-th inner order. Therefore, for the moment we *assume* that the j-th inner order has already been constructed for $j = 0,...,k$, that it is smooth and $\hat{u}_{j+1}^I \in R_{j,(\beta)}^I$ for every $\beta\in(0,\min\{\sqrt{f''(\pm1)}\})$. This assumption will be fulfilled later, when we apply an induction argument.

Then with the notation in Definition 5.2 it holds for all $\beta\in(0,\min\{\sqrt{f''(\pm1)}\})$:

For $j = 1,...,k+2$: $[\mathcal{O}(\varepsilon^j)$ in (5.3)$] \in f''(u_0^I)u_j^I + R_{j-2,(\beta)}^I \quad [\subseteq R_{j-1,(\beta)}^I$, if $j\le k+1]$,
For $j = 1,...,k+1$: $[\mathcal{O}(\varepsilon^j)$ in (5.4)$] \in \partial_r\tilde{g}|_{(0,s,t)}h_j + P_{j-1}^I \quad [\subseteq P_j^I$, if $j\le k]$,
For $j = 3,...,k+1$: $[\mathcal{O}(\varepsilon^j)$ in (5.4)$] \in \partial_r\tilde{g}|_{(0,s,t)}h_j + \partial_r^2\tilde{g}|_{(0,s,t)}(\rho+h_1)h_{j-1} + P_{j-2}^I$.

Now we consider all terms at order $\mathcal{O}(\varepsilon^k)$ for $k = 1,...,M-1$ in (AC1) for $u_\varepsilon = u_\varepsilon^I$ with the above expansions. Let $\beta\in(0,\min\{\sqrt{f''(\pm1)}\})$ be arbitrary. The $\mathcal{O}(\varepsilon^k)$-order in the term $\frac{1}{\varepsilon}(\partial_t r - \Delta r)\partial_\rho\hat{u}_\varepsilon^I$ is an element of

$$\begin{aligned} (\partial_t r - \Delta r)|_{\overline{X}_0(s,t)}\partial_\rho\hat{u}_{k+1}^I + \sum_{j=1}^k \partial_\rho\hat{u}_{k+1-j}^I P_j^I + \theta_0'(\rho)[\partial_r((\partial_t r - \Delta r)\circ\overline{X})|_{(0,s,t)}h_{k+1} + P_k^I]\\ \subseteq \theta_0'(\rho)\partial_r((\partial_t r - \Delta r)\circ\overline{X})|_{(0,s,t)}h_{k+1} + R_{k,(\beta)}^I, \end{aligned}$$

where we used Theorem 3.3 and that Γ evolves according to MCF. From the expression $\partial_\rho \hat{u}^I_\varepsilon[\partial_t h_\varepsilon - |\nabla s|^2 \partial_s^2 h_\varepsilon + (\partial_t s - \Delta s)\partial_s h_\varepsilon]$ we obtain at order $\mathcal{O}(\varepsilon^k)$ a term in

$$\begin{aligned}
\theta_0'(\rho)[\partial_t h_{k+1} - |\nabla s|^2|_{\overline{X}_0(s,t)}\partial_s^2 h_{k+1} + (\partial_t s - \Delta s)|_{\overline{X}_0(s,t)}\partial_s h_{k+1} + P^I_k] + \sum_{j=1}^k \partial_\rho \hat{u}^I_j P^I_{k+1-j}\\
\subseteq \theta_0'(\rho)[\partial_t h_{k+1} - |\nabla s|^2|_{\overline{X}_0(s,t)}\partial_s^2 h_{k+1} + (\partial_t s - \Delta s)|_{\overline{X}_0(s,t)}\partial_s h_{k+1}] + R^I_{k,(\beta)}.
\end{aligned}$$

The contribution from $\partial_s \hat{u}^I_\varepsilon(\partial_t s - \Delta s) + \partial_t \hat{u}^I_\varepsilon - |\nabla s|^2 \partial_s^2 \hat{u}^I_\varepsilon$ in $\mathcal{O}(\varepsilon^k)$ is $R^I_{k,(\beta)}$. Moreover, the $\mathcal{O}(\varepsilon^k)$-order in $\frac{1}{\varepsilon^2}\partial_\rho^2 \hat{u}^I_\varepsilon |\nabla r|^2$ is an element of

$$\begin{aligned}
&\theta_0''(\rho)[\partial_r(|\nabla r|^2 \circ \overline{X})|_{(0,s,t)} h_{k+2} + \partial_r^2(|\nabla r|^2 \circ \overline{X})|_{(0,s,t)}(\rho + h_1)h_{k+1} + P^I_k]\\
&+ 0 \cdot P^I_{k+1} + \sum_{j=1}^k \partial_\rho^2 \hat{u}^I_{k+2-j} P^I_j + \partial_\rho^2 \hat{u}^I_{k+2}|\nabla r|^2|_{\overline{X}_0(s,t)}\\
&\subseteq \theta_0''(\rho)\partial_r^2(|\nabla r|^2 \circ \overline{X})|_{(0,s,t)}(\rho + h_1)h_{k+1} + \partial_\rho^2 \hat{u}^I_{k+2} + R^I_{k,(\beta)},
\end{aligned}$$

where we used $\hat{u}^I_1 = 0$ and Theorem 3.3. From $\frac{2}{\varepsilon}\partial_s h_\varepsilon \nabla r \cdot \nabla s\, \partial_\rho^2 \hat{u}^I_\varepsilon$ we obtain a term in

$$\begin{aligned}
&2\partial_s h_1[\partial_r((\nabla r \cdot \nabla s) \circ \overline{X})|_{(0,s,t)} h_{k+1} + P^I_k]\theta_0''(\rho) + \partial_s h_1 R^I_{k,(\beta)} + R^I_{k,(\beta)}\\
&+ 2\partial_s h_{k+1}[\partial_r((\nabla r \cdot \nabla s) \circ \overline{X})|_{(0,s,t)}(\rho + h_1)]\theta_0''(\rho)\\
&\subseteq 2\partial_r((\nabla r \cdot \nabla s) \circ \overline{X})|_{(0,s,t)}[\partial_s h_1 h_{k+1} + \partial_s h_{k+1}(\rho + h_1)] + R^I_{k,(\beta)}.
\end{aligned}$$

Furthermore, the contribution of $\partial_\rho^2 \hat{u}^I_\varepsilon |\nabla s|^2 (\partial_s h_\varepsilon)^2$ in $\mathcal{O}(\varepsilon^k)$ is an element of

$$2\partial_s h_1 \partial_s h_{k+1} |\nabla s|^2|_{\overline{X}_0(s,t)}\theta_0''(\rho) + R^I_{k,(\beta)}.$$

Since $\partial_s \theta_0' = 0$, the term we get from $2\partial_\rho \partial_s \hat{u}^I_\varepsilon \nabla s \cdot \left[\frac{\nabla r}{\varepsilon} - \nabla s \partial_s h_\varepsilon\right]$ is contained in $R^I_{k,(\beta)}$. Finally, $\frac{1}{\varepsilon^2} f'(\hat{u}^I_\varepsilon)$ contributes a term in $f''(\theta_0)\hat{u}^I_{k+2} + R^I_{k,(\beta)}$ at order $\mathcal{O}(\varepsilon^k)$.

Altogether the $\mathcal{O}(\varepsilon^k)$-order in (AC1) for $u_\varepsilon = u^I_\varepsilon$ for $k \in \{1, ..., M-1\}$ cancels if

$$\begin{aligned}
-\mathcal{L}_0 \hat{u}^I_{k+2}(\rho, s, t) &= R_{k+1}(\rho, s, t), \qquad (5.10)\\
R_{k+1}(\rho, s, t) := \theta_0'(\rho) &\Big[-\partial_t h_{k+1} + |\nabla s|^2|_{\overline{X}_0(s,t)}\partial_s^2 h_{k+1} - (\partial_t s - \Delta s)|_{\overline{X}_0(s,t)}\partial_s h_{k+1}\\
&\quad + h_{k+1}\partial_r((\partial_t r - \Delta r) \circ \overline{X})|_{(0,s,t)}\Big]\\
+\theta_0''(\rho) &\Big[-(\rho + h_1)h_{k+1}\partial_r^2(|\nabla r|^2 \circ \overline{X})|_{(0,s,t)} - 2|\nabla s|^2|_{\overline{X}_0(s,t)}\partial_s h_1 \partial_s h_{k+1}\\
&\quad + 2[(\rho + h_1)\partial_s h_{k+1} + h_{k+1}\partial_s h_1]\partial_r((\nabla r \cdot \nabla s) \circ \overline{X})|_{(0,s,t)}\Big]\\
+\tilde{R}_k(\rho, s, t)&,
\end{aligned}$$

where $\tilde{R}_k \in R^I_{k,(\beta)}$. If h_{k+1} is smooth, then due to Theorem 4.4 equation (5.10) admits a unique bounded solution $\hat{u}^I_{k+2}$ with $\hat{u}^I_{k+2}(0, s, t) = 0$ if and only if $\int_{\mathbb{R}} R_{k+1}(\rho, s, t)\theta_0'(\rho)\, d\rho = 0$. Because of $\int_{\mathbb{R}} \theta_0''\theta_0' = 0$ the latter is equivalent to

$$\partial_t h_{k+1} - |\nabla s|^2|_{\overline{X}_0(s,t)}\partial_s^2 h_{k+1} + a_1 \partial_s h_{k+1} + a_0 h_{k+1} = f_k, \qquad (5.11)$$

where

$$f_k(s,t) := \int_{\mathbb{R}} \tilde{R}_k(\rho,s,t)\theta_0'(\rho)\,d\rho \frac{1}{\|\theta_0'\|^2_{L^2(\mathbb{R})}}$$

is a smooth function of (s,t) and depends only on the j-th inner orders for $0 \leq j \leq k$. Here a_0, a_1 are defined in (5.7)-(5.8). If h_{k+1} is smooth and solves (5.11), then we obtain from Theorem 4.4 a solution $\hat{u}^I_{k+2}$ to (5.10) such that $\hat{u}^I_{k+2} \in R^I_{k+1,(\beta)}$ for all $\beta \in (0, \min\{\sqrt{f''(\pm 1)}\})$.

Remark 5.4. For the explicit computation of order $\mathcal{O}(\varepsilon)$ cf. [AM], Section 3.1.4. This yields

$$\begin{aligned}\tilde{R}_1(\rho,s,t) = \theta_0'(\rho)\Bigg[&(\rho+h_1)[\partial_s^2 h_1 \partial_r(|\nabla s|^2 \circ \overline{X})|_{(0,s,t)} - \partial_s h_1 \partial_r((\partial_t s - \Delta s)\circ \overline{X})|_{(0,s,t)}] \\ &+ \frac{(\rho+h_1)^2}{2}\partial_r^2((\partial_t r - \Delta r)\circ \overline{X})|_{(0,s,t)}\Bigg] \\ +\theta_0''(\rho)\Bigg[&-\frac{(\rho+h_1)^3}{3!}\partial_r^3(|\nabla r|^2 \circ \overline{X})|_{(0,s,t)} - \frac{(\rho+h_1)^2}{2}\partial_r^2((\nabla r\cdot\nabla s)\circ \overline{X})|_{(0,s,t)} \\ &-(\rho+h_1)(\partial_s h_1)^2 \partial_r(|\nabla s|^2 \circ \overline{X})|_{(0,s,t)}\Bigg]\end{aligned}$$

as well as

$$\begin{aligned}f_1(s,t) =& \left(\frac{d_2}{d_1}+h_1\right)\left[\partial_s^2 h_1 \partial_r(|\nabla s|^2\circ \overline{X})|_{(0,s,t)} - \partial_s h_1 \partial_r((\partial_t s - \Delta s)\circ \overline{X})|_{(0,s,t)}\right] \\ &+ \frac{1}{2d_1}\left[d_3 + 2d_2 h_1 + h_1^2\right]\partial_r^2((\partial_t r - \Delta r)\circ \overline{X})|_{(0,s,t)} \\ &- \frac{1}{3!d_1}[d_6 + 3d_5 h_1 + 3d_4 h_1^2]\partial_r^3(|\nabla r|^2 \circ \overline{X})|_{(0,s,t)} \\ &- \frac{1}{d_1}[d_5 + 2d_4 h_1 + d_3 h_1^2]\frac{\partial_r^2((\nabla r\cdot \nabla s)\circ \overline{X})|_{(0,s,t)}}{2} - \frac{d_4}{d_1}(\partial_s h_1)^2 \partial_r(|\nabla s|^2 \circ \overline{X})|_{(0,s,t)}.\end{aligned}$$

In particular, the equations for h_2 and $\hat{u}^I_3$ are not homogeneous in general.

5.1.2 Contact Point Expansion of (AC) in 2D

In the contact point expansion we make the ansatz $u_\varepsilon = u^I_\varepsilon + u^{C\pm}_\varepsilon$ in $\Gamma(2\delta)$ near the contact points $p^\pm(t), t \in [0,T]$. For $u^{C\pm}_\varepsilon$ we combine the stretched-variable ansatz in the last Section 5.1.1 with scaling the tangential variable: With $s^\pm := \mp(s \mp 1)$ and $H^\pm_\varepsilon := \frac{s^\pm}{\varepsilon}$ we set

$$u^{C\pm}_\varepsilon := \sum_{j=1}^M \varepsilon^j u^{C\pm}_j, \quad u^{C\pm}_j(x,t) := \hat{u}^{C\pm}_j(\rho_\varepsilon(x,t), H^\pm_\varepsilon(x,t), t) \quad \text{for } (x,t) \in \overline{\Gamma(2\delta)},$$

where

$$\hat{u}^{C\pm}_j : \overline{\mathbb{R}^2_+} \times [0,T] \to \mathbb{R} : (\rho, H, t) \mapsto \hat{u}^{C\pm}_j(\rho, H, t)$$

for $j = 1, ..., M$. Moreover, we set $\hat{u}^{C\pm}_\varepsilon := \sum_{j=1}^M \varepsilon^j \hat{u}^{C\pm}_j$. To simplify the asymptotic expansion, we remark that later u^I_ε should solve the equation $\partial_t u^I_\varepsilon - \Delta u^I_\varepsilon + f'(u^I_\varepsilon)/\varepsilon^2 = 0$ approximately. Therefore instead of (AC1) for $u_\varepsilon = u^I_\varepsilon + u^{C\pm}_\varepsilon$, we will expand the difference

$$\partial_t u^{C\pm}_\varepsilon - \Delta u^{C\pm}_\varepsilon + \frac{1}{\varepsilon^2}\left[f'(u^I_\varepsilon + u^{C\pm}_\varepsilon) - f'(u^I_\varepsilon)\right] = 0 \tag{5.12}$$

into ε-series with coefficients in $(\rho_\varepsilon, H_\varepsilon^\pm, t)$. Equation (5.12) will be referred to as the "bulk equation" later. Here compared to the inner expansion we will only expand up to $\mathcal{O}(\varepsilon^{M-2})$ which later turns out to be sufficient. Moreover, we will expand the Neumann boundary condition (AC2) for $u_\varepsilon = u_\varepsilon^I + u_\varepsilon^{C\pm}$ into ε-series with coefficients in (ρ_ε, t) up to $\mathcal{O}(\varepsilon^{M-1})$. For the latter the lowest order $\mathcal{O}(\frac{1}{\varepsilon})$ will vanish due to the 90°-contact angle condition. The successive requirement that the coefficients in the expansions disappear yield t-dependent equations on $\mathbb{R}^2_+$ of type as in Subsection 4.2.1 (up to a t-dependent scaling in H). The corresponding solvability condition (4.5) will give us the boundary conditions for the height functions h_j.

In the following lemma we compute how the differential operators act on $u_\varepsilon^{C\pm}$.

Lemma 5.5. *Let $\overline{\mathbb{R}^2_+} \times [0,T] \ni (\rho, H, t) \mapsto \hat{w}(\rho, H, t) \in \mathbb{R}$ be sufficiently smooth and let $w : \overline{\Gamma(2\delta)} \to \mathbb{R}$ be defined by $w(x,t) := \hat{w}(\rho_\varepsilon(x,t), H_\varepsilon^\pm(x,t), t)$ for all $(x,t) \in \overline{\Gamma(2\delta)}$. Then*

$$\partial_t w = \partial_\rho \hat{w}\left[\frac{\partial_t r}{\varepsilon} - (\partial_t h_\varepsilon + \partial_t s \partial_s h_\varepsilon)\right] \mp \partial_H \hat{w}\frac{\partial_t s}{\varepsilon} + \partial_t \hat{w},$$
$$\nabla w = \partial_\rho \hat{w}\left[\frac{\nabla r}{\varepsilon} - \nabla s \partial_s h_\varepsilon\right] \mp \partial_H \hat{w}\frac{\nabla s}{\varepsilon},$$
$$\Delta w = \partial_\rho \hat{w}\left[\frac{\Delta r}{\varepsilon} - (\Delta s \partial_s h_\varepsilon + |\nabla s|^2 \partial_s^2 h_\varepsilon)\right] \mp \partial_H \hat{w}\frac{\Delta s}{\varepsilon} + \partial_H^2 \hat{w}\frac{|\nabla s|^2}{\varepsilon^2}$$
$$\mp 2\partial_\rho \partial_H \hat{w}\,\frac{\nabla s}{\varepsilon} \cdot \left[\frac{\nabla r}{\varepsilon} - \nabla s \partial_s h_\varepsilon\right] + \partial_\rho^2 \hat{w}\left|\frac{\nabla r}{\varepsilon} - \nabla s \partial_s h_\varepsilon\right|^2,$$

where the w-terms on the left hand side and derivatives of r or s are evaluated at (x,t), the h_ε-terms at $(s(x,t),t)$ and the $\hat{w}$-terms at $(\rho_\varepsilon(x,t), H_\varepsilon^\pm(x,t), t)$.

Proof. This can be directly shown using the chain rule. □

5.1.2.1 Contact Point Expansion: The Bulk Equation In (5.12) we have to expand the f'-part: If the $u_j^I, u_j^{C\pm}$ are bounded, we apply a Taylor expansion to obtain on $\overline{\Gamma(2\delta)}$ with $u_{M+1}^{C\pm} := 0$

$$f'(u_\varepsilon^I + u_\varepsilon^{C\pm}) = f'(\theta_0) + \sum_{k=1}^{M+2} \frac{1}{k!} f^{(k+1)}(\theta_0)\left[\sum_{j=1}^{M+1} \varepsilon^j (u_j^I + u_j^{C\pm})\right]^k + \mathcal{O}(\varepsilon^{M+3}). \quad (5.13)$$

Combining this with the expansion for $f'(u_\varepsilon^I)$ in (5.3) and using $u_1^I = 0$, the terms in the asymptotic expansion for $f'(u_\varepsilon^I + u_\varepsilon^{C\pm}) - f'(u_\varepsilon^I)$ are for $k = 1, ..., M+1$:

$\mathcal{O}(1)$: 0,

$\mathcal{O}(\varepsilon)$: $f''(\theta_0) u_1^{C\pm}$,

$\mathcal{O}(\varepsilon^k)$: $f''(\theta_0) u_k^{C\pm} +$ [some polynomial in $(u_1^I, ..., u_{k-1}^I, u_1^{C\pm}, ..., u_{k-1}^{C\pm})$ of order $\leq k$, where the coefficients are multiples of $f^{(3)}(\theta_0), ..., f^{(k+1)}(\theta_0)$ and every term contains a $u_j^{C\pm}$-factor].

Let $u_{M+2}^{C\pm} := 0$. Then the latter is also valid for $k = M+2$. The other explicit terms in $f'(u_\varepsilon^I + u_\varepsilon^{C\pm}) - f'(u_\varepsilon^I)$ are of order $\mathcal{O}(\varepsilon^{M+3})$.

Moreover, we have to expand terms in (5.12) appearing due to Lemma 5.5 that depend on (s,t) or (ρ, s, t), i.e. all the h_j-terms as well as the u_j^I-terms from the f'-expansion, respectively.

Therefore let $g_1 : I \times [0,T] \to \mathbb{R}$ or $g_1 : \mathbb{R} \times I \times [0,T] \to \mathbb{R}$ be smooth with bounded derivatives in s. Since $s = \pm 1 \mp \varepsilon H_\varepsilon^\pm$, we apply a Taylor expansion to a smooth extension to get uniformly

$$g_1|_{s=\pm 1\mp\varepsilon H} = g_1|_{s=\pm 1} + \sum_{k=1}^{M+2} (\mp\varepsilon H)^k \frac{\partial_s^k g_1|_{s=\pm 1}}{k!} + \mathcal{O}((\varepsilon H)^{M+3}) \quad \text{for } H \in [0, \frac{1}{\varepsilon}]. \quad (5.14)$$

Furthermore, we expand the terms depending on (x,t), i.e. all the derivatives of r and s. To this end let $g_2 : \overline{\Gamma(2\delta)} \to \mathbb{R}$ be smooth, then a Taylor expansion yields

$$\tilde{g}_2(r,s,t) := g_2(\overline{X}(r,s,t)) = \sum_{j+k=0}^{M+2} \frac{\partial_r^j \partial_s^k \tilde{g}_2|_{(0,\pm 1,t)}}{j!\,k!} r^j (s\mp 1)^k + \mathcal{O}(|(r, s\mp 1)|^{M+3}) \quad (5.15)$$

uniformly in $(r,s,t) \in [-2\delta, 2\delta] \times I \times [0,T]$. Later we insert

$$r = \varepsilon(\rho_\varepsilon(x,t) + h_\varepsilon(s(x,t),t)), \quad s = \pm 1 \mp \varepsilon H_\varepsilon^\pm(x,t) \quad \text{for } (x,t) \in \overline{\Gamma(2\delta)}$$

and expand h_ε with (5.14). Then $(\rho_\varepsilon, H_\varepsilon^\pm)$ are replaced by arbitrary $(\rho, H) \in \overline{\mathbb{R}^2_+}$ in order to derive suitable equations. The terms in the resulting expansion are for $k = 1, ..., M+2$:

$$\begin{aligned}
\mathcal{O}(1):&\quad g_2|_{\overline{p}^\pm(t)},\\
\mathcal{O}(\varepsilon):&\quad \partial_r \tilde{g}_2|_{(0,\pm 1,t)}(\rho + h_1|_{(\pm 1,t)}) + \partial_s \tilde{g}_2|_{(0,\pm 1,t)}(\mp H),\\
\mathcal{O}(\varepsilon^k):&\quad [\text{some polynomial in } (\rho, H, \partial_s^l h_j|_{(\pm 1,t)}), l = 0,...,k-1, j = 1,...,k \text{ of order} \le k,\\
&\quad\ \text{where the coefficients are multiples of } \partial_r^{l_1}\partial_s^{l_2}\tilde{g}_2|_{(0,\pm 1,t)}, l_1, l_2 \in \mathbb{N}_0, l_1 + l_2 \le k],
\end{aligned}$$

where we have defined $h_{M+1} = h_{M+2} = 0$ before. Note that $\mathcal{O}(\varepsilon)$ is not even needed explicitly. We just included it for the convenience of the reader. The other explicit terms in (5.14) can be estimated by ε^{M+3} times some polynomial in $(|\rho_\varepsilon|, H_\varepsilon^\pm)$. Later these terms and the remainder in (5.14) will be multiplied with exponentially decaying terms and they become $\mathcal{O}(\varepsilon^{M+3})$.

For the higher orders in the expansion (and also for the expansion of the Neumann boundary condition later) the following notation will be helpful:

Definition 5.6 (Notation for Contact Point Expansion of (AC) in 2D).

1. We call (θ_0, u_1^I) the *zero-th order* and $(h_j, u_{j+1}^I, u_j^{C\pm})$ the *j-th order* for $j = 1, ..., M$.

2. Let $k \in \{-1, ..., M+2\}$. We write $P_k^C(\rho, H)$ for the set of polynomials in (ρ, H) with smooth coefficients in $t \in [0,T]$ depending only on the h_j for $1 \le j \le \min\{k, M\}$. The sets $P_k^C(\rho)$ and $P_k^C(H)$ are defined analogously with (ρ, H) replaced by ρ and H, respectively.

3. Let $k \in \{-1, ..., M+2\}$ and $\beta, \gamma > 0$. Let $R_{k,(\beta,\gamma)}^C$ be the set of smooth functions $R : \overline{\mathbb{R}^2_+} \times [0,T] \to \mathbb{R}$ that depend only on the j-th orders for $0 \le j \le \min\{k, M\}$ and such that uniformly in (ρ, H, t):

$$|\partial_\rho^i \partial_H^l \partial_t^n R(\rho, H, t)| = \mathcal{O}(e^{-(\beta|\rho| + \gamma H)}) \quad \text{for all } i, l, n \in \mathbb{N}_0.$$

The set $R_{k,(\beta)}^C$ is defined analogously without the H-dependence.

Now we expand (5.12) with the above identities into ε-series with coefficients in $(\rho_\varepsilon, H_\varepsilon^\pm, t)$.

5.1.2.1.1 Bulk Equation: $\mathcal{O}(\varepsilon^{-1})$ The lowest order $\mathcal{O}(\frac{1}{\varepsilon})$ in (5.12) vanishes if

$$[-\Delta_t^\pm + f''(\theta_0(\rho))]\hat{u}_1^{C\pm}(\rho, H, t) + 0 \cdot \partial_H\partial_\rho\hat{u}_1^{C\pm}(\rho, H, t) = 0, \tag{5.16}$$

where $\Delta_t^\pm := \partial_\rho^2 + |\nabla s|^2|_{\overline{p}^\pm(t)}\partial_H^2$.

5.1.2.1.2 Bulk Equation: $\mathcal{O}(\varepsilon^{k-1})$ For $k = 1, ..., M-1$ we compute $\mathcal{O}(\varepsilon^{k-1})$ in (5.12) and derive an equation for $\hat{u}_{k+1}^{C\pm}$. Therefore we assume that the j-th order is constructed for all $j = 0, ..., k$, that it is smooth and that $\hat{u}_{j+1}^I \in R_{j,(\beta_1)}^I$ for all $\beta_1 \in (0, \min\{\sqrt{f''(\pm 1)}\})$ (bounded and all derivatives bounded would be enough here) and $\hat{u}_j^{C\pm} \in R_{j,(\beta,\gamma)}^C$ for every $\beta \in (0, \min\{\overline{\beta}(\gamma), \sqrt{f''(\pm 1)}\})$, $\gamma \in (0, \overline{\gamma})$, where $\overline{\beta}, \overline{\gamma}$ are as in Theorem 4.11.

Then with the notation as in Definition 5.6 it holds for all those (β, γ):

$$\begin{aligned}
\text{For } j = 1, ..., k+1: &\quad [\mathcal{O}(\varepsilon^j) \text{ in (5.13) minus (5.3)}] &&\in\quad f''(\theta_0(\rho))\hat{u}_j^{C\pm} + R_{j-1,(\beta,\gamma)}^C,\\
\text{For } i, j = 1, ..., k: &\quad [\mathcal{O}(\varepsilon^j) \text{ in (5.14) for } g_1 = g_1(h_i)] &&\in\quad P_i^C(H),\\
\text{For } j = 0, ..., k: &\quad [\mathcal{O}(\varepsilon^j) \text{ in (5.15)}] &&\in\quad P_j^C(\rho, H).
\end{aligned}$$

Now we compute $\mathcal{O}(\varepsilon^{k-1})$ for $k = 1, ..., M-1$ in (5.12). Let (β, γ) be as above and arbitrary. The f'-part yields a term in $f''(\theta_0(\rho))\hat{u}_{k+1}^{C\pm} + R_{k,(\beta,\gamma)}^C$. From $\frac{1}{\varepsilon}(\partial_t r - \Delta r)\partial_\rho\hat{u}_\varepsilon^{C\pm}$ we obtain a term in

$$\sum_{j=0}^{k} P_j^C(\rho, H)\partial_\rho\hat{u}_{k-j}^{C\pm} \subseteq R_{k,(\beta,\gamma)}^C.$$

Additionally, from $\partial_\rho\hat{u}_\varepsilon^{C\pm}[\partial_t h_\varepsilon + (\partial_t s - \Delta s)\partial_s h_\varepsilon + |\nabla s|^2\partial_s^2 h_\varepsilon]$ we get a contribution in

$$\sum_{j=0}^{k-1}\left[P_{j+1}^C(H) + \sum_{i=0}^{j} P_{i+1}^C(H)P_{j-i}^C(\rho, H)\right]\partial_\rho\hat{u}_{k-1-j}^{C\pm} \subseteq R_{k,(\beta,\gamma)}^C.$$

Analogously, $\frac{1}{\varepsilon}(\partial_t s - \Delta s)\partial_H\hat{u}_\varepsilon^{C\pm}$ and $\partial_t\hat{u}_\varepsilon^{C\pm}$ yield terms in $R_{k,(\beta,\gamma)}^C$. Moreover, the contribution of $\frac{1}{\varepsilon}\partial_s h_\varepsilon[\partial_\rho\partial_H\hat{u}_\varepsilon^{C\pm}|\nabla s|^2 + \partial_\rho^2\hat{u}_\varepsilon^{C\pm}(-2\nabla r \cdot \nabla s + |\nabla s|^2)]$ are elements of

$$\begin{aligned}
&\partial_s h_{k+1}\left[\partial_\rho\partial_H\hat{u}_0^{C\pm}|\nabla s|^2|_{\overline{p}^\pm(t)} + \partial_\rho^2\hat{u}_0^{C\pm}(-2\nabla r \cdot \nabla s|_{\overline{p}^\pm(t)} + |\nabla s|^2|_{\overline{p}^\pm(t)})\right]\\
&+\sum_{j=0}^{k-1}\left[P_{j+1}^C(H) + \sum_{i=0}^{j} P_{i+1}^C(H)P_{j-i}^C(\rho, H)\right](\partial_\rho\partial_H\hat{u}_{k-j}^{C\pm} + \partial_\rho\hat{u}_{k-j}^{C\pm}) \subseteq R_{k,(\beta,\gamma)}^C,
\end{aligned}$$

where we used $\hat{u}_0^{C\pm} = 0$. Finally, from $\frac{1}{\varepsilon^2}[-|\nabla r|^2\partial_\rho^2 - |\nabla s|^2\partial_H^2 \pm 2\nabla r \cdot \nabla s\partial_\rho\partial_H]\hat{u}_\varepsilon^{C\pm}$ we obtain similar as before a term contained in $-\Delta_t^\pm\hat{u}_{k+1}^{C\pm} + R_{k,(\beta,\gamma)}^C$.

Altogether the $\mathcal{O}(\varepsilon^{k-1})$-order in the expansion for the bulk equation (5.12) is zero if

$$[-\Delta_t^\pm + f''(\theta_0)]\hat{u}_{k+1}^{C\pm} = G_k^\pm(\rho, H, t), \tag{5.17}$$

where $G_k^\pm \in R_{k,(\beta,\gamma)}^C$.

Remark 5.7. The order $\mathcal{O}(1)$ is explicitly computed in [AM], Section 3.2.1. This implies:

$$
\begin{aligned}
G_1^{\pm}(\rho, H, t) = & -\frac{f^{(3)}(\theta_0)}{2!}(\hat{u}_1^{C\pm})^2 \pm \partial_H \hat{u}_1^{C\pm}(\partial_t s - \Delta s)|_{\overline{p}^{\pm}(t)} \\
& + (\rho + h_1|_{(\pm 1,t)})\partial_H^2 \hat{u}_1^{C\pm}\partial_r(|\nabla s|^2 \circ \overline{X})|_{(0,\pm 1,t)} \\
& \pm 2\partial_H \partial_\rho \hat{u}_1^{C\pm}\left[|\nabla s|^2|_{\overline{p}^{\pm}(t)}\partial_s h_1|_{(\pm 1,t)} - (\rho + h_1|_{(\pm 1,t)})\partial_r((\nabla r \cdot \nabla s) \circ \overline{X})|_{(0,\pm 1,t)}\right].
\end{aligned}
$$

To complement the equations (5.16) and (5.17) we need boundary conditions. These will be obtained from the expansion of the Neumann boundary condition in the next section.

5.1.2.2 Contact Point Expansion: The Neumann Boundary Condition We look at (AC2) for $u_\varepsilon = u_\varepsilon^I + u_\varepsilon^{C\pm}$, i.e. $N_{\partial\Omega} \cdot \nabla(u_\varepsilon^I + u_\varepsilon^{C\pm})|_{\partial Q_T} = 0$. Lemma 5.1 and Lemma 5.5 yield in $\overline{\Gamma(2\delta)}$

$$
\begin{aligned}
\nabla u_\varepsilon^I|_{(x,t)} &= \partial_\rho \hat{u}_\varepsilon^I|_{(\rho,s,t)}\left[\frac{\nabla r|_{(x,t)}}{\varepsilon} - \nabla s|_{(x,t)}\partial_s h_\varepsilon|_{(s,t)}\right] + \partial_s \hat{u}_\varepsilon^I|_{(\rho,s,t)}\,\nabla s|_{(x,t)}, \\
\nabla u_\varepsilon^{C\pm}|_{(x,t)} &= \partial_\rho \hat{u}_\varepsilon^{C\pm}|_{(\rho,H,t)}\left[\frac{\nabla r|_{(x,t)}}{\varepsilon} - \nabla s|_{(x,t)}\partial_s h_\varepsilon|_{(s,t)}\right] \mp \frac{\nabla s|_{(x,t)}}{\varepsilon}\partial_H \hat{u}_\varepsilon^{C\pm}|_{(\rho,H,t)},
\end{aligned}
$$

where $\rho = \rho_\varepsilon(x,t), H = H_\varepsilon^{\pm}(x,t)$ and $s = s(x,t)$. We evaluate at points $x = X(r, \pm 1, t)$, i.e. $H = 0$ and $s = \pm 1$.

For $g : \overline{\Gamma(2\delta)} \cap \partial Q_T \to \mathbb{R}$ smooth we use an analogous expansion as in (5.4) for $s = \pm 1$:

$$
\tilde{g}(r, \pm 1, t) := g(\overline{X}(r, \pm 1, t)) = \sum_{k=0}^{M+2} \frac{\partial_r^k \tilde{g}|_{(0,\pm 1,t)}}{k!} r^k + \mathcal{O}(|r|^{M+3}). \tag{5.18}
$$

Then we use $r = \varepsilon(\rho_\varepsilon + h_\varepsilon|_{(\pm 1,t)})$ and replace ρ_ε by an arbitrary $\rho \in \mathbb{R}$. Analogous to the inner expansion, the terms in the ε-expansion of (5.18) are for $k = 2, ..., M$:

$$
\begin{aligned}
\mathcal{O}(1): &\quad g|_{\overline{p}^{\pm}(t)}, \\
\mathcal{O}(\varepsilon): &\quad (\rho + h_1|_{(\pm 1,t)})\partial_r \tilde{g}|_{(0,\pm 1,t)}, \\
\mathcal{O}(\varepsilon^k): &\quad h_k|_{(\pm 1,t)}\partial_r \tilde{g}|_{(0,\pm 1,t)} + [\text{a polynomial in } (\rho, h_1|_{(\pm 1,t)}, ..., h_{k-1}|_{(\pm 1,t)}) \text{ of order } \leq k, \\
&\qquad\qquad \text{where coefficients are multiples of } (\partial_r^2 \tilde{g}, ..., \partial_r^k \tilde{g})|_{(0,\pm 1,t)}].
\end{aligned}
$$

The latter also holds for $k = M+1, M+2$ since $h_{M+1} = h_{M+2} = 0$ by definition. The other explicit terms in (5.18) are bounded by ε^{M+3} times some polynomial in $|\rho|$ if the h_j are bounded. Later, these terms and the $\mathcal{O}(|r|^{M+3})$-term in (5.18) for each choice of g will be multiplied with terms that decay exponentially in $|\rho|$. Then these remainder terms will become $\mathcal{O}(\varepsilon^{M+3})$.

In the following we expand the Neumann boundary condition into ε-series with coefficients in (ρ_ε, t) up to $\mathcal{O}(\varepsilon^{M-1})$.

5.1.2.2.1 Neumann Boundary Condition: $\mathcal{O}(\varepsilon^{-1})$ At the lowest order $\mathcal{O}(\frac{1}{\varepsilon})$ we obtain $(N_{\partial\Omega} \cdot \nabla r)|_{\overline{p}^{\pm}(t)}\theta_0'(\rho) = 0$. This holds true because we required a 90°-contact angle.

5.1.2.2.2 Neumann Boundary Condition: $\mathcal{O}(\varepsilon^0)$ The next order $\mathcal{O}(1)$ vanishes if

$$0 \cdot \partial_\rho \hat{u}_1^{C\pm}|_{H=0} + 0 \cdot \partial_\rho \hat{u}_1^I|_{s=\pm 1} + (N_{\partial\Omega} \cdot \nabla s)|_{\overline{p}^\pm(t)} \left[\mp \partial_H \hat{u}_1^{C\pm}|_{H=0} + \partial_s \theta_0(\rho)\right]$$
$$+ \theta_0'(\rho) \left[-(N_{\partial\Omega} \cdot \nabla s)|_{\overline{p}^\pm(t)} \partial_s h_1|_{(\pm 1,t)} + \partial_r((N_{\partial\Omega} \cdot \nabla r) \circ \overline{X})|_{(0,\pm 1,t)}(\rho + h_1|_{(\pm 1,t)})\right] = 0.$$

Therefore we require

$$\mp (N_{\partial\Omega} \cdot \nabla s)|_{\overline{p}^\pm(t)} \partial_H \hat{u}_1^{C\pm}|_{H=0}(\rho, t) = g_1^\pm(\rho, t), \tag{5.19}$$
$$g_1^\pm|_{(\rho,t)} := \theta_0'[(N_{\partial\Omega} \cdot \nabla s)|_{\overline{p}^\pm(t)} \partial_s h_1|_{(\pm 1,t)} - \partial_r((N_{\partial\Omega} \cdot \nabla r) \circ \overline{X})|_{(0,\pm 1,t)} h_1|_{(\pm 1,t)}] + \tilde{g}_0^\pm|_{(\rho,t)},$$

where $\tilde{g}_0^\pm(\rho, t) := -\rho\theta_0'(\rho)\partial_r((N_{\partial\Omega} \cdot \nabla r) \circ \overline{X})|_{(0,\pm 1,t)}$. We define

$$\overline{u}_j^{C\pm} : \overline{\mathbb{R}^2_+} \times [0, T] \to \mathbb{R} : (\rho, H, t) \mapsto \hat{u}_j^{C\pm}(\rho, |\nabla s|(\overline{p}^\pm(t))H, t) \quad \text{for } j = 1, ..., M. \tag{5.20}$$

Then equations (5.16) and (5.19) for $\hat{u}_1^{C\pm}$ are equivalent to

$$[-\Delta + f''(\theta_0(\rho))]\overline{u}_1^{C\pm} = 0, \tag{5.21}$$
$$-\partial_H \overline{u}_1^{C\pm}|_{H=0} = g_1^\pm(\rho, t), \tag{5.22}$$

where we used that $-(|\nabla s|/ \mp N_{\partial\Omega} \cdot \nabla s)|_{\overline{p}^\pm(t)}$ is well-defined and equal to 1 due to Theorem 3.3. Note that equality to 1 is not crucial. Any smooth factor bounded away from zero would be fine and could be treated by scaling the right hand side in (5.22). The corresponding solvability condition (4.5) is $\int_{\mathbb{R}} g_1^\pm(\rho, t)\theta_0'(\rho)\, d\rho = 0$. This gives a linear boundary condition for h_1:

$$b_1^\pm(t)\partial_s h_1|_{(\pm 1,t)} + b_0^\pm(t) h_1|_{(\pm 1,t)} = f_0^\pm(t) \quad \text{for } t \in [0, T], \tag{5.23}$$

where

$$b_1^\pm(t) := (N_{\partial\Omega} \cdot \nabla s)|_{\overline{p}^\pm(t)},$$
$$b_0^\pm(t) := -\partial_r((N_{\partial\Omega} \cdot \nabla r) \circ \overline{X})|_{(0,\pm 1,t)},$$
$$f_0^\pm(t) := \int_{\mathbb{R}} \theta_0'(\rho)\tilde{g}_0^\pm(\rho, t)\, d\rho / \|\theta_0'\|^2_{L^2(\mathbb{R})} = -b_0^\pm(t) \int_{\mathbb{R}} \theta_0'(\rho)^2 \rho\, d\rho / \|\theta_0'\|^2_{L^2(\mathbb{R})}$$

are smooth in $t \in [0, T]$. Together with the linear parabolic equation (5.6) for h_1 from Subsection 5.1.1.3, we have a time-dependent linear parabolic boundary value problem for h_1, where the initial value $h_1|_{t=0}$ is not specified yet.

Remark 5.8. If f is even, then so is θ_0' and hence $f_0^\pm = 0$. Therefore the boundary condition for h_1 is homogeneous and because of Remark 5.3 we can take $h_1 = 0$ in this case.

To obtain a smooth solution of (5.6), (5.23), certain compatibility conditions have to be fulfilled, cf. Lunardi, Sinestrari, von Wahl [LSW], Chapter 9. To overcome this problem, we extend the coefficients and right hand sides smoothly to $[-T, T]$ such that the coefficient in front of $\partial_s^2 h_1$ in (5.6) as well as the modulus of the coefficient in front of $\partial_s h_1$ in (5.23) is bounded from below by a $c_0 > 0$ and the right hand sides are zero for $t \leq -\frac{1}{2}T$. The extension can e.g. be done with the Stein Extension Theorem, see Leoni [Le], Theorem 13.17. Then for the initial value zero at $t = -T$ all compatibility conditions are fulfilled and we obtain a smooth solution on $[-T, T]$ by [LSW], Theorem 9.1. Restriction to $[0, T]$ yields a smooth solution h_1 on $[0, T]$.

Since h_1 is fixed now, we obtain $\hat{u}_2^I$ (solving (5.5)) from Section 5.1.1.3 such that $\hat{u}_2^I \in R^I_{1,(\beta_1)}$ for every $\beta_1 \in (0, \min\{\sqrt{f''(\pm 1)}\})$. Hence the first inner order is computed. Moreover, $g_1^\pm \in \hat{R}^I_{1,(\beta_1)}$ for all β_1 as above due to Theorem 4.1, cf. (5.19). Therefore Theorem 4.11 yields a unique smooth solution $\overline{u}_1^{C\pm}$ to (5.21)-(5.22) with the decay $\overline{u}_1^{C\pm} \in R^C_{1,(\beta,\gamma)}$ for all $\beta \in (0, \min\{\overline{\beta}(\gamma), \sqrt{f''(\pm 1)}\})$, $\gamma \in (0, \overline{\gamma})$, where $\overline{\beta}, \overline{\gamma}$ are as in Theorem 4.11. In particular the first order is determined.

5.1.2.2.3 Neumann Boundary Condition: $\mathcal{O}(\varepsilon^k)$ and Induction For $k = 1, ..., M-1$ we consider $\mathcal{O}(\varepsilon^k)$ in (AC2) for $u_\varepsilon = u_\varepsilon^I + u_\varepsilon^{C\pm}$ and derive equations for the $(k+1)$-th order. Therefore we assume the following induction hypothesis: suppose that the j-th order is constructed for all $j = 0, ..., k$, that it is smooth and that $\hat{u}_{j+1}^I \in R^I_{j,(\beta_1)}$ for all $\beta_1 \in (0, \min\{\sqrt{f''(\pm 1)}\})$ as well as $\hat{u}_j^{C\pm} \in R^C_{j,(\beta,\gamma)}$ for every $\beta \in (0, \min\{\overline{\beta}(\gamma), \sqrt{f''(\pm 1)}\})$, $\gamma \in (0, \overline{\gamma})$, where $\overline{\beta}, \overline{\gamma}$ are as in Theorem 4.11. The assumption is valid for $k = 1$ due to Section 5.1.2.2.2.

With the notation as in Definition 5.6 it holds for $j = 1, ..., k+1$:

$$[\mathcal{O}(\varepsilon^j) \text{ in } (5.18)] \quad \in \partial_r \tilde{g}|_{(0,\pm 1,t)} h_j|_{(\pm 1,t)} + P^C_{j-1}(\rho) \quad [\subseteq P^C_j(\rho), \text{ if } j \leq k].$$

In the following we compute $\mathcal{O}(\varepsilon^k)$ for $k = 1, ..., M-1$ in (AC2) for $u_\varepsilon = u_\varepsilon^I + u_\varepsilon^{C\pm}$. Therefore let (β, γ) be as above and arbitrary. From $\frac{1}{\varepsilon} N_{\partial\Omega} \cdot \nabla r [\partial_\rho \hat{u}_\varepsilon^{C\pm}(\rho, 0, t) + \partial_\rho \hat{u}_\varepsilon^I(\rho, \pm 1, t)]$ we get a term in

$$\begin{aligned} 0 \cdot [\partial_\rho \hat{u}_{k+1}^{C\pm}|_{(\rho,0,t)} + \partial_\rho \hat{u}_{k+1}^I|_{(\rho,\pm 1,t)}] + \sum_{j=1}^k P_j^{C\pm}(\rho) \left[\partial_\rho \hat{u}_{k+1-j}^{C\pm}|_{(\rho,0,t)} + \partial_\rho \hat{u}_{k+1-j}^I|_{(\rho,\pm 1,t)}\right] \\ + \theta_0'(\rho) \left[-b_0^\pm(t) h_{k+1}|_{(\pm 1,t)} + P_k^C(\rho)\right] \subseteq -\theta_0'(\rho) b_0^\pm(t) h_{k+1}|_{(\pm 1,t)} + R^C_{k,(\beta)}, \end{aligned}$$

where we used $(N_{\partial\Omega} \cdot \nabla r)|_{\overline{p}^\pm(t)} = 0$ and $\hat{u}_0^{C\pm} = 0$. Moreover, the contribution of the term $\partial_s h_\varepsilon|_{(\pm 1,t)} N_{\partial\Omega} \cdot \nabla s [\partial_\rho \hat{u}_\varepsilon^{C\pm}(\rho, 0, t) + \partial_\rho \hat{u}_\varepsilon^I(\rho, \pm 1, t)]$ is an element of

$$\begin{aligned} &\partial_s h_{k+1}|_{(\pm 1,t)} N_{\partial\Omega} \cdot \nabla s|_{\overline{p}^\pm(t)} \left[\partial_\rho \hat{u}_0^{C\pm}|_{(\rho,0,t)} + \theta_0'(\rho)\right] \\ &+ \sum_{j=1}^k \partial_s h_{k+1-j}|_{(\pm 1,t)} \sum_{i=0}^j P_i^C(\rho) \left[\partial_\rho \hat{u}_{j-i}^{C\pm}|_{(\rho,0,t)} + \partial_\rho \hat{u}_{j-i}^I|_{(\rho,\pm 1,t)}\right] \\ &\subseteq \partial_s h_{k+1}|_{(\pm 1,t)} N_{\partial\Omega} \cdot \nabla s|_{\overline{p}^\pm(t)} \theta_0'(\rho) + R^C_{k,(\beta)}. \end{aligned}$$

From $\partial_\rho \hat{u}_\varepsilon^I|_{(\rho,\pm 1,t)} N_{\partial\Omega} \cdot \nabla s$ we obtain a term in $\sum_{j=0}^k \partial_s \hat{u}_j^I|_{(\rho,\pm 1,t)} P^C_{k-j}(\rho) \subseteq R^C_{k,(\beta)}$. Finally, $\frac{1}{\varepsilon} \partial_H \hat{u}_\varepsilon^{C\pm}|_{(\rho,0,t)} N_{\partial\Omega} \cdot \nabla s$ yields due to $\hat{u}_0^{C\pm} = 0$:

$$\begin{aligned} \sum_{j=1}^k \partial_H \hat{u}_{k+1-j}^{C\pm}|_{(\rho,0,t)} P_j^C(\rho) + N_{\partial\Omega} \cdot \nabla s|_{\overline{p}^\pm(t)} \partial_H \hat{u}_{k+1}^{C\pm}|_{(\rho,0,t)} \\ \subseteq N_{\partial\Omega} \cdot \nabla s|_{\overline{p}^\pm(t)} \partial_H \hat{u}_{k+1}^{C\pm}|_{(\rho,0,t)} + R^C_{k,(\beta)}. \end{aligned}$$

Therefore the $\mathcal{O}(\varepsilon^k)$-order in the expansion of (AC2) for $u_\varepsilon = u_\varepsilon^I + u_\varepsilon^{C\pm}$ vanishes if

$$\mp (N_{\partial\Omega} \cdot \nabla s)|_{\overline{p}^\pm(t)} \partial_H \hat{u}_{k+1}^{C\pm}|_{H=0}(\rho, t) = g_{k+1}^\pm(\rho, t), \tag{5.24}$$

$$\begin{aligned} g_{k+1}^\pm(\rho, t) := \theta_0'(\rho) \left[(N_{\partial\Omega} \cdot \nabla s)|_{\overline{p}^\pm(t)} \partial_s h_{k+1}|_{(\pm 1,t)} - \partial_r((N_{\partial\Omega} \cdot \nabla r) \circ \overline{X})|_{(0,\pm 1,t)} h_{k+1}|_{(\pm 1,t)}\right] \\ + \tilde{g}_k^\pm(\rho, t), \end{aligned}$$

where $\tilde{g}_k^\pm \in R^C_{k,(\beta)}$ and therefore $g^\pm_{k+1} \in \hat{R}^I_{k+1,(\beta)} + R^C_{k,(\beta)}$, if h_{k+1} is smooth.

Remark 5.9. In Abels, Moser [AM], Section 3.2.2 we computed the order $\mathcal{O}(\varepsilon)$ in (AC2) for $u_\varepsilon = u^I_\varepsilon + u^{C\pm}_\varepsilon$. This implies

$$\begin{aligned}\tilde{g}_1^\pm(\rho,t) &= -\partial_\rho \hat{u}_1^{C\pm}|_{(\rho,0,t)}\partial_r((N_{\partial\Omega}\cdot\nabla r)\circ\overline{X})|_{(0,\pm1,t)}(\rho+h_1|_{(\pm1,t)})\\ &\quad \pm \partial_H \hat{u}_1^{C\pm}|_{(\rho,0,t)}\partial_r((N_{\partial\Omega}\cdot\nabla s)\circ\overline{X})|_{(0,\pm1,t)}(\rho+h_1|_{(\pm1,t)})\\ &\quad + N_{\partial\Omega}\cdot\nabla s|_{\overline{p}^\pm(t)}\partial_\rho\hat{u}_1^{C\pm}|_{(\rho,0,t)}\partial_s h_1|_{(\pm1,t)}\\ &\quad - \theta_0'(\rho)\left[\frac{1}{2}(\rho+h_1|_{(\pm1,t)})^2\partial_r^2((N_{\partial\Omega}\cdot\nabla r)\circ\overline{X})|_{(0,\pm1,t)}\right]\\ &\quad + \theta_0'(\rho)\left[(\rho+h_1|_{(\pm1,t)})\partial_s h_1|_{(\pm1,t)}\partial_r((N_{\partial\Omega}\cdot\nabla s)\circ\overline{X})|_{(0,\pm1,t)}\right].\end{aligned}$$

As in the last Section 5.1.2.2.2, the equations (5.17), (5.24) are equivalent to

$$[-\Delta + f''(\theta_0(\rho))]\overline{u}^{C\pm}_{k+1} = \overline{G}^\pm_k, \tag{5.25}$$

$$-\partial_H\overline{u}^{C\pm}_{k+1}|_{H=0} = g^\pm_{k+1}, \tag{5.26}$$

where $\overline{u}^{C\pm}_{k+1}$ was defined in (5.20) and $\overline{G}^\pm_k$ is defined analogously with the $G^\pm_k \in R^C_{k,(\beta,\gamma)}$ from (5.17) in Section 5.1.2.1.2. The corresponding compatibility condition (4.5), i.e.

$$\int_{\mathbb{R}^2_+}\overline{G}^\pm_k(\rho,H,t)\theta_0'(\rho)\,d(\rho,H) + \int_{\mathbb{R}} g^\pm_{k+1}(\rho,t)\theta_0'(\rho)\,d\rho = 0,$$

leads to a linear boundary condition for h_{k+1}:

$$b_1^\pm(t)\partial_s h_{k+1}|_{(\pm1,t)} + b_0^\pm(t)h_{k+1}|_{(\pm1,t)} = f_k^\pm(t) \quad \text{for } t\in[0,T], \tag{5.27}$$

where $b_0^\pm, b_1^\pm$ are defined after (5.23) and

$$f_k^\pm(t) := -\left[\int_{\mathbb{R}^2_+}\overline{G}^\pm_k(\rho,H,t)\theta_0'(\rho)\,d\rho + \int_{\mathbb{R}}\tilde{g}^\pm_k(\rho,t)\theta_0'(\rho)\,d\rho\right]\frac{1}{\|\theta_0'\|^2_{L^2(\mathbb{R})}}$$

is smooth in $t\in[0,T]$.

In an analogous way as in the last Section 5.1.2.2.2 we solve (5.11) from Section 5.1.1.4 together with (5.27) and get a smooth solution h_{k+1}. Therefore Section 5.1.1.4 yields $\hat{u}^I_{k+2}$ (solving (5.10)) with decay $\hat{u}^I_{k+2} \in R^I_{k+1,(\beta_1)}$ for all $\beta_1 \in (0,\min\{\sqrt{f''(\pm1)}\})$. Hence the $(k+1)$-th inner order is computed and it holds $G^\pm_k \in R^C_{k,(\beta,\gamma)}$ as well as $g^\pm_{k+1} \in \hat{R}^I_{k+1,(\beta)} + R^C_{k,(\beta)}$ for all $\beta \in (0,\min\{\overline{\beta}(\gamma),\sqrt{f''(\pm1)}\})$, $\gamma\in(0,\overline{\gamma})$, where $\overline{\beta},\overline{\gamma}$ are as in Theorem 4.11. Therefore due to Theorem 4.11 we obtain a unique smooth solution $\overline{u}^{C\pm}_{k+1}$ to (5.25)-(5.26) with the decay $\hat{u}^{C\pm}_{k+1} \in R^C_{k,(\beta,\gamma)}$ for all (β,γ) as above. In particular, the $(k+1)$-th order is determined.

Finally, by induction we have constructed the j-th order for all $j=0,...,k$, the h_j are smooth and $\hat{u}^I_{j+1} \in R^I_{j,(\beta_1)}$ for all $\beta_1 \in (0,\min\{\sqrt{f''(\pm1)}\})$ as well as $\hat{u}^{C\pm}_j \in R^C_{j,(\beta,\gamma)}$ for every $\beta \in (0,\min\{\overline{\beta}(\gamma),\sqrt{f''(\pm1)}\})$, $\gamma\in(0,\overline{\gamma})$, where $\overline{\beta},\overline{\gamma}$ are as in Theorem 4.11.

5.1.3 The Approximate Solution for (AC) in 2D

Let $N=2$ and $\Gamma:=(\Gamma_t)_{t\in[0,T]}$ be as in Section 3.1 with contact angle $\alpha=\frac{\pi}{2}$ and a solution to MCF in Ω. Moreover, let $\delta>0$ be such that the assertions of Theorem 3.3 hold for 2δ instead of δ and let r,s,μ_0 be as in the theorem. Let $M\in\mathbb{N}$, $M\geq 2$ be as in the beginning of Section 5.1. Finally, let $\eta:\mathbb{R}\to[0,1]$ be smooth with $\eta(r)=1$, $|r|\leq 1$ and $\eta(r)=0$, $|r|\geq 2$. For $\varepsilon>0$ set

$$u_\varepsilon^A:=\begin{cases}\eta(\frac{r}{\delta})\left[u_\varepsilon^I+\sum_\pm u_\varepsilon^{C\pm}\eta(\frac{s^\pm}{2\mu_0})\right]+(1-\eta(\frac{r}{\delta}))\mathrm{sign}(r) & \text{in } \overline{\Gamma(2\delta)},\\ \pm 1 & \text{in } Q_T^\pm\setminus\Gamma(2\delta),\end{cases}$$

where u_ε^I and $u_\varepsilon^{C\pm}$ were constructed in Sections 5.1.1-5.1.2 and $s^\pm=\mp(s\mp 1)$. Note that μ_0 is just used for clarity. Here one could also use e.g. 1 instead. This yields an approximate solution for (AC1)-(AC3) in the following sense:

Lemma 5.10. *The function u_ε^A is smooth, uniformly bounded with respect to x,t,ε and for the remainder $r_\varepsilon^A:=\partial_t u_\varepsilon^A-\Delta u_\varepsilon^A+\frac{1}{\varepsilon^2}f'(u_\varepsilon^A)$ in* (AC1) *and $s_\varepsilon^A:=\partial_{N_{\partial\Omega}}u_\varepsilon^A$ in* (AC2) *it holds*

$$\begin{aligned}|r_\varepsilon^A|&\leq C(\varepsilon^{M-1}e^{-c(|\rho_\varepsilon|+H_\varepsilon^\pm)}+\varepsilon^M e^{-c|\rho_\varepsilon|}+\varepsilon^{M+1}) && \text{in } \Gamma^\pm(2\delta,1),\\ r_\varepsilon^A&=0 && \text{in } Q_T\setminus\Gamma(2\delta),\\ |s_\varepsilon^A|&\leq C\varepsilon^M e^{-c|\rho_\varepsilon|} && \text{on } \partial Q_T\cap\Gamma(2\delta),\\ s_\varepsilon^A&=0 && \text{on } \partial Q_T\setminus\Gamma(2\delta)\end{aligned}$$

for $\varepsilon>0$ small and some $c,C>0$. Here ρ_ε is defined in (5.1) *and $H_\varepsilon^\pm$ was set as $\frac{s^\pm}{\varepsilon}$.*

Remark 5.11. The estimate also holds without the ε^{M+1}-term. This follows from a more precise consideration of the remainder terms in the Taylor expansions in Sections 5.1.1-5.1.2 and below. Moreover, one could also lower the number of terms needed in the Taylor expansions a little bit by looking closely at the construction in the previous sections. This would only be of interest if one considers hypersurfaces of class C^l for some large but finite l.

Proof. The second and the last equation are evident from the construction. Moreover, the rigorous Taylor expansions (5.3)-(5.4), (5.13)-(5.15) and (5.18) together with the remarks for the remainders and Sections 5.1.1-5.1.2 yield

$$\begin{aligned}|\partial_t u_\varepsilon^I-\Delta u_\varepsilon^I+\frac{1}{\varepsilon^2}f'(u_\varepsilon^I)|&\leq C(\varepsilon^M e^{-c|\rho_\varepsilon|}+\varepsilon^{M+1}) && \text{in } \Gamma(2\delta),\\ \left|\partial_t u_\varepsilon^{C\pm}-\Delta u_\varepsilon^{C\pm}+\frac{f'(u_\varepsilon^I+u_\varepsilon^{C\pm})-f'(u_\varepsilon^I)}{\varepsilon^2}\right|&\leq C(\varepsilon^{M-1}e^{-c(|\rho_\varepsilon|+H_\varepsilon^\pm)}+\varepsilon^{M+1}) && \text{in } \Gamma^\pm(2\delta,1),\\ |\partial_{N_{\partial\Omega}}(u_\varepsilon^I+u_\varepsilon^{C\pm})|&\leq C\varepsilon^M e^{-c|\rho_\varepsilon|} && \text{on } \Gamma^\pm(2\delta,1)\cap\partial Q_T.\end{aligned}$$

Therefore the estimate in the lemma holds for the remainder of u_ε^I in (AC1) on $\Gamma(2\delta)$ and for the remainder of $u_\varepsilon^I+u_\varepsilon^{C\pm}$ in (AC1) on $\Gamma^\pm(2\delta,1)$. In order to use this for u_ε^A, we have to deal with the mixed terms due to the cutoff-functions. First, we prove that the remainder of $\tilde{u}_\varepsilon^A:=u_\varepsilon^I+\sum_\pm u_\varepsilon^{C\pm}\eta(s^\pm/2\mu_0)$ in (AC1) on $\Gamma^\pm(2\delta,1)$ can be estimated as in the lemma. Note that on $\Gamma(2\delta,1-2\mu_0)$ and $\Gamma^\pm(2\delta,\mu_0)$ this follows from the above estimates. Moreover, due to

Taylor expansions it holds

$$
\begin{aligned}
f'(u^I_\varepsilon+\eta(\frac{s^\pm}{2\mu_0})u^{C\pm}_\varepsilon) &= f'(u^I_\varepsilon)+\mathcal{O}\left(\eta(\frac{s^\pm}{2\mu_0})|u^{C\pm}_\varepsilon|\right)\\
&= (1-\eta(\frac{s^\pm}{2\mu_0}))f'(u^I_\varepsilon)+\eta(\frac{s^\pm}{2\mu_0})f'(u^I_\varepsilon+u^{C\pm}_\varepsilon)+\mathcal{O}\left(\eta(\frac{s^\pm}{2\mu_0})|u^{C\pm}_\varepsilon|\right).
\end{aligned}
$$

Hence with the product rule for ∂_t and Δ we obtain

$$
\begin{aligned}
&\left[\text{l.h.s. in (AC1) for } u^I_\varepsilon+\eta(\frac{s^\pm}{2\mu_0})u^{C\pm}_\varepsilon\right]\\
&= (1-\eta(\frac{s^\pm}{2\mu_0}))\left[\text{l.h.s. in (AC1) for } u^I_\varepsilon\right]+\eta(\frac{s^\pm}{2\mu_0})\left[\text{l.h.s. in (AC1) for } u^I_\varepsilon+u^{C\pm}_\varepsilon\right]\\
&+u^{C\pm}_\varepsilon(\partial_t-\Delta)\left[\eta(\frac{s^\pm}{2\mu_0})\right]-2\nabla\left[\eta(\frac{s^\pm}{2\mu_0})\right]\cdot\nabla u^{C\pm}_\varepsilon+\frac{1}{\varepsilon^2}\mathcal{O}\left(\eta(\frac{s^\pm}{2\mu_0})|u^{C\pm}_\varepsilon|\right),
\end{aligned}
$$

where “l.h.s.” stands for “left hand side”. Hence the above estimates and the asymptotics of $\hat{u}^{C\pm}_j$ for $j=0,...,M$ yield the desired estimate for the remainder of $\tilde{u}^A_\varepsilon$ in (AC1) in $\Gamma^\pm(2\delta,1)$. Altogether the first estimate holds in $\Gamma^\pm(2\delta,1)\cap\Gamma(\delta)$. In $\Gamma^\pm(2\delta,1)\backslash\Gamma(\delta)$ we have again similar mixed terms as above due to the cutoff functions. With Taylor expansions we obtain in $Q_T^\pm\backslash\Gamma(\delta)$:

$$
f'(u^A_\varepsilon)=\mathcal{O}\left(\eta(\frac{r}{\delta})|\tilde{u}^A_\varepsilon\mp 1|\right),\quad \eta(\frac{r}{\delta})f'(\tilde{u}^A_\varepsilon)=\mathcal{O}\left(\eta(\frac{r}{\delta})|\tilde{u}^A_\varepsilon\mp 1|\right),
$$

where we used $f'(\pm1)=0$. Hence the product rule for ∂_t and Δ yields in $\Gamma^\pm(2\delta,1)\backslash\Gamma(\delta)$

$$
\begin{aligned}
\left[\text{l.h.s. in (AC1) for } u^A_\varepsilon\right] &= \eta(\frac{r}{\delta})\left[\text{l.h.s. in (AC1) for } \tilde{u}^A_\varepsilon\right]+(\tilde{u}^A_\varepsilon\mp1)(\partial_t-\Delta)\left[\eta(\frac{r}{\delta})\right]\\
&-2\nabla\left[\eta(\frac{r}{\delta})\right]\cdot\nabla\tilde{u}^A_\varepsilon+\frac{1}{\varepsilon^2}\mathcal{O}\left(\eta(\frac{r}{\delta})|\tilde{u}^A_\varepsilon\mp1|\right).
\end{aligned}
$$

Finally, the asymptotics of u^I_j and $u^{C\pm}_j$ for $j=0,...,M$ imply the estimate for r^A_ε.

It is left to prove the remaining assertion for s^A_ε. By definition it holds $u^A_\varepsilon=u^I_\varepsilon+u^{C\pm}_\varepsilon$ on $\Gamma^\pm(\delta,1)\cap\partial Q_T$. For the latter we already have an estimate of the Neumann derivative on $\Gamma^\pm(2\delta,1)\cap\partial Q_T$, see above. Again we have mixed terms for u^A_ε on $[\Gamma^\pm(2\delta,1)\cap\partial Q_T]\backslash\Gamma(\delta)$ because of the cutoff-functions. On the latter set it holds $\tilde{u}^A_\varepsilon=u^I_\varepsilon+u^{C\pm}_\varepsilon$ and

$$
\partial_{N_{\partial\Omega}}u^A_\varepsilon=(\tilde{u}^A_\varepsilon\mp1)\partial_{N_{\partial\Omega}}\left[\eta(\frac{r}{\varepsilon})\right]+\eta(\frac{r}{\varepsilon})\partial_{N_{\partial\Omega}}\tilde{u}^A_\varepsilon.
$$

Therefore the claim follows with the asymptotics of u^I_j and $u^{C\pm}_j$ for $j=0,...,M$. □

5.2 Asymptotic Expansion of (AC) in ND

Let $N\geq 2$, $\Omega\subseteq\mathbb{R}^N$ be as in Remark 1.1, 1. and $\Gamma:=(\Gamma_t)_{t\in[0,T]}$ be as in Section 3.1 with contact angle $\alpha=\frac{\pi}{2}$. We use the notation from Section 3.1 and 3.3. Moreover, let $\delta>0$ be such that the assertions of Theorem 3.7 hold for 2δ instead of δ. Finally, we assume that Γ evolves according to MCF. Based on Γ we construct a smooth approximate solution u_ε^A to (AC1)-(AC3) with analogous qualitative behaviour as in the 2-dimensional case, cf. the previous Section 5.1. The computations are similar to the latter case. Here I is replaced by the smooth hypersurface with boundary Σ. The most striking insight is that in the contact point expansion we also end up with model problems on the half space $\mathbb{R}^2_+$. Here elements of $\partial\Sigma$ enter as independent variables. For simplicity, we often use identical notation as in the 2D-case.

Let $M \in \mathbb{N}$ with $M \geq 2$. Then we consider height functions $h_j : \Sigma \times [0,T] \to \mathbb{R}$ for $j = 1, ..., M$ and we set $h_\varepsilon := \sum_{j=1}^M \varepsilon^{j-1} h_j$. Analogously as in the 2-dimensional case we define $h_{M+1} := h_{M+2} := 0$ and we introduce the scaled variable

$$\rho_\varepsilon(x,t) := \frac{r(x,t)}{\varepsilon} - h_\varepsilon(s(x,t),t) \quad \text{for } (x,t) \in \overline{\Gamma(2\delta)}. \tag{5.28}$$

In Section 5.2.1 we construct the inner expansion and in Section 5.2.2 the contact point expansion. Finally, in Section 5.2.3 we show that the construction yields a suitable approximate solution u_ε^A to (AC1)-(AC3).

5.2.1 Inner Expansion of (AC) in ND

For the inner expansion we consider the following ansatz: Let $\varepsilon > 0$ be small and

$$u_\varepsilon^I := \sum_{j=0}^{M+1} \varepsilon^j u_j^I, \quad u_j^I(x,t) := \hat{u}_j^I(\rho_\varepsilon(x,t), s(x,t), t) \quad \text{for } (x,t) \in \overline{\Gamma(2\delta)},$$

where

$$\hat{u}_j^I : \mathbb{R} \times \Sigma \times [0,T] \to \mathbb{R} : (\rho, s, t) \mapsto \hat{u}_j^I(\rho, s, t)$$

for $j = 0, ..., M+1$. Moreover, we set $u_{M+2}^I := 0$ and $\hat{u}_\varepsilon^I := \sum_{j=0}^{M+1} \varepsilon^j \hat{u}_j^I$. We will expand (AC1) for $u_\varepsilon = u_\varepsilon^I$ into ε-series with coefficients in (ρ_ε, s, t) up to $\mathcal{O}(\varepsilon^{M-1})$. This yields equations of analogous form as in Section 5.1.1, where I is replaced by Σ. Therefore we have to compute the action of the differential operators on u_ε^I.

In the following the surface gradient ∇_Σ (see Depner [D], Definition 2.21) for functions $g : \Sigma \to \mathbb{R}$ is viewed as a map $\nabla_\Sigma g : \Sigma \to \mathbb{R}^N$. Then we set $(\nabla_\Sigma)_i g := (\nabla_\Sigma g)_i$ for $i = 1, ..., N$. If g depends on other variables as well, the analogous definition applies.

Lemma 5.12. *Let $\varepsilon > 0$, $\hat{w} : \mathbb{R} \times \Sigma \times [0,T] \to \mathbb{R}$ be sufficiently smooth and $w : \overline{\Gamma(2\delta)} \to \mathbb{R}$ be defined by $w(x,t) := \hat{w}(\rho_\varepsilon(x,t), s(x,t), t)$ for all $(x,t) \in \overline{\Gamma(2\delta)}$. Then*

$$\begin{aligned}
\partial_t w &= \partial_\rho \hat{w} \left[\frac{\partial_t r}{\varepsilon} - (\partial_t h_\varepsilon + \partial_t s \cdot \nabla_\Sigma h_\varepsilon) \right] + \partial_t s \cdot \nabla_\Sigma \hat{w} + \partial_t \hat{w}, \\
\nabla w &= \partial_\rho \hat{w} \left[\frac{\nabla r}{\varepsilon} - (D_x s)^\top \nabla_\Sigma h_\varepsilon \right] + (D_x s)^\top \nabla_\Sigma \hat{w}, \\
\Delta w &= \partial_\rho \hat{w} \left[\frac{\Delta r}{\varepsilon} - \left(\Delta s \cdot \nabla_\Sigma h_\varepsilon + \sum_{i,l=1}^N \nabla s_i \cdot \nabla s_l (\nabla_\Sigma)_i (\nabla_\Sigma)_l h_\varepsilon \right) \right] \\
&\quad + \Delta s \cdot \nabla_\Sigma \hat{w} + \sum_{i,l=1}^N \nabla s_i \cdot \nabla s_l (\nabla_\Sigma)_i (\nabla_\Sigma)_l \hat{w} \\
&\quad + 2\left((D_x s)^\top \nabla_\Sigma \partial_\rho \hat{w} \right) \cdot \left[\frac{\nabla r}{\varepsilon} - (D_x s)^\top \nabla_\Sigma h_\varepsilon \right] + \partial_\rho^2 \hat{w} \left| \frac{\nabla r}{\varepsilon} - (D_x s)^\top \nabla_\Sigma h_\varepsilon \right|^2,
\end{aligned}$$

where the w-terms on the left hand side and derivatives of r or s are evaluated at $(x,t) \in \overline{\Gamma(2\delta)}$*, the h_ε-terms at* $(s(x,t),t)$ *and the $\hat{w}$-terms at* $(\rho_\varepsilon(x,t), s(x,t), t)$.

Remark 5.13.

1. The differential operator ∇_Σ commutes with other ones acting on different variables. This can be shown directly with the definitions or suitable extension arguments. For the latter there are similar ideas in the proof of Lemma 5.12.
2. Note the similarity to Lemma 5.1. The terms on the right hand side are more sophisticated, but are always of the same type as in the 2D-case. Therefore in the expansion they will contribute in the analogous way.

Proof of Lemma 5.12. Basically, this follows from the chain and product rules as well as from the properties of ∇_Σ. Let $g : \Sigma \to \mathbb{R}$ be C^1. Then $g \circ s : \overline{\Gamma(2\delta)} \to \mathbb{R}$. In order to compute $\nabla_x[g(s)]$, let $\overline{g} : \mathbb{R}^N \to \mathbb{R}$ and $\overline{s} : \mathbb{R}^{N+1} \to \mathbb{R}^N$ be smooth extensions of g and s, respectively. Such a $\overline{g}$ can be constructed via local extensions in submanifold charts and the existence of $\overline{s}$ follows from Theorem 3.7, 1. Then the chain rule yields

$$D_x(g(s))|_{(x,t)} = D\overline{g}|_{s(x,t)} D_x\overline{s}|_{(x,t)} = (\nabla_\Sigma g)^\top|_{s(x,t)} D_x s|_{(x,t)},$$

where we used $\nabla_\Sigma g|_s = P_{T_s\Sigma}\nabla\overline{g}|_s$ for all $s \in \Sigma$ due to Depner [D], Remark 2.22 as well as $\partial_{x_j} s|_{(x,t)} \in T_{s(x,t)}\Sigma$ for all $(x,t) \in \overline{\Gamma(2\delta)}$, cf. Theorem 3.7. Alternatively, one can also use the chain rule for differentials and the definition of the surface gradient. For the derivative in time this works analogously. Therefore it holds for all $(x,t) \in \overline{\Gamma(2\delta)}$:

$$\nabla_x[g(s)]|_{(x,t)} = (D_x s)^\top|_{(x,t)} \nabla_\Sigma g|_{s(x,t)} \quad \text{and} \quad \partial_t[g(s)]|_{(x,t)} = \partial_t s|_{(x,t)} \cdot \nabla_\Sigma g|_{s(x,t)}.$$

With similar arguments and the chain rule one can derive formulas for the first derivatives of functions of type $g(s(x,t),t)$ for $(x,t) \in \overline{\Gamma(2\delta)}$, where $g : \Sigma \times [0,T] \to \mathbb{R}$. In this case it holds

$$\begin{aligned}\frac{d}{dt}[g(s(x,.),.)]|_t &= \partial_t g|_{(s(x,t),t)} + \partial_t s|_{(x,t)} \cdot \nabla_\Sigma g|_{(s(x,t),t)},\\ \nabla_x[g(s(.,t),t)]|_x &= (D_x s)|_{(x,t)}^\top \nabla_\Sigma g|_{(s(x,t),t)}.\end{aligned}$$

Using this and similar arguments as before, one can derive formulas for the first derivatives of functions of the form $g(\rho_\varepsilon(x,t), s(x,t), t)$ for $(x,t) \in \overline{\Gamma(2\delta)}$ with $g : \mathbb{R} \times \Sigma \times [0,T] \to \mathbb{R}$. The latter are written in the lemma for $\hat{w}$ instead of g. Putting all those identities together and using the product rule in $\partial_{x_j}(\nabla w)_j$ for $j = 1, ..., N$, one obtains the formula for Δw. □

To expand the Allen-Cahn equation $\partial_t u_\varepsilon^I - \Delta u_\varepsilon^I + \frac{1}{\varepsilon^2} f'(u_\varepsilon^I) = 0$ into ε-series, we again use Taylor expansions. For the f'-part this is identically to the 2D-case: If the u_j^I are bounded, then

$$f'(u_\varepsilon^I) = f'(u_0^I) + \sum_{k=1}^{M+2} \frac{f^{(k+1)}(u_0^I)}{k!} \left[\sum_{j=1}^{M+1} u_j^I \varepsilon^j\right]^k + \mathcal{O}(\varepsilon^{M+3}) \quad \text{on } \overline{\Gamma(2\delta)} \tag{5.29}$$

and the same assertions after (5.3) hold for the explicit terms and the remainder terms in (5.29).

Moreover, we expand functions of $(x,t) \in \overline{\Gamma(2\delta)}$ into ε-series with a Taylor expansion via $r(x,t) = \varepsilon(\rho_\varepsilon(x,t) + h_\varepsilon(s(x,t),t))$ for $(x,t) \in \overline{\Gamma(2\delta)}$. Then again ρ_ε is replaced by $\rho \in \mathbb{R}$. For a smooth $g : \overline{\Gamma(2\delta)} \to \mathbb{R}$ the Taylor expansion yields for $r \in [-2\delta, 2\delta]$ uniformly in (s,t):

$$\tilde{g}(r,s,t) := g(\overline{X}(r,s,t)) = \sum_{k=0}^{M+2} \frac{\partial_r^k \tilde{g}|_{(0,s,t)}}{k!} r^k + \mathcal{O}(|r|^{M+3}) \tag{5.30}$$

and the properties for the explicit terms and the remainder terms below (5.4) are valid for (5.30).

For the higher orders in the expansion we use analogous definitions as in the 2D-case:

Definition 5.14 (Notation for Inner Expansion of (AC) in ND). 1. We call (θ_0, u_1^I) the *zero-th inner order* and (h_j, u_{j+1}^I) the *j-th inner order* for $j = 1, ..., M$.

2. Let $k \in \{-1, ..., M+2\}$ and $\beta > 0$. We denote with $R_{k,(\beta)}^I$ the set of smooth functions $R : \mathbb{R} \times \Sigma \times [0,T] \to \mathbb{R}$ that depend only on the j-th inner orders for $0 \leq j \leq \min\{k, M\}$ and satisfy uniformly in (ρ, s, t):

$$|\partial_\rho^i (\nabla_\Sigma)_{n_1} ... (\nabla_\Sigma)_{n_d} \partial_t^n R(\rho, s, t)| = \mathcal{O}(e^{-\beta|\rho|})$$

for all $n_1, ..., n_d \in \{1, ..., N\}$ and $d, i, l, n \in \mathbb{N}_0$.

3. For $k \in \{-1, ..., M+2\}$ and $\beta > 0$ the set $\hat{R}_{k,(\beta)}^I$ is defined analogously to $R_{k,(\beta)}^I$ with functions of type $R : \mathbb{R} \times \partial\Sigma \times [0,T] \to \mathbb{R}$ instead.

Now we expand (AC1) for $u_\varepsilon = u_\varepsilon^I$ into ε-series. This works analogously to the 2D-case, cf. Remark 5.13, 2. In the following (ρ, s, t) are always in $\mathbb{R} \times \Sigma \times [0,T]$ and sometimes omitted.

5.2.1.1 Inner Expansion: $\mathcal{O}(\varepsilon^{-2})$

The $\mathcal{O}(\frac{1}{\varepsilon^2})$-order is zero if

$$-|\nabla r|^2|_{\overline{X}_0(s,t)} \partial_\rho^2 \hat{u}_0^I(\rho, s, t) + f'(\hat{u}_0^I(\rho, s, t)) = 0.$$

Because of Theorem 3.7 we have $|\nabla r|^2|_{\overline{X}_0(s,t)} = 1$. For the same reasons as in the 2D-case we require $\hat{u}_0^I(0, s, t) = 0$ and $\lim_{\rho \to \pm\infty} \hat{u}_0^I(\rho, s, t) = \pm 1$. Hence because of Theorem 4.1 we set

$$\hat{u}_0^I(\rho, s, t) := \theta_0(\rho).$$

5.2.1.2 Inner Expansion: $\mathcal{O}(\varepsilon^{-1})$

The $\partial_t u$-part yields $\frac{1}{\varepsilon} \partial_t r|_{\overline{X}_0(s,t)} \theta_0'(\rho)$ and Δu:

$$\begin{aligned}
&\frac{1}{\varepsilon^2} \left[\partial_r (|\nabla r|^2 \circ \overline{X})|_{(0,s,t)} \varepsilon (\rho + h_1(s,t)) \theta_0''(\rho) + |\nabla r|^2|_{\overline{X}_0(s,t)} \varepsilon \partial_\rho^2 \hat{u}_1^I(\rho, s, t) \right] \\
&+ \frac{1}{\varepsilon} \left[\theta_0'(\rho) \Delta r|_{\overline{X}_0(s,t)} + 2(D_x s \nabla r)^\top|_{\overline{X}_0(s,t)} (\nabla_\Sigma \theta_0'(\rho) - \nabla_\Sigma h_1(s,t) \theta_0''(\rho)) \right] \\
&= \frac{1}{\varepsilon} \left[\partial_\rho^2 \hat{u}_1^I(\rho, s, t) + \theta_0'(\rho) \Delta r|_{\overline{X}_0(s,t)} \right],
\end{aligned}$$

where we used Theorem 3.7. Therefore the $\mathcal{O}(\frac{1}{\varepsilon})$-order cancels if

$$\mathcal{L}_0 \hat{u}_1^I(\rho, s, t) + \theta_0'(\rho)(\partial_t r - \Delta r)|_{\overline{X}_0(s,t)} = 0, \quad \text{where } \mathcal{L}_0 := -\partial_\rho^2 + f''(\theta_0).$$

Due to Theorem 4.4 this parameter-dependent ODE together with $\hat{u}_1^I(0, s, t) = 0$ and boundedness in ρ has a (unique) solution $\hat{u}_1^I$ if and only if $(\partial_t r - \Delta r)|_{\overline{X}_0(s,t)} = 0$. The latter is valid because it is equivalent to MCF for Γ by Theorem 3.7. Therefore we define $\hat{u}_1^I := 0$.

5.2.1.3 Inner Expansion: $\mathcal{O}(\varepsilon^0)$ From $\partial_t u$ we obtain

$$\begin{aligned}
&\frac{1}{\varepsilon}\left[\partial_t r|_{\overline{X}_0(s,t)}\varepsilon\partial_\rho \hat{u}_1^I + \partial_r(\partial_t r \circ \overline{X})|_{(0,s,t)}\varepsilon(\rho + h_1(s,t))\theta_0'(\rho)\right] \\
&+ \theta_0'(\rho)\left[-\partial_t h_1(s,t) - \partial_t s|_{\overline{X}_0(s,t)} \cdot \nabla_\Sigma h_1(s,t)\right] + \partial_t s|_{\overline{X}_0(s,t)} \cdot \nabla_\Sigma \theta_0 + \partial_t \theta_0(\rho) \\
&= \theta_0'(\rho)\left[(\rho + h_1(s,t))\partial_r(\partial_t r \circ \overline{X})|_{(0,s,t)} - \partial_t h_1(s,t) - \partial_t s|_{\overline{X}_0(s,t)} \cdot \nabla_\Sigma h_1(s,t)\right],
\end{aligned}$$

and from Δu:

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\theta_0''(\rho)\left[\varepsilon^2\frac{1}{2}(\rho + h_1)^2\partial_r^2(|\nabla r|^2 \circ \overline{X})|_{(0,s,t)} + \varepsilon^2 h_2 \partial_r(|\nabla r|^2 \circ \overline{X})|_{(0,s,t)}\right] \\
&+ \frac{1}{\varepsilon^2}\partial_\rho^2 \hat{u}_1^I \varepsilon^2(\rho + h_1)\partial_r(|\nabla r|^2 \circ \overline{X})|_{(0,s,t)} + \frac{1}{\varepsilon^2}|\nabla r|^2|_{\overline{X}_0(s,t)}\varepsilon^2\partial_\rho^2 \hat{u}_2^I \\
&+ \frac{1}{\varepsilon}\left[\theta_0'(\rho)\varepsilon(\rho + h_1)\partial_r(\Delta r \circ \overline{X})|_{(0,s,t)} + \varepsilon\partial_\rho \hat{u}_1^I \Delta r|_{\overline{X}_0(s,t)}\right] \\
&+ \frac{1}{\varepsilon}2(D_x s\nabla r)^\top|_{\overline{X}_0(s,t)}\left[\nabla_\Sigma \partial_\rho \hat{u}_1^I \varepsilon - \nabla_\Sigma h_1 \varepsilon \partial_\rho^2 \hat{u}_1^I - \varepsilon \nabla_\Sigma h_2 \theta_0''(\rho)\right] \\
&+ \frac{1}{\varepsilon}2\partial_r((D_x s\nabla r)^\top \circ \overline{X})|_{(0,s,t)}\varepsilon(\rho + h_1)\left[\nabla_\Sigma \theta_0'(\rho) - \nabla_\Sigma h_1 \theta_0''(\rho)\right] \\
&+ \Delta s|_{\overline{X}_0(s,t)} \cdot \nabla_\Sigma \theta_0(\rho) + \sum_{i,l=1}^N \nabla s_i \cdot \nabla s_l|_{\overline{X}_0(s,t)}(\nabla_\Sigma)_i(\nabla_\Sigma)_l \theta_0(\rho) \\
&- 2\nabla_\Sigma \theta_0'(\rho)^\top D_x s(D_x s)^\top|_{\overline{X}_0(s,t)}\nabla_\Sigma h_1 + \left|(D_x s)^\top|_{\overline{X}_0(s,t)}\nabla_\Sigma h_1\right|^2 \theta_0''(\rho) \\
&- \theta_0'(\rho)\left[\Delta s|_{\overline{X}_0(s,t)} \cdot \nabla_\Sigma h_1 + \sum_{i,l=1}^N \nabla s_i \cdot \nabla s_l|_{\overline{X}_0(s,t)}(\nabla_\Sigma)_i(\nabla_\Sigma)_l h_1\right].
\end{aligned}$$

Because of Theorem 3.7 the latter is the same as

$$\begin{aligned}
&\theta_0''(\rho)\left[\frac{1}{2}(\rho + h_1)^2\partial_r^2(|\nabla r|^2 \circ \overline{X})|_{(0,s,t)} + \left|(D_x s)^\top|_{\overline{X}_0(s,t)}\nabla_\Sigma h_1\right|^2\right] + \partial_\rho^2 \hat{u}_2^I \\
&+ \theta_0''(\rho)2\partial_r((D_x s\nabla r)^\top \circ \overline{X})|_{(0,s,t)}(\rho + h_1)(-\nabla_\Sigma h_1) + \theta_0'(\rho)(\rho + h_1)\partial_r(\Delta r \circ \overline{X})|_{(0,s,t)} \\
&- \theta_0'(\rho)\left[\Delta s|_{\overline{X}_0(s,t)} \cdot \nabla_\Sigma h_1 + \sum_{i,l=1}^N \nabla s_i \cdot \nabla s_l(\nabla_\Sigma)_i(\nabla_\Sigma)_l h_1\right].
\end{aligned}$$

Due to $\hat{u}_1^I = 0$, the f'-part contributes $f''(\theta_0)\hat{u}_2^I$. Hence for the cancellation of the $\mathcal{O}(1)$-term in the expansion for the Allen-Cahn equation (AC1) for $u_\varepsilon = u_\varepsilon^I$ we require

$$-\mathcal{L}_0 \hat{u}_2^I(\rho,s,t) = R_1(\rho,s,t), \tag{5.31}$$

$$\begin{aligned}
R_1(\rho,s,t) := \theta_0'(\rho)&\left[-\partial_t h_1 + \sum_{i,l=1}^N \nabla s_i \cdot \nabla s_l(\nabla_\Sigma)_i(\nabla_\Sigma)_l h_1\right. \\
&\left. +(\rho + h_1)\partial_r((\partial_t r - \Delta r) \circ \overline{X})|_{(0,s,t)} - (\partial_t s - \Delta s)|_{\overline{X}_0(s,t)} \cdot \nabla_\Sigma h_1\right] \\
+\theta_0''(\rho)&\left[-\frac{1}{2}(\rho + h_1)^2\partial_r^2(|\nabla r|^2 \circ \overline{X})|_{(0,s,t)}\right. \\
&\left. +2(\rho + h_1)\partial_r((D_x s\nabla r)^\top \circ \overline{X})|_{(0,s,t)}\nabla_\Sigma h_1 - \left|(D_x s)^\top|_{\overline{X}_0(s,t)}\nabla_\Sigma h_1\right|^2\right].
\end{aligned}$$

If h_1 is smooth, then R_1 is smooth and together with all derivatives decays exponentially in $|\rho|$ uniformly in (s,t) with rate β for every $\beta \in (0, \min\{\sqrt{f''(\pm 1)}\})$ because of Theorem 4.1. Therefore Theorem 4.4 applied in local coordinates for Σ yields that there is a unique bounded solution $\hat{u}_2^I$ to (5.31) together with $\hat{u}_2^I(0,s,t) = 0$ if and only if the compatibility condition $\int_{\mathbb{R}} R_1(\rho,s,t)\theta_0'(\rho)\,d\rho = 0$ holds. Since $\int_{\mathbb{R}} \theta_0'(\rho)\theta_0''(\rho)\,d\rho = 0$ due to integration by parts, the nonlinearities in h_1 drop out and we obtain a linear non-autonomous parabolic equation for h_1 on Σ with principal part $\partial_t - \sum_{i,l=1}^N \nabla s_i \cdot \nabla s_l|_{\overline{X}_0(s,t)}(\nabla_\Sigma)_i(\nabla_\Sigma)_l$:

$$\partial_t h_1 - \sum_{i,l=1}^N \nabla s_i \cdot \nabla s_l|_{\overline{X}_0(s,t)}(\nabla_\Sigma)_i(\nabla_\Sigma)_l h_1 + a_1 \cdot \nabla_\Sigma h_1 + a_0 h_1 = f_0 \tag{5.32}$$

in $\Sigma \times [0,T]$. Here, with $d_1, ..., d_5$ defined below (5.6), we have set for all $(s,t) \in \Sigma \times [0,T]$:

$$a_1(s,t) := (\partial_t s - \Delta s)|_{\overline{X}_0(s,t)} - 2\frac{d_4}{d_1}\partial_r((D_x s \nabla r)^\top \circ \overline{X})|_{(0,s,t)} \in \mathbb{R}^N, \tag{5.33}$$

$$a_0(s,t) := -\partial_r((\partial_t r - \Delta r) \circ \overline{X})|_{(0,s,t)} + \frac{d_4}{d_1}\partial_r^2(|\nabla r|^2 \circ \overline{X})|_{(0,s,t)} \in \mathbb{R}, \tag{5.34}$$

$$f_0(s,t) := \frac{d_2}{d_1}\partial_r((\partial_t r - \Delta r) \circ \overline{X})|_{(0,s,t)} - \frac{d_5}{2d_1}\partial_r^2(|\nabla r|^2 \circ \overline{X})|_{(0,s,t)} \in \mathbb{R}. \tag{5.35}$$

If h_1 is smooth and solves (5.32), then Theorem 4.4 (applied in local coordinates for Σ) yields a smooth solution $\hat{u}_2^I$ to (5.31) and we also get decay estimates. By compactness, we obtain $\hat{u}_2^I \in R_{1,(\beta)}^I$ for any $\beta \in (0, \min\{\sqrt{f''(\pm 1)}\})$ because of the following remark:

Remark 5.15. The norm of the entirety of derivatives in local coordinates up to any fixed order $d \in \mathbb{N}$ is equivalent to the norm of the collection of all ∇_Σ-derivatives up to order d on any compact subset of a chart domain. This can be shown inductively via local representations.

Remark 5.16. If f is even, then the equation (5.32) for h_1 is homogeneous.

5.2.1.4 Inner Expansion: $\mathcal{O}(\varepsilon^k)$ Let $k \in \{1, ..., M-1\}$ and suppose that the j-th inner order has already been constructed for $j = 0, ..., k$, that it is smooth and $\hat{u}_{j+1}^I \in R_{j,(\beta)}^I$ for every $\beta \in (0, \min\{\sqrt{f''(\pm 1)}\})$. Analogously to the 2D-case one can compute the $\mathcal{O}(\varepsilon^k)$-order in (AC1) for $u_\varepsilon = u_\varepsilon^I$ using the notation in Definition 5.14. This yields that $\mathcal{O}(\varepsilon^k)$ vanishes if

$$
\begin{aligned}
-\mathcal{L}_0 \hat{u}_{k+2}^I(\rho,s,t) &= R_{k+1}(\rho,s,t), && (5.36)\\
R_{k+1}(\rho,s,t) &:= \theta_0'(\rho)\Bigg[-\partial_t h_{k+1} + \sum_{i,l=1}^N \nabla s_i \cdot \nabla s_l|_{\overline{X}_0(s,t)}(\nabla_\Sigma)_i(\nabla_\Sigma)_j h_{k+1} \\
&\qquad -(\partial_t s - \Delta s)|_{\overline{X}_0(s,t)} \cdot \nabla_\Sigma h_{k+1} + h_{k+1}\partial_r((\partial_t r - \Delta r) \circ \overline{X})|_{(0,s,t)}\Bigg] \\
&\quad +\theta_0''(\rho)\Big[-(\rho + h_1)h_{k+1}\partial_r^2(|\nabla r|^2 \circ \overline{X})|_{(0,s,t)} \\
&\qquad - 2(\nabla_\Sigma h_1)^\top D_x s (D_x s)^\top|_{\overline{X}_0(s,t)}\nabla_\Sigma h_{k+1} \\
&\qquad +2\partial_r((D_x s \nabla r)^\top \circ \overline{X})|_{(0,s,t)}[(\rho + h_1)\nabla_\Sigma h_{k+1} + h_{k+1}\nabla_\Sigma h_1]\Big] \\
&\quad +\tilde{R}_k(\rho,s,t),
\end{aligned}
$$

where $\tilde{R}_k \in R^I_{k,(\beta)}$. If h_{k+1} is smooth, then due to Theorem 4.4 (applied in local coordinates for Σ) equation (5.36) has a unique bounded solution $\hat{u}^I_{k+2}$ with $\hat{u}^I_{k+2}(0,s,t) = 0$ if and only if $\int_{\mathbb{R}} R_{k+1}(\rho,s,t)\theta_0'(\rho)\,d\rho = 0$. Because of $\int_{\mathbb{R}} \theta_0''\theta_0' = 0$ the latter is equivalent to

$$\partial_t h_{k+1} - \sum_{i,l=1}^{N} \nabla s_i \cdot \nabla s_l|_{\overline{X_0(s,t)}}(\nabla_\Sigma)_i(\nabla_\Sigma)_l h_{k+1} + a_1 \cdot \nabla_\Sigma h_{k+1} + a_0 h_{k+1} = f_k, \tag{5.37}$$

where

$$f_k(s,t) := \int_{\mathbb{R}} \tilde{R}_k(\rho,s,t)\theta_0'(\rho)\,d\rho \frac{1}{\|\theta_0'\|^2_{L^2(\mathbb{R})}}$$

is a smooth function of (s,t) and depends only on the j-th inner orders for $0 \leq j \leq k$. Here a_0, a_1 are defined in (5.33)-(5.34). If h_{k+1} is smooth and solves (5.37), then Theorem 4.4 yields as in the last Section 5.2.1.3 a smooth solution $\hat{u}^I_{k+2}$ to (5.36) such that $\hat{u}^I_{k+2} \in R^I_{k+1,(\beta)}$ for all $\beta \in (0, \min\{\sqrt{f''(\pm 1)}\})$.

5.2.2 Contact Point Expansion of (AC) in ND

In the contact point expansion we proceed similarly as in the 2D-case. Here we have more contact points, namely all $X_0(\sigma,t)$, where $(\sigma,t) \in \partial\Sigma \times [0,T]$. We make the ansatz $u_\varepsilon = u^I_\varepsilon + u^C_\varepsilon$ in $\Gamma(2\delta)$ close to $X_0(\partial\Sigma \times [0,T])$. Therefore we use the mappings $Y : \partial\Sigma \times [0, 2\mu_1] \to R(Y) \subset \Sigma$ as well as

$$(\sigma, b) = Y^{-1} \circ s : \overline{\Gamma^C(2\delta, 2\mu_1)} \to \partial\Sigma \times [0, 2\mu_1]$$

from Section 3.3. Besides r we only scale b with ε. Let $H_\varepsilon := \frac{b}{\varepsilon}$ and

$$u^C_\varepsilon := \sum_{j=1}^{M} \varepsilon^j u^C_j, \quad u^C_j(x,t) := \hat{u}^C_j(\rho_\varepsilon(x,t), H_\varepsilon(x,t), \sigma(x,t), t) \quad \text{for } (x,t) \in \overline{\Gamma^C(2\delta, 2\mu_1)},$$

where

$$\hat{u}^C_j : \overline{\mathbb{R}^2_+} \times \partial\Sigma \times [0,T] \to \mathbb{R} : (\rho, H, \sigma, t) \mapsto \hat{u}^C_j(\rho, H, \sigma, t)$$

for $j = 1, ..., M$. Moreover, we set $u^C_{M+1} := u^C_{M+2} := 0$ and $\hat{u}^C_\varepsilon := \sum_{j=1}^{M} \varepsilon^j \hat{u}^C_j$. As in the 2D-case, instead of (AC1) for $u_\varepsilon = u^I_\varepsilon + u^C_\varepsilon$, we will expand

$$\partial_t u^C_\varepsilon - \Delta u^C_\varepsilon + \frac{1}{\varepsilon^2}\left[f'(u^I_\varepsilon + u^C_\varepsilon) - f'(u^I_\varepsilon)\right] = 0 \tag{5.38}$$

into ε-series with coefficients in $(\rho_\varepsilon, H_\varepsilon, \sigma, t)$. As in the 2D-case we call (5.38) the "bulk equation" and expand it up to $\mathcal{O}(\varepsilon^{M-2})$. Moreover, we will expand (AC2) for $u_\varepsilon = u^I_\varepsilon + u^C_\varepsilon$ into ε-series with coefficients in $(\rho_\varepsilon, \sigma, t)$ up to $\mathcal{O}(\varepsilon^{M-1})$. Altogether we end up with similar equations as in Section 5.1.2. Here besides $t \in [0,T]$ also points on $\partial\Sigma$ enter as independent variables in the model problems on $\mathbb{R}^2_+$. The solvability condition (4.5) yields the boundary conditions on $\partial\Sigma \times [0,T]$ for the height functions h_j.

For the expansion we calculate the action of the differential operators on u^C_ε in the next lemma. Here for $\nabla_{\partial\Sigma}$ and ∇_Σ we use similar conventions as in Lemma 5.12.

Lemma 5.17. *Let* $\overline{\mathbb{R}^2_+} \times \partial\Sigma \times [0,T] \ni (\rho, H, \sigma, t) \mapsto \hat{w}(\rho, H, \sigma, t) \in \mathbb{R}$ *be sufficiently smooth and let* $w : \overline{\Gamma^C(2\delta, 2\mu_1)} \to \mathbb{R} : (x,t) \mapsto \hat{w}(\rho_\varepsilon(x,t), H_\varepsilon(x,t), \sigma(x,t), t)$. *Then*

$$
\begin{aligned}
\partial_t w &= \partial_\rho \hat{w} \left[\frac{\partial_t r}{\varepsilon} - (\partial_t h_\varepsilon + \partial_t s \cdot \nabla_\Sigma h_\varepsilon) \right] + \partial_H \hat{w} \frac{\partial_t b}{\varepsilon} + \partial_t \sigma \cdot \nabla_{\partial\Sigma} \hat{w} + \partial_t \hat{w}, \\
\nabla w &= \partial_\rho \hat{w} \left[\frac{\nabla r}{\varepsilon} - (D_x s)^\top \nabla_\Sigma h_\varepsilon \right] + \partial_H \hat{w} \frac{\nabla b}{\varepsilon} + (D_x \sigma)^\top \nabla_{\partial\Sigma} \hat{w}, \\
\Delta w &= \partial_\rho \hat{w} \left[\frac{\Delta r}{\varepsilon} - \left(\Delta s \cdot \nabla_\Sigma h_\varepsilon + \sum_{i,l=1}^N \nabla s_i \cdot \nabla s_l (\nabla_\Sigma)_i (\nabla_\Sigma)_l h_\varepsilon \right) \right] + \partial_H \hat{w} \frac{\Delta b}{\varepsilon} \\
&\quad + \partial_H^2 \hat{w} \frac{|\nabla b|^2}{\varepsilon^2} + \partial_\rho^2 \hat{w} \left| \frac{\nabla r}{\varepsilon} - (D_x s)^\top \nabla_\Sigma h_\varepsilon \right|^2 + 2 \partial_\rho \partial_H \hat{w} \, \frac{\nabla b}{\varepsilon} \cdot \left[\frac{\nabla r}{\varepsilon} - (D_x s)^\top \nabla_\Sigma h_\varepsilon \right] \\
&\quad + 2 \left((D_x \sigma)^\top \nabla_{\partial\Sigma} \partial_\rho \hat{w} \right) \cdot \left[\frac{\nabla r}{\varepsilon} - (D_x s)^\top \nabla_\Sigma h_\varepsilon \right] + 2 \left((D_x \sigma)^\top \nabla_{\partial\Sigma} \partial_H \hat{w} \right) \cdot \frac{\nabla b}{\varepsilon} \\
&\quad + \Delta\sigma \cdot \nabla_{\partial\Sigma} \hat{w} + \sum_{i,l=1}^N \nabla \sigma_i \cdot \nabla \sigma_l (\nabla_{\partial\Sigma})_i (\nabla_{\partial\Sigma})_l \hat{w},
\end{aligned}
$$

where the w*-terms on the left hand side and derivatives of* r *or* s *are evaluated at* (x,t)*, the* h_ε*-terms at* $(s(x,t),t)$ *and the* $\hat{w}$*-terms at* $(\rho_\varepsilon(x,t), H_\varepsilon(x,t), \sigma(x,t), t)$.

Proof. This can be shown in a similar manner as in the proof of Lemma 5.12. □

Remark 5.18. The formulas in Lemma 5.17 without the $\nabla_{\partial\Sigma}$-terms directly correspond to Lemma 5.5. The structure of the new terms is similar to known ones whereas their ε-order is the same or higher. Later these will only contribute to lower order remainder terms in the expansion.

5.2.2.1 Contact Point Expansion: The Bulk Equation We expand the f'-part in (5.38): If the u_j^I, u_j^C are bounded, the Taylor expansion yields on $\overline{\Gamma(2\delta)}$

$$
f'(u_\varepsilon^I + u_\varepsilon^C) = f'(\theta_0) + \sum_{k=1}^{M+2} \frac{1}{k!} f^{(k+1)}(\theta_0) \left[\sum_{j=1}^{M+1} \varepsilon^j (u_j^I + u_j^C) \right]^k + \mathcal{O}(\varepsilon^{M+3})
$$

and as in the 2D-case one can combine the latter with the expansion for $f'(u_\varepsilon^I)$ in (5.29) to deduce analogous assertions as after (5.13) for the explicit terms and the remainder terms in the asymptotic expansion for $f'(u_\varepsilon^I + u_\varepsilon^C) - f'(u_\varepsilon^I)$.

Moreover, we expand terms arising from Lemma 5.17 in (5.38) that are functions of (s,t) or (ρ, s, t), i.e. all the h_j-terms and the u_j^I-terms from the f'-expansion, respectively, as well as the terms depending on (x,t), i.e. all the derivatives of r, b, s, σ. Therefore we consider smooth $g_1 : \Sigma \times [0,T] \to \mathbb{R}$ or $g_1 : \mathbb{R} \times \Sigma \times [0,T] \to \mathbb{R}$ such that $\tilde{g}_1 := g_1|_{s=Y}$ admits bounded derivatives in b, where $Y : \partial\Sigma \times [0, 2\mu_1] \to \Sigma : (\sigma, b) \mapsto Y(\sigma, b)$. Due to $b = \varepsilon H_\varepsilon$ we apply a Taylor expansion to obtain uniformly

$$
g_1|_{b=\varepsilon H} = \tilde{g}_1|_{b=0} + \sum_{k=1}^{M+2} (\varepsilon H)^k \frac{\partial_b^k \tilde{g}_1|_{b=0}}{k!} + \mathcal{O}((\varepsilon H)^{M+3}) \quad \text{for } H \in [0, \frac{2\mu_1}{\varepsilon}]. \tag{5.39}
$$

Furthermore, let $g_2 : \overline{\Gamma^C(2\delta, 2\mu_1)} \to \mathbb{R}$ be smooth. For convenience we define

$$\overline{X}_1 : [-2\delta, 2\delta] \times [0, 2\mu_1] \times \partial\Sigma \times [0,T] \to \overline{\Gamma^C(2\delta, 2\mu_1)} : (r, b, \sigma, t) \mapsto \overline{X}(r, Y(\sigma, b), t).$$

Then a Taylor expansion yields

$$\tilde{g}_2(r, b, \sigma, t) := g_2(\overline{X}_1(r, b, \sigma, t)) = \sum_{j+k=0}^{M+2} \frac{\partial_r^j \partial_s^k \tilde{g}_2|_{(0,\sigma,t)}}{j!\, k!} r^j b^k + \mathcal{O}(|(r,b)|^{M+3}) \tag{5.40}$$

uniformly in $(r, b, \sigma, t) \in [-2\delta, 2\delta] \times [0, 2\mu_1] \times \partial\Sigma \times [0,T]$. Later we evaluate at

$$r = \varepsilon(\rho_\varepsilon(x,t) + h_\varepsilon(s(x,t),t)), \quad b = \varepsilon H_\varepsilon(x,t), \quad \sigma = \sigma(x,t) \quad \text{for } (x,t) \in \overline{\Gamma^C(2\delta, 2\mu_1)}$$

and expand h_ε with (5.39). Then we replace $(\rho_\varepsilon, H_\varepsilon)$ by arbitrary $(\rho, H) \in \overline{\mathbb{R}^2_+}$. The terms in the resulting expansion are analogous as in the 2D-case, but we include the details for the convenience of the reader. For $k = 1, ..., M+2$ we obtain

$\mathcal{O}(1)$: $g_2|_{\overline{X}_0(\sigma,t)}$

$\mathcal{O}(\varepsilon)$: $\partial_r \tilde{g}_2|_{(0,\sigma,t)}(\rho + h_1|_{(\sigma,t)}) + \partial_b \tilde{g}_2|_{(0,\sigma,t)} H.$

$\mathcal{O}(\varepsilon^k)$: [some polynomial in $(\rho, H, \partial_b^l \tilde{h}_j|_{(\sigma,t)})$, $l = 0, ..., k-1$, $j = 1, ..., k$ of order $\leq k$, where the coefficients are multiples of $\partial_r^{l_1} \partial_b^{l_2} \tilde{g}_2|_{(0,\sigma,t)}$, $l_1, l_2 \in \mathbb{N}_0$, $l_1 + l_2 \leq k$],

where $h_{M+1} = h_{M+2} = 0$ by definition. Again the order $\mathcal{O}(\varepsilon)$ is not needed explicitly and just included for clarity. The other explicit terms in (5.39) are estimated by ε^{M+3} times some polynomial in $(|\rho_\varepsilon|, H_\varepsilon)$. In the end these terms and the remainder in (5.39) are multiplied with exponentially decaying terms and hence become $\mathcal{O}(\varepsilon^{M+3})$.

As in the 2D-case we use some notation for the higher orders in the expansion:

Definition 5.19 (**Notation for Contact Point Expansion of (AC) in ND**).

1. We denote with (θ_0, u_1^I) the *zero-th order* and with (h_j, u_{j+1}^I, u_j^C) the *j-th order* for $j = 1, ..., M$.
2. Let $k \in \{-1, ..., M+2\}$ and $\beta, \gamma > 0$. Then $R^C_{k,(\beta,\gamma)}$ denotes the set of smooth functions $R : \overline{\mathbb{R}^2_+} \times \partial\Sigma \times [0,T] \to \mathbb{R}$ depending only on the j-th orders for $0 \leq j \leq \min\{k, M\}$ and such that uniformly in (ρ, H, σ, t):
$$|\partial_\rho^i \partial_H^l (\nabla_{\partial\Sigma})_{n_1} ... (\nabla_{\partial\Sigma})_{n_d} \partial_t^n R(\rho, H, \sigma, t)| = \mathcal{O}(e^{-(\beta|\rho| + \gamma H)})$$
for all $n_1, ..., n_d \in \{1, ..., N\}$ and $d, i, l, n \in \mathbb{N}_0$.
3. For $k \in \{-1, ..., M+2\}$ and $\beta, \gamma > 0$ the set $R^C_{k,(\beta)}$ is defined analogous to $R^C_{k,(\beta,\gamma)}$ but without the H-dependence.

In the following we expand the bulk equation (5.38) with the above formulas into ε-series with coefficients in $(\rho_\varepsilon, H_\varepsilon, \sigma, t)$. This is analogous to the 2D-case, cf. Section 5.1.2.1.

5.2.2.1.1 Bulk Equation: $\mathcal{O}(\varepsilon^{-1})$ The lowest order $\mathcal{O}(\frac{1}{\varepsilon})$ in (5.38) vanishes if

$$[-\Delta^{\sigma,t} + f''(\theta_0(\rho))]\hat{u}_1^C(\rho, H, \sigma, t) = 0, \tag{5.41}$$

where $\Delta^{\sigma,t} := \partial_\rho^2 + |\nabla b|^2|_{\overline{X}_0(\sigma,t)} \partial_H^2$ and we used $\nabla r \cdot \nabla b|_{\overline{X}_0(\sigma,t)} = 0$ for all $(\sigma,t) \in \partial\Sigma \times [0,T]$.

