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Excluding and constructing of exotic group actions on spheres

A doctoral dissertation in mathematical sciences in the area of mathematics

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Wykluczanie i konstruowanie egzotycznych działań grup na sferach

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Abstract

In their article [25], Wu-Chung Hsiang and Wu Yi Hsiang write on pages 224 and 231 the following.

"Due to the existence of natural linear actions on Euclidean spaces, spheres and disks, it is quite fair to say that they are the best testing spaces in the study of differentiable transformation groups (...) We share the prevailing conviction that the study of differentiable actions on these best testing spaces is probably still the most important topic in transformation groups."

The thesis concerns exotic smooth actions of finite groups on manifolds. We focus on actions on spheres with one and two fixed points. The exoticism means that the actions are not equivalent to linear ones. An important problem is, on one hand, exlcuding of the exotic actions, and, on the other hand, the constructions of them. For two fixed point actions of a finite group G on a sphere, we focus on these cases where the tangent spaces at the two fixed points have non-isomorphic $\mathbb{R}G$-module structures.

The first subject of our research are exclusions of smooth one fixed point actions of finite groups on spheres. We develop a strategy of excluding of such actions on spheres of a given dimension. The strategy relies on homological properties of the fixed point data and intersection theory. We provide new algebraic conditions, sufficient to exclude one fixed point actions. We present an algorithm which, by verifying the appropriate sufficient conditions (both described in this thesis, and the conditions obtained earlier by Morimoto and Tamura [41] and Borowiecka and the author [5, 6]), allows us to exclude the actions in question. This algorithm, implemented in GAP [23], provides new exclusion results.

This thesis is also concerned with two fixed point actions on spheres having non-isomorphic $\mathbb{R}G$-module structures on the tangent spaces at the fixed points, which are defined by differentiation of the action. The question about the existence of such actions was raised by Smith [65] who asked whether for a finite group G acting smoothly on a sphere with exactly two fixed points, the $\mathbb{R}G$-module structures induced on the tangent spaces at the two fixed points, are always isomorphic to each other. There is a conjecture of Laitinen [29] which predicts the negative answer to the Smith question for groups satisfying certain algebraic conditions. Although not true in general, the conjecture holds for many families of finite groups. Still, the Laitinen Conjecture remains unsettled for various families of groups. Our main result of this part is indicating a new infinite family of finite groups for which the Laitinen Conjecture holds.

Abstrakt

W artykule [25], Wu-Chung Hsiang oraz Wu Yi Hsiang piszą na stronach 224 i 231 w sposób następujący.

„Ze względu na istnienie naturalnych działań liniowych na przestrzeniach euklidesowych, sferach i dyskach, można uczciwie powiedzieć, że są one najlepszymi przestrzeniami testowymi w badaniu różniczkowalnych grup przekształceń (...) Podzielamy powszechne przekonanie, że badanie różniczkowalnych działań na tych najlepszych przestrzeniach testowych jest prawdopodobnie ciągle najważniejszym zagadnieniem dotyczącym grup przekształceń."

Praca dotyczy egzotycznych gładkich działań grup skończonych na rozmaitościach. Skupiamy się na działaniach na sferach z jednym, bądź dwoma punktami stałymi. Wspomniana egzotyka oznacza, że działania nie są równoważne z liniowymi. Ważnym zagadnieniem jest, z jednej strony, wykluczanie egzotycznych działań, a z drugiej ich konstrukcja. W przypadku działań grupy skończonej G na sferze z dwoma punktami stałymi, rozważamy te przypadki, gdzie przestrzenie styczne w punktach stałych mają nieizomorficzne struktury $\mathbb{R}G$-modułów.

Pierwszym tematem naszych badań jest wykluczanie gładkich działań grup skończonych na sferach z jednym punktem stałym. Podajemy strategię wykluczania działań z jednym punktem stałym na sferach o zadanym wymiarze. Strategia ta polega na wykorzystaniu własności homologicznych danych dotyczących punktów stałych oraz użycia teorii przecięć. Podajemy nowe algebraiczne warunki, wystarczające do wykluczania działań z jednym punktem stałym. Przedstawiamy algorytm, który, poprzez weryfikację odpowiednich wystarczających warunków (opisanych zarówno w tej pracy, jak i warunków uzyskanych wcześniej przez Morimoto i Tamurę [41] oraz Borowiecką i autora [5,6]), pozwala nam wykluczyć rozważane działania. Wspomniany algorytm, zaimplementowany w języku GAP [23], daje nowe wyniki wykluczające.

Praca dotyczy również działań z dwoma punktami stałymi na sferach, dla których struktury $\mathbb{R}G$-modułów na przestrzeniach stycznych w punktach stałych zdefiniowane za pomocą różniczkowania działania nie są ze sobą izomorficzne. Pytanie dotyczące takich działań zostało zadane przez Smitha [65], który zapytał, czy dla grupy skończonej G działającej w sposób gładki na sferze z dokładnie dwoma punktami stałymi, struktury $\mathbb{R}G$-modułów zaindukowane na przestrzeniach stycznych w punktach stałych są izomorficzne. Hipoteza Laitinena [29] sugeruje negatywną odpowiedź na pytanie Smitha dla grup spełniających określone warunki algebraiczne. Chociaż wspomniana hipoteza nie jest prawdziwa w pełnej ogólności, zachodzi ona jednak dla szeregu grup skończonych. Hipoteza Laitinena pozostaje nierozstrzygnięta dla różnych rodzin grup. Naszym głównym wynikiem tej części rozprawy jest wskazanie nowej nieskończonej rodziny grup skończonych, dla których zachodzi hipoteza Laitinena.

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Chapter 1

Introduction

All groups occurring in the thesis are assumed to be finite. Since we deal mostly with smooth actions, without mentioning it explicitly, we assume that manifolds and group actions are smooth. If smoothness is not required, we note this explicitly. Throughout, unless stated otherwise, by an *R-homology sphere* (R is the coefficient ring), we mean any closed manifold M with the R-homology of a sphere. If $R = \mathbb{Z}$, we skip the coefficient ring and say briefly that M is a *homology sphere*. Analogously, a *homotopy sphere* is a closed manifold homotopy equivalent to a sphere.

Let G be a group acting on a manifold M. We are interested in the cases when the fixed point data of the action of G on M reveals some atypical properties. Our goal is to follow the spirit of the following question.

Assume G can act on a manifold M with specific properties, and with fixed point data satisfying some exotic conditions. What are the dimensions of manifolds M admitting actions of G, which satisfy the exotic conditions?

In this work, we deal with answering the question above for one fixed point actions on spheres. Apart from that, we consider also two fixed point actions of groups on spheres with distinct local behaviour around the two fixed points. In this case, we focus mostly on the existence of such actions, instead of indicating the dimensions of the spheres in question.

The groups admitting one fixed point actions on spheres are already characterized. The characterization is a combined work of several mathematicians through years 1977 − 1998, including Stein, Petrie, Laitinen, Morimoto, Oliver, and Pawałowski. Groups admitting one fixed point actions on spheres are known to be Oliver groups. The notion of Oliver group (coined by Laitinen and Morimoto) is recalled in [28]. According to Oliver [44], a finite group G has a smooth fixed point free action on a disk if and only if G is an Oliver group. Similarly, by the work of Laitinen and Morimoto [28], a finite group G has a smooth one fixed point action on a sphere if and only if G is an Oliver group. This allows us to use each of the group action property of G as the definition of an Oliver group.

The research on finding the dimensions of spheres which admit one fixed point actions of Oliver groups is getting more and more advanced lately. One should mention the remarkable result which is a combined work of Morimoto [35], Furuta [22], Buchdahl, Kwasik and Schultz [9]. This result states that if G acts with one fixed point on S^n, then $n \geq 6$. Due to the work of Bak, Katsushiro and Morimoto (see [3, 4, 42]), we can give the full classification of dimensions admitting such an exoticism for $G = A_5$, the alternating group on five letters. It turns out that A_5 admits one fixed point actions on S^n whenever $n \geq 6$. There is also a result of Borowiecka [5] which excludes effective one fixed point actions of G on S^8 for $G = \mathrm{SL}(2,5)$, the group of 2×2 matrices with entries in the field of 5 elements. The joint article of Borowiecka and the author [6] generalizes the methods used in [5] and applies this generalization for more Oliver groups and dimensions of spheres. In 2018, further exclusions were obtained by Morimoto and Tamura [41]. They showed that S_5, the symmetric group on five letters, does not admit a one fixed point actions on S^n whenever $n \in \{7, 8, 9, 13\}$. The analogous statement for $\mathrm{SL}(2,5)$ and $n \in \{6, 8, 9\}$ was also shown in [41]. In this thesis, we extend these exclusion results to increase their efficiency. We establish an algorithm for exclusion of one fixed point actions on spheres. This algorithm uses essentially our three new results from this thesis. Suppose Σ is a homology sphere with a group G acting on it and, for $H \leq G$, denote by $C(H)$ the connected component of the fixed point set Σ^H containing Σ^G in the case the fixed point set Σ^G is connected. Using these assumptions and notations, we can present the announced results below – the details are described in chapter 5.

Theorem 1.1. (cf. Theorem 5.3) *Suppose that H_1 and H_2 are non-Oliver subgroups of G which generate G and suppose P is a prime power order subgroup of $H_1 \cap H_2$. If there exists $x \in \Sigma^G$ with $\dim T_x(\Sigma^P) = 0$, then Σ^G is a two point set.*

Theorem 1.2. (cf. Theorem 5.8) *Assume Σ^G is connected. Suppose there exist subgroups $H_1, H_2 \leq G$ with $\langle H_1, H_2 \rangle = G$ such that the submanifold $C(H_i)$ is of positive dimension for $i = 1, 2$. Moreover, assume there is a p-subgroup $P \leq H_1 \cap H_2$ for some prime p such that*

$$\dim C(H_1) + \dim C(H_2) = \dim \Sigma^P$$

Suppose further that at least one of the following conditions hold.

(1) *P is of 2-power order.*

(2) *The orders of H_1 and H_2 are odd.*

(3) *P is normal in H_1 and H_2, and the orders of H_1/P and H_2/P are odd.*

Then Σ^G cannot consist of a single point.

Theorem 1.3. (cf. Theorem 5.10) *Assumme Σ^G decomposes into the connected components $C_1, ..., C_k$. Suppose there exist subgroups $H_1, H_2 \leq G$ with $\langle H_1, H_2 \rangle = G$ such that for any connected component C_j, $j = 1, ..., k$, the submanifold $C_j(H_i)$ which is the connected component of Σ^{H_i} containing C_j, is of positive dimension for $i = 1, 2$. Moreover, assume there is a p-subgroup $P \leq H_1 \cap H_2$ for some prime p such that for any $j = 1, ..., k$,*

$$\dim C_j(H_1) + \dim C_j(H_2) = \dim \Sigma^P.$$

Suppose further that at least one of the following conditions hold.

(1) *P is of order which is a power of 2.*

(2) *The orders of H_1 and H_2 are odd.*

(3) *P is normal in H_1 and H_2, and the orders of H_1/P and H_2/P are odd.*

Then Σ^G cannot consist of odd number of points.

In chapter 5, also, using the GAP software [23], we present new results obtained from the application of the exclusion algorithm. As a corollary from these exclusion results we can formulate the following.

Theorem 1.4. (cf. Theorem 5.29) *An Oliver group G cannot act on S^n with exactly one fixed point, provided:*

- $n = 6$ *and* $G \in \{C_3 \times S_4, C_3 \rtimes S_4, S_3 \times A_4, C_3 \rtimes F_7\}$,
- $n = 7$ *and* $G \in \{C_3 \rtimes S_4, S_3 \times A_4, S_5, C_3 \rtimes F_7\}$,
- $n = 8$ *and* $G \in \{C_3 \times S_4, S_3 \times A_4, C_3 \rtimes S_4, S_5, C_3 \rtimes F_7\}$,
- $n = 9$ *and* $G \in \{S_5, C_3 \rtimes F_7\}$,
- $n = 10$ *and* $G \in \{C_3 \rtimes S_4, S_3 \times A_4\}$.

The question concerning two fixed point actions on spheres mentioned at the beginning was firstly posed by Smith in 1960 [65, pp. 406, the footnote]. He asked whether for a group G acting on a sphere with exactly two fixed points, the $\mathbb{R}G$-module structures at the tangent spaces at the two fixed points (defined by the differentiation of the action) are always isomorphic. The Smith question can be exhaustively answered once we know the so called *Smith set* of G, defined for actions on (homotopy) spheres. Then, the answer to the Smith question for G depends only on the triviality of $\mathrm{Sm}(G)$, the Smith set of G – this answer is affirmative if and only if $\mathrm{Sm}(G) = 0$. A lot of interesting results concerning the structure of the Smith sets were obtained during the last 10 years. In favorable cases, we can show that the list of dimensions of spheres admitting exotic actions in the Smith question is finite. Our main result on the Smith question concerns a conjecture of Laitinen which predicts circumstances under which the answer to the Smith question is negative. This conjecture remains still unsettled for the so called *special Oliver groups*. We indicate a new infinite family of special Oliver groups for which this conjecture holds. The groups (denoted in this thesis by $G_{p,q}$) are certain semidirect products of cyclic groups of order q and dihedral groups of order $2pq$ for certain primes p and q. The main result can be stated as follows (for the precise statement of the Laitinen Conjecture, as well, as for the concept of being $\mathcal{P}$-matched and Smith equivalent, and the definition of the Laitinen number, $\lambda(G)$, we refer the reader to chapter 7).

Theorem 1.5. (cf. Theorem 7.3) *For any two odd primes p and q such that $q|(p-1)$, $G_{p,q}$ is a special Oliver group with $\lambda(G_{p,q}) \geq 2$, possessing pairs of non-isomorphic $\mathcal{P}$-matched Smith equivalent $\mathbb{R}G_{p,q}$-modules.*

The contents is organized as follows. In the first part, we setup the necessary theoretical background for understanding the most important results presented here. In the second chapter (the first one is the Introduction), we present the exposition of representation and intersection theories needed for our purposes. The first section is a collection of notions from representation theory. We cite here the result which states, for a given group G, that FG-modules are determined by their characters if F is the field of characteristic zero. Also, we indicate how to read off the characters of real irreducible representations from complex ones. Further, we cover the necessary ideas from intersection theory. This includes intersection product and intersection number. We conclude this section with a helpful relation between these two ideas. In the third chapter, we recall concepts concerning group actions on manifolds and point out several useful properties of these actions. The first section contains basic notions on this topic. We recall that fixed point sets of group actions on manifolds are their submanifolds. Next, we look closer at tangent spaces at fixed points. Using the actions by differentiation, the spaces can be equipped with group module structures, linking the world of group actions on manifolds with representation theory. We state the Slice Theorem [2] which asserts in particular that, if G acts on M with fixed point $p \in M$, then there exists a G-invariant open neighborhood of p which is G-equivariantly diffeomorphic to T_pM. Then, we look at linear actions on disks and spheres. We recall their properties in the context of atypical actions studied here. In a separate section, we recall the result of Smith concerning homological structure of fixed point sets for p-groups. The last section of the third chapter introduces the Riemannian structure on manifolds and uses it to prove that if a G-action on a manifold M with a fixed point p is effective, then the G-action on T_pM determined by its $\mathbb{R}G$-module structure, becomes effective as well.

The second part of the thesis deals with studying the dimensions of spheres admitting one fixed point actions of groups. The first chapter is a survey on groups admitting one fixed actions on spheres. In the first section, we present an algebraic characterization of Oliver groups and show examples of classes of groups which have this property. Next, we say some words about the history of answering of the question which groups admit one fixed point actions on spheres. We mention here the final result on this problem that a group G admits such an action if and only if G is an Oliver group. In the second part of this section, we give an overview of the results on dimensions of spheres admitting such exoticism for several Oliver groups. In the next chapter, we present a strategy of excluding of one fixed point actions on spheres. We present here an algorithm for excluding of one fixed point actions for a given Oliver group G and dimension $n \geq 0$. The algorithm uses three types of conditions sufficient for nonexistence of one fixed point actions on spheres. The first one comes from the study of the Euler characteristic of fixed point sets for certain subgroups. This restriction becomes effective, once we assume that particular fixed point sets are finite. In the next section, we formulate the second restriction condition. It utilizes the link between intersection form and the intersection number mentioned in the first part of the thesis. The third restriction is the generalization of results obtained by Morimoto and Tamura [41] for the cases of the symmetric and alternating groups on five letters, and the group $\mathrm{SL}(2,5)$. This restriction is essentially based on examination of subgroups of index two. In the another section, we focus on the additional restriction, once we assume that the action is effective. We use here the result that, in such a case, the tangent space at the fixed point is a faithful group module. We collect the restrictions to derive an algorithm for excluding of one fixed point actions on spheres. Finally, in a separate section, we present new results. By applying of the already mentioned exclusion algorithm, we show how to exclude one fixed point actions of Oliver groups on spheres of certain dimensions.

The last part of this thesis concerns the Smith question. In the first chapter, we provide the reader with an overview of the most important results concerning answering this question. We mention here classes of groups for which the answer to the Smith question is affirmative (respectively negative). We give here also definitions of algebraic concepts related to Smith sets. These ideas include, among others, the primary group and the reduced primary group. In favorable cases of groups, the reduced primary group turns out to be the subset of the Smith set, or even more, turns out to be equal to the Smith set. This allows us to establish lower bounds for the dimensions of spheres admitting exotic actions, as described in the Smith question. On the other hand, if we take for example $G = C_{2^n}$, the cyclic group of order 2^n (which is a group providing a negative to the Smith question), then the result of Bredon [7, Theorem II] states that there exists an integer $\mathrm{Br}(n) \geq 0$ such that for all $k \geq \mathrm{Br}(n)$, there does not exist an exotic action on S^k, as in the Smith question. This shows how the discussed answers may differ and that they depend strongly on the acting group structure itself. The last chapter contains the main results of this part of the thesis. We present here our latest result on the Laitinen Conjecture which predicts negative answer to the Smith question for Oliver groups satisfying certain algebraic properties. We indicate a new family of Oliver groups satisfying this conjecture (cf. Theorem 1.5 and Theorem 7.3). We also refer the reader to our article containing these results [32].

Part I

The Framework

Chapter 2

Preliminaries

In this chapter we collect results on representation theory of groups and intersection theory. We describe how to determine the real irreducible representations from complex ones. We remind the formula for fixed point dimensions and the characters of the induced representations. The main result from intersection theory mentioned here is the equality between the intersection number of two submanifolds (which is defined in a geometric manner) and the intersection product of cohomology classes determined by such submanifolds.

2.1 Representation theory

Here we recall some classical results from representation theory which will be useful for our purposes. In the first part, we present how characters determine representations for fields of characteristic zero. Then, we show, how to read off real irreducible group modules from complex ones.

Throughout this section, unless stated otherwise, let G be a group and F a field.

2.1.1 Characters over fields of characteristic zero

A very useful invariant of FG-modules constitute their characters. We recall here the proof of the theorem, that if F is of characteristic zero, then the characters determine FG-modules up to isomorphism. The proof of this fact can be found in classical books from representation theory, see for example [15]. Nevertheless, we present it here as well.

Let us remind the necessary facts for the proof of the theorem. First of them is the theorem of Maschke, see [27, 8.1. Theorem] – the version presented there assumes $F = \mathbb{R}, \mathbb{C}$ but the theorem works in a slightly more general version for fields of characteristic not dividing the group order.

Theorem 2.1. *Assume F is of characteristic not dividing $|G|$. Let V be an FG-module and U its submodule. Then, there exists a submodule $W \leq V$ such that $V = U \oplus W$.*

The proof remains the same, since the only point where the assumption about F comes into play is the fact that $\frac{1}{|G|}\sum_{g\in G} u = u$ for any $u \in U$ (if we didn't assume this, then char(F) could divide $|G|$ and, in such a case the sum on the left hand side would cancel out). In particular, we get the following corollary from the theorem of Maschke,

Corollary 2.2. *Let F be a field of characteristic zero and let V be an FG-module. Then, we can express V as the direct sum of irreducible FG-modules in the following way, which is unique up to isomorphism of FG-modules,*

$$V \cong a_1 V_1 \oplus ... \oplus a_k V_k,$$

where $V_1, ..., V_k$ are irreducible FG-modules, $a_1, ..., a_k$ are non-negative integers and mW denotes the m-fold direct sum of an FG-module W.

Now, let us denote by $g_1, ..., g_k \in G$ the representatives of all the distinct conjugacy classes of G. Let

$$\chi_1 = (\chi_1(g_1), ..., \chi_1(g_k)), ..., \chi_k = (\chi_k(g_1), ..., \chi_k(g_k)),$$

that is $\chi_1, ..., \chi_k$ are the vectors given by the characters of irreducible representations of G over F (we identify the characters with the vector of their values on conjugacy classes). Then, the folllowing holds.

Theorem 2.3. [15, (30.12) Theorem] *If* $\mathrm{char}(F) = 0$*, then the vectors $\chi_1, ..., \chi_k$ are linearly independent in F^k.*

As a corollary from the above theorem, we get the desired result that characters determine FG-modules. It is obvious from the definitions of characters that they are additive with respect to direct sums.

Corollary 2.4. [15, (30.14) Corollary] *Let U and V be two FG-modules, where F has characteristic zero. Denote by χ_U and χ_V the characters of U and V respectively. Then*

$$U \cong V \Leftrightarrow \chi_U = \chi_V.$$

The assumption concerning the characteristic of F cannot be omitted. It can be illustrated with the following example.

Example 2.5. Let $F = \mathbb{F}_p = \langle a \rangle$, the field on p letters for some prime p. Consider the representations ρ and τ given by $\rho(a) = I(mp)$ and $\tau(a) = I(np)$, where $m \neq n \in \mathbb{Z}_+$ and $I(k)$ denotes the $k \times k$ identity matrix. Obviously, ρ and τ are not equivalent and define thus non-isomorphic FG-modules. On the other hand, their traces are equal to zero, since we are in characteristic p.

2.1.2 Determining real irreducible representations from complex ones

By the previous subsection, we know that, for F with $\mathrm{char}(F) = 0$, FG-modules can be determined up to isomorphism by their characters. Therefore, instead of describing explicitly of irreducible real representations of G, it is sufficient for us to consider their characters. Here we recall, how to compute the characters of irreducible $\mathbb{R}G$-modules, once the irreducible $\mathbb{C}G$-modules are known. This is of great importance for us, since we have to work just with $\mathbb{R}G$-modules.

Let V be an irreducible $\mathbb{C}G$-module with character χ. There are three mutually exclusive possibilities (see [61, Proposition 38]).

(1) There is no nonzero invariant bilinear form on V. In this case $\chi(g)$ is not real for some $g \in G$. Moreover, the character $2\,\mathrm{Re}(\chi) = \chi + \overline{\chi}$ is the character of an irreducible $\mathbb{R}G$-module.

(2) There is a nonzero symmetric invariant bilinear form on V. Then χ is *realizable* over $\mathbb{R}$, that is, the $\mathbb{R}\mathbb{G}$-module V is isomorphic (as $\mathbb{C}G$-module) to some $\mathbb{R}G$-module.

(3) There exists a nonzero skew-symmetric invariant bilinear form on V – in such a case, χ is real but not realizable over $\mathbb{R}$ and 2χ is the character of some irreducible $\mathbb{R}G$-module.

Furthermore, every irreducible $\mathbb{R}G$-module can be obtained in one of the three ways described above, see [61, pp. 108].

It turns out that for the determination of irreducible $\mathbb{R}G$-modules, the concept of *Frobenius-Schur indicator* is very useful. It is defined below.

Definition 2.6. [61] *The **Frobenius-Schur indicator** of a $\mathbb{C}G$-module V with character χ is defined to be*

$$\iota(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g^2).$$

For irreducible $\mathbb{C}G$-modules, the Frobenius-Schur indicator can take one of the three values: 0, 1 or -1, see [61, pp. 109, Proposition 39]. The following theorem allows us to deduce existence of appropriate bilinear forms on irreducible $\mathbb{C}G$-modules with the help of the Frobenius-Schur indicator.

Theorem 2.7. [27, 23.16 Theorem] *Let V be an irreducible $\mathbb{C}G$-module with character χ. Then, the following statements hold.*

(1) $\iota(\chi) = 0$ *if and only if there is no nonzero invariant bilinear form on V,*

(2) $\iota(\chi) = 1$ *if and only if there exists a nonzero symmetric bilinear form on V,*

(3) $\iota(\chi) = -1$ *if and only if there is a nonzero skew-symmetric bilinear form on V.*

Thus, we may determine characters of irreducible $\mathbb{R}G$-modules from complex ones as follows. We look at all irreducible complex characters χ and compute their Frobenius-Schur indicators $\iota(\chi)$. These indicators contain information about which one of the characters from $\chi, 2\,\mathrm{Re}(\chi)$ is a character of an irreducible $\mathbb{R}G$-module. The set of so obtained characters (we omit repetitions – the characters of the form $2\,\mathrm{Re}(\chi)$ appear twice – once for the irreducible complex character χ and second for $\overline{\chi}$ which is irreducible as well) is the complete list of characters of real irreducible representations.

2.1.3 Three useful formulas

This subsection is a collection of results which have much utility for us and they involve characters of group representations. We recall here the formula for the dimension of fixed point sets for actions of subgroups of G on $\mathbb{R}G$-modules V. We mention also the easy to check character criterion for a module to be faithful. The last theorem we quote here concerns computation of the induced characters.

Let V be an $\mathbb{R}G$-module and $H \leq G$ be a subgroup of G. We would like to compute the dimension of the fixed point subspace V^H. Since V can be considered in a natural way as a $\mathbb{C}G$-module, the real dimension of V^H considered as a vector space over $\mathbb{R}$ is equal to the complex dimension of V^H considered as a vector space over $\mathbb{C}$. The latter is given by the following.

Theorem 2.8. [15,18] *The dimension of the fixed point set of the action of H on a $\mathbb{C}G$-module V is given by the formula*

$$\dim V^H = \frac{1}{|H|}\sum_{h\in H}\chi_V(h),$$

where χ_V denotes the character of V.