5.2.2.1.2 Bulk Equation: $\mathcal{O}(\varepsilon^{k-1})$ For $k = 1, ..., M-1$ we assume that the j-th order is constructed for $j = 0, ..., k$, that it is smooth and that it holds $\hat{u}^I_{j+1} \in R^I_{j,(\beta_1)}$ for every $\beta_1 \in (0, \min\{\sqrt{f''(\pm 1)}\})$ (bounded and all derivatives bounded is enough here) and we assume $\hat{u}^C_j \in R^C_{j,(\beta,\gamma)}$ for every $\beta \in (0, \min\{\overline{\beta}(\gamma), \sqrt{f''(\pm 1)}\})$, $\gamma \in (0, \overline{\gamma})$, where $\overline{\beta}, \overline{\gamma}$ are as in Theorem 4.11. Analogously to the 2D-case, cf. Section 5.1.2.1.2, the $\mathcal{O}(\varepsilon^{k-1})$-order in the expansion for the bulk equation (5.38) is zero if

$$[-\Delta^{\sigma,t} + f''(\theta_0)]\hat{u}^C_{k+1} = G_k(\rho, H, \sigma, t), \tag{5.42}$$

where $G_k \in R^C_{k,(\beta,\gamma)}$.

5.2.2.2 Contact Point Expansion: The Neumann Boundary Condition As in the 2D-case, the boundary conditions complementing (5.41) and (5.42) will be obtained from the expansion of the Neumann boundary condition (AC2) for $u_\varepsilon = u^I_\varepsilon + u^C_\varepsilon$, i.e. $N_{\partial\Omega} \cdot \nabla(u^I_\varepsilon + u^C_\varepsilon)|_{\partial Q_T} = 0$. Lemma 5.12 and Lemma 5.17 yield on $\overline{\Gamma^C(2\delta, 2\mu_1)}$

$$\nabla u^I_\varepsilon|_{(x,t)} = \partial_\rho \hat{u}^I_\varepsilon|_{(\rho,s,t)} \left[\frac{\nabla r|_{(x,t)}}{\varepsilon} - (D_x s)^\top|_{(x,t)} \nabla_\Sigma h_\varepsilon|_{(s,t)}\right] + (D_x s)^\top|_{(x,t)} \nabla_\Sigma \hat{u}^I_\varepsilon|_{(\rho,s,t)},$$

$$\begin{aligned}\nabla u^C_\varepsilon|_{(x,t)} &= \partial_\rho \hat{u}^C_\varepsilon|_{(\rho,H,\sigma,t)} \left[\frac{\nabla r|_{(x,t)}}{\varepsilon} - (D_x s)^\top|_{(x,t)} \nabla_\Sigma h_\varepsilon|_{(s,t)}\right] + \frac{\nabla b|_{(x,t)}}{\varepsilon} \partial_H \hat{u}^C_\varepsilon|_{(\rho,H,\sigma,t)} \\ &\quad + (D_x \sigma)^\top|_{(x,t)} \nabla_{\partial\Sigma} \hat{u}^C_\varepsilon|_{(\rho,H,\sigma,t)},\end{aligned}$$

where $\rho = \rho_\varepsilon(x,t), H = H_\varepsilon(x,t)$, $s = s(x,t)$ and $\sigma = \sigma(x,t)$. We consider the points $x = X(r,\sigma,t)$ for $(r,\sigma,t) \in [-2\delta, 2\delta] \times \partial\Sigma \times [0,T]$, in particular $H = 0$ and $s = \sigma$.

For $g : \overline{\Gamma(2\delta)} \cap \partial Q_T \to \mathbb{R}$ smooth we use a Taylor expansion similar to (5.30):

$$\tilde{g}(r,\sigma,t) := g(\overline{X}(r,\sigma,t)) = \sum_{k=0}^{M+2} \frac{\partial^k_r \tilde{g}|_{(0,\sigma,t)}}{k!} r^k + \mathcal{O}(|r|^{M+3}). \tag{5.43}$$

Then we insert $r = \varepsilon(\rho_\varepsilon + h_\varepsilon|_{(\sigma,t)})$ and replace ρ_ε by an arbitrary $\rho \in \mathbb{R}$. The analogous assertions as in the 2D-case after (5.18) hold for the explicit terms and the remainders in (5.43).

In the following we expand the Neumann boundary condition into ε-series with coefficients in $(\rho_\varepsilon, \sigma, t)$ up to the order $\mathcal{O}(\varepsilon^{M-1})$. This works analogously as in the 2D-case in Section 5.1.2.2.

5.2.2.2.1 Neumann Boundary Condition: $\mathcal{O}(\varepsilon^{-1})$ At the lowest order $\mathcal{O}(\frac{1}{\varepsilon})$ we have $(N_{\partial\Omega} \cdot \nabla r)|_{\overline{X}_0(\sigma,t)} \theta'_0(\rho) = 0$. This is valid due to the $90°$-contact angle condition.

5.2.2.2.2 Neumann Boundary Condition: $\mathcal{O}(\varepsilon^0)$ The order $\mathcal{O}(1)$ is zero if we require

$$(N_{\partial\Omega} \cdot \nabla b)|_{\overline{X}_0(\sigma,t)} \partial_H \hat{u}^C_1|_{H=0}(\rho,\sigma,t) = g_1(\rho,\sigma,t), \tag{5.44}$$

$$g_1|_{(\rho,\sigma,t)} := \theta'_0[(D_x s N_{\partial\Omega})^\top|_{\overline{X}_0(\sigma,t)} \nabla_\Sigma h_1|_{(\sigma,t)} - \partial_r((N_{\partial\Omega} \cdot \nabla r) \circ \overline{X})|_{(0,\sigma,t)} h_1|_{(\sigma,t)}] + \tilde{g}_0|_{(\rho,\sigma,t)},$$

where $\tilde{g}_0(\rho,\sigma,t) := -\rho\theta'_0(\rho)\partial_r((N_{\partial\Omega} \cdot \nabla r) \circ \overline{X})|_{(0,\sigma,t)}$. For $j = 1, ..., M$ let

$$\overline{u}^C_j : \overline{\mathbb{R}^2_+} \times \partial\Sigma \times [0,T] \to \mathbb{R} : (\rho, H, \sigma, t) \mapsto \hat{u}^C_j(\rho, |\nabla b|(\overline{X}_0(\sigma,t))H, \sigma, t). \tag{5.45}$$

Note that $|\nabla b|_{\overline{X}_0(\sigma,t)}| \geq c > 0$ and $|N_{\partial\Omega} \cdot \nabla b|_{\overline{X}_0(\sigma,t)}| \geq c > 0$ for all $(\sigma, t) \in \partial\Sigma \times [0, T]$ because of Theorem 3.7. Hence equations (5.41) and (5.44) for $\hat{u}_1^C$ are equivalent to

$$[-\Delta + f''(\theta_0(\rho))]\overline{u}_1^C = 0, \tag{5.46}$$

$$-\partial_H \overline{u}_1^C|_{H=0} = (|\nabla b|/N_{\partial\Omega} \cdot \nabla b)|_{\overline{X}_0(\sigma,t)} g_1(\rho, \sigma, t). \tag{5.47}$$

The solvability condition (4.5) belonging to (5.46)-(5.47) is

$$(|\nabla b|/N_{\partial\Omega} \cdot \nabla b)|_{\overline{X}_0(\sigma,t)} \int_{\mathbb{R}} g_1(\rho, \sigma, t)\theta_0'(\rho)\, d\rho = 0.$$

This yields a linear boundary condition for h_1:

$$b_1(\sigma, t) \cdot \nabla_\Sigma h_1|_{(\sigma,t)} + b_0(\sigma, t) h_1|_{(\sigma,t)} = f_0(\sigma, t) \quad \text{for } (\sigma, t) \in \partial\Sigma \times [0, T], \tag{5.48}$$

where

$$b_1(\sigma, t) := (|\nabla b|/N_{\partial\Omega} \cdot \nabla b)|_{\overline{X}_0(\sigma,t)} (D_x s N_{\partial\Omega})|_{\overline{X}_0(\sigma,t)} \in \mathbb{R}^N,$$
$$b_0(\sigma, t) := -(|\nabla b|/N_{\partial\Omega} \cdot \nabla b)|_{\overline{X}_0(\sigma,t)} \partial_r((N_{\partial\Omega} \cdot \nabla r) \circ \overline{X})|_{(0,\sigma,t)} \in \mathbb{R},$$
$$f_0(\sigma, t) := (|\nabla b|/N_{\partial\Omega} \cdot \nabla b)|_{\overline{X}_0(\sigma,t)} \int_{\mathbb{R}} \theta_0'(\rho)\tilde{g}_0(\rho, \sigma, t)\, d\rho / \|\theta_0'\|^2_{L^2(\mathbb{R})} \in \mathbb{R}$$

are smooth in $(\sigma, t) \in \partial\Sigma \times [0, T]$. Together with the linear parabolic equation (5.32) for h_1 from Subsection 5.2.1.3, we obtain a time-dependent linear parabolic boundary value problem for h_1, where the initial value $h_1|_{t=0}$ is not prescribed.

Remark 5.20. If f is even, then so is θ_0' and thus $f_0 = 0$. Therefore the boundary condition (5.48) for h_1 is homogeneous and because of Remark 5.16 we can choose $h_1 = 0$ in this case.

Now we solve (5.32) together with (5.48). We show that the principal part in (5.32) satisfies a suitable ellipticity condition and that (5.48) fulfils a non-tangentiality condition in local coordinates. Based on this one can show maximal regularity results in Hölder spaces (similar to Lunardi, Sinestrari, von Wahl [LSW]) and Sobolev spaces (similar to Prüss, Simonett [PS], Chapter 6.1-6.4 and Denk, Hieber, Prüss [DHP]) with typical localization procedures. This always involves compatibility conditions for the initial value. In our case these can be avoided via extension arguments as in Section 5.1.2.2.2. All these arguments involve many technical computations, but are in principle well-known. Therefore we refrain from going into details.

The Ellipticity Condition. Let $y : U \subseteq \Sigma \to V \subseteq \overline{\mathbb{R}^{N-1}_+}$ be a chart. Moreover, we denote with $(g_{ij})_{i,j=1}^{N-1} : V \to \mathbb{R}^{(N-1)\times(N-1)}$ the local representation of the Euclidean metric on Σ and let $(g^{kl})_{k,l=1}^{N-1}$ denote its pointwise inverse. Then one can show with local representations that for $h : U \to \mathbb{R}$ sufficiently smooth and $i, j = 1, ..., N$ it holds

$$\begin{aligned}[(\nabla_\Sigma)_i(\nabla_\Sigma)_j h] \circ y^{-1} = \sum_{p,q,k,l=1}^{N-1} & g^{pq} g^{kl} \partial_{v_q}(y^{-1})_i \partial_{v_l}(y^{-1})_j \partial_{v_p}\partial_{v_k}(h \circ y^{-1}) \\ & + g^{pq}\partial_{v_q}(y^{-1})_i \left[\partial_{v_p} g^{kl} \partial_{v_l}(y^{-1})_j + g^{kl}\partial_{v_p}\partial_{v_k}(y^{-1})_j\right] \partial_{v_k}(h \circ y^{-1}).\end{aligned}$$

Therefore the principal part of $\sum_{i,j=1}^N \nabla s_i \cdot \nabla s_j|_{\overline{X}_0(.,t)}(\nabla_\Sigma)_i(\nabla_\Sigma)_j$ for fixed $t \in [0,T]$ in the local coordinates with respect to y is given by

$$\sum_{p,k=1}^{N-1} A_{p,k}\partial_{v_p}\partial_{v_k}, \quad A_{p,k} := \sum_{i,j=1}^{N} \nabla s_i \cdot \nabla s_j|_{\overline{X}_0(y^{-1},t)} \sum_{q,l=1}^{N-1} g^{pq}g^{kl}\partial_{v_q}(y^{-1})_i\partial_{v_l}(y^{-1})_j.$$

For any appropriate ellipticity notion, it is enough to prove that $A := (A_{p,k})_{p,k=1}^{N-1}$ is uniformly positive definite on compact subsets of V. First we show that A is pointwise positive definite. Therefore we represent $\partial_{x_n} s|_{\overline{X}_0(y^{-1}(.),t)} = \sum_{\mu=1}^{N-1} \tilde{A}_{n,\mu}|_{(.,t)}\partial_{v_\mu}(y^{-1})$ on V for $n = 1, ..., N$ and we denote $\tilde{A} := (\tilde{A}_{n,\mu}|_{(.,t)})_{n,\mu=1}^{N,N-1} : V \to \mathbb{R}^{N\times(N-1)}$. Then it holds for all $p, k = 1, ..., N-1$

$$\begin{aligned} A_{p,k} &= \sum_{i,j,n=1}^{N} \sum_{\mu,\nu,q,l=1}^{N-1} \tilde{A}_{n,\mu}\tilde{A}_{n,\nu}g^{pq}g^{kl}\partial_{v_\mu}(y^{-1})_i\partial_{v_q}(y^{-1})_i\partial_{v_\nu}(y^{-1})_j\partial_{v_l}(y^{-1})_j \\ &= \sum_{n=1}^{N} \sum_{\mu,\nu,q,l=1}^{N-1} \tilde{A}_{n,\mu}\tilde{A}_{n,\nu}g^{pq}g^{kl}g_{\mu q}g_{\nu l} = \sum_{n=1}^{N} \sum_{\mu,\nu=1}^{N-1} \tilde{A}_{n,\mu}\tilde{A}_{n,\nu}\delta_\mu^p\delta_\nu^k = (\tilde{A}^\top\tilde{A})_{p,k}. \end{aligned}$$

Moreover, Theorem 3.7 yields that $(\partial_{x_n} s|_{\overline{X}_0(s,t)})_{n=1}^N$ generate $T_s\Sigma$ for all $s \in \Sigma$. Hence the matrix $\tilde{A}|_v \in \mathbb{R}^{N\times(N-1)}$ is injective for all $v \in V$. Finally, this yields

$$w^\top A w = |\tilde{A}w|^2 > 0 \quad \text{for all} \quad w \in \mathbb{R}^{N-1}.$$

Therefore A is pointwise positive definite. Since it is equivalent to prove the estimate for vectors on the sphere in $\mathbb{R}^{N-1}$, by compactness A is uniform positive definite on compact subsets of V. Altogether, the principal part in (5.32) satisfies a suitable ellipticity condition.

The Non-Tangentiality Condition. Let $y : U \subseteq \Sigma \to V \subseteq \overline{\mathbb{R}_+^{N-1}}$ be a chart with $U \cap \partial\Sigma \neq \emptyset$. Then for $h : U \to \mathbb{R}$ sufficiently smooth and fixed $t \in [0,T]$ it holds:

$$b_1(y^{-1},t)\cdot(\nabla_\Sigma h \circ y^{-1}) = \nabla_v(h\circ y^{-1}) \cdot \left[\sum_{l=1}^{N-1} g^{kl} b_1(y^{-1},t)\cdot\partial_{v_l}(y^{-1})\right]_{k=1}^{N-1}$$

on $V \cap (\mathbb{R}^{N-2} \times \{0\})$, where b_1 is defined below (5.48). Therefore the transformed boundary condition in the local coordinates with respect to y satisfies a non-tangentiality condition if $\sum_{l=1}^{N-1} g^{N-1,l} b_1(y^{-1},t)\cdot\partial_{v_l}(y^{-1}) \neq 0$ on the set $V \cap (\mathbb{R}^{N-2}\times\{0\})$. On the latter set we use the representation $b_1(y^{-1},t) = \sum_{n=1}^{N-1} B_n(y^{-1},t)\partial_{v_n}(y^{-1})$. Then the condition reads as $B_{N-1}(\sigma,t) \neq 0$ for all $(\sigma,t) \in (U\cap\partial\Sigma)\times[0,T]$. However, Theorem 3.7 yields $b_1(\sigma,t)\cdot\vec{n}_{\partial\Sigma}(\sigma) \neq 0$ for all $(\sigma,t) \in \partial\Sigma\times[0,T]$. Due to the properties of y, we know that $(\partial_{v_l}(y^{-1})|_{y(\sigma)})_{l=1}^{N-1}$ form a basis of $T_\sigma\Sigma$ and the first $N-2$ components are a basis of $T_\sigma\partial\Sigma$ for all $\sigma \in U \cap \partial\Sigma$. Because $\vec{n}_{\partial\Sigma}$ is orthogonal to $T_\sigma\partial\Sigma$, it necessarily holds $B_{N-1}(\sigma,t) \neq 0$ for all $(\sigma,t) \in (U\cap\partial\Sigma)\times[0,T]$. Therefore the boundary condition (5.48) satisfies a non-tangentiality condition in local coordinates.

Finally, we obtain a smooth solution h_1 to (5.32) and (5.48). Therefore $\hat{u}_2^I$ (solving (5.31)) is determined from Section 5.2.1.3 and it holds $\hat{u}_2^I \in R_{1,(\beta_1)}^I$ for every $\beta_1 \in (0, \min\{\sqrt{f''(\pm 1)}\})$. In particular the first inner order is computed. Moreover, it holds $g_1 \in \hat{R}_{1,(\beta_1)}^I$ for all β_1 as above because of Theorem 4.1. Hence with Theorem 4.11 (applied in local coordinates for $\partial\Sigma$) there is a unique smooth solution $\overline{u}_1^C$ to (5.46)-(5.47) and we get decay properties. By compactness and Remark 5.15 with $\partial\Sigma$ instead of Σ we obtain the decay property $\overline{u}_1^C \in R^C_{1,(\beta,\gamma)}$ for all $\beta \in (0, \min\{\overline{\beta}(\gamma), \sqrt{f''(\pm 1)}\})$, $\gamma \in (0, \overline{\gamma})$, where $\overline{\beta}, \overline{\gamma}$ are as in Theorem 4.11. Altogether the first order is determined.

5.2.2.2.3 Neumann Boundary Condition: $\mathcal{O}(\varepsilon^k)$ and Induction For $k = 1, ..., M-1$ we consider $\mathcal{O}(\varepsilon^k)$ in (AC2) for $u_\varepsilon = u_\varepsilon^I + u_\varepsilon^C$ and derive equations for the $(k+1)$-th order. We assume the following induction hypothesis: suppose that the j-th order already has been constructed for all $j = 0, ..., k$, that it is smooth and admits the decay $\hat{u}^I_{j+1} \in R^I_{j,(\beta_1)}$ for all $\beta_1 \in (0, \min\{\sqrt{f''(\pm 1)}\})$ as well as $\hat{u}^C_j \in R^C_{j,(\beta,\gamma)}$ for every $\beta \in (0, \min\{\overline{\beta}(\gamma), \sqrt{f''(\pm 1)}\})$, $\gamma \in (0, \overline{\gamma})$, where $\overline{\beta}, \overline{\gamma}$ are as in Theorem 4.11.

The assumption is valid for $k = 1$ due to Section 5.2.2.2.2. Analogously as in the 2D-case (cf. Section 5.1.2.2.2), one can show that the $\mathcal{O}(\varepsilon^k)$-order in (AC2) for $u_\varepsilon = u_\varepsilon^I + u_\varepsilon^C$ vanishes if

$$(N_{\partial\Omega} \cdot \nabla b)|_{\overline{X}_0(\sigma,t)} \partial_H \hat{u}^C_{k+1}|_{H=0}(\rho,\sigma,t) = g_{k+1}(\rho,\sigma,t), \tag{5.49}$$

$$\begin{aligned} g_{k+1}|_{(\rho,\sigma,t)} &:= \theta_0'(\rho)[(D_x s N_{\partial\Omega})^\top|_{\overline{X}_0(\sigma,t)} \nabla_\Sigma h_{k+1}|_{(\sigma,t)} - \partial_r((N_{\partial\Omega} \cdot \nabla r) \circ \overline{X})|_{(0,\sigma,t)} h_{k+1}|_{(\sigma,t)}] \\ &\quad + \tilde{g}_k(\rho,\sigma,t), \end{aligned}$$

where $\tilde{g}_k \in R^C_{k,(\beta)}$ and therefore $g_{k+1} \in \hat{R}^I_{k+1,(\beta)} + R^C_{k,(\beta)}$, if h_{k+1} is smooth.

As in the last Section 5.2.2.2.2, the equations (5.42), (5.49) are equivalent to

$$[-\Delta + f''(\theta_0(\rho))]\overline{u}^C_{k+1} = \overline{G}_k(\rho, H, \sigma, t), \tag{5.50}$$

$$-\partial_H \overline{u}^C_{k+1}|_{H=0} = (|\nabla b|/N_{\partial\Omega} \cdot \nabla b)|_{\overline{X}_0(\sigma,t)} g_{k+1}(\rho, H, \sigma, t), \tag{5.51}$$

where $\overline{u}^C_{k+1}$ was defined in (5.45) and $\overline{G}_k$ is defined analogously with the $G_k \in R^C_{k,(\beta,\gamma)}$ from Section 5.2.2.1.2. The corresponding compatibility condition (4.5), namely

$$\int_{\mathbb{R}^2_+} \overline{G}_k(\rho, H, \sigma, t)\theta_0'(\rho)\, d(\rho, H) + (|\nabla b|/N_{\partial\Omega} \cdot \nabla b)|_{\overline{X}_0(\sigma,t)} \int_{\mathbb{R}} g_{k+1}(\rho,\sigma,t)\theta_0'(\rho)\, d\rho = 0,$$

yields a linear boundary condition for h_{k+1}:

$$b_1(\sigma,t) \cdot \nabla_\Sigma h_{k+1}|_{(\sigma,t)} + b_0(\sigma,t) h_{k+1}|_{(\sigma,t)} = f^B_k(\sigma,t) \quad \text{for } (\sigma,t) \in \partial\Sigma \times [0,T], \tag{5.52}$$

where b_0, b_1 are defined below (5.48) and

$$f^B_k(\sigma,t) := -\frac{1}{\|\theta_0'\|^2_{L^2(\mathbb{R})}} \left[\int_{\mathbb{R}^2_+} \overline{G}_k|_{(\rho,H,\sigma,t)} \theta_0'(\rho)\, d\rho + \frac{|\nabla b|}{N_{\partial\Omega} \cdot \nabla b}\bigg|_{\overline{X}_0(\sigma,t)} \int_{\mathbb{R}} \tilde{g}_k|_{(\rho,\sigma,t)} \theta_0'(\rho)\, d\rho \right]$$

is smooth in $(\sigma,t) \in \partial\Sigma \times [0,T]$.

Because of the computations in the last Section 5.2.2.2.2 one can solve (5.37) from Section 5.2.1.4 together with (5.52) and get a smooth solution h_{k+1}. Therefore Section 5.2.1.4 yields $\hat{u}^I_{k+2}$ (solving (5.36)) with $\hat{u}^I_{k+2} \in R^I_{k+1,(\beta_1)}$ for all $\beta_1 \in (0, \min\{\sqrt{f''(\pm 1)}\})$. In particular the $(k+1)$-th inner order is computed and it holds $G_k \in R^C_{k,(\beta,\gamma)}$ as well as $g_{k+1} \in \hat{R}^I_{k+1,(\beta)} + R^C_{k,(\beta)}$ for all $\beta \in (0, \min\{\overline{\beta}(\gamma), \sqrt{f''(\pm 1)}\})$, $\gamma \in (0, \overline{\gamma})$, where $\overline{\beta}, \overline{\gamma}$ are as in Theorem 4.11. As in the last Section 5.2.2.2.2 we obtain a unique smooth solution $\overline{u}^C_{k+1}$ to (5.50)-(5.51) with the decay $\hat{u}^C_{k+1} \in R^C_{k,(\beta,\gamma)}$ for all (β,γ) as above. Altogether, the $(k+1)$-th order is constructed.

Finally, by induction the j-th order is determined for all $j = 0,...,k$, the h_j are smooth and $\hat{u}^I_{j+1} \in R^I_{j,(\beta_1)}$ for all $\beta_1 \in (0, \min\{\sqrt{f''(\pm 1)}\})$ as well as $\hat{u}^C_j \in R^C_{j,(\beta,\gamma)}$ for every $\beta \in (0, \min\{\overline{\beta}(\gamma), \sqrt{f''(\pm 1)}\})$, $\gamma \in (0, \overline{\gamma})$, where $\overline{\beta}, \overline{\gamma}$ are as in Theorem 4.11.

5.2.3 The Approximate Solution for (AC) in ND

Let $N \geq 2$ and $\Gamma := (\Gamma_t)_{t\in[0,T]}$ be as in Section 3.1 with contact angle $\alpha = \frac{\pi}{2}$ and a solution to MCF in Ω. Moreover, let $\delta > 0$ be such that the assertions of Theorem 3.7 hold for 2δ instead of δ and let r, s, b, σ, μ_1 be as in the theorem. Furthermore, let $M \in \mathbb{N}$, $M \geq 2$ be as in the beginning of Section 5.2. Let $\eta : \mathbb{R} \to [0,1]$ be smooth with $\eta(r) = 1$ for $|r| \leq 1$ and $\eta(r) = 0$ for $|r| \geq 2$. Then for $\varepsilon > 0$ we set

$$u^A_\varepsilon := \begin{cases} \eta(\frac{r}{\delta}) \left[u^I_\varepsilon + u^C_\varepsilon \eta(\frac{b}{\mu_1}) \right] + (1 - \eta(\frac{r}{\delta}))\mathrm{sign}(r) & \text{in } \overline{\Gamma(2\delta)}, \\ \pm 1 & \text{in } Q^\pm_T \setminus \Gamma(2\delta), \end{cases}$$

where u^I_ε and u^C_ε were constructed in Sections 5.2.1 and 5.2.2. Analogously to the 2D-case, cf. Section 5.1.3, one can show that this yields an approximate solution for (AC1)-(AC3) in the following sense:

Lemma 5.21. *The function u^A_ε is smooth, uniformly bounded with respect to x, t, ε and for the remainder $r^A_\varepsilon := \partial_t u^A_\varepsilon - \Delta u^A_\varepsilon + \frac{1}{\varepsilon^2} f'(u^A_\varepsilon)$ in* (AC1) *and $s^A_\varepsilon := \partial_{N_{\partial\Omega}} u^A_\varepsilon$ in* (AC2) *it holds*

$$\begin{aligned} |r^A_\varepsilon| &\leq C(\varepsilon^M e^{-c|\rho_\varepsilon|} + \varepsilon^{M+1}) && \text{in } \Gamma(2\delta, \mu_1), \\ |r^A_\varepsilon| &\leq C(\varepsilon^{M-1} e^{-c(|\rho_\varepsilon| + H_\varepsilon)} + \varepsilon^M e^{-c|\rho_\varepsilon|} + \varepsilon^{M+1}) && \text{in } \Gamma^C(2\delta, 2\mu_1), \\ r^A_\varepsilon &= 0 && \text{in } Q_T \setminus \Gamma(2\delta), \\ |s^A_\varepsilon| &\leq C\varepsilon^M e^{-c|\rho_\varepsilon|} && \text{on } \partial Q_T \cap \Gamma(2\delta), \\ s^A_\varepsilon &= 0 && \text{on } \partial Q_T \setminus \Gamma(2\delta) \end{aligned}$$

for $\varepsilon > 0$ small and some $c, C > 0$. Here ρ_ε is defined in (5.28) *and $H_\varepsilon = \frac{b}{\varepsilon}$.*

Remark 5.22. Analogous statements as in Remark 5.11 are valid.

5.3 Asymptotic Expansion of (vAC) in ND

Let $N \geq 2$, $\Omega \subseteq \mathbb{R}^N$, $\Gamma := (\Gamma_t)_{t\in[0,T]}$ and $\delta > 0$ be as in the beginning of Section 5.2, in particular Γ is a smooth solution to MCF with 90°-contact angle condition in Ω. Moreover, let $W : \mathbb{R}^m \to \mathbb{R}$ be as in Definition 1.4 and $\vec{u}_\pm$ be any distinct pair of minimizers of W. In this section we construct a smooth approximate solution $\vec{u}^A_\varepsilon$ to (vAC1)-(vAC3) with $\vec{u}^A_\varepsilon = \vec{u}_\pm$ in $Q^\pm_T \setminus \Gamma(2\delta)$, increasingly "steep" transition from $\vec{u}_-$ to $\vec{u}_+$ for $\varepsilon \to 0$ and such that $\{\vec{u}^A_\varepsilon = 0\}$ converges to Γ for $\varepsilon \to 0$. All computations are very similar to the ones in Section 5.2. We just have to incorporate vector-valued functions and for the appearing vector-valued model problems we use the corresponding solution theorems in Sections 4.3-4.4. For the latter we make the assumption $\dim\ker \check{L}_0 = 1$, where $\check{L}_0$ is as in Remark 4.28 for a solution $\vec{\theta}_0$ as in Theorem 4.26.

Let $M \in \mathbb{N}$ with $M \geq 2$. Again we introduce height functions $\check{h}_j : \Sigma \times [0,T] \to \mathbb{R}$ for $j = 1, ..., M$ and $\check{h}_\varepsilon := \sum_{j=1}^{M} \varepsilon^{j-1}\check{h}_j$. Moreover, we set $\check{h}_{M+1} := \check{h}_{M+2} := 0$ and we define the scaled variable

$$\check{\rho}_\varepsilon(x,t) := \frac{r(x,t)}{\varepsilon} - \check{h}_\varepsilon(s(x,t),t) \quad \text{for } (x,t) \in \overline{\Gamma(2\delta)}. \tag{5.53}$$

In Section 5.3.1 we construct the inner expansion and in Section 5.3.2 the contact point expansion. Finally, in Section 5.3.3 the result on the approximation error of $\vec{u}_\varepsilon^A$ can be found.

5.3.1 Inner Expansion of (vAC) in ND

For the inner expansion we consider the following ansatz: Let $\varepsilon > 0$ be small and

$$\vec{u}_\varepsilon^I := \sum_{j=0}^{M+1} \varepsilon^j \check{u}_j^I, \quad \vec{u}_j^I(x,t) := \check{u}_j^I(\rho_\varepsilon(x,t), s(x,t), t) \quad \text{for } (x,t) \in \overline{\Gamma(2\delta)},$$

where

$$\check{u}_j^I : \mathbb{R} \times \Sigma \times [0,T] \to \mathbb{R}^m : (\rho, s, t) \mapsto \check{u}_j^I(\rho, s, t)$$

for $j = 0, ..., M+1$. Moreover, we set $\vec{u}_{M+2}^I := 0$ and $\check{u}_\varepsilon^I := \sum_{j=0}^{M+1} \varepsilon^j \check{u}_j^I$. We will expand (vAC1) for $\vec{u}_\varepsilon = \vec{u}_\varepsilon^I$ into ε-series with coefficients in $(\check{\rho}_\varepsilon, s, t)$ up to $\mathcal{O}(\varepsilon^{M-1})$. This yields equations of analogous form as in the scalar case in Section 5.2.1. Therefore we have to compute the action of the differential operators on $\vec{u}_\varepsilon^I$.

In the following we use the same conventions as in Lemma 5.12. Moreover, for a sufficiently smooth $\vec{g} : \Sigma \to \mathbb{R}^m$ we set $D_\Sigma \vec{g} := (\nabla_\Sigma g_1, ..., \nabla_\Sigma g_m)^\top : \Sigma \to \mathbb{R}^{m \times N}$.

Lemma 5.23. *Let $\varepsilon > 0$, $\check{w} : \mathbb{R} \times \Sigma \times [0,T] \to \mathbb{R}^m$ be sufficiently smooth and $\vec{w} : \overline{\Gamma(2\delta)} \to \mathbb{R}^m$ be defined by $\vec{w}(x,t) := \check{w}(\check{\rho}_\varepsilon(x,t), s(x,t), t)$ for all $(x,t) \in \overline{\Gamma(2\delta)}$. Then it holds*

$$\begin{aligned}
\partial_t \vec{w} &= \partial_\rho \check{w} \left[\frac{\partial_t r}{\varepsilon} - (\partial_t \check{h}_\varepsilon + \partial_t s \cdot \nabla_\Sigma \check{h}_\varepsilon) \right] + D_\Sigma \check{w}\, \partial_t s + \partial_t \check{w}, \\
D_x \vec{w} &= \partial_\rho \check{w} \left[\frac{\nabla r}{\varepsilon} - (D_x s)^\top \nabla_\Sigma \check{h}_\varepsilon \right]^\top + D_\Sigma \check{w} D_x s, \\
\Delta \vec{w} &= \partial_\rho \check{w} \left[\frac{\Delta r}{\varepsilon} - \left(\Delta s \cdot \nabla_\Sigma \check{h}_\varepsilon + \sum_{i,l=1}^{N} \nabla s_i \cdot \nabla s_l (\nabla_\Sigma)_i (\nabla_\Sigma)_l \check{h}_\varepsilon \right) \right] \\
&\quad + D_\Sigma \check{w}\, \Delta s + \sum_{i,l=1}^{N} \nabla s_i \cdot \nabla s_l (\nabla_\Sigma)_i (\nabla_\Sigma)_l \check{w} \\
&\quad + 2 D_\Sigma \partial_\rho \check{w}\, D_x s \left[\frac{\nabla r}{\varepsilon} - (D_x s)^\top \nabla_\Sigma \check{h}_\varepsilon \right] + \partial_\rho^2 \check{w} \left| \frac{\nabla r}{\varepsilon} - (D_x s)^\top \nabla_\Sigma \check{h}_\varepsilon \right|^2,
\end{aligned}$$

where the $\vec{w}$-terms on the left hand side and derivatives of r or s are evaluated at $(x,t) \in \overline{\Gamma(2\delta)}$, the $\check{h}_\varepsilon$-terms at $(s(x,t),t)$ and the $\check{w}$-terms at $(\check{\rho}_\varepsilon(x,t), s(x,t), t)$.

Proof of Lemma 5.23. This follows from Lemma 5.12 applied to every component. □

For the expansion of (vAC1) for $\vec{u}_\varepsilon = \vec{u}^I_\varepsilon$ we use Taylor expansions again. For the ∇W-part this yields: If the $\vec{u}^I_j$ are bounded, then

$$\nabla W(\vec{u}^I_\varepsilon) = \nabla W(\vec{u}^I_0) + \sum_{\nu\in\mathbb{N}_0^m, |\nu|=1}^{M+2} \frac{\partial_y^\nu \nabla W(\vec{u}^I_0)}{\nu!} \left[\sum_{j=1}^{M+1} \vec{u}^I_j \varepsilon^j\right]^\nu + \mathcal{O}(\varepsilon^{M+3}) \text{ on } \overline{\Gamma(2\delta)}. \quad (5.54)$$

The terms in the ε-expansion that are needed explicitly are

$$\begin{aligned}
\mathcal{O}(1): &\quad \nabla W(\vec{u}^I_0),\\
\mathcal{O}(\varepsilon): &\quad D^2W(\vec{u}^I_0)\vec{u}^I_1,\\
\mathcal{O}(\varepsilon^2): &\quad D^2W(\vec{u}^I_0)\vec{u}^I_2 + \sum_{\nu\in\mathbb{N}_0^m, |\nu|=2} \frac{\partial_y^\nu \nabla W(\vec{u}^I_0)}{\nu!} \left[\vec{u}^I_j\right]^\nu.
\end{aligned}$$

For $k = 3, ..., M+2$ the order $\mathcal{O}(\varepsilon^k)$ is given by

$$\begin{aligned}
\mathcal{O}(\varepsilon^k): \quad D^2W(\vec{u}^I_0)\vec{u}^I_k + \quad &[\text{some polynomial in entries of } (\vec{u}^I_1, ..., \vec{u}^I_{k-1}) \text{ of order } \le k,\\
&\text{where the coefficients are multiples of } \partial_y^\nu \nabla W(\vec{u}^I_0),\\
&\nu \in \mathbb{N}_0^m, |\nu| = 2, ..., k \text{ and every term contains a } (\vec{u}^I_j)_n\text{-factor}].
\end{aligned}$$

The other explicit terms in (5.54) are of order $\mathcal{O}(\varepsilon^{M+3})$.

Moreover, we expand functions of $(x,t) \in \overline{\Gamma(2\delta)}$ into ε-series analogously to the scalar case, cf. the Taylor expansion (5.30) and the remarks there. We just replace h_j, ρ_ε by $\check{h}_j, \check{\rho}_\varepsilon$.

For the higher orders in the expansion we use analogous definitions as in the scalar case:

Definition 5.24 (Notation for Inner Expansion of (vAC)).

1. We call $(\vec{\theta}_0, \vec{u}^I_1)$ the *zero-th inner order* and $(\check{h}_j, \vec{u}^I_{j+1})$ the *j-th inner order* for $j = 1, ..., M$.
2. Let $k \in \{-1, ..., M+2\}$ and $\beta > 0$. We denote with $\check{R}^I_{k,(\beta)}$ the set of smooth vector-valued functions $\vec{R}: \mathbb{R} \times \Sigma \times [0,T] \to \mathbb{R}^m$ that depend only on the j-th inner orders for $0 \le j \le \min\{k, M\}$ and satisfy uniformly in (ρ, s, t):
$$|\partial_\rho^i (\nabla_\Sigma)_{n_1} ... (\nabla_\Sigma)_{n_d} \partial_t^n \vec{R}(\rho, s, t)| = \mathcal{O}(e^{-\beta|\rho|})$$
for all $n_1, ..., n_d \in \{1, ..., N\}$ and $d, i, l, n \in \mathbb{N}_0$.
3. For $k \in \{-1, ..., M+2\}$ and $\beta > 0$ the set $\tilde{R}^I_{k,(\beta)}$ is defined analogously to $\check{R}^I_{k,(\beta)}$ with functions of type $\vec{R}: \mathbb{R} \times \partial\Sigma \times [0,T] \to \mathbb{R}^m$ instead.

Now we expand (vAC1) for $\vec{u}_\varepsilon = \vec{u}^I_\varepsilon$ into ε-series. This is analogous to the scalar case in Section 5.2.1. In the following (ρ, s, t) are always in $\mathbb{R} \times \Sigma \times [0,T]$ and sometimes omitted.

5.3.1.1 Inner Expansion: $\mathcal{O}(\varepsilon^{-2})$ Using $|\nabla r|^2|_{\overline{X}_0(s,t)} = 1$ due to Theorem 3.7, the $\mathcal{O}(\frac{1}{\varepsilon^2})$-order is zero if

$$-\partial_\rho^2 \check{u}^I_0(\rho, s, t) + \nabla W(\check{u}^I_0(\rho, s, t)) = 0. \quad (5.55)$$

Since we want to connect the minima $\vec{u}_\pm$ of W, we require $\lim_{\rho\to\pm\infty}\check{u}_0^I(\rho,s,t)=\vec{u}_\pm$. Moreover, it is natural to ask for $R_{\vec{u}_-,\vec{u}_+}\check{u}_0^I|_{\rho=0}=\check{u}_0^I|_{\rho=0}$, since then $\check{u}_0^I|_{\rho=0}$ can be interpreted as being in the middle of the two phases $\vec{u}_\pm$. Here $R_{\vec{u}_-,\vec{u}_+}$ is as in Definition 1.4. By Theorem 4.26 there is a smooth $R_{\vec{u}_-,\vec{u}_+}$-odd $\vec{\theta}_0:\mathbb{R}\to\mathbb{R}^m$ such that $\check{u}_0^I(\rho,s,t):=\vec{\theta}_0(\rho)$ solves (5.55) and

$$\partial_z^l[\vec{\theta}_0-\vec{u}_\pm](\rho)=\mathcal{O}(e^{-\beta|\rho|})\quad \text{for } \rho\to\pm\infty \text{ and all } l\in\mathbb{N}_0,\beta\in\left(0,\sqrt{\lambda/2}\right),$$

where $\lambda>0$ is such that $D^2W(\vec{u}_\pm)\geq\lambda I$. Moreover, it holds $R_{\vec{u}_-,\vec{u}_+}\vec{\theta}_0'|_{\rho=0}\neq\vec{\theta}_0'|_{\rho=0}$.

5.3.1.2 Inner Expansion: $\mathcal{O}(\varepsilon^{-1})$ Analogously to the scalar case, cf. Section 5.2.1.2, it follows that the $\mathcal{O}(\frac{1}{\varepsilon})$-order cancels if

$$\check{\mathcal{L}}_0\check{u}_1^I(\rho,s,t)+\vec{\theta}_0'(\rho)(\partial_t r-\Delta r)|_{\overline{X}_0(s,t)}=0,\qquad \text{where } \check{\mathcal{L}}_0:=-\partial_\rho^2+D^2W(\vec{\theta}_0).$$

Moreover, it is natural to require that $R_{\vec{u}_-,\vec{u}_+}\check{u}_1^I|_{\rho=0}=\check{u}_1^I|_{\rho=0}$ since then heuristically $\check{u}_1^I|_{\rho=0}$ is in the middle of the two phases $\vec{u}_\pm$. Let us assume $\dim\ker\check{L}_0=1$ with respect to the spaces in (4.26), cf. Remark 4.28. Then due to Theorem 4.31 and Remarks 4.30, 4.32, 1. this parameter-dependent ODE together with the additional condition and suitable decay in $|\rho|$ has a unique solution $\check{u}_1^I$ if and only if $(\partial_t r-\Delta r)|_{\overline{X}_0(s,t)}=0$. The latter holds since it is equivalent to MCF for Γ by Theorem 3.7. Therefore we set $\check{u}_1^I:=0$.

5.3.1.3 Inner Expansion: $\mathcal{O}(\varepsilon^0)$ In the analogous way as in the scalar case, cf. Section 5.2.1.3, the $\mathcal{O}(1)$-term in the expansion cancels if we require

$$\begin{aligned}
-\check{\mathcal{L}}_0\check{u}_2^I(\rho,s,t)&=\vec{R}_1(\rho,s,t), \qquad (5.56)\\
\vec{R}_1(\rho,s,t):=\vec{\theta}_0'(\rho)&\Bigg[-\partial_t\check{h}_1+\sum_{i,l=1}^N\nabla s_i\cdot\nabla s_l(\nabla_\Sigma)_i(\nabla_\Sigma)_l\check{h}_1\\
&\quad+(\rho+\check{h}_1)\partial_r((\partial_t r-\Delta r)\circ\overline{X})|_{(0,s,t)}-(\partial_t s-\Delta s)|_{\overline{X}_0(s,t)}\cdot\nabla_\Sigma\check{h}_1\Bigg]\\
+\vec{\theta}_0''(\rho)&\Bigg[-\frac{1}{2}(\rho+\check{h}_1)^2\partial_r^2(|\nabla r|^2\circ\overline{X})|_{(0,s,t)}\\
&\quad+2(\rho+\check{h}_1)\partial_r((D_x s\nabla r)^\top\circ\overline{X})|_{(0,s,t)}\nabla_\Sigma\check{h}_1-\left|(D_x s)^\top|_{\overline{X}_0(s,t)}\nabla_\Sigma\check{h}_1\right|^2\Bigg].
\end{aligned}$$

If $\check{h}_1$ is smooth, then $\vec{R}_1$ is smooth and together with all derivatives decays exponentially in $|\rho|$ uniformly in (s,t) with rate β for every $\beta\in(0,\sqrt{\lambda/2})$ because of Theorem 4.26. Therefore Theorem 4.31 (applied in local coordinates for Σ) yields that there is a unique solution $\hat{u}_2^I$ to (5.56) together suitable regularity and decay as well as $R_{\vec{u}_-,\vec{u}_+}\check{u}_2^I|_{\rho=0}=\check{u}_2^I|_{\rho=0}$ if and only if $\int_\mathbb{R}\vec{R}_1(\rho,s,t)\cdot\vec{\theta}_0'(\rho)\,d\rho=0$. Because of integration by parts it holds $\int_\mathbb{R}\vec{\theta}_0'(\rho)\cdot\vec{\theta}_0''(\rho)\,d\rho=0$. Therefore the nonlinearities in $\check{h}_1$ cancel and we obtain a linear non-autonomous parabolic equation for $\check{h}_1$ on Σ:

$$\partial_t\check{h}_1-\sum_{i,l=1}^N\nabla s_i\cdot\nabla s_l|_{\overline{X}_0(s,t)}(\nabla_\Sigma)_i(\nabla_\Sigma)_l\check{h}_1+\check{a}_1\cdot\nabla_\Sigma\check{h}_1+\check{a}_0\check{h}_1=\check{f}_0 \qquad (5.57)$$

in $\Sigma \times [0,T]$. Here with

$$\check{d}_1 := \int_{\mathbb{R}} |\vec{\theta}_0'(\rho)|^2 \, d\rho, \qquad \check{d}_2 := \int_{\mathbb{R}} |\vec{\theta}_0'(\rho)|^2 \rho \, d\rho, \qquad \check{d}_3 := \int_{\mathbb{R}} |\vec{\theta}_0'(\rho)|^2 \rho^2 \, d\rho,$$
$$\check{d}_4 := \int_{\mathbb{R}} \vec{\theta}_0'(\rho) \cdot \vec{\theta}_0''(\rho) \rho \, d\rho, \quad \check{d}_5 := \int_{\mathbb{R}} \vec{\theta}_0'(\rho) \cdot \vec{\theta}_0''(\rho) \rho^2 \, d\rho, \quad \check{d}_6 := \int_{\mathbb{R}} \vec{\theta}_0'(\rho) \cdot \vec{\theta}_0''(\rho) \rho^3 \, d\rho,$$

we have defined for all $(s,t) \in \Sigma \times [0,T]$:

$$\check{a}_1(s,t) := (\partial_t s - \Delta s)|_{\overline{X}_0(s,t)} - 2\frac{\check{d}_4}{\check{d}_1} \partial_r ((D_x s \nabla r)^\top \circ \overline{X})|_{(0,s,t)} \in \mathbb{R}^N, \tag{5.58}$$

$$\check{a}_0(s,t) := -\partial_r ((\partial_t r - \Delta r) \circ \overline{X})|_{(0,s,t)} + \frac{\check{d}_4}{\check{d}_1} \partial_r^2 (|\nabla r|^2 \circ \overline{X})|_{(0,s,t)} \in \mathbb{R}, \tag{5.59}$$

$$\check{f}_0(s,t) := \frac{\check{d}_2}{\check{d}_1} \partial_r ((\partial_t r - \Delta r) \circ \overline{X})|_{(0,s,t)} - \frac{\check{d}_5}{2\check{d}_1} \partial_r^2 (|\nabla r|^2 \circ \overline{X})|_{(0,s,t)} \in \mathbb{R}. \tag{5.60}$$

Note that since $\vec{\theta}_0$ is $R_{\vec{u}_-,\vec{u}_+}$-odd and due to the isometry properties of $R_{\vec{u}_-,\vec{u}_+}$, it follows that $\check{d}_2 = \check{d}_5 = 0$ and hence also $\check{f}_0 = 0$. Therefore the equation (5.57) for $\check{h}_1$ is homogeneous. Note that this corresponds to the case of symmetric f in the scalar case, cf. Remark 5.16. This is due to the fact that we restricted to symmetric W in the vector-valued case.

If $\check{h}_1$ is smooth and solves (5.57), then Theorem 4.31 (applied in local coordinates for Σ) yields a smooth solution $\check{u}_2^I$ to (5.56) with $R_{\vec{u}_-,\vec{u}_+} \check{u}_2^I|_{\rho=0} = \check{u}_2^I|_{\rho=0}$ and we get decay estimates. With Remark 5.15 and compactness we obtain $\check{u}_2^I \in \check{R}^I_{1,(\beta)}$ for any $\beta \in (0, \min\{\sqrt{\lambda/2}, \check{\beta}_0\})$, where $\check{\beta}_0 > 0$ is as in Theorem 4.31.

5.3.1.4 Inner Expansion: $\mathcal{O}(\varepsilon^k)$ Let $k \in \{1, ..., M-1\}$ and suppose that the j-th inner order has already been constructed for $j = 0, ..., k$, that it is smooth and $\check{u}_{j+1}^I \in \check{R}^I_{j,(\beta)}$ for every $\beta \in (0, \min\{\sqrt{\lambda/2}, \check{\beta}_0\})$ with $\check{\beta}_0 > 0$ as in Theorem 4.31. Analogously to the scalar case one can compute the $\mathcal{O}(\varepsilon^k)$-order in (vAC1) for $\vec{u}_\varepsilon = \vec{u}_\varepsilon^I$. This yields that the order $\mathcal{O}(\varepsilon^k)$ is zero if

$$
\begin{aligned}
-\check{\mathcal{L}}_0 \check{u}_{k+2}^I(\rho,s,t) &= \vec{R}_{k+1}(\rho,s,t), \qquad (5.61)\\
\vec{R}_{k+1}(\rho,s,t) := \vec{\theta}_0'(\rho) \Bigg[&-\partial_t \check{h}_{k+1} + \sum_{i,l=1}^N \nabla s_i \cdot \nabla s_l|_{\overline{X}_0(s,t)} (\nabla_\Sigma)_i (\nabla_\Sigma)_j \check{h}_{k+1} \\
&-(\partial_t s - \Delta s)|_{\overline{X}_0(s,t)} \cdot \nabla_\Sigma \check{h}_{k+1} + \check{h}_{k+1} \partial_r ((\partial_t r - \Delta r) \circ \overline{X})|_{(0,s,t)} \Bigg] \\
+\vec{\theta}_0''(\rho) \Big[&-(\rho + \check{h}_1) \check{h}_{k+1} \partial_r^2 (|\nabla r|^2 \circ \overline{X})|_{(0,s,t)} \\
&- 2(\nabla_\Sigma \check{h}_1)^\top D_x s (D_x s)^\top|_{\overline{X}_0(s,t)} \nabla_\Sigma \check{h}_{k+1} \\
&+2\partial_r ((D_x s \nabla r)^\top \circ \overline{X})|_{(0,s,t)} [(\rho + \check{h}_1) \nabla_\Sigma \check{h}_{k+1} + \check{h}_{k+1} \nabla_\Sigma \check{h}_1] \Big] \\
+\check{R}_k(\rho,s,t)&,
\end{aligned}
$$

where $\check{R}_k \in \check{R}^I_{k,(\beta)}$. If $\check{h}_{k+1}$ is smooth, then due to Theorem 4.31 equation (5.61) admits a unique solution $\hat{u}_{k+2}^I$ with suitable regularity and decay as well as $R_{\vec{u}_-,\vec{u}_+} \check{u}_2^I|_{\rho=0} = \check{u}_2^I|_{\rho=0}$ if and only if $\int_{\mathbb{R}} \vec{R}_{k+1}(\rho,s,t)\cdot\vec{\theta}_0'(\rho)\,d\rho = 0$. Because of $\int_{\mathbb{R}} \vec{\theta}_0''\cdot\vec{\theta}_0' = 0$, the latter is equivalent to

$$\partial_t \check{h}_{k+1} - \sum_{i,l=1}^{N} \nabla s_i \cdot \nabla s_l|_{\overline{X}_0(s,t)} (\nabla_\Sigma)_i (\nabla_\Sigma)_l \check{h}_{k+1} + \check{a}_1 \cdot \nabla_\Sigma \check{h}_{k+1} + \check{a}_0 \check{h}_{k+1} = \check{f}_k, \tag{5.62}$$

where

$$\check{f}_k(s,t) := \int_{\mathbb{R}} \check{R}_k(\rho,s,t)\cdot\vec{\theta}_0'(\rho)\,d\rho/\|\vec{\theta}_0'\|^2_{L^2(\mathbb{R})^m}$$

is a smooth function of (s,t) and depends only on the j-th inner orders for $0 \leq j \leq k$. Here $\check{a}_0, \check{a}_1$ are defined in (5.58)-(5.59). If $\check{h}_{k+1}$ is smooth and solves (5.62), then Theorem 4.31 yields as in the last Section 5.3.1.3 a smooth solution $\check{u}^I_{k+2}$ to (5.61) such that $\check{u}^I_{k+2} \in \check{R}^I_{k+1,(\beta)}$ for all $\beta \in (0, \min\{\sqrt{\lambda/2}, \check{\beta}_0\})$.

5.3.2 Contact Point Expansion of (vAC) in ND

This is analogous to the scalar case, cf. Section 5.2.2. We make the ansatz $\vec{u}_\varepsilon = \vec{u}^I_\varepsilon + \vec{u}^C_\varepsilon$ in $\Gamma(2\delta)$ close to the contact points. Let $\sigma, b : \overline{\Gamma^C(2\delta, 2\mu_1)} \to \partial\Sigma \times [0, 2\mu_1]$ be as in Theorem 3.7. Then with $H_\varepsilon := \frac{b}{\varepsilon}$ we set

$$\vec{u}^C_\varepsilon := \sum_{j=1}^{M} \varepsilon^j \vec{u}^C_j, \quad \vec{u}^C_j(x,t) := \check{u}^C_j(\check{\rho}_\varepsilon(x,t), H_\varepsilon(x,t), \sigma(x,t), t) \quad \text{for } (x,t) \in \overline{\Gamma^C(2\delta, 2\mu_1)},$$

where

$$\check{u}^C_j : \overline{\mathbb{R}^2_+} \times \partial\Sigma \times [0,T] \to \mathbb{R}^m : (\rho, H, \sigma, t) \mapsto \check{u}^C_j(\rho, H, \sigma, t)$$

for $j = 1, ..., M$. Moreover, we define $\vec{u}^C_{M+1} := \vec{u}^C_{M+2} := 0$ and $\check{u}^C_\varepsilon := \sum_{j=1}^M \varepsilon^j \check{u}^C_j$. As in the scalar case, instead of (vAC1) for $\vec{u}_\varepsilon = \vec{u}^I_\varepsilon + \vec{u}^C_\varepsilon$, we will expand the "bulk equation"

$$\partial_t \vec{u}^C_\varepsilon - \Delta \vec{u}^C_\varepsilon + \frac{1}{\varepsilon^2}\left[\nabla W(\vec{u}^I_\varepsilon + \vec{u}^C_\varepsilon) - \nabla W(\vec{u}^I_\varepsilon)\right] = 0 \tag{5.63}$$

into ε-series with coefficients in $(\check{\rho}_\varepsilon, H_\varepsilon, \sigma, t)$ up to $\mathcal{O}(\varepsilon^{M-2})$. Moreover, we will expand (vAC2) for $\vec{u}_\varepsilon = \vec{u}^I_\varepsilon + \vec{u}^C_\varepsilon$ into ε-series with coefficients in $(\check{\rho}_\varepsilon, \sigma, t)$ up to $\mathcal{O}(\varepsilon^{M-1})$. Altogether we end up with analogous equations as in the scalar case. The solvability condition (4.31) will yield the boundary conditions on $\partial\Sigma \times [0,T]$ for the height functions $\check{h}_j$.

For the expansions we calculate the action of the differential operators on $\vec{u}^C_\varepsilon$ in the next lemma. Here we use the same conventions as in Lemma 5.23 and define $D_{\partial\Sigma}$ in an analogous way as D_Σ.

Lemma 5.25. *Let* $\overline{\mathbb{R}^2_+} \times \partial\Sigma \times [0,T] \ni (\rho, H, \sigma, t) \mapsto \check{w}(\rho, H, \sigma, t) \in \mathbb{R}^m$ *be sufficiently smooth* *and let* $\vec{w} : \overline{\Gamma^C(2\delta, 2\mu_1)} \to \mathbb{R}^m : (x,t) \mapsto \check{w}(\check{\rho}_\varepsilon(x,t), H_\varepsilon(x,t), \sigma(x,t), t)$. *Then*

$$\begin{aligned}
\partial_t \vec{w} &= \partial_\rho \check{w} \left[\frac{\partial_t r}{\varepsilon} - \left(\partial_t \check{h}_\varepsilon + \partial_t s \cdot \nabla_\Sigma \check{h}_\varepsilon \right) \right] + \partial_H \check{w} \frac{\partial_t b}{\varepsilon} + D_{\partial\Sigma} \check{w}\, \partial_t \sigma + \partial_t \check{w}, \\
D_x \vec{w} &= \partial_\rho \check{w} \left[\frac{\nabla r}{\varepsilon} - (D_x s)^\top \nabla_\Sigma \check{h}_\varepsilon \right]^\top + \partial_H \check{w} \left[\frac{\nabla b}{\varepsilon} \right]^\top + D_{\partial\Sigma} \check{w} D_x \sigma, \\
\Delta \vec{w} &= \partial_\rho \check{w} \left[\frac{\Delta r}{\varepsilon} - \left(\Delta s \cdot \nabla_\Sigma \check{h}_\varepsilon + \sum_{i,l=1}^{N} \nabla s_i \cdot \nabla s_l (\nabla_\Sigma)_i (\nabla_\Sigma)_l \check{h}_\varepsilon \right) \right] + \partial_H \check{w} \frac{\Delta b}{\varepsilon} \\
&\quad + \partial_H^2 \check{w} \frac{|\nabla b|^2}{\varepsilon^2} + \partial_\rho^2 \check{w} \left| \frac{\nabla r}{\varepsilon} - (D_x s)^\top \nabla_\Sigma \check{h}_\varepsilon \right|^2 + 2 \partial_\rho \partial_H \check{w} \frac{\nabla b}{\varepsilon} \cdot \left[\frac{\nabla r}{\varepsilon} - (D_x s)^\top \nabla_\Sigma \check{h}_\varepsilon \right] \\
&\quad + 2 D_{\partial\Sigma} \partial_\rho \check{w}\, D_x \sigma \left[\frac{\nabla r}{\varepsilon} - (D_x s)^\top \nabla_\Sigma \check{h}_\varepsilon \right] + 2 D_{\partial\Sigma} \partial_H \check{w}\, D_x \sigma \frac{\nabla b}{\varepsilon} \\
&\quad + D_{\partial\Sigma} \check{w}\, \Delta \sigma + \sum_{i,l=1}^{N} \nabla \sigma_i \cdot \nabla \sigma_l (\nabla_{\partial\Sigma})_i (\nabla_{\partial\Sigma})_l \check{w},
\end{aligned}$$

where the $\vec{w}$*-terms on the left hand side and derivatives of* r *or* s *are evaluated at* (x,t)*, the* $\check{h}_\varepsilon$*-terms at* $(s(x,t),t)$ *and the* $\check{w}$*-terms at* $(\check{\rho}_\varepsilon(x,t), H_\varepsilon(x,t), \sigma(x,t), t)$.

Proof. This can be shown by applying Lemma 5.17 to every component. □

5.3.2.1 Contact Point Expansion: The Bulk Equation We expand the ∇W-part in (5.63): If the $\vec{u}_j^I, \vec{u}_j^C$ are bounded, the Taylor expansion yields on $\overline{\Gamma(2\delta)}$

$$\nabla W(\vec{u}_\varepsilon^I + \vec{u}_\varepsilon^C) = \nabla W(\vec{\theta}_0) + \sum_{\nu \in \mathbb{N}_0^m, |\nu|=1}^{M+2} \frac{1}{\nu!} \partial_y^\nu \nabla W(\vec{\theta}_0) \left[\sum_{j=1}^{M+1} \varepsilon^j (\vec{u}_j^I + \vec{u}_j^C) \right]^\nu + \mathcal{O}(\varepsilon^{M+3}).$$

As in the scalar case one can combine the latter with the expansion for $\nabla W(\vec{u}_\varepsilon^I)$ in (5.54) and use $\vec{u}_1^I = 0$. This yields that the terms in the asymptotic expansion for $\nabla W(\vec{u}_\varepsilon^I + \vec{u}_\varepsilon^C) - \nabla W(\vec{u}_\varepsilon^I)$ are for $k = 1, ..., M+1$:

$$\begin{aligned}
\mathcal{O}(1):&\quad 0, \\
\mathcal{O}(\varepsilon):&\quad D^2 W(\vec{\theta}_0) \vec{u}_1^C, \\
\mathcal{O}(\varepsilon^k):&\quad D^2 W(\vec{\theta}_0) \vec{u}_k^C + \quad [\text{some polynomial in entries of } (\vec{u}_1^I, ..., \vec{u}_{k-1}^I, \vec{u}_1^C, ..., \vec{u}_{k-1}^C) \text{ of} \\
&\quad \text{order } \leq k, \text{ where the coefficients are multiples of } \partial_y^\nu \nabla W(\vec{\theta}_0), \\
&\quad \nu \in \mathbb{N}_0^m, |\nu| = 2, ..., k \text{ and every term contains a } (\vec{u}_j^C)_n\text{-factor}].
\end{aligned}$$

The other explicit terms in $\nabla W(\vec{u}_\varepsilon^I + \vec{u}_\varepsilon^C) - \nabla W(\vec{u}_\varepsilon^I)$ are of order $\mathcal{O}(\varepsilon^{M+3})$.

Functions of (s,t), (ρ, s, t) and (x,t) are expanded in the analogous way as in the scalar case, cf. (5.39)-(5.40) and the remarks there. We just replace h_j, ρ_ε by $\check{h}_j, \check{\rho}_\varepsilon$.

As in the scalar case we use some notation for the higher orders in the expansion:

Definition 5.26 (Notation for Contact Point Expansion of (vAC)). 1. We call $(\vec{\theta}_0, \vec{u}_1^I)$ the *zero-th order* and $(\check{h}_j, \vec{u}_{j+1}^I, \vec{u}_j^C)$ the *j-th order* for $j = 1, ..., M$.

2. Let $k \in \{-1, ..., M+2\}$ and $\beta, \gamma > 0$. Then $\check{R}^C_{k,(\beta,\gamma)}$ denotes the set of smooth functions $\vec{R} : \overline{\mathbb{R}^2_+} \times \partial\Sigma \times [0,T] \to \mathbb{R}^m$ depending only on the j-th orders for $0 \leq j \leq \min\{k, M\}$ and such that uniformly in (ρ, H, σ, t):
$$|\partial_\rho^i \partial_H^l (\nabla_{\partial\Sigma})_{n_1} ... (\nabla_{\partial\Sigma})_{n_d} \partial_t^n \vec{R}(\rho, H, \sigma, t)| = \mathcal{O}(e^{-(\beta|\rho| + \gamma H)})$$
for all $n_1, ..., n_d \in \{1, ..., N\}$ and $d, i, l, n \in \mathbb{N}_0$.
3. The set $\check{R}^C_{k,(\beta)}$ is defined in an analogous way without the H-dependence.

In the following we expand (5.63) into ε-series with coefficients in $(\check{\rho}_\varepsilon, H_\varepsilon, \sigma, t)$.

5.3.2.1.1 Bulk Equation: $\mathcal{O}(\varepsilon^{-1})$ The lowest order $\mathcal{O}(\frac{1}{\varepsilon})$ in (5.63) cancels if
$$[-\Delta^{\sigma,t} + D^2 W(\vec{\theta}_0(\rho))]\check{u}^C_1(\rho, H, \sigma, t) = 0, \tag{5.64}$$
where $\Delta^{\sigma,t} := \partial_\rho^2 + |\nabla b|^2|_{\overline{X}_0(\sigma,t)} \partial_H^2$ and we used $\nabla r \cdot \nabla b|_{\overline{X}_0(\sigma,t)} = 0$ for all $(\sigma, t) \in \partial\Sigma \times [0,T]$.

5.3.2.1.2 Bulk Equation: $\mathcal{O}(\varepsilon^{k-1})$ For $k = 1, ..., M-1$ we assume that the j-th order is constructed for all $j = 0, ..., k$, that it is smooth and that $\check{u}^I_{j+1} \in \check{R}^I_{j,(\beta)}$ (bounded and all derivatives bounded is enough here) and $\check{u}^C_j \in \check{R}^C_{j,(\beta,\gamma)}$ for every $\beta \in (0, \min\{\check{\beta}(\gamma), \sqrt{\lambda/2}, \check{\beta}_0\})$, $\gamma \in (0, \check{\gamma})$, where $\check{\beta}_0$ is from Theorem 4.31 and $\check{\beta}, \check{\gamma}$ are as in Theorem 4.36. With analogous computations as in the scalar case, the $\mathcal{O}(\varepsilon^{k-1})$-order in the expansion for the bulk equation (5.63) is zero if
$$[-\Delta^{\sigma,t} + D^2 W(\vec{\theta}_0)]\check{u}^C_{k+1} = \vec{G}_k(\rho, H, \sigma, t), \tag{5.65}$$
where $\vec{G}_k \in \check{R}^C_{k,(\beta,\gamma)}$.

5.3.2.2 Contact Point Expansion: The Neumann Boundary Condition As in the scalar case, the boundary conditions complementing (5.64)-(5.65) will be obtained from the expansion of the Neumann boundary condition (vAC2) for $\vec{u}_\varepsilon = \vec{u}^I_\varepsilon + \vec{u}^C_\varepsilon$, i.e. $D_x(\vec{u}^I_\varepsilon + \vec{u}^C_\varepsilon)|_{\partial Q_T} N_{\partial\Omega} = 0$. Lemma 5.23 and Lemma 5.25 yield on $\overline{\Gamma^C(2\delta, 2\mu_1)}$
$$D_x \vec{u}^I_\varepsilon|_{(x,t)} = \partial_\rho \check{u}^I_\varepsilon|_{(\rho,s,t)} \left[\frac{\nabla r|_{(x,t)}}{\varepsilon} - (D_x s)^\top|_{(x,t)} \nabla_\Sigma h_\varepsilon|_{(s,t)}\right]^\top + D_\Sigma \check{u}^I_\varepsilon|_{(\rho,s,t)} \, D_x s|_{(x,t)},$$
$$D_x \vec{u}^C_\varepsilon|_{(x,t)} = \partial_\rho \check{u}^C_\varepsilon|_{(\rho,H,\sigma,t)} \left[\frac{\nabla r|_{(x,t)}}{\varepsilon} - (D_x s)^\top|_{(x,t)} \nabla_\Sigma h_\varepsilon|_{(s,t)}\right]^\top$$
$$+ \partial_H \check{u}^C_\varepsilon|_{(\rho,H,\sigma,t)} \left[\frac{\nabla b|_{(x,t)}}{\varepsilon}\right]^\top + D_{\partial\Sigma} \check{u}^C_\varepsilon|_{(\rho,H,\sigma,t)} \, D_x \sigma|_{(x,t)},$$
where $\rho = \check{\rho}_\varepsilon(x,t), H = H_\varepsilon(x,t)$, $s = s(x,t)$ and $\sigma = \sigma(x,t)$. We consider the points $x = X(r, \sigma, t)$ for $(r, \sigma, t) \in [-2\delta, 2\delta] \times \partial\Sigma \times [0,T]$, in particular $H = 0$ and $s = \sigma$.

For $g : \overline{\Gamma(2\delta)} \cap \partial Q_T \to \mathbb{R}$ smooth we use an expansion as in the scalar case, cf. (5.43) and the remarks there. We just use $\check{h}_j, \check{\rho}_\varepsilon$ instead of h_j, ρ_ε.

In the following we expand the Neumann boundary condition into ε-series with coefficients in $(\check{\rho}_\varepsilon, \sigma, t)$ up to the order $\mathcal{O}(\varepsilon^{M-1})$.

5.3.2.2.1 Neumann Boundary Condition: $\mathcal{O}(\varepsilon^{-1})$ At the lowest order $\mathcal{O}(\frac{1}{\varepsilon})$ we have $(N_{\partial\Omega}\cdot\nabla r)|_{\overline{X}_0(\sigma,t)}\vec{\theta}_0'(\rho)=0$. This is valid due to the 90°-contact angle condition.

5.3.2.2.2 Neumann Boundary Condition: $\mathcal{O}(\varepsilon^{0})$ The order $\mathcal{O}(1)$ vanishes if

$$
\begin{aligned}
(N_{\partial\Omega}\cdot\nabla b)|_{\overline{X}_0(\sigma,t)}\partial_H\check{u}_1^C|_{H=0}(\rho,\sigma,t) &= \vec{g}_1(\rho,\sigma,t), && (5.66)\\
\vec{g}_1(\rho,\sigma,t) := \vec{\theta}_0'(\rho)[(D_x s N_{\partial\Omega})^\top|_{\overline{X}_0(\sigma,t)}\nabla_\Sigma\check{h}_1|_{(\sigma,t)} &- \partial_r((N_{\partial\Omega}\cdot\nabla r)\circ\overline{X})|_{(0,\sigma,t)}\check{h}_1|_{(\sigma,t)}]\\
&+\check{g}_0(\rho,\sigma,t),
\end{aligned}
$$

where $\check{g}_0(\rho,\sigma,t):=-\rho\vec{\theta}_0'(\rho)\partial_r((N_{\partial\Omega}\cdot\nabla r)\circ\overline{X})|_{(0,\sigma,t)}$. For $j=1,...,M$ let

$$
\underline{u}_j^C:\overline{\mathbb{R}^2_+}\times\partial\Sigma\times[0,T]\to\mathbb{R}^m:(\rho,H,\sigma,t)\mapsto\check{u}_j^C(\rho,|\nabla b|(\overline{X}_0(\sigma,t))H,\sigma,t). \tag{5.67}
$$

Due to Theorem 3.7 it holds $|\nabla b|_{\overline{X}_0(\sigma,t)}|\geq c>0$ and $|N_{\partial\Omega}\cdot\nabla b|_{\overline{X}_0(\sigma,t)}|\geq c>0$ for all $(\sigma,t)\in\partial\Sigma\times[0,T]$. Therefore (5.64) and (5.66) for $\check{u}_1^C$ are equivalent to

$$
\begin{aligned}
[-\Delta+D^2W(\vec{\theta}_0(\rho))]\underline{u}_1^C &= 0, && (5.68)\\
-\partial_H\underline{u}_1^C|_{H=0} &= (|\nabla b|/N_{\partial\Omega}\cdot\nabla b)|_{\overline{X}_0(\sigma,t)}\vec{g}_1(\rho,\sigma,t). && (5.69)
\end{aligned}
$$

The solvability condition (4.31) corresponding to (5.68)-(5.69) is

$$
(|\nabla b|/N_{\partial\Omega}\cdot\nabla b)|_{\overline{X}_0(\sigma,t)}\int_{\mathbb{R}}\vec{g}_1(\rho,\sigma,t)\cdot\vec{\theta}_0'(\rho)\,d\rho=0.
$$

Due to the symmetry properties of $\vec{\theta}_0$, the term coming from $\check{g}_0$ vanishes. Therefore the latter condition yields the following boundary condition for $\check{h}_1$:

$$
b_1(\sigma,t)\cdot\nabla_\Sigma\check{h}_1|_{(\sigma,t)}+b_0(\sigma,t)\check{h}_1|_{(\sigma,t)}=0\quad\text{for }(\sigma,t)\in\partial\Sigma\times[0,T], \tag{5.70}
$$

where b_1,b_0 are as in the scalar case, cf. the formulas below (5.48). Together with the linear parabolic equation (5.57) for $\check{h}_1$ from Subsection 5.3.1.3, we obtain a time-dependent linear parabolic boundary value problem for $\check{h}_1$, where the initial value $\check{h}_1|_{t=0}$ is not prescribed. However, since $\check{f}_0$ is zero, the equations for $\check{h}_1$ are homogeneous and therefore we can take $\check{h}_1=0$.

Hence we get $\check{u}_2^I$ from Section 5.3.1.3 with $\check{u}_2^I\in\check{R}^I_{1,(\beta_1)}$ for all $\beta_1\in(0,\min\{\sqrt{\lambda/2},\check{\beta}_0\})$, where $\check{\beta}_0>0$ is as in Theorem 4.31. In particular the first inner order is determined. Furthermore, we have $\vec{g}_1\in\tilde{R}^I_{1,(\beta_1)}$ for all $\beta_1\in(0,\sqrt{\lambda/2})$ due to Theorem 4.26. With Theorem 4.36 (applied in local coordinates for $\partial\Sigma$) there is a unique smooth solution $\underline{u}_1^C$ to (5.68)-(5.69) and we get decay properties. By compactness and Remark 5.15 with $\partial\Sigma$ instead of Σ we obtain the decay $\underline{u}_1^C\in\check{R}^C_{1,(\beta,\gamma)}$ for all $\beta\in(0,\min\{\check{\beta}(\gamma),\sqrt{\lambda/2}\})$, $\gamma\in(0,\check{\gamma})$, where $\check{\beta},\check{\gamma}$ are as in Theorem 4.36. Altogether we computed the first order.

5.3.2.2.3 Neumann Boundary Condition: $\mathcal{O}(\varepsilon^{k})$ **and Induction** For $k=1,...,M-1$ we compute $\mathcal{O}(\varepsilon^k)$ in (vAC2) for $\vec{u}_\varepsilon=\vec{u}^I_\varepsilon+\vec{u}^C_\varepsilon$ and obtain equations for the $(k+1)$-th order. We use the following induction hypothesis: assume that the j-th order is constructed for all $j=0,...,k$, that it is smooth and has the decay $\check{u}^I_{j+1}\in\check{R}^I_{j,(\beta_1)}$ for all $\beta_1\in(0,\min\{\sqrt{\lambda/2},\check{\beta}_0\})$ as well as

$\check{u}_j^C \in \check{R}^C_{j,(\beta,\gamma)}$ for all $\beta \in (0, \min\{\check{\beta}(\gamma), \sqrt{\lambda/2}, \check{\beta}_0\})$, $\gamma \in (0, \check{\gamma})$, where $\check{\beta}_0$ is from Theorem 4.31 and $\check{\beta}, \check{\gamma}$ are as in Theorem 4.36. The assumption holds for $k = 1$ due to Section 5.3.2.2.2.

Analogously as in the scalar case, the $\mathcal{O}(\varepsilon^k)$-order in (vAC2) for $\vec{u}_\varepsilon = \vec{u}^I_\varepsilon + \vec{u}^C_\varepsilon$ is zero if

$$(N_{\partial\Omega} \cdot \nabla b)|_{\overline{X}_0(\sigma,t)} \partial_H \check{u}^C_{k+1}|_{H=0}(\rho, \sigma, t) = \vec{g}_{k+1}(\rho, \sigma, t), \tag{5.71}$$

$$\begin{aligned}\vec{g}_{k+1}|_{(\rho,\sigma,t)} :=& \,\vec{\theta}_0'(\rho)[(D_x s N_{\partial\Omega})^\top|_{\overline{X}_0(\sigma,t)} \nabla_\Sigma \check{h}_{k+1}|_{(\sigma,t)} - \partial_r((N_{\partial\Omega} \cdot \nabla r) \circ \overline{X})|_{(0,\sigma,t)} \check{h}_{k+1}|_{(\sigma,t)}] \\ &+ \check{g}_k(\rho, \sigma, t),\end{aligned}$$

where $\check{g}_k \in \check{R}^C_{k,(\beta)}$ and hence $\vec{g}_{k+1} \in \tilde{R}^I_{k+1,(\beta)} + \check{R}^C_{k,(\beta)}$, if $\check{h}_{k+1}$ is smooth.

As in the last Section 5.3.2.2.2, the equations (5.65), (5.71) are equivalent to

$$[-\Delta + D^2 W(\vec{\theta}_0(\rho))] \underline{u}^C_{k+1} = \underline{G}_k(\rho, H, \sigma, t), \tag{5.72}$$

$$-\partial_H \underline{u}^C_{k+1}|_{H=0} = (|\nabla b|/N_{\partial\Omega} \cdot \nabla b)|_{\overline{X}_0(\sigma,t)} \vec{g}_{k+1}(\rho, H, \sigma, t), \tag{5.73}$$

where we defined $\underline{u}^C_{k+1}$ in (5.67) and $\underline{G}_k$ is set in the analogous way with the $\vec{G}_k \in \check{R}^C_{k,(\beta,\gamma)}$ from Section 5.3.2.1.2. The compatibility condition (4.31) for (5.72)-(5.73), i.e.

$$\int_{\mathbb{R}^2_+} \underline{G}_k(\rho, H, \sigma, t) \cdot \vec{\theta}_0'(\rho) \, d(\rho, H) + (|\nabla b|/N_{\partial\Omega} \cdot \nabla b)|_{\overline{X}_0(\sigma,t)} \int_{\mathbb{R}} \vec{g}_{k+1}(\rho, \sigma, t) \cdot \vec{\theta}_0'(\rho) \, d\rho = 0,$$

implies a linear boundary condition for $\check{h}_{k+1}$:

$$b_1(\sigma, t) \cdot \nabla_\Sigma \check{h}_{k+1}|_{(\sigma,t)} + b_0(\sigma, t) \check{h}_{k+1}|_{(\sigma,t)} = \check{f}^B_k(\sigma, t) \quad \text{for } (\sigma, t) \in \partial\Sigma \times [0, T], \tag{5.74}$$

where b_0, b_1 are defined below (5.48) and

$$\check{f}^B_k|_{(\sigma,t)} := \frac{-1}{\|\vec{\theta}_0'\|^2_{L^2(\mathbb{R})^m}} \left[\int_{\mathbb{R}^2_+} \underline{G}_k|_{(\rho,H,\sigma,t)} \cdot \vec{\theta}_0'|_\rho \, d\rho + \frac{|\nabla b|}{N_{\partial\Omega} \cdot \nabla b}\bigg|_{\overline{X}_0(\sigma,t)} \int_{\mathbb{R}} \check{g}_k|_{(\rho,\sigma,t)} \cdot \vec{\theta}_0'|_\rho \, d\rho \right]$$

is smooth in $(\sigma, t) \in \partial\Sigma \times [0, T]$.

Because of the remarks and computations in Section 5.2.2.2.2 we can solve (5.62) from Section 5.3.1.4 together with (5.74) and obtain a smooth solution $\check{h}_{k+1}$. Therefore Section 5.3.1.4 yields $\check{u}^I_{k+2}$ (solving (5.61)) with $\check{u}^I_{k+2} \in \check{R}^I_{k+1,(\beta_1)}$ for all $\beta_1 \in (0, \min\{\sqrt{\lambda/2}, \check{\beta}_0\})$. In particular the $(k+1)$-th inner order is computed and it holds $\vec{G}_k \in \check{R}^C_{k,(\beta,\gamma)}$ as well as $\vec{g}_{k+1} \in \tilde{R}^I_{k+1,(\beta)} + \check{R}^C_{k,(\beta)}$ for all $\beta \in (0, \min\{\check{\beta}(\gamma), \sqrt{\lambda/2}, \check{\beta}_0\})$, $\gamma \in (0, \check{\gamma})$. As in the last Section 5.3.2.2.2 we obtain a unique smooth solution $\underline{u}^C_{k+1}$ to (5.72)-(5.73) with the decay $\check{u}^C_{k+1} \in \check{R}^C_{k,(\beta,\gamma)}$ for all (β, γ) as above. Altogether, the $(k+1)$-th order is determined.

Finally, by induction the j-th order is constructed for all $j = 0, ..., k$, the $\check{h}_j$ are smooth and $\check{u}^I_{j+1} \in \check{R}^I_{j,(\beta_1)}$ for all for all $\beta_1 \in (0, \min\{\sqrt{\lambda/2}, \check{\beta}_0\})$ as well as $\check{u}^C_j \in \check{R}^C_{j,(\beta,\gamma)}$ for every $\beta \in (0, \min\{\check{\beta}(\gamma), \sqrt{\lambda/2}, \check{\beta}_0\})$, $\gamma \in (0, \check{\gamma})$.

5.3.3 The Approximate Solution for (vAC) in ND

Let $N \geq 2$ and $\Gamma = (\Gamma_t)_{t \in [0,T]}$ be as in Section 3.1 with contact angle $\alpha = \frac{\pi}{2}$ and a solution to MCF in Ω. Moreover, let $\delta > 0$ be such that the assertions of Theorem 3.7 hold for 2δ instead of

δ and let r, s, b, σ, μ_0 be as in the theorem. Furthermore, let $W : \mathbb{R}^m \to \mathbb{R}$ be as in Definition 1.4 and $\vec{u}_\pm$ be any distinct pair of minimizers of W. Moreover, let $M \in \mathbb{N}$, $M \geq 2$ be as in the beginning of Section 5.3. Let $\eta : \mathbb{R} \to [0,1]$ be smooth with $\eta(r) = 1$ for $|r| \leq 1$ and $\eta(r) = 0$ for $|r| \geq 2$. Then for $\varepsilon > 0$ we set

$$\vec{u}_\varepsilon^A := \begin{cases} \eta(\frac{r}{\delta}) \left[\vec{u}_\varepsilon^I + \vec{u}_\varepsilon^C \eta(\frac{b}{\mu_1}) \right] + (1 - \eta(\frac{r}{\delta})) \vec{u}_{\mathrm{sign}(r)} & \text{in } \overline{\Gamma(2\delta)}, \\ \vec{u}_\pm & \text{in } Q_T^\pm \setminus \Gamma(2\delta), \end{cases}$$

where $\vec{u}_\varepsilon^I$ and $\vec{u}_\varepsilon^C$ were constructed in Sections 5.3.1 and 5.3.2. Analogously as in Section 5.1.3 one can prove that $\vec{u}_\varepsilon^A$ is an approximate solution for (vAC1)-(vAC3) in the following sense:

Lemma 5.27. *The function $\vec{u}_\varepsilon^A$ is smooth, uniformly bounded with respect to x, t, ε and for the remainder $\vec{r}_\varepsilon^A := \partial_t \vec{u}_\varepsilon^A - \Delta \vec{u}_\varepsilon^A + \frac{1}{\varepsilon^2} \nabla W(\vec{u}_\varepsilon^A)$ in* (vAC1) *and $\vec{s}_\varepsilon^A := \partial_{N_{\partial\Omega}} \vec{u}_\varepsilon^A$ in* (vAC2) *it holds*

$$\begin{aligned} |\vec{r}_\varepsilon^A| &\leq C(\varepsilon^M e^{-c|\check{\rho}_\varepsilon|} + \varepsilon^{M+1}) && \text{in } \Gamma(2\delta, \mu_1), \\ |\vec{r}_\varepsilon^A| &\leq C(\varepsilon^{M-1} e^{-c(|\check{\rho}_\varepsilon| + H_\varepsilon)} + \varepsilon^M e^{-c|\check{\rho}_\varepsilon|} + \varepsilon^{M+1}) && \text{in } \Gamma^C(2\delta, 2\mu_1), \\ \vec{r}_\varepsilon^A &= 0 && \text{in } Q_T \setminus \Gamma(2\delta), \\ |\vec{s}_\varepsilon^A| &\leq C\varepsilon^M e^{-c|\check{\rho}_\varepsilon|} && \text{on } \partial Q_T \cap \Gamma(2\delta), \\ \vec{s}_\varepsilon^A &= 0 && \text{on } \partial Q_T \setminus \Gamma(2\delta) \end{aligned}$$

for $\varepsilon > 0$ small and some $c, C > 0$. Here $\check{\rho}_\varepsilon$ is defined in (5.53) *and $H_\varepsilon = \frac{b}{\varepsilon}$.*

Remark 5.28. The analogous assertions as in Remark 5.11 hold.

5.4 Asymptotic Expansion of (AC_α) in 2D

Let $N = 2$, $\Omega \subseteq \mathbb{R}^N$ be as in Remark 1.1, 1. and $\Gamma := (\Gamma_t)_{t \in [0,T]}$ be as in Section 3.1 with contact angle $\alpha \in (0, \pi)$. We use the notation as in Sections 3.1-3.2. Moreover, let $\delta > 0$ be such that the assertions of Theorem 3.3 hold for 2δ instead of δ. In particular $(r, s) : \overline{\Gamma(2\delta)} \to \overline{S_{\delta,\alpha}}$ are curvilinear coordinates that describe a neighbourhood $\overline{\Gamma(2\delta)}$ of Γ in $\overline{\Omega} \times [0, T]$. Here $\overline{S_{\delta,\alpha}}$ is the trapeze with width δ and angle α defined in (3.1). Again r can be viewed as a signed distance function and s has the role of a tangential projection, both with respect to an extension of Γ. See also Figure 1 and Figure 7. Finally, we assume that Γ evolves according to MCF. Based on Γ we construct a smooth approximate solution $u_{\varepsilon,\alpha}^A$ to (AC_α1)-(AC_α3) with analogous properties as in the $\frac{\pi}{2}$-case in Section 5.1. Here σ_α is chosen as in Definition 1.8 and we will have to restrict α to a small interval around $\frac{\pi}{2}$ in order to use the results in Section 4.2.2.

Roughly the idea is as follows. For the inner expansion in Section 5.4.1 we can use the calculations from the 90°-case in Section 5.1.1. This formally yields a suitable approximate solution of (AC_α1) on $\{(x,t) \in \overline{\Gamma(2\delta)} : s(x,t) \in I\}$, where $I = [-1, 1]$. It would not make sense to use the construction for $s \in I_\mu := [-1-\mu, 1+\mu]$ for some $\mu > 0$ since then the height functions would have to satisfy a parabolic equation on I_μ, but we want to impose boundary conditions at $s = \pm 1$ later. However, we can use smooth extensions from I to I_μ (or $\mathbb{R}$) of the inner expansion terms and the height functions obtained on I for some large $\mu > 0$. Then also the rescaled variable ρ_ε from the inner expansion is well-defined close to the contact points. But we can only use the estimate on the approximation error for the inner expansion for $s \in I$. Therefore we have to cut off in an appropriate way. If the latter is done ε-independent, then it is difficult to set up a straight-forward ansatz at the boundary points: For the contact point expansion it is natural to rescale $z_\alpha^\pm := -r\cos\alpha + (1\mp s)\sin\alpha$ which runs in $\mathbb{R}_+$. Since one has to match the inner and the contact point expansion in every ε-order, this would lead to ansatz functions in $(\rho, Z, t) \in \overline{\mathbb{R}^2_+} \times [0,T]$ having non-trivial asymptotic properties for $Z \to \infty$. However, when using Taylor expansions for (AC$_\alpha$1) this behaviour is a problem, since some of the appearing polynomials will not be multiplied with suitable decaying terms. Therefore the idea is to cut-off the inner expansion with appropriate functions depending on the ε-scaled variables. The contact point expansion is done in Section 5.4.2 and leads to the model problems on $\mathbb{R}^2_+$ we considered in Section 4.2.2. In order to use these results we will have to restrict to $\alpha \in \frac{\pi}{2} + [-\alpha_0, \alpha_0]$, where $\alpha_0 > 0$ is determined in Remark 5.33 below. The compatibility condition (4.17) will yield the boundary conditions for the height functions at $s = \pm 1$. Altogether for $\alpha \in \frac{\pi}{2} + [-\alpha_0, \alpha_0]$ we obtain a suitable approximate solution $u^A_{\varepsilon,\alpha}$ to (AC$_\alpha$1)-(AC$_\alpha$3), see Section 5.4.3 below.

Let $M \in \mathbb{N}$ with $M \geq 2$. For $j = 1, ..., M$ we introduce height functions

$$h_{j,\alpha} : I_\mu \times [0,T] \to \mathbb{R} \quad \text{and} \quad h_{\varepsilon,\alpha} := \sum_{j=1}^{M} \varepsilon^{j-1} h_{j,\alpha}$$

for some $\mu > 0$, where $I_\mu := [-1-\mu, 1+\mu]$. Furthermore, we set $h_{M+1,\alpha} := h_{M+2,\alpha} := 0$ and analogously to the 90°-case

$$\rho_{\varepsilon,\alpha}(x,t) := \frac{r(x,t)}{\varepsilon} - h_{\varepsilon,\alpha}(s(x,t),t) \quad \text{for } (x,t) \in \overline{\Gamma(2\delta)}. \tag{5.75}$$

If $\mu > 0$ is large enough, the latter is well-defined.

5.4.1 Inner Expansion of (AC$_\alpha$) in 2D

For the inner expansion we consider the same ansatz as in Section 5.1.1. Let $\varepsilon > 0$ be small and

$$u^I_{\varepsilon,\alpha} := \sum_{j=0}^{M+1} \varepsilon^j u^I_{j,\alpha}, \quad u^I_{j,\alpha}|_{(x,t)} := \hat{u}^I_{j,\alpha}(\rho_{\varepsilon,\alpha}|_{(x,t)}, s|_{(x,t)}, t) \quad \text{in } \{(x,t) \in \overline{\Gamma(2\delta)} : s|_{(x,t)} \in I\},$$

where

$$\hat{u}^I_{j,\alpha} : \mathbb{R} \times I \times [0,T] \to \mathbb{R} : (\rho, s, t) \mapsto \hat{u}^I_{j,\alpha}(\rho, s, t) \quad \text{for } j = 0, ..., M+1.$$

Moreover, we set $u^I_{M+2,\alpha} := 0$ and $\hat{u}^I_{\varepsilon,\alpha} := \sum_{j=0}^{M+1} \varepsilon^j \hat{u}^I_{j,\alpha}$. We use the following notation:

Definition 5.29 (Notation for Inner Expansion of (AC$_\alpha$) in 2D). 1. We call $(\theta_0, u^I_{1,\alpha})$ the *zero-th inner order* and $(h_{j,\alpha}, u^I_{j+1,\alpha})$ the *j-th inner order* for $j = 1, ..., M$.

2. Let $k \in \{-1, ..., M+2\}$ and $\beta > 0$. We denote with $R^I_{k,(\beta),\alpha}$ the set of smooth functions $R : \mathbb{R} \times I \times [0,T] \to \mathbb{R}$ that depend only on the j-th inner orders for $0 \leq j \leq \min\{k, M\}$ and satisfy uniformly in (ρ, s, t):

$$|\partial_\rho^i \partial_s^l \partial_t^n R(\rho, s, t)| = \mathcal{O}(e^{-\beta|\rho|}) \quad \text{for all } i, l, n \in \mathbb{N}_0.$$

Finally, $\hat{R}^I_{k,(\beta),\alpha}$ is defined analogously with functions $R : \mathbb{R} \times [0,T] \to \mathbb{R}$.

We expand (AC$_\alpha$1) for $u_{\varepsilon,\alpha} = u^I_{\varepsilon,\alpha}$ in the same way as in Section 5.1.1. This leads to

$$\hat{u}^I_{0,\alpha}(\rho,s,t) = \theta_0(\rho) \quad \text{and} \quad \hat{u}^I_{1,\alpha}(\rho,s,t) = 0, \tag{5.76}$$

cf. Sections 5.1.1.1-5.1.1.2. Moreover, from Sections 5.1.1.3-5.1.1.4 we obtain the following: Inductively, if for $k = 0, ..., M-1$ the j-th inner order for $j = 0, ..., k$ is known, smooth and $\hat{u}^I_{j+1,\alpha} \in R^I_{j,(\beta),\alpha}$ for every $\beta \in (0, \min\{\sqrt{f''(\pm 1)}\})$, then there is an equation for $h_{k+1,\alpha}$:

$$\partial_t h_{k+1,\alpha} - |\nabla s|^2|_{\overline{X}_0(s,t)} \partial_s^2 h_{k+1,\alpha} + a_1 \partial_s h_{k+1,\alpha} + a_0 h_{k+1,\alpha} = f_{k,\alpha} \quad \text{in } I \times [0,T], \tag{5.77}$$

where $f_{k,\alpha} : I \times [0,T]$ is a smooth function that can be explicitly computed from the j-th inner orders for $0 \le j \le k$ and a_0, a_1 are defined in (5.7)-(5.8). If $h_{k+1,\alpha}$ is smooth and solves (5.77), then we obtain $\hat{u}^I_{k+2,\alpha}$ as the solution of

$$-\mathcal{L}_0 \hat{u}^I_{k+2,\alpha}(\rho,s,t) = R_{k+1,\alpha}(\rho,s,t) \quad \text{for } (\rho,s,t) \in \mathbb{R} \times I \times [0,T], \tag{5.78}$$

where $\mathcal{L}_0 := -\partial_\rho^2 + f''(\theta_0)$ and $R_{k+1,\alpha} \in R^I_{k+1,(\beta),\alpha}$ can be explicitly computed from $h_{k+1,\alpha}$ and the j-th inner orders for $j = 0, ..., k$. Here (5.77) is the compatibility condition for (5.78) and Theorem 4.4 yields the solution $\hat{u}^I_{k+2,\alpha} \in R^I_{k+1,(\beta),\alpha}$ for all $\beta \in (0, \min\{\sqrt{f''(\pm 1)}\})$.

Remark 5.30. If f is even, then it holds $f_{0,\alpha} = 0$. This follows from Section 5.1.1.3.

Lemma 5.31. *If for $k = 0, ..., M$ the k-th inner orders are known, smooth, $\hat{u}^I_{k+1} \in R^I_{k,(\beta),\alpha}$ for some $\beta > 0$ and the equations* (5.76)-(5.78) *hold, then for some $c, C > 0$ we have*

$$\left| \partial_t u^I_{\varepsilon,\alpha} - \Delta u^I_{\varepsilon,\alpha} + \frac{1}{\varepsilon^2} f'(u^I_{\varepsilon,\alpha}) \right| \le C(\varepsilon^M e^{-c|\rho_{\varepsilon,\alpha}|} + \varepsilon^{M+1}) \quad \text{in } \{(x,t) \in \overline{\Gamma(2\delta)} : s|_{(x,t)} \in I\}.$$

Proof. This follows from the expansions and remainder estimates in Sections 5.1.1.1-5.1.1.4. □

5.4.2 Contact Point Expansion of (AC$_\alpha$) in 2D

For the contact point expansion we define $s^\pm := 1 \mp s$,

$$z^\pm_\alpha := -r\cos\alpha + s^\pm \sin\alpha \quad \text{and} \quad Z^\pm_{\varepsilon,\alpha} := \frac{z^\pm_\alpha}{\varepsilon} \quad \text{on } \overline{\Gamma(2\delta)}. \tag{5.79}$$

See also Figure 11 below. Note that

$$s = \pm 1 \mp s^\pm \quad \text{and} \quad s^\pm = \varepsilon \frac{1}{\sin\alpha} \left[Z^\pm_{\varepsilon,\alpha} + (\rho_{\varepsilon,\alpha} + h_{\varepsilon,\alpha}) \cos\alpha \right]. \tag{5.80}$$

This identity will be used later to expand s-dependent terms and it motivates us (see Remark 5.32 below) to define the following cut-off function for $u^I_{\varepsilon,\alpha}$: Let $\hat{\chi} : \mathbb{R} \to [0,1]$ be smooth with $\hat{\chi}(y) = 0$ for $y \le 1$ and $\hat{\chi}(y) = 1$ for $y \ge 2$. Then we set for some constant $H_0 \ge 0$

$$\hat{\chi}_\alpha(\rho, Z) := \hat{\chi}(Z)\, \hat{\chi}\left(\frac{1}{\sin\alpha} [Z + \rho\cos\alpha - H_0] \right) \quad \text{for all } (\rho, Z) \in \overline{\mathbb{R}^2_+}, \tag{5.81}$$

$$\chi_\alpha(x,t) := \hat{\chi}_\alpha(\rho_{\varepsilon,\alpha}(x,t), Z^\pm_{\varepsilon,\alpha}(x,t)) \quad \text{for all } (x,t) \in \overline{\Gamma(2\delta,1)}. \tag{5.82}$$

See Figure 11 below for a sketch.

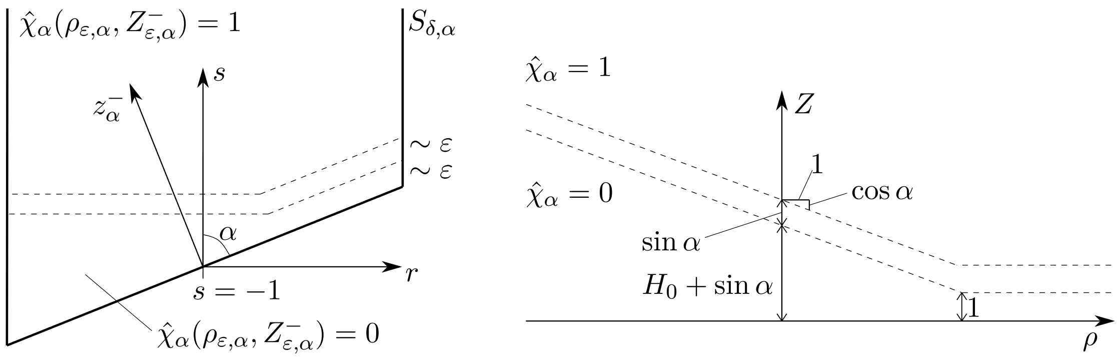

Figure 11: Coordinates and cut-off $\hat\chi_\alpha$.

For the contact point expansion we make the ansatz

$$u_{\varepsilon,\alpha} = \chi_\alpha u^I_{\varepsilon,\alpha} + u^{C\pm}_{\varepsilon,\alpha} \quad \text{in } \overline{\Gamma(2\delta)}$$

close to the contact points $p^\pm(t), t \in [0,T]$. Here we define $u^{C\pm}_{\varepsilon,\alpha} := \sum_{j=1}^M \varepsilon^j u^{C\pm}_{j,\alpha}$ and

$$u^{C\pm}_{j,\alpha}(x,t) := \hat u^{C\pm}_{j,\alpha}(\rho_{\varepsilon,\alpha}(x,t), Z^\pm_{\varepsilon,\alpha}(x,t), t)$$

for $(x,t) \in \overline{\Gamma(2\delta)}$, where

$$\hat u^{C\pm}_{j,\alpha} : \overline{\mathbb{R}^2_+} \times [0,T] \to \mathbb{R} : (\rho, Z, t) \mapsto \hat u^{C\pm}_{j,\alpha}(\rho, Z, t) \quad \text{ for } j = 1, ..., M.$$

Moreover, we set $\hat u^{C\pm}_{\varepsilon,\alpha} := \sum_{j=1}^M \varepsilon^j \hat u^{C\pm}_{j,\alpha}$ and $u^{C\pm}_{M+1,\alpha} := 0$.

Remark 5.32. 1. Note that $\hat\chi(Z^\pm_{\varepsilon,\alpha})$ is zero on $\overline{\Gamma(2\delta)}$ close to ∂Q_T. Therefore there will be no contribution of $u^I_{\varepsilon,\alpha}$ in the expansion of the boundary condition (AC$_\alpha$2). Nevertheless, this is just for aesthetic reasons. However, the second factor of $\hat\chi_\alpha$ in (5.81) is crucial. Namely, if $h_{1,\alpha}$ is known independently of χ_α, then we can take $H_0 := 2\|h_{1,\alpha}\|_\infty$. Then due to (5.80) it holds:

$$\frac{1}{\sin\alpha}[Z^\pm_{\varepsilon,\alpha} + \rho_{\varepsilon,\alpha}\cos\alpha - H_0] \geq 1 \quad \Rightarrow \quad \frac{s^\pm}{\varepsilon} \geq 1 + \frac{1}{\sin\alpha}[\cos\alpha\, h_{\varepsilon,\alpha} + H_0] \geq 0 \ \text{ in } \overline{\Gamma(2\delta)}$$

if $\varepsilon > 0$ is small depending on $\|h_{2,\alpha}\|_\infty, ..., \|h_{M,\alpha}\|_\infty$ and α. This is important since then values of $u^I_{\varepsilon,\alpha}$ are only used in the set $\{(x,t) \in \overline{\Gamma(2\delta)} : s|_{(x,t)} \in I\}$ on which we know that $u^I_{\varepsilon,\alpha}$ has appropriate decay and is (at the moment formally) a suitable approximate solution of (AC$_\alpha$1), cf. Lemma 5.31.

2. The ε-scaled cut-off function $\hat\chi(Z)\hat\chi(s^\pm/\varepsilon)$ should also work, but there are even more terms that have to be expanded.

To get an idea for the expansion of (AC$_\alpha$1) for $u_{\varepsilon,\alpha} = \chi_\alpha u^I_{\varepsilon,\alpha} + u^{C\pm}_{\varepsilon,\alpha}$ in $\overline{\Gamma(2\delta)}$, we rewrite

$$0 = (\partial_t - \Delta)\left[\chi_\alpha u^I_{\varepsilon,\alpha} + u^{C\pm}_{\varepsilon,\alpha}\right] + \frac{1}{\varepsilon^2} f'(\chi_\alpha u^I_{\varepsilon,\alpha} + u^{C\pm}_{\varepsilon,\alpha}) \tag{5.83}$$

$$= u^I_{\varepsilon,\alpha}(\partial_t - \Delta)\chi_\alpha + 2\nabla(u^I_{\varepsilon,\alpha}) \cdot \nabla(\chi_\alpha) + \chi_\alpha\left[(\partial_t - \Delta)u^I_{\varepsilon,\alpha} + \frac{1}{\varepsilon^2} f'(u^I_{\varepsilon,\alpha})\right] \tag{5.84}$$

$$+ (\partial_t - \Delta)u^{C\pm}_{\varepsilon,\alpha} + \frac{1}{\varepsilon^2}\left[f'(\chi_\alpha u^I_{\varepsilon,\alpha} + u^{C\pm}_{\varepsilon,\alpha}) - \chi_\alpha f'(u^I_{\varepsilon,\alpha})\right]. \tag{5.85}$$

Due to Remark 5.32, 1. and Lemma 5.31 it will be possible to control the last term in (5.84) rigorously in the end. Hence this term can be left out in the expansion of (5.83). Moreover, the lowest order will be important. Therefore we set

$$w_\alpha^{C\pm} := \chi_\alpha u_{0,\alpha}^I + u_{0,\alpha}^{C\pm} \quad \text{and} \quad \hat{w}_\alpha^{C\pm}(\rho, Z, t) := \hat{\chi}_\alpha(\rho, Z)\theta_0(\rho) + \hat{u}_{0,\alpha}^{C\pm}(\rho, Z, t),$$

$\tilde{u}_{\varepsilon,\alpha}^I := u_{\varepsilon,\alpha}^I - u_{0,\alpha}^I$, $\tilde{u}_{\varepsilon,\alpha}^{C\pm} := u_{\varepsilon,\alpha}^{C\pm} - u_{0,\alpha}^{C\pm}$ as well as $\hat{\tilde{u}}_{\varepsilon,\alpha}^{C\pm} := \hat{u}_{\varepsilon,\alpha}^{C\pm} - \hat{u}_{0,\alpha}^{C\pm}$. Then we rewrite (5.83)-(5.85) without the last term in (5.84) as follows:

$$\begin{aligned} 0 &= (\partial_t - \Delta)[w_\alpha^{C\pm} + \tilde{u}_{\varepsilon,\alpha}^{C\pm}] - \chi_\alpha(\partial_t - \Delta)u_{0,\alpha}^I + \tilde{u}_{\varepsilon,\alpha}^I(\partial_t - \Delta)\chi_\alpha \\ &\quad + 2\nabla(\tilde{u}_{\varepsilon,\alpha}^I) \cdot \nabla(\chi_\alpha) + \frac{1}{\varepsilon^2}\left[f'(w_\alpha^{C\pm} + \chi_\alpha \tilde{u}_{\varepsilon,\alpha}^I + \tilde{u}_{\varepsilon,\alpha}^{C\pm}) - \chi_\alpha f'(u_{\varepsilon,\alpha}^I)\right]. \end{aligned} \tag{5.86}$$

We will expand the "bulk equation" (5.86) in $\overline{\Gamma(2\delta)}$ into ε-series with coefficients in $(\rho_{\varepsilon,\alpha}, Z_{\varepsilon,\alpha}^\pm, t)$ up to $\mathcal{O}(\varepsilon^{M-2})$ and the nonlinear Robin boundary condition (AC$_\alpha$2) for $u_{\varepsilon,\alpha} = \chi_\alpha u_{\varepsilon,\alpha}^I + u_{\varepsilon,\alpha}^{C\pm}$ on $\partial Q_T \cap \overline{\Gamma(2\delta)}$ into ε-series with coefficients in $(\rho_{\varepsilon,\alpha}, t)$ up to $\mathcal{O}(\varepsilon^{M-1})$. Note that in order to yield a suitable approximate solution, the contact point expansion has to match the inner expansion. To this end we desire

$$\partial_\rho^i \partial_H^l \partial_t^n [\hat{w}_\alpha^{C\pm}(\rho, H, t) - \theta_0(\rho)] = \mathcal{O}(e^{-(\beta|\rho| + \gamma H)}), \tag{5.87}$$

$$\partial_\rho^i \partial_H^l \partial_t^n \hat{u}_{j,\alpha}^{C\pm}(\rho, H, t) = \mathcal{O}(e^{-(\beta|\rho| + \gamma H)}) \tag{5.88}$$

for $j = 1, ..., M$ and all $i, l, n \in \mathbb{N}_0$, for some $\beta, \gamma > 0$ possibly depending on j, i, l, n. Later we will use arbitrary $\beta \in [0, \beta_0)$ and $\gamma \in [\frac{\gamma_0}{2}, \gamma_0)$, where β_0, γ_0 are specified as follows:

Remark 5.33 (Decay Parameters β_0,γ_0, Angle α_0, Lowest Order v_α). We choose $\beta_0, \gamma_0 > 0$ as in Remark 4.19 and such that the inequality

$$\beta_0 + \gamma_0 \leq \min\{\sqrt{f''(\pm 1)}\} \tag{5.89}$$

holds. For these β_0, γ_0 we can use all the assertions in Section 4.2.2, in particular Theorems 4.18, 4.21 and 4.23 for the nonlinear problem. Hence we obtain an $\alpha_0 > 0$ from Theorem 4.21 and a solution v_α of (4.11)-(4.12) for all $\alpha \in \frac{\pi}{2} + [-\alpha_0, \alpha_0]$ such that $v_\alpha = \theta_0 + \hat{v}_\alpha$ and $\hat{v}_. : \frac{\pi}{2} + [-\alpha_0, \alpha_0] \to H^k_{(\beta_0,\gamma_0)}(\mathbb{R}^2_+)$ is Lipschitz-continuous for all $k \in \mathbb{N}_0$. Moreover, due to Theorem 4.25 the linearized problem (4.15)-(4.16) can be solved in Sobolev spaces with exponential weight with decay parameters $\beta \in [0, \beta_0]$, $\gamma \in [\frac{\gamma_0}{2}, \gamma_0]$. Note that every choice in this remark is independent of Ω and Γ.

The successive requirement that the coefficients in the expansions disappear will yield equations on $\mathbb{R}^2_+$ of the type as in Subsection 4.2.2. It will turn out that for $\hat{w}_\alpha^{C\pm} = v_\alpha$ the lowest order vanishes. The solvability condition (4.17) for the linear problems in the higher orders will yield the boundary conditions at $s = \pm 1$ for the height functions $h_{j,\alpha}$.

For the expansion we compute in the following lemma how the differential operators act on $(\rho_{\varepsilon,\alpha}, Z_{\varepsilon,\alpha}^\pm, t)$-dependent terms like $u_{\varepsilon,\alpha}^{C\pm}$, χ_α or $\chi_\alpha u_{0,\alpha}^I$.

Lemma 5.34. *Let $\overline{\mathbb{R}^2_+} \times [0,T] \ni (\rho, Z, t) \mapsto \hat{w}(\rho, Z, t) \in \mathbb{R}$ be sufficiently smooth and let $w : \overline{\Gamma(2\delta)} \to \mathbb{R}$ be defined by $w(x,t) := \hat{w}(\rho_{\varepsilon,\alpha}(x,t), Z^{\pm}_{\varepsilon,\alpha}(x,t), t)$ for all $(x,t) \in \overline{\Gamma(2\delta)}$. Then*

$$
\begin{aligned}
\partial_t w &= \partial_\rho \hat{w} \left[\frac{\partial_t r}{\varepsilon} - (\partial_t h_{\varepsilon,\alpha} + \partial_t s \partial_s h_{\varepsilon,\alpha}) \right] + \partial_Z \hat{w} \frac{\partial_t z^{\pm}_\alpha}{\varepsilon} + \partial_t \hat{w}, \\
\nabla w &= \partial_\rho \hat{w} \left[\frac{\nabla r}{\varepsilon} - \nabla s \partial_s h_{\varepsilon,\alpha} \right] + \partial_Z \hat{w} \frac{\nabla z^{\pm}_\alpha}{\varepsilon}, \\
\Delta w &= \partial_\rho \hat{w} \left[\frac{\Delta r}{\varepsilon} - (\Delta s \partial_s h_{\varepsilon,\alpha} + |\nabla s|^2 \partial_s^2 h_{\varepsilon,\alpha}) \right] + \partial_Z \hat{w} \frac{\Delta z^{\pm}_\alpha}{\varepsilon} + \partial_Z^2 \hat{w} \frac{|\nabla z^{\pm}_\alpha|^2}{\varepsilon^2} \\
&\quad + 2\partial_\rho \partial_Z \hat{w} \, \frac{\nabla z^{\pm}_\alpha}{\varepsilon} \cdot \left[\frac{\nabla r}{\varepsilon} - \nabla s \partial_s h_{\varepsilon,\alpha} \right] + \partial_\rho^2 \hat{w} \left| \frac{\nabla r}{\varepsilon} - \nabla s \partial_s h_{\varepsilon,\alpha} \right|^2,
\end{aligned}
$$

where the w-terms on the left hand side and derivatives of r or s are evaluated at (x,t), the $h_{\varepsilon,\alpha}$-terms at $(s(x,t),t)$ and the $\hat{w}$-terms at $(\rho_{\varepsilon,\alpha}(x,t), H^{\pm}_{\varepsilon,\alpha}(x,t), t)$.