The next result gives the necessary and sufficient condition for a $\mathbb{C}G$-module to be faithful. Obviously, this applies for $\mathbb{R}G$-modules as well, since we can consider them as $\mathbb{C}G$-modules too.

Theorem 2.9. [27, 13.11 Theorem] *Let V be a $\mathbb{C}G$-module with character χ. Then V is faithful if and only if the only element $g \in G$ with $\chi(g) = \chi(1)$ is $g = 1$.*

Since we will be considering induced representations, let us introduce the definition of induced modules. Suppose $H \leq G$ is a subgroup of a group G. Let F be a field and V an FH-module. Having an FH-module structure on a vector space V over F is the same as defining a linear action of H on V. We define a linear action of G on a vector space W which is a $[G : H]$-fold cartesian product of V in the following way. Assume $a_1, ..., a_k$ are representatives of the left cosets of H in G. Take $g \in G$ and $w = (v_1, ..., v_k) \in V^k = V^{[G:H]}$. Then, for any $i = 1, ..., k$, $ga_i = a_{\sigma(i)}h_i$ for uniquely determined $h_i \in H$ and a permutation $\sigma \in S_k$. The action of the element $g \in G$ on $w \in W$ is defined as follows.

$$gw = g(v_1, ..., v_k) = (h_{\sigma^{-1}(1)}v_{\sigma^{-1}(1)}, ..., h_{\sigma^{-1}(k)}v_{\sigma^{-1}(k)}).$$

The vector space W together with this linear action of G constitute the ***induced FG-module*** of V from H to G. We donote this FG-module by $\mathrm{Ind}_H^G(V)$. If χ is the character of V (in case $F = \mathbb{R}, \mathbb{C}$), then we denote by $\mathrm{Ind}_H^G(\chi)$ the character of $\mathrm{Ind}_H^G(V)$.

Below we cite the theorem on how to compute the characters from the ones we induce from.

Theorem 2.10. [27, 21.23 Proposition] *Let $H \leq G$ and χ be a character of a $\mathbb{C}H$-module V. Suppose $g \in G$ and denote by (g) its conjugacy class. Then, we have two possibilities.*

(1) *If $H \cap (g) = \emptyset$, then $\mathrm{Ind}_H^G(\chi)(g) = 0$.*

(2) *If $H \cap (g) \neq \emptyset$, then*

$$\mathrm{Ind}_H^G(\chi)(g) = |C_G(g)|\Big(\frac{\chi(h_1)}{|C_H(h_1)|} + \ldots + \frac{\chi(h_m)}{|C_H(h_m)|}\Big),$$

where $C_K(x)$ denotes the centralizer in K of x and $h_1, \ldots, h_m$ are the representatives of all the distinct conjugacy classes in H of the elements of the set $H \cap (g)$.

2.2 Intersection theory

In this section, we recall the concept of an intersection number of two submanifolds and its relationship to their (co)homological properties (this relationship involves the Kronecker pairing). As such, the intersection number turns out to be a homotopy invariant − homotopic configurations yield equal intersection numbers. In the first part we introduce the setup. Next, we focus on the case when the manifolds are oriented and then we formulate the relationship for the general case. We illustrate the theory with explicit computations.

2.2.1 The setup

We proceed in the spirit of [16]. Manifolds to be considered may have boundary and are assumed to be connected and compact unless stated otherwise.

Assume M is a manifold of dimension m and $A, B \subseteq M$ its submanifolds – B without boundary and A possibly with boundary contained in the boundary of M. Let a, b be the dimensions of A and B respectively and suppose that A and B are of complementary dimensions, that is, $a+b=m$. Moreover, assume that A and B are transverse in M, i.e. for any point $x \in A \cap B$, we have

$$\operatorname{span}(\{T_xA, T_xB\}) \cong T_xM.$$

2.2.2 The oriented case

Assume that A, B and M are oriented. Let us define the intersection number of A and B in M.

Take any $x \in A \cap B$. Since A, B and M are oriented, this means that the equivalence classes of bases of T_xA, T_xB and T_xM are chosen (two bases with the positive determinant of the change-of-base matrix between them are called equivalent). Let $\mathcal{B}_{x,A}$, $\mathcal{B}_{x,B}$ and $\mathcal{B}_{x,M}$ be representatives of these equivalence classes of bases of T_xA, T_xB and T_xM respectively. Denote by $\kappa(x)$ the sign of the change-of-base matrix from the ordered basis $\{\mathcal{B}_{x,A}, \mathcal{B}_{x,B}\}$ to $\mathcal{B}_{x,M}$.

Definition 2.11. *We define the* ***oriented intersection number*** *of A and B in M as*

$$\overline{A \cdot B} = \sum_{x \in A \cap B} \kappa(x).$$

Since A and B are oriented, we have the fundamental classes $\overline{[A, \partial A]} \in H_a(A, \partial A)$ and $\overline{[B]} \in H_b(B)$. They induce elements $\overline{[A, \partial A]}_M = (i_A)_*(\overline{[A, \partial A]}) \in H_a(M, \partial M)$ and $\overline{[B]}_M = (i_B)_*(\overline{[B]}) \in H_b(M)$, where $i_A \colon (A, \partial A) \subseteq (M, \partial M)$ and $i_B \colon B \subseteq M$ are the inclusions. Denote by $\overline{\alpha} \in H^b(M)$ and $\overline{\beta} \in H^a(M, \partial M)$ the Poincaré duals to $\overline{[A, \partial A]}_M$ and $\overline{[B]}_M$ respectively.

Theorem 2.12. [16, Theorem 10.32] *The oriented intersection number of A and B in M can be expressed via the Kronecker pairing in the following way,*

$$\overline{A \cdot B} = \langle \overline{\alpha} \cup \overline{\beta}, \overline{[M, \partial M]} \rangle,$$

where $\overline{[M, \partial M]} \in H_m(M, \partial M)$ is the fundamental class of M.

Remark 2.13. For any $k = 0, ..., m$ the bilinear form

$$\overline{\lambda} \colon H^k(M) \times H^{m-k}(M, \partial M) \xrightarrow{\cup} H^m(M, \partial M) \cong \mathbb{Z}$$

determined by the cup product is called the ***oriented intersection product*** *on M.*

2.2.3 The general case

Let us now turn to the general case, when we do not know whether A, B and M are orientable. We would like to obtain a similar relation to that from Theorem 2.12. It is possible to get an analogous correspondence if one considers homology and cohomology groups with coefficients in $\mathbb{Z}_2$.

Definition 2.14. *We define* ***unoriented intersection number*** *of A and B in M as*

$$A \cdot B = |A \cap B| \pmod 2.$$

Analogously as in the oriented case, we can consider the fundamental classes $[A, \partial A] \in H_a(A, \partial A; \mathbb{Z}_2)$ and $[B] \in H_b(B; \mathbb{Z}_2)$. These classes induce elements $[A, \partial A]_M = (i_A)_*([A, \partial A]) \in H_a(M, \partial M; \mathbb{Z}_2)$ and $[B]_M = (i_B)_*([B]) \in H_b(M; \mathbb{Z}_2)$. Let $\alpha \in H^b(M; \mathbb{Z}_2)$ and $\beta \in H^a(M, \partial M; \mathbb{Z}_2)$ be the Poincaré duals to $[A, \partial A]_M$ and $[B]_M$ respectively. The link between unoriented intersection number and (co)homological properties of A, B and M can be stated in the following way.

Theorem 2.15. [16, Exercise 180]

$$A \cdot B = \langle \alpha \cup \beta, [M, \partial M] \rangle$$

where $[M, \partial M] \in H_m(M, \partial M; \mathbb{Z}_2)$ denotes the fundamental class of M and $\langle \cdot, \cdot \rangle$ denotes the Kronecker pairing of cohomology and homology with coefficients in $\mathbb{Z}_2$.

Remark 2.16. Analogously as in Remark 2.13, we have for any $k = 0, ..., m$ the bilinear form

$$\lambda \colon H^k(M; \mathbb{Z}_2) \times H^{m-k}(M, \partial M; \mathbb{Z}_2) \xrightarrow{\cup} H^m(M, \partial M; \mathbb{Z}_2) \cong \mathbb{Z}_2.$$

We call the form above the ***unoriented intersection product*** *on M.*

The rest of this subsection is devoted to the proof of Theorem 2.15. The proof borrows substantially from the proof of Theorem 2.12 which can be found in [16].

We will need the Thom isomorphism theorem.

Theorem 2.17. [16, Theorem 10.28] *Let $n \geq 1$ and*

$$\mathbb{R}^n \hookrightarrow E \to B$$

be a real vector bundle with $E_0 \subset E$ being the complement of the zero section $B \to E$, $b \mapsto 0 \in E_b$, where E_b denotes the fiber over $b \in B$.

Then, there is a unique class $u \in H^n(E, E_0; \mathbb{Z}_2)$ such that the inclusion of pairs, $i \colon (E_b, (E_0)_b) \hookrightarrow (E, E_0)$ ($(E_0)_b$ denotes the fiber over b of the fiber bundle $E_0 \to B$), induces a map

$$H^n(E, E_0; \mathbb{Z}_2) \to H^n(E_b, (E_0)_b; \mathbb{Z}_2) \cong \mathbb{Z}_2$$

which takes u to the unique non-zero element. Moreover, u has the property that the cup product map

$$H^m(B; \mathbb{Z}_2) \to H^{n+m}(E, E_0; \mathbb{Z}_2), \quad a \mapsto a \cup u$$

is an isomorphism for all $m \geq 0$.

If we assume that $E \to B$ is orientable, then we can say even more. Namely, there exists a unique (up to sign) class $\overline{u} \in H^k(E, E_0; \mathbb{Z})$ such that the inclusion i induces the homomorphism

$$H^n(E, E_0; \mathbb{Z}) \to H^n(E_b, (E_0)_b; \mathbb{Z}) \cong \mathbb{Z}$$

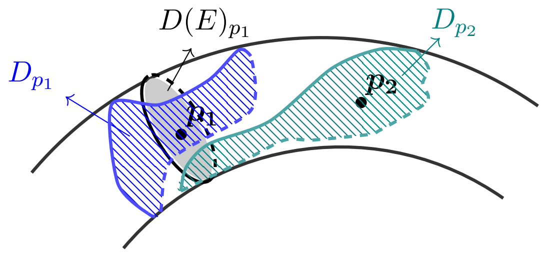
which takes $\overline{u}$ to the generator. Moreover, $\overline{u}$ has the property that the cup product map

$$H^m(B; \mathbb{Z}) \to H^{n+m}(E, E_0; \mathbb{Z}), \quad a \mapsto a \cup \overline{u}$$

is an isomorphism for all $m \geq 0$.

Proof of Theorem 2.15. Since all homology and cohomology groups which appear in the proof have coefficients in $\mathbb{Z}_2$, we omit the coefficient group.

Let $E \to B$ be the normal bundle to B in M. We know by the tubular neihgbourhood theorem [24] that E can be embedded as a neighbourhood of B in M, where B itself corresponds to the zero section. Give E a metric. Using the tubular neighbourhood theorem [24] and the fact that A and B are transverse in M, we can find $\varepsilon > 0$ small enough such that the disk bundle $D(E)$ of radius ε intersects A in precisely $|A \cap B|$ disks D_p, one for each $p \in A \cap B$. Each D_p is isotopic in $D(E)$ to the fiber $D(E)_p$. Making ε small enough if necessary, we can find an isotopy from $\sqcup_{p \in A \cap B} D_p$ to $\sqcup_{p \in A \cap B} D(E)_p$.



In this case two isotopies, of D_{p_1} to $D(E)_{p_1}$ and of D_{p_2} to $D(E)_{p_2}$ interfere – we have to shrink $D(E)$.

Let us remind the isotopy extension theorem.

Theorem 2.18. [24, p. 180, 1.3. Theorem] *Let $X \subseteq Y$ be a compact submanifold of a manifold Y and $H \colon X \times I \to Y$ an isotopy (i.e. H restricted to $\{X\} \times \{t\}$ is an embedding for any $t \in I$). If $H(X \times I) \subseteq Y \setminus \partial Y$, then H extends to a diffeotopy $Y \times I \to Y$ (i.e. isotopy being diffeomorphism when restricted to $\{Y\} \times \{t\}$ for any $t \in I$).*

Using this theorem, we can extend the isotopy from $\sqcup_{p \in A \cap B} D_p$ to $\sqcup_{p \in A \cap B} D(E)_p$ to a diffeotopy $M \times I \to M$. Notice that all the operations we performed to that moment did not affect the Kronecker pairing $\langle \alpha \cup \beta, [M, \partial M] \rangle$. Moreover, these perturbations did not change $A \cdot B$ as well. Therefore, we may assume that A intersects $D(E)$ precisely in the union of fibers over intersection points $p \in A \cap B$,

$$A \cap D(E) = \sqcup_{p \in A \cap B} D(E)_p.$$

Put $D = D(E)$ from now on. By the Thom isomorphism theorem 2.17, we can find a unique Thom class $u \in H^a(D, \partial D)$ such that

$$\cup u \colon H^k(B) \to H^{k+a}(D, \partial D)$$

is an isomorphism for all k so that the restriction of u to the fiber D_p satisfies $u|_{D_p} = [D_p, \partial D_p]^*$, where $[D_p, \partial D_p]^*$ is the Poincaré dual in the cohomology of $(D_p, \partial D_p)$ to the unique non-zero element $[D_p, \partial D_p] \in H_a(D_p, \partial D_p) \cong \mathbb{Z}_2$, that is

$$\langle u|_{D_p}, [D_p, \partial D_p] \rangle = 1.$$

Let $[B]^*_D \in H^a(D)$ be the Poincaré dual (in D) to $[B]_D = i_*([B]) \in H_b(D)$ ($i \colon B \subseteq D$ denotes the inclusion). From 2.2.3 we conclude that $[B]^*_D \cup u$ generates $H^m(D, \partial D)$, so $[B]^*_D \cup u = [D, \partial D]^*$.

Using the properties of $\cup$ and $\cap$-products and Kronecker pairing, we have

$$\begin{aligned} 1 &= \langle [D,\partial D]^*, [D,\partial D]\rangle = \langle [B]^*_D \cup u, [D,\partial D]\rangle \\ &= ([B]^*_D \cup u) \cap [D,\partial D] = [B]^*_D \cap (u \cap [D,\partial D]) \\ &= \langle [B]^*_D, u \cap [D,\partial D]\rangle, \end{aligned}$$

so $u \cap [D,\partial D] = [B]_D \in H_b(D)$.

Let us now prove the auxiliary

Proposition 2.19. *The inclusion* $i_1\colon (D,\partial D) \subseteq (M, M\setminus \operatorname{Int}(D))$ *induces isomorphisms in homology and cohomology.*

Proof. Let r be the radius of the disk bundle D of E. Take $r' > r$ such that the disk bundle D' of E with radius r' does not have any self-intersection points in M. Hence, the inclusion $D' \subseteq M$ is an embedding. Moreover, the inclusions $i\colon (D,\partial D) \subseteq (D',\partial D)$ and $\partial D \subseteq D'\setminus \operatorname{Int}(D)$ are homotopy equivalences. Therefore, the inclusion

$$i'\colon (D',\partial D) \subseteq (D', D'\setminus \operatorname{Int}(D))$$

induces isomorphisms in (co)homology. We show that the inclusion

$$i''\colon (D', D'\setminus \operatorname{Int}(D)) \subseteq (M, M\setminus \operatorname{Int}(D))$$

comes from an excision and thus induces isomorphisms in (co)homology. It would mean that i_1, being the composition $i''\circ i'\circ i$ induces such isomorphisms as well.

Set $X = M$, $A = M\setminus \operatorname{Int}(D)$ and $Z = M\setminus D'$. Then $D'\setminus \operatorname{Int}(D) = A\setminus Z$ and $D' = X\setminus Z$, the closure $\overline{Z} = M\setminus (D') \subseteq M\setminus D = \operatorname{Int}(A)$ and the inclusion $i''\colon (X\setminus Z, A\setminus Z) \subseteq (X,A)$ is an excision. Hence $(i'')_*\colon H_n(X\setminus Z, A\setminus Z) \to H_n(X,A)$ and $(i'')^*\colon H^n(X,A) \to H^n(X\setminus Z, A\setminus Z)$ are isomorphisms. ☐

It follows from the proposition above that the inclusion

$$i_1\colon (D,\partial D) \subseteq (M, M\setminus \operatorname{Int}(D))$$

induces isomorphisms in (co)homology. Hence, $H_n(M, M\setminus \operatorname{Int} D) \cong \mathbb{Z}_2$. Moreover, the inclusions $(M,\partial M) \hookrightarrow (M, M\setminus\{p\})$ and $(M, M\setminus \operatorname{Int} D) \hookrightarrow (M, M\setminus \{p\})$ induce isomorphisms in (co)homology. Thus, the inclusion

$$i_2\colon (M,\partial M) \subseteq (M, M\setminus \operatorname{Int}(D))$$

induces the isomorphism

$$(i_2)_*\colon H_n(M,\partial M) \to H_n(M, M\setminus \operatorname{Int}(D)).$$

Hence (since we do not have to bother with signs as the coefficient field is $\mathbb{Z}_2$),

$$(i_1)_*([D,\partial D]) = [M, M\setminus \operatorname{Int}(D)] = (i_2)_*([M,\partial M]).$$

Cap product is natural – in the sense that for any map of pairs $f\colon (X,C) \to (Y,D)$, the following diagram commutes

$$\begin{array}{ccc} H^k(X,C)\times H_m(X,C) & \xrightarrow{\ \cap\ } & H_{m-k}(X) \\ {\scriptstyle f^*}\big\uparrow \qquad {\scriptstyle f_*}\big\downarrow & & \big\downarrow{\scriptstyle f_*} \\ H^k(Y,D)\times H_m(Y,D) & \xrightarrow{\ \cap\ } & H_{m-k}(Y) \end{array}$$

(here the homomorphism $f_*\colon H_{m-k}(X) \to H_{m-k}(Y)$ is induced from the map f with forgetting the information about C and D, that is from the map $X \to Y$, $x \mapsto f(x)$). Fixing $c \in H_m(X,C)$, we get a commutative diagram

$$\begin{array}{ccc} H^k(X,C) & \xrightarrow{\ \cap c\ } & H_{m-k}(X) \\ {\scriptstyle f^*}\big\uparrow & & \big\downarrow{\scriptstyle f_*} \\ H^k(Y,D) & \xrightarrow{\ \cap f_*(c)\ } & H_{m-k}(Y). \end{array}$$

Hence, by the equality $\langle u|_{D_p}, [D_p, \partial D_p]\rangle = 1$, we get a commutative diagram

$$\begin{array}{ccc} H^k(D,\partial D) & \xrightarrow{\ \cap[D,\partial D]\ } & H_{m-k}(D) \\ {\scriptstyle i_1^*}\big\uparrow & & \big\downarrow{\scriptstyle (i_D)_*} \\ H^k(M, M\setminus \mathrm{Int}(D)) & \xrightarrow{\ \cap[M, M\setminus \mathrm{Int}(D)]\ } & H_{m-k}(M) \\ {\scriptstyle i_2^*}\big\downarrow & & \big\uparrow \mathrm{id} \\ H^k(M,\partial M) & \xrightarrow{\ \cap[M,\partial M]\ } & H_{m-k}(M), \end{array}$$

where $i_D\colon D \subseteq M$ is the inclusion. Let $j^* = i_2^* \circ (i_1^*)^{-1}\colon H^k(M,\partial M) \to H^k(D,\partial D)$. Take $x \in H^k(D,\partial D)$. It follows from the diagram above that

$$(i_D)_*(x \cap [D,\partial D]) = j^*(x) \cap [M,\partial M].$$

Hence, evaluating j^* on the Thom class, we o obtain

$$\begin{aligned} j^*(u) \cap [M,\partial M] &= (i_D)_*(u \cap [D,\partial D]) = \\ &= (i_D)_*([B]_D) = (i_B)_*([B]) \in H_b(M), \end{aligned}$$

where $i_B\colon B \subseteq M$ denotes the inclusion. Therefore, the Poincaré dual β to $(i_B)_*([B])$ in M is equal to $j^*(u)$.

We can think of j^* as being induced by the quotient map $j\colon M/\partial M \to D/\partial D$ (we use identifications $H^n(X,C) \cong H^n(X/C)$ and $H_n(X,C) \cong H_n(X/C)$, so we have a corresponding homomorphism $j_*\colon H_k(M,\partial M) \to H_k(D,\partial D)$. Recall that A is a submanifold of M of dimension a and the union of the disks D_p is the intersection of A with the disk bundle $D(E)$. If $[A,\partial A]_M \in H_a(M,\partial M)$ is the image of the fundamental class $[A,\partial A] \in H_a(A,\partial A)$ in $H_a(M,\partial M)$, then

$$j_*([A,\partial A]_M) = \sum_{p \in A\cap B} [D_p, \partial D_p].$$

Hence,

$$\begin{aligned}\langle \alpha \cup \beta, [M, \partial M]\rangle &= \langle \alpha \cup j^*(u), [M, \partial M]\rangle = (j^*(u) \cup \alpha) \cap [M, \partial M] \\ &= j^*(u) \cap (\alpha \cap [M, \partial M]) = j^*(u) \cap [A, \partial A]_M \\ &= \langle j^*(u), [A, \partial A]_M\rangle = \langle u, j_*([A, \partial A]_M)\rangle \\ &= \sum_{p \in A \cap B} \langle u, [D_p, \partial D_p]\rangle.\end{aligned}$$

The latter expression is equal, by the equality $\langle u|_{D_p}, [D_p, \partial D_p]\rangle = 1$, to the value

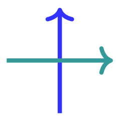
$$\sum_{p \in A \cap B} 1 = A \cdot B$$

and the proof is finished. □

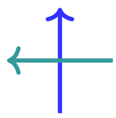
2.2.4 Examples of calculating of intersection numbers

This paragraph contains examples of calculations of intersection numbers. We will see, in particular, that every integer can be realized as the intersection number of some manifolds.

We use the notations from the previous subsections. In all examples to be considered the submanifolds are one-dimensional (either interval or circle). Let us mark the first submanifold (denoted by A) with blue color and the second one (denoted by B) with green color. When oriented intersection number is considered, we introduce the following two possible crossing types of such submanifolds.



type (1)



type (2)

The convention we use to calculate intersection numbers is that we add one when we encounter a crossing of type (1) and subtract one by crossings of type (2). We use the same symbols as introduced in subsections 2.2.2 and 2.2.3 (in particular, this concerns the Poincaré duals $\overline{\alpha}, \overline{\beta}$ and α, β). We can arrange orientations to be consistent with that convention in any of examples below.

Example 2.20. Take $M = D^2$, $A = D^1$ and $B = S^1$. Let A, B and M be arranged as follows.

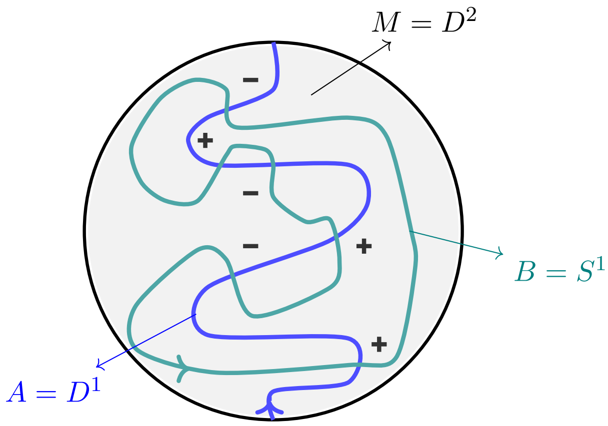


Figure 2.1: Calculating intersection number of D^1 and S^1 in D^2.

Then $\overline{A \cdot B} = 3 - 3 = 0$. In fact, there is no other possibility – the first cohomology groups, $H^1(D^2, \partial D^2) \cong H^1(S^2)$ and $H^1(D^2)$ are rivial, so the Poincaré duals, $\overline{\alpha}$ and $\overline{\beta}$ are zero and

$$\overline{A \cdot B} = \langle \overline{\alpha} \cup \overline{\beta}, [M, \partial M] \rangle = \langle 0 \cup 0, [M, \partial M] \rangle = 0$$

no matter how we choose the orientations.

Example 2.21. In this example we consider configurations of two circles in the torus with arbitrary intersection numbers.

To obtain intersection number 0, it just suffices to embed two non-intersecting copies of S^1 in the torus. Let us deal with the cases when intersection numbers are non-zero. Put $M = T^2$, the two-dimensional torus and $A = S^1$, $B = S^1$ be two circles embedded in it. We show how to embed A and B in M to obtain any integer n as their intersection number. Let us first consider the case when $n = 2$. Let A and B be embedded as in the following picture.

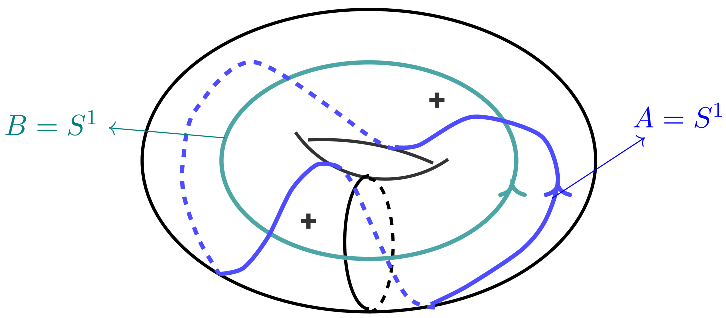


Figure 2.2: Calculating the intersection number of two circles in a torus (visualization).

In this situation the blue curve winded two times around the torus. This circle intersected twice with the green one in crossing of type (1) yielding the intersection number $n = 2$. Before we generalize this to any integer $n \neq 0$, let us look at the plane model of the situation from Figure 2.2.

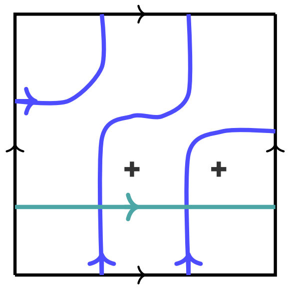


Figure 2.3: Calculating the intersection number of two circles in a torus (plane model).