Proof. This follows from the chain rule. □

Similar as in the 90°-case for the higher orders in the expansions we use the following notation:

Definition 5.35 (Notation for Contact Point Expansion of (AC$_\alpha$) in 2D).

1. We refer to the functions $(\theta_0, u^I_{1,\alpha}, w^{C\pm}_\alpha)$ as the *zero-th order* and $(h_{j,\alpha}, u^I_{j+1,\alpha}, u^{C\pm}_{j,\alpha})$ as the *j-th order*, where $j = 1, ..., M$.
2. Let $k \in \{-1, ..., M+2\}$. We write $P^C_{k,\alpha}(\rho, Z)$ for the set of polynomials in (ρ, Z) with smooth coefficients in $t \in [0,T]$ depending only on the $h_{j,\alpha}$ for $1 \leq j \leq \min\{k, M\}$. The sets $P^C_{k,\alpha}(\rho)$ and $P^C_{k,\alpha}(Z)$ are defined analogously with (ρ, Z) replaced by ρ and Z, respectively.
3. Let $k \in \{-1, ..., M+2\}$ and $\beta, \gamma > 0$. Let $R^C_{k,(\beta,\gamma),\alpha}$ be the set of smooth functions $R : \overline{\mathbb{R}^2_+} \times [0,T] \to \mathbb{R}$ that depend only on the j-th orders for $0 \leq j \leq \min\{k, M\}$ and such that uniformly in (ρ, Z, t):
$$
|\partial_\rho^i \partial_Z^l \partial_t^n R(\rho, Z, t)| = \mathcal{O}(e^{-(\beta|\rho| + \gamma Z)}) \quad \text{for all } i, l, n \in \mathbb{N}_0.
$$
The set $R^C_{k,(\beta),\alpha}$ is defined analogously without the Z-dependence.

5.4.2.1 Contact Point Expansion: The Bulk Equation We rewrite (5.86) in $\overline{\Gamma(2\delta)}$ with Lemma 5.34 as follows:

$$
\begin{aligned}
0 &= (\partial_\rho \hat{w}^{C\pm}_\alpha - \hat{\chi}_\alpha \theta_0') \left[\frac{\partial_t r - \Delta r}{\varepsilon} - (\partial_t h_{\varepsilon,\alpha} + |\nabla s|^2 \partial_s^2 h_{\varepsilon,\alpha} + (\partial_t s - \Delta s) \partial_s h_{\varepsilon,\alpha}) \right] \\
&+ \partial_Z \hat{w}^{C\pm}_\alpha \frac{(\partial_t - \Delta) z^{\pm}_\alpha}{\varepsilon} + \partial_t \hat{w}^{C\pm}_\alpha - \partial_Z^2 \hat{w}^{C\pm}_\alpha \frac{|\nabla z^{\pm}_\alpha|^2}{\varepsilon^2} \\
&- 2\partial_\rho \partial_Z \hat{w}^{C\pm}_\alpha \frac{\nabla z^{\pm}_\alpha}{\varepsilon} \cdot \left[\frac{\nabla r}{\varepsilon} - \nabla s \partial_s h_{\varepsilon,\alpha} \right] - (\partial_\rho^2 \hat{w}^{C\pm}_\alpha - \hat{\chi}_\alpha \theta_0'') \left| \frac{\nabla r}{\varepsilon} - \nabla s \partial_s h_{\varepsilon,\alpha} \right|^2 \\
&+ (\partial_t - \Delta) \tilde{u}^{C\pm}_{\varepsilon,\alpha} + \tilde{u}^I_{\varepsilon,\alpha} (\partial_t - \Delta) \chi_\alpha + 2\nabla(\tilde{u}^I_{\varepsilon,\alpha}) \cdot \nabla(\chi_\alpha) \\
&+ \frac{1}{\varepsilon^2} \left[f'(w^{C\pm}_\alpha + \chi_\alpha \tilde{u}^I_{\varepsilon,\alpha} + \tilde{u}^{C\pm}_{\varepsilon,\alpha}) - \chi_\alpha f'(u^I_{\varepsilon,\alpha}) \right],
\end{aligned}
\tag{5.90}
$$

where we use the conventions for evaluations as in Lemma 5.34. Later we will choose $\hat{w}_\alpha^{C\pm} = v_\alpha$ such that the lowest order in the ε-expansion vanishes, where v_α is from Remark 5.33. In (5.90) one can observe that the θ_0-contributions are crucial for the asymptotics as $Z \to \infty$ in the ε-expansion, since we want exponentially decaying terms in the expansion at each order.

In the following we specify how all the terms in (5.90) are expanded into ε-series. For the f'-parts: If the $u_{j,\alpha}^I, u_{j,\alpha}^{C\pm}$ are bounded, then Taylor expansions yield on $\overline{\Gamma(2\delta)}$

$$\begin{aligned} &f'(w_\alpha^{C\pm} + \chi_\alpha \tilde{u}_{\varepsilon,\alpha}^I + \tilde{u}_{\varepsilon,\alpha}^{C\pm}) \\ =&f'(w_\alpha^{C\pm}) + \sum_{k=1}^{M+2} \frac{1}{k!} f^{(k+1)}(w_\alpha^{C\pm}) \left[\sum_{j=1}^{M+1} \varepsilon^j (\chi_\alpha u_{j,\alpha}^I + u_{j,\alpha}^{C\pm}) \right]^k + \mathcal{O}(\varepsilon^{M+3}), \end{aligned} \tag{5.91}$$

as well as on $\{(x,t) \in \overline{\Gamma(2\delta)} : s|_{(x,t)} \in I\}$

$$f'(u_{\varepsilon,\alpha}^I) = f'(\theta_0) + \sum_{k=1}^{M+2} \frac{1}{k!} f^{(k+1)}(\theta_0) \left[\sum_{j=2}^{M+1} \varepsilon^j u_{j,\alpha}^I \right]^k + \mathcal{O}(\varepsilon^{M+3}), \tag{5.92}$$

where $u_{1,\alpha}^I = 0$ due to (5.76). Therefore the terms for $f'(w_\alpha^{C\pm} + \chi_\alpha \tilde{u}_{\varepsilon,\alpha}^I + \tilde{u}_{\varepsilon,\alpha}^{C\pm}) - \chi_\alpha f'(u_{\varepsilon,\alpha}^I)$ in the asymptotic expansion are for $k = 2, ..., M+2$:

$$\begin{aligned} \mathcal{O}(1): \quad & f'(w_\alpha^{C\pm}) - \chi_\alpha f'(\theta_0), \\ \mathcal{O}(\varepsilon): \quad & f''(w_\alpha^{C\pm}) u_{1,\alpha}^{C\pm} + \chi_\alpha [f''(w_\alpha^{C\pm}) - f''(\theta_0)] u_{1,\alpha}^I = f''(w_\alpha^{C\pm}) u_{1,\alpha}^{C\pm}, \\ \mathcal{O}(\varepsilon^k): \quad & f''(w_\alpha^{C\pm}) u_{k,\alpha}^{C\pm} + [\text{some polynomial in } (\chi_\alpha u_{1,\alpha}^I, ..., \chi_\alpha u_{k-1,\alpha}^I, u_{1,\alpha}^{C\pm}, ..., u_{k-1,\alpha}^{C\pm}) \\ & \qquad \text{of order } \leq k, \text{ where the coefficients are multiples of} \\ & \qquad f^{(3)}(w_\alpha^{C\pm}), ..., f^{(k+1)}(w_\alpha^{C\pm}) \text{ and every term admits a } u_{j,\alpha}^{C\pm}\text{-factor}] \\ & \quad + [\text{some polynomial in } (u_{1,\alpha}^I, ..., u_{k,\alpha}^I) \text{ of order } \leq k, \text{ where the} \\ & \qquad \text{coefficients are multiples of } \chi_\alpha^l f^{(l+1)}(w_\alpha^{C\pm}) - \chi_\alpha f^{(l+1)}(\theta_0), \\ & \qquad l = 1, ..., k+1, \text{and every term contains a } u_{j,\alpha}^I\text{-factor}]. \end{aligned}$$

The other explicit terms in $f'(w_\alpha^{C\pm} + \chi_\alpha \tilde{u}_{\varepsilon,\alpha}^I + \tilde{u}_{\varepsilon,\alpha}^{C\pm}) - \chi_\alpha f'(u_{\varepsilon,\alpha}^I)$ are of order $\mathcal{O}(\varepsilon^{M+3})$.

Moreover, we expand terms in (5.90)-(5.92) that depend on (s,t) or (ρ, s, t), namely all the $h_{j,\alpha}$-terms and the $u_{j,\alpha}^I$-terms, respectively. Such terms also appear because of Lemma 5.1 and Lemma 5.34 for $\chi_\alpha, \tilde{u}_{\varepsilon,\alpha}^I, \tilde{u}_{\varepsilon,\alpha}^{C\pm}$. Therefore we consider smooth $g_1 : I_\mu \times [0,T] \to \mathbb{R}$ or $g_1 : \mathbb{R} \times I_\mu \times [0,T] \to \mathbb{R}$ with bounded derivatives in s. Then with a Taylor expansion we obtain

$$g_1|_s = g_1|_{s=\pm 1} + \sum_{k=1}^{M+2} \frac{\partial_s^k g_1|_{s=\pm 1}}{k!} (s \mp 1)^k + \mathcal{O}(|s \mp 1|^{M+3}) \tag{5.93}$$

with a uniform remainder. Then because of (5.80) we replace

$$s \mp 1 = \mp \varepsilon \frac{1}{\sin \alpha} \left[Z + (\rho + h_{\varepsilon,\alpha}(s,t)) \cos \alpha \right]. \tag{5.94}$$

In particular $|s \mp 1| = \varepsilon \mathcal{O}(1 + |\rho| + Z)$, if the $h_{j,\alpha}$ are bounded. On the right hand side in (5.94) we again have the s-dependent term $h_{\varepsilon,\alpha}$, but the ε-order has increased by one. If the $h_{j,\alpha}$ are sufficiently regular, then we can use (5.93)-(5.94) for the $h_{\varepsilon,\alpha}$-term inductively. The latter is needed only finitely many times and yields an expansion of g_1 into an ε-series with coefficients in (ρ, Z, t) up to $\mathcal{O}(\varepsilon^{M+2})$. The terms are for $k = 1, ..., M+2$:

$$\begin{aligned}
\mathcal{O}(1):\quad & g_1|_{s=\pm 1},\\
\mathcal{O}(\varepsilon^k):\quad & [\text{some polynomial in } (\rho, Z, \partial_s^l h_{j,\alpha}|_{(\pm1,t)}), l = 0, ..., k-1, j = 1, ..., k \text{ of order } \le k,\\
& \quad \text{where the coefficients are multiples of } \partial_s^l g_1|_{s=\pm1}, l = 1, ..., k].
\end{aligned}$$

Finally, the remainder in the expansion of g_1 is of order $\varepsilon^{M+3}\mathcal{O}((1+|\rho|+Z)^{M+3})$. The latter will be multiplied with decaying terms later and becomes $\mathcal{O}(\varepsilon^{M+3})$.

Furthermore, we have to expand terms in (5.90) depending on (x,t) that appear after applying Lemma 5.1 and Lemma 5.34, more precisely the derivatives of r, s and $z_\alpha^\pm$. To this end let $g_2 : \overline{\Gamma(2\delta)} \to \mathbb{R}$ be smooth. Then a Taylor expansion yields uniformly in $(r,s,t) \in \overline{S_{\delta,\alpha}} \times [0,T]$

$$\tilde{g}_2(r,s,t) := g_2(\overline{X}(r,s,t)) = \sum_{j+k=0}^{M+2} \frac{\partial_r^j \partial_s^k \tilde{g}_2|_{(0,\pm1,t)}}{j!\,k!} r^j (s\mp 1)^k + \mathcal{O}(|(r, s\mp 1)|^{M+3}). \quad (5.95)$$

We substitute r by $\varepsilon(\rho + h_{\varepsilon,\alpha}(s,t))$ and $s \mp 1$ by (5.94). For the appearing s-dependent term $h_{\varepsilon,\alpha}$ we use the expansion for g_1 above. Hence we obtain an expansion of g_2 into ε-series with coefficients in (ρ, Z, t) up to $\mathcal{O}(\varepsilon^{M+2})$. The terms in the expansion are for $k = 1, ..., M+2$:

$$\begin{aligned}
\mathcal{O}(1):\quad & g_2|_{\overline{p}^\pm(t)},\\
\mathcal{O}(\varepsilon):\quad & \partial_r \tilde{g}_2|_{(0,\pm1,t)}(\rho + h_{1,\alpha}|_{(\pm1,t)}) \mp \partial_s \tilde{g}_2|_{(0,\pm1,t)} \frac{1}{\sin\alpha}\left[Z + (\rho + h_{1,\alpha}|_{(\pm1,t)})\cos\alpha\right],\\
\mathcal{O}(\varepsilon^k):\quad & h_{k,\alpha}|_{(\pm1,t)}\left[\partial_r \tilde{g}_2|_{(0,\pm1,t)} \mp \frac{\cos\alpha}{\sin\alpha}\partial_s \tilde{g}_2|_{(0,\pm1,t)}\right]\\
& +[\text{a polynomial in } (\rho, Z, \partial_s^l h_{j,\alpha}|_{(\pm1,t)}), l = 0, ..., k-1, j = 1, ..., k-1 \text{ of order} \le k,\\
& \quad \text{where the coefficients are multiples of } \partial_r^{l_1}\partial_s^{l_2}\tilde{g}_2|_{(0,\pm1,t)}, l_1, l_2 \in \mathbb{N}_0, l_1 + l_2 \le k].
\end{aligned}$$

Here in contrast to the case $\alpha = \frac{\pi}{2}$ in Section 5.1.2 we need the $\mathcal{O}(\varepsilon)$ explicitly. The remainder in the expansion for g_2 is $\varepsilon^{M+3}\mathcal{O}((1+|\rho|+Z)^{M+3})$. The latter will be multiplied with exponentially decaying terms later and becomes $\mathcal{O}(\varepsilon^{M+3})$.

Now we expand (5.90) with the above identities into ε-series with coefficients in $(\rho_{\varepsilon,\alpha}, Z_{\varepsilon,\alpha}^\pm, t)$.

5.4.2.1.1 Bulk Equation: $\mathcal{O}(\varepsilon^{-2})$ The lowest order $\mathcal{O}(\frac{1}{\varepsilon^2})$ in (5.90) vanishes if

$$\begin{aligned}
0 = &-\partial_Z^2 \hat{w}_\alpha^{C\pm}|\nabla z_\alpha^\pm|^2|_{\overline{p}^\pm(t)} - 2\partial_Z\partial_\rho \hat{w}_\alpha^{C\pm}\nabla z_\alpha^\pm \cdot \nabla r|_{\overline{p}^\pm(t)} - (\partial_\rho^2 \hat{w}_\alpha^{C\pm} - \hat{\chi}_\alpha \theta_0'')|\nabla r|^2|_{\overline{p}^\pm(t)}\\
&+ f'(\hat{w}_\alpha^{C\pm}) - \hat{\chi}_\alpha f'(\theta_0).
\end{aligned}$$

Now we use that due to Remark 3.2 and Theorem 3.3 it holds $|\nabla r|^2|_{\overline{p}^\pm(t)} = |\nabla s|^2|_{\overline{p}^\pm(t)} = 1$ as well as $\nabla r \cdot \nabla s|_{\overline{p}^\pm(t)} = 0$. Therefore with the definition (5.79) we obtain $|\nabla z_\alpha^\pm|^2|_{\overline{p}^\pm(t)} = 1$ and $\nabla z_\alpha^\pm \cdot \nabla r|_{\overline{p}^\pm(t)} = -\cos\alpha$. Hence because of $\theta_0'' = f'(\theta_0)$ the lowest order becomes

$$[-\partial_Z^2 + 2\cos\alpha\partial_Z\partial_\rho - \partial_\rho^2]\hat{w}_\alpha^{C\pm} + f'(\hat{w}_\alpha^{C\pm}) = 0 \quad \text{for } (\rho, Z, t) \in \overline{\mathbb{R}_+^2} \times [0,T]. \quad (5.96)$$

5.4.2.1.2 Bulk Equation: $\mathcal{O}(\varepsilon^{-1})$ The next order $\mathcal{O}(\frac{1}{\varepsilon})$ in (5.90) cancels if we require

$$\left[-\partial_Z^2 + 2\cos\alpha\partial_Z\partial_\rho - \partial_\rho^2 + f''(\hat{w}_\alpha^{C\pm})\right]\hat{u}_{1,\alpha}^{C\pm} = G_{1,\alpha}^{C\pm}, \tag{5.97}$$

$$\begin{aligned}
G_{1,\alpha}^{C\pm}(\rho,Z,t) := &- \left[\partial_\rho\hat{w}_\alpha^{C\pm} - \hat{\chi}_\alpha\theta_0'\right](\partial_t r - \Delta r)|_{\overline{p}^\pm(t)} - \partial_Z\hat{w}_\alpha^{C\pm}(\partial_t z_\alpha^\pm - \Delta z_\alpha^\pm)|_{\overline{p}^\pm(t)} \\
&- 2\partial_\rho\partial_Z\hat{w}_\alpha^{C\pm}(\nabla z_\alpha^\pm\cdot\nabla s)|_{\overline{p}^\pm(t)}\partial_s h_{1,\alpha}|_{(\pm1,t)} \\
&- \left[\partial_\rho^2\hat{w}_\alpha^{C\pm} - \hat{\chi}_\alpha\theta_0''\right]2(\nabla r\cdot\nabla s)|_{\overline{p}^\pm(t)}\partial_s h_{1,\alpha}|_{(\pm1,t)} \\
&+ \partial_Z^2\hat{w}_\alpha^{C\pm}\Big[\partial_r(|\nabla z_\alpha^\pm|^2\circ\overline{X})|_{(0,\pm1,t)}(\rho + h_{1,\alpha}|_{(\pm1,t)}) \\
&\qquad\mp\partial_s(|\nabla z_\alpha^\pm|^2\circ\overline{X})|_{(0,\pm1,t)}\frac{1}{\sin\alpha}[Z + (\rho + h_{1,\alpha}|_{(\pm1,t)})\cos\alpha]\Big] \\
&+ 2\partial_\rho\partial_Z\hat{w}_\alpha^{C\pm}\Big[\partial_r((\nabla z_\alpha^\pm\cdot\nabla r)\circ\overline{X})|_{(0,\pm1,t)}(\rho + h_{1,\alpha}|_{(\pm1,t)}) \\
&\qquad\mp\partial_s((\nabla z_\alpha^\pm\cdot\nabla r)\circ\overline{X})|_{(0,\pm1,t)}\frac{1}{\sin\alpha}[Z + (\rho + h_{1,\alpha}|_{(\pm1,t)})\cos\alpha]\Big] \\
&+ \left[\partial_\rho^2\hat{w}_\alpha^{C\pm} - \hat{\chi}_\alpha\theta_0''\right]\Big[\partial_r(|\nabla r|^2\circ\overline{X})|_{(0,\pm1,t)}(\rho + h_{1,\alpha}|_{(\pm1,t)}) \\
&\qquad\mp\partial_s(|\nabla r|^2\circ\overline{X})|_{(0,\pm1,t)}\frac{1}{\sin\alpha}[Z + (\rho + h_{1,\alpha}|_{(\pm1,t)})\cos\alpha]\Big].
\end{aligned}$$

Because of Remark 3.2 and Theorem 3.3 it follows that $(\partial_t r - \Delta r)|_{\overline{p}^\pm(t)} = (\nabla r\cdot\nabla s)|_{\overline{p}^\pm(t)} = 0$, $(\nabla z_\alpha^\pm\cdot\nabla s)|_{\overline{p}^\pm(t)} = \mp\sin\alpha$ and $\partial_r(|\nabla r|^2\circ\overline{X})|_{(0,\pm1,t)} = \partial_s(|\nabla r|^2\circ\overline{X})|_{(0,\pm1,t)} = 0$. Therefore

$$\begin{aligned}
G_{1,\alpha}^{C\pm}(\rho,Z,t) = &- \partial_Z\hat{w}_\alpha^{C\pm}(\partial_t z_\alpha^\pm - \Delta z_\alpha^\pm)|_{\overline{p}^\pm(t)} \pm 2\sin\alpha\,\partial_\rho\partial_Z\hat{w}_\alpha^{C\pm}\partial_s h_{1,\alpha}|_{(\pm1,t)} \\
&+ \partial_Z^2\hat{w}_\alpha^{C\pm}\Big[\partial_r(|\nabla z_\alpha^\pm|^2\circ\overline{X})|_{(0,\pm1,t)}(\rho + h_{1,\alpha}|_{(\pm1,t)}) \\
&\qquad\mp\partial_s(|\nabla z_\alpha^\pm|^2\circ\overline{X})|_{(0,\pm1,t)}\frac{1}{\sin\alpha}[Z + (\rho + h_{1,\alpha}|_{(\pm1,t)})\cos\alpha]\Big] \\
&+ 2\partial_\rho\partial_Z\hat{w}_\alpha^{C\pm}\Big[\partial_r((\nabla z_\alpha^\pm\cdot\nabla r)\circ\overline{X})|_{(0,\pm1,t)}(\rho + h_{1,\alpha}|_{(\pm1,t)}) \\
&\qquad\mp\partial_s((\nabla z_\alpha^\pm\cdot\nabla r)\circ\overline{X})|_{(0,\pm1,t)}\frac{1}{\sin\alpha}[Z + (\rho + h_{1,\alpha}|_{(\pm1,t)})\cos\alpha]\Big].
\end{aligned}$$

In particular $G_{1,\alpha}^{C\pm}$ is independent of $\hat{\chi}_\alpha$. This is important in order to choose H_0 and hence $\hat{\chi}_\alpha$ independently, see Remark 5.32, 1. For later use, we collect the $h_{1,\alpha}$-terms and write

$$\begin{aligned}
&G_{1,\alpha}^{C\pm}(\rho,Z,t) = \pm2\sin\alpha\,\partial_\rho\partial_Z\hat{w}_\alpha^{C\pm}\partial_s h_{1,\alpha}|_{(\pm1,t)} \\
&+ 2\partial_\rho\partial_Z\hat{w}_\alpha^{C\pm}\left[\partial_r((\nabla z_\alpha^\pm\cdot\nabla r)\circ\overline{X})|_{(0,\pm1,t)} \mp \frac{\cos\alpha}{\sin\alpha}\partial_s((\nabla z_\alpha^\pm\cdot\nabla r)\circ\overline{X})|_{(0,\pm1,t)}\right]h_{1,\alpha}|_{(\pm1,t)} \\
&+ \partial_Z^2\hat{w}_\alpha^{C\pm}\left[\partial_r(|\nabla z_\alpha^\pm|^2\circ\overline{X})|_{(0,\pm1,t)} \mp \frac{\cos\alpha}{\sin\alpha}\partial_s(|\nabla z_\alpha^\pm|^2\circ\overline{X})|_{(0,\pm1,t)}\right]h_{1,\alpha}|_{(\pm1,t)} + \tilde{G}_{0,\alpha}^{C\pm},
\end{aligned}$$

where $\tilde{G}_{0,\alpha}^{C\pm} \in R_{0,(\beta,\gamma),\alpha}^C$ for all $\beta \in [0,\beta_0)$, $\gamma \in [\frac{\gamma_0}{2},\gamma_0)$ provided that $\hat{w}_\alpha^{C\pm} - \theta_0 \in R_{0,(\beta,\gamma),\alpha}^C$ for all these β, γ. The latter corresponds to the matching condition (5.87).

5.4.2.1.3 Bulk Equation: $\mathcal{O}(\varepsilon^{k-1})$ For $k=1,...,M-1$ we compute $\mathcal{O}(\varepsilon^{k-1})$ in (5.90) and derive an equation for $\hat{u}^{C\pm}_{k+1,\alpha}$. Therefore suppose that the j-th order is constructed for all $j=0,...,k$, that it is smooth and that H_0 in $\hat{\chi}_\alpha$ is known. Additionally, let $\hat{u}^I_{j+1,\alpha} \in R^I_{j,(\beta_1),\alpha}$ for all $\beta_1 \in (0, \min\{\sqrt{f''(\pm 1)}\})$ and $j=0,...,k$. Note that in contrast to the 90°-case the decay is used at this point. Namely, with the inequality (5.89) for the decay parameters β_0, γ_0 and the decay for the $\hat{u}^I_{i,\alpha}$ it will be possible to control the contributions of $\partial^l_\rho \hat{w}^{C\pm}_\alpha - \hat{\chi}_\alpha \partial^l_\rho \theta_0$, $l=1,2$ and the new types of terms in the expansion of the f'-parts, cf. (5.91)-(5.92) below. Finally, we assume that $\hat{w}^{C\pm}_\alpha - \theta_0 \in R^C_{0,(\beta,\gamma),\alpha}$ as well as $\hat{u}^{C\pm}_{j,\alpha} \in R^C_{j,(\beta,\gamma),\alpha}$ for all $j=1,...,k$ and all $\beta \in [0,\beta_0)$, $\gamma \in [\frac{\gamma_0}{2}, \gamma_0)$. The latter corresponds to the matching conditions (5.87)-(5.88).

Then with the notation from Definition 5.35 it follows for all those (β,γ):

$$\begin{aligned}
\text{For } j=1,...,k+1: \quad & [\mathcal{O}(\varepsilon^j) \text{ in } [(5.91) - \chi_\alpha \cdot (5.92)]] && \in && f''(\hat{w}^{C\pm}_\alpha)\hat{u}^{C\pm}_{j,\alpha} + R^C_{j-1,(\beta,\gamma),\alpha}, \\
\text{For } i,j=1,...,k: \quad & [\mathcal{O}(\varepsilon^j) \text{ in (5.93) for } g_1 = g_1(h_i)] && \in && P^C_{\max\{i,j\},\alpha}(\rho, Z).
\end{aligned}$$

Moreover, for $j=1,...,k+1$ we obtain

$$[\mathcal{O}(\varepsilon^j) \text{ in (5.95)}] \quad \in \quad h_{j,\alpha}|_{(\pm 1,t)} \left[\partial_r \tilde{g}_2|_{(0,\pm 1,t)} \mp \frac{\cos\alpha}{\sin\alpha} \partial_s \tilde{g}_2|_{(0,\pm 1,t)} \right] + P^C_{j-1,\alpha}(\rho, Z).$$

All those identities can be verified with the remarks accompanying (5.91)-(5.95). The only contributions that are not straight-forward are the (finitely many) terms appearing in the expansion of the f'-parts that are of type $\chi^l_\alpha f^{(l+1)}(w^{C\pm}_\alpha) - \chi_\alpha f^{(l+1)}(\theta_0)$, $l \in \{1,...,j+1\}$ times a term in $\hat{R}^I_{j-1,(\beta_1),\alpha}$ for all $\beta_1 \in (0, \min\{f''(\pm 1)\})$ times some polynomial in $P^C_{j-1,\alpha}(\rho, Z)$. The latter appear due to the order $\mathcal{O}(\varepsilon^i)$ in the expansion of $u^I_{n,\alpha}$, where $i \in \{0,...,j-1\}$ and $n \in \{1,...,j\}$. We have to show that such terms are contained in $R^C_{j-1,(\beta,\gamma),\alpha}$ for all $\beta \in [0,\beta_0)$, $\gamma \in [\frac{\gamma_0}{2}, \gamma_0)$. On the set $\{\hat{\chi}_\alpha = 0\}$ there is nothing to prove. Moreover, on $\{\hat{\chi}_\alpha = 1\}$ we can use uniform continuity for f'-derivatives on compact sets and that $\hat{w}^{C\pm}_\alpha - \theta_0 \in R^C_{0,(\beta,\gamma),\alpha}$ for all (β,γ) as above to obtain the desired estimate. Finally, for the decay on the set $\Xi := \{\hat{\chi}_\alpha \in (0,1)\}$ we use $Z \le |\rho|$ for all $(\rho, Z) \in \Xi$ with $|\rho| + Z \ge \overline{R}$, where $\overline{R}$ is large depending on α, H_0. See also Figure 11. Then it follows that $\beta|\rho| + \gamma Z \le (\beta+\gamma)|\rho| \le \beta_1 |\rho|$ for all those (ρ, Z) and all $\beta + \gamma \le \beta_1$. Due to the inequality $\beta_0 + \gamma_0 \le \min\{f''(\pm 1)\}$ we obtain the claimed inclusion.

Now we compute $\mathcal{O}(\varepsilon^{k-1})$ for $k=1,...,M-1$ in (5.90). Let (β,γ) be as above and arbitrary. The f'-part yields a term in $f''(\hat{w}^{C\pm}_\alpha)\hat{u}^{C\pm}_{k+1,\alpha} + R^C_{k,(\beta,\gamma),\alpha}$. Moreover, note that

$$\partial^l_\rho \hat{w}^{C\pm}_\alpha - \hat{\chi}_\alpha \theta^{(l)}_0 \in \partial^l_\rho \hat{w}^{C\pm}_\alpha - \theta^{(l)}_0 + R^C_{0,(\beta,\gamma),\alpha} \subseteq R^C_{0,(\beta,\gamma),\alpha} \tag{5.98}$$

for all $l \in \mathbb{N}$ and all β, γ as above. The first inclusion can be shown with the decay of $\theta^{(l)}_0$ from Theorem 4.1 and analogous arguments as before since for $(\rho, Z) \in \{\hat{\chi}_\alpha \in [0,1)\}$ it holds $Z \le |\rho|$ if $|\rho| + Z$ is large. The second inclusion follows from $\hat{w}^{C\pm}_\alpha - \theta_0 \in R^C_{0,(\beta,\gamma),\alpha}$. Therefore the contribution at order $\mathcal{O}(\varepsilon^{k-1})$ from the first line in (5.90) is contained in

$$(\partial_\rho \hat{w}^{C\pm}_\alpha - \hat{\chi}_\alpha \theta'_0) \left[P^C_{k,\alpha}(\rho, Z) + \sum_{j=0}^{k-1} P^C_{j,\alpha}(\rho, Z) P^C_{k-1,\alpha}(\rho, Z) \right] \subseteq R^C_{k,(\beta,\gamma),\alpha}.$$

Analogously it follows that the $\partial_Z \hat{w}^{C\pm}_\alpha$-part yields an element of $R^C_{k,(\beta,\gamma),\alpha}$. Moreover, the term

$\partial_t \hat{w}_\alpha^{C\pm} \in R^C_{0,(\beta,\gamma),\alpha}$ only contributes to $\mathcal{O}(1)$. Furthermore, the $\partial_Z^2 \hat{w}_\alpha^{C\pm}$-part gives an element of

$$\begin{aligned}
&- \partial_Z^2 \hat{w}_\alpha^{C\pm} \left[h_{k+1,\alpha}|_{(\pm 1,t)} \left(\partial_r (|\nabla z_\alpha^\pm|^2 \circ \overline{X})|_{(0,\pm 1,t)} \mp \frac{\cos\alpha}{\sin\alpha} \partial_s (|\nabla z_\alpha^\pm|^2 \circ \overline{X})|_{(0,\pm 1,t)} \right) \right] \\
&+ \partial_Z^2 \hat{w}_\alpha^{C\pm} P^C_{k,\alpha}(\rho, Z),
\end{aligned}$$

where the last term is contained in $R^C_{k,(\beta,\gamma),\alpha}$. Similarly, the $\partial_\rho \partial_Z \hat{w}_\alpha^{C\pm}$-part yields a term in

$$\begin{aligned}
R^C_{k,(\beta,\gamma),\alpha} &- 2\partial_\rho \partial_Z \hat{w}_\alpha^{C\pm} \Big[\pm \sin\alpha \, \partial_s h_{k+1,\alpha}|_{(\pm 1,t)} \\
&+ h_{k+1,\alpha}|_{(\pm 1,t)} \left(\partial_r ((\nabla z_\alpha^\pm \cdot \nabla r) \circ \overline{X})|_{(0,\pm 1,t)} \mp \frac{\cos\alpha}{\sin\alpha} \partial_s ((\nabla z_\alpha^\pm \cdot \nabla r) \circ \overline{X})|_{(0,\pm 1,t)} \right) \Big].
\end{aligned}$$

Due to (5.98) and since $\nabla r \cdot \nabla s|_{\overline{p}^\pm(t)} = \partial_r (|\nabla r|^2 \circ \overline{X})|_{(0,\pm 1,t)} = \partial_s (|\nabla r|^2 \circ \overline{X})|_{(0,\pm 1,t)} = 0$, we get in an analogous way that the contribution at order $\mathcal{O}(\varepsilon^{k-1})$ from the $(\partial_\rho^2 \hat{w}_\alpha^{C\pm} - \hat{\chi}_\alpha \theta_0'')$-term in (5.90) is an element of $R^C_{k,(\beta,\gamma),\alpha}$. Moreover, the term $(\partial_t - \Delta)\tilde{u}^{C\pm}_{\varepsilon,\alpha}$ yields a contribution in

$$\left[-\partial_Z^2 + 2\cos\alpha \, \partial_\rho \partial_Z - \partial_\rho^2 \right] \hat{u}^{C\pm}_{k+1,\alpha} + R^C_{k,(\beta,\gamma),\alpha}.$$

For $\tilde{u}^I_{\varepsilon,\alpha}(\partial_t - \Delta)\chi_\alpha$ we use $u^I_{1,\alpha} = 0$, the decay of $\hat{u}^I_{j,\alpha}$ and that $Z \leq |\rho|$ on $\{\hat{\chi}_\alpha \in (0,1)\}$ if $|\rho| + Z$ is large. With the latter we obtain as above the decay of the products of the inner expansion terms with derivatives of $\hat{\chi}_\alpha$. Therefore we get a term in $R^C_{k,(\beta,\gamma),\alpha}$, where we note that $\hat{u}^I_{j,\alpha}$ counts to order $j-1$. Finally, with the same ideas we obtain that $2\nabla(\tilde{u}^I_{\varepsilon,\alpha}) \cdot \nabla(\chi_\alpha)$ also yields a contribution in $R^C_{k,(\beta,\gamma),\alpha}$.

Altogether the $\mathcal{O}(\varepsilon^{k-1})$-order in the expansion for the bulk equation (5.90) is zero if

$$\left[-\partial_Z^2 + 2\cos\alpha \partial_Z \partial_\rho - \partial_\rho^2 + f''(\hat{w}_\alpha^{C\pm}) \right] \hat{u}^{C\pm}_{k+1,\alpha} = G^{C\pm}_{k+1,\alpha}, \tag{5.99}$$

$$\begin{aligned}
G^{C\pm}_{k+1,\alpha} &:= 2\partial_\rho \partial_Z \hat{w}_\alpha^{C\pm} \Big[\pm \sin\alpha \, \partial_s h_{k+1,\alpha}|_{(\pm 1,t)} \\
&\quad + h_{k+1,\alpha}|_{(\pm 1,t)} \left(\partial_r ((\nabla z_\alpha^\pm \cdot \nabla r) \circ \overline{X})|_{(0,\pm 1,t)} \mp \frac{\cos\alpha}{\sin\alpha} \partial_s ((\nabla z_\alpha^\pm \cdot \nabla r) \circ \overline{X})|_{(0,\pm 1,t)} \right) \Big] \\
&+ \partial_Z^2 \hat{w}_\alpha^{C\pm} \, h_{k+1,\alpha}|_{(\pm 1,t)} \left[\partial_r (|\nabla z_\alpha^\pm|^2 \circ \overline{X})|_{(0,\pm 1,t)} \mp \frac{\cos\alpha}{\sin\alpha} \partial_s (|\nabla z_\alpha^\pm|^2 \circ \overline{X})|_{(0,\pm 1,t)} \right] + \tilde{G}^{C\pm}_{k,\alpha},
\end{aligned}$$

where $\tilde{G}^{C\pm}_{k,\alpha} \in R^C_{k,(\beta,\gamma),\alpha}$ for all $\beta \in [0,\beta_0)$, $\gamma \in [\frac{\gamma_0}{2}, \gamma_0)$.

From the expansion of the Robin boundary condition (AC$_\alpha$2) we will obtain boundary conditions for the equations (5.96), (5.97) and (5.99). This is done in the next section.

5.4.2.2 Contact Point Expansion: The Robin Boundary Condition We expand the nonlinear Robin-boundary condition (AC$_\alpha$2) for $u_{\varepsilon,\alpha} = \chi_\alpha u^I_{\varepsilon,\alpha} + u^{C\pm}_{\varepsilon,\alpha}$ on $\partial Q_T \cap \overline{\Gamma(2\delta)}$. Since χ_α is zero in an ε-dependent neighbourhood of ∂Q_T, the latter is the same as

$$N_{\partial\Omega} \cdot \nabla(w_\alpha^{C\pm} + \tilde{u}^{C\pm}_{\varepsilon,\alpha})|_{\partial Q_T} + \frac{1}{\varepsilon} \sigma'_\alpha(w_\alpha^{C\pm} + \tilde{u}^{C\pm}_{\varepsilon,\alpha})|_{\partial Q_T} = 0 \quad \text{on } \partial Q_T \cap \overline{\Gamma(2\delta)}. \tag{5.100}$$

Here on $\partial Q_T \cap \overline{\Gamma(2\delta)}$ it holds $z_\alpha^\pm = Z^\pm_{\varepsilon,\alpha} = 0$ and $s^\pm = \frac{\cos\alpha}{\sin\alpha} r$, cf. (5.79). Therefore we set

$$\overline{s}^\pm(r) := \pm 1 \mp \frac{\cos\alpha}{\sin\alpha} r, \quad \overline{X}_1^\pm|_{(r,t)} := \overline{X}|_{(r,\overline{s}^\pm(r),t)} \quad \text{for } (r,t) \in [-2\delta, 2\delta] \times [0,T]. \tag{5.101}$$

Then due to Lemma 5.34 the equation (5.100) is equivalent to

$$0 = N_{\partial\Omega}|_{\overline{X}_1^{\pm}(r,t)} \cdot \Bigg[(\partial_\rho \hat{w}_\alpha^{C\pm} + \partial_\rho \hat{\hat{u}}_{\varepsilon,\alpha}^{C\pm})|_{Z=0} \left(\frac{\nabla r|_{\overline{X}_1^{\pm}(r,t)}}{\varepsilon} - \nabla s|_{\overline{X}_1^{\pm}(r,t)} \partial_s h_{\varepsilon,\alpha}|_{(\overline{s}^{\pm}(r),t)} \right) + (\partial_Z \hat{w}_\alpha^{C\pm} + \partial_Z \hat{\hat{u}}_{\varepsilon,\alpha}^{C\pm})|_{Z=0} \frac{\nabla z_\alpha^{\pm}|_{\overline{X}_1^{\pm}(r,t)}}{\varepsilon} \Bigg] + \frac{1}{\varepsilon} \sigma_\alpha' (\hat{w}_\alpha^{C\pm} + \hat{\hat{u}}_{\varepsilon,\alpha}^{C\pm})|_{Z=0} \quad (5.102)$$

on $\partial Q_T \cap \overline{\Gamma(2\delta)}$, where $r = r(x,t)$ and $\rho = \rho_{\varepsilon,\alpha}(x,t)$.

In the following we determine how all the terms are expanded into ε-series up to $\mathcal{O}(\varepsilon^{M-1})$ with coefficients in (ρ, t). For the $h_{\varepsilon,\alpha}$-terms let $g_1 : I_\mu \times [0,T] \to \mathbb{R}$ be smooth. We use the rigorous Taylor expansion (5.93) and we replace $s \mp 1$ by (5.94) with $Z = 0$. Then the remarks and the assertions for the remainder terms below (5.93) are still valid when we formally set $Z = 0$. Concerning terms evaluated at $\overline{X}_1^{\pm}$, we consider $g_2 : \partial Q_T \cap \overline{\Gamma(2\delta)} \to \mathbb{R}$ smooth. A Taylor expansion yields for $(r,t) \in [-2\delta, 2\delta] \times [0,T]$:

$$\tilde{g}_2^{\pm}(r,t) := g_2(\overline{X}_1^{\pm}(r,t)) = \sum_{k=0}^{M+2} \frac{\partial_r^k \tilde{g}_2^{\pm}|_{(0,t)}}{k!} r^k + \mathcal{O}(|r|^{M+3}). \quad (5.103)$$

Then we use $r = \varepsilon(\rho_{\varepsilon,\alpha} + h_{\varepsilon,\alpha}|_{(s,t)})$ and expand $h_{\varepsilon,\alpha}$ as specified above. To this end the height functions need to be smooth enough. Similar to the expansion of the (x,t)-dependent terms in the bulk equation, cf. (5.95) below, the terms in the ε-expansion of (5.103) are for $k = 2, ..., M+2$:

$$\begin{aligned} \mathcal{O}(1): &\quad g_2|_{\overline{p}^{\pm}(t)}, \\ \mathcal{O}(\varepsilon): &\quad (\rho + h_{1,\alpha}|_{(\pm 1,t)}) \partial_r \tilde{g}_2^{\pm}|_{(0,t)}, \\ \mathcal{O}(\varepsilon^k): &\quad h_{k,\alpha}|_{(\pm 1,t)} \partial_r \tilde{g}_2^{\pm}|_{(0,t)} + [\text{some polynomial in } (\rho, \partial_s^l h_{j,\alpha}|_{(\pm 1,t)}), l = 0, ..., k-1, \\ &\qquad j = 1, ..., k-1 \text{ of order } \leq k, \text{ where} \\ &\qquad \text{the coefficients are multiples of } (\partial_r^2 \tilde{g}_2^{\pm}, ..., \partial_r^k \tilde{g}_2^{\pm})|_{(0,t)}]. \end{aligned}$$

The other explicit terms in (5.103) are bounded by ε^{M+3} times some polynomial in $|\rho|$ if the $h_{j,\alpha}$ are smooth. Later, these terms and the $\mathcal{O}(|r|^{M+3})$-remainder in (5.103) for each choice of g_2 will be multiplied with exponentially decaying terms in $|\rho|$. Then these terms are $\mathcal{O}(\varepsilon^{M+3})$. Finally, the σ_α-term is replaced via

$$\sigma_\alpha'(\hat{w}_\alpha^{C\pm} + \hat{\hat{u}}_{\varepsilon,\alpha}^{C\pm})|_{Z=0} = \sigma_\alpha'(\hat{w}_\alpha^{C\pm})|_{Z=0} + \sum_{k=1}^{M+2} \frac{1}{k!} \sigma_\alpha^{(k+1)}(\hat{w}_\alpha^{C\pm})|_{Z=0} (\hat{\hat{u}}_{\varepsilon,\alpha}^{C\pm}|_{Z=0})^k + \mathcal{O}(\varepsilon^{M+3}).$$

The terms in the ε-expansion are for $k = 2, ..., M+2$:

$$\begin{aligned} \mathcal{O}(1): &\quad \sigma_\alpha'(\hat{w}_\alpha^{C\pm}|_{Z=0}), \\ \mathcal{O}(\varepsilon): &\quad \sigma_\alpha''(\hat{w}_\alpha^{C\pm}) \hat{u}_{1,\alpha}^{C\pm}|_{Z=0}, \\ \mathcal{O}(\varepsilon^k): &\quad \sigma_\alpha''(\hat{w}_\alpha^{C\pm}) \hat{u}_{k,\alpha}^{C\pm}|_{Z=0} + [\text{a polynomial in } (\hat{u}_{1,\alpha}^{C\pm}|_{Z=0}, ..., \hat{u}_{k-1,\alpha}^{C\pm}|_{Z=0}) \text{ of order } \leq k, \\ &\qquad \text{where the coefficients are multiples of } \sigma_\alpha^{(3)}(\hat{w}_\alpha^{C\pm})|_{Z=0}, ..., \\ &\qquad \sigma_\alpha^{(k+1)}(\hat{w}_\alpha^{C\pm})|_{Z=0} \text{ and every term contains a } \hat{u}_{j,\alpha}^{C\pm}\text{-factor}]. \end{aligned}$$

The other explicit terms are of order $\mathcal{O}(\varepsilon^{M+3})$.

Now we can expand (5.102) into ε-series with coefficients in $(\rho_{\varepsilon,\alpha}, t)$ up to $\mathcal{O}(\varepsilon^{M-1})$.

5.4.2.2.1 Robin Boundary Condition: $\mathcal{O}(\varepsilon^{-1})$ At the lowest order $\mathcal{O}(\frac{1}{\varepsilon})$ we have

$$(N_{\partial\Omega}\cdot\nabla r)|_{\overline{p}^{\pm}(t)}\partial_\rho \hat{w}_\alpha^{C\pm}|_{Z=0} + (N_{\partial\Omega}\cdot\nabla z_\alpha^{\pm})|_{\overline{p}^{\pm}(t)}\partial_Z \hat{w}_\alpha^{C\pm}|_{Z=0} + \sigma_\alpha'(\hat{w}_\alpha^{C\pm}|_{Z=0}) = 0.$$

By construction it holds $N_{\partial\Omega}|_{\overline{p}^{\pm}(t)} = -\nabla z_\alpha^{\pm}|_{\overline{p}^{\pm}(t)}$, where $\nabla z_\alpha^{\pm}\cdot\nabla r|_{\overline{p}^{\pm}(t)} = -\cos\alpha$ and $|\nabla z_\alpha^{\pm}|^2|_{\overline{p}^{\pm}(t)} = 1$, cf. Section 5.4.2.1.1. Therefore the order $\mathcal{O}(\frac{1}{\varepsilon})$ vanishes if we require

$$[-\partial_Z + \cos\alpha\,\partial_\rho]\hat{w}_\alpha^{C\pm}|_{Z=0} + \sigma_\alpha'(\hat{w}_\alpha^{C\pm}|_{Z=0}) = 0. \tag{5.104}$$

The latter equation together with (5.96) is solved by $\hat{w}_\alpha^{C\pm} := v_\alpha$ for $\alpha \in \frac{\pi}{2} + [-\alpha_0, \alpha_0]$, where v_α and α_0 are as in Remark 5.33. In particular v_α is t-independent and $v_\alpha - \theta_0 \in R^C_{0,(\beta,\gamma),\alpha}$ for all $\beta \in [0,\beta_0]$, $\gamma \in [\frac{\gamma_0}{2}, \gamma_0]$.

5.4.2.2.2 Robin Boundary Condition: $\mathcal{O}(\varepsilon^0)$ The next order $\mathcal{O}(1)$ vanishes if

$$[-\partial_Z + \cos\alpha\,\partial_\rho + \sigma_\alpha''(v_\alpha|_{Z=0})]\hat{u}_1^{C\pm}|_{Z=0}(\rho,t) = g_{1,\alpha}^{C\pm}(\rho,t), \tag{5.105}$$

$$\begin{aligned} g_{1,\alpha}^{C\pm}(\rho,t) :=& -\partial_\rho v_\alpha|_{Z=0}\left[h_{1,\alpha}|_{(\pm1,t)}\partial_r((N_{\partial\Omega}\cdot\nabla r)\circ\overline{X}_1^{\pm})|_{(0,t)} \mp \sin\alpha\,\partial_s h_{1,\alpha}|_{(\pm1,t)}\right]\\ &-\partial_Z v_\alpha|_{Z=0}h_{1,\alpha}|_{(\pm1,t)}\partial_r((N_{\partial\Omega}\cdot\nabla z_\alpha^{\pm})\circ\overline{X}_1^{\pm})|_{(0,t)} + \tilde{g}_{0,\alpha}^{C\pm}(\rho,t),\end{aligned}$$

where $\tilde{g}_{0,\alpha}^{C\pm} \in R^C_{0,(\beta),\alpha}$ is given by

$$\tilde{g}_{0,\alpha}^{C\pm}(\rho,t) := -\rho\,\partial_\rho v_\alpha|_{Z=0}\partial_r((N_{\partial\Omega}\cdot\nabla r)\circ\overline{X}_1^{\pm})|_{(0,t)} - \rho\,\partial_Z v_\alpha|_{Z=0}\partial_r((N_{\partial\Omega}\cdot\nabla z_\alpha^{\pm})\circ\overline{X}_1^{\pm})|_{(0,t)}.$$

We solve this equation together with (5.97). If $h_{1,\alpha}$ is smooth and determined only from the 0-th order, then $G_{1,\alpha}^{C\pm} \in R^C_{0,(\beta,\gamma),\alpha}$ and $g_{1,\alpha}^{C\pm} \in R^C_{0,(\beta),\alpha}$ for all $\beta \in [0,\beta_0)$, $\gamma \in [\frac{\gamma_0}{2}, \gamma_0)$. Note that both are independent of χ_α. Therefore due to Remark 5.33 and Theorem 4.25 there is a unique smooth solution $\hat{u}_1^{C\pm}$ to (5.97) and (5.105) with the same decay as $G_{1,\alpha}^{C\pm}$ if and only if the compatibility condition (4.17) holds, i.e.

$$\int_{\mathbb{R}^2_+} G_{1,\alpha}^{C\pm}\partial_\rho v_\alpha\,d(\rho,Z) + \int_{\mathbb{R}} g_{1,\alpha}^{C\pm}\partial_\rho v_\alpha|_{Z=0}\,d\rho = 0.$$

The latter is equivalent to the following linear boundary condition for $h_{1,\alpha}$:

$$b_{1,\alpha}^{\pm}(t)\partial_s h_{1,\alpha}|_{(\pm1,t)} + b_{0,\alpha}^{\pm}(t)h_{1,\alpha}|_{(\pm1,t)} = f_{0,\alpha}^{\pm}(t) \quad \text{for } t\in[0,T], \tag{5.106}$$

where

$$\begin{aligned} b_{1,\alpha}^{\pm}(t) :=& \pm\sin\alpha\left[2\int_{\mathbb{R}^2_+}\partial_\rho\partial_Z v_\alpha\partial_\rho v_\alpha\,d(\rho,Z) + \int_{\mathbb{R}}(\partial_\rho v_\alpha)^2|_{Z=0}\,dZ\right],\\ b_{0,\alpha}^{\pm}(t) :=& \int_{\mathbb{R}^2_+}\partial_Z^2 v_\alpha\partial_\rho v_\alpha\,d(\rho,Z)\left[\partial_r(|\nabla z_\alpha^{\pm}|^2\circ\overline{X})|_{(0,\pm1,t)} \mp \frac{\cos\alpha}{\sin\alpha}\partial_s(|\nabla z_\alpha^{\pm}|^2\circ\overline{X})|_{(0,\pm1,t)}\right]\\ &+2\int_{\mathbb{R}^2_+}\partial_\rho\partial_Z v_\alpha\partial_\rho v_\alpha\left[\partial_r((\nabla z_\alpha^{\pm}\cdot\nabla r)\circ\overline{X})|_{(0,\pm1,t)} \mp \frac{\cos\alpha}{\sin\alpha}\partial_s((\nabla z_\alpha^{\pm}\cdot\nabla r)\circ\overline{X})|_{(0,\pm1,t)}\right]\\ &-\textstyle\int_{\mathbb{R}}(\partial_\rho v_\alpha)^2|_{Z=0}\partial_r((N_{\partial\Omega}\cdot\nabla r)\circ\overline{X}_1^{\pm})|_{(0,t)} - \int_{\mathbb{R}}\partial_Z v_\alpha\partial_\rho v_\alpha|_{Z=0}\partial_r((N_{\partial\Omega}\cdot\nabla z_\alpha^{\pm})\circ\overline{X}_1^{\pm})|_{(0,t)},\\ f_{0,\alpha}^{\pm}(t) :=& -\int_{\mathbb{R}^2_+}\tilde{G}_{0,\alpha}^{C\pm}\partial_\rho v_\alpha\,d(\rho,Z) - \int_{\mathbb{R}}\tilde{g}_{0,\alpha}^{C\pm}\partial_\rho v_\alpha|_{Z=0}\,d\rho\end{aligned}$$

are smooth in $t \in [0,T]$ and independent of χ_α, where $\tilde{G}^{C\pm}_{0,\alpha}$ is as in Section 5.4.2.1.2. Together with the linear parabolic equation (5.77) for $k=0$ from the inner expansion in Section 5.4.1, we obtain a time-dependent linear parabolic boundary value problem for $h_{1,\alpha}$. Here analogously to the 90°-case the initial value $h_{1,\alpha}|_{t=0}$ is not specified.

Remark 5.36. If f is even, then due to Remark 5.30 it holds $f_{0,\alpha}=0$ for the right hand side in (5.77) for $k=0$. However, in general it holds $f^{\pm}_{0,\alpha}\neq 0$. Therefore in contrast to the $\frac{\pi}{2}$-case a non-trivial $h_{1,\alpha}$ is needed except from special cases. This is due to the many terms appearing in $f^{\pm}_{0,\alpha}$. Also note that for v_α there are no symmetry properties available in general.

We solve the equations for $h_{1,\alpha}$ with Lunardi, Sinestrari, von Wahl [LSW], Chapter 9 in the analogous way as in Section 5.1.2.2.2. To this end we only need that all the coefficients are smooth and that $|\nabla s|^2|_{\overline{X}_0}$, $|b^{\pm}_{1,\alpha}|$ are bounded from below by a positive constant. Everything is known except the estimate for $b^{\pm}_{1,\alpha}$. However, the latter follows from the additional estimates in Remark 4.19. Hence we obtain a smooth solution $h_{1,\alpha}: I\times[0,T]\to\mathbb{R}$ to (5.77) with $k=0$ together with (5.106). We can extend $h_{1,\alpha}$ to a smooth function on $I_\mu\times[0,T]$ for example with the Stein Extension Theorem, see Leoni [Le], Theorem 13.17. Moreover, $h_{1,\alpha}$ is independent of χ_α. This enables us to define H_0 and χ_α according to Remark 5.32, 1. Furthermore, due to Section 5.4.1 we obtain $\hat{u}^I_{2,\alpha}$ (solving (5.78) for $k=0$) with $\hat{u}^I_{2,\alpha}\in R^I_{1,(\beta_1),\alpha}$ for all $\beta_1\in(0,\min\{\sqrt{f''(\pm1)}\})$. Therefore the first inner order is computed. Finally, Theorem 4.25 yields a unique smooth solution $\hat{u}^{C\pm}_{1,\alpha}$ to (5.97) and (5.105) such that $\hat{u}^{C\pm}_1\in R^C_{1,(\beta,\gamma),\alpha}$ for all $\beta\in[0,\beta_0)$, $\gamma\in[\frac{\gamma_0}{2},\gamma_0)$. Hence the first order is determined.

5.4.2.2.3 Robin Boundary Condition: $\mathcal{O}(\varepsilon^k)$ and Induction For $k=1,\dots,M-1$ we consider $\mathcal{O}(\varepsilon^k)$ in (5.102) and derive equations for the $(k+1)$-th order. Therefore we assume the following induction hypothesis: suppose that the j-th order is constructed for all $j=0,\dots,k$, that it is smooth and that H_0 and χ_α is known. Moreover, suppose that $\hat{u}^I_{j+1,\alpha}\in R^I_{j,(\beta_1),\alpha}$ for every $\beta_1\in(0,\min\{\sqrt{f''(\pm1)}\})$ and $j=0,\dots,k$. Finally, let $\hat{u}^{C\pm}_{j,\alpha}\in R^C_{j,(\beta,\gamma),\alpha}$ for all $\beta\in[0,\beta_0)$, $\gamma\in[\frac{\gamma_0}{2},\gamma_0)$ and $j=0,\dots,k$. The assumption holds for $k=1$ due to Section 5.4.2.2.2.

Then with the notation in Definition 5.35 we have

$$\begin{aligned}
\text{For } j=1,\dots,k+1: &\quad [\mathcal{O}(\varepsilon^j)\text{ in }\sigma'_\alpha(\hat{w}^{C\pm}_\alpha+\hat{\hat{u}}^{C\pm}_{\varepsilon,\alpha})|_{Z=0}] &\in&\quad \sigma''_\alpha(v_\alpha)\hat{u}^C_{k+1,\alpha}|_{Z=0}+R^C_{k,(\beta),\alpha},\\
\text{For } i,j=1,\dots,k: &\quad [\mathcal{O}(\varepsilon^j)\text{ for }g_1=g_1(h_i)|_{(\overline{s}^\pm(r),t)}] &\in&\quad P^C_{\max\{i,j\},\alpha}(\rho).
\end{aligned}$$

Moreover, for $j=1,\dots,k+1$ we obtain

$$[\mathcal{O}(\varepsilon^j)\text{ in (5.103)}]\quad \in h_{j,\alpha}|_{(\pm1,t)}\partial_r\tilde{g}^{\pm}_2|_{(0,t)}+P^C_{j-1,\alpha}(\rho)\quad [\subseteq P^C_{j,\alpha}(\rho),\text{ if } j\le k].$$

With this we can compute the order $\mathcal{O}(\varepsilon^k)$ in (5.102). Therefore let $\beta\in[0,\beta_0)$ be arbitrary. The contribution of $\frac{1}{\varepsilon}(\partial_\rho v_\alpha+\partial_\rho\hat{\hat{u}}^{C\pm}_{\varepsilon,\alpha})|_{Z=0}(N_{\partial\Omega}\cdot\nabla r)|_{\overline{X}^{\pm}_1(r,t)}$ yields a term in

$$\begin{aligned}
&\partial_\rho\hat{u}^{C\pm}_{k+1,\alpha}|_{Z=0}N_{\partial\Omega}\cdot\nabla r|_{\overline{p}^\pm(t)}+\partial_\rho v_\alpha|_{Z=0}h_{k+1,\alpha}|_{(\pm1,t)}\partial_r((N_{\partial\Omega}\cdot\nabla r)\circ\overline{X}^{\pm}_1)|_{(0,t)}\\
&+\sum_{j=1}^{k}P^C_{j,\alpha}(\rho)\partial_\rho\hat{u}^{C\pm}_{k+1-j}|_{Z=0},
\end{aligned}$$

where $N_{\partial\Omega}\cdot\nabla r|_{\overline{p}^\pm(t)} = \cos\alpha$ and the last sum is contained in $R^C_{k,(\beta),\alpha}$. Moreover, from the term $-(\partial_\rho v_\alpha + \partial_\rho \hat{\hat{u}}^{C\pm}_{\varepsilon,\alpha})|_{Z=0}(N_{\partial\Omega}\cdot\nabla s)|_{\overline{X}_1^\pm(r,t)}\partial_s h_{\varepsilon,\alpha}|_{(\overline{s}^\pm(r),t)}$ we obtain an element of

$$
\begin{aligned}
(\partial_s h_{k+1,\alpha}|_{(\pm1,t)} + P^C_{k,\alpha}(\rho))N_{\partial\Omega}\cdot\nabla s|_{\overline{p}^\pm(t)}\partial_\rho v_\alpha|_{Z=0} + P^C_{k,\alpha}(\rho)\sum_{j=0}^{k} P^C_{j,\alpha}(\rho)\partial_\rho \hat{u}^{C\pm}_{k-j,\alpha}&\\
\subseteq \pm\sin\alpha\,\partial_\rho v_\alpha|_{Z=0}\partial_s h_{k+1,\alpha}|_{(\pm1,t)} + R^C_{k,(\beta),\alpha}&.
\end{aligned}
$$

Analogously as before, $\frac{1}{\varepsilon}(\partial_Z v_\alpha + \partial_Z \hat{\hat{u}}^{C\pm}_{\varepsilon,\alpha})|_{Z=0}(N_{\partial\Omega}\cdot\nabla z^\pm_\alpha)|_{\overline{X}_1^\pm(r,t)}$ contributes a term in

$$
-\partial_Z \hat{u}^{C\pm}_{k+1,\alpha}|_{Z=0} + \partial_Z v_\alpha|_{Z=0} h_{k+1,\alpha}|_{(\pm1,t)}\partial_r((N_{\partial\Omega}\cdot\nabla z^\pm_\alpha)\circ\overline{X}_1^\pm)|_{(0,t)} + R^C_{k,(\beta),\alpha}.
$$

Finally, the term $\frac{1}{\varepsilon}\sigma'_\alpha(v_\alpha + \hat{\hat{u}}^{C\pm}_{\varepsilon,\alpha})|_{Z=0}$ gives an element in $\sigma''_\alpha(v_\alpha)\hat{u}^{C\pm}_{k+1,\alpha}|_{Z=0} + R^C_{k,(\beta),\alpha}$.

Altogether the $\mathcal{O}(\varepsilon^k)$-order in the expansion of (5.102) vanishes if

$$
\begin{aligned}
[-\partial_Z + \cos\alpha\,\partial_\rho + \sigma''_\alpha(v_\alpha|_{Z=0})]\hat{u}^{C\pm}_{k+1}|_{Z=0}(\rho,t) &= g^{C\pm}_{k+1,\alpha}(\rho,t), \qquad (5.107)\\
g^{C\pm}_{k+1,\alpha}(\rho,t) := &-\partial_\rho v_\alpha|_{Z=0}\Big[h_{k+1,\alpha}|_{(\pm1,t)}\partial_r((N_{\partial\Omega}\cdot\nabla r)\circ\overline{X}_1^\pm)|_{(0,t)} \mp \sin\alpha\,\partial_s h_{k+1,\alpha}|_{(\pm1,t)}\Big]\\
&-\partial_Z v_\alpha|_{Z=0}h_{k+1,\alpha}|_{(\pm1,t)}\partial_r((N_{\partial\Omega}\cdot\nabla z^\pm_\alpha)\circ\overline{X}_1^\pm)|_{(0,t)} + \tilde{g}^{C\pm}_{k,\alpha}(\rho,t),
\end{aligned}
$$

where $\tilde{g}^{C\pm}_{k,\alpha}\in R^C_{k,(\beta),\alpha}$. We solve the latter equation (5.107) together with (5.99).

The compatibility condition (4.17) yields the following linear boundary condition for $h_{k+1,\alpha}$:

$$
b^\pm_{1,\alpha}(t)\partial_s h_{k+1,\alpha}|_{(\pm1,t)} + b^\pm_{0,\alpha}(t)h_{k+1,\alpha}|_{(\pm1,t)} = f^\pm_{k,\alpha}(t) \quad \text{for } t\in[0,T], \qquad (5.108)
$$

where $b^\pm_{1,\alpha}, b^\pm_{0,\alpha}$ are the same as the ones after (5.106) and

$$
f^\pm_{k,\alpha}(t) := -\int_{\mathbb{R}^2_+}\tilde{G}^{C\pm}_{k,\alpha}\partial_\rho v_\alpha\,d(\rho,Z) - \int_{\mathbb{R}}\tilde{g}^{C\pm}_{k,\alpha}\partial_\rho v_\alpha|_{Z=0}\,d\rho
$$

is smooth in $t\in[0,T]$, where $\tilde{G}^{C\pm}_{k,\alpha}$ is as in Section 5.4.2.1.3.

The arguments in the last Section 5.4.2.2.2 yield a smooth solution $h_{k+1,\alpha}: I\times[0,T]\to\mathbb{R}$ of (5.77) from Section 5.4.1 together with (5.108). Again, the latter can be extended to a smooth function on $I_\mu\times[0,T]$. Moreover, Section 5.4.1 determines $\hat{u}^I_{k+2,\alpha}$ (solving (5.78)) with $\hat{u}^I_{k+2,\alpha}\in R^I_{k+1,(\beta_1),\alpha}$ for all $\beta_1\in(0,\min\{\sqrt{f''(\pm1)}\})$. Therefore the $(k+1)$-th inner order is computed. Moreover, it holds $G^{C\pm}_{k,\alpha}\in R^C_{k+1,(\beta,\gamma),\alpha}$ as well as $g^{C\pm}_{k+1}\in R^C_{k+1,(\beta),\alpha}$ for all $\beta\in[0,\beta_0), \gamma\in[\frac{\gamma_0}{2},\gamma_0)$ and they are independent of $\hat{u}^{C\pm}_{k+1,\alpha}$. Finally, Theorem 4.25 yields a unique smooth solution $\hat{u}^{C\pm}_{k+1,\alpha}$ to (5.99) and (5.107) such that $\hat{u}^{C\pm}_{k+1,\alpha}\in R^C_{k+1,(\beta,\gamma),\alpha}$ for all $\beta\in[0,\beta_0), \gamma\in[\frac{\gamma_0}{2},\gamma_0)$. Hence the $(k+1)$-th order is determined.

Finally, the j-th order is determined inductively for all $j=0,...,k$, the $h_{j,\alpha}$ are smooth and $\hat{u}^I_{j+1,\alpha}\in R^I_{j,(\beta_1),\alpha}$ for all $\beta_1\in(0,\min\{\sqrt{f''(\pm1)}\})$ as well as $\hat{u}^{C\pm}_{j,\alpha}\in R^C_{j,(\beta,\gamma),\alpha}$ for all $\beta\in[0,\beta_0), \gamma\in[\frac{\gamma_0}{2},\gamma_0)$.

5.4.3 The Approximate Solution for (AC$_\alpha$) in 2D

Let σ_α be as in Definition 1.8 and $\alpha_0 > 0$ be as in Remark 5.33. Moreover, let $N = 2$ and $\Gamma := (\Gamma_t)_{t\in[0,T]}$ be as in Section 3.1 with contact angle $\alpha \in \frac{\pi}{2} + [-\alpha_0, \alpha_0]$ and a solution to MCF in Ω. Additionally, let $\delta > 0$ be such that the assertions of Theorem 3.3 hold for 2δ instead of δ and let r, s be as in the theorem. Let $M \in \mathbb{N}$, $M \geq 2$ be as in the beginning of Section 5.4. Moreover, let $\delta_0 \in (0, \delta]$ be small such that $-2\delta_0 + \mu_0 \sin\alpha > 0$ and

$$s^\pm = \frac{1}{\sin\alpha}[z^\pm_\alpha + r\cos\alpha] \in (\frac{5}{4}, \frac{7}{4})\mu_0 \quad \text{for } z^\pm_\alpha \in [\frac{11}{8}, \frac{3}{2}]\mu_0 \sin\alpha \text{ and } |r| \leq \delta_0, \tag{5.109}$$

where μ_0 is from Theorem 3.3. Note that (5.109) is only needed in order to have a suitable partition of $\Gamma(\delta_0)$ for the spectral estimate later, cf. (6.66) below. Finally, let $\eta, \tilde{\eta} : \mathbb{R} \to [0, 1]$ be smooth with $\eta(r) = 1$ for $|r| \leq 1$, $\eta(r) = 0$ for $|r| \geq 2$ and $\tilde{\eta}(r) = 1$ for $r \leq 1$, $\tilde{\eta}(r) = 0$ for $r \geq 2$. Then we set

$$u^{B\pm}_{\varepsilon,\alpha} := \chi_\alpha u^I_{\varepsilon,\alpha} + u^{C\pm}_{\varepsilon,\alpha} = v_\alpha + \chi_\alpha \tilde{u}^I_{\varepsilon,\alpha} + \tilde{u}^{C\pm}_{\varepsilon,\alpha} \tag{5.110}$$

for $\varepsilon > 0$, where χ_α and v_α are evaluated at $(\rho_{\varepsilon,\alpha}, Z^\pm_{\varepsilon,\alpha})$. The appearing functions were constructed in Sections 5.4.1-5.4.2. Then we define

$$u^A_{\varepsilon,\alpha} := \begin{cases} \eta(\frac{r}{\delta_0}) \left[\tilde{\eta}(\frac{s^\pm}{\mu_0}) u^{B\pm}_{\varepsilon,\alpha} + (1 - \tilde{\eta}(\frac{s^\pm}{\delta_0})) u^I_{\varepsilon,\alpha}\right] + (1 - \eta(\frac{r}{\delta_0}))\operatorname{sign}(r) & \text{in } \overline{\Gamma^\pm(2\delta, 1)}, \\ \pm 1 & \text{in } Q^\pm_T \setminus \Gamma(2\delta), \end{cases}$$

where $s^\pm = \pm 1 \mp s$ and the sets were defined in Remark 3.4, 1. This yields an approximate solution for (AC$_\alpha$1)-(AC$_\alpha$3) in the following sense:

Lemma 5.37. *The function $u^A_{\varepsilon,\alpha}$ is smooth, uniformly bounded in x, t, ε and the remainders $r^A_{\varepsilon,\alpha} := (\partial_t - \Delta)u^A_{\varepsilon,\alpha} + \frac{1}{\varepsilon^2} f'(u^A_{\varepsilon,\alpha})$ and $s^A_{\varepsilon,\alpha} := \partial_{N_{\partial\Omega}} u^A_{\varepsilon,\alpha} + \frac{1}{\varepsilon}\sigma'_\alpha(u^A_{\varepsilon,\alpha})$ in* (AC$_\alpha$1)-(AC$_\alpha$2) *satisfy*

$$\begin{aligned} |r^A_{\varepsilon,\alpha}| &\leq C(\varepsilon^{M-1} e^{-c(|\rho_{\varepsilon,\alpha}| + Z^\pm_{\varepsilon,\alpha})} + \varepsilon^M e^{-c|\rho_{\varepsilon,\alpha}|} + \varepsilon^{M+1}) && \text{in } \Gamma^\pm(2\delta, 1), \\ r^A_{\varepsilon,\alpha} &= 0 && \text{in } Q_T \setminus \Gamma(2\delta), \\ |s^A_{\varepsilon,\alpha}| &\leq C\varepsilon^M e^{-c|\rho_{\varepsilon,\alpha}|} && \text{on } \partial Q_T \cap \Gamma(2\delta), \\ s^A_{\varepsilon,\alpha} &= 0 && \text{on } \partial Q_T \setminus \Gamma(2\delta) \end{aligned}$$

for $\varepsilon > 0$ small and some $c, C > 0$.

Remark 5.38. The analogous statements as in Remark 5.11 are true.

Proof. The proof is analogous to the one of Lemma 5.10 where the case $\alpha = \frac{\pi}{2}$ is shown. One verifies that the Taylor expansions and remainder estimates stated in Sections 5.1.1-5.1.2 hold rigorously. The main point left to show in the case $\alpha \neq \frac{\pi}{2}$ is the suitable convergence with respect to $\varepsilon \to 0$ (i.e. rates of type $e^{-c/\varepsilon}$ for the function and all derivatives) in the transition regions for the functions we glued together in the definition of $u^A_{\varepsilon,\alpha}$. With the latter, one can then estimate the mixed terms in $r^A_{\varepsilon,\alpha}$ and $s^A_{\varepsilon,\alpha}$ appearing due to the cutoff functions similar to the case $\alpha = \frac{\pi}{2}$. Because of (5.110) and the asymptotics of the appearing functions it is enough to prove

$$z^\pm_\alpha|_{(x,t)} \geq c > 0 \quad \text{and} \quad \chi_\alpha(\rho_{\varepsilon,\alpha}, Z^\pm_{\varepsilon,\alpha})|_{(x,t)} \equiv 1$$

for all $(x,t) \in \overline{\Gamma^\pm(2\delta,1)}$ with $|r(x,t)| \leq 2\delta^0$ and $s^\pm(x,t) \geq \mu_0$ as well as $\varepsilon > 0$ small. However, by the assumption on δ_0 it holds

$$z^\pm_\alpha|_{(x,t)} = |z^\pm_\alpha|_{(x,t)}| \geq -|r|_{(x,t)}\cos\alpha| + s^\pm|_{(x,t)}\sin\alpha \geq -2\delta_0 + \mu_0\sin\alpha > 0$$

for all those (x,t). Moreover, recall the definition (5.82) of $\hat{\chi}_\alpha$. Therefore it is left to show that

$$\frac{1}{\sin\alpha}[Z^\pm_{\varepsilon,\alpha}|_{(x,t)} + \rho_{\varepsilon,\alpha}|_{(x,t)}\cos\alpha - H_0] \geq 1$$

for all the (x,t) as above and $\varepsilon > 0$ small, where $H_0 = 2\|h_{1,\alpha}\|_\infty$, see Remark 5.32, 1. and the end of Section 5.4.2.2.2. The estimate follows from

$$\frac{1}{\sin\alpha}[Z^\pm_{\varepsilon,\alpha}|_{(x,t)} + \rho_{\varepsilon,\alpha}|_{(x,t)}\cos\alpha - H_0] = \frac{s^\pm|_{(x,t)}}{\varepsilon} - \frac{1}{\sin\alpha}[h_{\varepsilon,\alpha}|_{(s(x,t),t)}\cos\alpha + H_0]$$

for all the (x,t) as before and $\varepsilon > 0$. The second term is estimated by $4\|h_{1,\alpha}\|_\infty/\sin\alpha$ for $\varepsilon > 0$ small. Hence the lemma is proven. $\square$

6 Spectral Estimates

The second step in the method of de Mottoni and Schatzman [deMS] consists of estimating the difference of the exact and approximate solution. To this end one employs a Gronwall-type argument together with the idea of linearization at the approximate solution, since the structure of the latter is known in detail. In order to estimate all terms in a suitable way, it is important to have a spectral estimate for a linear operator corresponding to the diffuse interface model and the approximate solution, i.e. an estimate for the related bilinear form. The form of the linear operator and the estimate typically take into account the energy and the scalar product generating a gradient flow which is usually a part of the diffuse interface model.

For instance, in the situation of (AC1)-(AC3) the operator is given by

$$\mathcal{L}_{\varepsilon,t} := -\Delta + \frac{1}{\varepsilon^2} f''(u_\varepsilon^A(.,t)) \quad \text{on } \Omega$$

together with homogeneous Neumann boundary condition, where u_ε^A is from Section 5.2.3 (for $N = 2$ see also Section 5.1.3). We will show a spectral estimate of the following form: there are constants $c, C, \varepsilon_0 > 0$ such that for all $u \in H^1(\Omega)$ and $\varepsilon \in (0, \varepsilon_0]$

$$\int_\Omega |\nabla u|^2 + \frac{1}{\varepsilon^2} f''(u_\varepsilon^A(.,t))u^2 \geq -C\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega\setminus\Gamma_t(\delta))} + c\|\nabla_\tau u\|^2_{L^2(\Gamma_t(\delta))}, \quad (6.1)$$

where ∇_τ is a suitable tangential derivative defined in Remark 3.4, 2. for $N = 2$ and Remark 3.8, 2. for $N \geq 2$, respectively. The estimate (also without the two additional last terms in (6.1)) implies that the spectrum of $\mathcal{L}_{\varepsilon,t}$ is bounded from below by $-C$, where $\mathcal{L}_{\varepsilon,t}$ is viewed as an unbounded operator on $\{u \in H^2(\Omega) : \partial_{N_{\partial\Omega}} u = 0\}$ with values in $L^2(\Omega)$. $\mathcal{L}_{\varepsilon,t}$ is selfadjoint and has spectrum in $\mathbb{R}$ in this setting. This explains the name "spectral estimate".

Review of Spectral Estimates used within the Method of [deMS]. In the following we give a review about the development of spectral estimates used for the method by de Mottoni and Schatzman. The proof of the spectral estimate for $\Omega = \mathbb{R}^N$ without a boundary condition for the Allen-Cahn-equation and without the additional two terms on the right hand side in (6.1) was first executed by de Mottoni, Schatzman in [deMS] (for a different approximate solution). Their basic idea was to consider normal modes, i.e. the sets where the signed distance varies for a fixed tangential coordinate. By scaling and perturbation arguments, the spectral properties of the corresponding 1D-operators on finite intervals can essentially be reduced to the ones of the unperturbed operator $\mathcal{L}_0 := -\frac{d^2}{dz^2} + f''(\theta_0)$ on finite large intervals, where f, θ_0 are as in Section 4.1. One uses e.g. the properties of θ_0 from Theorem 4.1. However, the computations in [deMS] were quite complicated. They work with Rayleigh quotients and use a perturbation result for isolated eigenvalues of selfadjoint operators. Altogether, de Mottoni and Schatzman prove interesting and detailed spectral properties. But (6.1) can be obtained with less effort, which was shown by Chen in [C2], Theorem 2.3. There a general form of u_ε^A but without the height functions we used in the scaled variable (5.1) is allowed, but these can be included with the analogous proof, cf. Abels, Liu [AL] and for more details see Marquardt [Ma]. In Chen [C2] the part of the integral in (6.1) over the tubular neighbourhood of a closed hypersurface is transformed to a double integral over the tangential coordinate and the normal modes. Then the integrals over the normal modes are estimated a little bit more roughly (compared to [deMS]) from below via scaling and perturbation arguments which is also the idea in de Mottoni and Schatzman [deMS].

Still in the proof one has to pay great attention on the orders of ε. The two last additional terms in (6.1) in the case of a closed interface can be added by looking more closely at the integral transformation by Chen [C2]. This is a simple argument first observed by Abels, Liu [AL], but the additional terms can help to lower the required order for the approximate solution that is needed for the difference estimate to work and to optimize the result, cf. [AL] and Section 7 below, in particular Remark 7.6, 3. Based on the estimate for the Allen-Cahn operator, Chen [C2] also obtains spectral estimates for the Cahn-Hilliard operator (cf. also Marquardt [Ma]) and the phase field operator. Here the idea of splitting the H^1-space with respect to a subspace approximating the eigenvectors to the lowest eigenvalues is used to get further estimates, cf. Lemma 2.4 in [C2]. The subspace is roughly the one obtained by multiplying functions depending on the tangential variable with the eigenvectors to the lowest eigenvalue for the normal mode problems.

Overview and Strategy for the Spectral Estimates in this Section. Also for the spectral estimates in our cases, the control of perturbed 1D-operators on normal modes and large intervals will be a crucial ingredient. We show such estimates in the scalar-valued and vector-valued case similar as in Chen [C2] (and [AL],[Ma]). In the vector-valued case some new ideas are required to replace arguments with comparison principles etc. To this end we use contradiction arguments together with the properties of the unperturbed operator on $\mathbb{R}$ in Lemma 4.29 and other assertions in Kusche [Ku]. Moreover, in the works [C2], [AL], [Ma] the formulation of the 1D-problems was always linked to the situation of a given u^A_ε over an interface.

Here we will treat the 1D-problems separately in Section 6.1 for better readability. Therefore we introduce an abstract setting in 1D in Section 6.1.1 that is applicable in all our cases. We prove integral transformations and remainder estimates in Section 6.1.2. In Sections 6.1.3 and 6.1.4 we show the spectral estimates for (unperturbed and perturbed) operators in 1D in the scalar and vector-valued case on finite large intervals, respectively. In the appendix, Section 6.1.5, for Section 6.1 we summarize an abstract Fredholm Alternative that can be applied for all the cases in order to obtain discrete eigenvalues and orthonormal bases of eigenfunctions.

Equipped with this we prove the spectral estimates for all our cases. In Section 6.2 we show (6.1) for the 90°-contact angle situation for (AC) in 2D and in Section 6.3 for ND. As for the asymptotic expansions, the 2D-case can be included in the ND-case, but we decided to treat them separately since then the underlying ideas become more transparent. In Section 6.4 the vector-valued case with (vAC) is considered and finally in Section 6.5 we treat the situation of (AC_α) with contact angle α close to $\frac{\pi}{2}$.

At this point, let us motivate our approach. The approximate solution always has a specific structure. For parts away from the contact points/lines the estimate directly follows with the 1D-estimates from Section 6.1 and a transformation as in Chen [C2]. Therefore by an argument with a partition of unity one can reduce the spectral estimate to a corresponding one close to the contact points. Moreover, via Taylor expansions, one can replace the potential part by a term with simpler structure.

For the case of (AC) in 2D one can replace $\mathcal{L}_{\varepsilon,t}$ by

$$\mathcal{L}^{\pm}_{\varepsilon,t} := -\Delta + \frac{1}{\varepsilon^2} f''(\theta_0(\rho_\varepsilon(.,t))) + \frac{1}{\varepsilon} f^{(3)}(\theta_0(\rho_\varepsilon(.,t)))u_1^{C\pm}(\rho_\varepsilon, H^{\pm}_\varepsilon, t)$$

in $\Omega^{C\pm}_t := \Gamma^{\pm}_t(\delta, 2\mu_0)$. To get an idea for the proof of the spectral estimate in this case, let us first discard the curvilinear structure and the higher order term. Therefore we consider the simpler operator $\mathcal{L} := -\Delta + \frac{1}{\varepsilon^2} f''(\theta_0(\frac{r}{\varepsilon}))$ defined for functions in variables (r, s) on a rectangle $[-\delta, \delta] \times [0, \eta]$ with homogeneous Neumann boundary condition. One can obtain all the eigenvalues and eigenfunctions with a separation ansatz. Formally, because of Lemma 4.2 and a scaling argument, for small $\varepsilon > 0$ the eigenfunctions corresponding to the lowest eigenvalues should approximately have the form $a(s)\theta_0'(\frac{r}{\varepsilon})$ with $a : [0, \eta] \to \mathbb{R}$ and there should be a spectral gap. In $\mathcal{L}_{\varepsilon,t}^{\pm}$ there is ρ_ε instead of $\frac{r}{\varepsilon}$. Moreover, we have to deal with the $u_1^{C\pm}$-term at order $\frac{1}{\varepsilon}$ and we have to take into account the curvilinear structure of $\Omega_t^{C\pm}$. Therefore we refine $\theta_0'(\rho_\varepsilon(.,t))$ via a suitable ansatz to get an approximate first eigenfunction $\phi_\varepsilon^A(.,t)$. This will lead to the same model problem we have studied in Section 4.2.1. Then we define a subspace of $H^1(\Omega_t^{C\pm})$ consisting of tangential alterations $a(s(.,t))\phi_\varepsilon^A(.,t)$ for suitable $a : [0, 2\mu_0] \to \mathbb{R}$. Finally, we split $H^1(\Omega_t^{C\pm})$ orthogonally in $L^2(\Omega_t^{C\pm})$ with respect to this subspace and analyze the bilinear form corresponding to $\mathcal{L}_{\varepsilon,t}^{\pm}$ on every part. The basic splitting above is reminiscent of Lemma 2.4 in Chen [C2]. The more refined splitting with the construction of an approximate eigenfunction is similar to and motivated from Alikakos, Chen, Fusco [ACF].

In the other cases, the above idea is adapted correspondingly. To construct the approximate eigenfunctions we use the model problems considered in Section 4.

Remark 6.1. Note that a general reduction strategy in analogy to Chen [C2] also might work in our cases, i.e. the idea would be to reduce via perturbation arguments to the spectral properties of corresponding unperturbed operators on large domains approximating $\mathbb{R}^2_+$. However, that would require tedious estimates and the degeneracy is a difficulty, cf. the tangential alterations in the eigenfunctions on the rectangle before. Therefore we work with the simpler strategy above.

6.1 Preliminaries in 1D

6.1.1 The Setting

We consider an abstract setting in 1D that can be applied later to integrals over normal modes in all our cases. Therefore let $\delta > 0$ be fixed, $h_\varepsilon \in C^1([-\delta, \delta], \mathbb{R})$ for $\varepsilon > 0$ small such that

$$\|h_\varepsilon\|_{C_b^1([-\delta,\delta])} \leq C_0. \tag{6.2}$$

Then we set

$$r_\varepsilon : [-\delta, \delta] \to \mathbb{R} : r \mapsto r - \varepsilon h_\varepsilon(r) \quad \text{and} \quad \rho_\varepsilon := \frac{r_\varepsilon}{\varepsilon}. \tag{6.3}$$

At this point let us already note Remark 6.4, 1. below, where the correspondence to the application is explained.

Lemma 6.2. *There is an $\varepsilon_0 = \varepsilon_0(C_0) > 0$ such that*

1. *$r_\varepsilon : [-\tilde{\delta}, \tilde{\delta}] \to [-\tilde{\delta} - \varepsilon h_\varepsilon(-\tilde{\delta}), \tilde{\delta} - \varepsilon h_\varepsilon(\tilde{\delta})]$ is C^1 and invertible for all $\tilde{\delta} \in (0, \delta]$, $\varepsilon \in (0, \varepsilon_0]$. Moreover,*
$$\left|\frac{d}{dr} r_\varepsilon - 1\right| \leq C_0 \varepsilon \leq \frac{1}{2}, \quad \left|\frac{d}{d\tilde{r}}(r_\varepsilon^{-1}) - 1\right| \leq 2C_0\varepsilon$$
and $|r_\varepsilon^{-1}(\tilde{r})| \leq (1 + 2C_0\varepsilon)(|\tilde{r}| + \varepsilon|h_\varepsilon(0)|)$ for all $\tilde{r} \in r_\varepsilon([-\delta, \delta])$ and $\varepsilon \in (0, \varepsilon_0]$.
2. *If additionally $h_\varepsilon \in C^2([-\delta, \delta])$ with $\|\frac{d^2}{dr^2} h_\varepsilon\|_{C_b^0([-\delta,\delta])} \leq \tilde{C}_0$ for $\varepsilon \in (0, \varepsilon_1]$, $0 < \varepsilon_1 \leq \varepsilon_0$, then r_ε is C^2 for $\varepsilon \in (0, \varepsilon_1]$ and it holds*
$$\left|\frac{d^2}{dr^2} r_\varepsilon\right| \leq \tilde{C}_0 \varepsilon \quad \textit{and} \quad \left|\frac{d^2}{d\tilde{r}^2}(r_\varepsilon^{-1})\right| \leq 8\tilde{C}_0\varepsilon.$$

Proof. Ad 1. Since h_ε is C^1, this is also true for r_ε. Moreover, it holds

$$\frac{d}{dr}r_\varepsilon(r) = 1 - \varepsilon\frac{d}{dr}h_\varepsilon(r) \quad \text{and} \quad \left|\frac{d}{dr}r_\varepsilon(r) - 1\right| \le C_0\varepsilon \le \frac{1}{2}$$

for all $r \in [-\delta,\delta]$ if $\varepsilon \in (0,\varepsilon_0]$ and $\varepsilon_0 = \varepsilon_0(C_0) > 0$ is small. In particular, r_ε is strictly monotone and invertible on $[-\tilde{\delta},\tilde{\delta}]$ onto $[r_\varepsilon(-\tilde{\delta}), r_\varepsilon(\tilde{\delta})]$ for all $\tilde{\delta} \in (0,\delta]$. The inverse is also C^1 and

$$\left|\frac{d}{d\tilde{r}}(r_\varepsilon^{-1}) - 1\right| = \left|\frac{1 - \frac{d}{dr}r_\varepsilon(r_\varepsilon^{-1})}{\frac{d}{dr}r_\varepsilon(r_\varepsilon^{-1})}\right| \le 2C_0\varepsilon$$

due to $|\frac{d}{dr}r_\varepsilon(r_\varepsilon^{-1})| \ge \frac{1}{2}$. Finally, note that $r_\varepsilon(0) = -\varepsilon h_\varepsilon(0)$. This yields for all $\tilde{r} \in r_\varepsilon([-\delta,\delta])$:

$$r_\varepsilon^{-1}(\tilde{r}) = \int_{-\varepsilon h_\varepsilon(0)}^{\tilde{r}} \frac{d}{d\overline{r}}(r_\varepsilon^{-1})(\overline{r})\,d\overline{r}.$$

Since the modulus of the integrand is bounded by $1+2C_0\varepsilon$, we obtain the estimate for $|r_\varepsilon^{-1}|$. $\square_1$.

Ad 2. Let additionally $h_\varepsilon \in C^2([-\delta,\delta])$ with $\|\frac{d^2}{dr^2}h_\varepsilon\|_{C_b^0([-\delta,\delta])} \le \tilde{C}_0$ for $\varepsilon \in (0,\varepsilon_1]$. Then r_ε and r_ε^{-1} are C^2 for all $\varepsilon \in (0,\varepsilon_1]$. The estimates follow from

$$\frac{d^2}{dr^2}r_\varepsilon = -\varepsilon\frac{d^2}{dr^2}h_\varepsilon \quad \text{and} \quad \frac{d^2}{d\tilde{r}^2}(r_\varepsilon^{-1}) = -\frac{\frac{d^2}{dr^2}r_\varepsilon(r_\varepsilon^{-1})}{(\frac{d}{dr}r_\varepsilon(r_\varepsilon^{-1}))^3}$$

together with $|\frac{d}{dr}r_\varepsilon(r_\varepsilon^{-1})| \ge \frac{1}{2}$. $\square_2$.

In particular $\rho_\varepsilon : [-\delta,\delta] \to \frac{1}{\varepsilon}r_\varepsilon([-\delta,\delta])$ is C^1 for $\varepsilon \in (0,\varepsilon_0]$ and invertible with inverse

$$F_\varepsilon : \frac{1}{\varepsilon}r_\varepsilon([-\delta,\delta]) \to [-\delta,\delta] : z \mapsto r_\varepsilon^{-1}(\varepsilon z). \tag{6.4}$$

Finally, let $J \in C^2([-\delta,\delta],\mathbb{R})$ with

$$J \ge c_1 > 0 \quad \text{and} \quad \|J\|_{C_b^2([-\delta,\delta])} \le C_2. \tag{6.5}$$

Then we define $J_\varepsilon := J(F_\varepsilon) : \frac{1}{\varepsilon}r_\varepsilon([-\delta,\delta]) \to \mathbb{R}$ for $\varepsilon \in (0,\varepsilon_0]$.

Corollary 6.3. *Let $h_\varepsilon \in C^2([-\delta,\delta],\mathbb{R})$ with $\|h_\varepsilon\|_{C^2([-\delta,\delta])} \le \overline{C}_0$ for small $\varepsilon > 0$ and J be as above. Let $\varepsilon_0 = \varepsilon_0(\overline{C}_0) > 0$ be such that Lemma 6.2 holds. Then F_ε, J_ε are well-defined for $\varepsilon \in (0,\varepsilon_0]$, C^2 and we obtain for all $z \in \frac{1}{\varepsilon}r_\varepsilon([-\delta,\delta])$ the estimates*

$$|F_\varepsilon(z)| \le 2\varepsilon(|z| + \overline{C}_0), \quad \left|\frac{d}{dz}F_\varepsilon(z)\right| \le 2\varepsilon, \quad \left|\frac{d^2}{dz^2}F_\varepsilon(z)\right| \le 8\overline{C}_0\varepsilon^3,$$

$$c_1 \le J_\varepsilon(z) \le C_2, \quad \left|\frac{d}{dz}J_\varepsilon(z)\right| \le 2C_2\varepsilon, \quad \left|\frac{d^2}{dz^2}J_\varepsilon(z)\right| \le C(\overline{C}_0, C_2)\varepsilon^2.$$

Proof. This directly follows from Lemma 6.2 and the chain rule. $\square$

Remark 6.4. 1. In the applications later δ corresponds to the one from Theorems 3.3 and 3.7. Moreover, J will correlate to the determinant in Remarks 3.4, 3. and 3.8, 3. The h_ε, ρ_ε here stand for the height function and the rescaled normal variable in the asymptotic expansions. However, here h_ε depends on the normal variable. The latter will be constant in this setting for all cases except in the situation of an α-contact angle, $\alpha \neq \frac{\pi}{2}$. In this case one uses a transformation to $(r, z_\alpha^\pm)$-coordinates, see (5.79) for the definition of $z_\alpha^\pm$. Then one obtains an r-dependence for functions in the tangential variable s. In the abstract setting in this section additional variables like tangential ones or time do not appear. Therefore we prove uniform estimates with respect to the constants above.

2. Consider the situation of Corollary 6.3. Later it will be convenient to consider suitable symmetric subintervals of $\frac{1}{\varepsilon}r_\varepsilon([-\delta,\delta])$. Therefore note that for $\varepsilon_1 = \varepsilon_1(\delta, \overline{C}_0) > 0$ small it holds $\varepsilon_1 \leq \varepsilon_0$ and

$$[-\tilde{\delta}, \tilde{\delta}] \subseteq r_\varepsilon([-\delta,\delta]) \quad \text{for all } \tilde{\delta} \in (0, \frac{3\delta}{4}], \varepsilon \in (0, \varepsilon_1].$$

Hence $F_\varepsilon, J_\varepsilon$ are well-defined and the assertions of Corollary 6.3 hold on $\overline{I_{\varepsilon,\tilde{\delta}}}$ for all $\tilde{\delta} \in (0, \frac{3\delta}{4}]$ and $\varepsilon \in (0, \varepsilon_1]$, where $I_{\varepsilon,\tilde{\delta}} := (-\tilde{\delta}/\varepsilon, \tilde{\delta}/\varepsilon)$. Typically remainder terms on $\frac{1}{\varepsilon}r_\varepsilon([-\tilde{\delta},\tilde{\delta}]) \setminus I_{\varepsilon,\tilde{\delta}}$ will behave nicely and it is sufficient to prove finer estimates on $I_{\varepsilon,\tilde{\delta}}$. This simplifies some notation.

6.1.2 Transformations and Remainder Terms

Let $\delta, C_0 > 0$ and $h_\varepsilon \in C^1([-\delta,\delta],\mathbb{R})$ such that (6.2) holds for $\varepsilon > 0$ small. Moreover, let $r_\varepsilon, \rho_\varepsilon$ be as in (6.3) and $\varepsilon_0 = \varepsilon_0(C_0) > 0$ be such that Lemma 6.2 holds. Then F_ε as in (6.4) is well-defined. We obtain the following lemma for transformation arguments and estimates for remainder terms.

Lemma 6.5. *Let $\tilde{\varepsilon}_0 \in (0, \varepsilon_0]$ and $R_\varepsilon : \mathbb{R} \times [-\delta,\delta] \to \mathbb{R}$ be integrable for $\varepsilon \in (0, \tilde{\varepsilon}_0]$. Moreover, let $\mathcal{J} \subseteq [-\delta,\delta]$ be an interval.*

1. *For all $\varepsilon \in (0, \tilde{\varepsilon}_0]$ it holds*

$$\int_{\mathcal{J}} R_\varepsilon(\rho_\varepsilon(r), r)\, dr = \int_{r_\varepsilon(\mathcal{J})/\varepsilon} R_\varepsilon(z, F_\varepsilon(z)) \frac{d}{dz}F_\varepsilon(z)\, dz,$$

where $\frac{d}{dz}F_\varepsilon(z) = \varepsilon \frac{d}{d\tilde{r}}(r_\varepsilon^{-1})(\varepsilon z) > 0$ and $\left|\frac{d}{d\tilde{r}}(r_\varepsilon^{-1})(\varepsilon z) - 1\right| \leq 2C_0\varepsilon$ for all $z \in \frac{r_\varepsilon([-\delta,\delta])}{\varepsilon}$.

2. *If additionally $|R_\varepsilon(\rho, r)| \leq \overline{C}|r|^k e^{-\beta|\rho|}$ for all $(\rho, r) \in \mathbb{R} \times [-\delta,\delta]$ and some $k \geq 0$, $\overline{C}, \beta > 0$, then for all $\varepsilon \in (0, \tilde{\varepsilon}_0]$ it follows that with constants independent of $\mathcal{J}$ we have the estimate*

$$\int_{\mathcal{J}} |R_\varepsilon(\rho_\varepsilon(r), r)|\, dr \leq \overline{C} C(C_0, k, \beta)\varepsilon^{k+1}.$$

Proof. The first assertion follows from Lemma 6.2 and the transformation rule. Using the latter for the second part, we obtain for all $\varepsilon \in (0, \tilde{\varepsilon}_0]$ that

$$\int_{\mathcal{J}} |R_\varepsilon(\rho_\varepsilon(r), r)|\, dr \leq \overline{C}\varepsilon(1 + 2C_0\varepsilon) \int_{r_\varepsilon(\mathcal{J})/\varepsilon} |F_\varepsilon(z)|^k e^{-\beta|z|}\, dz.$$

Here it holds $|F_\varepsilon(z)| = |r_\varepsilon^{-1}(\varepsilon z)| \leq (1 + 2C_0\varepsilon)\varepsilon(|z| + C_0)$ for all $z \in r_\varepsilon([-\delta, \delta])/\varepsilon$ and $C_0\varepsilon \leq \frac{1}{2}$ due to Lemma 6.2, 1. This yields

$$\int_{\mathcal{J}} |R_\varepsilon(\rho_\varepsilon(r), r)|\, dr \leq \overline{C} 2^{k+1} \int_{\mathbb{R}} (|z| + C_0)^k e^{-\beta|z|}\, dz\, \varepsilon^{k+1}.$$

This shows the estimate with constants independent of $\mathcal{J}$. □

6.1.3 Spectral Estimates for Scalar-Valued Allen-Cahn-Type Operators in 1D

6.1.3.1 Unperturbed Scalar-Valued Allen-Cahn-Type Operators in 1D In Section 4.1.2 we obtained assertions for the spectrum of the operator $L_0 : H^2(\mathbb{R}, \mathbb{K}) \to L^2(\mathbb{R}, \mathbb{K}) : u \mapsto \mathcal{L}_0 u$, where $\mathcal{L}_0 := -\frac{d^2}{dz^2} + f''(\theta_0)$ and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Now we consider $\mathcal{L}_0$ on finite large intervals together with homogeneous Neumann boundary condition. Note that in the following we only use Theorem 4.1, but not Lemma 4.2.

Let $\tilde{\delta} > 0$ be fixed, $\varepsilon > 0$, $I_{\varepsilon,\tilde{\delta}} := (-\frac{\tilde{\delta}}{\varepsilon}, \frac{\tilde{\delta}}{\varepsilon})$ and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We will only need the case $\mathbb{K} = \mathbb{R}$, but $\mathbb{C}$ is included for more generality. Moreover, here one can reduce to $\tilde{\delta} = 1$ by scaling in ε. However, introducing the $\tilde{\delta}$ in the notation already here will simplify the notation later. We consider the unbounded operator

$$L_{0,\varepsilon} : H^2_N(I_{\varepsilon,\tilde{\delta}}, \mathbb{K}) \to L^2(I_{\varepsilon,\tilde{\delta}}, \mathbb{K}) : u \mapsto \mathcal{L}_0 u = \left[-\frac{d^2}{dz^2} + f''(\theta_0) \right] u,$$

where $H^2_N(I_{\varepsilon,\tilde{\delta}}, \mathbb{K})$ is the space of H^2-functions u on $I_{\varepsilon,\tilde{\delta}}$ satisfying the homogeneous Neumann boundary condition $\frac{d}{dz} u|_z = 0$ for $z = \pm\tilde{\delta}/\varepsilon$. The corresponding sesquilinearform is

$$B_{0,\varepsilon} : H^1(I_{\varepsilon,\tilde{\delta}}, \mathbb{K}) \times H^1(I_{\varepsilon,\tilde{\delta}}, \mathbb{K}) \to \mathbb{K} : (\Phi, \Psi) \mapsto \int_{I_{\varepsilon,\tilde{\delta}}} \frac{d}{dz}\Phi \overline{\frac{d}{dz}\Psi} + f''(\theta_0)\Phi\overline{\Psi}\, dz.$$

As in Chen [C2], Lemma 2.1 and Marquardt [Ma], Section 3.1 we obtain the following lemma:

Lemma 6.6. 1. *$L_{0,\varepsilon}$ is selfadjoint and the spectrum is given by discrete eigenvalues $(\lambda^k_{0,\varepsilon})_{k\in\mathbb{N}}$ in $\mathbb{R}$ with $\lambda^1_{0,\varepsilon} \leq \lambda^2_{0,\varepsilon} \leq ...$ and $\lambda^k_{0,\varepsilon} \overset{k\to\infty}{\longrightarrow} \infty$. Moreover, there is an orthonormal basis $(\Psi^k_{0,\varepsilon})_{k\in\mathbb{N}}$ of $L^2(I_\varepsilon, \mathbb{K})$ consisting of smooth $\mathbb{R}$-valued eigenfunctions $\Psi^k_{0,\varepsilon}$ to $\lambda^k_{0,\varepsilon}$.*

2. *$\lambda^1_{0,\varepsilon}$ is simple and the corresponding eigenfunction $\Psi^1_{0,\varepsilon}$ has a sign. We take $\Psi^1_{0,\varepsilon}$ positive.*

3. *Let $c_0 > 0$ be such that $\inf_{|z|\geq c_0} f''(\theta_0(z)) \geq \frac{3}{4}\min\{f''(\pm 1)\}$. Then for $\varepsilon > 0$ small and any normalized eigenfunction $\Psi_{0,\varepsilon}$ of $L_{0,\varepsilon}$ to an eigenvalue $\lambda_{0,\varepsilon} \leq \frac{1}{4}\min\{f''(\pm 1)\}$ it holds*
$$|\Psi_{0,\varepsilon}(z)| \leq Ce^{-|z|\sqrt{\min\{f''(\pm 1)\}/3}} \quad \text{for all } z \in I_{\varepsilon,\tilde{\delta}}, |z| \geq c_0 + 1,$$
where $C > 0$ only depends on c_0 and $\min\{f''(\pm 1)\}$.

4. *There is $\varepsilon_0 = \varepsilon_0(\tilde{\delta}) > 0$ small such that for all $\varepsilon \in (0, \varepsilon_0]$*
$$\lambda^1_{0,\varepsilon} = \inf_{\Psi \in H^1(I_{\varepsilon,\tilde{\delta}}), \|\Psi\|_{L^2}=1} B_{0,\varepsilon}(\Psi, \Psi) = B_{0,\varepsilon}(\Psi^1_{0,\varepsilon}, \Psi^1_{0,\varepsilon}), \quad |\lambda^1_{0,\varepsilon}| \leq Ce^{-\frac{3\tilde{\delta}\sqrt{\min\{f''(\pm 1)\}}}{2\varepsilon}},$$
where $C > 0$ is independent of $\tilde{\delta}$, ε.

5. *There is* $\nu_1 > 0$ *independent of* $\tilde{\delta}$, ε *and* $\varepsilon_0 = \varepsilon_0(\tilde{\delta}) > 0$ *small such that*
$$\lambda_{0,\varepsilon}^2 = \inf_{\Psi\in H^1(I_{\varepsilon,\tilde{\delta}}),\|\Psi\|_{L^2}=1,\Psi\perp_{L^2}\Psi_{0,\varepsilon}^1} B_{0,\varepsilon}(\Psi,\Psi) \geq \nu_1 \quad \textit{for all } \varepsilon \in (0,\varepsilon_0].$$

6. *Let* $\beta_\varepsilon := \|\theta_0'\|_{L^2(I_{\varepsilon,\tilde{\delta}})}^{-1}$ *and* $\Psi_{0,\varepsilon}^R := \Psi_{0,\varepsilon}^1 - \beta_\varepsilon\theta_0'$. *For* $\varepsilon_0 = \varepsilon_0(\tilde{\delta}) > 0$ *small and* $\varepsilon \in (0,\varepsilon_0]$ *we have*
$$\left\|\Psi_{0,\varepsilon}^R\right\|_{L^2(I_{\varepsilon,\tilde{\delta}})}^2 + \left\|\tfrac{d}{dz}\Psi_{0,\varepsilon}^R\right\|_{L^2(I_{\varepsilon,\tilde{\delta}})}^2 \leq Ce^{-\frac{3\tilde{\delta}\sqrt{\min\{f''(\pm 1)\}}}{2\varepsilon}},$$
where $C > 0$ *is independent of* $\tilde{\delta}$, ε.

Proof. By scaling in ε it is enough to consider the case $\tilde{\delta} = 1$. We set $I_\varepsilon := I_{\varepsilon,1}$.

Ad 1. $L_{0,\varepsilon}$ is selfadjoint because $L_{0,\varepsilon}$ is symmetric and the resolvent set nonempty. The latter follows from the Lax-Milgram Theorem applied to a constant shift of $B_{0,\varepsilon}$ due to $f''(\theta_0) \geq -C$. The spectral properties in 1. and the existence of the orthonormal basis follow from the abstract Fredholm alternative in Theorem 6.14 below (and standard regularity and integration by parts arguments) applied to
$$A_{0,\varepsilon} : H^1(I_\varepsilon,\mathbb{C}) \to H^1(I_\varepsilon,\mathbb{C})^* : u \mapsto [v \mapsto B_{0,\varepsilon}(u,v)],$$
where $H^1(I_\varepsilon,\mathbb{C})^*$ is the anti-dual space of $H^1(I_\varepsilon,\mathbb{C})$, i.e. the space of conjugate linear functionals on $H^1(I_\varepsilon,\mathbb{C})$. Note that here also in the case $\mathbb{K} = \mathbb{C}$ one can obtain an $\mathbb{R}$-valued orthonormal basis since for an eigenfunction u also $\overline{u}$ is an eigenfunction to the same eigenvalue and $u, \overline{u}$ are $\mathbb{C}$-linearly independent if and only if $\operatorname{Re} u, \operatorname{Im} u$ are $\mathbb{R}$-linearly independent. The latter can be seen with elementary arguments. $\square_1$.

Ad 2. The properties of $\lambda_{0,\varepsilon}^1$ and $\Psi_{0,\varepsilon}^1$ can be shown with the Krein-Rutman-Theorem and the maximum principle, cf. [Ma], Proposition 3.6, 2. $\square_2$.

Ad 3.-6. For the rest it is enough to consider the case $\mathbb{K} = \mathbb{R}$. The eigenvalues are the same for $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and the orthonormal basis of $\mathbb{R}$-valued eigenfunctions can be chosen to be the same. Moreover, the inf-characterizations are known, cf. e.g. [Ma], proof of Proposition 3.6. The other assertions can be deduced with the comparison principle, Theorem 4.1, the Harnack-inequality and the Hopf maximum principle, cf. [C2], Lemma 2.1 and [Ma], Proposition 3.7 and Lemma 3.8. For the proof of 6. see also the analogous computation in the proof of Lemma 6.6, 6. below in the vector-valued case. $\square$

6.1.3.2 Perturbed Scalar-Valued Allen-Cahn-Type Operators in 1D In this section we derive a result for perturbed and weighted operators in 1D. Let $\delta > 0$ and $h_\varepsilon, J \in C^2([-\delta,\delta],\mathbb{R})$ with $\|h_\varepsilon\|_{C^2([-\delta,\delta])} \leq \overline{C}_0$ for $\varepsilon > 0$ small and $c_1, C_2 > 0$ be such that (6.5) holds. Then let $\rho_\varepsilon, F_\varepsilon, J_\varepsilon$ for $\varepsilon > 0$ small be as in Section 6.1.1. We consider
$$\phi_\varepsilon : [-\delta,\delta] \to \mathbb{R} : r \mapsto \theta_0(\frac{r}{\varepsilon}) + \varepsilon p_\varepsilon \theta_1(\frac{r}{\varepsilon}) + q_\varepsilon(r)\varepsilon^2, \tag{6.6}$$
where $p_\varepsilon \in \mathbb{R}$ and $q_\varepsilon : [-\delta,\delta] \to \mathbb{R}$ is measurable with $|p_\varepsilon| + \frac{\varepsilon}{\varepsilon+|r|}|q_\varepsilon(r)| \leq C_3$ for all $r \in [-\delta,\delta]$, some $C_3 > 0$, and $\varepsilon > 0$ small. Moreover, let $\theta_1 \in L^\infty(\mathbb{R})$ with $\|\theta_1\|_\infty \leq C_4$ for a $C_4 > 0$ and
$$\int_{\mathbb{R}} f''(\theta_0)(\theta_0')^2\,\theta_1 = 0. \tag{6.7}$$

Let $\tilde{\delta} \in (0, \frac{3\delta}{4}]$ be fixed. Then F_ε, J_ε are well-defined on $\overline{I_{\varepsilon,\tilde{\delta}}}$ for $\varepsilon \in (0, \varepsilon_1(\delta, \overline{C}_0)]$ and Corollary 6.3 is applicable due to Remark 6.4, 2. We consider the operators

$$L_\varepsilon : H^2_N(I_{\varepsilon,\tilde{\delta}}, \mathbb{K}) \to L^2_{J_\varepsilon}(I_{\varepsilon,\tilde{\delta}}, \mathbb{K}) : u \mapsto \mathcal{L}_\varepsilon u := \left[-J_\varepsilon^{-1} \frac{d}{dz}\left(J_\varepsilon \frac{d}{dz}\right) + f''(\phi_\varepsilon(\varepsilon.))\right] u,$$

where $L^2_{J_\varepsilon}(I_{\varepsilon,\tilde{\delta}}, \mathbb{K})$ is the space of L^2-functions defined on $I_{\varepsilon,\tilde{\delta}}$ with the weight J_ε. We write $(.,.)_{J_\varepsilon}$, $\|.\|_{J_\varepsilon}$ and $\perp_{J_\varepsilon}$ for the corresponding scalar product, norm and orthogonal relation. The sesquilinearform associated to L_ε is given by

$$B_\varepsilon : H^1(I_{\varepsilon,\tilde{\delta}}, \mathbb{K}) \times H^1(I_{\varepsilon,\tilde{\delta}}, \mathbb{K}) \to \mathbb{K} : (\Phi, \Psi) \mapsto \int_{I_{\varepsilon,\tilde{\delta}}} \left[\frac{d}{dz}\Phi \overline{\frac{d}{dz}\Psi} + f''(\phi_\varepsilon(\varepsilon.))\Phi\overline{\Psi}\right] J_\varepsilon \, dz.$$

Again only $\mathbb{K} = \mathbb{R}$ is needed and $\mathbb{K} = \mathbb{C}$ is added for more generality. We obtain the analogue of Lemma 6.6, 1.-3.