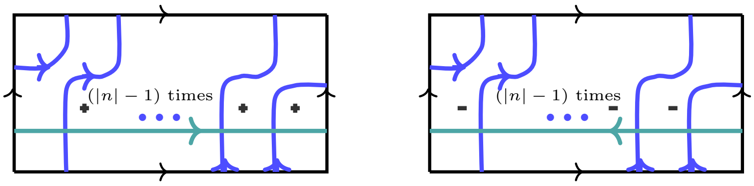


Figure 2.4: Obtaining non-zero integers as the intersection number of two circles in a torus.

We can generalize the case $n = 2$ by winding the blue curve $|n|$ times around the green and potentially changing the orientation on the green curve to obtain negative integer numbers.

Example 2.22. Let us see an example of the unoriented case. Take $M = \mathbb{R}P^2$, the real projective plane, and $A = S^1$ and $B = S^1$.

Since there is no choice for orientations in this case, we have have to consider the unoriented intersection number. If A and B are two non-intersecting circles in M, then their intersection number is 0. On the other hand, if A and B intersect in precisely one point, their intersection number is 1.

Chapter 3

Group actions on manifolds

This chapter contains a review of results on group actions on manifolds. We include here some classical theorems such as the tubular neighbourhood theorem in the equivariant setting and the Slice Theorem [2]. The chapter is concluded with the proof of the well-known fact that effective actions on manifolds induce faithful group module structures on tangent spaces at fixed points.

3.1 Basic properties

In this section we recall well-known properties of group actions on manifolds. The aim of the first paragraph is to show that the fixed point set has the structure of a manifold. In case of an isolated fixed point, using the Slice Theorem, we conclude that there is an equivariant diffeomorphism between a neighbourhood of the fixed point and the tangent space at this point. As a corollary, linear actions on spheres which have two isolated fixed points must have isomorphic representations on tangent spaces at the fixed points. Thus, the actions from the introduction, i.e. the actions with one fixed point on spheres and with two fixed points on spheres and non-equivalent group module structures at the tangent spaces at the fixed points, cannot be equivalent to linear actions. Therefore we justify the name exotic attributed to them.

3.1.1 Fixed point sets

Assume G acts on a manifold M (we denote it by $G \curvearrowright M$). Let us comment on the fact that M^G, the fixed point set of the action of G on M, is a submanifold of M. The main tool we use for this purpose is the following equivariant version of the tubular neighbourhood theorem.

Theorem 3.1. [8, 2.2. Theorem, p. 306] *If A is a closed invariant submanifold of M, then there exists a real G-vector bundle $E \to A$ and a G-equivariant diffeomorphism $f : E \to U$, where U is some open neighbourhood of A in M.*

Corollary 3.2. *The fixed point set M^G is a submanifold of M.*

To prove the corollary above, we apply Theorem 3.1 to the real vector bundle over a single point and then use the fact that the fixed point sets of group actions on vector spaces are their subspaces.

Let us introduce the following definition of the dimension of the fixed point set.

Definition 3.3. *The **dimension** of the fixed point set M^G (denoted by $\dim(M^G)$) is the maximum from the dimensions of connected components of M^G.*

If we assume further that M is compact, this will imply compactness of M^G. Indeed, if we take any $g \in G$, then the fixed point set $M^{\langle g \rangle}$ is closed (by our implicit assumption on the smoothness of the action). Since M^G is the intersection of such fixed point sets over all $g \in G$, we conclude that M^G is closed. As a closed subset of a compact space, it has to be compact as well.

3.1.2 Tangent spaces at fixed points

In this subsection we define the action on tangent spaces at fixed points induced from the action on the manifold. Since the action on the tangent space is given by the differential, it turns out to be a linear action and hence defines a real group module structure on the tangent space. Using the Slice Theorem, we can translate the local behaviour of actions on manifolds around fixed points to the induced actions on tangent spaces at these points.

Let G be a group acting on a manifold M, where each $g \in G$ acts by some diffeomorphism $\theta_g\colon M \to M$. Assume that $x \in M$ is fixed by all θ_g. In this case, the differentials $D(\theta_g)_x$ are linear automorphisms of T_xM, and we can define the operation $*$ on the tangent space as follows.

$$\begin{aligned} *\colon G \times T_xM &\longrightarrow T_xM \\ (g, v) &\mapsto g * v = D(\theta_g)_x(v). \end{aligned}$$

Since the differential is a linear map such that the differential of the composition is the composition of differentials, and the differential of the identity is the identity map on the tangent space, it follows $*$ defines a linear action of G on T_xM and thus an $\mathbb{R}G$-module structure on T_xM. With this, we can formulate the Slice Theorem now. It is a corollary from the equivariant tubular neighbourhood theorem (see Theorem 3.1) applied for the case when a G-invariant submanifold is a single point.

Theorem 3.4 (**Slice Theorem**). [2, Theorem I.2.1] *There exists a G-invariant neighbourhood U of x in M and a G-diffeomorphism $f\colon U \to T_xM$, where the action of G on T_xM is given by $*$.*

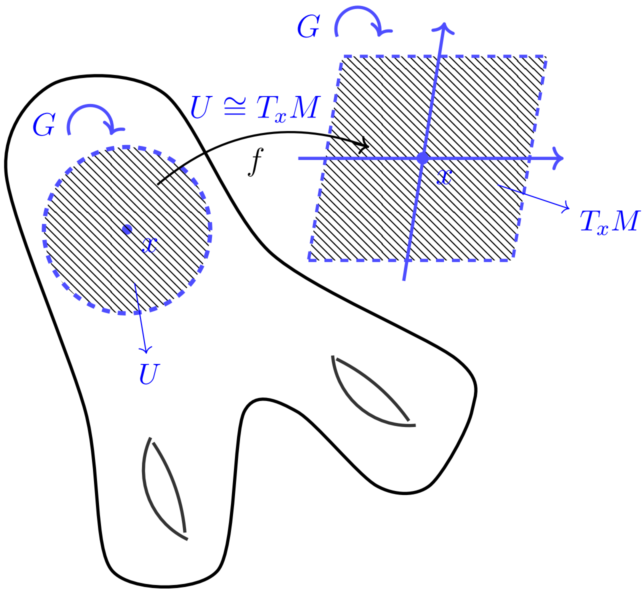


Figure 3.1: An illustration of the Slice Theorem. The equivariant diffeomorphism is given by the exponential map (this map is defined in section 3.3).

3.1.3 Linear actions on disks and spheres

Linear actions on disks and spheres behave in a more rigid manner than the general smooth actions. This rigidity prevents linear actions with one fixed point on spheres, fixed point free actions on disks and actions on spheres with exactly two fixed points and non-isomorphic tangent module structures at the fixed points. We explain this phenomenon in this subsection. Before we do this, we bring the definition of linear actions on disks and spheres and their properties.

Definition 3.5. *We say that a group G acts* ***linearly*** *on a disk D with $n = \dim D$ if there exists a real vector space V_D of dimension n with a linear action of G on this space such that $D = D(V_D)$ is the unit disk of V_D and this disk is the G-invariant subspace of V_D.*

Definition 3.6. *We say that a group G acts* ***linearly*** *on a sphere S with $n = \dim S$ if there exists a real vector space V_S of dimension $n+1$ with a linear action of G on this space such that $S = S(V_S)$ is the unit sphere of V_S and this sphere is the G-invariant subspace of V_S.*

In the definitions above, given a G-vector space V, $D(V)$ and $S(V)$ mean the representation disk and the representation sphere respectively. Moreover, both V_D and V_S are considered as G-manifolds with manifold structures given by the standard dot product.

Proposition 3.7. *Assume M is either a disk or a sphere. If G acts linearly on M with the fixed point set finite, then M^G consists of two points for M being a sphere and of a single point if M is a disk.*

Proof. Racall first the general property of fixed point sets of invariant subspaces.

Fact 3.8. *If X is a topological G-space and $A \subseteq X$ its invariant subspace, then $A^G = A \cap X^G$.*

Going back to the proof of Proposition 3.7, assume G acts linearly on M for some $n \geq 0$. We may identify M with its embedding into some real vector space V of dimension $m+1$ where

$m = \dim M$. It follows from Fact 3.8 the fixed point set M^G is the intersection of M with some vector subspace $W \leq V$ since the fixed point sets of linear actions on vector spaces are their subspaces. The Proposition 3.7 follows now because the intersections of spheres and disks embedded in real vector spaces are also spheres and disks respectively – if M^G is finite, then it has to consist of two points in case $M = S^n$ and of one point when $M = D^n$. □

We see therefore that linear actions on spheres with exactly one fixed point are not possible. This justifies the exoticism of such actions for the more general smooth case.

Let us now pay attention to linear actions on spheres with exactly two fixed points. We show that linearity forces the tangent modules at the fixed points to be isomorphic. Assume G acts linearly on $S = S^n$ for some $n \geq 0$. This means that $S = S(V)$ for some real vector space V endowed with a linear G-action. Note that $V = V_G \oplus V^G$, where V_G is the orthogonal complement to the fixed point subspace V^G. Suppose that G acts on S with two fixed points $x, y \in S \subseteq V$. It follows from Fact 3.8 that $\{x, y\} = S^G = S \cap V^G$. Since S is a sphere, the only possibility to obtain two point set as S^G is when V^G is a one dimensional subspace – it intersects then the sphere in two points. Hence $V = V_G \oplus \mathbf{1}_G$ and $S = S(V) = S(V_G \oplus \mathbf{1}_G)$, where $\mathbf{1}_G$ is the trivial $\mathbb{R}G$-module. Note that the $\mathbb{R}G$-module structures at tangent spaces $T_xS(V_G \oplus \mathbb{R})$ and $T_yS(V_G \oplus \mathbb{R})$ are isomorphic.

Summarizing, we can formulate the corollary below.

Corollary 3.9. *If G acts linearly on a sphere Σ with exactly two fixed points x and y, then the tangent spaces $T_x\Sigma$ and $T_y\Sigma$ are isomorphic as $\mathbb{R}G$-modules.*

3.2 Homological structure of fixed point sets for p-groups

Assume G is a group of order which is a power of a prime number p. Let us note first that if G is of prime power order, then it is not possible to obtain empty fixed point set. This follows from the Smith theory (see [62–64]). Smith proved that fixed point sets of prime power order groups on $\mathbb{Z}_p$-homology spheres and disks have rigid homological properties.

Theorem 3.10. [62–64] *If X is a topological G-space then the following statements hold.*

(1) *If X is has mod-p homology of a point, then so has X^G. In particular, X^G is nonempty.*

(2) *If X^G has mod-p homology of a sphere, then either so has X^G or X^G is empty.*

3.3 Effective actions and Riemannian manifolds

We recall here that effective actions on manifolds induce faithful group module structures on tangent spaces at fixed points. This can be proved using the existence of invariant Riemannian metric on equivariant manifolds.

Assume that $G \curvearrowright M$ is a group action on a compact manifold M. Then, the action $G \curvearrowright M$ is *properly discontinuous*, that is the set $A_K = \{g \in G | gK \cap K \neq \emptyset\}$ is finite for any compact $K \subseteq M$, since G is finite. Therefore we can find a G-invariant Riemannian metric on M.

Let $p \in M^G$ and suppose that the action of G on M is effective. Endow M with a G-invariant Riemannian metric. Denoting by $\exp_p \colon T_pM \to M$ the exponential map (sending

$v \in T_pM$ to $\gamma_v(1)$, where γ_v is the unique (maximal) geodesic with $\gamma_v(0) = p$ and $\gamma'(0) = v$) we have the following result.

Proposition 3.11. $\exp_p\colon T_pM \to M$ *is* G*-equivariant.*

Proof. Pick $g \in G$. From the definition of $G \curvearrowright M$, for any $x \in M$, we have $gx = \theta_g(x)$ for some diffeomorphism θ_g. Moreover, our metric is G-invariant, so θ_g preserves distance. By [52, p.143, Proposition 21. (2)] we have a commutative diagram

$$\begin{array}{ccc} T_pM & \xrightarrow{D\theta_{g,p}} & T_{\theta_g(p)}M = T_pM \\ \exp_p \big\downarrow & & \big\downarrow \exp_{\theta_g(p)} = \exp_p \\ M & \xrightarrow{\theta_g} & M. \end{array}$$

Summing up, from the commutativity of the diagram above, we get for any $v \in T_pM$

$$g \exp_p(v) = (\theta_g \circ \exp_p)(v) = (\exp_p \circ D\theta_{g,p})(v) = \exp_p(gv),$$

and $\exp_p$ is G-equivariant. □

Now, we are ready to prove the announced result.

Proposition 3.12. *The* $\mathbb{R}G$*-module structure induced on* T_pM *is faithful.*

Proof. We know that the structure of an $\mathbb{R}G$-module is induced on T_pM by differentials, $gv = D\theta_{g,p}(v)$ for any $v \in T_pM$, $g \in G$, where $G \curvearrowright M$ is given by a monomorphism $\theta\colon G \to \mathrm{Diff}(M)$, $g \mapsto \theta_g$.

Let us choose a G-invariant metric on M. Assume for the converse that $gv = v$ for some $1 \neq g \in G$ and any $v \in T_pM$. Since M is compact, it follows that it has to be geodesically complete [52, p. 118, Lemma 8.]. Then, the corollary from the proof of the Hopf-Rinow theorem [56, pp. 137-138, Theorem 16.] tells us that for any $q \in M$ there exists $v \in T_pM$ such that $\exp_p(v) = q$. Thus, $\exp_p\colon T_pM \to M$ is surjective. It follows by Proposition 3.11 that $\exp_p$ is G-equivariant.

Take any $x \in M$. Then, by the surjectivity of $\exp_p$, we infer that there exists $v_x \in T_pM$ for which $\exp_p(v_x) = x$. The G-equivariance of $\exp_p$ tells us in this case that

$$x = \exp_p(v_x) = \exp_p(gv_x) = g\exp_p(v_x) = gx.$$

Hence, g acts on M in a trivial way as well which contradicts the assumption on effectiveness of $G \curvearrowright M$.

Thus, it is not possible to exist $1 \neq g \in G$ with $gv = v$ for any $v \in T_pM$ what had to be proved. □

Part II

One fixed point actions on spheres

Chapter 4

Groups admitting one fixed point actions on spheres

We cover here the topic of groups admitting one fixed point actions on spheres. We recall briefly the history of finding the groups acting in that way. We finish with the theorem of Laitinen and Morimoto which characterizes groups admitting one fixed point actions on spheres. These groups turn out to be the *Oliver groups*. We end this chapter with a survey of results concerning the dimensions of spheres admitting one fixed point actions.

4.1 Oliver groups

Using the results of Oliver, Laitinen and Morimoto [28, 44, 45], we can introduce the following definition.

Definition 4.1. *A group G is an Oliver group if it admits fixed point free action on a disk.*

We provide an algebraic characterization of Oliver groups (see [44, 46–48]) and give examples of them.

The following theorem provides an algebraic necessary and sufficient condition for G to be Oliver.

Theorem 4.2. [44, 46–48] *G is an Oliver group if and only if G does not contain a sequence of subgroups $P \trianglelefteq H \trianglelefteq G$ such that P and G/H are of prime power order groups and H/P is a cyclic group.*

From the theorem above, we see that the first subclass of Oliver groups are nonsolvable groups. This class contains the smallest Oliver group which is A_5, the alternating group on 5 letters (this group has 60 elements), see [48]. Concerning solvable groups, the smallest such groups are of order 72. These groups are $A_4 \times S_3$ and $S_4 \times C_3$, the direct products of alternating groups on 4 letters with symmetric group on 3 letters and of symmetric group on 4 letters with cyclic group of order 3 respectively. Important subclass of solvable groups are abelian groups. Using Theorem 4.2 we conclude that an abelian group is Oliver if and only if it contains at least three noncyclic Sylow subgroups. Thus, the smallest commutative Oliver group is $C_2^2 \times C_3^2 \times C_5^2 \cong C_{30} \times C_{30}$ of order 900.

The class of Oliver groups is closed under the operation of taking overgroups. This follows from the definition of Oliver groups. Assume $H \leq G$ is an Oliver subgroup of G and D^n, $n \geq 0$, be an H-disk without fixed points. Inducing the action from H to G, we get a G-action on the $[G:H]$-fold Cartesian product $D^n \times ... \times D^n$ which is G-equivalent to the action of G on the disk $D = D^{[G:H]n}$. Since the induction preserves the fixed point sets, the G-action on D is without fixed points as well showing that G is an Oliver group.

4.2 Historical overview

We recall in this place the chronological development of results on groups admitting one fixed point actions on spheres. The research has been already completed in 1998 – the groups admitting such actions are precisely Oliver groups.

Before we begin with reminding the results, let us comment that every group G admitting one fixed point action on a sphere is an Oliver group. This follows directly from the Slice Theorem (see Theorem 3.4): cutting out a G-invariant neighbourhood of the fixed point from the sphere, we obtain a fixed point free G-action on a disk. Therefore, the whole effort came for the proof of the converse statement, that is every Oliver group admits one fixed point action on a sphere.

In 1946, Montgomery and Samelson [34] conjectured that it was unlikely for a group to act with one fixed point on a sphere. The first result denying this conjecture were actions of $\mathrm{SL}(2,5) \times C_n$ on S^7 for n such that $\gcd(120, n) = 1$. In particular, $\mathrm{SL}(2,5)$ can act with one fixed point on seven-dimensional sphere. In 1977, constructions of these actions were obtained by Stein [66]. Next important conclusion was due to Petrie [56] in 1982. He showed that any abelian Oliver group of odd order admits one fixed point action on some sphere.

The final answer on determining the groups which can act with one fixed point on spheres was established in '90s by three mathematicians: Laitinen, Morimoto and Pawałowski. They showed in their joint article [29] from 1995 that every nonsolvable group has a one fixed point action on some sphere. Eventually, this was generalized in 1998 to any Oliver group by Laitinen and Morimoto [28], yielding the following theorem.

Theorem 4.3. (Laitinen-Morimoto, [28]) *A group G can act with exactly one fixed point on a sphere if and only if G is an Oliver group.*

4.2.1 Dimensions of spheres

This paragraph contains a survey of results on establishing dimensions of spheres admitting one fixed point actions for given Oliver groups.

Let us focus first on the following question.

Question 4.4. *What is the lowest dimension d_{min} of a sphere on which there exists a one fixed point action of some Oliver group G?*

Obviously, one should not forget about the already mentioned result of Stein who constructed actions of $\mathrm{SL}(2,5) \times C_n$ on S^7 for n coprime to 120. This showed that the lowest dimension (d_{min}) of a sphere admitting such actions could be 7. Ten years later, in 1987, Morimoto proved [35] that the smallest Oliver group, that is A_5, can act on S^6 with one fixed point and thus lowered d_{min} to be at most 6. In 1989, Furuta [22] determined there nonexistence of one fixed point actions of groups on homotopy 4-spheres which preserved orientation. Independently, a similar result was obtained in the same year by De Michelis [17]. The paper of Morimoto [35] from 1987 showed that, once we have the exclusion result for 4-dimensional spheres, we can conclude that $d_{min} \neq 4$. The combined effort of Furuta, De Michelis and Morimoto limited d_{min} to be 5 or 6. The final answer to the Question 4.4 was given in 1990 – it was proved that $d_{min} = 6$. This ultimate step was done by Buchdahl, Kwasik and Schultz [9] who additionally were able to exclude 5 from the candidates for d_{min}.

The general question about dimensions of spheres admitting one fixed point actions of given Oliver groups remains, however, still unsolved. We have to our disposal partial results for certain Oliver groups or even some subclasses of them. Nevertheless, the complete answer seems to be quite far away since the admissible dimensions of spheres depend strongly on acting groups. The effort goes for establishing dimension lists for particular Oliver groups for one fixed point actions on spheres.

Lest us note the case of A_5. This is the only case of Oliver group with the complete list of dimensions of spheres determined. In the view of the fact that lowest possible dimension is 6, this group turns out to admit all possible dimensions n, that is $n \geq 6$. This conclusion was obtained by Bak and Morimoto in a sequence of papers on equivariant surgery. First, Morimoto [36] showed in 1987 that A_5 can act on S^n with one fixed point whenever $n = 12, 15, 16$ or $n \geq 18$. The joint work of Bak and Morimoto [3] from 1990 proved that A_5 admits one fixed point action on S^7. Further, the second author showed one year later that $n = 4k + 6$ or $n = 9 + 4k$, $k \geq 0$, are dimensions of spheres such actions, see [37]. This left us with dimension 8 to check – all other dimensions were positively verified. In 2005, the final step was done by Bak and Morimoto [4]. They confirmed existence of one fixed point A_5-action on S^8 and formulated the following.

Theorem 4.5. (Bak-Morimoto, [4]) A_5 *admits one fixed point action on* S^n *whenever* $n \geq 6$.

For a more general situation, the case of nonsolvable groups, the genaral construction of Laitinen, Morimoto and Pawałowski [29] called the "Deleting-inseting theorem" (which we shall consider in more details in the third park of this thesis) allowed to indicate admissible dimensions of spheres depending on a given nonsolvable group itself. Denoting by G^{sol} the smallest normal subgroup of a nonsolvable group G for which G/G^{sol} is solvable, they showed existence of one fixed point actions of G on spheres of dimensions $l(|G| - |G/G^{\mathrm{sol}}|)$ for any $l \geq 6$.

It turned out that for many Oliver groups the lists of admissible dimensions are more modest than for A_5. In 2018, Morimoto and Tamura excluded one fixed point actions on spheres for S_5 and $\mathrm{SL}(2,5)$ and dimensions $7, 8, 9, 13$ and $6, 8, 9$ respectively. Earlier there were obtained exclusions of effective one fixed point actions. In 2016, Borowiecka showed that $\mathrm{SL}(2,5)$ cannot act in that way on S^8. This result was generalized two years later in a joint article of Borowiecka with the author [6]. Using GAP [23] computations applied to the generalized exclusion method described in this thesis, we were able to exclude new dimensions varying within the set $\{6, 7, 8, 9, 10\}$ for most of Oliver groups of order up to 126.

Chapter 5

Exclusion algorithm

We present in this chapter a strategy of excluding of one fixed point actions on spheres. Assume we are given a group G and an $\mathbb{R}G$-module V. We would like to extract conditions from which we can deduce that there is no one fixed point action of G on S^n with $n = \dim V$ such that the $\mathbb{R}G$-module structure induced on the tangent space at at the fixed point is isomorphic to V.

The exclusion method can be divided into three stages. The first two stages are concerned with restricting the action to the family of certain subgroups. The first stage constitute restrictions coming from looking at finite fixed point sets for actions of certain subgroups. We make a remark how these restrictions simplify in the case of exclusions of group actions on disks with positive even number of fixed points. The second constraints are due to intersection properties of analogous higher-dimensional fixed point sets. These restrictions were developed by Agnieszka Borowiecka and the author in [5] and [6]. The third stage is the examination of index two subgroups and their fixed point properties. This method was developed first by Morimoto and Tamura in their joint paper from 2018, see [41]. They used it for exclusions of group actions on spheres with odd number of fixed points for the case of S_5 and $\mathrm{SL}(2,5)$. In this thesis we try, for the first time, to combine the aforementioned obstructions for one fixed point actions on spheres to increase efficiency of excluding of such exoticism. Using GAP [23] computations we were able to obtain new exclusion results.

5.1 Discrete fixed point set restriction

This section describes the first strategy which we shall use to exclude the existence of one fixed point actions on spheres. In fact, this strategy works for the exclusions of the more general case of actions with odd number of fixed points. For an Oliver group G acting on a homology sphere Σ, we try to find two non-Oliver subgroups which generate a given Oliver group under consideration. Once such subgroups H, K are found and there exists a prime power order subgroup $P \leq H \cap K$ such that Σ^P is finite, we can exclude the case that the considered action has odd number of fixed points.

After [41], let us introduce the following notation (variables denoted by p and q in the definitions below are primes or equal to 1).

- $\mathcal{G}_p^q$ – the class of groups G for which there exists a sequence of subgroups $P \trianglelefteq H \trianglelefteq G$ such that P is a p-group, G/H is a q-group and H/P is cyclic (apart from p and q prime we allow them to be one – in these cases H/P or G/H are the trivial groups).

- $\mathcal{G}^q = \bigcup_p \mathcal{G}_p^q$.
- $\mathcal{G} = \bigcup_{p,q} \mathcal{G}_p^q$.
- $\mathcal{G}_p^q(G)$ – the intersection of $\mathcal{G}_p^q$ with the set of all subgroups of G.
- $\mathcal{G}^q(G)$ – the intersection of $\mathcal{G}^q$ with the set of all subgroups of G.
- $\mathcal{G}(G)$ – the intersection of $\mathcal{G}$ with the set of all subgroups of G.

Remark 5.1. *Note that a group G is an Oliver if and only if it does not belong to $\mathcal{G}$.*

We will need a result of Morimoto and Tamura concering Euler characteristics of fixed point sets for certain subgroups.

Proposition 5.2. (Morimoto-Tamura, [41, Proposition 2.4.]) *If Σ is a homology sphere with an action of a group G and $x_0 \in \Sigma^G$, then the following hold.*

(1) $\chi(\Sigma^H) = 1 + (-1)^{\dim T_{x_0}(\Sigma^H)}$ *for any $H \in \mathcal{G}^1(G)$,*

(2) $\chi(\Sigma^H) \equiv 1 + (-1)^{\dim T_{x_0}(\Sigma^H)} \pmod q$ *for any $H \in \mathcal{G}^q(G)$.*

Using the proposition above, we can formulate now the main theorem allowing one to exclude the existence of one fixed point actions on spheres by the examination of fixed point sets of certain subgroups.

Theorem 5.3. (cf. Theorem 1.1) *Assume a group G acts on a homology sphere Σ. Suppose there exist non-Oliver subgroups $H_1, H_2 \leq G$ which generate G and that there is a prime power order subgroup $P \leq H_1 \cap H_2$. If there exists $x \in \Sigma^G$ with $\dim T_x(\Sigma^P) = 0$, then Σ^G is a two point set.*

Proof. Since P is a p-group, we deduce from the Smith theory that Σ^P is a $\mathbb{Z}_p$-homology sphere (see Theorem 3.10). Since Σ^P is finite, it consists of exactly two points. Thus, Σ^{H_1} and Σ^{H_2} are also finite and have Euler characteristics equal to their cardinalities.

Assume $\Sigma^G = \{x\}$. Therefore, the Euler charcteristics of Σ^{H_1} and Σ^{H_2} are at least 1. It follows then by Proposition 5.2 that $\chi(\Sigma^{H_i}) = 2$ for $i = 1, 2$. Since $G = \langle H_1, H_2 \rangle$, we have $\Sigma^G = \Sigma^{H_1} \cap \Sigma^{H_2} = \Sigma^P$, which contradicts the assumption that Σ^G consists of one point. □

Remark 5.4. We can apply the Smith Theory to the case of the actions on homology disks. More precisely, there is an almost straightforward sufficient condition to exclude the existence of actions of a group G on a homology disk Δ with positive even number of fixed points. Namely, if $x \in \Delta^G$ and there exists a subgroup $P \leq G$ of prime power order such that $\dim T_x \Delta^P = 0$, then Δ^G is a one point set. Indeed, from the Smith Theory, we know that Δ^P is a homology disk. Therefore Δ^P is a one point set, for $\dim T_x \Delta^P = 0$. Since $\emptyset \neq \Delta^G \subseteq \Delta^P$, we conclude that $\Delta^G = \Delta^P$ and Δ^G contains a single point. Note that, in comparison to the case of actions on homology spheres described in Theorem 5.3, we do not need the subgroups H_1 and H_2 of G.

5.2 Intersection number restriction

Let us focus here on the second type of obstructions to one fixed point actions on spheres. We look here for subgroups which satisfy similar algebraic properties as for the first exclusion technique (we do not need here to assume that they are not Oliver subgroups). The dimension conditions look, however, a bit different here – we are particularly interested in fixed point sets of positive dimensions which are transverse to each other. We show that this does not allow one fixed point actions on spheres.

In the second part of this section, we provide analogous results for the exclusions of actions on spheres with odd number of fixed points with the usage of intersection number.

5.2.1 One fixed point actions

Before we proceed, let us introduce some notation. Assume a group G acts on a manifold M with the connected fixed point set. For a subgroup $H \leq G$, we denote by $C(H)$ the connected component of M^H containing M^G. The following lemma (from the joint paper with Borowiecka [6]) gives the sufficient conditions for the transversality of the fixed point sets for subgroups.

Lemma 5.5. [6, Lemma 3.1.] *Assume that G acts on a manifold M with the connected fixed point set. Suppose there exist subgroups $H_1, H_2 \leq G$ and $H \leq H_1 \cap H_2$ such that*

(1) $\langle H_1, H_2 \rangle = G$,

(2) $\dim C(H_1) + \dim C(H_2) - \dim M^G = \dim C(H)$.

Then M^{H_1} and M^{H_2} are transverse in M^H.

Another ingredient of the exclusion theorem is the orientability of $\mathbb{Z}_p$-homology spheres for any prime p.

Lemma 5.6. *Any $\mathbb{Z}_p$-homology sphere is orientable.*

Proof. Suppose Σ is a $\mathbb{Z}_p$-homology sphere which is not orientable, i.e. $H_n(\Sigma;\mathbb{Z}) = 0$ for $n = \dim \Sigma$. By the Universal Coefficient Theorem the following sequence is exact for $k = 1, \ldots, n$

$$0 \to H_k(\Sigma;\mathbb{Z}) \otimes \mathbb{Z}_p \to H_k(\Sigma;\mathbb{Z}_p) \to \mathrm{Tor}(H_{k-1}(\Sigma;\mathbb{Z}), \mathbb{Z}_p) \to 0.$$

Since $H_k(\Sigma;\mathbb{Z}_p)$ vanish for $0 < k < n$, we have

$$H_k(\Sigma;\mathbb{Z}) \otimes \mathbb{Z}_p \cong \mathrm{Tor}(H_{k-1}(\Sigma;\mathbb{Z}), \mathbb{Z}_p) = 0$$

and consequently

$$H_k(\Sigma;\mathbb{Z}) \cong \mathbb{Z}_{q_{k,1}} \oplus \ldots \oplus \mathbb{Z}_{q_{k,l_k}},$$

where $\gcd(q_{k,i}, p) = 1$.

For $k = n$, we have $H_n(\Sigma;\mathbb{Z}) = 0$ by our assumption. Hence, by the exactness of the sequence

$$0 \longrightarrow H_n(\Sigma;\mathbb{Z}_p) \longrightarrow \mathrm{Tor}(H_{n-1}(\Sigma;\mathbb{Z}), \mathbb{Z}_p) \longrightarrow 0,$$

we conclude that

$$\mathrm{Tor}(H_{n-1}(\Sigma;\mathbb{Z}), \mathbb{Z}_p) \cong \mathbb{Z}_p.$$

On the other hand,

$$\begin{aligned}\mathrm{Tor}(H_{n-1}(\Sigma;\mathbb{Z}),\mathbb{Z}_p) &\cong \mathrm{Tor}(\mathbb{Z}_{q_{n-1,1}}\oplus\ldots\oplus\mathbb{Z}_{q_{n-1,l_{n-1}}},\mathbb{Z}_p)\\ &\cong \mathrm{Tor}(\mathbb{Z}_{q_{n-1,1}},\mathbb{Z}_p)\oplus\ldots\oplus\mathrm{Tor}(\mathbb{Z}_{q_{n-1,l_{n-1}}},\mathbb{Z}_p)=0.\end{aligned}$$

□

Corollary 5.7. *Assume Σ_p is a $\mathbb{Z}_p$-homology sphere for some prime number p. Let $\alpha \in H^a(\Sigma_p;\mathbb{Z})$ and $\beta \in H^b(\Sigma_p;\mathbb{Z})$ be such that $a, b > 0$ and $a + b = \dim\Sigma_p$. Then the cup product $\alpha\cup\beta \in H^{\dim\Sigma_p}(\Sigma_p;\mathbb{Z})$ is zero. The same holds if $p = 2$ and we substitute the coefficient ring by $\mathbb{Z}_2$.*

Proof. Suppose Σ_p is a $\mathbb{Z}_p$-homology sphere for some prime number p and α, β are as in the statement of the corollary. It follows from the proof of Lemma 5.6 that both $H^a(\Sigma_p;\mathbb{Z})$ and $H^b(\Sigma_p;\mathbb{Z})$ are finite. Thus, α and β are of finite orders and $\alpha\cup\beta \in H^{\dim\Sigma_p}(\Sigma_p;\mathbb{Z})$ is zero for Σ_p is orientable by Lemma 5.6 which means $H^{\dim\Sigma_p}(\Sigma_p;\mathbb{Z}) \cong \mathbb{Z}$.

In the case $p = 2$ and the coefficient ring $\mathbb{Z}_2$ the situation becomes trivial as $H^a(\Sigma_p;\mathbb{Z}_2) = H^b(\Sigma_p;\mathbb{Z}_2) = 0$. □

We can prove now the main theorem of this section. This is an improvement of [6, Theorem 3.2.].

Theorem 5.8. (cf. Theorem 1.2) *Let G be a group acting on a homology sphere Σ with the connected fixed point set. Suppose there exist subgroups $H_1, H_2 \leq G$ with $\langle H_1, H_2\rangle = G$ such that the submanifold $C(H_i)$ is of positive dimension for $i = 1, 2$. Moreover, assume there is a p-subgroup $P \leq H_1 \cap H_2$ for some prime p such that* [1]

$$\dim C(H_1) + \dim C(H_2) = \dim\Sigma^P$$

Suppose further that at least one of the following conditions hold.

(1) *P is of order which is a power of 2.*

(2) *The orders of H_1 and H_2 are odd.*

(3) *P is normal in H_1 and H_2, and the orders of H_1/P and H_2/P are odd.*

Then Σ^G cannot consist of a single point.

Proof. Suppose Σ^G is a one point set. Then, since the subgroups H_1 and H_2 generate G, $\Sigma^G = \Sigma^{H_1}\cap\Sigma^{H_2} = C(H_1)\cap C(H_2)$. Notice by Lemma 5.5 that Σ^{H_1} and Σ^{H_2} are transverse in Σ^P.

Assume that P is a 2-group. Since $C(H_1)$ and $C(H_2)$ are of complementary dimensions in Σ^P, it follows then that we have a well-defined unoriented intersection number of $C(H_1)$ and $C(H_2)$ in Σ^P. For $i = 1, 2$ denote by $[C(H_i)] \in H_{\dim C(H_i)}(C(H_i);\mathbb{Z}_2)$ the fundamental class. Consider the natural inclusions $C(H_i) \hookrightarrow \Sigma^P$ and identify these fundamental classes with their images in $H_{\dim C(H_i)}(\Sigma^P;\mathbb{Z}_2)$ induced from these inclusions. Let c_1 and c_2 be the corresponding classes under the Poincaré duality. Then, we get by Theorem 2.15

$$C(H_1)\cdot C(H_2) = \langle c_1\cup c_2, [\Sigma^P]\rangle. \tag{1}$$

[1] $\dim\Sigma^P$ is well-defined for from the Smith theory, we know that Σ^P is connected as a $\mathbb{Z}_p$-homology sphere.

We know by Corollary 5.7 that the cup product $c_1 \cup c_2$ is the zero element in $H^d(\Sigma^P; \mathbb{Z}_2)$. Thus, by (1), we get $C(H_1) \cdot C(H_2) = 0$ which is a contradiction.

Now, suppose the second or the third case from the assumptions of the Theorem holds. We shall prove the statement in these cases using the oriented intersection numbers in a similar way as the first case. The matter of proper definition of these numbers becomes a little bit more subtle, however. We have to ensure orientability of Σ^{H_1} and Σ^{H_2}. We know from the Smith Theory that Σ^P is a $\mathbb{Z}_p$-homology sphere. Hence, by Lemma 5.6, Σ^P is orientable. If the orders of H_1 and H_2 are odd, we conclude from [8, p. 175, 2.1 Theorem] that Σ^{H_1} and Σ^{H_2} are orientable. In the case when P is normal in H_1 and H_2 and the orders of H_1/P and H_2/P are odd, we notice that for $i = 1, 2$

$$\Sigma^{H_i} = (\Sigma^P)^{H_i/P},$$

and, since Σ^P is orientable, it suffices to apply the same argument as in the previous case to state the orientability of Σ^{H_1} and Σ^{H_2}. Therefore we have a well-defined intersection number of $C(H_1)$ and $C(H_2)$ in Σ^P. Analogously as in the $\mathbb{Z}_2$-case, for $i = 1, 2$, denote by $[C(H_i)] \in H_{\dim C(H_i)}(C(H_i); \mathbb{Z})$ the fundamental class and identify these fundamental classes with their images in $H_{\dim C(H_i)}(\Sigma^P; \mathbb{Z})$ induced from the natural inclusions. Let c_1 and c_2 be the corresponding classes under the Poincaré duality. It follows by Corollary 5.7 that $c_1 \cup c_2 \in H^{\dim \Sigma^P}(\Sigma^P)$ is zero. Hence, from Theorem 2.12 we have

$$\overline{C(H_1) \cdot C(H_2)} = \langle c_1 \cup c_2, [\Sigma^P] \rangle = \langle 0, [\Sigma^P] \rangle = 0.$$

On the other hand, since $|\Sigma^G| = 1$, we have $\overline{C(H_1) \cdot C(H_2)} = \pm 1$. A contradiction. □

5.2.2 Actions with odd number of fixed points

If we want to exclude the existence of actions with odd number of fixed points, we need a slight generalization of Lemma 5.5.

Lemma 5.9. *Assume G acts on a smooth manifold M with M^G decomposing into the connected components $C_1, ..., C_k$. Suppose there exist subgroups $H_1, H_2 \leq G$ and $H \leq H_1 \cap H_2$ such that for any connected component C_j, $j = 1, ..., k$, the following holds*

(1) $\langle H_1 \cup H_2 \rangle = G$,

(2) $\dim C_j(H_1) + \dim C_j(H_2) - \dim C_j = \dim C(H)$,

where, for a given $K \leq G$, $C_j(K)$ stands for the connected component of M^K containing C_j. Then M^{H_1} and M^{H_2} are transverse in M^H.

Proof. Since H_1 and H_2 generate G, we have $M^{H_1} \cap M^{H_2} = M^G$. Choose $x \in M^{H_1} \cap M^{H_2} = M^G$. We must show

$$\dim T_x M^{H_1} + \dim T_x M^{H_2} - \dim(T_x M^{H_1} \cap T_x M^{H_2}) = \dim T_x M^H. \quad (1)$$

Let $C_j(H_1)$ and $C_j(H_2)$ be the connected components of M^{H_1} and M^{H_2} respectively which contain x. The following equality holds from the dimension assumption applied to $C_j(H_1)$ and $C_j(H_2)$.

$$\dim T_x M^{H_1} + \dim T_x M^{H_2} - \dim(T_x M^{H_1} \cap T_x M^{H_2}) = \dim T_x M^H.$$

Therefore, it suffices to prove

$$T_x(M^{H_1} \cap M^{H_2}) = T_x M^{H_1} \cap T_x M^{H_2}$$

to get (1). The proof of the equality above is the same as presented in the proof of [6, Lemma 3.1.] and we refer the reader to it. □

Now, we are ready to prove an analogue of Theorem 5.8 concerning actions on spheres with odd number of fixed points.

Theorem 5.10. (cf. Theorem 1.3) *Let G be a group acting on a homology sphere Σ with nonempty Σ^G decomposing into the connected components $C_1, ..., C_k$. Suppose there exist subgroups $H_1, H_2 \leq G$ with $\langle H_1, H_2 \rangle = G$ such that for any connected component C_j, $j = 1, ..., k$, the submanifold $C_j(H_i)$ which is the connected component of Σ^{H_i} containing C_j, is of positive dimension for $i = 1, 2$. Moreover, assume there is a p-subgroup $P \leq H_1 \cap H_2$ for some prime p such that for any $j = 1, ..., k$*

$$\dim C_j(H_1) + \dim C_j(H_2) = \dim \Sigma^P.$$

Suppose further that at least one of the following conditions hold.

(1) *P is of order which is a power of 2.*

(2) *The orders of H_1 and H_2 are odd.*

(3) *P is normal in H_1 and H_2, and the orders of H_1/P and H_2/P are odd.*

Then Σ^G cannot consist of odd number of points.

Proof. Since the subgroups H_1 and H_2 generate G, we have $\Sigma^G = \Sigma^{H_1} \cap \Sigma^{H_2}$. Writing this with the usage of connected components, $C_j(H_i)$, we get

$$\Sigma^G = \bigcup_{j_1, j_2 = 1, \dots, k} C_{j_1}(H_1) \cap C_{j_2}(H_2) = \bigsqcup_{m=1,\dots,l} C_{j_m}(H_1) \cap C_{j_m}(H_2). \tag{1}$$

Assume Σ^G consists of odd number of points. It follows then by (1) that for some $j \in \{1, ..., k\}$, the intersection $C_j(H_1) \cap C_j(H_2)$ contains odd number of points. By Lemma 5.9, we conclude that Σ^{H_1} and Σ^{H_2} are transverse in Σ^P. Introducing the notation $C(H_1) = C_j(H_1)$ and $C(H_2) = C_j(H_2)$, we can repeat then the arguments from the proof of Theorem 5.8 – the only difference is that instead of getting ± 1 as the appropriate intersection number of $C(H_1)$ and $C(H_2)$ in Σ^P, we can infer that this number is odd – the proof still holds since the corresponding Kronecker pairings vanish exactly as in the proof of Theorem 5.8. □

Remark 5.11. In order to exclude the existence of one fixed point G-actions on spheres or G-actions on spheres with odd number of fixed points, it suffices to find subgroups $H_1, H_2 \leq G$ satisfying the assumptions of Theorem 5.8 or Theorem 5.10 respectively. In the case of one fixed point actions, this is because when Σ^G is disconnected, then it cannot consist of a single point, and, in the case of the actions with odd number of fixed points – when Σ^G is an empty set, then it cannot consist of odd number of points.

Verifying the dimension conditions from Theorem 5.10 for every connected component $C_j(H_i)$ may be quite troublesome. However, in favorable cases, we can infer that all $C_j(H_i)$'s have the same dimension. This involves examination of fixed point dimension properties for actions of p-subgroups where p can be any prime number.

Let G be a group and $P_1, ..., P_k$ be representatives of conjugacy classes of subgroups of G which contain subgroup of prime power order. Let $L = (V_1, ..., V_l)$ be a list of mutually non-isomorphic irreducible $\mathbb{R}G$-modules and put

$$D_L(G) = \begin{pmatrix} d_{1,1} & d_{1,2} & \dots & d_{1,l} \\ d_{2,1} & d_{2,2} & \dots & d_{2,l} \\ & \dots\dots\dots & & \\ d_{k,1} & d_{k,2} & \dots & d_{k,l} \end{pmatrix},$$

where $d_{i,j} = \dim V_j^{P_i}$. Assume G acts on a $\mathbb{Z}$-homology sphere Σ and suppose that for any $x \in \Sigma^G$, the tangential representation at x decomposes into the direct sum of irreducible $\mathbb{R}G$-modules from the set $\{V_1, ..., V_l\}$.

Proposition 5.12. *If* $\operatorname{rank}(D_L(G)) = l$*, then for any* $x, y \in \Sigma^G$*, the tangent spaces* $T_x\Sigma$ *and* $T_y\Sigma$ *are isomorphic as* $\mathbb{R}G$*-modules.*

Proof. Assume $\operatorname{rank}(D_L(G)) = l$ and take any $x, y \in \Sigma^G$. By assumption, we can express the $\mathbb{R}G$-module structures at the tangent spaces at x and y as follows.

$$T_x\Sigma \cong a_1V_1 \oplus ... \oplus a_lV_l$$

and

$$T_y\Sigma \cong b_1V_1 \oplus ... \oplus b_lV_l$$

for some non-negative integers $a_1, ..., a_l$ and $b_1, ..., b_l$ (recall that for an $\mathbb{R}G$-module V, mV denotes the m-fold direct sum of V). We must show that the above decompositions of tangent spaces are identical, that is $a_i = b_i$ for $i = 1, ..., l$.

By the Smith theory we know that Σ^{P_i} is a $\mathbb{Z}_{p_i}$-homology sphere, where $|P_i|$ is a power of a prime number p_i. Thus, Σ^{P_i}'s are connected or are two point sets. Thus, the dimension of Σ^{P_i} is equal to the fixed point dimensions $(T_x\Sigma)^{P_i}$ and $(T_y\Sigma)^{P_i}$, since either both x and y lie in Σ^{P_i} and $\dim(T_x\Sigma)^{P_i} = \dim(T_y\Sigma)^{P_i}$ or $\dim(T_x\Sigma)^{P_i} = \dim(T_y\Sigma)^{P_i} = 0$. Since $d_{i,j} = \dim V_j^{P_i}$, we can rewrite the equality $\dim(T_x\Sigma)^{P_i} = \dim(T_y\Sigma)^{P_i}$ in terms of the coefficients a_l, b_l and $d_{i,j}$ in the following way.

$$d_{i,1}a_1 + \ldots + d_{i,l}a_l = d_{i,1}b_1 + \ldots + d_{i,l}b_l.$$

Therefore, we get the following homogeneous equation system

$$\begin{cases} d_{1,1}(a_1 - b_1) + \ldots + d_{1,l}(a_l - b_l) = 0 \\ \qquad\qquad \dots\dots\dots \\ d_{k,1}(a_1 - b_1) + \ldots + d_{k,l}(a_l - b_l) = 0. \end{cases}$$

Since $\operatorname{rank}(D_L(G)) = l$, it follows from that the homogeneous system above has a unique solution $a_j - b_j = 0$ that is $a_j = b_j$ for all $1 \le j \le l$. □

In particular, we get.

Corollary 5.13. *If* $\operatorname{rank}(D_L(G)) = l$*, then for any* $x, y \in \Sigma^G$ *and* $H \le G$ *the dimensions of tangent spaces* $T_x\Sigma^H$ *and* $T_y\Sigma^H$ *are equal.*

Thus, if the assumptions of the corollary above are satisfied and L is the list of all nontrivial real irreducible $\mathbb{R}G$-modules, then it suffices to check the dimension condition for only one arbitrary chosen connected component $C_j(H_i)$ in Theorem 5.10 in order to state that Σ^G cannot consist of odd number of points.

Notation. In the case when L is the list of all irreducible $\mathbb{R}G$-modules, we put $D_G = D_L(G)$. Moreover, we denote by r_G the number of real irreducible representations of G diminished by one (since the trivial module cannot be the summand of a group module structure occuring at the tangent space at the unique fixed point for postulated one fixed point actions on spheres).

5.3 Index two restriction

In this section, we recall briefly the results of Morimoto and Tamura [41] which are based on the examination of index two subgroups and their fixed point sets. These results allow us to exclude group actions on spheres with one and odd number of fixed points.

Let G be a group and H its subgroup equal to the intersection of all subgroups of G of index at most two. The first restriction tells us that a group G cannot act with one fixed point on a sphere with positive-dimensional tangent space to the fixed point set of H at the fixed point of the G-action.

Lemma 5.14. (Morimoto, [38, Lemma 2.1]) *Let Σ be a $\mathbb{Z}$-homology sphere with a G-action. If Σ^G consists of a single point x_0, then* $\dim T_{x_0}\Sigma^H = 0$.

The second result asserts that index two subgroups satisfying certain algebraic conditions inherit the property of admitting actions on spheres with odd number of fixed points from the bigger group. This result goes back to [41, Proposition 2.10].

Lemma 5.15. *Let K be an index two subgroup of G satisfying the following conditions.*

(1) *Every element of K is of prime power order.*

(2) *There is no element of K of order divisible by* 8.

Then, if Σ^G is finite and contains odd number of points, the same holds for Σ^K.

Proof. The same as in [41]. □

5.4 Effective one fixed point actions on spheres

Problem 5.16. *For a given Oliver group G determine the dimensions of spheres which admit effective one fixed point actions of G.*

It is known that the list of groups admitting one fixed point actions on spheres and the analogous list of groups admitting effective actions are the same – both lists contain Oliver groups (see the paper of Laitinen and Morimoto [28]). Therefore, the problem of determining the dimensions of spheres on which a given Oliver group can act with one fixed point can be phrased in an effective and non-effective way. To author's knowledge, these problems may potentially differ. In particular, we do not know in general that if an Oliver group G acts with one fixed point on S^n, then it can act effectively in such a way on S^n as well.

Consider a specific dimension $n \geq 6$ on which we would like to exclude effective one fixed point action of a given Oliver group G. It follows from Proposition 3.12 that if such a G-action exists, then the $\mathbb{R}G$-module structure of the tangent space at the fixed point is faithful. If one is interested only in effective one fixed point actions, we may restrict our attention to faithful $\mathbb{R}G$-modules – it is important, however, to do this at the appropriate moment (we comment on this in the subsequent chapter, see Remark 5.27).

We can use the information about groups for which we could exclude effective one fixed point actions on spheres, to exclude the noneffective ones too. Assume G is (an Oliver) group and consider an $\mathbb{R}G$-module V of dimension n, which we would like to exclude as a candidate for the tangent module structure at the postulated fixed point of G-action on S^n. Denote by $\mathrm{Ker}(V)$ the kernel of V, that is the normal subgroup of G consisting of those elements $g \in G$ which act trivially on V. Note, that in case G acts with exactly one fixed point x_0 on S^n with tangent module, $T_{x_0}S^n$, isomorphic to V, then the quotient $G/\mathrm{Ker}(V)$ acts effectively on S^n and $(S^n)^{G/\mathrm{Ker}(V)} = \{x_0\}$ and the $G/\mathrm{Ker}(V)$-action on V is faithful (V can be considered as well as a faithful $\mathbb{R}(G/\mathrm{Ker}(V))$-module). Therefore, if we know that the group $G/\mathrm{Ker}(V)$ is not an Oliver group, or if it is not the case, then at least we could exclude faithful one fixed point actions of $G/\mathrm{Ker}(V)$ on S^n, then we can exclude the situation in which V occurs as the tangent module $T_{x_0}S^n$. Hence, we can refine our exclusion strategy in the general (non-faithful) case by such a consideration of kernels of modules for which we could not exclude one fixed point actions to this point.

5.5 Exclusion algorithm

In this section we present an algorithm for the exclusion of one fixed point actions of Oliver groups on spheres. Given an Oliver group G and a dimension $n \geq 6$, the algorithm checks sufficient conditions to exclude the existence of one fixed point action of G on S^n. The strategy is to consider every n-dimensional $\mathbb{R}G$-module V as a candidate for the $\mathbb{R}G$-module structure on tangent space to S^n at the postulated single fixed point x_0. We check the conditions from the restriction strategies from sections 5.1, 5.2 and 5.3 – any of those are satisfied, we know that V cannot occur as the $\mathbb{R}G$-module structure on $T_{x_0}S^n$. More precisely, our algorithm provides us with a list of $\mathbb{R}G$-modules which can potentially occur at $T_{x_0}S^n$ (that is which could not be excluded).

Let $n \geq 6$ and m be integers. We would like to apply the exclusion strategies for all Oliver groups of order up to m concerning actions on S^n. We divide the algorithm into two parts. The first is responsible for verifying the conditions for actions with odd number of fixed points. For a given group G (not necessarily an Oliver group), we consider all n-dimensional $\mathbb{R}G$-modules and establish $MN_{odd}(n, G)$, the list of those n-dimensional $\mathbb{R}G$-modules for which we were not able to exclude the existence of actions on S^n with odd number of fixed points and the tangent space at one of the fixed points isomorphic to such modules. We compute $MN_{odd}(n, G)$ for a certain list of groups G derived from the list of all Oliver groups of order up to m. An important aspect of this procedure is that when computing $MN_{odd}(n, G)$ we base already on the contents of $MN_{odd}(n, H)$ for certain index two subgroups H of G. It is therefore important to start with the groups of the smallest orders. The second part utilizes the exclusion results of the first part (the contents of $MN_{odd}(n, G)$ for all Oliver groups G of order up to m) and checks additionally the conditions applying exclusively to one fixed point actions. As a result, for every Oliver group of order up to m we obtain the list $MN_{one}(n, G)$ containing the $\mathbb{R}G$-modules which could not be excluded by either of the restriction strategies. Optionally, once we are interested in the faithful case, we compute at the end the lists $MNF_{one}(n,G) \subseteq MN_{one}(n,G)$ containing those $\mathbb{R}G$-modules from $MN_{one}(n,G)$ which are faithful.