Lemma 6.7. *1. L_ε is selfadjoint and the spectrum is given by a sequence of discrete eigenvalues $(\lambda_\varepsilon^k)_{k\in\mathbb{N}}$ in $\mathbb{R}$ with $\lambda_\varepsilon^1 \leq \lambda_\varepsilon^2 \leq \ldots$ and $\lambda_\varepsilon^k \overset{k\to\infty}{\longrightarrow} \infty$. Moreover, there is an orthonormal basis $(\Psi_\varepsilon^k)_{k\in\mathbb{N}}$ of $L^2_{J_\varepsilon}(I_{\varepsilon,\tilde{\delta}}, \mathbb{K})$ consisting of smooth $\mathbb{R}$-valued eigenfunctions Ψ_ε^k to λ_ε^k.*

2. λ_ε^1 is simple and the corresponding eigenfunction Ψ_ε^1 has a sign. We take Ψ_ε^1 positive.

3. Let $c_0 > 0$ be such that $\inf_{|z|\geq c_0} f''(\theta_0(z)) \geq \frac{3}{4}\min\{f''(\pm 1)\}$. There is an $\varepsilon_0 > 0$ (only depending on δ, $\tilde{\delta}$, $\overline{C}_0$, c_1, C_2, C_3, C_4) such that for all $\varepsilon \in (0, \varepsilon_0]$ and any normalized $\mathbb{R}$-valued eigenfunction Ψ_ε of L_ε to an eigenvalue $\lambda_\varepsilon \leq \frac{1}{4}\min\{f''(\pm 1)\}$ it holds

$$|\Psi_\varepsilon(z)| \leq Ce^{-|z|\sqrt{\min\{f''(\pm 1)\}/3}} \quad \text{for all } z \in I_{\varepsilon,\tilde{\delta}}, |z| \geq c_0 + 1,$$

where $C > 0$ only depends on c_0, $\min\{f''(\pm 1)\}$ and c_1.

Proof. This follows in the analogous way as in the unperturbed case, cf. the proof of Lemma 6.6, 1.-3. above. For 3. consider the proof of Proposition 3.7 in [Ma]. Here the abstract Fredholm alternative in Theorem 6.14 below is applied to

$$A_\varepsilon : H^1_{J_\varepsilon}(I_{\varepsilon,\tilde{\delta}}, \mathbb{C}) \to H^1_{J_\varepsilon}(I_{\varepsilon,\tilde{\delta}}, \mathbb{C})^* : u \mapsto [v \mapsto B_\varepsilon(u, v)],$$

where $H^1_{J_\varepsilon}(I_{\varepsilon,\tilde{\delta}}, \mathbb{C})$ is $H^1(I_{\varepsilon,\tilde{\delta}}, \mathbb{C})$ with the weight J_ε in the norm (both in the L^2-norm for the function and the derivative) and $H^1_{J_\varepsilon}(I_{\varepsilon,\tilde{\delta}}, \mathbb{C})^*$ is the anti-dual space. □

Now we obtain assertions that correspond to Lemma 6.6, 3.-6. in the unperturbed case.

Theorem 6.8. *There is an $\varepsilon_0 > 0$ only depending on δ, $\tilde{\delta}$, $\overline{C}_0$,c_1, C_2, C_3, C_4 and $C > 0$ only depending on $\tilde{\delta}$, $\overline{C}_0$, c_1, C_2, C_3, C_4 such that*

1. For $\varepsilon \in (0, \varepsilon_0]$ it holds

$$\lambda_\varepsilon^1 = \inf_{\Psi\in H^1(I_{\varepsilon,\tilde{\delta}},\mathbb{K}), \|\Psi\|_{J_\varepsilon}=1} B_\varepsilon(\Psi, \Psi) = B_\varepsilon(\Psi_\varepsilon^1, \Psi_\varepsilon^1), \quad |\lambda_\varepsilon^1| \leq C\varepsilon^2.$$

2. Let $\Psi_\varepsilon^R := \Psi_\varepsilon^1 - J(0)^{-\frac{1}{2}}\beta_\varepsilon\theta_0'$, where $\beta_\varepsilon = \|\theta_0'\|_{L^2(I_{\varepsilon,\tilde{\delta}})}$. Then for $\varepsilon \in (0, \varepsilon_0]$

$$\left\|\Psi_\varepsilon^R\right\|_{J_\varepsilon} + \left\|\tfrac{d}{dz}\Psi_\varepsilon^R\right\|_{J_\varepsilon} \leq C\varepsilon.$$

3. *With ν_1 from Lemma 6.6, 5. it holds for all $\varepsilon \in (0, \varepsilon_0]$*

$$\lambda_\varepsilon^2 = \inf_{\Psi \in H^1(I_{\varepsilon,\tilde{\delta}},\mathbb{K}), \|\Psi\|_{J_\varepsilon}=1, \Psi \perp_{J_\varepsilon} \Psi_\varepsilon^1} B_\varepsilon(\Psi, \Psi) \geq \nu_2 := \min\left\{\frac{\nu_1}{2}, \frac{\min\{f''(\pm 1)\}}{4}\right\} > 0.$$

Although the proof is analogous to the ones of Lemma 2.2, 1.-2. in [C2] and Lemma 3.8 in [Ma], we give some details for the convenience of the reader. This will also help to understand the vector-valued case later.

Proof. Note that it is enough to consider the case $\mathbb{K} = \mathbb{R}$. The eigenvalues are the same for $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and the orthonormal basis of $\mathbb{R}$-valued eigenfunctions can be chosen to be the same. The inf-characterizations can be shown as in the unperturbed case. For convenience, if we write "for ε small" in the following it is always meant "for all $\varepsilon \in (0, \varepsilon_0]$ for some $\varepsilon_0 > 0$ small only depending on δ, $\tilde{\delta}$, $\overline{C}_0$,c_1, C_2, C_3, C_4". Similarly, all appearing constants (also in $\mathcal{O}$-notation) below only depend on $\tilde{\delta}$, $\overline{C}_0$,c_1, C_2, C_3, C_4, but we do not explicitly state this.

First, we derive an identity for $B_\varepsilon(\Psi, \Psi)$ for all $\Psi \in H^1(I_{\varepsilon,\tilde{\delta}}, \mathbb{R})$. We define $\hat{\Psi} := J_\varepsilon^{1/2}\Psi$. Then

$$\frac{d}{dz}\Psi = -\frac{1}{2}J_\varepsilon^{-\frac{3}{2}}(\frac{d}{dz}J_\varepsilon)\hat{\Psi} + J_\varepsilon^{-\frac{1}{2}}\frac{d}{dz}\hat{\Psi}.$$

Therefore

$$B_\varepsilon(\Psi, \Psi) = \int_{I_{\varepsilon,\tilde{\delta}}} (\frac{d}{dz}\hat{\Psi})^2 + \left[f''(\phi_\varepsilon(\varepsilon.)) + \frac{1}{4}J_\varepsilon^{-2}(\frac{d}{dz}J_\varepsilon)^2\right]\hat{\Psi}^2 - J_\varepsilon^{-1}(\frac{d}{dz}J_\varepsilon)\frac{1}{2}\frac{d}{dz}(\hat{\Psi}^2)\,dz.$$

In order to use results from the unperturbed case, we would like to replace $f''(\phi_\varepsilon(\varepsilon.))$ by $f''(\theta_0)$. Therefore we use a Taylor expansion and obtain for all $|z| \leq \frac{\delta}{\varepsilon}$

$$\begin{aligned}&|f''(\phi_\varepsilon(\varepsilon z)) - f''(\theta_0(z)) - \varepsilon p_\varepsilon f'''(\theta_0(z))\theta_1(z)| \\ &\leq C|q_\varepsilon(\varepsilon z)|\varepsilon^2 + C|\varepsilon p_\varepsilon \theta_1(z) + q_\varepsilon(\varepsilon z)\varepsilon^2|^2 \leq C(1+|z|)\varepsilon^2.\end{aligned}$$

Rewriting the last term in the above identity for $B_\varepsilon(\Psi, \Psi)$ with integration by parts yields

$$B_\varepsilon(\Psi, \Psi) = B_{0,\varepsilon}(\hat{\Psi}, \hat{\Psi}) + \int_{I_{\varepsilon,\tilde{\delta}}} \left[\varepsilon p_\varepsilon f'''(\theta_0)\theta_1 + \tilde{q}_\varepsilon\right]\hat{\Psi}^2 - \frac{1}{2}\left[J_\varepsilon^{-1}(\frac{d}{dz}J_\varepsilon)\hat{\Psi}^2\right]_{z=-\frac{\tilde{\delta}}{\varepsilon}}^{\frac{\tilde{\delta}}{\varepsilon}}, \quad (6.8)$$

$$\tilde{q}_\varepsilon := f''(\phi_\varepsilon(\varepsilon.)) - f''(\theta_0) - f'''(\theta_0)\varepsilon p_\varepsilon \theta_1 + \frac{1}{4}\left(2J_\varepsilon^{-1}(\frac{d^2}{dz^2}J_\varepsilon) - J_\varepsilon^{-2}(\frac{d}{dz}J_\varepsilon)^2\right).$$

The first part of $\tilde{q}_\varepsilon$ is estimated above, the second part can be controlled with Corollary 6.3. This implies $|\tilde{q}_\varepsilon(z)| \leq C\varepsilon^2(1+|z|)$ for all $z \in I_{\varepsilon,\tilde{\delta}}$.

Ad 1. First we show an upper bound on λ_ε^1 using (6.8). To this end we consider $\Psi = J_\varepsilon^{-1/2}\beta_\varepsilon\theta_0'$. It holds $\|J_\varepsilon^{-1/2}\beta_\varepsilon\theta_0'\|_{J_\varepsilon} = 1$ due to the definitions. Hence with (6.8) and Corollary 6.3 we obtain

$$\lambda_\varepsilon^1 \leq \beta_\varepsilon^2\left[B_{0,\varepsilon}(\theta_0', \theta_0') + \varepsilon p_\varepsilon \int_{I_{\varepsilon,\tilde{\delta}}} f'''(\theta_0)\theta_1(\theta_0')^2 + C\varepsilon^2\int_{I_{\varepsilon,\tilde{\delta}}}(1+|z|)\theta_0'(z)^2\,dz + C\varepsilon e^{-c\tilde{\delta}/\varepsilon}\right].$$

The identity $\int_{\mathbb{R}}(\theta_0'')^2 + f''(\theta_0)(\theta_0')^2 = 0$ due to integration by parts, (6.7) and the decay for θ_0 and its derivatives from Theorem 4.1 imply $\lambda_\varepsilon^1 \leq C\varepsilon^2$ for ε small.

In particular, Lemma 6.7, 3. yields for ε small the decay

$$|\Psi_\varepsilon^1(z)| \leq Ce^{-\frac{\sqrt{\min\{f''(\pm 1)\}}}{2}|z|} \quad \text{for all } z \in I_{\varepsilon,\tilde{\delta}}, |z| \geq c_0 + 1,$$

where $c_0 > 0$ is such that $\inf_{|z|\geq c_0} f''(\theta_0(z)) \geq \frac{3}{4}\min\{f''(\pm 1)\}$.

Hence with (6.8) and estimates as before we get for ε small

$$\lambda_\varepsilon^1 = B_{0,\varepsilon}(\hat{\Psi}_\varepsilon^1, \hat{\Psi}_\varepsilon^1) + \varepsilon p_\varepsilon \int_{I_{\varepsilon,\tilde{\delta}}} f'''(\theta_0)\theta_1(\hat{\Psi}_\varepsilon^1)^2 \, dz + \mathcal{O}(\varepsilon^2).$$

In order to use (6.7), we note that $\Psi_{0,\varepsilon}^R := \Psi_{0,\varepsilon}^1 - \beta_\varepsilon \theta_0'$ satisfies good estimates due to Lemma 6.6, 6. Therefore we split

$$\hat{\Psi}_\varepsilon^1 = J_\varepsilon^{\frac{1}{2}} \Psi_\varepsilon^1 = a_\varepsilon \Psi_{0,\varepsilon}^1 + \Psi_\varepsilon^\perp \tag{6.9}$$

orthogonally in $L^2(I_{\varepsilon,\tilde{\delta}})$, where $a_\varepsilon := (\hat{\Psi}_\varepsilon^1, \Psi_{0,\varepsilon}^1)_{L^2(I_{\varepsilon,\tilde{\delta}})}$. It holds $|a_\varepsilon| \leq 1$ due to the Cauchy-Schwarz-Inequality and $a_\varepsilon^2 = 1 - \|\Psi_\varepsilon^\perp\|_{L^2(I_{\varepsilon,\tilde{\delta}})}^2$. Moreover, due to positivity of $\hat{\Psi}_\varepsilon^1$ and $\Psi_{0,\varepsilon}^1$ we have $a_\varepsilon > 0$. Note that the latter is only needed for the estimate of Ψ_ε^R later. Hence

$$\left| \int_{I_{\varepsilon,\tilde{\delta}}} f'''(\theta_0)\theta_1(\hat{\Psi}_\varepsilon^1)^2 \, dz \right| \leq a_\varepsilon^2 \left| \int_{I_{\varepsilon,\tilde{\delta}}} f'''(\theta_0)\theta_1(\Psi_{0,\varepsilon}^1)^2 \, dz \right| + C\|\Psi_\varepsilon^\perp\|_{L^2(I_{\varepsilon,\tilde{\delta}})},$$

where we used $\|\Psi_\varepsilon^\perp\|_{L^2(I_{\varepsilon,\tilde{\delta}})} \leq 1$. We substitute $\Psi_{0,\varepsilon}^1$ by $\Psi_{0,\varepsilon}^R + \beta_\varepsilon\theta_0'$. With (6.7), the decay for θ_0' from Theorem 4.1 and Lemma 6.6, 6. we obtain for ε small

$$\left| \int_{I_{\varepsilon,\tilde{\delta}}} f'''(\theta_0)\theta_1(\hat{\Psi}_\varepsilon^1)^2 \, dz \right| \leq C(e^{-c/\varepsilon} + \|\Psi_\varepsilon^\perp\|_{L^2(I_{\varepsilon,\tilde{\delta}})}) \leq \tilde{C}(\varepsilon + \|\Psi_\varepsilon^\perp\|_{L^2(I_{\varepsilon,\tilde{\delta}})}).$$

Moreover, due to integration by parts we have $B_{0,\varepsilon}(\Psi_{0,\varepsilon}^1, \Psi_\varepsilon^\perp) = 0$ and therefore

$$B_{0,\varepsilon}(\hat{\Psi}_\varepsilon^1, \hat{\Psi}_\varepsilon^1) = a_\varepsilon^2 B_{0,\varepsilon}(\Psi_{0,\varepsilon}^1, \Psi_{0,\varepsilon}^1) + B_{0,\varepsilon}(\Psi_\varepsilon^\perp, \Psi_\varepsilon^\perp) = a_\varepsilon^2 \lambda_{0,\varepsilon}^1 + B_{0,\varepsilon}(\Psi_\varepsilon^\perp, \Psi_\varepsilon^\perp).$$

Together with Lemma 6.6, 4.-5. this yields for ε small

$$C\varepsilon^2 \geq \lambda_\varepsilon^1 \geq \nu_1 \|\Psi_\varepsilon^\perp\|_{L^2(I_{\varepsilon,\tilde{\delta}})}^2 + \mathcal{O}(\varepsilon)\|\Psi_\varepsilon^\perp\|_{L^2(I_{\varepsilon,\tilde{\delta}})} + \mathcal{O}(\varepsilon^2) \geq \frac{\nu_1}{2}\|\Psi_\varepsilon^\perp\|_{L^2(I_{\varepsilon,\tilde{\delta}})}^2 - \tilde{C}\varepsilon^2,$$

where we used Young's inequality for the last estimate. This shows $\|\Psi_\varepsilon^\perp\|_{L^2(I_{\varepsilon,\tilde{\delta}})} = \mathcal{O}(\varepsilon)$ and hence $\lambda_\varepsilon^1 = \mathcal{O}(\varepsilon^2)$ for ε small. Moreover, we get $a_\varepsilon^2 = 1 + \mathcal{O}(\varepsilon^2)$ for ε small. $\square_1$.

Ad 2. The estimates above yield $|B_{0,\varepsilon}(\Psi_\varepsilon^\perp, \Psi_\varepsilon^\perp)| = \mathcal{O}(\varepsilon^2)$. Therefore $\|\Psi_\varepsilon^\perp\|_{L^2(I_{\varepsilon,\tilde{\delta}})} = \mathcal{O}(\varepsilon)$ and the definition of $B_{0,\varepsilon}$ imply $\|\frac{d}{dz}\Psi_\varepsilon^\perp\|_{L^2(I_{\varepsilon,\tilde{\delta}})} = \mathcal{O}(\varepsilon)$. Using this together with properties of $\Psi_{0,\varepsilon}^R = \Psi_{0,\varepsilon}^1 - \beta_\varepsilon\theta_0'$ from Lemma 6.6, 6. we will deduce estimates for $\Psi_\varepsilon^R := \Psi_\varepsilon^1 - J(0)^{-1/2}\beta_\varepsilon\theta_0'$. First, we use the splitting (6.9) and the definition of $\Psi_{0,\varepsilon}^R$ to rewrite

$$\Psi_\varepsilon^R = J_\varepsilon^{-\frac{1}{2}} \left[(a_\varepsilon - J_\varepsilon^{\frac{1}{2}} J(0)^{-\frac{1}{2}})\beta_\varepsilon\theta_0' + a_\varepsilon \Psi_{0,\varepsilon}^R + \Psi_\varepsilon^\perp \right].$$

With $a_\varepsilon^2 = 1 + \mathcal{O}(\varepsilon^2)$ for ε small, $a_\varepsilon > 0$ and $(a_\varepsilon - 1)(a_\varepsilon + 1) = a_\varepsilon^2 - 1$ we get $a_\varepsilon = 1 + \mathcal{O}(\varepsilon^2)$. Moreover, a Taylor expansion and Corollary 6.3 yields for all $z \in I_{\varepsilon,\tilde{\delta}}$

$$J_\varepsilon(z)^{\frac{1}{2}} = J(0)^{\frac{1}{2}} + \mathcal{O}(|F_\varepsilon(z)|) \quad \text{and} \quad |F_\varepsilon(z)| \leq C\varepsilon(|z|+1).$$

Therefore $|a_\varepsilon - J_\varepsilon(z)^{\frac{1}{2}} J(0)^{-\frac{1}{2}}| \leq \tilde{C}\varepsilon(|z|+1)$ for all $z \in I_{\varepsilon,\tilde{\delta}}$ and ε small. Together with the decay for θ_0' and the estimates for $\|\Psi_\varepsilon^\perp\|_{L^2(I_{\varepsilon,\tilde{\delta}})}$ and $\|\Psi_{0,\varepsilon}^R\|_{L^2(I_{\varepsilon,\tilde{\delta}})}$ we obtain $\|\Psi_\varepsilon^R\|_{J_\varepsilon} = \mathcal{O}(\varepsilon)$. Moreover, the derivative is given by

$$\frac{d}{dz}\Psi_\varepsilon^R = J_\varepsilon^{-\frac{1}{2}} \left[(a_\varepsilon - J_\varepsilon^{\frac{1}{2}} J(0)^{-\frac{1}{2}})\beta_\varepsilon \theta_0'' + (-\frac{1}{2} J_\varepsilon^{-\frac{1}{2}} \frac{d}{dz} J_\varepsilon J(0)^{-\frac{1}{2}})\beta_\varepsilon \theta_0' + a_\varepsilon \frac{d}{dz}\Psi_{0,\varepsilon}^R + \frac{d}{dz}\Psi_\varepsilon^\perp \right]$$
$$- \frac{1}{2} J_\varepsilon^{-1} (\frac{d}{dz} J_\varepsilon) \Psi_\varepsilon^R.$$

The estimates for $|a_\varepsilon - J_\varepsilon(z)^{\frac{1}{2}} J(0)^{-\frac{1}{2}}|$, $\|\Psi_\varepsilon^R\|_{J_\varepsilon}$, $\|\frac{d}{dz}\Psi_{0,\varepsilon}^R\|_{L^2}$ and $\|\frac{d}{dz}\Psi_\varepsilon^\perp\|_{L^2}$ as well as Corollary 6.3 yield $\|\frac{d}{dz}\Psi_\varepsilon^R\|_{J_\varepsilon} = \mathcal{O}(\varepsilon)$. $\square_2$.

Ad 3. Consider any normalized eigenfunction Ψ_ε^2 to λ_ε^2. If $\lambda_\varepsilon^2 \geq \frac{1}{4}\min\{f''(\pm 1)\}$, then there is nothing to show. Therefore we assume that $\lambda_\varepsilon^2 \leq \frac{1}{4}\min\{f''(\pm 1)\}$. Then Ψ_ε^2 satisfies the decay in Lemma 6.7, 3. and computations as before yield

$$\lambda_\varepsilon^2 = B_{0,\varepsilon}(\hat{\Psi}_\varepsilon^2, \hat{\Psi}_\varepsilon^2) + \varepsilon p_\varepsilon \int_{I_{\varepsilon,\tilde{\delta}}} f'''(\theta_0)\theta_1 (\hat{\Psi}_\varepsilon^2)^2 \, dz + \mathcal{O}(\varepsilon^2) = B_{0,\varepsilon}(\hat{\Psi}_\varepsilon^2, \hat{\Psi}_\varepsilon^2) + \mathcal{O}(\varepsilon).$$

Analogously as above we split

$$\hat{\Psi}_\varepsilon^2 = \tilde{a}_\varepsilon \Psi_{0,\varepsilon}^1 + \Psi_\varepsilon^{2,\perp}$$

orthogonally in $L^2(I_{\varepsilon,\tilde{\delta}})$ and obtain with Lemma 6.6, 5. that

$$\lambda_\varepsilon^2 = \tilde{a}_\varepsilon^2 \lambda_{0,\varepsilon}^1 + B_{0,\varepsilon}(\Psi_\varepsilon^{2,\perp}, \Psi_\varepsilon^{2,\perp}) + \mathcal{O}(\varepsilon) \geq \tilde{a}_\varepsilon^2 \lambda_{0,\varepsilon}^1 + \nu_1(1 - \tilde{a}_\varepsilon^2) - C\varepsilon.$$

In order to get an estimate for $\tilde{a}_\varepsilon = (\hat{\Psi}_\varepsilon^2, \Psi_{0,\varepsilon}^1)_{L^2(I_{\varepsilon,\tilde{\delta}})}$, note that $\hat{\Psi}_\varepsilon^1 \perp_{L^2} \hat{\Psi}_\varepsilon^2$. Therefore with the splitting (6.9) we obtain for ε small

$$|\tilde{a}_\varepsilon| = \left| -\frac{1}{a_\varepsilon} (\Psi_\varepsilon^\perp, \hat{\Psi}_\varepsilon^2)_{L^2(I_{\varepsilon,\tilde{\delta}})} \right| \leq C\|\Psi_\varepsilon^\perp\|_{L^2(I_{\varepsilon,\tilde{\delta}})} \leq C\varepsilon.$$

This yields $\lambda_\varepsilon^2 \geq \frac{\nu_1}{2}$ for ε small if $\lambda_\varepsilon^2 \geq \frac{1}{4}\min\{f''(\pm 1)\}$. $\square_3$.

6.1.4 Spectral Estimates for Vector-Valued Allen-Cahn-Type Operators in 1D

In the scalar case we frequently used theorems and estimates that are not available in the vector-valued case, e.g. the comparison principle, the Harnack-inequality and the Hopf maximum principle. Looking closely into the last Section 6.1.3, we observe that these arguments were used explicitly only for the proofs of Lemma 6.6, 2.-5. and Lemma 6.7, 2.-3. For the vector-valued case we have to adjust suitably. The goal is to obtain analogous assertions based on the operator $\check{\mathcal{L}}_0 := -\frac{d^2}{dz^2} + D^2W(\vec{\theta}_0)$, where $W : \mathbb{R}^m \to \mathbb{R}$ is as in Definition 1.4 and $\vec{\theta}_0$ is as in Remark 4.27, 1. In Lemma 4.29 we already showed properties of $\check{\mathcal{L}}_0$ viewed as an unbounded operator $\check{L}_0 : H^2(\mathbb{R},\mathbb{K})^m \to L^2(\mathbb{R},\mathbb{K})^m$ and we obtained a spectral gap provided $\dim\ker\check{L}_0 = 1$. Under this assumption we show analogous properties as in the scalar case in the last Section 6.1.3. To this end we use contradiction arguments and further assertions in Kusche [Ku], in particular [Ku], Chapter 1, where abstract vector-valued Sturm-Liouville operators are considered.

6.1.4.1 Unperturbed Vector-Valued Allen-Cahn-Type Operators in 1D We consider $\check{\mathcal{L}}_0$ on finite large intervals together with homogeneous Neumann boundary condition.

Let $\tilde{\delta} > 0$ fixed, $\varepsilon > 0$, $I_{\varepsilon,\tilde{\delta}} := (-\frac{\tilde{\delta}}{\varepsilon}, \frac{\tilde{\delta}}{\varepsilon})$ and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Again, we only need the case $\mathbb{K} = \mathbb{R}$ but $\mathbb{C}$ is included for more generality. We consider the unbounded operator

$$\check{L}_{0,\varepsilon} : H^2_N(I_{\varepsilon,\tilde{\delta}}, \mathbb{K})^m \to L^2(I_{\varepsilon,\tilde{\delta}}, \mathbb{K})^m : \vec{u} \mapsto \check{\mathcal{L}}_0 \vec{u} = \left[-\frac{d^2}{dz^2} + D^2 W(\vec{\theta}_0) \right] \vec{u}.$$

The associated sesquilinearform is $\check{B}_{0,\varepsilon} : H^1(I_{\varepsilon,\tilde{\delta}}, \mathbb{K})^m \times H^1(I_{\varepsilon,\tilde{\delta}}, \mathbb{K})^m \to \mathbb{K}$,

$$\check{B}_{0,\varepsilon}(\vec{\Phi}, \vec{\Psi}) := \int_{I_{\varepsilon,\tilde{\delta}}} (\frac{d}{dz}\vec{\Phi}, \frac{d}{dz}\vec{\Psi})_{\mathbb{K}^m} + (D^2 W(\vec{\theta}_0)\vec{\Phi}, \vec{\Psi})_{\mathbb{K}^m} \, dz.$$

We obtain the analogy of Lemma 6.6.

Lemma 6.9. *Assume* $\dim \ker \check{L}_0 = 1$, *cf. Remark 4.28. Then*

1. $\check{L}_{0,\varepsilon}$ *is selfadjoint and the spectrum is given by discrete eigenvalues* $(\check{\lambda}^k_{0,\varepsilon})_{k\in\mathbb{N}}$ *in* $\mathbb{R}$ *with* $\check{\lambda}^1_{0,\varepsilon} \leq \check{\lambda}^2_{0,\varepsilon} \leq \ldots$ *and* $\check{\lambda}^k_{0,\varepsilon} \overset{k\to\infty}{\longrightarrow} \infty$. *Moreover, there is an orthonormal basis* $(\vec{\Psi}^k_{0,\varepsilon})_{k\in\mathbb{N}}$ *of* $L^2(I_\varepsilon, \mathbb{K})^m$ *consisting of smooth* $\mathbb{R}^m$*-valued eigenfunctions* $\vec{\Psi}^k_{0,\varepsilon}$ *to* $\check{\lambda}^k_{0,\varepsilon}$.

2. $\check{\lambda}^1_{0,\varepsilon}$ *is simple for* $\varepsilon > 0$ *small.*

3. *For any normalized eigenfunction* $\vec{\Psi}_{0,\varepsilon}$ *to an eigenvalue* $\check{\lambda}_{0,\varepsilon} \leq \frac{1}{4}\min\{\sigma(D^2 W(\vec{u}_\pm))\}$ *of* $\check{L}_{0,\varepsilon}$ *and* $\varepsilon > 0$ *small it holds*

$$|\vec{\Psi}_{0,\varepsilon}(z)| \leq C e^{-|z|\sqrt{\min\{\sigma(D^2 W(\vec{u}_\pm))\}/6}} \quad \textit{for all } z \in I_{\varepsilon,\tilde{\delta}},$$

   where $C > 0$ *is independent of* ε, $\tilde{\delta}$.

4. *There is* $\check{\varepsilon}_0 = \check{\varepsilon}_0(\tilde{\delta}) > 0$ *small such that for all* $\varepsilon \in (0, \check{\varepsilon}_0]$

$$\check{\lambda}^1_{0,\varepsilon} = \inf_{\vec{\Psi}\in H^1(I_{\varepsilon,\tilde{\delta}})^m, \|\vec{\Psi}\|_{L^2}=1} \check{B}_{0,\varepsilon}(\vec{\Psi}, \vec{\Psi}) = \check{B}_{0,\varepsilon}(\vec{\Psi}^1_{0,\varepsilon}, \vec{\Psi}^1_{0,\varepsilon}) = \mathcal{O}(e^{-\frac{3\tilde{\delta}\sqrt{\min\{\sigma(D^2 W(\vec{u}_\pm))\}}}{2\sqrt{2}\varepsilon}}),$$

   where the constant in the $\mathcal{O}$*-estimate is independent of* $\tilde{\delta}$, ε.

5. *There is* $\check{\nu}_1 > 0$ *independent of* $\tilde{\delta}$, ε *and* $\check{\varepsilon}_0 = \check{\varepsilon}_0(\tilde{\delta}) > 0$ *small such that*

$$\check{\lambda}^2_{0,\varepsilon} = \inf_{\vec{\Psi}\in H^1(I_{\varepsilon,\tilde{\delta}})^m, \|\vec{\Psi}\|_{L^2}=1, \vec{\Psi}\perp\vec{\Psi}^1_{0,\varepsilon}} \check{B}_{0,\varepsilon}(\vec{\Psi}, \vec{\Psi}) \geq \check{\nu}_1 \quad \textit{for all } \varepsilon \in (0, \check{\varepsilon}_0].$$

6. *Let* $\check{\beta}_\varepsilon := \|\vec{\theta}_0'\|^{-1}_{L^2(I_{\varepsilon,\tilde{\delta}})^m}$. *For* $\check{\varepsilon}_0 = \check{\varepsilon}_0(\tilde{\delta}) > 0$ *small and* $\varepsilon \in (0, \check{\varepsilon}_0]$ *there are* $\check{c}_{0,\varepsilon} \in \{\pm 1\}$ *such that for* $\vec{\Psi}^R_{0,\varepsilon} := \check{c}_{0,\varepsilon}\vec{\Psi}^1_{0,\varepsilon} - \check{\beta}_\varepsilon \vec{\theta}_0'$ *we have*

$$\left\|\vec{\Psi}^R_{0,\varepsilon}\right\|^2_{L^2(I_{\varepsilon,\tilde{\delta}})^m} + \left\|\tfrac{d}{dz}\vec{\Psi}^R_{0,\varepsilon}\right\|^2_{L^2(I_{\varepsilon,\tilde{\delta}})^m} \leq C e^{-\frac{3\tilde{\delta}\sqrt{\min\{\sigma(D^2 W(\vec{u}_\pm))\}}}{2\sqrt{2}\varepsilon}},$$

   where $C > 0$ *is independent of* $\tilde{\delta}$, ε.

Remark 6.10. Lemma 6.9, 1. and 3.-4. also work without the assumption $\dim\ker\check{L}_0 = 1$, cf. [Ku], Lemma 1.1 and Lemma 2.1. However, one has to modify the decay parameters by some scalar factor independent of W. This is because the decay properties for $\vec{\theta}_0$ from Theorem 4.26 are better than the ones obtained from [Ku] for other eigenfunctions. More precisely, the maximal rate is $\sqrt{\min\{\sigma(D^2W(\vec{u}_\pm))\}/2}$ instead of $\sqrt{\min\{\sigma(D^2W(\vec{u}_\pm))\}}/2$. Nevertheless, the precise rates in Lemma 6.9, 3.-4. are not so important anyway.

Proof. By scaling in ε it is enough to consider the case $\tilde{\delta} = 1$. We set $I_\varepsilon := I_{\varepsilon,1}$.

Ad 1. This can be seen as in the scalar case, cf. the proof of Lemma 6.6, 1. Here the abstract Fredholm alternative in Theorem 6.14 below is used for

$$\check{A}_{0,\varepsilon} : H^1(I_\varepsilon, \mathbb{C}^m) \to H^1(I_\varepsilon, \mathbb{C}^m)^* : \vec{u} \mapsto [\vec{v} \mapsto \check{B}_{0,\varepsilon}(\vec{u}, \vec{v})],$$

where $H^1(I_\varepsilon, \mathbb{C}^m)^*$ is the anti-dual space. $\Box_1$.

Ad 2. Assume the contrary. Then there is a zero sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ and normalized, pairwise orthogonal eigenfunctions $\vec{\Psi}^1_{0,\varepsilon_n}$, $\vec{\Psi}^2_{0,\varepsilon_n}$ of $\check{L}_{0,\varepsilon_n}$ to the eigenvalue $\check{\lambda}^1_{0,\varepsilon_n}$ for all $n \in \mathbb{N}$. Now note that the upper bound on $\check{\lambda}^1_{0,\varepsilon}$ in 4. can be shown solely with the decay properties of $\vec{\theta}_0$ from Theorem 4.26, cf. [Ku], proof of Lemma 2.1, 1. Therefore due to [Ku], Lemma 1.2 (and its proof) there is a subsequence $(\varepsilon_{n_k})_{k\in\mathbb{N}}$ such that $\vec{\Psi}^j_{0,\varepsilon_{n_k}}$ converges uniformly in C^2 on compact subsets of $\mathbb{R}$ to a normalized eigenfunction $\vec{\Psi}^j_0 \in H^2(\mathbb{R}, \mathbb{K})^m \cap C^2(\mathbb{R}, \mathbb{K})^m$ to the eigenvalue 0 of $\check{L}_0$ for $j = 1, 2$. Because of [Ku], Lemma 1.1 all $\vec{\Psi}^j_{0,\varepsilon_n}$, $\vec{\Psi}^j_0$ for $n \in \mathbb{N}$ and $j = 1, 2$ satisfy uniform pointwise exponential bounds. Hence the Dominated Convergence Theorem yields that $\vec{\Psi}^1_0$ is orthogonal to $\vec{\Psi}^2_0$. This is a contradiction to $\dim\ker\check{L}_0 = 1$. $\Box_2$.

Ad 3. This follows from Kusche [Ku], Lemma 1.1. $\Box_3$.

Ad 4. The inf-characterization can be shown as in the scalar case. As mentioned in the proof of 2. above, the upper bound on $\check{\lambda}^1_{0,\varepsilon}$ follows from Theorem 4.26. [Ku], Lemma 1.2 and a contradiction argument yield $|(\vec{\Psi}^1_{0,\varepsilon}, \vec{\theta}_0')_{L^2(I_\varepsilon)^m}| \geq C > 0$ for ε small. Together with the uniform decay for eigenfunctions from 2. this implies the estimate, cf. also the proof of Lemma 2.1, 1. in [Ku]. $\Box_4$.

Ad 5. The inf-characterization follows as in the scalar case. For $\check{\nu}_0$ as in Lemma 4.29 let $\check{\nu}_1 := \min\{\frac{1}{2}\check{\nu}_0, \frac{1}{4}\min\{\sigma(D^2(\vec{u}_\pm))\}\} > 0$. Assume the estimate on $\check{\lambda}^2_{0,\varepsilon}$ does not hold with this $\check{\nu}_1$. Then there is a zero sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ such that $\check{\lambda}^2_{0,\varepsilon_n} < \check{\nu}_1$. Due to [Ku], Lemma 1.2, there is a subsequence $(\varepsilon_{n_k})_{k\in\mathbb{N}}$ such that some normalized eigenvectors $\vec{\Psi}^2_{0,\varepsilon_{n_k}}$ to $\check{\lambda}^2_{0,\varepsilon_{n_k}}$ converge uniformly in C^2 on compact subsets of $\mathbb{R}$ to an eigenfunction $\vec{\Psi}^2_0$ of $\check{L}_0$. Due to the assumption on $\check{\nu}_1$, the eigenvalue corresponding to $\vec{\Psi}^2_0$ is necessarily zero. In particular $\dim\ker\check{L}_0 = 1$ yields $(\vec{\Psi}^2_0, \vec{\theta}_0')_{L^2(\mathbb{R})^m} \neq 0$. On the other hand, since $\vec{\Psi}^2_{0,\varepsilon}$ and $\vec{\Psi}^1_{0,\varepsilon}$ are orthogonal in $L^2(I_\varepsilon)^m$, we obtain with Lemma 2.1 in [Ku], the Dominated Convergence Theorem applied to another subsequence using the uniform decay in 3. that $(\vec{\Psi}^2_0, \vec{\theta}_0')_{L^2(\mathbb{R})^m} = 0$. This is a contradiction. $\Box_5$.

Ad 6. The proof is analogous to the one of Marquardt [Ma], Lemma 3.8, 3. We decompose $\check{\beta}_\varepsilon\vec{\theta}_0' = \check{a}_{0,\varepsilon}\vec{\Psi}^1_{0,\varepsilon} + \vec{\Psi}^\perp_{0,\varepsilon}$ orthogonally in $L^2(I_\varepsilon)^m$, where $\check{\beta}_\varepsilon = \|\vec{\theta}_0'\|^{-1}_{L^2(I_\varepsilon)^m}$. Due to 4. and Theorem 4.26 we obtain with integration by parts for ε small that

$$\mathcal{O}(e^{-\frac{3\sqrt{\min\{\sigma(D^2W(\vec{u}_\pm))\}}}{2\sqrt{2}\varepsilon}}) = \check{B}_{0,\varepsilon}(\check{\beta}_\varepsilon\vec{\theta}_0', \check{\beta}_\varepsilon\vec{\theta}_0') \geq \check{a}^2_{0,\varepsilon}\check{\lambda}^1_{0,\varepsilon} + \check{\nu}_1\|\vec{\Psi}^\perp_{0,\varepsilon}\|^2_{L^2(I_\varepsilon)^m}.$$

Now note that $1=\check{a}_{0,\varepsilon}^2+\|\vec{\Psi}_{0,\varepsilon}^\perp\|_{L^2(I_\varepsilon)^m}^2$. Therefore 4.-5. and the above estimate yield for ε small

$$\|\vec{\Psi}_{0,\varepsilon}^\perp\|_{L^2(I_\varepsilon)^m}^2=\mathcal{O}(e^{-\frac{3\sqrt{\min\{\sigma(D^2W(\vec{u}_\pm))\}}}{2\sqrt{2}\varepsilon}}).$$

Hence with $1-\check{a}_{0,\varepsilon}^2=(1-\check{a}_{0,\varepsilon})(1+\check{a}_{0,\varepsilon})$ we obtain the estimate in the lemma for $\|\vec{\Psi}_{0,\varepsilon}^R\|_{L^2(I_\varepsilon)^m}$ if we set $\check{c}_{0,\varepsilon}:=\operatorname{sign}\check{a}_{0,\varepsilon}\in\{\pm1\}$. For notational simplicity assume w.l.o.g. $\check{c}_{0,\varepsilon}=1$, otherwise one can replace $\vec{\Psi}_{0,\varepsilon}^1$ by $\check{c}_{0,\varepsilon}\vec{\Psi}_{0,\varepsilon}^1$. Then it holds $\frac{d}{dz}\vec{\Psi}_{0,\varepsilon}^R=\frac{d}{dz}\vec{\Psi}_{0,\varepsilon}^1-\check{\beta}_\varepsilon\vec{\theta}_0''$ and

$$\left\|\frac{d}{dz}\vec{\Psi}_{0,\varepsilon}^R\right\|_{L^2(I_\varepsilon)^m}^2=\int_{I_\varepsilon}|\check{\beta}_\varepsilon\vec{\theta}_0''|^2-2\check{\beta}_\varepsilon\vec{\theta}_0''\cdot\frac{d}{dz}\vec{\Psi}_{0,\varepsilon}^1+\left|\frac{d}{dz}\vec{\Psi}_{0,\varepsilon}^1\right|^2dz.$$

The first term is a problem. Therefore we rewrite

$$-\int_{I_\varepsilon}\vec{\theta}_0''\cdot\frac{d}{dz}\vec{\Psi}_{0,\varepsilon}^1=\int_{I_\varepsilon}\vec{\theta}_0'\cdot\frac{d^2}{dz^2}\vec{\Psi}_{0,\varepsilon}^1,\quad\int_{I_\varepsilon}\left|\frac{d}{dz}\vec{\Psi}_{0,\varepsilon}^1\right|^2=\check{\lambda}_{0,\varepsilon}^1-\int_{I_\varepsilon}(D^2W(\vec{\theta}_0)\vec{\Psi}_{0,\varepsilon}^1,\vec{\Psi}_{0,\varepsilon}^1)_{\mathbb{R}^m}.$$

We use $\frac{d^2}{dz^2}\vec{\Psi}_{0,\varepsilon}^1=-\check{\lambda}_{0,\varepsilon}^1\vec{\Psi}_{0,\varepsilon}^1+D^2W(\vec{\theta}_0)\vec{\Psi}_{0,\varepsilon}^1$ and insert $\vec{\Psi}_{0,\varepsilon}^1=\vec{\Psi}_{0,\varepsilon}^R+\check{\beta}_\varepsilon\vec{\theta}_0'$ everywhere. Then integration by parts yields that the quadratic terms in $\vec{\theta}_0$ cancel up to an appropriately decaying term. Moreover, $\check{\lambda}_{0,\varepsilon}^1$ has the decay due to 4. and the other terms (without the one with $\check{\lambda}_{0,\varepsilon}^1$) where $\vec{\theta}_0'$ is combined with $\vec{\Psi}_{0,\varepsilon}^R$ cancel. Together with the estimate on the L^2-norm of $\vec{\Psi}$ we obtain the estimate for the derivative. $\Box_6$.

6.1.4.2 Perturbed Vector-Valued Allen-Cahn-Type Operators in 1D In this section we consider perturbed and weighted vector-valued operators in 1D. Let $\delta>0$ and $h_\varepsilon, J\in C^2([-\delta,\delta],\mathbb{R})$ with $\|h_\varepsilon\|_{C^2([-\delta,\delta])}\leq\overline{C}_0$ for $\varepsilon>0$ small and $c_1,C_2>0$ be such that (6.5) holds. Then let $\rho_\varepsilon,F_\varepsilon,J_\varepsilon$ for $\varepsilon>0$ small be as in Section 6.1.1. We define

$$\vec{\phi}_\varepsilon:[-\delta,\delta]\to\mathbb{R}^m:r\mapsto\vec{\theta}_0(\frac{r}{\varepsilon})+\varepsilon p_\varepsilon\vec{\theta}_1(\frac{r}{\varepsilon})+\vec{q}_\varepsilon(r)\varepsilon^2, \tag{6.10}$$

where $p_\varepsilon\in\mathbb{R}$ and $\vec{q}_\varepsilon:[-\delta,\delta]\to\mathbb{R}$ is measurable with $|p_\varepsilon|+\frac{\varepsilon}{\varepsilon+|r|}|\vec{q}_\varepsilon(r)|\leq\check{C}_3$ for $r\in[-\delta,\delta]$, a $\check{C}_3>0$, and $\varepsilon>0$ small. Moreover, let $\vec{\theta}_1\in L^\infty(\mathbb{R})^m$ with $\|\vec{\theta}_1\|_\infty\leq\check{C}_4$ for a $\check{C}_4>0$ and

$$\int_{\mathbb{R}}(\vec{\theta}_0',\sum_{\xi\in\mathbb{N}_0^m,|\xi|=1}\partial^\xi D^2W(\vec{\theta}_0)(\vec{\theta}_1)^\xi\vec{\theta}_0')_{\mathbb{R}^m}=0. \tag{6.11}$$

Let $\tilde{\delta}\in(0,\frac{3\delta}{4}]$ be fixed. Then F_ε, J_ε are well-defined on $\overline{I_{\varepsilon,\tilde{\delta}}}$ for $\varepsilon\in(0,\varepsilon_1(\delta,\overline{C}_0)]$ and Corollary 6.3 is applicable due to Remark 6.4, 2. We consider the operators

$$\check{L}_\varepsilon:H_N^2(I_{\varepsilon,\tilde{\delta}},\mathbb{K})^m\to L_{J_\varepsilon}^2(I_{\varepsilon,\tilde{\delta}},\mathbb{K})^m:\vec{u}\mapsto\check{\mathcal{L}}_\varepsilon\vec{u}:=\left[-J_\varepsilon^{-1}\frac{d}{dz}\left(J_\varepsilon\frac{d}{dz}\right)+D^2W(\vec{\phi}_\varepsilon(\varepsilon.))\right]\vec{u},$$

where $L_{J_\varepsilon}^2(I_{\varepsilon,\tilde{\delta}},\mathbb{K})^m$ is the space of $\mathbb{K}^m$-valued L^2-functions defined on $I_{\varepsilon,\tilde{\delta}}$ with the weight J_ε. We write $(.,.)_{J_\varepsilon}$, $\|.\|_{J_\varepsilon}$ and $\perp_{J_\varepsilon}$ for the corresponding scalar product, norm and orthogonal relation. Note that for convenience we use the same notation for the latter as in the scalar case. The sesquilinearform associated to $\check{L}_\varepsilon$ is given by $\check{B}_\varepsilon:H^1(I_{\varepsilon,\tilde{\delta}},\mathbb{K})^m\times H^1(I_{\varepsilon,\tilde{\delta}},\mathbb{K})^m\to\mathbb{K}$,

$$\check{B}_\varepsilon(\vec{\Phi},\vec{\Psi}):=\int_{I_{\varepsilon,\tilde{\delta}}}\left[(\frac{d}{dz}\vec{\Phi},\frac{d}{dz}\vec{\Psi})_{\mathbb{K}^m}+(D^2W(\vec{\phi}_\varepsilon(\varepsilon.))\vec{\Phi},\vec{\Psi})_{\mathbb{K}^m}\right]J_\varepsilon\,dz.$$

Again only $\mathbb{K} = \mathbb{R}$ is needed and $\mathbb{K} = \mathbb{C}$ is added for more generality. We obtain the analogue of Lemma 6.9, 1.-3.

Lemma 6.11. *1. $\check{L}_\varepsilon$ is selfadjoint and the spectrum is given by a sequence of discrete eigenvalues $(\check{\lambda}_\varepsilon^k)_{k\in\mathbb{N}}$ in $\mathbb{R}$ with $\check{\lambda}_\varepsilon^1 \leq \check{\lambda}_\varepsilon^2 \leq ...$ and $\check{\lambda}_\varepsilon^k \overset{k\to\infty}{\longrightarrow} \infty$. Moreover, there is an orthonormal basis $(\vec{\Psi}_\varepsilon^k)_{k\in\mathbb{N}}$ of $L^2_{J_\varepsilon}(I_{\varepsilon,\tilde{\delta}}, \mathbb{K})^m$ consisting of smooth $\mathbb{R}$-valued eigenfunctions $\vec{\Psi}_\varepsilon^k$ to $\check{\lambda}_\varepsilon^k$.*

2. *$\check{\lambda}_\varepsilon^1$ is simple for $\varepsilon > 0$ small.*

3. *There is an $\check{\varepsilon}_0 > 0$ (only depending on δ, $\tilde{\delta}$, $\overline{C}_0$, c_1, C_2, $\check{C}_3$, $\check{C}_4$) such that for all $\varepsilon \in (0, \varepsilon_0]$ and any normalized eigenfunction $\vec{\Psi}_\varepsilon$ of L_ε to an eigenvalue $\check{\lambda}_\varepsilon \leq \frac{1}{4}\min\{\sigma(D^2W(\vec{u}_\pm))\}$ it holds*
$$|\vec{\Psi}_\varepsilon(z)| \leq Ce^{-|z|\sqrt{\min\{\sigma(D^2W(\vec{u}_\pm))\}/6}} \quad \textit{for all } z \in I_{\varepsilon,\tilde{\delta}},$$
where $C > 0$ only depends on c_1.

Proof. We need some properties of the weight and the perturbation. Corollary 6.3 and the assumptions yield $|\frac{d}{dz}J_\varepsilon| \leq 2C_2\varepsilon$, $|J_\varepsilon - J(0)| \leq C(\overline{C}_0, C_2)\varepsilon$ in $I_{\varepsilon,\tilde{\delta}}$ and
$$|\vec{\phi}_\varepsilon(\varepsilon z) - \vec{\theta}_0(z)| \leq C(\delta, \check{C}_3, \check{C}_4)\varepsilon \quad \text{for all } z \in I_{\varepsilon,\tilde{\delta}}.$$
With these uniform estimates one can show that the abstract results in [Ku] are applicable. Hence the assertions follow in the analogous way as in the unperturbed case, cf. the proof of Lemma 6.9, 1.-3. above. Here the abstract Fredholm alternative in Theorem 6.14 below is applied to
$$\check{A}_\varepsilon : H^1_{J_\varepsilon}(I_{\varepsilon,\tilde{\delta}}, \mathbb{C}^m) \to H^1_{J_\varepsilon}(I_{\varepsilon,\tilde{\delta}}, \mathbb{C}^m)^* : \vec{u} \mapsto [\vec{v} \mapsto \check{B}_\varepsilon(\vec{u}, \vec{v})],$$
where $H^1_{J_\varepsilon}(I_{\varepsilon,\tilde{\delta}}, \mathbb{C}^m)$ is $H^1(I_{\varepsilon,\tilde{\delta}}, \mathbb{C}^m)$ with the weight J_ε in the norm and $H^1_{J_\varepsilon}(I_{\varepsilon,\tilde{\delta}}, \mathbb{C}^m)^*$ is the anti-dual space. □

Now we obtain the analogy to Theorem 6.8.

Theorem 6.12. *There is an $\check{\varepsilon}_0 > 0$ only depending on δ, $\tilde{\delta}$, $\overline{C}_0$,c_1, C_2, $\check{C}_3$, $\check{C}_4$ and $C > 0$ only depending on $\tilde{\delta}$, $\overline{C}_0$, c_1, C_2, $\check{C}_3$, $\check{C}_4$ such that*

1. *For $\varepsilon \in (0, \check{\varepsilon}_0]$ it holds*
$$\check{\lambda}_\varepsilon^1 = \inf_{\vec{\Psi}\in H^1(I_{\varepsilon,\tilde{\delta}},\mathbb{K})^m, \|\vec{\Psi}\|_{J_\varepsilon}=1} \check{B}_\varepsilon(\vec{\Psi}, \vec{\Psi}) = \check{B}_\varepsilon(\vec{\Psi}_\varepsilon^1, \vec{\Psi}_\varepsilon^1), \quad |\check{\lambda}_\varepsilon^1| \leq C\varepsilon^2.$$

2. *There are $\check{c}_\varepsilon \in \{\pm 1\}$ such that for $\vec{\Psi}_\varepsilon^R := \check{c}_\varepsilon\vec{\Psi}_\varepsilon^1 - J(0)^{-\frac{1}{2}}\check{\beta}_\varepsilon\vec{\theta}_0'$, where $\beta_\varepsilon = \|\vec{\theta}_0'\|_{L^2(I_{\varepsilon,\tilde{\delta}})^m}$, and $\varepsilon \in (0, \check{\varepsilon}_0]$ it holds*
$$\|\vec{\Psi}_\varepsilon^R\|_{J_\varepsilon} + \|\tfrac{d}{dz}\vec{\Psi}_\varepsilon^R\|_{J_\varepsilon} \leq C\varepsilon.$$

3. *With $\check{\nu}_1$ from Lemma 6.9, 5. it holds for all $\varepsilon \in (0, \check{\varepsilon}_0]$*
$$\check{\lambda}_\varepsilon^2 = \inf_{\vec{\Psi}\in H^1(I_{\varepsilon,\tilde{\delta}},\mathbb{K})^m, \|\vec{\Psi}\|_{J_\varepsilon}=1, \vec{\Psi}\perp_{J_\varepsilon}\vec{\Psi}_\varepsilon^1} B_\varepsilon(\vec{\Psi}, \vec{\Psi}) \geq \check{\nu}_2 := \min\left\{\frac{\check{\nu}_1}{2}, \frac{\sigma(D^2W(\vec{u}_\pm))}{4}\right\} > 0.$$

The proof is analogous to the scalar case, cf. Theorem 6.8.

Proof. It is enough to consider $\mathbb{K} = \mathbb{R}$. Moreover, the inf-characterizations can be shown as in the scalar case. Again, if we write "for ε small" in the following it is always meant "for all $\varepsilon \in (0, \check{\varepsilon}_0]$ for some $\check{\varepsilon}_0 > 0$ small only depending on δ, $\tilde{\delta}$, $\overline{C}_0$,c_1, C_2, $\check{C}_3$, $\check{C}_4$". Similarly, all appearing constants (also in $\mathcal{O}$-notation) below only depend on $\tilde{\delta}$, $\overline{C}_0$,c_1, C_2, $\check{C}_3$, $\check{C}_4$, but we do not explicitly state this.

As in the scalar case, we prove an identity for $\check{B}_\varepsilon(\Psi, \Psi)$ for all $\Psi \in H^1(I_{\varepsilon,\tilde{\delta}}, \mathbb{R})^m$ first. Let $\check{\Psi} := J_\varepsilon^{1/2}\vec{\Psi}$. Then

$$\frac{d}{dz}\vec{\Psi} = -\frac{1}{2}J_\varepsilon^{-\frac{3}{2}}(\frac{d}{dz}J_\varepsilon)\check{\Psi} + J_\varepsilon^{-\frac{1}{2}}\frac{d}{dz}\check{\Psi}.$$

Therefore

$$\check{B}_\varepsilon(\vec{\Psi}, \vec{\Psi}) = \int_{I_{\varepsilon,\tilde{\delta}}} |\frac{d}{dz}\check{\Psi}|^2 + \check{\Psi} \cdot \left[D^2W(\vec{\phi}_\varepsilon(\varepsilon.)) + \frac{1}{4}J_\varepsilon^{-2}(\frac{d}{dz}J_\varepsilon)^2\right]\check{\Psi} - J_\varepsilon^{-1}(\frac{d}{dz}J_\varepsilon)\frac{1}{2}\frac{d}{dz}|\check{\Psi}^2|.$$

To use the result from the unperturbed case, we replace $D^2(\vec{\phi}_\varepsilon(\varepsilon.))$ by $D^2W(\vec{\theta}_0)$. To this end we use a Taylor expansion and get for all $|z| \leq \frac{\delta}{\varepsilon}$

$$\begin{aligned}&\left|D^2W(\vec{\phi}_\varepsilon(\varepsilon z)) - D^2W(\vec{\theta}_0(z)) - \varepsilon p_\varepsilon \sum_{\xi\in\mathbb{N}_0^m,|\xi|=1} \partial^\xi D^2W(\vec{\theta}_0(z))(\vec{\theta}_1)^\xi(z)\right| \\ &\qquad \leq C|\vec{q}_\varepsilon(\varepsilon z)|\varepsilon^2 + C\varepsilon^2(|p_\varepsilon\vec{\theta}_1(z)| + \varepsilon|\vec{q}_\varepsilon(\varepsilon z)|)^2 \leq \tilde{C}(1+|z|)\varepsilon^2.\end{aligned}$$

We use integration by parts for the last term in the above identity for $\check{B}_\varepsilon(\vec{\Psi}, \vec{\Psi})$. This yields

$$\begin{aligned}\check{B}_\varepsilon(\vec{\Psi}, \vec{\Psi}) &= \check{B}_{0,\varepsilon}(\check{\Psi}, \check{\Psi}) - \frac{1}{2}\left[J_\varepsilon^{-1}(\frac{d}{dz}J_\varepsilon)\check{\Psi}^2\right]_{z=-\frac{\tilde{\delta}}{\varepsilon}}^{\frac{\tilde{\delta}}{\varepsilon}} \\ &\quad + \int_{I_{\varepsilon,\tilde{\delta}}} \check{\Psi} \cdot \left[\varepsilon p_\varepsilon \sum_{\xi\in\mathbb{N}_0^m,|\xi|=1} \partial^\xi D^2W(\vec{\theta}_0)(\vec{\theta}_1)^\xi + \check{q}_\varepsilon\right]\check{\Psi}\,dz,\end{aligned} \tag{6.12}$$

where

$$\begin{aligned}\check{q}_\varepsilon := D^2W(\vec{\phi}_\varepsilon(\varepsilon.)) - D^2W(\vec{\theta}_0) - \varepsilon p_\varepsilon \sum_{\xi\in\mathbb{N}_0^m,|\xi|=1} \partial^\xi D^2W(\vec{\theta}_0)(\vec{\theta}_1)^\xi \\ + \frac{1}{4}\left(2J_\varepsilon^{-1}(\frac{d^2}{dz^2}J_\varepsilon) - J_\varepsilon^{-2}(\frac{d}{dz}J_\varepsilon)^2\right)\mathrm{Id}_{\mathbb{R}^{m\times m}}.\end{aligned}$$

The first part of $\check{q}_\varepsilon$ is estimated above, for the second part we use Corollary 6.3. This yields $|\check{q}_\varepsilon(z)| \leq C\varepsilon^2(1+|z|)$ for all $z \in I_{\varepsilon,\tilde{\delta}}$.

Ad 1. First we prove an upper bound on $\check{\lambda}_\varepsilon^1$ with (6.12). Let $\vec{\Psi} = J_\varepsilon^{-1/2}\check{\beta}_\varepsilon\vec{\theta}_0'$. Then it holds $\|J_\varepsilon^{-1/2}\check{\beta}_\varepsilon\vec{\theta}_0'\|_{J_\varepsilon} = 1$. Therefore (6.12) and Corollary 6.3 yield

$$\begin{aligned}\check{\lambda}_\varepsilon^1 \leq \check{\beta}_\varepsilon^2 \Bigg[\check{B}_{0,\varepsilon}(\vec{\theta}_0', \vec{\theta}_0') + \varepsilon p_\varepsilon \int_{I_{\varepsilon,\tilde{\delta}}} (\vec{\theta}_0', \sum_{\xi\in\mathbb{N}_0^m,|\xi|=1} \partial^\xi D^2W(\vec{\theta}_0)(\vec{\theta}_1)^\xi\vec{\theta}_0')_{\mathbb{R}^m}\,dz \\ + C\varepsilon^2 \int_{I_{\varepsilon,\tilde{\delta}}} (1+|z|)|\vec{\theta}_0'(z)|^2\,dz + C\varepsilon e^{-c\tilde{\delta}/\varepsilon}\Bigg].\end{aligned}$$

It holds $\int_{\mathbb{R}} |\vec{\theta}_0''|^2 + \vec{\theta}_0' \cdot D^2W(\vec{\theta}_0)\vec{\theta}_0' = 0$ because of integration by parts. Together with (6.11) and the decay properties of $\vec{\theta}_0$ from Theorem 4.26 this implies $\check{\lambda}_\varepsilon^1 \leq C\varepsilon^2$ for ε small.

In particular, Lemma 6.11, 3. yields for ε small the decay

$$|\vec{\Psi}_\varepsilon^1(z)| \leq Ce^{-|z|\sqrt{\min\{\sigma(D^2W(\vec{u}_\pm))\}/6}} \quad \text{for all } z \in I_{\varepsilon,\tilde{\delta}}.$$

Hence (6.12) and estimates as before imply for ε small

$$\check{\lambda}_\varepsilon^1 = \check{B}_{0,\varepsilon}(\check{\Psi}_\varepsilon^1, \check{\Psi}_\varepsilon^1) + \varepsilon p_\varepsilon \int_{I_{\varepsilon,\tilde{\delta}}} (\check{\Psi}_\varepsilon^1, \sum_{\xi\in\mathbb{N}_0^m, |\xi|=1} \partial^\xi D^2W(\vec{\theta}_0)(\vec{\theta}_1)^\xi \check{\Psi}_\varepsilon^1)_{\mathbb{R}^m}\, dz + \mathcal{O}(\varepsilon^2).$$

To estimate the second term we will use (6.11). For notational convenience we assume w.l.o.g. that $\check{c}_{0,\varepsilon} = 1$ in Lemma 6.9, 6. If this is not the case then one can simply exchange $\vec{\Psi}_{0,\varepsilon}^1$. We split

$$\check{\Psi}_\varepsilon^1 = J_\varepsilon^{\frac{1}{2}} \vec{\Psi}_\varepsilon^1 = \check{a}_\varepsilon \vec{\Psi}_{0,\varepsilon}^1 + \vec{\Psi}_\varepsilon^\perp \tag{6.13}$$

orthogonally in $L^2(I_{\varepsilon,\tilde{\delta}})^m$, where $\check{a}_\varepsilon := (\check{\Psi}_\varepsilon^1, \vec{\Psi}_{0,\varepsilon}^1)_{L^2(I_{\varepsilon,\tilde{\delta}})^m}$. Due to the Cauchy-Schwarz-Inequality we have $|\check{a}_\varepsilon| \leq 1$. Moreover, it holds $\check{a}_\varepsilon^2 = 1 - \|\vec{\Psi}_\varepsilon^\perp\|^2_{L^2(I_{\varepsilon,\tilde{\delta}})^m}$. Hence

$$\begin{aligned}
&\left| \int_{I_{\varepsilon,\tilde{\delta}}} (\check{\Psi}_\varepsilon^1, \sum_{\xi\in\mathbb{N}_0^m, |\xi|=1} \partial^\xi D^2W(\vec{\theta}_0)(\vec{\theta}_1)^\xi \check{\Psi}_\varepsilon^1)_{\mathbb{R}^m}\, dz \right| \\
&\leq \check{a}_\varepsilon^2 \left| \int_{I_{\varepsilon,\tilde{\delta}}} (\vec{\Psi}_{0,\varepsilon}^1, \sum_{\xi\in\mathbb{N}_0^m, |\xi|=1} \partial^\xi D^2W(\vec{\theta}_0)(\vec{\theta}_1)^\xi \vec{\Psi}_{0,\varepsilon}^1)_{\mathbb{R}^m}\, dz \right| + C\|\vec{\Psi}_\varepsilon^\perp\|_{L^2(I_{\varepsilon,\tilde{\delta}})^m},
\end{aligned}$$

where we used $\|\vec{\Psi}_\varepsilon^\perp\|_{L^2(I_{\varepsilon,\tilde{\delta}})^m} \leq 1$. We insert $\vec{\Psi}_{0,\varepsilon}^1 = \vec{\Psi}_{0,\varepsilon}^R + \check{\beta}_\varepsilon \vec{\theta}_0'$. The assumption (6.11) on $\vec{\theta}_1$, the decay for $\vec{\theta}_0'$ from Theorem 4.26 and Lemma 6.9, 6. yield for ε small

$$\left| \int_{I_{\varepsilon,\tilde{\delta}}} (\check{\Psi}_\varepsilon^1, \sum_{\xi\in\mathbb{N}_0^m, |\xi|=1} \partial^\xi D^2W(\vec{\theta}_0)(\vec{\theta}_1)^\xi \check{\Psi}_\varepsilon^1)_{\mathbb{R}^m}\, dz \right| \leq C(e^{-c/\varepsilon} + \|\vec{\Psi}_\varepsilon^\perp\|_{L^2(I_{\varepsilon,\tilde{\delta}})^m}).$$

Moreover, integration by parts yields $\check{B}_{0,\varepsilon}(\vec{\Psi}_{0,\varepsilon}^1, \vec{\Psi}_\varepsilon^\perp) = 0$. Hence we obtain

$$\check{B}_{0,\varepsilon}(\check{\Psi}_\varepsilon^1, \check{\Psi}_\varepsilon^1) = \check{a}_\varepsilon^2 \check{B}_{0,\varepsilon}(\vec{\Psi}_{0,\varepsilon}^1, \vec{\Psi}_{0,\varepsilon}^1) + \check{B}_{0,\varepsilon}(\vec{\Psi}_\varepsilon^\perp, \vec{\Psi}_\varepsilon^\perp) = \check{a}_\varepsilon^2 \check{\lambda}_{0,\varepsilon}^1 + \check{B}_{0,\varepsilon}(\vec{\Psi}_\varepsilon^\perp, \vec{\Psi}_\varepsilon^\perp).$$

Therefore Lemma 6.9, 4.-5. implies for ε small

$$C\varepsilon^2 \geq \check{\lambda}_\varepsilon^1 \geq \check{\nu}_1 \|\vec{\Psi}_\varepsilon^\perp\|^2_{L^2(I_{\varepsilon,\tilde{\delta}})^m} + \mathcal{O}(\varepsilon)\|\vec{\Psi}_\varepsilon^\perp\|_{L^2(I_{\varepsilon,\tilde{\delta}})^m} + \mathcal{O}(\varepsilon^2) \geq \frac{\check{\nu}_1}{2}\|\vec{\Psi}_\varepsilon^\perp\|^2_{L^2(I_{\varepsilon,\tilde{\delta}})^m} - \tilde{C}\varepsilon^2,$$

where the last estimate follows from Young's inequality. Hence we obtain $\|\vec{\Psi}_\varepsilon^\perp\|_{L^2(I_{\varepsilon,\tilde{\delta}})^m} = \mathcal{O}(\varepsilon)$ and $\check{\lambda}_\varepsilon^1 = \mathcal{O}(\varepsilon^2)$ for ε small. Moreover, it holds $\check{a}_\varepsilon^2 = 1 + \mathcal{O}(\varepsilon^2)$ for ε small. $\Box_1$.

Ad 2. The estimates above also imply $|\check{B}_{0,\varepsilon}(\vec{\Psi}_\varepsilon^\perp, \vec{\Psi}_\varepsilon^\perp)| = \mathcal{O}(\varepsilon^2)$. Therefore the definition of $\check{B}_{0,\varepsilon}$ and $\|\vec{\Psi}_\varepsilon^\perp\|_{L^2(I_{\varepsilon,\tilde{\delta}})^m} = \mathcal{O}(\varepsilon)$ yield $\|\frac{d}{dz}\vec{\Psi}_\varepsilon^\perp\|_{L^2(I_{\varepsilon,\tilde{\delta}})^m} = \mathcal{O}(\varepsilon)$. Let $\check{c}_\varepsilon := \operatorname{sign} \check{a}_\varepsilon \in \{\pm 1\}$.

For notational simplicity let $\check{c}_\varepsilon = 1$. If this is not the case, one can modify $\vec{\Psi}^1_\varepsilon$. We consider $\vec{\Psi}^R_\varepsilon := \vec{\Psi}^1_\varepsilon - J(0)^{-1/2}\check{\beta}_\varepsilon\vec{\theta}_0'$. Then with the splitting (6.13) and $\vec{\Psi}^1_{0,\varepsilon} = \vec{\Psi}^R_{0,\varepsilon} + \check{\beta}_\varepsilon\vec{\theta}_0'$ it follows that

$$\vec{\Psi}^R_\varepsilon = J_\varepsilon^{-\frac{1}{2}}\left[(\check{a}_\varepsilon - J_\varepsilon^{\frac{1}{2}}J(0)^{-\frac{1}{2}})\check{\beta}_\varepsilon\vec{\theta}_0' + \check{a}_\varepsilon\vec{\Psi}^R_{0,\varepsilon} + \vec{\Psi}^\perp_\varepsilon\right].$$

As in the scalar case one can show with Corollary 6.3 that $|\check{a}_\varepsilon - J_\varepsilon(z)^{\frac{1}{2}}J(0)^{-\frac{1}{2}}| \leq C\varepsilon(|z|+1)$ for all $z \in I_{\varepsilon,\tilde{\delta}}$ and ε small. Then the decay properties of $\vec{\theta}_0'$ and the estimates for the L^2-norms of $\vec{\Psi}^\perp_\varepsilon$ and $\vec{\Psi}^R_{0,\varepsilon}$ yield $\|\vec{\Psi}^R_\varepsilon\|_{J_\varepsilon} = \mathcal{O}(\varepsilon)$. Finally, as in the scalar case one can directly compute and estimate the derivative, cf. the proof of Theorem 6.8, 2. $\square_2$.

Ad 3. Let $\vec{\Psi}^2_\varepsilon$ be a normalized eigenfunction to $\check{\lambda}^2_\varepsilon$. If $\check{\lambda}^2_\varepsilon \geq \frac{1}{4}\min\{\sigma(D^2W(\vec{u}_\pm))\}$, then we are done. Therefore let $\check{\lambda}^2_\varepsilon \leq \frac{1}{4}\min\{\sigma(D^2W(\vec{u}_\pm))\}$. Then $\vec{\Psi}^2_\varepsilon$ has the decay property in Lemma 6.11, 3. Hence computations as before yield

$$\check{\lambda}^2_\varepsilon = \check{B}_{0,\varepsilon}(\check{\Psi}^2_\varepsilon, \check{\Psi}^2_\varepsilon) + \varepsilon p_\varepsilon \int_{I_{\varepsilon,\tilde{\delta}}} (\check{\Psi}^2_\varepsilon, \sum_{\xi\in\mathbb{N}_0^m, |\xi|=1} \partial^\xi D^2W(\vec{\theta}_0)(\vec{\theta}_1)^\xi\check{\Psi}^2_\varepsilon)_{\mathbb{R}^m}\,dz + \mathcal{O}(\varepsilon^2)$$

and the second term is $\mathcal{O}(\varepsilon)$. Therefore analogous computations as in the scalar case yield $\check{\lambda}^2_\varepsilon \geq \frac{\nu_1}{2}$ provided that $\check{\lambda}^2_\varepsilon \leq \frac{1}{4}\min\{\sigma(D^2W(\vec{u}_\pm))\}$, cf. the proof of Theorem 6.8, 3. $\square_3$.

6.1.5 Appendix: An Abstract Fredholm Alternative

We use an abstract Fredholm Alternative in the setting of a Gelfand-Triple. The result is basically well-known, but hard to find in the literature in the form presented below. Therefore we state the result for the convenience of the reader. The presentation is taken directly from Abels [A], Section 6.3. First, let us recall the definition of a Gelfand-Triple:

Remark 6.13 (Gelfand-Triple). Let V, H be $\mathbb{K}$-Hilbert spaces such that there exists $i \in \mathcal{L}(V,H)$ injective with $i(V)$ dense in H. We write $(.,.)_V, (.,.)_H$ for the scalar product in V, H, respectively. Moreover, we identify V with the subspace $i(V)$ of H and write $V \subseteq H$. Let V^*, H^* be the *anti-dual space* of V, H, respectively, i.e. the space of all conjugate-linear functionals. We write $\langle .,.\rangle_{V^*,V}, \langle .,.\rangle_{H^*,H}$ for the dual product on $V^* \times V$, $H^* \times H$, respectively. Then due to the Riesz-Representation Theorem we can identify $H \cong H^*$ via $y \mapsto (y,.)_H$. Moreover, $i^*: H^* \to V^*: y^* \mapsto y^* \circ i$ defines $i^* \in \mathcal{L}(H^*, V^*)$ injective and we identify $H^* \subseteq V^*$. Hence

$$V \overset{i}{\hookrightarrow}_d H \cong H^* \overset{i^*}{\hookrightarrow} V^* \quad \text{and shortened} \quad V \subseteq H \cong H^* \subseteq V^*.$$

The triple (V, H, V^*) is then called *Gelfand-Triple*.

Theorem 6.14 (An Abstract Fredholm Alternative). *Let (V, H, V^*) be a Gelfand-Triple as in Remark 6.13 with $\mathbb{K} = \mathbb{C}$, let H be infinite dimensional and i compact. Moreover, let $A \in \mathcal{L}(V, V^*)$ be such that for some $c_0 > 0, c_1 \in \mathbb{R}$ it holds*

$$\begin{aligned} \mathrm{Re}\langle Au, u\rangle_{V^*,V} &\geq c_0\|u\|_V^2 - c_1\|u\|_H^2 && \text{for all } u \in V,\\ \langle Au, v\rangle_{V^*,V} &= \overline{\langle Av, u\rangle_{V^*,V}} && \text{for all } u, v \in V. \end{aligned}$$

Then there is a sequence of real numbers $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_k \overset{k\to\infty}{\longrightarrow} \infty$ and $e_k \in V, k \in \mathbb{N}$ such that $(e_k)_{k\in\mathbb{N}}$ is an orthonormal base of H and e_k is an eigenvector of A to the eigenvalue λ_k, i.e. $(\lambda_k - A)e_k = 0$ in V^*. Moreover:

1. *For all $\lambda \in \mathbb{C} \setminus \{\lambda_k : k \in \mathbb{N}\}$ and $f \in H$ there is a unique solution $u \in V$ of*
$$(\lambda - A)u = f \quad \text{in } V^*. \tag{6.14}$$
The solution u can be represented as
$$u = \sum_{k=1}^{\infty} \frac{1}{\lambda - \lambda_k}(f, e_k)_H e_k \quad \text{in } H.$$

2. *Let $\lambda \in \{\lambda_k : k \in \mathbb{N}\}$ and $J_\lambda := \{j \in \mathbb{N} : \lambda_j = \lambda\}$. Then for $f \in H$ there exists a solution $u \in V$ of* (6.14) *if and only if $(f, e_j)_H = 0$ for all $j \in J_\lambda$. If the latter holds, then all solutions of* (6.14) *can be represented as*
$$\sum_{k \in \mathbb{N} \setminus J_\lambda} \frac{1}{\lambda - \lambda_k}(f, e_k)_H e_k + \operatorname{span}\{e_j : j \in J_\lambda\}.$$

Proof. One applies the Spectral Theorem for Compact Self-Adjoint Operators to $(\mu - A)^{-1}$ for some $\mu \leq -c_1$ viewed as a bounded linear operator in H. The existence of the resolvent for these μ follows from the Lax-Milgram Theorem. For spectral theorems see Alt [Al], Theorem 11.9 and Theorem 12.12. A similar application can be found in Renardy, Rogers [RR], Section 9.3. □

6.2 Spectral Estimate for (AC) in 2D

In this section we prove the spectral estimate (6.1) in the case of the Allen-Cahn equation (AC1)-(AC3) with boundary contact in two dimensions. From the construction of the approximate solution in Section 5.1 we know the precise structure of u^A_ε. For the spectral estimate itself a more general structure is enough. In particular the h_j and $u^{C\pm}_j$ are not specified as in the asymptotic expansion, only the lower orders and less regularity are needed. However, for convenience we often use the same notation. In the following we describe the assumptions for this section.

Let $\Omega \subset \mathbb{R}^2$ and $\Gamma = (\Gamma_t)_{t \in [0,T]}$ for $T > 0$ be as in Section 3.1 for $N = 2$ with contact angle $\alpha = \frac{\pi}{2}$ (MCF not needed). Moreover, let $\delta > 0$ be such that Theorem 3.3 holds for 2δ instead of δ. Then let X_0, X, μ_0, r, s be as in Theorem 3.3 and recall the definition of some sets and of ∂_n, ∇_τ, J from Remark 3.4. Throughout this section we assume the following structure: We consider height functions h_1 and $h_2 = h_2(\varepsilon)$ for $\varepsilon > 0$ small. We assume (with a slight abuse of notation)
$$h_j \in B([0,T], C^0(I) \cap C^2(\hat{I})), j = 1, 2, \quad \text{where } \hat{I} := I \setminus (-1 + 2\mu_0, 1 - 2\mu_0).$$

Additionally, let $\overline{C}_0 > 0$ be such that $\|h_j\|_{B([0,T],C^0(I)\cap C^2(\hat{I}))} \leq \overline{C}_0$ for $j = 1, 2$. Then for $\varepsilon > 0$ small we define $h_\varepsilon := h_1 + \varepsilon h_2$ and introduce the stretched variables
$$\rho_\varepsilon := \frac{r - \varepsilon h_\varepsilon(s,t)}{\varepsilon}, \quad s^\pm := \mp(s \mp 1), \quad H^\pm_\varepsilon := \frac{s^\pm}{\varepsilon} \quad \text{in } \overline{\Gamma(2\delta)}.$$

Let $\hat{u}^{C\pm}_1 : \overline{\mathbb{R}^2_+} \times [0,T] \to \mathbb{R} : (\rho, H, t) \mapsto \hat{u}^{C\pm}_1(\rho, H, t)$ be in $B([0,T], H^2_{(0,\gamma)}(\mathbb{R}^2_+))$ for a $\gamma > 0$. Then we set
$$u^{C\pm}_1(x,t) := \hat{u}^{C\pm}_1(\rho_\varepsilon(x,t), H^\pm_\varepsilon(x,t), t) \quad \text{for } (x,t) \in \Gamma^\pm(\delta, 2\mu_0).$$

Moreover, for $\varepsilon > 0$ small let

$$u_\varepsilon^A = \begin{cases} \theta_0(\rho_\varepsilon) + \mathcal{O}(\varepsilon^2) & \text{in } \Gamma(\delta, \mu_0), \\ \theta_0(\rho_\varepsilon) + \varepsilon u_1^{C\pm} + \mathcal{O}(\varepsilon^2) & \text{in } \Gamma^\pm(\delta, 2\mu_0), \\ \pm 1 + \mathcal{O}(\varepsilon) & \text{in } Q_T^\pm \setminus \Gamma(\delta), \end{cases}$$

where θ_0 is the optimal profile from Theorem 4.1 and $\mathcal{O}(\varepsilon^k)$ are measurable functions bounded by $C\varepsilon^k$. Note that in the following we often use Lemma 2.10 without mentioning it.

Remark 6.15. One can also allow an additional term of the form $\varepsilon\theta_1(\rho_\varepsilon)p_\varepsilon(s,t)$ in u_ε^A on $\Gamma(\delta)$, where $p_\varepsilon \in B([0,T], C^0(I) \cap C^2(\hat{I}))$ satisfies a uniform estimate for ε small and

$$\theta_1 \in C_b^0(\mathbb{R}) \quad \text{with} \quad \int_{\mathbb{R}} f'''(\theta_0)\theta_1(\theta_0')^2 \, d\rho = 0. \tag{6.15}$$

For details see Remark 6.18, 2. below.

The spectral estimate in this situation reads as follows.

Theorem 6.16 (**Spectral Estimate for (AC) in 2D**)**.** *There are $\varepsilon_0, C, c_0 > 0$ independent of the h_j for fixed $\overline{C}_0$ such that for all $\varepsilon \in (0, \varepsilon_0], t \in [0,T]$ and $\psi \in H^1(\Omega)$ it holds*

$$\int_\Omega |\nabla\psi|^2 + \frac{1}{\varepsilon^2} f''(u_\varepsilon^A|_{(.,t)})\psi^2 \, dx \geq -C\|\psi\|^2_{L^2(\Omega)} + \|\nabla\psi\|^2_{L^2(\Omega\setminus\Gamma_t(\delta))} + c_0\|\nabla_\tau\psi\|^2_{L^2(\Gamma_t(\delta))}.$$

The main new difficulty compared to Theorem 2.3 in Chen [C2] and Theorem 2.13 in Abels, Liu [AL] is to prove a spectral estimate on $\Omega_t^{C\pm} := \Gamma_t^\pm(\delta, 2\mu_0)$, $t \in [0,T]$. This is the content of

Theorem 6.17. *There are $\tilde{\varepsilon}_0, C, \tilde{c}_0 > 0$ independent of the h_j for fixed $\overline{C}_0$ such that for all $\varepsilon \in (0, \tilde{\varepsilon}_0]$, $t \in [0,T]$ and $\psi \in H^1(\Omega_t^{C\pm})$ with $\psi|_{X(.,s,t)} = 0$ for a.e. $\mp(s \mp 1) \in [\frac{3}{2}\mu_0, 2\mu_0]$:*

$$\int_{\Omega_t^{C\pm}} |\nabla\psi|^2 + \frac{1}{\varepsilon^2} f''(u_\varepsilon^A|_{(.,t)})\psi^2 \, dx \geq -C\|\psi\|^2_{L^2(\Omega_t^{C\pm})} + \tilde{c}_0\|\nabla_\tau\psi\|^2_{L^2(\Omega_t^{C\pm})}.$$

The additional assumption on ψ is not needed but simplifies the proof, cf. Remark 6.18, 3. below. This version is enough to show Theorem 6.16:

Proof of Theorem 6.16. For $\varepsilon_0 > 0$ small and all $\varepsilon \in (0, \varepsilon_0]$ we have $f''(u_\varepsilon^A) \geq 0$ on $Q_T^\pm \setminus \Gamma(\delta)$. Therefore it is enough to prove the estimate in Theorem 6.16 for $\Gamma_t(\delta)$ instead of Ω. In the following we reduce to further subsets.

Due to Theorem 6.17 we have an estimate for $\Omega_t^\pm = \Gamma_t^\pm(\delta, 2\mu_0)$ instead of Ω. Moreover, the estimate holds for $\Gamma_t(\delta, \mu_0)$ instead of Ω with $c_0 = 1$. There our curvilinear coordinate system coincides with the orthogonal one, cf. Theorem 3.3. Hence one can transform the integral as in Abels, Liu [AL], proof of Theorem 2.13 with Lemma 6.5, 1. in the normal variable and then use Theorem 6.8, 1. This also works with the additional terms in Remark 6.15. For the convenience of the reader we give the details. Let $\psi \in H^1(\Gamma_t(\delta, \mu_0))$. Then $|\nabla\psi|^2 = |\nabla_\tau\psi|^2 + |\partial_n\psi|^2$ on $\Gamma_t(\delta, \mu_0)$ due to Remark 3.4, 2. Furthermore, because of Taylor's Theorem we can replace

u_ε^A by $\theta_0(\rho_\varepsilon)$ in the integral by paying some error $\tilde{C}\|\psi\|^2_{L^2(\Gamma_t(\delta,\mu_0))}$. Therefore an integral transformation with $X(.,t)$ yields

$$\int_{\Gamma_t(\delta,\mu_0)} |\nabla\psi|^2 + \frac{1}{\varepsilon^2} f''(u_\varepsilon^A|_{(.,t)})\psi^2\,dx \geq -\tilde{C}\|\psi\|^2_{L^2(\Gamma_t(\delta,\mu_0))} + \|\nabla_\tau\psi\|^2_{L^2(\Gamma_t(\delta,\mu_0))}$$
$$+ \int_{-1+\mu_0}^{1-\mu_0}\int_{-\delta}^{\delta}\left[|\partial_r\tilde{\psi}_t|^2 + \frac{1}{\varepsilon^2}f''(\theta_0(\rho_\varepsilon|_{\overline{X}(.,t)}))\tilde{\psi}_t^2\right] J_t\,dr\,ds,$$

where we have set $\tilde{\psi}_t := \psi|_{X(.,t)}$. Now we split the integral over the normal variable r into an integral over $(-\frac{3\delta}{4}, \frac{3\delta}{4}) + \varepsilon h_\varepsilon(s,t)$ and the complement for ε small. For the latter we can use $f''(\theta_0(\rho_\varepsilon|_{\overline{X}(.,t)})) \geq 0$ for ε small. For the remaining integral we apply a transformation with $z \mapsto F_{\varepsilon,s,t}(z) := \varepsilon(z + h_\varepsilon(s,t))$. Note that this fits into the setting from Section 6.1.1 with a constant h_ε there. We set $\Psi_{\varepsilon,s,t}(z) := \sqrt{\varepsilon}\tilde{\psi}_t(F_{\varepsilon,s,t}(z),s,t)$ and $J_{\varepsilon,s,t}(z) := J_t(F_{\varepsilon,s,t}(z),s)$. Then with $\frac{d}{dz}F_{\varepsilon,s,t} \equiv \varepsilon$ and Lemma 6.5, 1. it follows that

$$\int_{(-\frac{3\delta}{4},\frac{3\delta}{4})+\varepsilon h_\varepsilon(s,t)} \left[|\partial_r\tilde{\psi}_t(.,s)|^2 + \frac{1}{\varepsilon^2}f''(\theta_0(\rho_\varepsilon|_{\overline{X}(.,s,t)}))\tilde{\psi}_t(.,s)^2\right] J_t(.,s)\,dr$$
$$= \frac{1}{\varepsilon^2}\int_{I_{\varepsilon,\frac{3\delta}{4}}} \left[\left|\frac{d}{dz}\Psi_{\varepsilon,s,t}\right|^2 + f''(\theta_0)|\Psi_{\varepsilon,s,t}|^2\right] J_{\varepsilon,s,t}\,dz,$$

where $I_{\varepsilon,\frac{3\delta}{4}} := (-\frac{3\delta}{4\varepsilon}, \frac{3\delta}{4\varepsilon})$. Due to Theorem 6.8, 1. the last integral is estimated from below by

$$-C\int_{I_{\varepsilon,\frac{3\delta}{4}}} |\Psi_{\varepsilon,s,t}|^2 J_{\varepsilon,s,t}\,dz = -C\int_{(-\frac{3\delta}{4},\frac{3\delta}{4})+\varepsilon h_\varepsilon(s,t)} \tilde{\psi}_t(.,s)^2 J_t(.,s)\,dr$$

for ε small. Finally, the transformation back to x-coordinates yields the estimate on $\Gamma_t(\delta,\mu_0)$.

Now we put the three estimates together with a suitable partition of unity for

$$\Gamma_t(\delta) \subseteq \overline{\Gamma_t(\delta,\mu_0)} \cup \overline{\Gamma_t^-(\delta,2\mu_0)} \cup \overline{\Gamma_t^+(\delta,2\mu_0)}. \tag{6.16}$$

Therefore let $\eta_0, \eta_\pm : I \to [0,1]$ be a partition of unity subordinated to

$$[-1,-1+\frac{3}{2}\mu_0], \quad [-1+\mu_0, 1-\mu_0], \quad [1-\frac{3}{2}\mu_0, 1].$$

W.l.o.g. $\sqrt{\eta_j} \in C_b^\infty(I)$ for $j = 0,\pm$, otherwise we replace η_j by $\eta_j^2/\sum_{j=0,\pm}\eta_j^2$. Then

$$\tilde{\eta}_j(.,t) : \overline{\Gamma_t(\delta)} \to [0,1] : x \mapsto \eta_j(s(x,t)) \quad \text{for } j = 0,\pm$$

defines a partition of unity for (6.16) and $\chi_j(.,t) := \sqrt{\tilde{\eta}_j(.,t)} \in C_b^\infty(\overline{\Gamma_t(\delta)})$ for $j = 0,\pm$. For any $\psi \in H^1(\Gamma_t(\delta))$ it holds $\psi^2 = \sum_{j=0,\pm}(\chi_j(.,t)\psi)^2$ and

$$\nabla(\chi_j(.,t)\psi) = \nabla\chi_j(.,t)\psi + \chi_j(.,t)\nabla\psi,$$
$$|\nabla(\chi_j(.,t)\psi)|^2 = |\nabla\chi_j(.,t)|^2\psi^2 + 2\chi_j(.,t)\nabla\chi_j(.,t)\cdot\psi\nabla\psi + \chi_j(.,t)^2|\nabla\psi|^2.$$

Since $\sum_{j=0,\pm}\chi_j(.,t)^2 = 1$, we have $\sum_{j=0,\pm}\chi_j(.,t)\nabla\chi_j(.,t) = 0$ and therefore

$$\sum_{j=0,\pm}|\nabla(\chi_j(.,t)\psi)|^2 = |\nabla\psi|^2 + \psi^2\sum_{j=0,\pm}|\nabla\chi_j(.,t)|^2.$$

This identity also holds for ∇_τ instead of ∇ which can be proven analogously. We write

$$\int_{\Gamma_t(\delta)} |\nabla\psi|^2 + \frac{1}{\varepsilon^2} f''(u_\varepsilon^A|_{(.,t)})\psi^2\,dx = -\int_{\Gamma_t(\delta)} \psi^2 \sum_{j=0,\pm} |\nabla\chi_j(.,t)|^2\,dx$$
$$+ \sum_{j=0,\pm} \int_{\Gamma_t(\delta)} |\nabla(\chi_j(.,t)\psi)|^2 + \frac{1}{\varepsilon^2} f''(u_\varepsilon^A|_{(.,t)})(\chi_j(.,t)\psi)^2\,dx.$$

Using the spectral estimates on $\Gamma_t(\delta,\mu_0)$ and $\Gamma_t^\pm(\delta,2\mu_0)$, cf. the beginning of the proof, and that $|\nabla\chi_j(.,t)|$ and $|\nabla_\tau\chi_j(.,t)|$ are bounded on $\Gamma_t(\delta)$ uniformly in $t\in[0,T]$ due to $\sqrt{\eta_j}\in C_b^\infty(I)$, we obtain the spectral estimate in Theorem 6.16. $\square$

6.2.1 Outline for the Proof of the Spectral Estimate close to the Contact Points

For a motivation see the end of the introduction for Section 6. Because of $H^2(\mathbb{R}^2_+)\hookrightarrow C_b^0(\overline{\mathbb{R}^2_+})$ and a Taylor expansion it is enough to prove Theorem 6.17 for

$$\frac{1}{\varepsilon^2} f''(\theta_0|_{\rho_\varepsilon(.,t)}) + \frac{1}{\varepsilon} f'''(\theta_0|_{\rho_\varepsilon(.,t)}) u_1^{C\pm}(.,t)$$

instead of $\frac{1}{\varepsilon^2} f''(u_\varepsilon^A(.,t))$. We construct an approximation $\phi_\varepsilon^A(.,t)$ to the first eigenfunction of

$$\mathcal{L}_{\varepsilon,t}^\pm := -\Delta + \frac{1}{\varepsilon^2} f''(\theta_0|_{\rho_\varepsilon(.,t)}) + \frac{1}{\varepsilon} f'''(\theta_0|_{\rho_\varepsilon(.,t)}) u_1^{C\pm}(.,t) \quad \text{on } \Omega_t^{C\pm}$$

together with a homogeneous Neumann boundary condition and decompose

$$\tilde{H}^1(\Omega_t^{C\pm}) := \left\{\psi\in H^1(\Omega_t^{C\pm}) : \psi|_{X(.,s,t)} = 0 \text{ for a.e. } \mp(s\mp 1)\in[\tfrac{3}{2}\mu_0, 2\mu_0]\right\} \tag{6.17}$$

along the subspace of tangential alterations of $\phi_\varepsilon^A(.,t)$. Therefore we make the ansatz

$$\begin{aligned}
\phi_\varepsilon^A(.,t) &:= \frac{1}{\sqrt{\varepsilon}}[v_\varepsilon^I(.,t) + \varepsilon v_\varepsilon^{C\pm}(.,t)] && \text{on } \Omega_t^{C\pm},\\
v_\varepsilon^I(.,t) &:= \hat{v}^I(\rho_\varepsilon(.,t), s(.,t), t) := \theta_0'|_{\rho_\varepsilon(.,t)} q^\pm(s^\pm(.,t),t) && \text{on } \Omega_t^{C\pm},\\
v_\varepsilon^{C\pm}(.,t) &:= \hat{v}^{C\pm}(\rho_\varepsilon(.,t), H_\varepsilon^\pm(.,t), t) && \text{on } \Omega_t^{C\pm},
\end{aligned}$$

where $q^\pm : [0,2\mu_0]\times[0,T]\to\mathbb{R} : (\sigma,t)\mapsto q^\pm(\sigma,t)$ and $\hat{v}^{C\pm} : \overline{\mathbb{R}^2_+}\times[0,T]\to\mathbb{R}$. Here the $\frac{1}{\sqrt{\varepsilon}}$-factor is multiplied for a certain normalization later, see Lemma 6.20 below.

In Subsection 6.2.2 we expand $\mathcal{L}_{\varepsilon,t}^\pm\phi_\varepsilon^A(.,t)$ and $\partial_{N_{\partial\Omega}}\phi_\varepsilon^A(.,t)$ similarly as in Section 5.1 and choose $q^\pm$ and $\hat{v}^{C\pm}$ such that there is some cancellation. The $q^\pm$-term was introduced in order to fulfil the compatibility condition for the equations for $\hat{v}^{C\pm}$. Then in Subsection 6.2.3 we characterize the L^2-orthogonal splitting of $\tilde{H}^1(\Omega_t^{C\pm})$ with respect to the subspace

$$V_{\varepsilon,t}^\pm := \left\{\phi = a(s^\pm(.,t))\phi_\varepsilon^A(.,t) : a\in\tilde{H}^1(0,2\mu_0)\right\}, \tag{6.18}$$
$$\tilde{H}^1(0,2\mu_0) := \left\{a\in H^1(0,2\mu_0) : a = 0 \text{ on } [\tfrac{3}{2}\mu_0, 2\mu_0]\right\}. \tag{6.19}$$

Finally, in Subsection 6.2.4 we analyze the bilinear form $B_{\varepsilon,t}^\pm$ corresponding to $\mathcal{L}_{\varepsilon,t}^\pm$ on $V_{\varepsilon,t}^\pm\times V_{\varepsilon,t}^\pm$, $(V_{\varepsilon,t}^\pm)^\perp\times(V_{\varepsilon,t}^\pm)^\perp$ and $V_{\varepsilon,t}^\pm\times(V_{\varepsilon,t}^\pm)^\perp$. Here for $\phi,\psi\in H^1(\Omega_t^{C\pm})$ let

$$B_{\varepsilon,t}^\pm(\phi,\psi) := \int_{\Omega_t^{C\pm}} \nabla\phi\cdot\nabla\psi + \left[\frac{1}{\varepsilon^2} f''(\theta_0|_{\rho_\varepsilon(.,t)}) + \frac{1}{\varepsilon} f'''(\theta_0|_{\rho_\varepsilon(.,t)}) u_1^{C\pm}(.,t)\right]\phi\psi\,dx. \tag{6.20}$$

6.2.2 Asymptotic Expansion for the Approximate Eigenfunction

Asymptotic Expansion of $\sqrt{\varepsilon}\mathcal{L}^{\pm}_{\varepsilon,t}\phi^A_\varepsilon(.,t)$. First, we expand Δv^I_ε as in the inner expansion in Section 5.1.1. At the lowest order $\mathcal{O}(\frac{1}{\varepsilon^2})$ we obtain $\frac{1}{\varepsilon^2}|\nabla r|^2|_{\overline{X}_0(s,t)}\theta_0'''(\rho)q^\pm(s^\pm,t)$. In $\sqrt{\varepsilon}\mathcal{L}^{\pm}_{\varepsilon,t}\phi^A_\varepsilon(.,t)$ this cancels with $\frac{1}{\varepsilon^2}f''(\theta_0(\rho))\theta_0'(\rho)q^\pm(s^\pm,t)$. For the $\frac{1}{\varepsilon}$-order of Δv^I_ε we get

$$\frac{1}{\varepsilon}\theta_0'''(\rho)q^\pm|_{(s^\pm,t)}\left[(\rho+h_1)\partial_r(|\nabla r|^2\circ\overline{X})|_{(0,s,t)}-2(\nabla r\cdot\nabla s)|_{\overline{X}_0(s,t)}\partial_s h_1\right]$$
$$+\frac{1}{\varepsilon}\theta_0''(\rho)\left[\Delta r|_{\overline{X}_0(s,t)}q^\pm|_{(s^\pm,t)}\mp 2\partial_\sigma q^\pm|_{(s^\pm,t)}\nabla r\cdot\nabla s|_{\overline{X}_0(s,t)}\right]=\frac{1}{\varepsilon}\theta_0''(\rho)\Delta r|_{\overline{X}_0(s,t)}q^\pm|_{(s^\pm,t)}.$$

The term $\frac{1}{\varepsilon}\Delta r|_{\overline{X}_0(s,t)}q^\pm(s^\pm,t)\theta_0''(\rho)$ is left as a remainder. This seems bad, but there is still hope to gain a power in ε later since $\int_{\mathbb{R}}\theta_0''\theta_0'\,d\rho=0$. For the precise argument see (6.27).

Moreover, for $\varepsilon\Delta v^{C\pm}_\varepsilon$ we use the expansion in Section 5.1.2.1, but without applying a Taylor expansion for the h_j. This is because here we just need the lowest order and we assumed h_j to be less regular. More precisely, the (x,t)-terms in the formula for $\Delta v^{C\pm}_\varepsilon$ in Lemma 5.5 are expanded solely via (5.15). Therefore at the order $\mathcal{O}(\frac{1}{\varepsilon})$ we get $\frac{1}{\varepsilon}\Delta^\pm_t\hat{v}^{C\pm}$, where $\Delta^\pm_t:=\partial^2_\rho+|\nabla s|^2|_{\overline{p}^\pm(t)}\partial^2_H$. From the f-parts we have $\frac{1}{\varepsilon}f''(\theta_0(\rho))\hat{v}^{C\pm}+\frac{1}{\varepsilon}f'''(\theta_0(\rho))\hat{u}^{C\pm}_1\hat{v}^I$. In order to obtain an equation for $\hat{v}^{C\pm}$ in (ρ,H,t) we use a Taylor expansion for $q^\pm$:

$$q^\pm(\varepsilon H,t)=q^\pm(0,t)+\mathcal{O}(\varepsilon H)\quad\text{ for } H\in[0,\frac{2\mu_0}{\varepsilon}].$$

Therefore we require

$$\left[-\Delta^\pm_t+f''(\theta_0(\rho))\right]\hat{v}^{C\pm}=-f'''(\theta_0)\theta_0'|_\rho\hat{u}^{C\pm}_1|_{(\rho,H,t)}q^\pm|_{(0,t)}\quad\text{ in }\overline{\mathbb{R}^2_+}\times[0,T].\tag{6.21}$$

Asymptotic Expansion of $\sqrt{\varepsilon}\partial_{N_{\partial\Omega}}\phi^A_\varepsilon(.,t)$. We proceed as in Section 5.1.2.2. Note that in $\overline{\Omega^{C\pm}_t}$

$$\begin{aligned}\nabla v^I_\varepsilon&=q^\pm(s^\pm,t)\theta_0''(\rho_\varepsilon)\left[\frac{\nabla r}{\varepsilon}-\nabla s\partial_s h_\varepsilon|_{(s,t)}\right]\mp\partial_\sigma q(s^\pm,t)\theta_0'(\rho_\varepsilon)\nabla s,\\ \nabla v^{C\pm}_\varepsilon&=\partial_\rho\hat{v}^{C\pm}|_{(\rho_\varepsilon,H^\pm_\varepsilon,t)}\left[\frac{\nabla r}{\varepsilon}-\nabla s\partial_s h_\varepsilon|_{(s,t)}\right]\mp\frac{\nabla s}{\varepsilon}\partial_H\hat{v}^{C\pm}|_{(\rho_\varepsilon,H^\pm_\varepsilon,t)}.\end{aligned}\tag{6.22}$$

Hence at the lowest order $\mathcal{O}(\frac{1}{\varepsilon})$ in $\sqrt{\varepsilon}\partial_{N_{\partial\Omega}}\phi^A_\varepsilon(.,t)$ we obtain $\frac{1}{\varepsilon}(N_{\partial\Omega}\cdot\nabla r)|_{\overline{p}^\pm(t)}\theta_0''|_\rho q^\pm|_{(0,t)}$. This is zero because of the 90°-contact angle condition. The $\mathcal{O}(1)$-order is

$$q^\pm|_{(0,t)}\theta_0''(\rho)\left[(\rho+h_1|_{(\pm1,t)})\partial_r((N_{\partial\Omega}\cdot\nabla r)\circ\overline{X})|_{(0,\pm1,t)}-(N_{\partial\Omega}\cdot\nabla s)|_{\overline{p}^\pm(t)}\partial_s h_1|_{(\pm1,t)}\right]$$
$$\mp(N_{\partial\Omega}\cdot\nabla s)|_{\overline{p}^\pm(t)}\partial_\sigma q^\pm|_{(0,t)}\theta_0'(\rho)+0\cdot\partial_\rho\hat{v}^{C\pm}|_{H=0}\mp(N_{\partial\Omega}\cdot\nabla s)|_{\overline{p}^\pm(t)}\partial_H\hat{v}^{C\pm}|_{H=0}.$$

The cancellation is equivalent to

$$\mp(N_{\partial\Omega}\cdot\nabla s)|_{\overline{p}^\pm(t)}\partial_H\hat{v}^{C\pm}|_{H=0}=g^{C\pm}(\rho,t)\pm(N_{\partial\Omega}\cdot\nabla s)|_{\overline{p}^\pm(t)}\partial_\sigma q^\pm|_{(0,t)}\theta_0'(\rho),$$

where $g^{C\pm}(\rho,t)$ is given by

$$q^\pm|_{(0,t)}\theta_0''(\rho)\left[(N_{\partial\Omega}\cdot\nabla s)|_{\overline{p}^\pm(t)}\partial_s h_1|_{(\pm1,t)}-(\rho+h_1|_{(\pm1,t)})\partial_r((N_{\partial\Omega}\cdot\nabla r)\circ\overline{X})|_{(0,\pm1,t)}\right].$$

As in Section 5.1.2.2.2 this equation together with (6.21) is equivalent to

$$[-\Delta + f''(\theta_0)]\overline{v}^{C\pm} = -f'''(\theta_0)\theta_0'\overline{u}_1^{C\pm}q^{\pm}|_{(0,t)} \quad \text{in } \overline{\mathbb{R}^2_+}\times[0,T], \tag{6.23}$$

$$-\partial_H\overline{v}^{C\pm}|_{H=0} = g^{C\pm} \pm (N_{\partial\Omega}\cdot\nabla s)|_{\overline{p}^{\pm}(t)}\partial_\sigma q^{\pm}|_{(0,t)}\theta_0' \quad \text{in } \mathbb{R}\times[0,T], \tag{6.24}$$

where $\overline{v}^{C\pm}, \overline{u}_1^{C\pm} : \overline{\mathbb{R}^2_+}\times[0,T]\rightarrow\mathbb{R}$ are related to $\hat{v}^{C\pm}$ and $\hat{u}_1^{C\pm}$ as in (5.20), respectively. The right hand sides are in $B([0,T], H^2_{(\beta,\gamma)}(\mathbb{R}^2_+)\times H^{5/2}_{(\beta)}(\mathbb{R}))$ for some $\beta,\gamma>0$ provided that $q^{\pm}|_{(0,t)}$ and $\partial_\sigma q^{\pm}|_{(0,t)}$ are uniformly bounded. Because of $|(N_{\partial\Omega}\cdot\nabla s)|_{\overline{p}^{\pm}(t)}|\geq c>0$, the compatibility condition (4.5) corresponding to (6.23)-(6.24) determines $\partial_\sigma q^{\pm}(0,t)$ if e.g. $q^{\pm}(0,t)=1$.

With a simple ansatz and cutoff we can construct $q^{\pm}\in B([0,T], C^2([0,2\mu_0]))$ such that the solvability condition (4.5) for (6.23)-(6.24) holds and such that $q^{\pm}(0,t)=1, q^{\pm}(.,t)=1$ on $[\mu_0,2\mu_0]$ for all $t\in[0,T]$ and $\frac{1}{2}\leq q^{\pm}\leq 2$. As a consequence, Remark 4.12 yields a unique solution $\overline{v}^{C\pm}\in B([0,T],H^4_{(\beta,\gamma)}(\mathbb{R}^2_+))\hookrightarrow B([0,T],C^2_{(\beta,\gamma)}(\overline{\mathbb{R}^2_+}))$ of (6.23)-(6.24) for some $\beta,\gamma>0$. For the definition of the spaces see Definition 2.21.

Remark 6.18. 1. Consider the situation of Section 5.1. Then $\overline{u}_1^{C\pm}$ solves (5.21)-(5.22). By differentiating these equations with respect to ρ we obtain

$$[-\Delta + f''(\theta_0)]\partial_\rho\overline{u}_1^{C\pm} = -f'''(\theta_0)\theta_0'\overline{u}_1^{C\pm} \quad \text{in } \overline{\mathbb{R}^2_+}\times[0,T],$$

$$-\partial_H(\partial_\rho\overline{u}_1^{C\pm})|_{H=0} = g^{C\pm} - \partial_r((N_{\partial\Omega}\cdot\nabla r)\circ\overline{X})|_{(0,\pm1,t)}\theta_0' \quad \text{in } \mathbb{R}\times[0,T],$$

where $g^{C\pm}$ is as above with $q^{\pm}(0,t)=1$. Therefore in this case we can choose

$$\hat{v}^{C\pm} := \partial_\rho\hat{u}_1^{C\pm} \quad \text{and} \quad \partial_\sigma q^{\pm}(0,t) = \frac{\partial_r((N_{\partial\Omega}\cdot\nabla r)\circ\overline{X})|_{(0,\pm1,t)}}{\mp(N_{\partial\Omega}\cdot\nabla s)|_{\overline{p}^{\pm}(t)}}.$$

2. In the case of additional terms in u^A_ε as in Remark 6.15 in the operator $\mathcal{L}^{\pm}_{\varepsilon,t}$ there is an additional term $\frac{1}{\varepsilon}f'''(\theta_0|_{\rho_\varepsilon(.,t)})p_\varepsilon(s(.,t),t)\theta_1|_{\rho_\varepsilon(.,t)}$. Then in the ansatz for ϕ^A_ε we add $\varepsilon\hat{v}_1(\rho_\varepsilon(.,t))q^{\pm}_{1,\varepsilon}(s^{\pm}(.,t),t)$. Therefore in the asymptotic expansion of $\sqrt{\varepsilon}\mathcal{L}^{\pm}_{\varepsilon,t}\phi^A_\varepsilon(.,t)$ there are new terms at order $\mathcal{O}(\frac{1}{\varepsilon})$ we need to take care of, namely

$$\left[-\frac{1}{\varepsilon^2}\varepsilon\partial_\rho^2\hat{v}_1(\rho) + \frac{f''(\theta_0(\rho))}{\varepsilon^2}\varepsilon\hat{v}_1(\rho)\right]q^{\pm}_{1,\varepsilon}(s^{\pm},t) + \frac{1}{\varepsilon}[f'''(\theta_0)\theta_1\theta_0'](\rho)p_\varepsilon(s,t)q^{\pm}(s^{\pm},t).$$

Therefore we set $q^{\pm}_{1,\varepsilon}(\sigma,t) := p_\varepsilon(\pm1\mp\sigma,t)q^{\pm}(\sigma,t)$ and require

$$\left[-\partial_\rho^2 + f''(\theta_0)\right]\hat{v}_1 = -f'''(\theta_0)\theta_1\theta_0'.$$

Due to (6.15) this equation can be solved with Theorem 4.4, 1. This yields a unique solution $\hat{v}_1\in C^2_{(\beta)}(\mathbb{R})$ for some $\beta>0$. Then below analogous arguments apply.

3. The behaviour of $\phi^A_\varepsilon(x,t)$ for $x\in\Omega^{C\pm}_t$ with $s^{\pm}(x,t)\in[\frac{7}{4}\mu_0,2\mu_0]$ is not important because we only need $\psi\in\tilde{H}^1(\Omega^{C\pm}_t)$ in Theorem 6.17, where $\tilde{H}^1(\Omega^{C\pm}_t)$ is from (6.17).