We can summarize the whole process in the following simplified schema.

Part I

Input:

- $n \geq 6$ – an integer being the dimension of spheres to consider,
- $m \geq 0$ – an integer which is the maximum order of Oliver groups to consider.

Output:

- $MN_{odd}(n,G)$'s for G varying within the range of Oliver groups of order up to m.

Part II

Input:

- $n \geq 6$ – an integer being the dimension of spheres to consider,
- $m \geq 0$ – an integer which is the maximum order of Oliver groups to consider,
- $MN_{odd}(n,G)$'s for G varying within the range of Oliver groups of order up to m.

Output:

- $MN_{one}(n,G)$'s for G varying within the range of Oliver groups of order up to m,
- (optionally) $MNF_{one}(n,G)$'s for G varying within the range of Oliver groups of order up to m.

For all Oliver groups G of order up to m, we iterate over all n-dimensional $\mathbb{R}G$-modules which are contained in $MN_{odd}(n,G)$ and check the conditions for every of the considered restriction strategies. If these conditions are not satisfied for any of the strategies, then we add the considered $\mathbb{R}G$-modules to $MN_{one}(n,G)$.

Notation. For the convenience, let us denote by $MT_{one}(n,G)$ the list of n-dimensional $\mathbb{R}G$-modules which occur at the tangent space to S^n at the fixed point of some one fixed point action on S^n. Analogously, let $MT_{odd}(n,G)$ be the list of n-dimensional $\mathbb{R}G$-modules which occur at the tangent spaces to S^n at one of the points of some action on S^n with odd number of fixed points. Of course, $MT_{one}(n,G) \subseteq MN_{one}(n,G)$, $MT_{odd}(n,G) \subseteq MN_{odd}(n,G)$ and $MT_{one}(n,G) \subseteq MT_{odd}(n,G)$. The list containing all $\mathbb{R}G$-modules of dimension n will be denoted by $M(n,G)$.

To express our results more efficiently, we introduce further the following definitions. Throughout, we assume that G is a group.

Definition 5.17. *We call a triple $((P),(H_1),(H_2))$ of conjugacy classes of subgrops of G a* ***good triple of type A*** *if H_1, H_2 are non-Oliver subgroups of G, $\langle H_1, H_2\rangle = G$ and P is a prime power order subgroup of $H_1 \cap H_2$.*

Definition 5.18. *A triple $((P),(H_1),(H_2))$ of conjugacy classes of subgrops of G is called a* ***good triple of type B*** *if H_1, H_2 are any subgroups of G, $\langle H_1, H_2\rangle = G$ and P is a prime power order subgroup of $H_1 \cap H_2$ and at least one of the following conditions hold.*

(1) *P is of order which is a power of 2.*

(2) *The orders of H_1 and H_2 are odd.*

(3) *P is normal in H_1 and H_2, and the orders of H_1/P and H_2/P are odd.*

Let us denote by $\mathcal{A}(G)$ and $\mathcal{B}(G)$ the subsets of good triples of types A and B respectively. Moreover, we denote by $I_2(G)$ the subgroup being the intersection of all subgroups of index at most two of G and by $\mathcal{I}_2(G)$ the subset of all subgroups of G of index two satisfying the assumptions of Lemma 5.15.

5.5.1 The first part

We give here the pseudocodes of functions excluding the existence of actions on spheres with odd number of fixed points. Before we do that, we formulate the corollaries from exclusion strategies described earlier. This corollaries refer directly to group modules and translate the conditions of group actions on spheres to conditions concerning the group modules. Given a group G and an $\mathbb{R}G$-module V, we would like to establish conditions which would allow us to exclude V from $MT_{odd}(n, G)$.

Assume G is an Oliver group and V is an n-dimensional $\mathbb{R}G$-module we would like to exclude from $MT_{odd}(n, G)$.

Let us begin with the discrete fixed point set restriction. As a corollary from the Smith theory, we know that the fixed points sets of the actions of groups of prime power order on homology spheres are connected or finite. Thus their dimension can be expressed as the appropriate fixed point dimension of V.

Corollary 5.19. *Let $((P), (H_1), (H_2))$ be a good subgroup triple of type A of a group G. Suppose that* $\dim V^P = 0$. *Then $V \notin MT_{odd}(n, G)$.*

Proof. For the converse, assume $V \in MT_{odd}(n, G)$. Then there exists an action of G on S^n with odd number of fixed points. Moreover, for one of the fixed points, say x_0, we have $T_{x_0}S^n \cong V$ as $\mathbb{R}G$-modules. Since $\dim V^P = 0$, we conclude that $\dim T_{x_0}S^n = 0$. Thus, we get a contradiction by Theorem 5.3. □

Now, let us consider the intersection number restriction. Put $L = (V_1, ..., V_l)$ be the complete list of nontrivial irreducible $\mathbb{R}G$-modules (in this case $l = r_G$ and $D_L(G) = D_G$ – see the definitions on pages 37 and 38). If it turns out that $\operatorname{rank}(D_G) = r_G$, then we can formulate the restriction condition for V to be contained in $MT_{odd}(n, G)$.

Corollary 5.20. *Assume* $\operatorname{rank}(D_G) = r_G$ *and suppose there exists $((P), (H_1), (H_2))$, a good triple of type B, such that* $\dim V^{H_1}$ *and* $\dim V^{H_2}$ *are positive and*

$$\dim V^{H_1} + \dim V^{H_2} = \dim V^P.$$

Then $V \notin MT_{odd}(n, G)$.

Proof. Assume that there exists an action of G on S^n with odd number of fixed points and the fixed point x_0 for which $T_{x_0}S^n$ is isomorphic as $\mathbb{R}G$-module to V. Since $\operatorname{rank}(D_G) = r_G$, it follows by Corollary 5.13 that $(S^n)^{H_1}$, as well as $(S^n)^{H_2}$, has equal dimensions of its connected components. Thus, it follows that the assumptions of Theorem 5.10 are satisfied for the considered action of G on S^n. We conclude from this theorem that this action cannot have odd number of fixed points. A contradiction. □

In order to exclude an action of G on S^n with odd number of fixed points, we have to show that every n-dimensional $\mathbb{R}G$-module V does not belong to $MT_{odd}(n, G)$. Checking the assumptions of the two corollaries above can be made more efficient. It suffices to look at subgroup triples being the representatives of different good triples of both types. Indeed, the following holds.

Proposition 5.21. *If V is an $\mathbb{R}G$-module and H and H' are conjugate subgroups of G, then*

$$\dim V^{H'} = \dim V^H.$$

Proof. Since H' and H are conjugate, there exists an element $g \in G$ such that $H' = gHg^{-1}$. This means that any $h' \in H'$ can be expressed in as $h' = ghg^{-1}$ for the unique $h \in H$. On the other hand, by Theorem 2.8, we have

$$\dim V^{H'} = \frac{1}{|H'|} \sum_{h' \in H'} \chi_V(h') = \frac{1}{|H|} \sum_{h' \in H'} \chi_V(h'),$$

where χ_V denotes the character of V. Since for any $h' \in H'$ we have $h' = ghg^{-1}$ for the unique h, it follows then from the equality above that

$$\dim V^{H'} = \frac{1}{|H|} \sum_{h \in H} \chi_V(ghg^{-1}).$$

Characters are conjugate invariant functions from G to $\mathbb{R}$, so $\chi_V(ghg^{-1}) = \chi_V(h)$ for any $h \in H$. From this and Theorem 2.8 applied once again, we conclude that

$$\dim V^{H'} = \frac{1}{|H|} \sum_{h \in H} \chi_V(ghg^{-1}) = \frac{1}{|H|} \sum_{h \in H} \chi_V(h) = \dim V^H.$$

□

Index two subgroups restriction provides us also with an exclusion corollary involving only the group G. More precisely, as an application of Lemma 5.15 we get the following corollary.

Corollary 5.22. *Assume G contains a subgroup H such that $[G : H] = 2$, $MN_{odd}(n, H) = \emptyset$ and the following conditions hold.*

(1) *Every element of H is of prime power order.*

(2) *There is no element of H of order divisible by 8.*

Then $MN_{odd}(n, G) = \emptyset$.

Below, we present the pseudocode of the function which computes $MN_{odd}(n, G)$ for a given dimension n and an Oliver group G. Apart from n and G, this function requires an additional parameter L which is a list of groups, intended to be the groups for which we excluded actions on S^n with odd number of fixed points.

```
function ModulesNotExcludedOdd(n, G, L)
    for H ∈ I_2(G) do
        if H ∈ L then
            return ∅
    MN_odd(n, G) ← M(n, G)
    for V ∈ M(n, G) do
        for ((P), (H_1), (H_2)) ∈ A(G) do
            if dim V^P = 0 then
                MN_odd(n, G) ← MN_odd(n, G) \ {V}
        if rank(D_G) = r_G then
            for ((P), (H_1), (H_2)) ∈ B(G) do
                if dim V^{H_1} + dim V^{H_2} = dim V^P and
                  dim V^{H_1} · dim V^{H_2} > 0 then
                    MN_odd(n, G) ← MN_odd(n, G) \ {V}
    return MN_odd(n, G)
```

We would like to calculate $MN_{odd}(n, G)$ for all Oliver groups G of order up to a fixed integer m. We achieve this by computing $MN_{odd}(n, G)$ for Oliver groups G starting from those with the smallest orders. The following function computes the lists $MN_{odd}(n, G)$ for given $n \geq 6$ and all Oliver groups of order up to m (we denote this list by *oliverGroupsUpToOrderM* and assume it is already sorted according to the order).

```
function ModulesNotExcludedOddUpToOrder(n, m)
    result ← ∅
    groupsExcludedOdd ← ∅
    for G ∈ oliverGroupsUpToOrderM do
        result[G] ← ModulesNotExcludedOdd(n, G, groupsExcludedOdd)
        if result[G] = ∅ then
            groupsExcludedOdd ← groupsExcludedOdd ∪ {G}
    return result
```

Remark 5.23. In the procedure above we can in fact iterate over *all* groups G up to a given order (not necessarily Oliver groups) – the strategy works in the general setup. We present this version of pseudocode, however, since the computations involving all groups up to a given order become time consuming. One can define an arbitrary subsets of subgroups (up to a given order) to iterate over. This can give potentially better results.

5.5.2 The second part

We present here pseudocodes of functions concerning exclusions of one fixed point actions on spheres. As in the case of actions with odd number of fixed points, we formulate first the necessary corollaries from all the considered strategies in order to be able to exclude group modules from $MN_{one}(n, G)$ for a given dimension n and an Oliver group G. We use the computed lists $MN_{odd}(n, G)$ from the first part.

In order to be able to obtain additional exclusion results different from those obtained from the first part, we need to look at restrictions applying exclusively to one fixed point actions. Such restrictions are provided by Theorem 5.8 and Lemma 5.14. The following two corollaries from these two results allow us to exclude an $\mathbb{R}G$-module V from $MT_{one}(n, G)$ for a given dimension n and an Oliver group G.

Corollary 5.24. *Suppose there exists good triple of type B, $((P), (H_1), (H_2))$, such that* $\dim V^{H_1}$ *and* $\dim V^{H_2}$ *are positive and*

$$\dim V^{H_1} + \dim V^{H_2} = \dim V^P.$$

Then $V \notin MT_{one}(n, G)$.

Proof. A direct application of Theorem 5.8. □

Remark 5.25. It makes sense to use the theorem above only in the case when $\mathrm{rank}(D_G) < r_G$ since then we cannot apply Corollary 5.20.

Corollary 5.26. *If* $\dim V^{I_2(G)} > 0$*, then $V \notin MT_{one}(n, G)$.*

Proof. A direct application of Lemma 5.14. □

Having this, we are able to establish the main exclusion function. For all Oliver groups G of order up to a fixed integer m, this function checks the assumptions of the corollaries above and excludes $\mathbb{R}G$-modules from $MN_{one}(n, G)$ if these assumptions are satisfied for such modules. We use here the results from the first part – we can restrict our attention to $\mathbb{R}G$-modules belonging to $MN_{odd}(n, G)$. The pseudocode of the function is presented below.

function $ModulesNotExcludedOneUpToOrder(n,\ m)$
    $MN_{odd} \leftarrow ModulesNotExcludedOddUpToOrder(n, m)$
    $result \leftarrow ModulesNotExcludedOddUpToOrder(n, m)$
    for $G \in oliverGroupsUpToOrderM$ **do**
        for $V \in MN_{odd}[G]$ **do**
            for $((P), (H_1), (H_2)) \in \mathcal{B}(G)$ **do**
                if $\dim V^{H_1} + \dim V^{H_2} = \dim V^P$ **and**
                    $\dim V^{H_1} \cdot \dim V^{H_2} > 0$ **then**
                    $result[G] \leftarrow result[G] \setminus \{V\}$
            if $\dim V^{I_2(G)} > 0$ **then**
                $result[G] \leftarrow result[G] \setminus \{V\}$
        $MNF_{one}[G] \leftarrow \emptyset$
        for $V \in result[G]$ **do**
            if V is faithful **then**
                $MNF_{one}[G] \leftarrow MNF_{one}[G] \cup \{V\}$
    $Step1MN_{one} \leftarrow result$
    for $G \in oliverGroupsUpToOrderM$ **do**
        for $V \in Step1MN_{one}[G]$ **do**
            if $G/(\mathrm{Ker}(V))$ is not an Oliver group or, otherwise, $MNF_{one}[G]$ is empty **then**
                $result[G] \leftarrow result[G] \setminus \{V\}$
    return $result$

Algorithm 5.1: The exclusion algorithm – the function above computes the list of group modules for which we were not able to exclude the existence of one fixed point action. This list is computed for all Oliver groups G up to order m.

Remark 5.27. Note that in the function $ModulesNotExcludedOddUpToOrder(n, m)$ we apply the result of Morimoto and Tamura (see Lemma 5.15). Using this result, we can infer the nonexistence of actions of a group G on spheres with odd number of fixed points, once we know that there are no such actions for a suitable index two subgroup of G. This assumption, however, does not necessarily concern effective actions, therefore we can work with the result of Morimoto and Tamura in the general (possibly non-faithful) case only. Thus, when considering the faithful case, we can restrict our attention to faithful modules, but we have to do it in the appropriate moment, that is after iterating over all $\mathbb{R}G$-modules from $MN_{odd}[G]$.

Remark 5.28. In the the exclusion algorithm (Algorithm 5.1) we used also the excluded faithful one fixed point actions in order to exclude the general (non-faithful) case – this part begins from the line "$Step1MN_{one} \leftarrow result$". The explanation of this restrictions was described in the last paragraph of Section 5.4.

The implementation of the exclusion algorithm described in this section is available at [33].

5.6 Exclusion results

We collect here the results obtained from the application of the exclusion algorithm for certain Oliver groups. Next, we illustrate the work of exclusion algorithm on the example of S_5.

Theorem 5.29. *The following tables present the exclusion results for the first* 8 *Oliver groups* for dimensions of spheres varying from* 6 *to* 10*. The green color indicates that we could exclude a given odd or one fixed point action.*

- *dimension* $n = 6$

G	$\|M(n,G)\|$	$\|MN_{odd}(n,G)\|$	$\|MN_{one}(n,G)\|$	$\|MNF_{one}(n,G)\|$
$C_3 \times S_4$	37	12	0	0
$C_3 \rtimes S_4$	49	9	0	0
$S_3 \times A_4$	36	12	0	0
$\mathrm{SL}(2,5)$	3	3	3	0
S_5	6	1	1	1
$C_2 \times A_5$	19	11	7	4
$C_3 \rtimes F_7$	27	8	0	0

- *dimension* $n = 7$

G	$\|M(n,G)\|$	$\|MN_{odd}(n,G)\|$	$\|MN_{one}(n,G)\|$	$\|MNF_{one}(n,G)\|$
$C_3 \times S_4$	51	36	1	1
$C_3 \rtimes S_4$	69	17	0	0
$S_3 \times A_4$	50	36	0	0
$\mathrm{SL}(2,5)$	6	6	6	4
S_5	6	1	0	0
$C_2 \times A_5$	27	17	4	2
$C_3 \rtimes F_7$	27	23	0	0

- *dimension* $n = 8$

G	$\|M(n,G)\|$	$\|MN_{odd}(n,G)\|$	$\|MN_{one}(n,G)\|$	$\|MNF_{one}(n,G)\|$
$C_3 \times S_4$	88	42	0	0
$C_3 \rtimes S_4$	120	18	0	0
$S_3 \times A_4$	84	42	0	0
SL(2, 5)	9	5	2	0
S_5	9	1	0	0
$C_2 \times A_5$	38	25	4	2
$C_3 \rtimes F_7$	58	26	0	0

- *dimension* $n = 9$

G	$\|M(n,G)\|$	$\|MN_{odd}(n,G)\|$	$\|MN_{one}(n,G)\|$	$\|MNF_{one}(n,G)\|$
$C_3 \times S_4$	122	91	4	4
$C_3 \rtimes S_4$	166	45	4	4
$S_3 \times A_4$	116	89	4	4
SL(2, 5)	7	7	4	0
S_5	13	1	0	0
$C_2 \times A_5$	62	44	16	12
$C_3 \rtimes F_7$	58	53	0	0

- *dimension* $n = 10$

G	$\|M(n,G)\|$	$\|MN_{odd}(n,G)\|$	$\|MN_{one}(n,G)\|$	$\|MNF_{one}(n,G)\|$
$C_3 \times S_4$	191	109	2	2
$C_3 \rtimes S_4$	262	46	0	0
$S_3 \times A_4$	178	107	0	0
SL(2, 5)	10	10	9	6
S_5	18	3	2	2
$C_2 \times A_5$	85	63	14	11
$C_3 \rtimes F_7$	113	60	1	1

** The list includes the following groups (we add also their idies from the GAP* [23] *SmallGroup Library – the id is presented in brackets* [,]*):* $C_3 \times S_4 - -[72, 42]$, $C_3 \rtimes S_4 - [72, 43]$, $S_3 \times A_4 - [72, 44]$, $\mathrm{SL}(2, 5) - [120, 5]$, $S_5 - [120, 34]$, $C_2 \times A_5 - [120, 35]$, $C_3 \rtimes F_7 = [126, 9]$, *where* F_q, q *– prime power, denotes the Frobenius group* $\mathbb{F}_q \rtimes \mathbb{F}_q^\times$*. We exclude* A_5 *from the list since, by the work of Morimoto, we know that this group admits one fixed point actions on spheres of dimension* n *whenever* $n \geq 6$.

5.6.1 Example – the case of S_5

Consider the case $G = S_5 = \mathrm{SmallGroup}(120, 34)$ and $n = 7$. We show how the exclusion algorithms work in the general case when there is no restriction on the action to be faithful. It will turn out that $MN_{odd}(n, G)$ is non-zero but $MN_{one}(n, G)$ is zero. Thus, G does not admit one fixed point action on S^n if $n = 7$. Those results were obtained by GAP computations (see author's implementation [33]).

In general, the cojugacy classes of symmetric group S_n are determined by their *cycle lengths*. That is, each sequance of integers $(k_1, ..., k_m)$ such that $k_1 \geq ... \geq k_m$ and $k_1 + ... +$

$k_m = n$ determines the conjugacy class of S_n consisting of permutations wihch decompose into compositions of m cyclces of length $k_1, ..., k_m$ respectively. Thus, G has the following conjugacy classes.

- $c_1 = (\,\mathrm{id}\,)$ consisting of the identity element,
- $c_{2,1} = (\,(1\quad 2)\,)$ which contains 10 elements elements of order 2,
- $c_{2,2} = (\,(1\quad 2)\,(3\quad 4)\,)$ which contains 15 elements of order 2,
- $c_3 = (\,(1\quad 2\quad 3)\,)$ which contains 20 elements of order 3,
- $c_4 = (\,(1\quad 2\quad 3\quad 4)\,)$ which contains 30 elements of order 4,
- $c_5 = (\,(1\quad 2\quad 3\quad 4\quad 5)\,)$ which contains 24 elements of order 5,
- $c_6 = (\,(1\quad 2\quad 3)\,(4\quad 5)\,)$ which contains 20 elements of order 6.

We compute the real nontrivial irreducible characters of G.

	c_1	$c_{2,1}$	$c_{2,2}$	c_3	c_4	c_5	c_6
X_1	1	-1	1	1	-1	1	-1
$X_{4,1}$	4	-2	0	1	0	-1	1
$X_{4,2}$	4	2	0	1	0	-1	1
$X_{5,1}$	5	1	1	-1	-1	0	1
$X_{5,2}$	5	-1	1	-1	1	0	-1
X_6	6	0	-2	0	0	1	0

G has 19 conjugacy classes of subgroups.

- d_1 which unique representative is isomorphic to the trivial group,
- $d_{2,1}$ with representative $\langle(1\quad 2)\rangle$ isomorphic to C_2,
- $d_{2,2}$ with representative $\langle(1\quad 2)\,(3\quad 4)\rangle$ isomorphic to C_2,
- d_3 with representative $\langle(1\quad 2\quad 3)\rangle$ isomorphic to C_3,
- $d_{4,1}$ with representative $\langle(1\quad 3)\,(2\quad 4)\,,(1\quad 4)\,(2\quad 3)\rangle$ isomorphic to $C_2 \times C_2$,
- $d_{4,2}$ with representative $\langle(1\quad 2\quad 4\quad 3)\rangle$ isomorphic to C_4,
- $d_{4,3}$ with representative $\langle(1\quad 4)\,,(2\quad 3)\rangle$ isomorphic to $C_2 \times C_2$,
- d_5 with representative $\langle(1\quad 2\quad 3\quad 4\quad 5)\rangle$ isomorphic to C_5,
- $d_{6,1}$ with representative $\langle(1\quad 2)\,,(1\quad 2\quad 3)\rangle$ isomorphic to S_3,
- $d_{6,2}$ with representative $\langle(1\quad 3)\,(4\quad 5)\,,(1\quad 2\quad 3)\rangle$ isomorphic to S_3,
- $d_{6,3}$ with representative $\langle(1\quad 4)\,(2\quad 5\quad 3)\rangle$ isomorphic to C_6,
- d_8 with representative $\langle(3\quad 4)\,,(1\quad 4)\,(2\quad 3)\rangle$ isomorphic to D_8,
- d_{10} with representative $\langle(1\quad 3)\,(2\quad 4)\,,(1\quad 2\quad 5\quad 4\quad 3)\rangle$ isomorphic to D_{10},

- $d_{12,1}$ with representative $\langle (1\ 2\ 3), (1\ 4)(2\ 3) \rangle$ isomorphic to A_4,
- $d_{12,2}$ with representative $\langle (1\ 2), (3\ 5\ 4), (4\ 5) \rangle$ isomorphic to D_{12},
- d_{20} with representative $\langle (1\ 3)(4\ 5), (1\ 2\ 4\ 3) \rangle$ isomorphic to F_5, the Frobenius group of order 20,
- d_{24} with representative $\langle (1\ 3\ 2), (2\ 5\ 4), (1\ 2\ 4\ 3) \rangle$ isomorphic to S_4,
- d_{60} with representative $\langle (1\ 4)(2\ 3), (2\ 5\ 4) \rangle$ isomorphic A_5,
- d_{120} with the only representative isomorphic to G.

Note that there is only one subgroup H of G of index 2 which is isomorphic to A_5 and constitutes the class d_{60} (hence $I_2(G) = H$). The following table shows the fixed point dimensions of the nontrivial irreducible characters for all subgroups of G.

	d_1	$d_{2,1}$	$d_{2,2}$	d_3	$d_{4,1}$	$d_{4,2}$	$d_{4,3}$	d_5	$d_{6,1}$	$d_{6,2}$
X_1	1	0	1	1	1	0	0	1	0	1
$X_{4,1}$	4	1	2	2	1	1	0	0	0	1
$X_{4,2}$	4	3	2	2	1	1	2	0	2	1
$X_{5,1}$	5	3	3	1	2	1	2	1	1	1
$X_{5,2}$	5	2	3	1	2	2	1	1	0	1
X_6	6	3	2	2	0	1	1	2	1	0

	$d_{6,3}$	d_8	d_{10}	$d_{12,1}$	$d_{12,2}$	d_{20}	d_{24}	d_{60}	d_{120}
X_1	0	0	1	1	0	0	0	1	0
$X_{4,1}$	1	0	0	0	1	0	0	0	0
$X_{4,2}$	1	1	0	1	1	0	1	0	0
$X_{5,1}$	1	1	1	0	1	0	0	0	0
$X_{5,2}$	0	1	1	0	0	1	0	0	0
X_6	1	0	0	0	0	0	0	0	0

Put $L = \{X_1, X_{4,1}, X_{4,2}, X_{5,1}, X_{5,2}, X_6\}$. The conjugacy classes of subgroups of G containing p-subgroups are: d_1, $d_{2,1}$, $d_{2,2}$, d_3, $d_{4,1}$, $d_{4,2}$, $d_{4,3}$, d_5 and d_8. Therefore

$$D_L(G) = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 4 & 1 & 2 & 2 & 0 & 1 & 1 & 2 & 1 \\ 4 & 3 & 2 & 2 & 1 & 1 & 2 & 0 & 1 \\ 5 & 3 & 3 & 1 & 2 & 1 & 2 & 1 & 1 \\ 5 & 2 & 3 & 1 & 2 & 2 & 1 & 1 & 1 \\ 6 & 3 & 2 & 2 & 0 & 1 & 1 & 2 & 0 \end{pmatrix},$$

and $\mathrm{rank}(D_L(G)) = 6$ which is equal to the number of nontrivial real irreducible characters of G.