Lemma 6.19. *The function $\phi_\varepsilon^A(.,t)$ is $C^2(\overline{\Omega_t^{C\pm}})$ and satisfies uniformly in $t \in [0,T]$:*

$$\begin{aligned}
\left|\sqrt{\varepsilon}\mathcal{L}_{\varepsilon,t}^{\pm}\phi_\varepsilon^A(.,t) + \frac{1}{\varepsilon}\Delta r|_{\overline{X}_0(s(.,t),t)} q^{\pm}|_{(s^{\pm}(.,t),t)}\theta_0''(\rho_\varepsilon(.,t))\right| &\leq Ce^{-c|\rho_\varepsilon(.,t)|} && \text{in } \Omega_t^{C\pm},\\
\left|\sqrt{\varepsilon}N_{\partial\Omega_t^{C\pm}}\cdot\nabla\phi_\varepsilon^A(.,t)\right| &\leq C\varepsilon e^{-c|\rho_\varepsilon(.,t)|} && \text{on } \partial\Omega_t^{C\pm}\cap\partial\Omega,\\
\left|\sqrt{\varepsilon}N_{\partial\Omega_t^{C\pm}}\cdot\nabla\phi_\varepsilon^A(.,t)\right| &\leq Ce^{-c/\varepsilon} && \text{on } \partial\Omega_t^{C\pm}\setminus\Gamma_t(\delta).
\end{aligned}$$

Proof. The regularity property follows from the construction and the assumptions on the h_j, $j = 1,2$. Moreover, one can rigorously estimate the remainder terms in the expansions to get

$$\left|\mathcal{L}_{\varepsilon,t}^{\pm}v_\varepsilon^I(.,t) + \frac{1}{\varepsilon}\Delta r|_{\overline{X}_0(s,t)}q^{\pm}|_{(s^{\pm},t)}\theta_0''(\rho_\varepsilon)|_{(.,t)} - \frac{1}{\varepsilon}f'''(\theta_0(\rho_\varepsilon))u_1^{C\pm}v_\varepsilon^I|_{(.,t)}\right| \leq Ce^{-c|\rho_\varepsilon(.,t)|},$$

$$\left|\sqrt{\varepsilon}\mathcal{L}_{\varepsilon,t}^{\pm}\phi_\varepsilon^A(.,t) + \frac{1}{\varepsilon}\Delta r|_{\overline{X}_0(s,t)}q^{\pm}|_{(s^{\pm},t)}\theta_0''(\rho_\varepsilon)|_{(.,t)}\right| \leq Ce^{-c|\rho_\varepsilon(.,t)|} + Ce^{-c(|\rho_\varepsilon(.,t)|+H_\varepsilon^{\pm}(.,t))}.$$

This shows the first estimate. The second one also directly follows from a rigorous Taylor expansion. On $\Omega_t^{C\pm}\setminus\Gamma_t(\frac{\delta}{2})$ we know that $\phi_\varepsilon^A(.,t)$ together with derivatives up to second order are $\mathcal{O}(e^{-c/\varepsilon})$. This proves the claim. □

6.2.3 Notation for Transformations and the Splitting

We introduce the notation

$$X^{\pm} : [-\delta,\delta]\times[0,2\mu_0]\times[0,T] \to \bigcup_{t\in[0,T]}\overline{\Omega_t^{C\pm}} : (r,\sigma,t)\mapsto X(r,\pm1\mp\sigma,t)$$

and $\overline{X}^{\pm} := (X^{\pm},\mathrm{pr}_t)$. Analogously we define $X_0^{\pm}$ and $\overline{X}_0^{\pm}$. Moreover, for $t\in[0,T]$ let

$$\begin{aligned}
J_t^{\pm}(r,\sigma) &:= J_t(r,\pm1\mp\sigma) &&\text{for } (r,\sigma)\in[-\delta,\delta]\times[0,2\mu_0],\\
h_j^{\pm}(\sigma,t) &:= h_j(\pm1\mp\sigma,t) &&\text{for } \sigma\in[0,2\mu_0], j=1,2
\end{aligned}$$

and $h_\varepsilon^{\pm} := h_1^{\pm} + \varepsilon h_2^{\pm}$. For transformation arguments we set

$$F_{\varepsilon,\sigma,t}^{\pm}(z) := \varepsilon(z + h_\varepsilon^{\pm}(\sigma,t)) \quad\text{and}\quad \tilde{J}_{\varepsilon,\sigma,t}^{\pm}(z) := J_t^{\pm}(F_{\varepsilon,\sigma,t}^{\pm}(z),\sigma)$$

for $z\in[-\frac{\delta}{\varepsilon},\frac{\delta}{\varepsilon}] - h_\varepsilon^{\pm}(\sigma,t)$ and $\sigma\in[0,2\mu_0], t\in[0,T]$.

Now we characterize the splitting of $\tilde{H}^1(\Omega_t^{C\pm})$.

Lemma 6.20. *Let $\tilde{H}^1(\Omega_t^{C\pm})$, $V_{\varepsilon,t}^{\pm}$ and $\tilde{H}^1(0,2\mu_0)$ be as in* (6.17)-(6.19)*. Then*

1. *$V_{\varepsilon,t}^{\pm}$ is a subspace of $\tilde{H}^1(\Omega_t^{C\pm})$ and for $\varepsilon_0>0$ small there are $c_1, C_1>0$ such that*

$$c_1\|a\|_{L^2(0,2\mu_0)} \leq \|\psi\|_{L^2(\Omega_t^{C\pm})} \leq C_1\|a\|_{L^2(0,2\mu_0)}$$

for all $\psi = a(s^{\pm}(.,t))\phi_\varepsilon^A(.,t)\in V_{\varepsilon,t}^{\pm}$ and $\varepsilon\in(0,\varepsilon_0]$, $t\in[0,T]$.

2. *Let $(V_{\varepsilon,t}^{\pm})^{\perp}$ be the L^2-orthogonal complement of $V_{\varepsilon,t}^{\pm}$ in $\tilde{H}^1(\Omega_t^{C\pm})$. Then for $\psi\in\tilde{H}^1(\Omega_t^{C\pm})$:*

$$\psi\in(V_{\varepsilon,t}^{\pm})^{\perp} \quad\Leftrightarrow\quad \int_{-\delta}^{\delta}(\phi_\varepsilon^A(.,t)\psi)|_{X^{\pm}(r,\sigma,t)}J_t^{\pm}(r,\sigma)\,dr = 0 \quad \textit{for a.e. } \sigma\in(0,2\mu_0).$$

Moreover, $\tilde{H}^1(\Omega_t^{C\pm}) = V_{\varepsilon,t}^{\pm}\oplus(V_{\varepsilon,t}^{\pm})^{\perp}$ for all $\varepsilon\in(0,\varepsilon_0]$ and $\varepsilon_0>0$ small.

Proof. Ad 1. Lemma 6.19 yields that $\phi_\varepsilon^A(.,t)$ belongs to $C^2(\overline{\Omega_t^{C\pm}})$ for fixed $t\in[0,T]$. Moreover, it holds $a(s^\pm(.,t))\in H^1(\Omega_t^{C\pm})$ for all $a\in H^1(0,2\mu_0)$ with Lemma 2.10, 3. and Theorem 2.9. Therefore $V_{\varepsilon,t}^\pm$ is a subspace of $\tilde{H}^1(\Omega_t^{C\pm})$ due to the definition of $\tilde{H}^1(0,2\mu_0)$. Now we show the norm equivalence for $\varepsilon\in(0,\varepsilon_0]$ and $\varepsilon_0>0$ small. Therefore let $\psi=a(s^\pm(.,t))\phi_\varepsilon^A(.,t)\in V_{\varepsilon,t}^\pm$. Then the transformation rule and Fubini's Theorem imply

$$\|\psi\|^2_{L^2(\Omega_t^{C\pm})}=\int_0^{2\mu_0}a(\sigma)^2\int_{-\delta}^{\delta}(\phi_\varepsilon^A|_{\overline{X}^\pm(r,\sigma,t)})^2J_t^\pm(r,\sigma)\,dr\,d\sigma. \tag{6.25}$$

We compute the inner integral. The leading order term with respect to ε is $q^\pm(\sigma,t)^2$ times

$$\frac{1}{\varepsilon}\int_{-\delta}^{\delta}\theta_0'\left(\frac{r-\varepsilon h_\varepsilon^\pm(\sigma,t)}{\varepsilon}\right)^2J_t^\pm(r,\sigma)\,dr=\int_{-\frac{\delta}{\varepsilon}-h_\varepsilon^\pm(\sigma,t)}^{\frac{\delta}{\varepsilon}-h_\varepsilon^\pm(\sigma,t)}\theta_0'(z)^2\tilde{J}_{\varepsilon,\sigma,t}^\pm(z)\,dz, \tag{6.26}$$

where we used Lemma 6.5, 1. For $\varepsilon_0=\varepsilon_0(\overline{C}_0)>0$ small it holds $|\varepsilon h_\varepsilon^\pm|\leq\frac{\delta}{2}$. Moreover, there are constants $c,C>0$ such that $c\leq J,q^\pm\leq C$. Therefore the integral in (6.26) can be estimated from above and below by constants $\tilde{c},\tilde{C}>0$ independent of $t\in[0,T]$, $\varepsilon\in(0,\varepsilon_0]$. For the remainder in the inner integral in (6.25) we use Lemma 6.5 and obtain an estimate of the absolute value to $C\varepsilon$. For $\varepsilon_0>0$ small this shows the claim. $\square_{1.}$

Ad 2. Let $t\in[0,T]$ be fixed. By definition it holds

$$(V_{\varepsilon,t}^\pm)^\perp=\left\{\psi\in\tilde{H}^1(\Omega_t^{C\pm}):\int_{\Omega_t^{C\pm}}\psi a(s^\pm(.,t))\phi_\varepsilon^A(.,t)\,dx=0\text{ for all }a\in\tilde{H}^1(0,2\mu_0)\right\}.$$

The integral equals $\int_0^{2\mu_0}a(\sigma)\int_{-\delta}^{\delta}(\phi_\varepsilon^A(.,t)\psi)|_{X^\pm(r,\sigma,t)}J_t^\pm(r,\sigma)\,dr\,d\sigma$. Hence the Fundamental Theorem of Calculus of Variations yields the characterization. Moreover, by definition it holds $V_{\varepsilon,t}^\pm\cap(V_{\varepsilon,t}^\pm)^\perp=\{0\}$. It remains to show $V_{\varepsilon,t}^\pm+(V_{\varepsilon,t}^\pm)^\perp=\tilde{H}^1(\Omega_t^{C\pm})$. To this end we set

$$w_\varepsilon:[0,2\mu_0]\to\mathbb{R}:\sigma\mapsto\int_{-\delta}^{\delta}(\phi_\varepsilon^A|_{\overline{X}^\pm(r,\sigma,t)})^2J_t^\pm(r,\sigma)\,dr.$$

It holds $w_\varepsilon\in C^1([0,2\mu_0])$ and $w_\varepsilon\geq c>0$ for small ε due to the proof of the first part. Now let $\psi\in\tilde{H}^1(\Omega_t^{C\pm})$ be arbitrary. Then we define

$$a_\varepsilon:[0,2\mu_0]\to\mathbb{R}:\sigma\mapsto\frac{1}{w_\varepsilon(\sigma)}\int_{-\delta}^{\delta}(\phi_\varepsilon^A(.,t)\psi)|_{X^\pm(r,\sigma,t)}J_t^\pm(r,\sigma)\,dr.$$

Because of Lemma 2.10, 2. and since integration gives a bounded linear functional on $L^2(-\delta,\delta)$, it follows that $a_\varepsilon\in\tilde{H}^1(0,2\mu_0)$. For $\psi_\varepsilon^\perp:=\psi-a_\varepsilon(s^\pm(.,t))\phi_\varepsilon^A(.,t)\in\tilde{H}^1(\Omega_t^{C\pm})$ we have

$$\int_{-\delta}^{\delta}(\phi_\varepsilon^A(.,t)\psi_\varepsilon^\perp)|_{X^\pm(r,\sigma,t)}J_t^\pm(r,\sigma)\,dr=a_\varepsilon(\sigma)w_\varepsilon(\sigma)-a_\varepsilon(\sigma)w_\varepsilon(\sigma)=0$$

for a.e. $\sigma\in(0,2\mu_0)$. The integral characterization above shows $\psi_\varepsilon^\perp\in(V_{\varepsilon,t}^\pm)^\perp$. $\square_{2.}$

6.2.4 Analysis of the Bilinear Form

First we consider $B^{\pm}_{\varepsilon,t}$ on $V^{\pm}_{\varepsilon,t} \times V^{\pm}_{\varepsilon,t}$.

Lemma 6.21. *There are $\varepsilon_0, C, c > 0$ such that*

$$B^{\pm}_{\varepsilon,t}(\phi,\phi) \geq -C\|\phi\|^2_{L^2(\Omega^{C\pm}_t)} + c\|a\|^2_{H^1(0,2\mu_0)}$$

for all $\phi = a(s^{\pm}(.,t))\phi^A_\varepsilon(.,t) \in V^{\pm}_{\varepsilon,t}$ and $\varepsilon \in (0,\varepsilon_0], t \in [0,T]$.

Proof. Let ϕ be as in the lemma. We rewrite $B^{\pm}_{\varepsilon,t}(\phi,\phi)$ in order to use Lemma 6.19. To this end we compute $\nabla\phi = \nabla(a|_{s^{\pm}(.,t)})\phi^A_\varepsilon(.,t) + a|_{s^{\pm}(.,t)}\nabla\phi^A_\varepsilon(.,t)$ and

$$|\nabla\phi|^2 = |\nabla(a(s^{\pm}))\phi^A_\varepsilon|^2|_{(.,t)} + a^2(s^{\pm})|\nabla\phi^A_\varepsilon|^2|_{(.,t)} + \nabla(a^2(s^{\pm}))\cdot\nabla\phi^A_\varepsilon\phi^A_\varepsilon|_{(.,t)}.$$

Integration by parts shows

$$\begin{aligned}\int_{\Omega^{C\pm}_t}\left[\nabla(a^2(s^{\pm}))\cdot\nabla\phi^A_\varepsilon\phi^A_\varepsilon\right]|_{(.,t)}\,dx &= -\int_{\Omega^{C\pm}_t}\left[a^2(s^{\pm})(\Delta\phi^A_\varepsilon\,\phi^A_\varepsilon + |\nabla\phi^A_\varepsilon|^2)\right]|_{(.,t)}\,dx\\ &\quad + \int_{\partial\Omega^{C\pm}_t}\left[N_{\partial\Omega^{C\pm}_t}\cdot\nabla\phi^A_\varepsilon\,\mathrm{tr}(a^2(s^{\pm})\phi^A_\varepsilon|_{(.,t)})\right]d\mathcal{H}^1.\end{aligned}$$

Therefore we obtain

$$\begin{aligned}B^{\pm}_{\varepsilon,t}(\phi,\phi) &= \int_{\Omega^{C\pm}_t}|\nabla(a(s^{\pm}))\phi^A_\varepsilon|^2|_{(.,t)}\,dx + \int_{\Omega^{C\pm}_t}(a^2(s^{\pm})\phi^A_\varepsilon)|_{(.,t)}\mathcal{L}^{\pm}_{\varepsilon,t}\phi^A_\varepsilon|_{(.,t)}\,dx\\ &\quad + \int_{\partial\Omega^{C\pm}_t}\left[N_{\partial\Omega^{C\pm}_t}\cdot\nabla\phi^A_\varepsilon\,\mathrm{tr}(a^2(s^{\pm})\phi^A_\varepsilon|_{(.,t)})\right]d\mathcal{H}^1 =: (I) + (II) + (III).\end{aligned}$$

Ad (I). It holds $|\nabla(a(s^{\pm}(.,t)))|^2 = \left[|\nabla s|^2(a')^2(s^{\pm})\right]|_{(.,t)}$ and therefore

$$(I) = \int_0^{2\mu_0}(a')^2(\sigma)\int_{-\delta}^{\delta}\left[|\nabla s|^2(\phi^A_\varepsilon)^2\right]|_{\overline{X}^{\pm}(r,\sigma,t)}J^{\pm}_t(r,\sigma)\,dr\,d\sigma.$$

Since $|\nabla s|, J^{\pm}_t \geq c > 0$, we obtain as in the proof of Lemma 6.20, 1. that $(I) \geq c_0\|a'\|^2_{L^2(0,2\mu_0)}$ for a $c_0 > 0$ independent of $\phi \in V^{\pm}_{\varepsilon,t}$ and all $\varepsilon \in (0,\varepsilon_0], t \in [0,T]$, if $\varepsilon_0 > 0$ is small.

Ad (II). We write

$$(II) = \int_0^{2\mu_0}a^2(\sigma)\int_{-\delta}^{\delta}\phi^A_\varepsilon|_{\overline{X}^{\pm}(r,\sigma,t)}(\mathcal{L}^{\pm}_{\varepsilon,t}\phi^A_\varepsilon(.,t))|_{X^{\pm}(r,\sigma,t)}J^{\pm}_t(r,\sigma)\,dr\,d\sigma$$

and estimate the inner integral. Lemma 6.19 implies

$$\left|\sqrt{\varepsilon}\mathcal{L}^{\pm}_{\varepsilon,t}\phi^A_\varepsilon(.,t) + \frac{1}{\varepsilon}\Delta r|_{\overline{X}_0(s(.,t),t)}q^{\pm}(s^{\pm}(.,t),t)\theta_0''(\rho_\varepsilon(.,t))\right| \leq Ce^{-c|\rho_\varepsilon(.,t)|} \quad \text{in } \Omega^{C\pm}_t.$$

Therefore the most delicate term with respect to the ε-order in the inner integral in (II) is

$$\frac{1}{\varepsilon^2}\Delta r|_{\overline{X}^{\pm}_0(\sigma,t)}q^{\pm}(\sigma,t)^2\int_{-\delta}^{\delta}\theta_0''\theta_0'(\rho_\varepsilon|_{\overline{X}^{\pm}(r,\sigma,t)})J^{\pm}_t(r,\sigma)\,dr. \tag{6.27}$$

First we consider (6.27) with $J_t^\pm(0,\sigma)$ instead of $J_t^\pm(r,\sigma)$. Using Lemma 6.5, 1. and $\int_{\mathbb{R}} \theta_0''\theta_0' = 0$ we obtain that the modulus of the latter is estimated by

$$\frac{C}{\varepsilon}\left|\int_{(-\frac{\delta}{\varepsilon},\frac{\delta}{\varepsilon})-h_\varepsilon^\pm(\sigma,t)} \theta_0''\theta_0'\,dz\right| = \frac{C}{\varepsilon}\left|\int_{\mathbb{R}\setminus[(-\frac{\delta}{\varepsilon},\frac{\delta}{\varepsilon})-h_\varepsilon^\pm(\sigma,t)]} \theta_0''\theta_0'\,dz\right|.$$

Due to the exponential decay of θ_0', θ_0'' from Theorem 4.1 this can be estimated by $Ce^{-c/\varepsilon}$ for some constants $C, c > 0$ independent of σ, t and $\varepsilon \in (0,\varepsilon_0]$ if $\varepsilon_0 > 0$ is small. Moreover, it holds $|J_t^\pm(r,\sigma) - J_t^\pm(0,\sigma)| \leq \tilde{C}|r|$ with $\tilde{C} > 0$ independent of (r,σ,t). Hence the remaining terms in (6.27) and (II) can be controlled with Lemma 6.5. Altogether we obtain $|(II)| \leq C\|a\|^2_{L^2(0,2\mu_0)}$ with $C > 0$ independent of $\phi \in V_{\varepsilon,t}^\pm$ and all $\varepsilon \in (0,\varepsilon_0]$, $t \in [0,T]$ if $\varepsilon_0 > 0$ is small.

Ad (III). Using the well-known representation for the integral over curves and properties of the trace-operator we obtain

$$\begin{aligned}(III) = \sum_\pm \int_0^{2\mu_0} a^2(\sigma)\left[\phi_\varepsilon^A N_{\partial\Omega_t^{C\pm}} \cdot \nabla\phi_\varepsilon^A\right]|_{\overline{X}^\pm(\pm\delta,\sigma,t)} |\partial_\sigma X^\pm(\pm\delta,\sigma,t)|\,d\sigma \\ + a^2(0)\int_{-\delta}^{\delta}\left[\phi_\varepsilon^A N_{\partial\Omega_t^{C\pm}} \cdot \nabla\phi_\varepsilon^A\right]|_{\overline{X}^\pm(r,0,t)} |\partial_r X^\pm(r,0,t)|\,dr.\end{aligned}$$

With Lemma 6.19 and for the last integral Lemma 6.5 it follows that

$$|(III)| \leq Ce^{-c/\varepsilon}\|a\|^2_{L^2(0,2\mu_0)} + C\varepsilon a^2(0).$$

We estimate $a^2(0)$ via $H^1(0,2\mu_0) \hookrightarrow C_b^0([0,2\mu_0])$. Then Lemma 6.20, 1. yields the claim. □

Next we analyze $B_{\varepsilon,t}^\pm$ on $(V_{\varepsilon,t}^\pm)^\perp \times (V_{\varepsilon,t}^\pm)^\perp$.

Lemma 6.22. *There are* $\nu, \varepsilon_0 > 0$ *such that*

$$B_{\varepsilon,t}^\pm(\psi,\psi) \geq \nu\left[\frac{1}{\varepsilon^2}\|\psi\|^2_{L^2(\Omega_t^{C\pm})} + \|\nabla\psi\|^2_{L^2(\Omega_t^{C\pm})}\right]$$

for all $\psi \in (V_{\varepsilon,t}^\pm)^\perp$ *and* $\varepsilon \in (0,\varepsilon_0]$, $t \in [0,T]$.

Proof. It is enough to show the existence of $\tilde{\nu}, \tilde{\varepsilon}_0 > 0$ such that

$$\tilde{B}_{\varepsilon,t}^\pm(\psi,\psi) := \int_{\Omega_t^{C\pm}} |\nabla\psi|^2 + \frac{1}{\varepsilon^2}f''(\theta_0|_{\rho_\varepsilon(.,t)})\psi^2\,dx \geq \frac{\tilde{\nu}}{\varepsilon^2}\|\psi\|^2_{L^2(\Omega_t^{C\pm})} \tag{6.28}$$

for all $\psi \in (V_{\varepsilon,t}^\pm)^\perp$ and $\varepsilon \in (0,\tilde{\varepsilon}_0]$, $t \in [0,T]$. Then the claim follows because for $\beta > 0$ small

$$\begin{aligned}&B_{\varepsilon,t}^\pm(\psi,\psi) \geq \tilde{B}_{\varepsilon,t}^\pm(\psi,\psi) - \frac{C}{\varepsilon}\|\psi\|^2_{L^2(\Omega_t^{C\pm})} = (1-\beta+\beta)\tilde{B}_{\varepsilon,t}^\pm(\psi,\psi) - \frac{C}{\varepsilon}\|\psi\|^2_{L^2(\Omega_t^{C\pm})} \\ &\geq \tfrac{(1-\beta)\tilde{\nu}-C(\beta+\varepsilon)}{\varepsilon^2}\|\psi\|^2_{L^2(\Omega_t^{C\pm})} + \beta\|\nabla\psi\|^2_{L^2(\Omega_t^{C\pm})} \geq \nu\left[\frac{1}{\varepsilon^2}\|\psi\|^2_{L^2(\Omega_t^{C\pm})} + \|\nabla\psi\|^2_{L^2(\Omega_t^{C\pm})}\right]\end{aligned}$$

for all $\psi \in (V_{\varepsilon,t}^\pm)^\perp$ and $\varepsilon \in (0,\varepsilon_0]$, $t \in [0,T]$ provided that $\nu, \varepsilon_0 > 0$ are small.

In the following we prove (6.28) by reducing to Neumann boundary problems in normal direction. This is also the idea for the proof of the spectral estimate on $\Gamma_t(\delta, \mu_0)$, cf. the proof of Theorem 6.16 above. Therefore we define $\tilde{\psi}_t^\pm := \psi|_{X^\pm(.,t)}$ for $\psi \in (V_{\varepsilon,t}^\pm)^\perp$. It holds

$$|\nabla\psi|^2|_{X^\pm(.,t)} = (\nabla_{(r,s)}\tilde{\psi}_t^\pm)^\top \begin{pmatrix} |\nabla r|^2 & \mp\nabla r \cdot \nabla s \\ \mp\nabla r \cdot \nabla s & |\nabla s|^2 \end{pmatrix} |_{\overline{X}^\pm(.,t)} \nabla_{(r,s)}\tilde{\psi}_t^\pm.$$

Hence Theorem 3.3, a Taylor expansion and Young's inequality imply

$$|\nabla\psi|^2|_{X^\pm(.,t)} \geq (1 - Cr^2)(\partial_r\tilde{\psi}_t^\pm)^2 + c(\partial_\sigma\tilde{\psi}_t^\pm)^2 \tag{6.29}$$

for some $c, C > 0$. The second term will not be needed here. To get Cr^2 small enough (which will be precise later), we fix $\tilde{\delta} > 0$ small and estimate separately for r in

$$I_{\sigma,t}^{\pm,\varepsilon} := (-\tilde{\delta}, \tilde{\delta}) + \varepsilon h_\varepsilon^\pm(\sigma, t) \quad \text{and} \quad \hat{I}_{\sigma,t}^{\pm,\varepsilon} := (-\delta, \delta) \setminus I_{\sigma,t}^{\pm,\varepsilon}.$$

If $\varepsilon_0 = \varepsilon_0(\tilde{\delta}, \overline{C}_0) > 0$ is small, then for all $\varepsilon \in (0, \varepsilon_0]$ and $\sigma \in [0, 2\mu_0]$, $t \in [0, T]$ it holds

$$f''(\theta_0(\rho_\varepsilon|_{\overline{X}^\pm(r,\sigma,t)})) \geq c_0 > 0 \quad \text{for } r \in \hat{I}_{\sigma,t}^{\pm,\varepsilon}, \quad |r| \leq \tilde{\delta} + \varepsilon|h_\varepsilon^\pm(\sigma, t)| \leq 2\tilde{\delta} \quad \text{for } r \in I_{\sigma,t}^{\pm,\varepsilon}.$$

We set $\tilde{c} = \tilde{c}(\tilde{\delta}) := 4C\tilde{\delta}^2$, where C is as in (6.29). Then we obtain for $\varepsilon \in (0, \varepsilon_0]$, $t \in [0, T]$

$$\begin{aligned} \tilde{B}_{\varepsilon,t}^\pm(\psi, \psi) \geq &\int_0^{2\mu_0} \int_{\hat{I}_{\sigma,t}^{\pm,\varepsilon}} \frac{c_0}{\varepsilon^2}(\tilde{\psi}_t^\pm)^2 J_t^\pm|_{(r,\sigma)}\, dr\, d\sigma \\ &+ \int_0^{2\mu_0} \int_{I_{\sigma,t}^{\pm,\varepsilon}} \left[(1 - \tilde{c})(\partial_r\tilde{\psi}_t^\pm)^2 + \frac{1}{\varepsilon^2} f''(\theta_0(\rho_\varepsilon|_{\overline{X}^\pm(.,t)}))(\tilde{\psi}_t^\pm)^2\right] J_t^\pm|_{(r,\sigma)}\, dr\, d\sigma. \end{aligned}$$

Lemma 6.5, 1. yields that the second inner integral equals $1/\varepsilon^2$ times

$$B_{\varepsilon,\sigma,t}^{\pm,\tilde{c}}(\Psi_{\varepsilon,\sigma,t}^\pm, \Psi_{\varepsilon,\sigma,t}^\pm) := \int_{I_{\varepsilon,\tilde{\delta}}} \left[(1 - \tilde{c})(\frac{d}{dz}\Psi_{\varepsilon,\sigma,t}^\pm)^2 + f''(\theta_0(z))(\Psi_{\varepsilon,\sigma,t}^\pm)^2\right] \tilde{J}_{\varepsilon,\sigma,t}^\pm\, dz, \tag{6.30}$$

where we used the notation from the beginning of Section 6.2.3 and we set $I_{\varepsilon,\tilde{\delta}} := (-\frac{\tilde{\delta}}{\varepsilon}, \frac{\tilde{\delta}}{\varepsilon})$ and $\Psi_{\varepsilon,\sigma,t}^\pm := \sqrt{\varepsilon}\tilde{\psi}_t^\pm(F_{\varepsilon,\sigma,t}^\pm(.), \sigma)$. Therefore (6.28) follows if we show with the same c_0 as above

$$B_{\varepsilon,\sigma,t}^{\pm,\tilde{c}}(\Psi_{\varepsilon,\sigma,t}^\pm, \Psi_{\varepsilon,\sigma,t}^\pm) \geq \overline{c}\|\Psi_{\varepsilon,\sigma,t}^\pm\|_{L^2(I_{\varepsilon,\tilde{\delta}}, \tilde{J}_{\varepsilon,\sigma,t}^\pm)}^2 - \frac{c_0}{2}\|\tilde{\psi}_t^\pm(.,\sigma)\|_{L^2(\hat{I}_{\sigma,t}^{\pm,\varepsilon}, J_t^\pm(.,\sigma))}^2 \tag{6.31}$$

for $\varepsilon \in (0, \varepsilon_0]$, a.e. $\sigma \in [0, 2\mu_0]$ and all $t \in [0, T]$ with some $\varepsilon_0, \overline{c} > 0$ independent of ε, σ, t.

Here $L^2(I_{\varepsilon,\tilde{\delta}}, \tilde{J}_{\varepsilon,\sigma,t}^\pm)$ is the space of L^2-functions on $I_{\varepsilon,\tilde{\delta}}$ with respect to the weight $\tilde{J}_{\varepsilon,\sigma,t}^\pm$. For simplicity we denote the scalar-product in $L^2(I_{\varepsilon,\tilde{\delta}}, \tilde{J}_{\varepsilon,\sigma,t}^\pm)$ by $(.,.)_{\varepsilon,\sigma,t}$ and the norm with $\|.\|_{\varepsilon,\sigma,t}$. For the proof of (6.31) we need results for $B_{\varepsilon,\sigma,t}^{\pm,0}$. The latter is defined according to (6.30) for 0 instead of $\tilde{c}$. With respect to $(.,.)_{\varepsilon,\sigma,t}$, $B_{\varepsilon,\sigma,t}^{\pm,0}$ is the bilinear form corresponding to

$$\mathcal{L}_{\varepsilon,\sigma,t}^{\pm,0} := -(\tilde{J}_{\varepsilon,\sigma,t}^\pm)^{-1}\frac{d}{dz}\left(\tilde{J}_{\varepsilon,\sigma,t}^\pm\frac{d}{dz}\right) + f''(\theta_0)$$

on $H^2(I_{\varepsilon,\tilde{\delta}})$ with homogeneous Neumann boundary condition. For the latter the results from Section 6.1.3.2 are applicable.

Proof of (6.31). The integral characterization for $\psi \in (V_{\varepsilon,t}^{\pm})^{\perp}$ in Lemma 6.20, 2. yields

$$\left|\int_{I_{\sigma,t}^{\pm,\varepsilon}} (\phi_\varepsilon^A(.,t)\psi)|_{X^{\pm}(r,\sigma,t)} J_t^{\pm}(r,\sigma)\, dr\right| \leq C(\tilde{\delta}) e^{-c\tilde{\delta}/\varepsilon} \|\tilde{\psi}_t^{\pm}(.,\sigma)\|_{L^2(\hat{I}_{\sigma,t}^{\pm,\varepsilon}, J_t^{\pm}(.,\sigma))}$$

for ε small, a.e. $\sigma \in [0, 2\mu_0]$ and all $t \in [0,T]$. The lowest order term in the integral on the left is

$$\frac{1}{\sqrt{\varepsilon}} q^{\pm}(\sigma,t) \int_{I_{\sigma,t}^{\pm,\varepsilon}} \theta_0'(\rho_\varepsilon|_{\overline{X}^{\pm}(r,\sigma,t)})(\tilde{\psi}_t^{\pm} J_t^{\pm})(r,\sigma)\, dr = q^{\pm}(\sigma,t)(\Psi_{\varepsilon,\sigma,t}^{\pm}, \theta_0')_{\varepsilon,\sigma,t}.$$

The remaining term in the integral due to ϕ_ε^A can be estimated with Hölder's inequality, the decay of $\hat{v}^{C\pm}$ and Lemma 6.5. Hence we obtain because of $0 < c \leq q^{\pm} \leq C$ that for ε small

$$|(\Psi_{\varepsilon,\sigma,t}^{\pm}, \theta_0')_{\varepsilon,\sigma,t}| \leq C\varepsilon \|\Psi_{\varepsilon,\sigma,t}^{\pm}\|_{\varepsilon,\sigma,t} + C(\tilde{\delta}) e^{-c\tilde{\delta}/\varepsilon} \|\tilde{\psi}_t^{\pm}(.,\sigma)\|_{L^2(\hat{I}_{\sigma,t}^{\pm,\varepsilon}, J_t^{\pm}(.,\sigma))}.$$

Therefore Theorem 6.8, 2. and uniform bounds for $q^{\pm}$, $J_t^{\pm}$ imply for the positive normalized eigenfunction $\Psi_{\varepsilon,\sigma,t}^{\pm,1}$ to the first eigenvalue $\lambda_{\varepsilon,\sigma,t}^{\pm,1}$ of $\mathcal{L}_{\varepsilon,\sigma,t}^{\pm,0}$ that

$$|(\Psi_{\varepsilon,\sigma,t}^{\pm}, \Psi_{\varepsilon,\sigma,t}^{\pm,1})_{\varepsilon,\sigma,t}| \leq C(\tilde{\delta})\varepsilon \|\Psi_{\varepsilon,\sigma,t}^{\pm}\|_{\varepsilon,\sigma,t} + C(\tilde{\delta}) e^{-c\tilde{\delta}/\varepsilon} \|\tilde{\psi}_t^{\pm}(.,\sigma)\|_{L^2(\hat{I}_{\sigma,t}^{\pm,\varepsilon}, J_t^{\pm}(.,\sigma))} \tag{6.32}$$

for a.e. $\sigma \in [0, 2\mu_0]$, all $t \in [0,T]$ and $\varepsilon \in (0, \varepsilon_0]$, if $\varepsilon_0 > 0$ is small.

Now we decompose $\Psi_{\varepsilon,\sigma,t}^{\pm}$ orthogonally in $L^2(I_{\varepsilon,\tilde{\delta}}, \tilde{J}_{\varepsilon,\sigma,t}^{\pm})$. With some $\Psi_{\varepsilon,\sigma,t}^{\pm,\perp} \in H^1(I_{\varepsilon,\tilde{\delta}})$ we have

$$\Psi_{\varepsilon,\sigma,t}^{\pm} = (\Psi_{\varepsilon,\sigma,t}^{\pm}, \Psi_{\varepsilon,\sigma,t}^{\pm,1})_{\varepsilon,\sigma,t} \Psi_{\varepsilon,\sigma,t}^{\pm,1} + \Psi_{\varepsilon,\sigma,t}^{\pm,\perp}.$$

Taking $\|.\|_{\varepsilon,\sigma,t}^2$ in this identity yields $\|\Psi_{\varepsilon,\sigma,t}^{\pm,\perp}\|_{\varepsilon,\sigma,t}^2 = \|\Psi_{\varepsilon,\sigma,t}^{\pm}\|_{\varepsilon,\sigma,t}^2 - |(\Psi_{\varepsilon,\sigma,t}^{\pm}, \Psi_{\varepsilon,\sigma,t}^{\pm,1})_{\varepsilon,\sigma,t}|^2$. Then

$$B_{\varepsilon,\sigma,t}^{\pm,\tilde{c}}(\Psi_{\varepsilon,\sigma,t}^{\pm}, \Psi_{\varepsilon,\sigma,t}^{\pm}) = (1-\tilde{c}) B_{\varepsilon,\sigma,t}^{\pm,0}(\Psi_{\varepsilon,\sigma,t}^{\pm}, \Psi_{\varepsilon,\sigma,t}^{\pm}) + \tilde{c} \int_{I_{\varepsilon,\tilde{\delta}}} f''(\theta_0)(\Psi_{\varepsilon,\sigma,t}^{\pm})^2 \tilde{J}_{\varepsilon,\sigma,t}^{\pm}\, dz.$$

The last part is estimated by $\tilde{c}\,\sup_{z\in\mathbb{R}} |f''(\theta_0(z))|\, \|\Psi_{\varepsilon,\sigma,t}^{\pm}\|_{\varepsilon,\sigma,t}^2$. Moreover,

$$B_{\varepsilon,\sigma,t}^{\pm,0}(\Psi_{\varepsilon,\sigma,t}^{\pm}, \Psi_{\varepsilon,\sigma,t}^{\pm}) = \lambda_{\varepsilon,\sigma,t}^{\pm,1} |(\Psi_{\varepsilon,\sigma,t}^{\pm}, \Psi_{\varepsilon,\sigma,t}^{\pm,1})_{\varepsilon,\sigma,t}|^2 + B_{\varepsilon,\sigma,t}^{\pm,0}(\Psi_{\varepsilon,\sigma,t}^{\pm,\perp}, \Psi_{\varepsilon,\sigma,t}^{\pm,\perp}),$$

where we used $B_{\varepsilon,\sigma,t}^{\pm,0}(\Psi_{\varepsilon,\sigma,t}^{\pm,1}, \Psi_{\varepsilon,\sigma,t}^{\pm,\perp}) = 0$ due to integration by parts. Theorem 6.8, 1. and 3. imply $\lambda_{\varepsilon,\sigma,t}^{\pm,1} = \mathcal{O}(\varepsilon^2)$ and $B_{\varepsilon,\sigma,t}^{\pm,0}(\Psi_{\varepsilon,\sigma,t}^{\pm,\perp}, \Psi_{\varepsilon,\sigma,t}^{\pm,\perp}) \geq \nu_2 \|\Psi_{\varepsilon,\sigma,t}^{\pm,\perp}\|_{\varepsilon,\sigma,t}^2$. Altogether we obtain if $\tilde{c}(\tilde{\delta}) \leq 1$:

$$\begin{aligned} B_{\varepsilon,\sigma,t}^{\pm,\tilde{c}}(\Psi_{\varepsilon,\sigma,t}^{\pm}, \Psi_{\varepsilon,\sigma,t}^{\pm}) \geq \|\Psi_{\varepsilon,\sigma,t}^{\pm}\|_{\varepsilon,\sigma,t}^2 & \left[\nu_2(1-\tilde{c}(\tilde{\delta})) - \tilde{c}(\tilde{\delta}) \sup_{z\in\mathbb{R}} |f''(\theta_0(z))|\right] \\ & -(1-\tilde{c}(\tilde{\delta}))(\mathcal{O}(\varepsilon^2)+\nu_2) \left|(\Psi_{\varepsilon,\sigma,t}^{\pm}, \Psi_{\varepsilon,\sigma,t}^{\pm,1})_{\varepsilon,\sigma,t}\right|^2 \end{aligned}$$

for a.e. $\sigma \in [0, 2\mu_0]$, all $t \in [0,T]$ and $\varepsilon \in (0, \varepsilon_0]$ if $\varepsilon_0 = \varepsilon_0(\tilde{\delta}, \overline{C}_0) > 0$ is small. If $\tilde{\delta} > 0$ and therefore $\tilde{c} = \tilde{c}(\tilde{\delta}) = 4C\tilde{\delta}^2$ was chosen small enough before, the term in the square brackets is estimated from below by $\nu_2/2$. Here it is important that ν_2 in Theorem 6.8 is independent of $\tilde{\delta}$. Together with (6.32) this shows (6.31) for small $\varepsilon_0 > 0$ and thus Lemma 6.22. $\square$

For $B_{\varepsilon,t}^{\pm}$ on $V_{\varepsilon,t}^{\pm} \times (V_{\varepsilon,t}^{\pm})^{\perp}$ we have

Lemma 6.23. *There are $\varepsilon_0, C > 0$ such that*

$$|B^{\pm}_{\varepsilon,t}(\phi,\psi)| \leq \frac{C}{\varepsilon}\|\phi\|_{L^2(\Omega_t^{C\pm})}\|\psi\|_{L^2(\Omega_t^{C\pm})} + \frac{1}{4}B^{\pm}_{\varepsilon,t}(\psi,\psi) + C\varepsilon^2\|a\|^2_{H^1(0,2\mu_0)}$$

for all $\phi = a(s^{\pm}(.,t))\phi^A_\varepsilon(.,t) \in V^{\pm}_{\varepsilon,t}$, $\psi \in (V^{\pm}_{\varepsilon,t})^\perp$ and $\varepsilon \in (0,\varepsilon_0]$, $t \in [0,T]$.

For the proof we need an auxiliary estimate.

Lemma 6.24. *Let $\overline{\varepsilon} > 0$ be fixed. Then there is a $\overline{C} > 0$ (independent of ψ, ε, t) such that*

$$\|\mathrm{tr}\,\psi\|^2_{L^2(\partial\Omega_t^{C\pm})} \leq \overline{C}\left[\varepsilon\|\nabla\psi\|^2_{L^2(\Omega_t^{C\pm})} + \frac{1}{\varepsilon}\|\psi\|^2_{L^2(\Omega_t^{C\pm})}\right]$$

for all $\psi \in H^1(\Omega_t^{C\pm})$ and $\varepsilon \in (0,\overline{\varepsilon}]$, $t \in [0,T]$.

Proof. It is enough to show the estimate for $S := (-\delta,\delta)\times(0,2\mu_0)$ instead of $\Omega_t^{C\pm}$ since all the appearing terms are equivalent to the transformed ones under $X^{\pm}(.,t)$ uniformly in $t \in [0,T]$. For the S-case we use the idea from Evans [Ev], 5.10, problem 7. Let $\vec{w} \in C^1(\overline{S})^2$ with $\vec{w}|_{\partial S}\cdot N_{\partial S} \geq 1$. One can directly construct such a $\vec{w}$. Then the Gauß Theorem yields

$$\|\mathrm{tr}\,\psi\|^2_{L^2(\partial S)} \leq \int_{\partial S}\psi^2\vec{w}\cdot N_{\partial S}\,d\mathcal{H}^1 = \int_S \mathrm{div}(\psi^2\vec{w})\,dx = \int_S \psi^2\mathrm{div}\,\vec{w} + 2\psi\vec{w}\cdot\nabla\psi\,dx.$$

Therefore

$$\|\mathrm{tr}\,\psi\|^2_{L^2(\partial S)} \leq C\left[\|\psi\|^2_{L^2(S)} + \varepsilon\|\nabla\psi\|^2_{L^2(S)} + \frac{1}{\varepsilon}\|\psi\|^2_{L^2(S)}\right]$$

by the Young Inequality. Hence the claim follows due to $1 \leq \overline{\varepsilon}/\varepsilon$. □

Proof of Lemma 6.23. We rewrite $B^{\pm}_{\varepsilon,t}(\phi,\psi)$ in order to use Lemma 6.19 and Lemma 6.20. It holds $\nabla\phi = \nabla(a(s^{\pm}(.,t)))\phi^A_\varepsilon + a(s^{\pm}(.,t))\nabla\phi^A_\varepsilon(.,t)$ and integration by parts yields

$$\int_{\Omega_t^{C\pm}} a(s^{\pm})\nabla\phi^A_\varepsilon|_{(.,t)}\cdot\nabla\psi\,dx = -\int_{\Omega_t^{C\pm}}\left[\nabla(a(s^{\pm}))\cdot\nabla\phi^A_\varepsilon + a(s^{\pm})\Delta\phi^A_\varepsilon|_{(.,t)}\right]\psi\,dx$$
$$+\int_{\partial\Omega_t^{C\pm}} N_{\partial\Omega_t^{C\pm}}\cdot\nabla\phi^A_\varepsilon|_{(.,t)}\mathrm{tr}\left[a(s^{\pm}(.,t))\psi\right]\,d\mathcal{H}^1.$$

Therefore we obtain

$$B^{\pm}_{\varepsilon,t}(\phi,\psi) = \int_{\Omega_t^{C\pm}} a(s^{\pm})|_{(.,t)}\psi\mathcal{L}^{\pm}_{\varepsilon,t}\phi^A_\varepsilon|_{(.,t)}\,dx + \int_{\partial\Omega_t^{C\pm}} N_{\partial\Omega_t^{C\pm}}\cdot\nabla\phi^A_\varepsilon|_{(.,t)}\mathrm{tr}\left[a(s^{\pm}(.,t))\psi\right]\,d\mathcal{H}^1$$
$$+\int_{\Omega_t^{C\pm}}\nabla(a(s^{\pm}))|_{(.,t)}\cdot\left[\phi^A_\varepsilon|_{(.,t)}\nabla\psi - \nabla\phi^A_\varepsilon|_{(.,t)}\psi\right]\,dx =: (I) + (II) + (III).$$

Ad (I). The Hölder Inequality yields $|(I)| \leq \|a(s^{\pm}|_{(.,t)})\mathcal{L}^{\pm}_{\varepsilon,t}\phi^A_\varepsilon(.,t)\|_{L^2(\Omega_t^{C\pm})}\|\psi\|_{L^2(\Omega_t^{C\pm})}$, where

$$\|a(s^{\pm}|_{(.,t)})\mathcal{L}^{\pm}_{\varepsilon,t}\phi^A_\varepsilon|_{(.,t)}\|^2_{L^2(\Omega_t^{C\pm})} = \int_0^{2\mu_0} a^2(\sigma)\int_{-\delta}^{\delta}(\mathcal{L}^{\pm}_{\varepsilon,t}\phi^A_\varepsilon(.,t))^2|_{X^{\pm}(r,\sigma,t)}\,J^{\pm}_t(r,\sigma)\,dr\,d\sigma.$$

Due to Lemma 6.19 and $|d^2-\tilde{d}^2| \leq |d+\tilde{d}|(|d+\tilde{d}|+2|\tilde{d}|)$ for $d,\tilde{d}\in\mathbb{R}$ it holds

$$\left|\varepsilon(\mathcal{L}^{\pm}_{\varepsilon,t}\phi^A_\varepsilon|_{(.,t)})^2 - \frac{1}{\varepsilon^2}\left[\Delta r|_{\overline{X}_0(s(.,t),t)}q^{\pm}|_{(s^{\pm}(.,t),t)}\theta_0''|_{\rho_\varepsilon(.,t)}\right]^2\right|$$
$$\leq Ce^{-c|\rho_\varepsilon(.,t)|}(Ce^{-c|\rho_\varepsilon(.,t)|} + \frac{2}{\varepsilon}|\Delta r||_{\overline{X}_0(s(.,t),t)}q^{\pm}|_{(s^{\pm}(.,t),t)}) \leq \frac{\tilde{C}}{\varepsilon}e^{-c|\rho_\varepsilon(.,t)|}.$$

Hence Lemma 6.5 yields that the inner integral is estimated by C/ε^2 and Lemma 6.20, 1. implies

$$|(I)| \leq \frac{C}{\varepsilon}\|a\|_{L^2(0,2\mu_0)}\|\psi\|_{L^2(\Omega_t^{C\pm})} \leq \frac{\tilde{C}}{\varepsilon}\|\phi\|_{L^2(\Omega_t^{C\pm})}\|\psi\|_{L^2(\Omega_t^{C\pm})}$$

for all $t \in [0,T]$ and $\varepsilon \in (0,\varepsilon_0]$ if $\varepsilon_0 > 0$ is small.

Ad (II). Due to Hölder's inequality we have

$$|(II)| \leq \|\mathrm{tr}\,\psi\|_{L^2(\partial\Omega_t^{C\pm})}\|\mathrm{tr}(a(s^\pm|_{(.,t)}))N_{\partial\Omega_t^{C\pm}} \cdot \nabla\phi_\varepsilon^A|_{(.,t)}\|_{L^2(\partial\Omega_t^{C\pm})}.$$

For the second integral we use the representation of integrals over curves, cf. also the estimate of (III) in the proof of Lemma 6.21. Then Lemma 6.19 and Lemma 6.5 imply

$$\|a|_{s^\pm(.,t)}N_{\partial\Omega_t^{C\pm}} \cdot \nabla\phi_\varepsilon^A|_{(.,t)}\|_{L^2(\partial\Omega_t^{C\pm})} \leq C\varepsilon|a(0)| + Ce^{-c/\varepsilon}\|a\|_{L^2(0,2\mu_0)} \leq C\varepsilon\|a\|_{H^1(0,2\mu_0)}.$$

To estimate $\|\mathrm{tr}\,\psi\|_{L^2(\partial\Omega_t^{C\pm})}$ we use Lemma 6.24. Then Young's inequality and Lemma 6.22 yield

$$|(II)| \leq \frac{\nu}{8\varepsilon\overline{C}}\|\mathrm{tr}\,\psi\|^2_{L^2(\partial\Omega_t^{C\pm})} + \tilde{C}\varepsilon^3\|a\|^2_{H^1(0,2\mu_0)} \leq \frac{1}{8}B^\pm_{\varepsilon,t}(\psi,\psi) + \tilde{C}\varepsilon^3\|a\|^2_{H^1(0,2\mu_0)},$$

where $\overline{C}$ is as in Lemma 6.24.

Ad (III). It holds $(III) = \int_0^{2\mu_0} a'(\sigma)g_t^\pm(\sigma)\,d\sigma$ with

$$g_t^\pm(\sigma) := \mp\int_{-\delta}^{\delta}\nabla s|_{\overline{X}^\pm(r,\sigma,t)} \cdot \left[\phi_\varepsilon^A(.,t)\nabla\psi - \nabla\phi_\varepsilon^A(.,t)\psi\right]|_{X^\pm(r,\sigma,t)}\,J_t^\pm(r,\sigma)\,dr.$$

We substitute $\nabla\psi|_{X^\pm(.,t)} = \nabla r|_{\overline{X}^\pm(.,t)}\partial_r\tilde{\psi}_t^\pm \mp \nabla s|_{\overline{X}^\pm(.,t)}\partial_\sigma\tilde{\psi}_t^\pm$ with $\tilde{\psi}_t^\pm := \psi|_{X^\pm(.,t)}$. In order to control the $\partial_\sigma\tilde{\psi}_t^\pm$-term in $g_t^\pm$ we use

$$\left||\nabla s|^2|_{\overline{X}^\pm(r,\sigma,t)} - |\nabla s|^2|_{\overline{X}_0^\pm(\sigma,t)}\right| \leq C|r|.$$

Therefore $|g_t^\pm(\sigma)|$ is for a.e. $\sigma \in [0,2\mu_0]$ estimated by

$$\begin{aligned}|\nabla s|^2|_{\overline{X}_0^\pm(\sigma,t)}&\left|\int_{-\delta}^{\delta}\left[\phi_\varepsilon^A|_{\overline{X}^\pm(.,t)}\partial_\sigma\tilde{\psi}_t^\pm J_t^\pm\right]|_{(r,\sigma)}\,dr\right| + \int_{-\delta}^{\delta}\left|(\tilde{\psi}_t^\pm J_t^\pm)|_{(r,\sigma)}\nabla s \cdot \nabla\phi_\varepsilon^A|_{\overline{X}^\pm(r,\sigma,t)}\right|\,dr\\ &+ \int_{-\delta}^{\delta}\left[C\left|r\,\partial_\sigma\tilde{\psi}_t^\pm|_{(r,\sigma)}\right| + \left|\nabla r \cdot \nabla s|_{\overline{X}^\pm(r,\sigma,t)}\partial_r\tilde{\psi}_t^\pm|_{(r,\sigma)}\right|\right] \cdot \left|\phi_\varepsilon^A|_{\overline{X}^\pm(r,\sigma,t)}J_t^\pm|_{(r,\sigma)}\right|\,dr.\end{aligned}$$

We rewrite the first term with the aid of $\psi \in (V_{\varepsilon,t}^\pm)^\perp$. Because of Lemma 2.10 and since integration gives a bounded linear operator on $L^2(-\delta,\delta)$, we can differentiate the identity in Lemma 6.20, 2. and use the product rule. Therefore the first term is estimated by

$$C\left|\int_{-\delta}^{\delta}\left[\left(\partial_\sigma(\phi_\varepsilon^A|_{\overline{X}^\pm(.,t)})J_t^\pm + \phi_\varepsilon^A|_{\overline{X}^\pm(.,t)}\partial_\sigma J_t^\pm\right)\tilde{\psi}_t^\pm\right]|_{(r,\sigma)}\,dr\right|.$$

Now we use the structure of ϕ^A_ε. Due to (6.22) we have in $\Omega^{C\pm}_t$

$$\nabla\phi^A_\varepsilon = \frac{1}{\sqrt{\varepsilon}}\left(\theta_0''(\rho_\varepsilon)q^\pm(s^\pm,t)+\varepsilon\partial_\rho\hat{v}^{C\pm}|_{(\rho_\varepsilon,H^\pm_\varepsilon,t)}\right)\left[\frac{\nabla r}{\varepsilon}-\nabla s\partial_s h_\varepsilon(s,t)\right] \\ \mp\frac{1}{\sqrt{\varepsilon}}\nabla s\left[\partial_\sigma q^\pm(s^\pm,t)\theta_0'(\rho_\varepsilon)+\partial_H\hat{v}^{C\pm}|_{(\rho_\varepsilon,H^\pm_\varepsilon,t)}\right].$$

Moreover, one can directly compute for $(r,\sigma)\in[-\delta,\delta]\times[0,2\mu_0]$ that

$$\partial_\sigma(\phi^A_\varepsilon|_{\overline{X}^\pm}) = -\frac{1}{\sqrt{\varepsilon}}\partial_\sigma h^\pm_\varepsilon|_{(\sigma,t)}\left[\theta_0''(\rho_\varepsilon)q^\pm(s^\pm,t)+\varepsilon\partial_\rho\hat{v}^{C\pm}(\rho_\varepsilon,H^\pm_\varepsilon,t)\right]|_{\overline{X}^\pm} \\ +\frac{1}{\sqrt{\varepsilon}}\left[\partial_H\hat{v}^{C\pm}(\rho_\varepsilon,H^\pm_\varepsilon,t)+\partial_\sigma q^\pm(s^\pm,t)\theta_0'(\rho_\varepsilon)\right]|_{\overline{X}^\pm}. \tag{6.33}$$

We estimate all the appearing terms in the above estimate for $|g^\pm_t(\sigma)|$ after inserting ϕ^A_ε, $\nabla\phi^A_\varepsilon$ and $\partial_\sigma(\phi^A_\varepsilon|_{\overline{X}^\pm})$. We apply Hölder's inequality, Lemma 6.5 and $\nabla r\cdot\nabla s|_{\overline{X}^\pm_0(\sigma,t)}=0$. Hence

$$|g^\pm_t(\sigma)|\leq C\|\tilde{\psi}^\pm_t(.,\sigma)\|_{L^2(-\delta,\delta;J^\pm_t(.,\sigma))}+C\varepsilon\|\nabla_{(r,\sigma)}\tilde{\psi}^\pm_t(.,\sigma)\|_{L^2(-\delta,\delta;J^\pm_t(.,\sigma))}$$

for a.e. $\sigma\in[0,2\mu_0]$. Since $|\nabla_{(r,\sigma)}\tilde{\psi}^\pm_t|\leq C|\nabla\psi|_{X^\pm(.,t)}|$, we obtain

$$|(III)|\leq C\|a'\|_{L^2(0,2\mu_0)}(\|\psi\|_{L^2(\Omega^{C\pm}_t)}+\varepsilon\|\nabla\psi\|_{L^2(\Omega^{C\pm}_t)}) \\ \leq C\varepsilon^2\|a'\|^2_{L^2(0,2\mu_0)}+\frac{\nu}{8}\left[\frac{1}{\varepsilon^2}\|\psi\|^2_{L^2(\Omega^{C\pm}_t)}+\|\nabla\psi\|^2_{L^2(\Omega^{C\pm}_t)}\right],$$

where we used Young's inequality in the second step and ν is as in Lemma 6.22. The last term is dominated by $\frac{1}{8}B^\pm_{\varepsilon,t}(\psi,\psi)$ because of Lemma 6.22. Altogether we have shown Lemma 6.23. $\square$

Finally, we combine Lemma 6.21-6.23.

Theorem 6.25. *There are $\varepsilon_0, C, c_0>0$ such that*

$$B^\pm_{\varepsilon,t}(\psi,\psi)\geq -C\|\psi\|^2_{L^2(\Omega^{C\pm}_t)}+c_0\|\nabla_\tau\psi\|^2_{L^2(\Omega^{C\pm}_t)}$$

for all $\varepsilon\in(0,\varepsilon_0], t\in[0,T]$ and $\psi\in H^1(\Omega^{C\pm}_t)$ with $\psi|_{X(.,s,t)}=0$ f.a.e. $\mp(s\mp1)\in[\frac{3}{2}\mu_0,2\mu_0]$.

Remark 6.26.
1. The estimate can be refined, cf. the proof below.
2. Theorem 6.25 directly implies Theorem 6.17, cf. the beginning of Section 6.2.1.

Proof of Theorem 6.25. Because of Lemma 6.20 any $\psi\in\tilde{H}^1(\Omega^{C\pm}_t)$ can be uniquely written as

$$\psi=\phi+\phi^\perp\quad\text{ with }\phi=[a(s^\pm)\phi^A_\varepsilon]|_{(.,t)}\in V^\pm_{\varepsilon,t}\text{ and }\phi^\perp\in(V^\pm_{\varepsilon,t})^\perp.$$

Lemma 6.21 and Lemma 6.23 imply for $\varepsilon_0>0$ small and all $\varepsilon\in(0,\varepsilon_0]$, $t\in[0,T]$ that

$$B^\pm_{\varepsilon,t}(\psi,\psi)=B^\pm_{\varepsilon,t}(\phi,\phi)+2B^\pm_{\varepsilon,t}(\phi,\phi^\perp)+B^\pm_{\varepsilon,t}(\phi^\perp,\phi^\perp) \\ \geq -C\|\phi\|^2_{L^2(\Omega^{C\pm}_t)}+(c_0-C\varepsilon^2)\|a\|^2_{H^1(0,2\mu_0)}-\frac{C}{\varepsilon}\|\phi\|_{L^2(\Omega^{C\pm}_t)}\|\phi^\perp\|_{L^2(\Omega^{C\pm}_t)}+\frac{B^\pm_{\varepsilon,t}(\phi^\perp,\phi^\perp)}{2}.$$

The third term is controlled via Young's inequality to $\frac{\nu}{4\varepsilon^2}\|\phi^\perp\|^2_{L^2(\Omega_t^{C\pm})}+\tilde{C}\|\phi\|^2_{L^2(\Omega_t^{C\pm})}$, where ν is as in Lemma 6.22. Therefore the latter yields for $\varepsilon_0>0$ small and $\varepsilon\in(0,\varepsilon_0]$, $t\in[0,T]$ that

$$B^\pm_{\varepsilon,t}(\psi,\psi)\geq -C\|\phi\|^2_{L^2(\Omega_t^{C\pm})}+\frac{\nu}{4\varepsilon^2}\|\phi^\perp\|^2_{L^2(\Omega_t^{C\pm})}+\frac{c_0}{2}\|a\|^2_{H^1(0,2\mu_0)}+\frac{\nu}{2}\|\nabla(\phi^\perp)\|^2_{L^2(\Omega_t^{C\pm})}.$$

It remains to include the $\nabla_\tau\psi$-term in the estimate. By the triangle inequality we have

$$\|\nabla_\tau\psi\|_{L^2(\Omega_t^{C\pm})}\leq\|\nabla_\tau\phi\|_{L^2(\Omega_t^{C\pm})}+\|\nabla_\tau(\phi^\perp)\|_{L^2(\Omega_t^{C\pm})}.$$

Inserting the definition of ∇_τ from Remark 3.4, 2. we obtain that $\nabla_\tau\phi|_{X^\pm(.,t)}$ equals

$$\mp\nabla s|_{\overline{X}^\pm(.,t)}\partial_\sigma(\phi|_{X^\pm(.,t)})=\mp\nabla s|_{\overline{X}^\pm(.,t)}\left[a'(\sigma)\phi^A_\varepsilon|_{\overline{X}^\pm(.,t)}+a(\sigma)\partial_\sigma(\phi^A_\varepsilon|_{\overline{X}^\pm(.,t)})\right].$$

We already computed $\partial_\sigma(\phi^A_\varepsilon|_{\overline{X}^\pm})$ in (6.33). Using the transformation rule, the Fubini Theorem and Lemma 6.5 it follows that $\|\nabla_\tau\phi\|_{L^2(\Omega_t^{C\pm})}\leq C\|a\|_{H^1(0,2\mu_0)}$. Moreover, the uniform boundedness of $|\nabla s|$ and $|\nabla_{(r,\sigma)}(\phi^\perp|_{X^\pm(.,t)})|\leq C|\nabla(\phi^\perp)|_{X^\pm(.,t)}|$ yields the estimate $\|\nabla_\tau(\phi^\perp)\|_{L^2(\Omega_t^{C\pm})}\leq C\|\nabla(\phi^\perp)\|_{L^2(\Omega_t^{C\pm})}$. Therefore we obtain

$$\|\nabla_\tau\psi\|^2_{L^2(\Omega_t^{C\pm})}\leq C(\|a\|^2_{H^1(0,2\mu_0)}+\|\nabla(\phi^\perp)\|^2_{L^2(\Omega_t^{C\pm})}).$$

Finally, together with the above estimate for $B^\pm_{\varepsilon,t}$ this yields the claim. □

6.3 Spectral Estimate for (AC) in ND

In this section we show the spectral estimate (6.1) for the Allen-Cahn equation (AC1)-(AC3) when the diffuse interface meets the boundary in the case of N dimensions, $N\geq 2$. This works in the analogous way as in the 2D-case in the last Section 6.2 but some computations are more technical. For convenience we often use the same notation. The construction of the approximate solution in Section 5.2 yields the precise structure of u^A_ε, but as in the 2D-case a more general structure is enough for the spectral estimate. Now we state the assumptions for this section.

Let $\Omega\subset\mathbb{R}^N$ and $\Gamma=(\Gamma_t)_{t\in[0,T]}$ for $T>0$ be as in Section 3.1 for $N\geq 2$ with contact angle $\alpha=\frac{\pi}{2}$ (MCF not needed). Moreover, we consider $\delta>0$ such that Theorem 3.7 holds for 2δ instead of δ. In the following we use the same notation for $\vec{n}_{\partial\Sigma},\vec{n}_{\partial\Gamma},Y,X_0,X,\mu_0,\mu_1,r,s,\sigma,b$ as in Theorem 3.7. Furthermore, we use the definitions of some sets and of ∂_n, ∇_τ, J from Remark 3.8. In this section we assume for the height functions h_1 and $h_2=h_2(\varepsilon)$ (with a slight abuse of notation) that

$$h_j\in B([0,T],C^0(\Sigma)\cap C^2(\hat{\Sigma})),j=1,2,\quad\hat{\Sigma}:=Y(\partial\Sigma\times[0,2\mu_0]),\quad C^2(\hat{\Sigma}):=C^2(\overline{\hat{\Sigma}^\circ}).$$

Moreover, let $\overline{C}_0>0$ be such that $\|h_j\|_{B([0,T],C^0(\Sigma)\cap C^2(\hat{\Sigma}))}\leq\overline{C}_0$ for $j=1,2$. Then we define $h_\varepsilon:=h_1+\varepsilon h_2$ for $\varepsilon>0$ small and introduce the scaled variables

$$\rho_\varepsilon:=\frac{r-\varepsilon h_\varepsilon(s,t)}{\varepsilon}\quad\text{in }\overline{\Gamma(2\delta)},\quad H_\varepsilon:=\frac{b}{\varepsilon}\quad\text{in }\overline{\Gamma^C(2\delta,2\mu_1)}.$$

Furthermore, let $\hat{u}^C_1:\overline{\mathbb{R}^2_+}\times\partial\Sigma\times[0,T]\to\mathbb{R}:(\rho,H,\sigma,t)\mapsto\hat{u}^C_1(\rho,H,\sigma,t)$ be in the space $B([0,T];C^2(\partial\Sigma,H^2_{(0,\gamma)}(\mathbb{R}^2_+)))$ for some $\gamma>0$. Then we define

$$u^C_1(x,t):=\hat{u}^C_1(\rho_\varepsilon(x,t),H_\varepsilon(x,t),\sigma(x,t),t)\quad\text{for }(x,t)\in\overline{\Gamma^C(2\delta,2\mu_1)}.$$

For $\varepsilon > 0$ small let

$$u_\varepsilon^A = \begin{cases} \theta_0(\rho_\varepsilon) + \mathcal{O}(\varepsilon^2) & \text{in } \Gamma(\delta, \mu_0), \\ \theta_0(\rho_\varepsilon) + \varepsilon u_1^C + \mathcal{O}(\varepsilon^2) & \text{in } \Gamma^C(\delta, 2\mu_0), \\ \pm 1 + \mathcal{O}(\varepsilon) & \text{in } Q_T^\pm \setminus \Gamma(\delta), \end{cases}$$

where θ_0 is from Theorem 4.1 and $\mathcal{O}(\varepsilon^k)$ are measurable functions bounded by $C\varepsilon^k$.

Remark 6.27. It is also possible to include an additional term of the form $\varepsilon\theta_1(\rho_\varepsilon)p_\varepsilon(s,t)$ in u_ε^A on $\Gamma(\delta)$, where $p_\varepsilon \in B([0,T], C^0(\Sigma) \cap C^2(\hat{\Sigma}))$ fulfils a uniform estimate for ε small and

$$\theta_1 \in C_b^0(\mathbb{R}) \quad \text{with} \quad \int_{\mathbb{R}} f'''(\theta_0)\theta_1(\theta_0')^2 \, d\rho = 0. \tag{6.34}$$

See Remark 6.31, 2. below for details.

We obtain the following spectral estimate:

Theorem 6.28 (Spectral Estimate for (AC) in ND). *There are $\varepsilon_0, C, c_0 > 0$ independent of the h_j for fixed $\overline{C}_0$ such that for all $\varepsilon \in (0, \varepsilon_0], t \in [0,T]$ and $\psi \in H^1(\Omega)$ it holds*

$$\int_\Omega |\nabla\psi|^2 + \frac{1}{\varepsilon^2} f''(u_\varepsilon^A(.,t))\psi^2 \, dx \geq -C\|\psi\|_{L^2(\Omega)}^2 + \|\nabla\psi\|_{L^2(\Omega\setminus\Gamma_t(\delta))}^2 + c_0\|\nabla_\tau\psi\|_{L^2(\Gamma_t(\delta))}^2.$$

As in the 2D-case we separately prove a spectral estimate on

$$\Omega_t^C := \Gamma_t^C(\delta, 2\mu_0) = X((-\delta,\delta) \times \hat{\Sigma}^\circ \times \{t\}) \quad \text{for } t \in [0,T].$$

In the following we need several properties of Sobolev spaces on the appearing sets.

Remark 6.29. The results of Sections 2.2.1-2.2.3 can be applied for sets such as $U = \Sigma^\circ$, $\hat{\Sigma}^\circ$, $(-\delta,\delta) \times \Sigma^\circ$, $(-\delta,\delta) \times \hat{\Sigma}^\circ$, $\Gamma(\delta)$, $\Gamma_t(\delta)$ and Ω_t^C for all $t \in [0,T]$. These sets can all be viewed as an open subset of a smooth compact Riemannian submanifold of some $\mathbb{R}^n$ with the Euclidean metric and they have Lipschitz boundary. For Σ°, $(-\delta,\delta) \times \Sigma^\circ$ etc. one can simply consider local charts. For $\Gamma(\delta)$, $\Gamma_t(\delta)$, Ω_t^C and similar sets one can show this with the extension of $\overline{X}$ to a diffeomorphism due to Theorem 3.7 and Remark 2.19. In particular

1. We can transform integrals via $\overline{X}$ and $X(.,t)$ for $t \in [0,T]$ with the usual transformation formula due to Theorem 2.6 with the factor J and J_t, $t \in [0,T]$ from Remark 3.8, 3.

2. Density and trace theorems for Sobolev spaces on the above sets hold due to Theorem 2.13, Theorem 2.20 and we can use integration by parts on Ω_t^C because of Theorem 2.13, 4.

3. $H^k(\Gamma_t(\delta)) \cong H^k((-\delta,\delta) \times \Sigma^\circ)$ and $H^k(\Omega_t^C) \cong H^k((-\delta,\delta) \times \hat{\Sigma}^\circ)$ etc. for $k \in \mathbb{N}_0$ via $X(.,t)$ for all $t \in [0,T]$ due to Theorem 2.16. Therefore Corollary 3.10, 1.-2. carries over to H^1-functions. In particular the gradients are pointwise a.e. uniformly equivalent. Moreover, note that for $k = 0$, i.e. L^2-spaces, the operator norms of the transformations can be estimated uniformly in $t \in [0,T]$ because of Theorem 2.6 and Remark 3.8, 3. Hence this also holds for the L^2-norms of the gradients and due to Lemma 2.15, 3. also for $k = 1$.

4. Lemma 2.17 yields $L^2((-\delta,\delta)\times\Sigma)\cong L^2(-\delta,\delta,L^2(\Sigma))\cong L^2(\Sigma,L^2(-\delta,\delta))$ as well as

$$\begin{aligned}H^1((-\delta,\delta)\times\Sigma^\circ)&\cong H^1(-\delta,\delta,L^2(\Sigma^\circ))\cap L^2(-\delta,\delta,H^1(\Sigma^\circ))\\&\cong H^1(\Sigma^\circ,L^2(-\delta,\delta))\cap L^2(\Sigma^\circ,H^1(-\delta,\delta))\end{aligned}$$

and the derivatives $\nabla_\Sigma:=\nabla_{\Sigma^\circ}$ and ∂_r are compatible. The analogous assertion holds for $\hat\Sigma$ instead of Σ and similar sets.

The spectral estimate on Ω_t^C is as follows:

Theorem 6.30. *There are $\tilde\varepsilon_0, C, \tilde c_0>0$ independent of the h_j for fixed $\overline{C}_0$ such that for all $\varepsilon\in(0,\tilde\varepsilon_0]$, $t\in[0,T]$ and $\psi\in H^1(\Omega_t^C)$ with $\psi|_{X(.,s,t)}=0$ for a.e. $s\in Y(\partial\Sigma\times[\frac{3}{2}\mu_0,2\mu_0])$:*

$$\int_{\Omega_t^C}|\nabla\psi|^2+\frac{1}{\varepsilon^2}f''(u_\varepsilon^A(.,t))\psi^2\,dx\geq -C\|\psi\|^2_{L^2(\Omega_t^C)}+\tilde c_0\|\nabla_\tau\psi\|^2_{L^2(\Omega_t^C)}.$$

The additional assumption on ψ is not needed but simplifies the proof, cf. Remark 6.31, 3. below. The latter is enough to show Theorem 6.28:

Proof of Theorem 6.28. For $\varepsilon_0>0$ small and all $\varepsilon\in(0,\varepsilon_0]$ it holds $f''(u_\varepsilon^A)\geq 0$ on $Q_T^\pm\setminus\Gamma(\delta)$. Therefore it is enough to show the estimate in Theorem 6.28 for $\Gamma_t(\delta)$ instead of Ω. On $\Gamma_t(\delta)$ we reduce to further subsets.

Due to Theorem 6.30 we have an estimate for $\Omega_t^C=\Gamma_t^C(\delta,2\mu_0)$ instead of Ω. Moreover, the estimate holds for $\Gamma_t(\delta,\mu_0)$ instead of Ω with $c_0=1$. There our curvilinear coordinate system is the usual tubular neighbourhood coordinate system, cf. Theorem 3.7. Let $\psi\in H^1(\Gamma_t(\delta,\mu_0))$. Then $|\nabla\psi|^2=|\partial_n\psi|^2+|\nabla_\tau\psi|^2$ on $\Gamma_t(\delta,\mu_0)$ due to Corollary 3.10, 2. and Remark 6.29, 3. Due to Taylor's Theorem we can replace $u_\varepsilon^A(.,t)$ by $\theta_0(\rho_\varepsilon(.,t))$ in the integral. Therefore an integral transformation with $X(.,t)$, cf. Remark 6.29, 1., and the Fubini Theorem yield for $\tilde\psi_t:=\psi|_{X(.,t)}$

$$\begin{aligned}\int_{\Gamma_t(\delta,\mu_0)}|\nabla\psi|^2+\frac{1}{\varepsilon^2}f''(u_\varepsilon^A(.,t))\psi^2\,dx&\geq-\tilde C\|\psi\|^2_{L^2(\Gamma_t(\delta,\mu_0))}+\|\nabla_\tau\psi\|^2_{L^2(\Gamma_t(\delta,\mu_0))}\\&+\int_{\Sigma\setminus[Y(\partial\Sigma\times[0,\mu_0])]}\int_{-\delta}^{\delta}\left[|\partial_r\tilde\psi_t|^2+\frac{1}{\varepsilon^2}f''(\theta_0(\rho_\varepsilon|_{\overline{X}(.,t)}))\tilde\psi_t^2\right]J_t\,dr\,d\mathcal{H}^{N-1}(s).\end{aligned}$$

Due to Remark 6.29, 4. we can estimate the inner integral in the analogous way as in the 2D-case, cf. the proof of Theorem 6.16 in the last section. Note that the results of Section 6.1.1 are again applicable with a constant h_ε there, in particular we use the transformations in Lemma 6.5, 1. and Theorem 6.8, 1. This yields the estimate for $\Gamma_t(\delta,\mu_0)$ instead of Ω with $c_0=1$.

Finally, we combine the above estimates with a suitable partition of unity for

$$\Gamma_t(\delta)\subseteq\overline{\Gamma_t(\delta,\mu_0)}\cup\overline{\Gamma_t^C(\delta,2\mu_0)}.\tag{6.35}$$

The arguments are similar to the 2D-case, cf. the proof of Theorem 6.16. Here one uses b from Theorem 3.7, 4. to construct the cut-off functions. This shows the estimate. □

6.3.1 Outline for the Proof of the Spectral Estimate close to the Contact Points

We proceed in the analogous way as in the 2D-case, cf. Section 6.2.1. For the proof of Theorem 6.30 we can replace $\frac{1}{\varepsilon^2}f''(u_\varepsilon^A(.,t))$ by

$$\frac{1}{\varepsilon^2}f''(\theta_0|_{\rho_\varepsilon(.,t)})+\frac{1}{\varepsilon}f'''(\theta_0|_{\rho_\varepsilon(.,t)})u_1^C|_{(.,t)}$$

due to a Taylor expansion. We construct an approximation $\phi_\varepsilon^A(.,t)$ to the first eigenfunction of

$$\mathcal{L}_{\varepsilon,t}^C:=-\Delta+\frac{1}{\varepsilon^2}f''(\theta_0|_{\rho_\varepsilon(.,t)})+\frac{1}{\varepsilon}f'''(\theta_0|_{\rho_\varepsilon(.,t)})u_1^C|_{(.,t)}\quad\text{on }\Omega_t^C$$

together with a homogeneous Neumann boundary condition. Then we split

$$\tilde{H}^1(\Omega_t^C):=\left\{\psi\in H^1(\Omega_t^C):\psi|_{X(.,s,t)}=0\text{ for a.e. }s\in Y(\partial\Sigma\times[\tfrac{3}{2}\mu_0,2\mu_0])\right\}\tag{6.36}$$

with respect to the subspace of tangential alterations of $\phi_\varepsilon^A(.,t)$. Therefore we set up the ansatz

$$\begin{aligned}\phi_\varepsilon^A(.,t)&:=\frac{1}{\sqrt{\varepsilon}}[v_\varepsilon^I(.,t)+\varepsilon v_\varepsilon^C(.,t)] &&\text{on }\Omega_t^C,\\ v_\varepsilon^I(.,t)&:=\hat{v}^I(\rho_\varepsilon(.,t),s(.,t),t):=\theta_0'|_{\rho_\varepsilon(.,t)}\,q(s(.,t),t) &&\text{on }\Omega_t^C,\\ v_\varepsilon^C(.,t)&:=\hat{v}^C(\rho_\varepsilon(.,t),H_\varepsilon(.,t),\sigma(.,t),t) &&\text{on }\Omega_t^C,\end{aligned}$$

where $q:\hat{\Sigma}\times[0,T]\to\mathbb{R}$ and $\hat{v}^C:\overline{\mathbb{R}^2_+}\times\partial\Sigma\times[0,T]\to\mathbb{R}$. The $\frac{1}{\sqrt{\varepsilon}}$-factor normalizes in a suitable way, see Lemma 6.33 below.

In Subsection 6.3.2 we expand $\mathcal{L}_{\varepsilon,t}^C\phi_\varepsilon^A(.,t)$ and $\partial_{N_{\partial\Omega}}\phi_\varepsilon^A(.,t)$ similarly as in Section 5.2 and choose q and $\hat{v}^C$ appropriately. The q-term will be used to enforce the compatibility condition for the equations for $\hat{v}^C$. In Subsection 6.3.3 we characterize the L^2-orthogonal splitting of $\tilde{H}^1(\Omega_t^C)$ with respect to the subspace

$$V_{\varepsilon,t}:=\left\{\phi=a(s(.,t))\phi_\varepsilon^A(.,t):a\in\tilde{H}^1(\hat{\Sigma}^\circ)\right\},\tag{6.37}$$

$$\tilde{H}^1(\hat{\Sigma}^\circ):=\left\{a\in H^1(\hat{\Sigma}^\circ):a|_{Y(.,b)}=0\text{ for a.e. }b\in[\tfrac{3}{2}\mu_0,2\mu_0]\right\}.\tag{6.38}$$

Finally, in Subsection 6.3.4 we analyze the bilinear form $B_{\varepsilon,t}^C$ corresponding to $\mathcal{L}_{\varepsilon,t}^C$ on $V_{\varepsilon,t}\times V_{\varepsilon,t}$, $V_{\varepsilon,t}^\perp\times V_{\varepsilon,t}^\perp$ and $V_{\varepsilon,t}\times V_{\varepsilon,t}^\perp$. Here for $\phi,\psi\in H^1(\Omega_t^C)$ we set

$$B_{\varepsilon,t}^C(\phi,\psi):=\int_{\Omega_t^C}\nabla\phi\cdot\nabla\psi+\left[\frac{1}{\varepsilon^2}f''(\theta_0|_{\rho_\varepsilon(.,t)})+\frac{1}{\varepsilon}f'''(\theta_0|_{\rho_\varepsilon(.,t)})u_1^C(.,t)\right]\phi\psi\,dx.\tag{6.39}$$

6.3.2 Asymptotic Expansion for the Approximate Eigenfunction

Asymptotic Expansion of $\sqrt{\varepsilon}\mathcal{L}_{\varepsilon,t}^C\phi_\varepsilon^A(.,t)$. First, we expand Δv_ε^I as in the inner expansion in Section 5.2.1. The lowest order $\mathcal{O}(\frac{1}{\varepsilon^2})$ is given by $\frac{1}{\varepsilon^2}|\nabla r|^2|_{\overline{X}_0(s,t)}\theta_0'''(\rho)q(s,t)=\frac{1}{\varepsilon^2}\theta_0'''(\rho)q(s,t)$. In $\sqrt{\varepsilon}\mathcal{L}_{\varepsilon,t}^C\phi_\varepsilon^A(.,t)$ this cancels with $\frac{1}{\varepsilon^2}f''(\theta_0(\rho))\theta_0'(\rho)q(s,t)$. For the $\frac{1}{\varepsilon}$-order of Δv_ε^I we get

$$\begin{aligned}&\frac{1}{\varepsilon}\theta_0'''(\rho)q(s,t)\left[(\rho+h_1)\partial_r(|\nabla r|^2\circ\overline{X})|_{(0,s,t)}-2(D_xs\nabla r)^\top|_{\overline{X}_0(s,t)}\nabla_\Sigma h_1\right]\\&+\frac{1}{\varepsilon}\theta_0''(\rho)\left[\Delta r|_{\overline{X}_0(s,t)}q(s,t)+2(D_xs\nabla r)^\top|_{\overline{X}_0(s,t)}\nabla_\Sigma q(s,t)\right]=\frac{1}{\varepsilon}\theta_0''(\rho)\Delta r|_{\overline{X}_0(s,t)}q(s,t).\end{aligned}$$

We leave $\frac{1}{\varepsilon}\Delta r|_{\overline{X}_0(s,t)} q(s,t)\theta_0''(\rho)$ as a remainder.

For $\varepsilon\Delta v_\varepsilon^C$ we apply the expansion in Section 5.2.2.1, but without using a Taylor expansion for the h_j since we just need the lowest order and the h_j are assumed to be less regular. More precisely, the (x,t)-terms in the formula for Δv_ε^C in Lemma 5.17 are expanded only with (5.40). At the lowest order $\mathcal{O}(\frac{1}{\varepsilon})$ we obtain $\frac{1}{\varepsilon}\Delta^{\sigma,t}\hat{v}^C$, where $\Delta^{\sigma,t} := \partial_\rho^2 + |\nabla b|^2|_{\overline{X}_0(\sigma,t)}\partial_H^2$ for $(\sigma,t)\in\partial\Sigma\times[0,T]$. Moreover, the f-parts yield $\frac{1}{\varepsilon}f''(\theta_0(\rho))\hat{v}^C + \frac{1}{\varepsilon}f'''(\theta_0(\rho))\hat{u}_1^C\hat{v}^I$. To get an equation for $\hat{v}^C$ in (ρ,H,σ,t) we apply a Taylor expansion for $q(Y(\sigma,.),t)|_{[0,2\mu_0]}$:

$$q(Y(\sigma,\varepsilon H),t) = q(\sigma,t) + \mathcal{O}(\varepsilon H) \quad \text{for } (\sigma,\varepsilon H)\in\partial\Sigma\times[0,\frac{2\mu_0}{\varepsilon}].$$

Therefore we require

$$\left[-\Delta^{\sigma,t} + f''(\theta_0(\rho))\right]\hat{v}^C = -f'''(\theta_0)\theta_0'|_\rho \hat{u}_1^C|_{(\rho,H,\sigma,t)} q|_{(\sigma,t)} \quad \text{in } \overline{\mathbb{R}^2_+}\times\partial\Sigma\times[0,T]. \tag{6.40}$$

Asymptotic Expansion of $\sqrt{\varepsilon}\partial_{N_{\partial\Omega}}\phi_\varepsilon^A(.,t)$. We proceed as in Section 5.2.2.2. Note that in $\overline{\Omega_t^C}$

$$\begin{aligned}
\nabla v_\varepsilon^I &= q|_{(s,t)}\theta_0''(\rho_\varepsilon)\left[\frac{\nabla r}{\varepsilon} - (D_x s)^\top \nabla_\Sigma h_\varepsilon|_{(s,t)}\right] + \theta_0'(\rho_\varepsilon)(D_x s)^\top\nabla_\Sigma q|_{(s,t)},\\
\nabla v_\varepsilon^C &= \partial_\rho \hat{v}^C\left[\frac{\nabla r}{\varepsilon} - (D_x s)^\top \nabla_\Sigma h_\varepsilon|_{(s,t)}\right] + \frac{\nabla b}{\varepsilon}\partial_H\hat{v}^C + (D_x\sigma)^\top\nabla_{\partial\Sigma}\hat{v}^C,
\end{aligned} \tag{6.41}$$

where the $\hat{v}^C$-terms are evaluated at $(\rho_\varepsilon, H_\varepsilon,\sigma,t)$. The lowest order $\mathcal{O}(\frac{1}{\varepsilon})$ in $\sqrt{\varepsilon}\partial_{N_{\partial\Omega}}\phi_\varepsilon^A(.,t)$ is $\frac{1}{\varepsilon}(N_{\partial\Omega}\cdot\nabla r)|_{\overline{X}_0(\sigma,t)}\theta_0''(\rho)q|_{(\sigma,t)} = 0$ due to the 90°-contact angle condition. At $\mathcal{O}(1)$ we get

$$\begin{aligned}
&q|_{(\sigma,t)}\theta_0''(\rho)\left[(\rho + h_1|_{(\sigma,t)})\partial_r((N_{\partial\Omega}\cdot\nabla r)\circ\overline{X})|_{(0,\sigma,t)} - (D_x s N_{\partial\Omega})^\top|_{\overline{X}_0(\sigma,t)}\nabla_\Sigma h_1|_{(\sigma,t)}\right]\\
&+(D_x s N_{\partial\Omega})^\top|_{\overline{X}_0(\sigma,t)}\nabla_\Sigma q|_{(\sigma,t)}\theta_0'(\rho) + 0\cdot\partial_\rho\hat{v}^C|_{H=0} + (N_{\partial\Omega}\cdot\nabla b)|_{\overline{X}_0(\sigma,t)}\partial_H\hat{v}^C|_{H=0}.
\end{aligned}$$

This vanishes if and only if

$$\begin{aligned}
&(N_{\partial\Omega}\cdot\nabla b)|_{\overline{X}_0(\sigma,t)}\partial_H\hat{v}^C|_{H=0} = -(D_x s N_{\partial\Omega})^\top|_{\overline{X}_0(\sigma,t)}\nabla_\Sigma q|_{(\sigma,t)}\theta_0'(\rho)\\
&+ q|_{(\sigma,t)}\theta_0''(\rho)\left[(D_x s N_{\partial\Omega})^\top|_{\overline{X}_0(\sigma,t)}\nabla_\Sigma h_1|_{(\sigma,t)} - (\rho + h_1|_{(\sigma,t)})\partial_r((N_{\partial\Omega}\cdot\nabla r)\circ\overline{X})|_{(0,\sigma,t)}\right].
\end{aligned}$$

Here note that for the desired regularity of $\hat{v}^C$ the term with $\nabla_\Sigma h_1$ is not good enough. One option is to require additionally $\nabla_\Sigma h_1|_{\partial\Sigma\times[0,T]}\in B([0,T],C^2(\partial\Sigma))$. However, we can also leave the term as a remainder similar to the one in $\sqrt{\varepsilon}\mathcal{L}_{\varepsilon,t}^C\phi_\varepsilon^A(.,t)$. Hence due to (6.40) we require

$$[-\Delta + f''(\theta_0)]\overline{v}^C = -f'''(\theta_0)\theta_0'\overline{u}_1^C q|_{(\sigma,t)} \qquad \text{in } \overline{\mathbb{R}^2_+}\times\partial\Sigma\times[0,T], \tag{6.42}$$

$$-\partial_H\overline{v}^C|_{H=0} = (|\nabla b|/N_{\partial\Omega}\cdot\nabla b)|_{\overline{X}_0(\sigma,t)}g^C \qquad \text{in } \mathbb{R}\times\partial\Sigma\times[0,T], \tag{6.43}$$

where $\overline{v}^C, \overline{u}_1^C : \overline{\mathbb{R}^2_+}\times\partial\Sigma\times[0,T]\to\mathbb{R}$ correspond to $\hat{v}^C$ and $\hat{u}_1^C$ in the same way as in (5.45) in Section 5.2.2.2.2, respectively, and we define $g^C(\rho,\sigma,t)$ for all $(\rho,\sigma,t)\in\mathbb{R}\times\partial\Sigma\times[0,T]$ as

$$-q|_{(\sigma,t)}\theta_0''(\rho)(\rho + h_1|_{(\sigma,t)})\partial_r((N_{\partial\Omega}\cdot\nabla r)\circ\overline{X})|_{(0,\sigma,t)} - (D_x s N_{\partial\Omega})^\top|_{\overline{X}_0(\sigma,t)}\nabla_\Sigma q|_{(\sigma,t)}\theta_0'(\rho).$$

The right hand sides in (6.42)-(6.43) are contained in $B([0,T];C^2(\partial\Sigma,H^2_{(\beta,\gamma)}(\mathbb{R}^2_+)\times H^{5/2}_{(\beta)}(\mathbb{R})))$ for some $\beta,\gamma>0$ provided that $(q,\nabla_\Sigma q)|_{\partial\Sigma\times[0,T]}\in B([0,T],C^2(\partial\Sigma))^{1+N}$. We require $q=1$ on $\partial\Sigma\times[0,T]$. Then the compatibility condition (4.5) associated to (6.42)-(6.43) is equivalent to

$$-(D_xsN_{\partial\Omega})^\top|_{\overline{X}_0(\sigma,t)}\nabla_\Sigma q|_{(\sigma,t)}=\hat{g}^C|_{(\sigma,t)},\tag{6.44}$$

$$\hat{g}^C|_{(\sigma,t)}:=\frac{1}{\|\theta_0'\|^2_{L^2(\mathbb{R})}}\Big[\partial_r((N_{\partial\Omega}\cdot\nabla r)\circ\overline{X})|_{(0,\sigma,t)}\int_{\mathbb{R}}\rho\theta_0'(\rho)\theta_0''(\rho)\,d\rho$$
$$+\frac{N_{\partial\Omega}\cdot\nabla b}{|\nabla b|}\Big|_{\overline{X}_0(\sigma,t)}\int_{\mathbb{R}^2_+}f'''(\theta_0(\rho))\theta_0'(\rho)^2\overline{u}_1^C|_{(\rho,H,\sigma,t)}\,d(\rho,H)\Big].$$

Note that $\hat{g}^C\in B([0,T],C^2(\partial\Sigma))$.

To construct q we consider $\tilde{q}:\partial\Sigma\times[0,2\mu_0]\times[0,T]\to\mathbb{R}:(\sigma,b,t)\mapsto q(Y(\sigma,b),t)$. Then due to $Y(.,0)=\mathrm{id}_{\partial\Sigma}$ and (3.5) it holds

$$\nabla_\Sigma q|_{(\sigma,t)}\cdot(\vec{v}-\vec{n}_{\partial\Sigma}|_\sigma w)=\nabla_\Sigma q|_{(\sigma,t)}\cdot d_{(\sigma,0)}Y(\vec{v},w)=\nabla_{\partial\Sigma}[\tilde{q}(.,0,t)]|_\sigma\cdot\vec{v}+w\,\partial_b\tilde{q}(\sigma,0,t)$$

for all $(\vec{v},w)\in T_\sigma\partial\Sigma\times\mathbb{R}$ and $(\sigma,t)\in\partial\Sigma\times[0,T]$. If $q=1$ on $\partial\Sigma\times[0,T]$, then we obtain $\nabla_{\partial\Sigma}[\tilde{q}(.,0,t)]|_\sigma=0$ and therefore

$$\nabla_\Sigma q|_{(\sigma,t)}=-\vec{n}_{\partial\Sigma}|_\sigma\partial_b\tilde{q}(\sigma,0,t)\quad\text{for all }(\sigma,t)\in\partial\Sigma\times[0,T].\tag{6.45}$$

Note that $|(D_xsN_{\partial\Omega})|_{\overline{X}_0(\sigma,t)}\cdot\vec{n}_{\partial\Sigma}|_\sigma|\geq c>0$ for all $(\sigma,t)\in\partial\Sigma\times[0,T]$ due to Theorem 3.7, 4. Hence with a simple ansatz and cutoff we can construct $q\in B([0,T],C^2(\hat{\Sigma}))$ such that $q=1$ on $(\partial\Sigma\cup Y(\partial\Sigma,[\mu_0,2\mu_0]))\times[0,T]$ and $c\leq q\leq C$ for some $c,C>0$ and such that (6.44) holds. Together with (6.45) the latter yields $\nabla_\Sigma q|_{\partial\Sigma\times[0,T]}\in B([0,T],C^2(\partial\Sigma))$. Therefore Remark 4.12 yields a unique solution of (6.42)-(6.43) such that for some $\beta,\gamma>0$

$$\overline{v}^C\in B([0,T];C^2(\partial\Sigma,H^4_{(\beta,\gamma)}(\mathbb{R}^2_+)))\hookrightarrow B([0,T];C^2(\partial\Sigma,C^2_{(\beta,\gamma)}(\overline{\mathbb{R}^2_+}))).$$

Remark 6.31.
1. Consider the situation of Section 5.2. Then h_1 is smooth and the $\nabla_\Sigma h_1$-term can be included above. Moreover, $\overline{u}_1^C$ is smooth and solves (5.46)-(5.47). Hence analogously to the 2D-case, cf. Remark 6.18, 1., we can use $\hat{v}^C:=\partial_\rho\overline{u}_1^C$ in this situation.
2. In the case of additional terms in u^A_ε as in Remark 6.27 one can proceed analogously as in the 2D-case, cf. Remark 6.18, 2.
3. The behaviour of $\phi^A_\varepsilon(x,t)$ for $x\in\Omega^C_t$ with $b(x,t)\in[\frac{7}{4}\mu_0,2\mu_0]$ is not important because we only consider $\psi\in\tilde{H}^1(\Omega^C_t)$ in Theorem 6.30, where $\tilde{H}^1(\Omega^C_t)$ was defined in (6.36).

Lemma 6.32. *The function $\phi^A_\varepsilon(.,t)$ is $C^2(\overline{\Omega^C_t})$ and satisfies uniformly in $t\in[0,T]$:*

$$\Big|\sqrt{\varepsilon}\mathcal{L}^C_{\varepsilon,t}\phi^A_\varepsilon(.,t)+\frac{1}{\varepsilon}\Delta r|_{\overline{X}_0(s(.,t),t)}q|_{(s(.,t),t)}\theta_0''|_{\rho_\varepsilon(.,t)}\Big|\leq Ce^{-c|\rho_\varepsilon(.,t)|}\quad\textit{in }\Omega^C_t,$$
$$\Big|\sqrt{\varepsilon}\partial_{N_{\partial\Omega}}\phi^A_\varepsilon|_{(.,t)}+D_xsN_{\partial\Omega}|_{\overline{X}_0(\sigma(.,t),t)}\cdot\nabla_\Sigma h_1|_{(\sigma(.,t),t)}\theta_0''\Big|\leq C\varepsilon e^{-c|\rho_\varepsilon(.,t)|}\quad\textit{on }\partial\Omega^C_t\cap\partial\Omega,$$
$$\Big|\sqrt{\varepsilon}N_{\partial\Omega^C_t}\cdot\nabla\phi^A_\varepsilon|_{(.,t)}\Big|\leq Ce^{-c/\varepsilon}\quad\textit{on }\partial\Omega^C_t\setminus\Gamma_t(\delta).$$

Proof. The regularity for ϕ^A_ε is obtained from the construction. The estimates follow from rigorous estimates for the remainder terms in the expansions above and the decay properties of the involved terms, cf. the proof of Lemma 6.19 in the 2D-case. □

6.3.3 The Splitting

Similar as in the 2D-case we show a characterization for the splitting of $\tilde{H}^1(\Omega_t^C)$.

Lemma 6.33. *Let $\tilde{H}^1(\Omega_t^C)$, $V_{\varepsilon,t}$ and $\tilde{H}^1(\hat{\Sigma}^\circ)$ be as in* (6.36)-(6.38). *Then*

1. *$V_{\varepsilon,t}$ is a subspace of $\tilde{H}^1(\Omega_t^C)$ and for $\varepsilon_0 > 0$ small there are $c_1, C_1 > 0$ such that*
$$c_1\|a\|_{L^2(\hat{\Sigma})} \leq \|\psi\|_{L^2(\Omega_t^C)} \leq C_1\|a\|_{L^2(\hat{\Sigma})}$$
for all $\psi = a(s(.,t))\phi_\varepsilon^A(.,t) \in V_{\varepsilon,t}$ and $\varepsilon \in (0,\varepsilon_0]$, $t \in [0,T]$.

2. *Let $V_{\varepsilon,t}^\perp$ be the L^2-orthogonal complement of $V_{\varepsilon,t}$ in $\tilde{H}^1(\Omega_t^C)$. Then for $\psi \in \tilde{H}^1(\Omega_t^C)$:*
$$\psi \in V_{\varepsilon,t}^\perp \quad \Leftrightarrow \quad \int_{-\delta}^{\delta} (\phi_\varepsilon^A(.,t)\psi)|_{X(r,s,t)} J_t(r,s)\, dr = 0 \quad \text{for a.e. } s \in \hat{\Sigma}.$$
Moreover, $\tilde{H}^1(\Omega_t^C) = V_{\varepsilon,t} \oplus V_{\varepsilon,t}^\perp$ for all $\varepsilon \in (0,\varepsilon_0]$ and $\varepsilon_0 > 0$ small.

Proof. Ad 1. It holds $\phi_\varepsilon^A(.,t) \in C^2(\overline{\Omega_t^C})$ for fixed $t \in [0,T]$ due to Lemma 6.32. Moreover, $a(s(.,t)) \in H^1(\Omega_t^C)$ for all $a \in H^1(\hat{\Sigma}^\circ)$ because of Remark 6.29, 3.-4. Therefore $V_{\varepsilon,t}$ is a subspace of $\tilde{H}^1(\Omega_t^C)$. Now let $\psi = a(s(.,t))\phi_\varepsilon^A(.,t) \in V_{\varepsilon,t}$ be arbitrary. Then the transformation rule, cf. Remark 6.29, 1., and the Fubini Theorem yield

$$\|\psi\|_{L^2(\Omega_t^C)}^2 = \int_{\hat{\Sigma}} a(s)^2 \int_{-\delta}^{\delta} (\phi_\varepsilon^A|_{\overline{X}(r,s,t)})^2 J_t(r,s)\, dr\, d\mathcal{H}^{N-1}(s). \tag{6.46}$$

Since there are $c, C > 0$ with $c \leq J, q \leq C$, we can transform and estimate the inner integral with Lemma 6.5 analogously to the 2D-case, cf. the proof of Lemma 6.20, 1. $\square_{1.}$

Ad 2. Let $t \in [0,T]$ be fixed. By definition

$$V_{\varepsilon,t}^\perp = \left\{\psi \in \tilde{H}^1(\Omega_t^C) : \int_{\Omega_t^C} \psi\, a(s(.,t))\phi_\varepsilon^A(.,t)\, dx = 0 \text{ for all } a \in \tilde{H}^1(\hat{\Sigma}^\circ)\right\}.$$

The integral equals $\int_{\hat{\Sigma}} a(s) \int_{-\delta}^{\delta} (\phi_\varepsilon^A(.,t)\psi)|_{X(r,s,t)} J_t(r,s)\, dr\, d\mathcal{H}^{N-1}(s)$ due to the transformation rule and the Fubini Theorem. Therefore with the Fundamental Theorem of Calculus of Variations we obtain the characterization. Since by definition $V_{\varepsilon,t} \cap V_{\varepsilon,t}^\perp = \{0\}$, it remains to show $V_{\varepsilon,t} + V_{\varepsilon,t}^\perp = \tilde{H}^1(\Omega_t^C)$. We define

$$w_\varepsilon : \hat{\Sigma} \to \mathbb{R} : s \mapsto \int_{-\delta}^{\delta} (\phi_\varepsilon^A|_{\overline{X}(r,s,t)})^2 J_t(r,s)\, dr.$$

It holds $w_\varepsilon \in C^1(\hat{\Sigma})$ and with Lemma 6.5 one can prove $w_\varepsilon \geq c > 0$ for small ε. Now let $\psi \in \tilde{H}^1(\Omega_t^C)$ be arbitrary. Then we set

$$a_\varepsilon : \hat{\Sigma} \to \mathbb{R} : s \mapsto \frac{1}{w_\varepsilon(s)} \int_{-\delta}^{\delta} (\phi_\varepsilon^A(.,t)\psi)|_{X(r,s,t)} J_t(r,s)\, dr.$$

Due to Remark 6.29, 3.-4. and since integration yields a bounded linear functional on $L^2(-\delta,\delta)$, we obtain $a_\varepsilon \in \tilde{H}^1(\hat{\Sigma}^\circ)$. For $\psi_\varepsilon^\perp := \psi - a_\varepsilon(s(.,t))\phi_\varepsilon^A(.,t) \in \tilde{H}^1(\Omega_t^C)$ it holds

$$\int_{-\delta}^{\delta} (\phi_\varepsilon^A(.,t)\psi_\varepsilon^\perp)|_{X(r,s,t)} J_t(r,s)\, dr = a_\varepsilon(s)w_\varepsilon(s) - a_\varepsilon(s)w_\varepsilon(s) = 0$$

for a.e. $s \in \hat{\Sigma}$. Therefore by the integral characterization above we obtain $\psi_\varepsilon^\perp \in V_{\varepsilon,t}^\perp$. $\square_{2.}$

6.3.4 Analysis of the Bilinear Form

First we analyze $B^C_{\varepsilon,t}$ on $V_{\varepsilon,t} \times V_{\varepsilon,t}$.

Lemma 6.34. *There are $\varepsilon_0, C, c > 0$ such that*

$$B^C_{\varepsilon,t}(\phi,\phi) \geq -C\|\phi\|^2_{L^2(\Omega^C_t)} + c\|a\|^2_{H^1(\hat{\Sigma}^\circ)}$$

for all $\phi = a(s(.,t))\phi^A_\varepsilon(.,t) \in V_{\varepsilon,t}$ and $\varepsilon \in (0,\varepsilon_0], t \in [0,T]$.

Proof. Let ϕ be as in the lemma. We rewrite $B^C_{\varepsilon,t}(\phi,\phi)$ in order to use Lemma 6.32. Therefore we compute $\nabla\phi = \nabla(a|_{s(.,t)})\phi^A_\varepsilon(.,t) + a|_{s(.,t)}\nabla\phi^A_\varepsilon(.,t)$ and

$$|\nabla\phi|^2 = |\nabla(a(s))\phi^A_\varepsilon|^2|_{(.,t)} + a^2(s)|\nabla\phi^A_\varepsilon|^2|_{(.,t)} + \nabla(a^2(s)) \cdot \nabla\phi^A_\varepsilon\phi^A_\varepsilon|_{(.,t)}.$$

Due to Remark 6.29, 2. we can use integration by parts on Ω^C_t. Therefore

$$\begin{aligned}\int_{\Omega^C_t} \left[\nabla(a^2(s)) \cdot \nabla\phi^A_\varepsilon\phi^A_\varepsilon\right]|_{(.,t)}\,dx = &-\int_{\Omega^C_t}\left[a^2(s)(\Delta\phi^A_\varepsilon\,\phi^A_\varepsilon + |\nabla\phi^A_\varepsilon|^2)\right]|_{(.,t)}\,dx \\ &+ \int_{\partial\Omega^C_t}\left[N_{\partial\Omega^C_t}\cdot\nabla\phi^A_\varepsilon\,\mathrm{tr}(a^2(s)\phi^A_\varepsilon|_{(.,t)})\right]\,d\mathcal{H}^{N-1}.\end{aligned}$$

Therefore we obtain

$$\begin{aligned}B^C_{\varepsilon,t}(\phi,\phi) = &\int_{\Omega^C_t}|\nabla(a(s))\phi^A_\varepsilon|^2|_{(.,t)}\,dx + \int_{\Omega^C_t}(a^2(s)\phi^A_\varepsilon)|_{(.,t)}\mathcal{L}^C_{\varepsilon,t}\phi^A_\varepsilon|_{(.,t)}\,dx \\ &+ \int_{\partial\Omega^C_t}\left[N_{\partial\Omega^C_t}\cdot\nabla\phi^A_\varepsilon\,\mathrm{tr}(a^2(s)\phi^A_\varepsilon|_{(.,t)})\right]\,d\mathcal{H}^{N-1} =: (I) + (II) + (III).\end{aligned}$$

Ad (I). Because of $\nabla(a|_{s(.,t)}) = (D_x s)^\top|_{\overline{X}(.,t)}\nabla_{\hat{\Sigma}}a|_{s(.,t)}$ and Theorem 3.7, 3. it follows that $|\nabla(a|_{s(.,t)})|^2 \geq c|\nabla_{\hat{\Sigma}}a|^2|_{s(.,t)}$. Therefore

$$(I) \geq \int_{\hat{\Sigma}}|\nabla_{\hat{\Sigma}}a|^2|_s\int_{-\delta}^{\delta}(\phi^A_\varepsilon)^2|_{\overline{X}(r,s,t)}J_t(r,s)\,dr\,d\mathcal{H}^{N-1}(s).$$

The proof of Lemma 6.33, 1. yields that the inner integral is estimated from below by a uniform positive constant. Hence $(I) \geq c_0\|\nabla_{\hat{\Sigma}}a\|^2_{L^2(\hat{\Sigma})}$ for a $c_0 > 0$ independent of $\phi \in V_{\varepsilon,t}$ and all $\varepsilon \in (0,\varepsilon_0]$, $t \in [0,T]$, if $\varepsilon_0 > 0$ is small. Lemma 2.15, 3. and Lemma 6.33 yield the claim.

Ad (II). We write

$$(II) = \int_{\hat{\Sigma}}a^2(s)\int_{-\delta}^{\delta}\phi^A_\varepsilon|_{\overline{X}(r,s,t)}(\mathcal{L}^C_{\varepsilon,t}\phi^A_\varepsilon(.,t))|_{X(r,s,t)}J_t(r,s)\,dr\,d\mathcal{H}^{N-1}(s)$$

and estimate the inner integral. Lemma 6.32 yields

$$\left|\sqrt{\varepsilon}\mathcal{L}^C_{\varepsilon,t}\phi^A_\varepsilon(.,t) + \frac{1}{\varepsilon}\Delta r|_{\overline{X}_0(s(.,t),t)}q(s(.,t),t)\theta_0''(\rho_\varepsilon(.,t))\right| \leq Ce^{-c|\rho_\varepsilon(.,t)|} \quad \text{in } \Omega^C_t.$$

The lowest ε-order term in the inner integral in (II) is

$$\frac{1}{\varepsilon^2}\Delta r|_{\overline{X}_0(s,t)}q(s,t)^2\int_{-\delta}^{\delta}\theta_0''\theta_0'(\rho_\varepsilon|_{\overline{X}(r,s,t)})J_t(r,s)\,dr. \tag{6.47}$$

It holds $|J_t(r,s)-J_t(0,s)| \leq \tilde{C}|r|$ with $\tilde{C}>0$ independent of (r,s,t). Due to Lemma 6.5, 1. and $\int_{\mathbb{R}} \theta_0''\theta_0'\,dz = 0$ the term (6.47) with $J_t(0,s)$ instead of $J_t(r,s)$ can be estimated by a constant $C>0$ independent of s,t and $\varepsilon \in (0,\varepsilon_0]$. The remaining terms in (6.47) and (II) can be controlled with Lemma 6.5. Altogether we obtain $|(II)| \leq C\|a\|^2_{L^2(\hat{\Sigma})}$ with $C>0$ independent of $\phi \in V_{\varepsilon,t}$ and all $\varepsilon \in (0,\varepsilon_0]$, $t \in [0,T]$ if $\varepsilon_0 > 0$ is small.

Ad (III). We transform the integral over $\partial\Omega_t^C$ to the boundary of $(-\delta,\delta)\times\hat{\Sigma}^\circ$ with the aid of Theorem 2.6 and Theorem 3.7. This makes sense because of Remark 2.12, 3. Note that the traces transform naturally by a density argument (possible due to Remark 6.29, 2.). Therefore we obtain

$$(III) = \sum_{\pm} \int_{\hat{\Sigma}} a^2(s)[\phi_\varepsilon^A N_{\partial\Omega_t^C}\cdot\nabla\phi_\varepsilon^A]|_{\overline{X}(\pm\delta,s,t)}|\det d_s[X(\pm\delta,.,t)]|\,d\mathcal{H}^{N-1}(s)$$
$$+\int_{\partial\Sigma} \operatorname{tr} a^2|_\sigma \int_{-\delta}^{\delta} [\phi_\varepsilon^A \partial_{N_{\partial\Omega}}\phi_\varepsilon^A]|_{\overline{X}(r,Y(\sigma,0),t)}|\det d_{(r,\sigma)}[X(.,Y(.,0),t)]|\,dr\,d\mathcal{H}^{N-2}(\sigma).$$

We apply Lemma 6.32 and for the last integral we use Lemma 6.5 and $\int_{\mathbb{R}}\theta_0'\theta_0'' = 0$. This yields

$$|(III)| \leq Ce^{-c/\varepsilon}\|a\|^2_{L^2(\hat{\Sigma})} + C\varepsilon\|\operatorname{tr} a|_{\partial\Sigma}\|^2_{L^2(\partial\Sigma)}.$$

Theorem 2.20 implies $\|\operatorname{tr} a|_{\partial\Sigma}\|_{L^2(\partial\Sigma)} \leq C\|a\|_{H^1(\hat{\Sigma}^0)}$ and Lemma 6.33, 1. yields the claim. □

Next we consider $B^C_{\varepsilon,t}$ on $V^\perp_{\varepsilon,t}\times V^\perp_{\varepsilon,t}$.

Lemma 6.35. *There are* $\nu,\varepsilon_0>0$ *such that*

$$B^C_{\varepsilon,t}(\psi,\psi) \geq \nu\left[\frac{1}{\varepsilon^2}\|\psi\|^2_{L^2(\Omega_t^C)} + \|\nabla\psi\|^2_{L^2(\Omega_t^C)}\right]$$

for all $\psi \in V^\perp_{\varepsilon,t}$ *and* $\varepsilon \in (0,\varepsilon_0]$, $t\in[0,T]$.