Let us analyze now all possible characters X which can occur as characters of tangent module to S^n at the fixed point (denoted by V) and exclude the corresponding postulated one fixed point actions. There are 6 such characters.

- $X = 7X_1$; take

$$\begin{aligned} H_1 &= \langle (2\ \ 3), (1\ \ 4) \rangle \in d_{4,3}, \\ H_2 &= \langle (2\ \ 3), (2\ \ 4)(3\ \ 5) \rangle \in d_8, \\ P &= \langle (2\ \ 3) \rangle \in d_{2,1}. \end{aligned}$$

Then $\langle H_1, H_2 \rangle = G$ and $\dim V^P = 0$ and, by Corollary 5.19, we conclude that $V \notin MN_{odd}(G)$.

- $X = 3X_1 + X_{4,1}$; take

$$\begin{aligned} H_1 &= \langle (2\ \ 3), (2\ \ 4)(3\ \ 5) \rangle \in d_8, \\ H_2 &= \langle (4\ \ 5), (1\ \ 2\ \ 3), (1\ \ 2) \rangle \in d_{12}, \\ P &= \langle (4\ \ 5), (2\ \ 3) \rangle \in d_{4,3}. \end{aligned}$$

As in the previous case, $\langle H_1, H_2 \rangle = G$ and $\dim V^P = 0$, and $V \notin MN_{odd}(G)$.

- $X = 3X_1 + X_{4,2}$; take

$$\begin{aligned} H_1 &= \langle (1\ \ 2) \rangle \in d_{2,1}, \\ H_2 &= \langle (2\ \ 5)(3\ \ 4), (2\ \ 4)(3\ \ 5) \rangle \in d_{4,1}, \\ P &= \langle \mathrm{id} \rangle \in d_1. \end{aligned}$$

Then $\langle H_1, H_2 \rangle = G$ and $\dim V^{H_1} + \dim V^{H_2} = 3 + 4 = 7 = \dim V^P$. Therefore, $V \notin MN_{odd}(G)$ by Corollary 5.20.

- $X = 2X_1 + X_{5,1}$; take H_1, H_2 and P from the previous case. Similarly, $\dim V^{H_1} + \dim V^{H_2} = 3 + 4 = 7 = \dim V^P$, and $V \notin MN_{odd}(G)$.

- $X = 2X_1 + X_{5,2}$; take

$$\begin{aligned} H_1 &= \langle (2\ \ 4)(3\ \ 5) \rangle \in d_{2,2}, \\ H_2 &= \langle (1\ \ 2\ \ 3\ \ 4) \rangle \in d_{4,2}, \\ P &= \langle \mathrm{id} \rangle \in d_1. \end{aligned}$$

Then $\langle H_1, H_2 \rangle = G$ and $\dim V^{H_1} + \dim V^{H_2} = 5 + 2 = 7 = \dim V^P$ and, again, $V \notin MN_{odd}(G)$ by Corollary 5.20.

- $X = X_1 + X_6$; then $\dim V^{I_2(G)} = 1$ and $V \notin MN_{one}(G)$ by Corollary 5.26. However, we cannot state that $V \notin MN_{odd}(G)$, since Corollary 5.26 concerns exclusively one fixed point actions on spheres.

Part III

Two fixed point actions on spheres

Chapter 6

Answering the Smith question

This chapter contains a survey of results concerning the question formulated by Smith in 1960.

Question 6.1. [65, footnote on p. 406] *Assume a group G acts with two fixed points x and y on a sphere S^n, $n \geq 0$. Are the tangent spaces T_xS^n and T_yS^n isomorphic as $\mathbb{R}G$-modules?*

First we present the cases of actions when the answer to the question is affirmative. Further, we describe negative answers to the question between year 1978 and 1992. Since more recent results are connected to two fixed point actions on disks, we cover this topic in a separate section. Afterwards, we return to the Smith problem and describe some aspects concerning computations of *Smith sets* which are fundamental concept of the modern theory. This part of the thesis is based on the recent survey of Pawałowski [51].

For convenience, an action of a group G on a homotopy sphere Σ of dimension n with exactly two fixed points and non-isomorphic tangent module structures at the two fixed points will be refered as *Smith exotic (G, n)-action*. We may shorten it to just *Smith exotic G-action* or even *Smith exotic action* if a group G and dimension n can be inferred from the context. We also call Σ a *Smith exotic G-sphere.*

6.1 Affirmative answers

In this section we present the cases of groups for which the affirmative answer to the Smith question is known, in chronological order.

One of the first answers was given by Atiyah and Bott in 1968 [1]. They excluded the existence of Smith exotic actions of cyclic groups of odd prime orders. Two years earlier Milnor [31] provided an affirmative answer for all groups in the special case of semi-free actions (that is, actions with isotropy subgroups being trivial or the whole group). In 1969 an interesting property was discovered by Bredon [7, Theorem II, p. 518] for cyclic groups of order being the power of 2. We obtain the following corollaries from the result of Bredon.

Theorem 6.2. [51, Theorem 1.3.1] *Assume G is a cyclic 2-group G acting on a homotopy sphere Σ of dimension n. Then, for any two points $x, y \in \Sigma^G$, the difference $T_x\Sigma - T_y\Sigma$ is divisible by $2^{f(n)}$ for some unbounded increasing function $f\colon \mathbb{N} \to \mathbb{N}$.*

Corollary 6.3. [51, Corollary 1.3.2] *Assume G is a cyclic group of order 2^k with $k \geq 1$. Then, there exists an integer $\mathrm{Br}(k)$ with the property that for any G-action on a homotopy sphere Σ of* *dimension at least* $\mathrm{Br}(k)$ *and arbitrary fixed points* $x, y \in \Sigma^G$*, the tangent spaces* $T_x\Sigma$ *and* $T_y\Sigma$ *are isomorphic as* $\mathbb{R}G$*-modules.*

From the corollary above, we conclude that there does not exists Smith exotic C_{2^k}-spheres of dimensions greater or equal $\mathrm{Br}(k)$.

Using the results of Atiyah and Bott [1], Sanchez [60] was able to obtain a result concerning actions of odd order groups satisfying some restrictions for fixed point sets of the actions of subgroups.

Theorem 6.4. (Sanchez, 1976, [60, Theorem 4.10]) *Assume a group* G *of odd order acts on a homology sphere* Σ *with two fixed points* x *and* y*. Suppose that for any* $H \leq G$ *we have* $\Sigma^H = \{x, y\}$ *or* Σ^H *is connected. Then* $T_x\Sigma$ *and* $T_y\Sigma$ *are isomorphic as* $\mathbb{R}G$*-modules.*

We can prove the following.

Corollary 6.5. [51, Corollary 1.4.2] *Assume that every element of a group* G *is of odd prime power order. If* G *acts on a homology sphere with* $\Sigma^G = x, y$*, then* $T_x\Sigma$ *and* $T_y\Sigma$ *are isomorphic as* $\mathbb{R}G$*-modules.*

For the next affirmative result, let us recall that a group G acts *pseudofreely* on a manifold M if for any $g \in G$ the fixed point set $M^{\langle g\rangle}$ is a discrete subset of M.

Theorem 6.6. (Illman, 1982, [26]) *There are no pseudofree Smith exotic* (G, n)*-actions for any group* G *and* $n \geq 5$*.*

A similar statement concerning Smith exotic C_{2n}-actions on odd-dimensional spheres for odd n was proved in 1988 by Suh.

Theorem 6.7. (Suh, 1988, [68, Corollary B]) *For any odd positive integers* n *and* k *there does not exist Smith exotic* $(C_{2n}, 2k+1)$*-actions.*

6.2 Negative answers

In the years 1978-1985 Dovermann, Petrie and Randall developed the equivariant surgery programme with the aim to construct group actions on manifolds with prescribed homotopy type (see [20,21,53–59] for the details). Their methods allowed to obtain the following negative results

Theorem 6.8. (Petrie-Randall, 1985, [58, Theorem A']) *If* G *is an abelian group of odd order with at least four noncyclic Sylow subgroups, then there is a Smith exotic* G*-action.*

In the meantime, between 1980 and 1982, Cappell and Shaneson [10,11] were condutcing rearearch concerning Smith problem for cyclic groups.

Theorem 6.9. (Cappell-Shaneson, 1980, [10, Theorem 2]) *If* $G = C_{4k}$ *where* $k \geq 2$*, then there exist Smith exotic* $(G, 9)$*-actions.*

Theorem 6.10. (Cappell-Shaneson, 1980, [10, Corollary 1A]) *Let* $G = C_n$*, where* $n \neq 4$ *is divisible by* 4 *and by at least one odd number greater than* 1*. There exist Smith exotic* $(G, 2l+1)$*-actions for any* $l \geq 4$*.*

Theorem 6.9 provides an exotic Smith action of C_8 on S^9. It can be proved using character theory arguments that C_8 is the group admitting Smith exotic actions of minimal order. Moreover, it belongs to the folklore knowledge that 9 is the minimal dimension of the Smith exotic sphere.

Further results of Dovermann, Cho, Suh and Washington from the period 1984 – 1989 showed that for certain cyclic, dihedral and quaternion groups, there exist Smith exotic actions on homotopy spheres, see [12–14, 19, 67]. The following results were obtained by the induction procedure for appropriate group modules developed by Cho [14].

Theorem 6.11. (Cho, 1985, [12]) *The generalized quaternion group of order* 2^{k+1}, $Q_k = \langle a, b | b^2 = a^{2^{k-1}}, a^{2^k} = 1, bab^{-1} = a^{-1} \rangle$, *admits Smith exotic actions provided* $k \geq 19$.

Theorem 6.12. (Cho, 1988, [13, Theorem A]) *There exists a Smith exotic action of the dihedral group of order* 2^m, *for a sufficiently large* m.

6.3 Two fixed point actions on disks

Smith exotic actions are closely related to two fixed point actions on disks with non-isomorphic group module structures on the tangent spaces at the fixed points. Since there shall be no confusion, we call these actions just *exotic* two fixed point actions on disks. In this section, we recall a few results concerning such actions. Throughout the following two sections, we assume that G is a group.

Definition 6.13. [51, Definition 2.4.2] *We call two* $\mathbb{R}G$*-modules* U *and* V ***Oliver-equivalent*** *if there exists an exotic two fixed action on a disk with tangent spaces at the two fixed points isomorphic to* U *and* V *respectively.*

Let us denote by $\mathrm{RO}(G)$ the real representation group of a group G, that is the abelian group which is the result of applying the Grothendieck construcion to the semigroup of all $\mathbb{R}G$-modules with the operation of direct sum. For a given subgroup $H \leq G$ and an $\mathbb{R}G$-module V, let us denote by $\mathrm{Res}^G_H(V)$ the module V considered as the $\mathbb{R}H$-module, that is the *restriction* of the G-action on V to the H-action on V. This induces the *restriction homomorphism*, $\mathrm{Res}^G_H \colon \mathrm{RO}(G) \to \mathrm{RO}(G)$, $U - V \mapsto \mathrm{Res}^G_H(U) - \mathrm{Res}^G_H(V)$.

Definition 6.14. *The* ***primary group*** *of* G, *denoted by* $\mathrm{PO}(G)$, *is the subgroup of all elements* $U - V \in \mathrm{RO}(G)$ *such that* $\mathrm{Res}^G_P(U) \cong \mathrm{Res}^G_P(V)$ *for any subgroup* $P \leq G$ *of prime power order. The* ***reduced primary group*** *of* G, *denoted by* $\widetilde{\mathrm{PO}}(G)$, *is the subgroup of* $\mathrm{PO}(G)$ *consisting of all elements* $U - V \in \mathrm{PO}(G)$ *such that* $U^G = V^G = \{0\}$.

It follows from the Smith theory that for any Oliver-equivalent $\mathbb{R}G$-modules U and V we have $U - V \in \widetilde{\mathrm{PO}}(G)$. The second article of Oliver provides also the converse result.

Theorem 6.15. (Oliver, [43, Theorem 0.4]) *If* $U - V \in \widetilde{\mathrm{PO}}(G)$ *for some* $\mathbb{R}G$*-modules* U *and* V, *then there exists an* $\mathbb{R}G$*-module* W *such that* $U \oplus W$ *and* $V \oplus W$ *are Oliver-equivalent.*

Corollary 6.16. [51, Theorem 2.4.3] *An element of* $\mathrm{RO}(G)$ *is of the form* $U - V$ *for some Oliver-equivalent* $\mathbb{R}G$*-modules* U *and* V *if and only if* $U - V \in \widetilde{\mathrm{PO}}(G)$.

It turns out that the rank of the reduced primary group can be expressed in terms of *Laitinen number*. Before we do that, let us recall the definition of a *real conjugacy class* and of the Laitinen number.

Definition 6.17. *The* ***real conjugacy class*** *of an element $g \in G$ is the union of conjugacy classes $(g)^{\pm} = (g) \cup (g^{-1})$. The* ***Laitinen number*** *of G, denoted by $\lambda(G)$, is the number of distinct real conjugacy classes of G whose representatives are not of prime power order.*

The result concerning the rank of primary groups reads as follows.

Theorem 6.18. (Laitinen-Pawałowski, [30, Lemma 2.1]) *The following holds.*

(1) *If $\lambda(G) = 0$, then $\mathrm{PO}(G) = 0$.*

(2) *If $\lambda(G) \geq 1$, then $\mathrm{rank}(\mathrm{PO}(G)) = \lambda(G)$ and $\mathrm{rank}(\widetilde{\mathrm{PO}}(G)) = \lambda(G) - 1$.*

6.4 Smith sets

The central object related to the modern approaches to the Smith question is the *(primary) Smith set.*

Definition 6.19. *The* ***Smith set*** *of a group G, denoted $\mathrm{Sm}(G)$, is the set of elements $U - V \in \mathrm{RO}(G)$ such that there exists a two fixed point action on some homotopy sphere with the $\mathbb{R}G$-modules structures at the tangent spaces at the two fixed points isomorphic to $U \oplus W$ and $V \oplus W$ respectively for some $\mathbb{R}G$-module W.*

Definition 6.20. *The* ***primary Smith set*** *of G is the intersection of $\mathrm{Sm}(G)$ and $\widetilde{\mathrm{PO}}(G)$. We denote it by $\mathrm{PSm}(G)$.*

The following theorem provides sufficient conditions for a group to have equal ordinary and primary Smith sets.

Theorem 6.21. [30, p. 304] [49, p. 853] [50, Definition 2.4] *If G has no element of order 8 or, otherwise, if for any element $g \in G$ of order 2^k, $k \geq 3$, then $\mathrm{Sm}(G) = \mathrm{PSm}(G)$. Moreover, $\mathrm{Sm}(G) = \mathrm{PSm}(G)$ if at least one of the following conditions hold.*

(1) *g or g^{-1} is conjugate to an odd power of g.*

(2) *$\dim W^{\langle g \rangle} > 0$ for every irreducible $\mathbb{R}G$-module W.*

On the other hand, if G is a perfect group, then the primary Smith set of G equals the reduced primary group, $\widetilde{\mathrm{PO}}(G)$, see [51, Corollary 4.1.6].

Corollary 6.22. *If G is a perfect group satisfying the assumptions of Theorem 6.21, then $\mathrm{Sm}(G) = \widetilde{\mathrm{PO}}(G)$.*

For a perfect group G, the Smith set, $\mathrm{Sm}(G)$, is trivial (consists of one element) whenever $\lambda(G) \in \{0, 1\}$ and it is infinite if $\lambda(G) \geq 2$. Therefore there exists Smith exotic G-action if and only if $\lambda(G) \geq 2$.

Remark 6.23. In the case a group G satisfies the assumptions of the corollary above, we have a natural lower bound for dimension of spheres admitting Smith exotic actions of G. This lower bound is the number $\min\{\dim U | U - V \in \widetilde{\mathrm{PO}}(G)\}$.

For more detailed survey on results concerning the Smith sets, we refer the reader to [51].

Chapter 7

New family of groups satisfying the Laitinen Conjecture

In this chapter we study the *Laitinen Conjecture* which proposes negative answers to the Smith question. We present the result concerning a new infinite family of groups which satisfy this conjecture. The contents of this chapter is also contained in our latest publication [32]. Let us provide the reader with the necessary background now.

Assume G is a group. Let us call two $\mathbb{R}G$-modules U and V *Smith equivalent* if $U \cong T_x(\Sigma)$ and $V \cong T_y(\Sigma)$ as $\mathbb{R}G$-modules, for a smooth action of G on a homotopy sphere Σ with exactly two fixed points x and y. We say that the *Laitinen Condition* is satisfied for G acting smoothly on a homotopy sphere Σ with $\Sigma^G = \{x, y\}$, if Σ^g is connected for any $g \in G$ of order 2^k, where $k \geq 3$.

The Laitinen Conjecture proposes negative answers to the Smith question concerning actions on homotopy spheres. The conjecture reads as follows.

Conjecture 7.1. [30, Appendix] *If G is an Oliver group with $\lambda(G) \geq 2$, then there exist non-isomorphic $\mathbb{R}G$-modules U and V which are Smith equivalent and the action of G on the homotopy sphere in question satisfies the Laitinen Condition.*

The converse conclusion is always true [30] and Conjecture 7.1 is known to be true in the following cases, [51].

(1) G is of odd order (and thus, by the Feit-Thompson Theorem, G is solvable).

(2) G has a cyclic quotient of odd composite order (for example, G is a nilpotent group with three or more noncyclic Sylow subgroups).

(3) G is a nonsolvable group not isomorphic to $\mathrm{Aut}(A_6)$ (in the case where $G = \mathrm{Aut}(A_6)$, the Laitinen Conjecture is false by [38]).

(4) G satisfies the Sumi G^{nil}-condition (the condition is defined below).

For a prime p, let us use the notation $\mathcal{O}^p(G)$ for the smallest normal subgroup of G with $G/\mathcal{O}^p(G)$ a p-group. A subgroup H of a group G is called *large* if $\mathcal{O}^p(G) \leq H$ for some prime p. We denote by $\mathcal{L}(G)$ the family of all large subgroups of G. Let us call G a *gap group* if there exists an $\mathbb{R}G$-module V such that for any $P < H \leq G$ with P of prime power order, we have $\dim V^P > 2 \dim V^H$ and for any $L \in \mathcal{L}(G)$, $\dim V^L = 0$ holds. Denote by G^{nil} the smallest normal subgroup of G

such that G/G^{nil} is nilpotent. We say that G satisfies the *Sumi G^{nil}-condition* if there exist two elements $a, b \in G$ of composite order which are not real conjugate in G, the equality $aG^{\text{nil}} = bG^{\text{nil}}$ holds and at least one of the following statements holds.

- $|a|$ and $|b|$ are even and the involutions of the cyclic subgroups $\langle a \rangle$ and $\langle b \rangle$ are conjugate in G.
- a and b belong to the same gap subgroup of G.

Therefore, in checking the Laitinen Conjecture, we shall focus on finite solvable Oliver groups G of even order, such that each cyclic quotient of G is either of even or of prime power order and G does not satisfy the Sumi G^{nil}-condition. We refer to such a group G as a *special Oliver group*. In general, however, Conjecture 7.1 is not true. It fails for example for $G = \text{Aut}(A_6)$ [38] or $G = S_3 \times A_4$ (see [50] for more counterexamples).

We say that two $\mathbb{R}G$-modules U and V are *$\mathcal{P}$-matched* if for any subgroup $P \leq G$ of prime power order, the restrictions $\text{Res}^G_P(U)$ and $\text{Res}^G_P(V)$ are isomorphic as P-modules. In other words, U and V are $\mathcal{P}$-matched if and only if $U - V \in \text{PO}(G)$.

In 2018, Pawałowski [51] proposed the following problem.

Problem 7.2. *For which special Oliver groups G with $\lambda(G) \geq 2$, there exist pairs of $\mathcal{P}$-matched and Smith equivalent $\mathbb{R}G$-modules which are not isomorphic to each other?*

Some examples of special Oliver groups G with $\lambda(G) \geq 2$ such that no $\mathbb{R}G$-modules in question exist were already given in [50]. We present here a certain infinite family of special Oliver groups with primary numbers at least 2, possessing pairs of $\mathcal{P}$-matched Smith equivalent $\mathbb{R}G$-modules which are not isomorphic.

Suppose p and q are odd prime numbers such that $q|(p-1)$. Let D_{2pq} be the dihedral group of order $2pq$ and C_q be the cyclic group of order q. These groups have the following presentations.

$$D_{2pq} = \langle a, b | a^{pq} = b^2 = 1, bab = a^{-1} \rangle \quad \text{and} \quad C_q = \langle c | c^q = 1 \rangle.$$

Let v be a primitive root modulo p which is not divisible by q (in case $q|v$, just take $p+v$ instead of v which is also a primitive root modulo p). Let ω be the remainder of $v^{(p-1)(q-1)/q}$ modulo pq. Note that $\omega \equiv 1 \pmod{q}$ and the order of ω modulo p is q. Therefore $\omega \not\equiv 1 \pmod{pq}$ and $\omega^q \equiv 1 \pmod{pq}$ by the Chinese Reminder Theorem. Consider the automorphism τ of D_{2pq} given by $\tau(a) = a^\omega$ and $\tau(b) = b$. The order of τ is q. Thus, we have a homomorphism $\varphi \colon C_q \to \text{Aut}(D_{2pq})$, $c \mapsto \tau$. Define $G_{p,q}$ as the following semidirect product.

$$G_{p,q} = D_{2pq} \rtimes_\varphi C_q$$

The main theorem of this chapter can be stated as follows.

Theorem 7.3. (cf. Theorem 1.5) *For any two odd primes p and q such that $q|(p-1)$, $G_{p,q}$ is a special Oliver group with $\lambda(G_{p,q}) \geq 2$, possessing pairs of non-isomorphic $\mathcal{P}$-matched Smith equivalent $\mathbb{R}G_{p,q}$-modules.*

Remark 7.4. Note that the theorem above confirms the Laitinen Conjecture for $G_{p,q}$'s since the Laitinen Condition is naturally satisfied due to the lack of elements of order divisible by 8 in $G_{p,q}$'s.

Remark 7.5. In the case where $q = 2$, $N = \{(a^{qs}, 1)|s = 0, ..., p-1\}$ is a normal subgroup of $G_{p,q}$ isomorphic to the cyclic group of order p, such that the quotient $G_{p,q}/N$ is a 2-group. Thus, $G_{p,q}$ is not an Oliver group. Moreover, any nontrivial element of $G_{p,q}$ is of order 2, 4, or p, where p is an odd prime. Therefore, by elementary character theory arguments and the result of Atiyah and Bott [1, Thm. 7.15], any two Smith equivalent $\mathbb{R}G_{p,q}$-modules are isomorphic.

Fix odd primes p and q such that $q|(p-1)$. For a better presentation of the material, let us introduce additionally the following symbols and concepts (G denotes a finite group).

- An $\mathbb{R}G$-module V is said to satisfy the *weak gap condition* if for any $P < H \leq G$ such that P is of prime power order, we have $\dim V^P \geq 2 \dim V^H$.
- $\mathrm{PO}^{\mathcal{L}}_{\mathrm{w}}(G)$ – the subgroup of $\mathrm{PO}(G)$ containing elements which can be written as $U - V$ for some $\mathbb{R}G$-modules U and V satisfying the weak gap condition and such that $\dim W^L = 0$ for any $L \in \mathcal{L}(G)$ and $W = U, V$.
- N_{pq^2} – the unique subgroup of $G_{p,q}$ of index 2.
- $\mathrm{Ind}_H^G \colon \mathrm{PO}(H) \to \mathrm{PO}(G)$ – the induction homomorphism defined for any subgroup $H \leq G$ by the formula $U - V \mapsto \mathrm{Ind}_H^G(U) - \mathrm{Ind}_H^G(V)$, where $\mathrm{Ind}_H^G(W)$ denotes the induced $\mathbb{R}G$-module from the $\mathbb{R}H$-module W. This is a well-defined map since, if U and V are $\mathcal{P}$-matched $\mathbb{R}H$-modules, then so are $\mathrm{Ind}_H^G(U) - \mathrm{Ind}_H^G(V)$ as $\mathbb{R}G$-modules (we comment on this fact in the subsequent part).

The chapter is organized as follows. First, we show that $\mathrm{PO}^{\mathcal{L}}_{\mathrm{w}}(N_{pq^2}) \neq 0$. In the next section, we prove that $G_{p,q}$ is a special Oliver group with $\lambda(G_{p,q}) \geq 2$. The third section provides, for any finite groups $H \leq G$, the necessary and sufficient condition for $\mathrm{Ind}_H^G \colon \mathrm{PO}(H) \to \mathrm{PO}(G)$ to be a monomorphism. Finally, we prove Theorem 7.3 using the properties of the induction from N_{pq^2} to $G_{p,q}$.

7.1 Nontriviality of a specific primary group

Note that $N_{pq^2} = \{(a^l, c^m)|l = 0, ..., pq-1, m = 0, ..., q-1\}$. Recall that $G_{p,q}$ is the semidirect product of D_{2pq} and C_q defined by the homomorphism $\varphi \colon C_q \to \mathrm{Aut}(D_{2pq})$ which sends the generator c of C_q to the automorphism τ of D_{2pq} such that $\tau(a) = a^\omega$ and $\tau(b) = b$, where ω is so chosen number from the set $\{1, ..., pq-1\}$ that $\omega \not\equiv 1 \pmod{pq}$ and $\omega^q \equiv 1 \pmod{pq}$. Thus, we have

$$(1, c)(a, 1)(1, c)^{-1} = (1, c)(a, 1)(1, c^{-1}) = (1, c)(a, c^{-1}) = (a^\omega, 1).$$

Thus, under the identifications $a \leftrightarrow (a, 1)$ and $c \leftrightarrow (1, c)$, N_{pq^2} can be presented as

$$N_{pq^2} = \langle a, c | a^{pq} = c^q = 1, cac^{-1} = a^\omega \rangle.$$

Let

$$N'_{pq^2} = \langle \alpha, \beta, \gamma | \alpha^q = \beta^p = \gamma^q = 1, \gamma\beta\gamma^{-1} = \beta^\omega, \alpha\beta = \beta\alpha, \alpha\gamma = \gamma\alpha \rangle.$$

Then N'_{pq^2} is isomorphic to the direct product of $C_q = \langle \alpha | \alpha^q = 1 \rangle$ with the Frobenius group $F_{p,q}$ generated by β and γ.

Lemma 7.6. *Let $f \colon N'_{pq^2} \to N_{pq^2}$ be given by $f(\alpha) = a^p$, $f(\beta) = a^q$ and $f(\gamma) = c$. Then f is a group isomorphism.*

Proof. Note that f is a well-defined group homomorphism. Indeed, $f(\alpha^q) = a^{pq} = 1$, $f(\beta^p) = a^{pq} = 1$, $f(\gamma^q) = c^q = 1$, $f(\gamma\beta\gamma^{-1}) = ca^q c^{-1} = (cac^{-1})^q = a^{\omega q} = f(\beta^\omega)$, $f(\alpha\beta) = a^{p+q} = a^{q+p} = f(\beta\alpha)$, $f(\gamma\alpha\gamma^{-1}) = ca^p c^{-1} = a^{p\omega} = a^p = f(\alpha)$. The equality $a^{p\omega} = a^p$ follows from the fact that $pq | p(\omega - 1)$ since $\omega \equiv 1 \pmod q$.

Take any $a^l c^m \in N_{pq^2}$. Since p and q are different primes, we can find $x, y \in \mathbb{Z}$ such that $1 = xp + yq$ and

$$f(\alpha^{lx}\beta^{ly}\gamma^m) = a^{plx}a^{qly}c^m = a^{l(xp+yq)}c^m = a^l c^m.$$

Hence f is surjective. Let us prove that it is injective as well. Suppose $f(\alpha^x\beta^y c^m) = 1$. Then $a^{px+qy}c^m = 1$ which is the case only if $pq | (px + qy)$ and m is divisible by q. Since $p|px$ and $q|qy$, it follows that $p|qy$ and $q|px$ and this means $p|y$ and $q|x$. As a consequence, $\alpha^x\beta^y\gamma^m = 1$ and f has the trivial kernel. □

Put $u \equiv \omega \pmod p$ and, according to our assumptions, let $p-1 = qr$. Assume $v_1, ..., v_r$ are the representatives of the cosets of $\langle u \rangle$ in the multiplicative group $\mathbb{Z}_p^*$. Let $(\beta^{v_j}) = \{\beta^{v_j s} | s \in \langle u \rangle\}$ and $(\gamma^n) = \{\beta^m\gamma^n | 0 \le m \le p-1\}$ for all $1 \le j \le r$ and $1 \le n \le q-1$. Following [27, 25.10 Theorem] the conjugacy classes of $F_{p,q}$ are as follows.

class	(1)	(β^{v_j})	(γ^n)
representative order	1	p	q
size	1	q	p
# of classes of a given type	1	r	$q-1$

Table 7.1: Conjugacy classes of $F_{p,q}$.

Let $\sigma_{t,x} = \sum_{s\in\langle u\rangle} \zeta_p^{v_t x s}$ for $x = 0, ..., p-1$, $t = 1, ..., r$, where $\zeta_p = e^{2\pi i}$. We have r nonlinear irreducible characters of $F_{p,q}$ given by $\chi_t(\beta^x) = \sigma_{t,x}$ and $\chi_t(\gamma^n) = 0$ for $x = 0, ..., p-1$, $t = 1, ..., r$ and $n = 1, ..., q-1$. They are presented in the table below.

	(1)	(β^{v_j})	(γ^n)
χ_1	q	σ_{1,v_j}	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
χ_r	q	σ_{r,v_j}	0

Table 7.2: Nonlinear irreducible characters of $F_{p,q}$.

The following table contains the nonlinear irreducible characters of $N_{pq^2} \cong C_q \times F_{p,q}$.

g	$(1,1)$	$(1,\beta^{v_j})$	$(1,\gamma^n)$	(α^l,β^{v_j})	(α^l,γ^n)	$(\alpha^l,1)$
$\lvert g\rvert$	1	p	q	pq	q	q
$\lvert(g)\rvert$	1	q	p	q	p	1
# (g)	1	r	$q-1$	$(q-1)r$	$(q-1)^2$	$q-1$
$\psi_{s,t} = \rho_s \times \chi_t$	q	σ_{t,v_j}	0	$\zeta_q^{ls}\sigma_{t,v_j}$	0	$q\zeta_q^{ls}$

Table 7.3: Nonlinear irreducible characters of N_{pq^2}.

Let $N_p = \{(1, \beta^s) | s = 0, ..., p-1\}$. Obviously, N_p is a normal subgroup of N_{pq^2} isomorphic to C_p.

Lemma 7.7. *$\mathcal{O}^q(N_{pq^2}) = N_p$ and $\mathcal{O}^p(N_{pq^2}) = N_{pq^2}$. As a result, all $L \in \mathcal{L}(N_{pq^2})$ contain N_p as a subgroup.*

Proof. It is obvious that $\mathcal{O}^q(N_{pq^2}) = N_p$. We show that there is no normal subgroup of N_{pq^2} of order q^2 which would conclude the proof.

Suppose for the converse that N is a normal subgroup of N_{pq^2} of order q^2. There exists $g \in N$ of order q. Since $N \trianglelefteq N_{pq^2}$, we have $(g) \subseteq N$. We know by Table 7.3 that g belongs to one of the following conjugacy classes: $((1, \gamma^n))$, $((\alpha^l, \gamma^n))$ or $((\alpha^l, 1))$. Suppose $g \in ((1, \gamma^{n_0}))$ for some $n_0 \in \{1, ..., q-1\}$. Since $\{(1, \gamma^n) | n = 0, ..., q-1\} = \langle (1, \gamma^{n_0}) \rangle \leq N$, it follows that each class $((1, \gamma^n))$ is contained in N. This yields at least $p(q-1) > q^2$ elements in N. A contradiction. Let $g \in ((\alpha^{l_0}, \gamma^{n_0}))$ for some $l_0, n_0 \in \{1, ..., q-1\}$. Then, similarly as before, considering $\langle (\alpha^{l_0}, \gamma^{n_0}) \rangle \leq N$ yields at least $p(q-1) > q^2$ elements in N and we can exclude this case as well. Thus, all elements of order q of N belong to one of the classes $((\alpha^l, 1))$. From Table 7.3 follows that there are $q-1$ elements in these classes altogether. Moreover, every element of N different from the identity is of order q. This yields $|N| = q$ which is also a contradiction. □

Since characters of any group G determine FG-modules up to isomorphism for $F = \mathbb{R}, \mathbb{C}$, we shall use the same symbols for the characters and the FG-modules determined by them. Moreover, if χ is the character of G determined by some FG-module, then by $\dim \chi^H$ we mean the fixed point dimension over F for a subgroup H acting on this FG-module. Note that in case χ is a character of some $\mathbb{R}G$-module then all such fixed point dimensions over $\mathbb{R}$ are equal as considered over $\mathbb{C}$ – we can treat χ as a character of a $\mathbb{C}G$-module as well.

Lemma 7.8. *Let $s \neq 0$ and H be a subgroup of N_{pq^2} of order p or q^2. Then $\dim \psi_{s,t}^H = 0$ for any $t = 1, ..., r$.*

Proof. Suppose $|H| = p$. Then $H = N_p$ and it follows from Table 7.3 that

$$\begin{aligned}\dim \psi_{s,t}^H &= \frac{1}{|H|} \sum_{h \in H} \psi_{s,t}(h) = \frac{1}{p}\Big(q + \sum_{x=1}^{p-1} \sigma_{t,x}\Big) = \frac{1}{p}\Big(q + \sum_{x=1}^{p-1} \sum_{s \in \langle u \rangle} e^{2\pi i v_t x s/p}\Big) \\ &= \frac{1}{p}\Big(q + \sum_{s \in \langle u \rangle} \sum_{x=1}^{p-1} e^{2\pi i v_t x s/p}\Big) = \frac{1}{p}\Big(q + \sum_{s \in \langle u \rangle} (-1)\Big) = 0.\end{aligned}$$

(The last equality holds since there are q elements in $\langle u \rangle$ by the definition of u).

If $|H| = q^2$, then, since the only nonzero values of $\psi_{s,t}$ on elements of order q are taken for the classes $(\alpha^l, 1)$, it follows that

$$\dim \psi_{s,t}^H < \frac{1}{q^2}\Big(q + \sum_{l=1}^{q-1} |q\zeta_q^{ls}|\Big) = \frac{1}{q^2}(q + q(q-1)) = 1$$

and $\dim \psi_{s,t}^H = 0$. □

Corollary 7.9. *If $s \neq 0$, then $2\,\mathrm{Re}\,\psi_{s,t}$ is an $\mathbb{R}N_{pq^2}$-module satisfying the weak gap condition and such that $\dim(2\,\mathrm{Re}\,\psi_{s,t})^L = 0$ for any $L \in \mathcal{L}(N_{pq^2})$.*

Proof. From the properties of real and complex irreducible representations, we know that $2\operatorname{Re}\psi_{s,t}$ is the character of a real irreducible N_{pq^2}-module since $\psi_{s,t}$ is not real-valued. Take any $L \in \mathcal{L}(N_{pq^2})$. We know by Lemma 7.7 that $N_p \leq L$. Thus, by Lemma 7.8, we get

$$\dim(2\operatorname{Re}\psi_{s,t})^L = \dim(\psi_{s,t} + \overline{\psi_{s,t}})^L = 2\dim\psi_{s,t}^L \leq 2\dim\psi_{s,t}^{N_p} = 0.$$

It remains to show that $2\operatorname{Re}\psi_{s,t}$ satisfies the weak gap condition. By means of Lemma 7.8, this boils down to proving that

$$\dim(2\operatorname{Re}\psi_{s,t}) \geq 2\dim(2\operatorname{Re}\psi_{s,t})^H$$

for any subgroup $H \leq N_{pq^2}$ of order q. Then there exists $h \in H$ which has one of the forms: $h = (1,\gamma^n)$, $h = (\alpha^l,\gamma^n)$ or $h = (\alpha^l, 1)$ for some $l, n \in \{1, ..., q-1\}$. Since $\psi_{s,t}(1,\gamma^n) = \psi_{s,t}(\alpha^l,\gamma^n) = 0$ by Table 7.3, we have in the first and in the second case

$$2\dim(2\operatorname{Re}\psi_{s,t})^H = 2\cdot 2\cdot\dim\psi_{s,t}^H = 4\cdot\frac{1}{q}\cdot q = 4.$$

In the second case, since $\psi_{s,t}(\alpha^l, 1) = q\zeta_q^{ls}$ and $s \neq 0$, it follows that

$$2\dim(2\operatorname{Re}\psi_{s,t})^H = 2\cdot 2\cdot\dim\psi_{s,t}^H = 4\cdot\frac{1}{q}(q + q\sum_{l=1}^{q-1}\zeta_q^{ls}) = 4\cdot\frac{1}{q}(q + q\cdot(-1)) = 0.$$

Thus, we get

$$\dim(2\operatorname{Re}\psi_{s,t}) = 2q > 4 \geq 2\dim(2\operatorname{Re}\psi_{s,t})^H.$$

□

Lemma 7.10. *Let $s \neq 0$. Then, for any $t = 1, ..., r$, the $\mathbb{R}G$-modules $U = 2\operatorname{Re}\psi_{s,t}$ and $V = 2\operatorname{Re}\psi_{q-s,t}$ are not isomorphic and $\mathcal{P}$-matched.*

Proof. It follows from Table 7.3 that U and V are $\mathcal{P}$-matched. Note that $U = \rho_s \times \chi_t$ and $V = \overline{\rho_s} \times \chi_t$. By the similar computation as in the proof of Lemma 7.8, we establish $\iota(\chi_t)$, the Frobenius-Schur indicator of character χ_t.

$$\begin{aligned}\iota(\chi_t) &= \frac{1}{|F_{p,q}|}\sum_{g\in F_{p,q}}\chi_t(g^2) = \frac{1}{pq}\Big(q + \sum_{|g|=p}\chi_t(g^2)\Big) = \frac{1}{pq}\Big(q + \sum_{|g|=p}\chi_t(g)\Big)\\ &= \frac{1}{pq}\Big(q + \sum_{x=1}^{p-1}\sigma_{t,x}\Big) = 0\end{aligned}$$

for from the proof of Lemma 7.8 we know that $\sum_{x=1}^{p-1}\sigma_{t,x} = -q$. Thus, χ_t is not real-valued and we can take $x = 0, ..., p-1$ such that $\operatorname{Im}(\chi_t(\beta^x)) \neq 0$. Now, take $l = 1, ..., q-1$ and put $g = (\alpha^l, \beta^x)$. Clearly, g is an element of order pq. Then, $U(g)$, the character of U evaluated on g is equal to the number

$$\begin{aligned}U(g) &= 2\operatorname{Re}\psi_{s,t}(g) = 2\operatorname{Re}(\rho_s(\alpha^l)\chi_t(\beta^x))\\ &= 2(\operatorname{Re}(\rho_s(\alpha^l))\operatorname{Re}(\chi_t(\beta^x)) - \operatorname{Im}(\rho_s(\alpha^l))\operatorname{Im}(\chi_t(\beta^x))).\end{aligned}$$

Analogously, $V(g)$, the character of V evaluated on g is equal to

$$\begin{aligned}V(g) &= 2\operatorname{Re}\psi_{q-s,t}(g) = 2\operatorname{Re}(\overline{\rho_s(\alpha^l)}\chi_t(\beta^x))\\ &= 2(\operatorname{Re}(\rho_s(\alpha^l))\operatorname{Re}(\chi_t(\beta^x)) + \operatorname{Im}(\rho_s(\alpha^l))\operatorname{Im}(\chi_t(\beta^x)))\\ &\neq U(g).\end{aligned}$$

□

Corollary 7.11. $\mathrm{PO}_{\mathrm{w}}^{\mathcal{L}}(N_{pq^2})$ *is nonzero.*

7.2 $G_{p,q}$ is a special Oliver group with $\lambda(G_{p,q}) \geq 2$

We divide the material contained in this section into three parts. In the first, we determine conjugacy classes of $G_{p,q}$. Using this, we show that $\lambda(G_{p,q}) \geq 2$. In the next part, we establish all normal subgroups of $G_{p,q}$ and infer the necessary information concerning the quotients of $G_{p,q}$. Finally, we use the performed computations to prove that $G_{p,q}$ is an example of a special Oliver group.

7.2.1 Conjugacy classes of $G_{p,q}$

Any element of $G_{p,q}$ is either of the form $x_1 = (ba^l, c^m)$ or $x_2 = (a^l, c^m)$ for some $l = 0, ..., pq-1$ and $m = 0, ..., q-1$. In the first case, its inverse $x_1^{-1} = (ba^{l\omega^{-m}}, c^{-m})$, while in the second $x_2^{-1} = (a^{-l\omega^{-m}}, c^{-m})$.

Let $g \in G_{p,q}$. We have the following possibilities.

(1) $g = (ba^{l_0}, c^{m_0})$. Then

$$x_1 g x_1^{-1} = (ba^{l(1+\omega^{m_0})-l_0\omega^m}, c^{m_0}) \quad \text{and} \quad x_2 g x_2^{-1} = (ba^{-l(1+\omega^{m_0})+l_0\omega^m}, c^{m_0}).$$

Note that the expression $l(1+\omega^{m_0}) - l_0\omega^m$ can take any remainder modulo pq. Since $\omega \equiv 1 \pmod q$, it follows that $l(1+\omega^{m_0}) - l_0\omega^m \equiv 2l - l_0 \pmod q$ and substituting subsequent values for $l = 0, ..., pq-1$, we can obtain any pair of remainders of $l(1+\omega^{m_0}) - l_0$ modulo p and q (this follows since $1+\omega^{m_0}$ cannot be divisible by p for $\omega^{m_0} \not\equiv 1 \pmod p$ because $\omega^{m_0} \equiv 1 \pmod p$ holds if and only if m_0 is divisible by $(p-1)/2$ which is not possible if $m_0 \in \{1, ..., q-1\}$). We conclude then from the Chinese Remainder Theorem, that for any $l' = 0, ..., pq-1$, there exist $l = 0, ..., pq-1$ such that $l(1+\omega^{m_0}) - l_0\omega^0 = l(1+\omega^{m_0}) - l_0 \equiv l' \pmod{pq}$. Therefore

$$(g) = \{(ba^l, c^{m_0}) | l = 0, ..., pq-1\}.$$

Note that $(b, c^{m_0})^n = (b^n, c^{nm_0})$. Hence $|g| = 2q$ if $m_0 \neq 0$ and $|g| = 2$ if $m_0 = 0$.

(2) $g = (a^{l_0}, c^{m_0})$, where $m_0 \neq 0$. Then

$$x_1 g x_1^{-1} = (a^{l(\omega^{m_0}-1)-l_0\omega^m}, c^{m_0}) \quad \text{and} \quad x_2 g x_2^{-1} = (a^{-l(\omega^{m_0}-1)+l_0\omega^m}, c^{m_0}).$$

We have $l(\omega^{m_0}-1) - l_0\omega^m \equiv -l_0 \pmod q$ and substituting subsequent values for l, we can achieve all remainders modulo p of $l(\omega^{m_0}-1) - l_0\omega^m$ ($\omega^{m_0} \not\equiv 1 \pmod p$ since otherwise m_0 had to be divisible by $p-1$ which is not possible in the considered case). If r_0 is the remainder modulo q of l_0, it follows then that

$$(g) = \{(a^{r_0+lq}, c^{m_0}), (a^{-r_0+lq}, c^{m_0}) | l = 0, ..., p-1\}.$$

For any $n \geq 0$

$$g^n = (a^{l_0(1+\omega^{m_0}+...+\omega^{(n-1)m_0})}, c^{m_0}) = (a^{l_0 \cdot \frac{1-\omega^{nm_0}}{1-\omega^{m_0}}}, c^{nm_0}).$$

Thus $q||g|$. On the other hand $p|\frac{1-\omega^{qm_0}}{1-\omega^{m_0}}$ since $1-\omega^{qm_0}$ is divisible by pq and $p \nmid 1-\omega^{m_0}$. Moreover, $1+\omega^{m_0}+...+\omega^{(q-1)m_0} \equiv q \equiv 0 \pmod q$, so $pq|\frac{1-\omega^{qm_0}}{1-\omega^{m_0}}$ and thus $g^q = (1,1)$ from which we conclude $|g| = q$.

(3) $g = (a^{l_0}, 1)$. The computations of conjugacy class elements reduce then to

$$x_1 g x_1^{-1} = (a^{-l_0\omega^m}, 1) \quad \text{and} \quad x_2 g x_2^{-1} = (a^{l_0\omega^m}, 1).$$

If $p \nmid l_0$, then all the numbers from the set $S_{l_0} = \{\pm l_0\omega^m | m = 0, ..., q-1\}$ give distinct remainders modulo p – this follows from the definition of ω. Thus, we have $(p-1)/2$ such conjugacy classes, each with $2q$ elements and

$$(g) = \{(a^{l_0\omega^m}, 1), (a^{-l_0\omega^m}, 1) | m = 0, ..., q-1\}$$

Moreover, for any $n \geq 0$, $g^n = (a^{nl_0}, 1)$, so $|g| = pq$ if $q \nmid l_0$ and $|g| = p$ if $q|l_0$.

If $p|l_0$ and $q \nmid l_0$, then the set S_{l_0} reduces to two elements, $(a^{l_0}, 1)$ and $(a^{-l_0}, 1)$. We have $(q-1)/2$ such classes and

$$(g) = \{(a^{l_0}, 1), (a^{-l_0}, 1)\} \quad \text{and} \quad |g| = q.$$

Finally, the last class left is the class of the identity element, $(g) = \{(1, 1)\}$.

The following table summarizes the information about the conjugacy classes of $G_{p,q}$ and orders of its elements (recall that $r = (p-1)/q$).

g	$(1,1)$	B	E_s	C_m	$D_{s,m}$	F_s	B_m	A_l
$\|g\|$	1	2	p	q	q	q	$2q$	pq
$\|(g)\|$	1	pq	$2q$	p	$2p$	2	pq	$2q$
$\#\ (g)$	1	1	$\frac{1}{2}r$	$q-1$	$\frac{1}{2}(q-1)^2$	$\frac{1}{2}(q-1)$	$q-1$	$\frac{1}{2}(q-1)r$

Table 7.4: Conjugacy classes of $G_{p,q}$.

where

$$\begin{aligned} B &= (b,1), & E_s &= (a^{qs},1), s = 1,...,p-1, \\ C_m &= (1,c^m), m = 1,...,q-1, & D_{s,m} &= (a^s,c^m), m = 1,...,q-1, q \nmid s, \\ F_s &= (a^{ps},1), s = 1,...,q-1, & B_m &= (b,c^m), m = 1,...,q-1, \\ A_l &= (a^l,1), p,q \nmid l. \end{aligned}$$

Lemma 7.12. $\lambda(G_{p,q}) = \frac{1}{2}(q-1)(r+1)$. *Thus* $\lambda(G_{p,q}) \geq 2$.

Proof. We establish first the real conjugacy classes of $G_{p,q}$ whose elements are not of prime power order. Let $g \in G$ be such an element. Obviously, we can consider only those g which are the distinguished representatives of conjugacy classes. It follows from Table 7.4 that $g \in (B_m)$ or $g \in (A_l)$ for some $m = 1, ..., q-1$ and l not divisible by p and q. In the first case, $g = (b, c^m)$ and $g^{-1} = (b, c^{-m})$, so $(g) \neq (g^{-1})$. This yields $(q-1)/2$ real conjugacy classes of the form $(B_m)^{\pm} = (B_m) \cup (B_{q-m})$ for any $m = 1, ..., (q-1)/2$. In case $g \in (A_l)$, we have $g = (a^l, 1)$ and $g^{-1} = (a^{-l}, 1)$ and g is conjugate to g^{-1},

$$(b,1)(a^l,1)(b,1)^{-1} = (a^{-l},1).$$

Thus, each of the classes (A_l) constitute the real conjugacy class. Therefore

$$\lambda(G_{p,q}) = \frac{1}{2}(q-1) + \frac{1}{2}(q-1)r = \frac{1}{2}(q-1)(r+1).$$

□

7.2.2 Normal subgroups and quotients of $G_{p,q}$

Lemma 7.13. *If $N \trianglelefteq G_{p,q}$, then $|N| \in \{1, p, q, pq, 2pq, pq^2, 2pq^2\}$.*

Proof. $|G|$ has the following set of divisors

$$\{1, 2, p, q, 2p, 2q, pq, q^2, 2pq, 2q^2, pq^2, 2pq^2\}.$$

We show that $|N| \notin \{2, 2p, 2q, q^2, 2q^2\}$. Assume $2||N|$. Then there is some element of order 2 in N. Since N is a normal subgroup of $G_{p,q}$, it follows from Table 7.4 that $(B) \subseteq N$ and thus $|N| \geq pq$. Observe that $pq > 2, 2p, 2q, 2q^2$. Hence $|N| \notin \{2, 2p, 2q, 2q^2\}$.

Now, suppose $|N| = q^2$. We conclude from Table 7.4 that $N \trianglelefteq N_{pq^2}$. However, this possibility was already excluded in the proof of Lemma 7.7. □

Consider the following subgroups of $G_{p,q}$.

$$N_{2pq} = \{(b^{\varepsilon}a^l, 1)|\varepsilon = 0, 1, l = 0, ..., pq-1\} \quad N_p = \{(a^{qs}, 1)|s = 0, ..., p-1\}$$
$$N_q = \{(a^{ps}, 1)|s = 0, ..., q-1\} \quad N^1_{pq} = \{(a^l, 1)|l = 0, ..., pq-1\}$$

and

$$N^2_{pq} = \{(a^{qs}, c^m)|s = 0, ..., p-1, m = 0, ..., q-1\}.$$

Lemma 7.14. *$N_{pq^2}, N_{2pq}, N^1_{pq}, N^2_{pq}, N_p$ and N_q are the only proper normal subgroups of $G_{p,q}$. Moreover, $N_{2pq} \cong D_{2pq}$, $N^1_{pq} \cong C_{pq}$, $N^2_{pq} \cong F_{p,q}$, $N_p \cong C_p$ and $N_q \cong C_q$.*

Proof. It follows from Table 7.4 that all the subgroups mentioned in the Lemma consist of the whole conjugacy classes and thus are normal. Clearly, $N_{2pq} \cong D_{2pq}$, $N^1_{pq} \cong C_{pq}$, $N^2_{pq} \cong F_{p,q}$ (since N^2_{pq} is not abelian and the unique nonabelian group of order pq is $F_{p,q}$), $N_p \cong C_p$ and $N_q \cong C_q$. We show that there are no other proper normal subgroups in $G_{p,q}$.

Assume for the converse that there exists a proper normal subgroup N of $G_{p,q}$ such that $N \notin \{N_p, N_q, N^1_{pq}, N^2_{pq}, N_{2pq}, N_{pq^2}\}$. From Lemma 7.13, we have

$$|N| \in \{p, q, pq, 2pq, pq^2\}.$$

If $|N| = pq^2$, then the only possibility is $N = N_{pq^2}$ which is a contradiction.

Suppose $|N| = 2pq$. Then there exists an element of order 2 contained in N. Thus, $(B) \subseteq N$. Since $N \neq N_{2pq}$, it follows that $g = (x, c^m) \in N$ for some $x \in D_{2pq}$ and $m \neq 0$. Since $(1, 1) \in N$, $(B) \subseteq N$ and $|\{(1, 1)\} \cup (B)| = pq + 1$, it follows that $|(g)| < pq$. We conclude then from Table 7.4 that $g \in (C_m)$ or $g \in (D_{s,m})$ for some s not divisible by q. Thus $C_m \in N$ or $D_{s,m} \in N$. Suppose $D_{s,m} \in N$. Then, for $n \geq 1$,

$$D^n_{s,m} = (a^{s(1+\omega^m+...+\omega^{(n-1)m})}, c^{nm})$$

and $D^n_{s,m} = D_{s_n,(nm \pmod q)}$ for any $n = 1, ..., q-1$, where $s_n \neq 0$ and $(nm \pmod q)$ denotes the remainder of nm modulo q. Hence $S = (D_{s,m}) \cup (D_{s_2,(2m \pmod q)}) \cup ... \cup (D_{s_{q-1},((q-1)m \pmod q)}) \subseteq N$. However, $|S| = 2p(q-1) > pq$. A contradiction. This means that $C_m \in N$. Therefore $\langle C_m \rangle \leq N$ and thus, for any $m = 1, ..., q-1$, $(C_m) \subseteq N$. On the other hand, $|(C_1) \cup ... \cup (C_{q-1})| = p(q-1) < pq-1$ which means that $D_{s,m} \in N$ for some $m \neq 0$ and $q \nmid s$ which we have already excluded.