Proof. It is enough to show that there are $\tilde{\nu},\tilde{\varepsilon}_0>0$ such that

$$\tilde{B}_{\varepsilon,t}(\psi,\psi) := \int_{\Omega_t^C} |\nabla\psi|^2 + \frac{1}{\varepsilon^2}f''(\theta_0|_{\rho_\varepsilon(.,t)})\psi^2\,dx \geq \frac{\tilde{\nu}}{\varepsilon^2}\|\psi\|^2_{L^2(\Omega_t^C)} \tag{6.48}$$

for all $\psi \in V^\perp_{\varepsilon,t}$ and $\varepsilon\in(0,\tilde{\varepsilon}_0]$, $t\in[0,T]$. Then the claim follows analogously as in the 2D-case, cf. the proof of Lemma 6.22.

Analogously to the 2D-case we show (6.48) by reducing to Neumann boundary problems in normal direction. To this end let $\tilde{\psi}_t := \psi|_{X(.,t)}$ for $\psi\in V^\perp_{\varepsilon,t}$. Then $\tilde{\psi}_t \in H^1((-\delta,\delta)\times\hat{\Sigma}^\circ)$ and

$$\nabla\psi|_{X(.,t)} = \nabla r|_{\overline{X}(.,t)}\partial_r\tilde{\psi}_t + (D_xs)^\top|_{\overline{X}(.,t)}\nabla_{\hat{\Sigma}}\tilde{\psi}_t$$

in Ω_t^C due to Corollary 3.10 and Remark 6.29, 3.-4. Therefore

$$|\nabla\psi|^2|_{X(.,t)} = (\partial_r,\nabla_{\hat{\Sigma}})^\top\tilde{\psi}_t\begin{pmatrix}|\nabla r|^2 & (D_xs\nabla r)^\top\\ D_xs\nabla r & D_xs(D_xs)^\top\end{pmatrix}|_{\overline{X}(.,t)}\begin{pmatrix}\partial_r\\ \nabla_{\hat{\Sigma}}\end{pmatrix}\tilde{\psi}_t$$

in Ω_t^C. Theorem 3.7, a Taylor expansion and Young's inequality imply

$$|\nabla\psi|^2|_{X(.,t)} \geq (1-Cr^2)(\partial_r\tilde{\psi}_t)^2 + c|\nabla_{\hat{\Sigma}}\tilde{\psi}_t|^2 \tag{6.49}$$

in Ω_t^C for some $c, C > 0$. The second term is not needed here. In order to get Cr^2 small enough (which will be precise later), we fix $\tilde{\delta} > 0$ small and estimate separately for r in

$$I_{s,t}^\varepsilon := (-\tilde{\delta}, \tilde{\delta}) + \varepsilon h_\varepsilon(s,t) \quad \text{and} \quad \hat{I}_{s,t}^\varepsilon := (-\delta, \delta) \setminus I_{s,t}^\varepsilon.$$

If $\varepsilon_0 = \varepsilon_0(\tilde{\delta}, \overline{C}_0) > 0$ is small, then for all $\varepsilon \in (0, \varepsilon_0]$ and $s \in \hat{\Sigma}$, $t \in [0,T]$ we have

$$f''(\theta_0(\rho_\varepsilon|_{\overline{X}(r,s,t)})) \geq c_0 > 0 \quad \text{for } r \in \hat{I}_{s,t}^\varepsilon, \quad |r| \leq \tilde{\delta} + \varepsilon |h_\varepsilon(s,t)| \leq 2\tilde{\delta} \quad \text{for } r \in I_{s,t}^\varepsilon.$$

Let $\tilde{c} = \tilde{c}(\tilde{\delta}) := 4C\tilde{\delta}^2$ with C from (6.49). Then for all $\varepsilon \in (0, \varepsilon_0]$ it holds

$$\begin{aligned} \tilde{B}_{\varepsilon,t}^C(\psi,\psi) \geq & \int_{\hat{\Sigma}} \int_{\hat{I}_{s,t}^\varepsilon} \frac{c_0}{\varepsilon^2} \tilde{\psi}_t^2 J_t|_{(r,s)} \, dr \, d\mathcal{H}^{N-1}(s) \\ & + \int_{\hat{\Sigma}} \int_{I_{s,t}^\varepsilon} \left[(1-\tilde{c})(\partial_r \tilde{\psi}_t)^2 + \frac{1}{\varepsilon^2} f''(\theta_0(\rho_\varepsilon|_{\overline{X}(.,t)})) \tilde{\psi}_t^2 \right] J_t|_{(r,s)} \, dr \, d\mathcal{H}^{N-1}(s). \end{aligned}$$

We set $F_{\varepsilon,s,t}(z) := \varepsilon(z + h_\varepsilon(s,t))$ and $\tilde{J}_{\varepsilon,s,t}(z) := J_t(F_{\varepsilon,s,t}(z), s)$ for all $z \in [-\frac{\delta}{\varepsilon}, \frac{\delta}{\varepsilon}] - h_\varepsilon(s,t)$ and $(s,t) \in \Sigma \times [0,T]$. Moreover, let $I_{\varepsilon,\tilde{\delta}} := (-\frac{\tilde{\delta}}{\varepsilon}, \frac{\tilde{\delta}}{\varepsilon})$ and $\Psi_{\varepsilon,s,t} := \sqrt{\varepsilon}\tilde{\psi}_t(F_{\varepsilon,s,t}(.), s)$. Due to Remark 6.29 it holds $\Psi_{\varepsilon,s,t} \in H^1(I_{\varepsilon,\tilde{\delta}})$ for a.e. $s \in \hat{\Sigma}$ and all $t \in [0,T]$ and together with Lemma 6.5, 1. we obtain that the second inner integral in the estimate above equals $1/\varepsilon^2$ times

$$B_{\varepsilon,s,t}^{\tilde{c}}(\Psi_{\varepsilon,s,t}, \Psi_{\varepsilon,s,t}) := \int_{I_{\varepsilon,\tilde{\delta}}} \left[(1-\tilde{c})(\frac{d}{dz}\Psi_{\varepsilon,s,t})^2 + f''(\theta_0(z))(\Psi_{\varepsilon,s,t})^2 \right] \tilde{J}_{\varepsilon,s,t} \, dz$$

for a.e. $s \in \hat{\Sigma}$ and all $t \in [0,T]$. Therefore (6.48) follows if we show with the same c_0 as above

$$B_{\varepsilon,s,t}^{\tilde{c}}(\Psi_{\varepsilon,s,t}, \Psi_{\varepsilon,s,t}) \geq \overline{c} \|\Psi_{\varepsilon,s,t}\|_{L^2(I_{\varepsilon,\tilde{\delta}}, \tilde{J}_{\varepsilon,s,t})}^2 - \frac{c_0}{2} \|\tilde{\psi}_t(.,s)\|_{L^2(\hat{I}_{s,t}^\varepsilon, J_t(.,s))}^2 \tag{6.50}$$

for $\varepsilon \in (0, \varepsilon_0]$, a.e. $s \in \hat{\Sigma}$ and all $t \in [0,T]$ with some $\varepsilon_0, \overline{c} > 0$ independent of ε, s, t.

The estimate (6.50) can be proven for appropriately small $\tilde{\delta}$ in the analogous way as in the 2D-case, cf. the proof of Lemma 6.22. One uses the integral characterization for $\psi \in V_{\varepsilon,t}^\perp$ from Lemma 6.33, 2. and results from Section 6.1.3.2 for the operator

$$\mathcal{L}_{\varepsilon,s,t}^0 := -(\tilde{J}_{\varepsilon,s,t})^{-1} \frac{d}{dz} \left(\tilde{J}_{\varepsilon,s,t} \frac{d}{dz} \right) + f''(\theta_0)$$

on $H^2(I_{\varepsilon,\tilde{\delta}})$ with homogeneous Neumann boundary condition, in particular Theorem 6.8. $\square$

For $B_{\varepsilon,t}^C$ on $V_{\varepsilon,t} \times V_{\varepsilon,t}^\perp$ it holds

Lemma 6.36. *There are $\varepsilon_0, C > 0$ such that*

$$|B_{\varepsilon,t}^C(\phi,\psi)| \leq \frac{C}{\varepsilon} \|\phi\|_{L^2(\Omega_t^C)} \|\psi\|_{L^2(\Omega_t^C)} + \frac{1}{4} B_{\varepsilon,t}^C(\psi,\psi) + C\varepsilon \|a\|_{H^1(\hat{\Sigma}^\circ)}^2$$

for all $\phi = a(s(.,t))\phi_\varepsilon^A(.,t) \in V_{\varepsilon,t}$, $\psi \in V_{\varepsilon,t}^\perp$ and $\varepsilon \in (0, \varepsilon_0]$, $t \in [0,T]$.

First we prove the following auxiliary estimate.

Lemma 6.37. *Let $\overline{\varepsilon} > 0$ be fixed. Then there is a $\overline{C} > 0$ (independent of ψ, ε, t) such that*

$$\|\mathrm{tr}\,\psi\|^2_{L^2(\partial\Omega_t^C)} \leq \overline{C}\left[\varepsilon\|\nabla\psi\|^2_{L^2(\Omega_t^C)} + \frac{1}{\varepsilon}\|\psi\|^2_{L^2(\Omega_t^C)}\right]$$

for all $\psi \in H^1(\Omega_t^C)$ and $\varepsilon \in (0, \overline{\varepsilon}]$, $t \in [0,T]$.

Proof. Because of Remark 6.29, Theorem 2.6 and Theorem 3.7 it is equivalent to prove the estimate for $S := (-\delta,\delta) \times \hat{\Sigma}^\circ$ instead of Ω_t^C and $\nabla_S = (\partial_r, \nabla_{\hat{\Sigma}})$ instead of ∇. For the S-case we use the idea from Evans [Ev], 5.10, problem 7. Note that S is a smooth manifold with thin singular set in the sense of Amann, Escher [AE], Chapter 3.1. Therefore the outer unit normal $N_{\partial S}$ is defined $\mathcal{H}^{N-1}$-a.e. on ∂S and the Gauß-Theorem holds for C^2-vector fields on $\overline{S}$ due to [AE], Theorem XII.3.15 and Remark XII.3.16(c). Let $w_1 \in C^2([-\delta,\delta])$ with $w_1|_{\pm\delta} = \pm 1$ and w_2 be a C^2-vector field on $\hat{\Sigma}$ such that $\vec{w}_2|_{\partial\hat{\Sigma}} = N_{\partial\hat{\Sigma}}$. Then

$$\vec{w} : \overline{S} = [-\delta,\delta] \times \hat{\Sigma} \to \mathbb{R}^{N+1} : (r,s) \mapsto (w_1(r), 0) + (0, \vec{w}_2(s))$$

is a C^2-vector field on $\overline{S}$ such that $\vec{w}\cdot N_{\partial S} \geq 1$ holds $\mathcal{H}^{N-1}$-a.e. on ∂S. Hence for all $\psi \in C^2(\overline{S})$:

$$\|\mathrm{tr}\,\psi\|^2_{L^2(\partial S)} \leq \int_{\partial S} \psi^2 \vec{w}\cdot N_{\partial S}\, d\mathcal{H}^{N-1} = \int_S \mathrm{div}_S(\psi^2\vec{w})\, d\mathcal{H}^N = \int_S \psi^2 \mathrm{div}_S\vec{w} + 2\psi\vec{w}\cdot\nabla_S\psi\, d\mathcal{H}^N.$$

Therefore Young's inequality and $1 \leq \overline{\varepsilon}/\varepsilon$ yields

$$\|\mathrm{tr}\,\psi\|^2_{L^2(\partial S)} \leq C\left[\varepsilon\|\nabla_S\psi\|^2_{L^2(S)} + (1+\frac{1}{\varepsilon})\|\psi\|^2_{L^2(S)}\right] \leq \overline{C}\left[\varepsilon\|\nabla_S\psi\|^2_{L^2(\Omega_t^C)} + \frac{1}{\varepsilon}\|\psi\|^2_{L^2(\Omega_t^C)}\right]$$

for all $\psi \in C^2(\overline{S})$ and $\varepsilon \in (0,\overline{\varepsilon}]$, where $\overline{C} > 0$ is independent of ψ, ε. Hence the estimate also follows for all $\psi \in H^1(S)$ via density due to Remark 6.29 and Theorem 2.20. $\square$

Proof of Lemma 6.36. We rewrite $B^C_{\varepsilon,t}(\phi,\psi)$ in order to use Lemma 6.32 and Lemma 6.33. It holds $\nabla\phi = \nabla(a|_{s(.,t)})\phi^A_\varepsilon + a|_{s(.,t)}\nabla\phi^A_\varepsilon|_{(.,t)}$ and integration by parts, cf. Remark 6.29, 2., yields

$$\begin{aligned}\int_{\Omega_t^C} a(s)\nabla\phi^A_\varepsilon|_{(.,t)}\cdot\nabla\psi\, dx = &-\int_{\Omega_t^C}\left[\nabla(a(s))\cdot\nabla\phi^A_\varepsilon + a(s)\Delta\phi^A_\varepsilon|_{(.,t)}\right]\psi\, dx \\ &+ \int_{\partial\Omega_t^C} N_{\partial\Omega_t^C}\cdot\nabla\phi^A_\varepsilon|_{(.,t)}\mathrm{tr}\left[a(s(.,t))\psi\right]\, d\mathcal{H}^{N-1}.\end{aligned}$$

Therefore we obtain

$$\begin{aligned}B^C_{\varepsilon,t}(\phi,\psi) = &\int_{\Omega_t^C} a(s)|_{(.,t)}\psi\mathcal{L}^C_{\varepsilon,t}\phi^A_\varepsilon|_{(.,t)}\, dx + \int_{\partial\Omega_t^C} N_{\partial\Omega_t^C}\cdot\nabla\phi^A_\varepsilon|_{(.,t)}\mathrm{tr}\left[a(s(.,t))\psi\right]\, d\mathcal{H}^{N-1} \\ &+ \int_{\Omega_t^C}\nabla(a(s))|_{(.,t)}\cdot\left[\phi^A_\varepsilon|_{(.,t)}\nabla\psi - \nabla\phi^A_\varepsilon|_{(.,t)}\psi\right]\, dx =: (I) + (II) + (III).\end{aligned}$$

Ad (I). The Hölder Inequality yields $|(I)| \leq \|a(s|_{(.,t)})\mathcal{L}^C_{\varepsilon,t}\phi^A_\varepsilon(.,t)\|_{L^2(\Omega_t^C)}\|\psi\|_{L^2(\Omega_t^C)}$, where

$$\|a(s|_{(.,t)})\mathcal{L}^C_{\varepsilon,t}\phi^A_\varepsilon(.,t)\|^2_{L^2(\Omega_t^C)} = \int_{\hat{\Sigma}} a^2(s)\int_{-\delta}^{\delta}(\mathcal{L}^C_{\varepsilon,t}\phi^A_\varepsilon(.,t))^2|_{X(r,s,t)}\, J_t(r,s)\, dr\, d\mathcal{H}^{N-1}(s)$$

due to Theorem 2.6. By Lemma 6.32 and Lemma 6.5 the inner integral is estimated by $\frac{C}{\varepsilon^2}$, cf. also the estimate of (I) in the proof of Lemma 6.23 for more details. Hence Lemma 6.33, 1. yields

$$|(I)| \leq \frac{C}{\varepsilon}\|a\|_{L^2(\hat{\Sigma})}\|\psi\|_{L^2(\Omega_t^C)} \leq \frac{\tilde{C}}{\varepsilon}\|\phi\|_{L^2(\Omega_t^C)}\|\psi\|_{L^2(\Omega_t^C)}$$

for all $t \in [0,T]$ and $\varepsilon \in (0,\varepsilon_0]$ if $\varepsilon_0 > 0$ is small.

Ad (II). The Hölder Inequality yields

$$|(II)| \leq \|\mathrm{tr}\,\psi\|_{L^2(\partial\Omega_t^C)}\|\mathrm{tr}(a(s|_{(.,t)}))N_{\partial\Omega_t^C}\cdot\nabla\phi_\varepsilon^A|_{(.,t)}\|_{L^2(\partial\Omega_t^C)}.$$

We transform the second integral as in the estimate of (III) in the proof of Lemma 6.34 with Theorem 2.6 and Remark 6.29. Then Lemma 6.32, Lemma 6.5 and Theorem 2.20 yield

$$\|a(s)N_{\partial\Omega_t^C}\cdot\nabla\phi_\varepsilon^A|_{(.,t)}\|_{L^2(\partial\Omega_t^C)} \leq C\|\mathrm{tr}\,a|_{\partial\Sigma}\|_{L^2(\partial\Sigma)} + Ce^{-c/\varepsilon}\|a\|_{L^2(\hat{\Sigma})} \leq C\|a\|_{H^1(\hat{\Sigma}^\circ)}.$$

We estimate $\|\mathrm{tr}\,\psi\|_{L^2(\partial\Omega_t^C)}$ with Lemma 6.37. Hence Young's inequality and Lemma 6.35 imply

$$|(II)| \leq \frac{\nu}{8\varepsilon\overline{C}}\|\mathrm{tr}\,\psi\|^2_{L^2(\partial\Omega_t^C)} + \tilde{C}\varepsilon\|a\|^2_{H^1(\hat{\Sigma}^\circ)} \leq \frac{1}{8}B^C_{\varepsilon,t}(\psi,\psi) + \tilde{C}\varepsilon\|a\|^2_{H^1(\hat{\Sigma}^\circ)},$$

where $\overline{C}$ is as in Lemma 6.37.

Ad (III). With Remark 6.29, 1. we can transform $(III) = \int_{\hat{\Sigma}}\nabla_{\hat{\Sigma}}a(s)\cdot g_t(s)\,d\mathcal{H}^{N-1}(s)$, where

$$g_t(s) := \int_{-\delta}^{\delta} D_x s|_{\overline{X}(r,s,t)}\left[\phi_\varepsilon^A(.,t)\nabla\psi - \nabla\phi_\varepsilon^A(.,t)\psi\right]|_{X(r,s,t)}\,J_t(r,s)\,dr.$$

It holds $\nabla\psi|_{X(.,t)} = \nabla r|_{\overline{X}(.,t)}\partial_r\tilde{\psi}_t + (D_x s)^\top|_{\overline{X}(.,t)}\nabla_{\hat{\Sigma}}\tilde{\psi}_t$ with $\tilde{\psi}_t := \psi|_{X(.,t)}$ in Ω_t^C because of Remark 6.29, 3. and Corollary 3.10. For the $\nabla_{\hat{\Sigma}}\tilde{\psi}_t$-term in g_t we use

$$\left|D_x s(D_x s)^\top|_{\overline{X}(r,s,t)} - D_x s(D_x s)^\top|_{\overline{X}(0,s,t)}\right| \leq C|r|.$$

Therefore $|g_t(s)|$ is for a.e. $s \in \hat{\Sigma}$ estimated by

$$|D_x s(D_x s)^\top|_{\overline{X}_0(s,t)}\left|\int_{-\delta}^{\delta}\left[\phi_\varepsilon^A|_{\overline{X}(.,t)}\nabla_{\hat{\Sigma}}\tilde{\psi}_t J_t\right]|_{(r,s)}\,dr\right| + \int_{-\delta}^{\delta}\left|(\tilde{\psi}_t J_t)|_{(r,s)}D_x s\nabla\phi_\varepsilon^A|_{\overline{X}(r,s,t)}\right|\,dr$$
$$+\int_{-\delta}^{\delta}\left[C\left|r\,\nabla_{\hat{\Sigma}}\tilde{\psi}_t|_{(r,s)}\right| + \left|D_x s\nabla r|_{\overline{X}(r,s,t)}\partial_r\tilde{\psi}_t|_{(r,s)}\right|\right]\cdot\left|\phi_\varepsilon^A|_{\overline{X}(r,s,t)}J_t|_{(r,s)}\right|\,dr.$$

We use $\psi \in V_{\varepsilon,t}^\perp$ to estimate the first term. Due to Lemma 2.17 and since integration gives a bounded linear operator on $L^2(-\delta,\delta)$, we can apply $\nabla_{\hat{\Sigma}}$ to the identity in Lemma 6.33, 2. and commute integration with $\nabla_{\hat{\Sigma}}$. Therefore the first term is bounded by

$$C\left|\int_{-\delta}^{\delta}\left[\left(\nabla_{\hat{\Sigma}}(\phi_\varepsilon^A|_{\overline{X}(.,t)})J_t + \phi_\varepsilon^A|_{\overline{X}(.,t)}\nabla_{\hat{\Sigma}}J_t\right)\tilde{\psi}_t\right]|_{(r,s)}\,dr\right|$$

for a.e. $s \in \hat{\Sigma}$. Now we make use of the structure of ϕ_ε^A. Due to (6.41) it holds in Ω_t^C

$$\nabla\phi_\varepsilon^A|_{(.,t)} = \frac{1}{\sqrt{\varepsilon}}\left(\theta_0''|_{\rho_\varepsilon} q|_{(s,t)} + \varepsilon\partial_\rho\hat{v}^C|_{(\rho_\varepsilon,H_\varepsilon,\sigma,t)}\right)\left[\frac{\nabla r}{\varepsilon} - (D_x s)^\top \nabla_{\hat{\Sigma}} h_\varepsilon|_{(s,t)}\right]$$
$$+ \frac{1}{\sqrt{\varepsilon}}\left[(D_x s)^\top \nabla_{\hat{\Sigma}} q|_{(s,t)}\theta_0'|_{\rho_\varepsilon} + \nabla b\,\partial_H \hat{v}^C|_{(\rho_\varepsilon,H_\varepsilon,\sigma,t)}\right] + \sqrt{\varepsilon}(D_x\sigma)^\top \nabla_{\partial\Sigma}\hat{v}^C|_{(\rho_\varepsilon,H_\varepsilon,\sigma,t)},$$

where all terms are evaluated at $(.,t)$. Moreover, instead of $\nabla_{\hat{\Sigma}}(\phi_\varepsilon^A|_{\overline{X}})$ it is equivalent to estimate $\nabla_\tau\phi_\varepsilon^A|_{\overline{X}} = \nabla\phi_\varepsilon^A|_{\overline{X}} - \nabla r\partial_r(\phi_\varepsilon^A|_{\overline{X}})$ due to Corollary 3.10. The latter identity yields

$$\begin{aligned}\nabla_\tau\phi_\varepsilon^A|_{\overline{X}} &= -\frac{1}{\sqrt{\varepsilon}}\left(\theta_0''|_{\rho_\varepsilon} q|_{(s,t)} + \varepsilon\partial_\rho\hat{v}^C|_{(\rho_\varepsilon,H_\varepsilon,\sigma,t)}\right)(D_x s)^\top \nabla_{\hat{\Sigma}} h_\varepsilon|_{(s,t)}\\ &+\frac{1}{\sqrt{\varepsilon}}\left[(D_x s)^\top \nabla_{\hat{\Sigma}} q|_{(s,t)}\theta_0'|_{\rho_\varepsilon} + \nabla b\,\partial_H \hat{v}^C|_{(\rho_\varepsilon,H_\varepsilon,\sigma,t)}\right] + \sqrt{\varepsilon}(D_x\sigma)^\top \nabla_{\partial\Sigma}\hat{v}^C|_{(\rho_\varepsilon,H_\varepsilon,\sigma,t)},\end{aligned} \tag{6.51}$$

where the terms on the right hand side are evaluated at $\overline{X}$. Note that compared to $\nabla\phi_\varepsilon^A$ the ε-order is better by one. Therefore we can control the terms in the above estimate for $|g_t(s)|$. The Hölder Inequality, Lemma 6.5 and $D_x s\nabla r|_{\overline{X}_0(s,t)} = 0$ yield for a.e. $s \in \hat{\Sigma}$

$$|g_t(s)| \leq C\|\tilde{\psi}_t(.,s)\|_{L^2(-\delta,\delta;J_t(.,s))} + C\varepsilon\|(\partial_r,\nabla_{\hat{\Sigma}})\tilde{\psi}_t(.,s)\|_{L^2(-\delta,\delta;J_t(.,s))}.$$

Due to Remark 6.29 and Corollary 3.10 it holds $|(\partial_r,\nabla_{\hat{\Sigma}})\tilde{\psi}_t| \leq C|\nabla\psi|_{X(.,t)}|$. Therefore

$$\begin{aligned}|(III)| &\leq C\|\nabla_{\hat{\Sigma}} a\|_{L^2(\hat{\Sigma})}(\|\psi\|_{L^2(\Omega_t^C)} + \varepsilon\|\nabla\psi\|_{L^2(\Omega_t^C)})\\ &\leq C\varepsilon^2\|\nabla_{\hat{\Sigma}} a\|^2_{L^2(\hat{\Sigma})} + \frac{\nu}{8}\left[\frac{1}{\varepsilon^2}\|\psi\|^2_{L^2(\Omega_t^C)} + \|\nabla\psi\|^2_{L^2(\Omega_t^C)}\right],\end{aligned}$$

where we used Young's inequality in the second step and ν is as in Lemma 6.35. The last term is estimated by $\frac{1}{8}B^C_{\varepsilon,t}(\psi,\psi)$ due to Lemma 6.35. With Lemma 2.15 the claim follows. ☐

Finally, we put together Lemma 6.34-6.36.

Theorem 6.38. *There are $\varepsilon_0, C, c_0 > 0$ such that for all $\varepsilon \in (0,\varepsilon_0]$, $t \in [0,T]$ and $\psi \in H^1(\Omega_t^C)$ with $\psi|_{X(.,s,t)} = 0$ for a.e. $s \in Y(\partial\Sigma \times [\frac{3}{2}\mu_0, 2\mu_0])$ it holds*

$$B^C_{\varepsilon,t}(\psi,\psi) \geq -C\|\psi\|^2_{L^2(\Omega_t^C)} + c_0\|\nabla_\tau\psi\|^2_{L^2(\Omega_t^C)}.$$

Remark 6.39.

1. The estimate can be refined, cf. the proof below.
2. Theorem 6.38 directly yields Theorem 6.30, cf. the beginning of Section 6.3.1.

Proof of Theorem 6.38. Due to Lemma 6.33 we can uniquely represent any $\psi \in \tilde{H}^1(\Omega_t^C)$ as

$$\psi = \phi + \phi^\perp \quad \text{with } \phi = [a(s)\phi_\varepsilon^A]|_{(.,t)} \in V_{\varepsilon,t} \text{ and } \phi^\perp \in V_{\varepsilon,t}^\perp.$$

Lemma 6.34 and Lemma 6.36 yield for $\varepsilon_0 > 0$ small and all $\varepsilon \in (0,\varepsilon_0]$, $t \in [0,T]$ that

$$\begin{aligned}B^C_{\varepsilon,t}(\psi,\psi) &= B^C_{\varepsilon,t}(\phi,\phi) + 2B^C_{\varepsilon,t}(\phi,\phi^\perp) + B^C_{\varepsilon,t}(\phi^\perp,\phi^\perp)\\ &\geq -C\|\phi\|^2_{L^2(\Omega_t^C)} + (c_0 - C\varepsilon)\|a\|^2_{H^1(\hat{\Sigma}^\circ)} - \frac{C}{\varepsilon}\|\phi\|_{L^2(\Omega_t^C)}\|\phi^\perp\|_{L^2(\Omega_t^C)} + \frac{B^C_{\varepsilon,t}(\phi^\perp,\phi^\perp)}{2}.\end{aligned}$$

The third term is estimated by $\frac{\nu}{4\varepsilon^2}\|\phi^\perp\|^2_{L^2(\Omega_t^C)} + \tilde{C}\|\phi\|^2_{L^2(\Omega_t^C)}$ due to Young's inequality, where ν is as in Lemma 6.35. Hence we obtain

$$B^C_{\varepsilon,t}(\psi,\psi) \geq -C\|\phi\|^2_{L^2(\Omega_t^C)} + \frac{\nu}{4\varepsilon^2}\|\phi^\perp\|^2_{L^2(\Omega_t^C)} + \frac{c_0}{2}\|a\|^2_{H^1(\hat{\Sigma}^\circ)} + \frac{\nu}{2}\|\nabla(\phi^\perp)\|^2_{L^2(\Omega_t^C)}$$

for all $\varepsilon \in (0,\varepsilon_0]$ and $t \in [0,T]$, where $\varepsilon_0 > 0$ is small.

It remains to include the $\nabla_\tau\psi$-term in the estimate. Because of the triangle inequality it holds $\|\nabla_\tau\psi\|_{L^2(\Omega_t^C)} \leq \|\nabla_\tau\phi\|_{L^2(\Omega_t^C)} + \|\nabla_\tau(\phi^\perp)\|_{L^2(\Omega_t^C)}$. Here

$$\nabla_\tau\phi|_{X(r,s,t)} = (D_x s)^\top|_{\overline{X}(r,s,t)}\nabla_{\hat{\Sigma}} a|_s \phi^A_\varepsilon|_{\overline{X}(r,s,t)} + a|_s \nabla_\tau \phi^A_\varepsilon|_{\overline{X}(r,s,t)}.$$

We already computed $\nabla_\tau\phi^A_\varepsilon$ in (6.51). An integral transformation with Remark 6.29, 1., the Fubini Theorem, Lemma 6.5 and Lemma 2.15, 3. yield $\|\nabla_\tau\phi\|_{L^2(\Omega_t^C)} \leq C\|a\|_{H^1(\hat{\Sigma}^\circ)}$. Moreover, Remark 6.29, 3. and Corollary 3.10 imply $\|\nabla_\tau(\phi^\perp)\|_{L^2(\Omega_t^C)} \leq C\|\nabla(\phi^\perp)\|_{L^2(\Omega_t^C)}$. Therefore

$$\|\nabla_\tau\psi\|^2_{L^2(\Omega_t^C)} \leq C(\|a\|^2_{H^1(\hat{\Sigma}^\circ)} + \|\nabla(\phi^\perp)\|^2_{L^2(\Omega_t^C)}).$$

Finally, together with the above estimate for $B^C_{\varepsilon,t}$ this shows the claim. $\square$

6.4 Spectral Estimate for (vAC)

In this section we prove the analogy of the spectral estimate (6.1) from the scalar case for the vector-valued Allen-Cahn equation (vAC1)-(vAC3) when the diffuse interface meets the boundary in the case of N dimensions, $N \geq 2$. The procedure is analogous to the scalar case in the last Section 6.3. The coordinates are the same, in particular we can use Remark 6.29. Hence the only new difficulty is that we have to consider the potential $W : \mathbb{R}^m \to \mathbb{R}$ from Definition 1.4 and vector-valued functions, i.e. the image space is $\mathbb{R}^m$ instead of $\mathbb{R}$. However, we already laid all the foundations to adapt the arguments from the scalar case. Under the assumption in Remark 4.28 we solved the model problems for the vector-valued case in Sections 4.3-4.4 and proved spectral properties for vector-valued Allen-Cahn-type operators in 1D in Section 6.1.4. We have a specific $\vec{u}^A_\varepsilon$ in mind, cf. Section 5.3.3. Nevertheless, as in the scalar case a more general form is enough to prove the spectral estimate. In the following we fix the assumptions for this section.

Let $\Omega \subset \mathbb{R}^N$ and $\Gamma = (\Gamma_t)_{t\in[0,T]}$ for $T > 0$ be as in Section 3.1 for $N \geq 2$ with contact angle $\alpha = \frac{\pi}{2}$ (MCF not needed). Moreover, let $\delta > 0$ be such that Theorem 3.7 holds for 2δ instead of δ. We use the notation for $\vec{n}_{\partial\Sigma}, \vec{n}_{\partial\Gamma}, Y, X_0, X, \mu_0, \mu_1, r, s, \sigma, b$ as in Theorem 3.7 and the definitions of several sets and of ∂_n, ∇_τ, J from Remark 3.8. Here for suitable $\mathbb{R}^m$-valued functions $\vec{\psi}$ we define $\partial_n\vec{\psi}$ and $\nabla_\tau\vec{\psi}$ component-wise. More precisely $\partial_n\vec{\psi} := (\partial_n\psi_1, ..., \partial_n\psi_m)$ and $\nabla_\tau\vec{\psi} := ((\nabla_\tau\psi_1)^\top, ..., (\nabla_\tau\psi_m)^\top)$. Note that Corollary 3.10 carries over to $\mathbb{R}^m$-valued functions, in particular

$$\nabla\vec{\psi} := (D_x\vec{\psi})^\top = \nabla r \partial_n\vec{\psi} + \nabla_\tau\vec{\psi}.$$

We consider height functions $\check{h}_1$ and $\check{h}_2 = \check{h}_2(\varepsilon)$ and assume (with a slight abuse of notation)

$$\check{h}_j \in B([0,T], C^0(\Sigma) \cap C^2(\hat{\Sigma})), j = 1,2, \quad \hat{\Sigma} := Y(\partial\Sigma \times [0,2\mu_0]), \quad C^2(\hat{\Sigma}) := C^2(\overline{\hat{\Sigma}^\circ}).$$

Moreover, consider $\check{C}_0 > 0$ such that $\|\check{h}_j\|_{B([0,T],C^0(\Sigma)\cap C^2(\hat{\Sigma}))} \leq \check{C}_0$ for $j = 1,2$. We set $\check{h}_\varepsilon := \check{h}_1 + \varepsilon\check{h}_2$ for $\varepsilon > 0$ small and introduce the scaled variables

$$\check{\rho}_\varepsilon := \frac{r - \varepsilon\check{h}_\varepsilon(s,t)}{\varepsilon} \quad \text{in } \overline{\Gamma(2\delta)}, \quad H_\varepsilon := \frac{b}{\varepsilon} \quad \text{in } \overline{\Gamma^C(2\delta, 2\mu_1)}.$$

For $W:\mathbb{R}^m\to\mathbb{R}$ as in Definition 1.4 and any fixed distinct pair $\vec{u}_\pm$ of minimizers of W let $\vec{\theta}_0$ be as in Remark 4.27, 1. We make the assumption $\dim\ker\check{L}_0=1$, cf. Remark 4.28, where $\check{L}_0$ is as in (4.26). Moreover, let $\check{u}_1^C:\overline{\mathbb{R}^2_+}\times\partial\Sigma\times[0,T]\to\mathbb{R}^m:(\rho,H,\sigma,t)\mapsto\check{u}_1^C(\rho,H,\sigma,t)$ be in the space $B([0,T];C^2(\partial\Sigma,H^2_{(0,\gamma)}(\mathbb{R}^2_+)^m))$ for some $\gamma>0$. Then we define

$$\vec{u}_1^C(x,t):=\check{u}_1^C(\check{\rho}_\varepsilon(x,t),H_\varepsilon(x,t),\sigma(x,t),t)\quad\text{ for }(x,t)\in\overline{\Gamma^C(2\delta,2\mu_1)}.$$

For $\varepsilon>0$ small we consider

$$\vec{u}^A_\varepsilon=\begin{cases}\vec{\theta}_0(\check{\rho}_\varepsilon)+\mathcal{O}(\varepsilon^2) & \text{in }\Gamma(\delta,\mu_0),\\ \vec{\theta}_0(\check{\rho}_\varepsilon)+\varepsilon\vec{u}_1^C+\mathcal{O}(\varepsilon^2) & \text{in }\Gamma^C(\delta,2\mu_0),\\ \vec{u}_\pm+\mathcal{O}(\varepsilon) & \text{in }Q_T^\pm\setminus\Gamma(\delta),\end{cases}$$

where $\mathcal{O}(\varepsilon^k)$ are $\mathbb{R}^m$-valued measurable functions bounded by $C\varepsilon^k$.

Remark 6.40. We can also include an additional term of the form $\varepsilon\vec{\theta}_1(\check{\rho}_\varepsilon)\check{p}_\varepsilon(s,t)$ in $\vec{u}^A_\varepsilon$ on $\Gamma(\delta)$, where $\check{p}_\varepsilon\in B([0,T],C^0(\Sigma)\cap C^2(\hat{\Sigma}))$ satisfies a uniform estimate for ε small and

$$\vec{\theta}_1\in C^0_b(\mathbb{R})^m\quad\text{ with }\quad\int_{\mathbb{R}}(\vec{\theta}_0',\sum_{\xi\in\mathbb{N}_0^m,|\xi|=1}\partial^\xi D^2W(\vec{\theta}_0)(\vec{\theta}_1)^\xi\vec{\theta}_0')_{\mathbb{R}^m}=0.\tag{6.52}$$

See Remark 6.43, 2. below for details.

We obtain the following spectral estimate:

Theorem 6.41 (Spectral Estimate for (vAC) in ND). *There are $\varepsilon_0,C,c_0>0$ independent of the $\check{h}_j$ for fixed $\check{C}_0$ such that for all $\varepsilon\in(0,\varepsilon_0],t\in[0,T]$ and $\vec{\psi}\in H^1(\Omega)^m$ it holds*

$$\begin{aligned}\int_\Omega|\nabla\vec{\psi}|^2+\frac{1}{\varepsilon^2}(\vec{\psi},D^2W(\vec{u}^A_\varepsilon(.,t))\vec{\psi})_{\mathbb{R}^m}\,dx&\\ \geq -C\|\vec{\psi}\|^2_{L^2(\Omega)^m}+\|\nabla\vec{\psi}\|^2_{L^2(\Omega\setminus\Gamma_t(\delta))^{N\times m}}&+c_0\|\nabla_\tau\vec{\psi}\|^2_{L^2(\Gamma_t(\delta))^{N\times m}}.\end{aligned}$$

We prove a spectral estimate on $\Omega_t^C:=\Gamma_t^C(\delta,2\mu_0)=X((-\delta,\delta)\times\hat{\Sigma}^\circ)$ for $t\in[0,T]$.

Theorem 6.42. *There are $\check{\varepsilon}_0,\check{C},\check{c}_0>0$ independent of the $\check{h}_j$ for fixed $\check{C}_0$ such that for all $\varepsilon\in(0,\check{\varepsilon}_0]$, $t\in[0,T]$ and $\vec{\psi}\in H^1(\Omega_t^C)^m$ with $\vec{\psi}|_{X(.,s,t)}=0$ for a.e. $s\in Y(\partial\Sigma\times[\frac{3}{2}\mu_0,2\mu_0])$:*

$$\int_{\Omega_t^C}|\nabla\vec{\psi}|^2+\frac{1}{\varepsilon^2}(\vec{\psi},D^2W(\vec{u}^A_\varepsilon(.,t))\vec{\psi})_{\mathbb{R}^m}\,dx\geq-\check{C}\|\vec{\psi}\|^2_{L^2(\Omega_t^C)^m}+\check{c}_0\|\nabla_\tau\vec{\psi}\|^2_{L^2(\Omega_t^C)^{N\times m}}.$$

The extra assumption on $\vec{\psi}$ is not needed but simplifies the proof, cf. Remark 6.43, 3. below.

Proof of Theorem 6.41. Note that $D^2W(\vec{u}^A_\varepsilon)$ is positive (semi-)definite on $Q_T^\pm\setminus\Gamma(\delta)$ for all $\varepsilon\in(0,\varepsilon_0]$ if $\varepsilon_0>0$ is small. Therefore it is enough to show the estimate in Theorem 6.41 for $\Gamma_t(\delta)$ instead of Ω. This can be done with Theorem 6.42 and Theorem 6.12, 1. in the analogous way as in the scalar case, cf. the proof of Theorem 6.28. □

6.4.1 Outline for the Proof of the Spectral Estimate close to the Contact Points

For the proof of Theorem 6.42 we can replace $\frac{1}{\varepsilon^2}D^2W(\vec{u}^A_\varepsilon(.,t))$ by

$$\frac{1}{\varepsilon^2}D^2W(\vec{\theta}_0|_{\check{\rho}_\varepsilon(.,t)})+\frac{1}{\varepsilon}\sum_{\xi\in\mathbb{N}_0^m,|\xi|=1}\partial^\xi D^2W(\vec{\theta}_0|_{\check{\rho}_\varepsilon(.,t)})(\vec{u}^C_1|_{(.,t)})^\xi$$

with a Taylor expansion. We construct an approximation $\vec{\phi}^A_\varepsilon(.,t)$ to the first eigenfunction of

$$\check{\mathcal{L}}^C_{\varepsilon,t}:=-\Delta+\frac{1}{\varepsilon^2}D^2W(\vec{\theta}_0|_{\check{\rho}_\varepsilon(.,t)})+\frac{1}{\varepsilon}\sum_{\xi\in\mathbb{N}_0^m,|\xi|=1}\partial^\xi D^2W(\vec{\theta}_0|_{\check{\rho}_\varepsilon(.,t)})(\vec{u}^C_1|_{(.,t)})^\xi\quad\text{on }\Omega^C_t$$

together with a homogeneous Neumann boundary condition. Then we split

$$\check{H}^1(\Omega^C_t):=\left\{\vec{\psi}\in H^1(\Omega^C_t)^m:\vec{\psi}|_{X(.,s,t)}=0\text{ for a.e. }s\in Y(\partial\Sigma\times[\frac{3}{2}\mu_0,2\mu_0])\right\}\tag{6.53}$$

along the subspace of tangential alterations of $\vec{\phi}^A_\varepsilon(.,t)$. Therefore we consider the ansatz

$$\begin{aligned}\vec{\phi}^A_\varepsilon(.,t)&:=\frac{1}{\sqrt{\varepsilon}}[\vec{v}^I_\varepsilon(.,t)+\varepsilon\vec{v}^C_\varepsilon(.,t)] &&\text{on }\Omega^C_t,\\ \vec{v}^I_\varepsilon(.,t)&:=\check{v}^I(\check{\rho}_\varepsilon(.,t),s(.,t),t):=\vec{\theta}_0'(\check{\rho}_\varepsilon(.,t))\,\check{q}(s(.,t),t) &&\text{on }\Omega^C_t,\\ \vec{v}^C_\varepsilon(.,t)&:=\check{v}^C(\check{\rho}_\varepsilon(.,t),H_\varepsilon(.,t),\sigma(.,t),t) &&\text{on }\Omega^C_t,\end{aligned}$$

where $\check{q}:\hat{\Sigma}\times[0,T]\to\mathbb{R}$ and $\check{v}^C:\overline{\mathbb{R}^2_+}\times\partial\Sigma\times[0,T]\to\mathbb{R}^m$. The $\frac{1}{\sqrt{\varepsilon}}$-factor is used for a normalization, see Lemma 6.45 below.

In Subsection 6.4.2 we expand $\check{\mathcal{L}}^C_{\varepsilon,t}\vec{\phi}^A_\varepsilon(.,t)$ and $\partial_{N_{\partial\Omega}}\vec{\phi}^A_\varepsilon(.,t)$ in a similar way as in Section 5.3 and choose $\check{q}$ and $\check{v}^C$ suitably. The $\check{q}$-term is introduced in order to enforce the compatibility condition for the equations for $\check{v}^C$. In Subsection 6.4.3 we characterize the $L^2(\Omega^C_t)^m$-orthogonal splitting of $\check{H}^1(\Omega^C_t)^m$ with respect to the subspace

$$\check{V}_{\varepsilon,t}:=\left\{\vec{\phi}=\check{a}(s(.,t))\vec{\phi}^A_\varepsilon(.,t):\check{a}\in\check{H}^1(\hat{\Sigma}^\circ)\right\},\tag{6.54}$$

$$\check{H}^1(\hat{\Sigma}^\circ):=\left\{\check{a}\in H^1(\hat{\Sigma}^\circ):\check{a}|_{Y(.,b)}=0\text{ for a.e. }b\in[\frac{3}{2}\mu_0,2\mu_0]\right\}.\tag{6.55}$$

Finally, in Subsection 6.4.4 we prove estimates for the bilinear form $\check{B}^C_{\varepsilon,t}$ associated to $\check{\mathcal{L}}^C_{\varepsilon,t}$ on $\check{V}_{\varepsilon,t}\times\check{V}_{\varepsilon,t}$, $\check{V}^\perp_{\varepsilon,t}\times\check{V}^\perp_{\varepsilon,t}$ and $\check{V}_{\varepsilon,t}\times\check{V}^\perp_{\varepsilon,t}$. Here for $\vec{\phi},\vec{\psi}\in H^1(\Omega^C_t)^m$ we set

$$\begin{aligned}\check{B}^C_{\varepsilon,t}(\vec{\phi},\vec{\psi})&:=\int_{\Omega^C_t}\nabla\vec{\phi}:\nabla\vec{\psi}\,dx\\ &+\int_{\Omega^C_t}(\vec{\phi},\left[\frac{1}{\varepsilon^2}D^2W(\vec{\theta}_0|_{\check{\rho}_\varepsilon(.,t)})+\frac{1}{\varepsilon}\sum_{\xi\in\mathbb{N}_0^m,|\xi|=1}\partial^\xi D^2W(\vec{\theta}_0|_{\check{\rho}_\varepsilon(.,t)})(\vec{u}^C_1|_{(.,t)})^\xi\right]\vec{\psi})_{\mathbb{R}^m}\,dx.\end{aligned}\tag{6.56}$$

6.4.2 Asymptotic Expansion for the Approximate Eigenfunction

Asymptotic Expansion of $\sqrt{\varepsilon}\check{\mathcal{L}}^C_{\varepsilon,t}\vec{\phi}^A_\varepsilon(.,t)$. First, we expand $\Delta\vec{v}^I_\varepsilon$ as in the inner expansion in Section 5.3.1. The lowest order $\mathcal{O}(\frac{1}{\varepsilon^2})$ equals $\frac{1}{\varepsilon^2}|\nabla r|^2|_{\overline{X}_0(s,t)}\vec{\theta}_0'''(\rho)\check{q}(s,t) = \frac{1}{\varepsilon^2}\vec{\theta}_0'''(\rho)\check{q}(s,t)$. In $\sqrt{\varepsilon}\check{\mathcal{L}}^C_{\varepsilon,t}\vec{\phi}^A_\varepsilon(.,t)$ the latter cancels with $\frac{1}{\varepsilon^2}D^2W(\vec{\theta}_0(\rho))\vec{\theta}_0'(\rho)\check{q}(s,t)$. At $\mathcal{O}(\frac{1}{\varepsilon})$ in $\Delta\vec{v}^I_\varepsilon$ we obtain

$$\begin{aligned}&\frac{1}{\varepsilon}\vec{\theta}_0'''(\rho)\check{q}(s,t)\left[(\rho+\check{h}_1)\partial_r(|\nabla r|^2\circ\overline{X})|_{(0,s,t)} - 2(D_x s\nabla r)^\top|_{\overline{X}_0(s,t)}\nabla_\Sigma\check{h}_1\right]\\ &+\frac{1}{\varepsilon}\vec{\theta}_0''(\rho)\left[\Delta r|_{\overline{X}_0(s,t)}\check{q}(s,t) + 2(D_x s\nabla r)^\top|_{\overline{X}_0(s,t)}\nabla_\Sigma\check{q}(s,t)\right] = \frac{1}{\varepsilon}\vec{\theta}_0''(\rho)\Delta r|_{\overline{X}_0(s,t)}\check{q}(s,t).\end{aligned}$$

The term $\frac{1}{\varepsilon}\Delta r|_{\overline{X}_0(s,t)}\check{q}(s,t)\vec{\theta}_0''(\rho)$ is left as a remainder.

For $\varepsilon\Delta\vec{v}^C_\varepsilon$ we use the expansion in Section 5.3.2.1, but without using a Taylor expansion for the $\check{h}_j$ because we only need the lowest order and we intended to reduce the regularity assumption on the $\check{h}_j$. More precisely, the (x,t)-terms in the formula for $\Delta\vec{v}^C_\varepsilon$ in Lemma 5.25 are expanded solely with (5.40). At the lowest order $\mathcal{O}(\frac{1}{\varepsilon})$ we get $\frac{1}{\varepsilon}\Delta^{\sigma,t}\check{v}^C$, where $\Delta^{\sigma,t} := \partial_\rho^2 + |\nabla b|^2|_{\overline{X}_0(\sigma,t)}\partial_H^2$ for $(\sigma,t)\in\partial\Sigma\times[0,T]$. Moreover, the W-parts yield

$$\frac{1}{\varepsilon}D^2W(\vec{\theta}_0(\rho))\check{v}^C + \frac{1}{\varepsilon}\sum_{\xi\in\mathbb{N}_0^m,|\xi|=1}\partial^\xi D^2W(\vec{\theta}_0(\rho))(\check{u}_1^C)^\xi\check{v}^I.$$

To obtain an equation for $\check{v}^C$ in (ρ,H,σ,t) we use a Taylor expansion for $\check{q}(Y(\sigma,.),t)|_{[0,2\mu_0]}$:

$$\check{q}(Y(\sigma,\varepsilon H),t) = \check{q}(\sigma,t) + \mathcal{O}(\varepsilon H) \quad \text{for } (\sigma,\varepsilon H)\in\partial\Sigma\times[0,\frac{2\mu_0}{\varepsilon}].$$

Therefore we require in $\overline{\mathbb{R}^2_+}\times\partial\Sigma\times[0,T]$:

$$\left[-\Delta^{\sigma,t} + D^2W(\vec{\theta}_0(\rho))\right]\check{v}^C = -\sum_{\xi\in\mathbb{N}_0^m,|\xi|=1}\partial^\xi D^2W(\vec{\theta}_0(\rho))(\check{u}_1^C|_{(\rho,H,\sigma,t)})^\xi\vec{\theta}_0'|_\rho\check{q}|_{(\sigma,t)}. \quad (6.57)$$

Asymptotic Expansion of $\sqrt{\varepsilon}\partial_{N_{\partial\Omega}}\vec{\phi}^A_\varepsilon(.,t)$. We expand as in Section 5.3.2.2. Note that in $\overline{\Omega^C_t}$

$$\begin{aligned}D_x\vec{v}^I_\varepsilon &= \check{q}|_{(s,t)}\vec{\theta}_0''(\check{\rho}_\varepsilon)\left[\frac{\nabla r}{\varepsilon} - (D_x s)^\top\nabla_\Sigma\check{h}_\varepsilon|_{(s,t)}\right]^\top + \vec{\theta}_0'(\check{\rho}_\varepsilon)(\nabla_\Sigma\check{q}|_{(s,t)})^\top D_x s,\\ D_x\vec{v}^C_\varepsilon &= \partial_\rho\check{v}^C\left[\frac{\nabla r}{\varepsilon} - (D_x s)^\top\nabla_\Sigma\check{h}_\varepsilon|_{(s,t)}\right]^\top + \partial_H\check{v}^C\left[\frac{\nabla b}{\varepsilon}\right]^\top + D_{\partial\Sigma}\check{v}^C D_x\sigma,\end{aligned} \quad (6.58)$$

where the $\check{v}^C$-terms are evaluated at $(\check{\rho}_\varepsilon, H_\varepsilon,\sigma,t)$. In $\sqrt{\varepsilon}\partial_{N_{\partial\Omega}}\vec{\phi}^A_\varepsilon(.,t) = \sqrt{\varepsilon}D_x\vec{\phi}^A_\varepsilon N_{\partial\Omega}$ the lowest order is $\mathcal{O}(\frac{1}{\varepsilon})$ and given by $\frac{1}{\varepsilon}(N_{\partial\Omega}\cdot\nabla r)|_{\overline{X}_0(\sigma,t)}\vec{\theta}_0''(\rho)\check{q}|_{(\sigma,t)} = 0$ due to the 90°-contact angle condition. At $\mathcal{O}(1)$ we obtain

$$\begin{aligned}&\check{q}|_{(\sigma,t)}\vec{\theta}_0''(\rho)\left[(\rho+\check{h}_1|_{(\sigma,t)})\partial_r((N_{\partial\Omega}\cdot\nabla r)\circ\overline{X})|_{(0,\sigma,t)} - (D_x sN_{\partial\Omega})^\top|_{\overline{X}_0(\sigma,t)}\nabla_\Sigma\check{h}_1|_{(\sigma,t)}\right]\\ &+(D_x sN_{\partial\Omega})^\top|_{\overline{X}_0(\sigma,t)}\nabla_\Sigma\check{q}|_{(\sigma,t)}\vec{\theta}_0'(\rho) + 0\cdot\partial_\rho\check{v}^C|_{H=0} + (N_{\partial\Omega}\cdot\nabla b)|_{\overline{X}_0(\sigma,t)}\partial_H\check{v}^C|_{H=0}.\end{aligned}$$

The latter is zero if and only if

$$\begin{aligned}&(N_{\partial\Omega}\cdot\nabla b)|_{\overline{X}_0(\sigma,t)}\partial_H\check{v}^C|_{H=0} = -(D_x sN_{\partial\Omega})^\top|_{\overline{X}_0(\sigma,t)}\nabla_\Sigma\check{q}|_{(\sigma,t)}\vec{\theta}_0'(\rho)\\ &+\check{q}|_{(\sigma,t)}\vec{\theta}_0''(\rho)\left[(D_x sN_{\partial\Omega})^\top|_{\overline{X}_0(\sigma,t)}\nabla_\Sigma\check{h}_1|_{(\sigma,t)} - (\rho+\check{h}_1|_{(\sigma,t)})\partial_r((N_{\partial\Omega}\cdot\nabla r)\circ\overline{X})|_{(0,\sigma,t)}\right].\end{aligned}$$

Analogously to the scalar case, cf. Section 6.3.2, we leave the term with $\nabla_\Sigma \check{h}_1$ as a remainder in order to lower the needed regularity for $\check{h}_1$. Therefore due to (6.57) we require

$$[-\Delta + D^2W(\vec{\theta}_0)]\underline{v}^C = -\sum_{\xi\in\mathbb{N}_0^m,|\xi|=1} \partial^\xi D^2W(\vec{\theta}_0)(\underline{u}_1^C)^\xi \vec{\theta}_0' \check{q}|_{(\sigma,t)} \quad \text{in } \overline{\mathbb{R}^2_+}\times\partial\Sigma\times[0,T], \tag{6.59}$$

$$-\partial_H \underline{v}^C|_{H=0} = (|\nabla b|/N_{\partial\Omega}\cdot\nabla b)|_{\overline{X}_0(\sigma,t)}\vec{g}^C \quad \text{in } \mathbb{R}\times\partial\Sigma\times[0,T], \tag{6.60}$$

where $\underline{v}^C, \underline{u}_1^C : \overline{\mathbb{R}^2_+}\times\partial\Sigma\times[0,T]\to\mathbb{R}$ are associated to $\check{v}^C$ and $\check{u}_1^C$ analogous to (5.67) in Section 5.3.2.2.2, respectively, and we define $\vec{g}^C(\rho,\sigma,t)$ for all $(\rho,\sigma,t)\in\mathbb{R}\times\partial\Sigma\times[0,T]$ as

$$-\check{q}|_{(\sigma,t)}\vec{\theta}_0''(\rho)(\rho+\check{h}_1|_{(\sigma,t)})\partial_r((N_{\partial\Omega}\cdot\nabla r)\circ\overline{X})|_{(0,\sigma,t)} - (D_x s N_{\partial\Omega})^\top|_{\overline{X}_0(\sigma,t)}\nabla_\Sigma\check{q}|_{(\sigma,t)}\vec{\theta}_0'(\rho).$$

The right hand sides in (6.59)-(6.60) are of class $B([0,T];C^2(\partial\Sigma,H^2_{(\beta,\gamma)}(\mathbb{R}^2_+)^m\times H^{5/2}_{(\beta)}(\mathbb{R})^m))$ for some $\beta,\gamma>0$ if $(\check{q},\nabla_\Sigma\check{q})|_{\partial\Sigma\times[0,T]}\in B([0,T],C^2(\partial\Sigma))^{1+N}$. For simplicity we require $\check{q}=1$ on $\partial\Sigma\times[0,T]$. Then the compatibility condition (4.31) for (6.59)-(6.60) is equivalent to

$$-(D_x s N_{\partial\Omega})^\top|_{\overline{X}_0(\sigma,t)}\nabla_\Sigma\check{q}|_{(\sigma,t)} = \check{g}^C|_{(\sigma,t)}, \tag{6.61}$$

$$\check{g}^C|_{(\sigma,t)} := \frac{1}{\|\vec{\theta}_0'\|^2_{L^2(\mathbb{R})^m}}\Big[\partial_r((N_{\partial\Omega}\cdot\nabla r)\circ\overline{X})|_{(0,\sigma,t)}\int_{\mathbb{R}}\rho\vec{\theta}_0'(\rho)\cdot\vec{\theta}_0''(\rho)\,d\rho$$
$$+\frac{N_{\partial\Omega}\cdot\nabla b}{|\nabla b|}\Big|_{\overline{X}_0(\sigma,t)}\int_{\mathbb{R}^2_+}(\vec{\theta}_0', \sum_{\xi\in\mathbb{N}_0^m,|\xi|=1}\partial^\xi D^2W(\vec{\theta}_0)(\underline{u}_1^C|_{(\rho,H,\sigma,t)})^\xi\vec{\theta}_0')_{\mathbb{R}^m}\,d(\rho,H)\Big].$$

Here because of the assumptions it holds $\check{g}^C\in B([0,T],C^2(\partial\Sigma))$.

Analogously to the scalar case, cf. Section 6.3.2, it is possible to construct $\check{q}\in B([0,T],C^2(\hat{\Sigma}))$ with $\nabla_\Sigma\check{q}|_{\partial\Sigma\times[0,T]}\in B([0,T],C^2(\partial\Sigma))$ and $\check{q}=1$ on $(\partial\Sigma\cup Y(\partial\Sigma,[\mu_0,2\mu_0]))\times[0,T]$ and such that $c\le\check{q}\le C$ for some $c,C>0$ and (6.61) holds. Hence Remark 4.12 yields a unique solution of (6.59)-(6.60) such that for some $\beta,\gamma>0$

$$\underline{v}^C\in B([0,T];C^2(\partial\Sigma,H^4_{(\beta,\gamma)}(\mathbb{R}^2_+)^m))\hookrightarrow B([0,T];C^2(\partial\Sigma,C^2_{(\beta,\gamma)}(\overline{\mathbb{R}^2_+})^m)).$$

Remark 6.43.

1. Consider the situation of Section 5.3. Then $\check{h}_1$ is smooth and the $\nabla_\Sigma\check{h}_1$-term can be included above. Moreover, $\underline{u}_1^C$ is smooth and solves (5.68)-(5.69). However note that in general it is not valid to use $\check{v}^C := \partial_\rho\underline{u}_1^C$ in analogy to Remark 6.31, 1. in the scalar case. This is because in general

$$\partial_\rho(D^2W(\vec{\theta}_0))\underline{u}_1^C = \sum_{\xi\in\mathbb{N}_0^m,|\xi|=1}\partial_\xi D^2W(\vec{\theta}_0)(\vec{\theta}_0')^\xi\underline{u}_1^C \neq \sum_{\xi\in\mathbb{N}_0^m,|\xi|=1}\partial_\xi D^2W(\vec{\theta}_0)(\underline{u}_1^C)^\xi\vec{\theta}_0'.$$

2. In the case of additional terms in $\vec{u}^A_\varepsilon$ as in Remark 6.40 one can proceed analogously as in the scalar 2D-case, cf. Remark 6.18, 2. More precisely, there is an additional term

$$\frac{1}{\varepsilon}\sum_{\xi\in\mathbb{N}_0^m,|\xi|=1}\partial^\xi D^2W(\vec{\theta}_0|_{\check{\rho}_\varepsilon(.,t)})(\vec{\theta}_1|_{\check{\rho}_\varepsilon(.,t)})^\xi\check{p}_\varepsilon(s(.,t),t)$$

in the operator $\check{\mathcal{L}}^C_{\varepsilon,t}$. Therefore in the ansatz for $\vec{\phi}^A_\varepsilon$ we add $\varepsilon\check{v}_1(\check{\rho}_\varepsilon(.,t))\check{q}_{1,\varepsilon}(s(.,t),t)$, where $\check{v}_1:\mathbb{R}\to\mathbb{R}^m$ and $\check{q}_{1,\varepsilon}$ is $\mathbb{R}$-valued. Hence in the asymptotic expansion of

$\sqrt{\varepsilon}\check{\mathcal{L}}^C_{\varepsilon,t}\vec{\phi}^A_\varepsilon(.,t)$ new terms appear at order $\mathcal{O}(\frac{1}{\varepsilon})$, namely

$$\frac{1}{\varepsilon}[-\partial_\rho^2 + D^2W(\vec{\theta}_0)]\check{v}_1|_\rho \check{q}_{1,\varepsilon}|_{(s,t)} + \frac{1}{\varepsilon}\sum_{\xi\in\mathbb{N}_0^m,|\xi|=1} \partial^\xi D^2W(\vec{\theta}_0)(\vec{\theta}_1)^\xi \vec{\theta}_0'|_\rho(\check{p}_\varepsilon \check{q})|_{(s,t)}.$$

Therefore we define $\check{q}_{1,\varepsilon} := \check{p}_\varepsilon \check{q}$ and look for a solution of

$$[-\partial_\rho^2 + D^2W(\vec{\theta}_0)]\check{v}_1 = -\sum_{\xi\in\mathbb{N}_0^m,|\xi|=1} \partial^\xi D^2W(\vec{\theta}_0)(\vec{\theta}_1)^\xi \vec{\theta}_0'.$$

The right hand side is an element of $C^0_{(\beta)}(\mathbb{R})^m$ for some $\beta > 0$ and (6.52) holds. Therefore Theorem 4.31 yields a unique solution $\check{v}_1 \in H^2_{(\beta)}(\mathbb{R})^m$ for possibly smaller β. With embeddings and the equation it follows that $\check{v}_1 \in C^2_{(\beta)}(\mathbb{R})^m$ for some $\beta > 0$. Then below analogous arguments can be used.

3. The behaviour of $\vec{\phi}^A_\varepsilon(x,t)$ for $x \in \Omega_t^C$ with $b(x,t) \in [\frac{7}{4}\mu_0, 2\mu_0]$ does not matter because we only have to consider $\vec{\psi} \in \check{H}^1(\Omega_t^C)$ in Theorem 6.42, where $\check{H}^1(\Omega_t^C)$ is from (6.53).

Lemma 6.44. *The function $\vec{\phi}^A_\varepsilon(.,t)$ is $C^2(\overline{\Omega_t^C})^m$ and satisfies uniformly in $t \in [0,T]$:*

$$\left|\sqrt{\varepsilon}\check{\mathcal{L}}^C_{\varepsilon,t}\vec{\phi}^A_\varepsilon(.,t) + \frac{1}{\varepsilon}\Delta r|_{\overline{X}_0(s(.,t),t)}\check{q}|_{(s(.,t),t)}\vec{\theta}_0''|_{\check{\rho}_\varepsilon(.,t)}\right| \le Ce^{-c|\check{\rho}_\varepsilon(.,t)|} \quad \text{in } \Omega_t^C,$$

$$\left|\sqrt{\varepsilon}D_x\vec{\phi}^A_\varepsilon|_{(.,t)}N_{\partial\Omega_t^C} + [(D_x sN_{\partial\Omega})^\top|_{\overline{X}_0}\nabla_\Sigma \check{h}_1]|_{(\sigma(.,t),t)}\vec{\theta}_0''\right| \le C\varepsilon e^{-c|\check{\rho}_\varepsilon(.,t)|} \quad \text{on } \partial\Omega_t^C \cap \partial\Omega,$$

$$\left|\sqrt{\varepsilon}D_x\vec{\phi}^A_\varepsilon|_{(.,t)}N_{\partial\Omega_t^C}\right| \le Ce^{-c/\varepsilon} \quad \text{on } \partial\Omega_t^C \setminus \Gamma_t(\delta).$$

Proof. The construction yields the regularity for $\vec{\phi}^A_\varepsilon$ and rigorous remainder estimates for the Taylor expansions above imply the estimates, cf. Lemma 6.19 in the scalar 2D-case. □

6.4.3 The Splitting

Analogously to the scalar case we use a characterization for the splitting of $\check{H}^1(\Omega_t^C)$.

Lemma 6.45. *Let $\check{H}^1(\Omega_t^C)$, $\check{V}_{\varepsilon,t}$ and $\check{H}^1(\hat{\Sigma}^\circ)$ be as in* (6.53)-(6.55). *Then*

1. *$\check{V}_{\varepsilon,t}$ is a subspace of $\check{H}^1(\Omega_t^C)$ and for $\varepsilon_0 > 0$ small there are $\check{c}_1, \check{C}_1 > 0$ such that*

$$\check{c}_1\|\check{a}\|_{L^2(\hat{\Sigma})} \le \|\vec{\psi}\|_{L^2(\Omega_t^C)^m} \le \check{C}_1\|\check{a}\|_{L^2(\hat{\Sigma})}$$

for all $\vec{\psi} = \check{a}(s(.,t))\vec{\phi}^A_\varepsilon(.,t) \in \check{V}_{\varepsilon,t}$ and $\varepsilon \in (0,\varepsilon_0]$, $t \in [0,T]$.

2. *Let $\check{V}^\perp_{\varepsilon,t}$ be the L^2-orthogonal complement of $\check{V}_{\varepsilon,t}$ in $\check{H}^1(\Omega_t^C)$. Then for $\vec{\psi} \in \check{H}^1(\Omega_t^C)$:*

$$\vec{\psi} \in \check{V}^\perp_{\varepsilon,t} \quad \Leftrightarrow \quad \int_{-\delta}^{\delta} (\vec{\phi}^A_\varepsilon(.,t)\cdot\vec{\psi})|_{X(r,s,t)} J_t(r,s)\,dr = 0 \quad \text{for a.e. } s \in \hat{\Sigma}.$$

Moreover, $\check{H}^1(\Omega_t^C) = \check{V}_{\varepsilon,t} \oplus \check{V}^\perp_{\varepsilon,t}$ for all $\varepsilon \in (0,\varepsilon_0]$ and $\varepsilon_0 > 0$ small.

Proof. This follows in the analogous way as in the scalar case, cf. the proof of Lemma 6.33. Here note that $\vec{\theta}_0'(0) \ne 0$ and therefore $\int_{\mathbb{R}} |\vec{\theta}_0'|^2 > 0$ due to Theorem 4.26. □

6.4.4 Analysis of the Bilinear Form

First we consider $\check{B}^C_{\varepsilon,t}$ on $\check{V}_{\varepsilon,t} \times \check{V}_{\varepsilon,t}$.

Lemma 6.46. *There are $\varepsilon_0, C, c > 0$ such that*

$$\check{B}^C_{\varepsilon,t}(\vec{\phi},\vec{\phi}) \geq -C\|\vec{\phi}\|^2_{L^2(\Omega^C_t)^m} + c\|\check{a}\|^2_{H^1(\hat{\Sigma}^\circ)}$$

for all $\vec{\phi} = \check{a}(s(.,t))\vec{\phi}^A_\varepsilon(.,t) \in \check{V}_{\varepsilon,t}$ and $\varepsilon \in (0,\varepsilon_0], t \in [0,T]$.

Proof. Let $\vec{\phi}$ be as in the lemma. For $j = 1, ..., m$ we denote with ϕ_j, $\phi^A_{\varepsilon,j}$ the j-th component of $\vec{\phi}$, $\vec{\phi}^A_\varepsilon$, respectively. Then $\nabla\phi_j = \nabla(\check{a}|_{s(.,t)})\phi^A_{\varepsilon,j}(.,t) + \check{a}|_{s(.,t)}\nabla(\phi^A_{\varepsilon,j})(.,t)$ for $j = 1, ..., m$ and

$$|\nabla\vec{\phi}|^2 = |\nabla(\check{a}(s))|^2|\vec{\phi}^A_\varepsilon|^2|_{(.,t)} + \check{a}^2(s)|\nabla\vec{\phi}^A_\varepsilon|^2|_{(.,t)} + \sum_{j=1}^m \nabla(\check{a}^2(s)) \cdot \nabla(\phi^A_{\varepsilon,j})\phi^A_{\varepsilon,j}|_{(.,t)}.$$

Integration by parts, cf. Remark 6.29, 2., yields

$$\begin{aligned}\sum_{j=1}^m \int_{\Omega^C_t} \left[\nabla(\check{a}^2(s)) \cdot \nabla(\phi^A_{\varepsilon,j})\phi^A_{\varepsilon,j}\right]|_{(.,t)}\,dx &= -\int_{\Omega^C_t}\left[\check{a}^2(s)(\Delta\vec{\phi}^A_\varepsilon \cdot \vec{\phi}^A_\varepsilon + |\nabla\vec{\phi}^A_\varepsilon|^2)\right]|_{(.,t)}\,dx \\ &\quad + \int_{\partial\Omega^C_t}\left[D_x\vec{\phi}^A_\varepsilon N_{\partial\Omega^C_t} \cdot \operatorname{tr}(\check{a}^2(s)\vec{\phi}^A_\varepsilon|_{(.,t)})\right]\,d\mathcal{H}^{N-1}.\end{aligned}$$

Therefore we obtain

$$\begin{aligned}\check{B}^C_{\varepsilon,t}(\vec{\phi},\vec{\phi}) &= \int_{\Omega^C_t}|\nabla(\check{a}(s))|^2|\vec{\phi}^A_\varepsilon|^2|_{(.,t)}\,dx + \int_{\Omega^C_t}(\check{a}^2(s)\vec{\phi}^A_\varepsilon)|_{(.,t)} \cdot \check{\mathcal{L}}^C_{\varepsilon,t}\vec{\phi}^A_\varepsilon|_{(.,t)}\,dx \\ &\quad + \int_{\partial\Omega^C_t}\left[D_x\vec{\phi}^A_\varepsilon N_{\partial\Omega^C_t} \cdot \operatorname{tr}(\check{a}^2(s)\vec{\phi}^A_\varepsilon|_{(.,t)})\right]\,d\mathcal{H}^{N-1} =: (I) + (II) + (III).\end{aligned}$$

Due to integration by parts it holds $\int_{\mathbb{R}}\vec{\theta}_0' \cdot \vec{\theta}_0'' = 0$. Therefore one can estimate (I)-(III) in the analogous way as in the scalar case with Lemma 6.44, cf. the proof of Lemma 6.34. □

Next we analyze $\check{B}^C_{\varepsilon,t}$ on $\check{V}^\perp_{\varepsilon,t} \times \check{V}^\perp_{\varepsilon,t}$.

Lemma 6.47. *There are $\check{\nu}, \check{\varepsilon}_0 > 0$ such that*

$$\check{B}^C_{\varepsilon,t}(\vec{\psi},\vec{\psi}) \geq \check{\nu}\left[\frac{1}{\varepsilon^2}\|\vec{\psi}\|^2_{L^2(\Omega^C_t)^m} + \|\nabla\vec{\psi}\|^2_{L^2(\Omega^C_t)^{N\times m}}\right]$$

for all $\vec{\psi} \in V^\perp_{\varepsilon,t}$ and $\varepsilon \in (0,\check{\varepsilon}_0]$, $t \in [0,T]$.

Proof. As in the scalar case, cf. the proof of Lemma 6.22, it is enough to show for some $\nu, \varepsilon_0 > 0$

$$\check{B}_{\varepsilon,t}(\vec{\psi},\vec{\psi}) := \int_{\Omega^C_t}|\nabla\vec{\psi}|^2 + \frac{1}{\varepsilon^2}(\vec{\psi}, D^2W(\vec{\theta}_0|_{\check{\rho}_\varepsilon(.,t)})\vec{\psi})_{\mathbb{R}^m}\,dx \geq \frac{\nu}{\varepsilon^2}\|\vec{\psi}\|^2_{L^2(\Omega^C_t)^m} \tag{6.62}$$

for all $\vec{\psi} \in \check{V}^\perp_{\varepsilon,t}$ and $\varepsilon \in (0,\varepsilon_0]$, $t \in [0,T]$.

Analogously to the scalar case we prove (6.62) by reducing to Neumann boundary problems in normal direction. Therefore let $\check{\psi}_t := \vec{\psi}|_{X(.,t)}$ for $\vec{\psi} \in \check{V}_{\varepsilon,t}^{\perp}$. Then $\check{\psi}_t \in H^1((-\delta,\delta)\times\hat{\Sigma}^\circ)^m$ and

$$|\nabla\vec{\psi}|^2|_{X(.,t)} \geq (1-Cr^2)|\partial_r\check{\psi}_t|^2 + c|\nabla_{\hat{\Sigma}}\check{\psi}_t|^2 \tag{6.63}$$

in Ω_t^C for some $c, C > 0$ due to (6.49) for every component. We do not use the second term here. To get Cr^2 small enough (which will be precise later), we fix $\check{\delta} > 0$ small and estimate separately for r in

$$I_{s,t}^\varepsilon := (-\check{\delta},\check{\delta}) + \varepsilon\check{h}_\varepsilon(s,t) \quad \text{and} \quad \check{I}_{s,t}^\varepsilon := (-\delta,\delta)\setminus I_{s,t}^\varepsilon.$$

If $\varepsilon_0 = \varepsilon_0(\check{\delta},\check{C}_0) > 0$ is small, then for all $\varepsilon \in (0,\varepsilon_0]$ and $s \in \hat{\Sigma}$, $t \in [0,T]$ it holds

$$D^2W(\vec{\theta}_0(\check{\rho}_\varepsilon|_{\overline{X}(r,s,t)})) \geq \check{c}_0 I \quad \text{for } r \in \check{I}_{s,t}^\varepsilon, \quad |r| \leq \check{\delta} + \varepsilon|\check{h}_\varepsilon(s,t)| \leq 2\check{\delta} \quad \text{for } r \in I_{s,t}^\varepsilon,$$

where $\check{c}_0 > 0$. Let $\check{c} = \check{c}(\check{\delta}) := 4C\check{\delta}^2$ with C from (6.63). Then for all $\varepsilon \in (0,\varepsilon_0]$ we obtain

$$\check{B}_{\varepsilon,t}^C(\vec{\psi},\vec{\psi}) \geq \int_{\hat{\Sigma}}\int_{\check{I}_{s,t}^\varepsilon} \frac{\check{c}_0}{\varepsilon^2}|\check{\psi}_t|^2 J_t|_{(r,s)}\,dr\,d\mathcal{H}^{N-1}(s)$$
$$+ \int_{\hat{\Sigma}}\int_{I_{s,t}^\varepsilon}\left[(1-\check{c})|\partial_r\check{\psi}_t|^2 + \frac{1}{\varepsilon^2}(\check{\psi}_t, D^2W(\vec{\theta}_0(\check{\rho}_\varepsilon|_{\overline{X}(.,t)}))\check{\psi}_t)_{\mathbb{R}^m}\right] J_t|_{(r,s)}\,dr\,d\mathcal{H}^{N-1}(s).$$

We set $\check{F}_{\varepsilon,s,t}(z) := \varepsilon(z+\check{h}_\varepsilon(s,t))$ and $\check{J}_{\varepsilon,s,t}(z) := J_t(\check{F}_{\varepsilon,s,t}(z),s)$ for all $z \in [-\frac{\delta}{\varepsilon},\frac{\delta}{\varepsilon}]-\check{h}_\varepsilon(s,t)$ and $(s,t) \in \Sigma\times[0,T]$. Moreover, we define $I_{\varepsilon,\check{\delta}} := (-\frac{\check{\delta}}{\varepsilon},\frac{\check{\delta}}{\varepsilon})$ and $\vec{\Psi}_{\varepsilon,s,t} := \sqrt{\varepsilon}\check{\psi}_t(\check{F}_{\varepsilon,s,t}(.),s)$. Due to Remark 6.29 it holds $\vec{\Psi}_{\varepsilon,s,t} \in H^1(I_{\varepsilon,\check{\delta}})^m$ for a.e. $s \in \hat{\Sigma}$ and all $t \in [0,T]$. Together with Lemma 6.5, 1. it follows that the second inner integral in the estimate above equals $1/\varepsilon^2$ times

$$\check{B}_{\varepsilon,s,t}^{\check{c}}(\vec{\Psi}_{\varepsilon,s,t},\vec{\Psi}_{\varepsilon,s,t}) := \int_{I_{\varepsilon,\check{\delta}}}\left[(1-\check{c})|\frac{d}{dz}\vec{\Psi}_{\varepsilon,s,t}|^2 + (\vec{\Psi}_{\varepsilon,s,t}, D^2W(\vec{\theta}_0(z))\vec{\Psi}_{\varepsilon,s,t})_{\mathbb{R}^m}\right]\check{J}_{\varepsilon,s,t}\,dz$$

for a.e. $s \in \hat{\Sigma}$ and all $t \in [0,T]$. Therefore (6.62) follows if we show with the same $\check{c}_0$ as above

$$\check{B}_{\varepsilon,s,t}^{\check{c}}(\vec{\Psi}_{\varepsilon,s,t},\vec{\Psi}_{\varepsilon,s,t}) \geq \overline{c}\|\vec{\Psi}_{\varepsilon,s,t}\|_{L^2(I_{\varepsilon,\check{\delta}},\check{J}_{\varepsilon,s,t})^m}^2 - \frac{\check{c}_0}{2}\|\check{\psi}_t(.,s)\|_{L^2(\check{I}_{s,t}^\varepsilon,J_t(.,s))^m}^2 \tag{6.64}$$

for $\varepsilon \in (0,\varepsilon_0]$, a.e. $s \in \hat{\Sigma}$ and all $t \in [0,T]$ with some $\varepsilon_0, \overline{c} > 0$ independent of ε, s, t.

The estimate (6.64) follows for suitable small $\check{\delta}$ analogously as in the scalar 2D-case, cf. the proof of Lemma 6.22. One uses the integral characterization for $\vec{\psi} \in \check{V}_{\varepsilon,t}^{\perp}$ due to Lemma 6.45, 2. and spectral properties for the operator

$$\check{\mathcal{L}}_{\varepsilon,s,t}^0 := -(\check{J}_{\varepsilon,s,t})^{-1}\frac{d}{dz}\left(\check{J}_{\varepsilon,s,t}\frac{d}{dz}\right) + D^2W(\vec{\theta}_0)$$

on $H^2(I_{\varepsilon,\check{\delta}})^m$ with homogeneous Neumann boundary condition, see Section 6.1.4.2 and in particular Theorem 6.12. $\square$

For $\check{B}_{\varepsilon,t}^C$ on $\check{V}_{\varepsilon,t} \times \check{V}_{\varepsilon,t}^{\perp}$ we obtain

Lemma 6.48. *There are $\varepsilon_0, C > 0$ such that*

$$|\check{B}^C_{\varepsilon,t}(\vec{\phi},\vec{\psi})| \leq \frac{C}{\varepsilon}\|\vec{\phi}\|_{L^2(\Omega^C_t)^m}\|\vec{\psi}\|_{L^2(\Omega^C_t)^m} + \frac{1}{4}\check{B}^C_{\varepsilon,t}(\vec{\psi},\vec{\psi}) + C\varepsilon\|\check{a}\|^2_{H^1(\hat{\Sigma}^\circ)}$$

for all $\vec{\phi} = \check{a}(s(.,t))\vec{\phi}^A_\varepsilon(.,t) \in \check{V}_{\varepsilon,t}$, $\vec{\psi} \in \check{V}^\perp_{\varepsilon,t}$ and $\varepsilon \in (0,\varepsilon_0]$, $t \in [0,T]$.

Proof. We rewrite $\check{B}^C_{\varepsilon,t}(\vec{\phi},\vec{\psi})$ in order to use Lemma 6.44 and Lemma 6.45. Using notation as in the beginning of the proof of Lemma 6.46 we obtain with integration by parts

$$\begin{aligned}\int_{\Omega^C_t} \check{a}(s)\nabla\vec{\phi}^A_\varepsilon|_{(.,t)} : \nabla\vec{\psi}\,dx = &-\sum_{j=1}^m \int_{\Omega^C_t}\left[\nabla(\check{a}(s))\cdot\nabla\phi^A_{\varepsilon,j} + \check{a}(s)\Delta\phi^A_{\varepsilon,j}|_{(.,t)}\right]\psi_j\,dx \\ &+ \int_{\partial\Omega^C_t} D_x\phi^A_\varepsilon|_{(.,t)}N_{\partial\Omega^C_t}\cdot \mathrm{tr}\left[\check{a}(s(.,t))\vec{\psi}\right]\,d\mathcal{H}^{N-1}.\end{aligned}$$

Therefore it follows that

$$\begin{aligned}\check{B}^C_{\varepsilon,t}(\vec{\phi},\vec{\psi}) = &\int_{\Omega^C_t}\check{a}(s)|_{(.,t)}\vec{\psi}\cdot\check{\mathcal{L}}^C_{\varepsilon,t}\vec{\phi}^A_\varepsilon|_{(.,t)}\,dx + \int_{\partial\Omega^C_t} D_x\phi^A_\varepsilon|_{(.,t)}N_{\partial\Omega^C_t}\cdot\mathrm{tr}\left[\check{a}(s(.,t))\vec{\psi}\right]\,d\mathcal{H}^{N-1} \\ &+ \int_{\Omega^C_t}\nabla(\check{a}(s))|_{(.,t)}\cdot\left[\sum_{j=1}^m \nabla\psi_j\,\phi^A_{\varepsilon,j}|_{(.,t)} - \psi_j\nabla\phi^A_{\varepsilon,j}|_{(.,t)}\right]dx =: (I)+(II)+(III).\end{aligned}$$

The terms (I)-(III) can be estimated in the analogous way as in the scalar case, cf. the proof of Lemma 6.36. The most important ingredients for the estimate of (I) and (II) are Lemma 6.44 and Lemma 6.37. For (III) one essentially uses the integral characterization for $\vec{\psi} \in \check{V}^\perp_{\varepsilon,t}$ from Lemma 6.45, 2. (by differentiating it) and the structure of $\vec{\phi}^A_\varepsilon$. □

Finally, we combine Lemma 6.46-6.48.

Theorem 6.49. *There are $\check{\varepsilon}_0, \check{C}, \check{c}_0 > 0$ such that for all $\varepsilon \in (0,\check{\varepsilon}_0]$, $t \in [0,T]$ and every $\vec{\psi} \in H^1(\Omega^C_t)^m$ with $\vec{\psi}|_{X(.,s,t)} = 0$ for a.e. $s \in Y(\partial\Sigma \times [\frac{3}{2}\mu_0, 2\mu_0])$ it holds*

$$\check{B}^C_{\varepsilon,t}(\vec{\psi},\vec{\psi}) \geq -C\|\vec{\psi}\|^2_{L^2(\Omega^C_t)^m} + c_0\|\nabla_\tau\vec{\psi}\|^2_{L^2(\Omega^C_t)^{N\times m}}.$$

Remark 6.50.

1. The estimate can be refined, cf. the proof below.
2. Theorem 6.49 directly yields Theorem 6.42, cf. the beginning of Section 6.4.1.

Proof of Theorem 6.49. Let $\vec{\psi} \in \check{H}^1(\Omega^C_t)^m$. Because of Lemma 6.45 we can uniquely write

$$\vec{\psi} = \vec{\phi} + \vec{\phi}^\perp \quad \text{with } \vec{\phi} = [\check{a}(s)\vec{\phi}^A_\varepsilon]|_{(.,t)} \in \check{V}_{\varepsilon,t} \text{ and } \vec{\phi}^\perp \in \check{V}^\perp_{\varepsilon,t}.$$

Analogously to the scalar case, cf. the proof of Theorem 6.38, we obtain from Lemma 6.46-6.48:

$$\check{B}^C_{\varepsilon,t}(\vec{\psi},\vec{\psi}) \geq -C\|\vec{\phi}\|^2_{L^2(\Omega^C_t)^m} + \frac{\check{\nu}}{4\varepsilon^2}\|\vec{\phi}^\perp\|^2_{L^2(\Omega^C_t)^m} + \frac{c_0}{2}\|\check{a}\|^2_{H^1(\hat{\Sigma}^\circ)} + \frac{\check{\nu}}{2}\|\nabla(\vec{\phi}^\perp)\|^2_{L^2(\Omega^C_t)^{N\times m}}$$

for all $\varepsilon \in (0,\varepsilon_0]$ and $t \in [0,T]$, where $\varepsilon_0 > 0$ is small and $\check{\nu}$ is as in Lemma 6.47. Moreover, as in the scalar case it follows that

$$\|\nabla_\tau\vec{\psi}\|^2_{L^2(\Omega^C_t)^m} \leq C(\|\check{a}\|^2_{H^1(\hat{\Sigma}^\circ)} + \|\nabla(\vec{\phi}^\perp)\|^2_{L^2(\Omega^C_t)^{N\times m}}).$$

Together with the estimate for $\check{B}^C_{\varepsilon,t}$ this shows the claim. □

6.5 Spectral Estimate for (AC$_\alpha$) in 2D

In this section we prove a spectral estimate for the Allen-Cahn equation with nonlinear Robin boundary condition (AC$_\alpha$1)-(AC$_\alpha$3) in the case of boundary contact in 2D. Let us state the assumptions for this section. For convenience we use the same notation as in Section 5.4 and occasionally the same notation as in Section 6.2.

We consider $\beta_0, \gamma_0, \alpha_0 > 0$, $\alpha \in \frac{\pi}{2} + [-\alpha_0, \alpha_0]$ and v_α as in Remark 5.33. Let $\Omega \subset \mathbb{R}^2$ and $\Gamma = (\Gamma_t)_{t\in[0,T]}$ for $T > 0$ be as in Section 3.2 for $N = 2$ and with contact angle α (MCF not needed). Moreover, let $\delta_1 > 0$ be such that Theorem 3.3 holds for $2\delta_1$ instead of δ. We use the notation from Theorem 3.3 and Remark 3.4. Furthermore, we consider height functions $h_{1,\alpha}$ and $h_{2,\alpha} = h_{2,\alpha}(\varepsilon)$ for $\varepsilon > 0$ small. We assume (with a slight abuse of notation)

$$h_{j,\alpha} \in B([0,T], C^0(I_\mu) \cap C^2(\hat{I}_\mu)) \quad \text{for } j = 1, 2,$$

where $I_\mu := [-1-\mu, 1+\mu]$ for some $\mu > 0$ large and $\hat{I}_\mu := I_\mu \setminus (-1+2\mu_0, 1-2\mu_0)$. Moreover, let $\overline{C}_0 > 0$ be such that $\|h_{j,\alpha}\|_{B([0,T],C^0(I_\mu)\cap C^2(\hat{I}_\mu))} \leq \overline{C}_0$ for $j = 1, 2$. Then for $\varepsilon > 0$ small we define $h_{\varepsilon,\alpha} := h_{1,\alpha} + \varepsilon h_{2,\alpha}$ and introduce the stretched variables

$$\rho_{\varepsilon,\alpha} := \frac{r - \varepsilon h_{\varepsilon,\alpha}(s,t)}{\varepsilon}, \quad Z^\pm_{\varepsilon,\alpha} := \frac{z^\pm_\alpha}{\varepsilon} \quad \text{in } \overline{\Gamma(2\delta)},$$

where $z^\pm_\alpha = -r\cos\alpha + (1 \mp s)\sin\alpha$ is as in (5.79). Moreover, let $\delta_0 \in (0, \delta]$ be small such that (5.109) holds. We set $\hat{\mu}_0 := \frac{11}{8}\mu_0 \sin\alpha$ and $\tilde{\mu}_0 := \frac{3}{2}\mu_0 \sin\alpha$ as well as for $t \in [0,T]$:

$$\Omega^{C\pm}_t := \{x \in \Gamma_t(\delta_0) : z^\pm_\alpha(x,t) \in (0, \tilde{\mu}_0)\}.$$

Let $\hat{u}^{C\pm}_{1,\alpha} : \overline{\mathbb{R}^2_+} \times [0,T] \to \mathbb{R} : (\rho, Z, t) \mapsto \hat{u}^{C\pm}_{1,\alpha}(\rho, Z, t)$ be in $B([0,T], H^2_{(0,\frac{\gamma_0}{2})}(\mathbb{R}^2_+))$. Then we set

$$u^{C\pm}_{1,\alpha}(x,t) := \hat{u}^{C\pm}_{1,\alpha}(\rho_{\varepsilon,\alpha}(x,t), Z^\pm_{\varepsilon,\alpha}(x,t), t) \quad \text{for } (x,t) \in \Omega^{C\pm}_t.$$

For $\varepsilon > 0$ small let

$$u^A_{\varepsilon,\alpha} = \begin{cases} \theta_0(\rho_{\varepsilon,\alpha}) + \mathcal{O}(\varepsilon^2) & \text{in } \Gamma(\delta_0, \mu_0), \\ v_\alpha(\rho_{\varepsilon,\alpha}, Z^\pm_{\varepsilon,\alpha}) + \varepsilon u^{C\pm}_{1,\alpha} + \mathcal{O}(\varepsilon^2) & \text{in } \Omega^{C\pm}_t, \\ \pm 1 + \mathcal{O}(\varepsilon) & \text{in } Q^\pm_T \setminus \Gamma(\delta_0), \end{cases}$$

where θ_0 is as in Theorem 4.1, v_α is as in Remark 5.33 and $\mathcal{O}(\varepsilon^k)$ are continuous[11] functions bounded by $C\varepsilon^k$. For convenience[12] we do not consider further ε-order terms similar to Remark 6.15. In this situation we prove

Theorem 6.51 (Spectral Estimate for (AC$_\alpha$) in 2D). *There is an $\overline{\alpha}_0 \in (0, \alpha_0]$ independent of Ω, Γ such that, if $\alpha \in \frac{\pi}{2} + [-\overline{\alpha}_0, \overline{\alpha}_0]$, then there are $\varepsilon_0, C, c_0 > 0$ independent of the $h_{j,\alpha}$ for fixed $\overline{C}_0$ such that for all $\varepsilon \in (0, \varepsilon_0], t \in [0,T]$ and $\psi \in H^1(\Omega)$ it holds*

$$\begin{aligned} \int_\Omega |\nabla\psi|^2 + \frac{1}{\varepsilon^2} f''(u^A_{\varepsilon,\alpha}|_{(.,t)})\psi^2 \, dx + \int_{\partial\Omega} \frac{1}{\varepsilon} \sigma''_\alpha(u^A_{\varepsilon,\alpha}|_{(.,t)})(\operatorname{tr}\psi)^2 \, d\mathcal{H}^1 \\ \geq -C\|\psi\|^2_{L^2(\Omega)} + \|\nabla\psi\|^2_{L^2(\Omega\setminus\Gamma_t(\delta_0))} + c_0\varepsilon\|\nabla_\tau\psi\|^2_{L^2(\Gamma_t(\delta_0))}. \end{aligned} \tag{6.65}$$

[11] For evaluation on the boundary.

[12] It would be tedious to include such terms in the asymptotic expansion for the approximate eigenfunction below and probably a cut-off structure similar to Section 5.4.2 is needed.

Remark 6.52. 1. To cover all possible angles α, i.e. $\overline{\alpha}_0 = \alpha_0$, for which we solved the model problems on the half space appropriately, cf. Remark 5.33, one could try a strategy similar to the 1D-estimates in Chen [C2]. But presently it is not clear how this should work precisely even for the case $\alpha = \frac{\pi}{2}$, see also Remark 6.1. Since α_0 is in general small anyway, we chose the most convenient way by sticking to the proof of the case $\alpha = \frac{\pi}{2}$ in Section 6.2. However, we have to be careful in order to obtain an $\overline{\alpha}_0$ independent of Ω, Γ.

2. Compared to the case $\alpha = \frac{\pi}{2}$ in Theorem 6.16 in Section 6.2 we can only show the weaker estimate (6.65) with an ε-factor in front of the ∇_τ-term. The reason is that the arguments in Section 6.2 can be adapted except for the estimate of $\nabla_\tau\phi$ in the proof of Theorem 6.25. In the present situation there will be a term roughly of the form
$$\frac{1}{\varepsilon^3}\int_0^{\tilde{\mu}_0} a(z)^2 \int_{-\delta_0}^{\delta_0} (\partial_Z\partial_\rho v_\alpha)^2|_{(\frac{r}{\varepsilon},\frac{z}{\varepsilon})}\, dr\, dz.$$
We can control the latter only with an $\mathcal{O}(\varepsilon^{-1})$-term, e.g. $\frac{1}{\varepsilon}C\|a\|^2_{H^1(0,\tilde{\mu}_0)}$, cf. (6.80) and the proof of Theorem 6.61. Therefore we need the additional ε. Nevertheless, the estimate still gives some control on the ∇_τ-term.

3. Note that in this section z corresponds to $z^\pm_\alpha$, whereas in previous sections z was an integral variable on $\mathbb{R}$. Here we use ρ for the latter instead.

Similar to the case $\alpha = \frac{\pi}{2}$ considered in Section 6.2 the main task is the proof of a spectral estimate close to the contact points:

Theorem 6.53. *There is an $\overline{\alpha}_0 \in (0, \alpha_0]$ independent of Ω, Γ such that, if $\alpha \in \frac{\pi}{2} + [-\overline{\alpha}_0, \overline{\alpha}_0]$, then there are $\tilde{\varepsilon}_0, C, \tilde{c}_0 > 0$ independent of the $h_{j,\alpha}$ for fixed $\overline{C}_0$ such that for all $\varepsilon \in (0, \tilde{\varepsilon}_0]$, $t \in [0, T]$ and $\psi \in H^1(\Omega^{C\pm}_t)$ with $\psi(x) = 0$ for a.e. $x \in \Omega^{C\pm}_t$ with $z^\pm_\alpha(x,t) \geq \hat{\mu}_0$ it holds:*
$$\int_{\Omega^{C\pm}_t} |\nabla\psi|^2 + \frac{1}{\varepsilon^2} f''(u^A_{\varepsilon,\alpha}|_{(.,t)})\psi^2\, dx + \int_{\partial\Omega\cap\partial\Omega^{C\pm}_t} \frac{1}{\varepsilon}\sigma''_\alpha(u^A_{\varepsilon,\alpha}|_{(.,t)})(\operatorname{tr}\psi)^2\, d\mathcal{H}^1$$
$$\geq -C\|\psi\|^2_{L^2(\Omega^{C\pm}_t)} + \tilde{c}_0\varepsilon\|\nabla_\tau\psi\|^2_{L^2(\Omega^{C\pm}_t)}.$$

This is sufficient to prove Theorem 6.51:

Proof of Theorem 6.51. For $\varepsilon_0 > 0$ small and all $\varepsilon \in (0, \varepsilon_0]$ it holds $f''(u^A_{\varepsilon,\alpha}) \geq 0$ in $Q^\pm_T \setminus \Gamma(\delta_0)$ and $\sigma''_\alpha(u^A_{\varepsilon,\alpha}) = 0$ on $\partial\Omega \setminus \partial\Omega^{C\pm}_t$. Therefore it is enough to prove the estimate in Theorem 6.51 for $\Gamma_t(\delta_0)$ instead of Ω. Analogously to the case $\alpha = \frac{\pi}{2}$, cf. the proof of Theorem 6.16, we reduce to further subsets. The estimate holds for $\Gamma_t(\delta_0, \mu_0)$ instead of Ω with 1 in front of the ∇_τ-term and without the boundary term. The latter was already proven in the case $\alpha = \frac{\pi}{2}$, cf. the proof of Theorem 6.16. Finally, one can combine this with Theorem 6.53 similar as in the case $\alpha = \frac{\pi}{2}$ with a suitable partition of unity for
$$\Gamma_t(\delta_0) \subseteq \overline{\Gamma_t(\delta_0, \mu_0)} \cup \bigcup_\pm \overline{\Gamma^\pm_t(\delta_0, 5\mu_0/4)}, \tag{6.66}$$
cf. the proof of Theorem 6.16. This is possible since
$$\Gamma^\pm_t(\delta_0, \frac{5}{4}\mu_0) \subseteq \{x \in \Gamma_t(\delta_0) : z^\pm_\alpha(x,t) \in (0, \hat{\mu}_0)\} \subseteq \Omega^{C\pm}_t \subseteq \Gamma^\pm_t(\delta_0, \frac{7}{4}\mu_0)$$
for $t \in [0, T]$ due to (5.109). $\square$

6.5.1 Outline for the Proof of the Spectral Estimate close to the Contact Points

Because of a Taylor expansion it is enough to prove Theorem 6.53 for

$$\frac{1}{\varepsilon^2} f''(v_\alpha|_{(\rho_{\varepsilon,\alpha}(.,t),Z^\pm_{\varepsilon,\alpha}(.,t))}) + \frac{1}{\varepsilon} f'''(v_\alpha|_{(\rho_{\varepsilon,\alpha}(.,t),Z^\pm_{\varepsilon,\alpha}(.,t))}) u^{C\pm}_{1,\alpha}(.,t)$$

instead of $\frac{1}{\varepsilon^2} f''(u^A_{\varepsilon,\alpha}(.,t))$. Moreover, we can replace $\frac{1}{\varepsilon}\sigma''_\alpha(u^A_{\varepsilon,\alpha}(.,t))$ by

$$\frac{1}{\varepsilon}\sigma''_\alpha(v_\alpha|_{(\rho_{\varepsilon,\alpha}(.,t),0)}) + \sigma'''_\alpha(v_\alpha|_{(\rho_{\varepsilon,\alpha}(.,t),0)}) u^{C\pm}_{1,\alpha}(.,t)$$

due to Young's inequality and the second estimate in the following lemma.

Lemma 6.54. *There is a $\overline{C}_1 > 0$ (independent of ψ, t) such that*

$$\begin{aligned}
\|\mathrm{tr}\,\psi\|^2_{L^2(\partial\Omega^{C\pm}_t)} &\leq \overline{C}_1 \left[\|\psi\|^2_{L^2(\Omega^{C\pm}_t)} + \|\nabla\psi\|_{L^2(\Omega^{C\pm}_t)}\|\psi\|_{L^2(\Omega^{C\pm}_t)}\right],\\
\|\mathrm{tr}\,\psi\|^2_{L^2(\partial\Omega\cap\partial\Omega^{C\pm}_t)} &\leq \overline{C}_1 \left[\|\psi\|^2_{L^2(\Omega^{C\pm}_t)} + \|\nabla_\tau\psi\|_{L^2(\Omega^{C\pm}_t)}\|\psi\|_{L^2(\Omega^{C\pm}_t)}\right]
\end{aligned}$$

for all $\psi \in H^1(\Omega^{C\pm}_t)$ and $t \in [0,T]$.

Proof. The first estimate follows analogously to Lemma 6.24. For convenience we do not go into details. The proof of the second estimate is similar. First note that it is equivalent to prove the estimate for $S := \{(r,s) \in \mathbb{R}^2 : r \in (-\delta_0,\delta_0), s \in \pm[1-\frac{3}{2}\mu_0, 1] \mp \frac{\cos\alpha}{\sin\alpha} r\}$, $S^\pm_{\delta_0,\alpha}$ instead of $\Omega^{C\pm}_t, \partial\Omega\cap\partial\Omega^{C\pm}_t$ as well as ∂_s instead of ∇_τ. For the latter we use the same idea as in the proof of Lemma 6.24. Here $\vec{w} \in C^1(\overline{S})^2$ with $\vec{w}\cdot N_{\partial S} \geq 1$ on $S^\pm_{\delta_0,\alpha}$ and $\vec{w}\cdot N_{\partial S} = 0$ on $\partial S \setminus S^\pm_{\delta_0,\alpha}$ as well as $w_1 = 0$ yields the claim. □

We construct an approximation $\phi^A_{\varepsilon,\alpha}(.,t)$ to the first eigenfunction of

$$\mathcal{L}^\pm_{\varepsilon,t} := -\Delta + \frac{1}{\varepsilon^2} f''(v_\alpha|_{(\rho_{\varepsilon,\alpha}(.,t),Z^\pm_{\varepsilon,\alpha}(.,t))}) + \frac{1}{\varepsilon} f'''(v_\alpha|_{(\rho_{\varepsilon,\alpha}(.,t),Z^\pm_{\varepsilon,\alpha}(.,t))}) u^{C\pm}_{1,\alpha}(.,t) \quad \text{on } \Omega^{C\pm}_t$$

together with the linear Robin boundary condition $\mathcal{N}^\pm_{\varepsilon,t} u = 0$ on $\partial\Omega^{C\pm}_t$, where

$$\mathcal{N}^\pm_{\varepsilon,t} u := \left[N_{\partial\Omega^{C\pm}_t}\cdot\nabla + \chi_{\partial\Omega}\left[\frac{1}{\varepsilon}\sigma''_\alpha(v_\alpha|_{(\rho_{\varepsilon,\alpha}(.,t),0)}) + \sigma'''_\alpha(v_\alpha|_{(\rho_{\varepsilon,\alpha}(.,t),0)}) u^{C\pm}_{1,\alpha}|_{(.,t)}\right]\right] u \quad \text{on } \partial\Omega^{C\pm}_t.$$

Here $\chi_{\partial\Omega}$ is the characteristic function of $\partial\Omega$. In analogy to the case $\alpha = \frac{\pi}{2}$, cf. Section 6.2.1, we use the ansatz

$$\begin{aligned}
\phi^A_{\varepsilon,\alpha}(.,t) &:= \frac{1}{\sqrt{\varepsilon}}[v^{C\pm}_{\varepsilon,0}(.,t) + \varepsilon v^{C\pm}_{\varepsilon,1}(.,t)] && \text{on } \Omega^{C\pm}_t,\\
v^{C\pm}_{\varepsilon,0}(.,t) &:= \hat{v}^{C\pm}_0|_{(\rho_{\varepsilon,\alpha}(.,t),Z^\pm_{\varepsilon,\alpha}(.,t),z^\pm_\alpha(.,t),t)} := q^\pm|_{(z^\pm_\alpha(.,t),t)}\partial_\rho v_\alpha|_{(\rho_{\varepsilon,\alpha}(.,t),Z^\pm_{\varepsilon,\alpha}(.,t))} && \text{on } \Omega^{C\pm}_t,\\
v^{C\pm}_{\varepsilon,1}(.,t) &:= \hat{v}^{C\pm}_1|_{(\rho_{\varepsilon,\alpha}(.,t),Z^\pm_{\varepsilon,\alpha}(.,t),t)} && \text{on } \Omega^{C\pm}_t,
\end{aligned}$$

where $q^\pm : [0,\tilde{\mu}_0]\times[0,T] \to \mathbb{R} : (z,t) \mapsto q^\pm(z,t)$ and $\hat{v}^{C\pm}_0 : \overline{\mathbb{R}^2_+}\times[0,\tilde{\mu}_0]\times[0,T] \to \mathbb{R}$ as well as $\hat{v}^{C\pm}_1 : \overline{\mathbb{R}^2_+}\times[0,T] \to \mathbb{R}$.

In Subsection 6.5.2 we expand $\mathcal{L}^{\pm}_{\varepsilon,t}\phi^A_{\varepsilon,\alpha}(.,t)$ and $\mathcal{N}^{\pm}_{\varepsilon,t}\phi^A_{\varepsilon,\alpha}(.,t)$ with similar ideas as in Section 5.4.2 and choose $q^\pm$ and $\hat{v}_1^{C\pm}$ appropriately. The $q^\pm$-term is used to enforce the compatibility condition for the equations for $\hat{v}_1^{C\pm}$. Then in Subsection 6.5.3 we decompose

$$\tilde{H}^1(\Omega_t^{C\pm}) := \left\{\psi \in H^1(\Omega_t^{C\pm}) : \psi(x) = 0 \text{ for a.e. } x \in \Omega_t^{C\pm} \text{ with } z_\alpha^\pm(x,t) \geq \hat{\mu}_0\right\} \tag{6.67}$$

in a suitable way. To this end we consider

$$\hat{V}^{\pm}_{\varepsilon,t} := \left\{\phi = a(z^\pm_\alpha(.,t))\phi^A_{\varepsilon,\alpha}(.,t) : a \in \hat{H}^1(0,\tilde{\mu}_0)\right\}, \tag{6.68}$$

$$\hat{H}^1(0,\tilde{\mu}_0) := \left\{a \in H^1(0,\tilde{\mu}_0) : a = 0 \text{ on } [\hat{\mu}_0,\tilde{\mu}_0], a(0) = 0\right\}. \tag{6.69}$$

Finally, in Subsection 6.5.4 we analyze the bilinear form $B^\pm_{\varepsilon,t}$ corresponding to $\mathcal{L}^\pm_{\varepsilon,t}$ on $\hat{V}^\pm_{\varepsilon,t} \times \hat{V}^\pm_{\varepsilon,t}$, $(\hat{V}^\pm_{\varepsilon,t})^\perp \times (\hat{V}^\pm_{\varepsilon,t})^\perp$ and $\hat{V}^\pm_{\varepsilon,t} \times (\hat{V}^\pm_{\varepsilon,t})^\perp$. Here for $\phi,\psi \in H^1(\Omega_t^{C\pm})$ let

$$\begin{aligned} B^\pm_{\varepsilon,t}(\phi,\psi) := &\int_{\partial\Omega\cap\partial\Omega_t^{C\pm}} \left[\frac{1}{\varepsilon}\sigma''_\alpha(v_\alpha|_{(\rho_{\varepsilon,\alpha},0)}) + \sigma'''_\alpha(v_\alpha|_{(\rho_{\varepsilon,\alpha},0)})u^{C\pm}_{1,\alpha}\right]|_{(.,t)} \operatorname{tr}\phi \operatorname{tr}\psi \, d\mathcal{H}^1 \\ &+ \int_{\Omega_t^{C\pm}} \nabla\phi\cdot\nabla\psi + \left[\frac{1}{\varepsilon^2}f''(v_\alpha|_{(\rho_{\varepsilon,\alpha},Z^\pm_{\varepsilon,\alpha})}) + \frac{1}{\varepsilon}f'''(v_\alpha|_{(\rho_{\varepsilon,\alpha},Z^\pm_{\varepsilon,\alpha})})u^{C\pm}_{1,\alpha}\right]|_{(.,t)}\phi\psi\, dx. \end{aligned} \tag{6.70}$$

6.5.2 Asymptotic Expansion for the Approximate Eigenfunction

Asymptotic Expansion of $\sqrt{\varepsilon}\mathcal{L}^\pm_{\varepsilon,t}\phi^A_{\varepsilon,\alpha}(.,t)$. In $\sqrt{\varepsilon}\Delta\phi^A_{\varepsilon,\alpha}(.,t)$ there are some additional terms due to $q^\pm$ compared to the formula in Lemma 5.34. More precisely, via direct computation we get

$$\begin{aligned} \sqrt{\varepsilon}\Delta\phi^A_{\varepsilon,\alpha} = &(q^\pm\partial_\rho^2 v_\alpha + \varepsilon\partial_\rho\hat{v}_1^{C\pm})\left[\frac{\Delta r}{\varepsilon} - (\Delta s\partial_s h_{\varepsilon,\alpha} + |\nabla s|^2\partial_s^2 h_{\varepsilon,\alpha})\right] \\ &+ (q^\pm\partial_Z\partial_\rho v_\alpha + \varepsilon\partial_Z\hat{v}_1^{C\pm})\frac{\Delta z^\pm_\alpha}{\varepsilon} + (q^\pm\partial_Z^2\partial_\rho v_\alpha + \varepsilon\partial_Z^2\hat{v}_1^{C\pm})\frac{|\nabla z^\pm_\alpha|^2}{\varepsilon^2} \\ &+ 2(q^\pm\partial_Z\partial_\rho^2 v_\alpha + \varepsilon\partial_Z\partial_\rho\hat{v}_1^{C\pm})\frac{\nabla z^\pm_\alpha}{\varepsilon}\cdot\left[\frac{\nabla r}{\varepsilon} - \nabla s\partial_s h_{\varepsilon,\alpha}\right] \\ &+ (q^\pm\partial_\rho^3 v_\alpha + \varepsilon\partial_\rho^2\hat{v}_1^{C\pm})\left|\frac{\nabla r}{\varepsilon} - \nabla s\partial_s h_{\varepsilon,\alpha}\right|^2 \\ &+ 2\partial_z q^\pm\partial_Z\partial_\rho v_\alpha\frac{|\nabla z^\pm_\alpha|^2}{\varepsilon} + 2\partial_z q^\pm\partial_\rho^2 v_\alpha\nabla z^\pm_\alpha\cdot\left[\frac{\nabla r}{\varepsilon} - \nabla s\partial_s h_{\varepsilon,\alpha}\right] \\ &+ \partial_z q^\pm\partial_\rho v_\alpha\Delta z^\pm_\alpha + \partial_z^2 q^\pm\partial_\rho v_\alpha|\nabla z^\pm_\alpha|^2, \end{aligned}$$

with evaluations as in Lemma 5.34 except that the $q^\pm$-terms are evaluated at $(z^\pm_\alpha(x,t),t)$.

The difficulty in expanding $\sqrt{\varepsilon}\Delta\phi^A_{\varepsilon,\alpha}$ is that for the v_α-terms without a derivative in Z, i.e. terms with the factors $\partial_\rho^k v_\alpha$, $k=1,2,3$, we do not have exponential decay estimates with respect to Z. Therefore we have to expand the corresponding factors in a more subtle way than in Section 5.4.2.1, where this problem was solved with a suitable ansatz. However, we only need the expansion up to order $\frac{1}{\varepsilon}$ and for the remainder terms a decay in normal direction as in Lemma 6.19 in the case $\alpha = \frac{\pi}{2}$ will be enough. Therefore we do not need to expand terms of order ε^0 in the formula for $\sqrt{\varepsilon}\Delta\phi^A_{\varepsilon,\alpha}$ above, in particular higher regularity for $\partial_s^2 h_{\varepsilon,\alpha}$ is not necessary. This leads to the following expansion procedure for $\sqrt{\varepsilon}\mathcal{L}^\pm_{\varepsilon,t}\phi^A_{\varepsilon,\alpha}(.,t)$:

Terms of order $\mathcal{O}(1)$ are not expanded. The other terms are expanded as follows. For (x,t)-terms that are not multiplied by a term with a Z-derivative we only use a Taylor-expansion in normal direction analogous to (5.4) and replace r by $\varepsilon(\rho + h_{\varepsilon,\alpha}(s,t))$. But then we leave untouched all the appearing $h_{\varepsilon,\alpha}$-terms that are not multiplied with a term including a Z-derivative. Moreover, for all the other (x,t)-terms we apply the full Taylor expansion (5.95) and replace r as above and $s \mp 1$ via (5.94). Then we rewrite the $h_{\varepsilon,\alpha}$-terms that are multiplied by a term with a Z-derivative via

$$\partial_s^k h_{\varepsilon,\alpha}|_{(s(x,t),t)} = \partial_s^k h_{\varepsilon,\alpha}|_{(\pm 1 \mp \frac{1}{\sin\alpha} z_\alpha^\pm(x,t),t)} = \partial_s^k h_{\varepsilon,\alpha}|_{(\pm 1,t)} + \mathcal{O}(|z_\alpha^\pm|_{(x,t)}|) \quad \text{for } k = 0,1.$$

Regarding the $q^\pm$-terms we only rewrite the ones in front of the $\partial_Z \partial_\rho v_\alpha$, the ones multiplied with the ε-orders of $|\nabla z_\alpha^\pm|^2$ or $\nabla r \cdot \nabla z_\alpha^\pm$ as well as the one multiplied with $f^{(3)}(v_\alpha)$ via the formula $\partial_z^k q^\pm(z,t) = \partial_z^k q^\pm(0,t) + \mathcal{O}(|z|)$ for $k = 0,1$. Note that the remainder term stemming from the $f^{(3)}(v_\alpha)$-term can be controlled because $\hat{u}_{1,\alpha}^{C\pm} \partial_\rho v_\alpha$ has appropriate decay. The z-remainders will only contribute to order ε^0 in the expansion of $\sqrt{\varepsilon}\Delta\phi_{\varepsilon,\alpha}^A$ due to $z_\alpha^\pm = \varepsilon Z_{\varepsilon,\alpha}^\pm$.

At the lowest order $\mathcal{O}(\frac{1}{\varepsilon^2})$ in $\sqrt{\varepsilon}\mathcal{L}_{\varepsilon,t}^\pm \phi_{\varepsilon,\alpha}^A(.,t)$ we obtain

$$\frac{1}{\varepsilon^2} q^\pm(z,t) \left[-\partial_Z^2 + 2\cos\alpha \partial_\rho \partial_Z - \partial_\rho^2 + f''(v_\alpha)\right] \partial_\rho v_\alpha = 0$$

due to (4.11). For the $\frac{1}{\varepsilon}$-order we get

$$\begin{aligned}
&\frac{1}{\varepsilon}\left[-\partial_Z^2 + 2\cos\alpha\partial_\rho\partial_Z - \partial_\rho^2 + f''(v_\alpha)\right]\hat{v}_1^{C\pm} + \frac{1}{\varepsilon} q^\pm(0,t) f^{(3)}(v_\alpha)\hat{u}_{1,\alpha}^{C\pm}\partial_\rho v_\alpha \\
&- \frac{1}{\varepsilon} q^\pm(z,t)\partial_\rho^2 v_\alpha \Delta r|_{\overline{X}_0(s,t)} - \frac{1}{\varepsilon} q^\pm(0,t)\partial_Z\partial_\rho v_\alpha \Delta z_\alpha^\pm|_{\overline{p}^\pm(t)} \\
&- \frac{1}{\varepsilon^2} q^\pm(0,t)\partial_Z^2\partial_\rho v_\alpha \left[\partial_r(|\nabla z_\alpha^\pm|^2 \circ \overline{X})|_{(0,\pm 1,t)}\varepsilon(\rho + h_{1,\alpha}|_{(\pm 1,t)})\right. \\
&\qquad\qquad \left. + \partial_s(|\nabla z_\alpha^\pm|^2 \circ \overline{X})|_{(0,\pm 1,t)}(\mp\varepsilon)\frac{1}{\sin\alpha}[Z + \cos\alpha(\rho + h_{1,\alpha}|_{(\pm 1,t)})]\right] \\
&- 2q^\pm(0,t)\partial_Z\partial_\rho^2 v_\alpha \left[\partial_r((\nabla r \cdot \nabla z_\alpha^\pm) \circ \overline{X})|_{(0,\pm 1,t)}\varepsilon(\rho + h_{1,\alpha}|_{(\pm 1,t)})\right. \\
&\left. + \partial_s((\nabla r \cdot \nabla z_\alpha^\pm) \circ \overline{X})|_{(0,\pm 1,t)}(\mp\varepsilon)\frac{1}{\sin\alpha}[Z + \cos\alpha(\rho + h_{1,\alpha}|_{(\pm 1,t)})] - \frac{\mp\sin\alpha}{\varepsilon}\partial_s h_{1,\alpha}|_{(\pm 1,t)}\right] \\
&+ q^\pm(z,t)\partial_\rho^3 v_\alpha \left[\partial_r(|\nabla r|^2 \circ \overline{X})|_{(0,s,t)}\varepsilon(\rho + h_1|_{(s,t)}) - \nabla r \cdot \nabla s|_{\overline{X}_0(s,t)}\partial_s h_{1,\alpha}|_{(s,t)}\right] \\
&- \frac{2}{\varepsilon}\partial_z q^\pm(0,t)\partial_Z\partial_\rho v_\alpha - \frac{2}{\varepsilon}\partial_z q^\pm(z,t)\partial_\rho^2 v_\alpha(\nabla r \cdot \nabla z_\alpha^\pm)|_{\overline{X}_0(s,t)}.
\end{aligned}$$

Here the penultimate line vanishes due to Theorem 3.3. Moreover, $(\nabla r \cdot \nabla z_\alpha^\pm)|_{\overline{X}_0(s,t)} = -\cos\alpha$. We leave the two $\partial_\rho^2 v_\alpha$-terms as a remainder. Analogously to the case $\alpha = \frac{\pi}{2}$ we will be able to improve the ε-order of these terms due to $\int_{\mathbb{R}}(\partial_\rho^2 v_\alpha \partial_\rho v_\alpha)|_{(\rho,Z)}\, d\rho = 0$ for $Z \geq 0$, see the estimate of (II) in the proof of Lemma 6.57. Moreover, we require the other terms to add up to zero. This gives the following equation for $\hat{v}_1^{C\pm}$ on $\overline{\mathbb{R}^2_+} \times [0,T]$:

$$\begin{aligned}
&\left[-\partial_Z^2 + 2\cos\alpha\partial_\rho\partial_Z - \partial_\rho^2 + f''(v_\alpha)\right]\hat{v}_1^{C\pm} \\
&= -q^\pm(0,t)f^{(3)}(v_\alpha)\hat{u}_{1,\alpha}^{C\pm}\partial_\rho v_\alpha + \partial_Z\partial_\rho v_\alpha\left[q^\pm(0,t)\Delta z_\alpha^\pm|_{\overline{p}^\pm(t)} + 2\partial_z q^\pm(0,t)\right] \\
&+ q^\pm(0,t)\partial_Z^2\partial_\rho v_\alpha\Bigg[\partial_r(|\nabla z_\alpha^\pm|^2\circ\overline{X})|_{(0,\pm1,t)}(\rho + h_{1,\alpha}|_{(\pm1,t)}) \\
&\qquad\qquad + \partial_s(|\nabla z_\alpha^\pm|^2\circ\overline{X})|_{(0,\pm1,t)}\frac{\mp1}{\sin\alpha}[Z + \cos\alpha(\rho + h_{1,\alpha}|_{(\pm1,t)})]\Bigg] \\
&+ 2q^\pm(0,t)\partial_Z\partial_\rho^2 v_\alpha\Bigg[\partial_r((\nabla r\cdot\nabla z_\alpha^\pm)\circ\overline{X})|_{(0,\pm1,t)}(\rho + h_{1,\alpha}|_{(\pm1,t)}) \\
&+ \partial_s((\nabla r\cdot\nabla z_\alpha^\pm)\circ\overline{X})|_{(0,\pm1,t)}\frac{\mp1}{\sin\alpha}[Z + \cos\alpha(\rho + h_{1,\alpha}|_{(\pm1,t)})] \pm \sin\alpha\partial_s h_{1,\alpha}|_{(\pm1,t)}\Bigg].
\end{aligned}$$

Asymptotic Expansion of $\sqrt{\varepsilon}\mathcal{N}_{\varepsilon,t}^\pm\phi_{\varepsilon,\alpha}^A(.,t)$ on $\partial\Omega\cap\partial\Omega_t^{C\pm}$. In $\overline{\Omega_t^{C\pm}}$ it holds

$$\begin{aligned}
\sqrt{\varepsilon}\nabla\phi_{\varepsilon,\alpha}^A &= \partial_z q^\pm\partial_\rho v_\alpha\nabla z_\alpha^\pm + (q^\pm\partial_Z\partial_\rho v_\alpha + \varepsilon\partial_Z\hat{v}_1^{C\pm})\frac{\nabla z_\alpha^\pm}{\varepsilon} \\
&+ (q^\pm\partial_\rho^2 v_\alpha + \varepsilon\partial_\rho\hat{v}_1^{C\pm})\left[\frac{\nabla r}{\varepsilon} - \nabla s\partial_s h_{\varepsilon,\alpha}\right]
\end{aligned} \tag{6.71}$$

with evaluations as in Lemma 5.34 except that the $q^\pm$-terms are evaluated at $(z_\alpha^\pm(x,t),t)$. In $\sqrt{\varepsilon}N_{\partial\Omega}\cdot\nabla\phi_{\varepsilon,\alpha}^A$ the $q^\pm$-terms are evaluated at $z=0$. Moreover, we expand the (x,t)-terms via (5.103) and insert $r = \varepsilon(\rho + h_{\varepsilon,\alpha}(s,t))$. Note that there are no $h_{\varepsilon,\alpha}$-terms in the lowest order and we only have to expand up to $\mathcal{O}(\varepsilon^0)$. Therefore we use $\partial_s^k h_{\varepsilon,\alpha}|_{(s,t)} = \partial_s^k h_{\varepsilon,\alpha}|_{(\pm1,t)} + \mathcal{O}(|s\mp1|)$ for $k=0,1$ and replace $s\mp1$ by (5.94) with $Z=0$.

At the lowest order $\mathcal{O}(\frac{1}{\varepsilon})$ in $\sqrt{\varepsilon}\mathcal{N}_{\varepsilon,t}^\pm\phi_{\varepsilon,\alpha}^A(.,t)$ we obtain

$$\frac{1}{\varepsilon}q^\pm(0,t)\left[-\partial_Z + \cos\alpha\partial_\rho + \sigma_\alpha''(v_\alpha|_{Z=0})\right]\partial_\rho v_\alpha|_{Z=0}$$

due to $N_{\partial\Omega} = -\nabla z_\alpha^\pm|_{\overline{p}^\pm(t)}$ and $N_{\partial\Omega}\cdot\nabla r|_{\overline{p}^\pm(t)} = \cos\alpha$, cf. Section 5.4.2.2.1. This is zero because of (4.12). The $\mathcal{O}(1)$-order equals

$$\begin{aligned}
&\left[-\partial_Z + \cos\alpha\partial_\rho + \sigma_\alpha''(v_\alpha|_{Z=0})\right]\hat{v}_1^{C\pm}|_{Z=0} + q^\pm(0,t)\sigma_\alpha'''(v_\alpha|_{Z=0})u_{1,\alpha}^{C\pm}|_{Z=0}\partial_\rho v_\alpha|_{Z=0} \\
&- \partial_z q^\pm(0,t)\partial_\rho v_\alpha|_{Z=0} + q^\pm(0,t)\partial_Z\partial_\rho v_\alpha|_{Z=0}\left[(\rho + h_{1,\alpha}|_{(\pm1,t)})\partial_r((N_{\partial\Omega}\cdot\nabla z_\alpha^\pm)\circ\overline{X}_1^\pm)|_{(0,t)}\right] \\
&+ q^\pm(0,t)\partial_\rho^2 v_\alpha|_{Z=0}\left[(\rho + h_{1,\alpha}|_{(\pm1,t)})\partial_r((N_{\partial\Omega}\cdot\nabla r)\circ\overline{X}_1^\pm)|_{(0,t)} \mp \sin\alpha\partial_s h_{1,\alpha}|_{(\pm1,t)}\right],
\end{aligned}$$

where $\overline{X}_1^\pm$ is defined as in (5.101). We require that this term vanishes. This yields a boundary condition for $\hat{v}_1^{C\pm}$ on $\partial\mathbb{R}^2_+\times[0,T]$.

Together with the equation derived in the asymptotic expansion of $\sqrt{\varepsilon}\mathcal{L}_{\varepsilon,t}^\pm\phi_{\varepsilon,\alpha}^A(.,t)$ we have equations of type (4.15)-(4.16) with the additional parameter $t\in[0,T]$ for $\hat{v}_1^{C\pm}$. Because of $\alpha\in\frac{\pi}{2}+[-\alpha_0,\alpha_0]$, we have solution theorems due to Remark 5.33 and Theorem 4.25. Note that the corresponding right hand sides are contained in $B([0,T];H^2_{(\beta,\frac{\gamma_0}{2})}(\mathbb{R}^2_+)\times H^{5/2}_{(\beta)}(\mathbb{R}))$ for some $\beta>0$ provided that $q^\pm\in B([0,T],C^1([0,\tilde{\mu}_0]))$. Hence under this condition on $q^\pm$ we obtain a unique solution $\hat{v}_1^{C\pm}\in B([0,T];H^4_{(\beta,\frac{\gamma_0}{2})}(\mathbb{R}^2_+))\hookrightarrow B([0,T];C^2_{(\beta,\frac{\gamma_0}{2})}(\overline{\mathbb{R}^2_+}))$ for some possibly smaller $\beta > 0$ if and only if (4.17) holds for the associated right hand sides. The latter is equivalent to an equation for $q^\pm$ only involving $q^\pm(0,t)$ and $\partial_z q^\pm(0,t)$ linearly. Moreover, note that the only terms where $\partial_z q^\pm(0,t)$ enters are

$$\partial_z q^\pm(0,t)\left[2\int_{\mathbb{R}^2_+}\partial_Z\partial_\rho v_\alpha\partial_\rho v_\alpha\, d(\rho,Z) + \int_{\mathbb{R}}(\partial_\rho v_\alpha)^2|_{Z=0}\, d\rho\right].$$

Because of the estimates in Remark 4.19 it follows that $\partial_z q^\pm(0,t)$ is a determined bounded function on $[0,T]$ if for example $q^\pm(0,t) = 1$ for $t \in [0,T]$. Therefore with a simple ansatz and cutoff we can construct $q^\pm \in B([0,T], C^2([0,2\mu_0]))$ such that (4.17) holds as well as $q^\pm(0,t) = 1, q^\pm(.,t) = 1$ on $[\hat{\mu}_0, \tilde{\mu}_0]$ for all $t \in [0,T]$ and $\frac{1}{2} \leq q^\pm \leq 2$.

Lemma 6.55. *The function $\phi^A_{\varepsilon,\alpha}(.,t)$ is $C^2(\overline{\Omega^{C\pm}_t})$ and satisfies uniformly in $t \in [0,T]$:*

$$\begin{aligned}
\left|\sqrt{\varepsilon}\mathcal{L}^\pm_{\varepsilon,t}\phi^A_{\varepsilon,\alpha}(.,t) + \frac{1}{\varepsilon}\tilde{q}^\pm(.,t)\right| &\leq Ce^{-c|\rho_{\varepsilon,\alpha}(.,t)|} && \textit{in } \Omega^{C\pm}_t,\\
\left|\sqrt{\varepsilon}\mathcal{N}^\pm_{\varepsilon,t}\phi^A_{\varepsilon,\alpha}(.,t)\right| &\leq C\varepsilon e^{-c|\rho_{\varepsilon,\alpha}(.,t)|} && \textit{on } \partial\Omega^{C\pm}_t \cap \partial\Omega,\\
\left|\sqrt{\varepsilon}\mathcal{N}^\pm_{\varepsilon,t}\phi^A_{\varepsilon,\alpha}(.,t)\right| &\leq Ce^{-c/\varepsilon} && \textit{on } \partial\Omega^{C\pm}_t \setminus \Gamma_t(\frac{\delta_0}{2}),
\end{aligned}$$

where we have set

$$\tilde{q}^\pm(.,t) := [\Delta r|_{\overline{X}_0(s(.,t),t)} q^\pm|_{(z^\pm_\alpha(.,t),t)} - 2\cos\alpha\partial_z q^\pm|_{(z^\pm_\alpha(.,t),t)}]\partial^2_\rho v_\alpha|_{(\rho_{\varepsilon,\alpha}(.,t),Z^\pm_{\varepsilon,\alpha}(.,t))}.$$

Proof. The assertions follow from the construction and rigorous remainder estimates for the expansions above. Note that no Z-terms are multiplied with $\partial^2_\rho v_\alpha$, $\partial^3_\rho v_\alpha$. □

6.5.3 Notation for Transformations and the Splitting

We introduce the notation

$$X^\pm : [-\delta_0,\delta_0]\times[0,\tilde{\mu}_0]\times[0,T] \to \bigcup_{t\in[0,T]}\overline{\Omega^{C\pm}_t} : (r,z,t) \mapsto X(r, \pm 1 \mp \frac{1}{\sin\alpha}[z+\cos\alpha\, r], t)$$

and $\overline{X}^\pm := (X^\pm, \mathrm{pr}_t)$. Here note that $(X^\pm(.,t))^{-1} = (r, z^\pm_\alpha)(.,t)$. Furthermore, we set $X^\pm_0 := X^\pm(0,.,.)$ and $\overline{X}^\pm_0 := \overline{X}^\pm(0,.,.)$. Moreover, let

$$\begin{aligned}
J^\pm_t(r,z) &:= |\det D_{(r,z)}X^\pm(r,z,t)| = J_t\left(r, \pm 1 \mp \frac{1}{\sin\alpha}[z+\cos\alpha\, r]\right)\frac{1}{\sin\alpha},\\
\tilde{h}^\pm_{j,\alpha}(r,z,t) &:= h_{j,\alpha}(\pm 1 \mp \frac{1}{\sin\alpha}[z+\cos\alpha\, r], t)
\end{aligned}$$

for $(r,z,t) \in [-\delta_0,\delta_0]\times[0,\tilde{\mu}_0]\times[0,T]$ and $j = 1,2$ as well as $\tilde{h}^\pm_{\varepsilon,\alpha} := \tilde{h}^\pm_{1,\alpha} + \varepsilon\tilde{h}^\pm_{2,\alpha}$. Integrals over $\Omega^{C\pm}_t$ can be transformed to $(-\delta_0,\delta_0)\times(0,\tilde{\mu}_0)$ via $X^\pm(.,t)$ for $t \in [0,T]$, where the determinant factor is given by $J^\pm_t$. Hereby $\rho_{\varepsilon,\alpha}(.,t)$ transforms to

$$\rho_{\varepsilon,\alpha}|_{\overline{X}^\pm(r,z,t)} = \frac{r - \varepsilon\tilde{h}^\pm_{\varepsilon,\alpha}(r,z,t)}{\varepsilon}.$$

After applying the Fubini Theorem we can use the results from Section 6.1 for fixed z. We set $r^\pm_{\varepsilon,z,t} : [-\delta_0, \delta_0] \to \mathbb{R} : r \mapsto r - \varepsilon\tilde{h}^\pm_{\varepsilon,\alpha}(r, z, t)$ for all $z \in [0, \tilde{\mu}_0]$ and $t \in [0, T]$. Then due to Section 6.1.1

$$F^\pm_{\varepsilon,z,t} : \frac{1}{\varepsilon} r^\pm_{\varepsilon,z,t}([-\delta_0, \delta_0]) \to [-\delta_0, \delta_0] : \rho \mapsto (r^\pm_{\varepsilon,z,t})^{-1}(\varepsilon\rho)$$

is well-defined for all z, t as above if $\varepsilon \in (0, \varepsilon_0]$ for some $\varepsilon_0 > 0$ independent of z, t. Finally, we set $\tilde{J}^\pm_{\varepsilon,z,t} := J^\pm_t(F^\pm_{\varepsilon,z,t}(.), z)$ for $z \in [0, \tilde{\mu}_0]$ and $t \in [0, T]$.

Now we characterize the splitting of $\hat{H}^1(\Omega^{C\pm}_t)$.

Lemma 6.56. *Let $\hat{H}^1(\Omega^{C\pm}_t)$, $\hat{V}^\pm_{\varepsilon,t}$ and $\hat{H}^1(0, \tilde{\mu}_0)$ be as in* (6.67)-(6.69). *Then*

1. *$\hat{V}^\pm_{\varepsilon,t}$ is a subspace of $\hat{H}^1(\Omega^{C\pm}_t)$ and for $\varepsilon_0 > 0$ small there are $c_1, C_1 > 0$ such that*
$$c_1\|a\|_{L^2(0,\tilde{\mu}_0)} \leq \|\phi\|_{L^2(\Omega^{C\pm}_t)} \leq C_1\|a\|_{L^2(0,\tilde{\mu}_0)}$$
for all $\phi = a(z^\pm_\alpha(., t))\phi^A_{\varepsilon,\alpha}(., t) \in \hat{V}^\pm_{\varepsilon,t}$ and $\varepsilon \in (0, \varepsilon_0]$, $t \in [0, T]$.

2. *Let $(\hat{V}^\pm_{\varepsilon,t})^\perp$ be the L^2-orthogonal complement of $\hat{V}^\pm_{\varepsilon,t}$ in $\hat{H}^1(\Omega^{C\pm}_t)$. Then for $\psi \in \hat{H}^1(\Omega^{C\pm}_t)$:*
$$\psi \in (\hat{V}^\pm_{\varepsilon,t})^\perp \quad \Leftrightarrow \quad \int_{-\delta_0}^{\delta_0} (\phi^A_{\varepsilon,\alpha}(., t)\psi)|_{X^\pm(r,z,t)} J^\pm_t(r, z)\, dr = 0 \quad \textit{for a.e. } z \in (0, \tilde{\mu}_0).$$
Moreover, $\hat{H}^1(\Omega^{C\pm}_t) = \hat{V}^\pm_{\varepsilon,t} \oplus (\hat{V}^\pm_{\varepsilon,t})^\perp$ for all $\varepsilon \in (0, \varepsilon_0]$ and $\varepsilon_0 > 0$ small.

Proof. Ad 1. Analogously to the case $\alpha = \frac{\pi}{2}$ it follows that $a(z^\pm_\alpha(., t)) \in H^1(\Omega^{C\pm}_t)$ for all $a \in H^1(0, \tilde{\mu}_0)$, cf. the proof of Lemma 6.20, 1. Therefore $\hat{V}^\pm_{\varepsilon,t}$ is a subspace of $\hat{H}^1(\Omega^{C\pm}_t)$. Now we show the norm equivalence for $\varepsilon \in (0, \varepsilon_0]$ and $\varepsilon_0 > 0$ small. To this end we consider $\psi = a(z^\pm_\alpha(., t))\phi^A_{\varepsilon,\alpha}(., t) \in \hat{V}^\pm_{\varepsilon,t}$. Then the transformation rule and Fubini's Theorem imply

$$\|\psi\|^2_{L^2(\Omega^{C\pm}_t)} = \int_0^{\tilde{\mu}_0} a(z)^2 \int_{-\delta_0}^{\delta_0} (\phi^A_{\varepsilon,\alpha}|_{\overline{X}^\pm(r,z,t)})^2 J^\pm_t(r, z)\, dr\, dz. \tag{6.72}$$

The leading order term with respect to ε in the inner integral is $\frac{1}{\varepsilon}q^\pm(z, t)^2$ times

$$\int_{-\delta_0}^{\delta_0} (\partial_\rho v_\alpha)^2|_{(\rho_{\varepsilon,\alpha}|_{\overline{X}^\pm(r,z,t)}, \frac{z}{\varepsilon})} J^\pm_t|_{(r,z)}\, dr = \int_{r_{\varepsilon,z,t}([-\delta_0,\delta_0])/\varepsilon} (\frac{d}{d\rho}F^\pm_{\varepsilon,z,t})|_\rho (\partial_\rho v_\alpha)^2|_{(\rho,\frac{z}{\varepsilon})} \tilde{J}^\pm_{\varepsilon,z,t}|_\rho\, d\rho,$$

where we used Lemma 6.5, 1. Because of Remark 6.4, 2., the decay of $\partial_\rho v_\alpha$, the estimate $0 < \frac{d}{d\rho}F^\pm_{\varepsilon,z,t} = \varepsilon\mathcal{O}(1)$ due to Lemma 6.5, Remark 4.19 and $c \leq J, q^\pm \leq C$ for some $c, C > 0$, it follows that the above integral can be estimated from above and below by constants $\tilde{c}, \tilde{C} > 0$ independent of $t \in [0, T]$, $\varepsilon \in (0, \varepsilon_0]$ provided that $\varepsilon_0 = \varepsilon_0(\overline{C}_0) > 0$ is small. For the remainder in the inner integral in (6.72) we use Lemma 6.5 and obtain an estimate of the absolute value to $C\varepsilon$. For $\varepsilon_0 > 0$ small this shows the claim. $\Box_1$.

Ad 2. Let $t \in [0, T]$ be fixed. By definition it holds

$$(\hat{V}^\pm_{\varepsilon,t})^\perp = \left\{\psi \in \hat{H}^1(\Omega^{C\pm}_t) : \int_{\Omega^{C\pm}_t} \psi a(z^\pm_\alpha(., t))\phi^A_{\varepsilon,\alpha}(., t)\, dx = 0 \text{ for all } a \in \hat{H}^1(0, \tilde{\mu}_0)\right\}.$$

The integral equals $\int_0^{\tilde{\mu}_0} a(z) \int_{-\delta_0}^{\delta_0} (\phi^A_{\varepsilon,\alpha}(., t)\psi)|_{X^\pm(r,z,t)} J^\pm_t(r, z)\, dr\, dz$. Therefore the Fundamental Theorem of Calculus of Variations yields the characterization. Moreover, by definition it holds $\hat{V}^\pm_{\varepsilon,t} \cap (\hat{V}^\pm_{\varepsilon,t})^\perp = \{0\}$. It is left to prove $\hat{V}^\pm_{\varepsilon,t} + (\hat{V}^\pm_{\varepsilon,t})^\perp = \hat{H}^1(\Omega^{C\pm}_t)$. Due to the proof of the first part this follows in the analogous way as in the case $\alpha = \frac{\pi}{2}$, cf. the proof of Lemma 6.20, 2. $\Box_2$.

6.5.4 Analysis of the Bilinear Form

First we consider $B^{\pm}_{\varepsilon,t}$ on $\hat{V}^{\pm}_{\varepsilon,t} \times \hat{V}^{\pm}_{\varepsilon,t}$.

Lemma 6.57. *There are $\varepsilon_0, C > 0$ such that*

$$B^{\pm}_{\varepsilon,t}(\phi,\phi) \geq -C\|\phi\|^2_{L^2(\Omega^{C\pm}_t)} + \overline{c}\|a\|^2_{H^1(0,\tilde{\mu}_0)}, \quad \overline{c} := \frac{1}{2}\|\theta_0'\|^2_{L^2(\mathbb{R})}$$

for all $\phi = a(z^{\pm}_\alpha(.,t))\phi^A_{\varepsilon,\alpha}(.,t) \in \hat{V}^{\pm}_{\varepsilon,t}$ and $\varepsilon \in (0,\varepsilon_0], t \in [0,T]$.

Proof. Let ϕ be as in the lemma. With the analogous computation as in the case $\alpha = \frac{\pi}{2}$, cf. the proof of Lemma 6.21, it follows that

$$\begin{aligned} B^{\pm}_{\varepsilon,t}(\phi,\phi) = &\int_{\Omega^{C\pm}_t} |\nabla(a(z^{\pm}_\alpha))\phi^A_{\varepsilon,\alpha}|^2|_{(.,t)}\,dx + \int_{\Omega^{C\pm}_t} (a^2(z^{\pm}_\alpha)\phi^A_{\varepsilon,\alpha})|_{(.,t)}\mathcal{L}^{\pm}_{\varepsilon,t}\phi^A_{\varepsilon,\alpha}|_{(.,t)}\,dx \\ &+ \int_{\partial\Omega^{C\pm}_t} \left[\mathcal{N}^{\pm}_{\varepsilon,t}\phi^A_{\varepsilon,\alpha}|_{(.,t)}\operatorname{tr}(a^2(z^{\pm}_\alpha)\phi^A_{\varepsilon,\alpha}|_{(.,t)})\right] d\mathcal{H}^1 =: (I) + (II) + (III). \end{aligned}$$

Ad (I). It holds $|\nabla(a(z^{\pm}_\alpha(.,t)))|^2 = \left[|\nabla z^{\pm}_\alpha|^2(a')^2(z^{\pm}_\alpha)\right]|_{(.,t)}$ and therefore

$$(I) = \int_0^{\tilde{\mu}_0}(a')^2(z)\int_{-\delta_0}^{\delta_0}\left[|\nabla z^{\pm}_\alpha|^2(\phi^A_{\varepsilon,\alpha})^2\right]|_{\overline{X}^{\pm}(r,z,t)} J^{\pm}_t(r,z)\,dr\,dz.$$

Note that $|\nabla z^{\pm}_\alpha|^2|_{\overline{X}^{\pm}(0,z,t)} = 1$ and $J^{\pm}_t(0,t) = 1$ due to Remark 3.2, Theorem 3.3, Remark 3.4, 3. and (5.79). Therefore the Taylor Theorem yields that these terms are $1+\mathcal{O}(|r|)$. Moreover, Remark 4.19 yields $\int_{\mathbb{R}}(\partial_\rho v_\alpha)^2(\rho,Z)\,d\rho \geq \frac{3}{4}\|\theta_0'\|^2_{L^2(\mathbb{R})}$ for $Z \geq 0$. With Lemma 6.5, exponential decay estimates and Remark 6.4, 2. it follows that the inner integral in (I) is estimated from below by $\frac{2}{3}\|\theta_0'\|^2_{L^2(\mathbb{R})}$ for all $\varepsilon \in (0,\varepsilon_0], t \in [0,T]$, if $\varepsilon_0 > 0$ is small.

Ad (II). It holds

$$(II) = \int_0^{\tilde{\mu}_0} a^2(z)\int_{-\delta_0}^{\delta_0}\phi^A_{\varepsilon,\alpha}|_{\overline{X}^{\pm}(r,z,t)}(\mathcal{L}^{\pm}_{\varepsilon,t}\phi^A_{\varepsilon,\alpha}(.,t))|_{X^{\pm}(r,z,t)}J^{\pm}_t(r,z)\,dr\,dz.$$

We estimate the inner integral. Lemma 6.55 implies

$$\begin{aligned} &\left|\sqrt{\varepsilon}\mathcal{L}^{\pm}_{\varepsilon,t}\phi^A_{\varepsilon,\alpha}(.,t))|_{X^{\pm}(r,z,t)} + \frac{1}{\varepsilon}[\Delta r|_{\overline{X}^{\pm}_0(z,t)}q^{\pm}|_{(z,t)} - 2\cos\alpha\partial_z q^{\pm}|_{(z,t)}]\partial^2_\rho v_\alpha(\rho_{\varepsilon,\alpha}|_{\overline{X}^{\pm}(r,z,t)},\frac{z}{\varepsilon})\right| \\ &\leq Ce^{-c|\rho_{\varepsilon,\alpha}(\overline{X}^{\pm}(r,z,t))|} \quad \text{for } (r,z) \in [-\delta_0,\delta_0]\times[0,\tilde{\mu}_0]. \end{aligned}$$

Analogously to the case $\alpha = \frac{\pi}{2}$ we estimate (II), cf. the proof of Lemma 6.21. More precisely, using Lemma 6.5, $\int_{\mathbb{R}}(\partial^2_\rho v_\alpha\partial_\rho v_\alpha)(\rho,Z)\,d\rho = 0$ for all $Z \geq 0$ due to integration by parts and $J_t(r,z) = J_t(0,z) + \mathcal{O}(|r|)$, we obtain $|(II)| \leq C\|a\|^2_{L^2(0,\tilde{\mu}_0)}$ with $C > 0$ independent of $\phi \in \hat{V}^{\pm}_{\varepsilon,t}$ and all $\varepsilon \in (0,\varepsilon_0], t \in [0,T]$ if $\varepsilon_0 > 0$ is small.

Ad (III). The representation for line integrals and properties of the trace operator imply

$$\begin{aligned} (III) = &\sum_{\pm}\int_0^{\tilde{\mu}_0} a^2(z)\left[\phi^A_{\varepsilon,\alpha}\mathcal{N}^{\pm}_{\varepsilon,t}\phi^A_{\varepsilon,\alpha}\right]|_{\overline{X}^{\pm}(\pm\delta_0,z,t)}|\partial_z X^{\pm}(\pm\delta_0,z,t)|\,dz \\ &+ a^2(0)\int_{-\delta_0}^{\delta_0}\left[\phi^A_{\varepsilon,\alpha}\mathcal{N}^{\pm}_{\varepsilon,t}\phi^A_{\varepsilon,\alpha}\right]|_{\overline{X}^{\pm}(r,0,t)}|\partial_r X^{\pm}(r,0,t)|\,dr. \end{aligned}$$

Using Lemma 6.55 and for the last integral Lemma 6.5 we obtain

$$|(III)| \leq Ce^{-c/\varepsilon}\|a\|^2_{L^2(0,\tilde{\mu}_0)} + C\varepsilon a^2(0).$$

Due to $H^1(0,\tilde{\mu}_0) \hookrightarrow C^0_b([0,\tilde{\mu}_0])$, the claim follows with Lemma 6.56, 1. $\square$

Next we analyze $B^\pm_{\varepsilon,t}$ on $(\hat{V}^\pm_{\varepsilon,t})^\perp \times (\hat{V}^\pm_{\varepsilon,t})^\perp$. To this end we need the following auxiliary lemma:

Lemma 6.58. *There is a $\overline{C}_2 > 0$ independent of $\Omega, \Gamma, \alpha \in \frac{\pi}{2} + [-\alpha_0, \alpha_0]$ and some $\tilde{\delta}_0 > 0$ with*

$$\begin{aligned}\|\mathrm{tr}\,\psi\|^2_{L^2(\partial\Omega\cap\partial\Omega^{C\pm}_{t,\delta})} &\leq \overline{C}_2 \left[\|\psi\|^2_{L^2(\Omega^{C\pm}_{t,\delta})} + \|\nabla_\tau\psi\|_{L^2(\Omega^{C\pm}_{t,\delta})}\|\psi\|_{L^2(\Omega^{C\pm}_{t,\delta})}\right],\\ \|\nabla_\tau\psi\|_{L^2(\Omega^{C\pm}_{t,\delta})} &\leq 2\|\nabla\psi\|_{L^2(\Omega^{C\pm}_{t,\delta})}\end{aligned}$$

for all $\psi \in H^1(\Omega^{C\pm}_{t,\delta})$ and $t \in [0,T]$, $\delta \in (0,\tilde{\delta}_0]$, where $\Omega^{C\pm}_{t,\delta} := \Omega^{C\pm}_t \cap \Gamma_t(\delta)$.

Proof. The proof is analogous to the one of Lemma 6.54, but we have to be careful in order to obtain constants independent of Ω, Γ and $\alpha \in \frac{\pi}{2} + [-\alpha_0, \alpha_0]$. Note that with S from the proof of Lemma 6.54 the first estimate for $S \cap [(-\delta,\delta)\times\mathbb{R}]$, $S^\pm_{\delta,\alpha}$, ∂_s instead of $\Omega^{C\pm}_{t,\delta}$, $\partial\Omega\cap\partial\Omega^{C\pm}_{t,\delta}$, ∇_τ holds with a uniform constant independent of $\alpha \in \frac{\pi}{2} + [-\alpha_0, \alpha_0]$. This follows as in the proof of Lemma 6.54 since $\vec{w}$ can be chosen in a uniform way for all those α. Moreover,

$$\int_{\partial\Omega\cap\partial\Omega^{C\pm}_{t,\delta}} |\mathrm{tr}\psi|^2 \, d\mathcal{H}^1 = \int_{S^\pm_{\delta,\alpha}} |\mathrm{tr}\psi|^2|_{X(.,t)}|\det d_.(X(.,t)|_{S^\pm_{\delta,\alpha}})| \, d\mathcal{H}^1.$$

Let $\gamma^\pm : (-\delta,\delta) \to S^\pm_{\delta,\alpha} : r \mapsto (r, s^\pm(r))$ with $s^\pm$ as in (3.2). It holds

$$\left| d_{\gamma^\pm(r)}[X(.,t)|_{S^\pm_{\delta,\alpha}}]\left(\frac{(\gamma^\pm)'(r)}{|(\gamma^\pm)'(r)|}\right)\right| = \frac{1}{|(\gamma^\pm)'(r)|}|\partial_r X^\pm(r,0,t)|$$

and $|\partial_r X^\pm(r,0,t)| = |\partial_r X^\pm(0,0,t)| + \mathcal{O}(|r|)$, where

$$\partial_r X^\pm(0,0,t) = D_{(r,s)}X|_{(0,t)}\begin{pmatrix}1\\ \mp\cos\alpha/\sin\alpha\end{pmatrix} = \mathrm{Id}\cdot(\gamma^\pm)'(r)$$

due to Remark 3.2 and Theorem 3.3. This shows $|\det d_.(X(.,t)|_{S^\pm_{\delta,\alpha}})| \leq 1 + C(\Gamma)\delta$. Additionally, integrals over $S \cap [(-\delta,\delta)\times\mathbb{R}]$ are transformed to $\Omega^{C\pm}_{t,\delta}$ via $X(.,t)$ with the determinant factor J_t, where $J_t(r,s) = 1 + \mathcal{O}(|\delta|)$ in $\Omega^{C\pm}_{t,\delta}$ because of Remark 3.2 and Remark 3.4, 3. Altogether we obtain the first estimate. For the second one we use $|\nabla s|^2 = 1 + \mathcal{O}(|\delta|)$ in $\Omega^{C\pm}_{t,\delta}$ due to Remark 3.2 and

$$|\nabla\psi|^2|_{X(.,t)} \geq (1 - C(\Gamma)\delta)\partial_s(\psi|_{X(.,t)})^2 \quad \text{in } \Omega^{C\pm}_{t,\delta}.$$

The latter follows analogously to (6.29) in the case $\alpha = \frac{\pi}{2}$. This yields the claim. $\square$

Lemma 6.59. *There are $\hat{\alpha}_0, \nu > 0$ independent of Ω, Γ such that, if $\alpha \in \frac{\pi}{2} + [-\hat{\alpha}_0, \hat{\alpha}_0]$, then there is an $\varepsilon_0 > 0$ such that for all $\psi \in (\hat{V}^\pm_{\varepsilon,t})^\perp$ and $\varepsilon \in (0,\varepsilon_0]$, $t \in [0,T]$ it holds*

$$B^\pm_{\varepsilon,t}(\psi,\psi) \geq \nu\left[\frac{1}{\varepsilon^2}\|\psi\|^2_{L^2(\Omega^{C\pm}_t)} + \|\nabla\psi\|^2_{L^2(\Omega^{C\pm}_t)}\right].$$

Proof. First we prove that it is sufficient to show the existence of $\tilde{\alpha}_0, \tilde{\nu} > 0$ independent of Ω, Γ, α and the existence of some $\tilde{\varepsilon}_0 > 0$ such that if $\alpha \in \frac{\pi}{2} + [-\tilde{\alpha}_0, \tilde{\alpha}_0]$, then

$$\tilde{B}^{\pm}_{\varepsilon,t}(\psi,\psi) := \int_{\Omega_t^{C\pm}} |\nabla\psi|^2 + \frac{1}{\varepsilon^2} f''(\theta_0|_{\rho_{\varepsilon,\alpha}(.,t)})\psi^2 \, dx \geq \frac{\tilde{\nu}}{\varepsilon^2}\|\psi\|^2_{L^2(\Omega_t^{C\pm})} \tag{6.73}$$

for all $\psi \in (\hat{V}^{\pm}_{\varepsilon,t})^\perp$ and $\varepsilon \in (0, \tilde{\varepsilon}_0]$, $t \in [0,T]$. In order to show with (6.73) the estimate in the lemma note that due to Remark 4.19 and Definition 1.8 there is a $\overline{C} > 0$ independent of Ω, Γ and $\alpha \in \frac{\pi}{2} + [-\alpha_0, \alpha_0]$ such that

$$|f''(v_\alpha(\rho, Z)) - f''(\theta_0(\rho))| \leq \overline{C}|\alpha - \frac{\pi}{2}| \text{ for all } (\rho, Z) \in \overline{\mathbb{R}^2_+} \quad \text{and} \quad |\sigma''_\alpha| + |\sigma'''_\alpha| \leq \overline{C}|\alpha - \frac{\pi}{2}|.$$

Moreover, to control the $\frac{1}{\varepsilon}\sigma''_\alpha$-term in $B^{\pm}_{\varepsilon,t}$ we use Lemma 6.58 and $\sigma''_\alpha(v_\alpha(\rho_{\varepsilon,\alpha}(.,t), 0)) = 0$ in $\Omega_t^{C\pm} \setminus \Gamma(\tilde{\delta}_0)$ for ε small because of Definition 1.8 and Remark 4.19, where $\tilde{\delta}_0$ is as in Lemma 6.58. For the σ'''_α-term we use Lemma 6.54. Therefore $|B^{\pm}_{\varepsilon,t}(\psi,\psi) - \tilde{B}^{\pm}_{\varepsilon,t}(\psi,\psi)|$ is estimated by

$$\frac{\overline{C}|\alpha - \frac{\pi}{2}| + C\varepsilon}{\varepsilon^2}\|\psi\|^2_{L^2(\Omega_t^{C\pm})} + \frac{2\overline{C}\,\overline{C}_2|\alpha - \frac{\pi}{2}| + C\varepsilon}{\varepsilon}\|\psi\|_{L^2(\Omega_t^{C\pm})}\|\nabla\psi\|_{L^2(\Omega_t^{C\pm})}$$

for all $\psi \in (\hat{V}^{\pm}_{\varepsilon,t})^\perp$ and $\varepsilon \in (0, \varepsilon_0]$, $t \in [0,T]$ if $\varepsilon_0 > 0$ is small. Let $\alpha \in \frac{\pi}{2} + [-\tilde{\alpha}_0, \tilde{\alpha}_0]$. Then for $\beta \in (0,1)$ it follows with Young's inequality that

$$\begin{aligned} B^{\pm}_{\varepsilon,t}(\psi,\psi) &\geq (1 - \beta + \beta)\tilde{B}^{\pm}_{\varepsilon,t}(\psi,\psi) - |B^{\pm}_{\varepsilon,t}(\psi,\psi) - \tilde{B}^{\pm}_{\varepsilon,t}(\psi,\psi)| \\ &\geq \frac{(1-\beta)\tilde{\nu} - \beta\sup_{\mathbb{R}}|f''(\theta_0)| - \overline{C}\,(\overline{C}_2 + 1)|\alpha - \frac{\pi}{2}| - C\varepsilon}{\varepsilon^2}\|\psi\|^2_{L^2(\Omega_t^{C\pm})} \\ &\quad + (\beta - \overline{C}\,\overline{C}_2|\alpha - \frac{\pi}{2}| - C\varepsilon)\|\nabla\psi\|^2_{L^2(\Omega_t^{C\pm})} \end{aligned}$$

for all $\psi \in (\hat{V}^{\pm}_{\varepsilon,t})^\perp$ and $\varepsilon \in (0, \varepsilon_0]$, $t \in [0,T]$. We choose $\beta := \frac{1}{4}\min\{1, \tilde{\nu}/\sup_{\mathbb{R}}|f''(\theta_0)|\}$ and then $\hat{\alpha}_0 > 0$ small such that

$$\frac{\tilde{\nu}}{2} - \overline{C}\,(\overline{C}_2 + 1)\hat{\alpha}_0 \geq \frac{\tilde{\nu}}{4} \quad \text{and} \quad \beta - \overline{C}\,\overline{C}_2\hat{\alpha}_0 \geq \frac{\beta}{2}.$$

Therefore the claim follows with $\nu := \min\{\frac{\tilde{\nu}}{8}, \frac{\beta}{4}\}$ provided that $\varepsilon_0 > 0$ is small.

In the following we prove (6.73) with similar ideas as in the case $\alpha = \frac{\pi}{2}$, cf. the proof of Lemma 6.22. Let $\tilde{\psi}^{\pm}_t := \psi|_{X^{\pm}(.,t)}$ for $\psi \in (\hat{V}^{\pm}_{\varepsilon,t})^\perp$. Because of the chain rule we obtain $\nabla\psi|_{X^{\pm}(.,t)} = \nabla r|_{\overline{X}^{\pm}(.,t)}\partial_r\tilde{\psi}^{\pm}_t + \nabla z^{\pm}_\alpha|_{\overline{X}^{\pm}(.,t)}\partial_z\tilde{\psi}^{\pm}_t$ and therefore

$$|\nabla\psi|^2|_{X^{\pm}(.,t)} = (\nabla_{(r,z)}\tilde{\psi}^{\pm}_t)^\top \begin{pmatrix} |\nabla r|^2 & \nabla r \cdot \nabla z^{\pm}_\alpha \\ \nabla r \cdot \nabla z^{\pm}_\alpha & |\nabla z^{\pm}_\alpha|^2 \end{pmatrix} |_{\overline{X}^{\pm}(.,t)}\nabla_{(r,z)}\tilde{\psi}^{\pm}_t,$$

where $|\nabla r|^2 = 1 + \mathcal{O}(|r|^2)$, $|\nabla r \cdot \nabla z^{\pm}_\alpha| = |\cos\alpha| + \mathcal{O}(|r|)$ and $|\nabla s|^2 = 1 + \mathcal{O}(|r|)$ due to Remark 3.2, Theorem 3.3 and Taylor's Theorem. Therefore Young's inequality yields

$$|\nabla\psi|^2|_{X^{\pm}(.,t)} \geq (1 - \overline{C}_3|\alpha - \frac{\pi}{2}| - C|r|)\left[(\partial_r\tilde{\psi}^{\pm}_t)^2 + (\partial_z\tilde{\psi}^{\pm}_t)^2\right] \tag{6.74}$$

with $\overline{C}_3 > 0$ independent of Ω, Γ and $\alpha \in \frac{\pi}{2} + [-\alpha_0, \alpha_0]$. To get $C|r|$ small enough (which will be precise later), we fix $\tilde{\delta} > 0$ small and estimate separately for r in

$$I_{z,t}^{\pm,\varepsilon} := (r_{\varepsilon,z,t}^{\pm})^{-1}[(-\tilde{\delta}, \tilde{\delta})] \quad \text{and} \quad \hat{I}_{z,t}^{\pm,\varepsilon} := (-\delta_0, \delta_0) \setminus I_{z,t}^{\pm,\varepsilon}. \tag{6.75}$$

If $\varepsilon_0 = \varepsilon_0(\tilde{\delta}, \overline{C}_0) > 0$ is small, then for all $\varepsilon \in (0, \varepsilon_0]$ and $z \in [0, \tilde{\mu}_0]$, $t \in [0, T]$ it holds

$$f''(\theta_0(\rho_{\varepsilon,\alpha}|_{\overline{X}^{\pm}(r,z,t)})) \geq c_0 := \frac{1}{2}\min\{f''(\pm 1)\} > 0 \quad \text{for } r \in \hat{I}_{z,t}^{\pm,\varepsilon}, \quad |r| \leq 2\tilde{\delta} \quad \text{for } r \in I_{z,t}^{\pm,\varepsilon}$$

where we used Remark 6.4, 2. for the first estimate and Lemma 6.2, 1. for the second one. With $\overline{C}_3, C$ as in (6.74) we define $\tilde{c} = \tilde{c}(\alpha, \tilde{\delta}) := \overline{C}_3|\alpha - \frac{\pi}{2}| + 2C\tilde{\delta}$. For $\varepsilon \in (0, \varepsilon_0]$, $t \in [0, T]$ we get

$$\begin{aligned}\tilde{B}_{\varepsilon,t}^{\pm}(\psi, \psi) \geq & \int_0^{\tilde{\mu}_0} \int_{\hat{I}_{z,t}^{\pm,\varepsilon}} \frac{c_0}{\varepsilon^2}(\tilde{\psi}_t^{\pm})^2 J_t^{\pm}|_{(r,z)}\, dr\, dz + \int_0^{\tilde{\mu}_0} \int_{I_{z,t}^{\pm,\varepsilon}} (1-\tilde{c})(\partial_z \tilde{\psi}_t^{\pm})^2 J_t^{\pm}|_{(r,z)}\, dr\, dz \\ &+ \int_0^{\tilde{\mu}_0} \int_{I_{z,t}^{\pm,\varepsilon}} \left[(1-\tilde{c})(\partial_r \tilde{\psi}_t^{\pm})^2 + \frac{1}{\varepsilon^2} f''(\theta_0(\rho_{\varepsilon,\alpha}|_{\overline{X}^{\pm}(.,t)}))(\tilde{\psi}_t^{\pm})^2\right] J_t^{\pm}|_{(r,z)}\, dr\, dz.\end{aligned}$$

We use the notation from the beginning of Section 6.5.3. Lemma 6.5, 1. yields that the inner integral in the second line equals $1/\varepsilon^2$ times

$$B_{\varepsilon,z,t}^{\pm,\tilde{c}}(\Psi_{\varepsilon,z,t}^{\pm}, \Psi_{\varepsilon,z,t}^{\pm}) := \int_{I_{\varepsilon,\tilde{\delta}}} \left[(1-\tilde{c})(\frac{d}{dz}\Psi_{\varepsilon,z,t}^{\pm})^2 + f''(\theta_0(z))(\Psi_{\varepsilon,z,t}^{\pm})^2\right] \tilde{J}_{\varepsilon,z,t}^{\pm}\, dz, \tag{6.76}$$

where we set $I_{\varepsilon,\tilde{\delta}} := (-\frac{\tilde{\delta}}{\varepsilon}, \frac{\tilde{\delta}}{\varepsilon})$ and $\Psi_{\varepsilon,z,t}^{\pm} := \sqrt{\varepsilon}\tilde{\psi}_t^{\pm}(F_{\varepsilon,z,t}^{\pm}(.), z)$. Hence (6.73) follows if we show for $\tilde{\alpha}_0 > 0$ small independent of Ω, Γ and $\tilde{\delta} > 0$ small, that $\tilde{c} \leq 1$ and with the c_0 from above

$$B_{\varepsilon,z,t}^{\pm,\tilde{c}}(\Psi_{\varepsilon,z,t}^{\pm}, \Psi_{\varepsilon,z,t}^{\pm}) \geq \overline{\nu}\|\Psi_{\varepsilon,z,t}^{\pm}\|_{L^2(I_{\varepsilon,\tilde{\delta}}, \tilde{J}_{\varepsilon,z,t}^{\pm})}^2 - \frac{c_0}{2}\|\tilde{\psi}_t^{\pm}(.,z)\|_{L^2(\hat{I}_{z,t}^{\pm,\varepsilon}, J_t^{\pm}(.,z))}^2 \tag{6.77}$$

for $\varepsilon \in (0, \varepsilon_0]$, a.e. $z \in [0, \tilde{\mu}_0]$ and all $t \in [0, T]$ with some $\varepsilon_0 > 0$ independent of ε, z, t and $\overline{\nu} > 0$ independent of $\Omega, \Gamma, \alpha, \tilde{\delta}, \varepsilon_0, \varepsilon, z, t$ provided that $\alpha \in \frac{\pi}{2} + [-\tilde{\alpha}_0, \tilde{\alpha}_0]$.

Here $L^2(I_{\varepsilon,\tilde{\delta}}, \tilde{J}_{\varepsilon,z,t}^{\pm})$ is the space of L^2-functions on $I_{\varepsilon,\tilde{\delta}}$ with respect to the weight $\tilde{J}_{\varepsilon,z,t}^{\pm}$. We denote the scalar-product in $L^2(I_{\varepsilon,\tilde{\delta}}, \tilde{J}_{\varepsilon,z,t}^{\pm})$ by $(.,.)_{\varepsilon,z,t}$ and the norm with $\|.\|_{\varepsilon,z,t}$. For the proof of (6.77) we need properties of $B_{\varepsilon,z,t}^{\pm,0}$. The latter is defined as in (6.76) with $\tilde{c}$ replaced by 0. With respect to $(.,.)_{\varepsilon,z,t}$, $B_{\varepsilon,z,t}^{\pm,0}$ is the bilinear form associated to

$$\mathcal{L}_{\varepsilon,z,t}^{\pm,0} := -(\tilde{J}_{\varepsilon,z,t}^{\pm})^{-1}\frac{d}{dz}\left(\tilde{J}_{\varepsilon,z,t}^{\pm}\frac{d}{dz}\right) + f''(\theta_0)$$

on $H^2(I_{\varepsilon,\tilde{\delta}})$ with homogeneous Neumann boundary condition. In this situation we can apply the results in Section 6.1.3.2.

Proof of (6.77). The integral characterization for $\psi \in (\hat{V}_{\varepsilon,t}^{\pm})^{\perp}$ in Lemma 6.56, 2. yields

$$\left|\int_{I_{z,t}^{\pm,\varepsilon}} (\phi_{\varepsilon,\alpha}^A(.,t)\psi)|_{X^{\pm}(r,z,t)} J_t^{\pm}(r,z)\, dr\right| \leq C(\tilde{\delta})e^{-c\tilde{\delta}/\varepsilon}\|\tilde{\psi}_t^{\pm}(.,z)\|_{L^2(\hat{I}_{z,t}^{\pm,\varepsilon}, J_t^{\pm}(.,z))}$$

for ε small, a.e. $z \in [0, \tilde{\mu}_0]$ and all $t \in [0, T]$. The lowest order term in the integral is

$$\frac{1}{\sqrt{\varepsilon}} q^\pm(z,t) \int_{I^{\pm,\varepsilon}_{z,t}} \partial_\rho v_\alpha(\rho_{\varepsilon,\alpha}|_{\overline{X}^\pm(r,z,t)}, \frac{z}{\varepsilon})(\tilde{\psi}^\pm_t J^\pm_t)(r,z)\, dr = q^\pm(z,t)(\Psi^\pm_{\varepsilon,z,t}, \partial_\rho v_\alpha(.,\frac{z}{\varepsilon}))_{\varepsilon,z,t}.$$

The remaining term in the integral due to $\phi^A_{\varepsilon,\alpha}$ can be estimated with the Hölder inequality, the decay of $\hat{v}^{C\pm}_1$ and Lemma 6.5 by $C\varepsilon\|\Psi^\pm_{\varepsilon,z,t}\|_{\varepsilon,z,t}$. Moreover, due to Remark 4.19 it holds

$$|\partial_\rho v_\alpha(\rho, Z) - \theta_0'(\rho)| \leq \overline{C}_4|\alpha - \frac{\pi}{2}|e^{-\beta_0|\rho|} \quad \text{for all } (\rho, Z) \in \overline{\mathbb{R}^2_+}$$

and some $\overline{C}_4 > 0$ independent of Ω, Γ, α. Together with

$$\tilde{J}^\pm_{\varepsilon,z,t}(\rho) = J_t(F^\pm_{\varepsilon,z,t}(\rho), z) = J_t(0,z) + \mathcal{O}(|F^\pm_{\varepsilon,z,t}(\rho)|) = 1 + \mathcal{O}(\varepsilon(|\rho| + C))$$

because of Remark 3.2, Remark 3.4, 3. and Corollary 6.3, we obtain

$$|(\Psi^\pm_{\varepsilon,z,t}, \partial_\rho v_\alpha(.,\frac{z}{\varepsilon}))_{\varepsilon,z,t} - (\Psi^\pm_{\varepsilon,z,t}, \theta_0')_{\varepsilon,z,t}| \leq (\overline{C}_4|\alpha - \frac{\pi}{2}| + C\varepsilon)\|\Psi^\pm_{\varepsilon,z,t}\|_{\varepsilon,z,t}$$

for some $\overline{C}_4 > 0$ independent of Ω, Γ, α. Using $0 < \frac{1}{2} \leq q^\pm \leq 2$ we obtain altogether

$$|(\Psi^\pm_{\varepsilon,z,t}, \theta_0')_{\varepsilon,z,t}| \leq (\overline{C}_4|\alpha - \frac{\pi}{2}| + C\varepsilon)\|\Psi^\pm_{\varepsilon,z,t}\|_{\varepsilon,z,t} + C(\tilde{\delta})e^{-c\tilde{\delta}/\varepsilon}\|\tilde{\psi}^\pm_t(.,z)\|_{L^2(\hat{I}^{\pm,\varepsilon}_{z,t}, J^\pm_t(.,z))}$$

for ε small. Theorem 6.8, 2. and uniform bounds for $q^\pm$, $J^\pm_t$ yield for the positive normalized eigenfunction $\Psi^{\pm,1}_{\varepsilon,z,t}$ to the first eigenvalue $\lambda^{\pm,1}_{\varepsilon,z,t}$ of $\mathcal{L}^{\pm,0}_{\varepsilon,z,t}$ the estimate

$$\begin{aligned} |(\Psi^\pm_{\varepsilon,z,t}, \Psi^{\pm,1}_{\varepsilon,z,t})_{\varepsilon,z,t}| &\leq (\overline{C}_4|\alpha - \frac{\pi}{2}| + C(\tilde{\delta})\varepsilon)\|\Psi^\pm_{\varepsilon,z,t}\|_{\varepsilon,z,t} \\ &\quad + C(\tilde{\delta})e^{-c\tilde{\delta}/\varepsilon}\|\tilde{\psi}^\pm_t(.,z)\|_{L^2(\hat{I}^{\pm,\varepsilon}_{z,t}, J^\pm_t(.,z))} \end{aligned} \tag{6.78}$$

for a.e. $z \in [0, \tilde{\mu}_0]$, all $t \in [0, T]$ and $\varepsilon \in (0, \varepsilon_0]$, if $\varepsilon_0 > 0$ is small.

With the analogous computation as in the case $\alpha = \frac{\pi}{2}$, cf. the proof of Lemma 6.22, it follows from Theorem 6.8, 1. and 3. that, if $\tilde{c}(\alpha, \tilde{\delta}) = \overline{C}_3|\alpha - \frac{\pi}{2}| + 2C\tilde{\delta} \leq 1$, then it holds

$$\begin{aligned} B^{\pm,\tilde{c}}_{\varepsilon,z,t}(\Psi^\pm_{\varepsilon,z,t}, \Psi^\pm_{\varepsilon,z,t}) \geq \|\Psi^\pm_{\varepsilon,z,t}\|^2_{\varepsilon,z,t} &\left[\nu_2(1 - \tilde{c}(\alpha,\tilde{\delta})) - \tilde{c}(\alpha,\tilde{\delta}) \sup_{\rho\in\mathbb{R}} |f''(\theta_0(\rho))|\right] \\ &- (1 - \tilde{c}(\alpha,\tilde{\delta}))(\mathcal{O}(\varepsilon^2) + \nu_2)\left|(\Psi^\pm_{\varepsilon,z,t}, \Psi^{\pm,1}_{\varepsilon,z,t})_{\varepsilon,z,t}\right|^2 \end{aligned}$$

for a.e. $z \in [0, \tilde{\mu}_0]$, all $t \in [0, T]$ and $\varepsilon \in (0, \varepsilon_0]$ if $\varepsilon_0 = \varepsilon_0(\tilde{\delta}, \overline{C}_0) > 0$ is small. We combine this with (6.78) in order to show (6.77). Note that ν_2 from Theorem 6.8 does not depend on $\alpha, \tilde{\delta}$. Therefore we can first choose $\tilde{\alpha}_0 > 0$ small such that

$$\overline{C}_3\tilde{\alpha}_0 \leq \frac{1}{4}, \quad (\nu_2 + \sup_{\rho\in\mathbb{R}} |f''(\theta_0(\rho))|)\overline{C}_3\tilde{\alpha}_0 \leq \frac{\nu_2}{4} \quad \text{and} \quad \frac{1}{2}\overline{C}_4^2\tilde{\alpha}_0^2 \leq \frac{1}{4}.$$

Note that this can be achieved independent of Ω, Γ. Then let $\tilde{\delta} > 0$ be small such that

$$2C\tilde{\delta} \leq \frac{1}{2} \quad \text{and} \quad (\nu_2 + \sup_{\rho\in\mathbb{R}} |f''(\theta_0(\rho))|)2C\tilde{\delta} \leq \frac{\nu_2}{4}.$$

These estimates imply $\tilde{c}(\alpha, \tilde{\delta}) \leq \frac{1}{2}$ and that the term in the square brackets above is estimated from below by $\frac{\nu_2}{2}$. Finally, we can choose $\varepsilon_0 > 0$ small such that (6.77) holds with $\overline{\nu} = \frac{\nu_2}{8}$. Finally, altogether we have proven Lemma 6.59. □

For $B^\pm_{\varepsilon,t}$ on $\hat{V}^\pm_{\varepsilon,t} \times (\hat{V}^\pm_{\varepsilon,t})^\perp$ we obtain

Lemma 6.60. *There is an $\overline{\alpha}_0 > 0$ independent of Ω, Γ such that, if $\alpha \in \frac{\pi}{2} + [-\overline{\alpha}_0, \overline{\alpha}_0]$, then there are $\varepsilon_0, C > 0$ such that*

$$|B^\pm_{\varepsilon,t}(\phi,\psi)| \leq \frac{C}{\varepsilon}\|\phi\|_{L^2(\Omega^{C\pm}_t)}\|\psi\|_{L^2(\Omega^{C\pm}_t)} + \frac{1}{4}B^\pm_{\varepsilon,t}(\psi,\psi) + (\frac{\overline{c}}{4} + C\varepsilon^2)\|a\|^2_{H^1(0,\tilde{\mu}_0)}$$

for all $\phi = a(z^\pm_\alpha(.,t))\phi^A_{\varepsilon,\alpha}(.,t) \in \hat{V}^\pm_{\varepsilon,t}$, $\psi \in (\hat{V}^\pm_{\varepsilon,t})^\perp$ and $\varepsilon \in (0,\varepsilon_0]$, $t \in [0,T]$, where the constant $\overline{c} = \frac{1}{2}\|\theta_0'\|^2_{L^2(\mathbb{R})}$ is as in Lemma 6.57.

Proof. The analogous computation as in the case $\alpha = \frac{\pi}{2}$, cf. the proof of Lemma 6.23, yields

$$\begin{aligned} B^\pm_{\varepsilon,t}(\phi,\psi) = &\int_{\Omega^{C\pm}_t} a(z^\pm_\alpha)|_{(.,t)}\psi\mathcal{L}^\pm_{\varepsilon,t}\phi^A_{\varepsilon,\alpha}|_{(.,t)}\,dx + \int_{\partial\Omega^{C\pm}_t}\mathcal{N}^\pm_{\varepsilon,t}\phi^A_{\varepsilon,\alpha}|_{(.,t)}\mathrm{tr}\left[a(z^\pm_\alpha(.,t))\psi\right]\,d\mathcal{H}^1\\ &+ \int_{\Omega^{C\pm}_t}\nabla(a(z^\pm_\alpha))|_{(.,t)}\cdot\left[\phi^A_{\varepsilon,\alpha}|_{(.,t)}\nabla\psi - \nabla\phi^A_{\varepsilon,\alpha}|_{(.,t)}\psi\right]\,dx =: (I) + (II) + (III).\end{aligned}$$

Ad (I). It holds $|(I)| \leq \|a(z^\pm_\alpha|_{(.,t)})\mathcal{L}^\pm_{\varepsilon,t}\phi^A_{\varepsilon,\alpha}(.,t)\|_{L^2(\Omega^{C\pm}_t)}\|\psi\|_{L^2(\Omega^{C\pm}_t)}$ because of the Hölder Inequality, where

$$\|a(z^\pm_\alpha|_{(.,t)})\mathcal{L}^\pm_{\varepsilon,t}\phi^A_{\varepsilon,\alpha}|_{(.,t)}\|^2_{L^2(\Omega^{C\pm}_t)} = \int_0^{\tilde{\mu}_0} a^2(z)\int_{-\delta_0}^{\delta_0}(\mathcal{L}^\pm_{\varepsilon,t}\phi^A_{\varepsilon,\alpha}(.,t))^2|_{X^\pm(r,z,t)}\,J^\pm_t(r,z)\,dr\,dz.$$

Analogously to the case $\alpha = \frac{\pi}{2}$, cf. the proof of Lemma 6.23, we obtain from Lemma 6.55 that $(\mathcal{L}^\pm_{\varepsilon,t}\phi^A_{\varepsilon,\alpha}|_{(.,t)})^2$ is estimated by

$$\frac{1}{\varepsilon^3}\left|[\Delta r|_{\overline{X}_0(s(.,t),t)}q^\pm|_{(z^\pm_\alpha(.,t),t)} - 2\cos\alpha\partial_z q^\pm|_{(z^\pm_\alpha(.,t),t)}]\partial^2_\rho v_\alpha|_{(\rho_{\varepsilon,\alpha}(.,t),Z^\pm_{\varepsilon,\alpha}(.,t))}\right|^2 + \frac{\tilde{C}}{\varepsilon^2}e^{-c|\rho_{\varepsilon,\alpha}(.,t)|}.$$

Therefore Lemma 6.5 implies that the inner integral above is estimated by C/ε^2 and because of Lemma 6.56, 1. we get

$$|(I)| \leq \frac{C}{\varepsilon}\|a\|_{L^2(0,\tilde{\mu}_0)}\|\psi\|_{L^2(\Omega^{C\pm}_t)} \leq \frac{\tilde{C}}{\varepsilon}\|\phi\|_{L^2(\Omega^{C\pm}_t)}\|\psi\|_{L^2(\Omega^{C\pm}_t)}$$

for all $t \in [0,T]$ and $\varepsilon \in (0,\varepsilon_0]$ if $\varepsilon_0 > 0$ is small.

Ad (II). Because of Hölder's inequality we obtain

$$|(II)| \leq \|\mathrm{tr}\,\psi\|_{L^2(\partial\Omega^{C\pm}_t)}\|\mathrm{tr}(a(z^\pm_\alpha|_{(.,t)}))\mathcal{N}^\pm_{\varepsilon,t}\phi^A_{\varepsilon,\alpha}|_{(.,t)}\|_{L^2(\partial\Omega^{C\pm}_t)}.$$

For the second integral we use the representation of integrals over curves, cf. also the estimate of (III) in the proof of Lemma 6.57. Then Lemma 6.55 and Lemma 6.5 yield

$$\|a(z^\pm_\alpha)\mathcal{N}^\pm_{\varepsilon,t}\phi^A_{\varepsilon,\alpha}|_{(.,t)}\|_{L^2(\partial\Omega^{C\pm}_t)} \leq C\varepsilon|a(0)| + Ce^{-c/\varepsilon}\|a\|_{L^2(0,\tilde{\mu}_0)} \leq C\varepsilon\|a\|_{H^1(0,\tilde{\mu}_0)}.$$

We estimate $\|\mathrm{tr}\,\psi\|_{L^2(\partial\Omega^{C\pm}_t)}$ with Lemma 6.54. Then Young's inequality and Lemma 6.59 imply

$$|(II)| \leq \frac{\nu}{8\varepsilon\overline{C}_1}\|\mathrm{tr}\,\psi\|^2_{L^2(\partial\Omega^{C\pm}_t)} + \tilde{C}\varepsilon^3\|a\|^2_{H^1(0,\tilde{\mu}_0)} \leq \frac{1}{8}B^\pm_{\varepsilon,t}(\psi,\psi) + \tilde{C}\varepsilon^3\|a\|^2_{H^1(0,\tilde{\mu}_0)},$$

where $\overline{C}_1$ is as in Lemma 6.54.

Ad (III). We proceed in the analogous way as in the case $\alpha = \frac{\pi}{2}$, cf. the estimate of (III) in the proof of Lemma 6.23. However, there are some new terms due to $\nabla r \cdot \nabla z_\alpha^\pm$ and since $\partial_\rho v_\alpha$ depends on Z. It holds $(III) = \int_0^{\tilde{\mu}_0} a'(z) g_t^\pm(z)\, dz$ with

$$g_t^\pm(z) := \int_{-\delta_0}^{\delta_0} \nabla z_\alpha^\pm|_{\overline{X}^\pm(r,z,t)} \cdot \left[\phi_{\varepsilon,\alpha}^A(.,t)\nabla\psi - \nabla\phi_{\varepsilon,\alpha}^A(.,t)\psi\right]|_{X^\pm(r,z,t)}\, J_t^\pm(r,z)\, dr.$$

We insert $\nabla\psi|_{X^\pm(.,t)} = \nabla r|_{\overline{X}^\pm(.,t)}\partial_r\tilde{\psi}_t^\pm + \nabla z_\alpha^\pm|_{\overline{X}^\pm(.,t)}\partial_z\tilde{\psi}_t^\pm$ with $\tilde{\psi}_t^\pm := \psi|_{X^\pm(.,t)}$. For the $\partial_z\tilde{\psi}_t^\pm$-term in $g_t^\pm$ we use $|\nabla z_\alpha^\pm|^2|_{\overline{X}^\pm(r,z,t)} = 1 + \mathcal{O}(|r|)$ due to Remark 3.2 and Theorem 3.3. Therefore $|g_t^\pm(z)|$ is for a.e. $z \in [0,\tilde{\mu}_0]$ and all $t \in [0,T]$ estimated by

$$\begin{aligned}
&\left|\int_{-\delta_0}^{\delta_0} \left[\phi_{\varepsilon,\alpha}^A|_{\overline{X}^\pm(.,t)}\partial_z\tilde{\psi}_t^\pm J_t^\pm\right]|_{(r,z)}\, dr\right| + \int_{-\delta_0}^{\delta_0} \left|(\tilde{\psi}_t^\pm J_t^\pm)|_{(r,z)}\nabla z_\alpha^\pm \cdot \nabla\phi_{\varepsilon,\alpha}^A|_{\overline{X}^\pm(r,z,t)}\right|\, dr\\
&+ \int_{-\delta_0}^{\delta_0} \left[C\left|r\,\partial_z\tilde{\psi}_t^\pm|_{(r,z)}\right| + \left|\nabla r\cdot\nabla z_\alpha^\pm|_{\overline{X}^\pm(r,z,t)}\partial_r\tilde{\psi}_t^\pm|_{(r,z)}\right|\right]\cdot\left|\phi_{\varepsilon,\alpha}^A|_{\overline{X}^\pm(r,z,t)}J_t^\pm|_{(r,z)}\right|\, dr.
\end{aligned}$$

We use $\psi \in (\hat{V}_{\varepsilon,t}^\pm)^\perp$ to rewrite the first term. With Lemma 2.10 and since integration yields a bounded linear operator on $L^2(-\delta,\delta)$, we can differentiate the identity in Lemma 6.56, 2. and use the product rule. Hence the first term is estimated by

$$\left|\int_{-\delta_0}^{\delta_0} \left[\left(\partial_z(\phi_{\varepsilon,\alpha}^A|_{\overline{X}^\pm(.,t)})J_t^\pm + \phi_{\varepsilon,\alpha}^A|_{\overline{X}^\pm(.,t)}\partial_z J_t^\pm\right)\tilde{\psi}_t^\pm\right]|_{(r,z)}\, dr\right|.$$

Now we use the structure of $\phi_{\varepsilon,\alpha}^A$. In (6.71) we computed $\nabla\phi_{\varepsilon,\alpha}^A$ in $\Omega_t^{C\pm}$. Moreover, it holds

$$\begin{aligned}
\sqrt{\varepsilon}\partial_z(\phi_{\varepsilon,\alpha}^A|_{\overline{X}^\pm}) &= -\partial_z\tilde{h}_\varepsilon^\pm\left[q^\pm|_{(z_\alpha^\pm,t)}\partial_\rho^2 v_\alpha|_{(\rho_{\varepsilon,\alpha},Z_{\varepsilon,\alpha}^\pm)} + \varepsilon\partial_\rho\hat{v}_1^{C\pm}|_{(\rho_{\varepsilon,\alpha},Z_{\varepsilon,\alpha}^\pm,t)}\right]|_{\overline{X}^\pm} \qquad (6.79)\\
&+ \left[\partial_z q^\pm|_{(z_\alpha^\pm,t)}\partial_\rho v_\alpha|_{(\rho_{\varepsilon,\alpha},Z_{\varepsilon,\alpha}^\pm)} + \frac{1}{\varepsilon}q^\pm|_{(z_\alpha^\pm,t)}\partial_Z\partial_\rho v_\alpha|_{(\rho_{\varepsilon,\alpha},Z_{\varepsilon,\alpha}^\pm)} + \partial_Z\hat{v}_1^{C\pm}|_{(\rho_{\varepsilon,\alpha},Z_{\varepsilon,\alpha}^\pm,t)}\right]|_{\overline{X}^\pm}
\end{aligned}$$

for all $(r,z) \in [-\delta_0,\delta_0]\times[0,\tilde{\mu}_0]$. Consider the estimate for $|g_t^\pm(z)|$ after inserting $\phi_{\varepsilon,\alpha}^A$, $\nabla\phi_{\varepsilon,\alpha}^A$ and $\partial_z(\phi_{\varepsilon,\alpha}^A|_{\overline{X}^\pm})$. Then there are four new critical terms compared to the case $\alpha = \frac{\pi}{2}$, cf. the estimate of (III) in the proof of Lemma 6.23. First, from $\partial_z(\phi_{\varepsilon,\alpha}^A|_{\overline{X}^\pm(.,t)})$ there is the contribution

$$\frac{1}{\varepsilon^{3/2}}|q^\pm|_{(z,t)}|\left|\int_{-\delta_0}^{\delta_0} \partial_Z\partial_\rho v_\alpha|_{(\rho_{\varepsilon,\alpha}(\overline{X}^\pm(.,t)),\frac{z}{\varepsilon})}[\tilde{\psi}_t^\pm J_t^\pm](r,z)\, dr\right|.$$

Moreover, due to $|\nabla z_\alpha^\pm|^2 = 1 + \mathcal{O}(|r|)$ and $\nabla r\cdot\nabla z_\alpha^\pm = -\cos\alpha + \mathcal{O}(|r|)$ we get from the $\nabla\phi_{\varepsilon,\alpha}^A$-term the two remainders

$$\frac{1}{\varepsilon^{3/2}}|q^\pm(z,t)|\int_{-\delta_0}^{\delta_0} \left[|\partial_Z\partial_\rho v_\alpha|_{(\rho_{\varepsilon,\alpha}(\overline{X}^\pm(.,t)),\frac{z}{\varepsilon})}| + |\cos\alpha\,\partial_\rho v_\alpha|_{(\rho_{\varepsilon,\alpha}(\overline{X}^\pm(.,t)),\frac{z}{\varepsilon})}|\right]|\tilde{\psi}_t^\pm J_t^\pm|(r,z)\, dr.$$

Finally, the $\nabla r\cdot\nabla z_\alpha^\pm$ multiplied by $\partial_r\tilde{\psi}_t^\pm$ yields the term

$$\frac{1}{\sqrt{\varepsilon}}|q^\pm(z,t)|\int_{-\delta_0}^{\delta_0} |\cos\alpha\partial_\rho v_\alpha|_{(\rho_{\varepsilon,\alpha}(\overline{X}^\pm(.,t)),\frac{z}{\varepsilon})}||\partial_r\tilde{\psi}_t^\pm J_t^\pm|(r,z)\, dr.$$

For the $\partial_Z\partial_\rho v_\alpha$-terms we use

$$|\partial_Z\partial_\rho v_\alpha(\rho,Z)| \leq \overline{C}|\alpha-\frac{\pi}{2}|e^{-\beta_0|\rho|} \quad \text{for all } (\rho,Z)\in\overline{\mathbb{R}^2_+}$$

because of Remark 5.33, where $\overline{C},\beta_0>0$ are independent of Ω,Γ,α. Moreover, we split the last integral with $\partial_r\tilde{\psi}_t^\pm$ as in the proof of Lemma 6.59, cf. (6.75), and we use (6.74). Note that with $\hat{\alpha}_0$ and $\tilde{\delta}$ as in the proof of Lemma 6.59, the prefactor in (6.74) is contained in $[\frac{1}{2},1]$ provided that $\alpha\in\frac{\pi}{2}+[-\hat{\alpha}_0,\hat{\alpha_0}]$ and $|r|\leq 2\tilde{\delta}$. Therefore the Hölder Inequality, Lemma 6.5, Remark 4.19, $\frac{1}{2}\leq q^\pm\leq 2$ and $J_t^\pm=1+\mathcal{O}(|r|)$ yield

$$|g_t^\pm(z)|\leq(\frac{1}{\varepsilon}\overline{C}_5|\alpha-\frac{\pi}{2}|+C)\left[\|\tilde{\psi}_t^\pm(.,z)\|_{L^2(-\delta_0,\delta_0;J_t^\pm(.,z))}+\varepsilon\|\nabla\psi|_{X^\pm(.,z)}\|_{L^2(-\delta_0,\delta_0;J_t^\pm(.,z))}\right]$$

for a.e. $z\in[0,\tilde{\mu}_0]$ and some $\overline{C}_5>0$ independent of Ω,Γ and $\alpha\in\frac{\pi}{2}+[-\hat{\alpha}_0,\hat{\alpha}_0]$. Therefore the Hölder Inequality and Young Inequality yield

$$|(III)|\leq(\overline{C}_6|\alpha-\frac{\pi}{2}|+C\varepsilon^2)\|a'\|^2_{L^2(0,\tilde{\mu}_0)}+\frac{\nu}{8}\left[\frac{1}{\varepsilon^2}\|\psi\|^2_{L^2(\Omega_t^{C\pm})}+\|\nabla\psi\|^2_{L^2(\Omega_t^{C\pm})}\right]$$

for some $\overline{C}_6>0$ independent of Ω,Γ, where ν is as in Lemma 6.59. The last term is dominated by $\frac{1}{8}B^\pm_{\varepsilon,t}(\psi,\psi)$ due to Lemma 6.59. Finally, we can choose $\overline{\alpha}_0>0$ small independent of Ω,Γ such that $\overline{C}_6\overline{\alpha}_0\leq\frac{\overline{c}}{4}$, where $\overline{c}$ is as in Lemma 6.57. This shows the claim. □

Finally, we combine Lemma 6.57-6.60.

Theorem 6.61. *There is an $\overline{\alpha}_0>0$ independent of Ω,Γ such that, if $\alpha\in\frac{\pi}{2}+[-\overline{\alpha}_0,\overline{\alpha}_0]$, then there are $\varepsilon_0,C,c_0>0$ such that for all $\varepsilon\in(0,\varepsilon_0]$, $t\in[0,T]$ and $\psi\in H^1(\Omega_t^{C\pm})$ with $\psi(x)=0$ for a.e. $x\in\Omega_t^{C\pm}$ with $z_\alpha^\pm(x,t)\geq\hat{\mu}_0$ it holds*

$$B^\pm_{\varepsilon,t}(\psi,\psi)\geq -C\|\psi\|^2_{L^2(\Omega_t^{C\pm})}+c_0\varepsilon\|\nabla_\tau\psi\|^2_{L^2(\Omega_t^{C\pm})}.$$

Remark 6.62.

1. The estimate can be refined, cf. the proof below.
2. Theorem 6.61 directly implies Theorem 6.53, cf. the beginning of Section 6.5.1.

Proof of Theorem 6.61. Let $t\in[0,T]$ and $\psi\in\hat{H}^1(\Omega_t^{C\pm})$. Due to Lemma 6.56 we can uniquely write

$$\psi=\phi+\phi^\perp \quad \text{with } \phi=[a(z_\alpha^\pm)\phi^A_{\varepsilon,\alpha}]|_{(.,t)}\in\hat{V}^\pm_{\varepsilon,t} \text{ and } \phi^\perp\in(\hat{V}^\pm_{\varepsilon,t})^\perp.$$

Analogously to the case $\alpha=\frac{\pi}{2}$, cf. the proof of Theorem 6.25 we obtain from Lemma 6.57, Lemma 6.60 and Lemma 6.59 that there are $C,\varepsilon_0>0$ independent of ψ,ε,t such that for all $\varepsilon\in(0,\varepsilon_0]$ it holds

$$B^\pm_{\varepsilon,t}(\psi,\psi)\geq -C\|\phi\|^2_{L^2(\Omega_t^{C\pm})}+\frac{\nu}{4\varepsilon^2}\|\phi^\perp\|^2_{L^2(\Omega_t^{C\pm})}+\frac{\overline{c}}{4}\|a\|^2_{H^1(0,\tilde{\mu}_0)}+\frac{\nu}{2}\|\nabla(\phi^\perp)\|^2_{L^2(\Omega_t^{C\pm})}.$$

It remains to include the $\nabla_\tau\psi$-term in the estimate. By the triangle inequality we have

$$\|\nabla_\tau\psi\|_{L^2(\Omega_t^{C\pm})}\leq\|\nabla_\tau\phi\|_{L^2(\Omega_t^{C\pm})}+\|\nabla_\tau(\phi^\perp)\|_{L^2(\Omega_t^{C\pm})}.$$

Theorem 3.3 yields $\|\nabla_\tau(\phi^\perp)\|_{L^2(\Omega_t^{C\pm})} \leq C\|\nabla(\phi^\perp)\|_{L^2(\Omega_t^{C\pm})}$. Moreover, by definition

$$\nabla_\tau\psi|_{X(r,s,t)} = \nabla s|_{\overline{X}(r,s,t)}\partial_s(\phi|_{X(r,s,t)}) = \nabla s|_{\overline{X}(r,s,t)}\partial_s(a(z_\alpha^\pm|_{X(r,s,t)})\phi^A_{\varepsilon,\alpha}|_{\overline{X}(r,s,t)}).$$

Note that $\partial_s(z_\alpha^\pm|_{\overline{X}(r,s,t)}) = \mp\sin\alpha$ due to the definition (5.79) of $z_\alpha^\pm$ and therefore

$$\begin{aligned}\sqrt{\varepsilon}\partial_s(\phi^A_{\varepsilon,\alpha}|_{\overline{X}}) &= (\mp\sin\alpha)\partial_z q^\pm(z_\alpha^\pm|_{\overline{X}},t)\partial_\rho v_\alpha(\rho_{\varepsilon,\alpha},Z^\pm_{\varepsilon,\alpha})|_{\overline{X}}\\ &\quad + q^\pm(z_\alpha^\pm|_{\overline{X}},t)\left[-\partial_s h_{\varepsilon,\alpha}\partial_\rho^2 v_\alpha(\rho_{\varepsilon,\alpha},Z^\pm_{\varepsilon,\alpha})|_{\overline{X}} + \frac{\mp\sin\alpha}{\varepsilon}\partial_Z\partial_\rho v_\alpha(\rho_{\varepsilon,\alpha},Z^\pm_{\varepsilon,\alpha})|_{\overline{X}}\right]\\ &\quad + \varepsilon\left[-\partial_s h_{\varepsilon,\alpha}\partial_\rho\hat{v}_1^{C\pm}(\rho_{\varepsilon,\alpha},Z^\pm_{\varepsilon,\alpha},t)|_{\overline{X}} + \frac{\mp\sin\alpha}{\varepsilon}\partial_Z\hat{v}_1^{C\pm}(\rho_{\varepsilon,\alpha},Z^\pm_{\varepsilon,\alpha},t)|_{\overline{X}}\right],\end{aligned}$$

where the $h_{\varepsilon,\alpha}$-terms are evaluated at (s,t). We estimate all appearing terms in $\|\nabla_\tau\phi\|^2_{L^2(\Omega_t^{C\pm})}$ using several times $(d+\tilde{d})^2 \leq 2(d^2+\tilde{d}^2)$ for $d,\tilde{d}\geq 0$. All terms are multiplied by a $\frac{1}{\varepsilon}$-factor (or better) except the $\frac{1}{\varepsilon^3}|\partial_Z\partial_\rho v_\alpha|^2$-term. Let us first estimate all terms except the latter one. We transform to (r,z)-coordinates, use the Fubini Theorem and Lemma 6.5. Then these terms are controlled by $C\|a\|^2_{H^1(0,\tilde{\mu}_0)}$. Now we consider the $\partial_Z\partial_\rho v_\alpha$-term. The latter is estimated by

$$C\frac{1}{\varepsilon^3}\int_0^{\tilde{\mu}_0} a^2(z)\int_{-\delta_0}^{\delta_0}|\partial_Z\partial_\rho v_\alpha(\rho_{\varepsilon,\alpha}|_{\overline{X}^\pm(r,z,t)},\frac{z}{\varepsilon})|^2\,dr\,dz. \tag{6.80}$$

We use $|a(z)|\leq C\|a\|_{H^1(0,\tilde{\mu}_0)}$ for all $z\in[0,\tilde{\mu}_0]$ due to the Fundamental Theorem and

$$|\partial_Z\partial_\rho v_\alpha(\rho,Z)| \leq Ce^{-\beta_0|\rho|-\gamma_0 Z} \quad \text{for all } (\rho,Z)\in\overline{\mathbb{R}^2_+}$$

because of Remark 5.33. Therefore Lemma 6.5 for the inner integral and another scaling argument for the z-integral yields that (6.80) is estimated by $C\frac{1}{\varepsilon}\|a\|^2_{H^1(0,\tilde{\mu}_0)}$. Finally, together with the above estimate for $B^\pm_{\varepsilon,t}$ this yields the claim. □

7 Difference Estimates and Proofs of the Convergence Theorems

In this section we estimate the difference of the exact and approximate solutions in all our cases. This is the second step in the method by de Mottoni and Schatzman [deMS]. In general, the procedure in the application of the method always consists of variants of Gronwall-type arguments. One considers the difference of the diffuse interface equation for the exact and the approximate solution, multiplies with suitable functions and integrates in space and time. The resulting terms have to be estimated in a suitable way. Here the major ingredient always is the spectral estimate. The approximate solution has to be designed in such a way that it allows for such an estimate and such that it fulfils the diffuse interface equation up to some sufficiently small remainder terms. In our cases this is provided by the construction of the approximate solution in Section 5 and the spectral estimates in Section 6. Apart from that, typically one has to control certain nonlinear terms stemming from differences of potential terms. First, one usually estimates the latter with the Taylor Theorem and then applies suitable interpolation inequalities or Gagliardo-Nirenberg estimates. For Allen-Cahn type models one can typically use uniform boundedness in ε for the exact solution, see de Mottoni, Schatzman [deMS], Section 6 for the standard Allen-Cahn equation as well as Abels, Liu [AL], Remark 1.2 and [AL], Section 5.2 for the Allen-Cahn equation coupled with the Stokes system.

We will use the same idea in all our cases. Therefore as preparation we prove a uniform a priori bound for exact classical solutions of (AC) and (AC$_\alpha$) in Section 7.1.1 and for exact classical solutions of (vAC) in Section 7.1.2. Moreover, we recall some Gagliardo-Nirenberg estimates in Section 7.1.3. Then we consider the case of the Allen-Cahn equation (AC) in ND, $N \geq 2$, in the situation of boundary contact in Section 7.2. In Section 7.3 we look at the vector-valued Allen-Cahn equation (vAC) with boundary contact in ND. Finally in Section 7.4 the case of the Allen-Cahn equation with non-linear Robin boundary condition (AC$_\alpha$) in the situation of boundary contact in 2D is done.

7.1 Preliminaries

7.1.1 Uniform A Priori Bound for Classical Solutions of (AC) and (AC$_\alpha$)

Let N, Ω, Q_T, ∂Q_T be as in Remark 1.1, 1. and $\varepsilon > 0$. We prove uniform boundedness estimates for classical solutions of the scalar equations (AC) and (AC$_\alpha$). Since (AC$_\alpha$) is the same as (AC) if $\sigma_\alpha' = 0$, it is enough to consider (AC$_\alpha$). Let f be as in (1.1) and $R_0 \geq 1$ such that the condition (1.2) for f' holds. Moreover, let $\alpha \in (0, \pi)$ and $\sigma_\alpha : \mathbb{R} \to \mathbb{R}$ be smooth with $\operatorname{supp} \sigma_\alpha' \subset (-1, 1)$.

Lemma 7.1. *Let $u_{0,\varepsilon,\alpha} \in C^0(\overline{\Omega})$ and $u_{\varepsilon,\alpha} \in C^0(\overline{Q_T}) \cap C^1(\overline{\Omega} \times (0,T]) \cap C^2(\Omega \times (0,T])$ be a solution of* (AC$_\alpha$1)-(AC$_\alpha$3)*. Then*

$$\|u_{\varepsilon,\alpha}\|_{L^\infty(Q_T)} \leq \max\{R_0, \|u_{0,\varepsilon,\alpha}\|_{L^\infty(\Omega)}\}.$$

Proof. We use a contradiction argument and ideas from the proof of the weak maximum principle for parabolic equations, cf. Renardy, Rogers [RR], Theorem 4.25. Variants of the proof may also work. We have chosen a proof that can be directly generalized to the vector-valued case, see below. Assume $\|u_{\varepsilon,\alpha}\|_{L^\infty(Q_T)} > \max\{R_0, \|u_{0,\varepsilon,\alpha}\|_{L^\infty(\Omega)}\}$. We consider $u_{\varepsilon,\alpha,\beta} := |u_{\varepsilon,\alpha}|^2 + \beta e^{-t}$ and

$u_{0,\varepsilon,\alpha,\beta} := |u_{0,\varepsilon,\alpha}|^2 + \beta$ for $\beta > 0$. Then for $\beta > 0$ small

$$\|u_{\varepsilon,\alpha,\beta}\|_{L^\infty(Q_T)} > \beta + \max\{R_0^2, \|u_{0,\varepsilon,\alpha,\beta}\|_{L^\infty(\Omega)}\}. \tag{7.1}$$

Because of $u_{\varepsilon,\alpha,\beta} \in C^0(\overline{Q_T})$, it follows that the maximum of $u_{\varepsilon,\alpha,\beta} = |u_{\varepsilon,\alpha,\beta}|$ is attained in some $(x_0,t_0) \in \overline{Q_T}$. Due to (7.1) it holds

$$|u_{\varepsilon,\alpha}|^2|_{(x_0,t_0)} = u_{\varepsilon,\alpha,\beta}|_{(x_0,t_0)} - \beta e^{-t_0} > \beta + R_0^2 - \beta = R_0^2. \tag{7.2}$$

Hence $f'(u_{\varepsilon,\alpha}|_{(x_0,t_0)})u_{\varepsilon,\alpha}|_{(x_0,t_0)} \geq 0$ due to (1.2). If $(x_0,t_0) \in \Omega \times (0,T]$, we get from (AC$_\alpha$1):

$$\begin{aligned}(\partial_t - \Delta)u_{\varepsilon,\alpha,\beta}|_{(x_0,t_0)} &= -\beta e^{-t_0} + 2u_{\varepsilon,\alpha}(\partial_t u_{\varepsilon,\alpha} - \Delta u_{\varepsilon,\alpha})|_{(x_0,t_0)} - 2|\nabla u_{\varepsilon,\alpha}|^2|_{(x_0,t_0)} \\ &= -\beta e^{-t_0} - 2\left[u_{\varepsilon,\alpha}|_{(x_0,t_0)}\frac{1}{\varepsilon^2}f'(u_{\varepsilon,\alpha}|_{(x_0,t_0)}) + |\nabla u_{\varepsilon,\alpha}|^2|_{(x_0,t_0)}\right] \\ &\leq -\beta e^{-t_0} < 0.\end{aligned}$$

If $(x_0,t_0) \in \overline{\Omega} \times \{0\}$, we get a contradiction from (7.1) and since $u_{\varepsilon,\alpha,\beta}|_{t=0} = u_{0,\varepsilon,\alpha,\beta}$ due to (AC$_\alpha$3). In the case $(x_0,t_0) \in \Omega \times (0,T]$ we obtain a contradiction to $(\partial_t - \Delta)u_{\varepsilon,\alpha,\beta}|_{(x_0,t_0)} \geq 0$ as in the proof of the weak maximum principle, cf. Renardy, Rogers [RR], Theorem 4.25.

Finally, let $(x_0,t_0) \in \partial\Omega \times (0,T]$. With the Hopf Lemma (cf. Gilbarg, Trudinger [GD], Lemma 3.4) we deduce a contradiction to the boundary condition (AC$_\alpha$2). The above consideration yields $u_{\varepsilon,\alpha,\beta}|_{(x_0,t_0)} > u_{\varepsilon,\alpha,\beta}|_{(x,t)}$ for all $(x,t) \in \Omega \times (0,T]$. Additionally, because of continuity and (7.2) it holds $|u_{\varepsilon,\alpha}(x,t)| > R_0$ for all $(x,t) \in B_\eta(x_0,t_0) \cap (\Omega \times (0,T])$ and $\eta > 0$ small. Hence as above $(\partial_t - \Delta)u_{\varepsilon,\alpha,\beta}|_{(x,t)} \leq -\beta e^{-t} < 0$ for these (x,t). Moreover, since (x_0,t_0) is a maximum of $u_{\varepsilon,\alpha,\beta}$, it follows that $\partial_t u_{\varepsilon,\alpha,\beta}|_{(x_0,t_0)} \geq 0$. Therefore continuity of $\partial_t u_{\varepsilon,\alpha,\beta}$ yields $\Delta u_{\varepsilon,\alpha,\beta}|_{(x,t)} < 0$ for all $(x,t) \in B_\eta(x_0,t_0) \cap (\Omega \times (0,T])$ and $\eta > 0$ small. Hence the Hopf Lemma is applicable on $B_\eta(x_0) \cap \Omega$ and yields $N_{\partial\Omega} \cdot \nabla u_{\varepsilon,\alpha,\beta}|_{(x_0,t_0)} > 0$. This gives a contradiction to (AC$_\alpha$2) because of $\nabla u_{\varepsilon,\alpha,\beta} = 2u_{\varepsilon,\alpha}\nabla u_{\varepsilon,\alpha}$ and $\sigma_\alpha'(u_{\varepsilon,\alpha}|_{(x_0,t_0)}) = 0$ due to $\operatorname{supp}\sigma_\alpha' \subset (-1,1)$ and (7.2). Finally, we have considered all possible cases and obtained a contradiction. Hence the lemma is proven. □

7.1.2 Uniform A Priori Bound for Classical Solutions of (vAC)

Let $N, \Omega, Q_T, \partial Q_T$ be as in Remark 1.1, 1. and $\varepsilon > 0$. Moreover, let $m \in \mathbb{N}$ and $W : \mathbb{R}^m \to \mathbb{R}$ be as in Definition 1.4. We prove uniform boundedness estimates for classical solutions of (vAC). This works analogously to the proof of Lemma 7.1 for the scalar case in the last section.

Lemma 7.2. *Let $\vec{u}_{0,\varepsilon} \in C^0(\overline{\Omega})$ and $\vec{u}_\varepsilon \in C^0(\overline{Q_T})^m \cap C^1(\overline{\Omega} \times (0,T])^m \cap C^2(\Omega \times (0,T])^m$ be a solution of* (vAC1)-(vAC3). *Then with $\check{R}_0 > 0$ as in Definition 1.4 it holds*

$$\|\vec{u}_\varepsilon\|_{L^\infty(Q_T,\mathbb{R}^m)} \leq \max\{\check{R}_0, \|\vec{u}_{0,\varepsilon}\|_{L^\infty(\Omega,\mathbb{R}^m)}\}.$$

Proof. Assume $\|\vec{u}_\varepsilon\|_{L^\infty(Q_T,\mathbb{R}^m)} > \max\{\check{R}_0, \|\vec{u}_{0,\varepsilon}\|_{L^\infty(\Omega,\mathbb{R}^m)}\}$. Let $\check{u}_{\varepsilon,\beta} := |\vec{u}_\varepsilon|^2 + \beta e^{-t}$ and $\check{u}_{0,\varepsilon,\beta} := |\vec{u}_{0,\varepsilon}|^2 + \beta$ for $\beta > 0$. Then for $\beta > 0$ small

$$\|\check{u}_{\varepsilon,\beta}\|_{L^\infty(Q_T)} > \beta + \max\{\check{R}_0^2, \|\check{u}_{0,\varepsilon,\beta}\|_{L^\infty(\Omega)}\}. \tag{7.3}$$

The maximum of $\check{u}_{\varepsilon,\beta} = |\check{u}_{\varepsilon,\beta}|$ is attained in some $(x_0, t_0) \in \overline{Q_T}$ due to $\vec{u}_\varepsilon \in C^0(\overline{Q_T})^m$. Then inequality (7.3) yields

$$|\vec{u}_\varepsilon|^2|_{(x_0,t_0)} = \check{u}_{\varepsilon,\beta}|_{(x_0,t_0)} - \beta e^{-t_0} > \beta + \check{R}_0^2 - \beta = \check{R}_0^2. \tag{7.4}$$

By the assumption on W in Definition 1.4 it follows that $\vec{u}_\varepsilon|_{(x_0,t_0)} \cdot \nabla W(\vec{u}_\varepsilon|_{(x_0,t_0)}) \geq 0$. Hence equation (vAC1) yields in the case $(x_0, t_0) \in \Omega \times (0, T]$ that

$$\begin{aligned}(\partial_t - \Delta)\check{u}_{\varepsilon,\beta}|_{(x_0,t_0)} &= -\beta e^{-t_0} + 2\vec{u}_\varepsilon \cdot (\partial_t \vec{u}_\varepsilon - \Delta \vec{u}_\varepsilon)|_{(x_0,t_0)} - 2|\nabla \vec{u}_\varepsilon|^2|_{(x_0,t_0)} \\ &= -\beta e^{-t_0} - 2\left[\vec{u}_\varepsilon|_{(x_0,t_0)} \cdot \frac{1}{\varepsilon^2}\nabla W(\vec{u}_\varepsilon|_{(x_0,t_0)}) + |\nabla \vec{u}_\varepsilon|^2|_{(x_0,t_0)}\right] \\ &\leq -\beta e^{-t_0} < 0.\end{aligned}$$

In the case $(x_0, t_0) \in \overline{\Omega} \times \{0\}$ the contradiction follows from (7.3) and $\check{u}_{\varepsilon,\beta}|_{t=0} = \check{u}_{0,\varepsilon,\beta}$ because of (vAC3). In the case $(x_0, t_0) \in \Omega \times (0, T]$ we obtain a contradiction to $(\partial_t - \Delta)\check{u}_{\varepsilon,\beta}|_{(x_0,t_0)} \geq 0$ as in the proof of the weak maximum principle, cf. Renardy, Rogers [RR], Theorem 4.25.

Finally, let $(x_0, t_0) \in \partial\Omega \times (0, T]$. The above consideration yields $\check{u}_{\varepsilon,\beta}|_{(x_0,t_0)} > \check{u}_{\varepsilon,\beta}|_{(x,t)}$ for all $(x, t) \in \Omega \times (0, T]$. Moreover, due to continuity and (7.4) we obtain $|\vec{u}_\varepsilon(x, t)| > \check{R}_0$ for all $(x, t) \in B_\eta(x_0, t_0) \cap (\Omega \times (0, T])$ and $\eta > 0$ small. Therefore as above it follows that $(\partial_t - \Delta)\check{u}_{\varepsilon,\beta}|_{(x,t)} \leq -\beta e^{-t} < 0$ for these (x, t). Furthermore, it holds $\partial_t \check{u}_{\varepsilon,\beta}|_{(x_0,t_0)} \geq 0$ because (x_0, t_0) is a maximum of $\check{u}_{\varepsilon,\beta}$. By continuity of $\partial_t \check{u}_{\varepsilon,\beta}$ we obtain $\Delta \check{u}_{\varepsilon,\beta}|_{(x,t)} < 0$ for all $(x, t) \in B_\eta(x_0, t_0) \cap (\Omega \times (0, T])$ and $\eta > 0$ small. Therefore the Hopf Lemma (cf. Gilbarg, Trudinger [GD], Lemma 3.4) can be applied on $B_\eta(x_0) \cap \Omega$ and yields $N_{\partial\Omega} \cdot \nabla \check{u}_{\varepsilon,\beta}|_{(x_0,t_0)} > 0$. Here $\nabla \check{u}_{\varepsilon,\beta} = \sum_{j=1}^m \nabla(|u_{\varepsilon,j}|^2) = 2\sum_{j=1}^m u_{\varepsilon,j} \nabla u_{\varepsilon,j}$. Therefore we obtain a contradiction to the boundary condition (vAC2). Finally, this yields the lemma. □

7.1.3 Gagliardo-Nirenberg Inequalities

Let us recall some Gagliardo-Nirenberg inequalities.

Lemma 7.3 (Gagliardo-Nirenberg Inequality). *Let $n \in \mathbb{N}$, $1 \leq p, q, r \leq \infty$ and $\theta \in [0, 1]$ such that*

$$\theta\left(\frac{1}{p} - \frac{1}{n}\right) + \frac{1-\theta}{q} = \frac{1}{r},$$

where $\frac{1}{\infty} := 0$. Moreover, if $p = n > 1$, then we assume $\theta < 1$. Then

$$\|u\|_{L^r(\mathbb{R}^n)} \leq c\|u\|_{L^q(\mathbb{R}^n)}^{1-\theta} \|\nabla u\|_{L^p(\mathbb{R}^n)}^\theta$$

for all $u \in L^q(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n)$ and a constant $c = c(n, p, q, r) > 0$.

Proof. See Leoni [Le], Theorem 12.83. □

Remark 7.4. With suitable extension operators the estimate in Lemma 7.3 carries over to domains with uniform Lipschitz boundary if $\|\nabla u\|_{L^p(\mathbb{R}^n)}$ in the estimate is replaced by $\|u\|_{W^{1,p}(\mathbb{R}^n)}$ and the constant in the estimate depends on n, p, q, r and the operator norm of the extension operator. For the existence of such extension operators (going back to Stein) see Leoni [Le], Theorem 13.8 and Theorem 13.17. Note that the operator norms in [Le] are estimated solely in terms of the usual parameters and the geometrical quantities of Ω and $\partial\Omega$. In particular if the geometrical quantities can be controlled in a uniform way, the operator norms and the constants in the above Gagliardo-Nirenberg inequalities can be taken uniformly with respect to Ω.

7.2 Difference Estimate and Proof of the Convergence Theorem for (AC) in ND

We prove in Section 7.2.1 a rather abstract estimate for the difference of exact solutions and suitable approximate solutions for the Allen-Cahn equation (AC1)-(AC3) in ND. Then in Section 7.2.2 we show the Theorem 1.2 about convergence by verifying the requirements for the difference estimate applied to the approximate solution from Section 5.2.3. For $N = 2$ also the approximate solution from Section 5.1.3 works.

7.2.1 Difference Estimate

Theorem 7.5 (**Difference Estimate for (AC)**). *Let $N \geq 2$, Ω, Q_T and ∂Q_T be as in Remark 1.1, 1. Moreover, let $\Gamma = (\Gamma_t)_{t\in[0,T_0]}$ for some $T_0 > 0$ be as in Section 3.3 and $\delta > 0$ be such that Theorem 3.7 holds for 2δ instead of δ. We use the notation for $\Gamma_t(\delta)$, $\Gamma(\delta)$, ∇_τ and ∂_n from Remark 3.8. Additionally, let f satisfy* (1.1)-(1.2).

Moreover, let $\varepsilon_0 > 0$, $u_\varepsilon^A \in C^2(\overline{Q_{T_0}})$, $u_{0,\varepsilon} \in C^2(\overline{\Omega})$ with $\partial_{N_{\partial\Omega}} u_{0,\varepsilon} = 0$ on $\partial\Omega$ and let $u_\varepsilon \in C^2(\overline{Q_{T_0}})$ be exact solutions to (AC1)-(AC3) *with $u_{0,\varepsilon}$ in* (AC3) *for $\varepsilon \in (0, \varepsilon_0]$.*

For some $R > 0$ and $M \in \mathbb{N}, M \geq k(N) := \max\{2, \frac{N}{2}\}$ we impose the following conditions:

1. Uniform Boundedness: $\sup_{\varepsilon\in(0,\varepsilon_0]} \|u_\varepsilon^A\|_{L^\infty(Q_{T_0})} + \|u_{0,\varepsilon}\|_{L^\infty(\Omega)} < \infty$.
2. Spectral Estimate: *There are $c_0, C > 0$ such that*
$$\int_\Omega |\nabla\psi|^2 + \frac{1}{\varepsilon^2} f''(u_\varepsilon^A(.,t))\psi^2\, dx \geq -C\|\psi\|^2_{L^2(\Omega)} + \|\nabla\psi\|^2_{L^2(\Omega\setminus\Gamma_t(\delta))} + c_0\|\nabla_\tau\psi\|^2_{L^2(\Gamma_t(\delta))}$$
for all $\psi \in H^1(\Omega)$ and $\varepsilon \in (0, \varepsilon_0], t \in [0, T_0]$.
3. Approximate Solution: *For the remainders*
$$r_\varepsilon^A := \partial_t u_\varepsilon^A - \Delta u_\varepsilon^A + \frac{1}{\varepsilon^2} f'(u_\varepsilon^A) \quad \text{and} \quad s_\varepsilon^A := \partial_{N_{\partial\Omega}} u_\varepsilon^A$$
in (AC1)-(AC2) *for u_ε^A and the difference $\overline{u}_\varepsilon := u_\varepsilon - u_\varepsilon^A$ it holds*
$$\begin{aligned}&\left|\int_{\partial\Omega} s_\varepsilon^A \mathrm{tr}\, \overline{u}_\varepsilon(t)\, d\mathcal{H}^{N-1} + \int_\Omega r_\varepsilon^A \overline{u}_\varepsilon(t)\, dx\right| \\ &\leq C\varepsilon^{M+\frac{1}{2}}(\|\overline{u}_\varepsilon(t)\|_{L^2(\Omega)} + \|\nabla_\tau \overline{u}_\varepsilon(t)\|_{L^2(\Gamma_t(\delta))} + \|\nabla\overline{u}_\varepsilon(t)\|_{L^2(\Omega\setminus\Gamma_t(\delta))})\end{aligned} \tag{7.5}$$
for all $\varepsilon \in (0, \varepsilon_0]$ and $T \in (0, T_0]$.
4. Well-Prepared Initial Data: *For all $\varepsilon \in (0, \varepsilon_0]$ it holds*
$$\|u_{0,\varepsilon} - u_\varepsilon^A|_{t=0}\|_{L^2(\Omega)} \leq R\varepsilon^{M+\frac{1}{2}}. \tag{7.6}$$

Then we obtain

1. *Let $M > k(N)$. Then there are $\beta, \varepsilon_1 > 0$ such that for $g_\beta(t) := e^{-\beta t}$ it holds*
$$\begin{aligned}\sup_{t\in[0,T]} \|g_\beta \overline{u}_\varepsilon(t)\|^2_{L^2(\Omega)} + \|g_\beta \nabla\overline{u}_\varepsilon\|^2_{L^2(Q_T\setminus\Gamma(\delta))} &\leq 2R^2\varepsilon^{2M+1}, \\ c_0\|g_\beta\nabla_\tau\overline{u}_\varepsilon\|^2_{L^2(Q_T\cap\Gamma(\delta))} + \varepsilon^2\|g_\beta\partial_n\overline{u}_\varepsilon\|^2_{L^2(Q_T\cap\Gamma(\delta))} &\leq 2R^2\varepsilon^{2M+1}\end{aligned} \tag{7.7}$$
for all $\varepsilon \in (0, \varepsilon_1]$ and $T \in (0, T_0]$.

2. *Let $k(N) \in \mathbb{N}$ and $M = k(N)$. Let* (7.5) *hold for some $\tilde{M} > M$ instead of M. Then there are $\beta, \tilde{R}, \varepsilon_1 > 0$ such that, if* (7.6) *holds for $\tilde{R}$ instead of R, then* (7.7) *for $\tilde{R}$ instead of R is valid for all $\varepsilon \in (0, \varepsilon_1], T \in (0, T_0]$.*

3. *Let $N \in \{2,3\}$ and $M = 2(= k(N))$. Then there are $\varepsilon_1, T_1 > 0$ such that* (7.7) *holds for $\beta = 0$ and for all $\varepsilon \in (0, \varepsilon_1], T \in (0, T_1]$.*

Remark 7.6. 1. The parameter M corresponds to the order of the approximate solution in Section 5.2 and, for $N = 2$, Section 5.1.

2. The parameter β was introduced in order to obtain a result valid for all times $T \in (0, T_0]$.

3. Note that weaker requirements in the theorem also work, e.g. when one does not have the two additional terms on the right hand side of the spectral estimate or only an estimate with the full H^1-norm on the right hand side in (7.5). Moreover, a slightly less involved proof is also possible, see e.g. Remark 7.8 below. However, then the result is also weaker, in particular the somewhat critical order $k(N)$ for M could be increased and the ε-orders in (7.7) could be weakened. This is because the H^1-norm can be controlled with the spectral term but one has to pay ε^{-2} times the L^2-norm. Nevertheless, we intended to give an optimal result, also having in mind e.g. couplings with other equations like the Stokes system as in Abels, Liu [AL], where a low number of terms in the ansatz for the approximate solution is convenient.

4. That the parameter $k(N)$ is critical for M in our proof can be seen at (7.13) in the proof below. This is due to an estimate of a cubic term, see (7.12) and Lemma 7.7 below. The results 2.-3. are the best we could prove for the critical case $M = k(N) \in \mathbb{N}$. This situation is difficult because in the estimates there will be a term of order larger than 2 in R and a linear term in R, but the desired order is 2 in R. The linear term in R will enter due to (7.5), see (7.13) below. For the parameter β there is a similar problem in the critical case.

Proof of Theorem 7.5. The continuity of the objects on the left hand side in (7.7) yields that

$$T_{\varepsilon,\beta,R} := \sup\{\tilde{T} \in (0, T_0] : \text{(7.7) holds for } \varepsilon, R \text{ and all } T \in (0, \tilde{T}]\} \tag{7.8}$$

is well-defined for all $\varepsilon \in (0, \varepsilon_0], \beta \geq 0$ and $T_{\varepsilon,\beta,R} > 0$. In the different cases we have to show:

1. If $M > k(N)$, then there are $\beta, \varepsilon_1 > 0$ such that $T_{\varepsilon,\beta,R} = T_0$ for all $\varepsilon \in (0, \varepsilon_1]$.

2. If $k(N) \in \mathbb{N}$ and $M = k(N)$, then there are $\beta, \tilde{R}, \varepsilon_1 > 0$ such that $T_{\varepsilon,\beta,\tilde{R}} = T_0$ provided that $\varepsilon \in (0, \varepsilon_1]$ and (7.5) is true for some $\tilde{M} > M$ instead of M and (7.6) is valid with R replaced by $\tilde{R}$.

3. If $N \in \{2,3\}$, $M = 2$, then there are $T_1, \varepsilon_1 > 0$ such that $T_{\varepsilon,0,R} \geq T_1$ for all $\varepsilon \in (0, \varepsilon_1]$.

We carry out a general computation first and return back to the different cases later. The difference of the left hand sides in (AC1) for u_ε and u_ε^A yields

$$\left[\partial_t - \Delta + \frac{1}{\varepsilon^2} f''(u_\varepsilon^A)\right] \overline{u}_\varepsilon = -r_\varepsilon^A - r_\varepsilon(u_\varepsilon, u_\varepsilon^A), \tag{7.9}$$

where $r_\varepsilon(u_\varepsilon, u_\varepsilon^A) := \frac{1}{\varepsilon^2}\left[f'(u_\varepsilon) - f'(u_\varepsilon^A) - f''(u_\varepsilon^A)\overline{u}_\varepsilon\right]$. We multiply (7.9) by $g_\beta^2\overline{u}_\varepsilon$ and integrate over Q_T for $T \in (0, T_{\varepsilon,\beta,R}]$, where $\varepsilon \in (0, \varepsilon_0]$ and $\beta \geq 0$ are fixed. This yields

$$\int_0^T g_\beta^2 \int_\Omega \overline{u}_\varepsilon \left[\partial_t - \Delta + \frac{1}{\varepsilon^2}f''(u_\varepsilon^A)\right]\overline{u}_\varepsilon \, dx\, dt = -\int_0^T g_\beta^2 \int_\Omega [r_\varepsilon^A + r_\varepsilon(u_\varepsilon, u_\varepsilon^A)]\overline{u}_\varepsilon \, dx\, dt \quad (7.10)$$

for all $T \in (0, T_{\varepsilon,\beta,R}]$, $\varepsilon \in (0, \varepsilon_0]$ and $\beta \geq 0$. We have to estimate all terms in a suitable way. First, $\frac{1}{2}\partial_t|\overline{u}_\varepsilon|^2 = \overline{u}_\varepsilon\partial_t\overline{u}_\varepsilon$, integration by parts in time and $\partial_t g_\beta = -\beta g_\beta$ imply

$$\int_0^T \int_\Omega g_\beta^2 \partial_t\overline{u}_\varepsilon\overline{u}_\varepsilon \, dx\, dt = \frac{1}{2}g_\beta(T)^2\|\overline{u}_\varepsilon(T)\|_{L^2(\Omega)}^2 - \frac{1}{2}\|\overline{u}_\varepsilon(0)\|_{L^2(\Omega)}^2 + \beta\int_0^T g_\beta^2\|\overline{u}_\varepsilon\|_{L^2(\Omega)}^2\, dt,$$

where $\|\overline{u}_\varepsilon(0)\|_{L^2(\Omega)}^2 \leq R^2\varepsilon^{2M+1}$ due to (7.6) ("well-prepared initial data"). For the other term on the left hand side in (7.10) we use integration by parts in space. This yields

$$\begin{aligned}
&\int_0^T g_\beta^2 \int_\Omega \overline{u}_\varepsilon \left[-\Delta + \frac{1}{\varepsilon^2}f''(u_\varepsilon^A)\right]\overline{u}_\varepsilon \, dx\, dt \\
&= \int_0^T g_\beta^2 \int_\Omega |\nabla\overline{u}_\varepsilon|^2 + \frac{1}{\varepsilon^2}f''(u_\varepsilon^A)\overline{u}_\varepsilon^2 \, dx\, dt + \int_0^T g_\beta^2 \int_{\partial\Omega} s_\varepsilon^A \operatorname{tr}\overline{u}_\varepsilon \, d\mathcal{H}^{N-1}\, dt.
\end{aligned}$$

With requirement 2. ("spectral estimate") in the theorem it follows that the first integral on the right hand side in the latter equation is bounded from below by

$$-C\int_0^T g_\beta^2\|\overline{u}_\varepsilon\|_{L^2(\Omega)}^2\, dt + \|g_\beta\nabla\overline{u}_\varepsilon\|_{L^2(Q_T\setminus\Gamma(\delta))}^2 + c_0\|g_\beta\nabla_\tau\overline{u}_\varepsilon\|_{L^2(Q_T\cap\Gamma(\delta))}^2.$$

For the remainder terms involving r_ε^A and s_ε^A we use (7.5) ("approximate solution"). This yields

$$\left|\int_0^T g_\beta^2 \left[\int_{\partial\Omega} s_\varepsilon^A \operatorname{tr}\overline{u}_\varepsilon(t)\, d\mathcal{H}^{N-1} + \int_\Omega r_\varepsilon^A\overline{u}_\varepsilon(t)\, dx\right] dt\right| \leq \overline{C}_1 R\|g_\beta\|_{L^2(0,T)}\varepsilon^{2M+1}$$

due to (7.7) for all $T \in (0, T_{\varepsilon,\beta,R}]$ and $\varepsilon \in (0, \varepsilon_0]$, where we used $\|g_\beta\|_{L^1(0,T)} \leq \sqrt{T_0}\|g_\beta\|_{L^2(0,T)}$.

In the following we estimate the r_ε-term in (7.10). The requirement 1. ("uniform boundedness") in the theorem and Lemma 7.1 yield

$$\sup_{\varepsilon\in(0,\varepsilon_0]}\left[\|u_\varepsilon\|_{L^\infty(Q_{T_0})} + \|u_\varepsilon^A\|_{L^\infty(Q_{T_0})}\right] < \infty. \quad (7.11)$$

Therefore we can apply the Taylor Theorem and obtain

$$\left|\int_0^T g_\beta^2 \int_\Omega r_\varepsilon(u_\varepsilon, u_\varepsilon^A)\overline{u}_\varepsilon \, dx\, dt\right| \leq \frac{C}{\varepsilon^2}\int_0^T g_\beta^2\|\overline{u}_\varepsilon\|_{L^3(\Omega)}^3\, dt. \quad (7.12)$$

In order to estimate the latter we use a standard Gagliardo-Nirenberg Inequality on $\Omega \setminus \Gamma_t(\delta)$ but on $\Gamma_t(\delta)$ we apply such inequalities in tangential and normal direction. This is similar to Abels, Liu [AL], Lemma 5.3. The idea is to get a finer estimate and account for the fact that the estimate for $\nabla_\tau\overline{u}_\varepsilon$ in (7.7) is better than that for $\partial_n\overline{u}_\varepsilon$. However, note that if N is too large, estimating the full L^3-norm for $\overline{u}_\varepsilon$ will not work because of the requirements for the Gagliardo-Nirenberg Inequality or because we only have L^2-estimates in (7.7) for $\nabla_\tau\overline{u}_\varepsilon$ and $\partial_n\overline{u}_\varepsilon$. Therefore we use the uniform boundedness (7.11) to lower the exponent. However, this will also decrease the resulting ε-order. Therefore we try to find the largest possible parameter. The estimates are lengthy and we decided to postpone them, see below. The result is

Lemma 7.7. *Under the assumptions in Theorem 7.5 it holds*

$$\left|\int_0^T g_\beta^2 \int_\Omega r_\varepsilon(u_\varepsilon, u_\varepsilon^A)\overline{u}_\varepsilon \, dx\, dt\right| \leq CR^{2+K(N)}\varepsilon^{2M+1}\varepsilon^{K(N)(M-k(N))}\|g_\beta^{-K(N)}\|_{L^{\frac{4}{4-\min\{4,N\}}}(0,T)}$$

for all $T \in (0, T_{\varepsilon,\beta,R}]$ *and* $\varepsilon \in (0, \varepsilon_0]$*, where* $T_{\varepsilon,\beta,R}$ *is as in* (7.8)*,* $k(N) = \max\{2, \frac{N}{2}\}$ *and* $K(N) := \min\{1, \frac{4}{N}\} \in (0, 1]$.

Remark 7.8. Note that one could apply the same standard Gagliardo-Nirenberg Inequality used for $\Omega \setminus \Gamma_t(\delta)$ also for whole Ω, but then the estimate is weaker and the minimal order $k(N)$ for M that is required for the difference estimate to work increases.

It remains to estimate $\partial_n \overline{u}_\varepsilon$. Therefore we use that due to Corollary 3.10

$$\begin{aligned}\varepsilon^2\|g_\beta \partial_n \overline{u}_\varepsilon\|^2_{L^2(Q_T\cap\Gamma(\delta))} \leq C\varepsilon^2 \int_0^T g_\beta^2 \int_\Omega |\nabla \overline{u}_\varepsilon|^2 + \frac{1}{\varepsilon^2} f''(u_\varepsilon^A)(\overline{u}_\varepsilon)^2 \, dx\, dt \\ + C \sup_{\varepsilon\in(0,\varepsilon_0]} \|f''(u_\varepsilon^A)\|_{L^\infty(Q_{T_0})} \int_0^T g_\beta^2 \|\overline{u}_\varepsilon(t)\|^2_{L^2(\Omega)}\, dt\end{aligned}$$

with a constant $C > 0$ independent of ε, T and R. The first term is absorbed with $\frac{1}{2}$ of the spectral term above if $\varepsilon \in (0, \varepsilon_1]$ and $\varepsilon_1 > 0$ is small (independent of T, R). Finally, all terms are estimated and we obtain

$$\begin{aligned}&\|\frac{g_\beta \overline{u}_\varepsilon}{2}|_T\|^2_{L^2(\Omega)} + \|\frac{g_\beta}{2}\nabla \overline{u}_\varepsilon\|^2_{L^2(Q_T\setminus\Gamma(\delta))} + \frac{c_0}{2}\|g_\beta \nabla_\tau \overline{u}_\varepsilon\|^2_{L^2(Q_T\cap\Gamma(\delta))} + \frac{\varepsilon^2}{2}\|g_\beta \partial_n \overline{u}_\varepsilon\|^2_{L^2(Q_T\cap\Gamma(\delta))} \\ &\leq \frac{R^2}{2}\varepsilon^{2M+1} + \int_0^T (-\beta + \overline{C}_0) g_\beta^2 \|\overline{u}_\varepsilon(t)\|^2_{L^2(\Omega)}\, dt + \overline{C}_1 R\varepsilon^{2M+1}\|g_\beta\|_{L^2(0,T)} \\ &+ CR^{2+K(N)}\varepsilon^{2M+1}\varepsilon^{K(N)(M-k(N))}\|g_\beta^{-K(N)}\|_{L^{\frac{4}{4-\min\{4,N\}}}(0,T)}\end{aligned} \tag{7.13}$$

for all $T \in (0, T_{\varepsilon,\beta,R}]$, $\varepsilon \in (0, \varepsilon_1]$ and constants $\overline{C}_0, \overline{C}_1, C > 0$ independent of ε, T, R, where $k(N) = \max\{2, \frac{N}{2}\}$ and $K(N) = \min\{1, \frac{4}{N}\}$. Now we consider the cases in the theorem.

Ad 1. If $M > k(N)$, then we choose $\beta \geq \overline{C}_0$ large such that $\overline{C}_1 R\|g_\beta\|_{L^2(0,T_0)} \leq \frac{R^2}{8}$. Then (7.13) is estimated by $\frac{3}{4}R^2\varepsilon^{2M+1}$ for all $T \in (0, T_{\varepsilon,\beta,R}]$ and $\varepsilon \in (0, \varepsilon_1]$, if $\varepsilon_1 > 0$ is small. By contradiction and continuity this shows $T_{\varepsilon,\beta,R} = T_0$ for all $\varepsilon \in (0, \varepsilon_1]$. $\square_1$.

Ad 2. Let $k(N) \in \mathbb{N}$, $M = k(N)$ and let (7.5) hold for some $\tilde{M} > M$ instead of M. Then the term in (7.13) where R enters linearly is improved by a factor $\varepsilon^{\tilde{M}-M}$. Let $\beta \geq \overline{C}_0$ be fixed. Now we can first choose $R > 0$ small such that the $R^{2+K(N)}$-term in (7.13) is bounded by $\frac{1}{8}R^2\varepsilon^{2M+1}$. Then $\varepsilon_1 > 0$ can be taken small such that (7.13) is estimated by $\frac{3}{4}R^2\varepsilon^{2M+1}$ for all $T \in (0, T_{\varepsilon,\beta,R}]$ and $\varepsilon \in (0, \varepsilon_1]$. By contradiction we get $T_{\varepsilon,\beta,R} = T_0$ for all $\varepsilon \in (0, \varepsilon_1]$. $\square_2$.

Ad 3. Finally, let $N \in \{2, 3\}$, $M = 2$ and $\beta = 0$. Then (7.13) is dominated by

$$\left[\frac{R^2}{2} + CR^2T + CRT^{\frac{1}{2}} + CR^3T^{\frac{4-N}{4}}\right]\varepsilon^{2M+1}.$$

Due to $\frac{4-N}{4} > 0$ there are $\varepsilon_1, T_1 > 0$ such that this is bounded by $\frac{3}{4}R^2\varepsilon^{2M+1}$ for every $T \in (0, \min(T_{\varepsilon,\beta,R}, T_1))$ and $\varepsilon \in (0, \varepsilon_1]$. Therefore $T_{\varepsilon,0,R} \geq T_1$ for all $\varepsilon \in (0, \varepsilon_1]$. $\square_3$.

Proof of Lemma 7.7. First, let us estimate (7.12) for $\Omega \setminus \Gamma_t(\delta)$ instead of Ω. The Gagliardo-Nirenberg Inequality in Lemma 7.3 and Remark 7.4 imply for $2 \leq N \leq 6$

$$\|\overline{u}_\varepsilon(t)\|^3_{L^3(\Omega\setminus\Gamma_t(\delta))} \leq C\|\overline{u}_\varepsilon(t)\|^{3(1-\frac{N}{6})}_{L^2(\Omega\setminus\Gamma_t(\delta))}\|\overline{u}_\varepsilon(t)\|^{3\frac{N}{6}}_{H^1(\Omega\setminus\Gamma_t(\delta))},$$

where the constant C is independent of $t \in [0, T_0]$ because of Remark 7.4 and since $\Omega \setminus \Gamma_t(\delta)$ has a Lipschitz-boundary uniformly in $t \in [0, T_0]$, cf. Remark 6.29. For $3\frac{N}{6} > 2$, i.e. $N = 5, 6$, the right hand side can not be controlled with (7.7). But for $N \in \{2, 3, 4\}$ we can use this estimate and obtain that (7.12) for $\Omega \setminus \Gamma_t(\delta)$ instead of Ω is estimated by

$$\begin{aligned}&\frac{C}{\varepsilon^2}\int_0^T g_\beta^{-1}\Big[g_\beta^3\|\overline{u}_\varepsilon(t)\|^3_{L^2(\Omega\setminus\Gamma_t(\delta))} + g_\beta^{3-\frac{N}{2}}\|\overline{u}_\varepsilon\|^{3-\frac{N}{2}}_{L^2(\Omega\setminus\Gamma_t(\delta))} g_\beta^{\frac{N}{2}}\|\nabla\overline{u}_\varepsilon(t)\|^{\frac{N}{2}}_{L^2(\Omega\setminus\Gamma_t(\delta))}\Big]dt\\ &\leq CR^3\varepsilon^{-2+3(M+\frac{1}{2})}\left[\|g_\beta^{-1}\|_{L^1(0,T)} + \|g_\beta^{-1}\|_{L^{\frac{4}{4-N}}(0,T)}\right]\end{aligned} \tag{7.14}$$

for all $T \in (0, T_{\varepsilon,\beta,R}]$ and $\varepsilon \in (0, \varepsilon_0]$, where we used (7.7) and the Hölder Inequality with exponents ∞, $\frac{4}{N}$ and $\frac{4}{4-N}$ for the second term. Now let $N \geq 5$. Then we consider $r(N) \in (2, 3]$ (to be determined later) and estimate with the uniform boundedness from (7.11)

$$\|\overline{u}_\varepsilon(t)\|^3_{L^3(\Omega\setminus\Gamma_t(\delta))} \leq C\|\overline{u}_\varepsilon(t)\|^{r(N)}_{L^{r(N)}(\Omega\setminus\Gamma_t(\delta))}.$$

By Lemma 7.3 the Gagliardo-Nirenberg Inequality is applicable for all $\theta = \theta(N) \in [0, 1]$ with

$$\theta\left(\frac{1}{2} - \frac{1}{N}\right) + \frac{1-\theta}{2} = \frac{1}{r(N)} \quad \Leftrightarrow \quad \theta = N\left(\frac{1}{2} - \frac{1}{r(N)}\right).$$

The condition $\theta \in [0, 1]$ restricts $r(N)$ to be in $(2, 2 + \frac{4}{N-2}]$. Then Lemma 7.3 and Remark 7.4 yield

$$\|\overline{u}_\varepsilon(t)\|^{r(N)}_{L^{r(N)}(\Omega\setminus\Gamma_t(\delta))} \leq C\|\overline{u}_\varepsilon(t)\|^{r(N)(1-\theta)}_{L^2(\Omega\setminus\Gamma_t(\delta))}\|\overline{u}_\varepsilon(t)\|^{r(N)\theta}_{H^1(\Omega\setminus\Gamma_t(\delta))}.$$

Because we have to use (7.7) to control the right hand side, we require $r(N)\theta \leq 2$. This is equivalent to

$$N\left(\frac{r(N)}{2} - 1\right) \leq 2 \quad \Leftrightarrow \quad r(N) \leq 2 + \frac{4}{N}.$$

Since $2 + \frac{4}{N} \leq \min\{3, 2 + \frac{4}{N-2}\}$ for $N \geq 5$, we can take $r(N) := 2 + \frac{4}{N}$ for $N \geq 5$. Therefore $r(N)\theta = 2$ and $r(N)(1-\theta) = r(N) - r(N)\theta = \frac{4}{N}$. Hence for $N \geq 5$ we obtain the estimate

$$\|\overline{u}_\varepsilon(t)\|^3_{L^3(\Omega\setminus\Gamma_t(\delta))} \leq C\|\overline{u}_\varepsilon(t)\|^{\frac{4}{N}}_{L^2(\Omega\setminus\Gamma_t(\delta))}\|\overline{u}_\varepsilon(t)\|^2_{H^1(\Omega\setminus\Gamma_t(\delta))}.$$

Note that for $N = 4$ the calculation also works but yields the same as before. Hence using (7.7) we obtain for $N \geq 5$ that (7.12) with Ω replaced by $\Omega \setminus \Gamma_t(\delta)$ is estimated via

$$\begin{aligned}&\frac{C}{\varepsilon^2}\int_0^T g_\beta^{-\frac{4}{N}}\left[g_\beta^{2+\frac{4}{N}}\|\overline{u}_\varepsilon(t)\|^{2+\frac{4}{N}}_{L^2(\Omega\setminus\Gamma_t(\delta))} + g_\beta^{\frac{4}{N}}\|\overline{u}_\varepsilon(t)\|^{\frac{4}{N}}_{L^2(\Omega\setminus\Gamma_t(\delta))} g_\beta^2\|\nabla\overline{u}_\varepsilon(t)\|^2_{L^2(\Omega\setminus\Gamma_t(\delta))}\right]dt\\ &\leq CR^{2+\frac{4}{N}}\varepsilon^{-2+(2+\frac{4}{N})(M+\frac{1}{2})}\left[\|g_\beta^{-\frac{4}{N}}\|_{L^1(0,T)} + \|g_\beta^{-\frac{4}{N}}\|_{L^\infty(0,T)}\right]\end{aligned} \tag{7.15}$$

for all $T \in (0, T_{\varepsilon,\beta,R}]$ and $\varepsilon \in (0, \varepsilon_0]$.

Next we estimate (7.12) for $\Gamma_t(\delta)$ instead of Ω. First we transform the integral to $(-\delta,\delta)\times\Sigma$ via $\overline{X}$. This yields

$$\int_0^T g_\beta^2\|\overline{u}_\varepsilon\|_{L^3(\Omega)}^3\,dt=\int_0^T g_\beta^2\int_{-\delta}^{\delta}\int_\Sigma |\overline{u}_\varepsilon|_{\overline{X}(r,s,t)}|^3 J_t(r,s)\,d\mathcal{H}^{N-1}(s)\,dr\,dt.$$

Here $0<c\leq J_t\leq C$ with $c,C>0$ independent of t by Remark 3.8, 3. Because of Lemma 2.15 and Remark 7.4 we can use the Gagliardo-Nirenberg Inequality for Σ with $N-1$ instead of N (and full $W^{1,p}$-norm on the right hand side). First we consider $N\in\{2,3,4\}$ since this was a special case for the estimate on $\Omega\setminus\Gamma_t(\delta)$, too. Then

$$\int_0^T g_\beta^2\int_{-\delta}^{\delta}\|\overline{u}_\varepsilon|_{\overline{X}(r,.,t)}\|_{L^3(\Sigma)}^3\,dr\,dt\leq\int_0^T Cg_\beta^2\int_{-\delta}^{\delta}\|\overline{u}_\varepsilon|_{\overline{X}(r,.,t)}\|_{L^2(\Sigma)}^{\frac{7-N}{2}}\|\overline{u}_\varepsilon|_{\overline{X}(r,.,t)}\|_{H^1(\Sigma)}^{\frac{N-1}{2}}\,dr\,dt.$$

By Lemma 2.17 we can use the Hölder Inequality with exponents $\frac{4}{5-N}$ and $\frac{4}{N-1}$. Therefore

$$\int_{-\delta}^{\delta}\|\overline{u}_\varepsilon|_{\overline{X}(r,.,t)}\|_{L^3(\Sigma)}^3\,dr\leq C\|\overline{u}_\varepsilon|_{\overline{X}(.,t)}\|_{L^{2\frac{7-N}{5-N}}(-\delta,\delta,L^2(\Sigma))}^{\frac{7-N}{2}}\|\overline{u}_\varepsilon|_{\overline{X}(.,t)}\|_{L^2(-\delta,\delta,H^1(\Sigma))}^{\frac{N-1}{2}}.$$

Here $2\frac{7-N}{5-N}\in(2,\infty)$. Hence for the first term we can use the Gagliardo-Nirenberg Inequality for $(-\delta,\delta)$. The condition on the intermediate parameter $\theta=\theta(N)\in[0,1]$ is

$$\theta\left(\frac{1}{2}-1\right)+\frac{1-\theta}{2}=\frac{5-N}{2(7-N)}\quad\Leftrightarrow\quad\theta=\frac{1}{7-N}.$$

Hence we obtain

$$\|\overline{u}_\varepsilon|_{\overline{X}(.,t)}\|_{L^{2\frac{7-N}{5-N}}(-\delta,\delta,L^2(\Sigma))}^{\frac{7-N}{2}}\leq C\|\overline{u}_\varepsilon|_{\overline{X}(.,t)}\|_{L^2(-\delta,\delta,L^2(\Sigma))}^{\frac{6-N}{2}}\|\overline{u}_\varepsilon|_{\overline{X}(.,t)}\|_{H^1(-\delta,\delta,L^2(\Sigma))}^{\frac{1}{2}}.$$

Hence (7.12) for $\Gamma_t(\delta)$ instead of Ω and $N\in\{2,3,4\}$ is estimated by

$$\frac{C}{\varepsilon^2}\int_0^T g_\beta^2\|\overline{u}_\varepsilon|_{\overline{X}(.,t)}\|_{L^2(-\delta,\delta,L^2(\Sigma))}^{\frac{6-N}{2}}\|\overline{u}_\varepsilon|_{\overline{X}(.,t)}\|_{H^1(-\delta,\delta,L^2(\Sigma))}^{\frac{1}{2}}\|\overline{u}_\varepsilon|_{\overline{X}(.,t)}\|_{L^2(-\delta,\delta,H^1(\Sigma))}^{\frac{N-1}{2}}\,dt.\tag{7.16}$$

Moreover, because of Lemma 2.15, Lemma 2.17 and Corollary 3.10, 1. it holds

$$\begin{aligned}\|\overline{u}_\varepsilon|_{\overline{X}(.,t)}\|_{H^1(-\delta,\delta,L^2(\Sigma))}&\leq C(\|\overline{u}_\varepsilon|_{\overline{X}(.,t)}\|_{L^2((-\delta,\delta)\times\Sigma)}+\|\partial_n\overline{u}_\varepsilon|_{\overline{X}(.,t)}\|_{L^2((-\delta,\delta)\times\Sigma)}),\\ \|\overline{u}_\varepsilon|_{\overline{X}(.,t)}\|_{L^2(-\delta,\delta,H^1(\Sigma))}&\leq C(\|\overline{u}_\varepsilon|_{\overline{X}(.,t)}\|_{L^2((-\delta,\delta)\times\Sigma)}+\|\nabla_\tau\overline{u}_\varepsilon|_{\overline{X}(.,t)}\|_{L^2((-\delta,\delta)\times\Sigma)}).\end{aligned}$$

For the product term in (7.16) involving both $\partial_n\overline{u}_\varepsilon$ and $\nabla_\tau\overline{u}_\varepsilon$ as factors we apply the Hölder Inequality with exponents $\frac{4}{4-N}$, 4 and $\frac{4}{N-1}$. For the term with $\partial_n\overline{u}_\varepsilon$ we use the exponents $\frac{4}{3}$, 4 and for the one with ∇_τ we use $\frac{4}{5-N}$, $\frac{4}{N-1}$. Altogether (7.12) for $\Gamma_t(\delta)$ instead of Ω and $N\in\{2,3,4\}$ is controlled by

$$CR^3\varepsilon^{3M-1}\Big[\|g_\beta^{-1}\|_{L^1(0,T)}+\|g_\beta^{-1}\|_{L^{\frac{4}{5-N}}(0,T)}+\|g_\beta^{-1}\|_{L^{\frac{4}{3}}(0,T)}+\|g_\beta^{-1}\|_{L^{\frac{4}{4-N}}(0,T)}\Big]\tag{7.17}$$

for all $T\in(0,T_{\varepsilon,\beta,R}]$ and $\varepsilon\in(0,\varepsilon_0]$, where we used (7.7) and $\varepsilon\leq\varepsilon_0$ for the terms that possess a higher ε-order. Now let $N\geq 5$. Then with (7.11) we estimate

$$\|\overline{u}_\varepsilon(t)\|_{L^3(\Gamma_t(\delta))}^3\leq\|\overline{u}_\varepsilon(t)\|_{L^{\tilde{r}(N)}(\Gamma_t(\delta))}^{\tilde{r}(N)}$$

for some $\tilde{r}(N) \in (2,3]$. We seek the maximal $\tilde{r}(N)$ such that similar calculations as above work. It will turn out that $\tilde{r}(N) = 2 + \frac{4}{N}$ is optimal. Note that the latter also was the best exponent for the estimate on $\Omega \setminus \Gamma_t(\delta)$ above in the case $N \geq 5$. But let us carry out the calculations with a general $\tilde{r}(N)$. The Gagliardo-Nirenberg Inequality on Σ is applicable for $\theta = \theta(N) \in [0,1]$ with

$$\theta\left(\frac{1}{2} - \frac{1}{N-1}\right) + \frac{1-\theta}{2} = \frac{1}{\tilde{r}(N)} \quad \Leftrightarrow \quad \theta = (N-1)\left(\frac{1}{2} - \frac{1}{\tilde{r}(N)}\right).$$

This restricts $\tilde{r}(N)$ to be in $(2, 2+\frac{4}{N-3}]$. Therefore for such $\tilde{r}(N)$ we obtain the estimate

$$\int_0^T g_\beta^2 \int_{-\delta}^{\delta} \|\overline{u}_\varepsilon|_{\overline{X}(r,.,t)}\|_{L^{\tilde{r}(N)}(\Sigma)}^{\tilde{r}(N)} \,dr\,dt \leq \int_0^T C g_\beta^2 \int_{-\delta}^{\delta} \|\overline{u}_\varepsilon|_{\overline{X}(r,.,t)}\|_{L^2(\Sigma)}^{q(N)} \|\overline{u}_\varepsilon|_{\overline{X}(r,.,t)}\|_{H^1(\Sigma)}^{p(N)} \,dr\,dt,$$

where $p(N) := \theta\tilde{r}(N) = (N-1)\left(\frac{\tilde{r}(N)}{2} - 1\right)$ and $q(N) := (1-\theta)\tilde{r}(N) = \tilde{r}(N) - p(N)$. The next step is to use the Hölder Inequality on $(-\delta,\delta)$. To this end we need $p(N) \leq 2$. This is equivalent to $\tilde{r}(N) \leq 2 + \frac{4}{N-1}$. Hence for these $\tilde{r}(N)$ we can use the Hölder Inequality with exponents $\frac{2}{2-p(N)}, \frac{2}{p(N)}$ and obtain

$$\int_{-\delta}^{\delta} \|\overline{u}_\varepsilon|_{\overline{X}(r,.,t)}\|_{L^{\tilde{r}(N)}(\Sigma)}^{\tilde{r}(N)} \,dr \leq C \|\overline{u}_\varepsilon|_{\overline{X}(.,t)}\|_{L^{y(N)}(-\delta,\delta,L^2(\Sigma))}^{q(N)} \|\overline{u}_\varepsilon|_{\overline{X}(.,t)}\|_{L^2(-\delta,\delta,H^1(\Sigma))}^{p(N)},$$

where we have set $y(N) := \frac{2q(N)}{2-p(N)}$. Note that $y(N) = \frac{2\tilde{r}(N)-2p(N)}{2-p(N)} > 2$. Therefore we can use the Gagliardo-Nirenberg Inequality on $(-\delta,\delta)$. The condition for $\theta = \theta(N) \in [0,1]$ is

$$\theta\left(\frac{1}{2} - 1\right) + \frac{\theta}{2} = \frac{1}{y(N)} \quad \Leftrightarrow \quad \theta = \frac{1}{2} - \frac{1}{y(N)} = \frac{q(N)+p(N)-2}{2q(N)} = \frac{\tilde{r}(N)-2}{2q(N)}.$$

Hence we obtain

$$\|\overline{u}_\varepsilon|_{\overline{X}(.,t)}\|_{L^{y(N)}(-\delta,\delta,L^2(\Sigma))}^{q(N)} \leq C \|\overline{u}_\varepsilon|_{\overline{X}(.,t)}\|_{L^2(-\delta,\delta,L^2(\Sigma))}^{q(N)+1-\frac{\tilde{r}(N)}{2}} \|\overline{u}_\varepsilon|_{\overline{X}(.,t)}\|_{H^1(-\delta,\delta,L^2(\Sigma))}^{\frac{\tilde{r}(N)}{2}-1}.$$

Therefore (7.12) for $\Gamma_t(\delta)$ instead of Ω and $N \geq 5$ is estimated by

$$\frac{C}{\varepsilon^2} \int_0^T g_\beta^2 \|\overline{u}_\varepsilon|_{\overline{X}(.,t)}\|_{L^2(-\delta,\delta,L^2(\Sigma))}^{1+q(N)-\frac{\tilde{r}(N)}{2}} \|\overline{u}_\varepsilon|_{\overline{X}(.,t)}\|_{H^1(-\delta,\delta,L^2(\Sigma))}^{\frac{\tilde{r}(N)}{2}-1} \|\overline{u}_\varepsilon|_{\overline{X}(.,t)}\|_{L^2(-\delta,\delta,H^1(\Sigma))}^{p(N)} \,dt. \quad (7.18)$$

We can estimate the terms using ∂_n and ∇_τ, see below (7.16). The last step is to use the Hölder Inequality in time. For the product of the $\partial_n\overline{u}_\varepsilon$-term and the $\nabla_\tau\overline{u}_\varepsilon$-term we want to use the Hölder Inequality with exponents $2\frac{2}{\tilde{r}(N)-2}, \frac{2}{p(N)}$. This gives the following condition for $\tilde{r}(N)$:

$$\frac{p(N)}{2} + \frac{\tilde{r}(N)-2}{2} \leq 2 \quad \Leftrightarrow \quad \tilde{r}(N) \leq 2 + \frac{4}{N}.$$

For $\tilde{r}(N) = 2 + \frac{4}{N}$ all the calculations work and $p(N) = 2 - \frac{2}{N} = 2\frac{N-1}{N}$, $q(N) = \frac{6}{N}$ as well as $1 + q(N) - \frac{\tilde{r}(N)}{2} = \frac{4}{N}$ and $\frac{\tilde{r}(N)}{2} - 1 = \frac{2}{N}$. Altogether (7.12) for $\Gamma_t(\delta)$ instead of Ω and $N \geq 5$ is controlled by $CR^{2+\frac{4}{N}}$ times

$$\varepsilon^{-2+(2+\frac{4}{N})M+1}\left[\|g_\beta^{-\frac{4}{N}}\|_{L^1(0,T)} + \|g_\beta^{-\frac{4}{N}}\|_{L^N(0,T)} + \|g_\beta^{-\frac{4}{N}}\|_{L^{\frac{N}{N-1}}(0,T)} + \|g_\beta^{-\frac{4}{N}}\|_{L^\infty(0,T)}\right] \quad (7.19)$$

for all $T \in (0, T_{\varepsilon,\beta,R}]$ and $\varepsilon \in (0, \varepsilon_0]$, where we used (7.7) and $\varepsilon \leq \varepsilon_0$.

Finally, we collect the above estimates. To reduce the number of g_β-terms we apply the embedding $L^p(0,t) \hookrightarrow L^q(0,t)$ for all $1 \leq q \leq p \leq \infty$ and $0 < t \leq T_0$ with embedding constant independent of t. Therefore if $N \in \{2,3,4\}$, then by (7.14), (7.17), and if $N \geq 5$, then by (7.15), (7.19), we get

$$\left|\int_0^T g_\beta^2 \int_\Omega r_\varepsilon(u_\varepsilon, u_\varepsilon^A)\overline{u}_\varepsilon \, dx \, dt\right| \leq \begin{cases} CR^3\varepsilon^{3M-1}\|g_\beta^{-1}\|_{L^{\frac{4}{4-N}}(0,T)} & \text{for } N \in \{2,3,4\}, \\ CR^{2+\frac{4}{N}}\varepsilon^{(2+\frac{4}{N})M-1}\|g_\beta^{-\frac{4}{N}}\|_{L^\infty(0,T)} & \text{for } N \geq 4. \end{cases}$$

This shows Lemma 7.7. □

The proof of Theorem 7.5 is completed.

7.2.2 Proof of Theorem 1.2

Let $N \geq 2$, Ω, Q_T and ∂Q_T be as in Remark 1.1, 1. Moreover, let $\Gamma = (\Gamma_t)_{t\in[0,T_0]}$ for some $T_0 > 0$ be a smooth solution to MCF with 90°-contact angle condition parametrized as in Section 3.1 and let $\delta > 0$ be such that Theorem 3.7 holds for 2δ instead of δ. We use the notation from Section 3.1 and Section 3.3. Let $M \in \mathbb{N}$ with $M \geq 2$ and denote with $(u_\varepsilon^A)_{\varepsilon>0}$ the approximate solution on $\overline{Q_{T_0}}$ defined in in Section 5.2.3 (which we obtained from asymptotic expansions in Section 5.2) and let $\varepsilon_0 > 0$ be such that Lemma 5.21 ("remainder estimate") holds for $\varepsilon \in (0, \varepsilon_0]$. The property $\lim_{\varepsilon\to 0} u_\varepsilon^A = \pm 1$ uniformly on compact subsets of $Q_{T_0}^\pm$ follows from the construction in Section 5.2. For $N = 2$ the approximate solution from Section 5.1.3 works analogously, too, but we treat the general case directly.

Note that $g_\beta(t) := e^{-\beta t}$ for fixed β is trapped between uniform positive constants for all $t \in [0, T_0]$. Therefore Theorem 1.2 follows immediately from Theorem 7.5 if we show the conditions 1.-4. in Theorem 7.5. The requirement 1. ("uniform boundedness") is fulfilled due to Lemma 5.21 for u_ε^A and for $u_{0,\varepsilon}$ this is an assumption in Theorem 1.2. Condition 2. ("spectral estimate") is valid because of Theorem 6.28. Requirement 4. ("well prepared initial data") is a condition on $u_{0,\varepsilon}$ and assumed in Theorem 1.2. It remains to prove 3. ("approximate solution").

First we estimate the boundary term in (7.5). Lemma 5.10 yields $s_\varepsilon^A = 0$ on $\partial\Omega \setminus \Gamma_t(2\delta)$ and $|s_\varepsilon^A| \leq C\varepsilon^M e^{-c|\rho_\varepsilon|}$, where ρ_ε is defined in (5.28). Therefore

$$\left|\int_{\partial\Omega} s_\varepsilon^A \mathrm{tr}\,\overline{u}_\varepsilon(t)\, d\mathcal{H}^{N-1}\right| \leq \|s_\varepsilon^A\|_{L^2(\partial\Omega\cap\Gamma_t(2\delta))}\|\mathrm{tr}\,\overline{u}_\varepsilon(t)\|_{L^2(\partial\Omega\cap\Gamma_t(2\delta))}.$$

Here by the substitution rule in Theorem 2.6 it holds

$$\|s_\varepsilon^A\|^2_{L^2(\partial\Omega\cap\Gamma_t(2\delta))} = \int_{\partial\Sigma}\int_{-2\delta}^{2\delta} |s_\varepsilon^A|^2|_{\overline{X}(r,Y(\sigma,0),t)}|\det d_{(r,\sigma)}[X(.,Y(.,0),t)]|\, dr\, d\mathcal{H}^{N-2}(\sigma).$$

With a scaling argument this is estimated by $C\varepsilon^{2M+1}$, see Lemma 6.5. Moreover, one can prove

$$\|\mathrm{tr}\,\overline{u}_\varepsilon(t)\|_{L^2(\partial\Omega\cap\Gamma_t(2\delta))} \leq C(\|\overline{u}_\varepsilon(t)\|_{L^2(\Gamma_t(2\delta))} + \|\nabla_\tau\overline{u}_\varepsilon(t)\|_{L^2(\Gamma_t(2\delta))}).$$

This can be shown with a similar idea as in the proof of Lemma 6.37. Here one uses $\vec{w}$ in the proof of the latter with $w_1 := 0$ there and then Corollary 3.10 to estimate $|\nabla_\Sigma(\overline{u}_\varepsilon|_{\overline{X}})| \leq C|\nabla_\tau\overline{u}_\varepsilon|_{\overline{X}}|$. Moreover, $|\nabla_\tau\overline{u}_\varepsilon| \leq C|\nabla\overline{u}_\varepsilon|$ by Corollary 3.10 and the estimate for the s_ε^A-term in (7.5) follows.

Finally, we estimate the r_ε^A-term in (7.5). Lemma 5.21 yields $r_\varepsilon^A = 0$ in $\Omega \setminus \Gamma_t(2\delta)$ and

$$\begin{aligned} |r_\varepsilon^A| &\leq C(\varepsilon^M e^{-c|\rho_\varepsilon|} + \varepsilon^{M+1}) && \text{in } \Gamma(2\delta, \mu_1), \\ |r_\varepsilon^A| &\leq C(\varepsilon^{M-1} e^{-c(|\rho_\varepsilon| + H_\varepsilon)} + \varepsilon^M e^{-c|\rho_\varepsilon|} + \varepsilon^{M+1}) && \text{in } \Gamma^C(2\delta, 2\mu_1). \end{aligned}$$

The substitution rule in Theorem 2.6 implies

$$\left| \int_\Omega r_\varepsilon^A \overline{u}_\varepsilon(t)\, dx \right| \leq \int_{\Gamma_t(2\delta)} |r_\varepsilon^A \overline{u}_\varepsilon(t)|\, dx = \int_\Sigma \int_{-2\delta}^{2\delta} |r_\varepsilon^A \overline{u}_\varepsilon|_{\overline{X}(r,s,t)} |J_t(r,s)\, dr\, d\mathcal{H}^{N-1}(s),$$

where J_t is uniformly bounded in $t \in [0, T_0]$ by Remark 3.4, 3. We split the integral over Σ in integrals over $\Sigma \setminus Y(\partial\Sigma \times [0, 2\mu_1])$ and $Y(\partial\Sigma \times [0, 2\mu_1])$. For both we use the Hölder Inequality with exponents 2, 2 for the inner integral. With a scaling argument, cf. Lemma 6.5, and Hölder's inequality the integral over $\Sigma \setminus Y(\partial\Sigma \times [0, 2\mu_1])$ is estimated by

$$C\varepsilon^{M+\frac{1}{2}} \int_{\Sigma \setminus Y(\partial\Sigma \times [0, 2\mu_1])} \|\overline{u}_\varepsilon|_{\overline{X}(.,s,t)}\|_{L^2(-2\delta, 2\delta)}\, d\mathcal{H}^{N-1}(s) \leq C\varepsilon^{M+\frac{1}{2}} \|\overline{u}_\varepsilon(t)\|_{L^2(\Gamma_t(2\delta, 2\mu_1))}.$$

Moreover, by the substitution rule in Theorem 2.6 the integral over $Y(\partial\Sigma \times [0, 2\mu_1])$ is controlled via

$$C\varepsilon^{M-\frac{1}{2}} \int_{\partial\Sigma} \int_0^{2\mu_1} \|\overline{u}_\varepsilon|_{\overline{X}(.,Y(\sigma,b),t)}\|_{L^2(-2\delta,2\delta)} (e^{-c\frac{b}{\varepsilon}} + \varepsilon)\, db\, d\mathcal{H}^{N-2}(\sigma).$$

For the $\overline{u}_\varepsilon$-term we use $H^1(0, 2\mu_1, L^2(-2\delta, 2\delta)) \hookrightarrow L^\infty(0, 2\mu_1, L^2(-2\delta, 2\delta))$. Moreover, a scaling argument yields $\int_0^{2\mu_1} e^{-cb/\varepsilon}\, db \leq C\varepsilon$. Hence the above term is estimated by

$$C\varepsilon^{M+\frac{1}{2}} \int_{\partial\Sigma} \|\overline{u}_\varepsilon|_{\overline{X}(.,Y(\sigma,.),t)}\|_{L^2(-2\delta,2\delta,H^1(0,2\mu_1))}\, d\mathcal{H}^{N-2}(\sigma).$$

Note that due to Lemma 2.10 the expression $\|\overline{u}_\varepsilon|_{\overline{X}(.,Y(\sigma,.),t)}\|_{L^2(-2\delta,2\delta,H^1(0,2\mu_1))}$ equals

$$\|\overline{u}_\varepsilon|_{\overline{X}(.,Y(\sigma,.),t)}\|_{L^2((-2\delta,2\delta)\times(0,2\mu_1))} + \|\partial_b \overline{u}_\varepsilon|_{\overline{X}(.,Y(\sigma,.),t)}\|_{L^2((-2\delta,2\delta)\times(0,2\mu_1))}$$

and Corollary 3.10 yields

$$\|\partial_b \overline{u}_\varepsilon|_{\overline{X}(.,Y(\sigma,.),t)}\|_{L^2((-2\delta,2\delta)\times(0,2\mu_1))} \leq C\|\nabla_\tau \overline{u}_\varepsilon|_{\overline{X}(.,Y(\sigma,.),t)}\|_{L^2((-2\delta,2\delta)\times(0,2\mu_1))}.$$

Finally, by Theorem 2.6, Lemma 2.10 and Hölder's inequality we obtain

$$\left| \int_\Omega r_\varepsilon^A \overline{u}_\varepsilon(t)\, dx \right| \leq C\varepsilon^{M+\frac{1}{2}} (\|\overline{u}_\varepsilon(t)\|_{L^2(\Gamma_t(2\delta))} + \|\nabla_\tau \overline{u}_\varepsilon(t)\|_{L^2(\Gamma_t(2\delta))}).$$

The estimate $|\nabla_\tau \overline{u}_\varepsilon| \leq C|\nabla \overline{u}_\varepsilon|$ due to Corollary 3.10 yields (7.5). Therefore Theorem 1.2 follows from the difference estimates in Theorem 7.5. □

7.3 Difference Estimate and Proof of the Convergence Theorem for (vAC) in ND

We show in Section 7.3.1 the difference estimate for exact and suitable approximate solutions for the vector-valued Allen-Cahn equation (vAC1)-(vAC3). Then in Section 7.3.2 we prove the Theorem 1.6 about convergence by checking the requirements for the difference estimate applied to the approximate solution from Section 5.3.3. All computations are analogous to the scalar case in the last Section 7.2.

7.3.1 Difference Estimate

Theorem 7.9 (Difference Estimate for (vAC)). *Let $N \geq 2$, Ω, Q_T and ∂Q_T be as in Remark 1.1, 1. Moreover, let $\Gamma = (\Gamma_t)_{t\in[0,T_0]}$ for some $T_0 > 0$ be as in Section 3.3 and $\delta > 0$ be such that Theorem 3.7 holds for 2δ instead of δ. We use the notation for $\Gamma_t(\delta)$, $\Gamma(\delta)$, ∇_τ and ∂_n from Remark 3.8. Additionally, let $W : \mathbb{R}^m \to \mathbb{R}$ be as in Definition 1.4.*

Moreover, let $\check{\varepsilon}_0 > 0$, $\vec{u}^A_\varepsilon \in C^2(\overline{Q_{T_0}})^m$, $\vec{u}_{0,\varepsilon} \in C^2(\overline{\Omega})^m$ with $\partial_{N_{\partial\Omega}}\vec{u}_{0,\varepsilon} = 0$ on $\partial\Omega$ and let $\vec{u}_\varepsilon \in C^2(\overline{Q_{T_0}})^m$ be exact solutions to (vAC1)-(vAC3) *with $\vec{u}_{0,\varepsilon}$ in* (vAC3) *for $\varepsilon \in (0, \check{\varepsilon}_0]$.*

For some $R > 0$ and $M \in \mathbb{N}, M \geq k(N) := \max\{2, \frac{N}{2}\}$ we impose the following conditions:

1. Uniform Boundedness: $\sup_{\varepsilon\in(0,\check{\varepsilon}_0]} \|\vec{u}^A_\varepsilon\|_{L^\infty(Q_{T_0})^m} + \|\vec{u}_{0,\varepsilon}\|_{L^\infty(\Omega)^m} < \infty$.
2. Spectral Estimate: *There are $\check{c}_0, \check{C} > 0$ such that*
$$\int_\Omega |\nabla\vec{\psi}|^2 + \frac{1}{\varepsilon^2}(\vec{\psi}, D^2W(\vec{u}^A_\varepsilon(.,t))\vec{\psi})_{\mathbb{R}^m}\, dx \geq -\check{C}\|\vec{\psi}\|^2_{L^2(\Omega)^m} + \|\nabla\vec{\psi}\|^2_{L^2(\Omega\setminus\Gamma_t(\delta))^{N\times m}} + \check{c}_0\|\nabla_\tau\vec{\psi}\|^2_{L^2(\Gamma_t(\delta))^{N\times m}}$$
for all $\vec{\psi} \in H^1(\Omega)^m$ and $\varepsilon \in (0, \check{\varepsilon}_0], t \in [0, T_0]$.
3. Approximate Solution: *For the remainders*
$$\vec{r}^A_\varepsilon := \partial_t\vec{u}^A_\varepsilon - \Delta\vec{u}^A_\varepsilon + \frac{1}{\varepsilon^2}\nabla W(\vec{u}^A_\varepsilon) \quad \text{and} \quad \vec{s}^A_\varepsilon := \partial_{N_{\partial\Omega}}\vec{u}^A_\varepsilon$$
in (vAC1)-(vAC2) *for $\vec{u}^A_\varepsilon$ and the difference $\underline{u}_\varepsilon := \vec{u}_\varepsilon - \vec{u}^A_\varepsilon$ it holds*
$$\begin{aligned}&\left|\int_{\partial\Omega} \vec{s}^A_\varepsilon \cdot \operatorname{tr}\underline{u}_\varepsilon(t)\, d\mathcal{H}^{N-1} + \int_\Omega \vec{r}^A_\varepsilon \cdot \underline{u}_\varepsilon(t)\, dx\right| \\ &\leq C\varepsilon^{M+\frac{1}{2}}(\|\underline{u}_\varepsilon(t)\|_{L^2(\Omega)^m} + \|\nabla_\tau\underline{u}_\varepsilon(t)\|_{L^2(\Gamma_t(\delta))^{N\times m}} + \|\nabla\underline{u}_\varepsilon(t)\|_{L^2(\Omega\setminus\Gamma_t(\delta))^{N\times m}})\end{aligned} \tag{7.20}$$
for all $\varepsilon \in (0, \check{\varepsilon}_0]$ and $T \in (0, T_0]$.
4. Well-Prepared Initial Data: *For all $\varepsilon \in (0, \check{\varepsilon}_0]$ it holds*
$$\|\vec{u}_{0,\varepsilon} - \vec{u}^A_\varepsilon|_{t=0}\|_{L^2(\Omega)^m} \leq R\varepsilon^{M+\frac{1}{2}}. \tag{7.21}$$

Then we obtain

1. *Let $M > k(N)$. Then there are $\beta, \check{\varepsilon}_1 > 0$ such that for $g_\beta(t) := e^{-\beta t}$ it holds*
$$\begin{aligned}\sup_{t\in[0,T]} \|g_\beta\underline{u}_\varepsilon(t)\|^2_{L^2(\Omega)^m} + \|g_\beta\nabla\underline{u}_\varepsilon\|^2_{L^2(Q_T\setminus\Gamma(\delta))^{N\times m}} &\leq 2R^2\varepsilon^{2M+1}, \\ \check{c}_0\|g_\beta\nabla_\tau\underline{u}_\varepsilon\|^2_{L^2(Q_T\cap\Gamma(\delta))^{N\times m}} + \varepsilon^2\|g_\beta\partial_n\underline{u}_\varepsilon\|^2_{L^2(Q_T\cap\Gamma(\delta))^m} &\leq 2R^2\varepsilon^{2M+1}\end{aligned} \tag{7.22}$$
for all $\varepsilon \in (0, \check{\varepsilon}_1]$ and $T \in (0, T_0]$.
2. *Let $k(N) \in \mathbb{N}$ and $M = k(N)$. Let* (7.20) *hold for some $\check{M} > M$ instead of M. Then there are $\beta, \check{R}, \check{\varepsilon}_1 > 0$ such that, if* (7.21) *holds for $\check{R}$ instead of R, then* (7.22) *for $\check{R}$ instead of R is valid for all $\varepsilon \in (0, \check{\varepsilon}_1], T \in (0, T_0]$.*

3. *Let* $N \in \{2,3\}$ *and* $M = 2(= k(N))$. *Then there are* $\check{\varepsilon}_1, \check{T}_1 > 0$ *such that* (7.22) *holds for* $\beta = 0$ *and for all* $\varepsilon \in (0, \check{\varepsilon}_1], T \in (0, \check{T}_1]$.

Remark 7.10. 1. The parameter M corresponds to the order of the approximate solution constructed in Section 5.3.

2. The comments for the scalar case in Remark 7.6, 2.-4., on the role of the parameters β, $k(N)$ and weaker requirements in the theorem, hold analogously for the vector-valued case. More precisely, see (7.28) below. The critical order $k(N)$ is the same as in the scalar case because we use the same Gagliardo-Nirenberg estimates that were used in the scalar case in the proof of Lemma 7.7.

Proof of Theorem 7.9. The continuity of the objects on the left hand side in (7.22) yields that

$$\check{T}_{\varepsilon,\beta,R} := \sup \{\tilde{T} \in (0, T_0] : (7.22) \text{ holds for } \varepsilon, R \text{ and all } T \in (0, \tilde{T}]\} \tag{7.23}$$

is well-defined for all $\varepsilon \in (0, \check{\varepsilon}_0], \beta \geq 0$ and $\check{T}_{\varepsilon,\beta,R} > 0$. In the different cases we have to show:

1. If $M > k(N)$, then there are $\beta, \check{\varepsilon}_1 > 0$ such that $\check{T}_{\varepsilon,\beta,R} = T_0$ for all $\varepsilon \in (0, \check{\varepsilon}_1]$.
2. If $M = k(N) \in \mathbb{N}$, then there are $\beta, \check{R}, \check{\varepsilon}_1 > 0$ such that $\check{T}_{\varepsilon,\beta,\check{R}} = T_0$ provided that $\varepsilon \in (0, \check{\varepsilon}_1]$ and (7.20) is true for some $\check{M} > M$ instead of M and (7.21) is valid with R replaced by $\check{R}$.
3. If $N \in \{2,3\}$, $M = 2$, then there are $\check{T}_1, \check{\varepsilon}_1 > 0$ such that $\check{T}_{\varepsilon,0,R} \geq \check{T}_1$ for all $\varepsilon \in (0, \check{\varepsilon}_1]$.

We do a general computation first and consider the specific cases later. The difference of the left hand sides in (vAC1) for $\vec{u}_\varepsilon$ and $\vec{u}_\varepsilon^A$ yields

$$\left[\partial_t - \Delta + \frac{1}{\varepsilon^2} D^2 W(\vec{u}_\varepsilon^A)\right] \underline{u}_\varepsilon = -\vec{r}_\varepsilon^A - \vec{r}_\varepsilon(\vec{u}_\varepsilon, \vec{u}_\varepsilon^A), \tag{7.24}$$

where $\vec{r}_\varepsilon(\vec{u}_\varepsilon, \vec{u}_\varepsilon^A) := \frac{1}{\varepsilon^2}\left[\nabla W(\vec{u}_\varepsilon) - \nabla W(\vec{u}_\varepsilon^A) - D^2 W(\vec{u}_\varepsilon^A)\underline{u}_\varepsilon\right]$. We multiply (7.24) by $g_\beta^2 \underline{u}_\varepsilon$ and integrate over Q_T for $T \in (0, \check{T}_{\varepsilon,\beta,R}]$, where $\varepsilon \in (0, \check{\varepsilon}_0]$ and $\beta \geq 0$ are fixed. This implies

$$\begin{aligned} &\int_0^T g_\beta^2 \int_\Omega \underline{u}_\varepsilon \cdot \left[\partial_t - \Delta + \frac{1}{\varepsilon^2} D^2 W(\vec{u}_\varepsilon^A)\right] \underline{u}_\varepsilon \, dx \, dt \\ &\qquad = -\int_0^T g_\beta^2 \int_\Omega [\vec{r}_\varepsilon^A + \vec{r}_\varepsilon(\vec{u}_\varepsilon, \vec{u}_\varepsilon^A)] \cdot \underline{u}_\varepsilon \, dx \, dt \end{aligned} \tag{7.25}$$

for all $T \in (0, \check{T}_{\varepsilon,\beta,R}]$, $\varepsilon \in (0, \check{\varepsilon}_0]$ and $\beta \geq 0$. We estimate all terms. First, $\frac{1}{2}\partial_t |\underline{u}_\varepsilon|^2 = \underline{u}_\varepsilon \cdot \partial_t \underline{u}_\varepsilon$, integration by parts in time and $\partial_t g_\beta = -\beta g_\beta$ yield

$$\int_0^T \int_\Omega g_\beta^2 \partial_t \underline{u}_\varepsilon \cdot \underline{u}_\varepsilon \, dx \, dt = \frac{1}{2} g_\beta(T)^2 \|\underline{u}_\varepsilon(T)\|_{L^2(\Omega)^m}^2 - \frac{1}{2}\|\underline{u}_\varepsilon(0)\|_{L^2(\Omega)^m}^2 + \beta \int_0^T g_\beta^2 \|\underline{u}_\varepsilon\|_{L^2(\Omega)^m}^2 \, dt,$$

where $\|\underline{u}_\varepsilon(0)\|_{L^2(\Omega)^m}^2 \leq R^2 \varepsilon^{2M+1}$ because of (7.21) ("well-prepared initial data"). For the other term on the left hand side in (7.25) we use integration by parts in space. This yields

$$\begin{aligned} &\int_0^T g_\beta^2 \int_\Omega \underline{u}_\varepsilon \cdot \left[-\Delta + \frac{1}{\varepsilon^2} D^2 W(\vec{u}_\varepsilon^A)\right] \underline{u}_\varepsilon \, dx \, dt \\ &= \int_0^T g_\beta^2 \int_\Omega |\nabla \underline{u}_\varepsilon|^2 + \frac{1}{\varepsilon^2}(\underline{u}_\varepsilon, D^2 W(\vec{u}_\varepsilon^A(.,t))\underline{u}_\varepsilon)_{\mathbb{R}^m} \, dx \, dt + \int_0^T g_\beta^2 \int_{\partial\Omega} \vec{s}_\varepsilon^A \cdot \operatorname{tr} \underline{u}_\varepsilon \, d\mathcal{H}^{N-1} \, dt. \end{aligned}$$

Using requirement 2. ("spectral estimate") in the theorem we obtain that the first integral on the right hand side of the latter equation is bounded from below by

$$-\check{C}\|\underline{u}_\varepsilon\|^2_{L^2(\Omega)^m} + \|\nabla\underline{u}_\varepsilon\|^2_{L^2(\Omega\setminus\Gamma_t(\delta))^{N\times m}} + \check{c}_0\|\nabla_\tau\underline{u}_\varepsilon\|^2_{L^2(\Gamma_t(\delta))^{N\times m}}.$$

For the remainder terms involving $\vec{r}^A_\varepsilon$ and $\vec{s}^A_\varepsilon$ we apply (7.20) ("approximate solution"). Hence

$$\left|\int_0^T g_\beta^2\left[\int_{\partial\Omega}\vec{s}^A_\varepsilon\cdot\operatorname{tr}\underline{u}_\varepsilon(t)\,d\mathcal{H}^{N-1} + \int_\Omega \vec{r}^A_\varepsilon\cdot\underline{u}_\varepsilon(t)\,dx\right]dt\right| \leq \overline{C}_1 R\|g_\beta\|_{L^2(0,T)}\varepsilon^{2M+1}$$

due to (7.22) for all $T\in(0,\check{T}_{\varepsilon,\beta,R}]$, $\varepsilon\in(0,\check{\varepsilon}_0]$, where $\|g_\beta\|_{L^1(0,T)}\leq\sqrt{T_0}\|g_\beta\|_{L^2(0,T)}$ is used.

Now we estimate the $\vec{r}_\varepsilon$-term in (7.25). The requirement 1. ("uniform boundedness") in the theorem and Lemma 7.2 yield

$$\sup_{\varepsilon\in(0,\check{\varepsilon}_0]}\left[\|\vec{u}_\varepsilon\|_{L^\infty(Q_{T_0})^m} + \|\vec{u}^A_\varepsilon\|_{L^\infty(Q_{T_0})^m}\right] < \infty. \tag{7.26}$$

Therefore the Taylor Theorem yields

$$\left|\int_0^T g_\beta^2\int_\Omega \vec{r}_\varepsilon(\vec{u}_\varepsilon,\vec{u}^A_\varepsilon)\cdot\underline{u}_\varepsilon\,dx\,dt\right| \leq \frac{C}{\varepsilon^2}\int_0^T g_\beta^2\|\underline{u}_\varepsilon\|^3_{L^3(\Omega)^m}\,dt. \tag{7.27}$$

This term can be estimated in the analogous way as in the scalar case with Gagliardo-Nirenberg inequalities on $\Omega\setminus\Gamma_t(\delta)$ and $\Gamma_t(\delta)$, cf. the proof of Lemma 7.7. This yields

$$\left|\int_0^T g_\beta^2\int_\Omega \vec{r}_\varepsilon(\vec{u}_\varepsilon,\vec{u}^A_\varepsilon)\cdot\underline{u}_\varepsilon\,dx\,dt\right| \leq CR^{2+K(N)}\varepsilon^{2M+1}\varepsilon^{K(N)(M-k(N))}\|g_\beta^{-K(N)}\|_{L^{\frac{4}{4-\min\{4,N\}}}(0,T)}$$

for all $T\in(0,\check{T}_{\varepsilon,\beta,R}]$ and $\varepsilon\in(0,\check{\varepsilon}_0]$, where $K(N):=\min\{1,\frac{4}{N}\}\in(0,1]$.

In order to control $\partial_n\underline{u}_\varepsilon$ we use Corollary 3.10 and obtain

$$\begin{aligned}\varepsilon^2\|g_\beta\partial_n\underline{u}_\varepsilon\|^2_{L^2(Q_T\cap\Gamma(\delta))^m} &\leq C\varepsilon^2\int_0^T g_\beta^2\int_\Omega|\nabla\underline{u}_\varepsilon|^2 + \frac{1}{\varepsilon^2}(\underline{u}_\varepsilon, D^2W(\vec{u}^A_\varepsilon(.,t))\underline{u}_\varepsilon)_{\mathbb{R}^m}\,dx\,dt\\ &\quad + C\sup_{\varepsilon\in(0,\check{\varepsilon}_0]}\|D^2W(\vec{u}^A_\varepsilon)\|_{L^\infty(Q_{T_0})^{m\times m}}\int_0^T g_\beta^2\|\underline{u}_\varepsilon(t)\|^2_{L^2(\Omega)^m}\,dt\end{aligned}$$

with a constant $C>0$ independent of ε, T and R. The first term is absorbed with $\frac{1}{2}$ of the spectral term above if $\varepsilon\in(0,\check{\varepsilon}_1]$ and $\check{\varepsilon}_1>0$ is small (independent of T, R). Altogether we obtain

$$\begin{aligned}&\frac{1}{2}g_\beta(T)\|\underline{u}_\varepsilon(T)\|^2_{L^2(\Omega)^m} + \frac{1}{2}\|g_\beta\nabla\underline{u}_\varepsilon\|^2_{L^2(Q_T\setminus\Gamma(\delta))^{N\times m}}\\ &+\frac{\check{c}_0}{2}\|g_\beta\nabla_\tau\underline{u}_\varepsilon\|^2_{L^2(Q_T\cap\Gamma(\delta))^{N\times m}} + \frac{1}{2}\varepsilon^2\|g_\beta\partial_n\underline{u}_\varepsilon\|^2_{L^2(Q_T\cap\Gamma(\delta))^m}\\ &\leq\frac{R^2}{2}\varepsilon^{2M+1} + \int_0^T(-\beta+\check{C}_0)g_\beta^2\|\underline{u}_\varepsilon(t)\|^2_{L^2(\Omega)^m}\,dt + \check{C}_1R\varepsilon^{2M+1}\|g_\beta\|_{L^2(0,T)}\\ &+CR^{2+K(N)}\varepsilon^{2M+1}\varepsilon^{K(N)(M-k(N))}\|g_\beta^{-K(N)}\|_{L^{\frac{4}{4-\min\{4,N\}}}(0,T)}\end{aligned} \tag{7.28}$$

for all $T\in(0,\check{T}_{\varepsilon,\beta,R}]$, $\varepsilon\in(0,\check{\varepsilon}_1]$ and constants $\check{C}_0,\check{C}_1,C>0$ independent of ε,T,R, where $k(N)=\max\{2,\frac{N}{2}\}$ and $K(N)=\min\{1,\frac{4}{N}\}$. Now we consider the cases in the theorem.

Ad 1. If $M > k(N)$, then we choose $\beta \geq \check{C}_0$ large such that $\check{C}_1 R \|g_\beta\|_{L^2(0,T_0)} \leq \frac{R^2}{8}$. Then (7.28) is estimated by $\frac{3}{4}R^2\varepsilon^{2M+1}$ for all $T \in (0, \check{T}_{\varepsilon,\beta,R}]$ and $\varepsilon \in (0, \check{\varepsilon}_1]$, if $\check{\varepsilon}_1 > 0$ is small. Via contradiction and continuity this proves $\check{T}_{\varepsilon,\beta,R} = T_0$ for all $\varepsilon \in (0, \check{\varepsilon}_1]$. $\square_{1.}$

Ad 2. Let $M = k(N) \in \mathbb{N}$ and let (7.20) hold for some $\check{M} > M$ instead of M. Then the term in (7.28) where R enters linearly is improved by a factor $\varepsilon^{\check{M}-M}$. Let $\beta \geq \check{C}_0$ be fixed. Now we first choose $R > 0$ small such that the $R^{2+K(N)}$-term in (7.28) is estimated by $\frac{1}{8}R^2\varepsilon^{2M+1}$. Then $\check{\varepsilon}_1 > 0$ can be taken small such that (7.28) is bounded by $\frac{3}{4}R^2\varepsilon^{2M+1}$ for all $T \in (0, \check{T}_{\varepsilon,\beta,R}]$ and $\varepsilon \in (0, \check{\varepsilon}_1]$. By contradiction and continuity we get $\check{T}_{\varepsilon,\beta,R} = T_0$ for all $\varepsilon \in (0, \check{\varepsilon}_1]$. $\square_{2.}$

Ad 3. Finally, let $N \in \{2, 3\}$, $M = 2$ and $\beta = 0$. Then (7.28) is estimated by

$$\left[\frac{R^2}{2} + CR^2T + CRT^{\frac{1}{2}} + CR^3T^{\frac{4-N}{4}}\right]\varepsilon^{2M+1}.$$

Due to $\frac{4-N}{4} > 0$ there are $\check{\varepsilon}_1, \check{T}_1 > 0$ such that the latter is bounded by $\frac{3}{4}R^2\varepsilon^{2M+1}$ for every $T \in (0, \min(T_{\varepsilon,\beta,R}, \check{T}_1)]$ and $\varepsilon \in (0, \check{\varepsilon}_1]$. Therefore $\check{T}_{\varepsilon,0,R} \geq \check{T}_1$ for all $\varepsilon \in (0, \check{\varepsilon}_1]$. $\square_{3.}$

The proof of Theorem 7.9 is completed. $\square$

7.3.2 Proof of Theorem 1.6

Let $N \geq 2$, Ω, Q_T and ∂Q_T be as in Remark 1.1, 1. Let $W : \mathbb{R}^m \to \mathbb{R}$ be as in Definition 1.4 and $\vec{u}_\pm$ be any distinct pair of minimizers of W. Moreover, let $\Gamma = (\Gamma_t)_{t\in[0,T_0]}$ for some $T_0 > 0$ be a smooth solution to MCF with 90°-contact angle condition parametrized as in Section 3.1 and let $\delta > 0$ be such that Theorem 3.7 holds for 2δ instead of δ. We use the notation from Sections 3.1 and Section 3.3. Let $M \in \mathbb{N}$ with $M \geq 2$ and denote with $(\vec{u}_\varepsilon^A)_{\varepsilon>0}$ the approximate solution on $\overline{Q_{T_0}}$ from Section 5.3.3 (that was constructed with asymptotic expansions in Section 5.3) and let $\check{\varepsilon}_0 > 0$ be such that Lemma 5.27 ("remainder estimate") holds for $\varepsilon \in (0, \check{\varepsilon}_0]$. The property $\lim_{\varepsilon\to 0} \vec{u}_\varepsilon^A = \vec{u}_\pm$ uniformly on compact subsets of $Q_{T_0}^\pm$ follows from Section 5.3.

Theorem 1.6 follows directly from Theorem 7.9 if we prove the conditions 1.-4. in Theorem 7.9. The requirement 1. ("uniform boundedness") is satisfied because of Lemma 5.27 for $\vec{u}_\varepsilon^A$ and for $\vec{u}_{0,\varepsilon}$ this is an assumption in Theorem 1.6. Condition 2. ("spectral estimate") is precisely the assertion in Theorem 6.41. Requirement 4. ("well prepared initial data") is a condition on $\vec{u}_{0,\varepsilon}$ and assumed in Theorem 1.6. It remains to prove 3. ("approximate solution"). This can be done in the analogous way as in the scalar case, cf. the proof of Theorem 1.2 in Section 7.2.2. Basically one uses suitable integral transformations, Hölder estimates, transformation arguments like in Lemma 6.5, the properties of $\vec{r}_\varepsilon^A$, $\vec{s}_\varepsilon^A$ from Lemma 5.27 as well as the comparison of several differential operators in Corollary 3.10. Since the computations are completely analogous to the scalar case, we refrain from going into details. $\square$

7.4 Difference Estimate and Proof of the Convergence Theorem for (AC$_\alpha$) in 2D

We prove in Section 7.4.1 the difference estimate for exact solutions and suitable approximate solutions for the Allen-Cahn equation with nonlinear Robin-boundary condition (AC$_\alpha$1)-(AC$_\alpha$3) in 2D. Then we show in Section 7.4.2 the Theorem 1.9 about convergence by verifying the requirements for the difference estimate applied to the approximate solution from Section 5.4.3.

7.4.1 Difference Estimate

Theorem 7.11 (Difference Estimate for (AC_α)). *Let $N=2$, Ω, Q_T and ∂Q_T be as in Remark 1.1, 1. Moreover, let $\Gamma=(\Gamma_t)_{t\in[0,T_0]}$ for some $T_0>0$ be as in Section 3.2 with contact angle $\alpha\in(0,\pi)$ and $\delta>0$ be such that Theorem 3.3 holds for 2δ instead of δ. We use the notation for $\Gamma_t(\delta)$, $\Gamma(\delta)$, ∇_τ and ∂_n from Remark 3.4. Additionally, let f satisfy* (1.1)-(1.2) *and σ_α for $\alpha\in(0,\pi)$ be as in Definition 1.8.*

Moreover, let $\varepsilon_0>0$, $u^A_{\varepsilon,\alpha}\in C^2(\overline{Q_{T_0}})$, $u_{0,\varepsilon,\alpha}\in C^2(\overline{\Omega})$ with $\partial_{N_{\partial\Omega}}u_{0,\varepsilon,\alpha}+\frac{1}{\varepsilon}\sigma'_\alpha(u_{0,\varepsilon,\alpha})=0$ on $\partial\Omega$ and $u_{\varepsilon,\alpha}\in C^2(\overline{Q_{T_0}})$ be exact solutions to (AC_α1)-(AC_α3) *with $u_{0,\varepsilon,\alpha}$ in* (AC_α3)*, $\varepsilon\in(0,\varepsilon_0]$.*

For some $R>0$, $M\in\mathbb{N}, M\geq 3$ and some $\delta_0\in(0,\delta]$ we impose the following conditions:

1. Uniform Boundedness: $\sup_{\varepsilon\in(0,\varepsilon_0]}\|u^A_{\varepsilon,\alpha}\|_{L^\infty(Q_{T_0})}+\|u_{0,\varepsilon,\alpha}\|_{L^\infty(\Omega)}<\infty$.
2. Spectral Estimate: *There are $c_0,C>0$ such that*

$$\int_\Omega|\nabla\psi|^2+\frac{1}{\varepsilon^2}f''(u^A_{\varepsilon,\alpha}|_{(.,t)})\psi^2\,dx+\int_{\partial\Omega}\frac{1}{\varepsilon}\sigma''_\alpha(u^A_{\varepsilon,\alpha}|_{(.,t)})(\mathrm{tr}\,\psi)^2\,d\mathcal{H}^1$$
$$\geq -C\|\psi\|^2_{L^2(\Omega)}+\|\nabla\psi\|^2_{L^2(\Omega\setminus\Gamma_t(\delta_0))}+c_0\varepsilon\|\nabla_\tau\psi\|^2_{L^2(\Gamma_t(\delta_0))}$$

for all $\psi\in H^1(\Omega)$ and $\varepsilon\in(0,\varepsilon_0],t\in[0,T_0]$.

3. Approximate Solution: *For the remainders*

$$r^A_{\varepsilon,\alpha}:=\partial_t u^A_{\varepsilon,\alpha}-\Delta u^A_{\varepsilon,\alpha}+\frac{1}{\varepsilon^2}f'(u^A_{\varepsilon,\alpha})\quad\text{and}\quad s^A_{\varepsilon,\alpha}:=\partial_{N_{\partial\Omega}}u^A_{\varepsilon,\alpha}+\frac{1}{\varepsilon}\sigma'_\alpha(u^A_{\varepsilon,\alpha})$$

in (AC_α1)-(AC_α2) *for $u^A_{\varepsilon,\alpha}$ and the difference $\overline{u}_{\varepsilon,\alpha}:=u_{\varepsilon,\alpha}-u^A_{\varepsilon,\alpha}$ it holds*

$$\begin{aligned}&\left|\int_{\partial\Omega}s^A_{\varepsilon,\alpha}\mathrm{tr}\,\overline{u}_{\varepsilon,\alpha}(t)\,d\mathcal{H}^1+\int_\Omega r^A_{\varepsilon,\alpha}\overline{u}_{\varepsilon,\alpha}(t)\,dx\right|\\&\leq C\varepsilon^{M+\frac{1}{2}}(\|\overline{u}_{\varepsilon,\alpha}(t)\|_{L^2(\Omega)}+\|\nabla_\tau\overline{u}_{\varepsilon,\alpha}(t)\|_{L^2(\Gamma_t(\delta_0))}+\|\nabla\overline{u}_{\varepsilon,\alpha}(t)\|_{L^2(\Omega\setminus\Gamma_t(\delta_0))})\end{aligned}\tag{7.29}$$

for all $\varepsilon\in(0,\varepsilon_0]$ and $T\in(0,T_0]$.

4. Well-Prepared Initial Data: *For all $\varepsilon\in(0,\varepsilon_0]$ it holds*

$$\|u_{0,\varepsilon,\alpha}-u^A_{\varepsilon,\alpha}|_{t=0}\|_{L^2(\Omega)}\leq R\varepsilon^M.\tag{7.30}$$

Then we obtain

1. *Let $M>3$. Then there are $\beta,\varepsilon_1>0$ such that for $g_\beta(t):=e^{-\beta t}$ it holds*

$$\begin{aligned}\sup_{t\in[0,T]}\|g_\beta\overline{u}_{\varepsilon,\alpha}(t)\|^2_{L^2(\Omega)}+\|g_\beta\nabla\overline{u}_{\varepsilon,\alpha}\|^2_{L^2(Q_T\setminus\Gamma(\delta_0))}&\leq 2R^2\varepsilon^{2M},\\c_0\varepsilon\|g_\beta\nabla_\tau\overline{u}_{\varepsilon,\alpha}\|^2_{L^2(Q_T\cap\Gamma(\delta_0))}+\varepsilon^2\|g_\beta\partial_n\overline{u}_{\varepsilon,\alpha}\|^2_{L^2(Q_T\cap\Gamma(\delta_0))}&\leq 2R^2\varepsilon^{2M}\end{aligned}\tag{7.31}$$

for all $\varepsilon\in(0,\varepsilon_1]$ and $T\in(0,T_0]$.

2. *Let* $M = 3$ *and* (7.29) *hold for some* $\tilde{M} > M$ *instead of* M*. Then there are* $\beta, \tilde{R}, \varepsilon_1 > 0$ *such that, if* (7.30) *holds for* $\tilde{R}$ *instead of* R*, then* (7.31) *for* $\tilde{R}$ *instead of* R *is valid for all* $\varepsilon \in (0, \varepsilon_1], T \in (0, T_0]$.

3. *Let* $M = 3$*. Then there are* $\varepsilon_1, T_1 > 0$ *such that* (7.31) *holds for* $\beta = 0$ *and for all* $\varepsilon \in (0, \varepsilon_1], T \in (0, T_1]$.

Remark 7.12. 1. The parameter M corresponds to the order of the approximate solution in Section 5.4. The δ_0 is introduced because in the application of Theorem 7.11 later we use the spectral estimate in Theorem 6.51. There δ_0 was chosen small in order to have (5.109).

2. The comments for the case $\alpha = \frac{\pi}{2}$ in Remark 7.6, 2.-4., on the role of the parameters β, $k(N)$ and weaker requirements in the theorem, hold analogously for the case $\alpha \neq \frac{\pi}{2}$. More precisely, see (7.38) below. The critical order for M is increased by one and the estimate (7.31) is slightly weaker compared to Theorem 7.5. This is because we only have a spectral estimate with the ε-factor in front of the ∇_τ-term.

3. In the proof of Lemma 7.7 for the case $\alpha = \frac{\pi}{2}$ we applied Gagliardo-Nirenberg inequalities for the integral on $\Gamma_t(\delta)$ subsequently in tangential and normal direction. These estimates are difficult to adapt for the case $\alpha \neq \frac{\pi}{2}$ because the relevant domain is a trapeze, not a rectangle. Even if this works, the possible increase in the ε-order is just $\frac{1}{4}$ and thus does not lower the critical integer order for M. Therefore we use a standard Gagliardo-Nirenberg Inequality on whole Ω, see the computation after (7.37) below.

Proof of Theorem 7.11. For the proof we can assume w.l.o.g. $\delta_0 = \delta$, otherwise one can simply shrink δ. The continuity of the objects on the left hand side in (7.31) implies that

$$T_{\varepsilon,\beta,R} := \sup \{\tilde{T} \in (0, T_0] : \text{(7.31) holds for } \varepsilon, R \text{ and all } T \in (0, \tilde{T}]\} \tag{7.32}$$

is well-defined for all $\varepsilon \in (0, \varepsilon_0], \beta \geq 0$ and $T_{\varepsilon,\beta,R} > 0$. In the different cases we have to prove:

1. If $M > 3$, then there exist $\beta, \varepsilon_1 > 0$ such that $T_{\varepsilon,\beta,R} = T_0$ for all $\varepsilon \in (0, \varepsilon_1]$.
2. If $M = 3$, then there are $\beta, \tilde{R}, \varepsilon_1 > 0$ such that $T_{\varepsilon,\beta,\tilde{R}} = T_0$ provided that $\varepsilon \in (0, \varepsilon_1]$ and (7.29) is true for some $\tilde{M} > 3$ instead of M and (7.30) is valid with R replaced by $\tilde{R}$.
3. If $M = 3$, then there are $T_1, \varepsilon_1 > 0$ such that $T_{\varepsilon,0,R} \geq T_1$ for all $\varepsilon \in (0, \varepsilon_1]$.

We carry out a general computation first and consider the different cases later. The difference of the left hand sides in (AC$_\alpha$1) for $u_{\varepsilon,\alpha}$ and $u^A_{\varepsilon,\alpha}$ yields

$$\left[\partial_t - \Delta + \frac{1}{\varepsilon^2} f''(u^A_{\varepsilon,\alpha})\right] \overline{u}_{\varepsilon,\alpha} = -r^A_{\varepsilon,\alpha} - r_\varepsilon(u_{\varepsilon,\alpha}, u^A_{\varepsilon,\alpha}), \tag{7.33}$$

where $r_\varepsilon(u_{\varepsilon,\alpha}, u^A_{\varepsilon,\alpha}) := \frac{1}{\varepsilon^2}\left[f'(u_{\varepsilon,\alpha}) - f'(u^A_{\varepsilon,\alpha}) - f''(u^A_{\varepsilon,\alpha})\overline{u}_{\varepsilon,\alpha}\right]$. We multiply (7.33) by $g^2_\beta \overline{u}_{\varepsilon,\alpha}$ and integrate over Q_T for $T \in (0, T_{\varepsilon,\beta,R}]$, where $\varepsilon \in (0, \varepsilon_0]$ and $\beta \geq 0$ are fixed. This yields

$$\int_0^T g^2_\beta \int_\Omega \overline{u}_{\varepsilon,\alpha} \left[\partial_t - \Delta + \frac{1}{\varepsilon^2} f''(u^A_{\varepsilon,\alpha})\right] \overline{u}_{\varepsilon,\alpha} = -\int_0^T g^2_\beta \int_\Omega [r^A_{\varepsilon,\alpha} + r_\varepsilon(u_{\varepsilon,\alpha}, u^A_{\varepsilon,\alpha})]\overline{u}_{\varepsilon,\alpha} \tag{7.34}$$

for all $T \in (0, T_{\varepsilon,\beta,R}]$, $\varepsilon \in (0, \varepsilon_0]$ and $\beta \geq 0$. We estimate all terms appropriately. Because of $\frac{1}{2}\partial_t|\overline{u}_{\varepsilon,\alpha}|^2 = \overline{u}_{\varepsilon,\alpha}\partial_t\overline{u}_{\varepsilon,\alpha}$, integration by parts in time and $\partial_t g_\beta = -\beta g_\beta$ we get

$$\int_0^T \int_\Omega g_\beta^2 \partial_t \overline{u}_{\varepsilon,\alpha}\overline{u}_{\varepsilon,\alpha}\, dx\, dt = \frac{1}{2} g_\beta|_T^2 \|\overline{u}_{\varepsilon,\alpha}|_T\|_{L^2(\Omega)}^2 - \frac{1}{2}\|\overline{u}_{\varepsilon,\alpha}(0)\|_{L^2(\Omega)}^2 + \beta \int_0^T g_\beta^2 \|\overline{u}_{\varepsilon,\alpha}\|_{L^2(\Omega)}^2\, dt,$$

where $\|\overline{u}_{\varepsilon,\alpha}(0)\|_{L^2(\Omega)}^2 \leq R^2\varepsilon^{2M}$ due to (7.30) ("well-prepared initial data"). For the other term on the left hand side in (7.34) we use integration by parts in space. This yields

$$\begin{aligned}
&\int_0^T g_\beta^2 \int_\Omega \overline{u}_{\varepsilon,\alpha}\left[-\Delta + \frac{1}{\varepsilon^2} f''(u_{\varepsilon,\alpha}^A)\right]\overline{u}_{\varepsilon,\alpha}\, dx\, dt \\
&= \int_0^T g_\beta^2 \left[\int_\Omega |\nabla \overline{u}_{\varepsilon,\alpha}|^2 + \frac{1}{\varepsilon^2} f''(u_{\varepsilon,\alpha}^A)\overline{u}_{\varepsilon,\alpha}^2\, dx + \int_{\partial\Omega} \frac{1}{\varepsilon}\sigma_\alpha''(u_{\varepsilon,\alpha}^A)(\operatorname{tr}\overline{u}_{\varepsilon,\alpha})^2\, d\mathcal{H}^1\right] dt \\
&+ \int_0^T g_\beta^2 \int_{\partial\Omega} \left[s_{\varepsilon,\alpha}^A + s_{\varepsilon,\alpha}(u_{\varepsilon,\alpha}, u_{\varepsilon,\alpha}^A)\right] \operatorname{tr}\overline{u}_{\varepsilon,\alpha}\, d\mathcal{H}^1\, dt,
\end{aligned} \tag{7.35}$$

where we have set $s_{\varepsilon,\alpha}(u_{\varepsilon,\alpha}, u_{\varepsilon,\alpha}^A) := \frac{1}{\varepsilon}\left[\sigma_\alpha'(u_{\varepsilon,\alpha}) - \sigma_\alpha'(u_{\varepsilon,\alpha}^A) - \sigma_\alpha''(u_{\varepsilon,\alpha}^A)\overline{u}_{\varepsilon,\alpha}\right]|_{\partial\Omega}$. With requirement 2. ("spectral estimate") in the theorem it follows that the first integral on the right hand side in the latter equation is bounded from below by

$$-C\int_0^T g_\beta^2 \|\overline{u}_{\varepsilon,\alpha}\|_{L^2(\Omega)}^2\, dt + \|g_\beta \nabla \overline{u}_{\varepsilon,\alpha}\|_{L^2(Q_T\setminus\Gamma(\delta))}^2 + c_0\varepsilon\|g_\beta \nabla_\tau \overline{u}_{\varepsilon,\alpha}\|_{L^2(Q_T\cap\Gamma(\delta))}^2.$$

For the remainder terms involving $r_{\varepsilon,\alpha}^A$ and $s_{\varepsilon,\alpha}^A$ we use (7.29) ("approximate solution"). Hence

$$\left|\int_0^T g_\beta^2 \left[\int_{\partial\Omega} s_{\varepsilon,\alpha}^A \operatorname{tr}\overline{u}_{\varepsilon,\alpha}(t)\, d\mathcal{H}^1 + \int_\Omega r_{\varepsilon,\alpha}^A \overline{u}_{\varepsilon,\alpha}(t)\, dx\right] dt\right| \leq \overline{C}_1 R \|g_\beta\|_{L^2(0,T)}\varepsilon^{2M}$$

because of (7.31) for all $T \in (0, T_{\varepsilon,\beta,R}]$ and $\varepsilon \in (0, \varepsilon_0]$, where we have used the Hölder Inequality to estimate $\|g_\beta\|_{L^1(0,T)} \leq \sqrt{T_0}\|g_\beta\|_{L^2(0,T)}$.

In the following we estimate the r_ε-term in (7.34) and the $s_{\varepsilon,\alpha}$-term in (7.35). The requirement 1. ("uniform boundedness") in the theorem and Lemma 7.1 yield

$$\sup_{\varepsilon\in(0,\varepsilon_0]} \left[\|u_{\varepsilon,\alpha}\|_{L^\infty(Q_{T_0})} + \|u_{\varepsilon,\alpha}^A\|_{L^\infty(Q_{T_0})}\right] < \infty. \tag{7.36}$$

Therefore we can apply the Taylor Theorem and obtain

$$\begin{aligned}
&\left|\int_0^T g_\beta^2 \left[\int_\Omega r_\varepsilon(u_{\varepsilon,\alpha}, u_{\varepsilon,\alpha}^A)\overline{u}_{\varepsilon,\alpha}\, dx + \int_{\partial\Omega} s_{\varepsilon,\alpha}(u_{\varepsilon,\alpha}, u_{\varepsilon,\alpha}^A)\operatorname{tr}\overline{u}_{\varepsilon,\alpha}\, d\mathcal{H}^1\right] dt\right| \\
&\leq C\int_0^T g_\beta^2 \left[\frac{1}{\varepsilon^2}\|\overline{u}_{\varepsilon,\alpha}\|_{L^3(\Omega)}^3 + \frac{1}{\varepsilon}\|\operatorname{tr}\overline{u}_{\varepsilon,\alpha}\|_{L^3(\partial\Omega)}^3\right] dt.
\end{aligned} \tag{7.37}$$

For the estimate of the $L^3(\Omega)$-norm we use a standard Gagliardo-Nirenberg Inequality on Ω, see Lemma 7.3 and Remark 7.4. This yields due to (7.31)

$$\int_0^T \frac{g_\beta^2}{\varepsilon^2}\|\overline{u}_{\varepsilon,\alpha}\|_{L^3(\Omega)}^3\, dt \leq \int_0^T \frac{g_\beta^2}{\varepsilon^2}\|\overline{u}_{\varepsilon,\alpha}\|_{L^2(\Omega)}^2\|\overline{u}_{\varepsilon,\alpha}\|_{H^1(\Omega)}\, dt \leq CR^3\varepsilon^{2M}\varepsilon^{M-3}\|g_\beta^{-1}\|_{L^2(0,T)}$$

for all $T \in (0, T_{\varepsilon,\beta,R}]$ and $\varepsilon \in (0, \varepsilon_0]$. For the $L^3(\partial\Omega)$-norm in (7.37) we use the idea from Evans [Ev], 5.10, problem 7 again. Let $\vec{w} \in C^1(\overline{\Omega})$ with $\vec{w} \cdot N_{\partial\Omega} \geq 1$. Then because of $|\overline{u}_{\varepsilon,\alpha}|^3(t) \in C^1(\overline{\Omega})$ with $\nabla(|\overline{u}_{\varepsilon,\alpha}|^3)(t) = 3\mathrm{sign}(\overline{u}_{\varepsilon,\alpha})|\overline{u}_{\varepsilon,\alpha}|^2\nabla\overline{u}_{\varepsilon,\alpha}(t)$ we get

$$\|\mathrm{tr}\,\overline{u}_{\varepsilon,\alpha}|_t\|^3_{L^3(\partial\Omega)} \leq \int_{\partial\Omega} \vec{w}\cdot N_{\partial\Omega}|\overline{u}_{\varepsilon,\alpha}|^3|_t\,d\mathcal{H}^1 \leq \int_\Omega \Big[|\mathrm{div}\vec{w}||\overline{u}_{\varepsilon,\alpha}|^3 + 3|\overline{u}_{\varepsilon,\alpha}|^2|\nabla\overline{u}_{\varepsilon,\alpha}\cdot\vec{w}|\Big]|_t dx$$
$$\leq C[\|\overline{u}_{\varepsilon,\alpha}\|^3_{L^3(\Omega)} + \|\overline{u}_{\varepsilon,\alpha}\|^2_{L^4(\Omega)}\|\nabla\overline{u}_{\varepsilon,\alpha}\|_{L^2(\Omega)}]|_t \leq C[\|\overline{u}_{\varepsilon,\alpha}\|^3_{L^3(\Omega)} + \|\overline{u}_{\varepsilon,\alpha}\|_{L^2(\Omega)}\|\overline{u}_{\varepsilon,\alpha}\|^2_{H^1(\Omega)}]|_t,$$

where we used the Gagliardo-Nirenberg Inequality for the $L^4(\Omega)$-norm, see Lemma 7.3 and Remark 7.4. With $|\nabla\overline{u}_{\varepsilon,\alpha}| \leq C(|\partial_n\overline{u}_{\varepsilon,\alpha}| + |\nabla_\tau\overline{u}_{\varepsilon,\alpha}|)$ and (7.31) we obtain

$$\frac{1}{\varepsilon}\int_0^T g_\beta^2\|\mathrm{tr}\,\overline{u}_{\varepsilon,\alpha}\|^3_{L^3(\partial\Omega)}\,dt \leq CR^3\varepsilon^{2M}\varepsilon^{M-3}\|g_\beta^{-1}\|_{L^2(0,T)}$$

for all $T \in (0, T_{\varepsilon,\beta,R}]$ and $\varepsilon \in (0, \varepsilon_0]$.

It remains to estimate $\partial_n\overline{u}_{\varepsilon,\alpha}$. To this end we use $|\partial_n\overline{u}_{\varepsilon,\alpha}| \leq C|\nabla\overline{u}_{\varepsilon,\alpha}|$ and

$$\varepsilon^2\|g_\beta\partial_n\overline{u}_{\varepsilon,\alpha}\|^2_{L^2(Q_T\cap\Gamma(\delta))} \leq C\int_0^T g_\beta^2\left[\|\overline{u}_{\varepsilon,\alpha}\|^2_{L^2(\Omega)} + \varepsilon\|\mathrm{tr}\,\overline{u}_{\varepsilon,\alpha}\|^2_{L^2(\partial\Omega)}\right]dt$$
$$+ C\varepsilon^2\int_0^T g_\beta^2\left[\int_\Omega |\nabla\overline{u}_{\varepsilon,\alpha}|^2 + \frac{1}{\varepsilon^2}f''(u^A_{\varepsilon,\alpha})(\overline{u}_{\varepsilon,\alpha})^2\,dx + \int_{\partial\Omega}\frac{1}{\varepsilon}\sigma''_\alpha(u^A_{\varepsilon,\alpha})(\mathrm{tr}\,\overline{u}_{\varepsilon,\alpha})^2\,d\mathcal{H}^1\right]dt$$

with a constant $C > 0$ independent of ε, T and R. The second line is absorbed with $\frac{1}{2}$ of the spectral term above if $\varepsilon \in (0, \varepsilon_1]$ and $\varepsilon_1 > 0$ is small (independent of T, R). Moreover, for the $\|\mathrm{tr}\,\overline{u}_{\varepsilon,\alpha}\|^2_{L^2(\partial\Omega)}$-term we use the analogous idea that we applied for the estimate of $\|\mathrm{tr}\,\overline{u}_{\varepsilon,\alpha}\|^3_{L^3(\partial\Omega)}$ above. This yields

$$\varepsilon\int_0^T g_\beta^2\|\mathrm{tr}\,\overline{u}_{\varepsilon,\alpha}\|^2_{L^2(\partial\Omega)}\,dt \leq \varepsilon\int_0^T g_\beta^2\|\overline{u}_{\varepsilon,\alpha}\|_{L^2(\Omega)}\|\overline{u}_{\varepsilon,\alpha}\|_{H^1(\Omega)}\,dt.$$

Here note that because of (7.7) it follows that for all $T \in (0, T_{\varepsilon,\beta,R}]$ and $\varepsilon \in (0, \varepsilon_0]$

$$\varepsilon^2\int_0^T g_\beta^2\|\overline{u}_{\varepsilon,\alpha}\|^2_{H^1(\Omega)}\,dt \leq CR^2\varepsilon^{2M}.$$

Therefore with the Young Inequality the contribution of the $\|\mathrm{tr}\,\overline{u}_{\varepsilon,\alpha}\|^2_{L^2(\partial\Omega)}$-term is controlled by

$$\tilde{C}\int_0^T g_\beta^2\|\overline{u}_{\varepsilon,\alpha}\|^2_{L^2(\Omega)}\,dt + \frac{1}{8}R^2\varepsilon^{2M}$$

for all $T \in (0, T_{\varepsilon,\beta,R}]$ and $\varepsilon \in (0, \varepsilon_0]$ with some $\tilde{C} > 0$ large.

Finally, all terms are estimated and we obtain

$$\begin{aligned}
&\frac{1}{2}g_\beta(T)\|\overline{u}_{\varepsilon,\alpha}(T)\|^2_{L^2(\Omega)} + \frac{1}{2}\|g_\beta\nabla\overline{u}_{\varepsilon,\alpha}\|^2_{L^2(Q_T\setminus\Gamma(\delta))}\\
&+ \frac{1}{2}c_0\varepsilon\|g_\beta\nabla_\tau\overline{u}_{\varepsilon,\alpha}\|^2_{L^2(Q_T\cap\Gamma(\delta))} + \frac{1}{2}\varepsilon^2\|g_\beta\partial_n\overline{u}_{\varepsilon,\alpha}\|^2_{L^2(Q_T\cap\Gamma(\delta))}\\
&\leq (\frac{1}{2}+\frac{1}{8})R^2\varepsilon^{2M} + \int_0^T(-\beta+\overline{C}_0)g_\beta^2\|\overline{u}_{\varepsilon,\alpha}(t)\|^2_{L^2(\Omega)}\,dt + \overline{C}_1R\varepsilon^{2M}\|g_\beta\|_{L^2(0,T)}\\
&+ CR^3\varepsilon^{2M}\varepsilon^{M-3}\|g_\beta^{-1}\|_{L^2(0,T)}
\end{aligned} \tag{7.38}$$

for all $T \in (0, T_{\varepsilon,\beta,R}]$, $\varepsilon \in (0, \varepsilon_1]$ and constants $\overline{C}_0, \overline{C}_1, C > 0$ independent of ε, T, R. Now we consider the cases in the theorem.

Ad 1. If $M > 3$, then we choose $\beta \geq \overline{C}_0$ large such that $\overline{C}_1 R \|g_\beta\|_{L^2(0,T_0)} \leq \frac{R^2}{8}$. Therefore (7.38) is estimated by $\frac{7}{8}R^2\varepsilon^{2M}$ for all $T \in (0, T_{\varepsilon,\beta,R}]$ and $\varepsilon \in (0, \varepsilon_1]$, if $\varepsilon_1 > 0$ is small. By contradiction and continuity this yields $T_{\varepsilon,\beta,R} = T_0$ for all $\varepsilon \in (0, \varepsilon_1]$. $\Box_1$.

Ad 2. Let $M = 3$ and let (7.29) hold for some $\tilde{M} > M$ instead of M. Then the term in (7.38) where R enters linearly is improved by a factor $\varepsilon^{\tilde{M}-M}$. We fix $\beta \geq \overline{C}_0$ and choose $R > 0$ small such that the R^3-term in (7.38) is bounded by $\frac{1}{8}R^2\varepsilon^{2M}$. Then $\varepsilon_1 > 0$ can be taken small such that (7.38) is estimated by $\frac{7}{8}R^2\varepsilon^{2M}$ for all $T \in (0, T_{\varepsilon,\beta,R}]$ and $\varepsilon \in (0, \varepsilon_1]$. Via contradiction and continuity we obtain $T_{\varepsilon,\beta,R} = T_0$ for all $\varepsilon \in (0, \varepsilon_1]$. $\Box_2$.

Ad 3. Let $M = 3$ and $\beta = 0$. Then (7.38) is controlled by

$$\left[(\frac{1}{2} + \frac{1}{8})R^2 + CR^2T + CRT^{\frac{1}{2}} + CR^3T^{\frac{1}{2}}\right]\varepsilon^{2M}.$$

There are $\varepsilon_1, T_1 > 0$ such that this is bounded by $\frac{7}{8}R^2\varepsilon^{2M}$ for every $T \in (0, \min(T_{\varepsilon,\beta,R}, T_1)]$ and $\varepsilon \in (0, \varepsilon_1]$. Hence $T_{\varepsilon,0,R} \geq T_1$ for all $\varepsilon \in (0, \varepsilon_1]$. $\Box_3$.

The proof of Theorem 7.11 is completed. $\Box$

7.4.2 Proof of Theorem 1.9

Let f satisfy (1.1)-(1.2) and σ_α for $\alpha \in (0, \pi)$ be as in Definition 1.8. Then let $\alpha_0 > 0$ be as in Remark 5.33 and $\overline{\alpha}_0 \in (0, \alpha_0]$ such that Theorem 6.51 holds. Moreover, let $N = 2$, Ω, Q_T and ∂Q_T be as in Remark 1.1, 1. Additionally, let $\Gamma = (\Gamma_t)_{t\in[0,T_0]}$ for some $T_0 > 0$ be a smooth solution to MCF with α-contact angle condition parametrized as in Section 3.1 for some $\alpha \in \frac{\pi}{2} + [-\overline{\alpha}_0, \overline{\alpha}_0]$ and let $\delta > 0$ be such that Theorem 3.3 holds for 2δ instead of δ. We use the notation from Section 3.1 and Section 3.2. Furthermore, let $\delta_0 \in (0, \delta]$ be such that (5.109) holds. Moreover, let $M \in \mathbb{N}$ with $M \geq 3$ and denote with $(u^A_{\varepsilon,\alpha})_{\varepsilon>0}$ the approximate solution on $\overline{Q_{T_0}}$ defined in in Section 5.4.3 (which we obtained from asymptotic expansions in Section 5.4) and let $\varepsilon_0 > 0$ be such that Lemma 5.37 ("remainder estimate") holds for $\varepsilon \in (0, \varepsilon_0]$. The property $\lim_{\varepsilon\to 0} u^A_{\varepsilon,\alpha} = \pm 1$ uniformly on compact subsets of $Q^\pm_{T_0}$ follows from the construction in Section 5.4.

Theorem 1.9 follows directly from Theorem 7.11 if we prove the conditions 1.-4. in Theorem 7.11. The requirement 1. ("uniform boundedness") is satisfied because of Lemma 5.37 for $u^A_{\varepsilon,\alpha}$ and for $u_{0,\varepsilon,\alpha}$ this is an assumption in Theorem 1.9. Condition 2. ("spectral estimate") holds due to Theorem 6.51. Requirement 4. ("well prepared initial data") is a condition on $u_{0,\varepsilon,\alpha}$ and assumed in Theorem 1.9. It is left to prove 3. ("approximate solution"). This is similar to the case $\alpha = \frac{\pi}{2}$, cf. the proof of Theorem 1.2 in Section 7.2.2.

First we consider the boundary term in (7.29). Lemma 5.37 yields $s^A_{\varepsilon,\alpha} = 0$ on $\partial\Omega \setminus \Gamma_t(2\delta)$ and $|s^A_{\varepsilon,\alpha}| \leq C\varepsilon^M e^{-c|\rho_{\varepsilon,\alpha}|}$, where $\rho_{\varepsilon,\alpha}$ is defined in (5.75). Therefore

$$\left|\int_{\partial\Omega} s^A_{\varepsilon,\alpha} \mathrm{tr}\, \overline{u}_{\varepsilon,\alpha}(t)\, d\mathcal{H}^1\right| \leq \|s^A_{\varepsilon,\alpha}\|_{L^2(\partial\Omega\cap\Gamma_t(2\delta))} \|\mathrm{tr}\, \overline{u}_{\varepsilon,\alpha}(t)\|_{L^2(\partial\Omega\cap\Gamma_t(2\delta))}.$$

Due to the substitution rule in Theorem 2.6 and a scaling argument with Lemma 6.5 we obtain $\|s^A_{\varepsilon,\alpha}\|_{L^2(\partial\Omega\cap\Gamma_t(2\delta))} \leq C\varepsilon^{M+\frac{1}{2}}$. Moreover, analogously to Lemma 6.54 it follows that

$$\|\mathrm{tr}\, \overline{u}_{\varepsilon,\alpha}(t)\|_{L^2(\partial\Omega\cap\Gamma_t(2\delta))} \leq C(\|\overline{u}_{\varepsilon,\alpha}(t)\|_{L^2(\Gamma_t(2\delta))} + \|\nabla_\tau \overline{u}_{\varepsilon,\alpha}(t)\|_{L^2(\Gamma_t(2\delta))}).$$

Because of $|\nabla_\tau \overline{u}_{\varepsilon,\alpha}| \leq C|\nabla \overline{u}_{\varepsilon,\alpha}|$, the estimate for the $s^A_{\varepsilon,\alpha}$-term in (7.29) follows.

Finally, we estimate the $r^A_{\varepsilon,\alpha}$-term in (7.29). Lemma 5.37 yields $r^A_{\varepsilon,\alpha} = 0$ in $\Omega \setminus \Gamma_t(2\delta)$ and

$$|r^A_{\varepsilon,\alpha}| \leq C(\varepsilon^{M-1} e^{-c(|\rho_{\varepsilon,\alpha}|+Z^\pm_{\varepsilon,\alpha})} + \varepsilon^M e^{-c|\rho_{\varepsilon,\alpha}|} + \varepsilon^{M+1}) \quad \text{in } \Gamma^\pm(2\delta,1).$$

An integral transformation yields

$$\left|\int_\Omega r^A_{\varepsilon,\alpha} \overline{u}_{\varepsilon,\alpha}(t)\, dx\right| \leq \int_{\Gamma_t(2\delta)} |r^A_{\varepsilon,\alpha} \overline{u}_{\varepsilon,\alpha}(t)|\, dx = \int_{S_{2\delta,\alpha}} |r^A_{\varepsilon,\alpha} \overline{u}_{\varepsilon,\alpha}|_{\overline{X}(r,s,t)}| J_t(r,s)\, d(r,s),$$

where $S_{2\delta,\alpha}$ is as in (3.1) and J_t is uniformly bounded in $t \in [0,T_0]$ by Remark 3.4, 3. We choose $\mu > 0$ such that for $I_\mu := (-1-\mu, 1+\mu)$ it holds $S_{2\delta,\alpha} \subseteq (-2\delta, 2\delta) \times I_\mu$. Moreover, we denote with $e_0(\overline{u}_{\varepsilon,\alpha}|_{\overline{X}})$ the extension of $\overline{u}_{\varepsilon,\alpha}|_{\overline{X}}$ by zero to $(-2\delta,2\delta) \times I_\mu$. With a scaling argument and $|\frac{r}{\varepsilon}| + |\frac{s^\pm}{\varepsilon}| \leq C(|\rho_{\varepsilon,\alpha}| + Z^\pm_{\varepsilon,\alpha} + 1)$ because of (5.80) it follows that

$$\left|\int_\Omega r^A_{\varepsilon,\alpha} \overline{u}_{\varepsilon,\alpha}(t)\, dx\right| \leq C\varepsilon^{M-\frac{1}{2}} \int_{-1-\mu}^{1+\mu} \|e_0(\overline{u}_{\varepsilon,\alpha}|_{\overline{X}})(.,s,t)\|_{L^2(-2\delta,2\delta)} \left[\sum_\pm e^{-c\frac{|s\mp 1|}{\varepsilon}} + \varepsilon\right] ds.$$

Note that $H^1(-s,s,B) \hookrightarrow L^\infty(-s,s,B)$ for all $s \in [1,1+\mu]$ and any Banach space B with uniform embedding constant. For B we use L^2-spaces over suitable intervals $I(s)$ (possibly empty for $|s|$ large) with $\bigcup_{s\in(1,1+\mu)} I(s) \times (-s,s) = (S_{2\delta,\alpha})^\circ$. Hence Lemma 2.10 yields

$$\|e_0(\overline{u}_{\varepsilon,\alpha}|_{\overline{X}})(.,t)\|_{L^\infty(I_\mu, L^2(-2\delta,2\delta))} \leq C(\|\overline{u}_{\varepsilon,\alpha}|_{\overline{X}}\|_{L^2(S_{2\delta,\alpha})} + \|\nabla_\tau \overline{u}_{\varepsilon,\alpha}|_{\overline{X}}\|_{L^2(S_{2\delta,\alpha})}).$$

With a scaling argument for the exponential term and an integral transformation we obtain

$$\left|\int_\Omega r^A_{\varepsilon,\alpha} \overline{u}_{\varepsilon,\alpha}(t)\, dx\right| \leq C\varepsilon^{M+\frac{1}{2}} (\|\overline{u}_{\varepsilon,\alpha}(t)\|_{L^2(\Gamma_t(2\delta))} + \|\nabla_\tau \overline{u}_{\varepsilon,\alpha}(t)\|_{L^2(\Gamma_t(2\delta))}).$$

Since $|\nabla_\tau \overline{u}_{\varepsilon,\alpha}| \leq C|\nabla \overline{u}_{\varepsilon,\alpha}|$, this shows (7.29). Hence Theorem 1.9 is proven. □

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