Assume $|N| = pq$. Then N has no element of order 2 and, since $N \neq N^1_{pq}$, $C_m \in N$ or $D_{s,m} \in N$ for some $m \neq 0$ and $q \nmid s$. The latter case implies $|N| > pq$. Thus $C_m \in N$. Suppose that one of the elements A_l or F_s is contained in N for some $p, q \nmid l$ and $s = 1, ..., q-1$. If $A_l \in N$, we obtain a contradiction for this leads to $|N| > pq$ (for $\langle A_l \rangle = N^1_{pq}$). If $F_s \in N$, then $(F_1) \cup ... \cup (F_{q-1}) \subseteq N$. On the other hand, there exist an element of order p in N and we conclude from Table 7.4 that $(E_1) \cup ... \cup (E_{p-1}) \subseteq N$. Thus,

$$\begin{aligned} |N| &\geq |\{(1,1)\} \cup \bigcup_{r=1}^{q-1} (C_r) \cup \bigcup_{s=1}^{q-1} (F_s) \cup \bigcup_{t=1}^{p-1} (E_t)| \\ &= 1 + p(q-1) + 2 \cdot \frac{1}{2}(q-1) + 2q \cdot \frac{1}{2} r = pq + q - 1 > pq. \end{aligned}$$

Thus, we obtain a contradiction. Hence

$$N = \{(1,1)\} \cup (C_1) \cup ... \cup (C_{q-1}) \cup (E_1) \cup ... \cup (E_{p-1}) = N^2_{p,q}$$

which contradicts our assumption.

Let $|N| = q$ and $g \in N$ be an element of order q. If $g \in (C_m)$ or $g \in (D_{s,m})$ for some $m \neq 0$ and $q \nmid s$, we conclude from Table 7.4 that this implies $|N| > q$. Thus, $g \in (F_s)$ which means that it is impossible that $N \neq N_q$.

If $|N| = p$, Table 7.4 leads immediately to a contradiction. □

Corollary 7.15. *$\mathcal{O}^p(G_{p,q}) = G_{p,q}$, $\mathcal{O}^q(G_{p,q}) = N_{2pq}$ and $\mathcal{O}^2(G_{p,q}) = N_{pq^2}$. Therefore, we have $\mathcal{L}(G_{p,q}) = \{N_{2pq}, N_{pq^2}, G_{p,q}\}$.*

Since $G_{p,q}$ is the semidirect product of D_{2pq} and C_q, it can be presented as follows.

$$G_{p,q} = \langle a, b, c | a^{pq} = b^2 = 1, bab^{-1} = a^{-1}, c^q, cac^{-1} = a^{\omega}, cbc^{-1} = b \rangle$$

Thus, we can identify a with $(a,1)$, b with $(b,1)$ and c with $(1,c)$.

Lemma 7.16. *$G_{p,q}/N^1_{pq} \cong C_{2q}$, $G_{p,q}/N^2_{pq} \cong D_{2q}$, $G_{p,q}/N_p \cong C_q \times D_{2q}$ and $G_{p,q}/N_q$ is a group not of prime power order which is not nilpotent.*

Proof. Define $\varphi^1_{pq} \colon G_{p,q} \to C_{2q} = \langle d | d^{2q} = 1 \rangle$ by $\varphi^1_{pq}(a) = 1$, $\varphi^1_{pq}(b) = d^q$, $\varphi^1_{pq}(c) = d^2$. Obviously

$$\varphi^1_{pq}(a^{pq}) = \varphi^1_{pq}(b^2) = \varphi(c^q) = \varphi^1_{pq}(baba) = 1$$

and

$$\varphi^1_{pq}(cac^{-1}) = \varphi^1_{pq}(a^{\omega}) = \varphi^1_{pq}(cbc^{-1}b^{-1}) = 1$$

and φ^1_{pq} is a well-defined group homomorphism. It is easy to observe that $\operatorname{Ker} \varphi^1_{pq} = N^1_{pq}$ and that φ^1_{pq} is surjective. Thus $G_{p,q}/N^1_{p,q} \cong C_{2q}$.

Let $\varphi^2_{pq} \colon G_{p,q} \to D_{2q} = \langle d, e | d^q = e^2 = 1, ede = d^{-1} \rangle$ be given by $\varphi^2_{pq}(a) = d$, $\varphi^2_{pq}(b) = e$ and $\varphi^2_{pq}(c) = 1$. We have

$$\varphi^2_{pq}(a^{pq}) = \varphi^2_{pq}(b^2) = \varphi(c^q) = \varphi^2_{pq}(baba) = \varphi^2_{pq}(cbc^{-1}b^{-1}) = 1$$

and

$$\varphi^2_{pq}(cac^{-1}) = d = d^{\omega} = \varphi^2_{pq}(a^{\omega}).$$

Thus φ^2_{pq} is a well-defined epimorphism. Moreover,

$$b^\varepsilon a^l c^m \in \operatorname{Ker}\varphi^2_{pq} \Leftrightarrow e^\varepsilon d^l = 1 \Leftrightarrow \varepsilon = 0, q|l, m = 0, ..., q-1$$

and $\operatorname{Ker}\varphi^2_{pq} = \{(a^{qs}, c^m) | s = 0, ..., p-1, m = 0, ..., q-1\} = N^2_{pq}$.

Put $\varphi_p \colon G_{p,q} \to C_q \times D_{2q} = \langle d | d^q = 1\rangle \times \langle e, f | e^q = f^2 = 1, fef = e^{-1}\rangle$, $\varphi_p(a) = (1, e)$, $\varphi_p(b) = (1, f)$ and $\varphi_p(c) = (d, 1)$. Obviously,

$$\varphi_p(a^{pq}) = \varphi_p(b^2) = \varphi_p(c^q) = \varphi_p(baba) = \varphi_p(cbc^{-1}b^{-1}) = (1, 1).$$

Moreover, $\varphi_p(cac^{-1}) = (1, e) = (1, e^\omega) = \varphi_p(a^\omega)$ since $i \equiv 1 \pmod q$. Since $\omega \equiv 1 \pmod q$, it follows that $(1, e^\omega) = (1, e)$. Hence φ_p is a well-defined homomorphism. Obviously, φ_p is surjective and $b^\varepsilon a^l c^m \in \operatorname{Ker}(\varphi_p)$ if and only if l and m are divisible by q and $\varepsilon = 0$. Therefore $\operatorname{Ker}(\varphi_p) = \{a^{qs} | s = 0, ..., p-1\} = N_p$.

Since the order of $G_{p,q}/N_q$ equals $2pq$, $G_{p,q}/N_q$ is not of prime power order. Suppose for the converse that $G_{p,q}/N_q$ is nilpotent. This means that $G_{p,q}/N_q$ is the direct product of its Sylows and, since $|G_{p,q}/N_q|$ is the product of three distinct primes, we conclude that $G_{p,q}/N_q$ has to be cyclic. Let dN_q be the generator of $G_{p,q}/N_q$. If $d = (ba^l, c^m)$ for some $l = 0, ..., pq-1$ and $m = 0, ..., q-1$, then $|d| \leq 2q$ and it follows that $(dN_q)^{2q} = 1$ in $G_{p,q}/N_q$. This contradicts that $G_{p,q}/N_q$ is of order pq. Thus, $d = (a^l, c^m)$. If $m \neq 0$, then $d^q = (1, 1)$ and we obtain a contradiction. Hence, $d = (a^l, 1)$. In this case, however, $d^p = (a^{pl}, 1) \in N_q$ which leads, again, to a contradiction. □

7.2.3 $G_{p,q}$ is a special Oliver group

We will need the following results of Sumi.

Theorem 7.17. [71, Theorem 1.2] *Let G be a group with no large subgroup of prime power order. Moreover, suppose that $[G\colon \mathcal{O}^2(G)] = 2$ and $\mathcal{O}^{p_0}(G) \neq G$ for a unique odd prime p_0 and that G does not have an element of order divisible by 4 and there is an element $g \in G$ of order 2 not belonging to $\mathcal{O}^2(G)$ such that $2|\mathcal{O}^2(C_G(g))| \geq |C_G(g)|$ and $\mathcal{O}^2(C_G(g))$ is a p_0-group. Then G is not a gap group.*

Lemma 7.18. [70, p.35, first paragraph] *If G is a group which has a large subgroup of prime power order, then G is not a gap group.*

Lemma 7.19. [69, pp. 982,984] *For any $n \geq 3$, the dihedral group D_{2n} is not a gap group.*

Now, we can prove the following.

Lemma 7.20. *N^1_{pq}, N_{2pq}, N_{pq^2} and $G_{p,q}$ are not gap groups.*

Proof. Let us prove that $G_{p,q}$ is not a gap group by means of Theorem 7.17. By Corollary 7.15 and the fact that $G_{p,q}$ does not have an element of order divisible by 4, it suffices to show that there exists an element $g \in G_{p,q}$ of order 2 not belonging to $\mathcal{O}^2(G_{p,q}) = N_{pq^2}$ such that $2|\mathcal{O}^2(C_{G_{p,q}}(g))| \geq |C_{G_{p,q}}(g)|$ and $\mathcal{O}^2(C_{G_{p,q}}(g))$ is a q-group. We show that this holds for $g = (b, 1)$. We have

$$(b^\varepsilon a^l, c^m)(b, 1) = (b, 1)(b^\varepsilon a^l, c^m) \Leftrightarrow (b^\varepsilon a^l b, c^m) = (b^{1+\varepsilon} a^l, c^m)$$

which holds if and only if $l = 0$. Thus $C_{G_{p,q}}(g) = \{(b^\varepsilon, c^m) | \varepsilon = 0, 1, m = 0, ..., q-1\} \cong C_{2q}$. Obviously $\mathcal{O}^2(C_{2q}) \cong C_q$ is a q-group and the inequality $2|\mathcal{O}^2(C_{G_{p,q}}(g))| \geq |C_{G_{p,q}}(g)|$ holds. Hence $G_{p,q}$ is not a gap group.

Note that both N^1_{pq} and N_{pq^2} contain N_p as a normal subgroup (since $N_p \trianglelefteq G_{p,q}$). Thus $\mathcal{O}^q(N^1_{pq}) = \mathcal{O}^q(N_{pq^2}) = N_p$ and N_p is a large subgroup for both N^1_{pq} and N_{pq^2}. Hence, we get from Lemma 7.18 that N^1_{pq} and N_{pq^2} are not gap groups.

The statement for N_{2pq} is the direct corollary from Lemma 7.19. □

Lemma 7.21. *$G_{p,q}$ has no cyclic quotient of odd composite order and $G_{p,q}$ does not satisfy the Sumi $G^{\mathrm{nil}}_{p,q}$-condition.*

Proof. It follows by Lemma 7.14 and Lemma 7.16 that $G_{p,q}$ has no cyclic quotient of odd composite order. It follows by Lemma 7.16 that $G^{\mathrm{nil}}_{p,q} = N^1_{pq}$. Assume $xN^1_{pq} = yN^1_{pq}$ for some elements $x, y \in G_{p,q}$ of even order. This means that $x = (ba^l, c^m)$ and $y = (ba^{l'}, c^{m'})$ for some $l, l' = 0, ..., pq-1$ and $m, m' = 0, ..., q-1$ and

$$xy^{-1} = (a^{l'\omega^{m-m'}-l}, c^{m-m'}) \in N^1_{pq}.$$

Thus $m' = m$ and $(x) = (y)$ by Table 7.4.

Suppose there exist $x', y' \in G_{p,q}$ of composite order such that one of them, say x', is of odd order. Then $x' \in N^1_{pq}$ by Table 7.4. Thus, the only subgroups of $G_{p,q}$ which can contain both x' and y' must have N^1_{pq} as a subgroup. These subgroups are precisely N^1_{pq}, N_{2pq}, N_{pq^2} and $G_{p,q}$. We showed in Lemma 7.20 that they are not gap groups. This shows that $G_{p,q}$ does not satisfy the Sumi $G^{\mathrm{nil}}_{p,q}$-condition. □

Lemma 7.22. *N_{pq^2} has no normal subgroup P of prime power order such that the quotient N_{pq^2}/P is cyclic. The same statement holds for N_{2pq}.*

Proof. Suppose that $P \trianglelefteq N_{pq^2}$ and N_{pq^2}/P is cyclic. Then $|P| \in \{1, p, q, q^2\}$. If $|P| = 1$, then $N_{pq^2}/P \cong N_{pq^2}$ and we obtain a contradiction. The case $|P| = q^2$ is not possible by the proof of Lemma 7.7.

Let $|P| = q$ and $N_{pq^2}/P = \langle gP \rangle$ for some $g \in N_{pq^2}$. If $g \notin (A_l)$ for any l not divisible by p and q, then, by Table 7.4, $|g| \leq p$ which is a contradiction since $|N_{pq^2}/P| = pq$. Suppose $g \in (A_l)$ for some l not divisible by p and q. It follows by Table 7.4 that precisely one of the elements C_1, F_1 or $D_{s,m}$, for some s and m not divisible by q, is the generator of P. It follows by Table 7.3 that C_1 cannot be the generator of P. Indeed, by Lemma 7.6, C_1 corresponds in N'_{pq^2} to $(1, \gamma)$, which yields at least $p(q-1)$ elements in P (P contains the whole conjugacy classes) which is a contradiction. In the case $F_1 = (a^p, 1)$ generates P, notice that $g^n \in \langle F_1 \rangle = P$ if and only if $p|n$ which yields $|N_{pq^2}/P| = p$ and a contradiction. Thus $P = \langle D_{s,m} \rangle$. Since $D_{s,m}$ corresponds in N_{pq^2} to (α^l, γ^m) for some $l \in \{1, ..., q-1\}$, it follows from Table 7.3 that $|P| > q$ which is a contradiction.

Suppose $|P| = p$. In this case, however, it follows from Lemma 7.6 and Table 7.3 that $P = N_p$. Suppose $N_{pq^2}/P = \langle (a^l, c^m)P \rangle$. As before, $|N_{pq^2}/P| \leq q$ in case $m \neq 0$, which is not possible. If $N_{pq^2}/P = \langle (a^l, 1)P \rangle$, then $(a^l, 1)^q = (a^{ql}, 1) \in N_p$ and, again, $|N_{pq^2}/P| \leq q$.

Assume that there exists $P \trianglelefteq N_{2pq}$ of prime power order such that N_{2pq}/P is cyclic. Then $|P| \in \{1, 2, p, q\}$. Obviously, P cannot be the trivial subgroup and, since there is no normal subgroup of order 2 in N_{2pq}, it follows that $|P| \in \{p, q\}$. This means that P is a subgroup of N^1_{pq}. If $|P| = p$, then $P = \{(a^{qs}, 1) | s = 0, ..., p-1\}$. Since $|(ba^l, 1)| = 2$ for any $l = 0, ..., pq-1$, it follows that $N_{2pq}/P = \langle (a^l,1)P \rangle$. Suppose $(a^l,1)^n P = (ba^{l'},1)P$ for some $n \geq 0$ and $l' = 0,...,pq-1$. This means that $(ba^{l'-nl},1) = (ba^{l'},1)(a^l,1)^{-n} \in P$. A contradiction which implies that N_{2pq}/P is not cyclic. The case $|P| = q$ is analogous. □

Lemma 7.23. *$G_{p,q}$ is a special Oliver group.*

Proof. Obviously, $G_{p,q}$ is not of odd order. Since D_{2pq} and C_q are solvable groups, it follows that $G_{p,q}$, as the semidirect product of D_{2pq} and C_q, is solvable as well. Moreover, by Lemma 7.21, we know that $G_{p,q}$ has no cyclic quotient of odd composite order and does not satisfy the Sumi $G_{p,q}^{\text{nil}}$-condition. Thus, we only have to show that $G_{p,q}$ is an Oliver group. Suppose for the converse that this is not true. Then, there exist subgroups $P \trianglelefteq H \trianglelefteq G_{p,q}$ such that $G_{p,q}/H$ and P are of prime power orders and H/P is cyclic. Then, by Lemma 7.13, $|H| \in \{2pq, pq^2, 2pq^2\}$ and thus, by Lemma 7.14, $H \in \{N_{2pq}, N_{pq^2}, G_{p,q}\}$. However, by Lemmas 7.14, 7.16 and 7.22, it follows that neither of the groups N_{2pq}, N_{pq^2} and $G_{p,q}$ has a normal subgroup of prime power order such that the quotient by it is cyclic. This concludes the proof. □

7.3 Injectivity of the induction for primary groups

Assume H is a subgroup of a group G and consider the induction homomorphism

$$\text{Ind}_H^G\colon \text{RO}(H) \to \text{RO}(G),\ U - V \mapsto \text{Ind}_H^G(U) - \text{Ind}_H^G(V)$$

and its restriction to $\text{PO}(H)$, that is

$$\text{Ind}_H^G\colon \text{PO}(H) \to \text{PO}(G).$$

Let s denote the number of real conjugacy classes of G which have nonzero intersection with H and whose elements are not of prime power order. Put $t = \lambda(H) = \text{rank}(\text{PO}(H))$ and let m be the number of real conjugacy classes of G. Obviously, $s \leq t$.

Consider the image $\text{Im}(\text{Ind}_H^G\colon \text{PO}(H) \to \text{PO}(G))$. Clearly, it is a torsion-free subgroup of $\text{PO}(G)$ and let r be its rank. Thus, $\text{Ind}_H^G\colon \text{PO}(H) \to \text{PO}(G)$ is a monomorphism if and only if $r = t$.

If A is a matrix with entries in the field K, then we denote by $\text{rank}_K(A)$ the rank of A over K.

Lemma 7.24. *Assume $A \in \text{GL}(n, \mathbb{C})$ is of finite order. Then $\text{tr}(A^{-1}) = \overline{\text{tr}(A)}$.*

Lemma 7.25. *The rank of $\text{Im}(\text{Ind}_H^G\colon \text{PO}(H) \to \text{PO}(G))$ is at most s, that is $r \leq s$.*

Proof. Pick the bases $\epsilon = \{\varepsilon_1, ..., \varepsilon_t\}$ and $\epsilon' = \{\varepsilon'_1, ..., \varepsilon'_{t'}\}$ of $\text{PO}(H)$ and $\text{PO}(G)$ respectively ($t' = \lambda(G)$). We use the column convention for elements from $\text{PO}(H)$ and $\text{PO}(G)$ – we represent them as $t \times 1$ and $t' \times 1$ vectors respectively, where the coordinates are given by the bases ϵ and ϵ' accordingly. The induction map is a linear map and denote by M its matrix form in bases ε and ε'.

Let $(g_1)^{\pm}, ..., (g_{t'})^{\pm}$ be the ordered list of all real conjugacy classes of G whose elements are not of prime power order and let χ be the map which evaluates the characters of the elements of $\text{PO}(G)$ on the classes $(g_i)^{\pm}$ for $i = 1, ..., t'$. Note by Lemma 7.24 that χ is well-defined and

$$\chi\colon \text{PO}(G) \to \mathbb{R}^{t'}$$

$$\varepsilon'_j \mapsto \begin{pmatrix} \varepsilon'_j(g_1) \\ \vdots \\ \varepsilon'_j(g_{t'}) \end{pmatrix}.$$

Let $X = (\chi_{ij})_{1 \le i,j \le t'}$ be the matrix of χ, that is $\chi_{ij} = \varepsilon'_j(g_i)$. Clearly, $\mathrm{rank}_{\mathbb{R}}(X) = t'$. Consider the composition

$$\chi \circ \mathrm{Ind}_H^G \colon \mathrm{PO}(H) \xrightarrow{\mathrm{Ind}_H^G} \mathrm{PO}(G) \xrightarrow{\chi} \mathbb{R}^{t'}.$$

The matrix of $\chi \circ \mathrm{Ind}_H^G$ is a $t' \times t$ matrix $A = (a_{ij})_{1 \le i \le t', 1 \le j \le t}$ given by $A = XM$. Thus, $a_{ij} = \mathrm{Ind}_H^G(\varepsilon_j)(g_i)$ for $1 \le i \le t'$ and $1 \le j \le t$. It follows from Theorem 2.10 that $\mathrm{rank}_{\mathbb{R}}(A) \le s$. On the other hand, since $\mathrm{rank}_{\mathbb{R}}(X) = t'$, it follows that $\mathrm{rank}_{\mathbb{R}}(M) = \mathrm{rank}_{\mathbb{R}}(A)$. Therefore $\mathrm{rank}_{\mathbb{R}}(M) \le s$.

Now, since M is an integer matrix and $\mathbb{R}$ is an extension of $\mathbb{Q}$, we conclude that the real rank of M equals its rational rank, that is $\mathrm{rank}_{\mathbb{R}}(M) = \mathrm{rank}_{\mathbb{Q}}(M) = r'$. We show that $r = r'$ which would mean that $r \le s$ and would complete the proof.

Obviously, $r \ge r'$. Let $V = \langle \varepsilon'_1, ..., \varepsilon'_{t'} \rangle$. Take any $r' + 1$ elements $v_1, ..., v_{r'+1}$ from $\mathrm{Im}(\mathrm{Ind}_H^G \colon \mathrm{PO}(H) \to \mathrm{PO}(G))$. They can be considered as vectors from V. Note that they are linearly dependent (over $\mathbb{Q}$), since the dimension of $\mathrm{Im}(\mathrm{Ind}_H^G \colon \mathrm{PO}(H) \to \mathrm{PO}(G))$ considered as a subspace of V equals r'. Let

$$\alpha_1 v_1 + ... + \alpha_{r'+1} v_{r'+1} = 0 \tag{1}$$

be a nontrivial combination. Suppose $\{\alpha_{i_1}, ..., \alpha_{i_k}\}$ is the set of all nonzero coefficients and $\alpha_{i_j} = p_j/q_j$, where $p_j, q_j \in \mathbb{Z} \setminus \{0\}$ for $1 \le j \le k$. Multiplying both sides of equality (1) by $q_1 ... q_k$, we get a nontrivial integer combination of v_j's. Thus $r \le r'$ and, as a result, $r = r'$. □

Lemma 7.26. $\mathrm{Ind}_H^G \colon \mathrm{PO}(H) \to \mathrm{PO}(G)$ *is a monomorphism if and only if* $(h)_G^{\pm} \cap H = (h)_H^{\pm}$ *for any* $h \in H$ *not of prime power order.*

Proof. Suppose that for any $h \in H$ not of prime power order we have $(h)_G^{\pm} \cap H = (h)_H^{\pm}$. Let x_1 and x_2 be two different elements of $\mathrm{PO}(H)$. We must show that $\mathrm{Ind}_H^G(x_1) \ne \mathrm{Ind}_H^G(x_2)$. There exists $h \in H$ not of prime power order with $x_1(h) \ne x_2(h)$. We have two possibilities. The first one is when $(h)_H = (h^{-1})_H = (h)_G^{\pm} \cap H$. Thus

$$(h)_G \cap H = (h^{-1})_G \cap H = (h)_G^{\pm} \cap H = (h)_H^{\pm} = (h)_H$$

and it follows by Theorem 2.10 that

$$\mathrm{Ind}_H^G(x_1)(h) = \tfrac{|C_G(h)|}{|C_H(h)|} x_1(h) \quad \text{and} \quad \mathrm{Ind}_H^G(x_2)(h) = \tfrac{|C_G(h)|}{|C_H(h)|} x_2(h).$$

Therefore $\mathrm{Ind}_H^G(x_1)(h) \ne \mathrm{Ind}_H^G(x_2)(h)$ since $x_1(h) \ne x_2(h)$. In the second possibility, we have $(h)_H \ne (h^{-1})_H$. If $(h)_G \cap H = (h)_H$, we have already proved the assertion. Assume $(h)_H \subsetneq (h)_G \cap H$. Note that

$$((h)_G \cap H) \cup ((h^{-1})_G \cap H) = (h)_G^{\pm} \cap H = (h)_H \cup (h^{-1})_H.$$

Clearly $(h^{-1})_H \subseteq (h^{-1})_G \cap H$, which in connection with $(h)_H \subsetneq (h)_G \cap H$ gives from the equalities above $(h)_G \cap H = (h^{-1})_G \cap H = (h)_H \cup (h^{-1})_H$. Note that $|C_H(h)| = |C_H(h^{-1})|$. Thus by Theorem 2.10 and Lemma 7.24, we get

$$\mathrm{Ind}_H^G(x_1)(h) = 2\tfrac{|C_G(h)|}{|C_H(h)|} x_1(h) \quad \text{and} \quad \mathrm{Ind}_H^G(x_2)(h) = 2\tfrac{|C_G(h)|}{|C_H(h)|} x_2(h).$$

Thus $\mathrm{Ind}_H^G(x_1)(h) \neq \mathrm{Ind}_H^G(x_2)(h)$.

We prove now the converse. Suppose $\mathrm{Ind}_H^G\colon \mathrm{PO}(H) \to \mathrm{PO}(G)$ is a monomorphism. Assume for the contrary that there exists $h \in H$ not of prime power order with $(h)_G^{\pm} \cap H \neq (h)_H^{\pm}$. Then $(h)_H^{\pm} \subsetneq (h)_G^{\pm} \cap H$ and thus $s < t$. Hence, it follows by Lemma 7.25 that we have $\mathrm{rank}(\mathrm{Im}(\mathrm{Ind}_H^G\colon \mathrm{PO}(H) \to \mathrm{PO}(G))) < t$ and $\mathrm{Ind}_H^G\colon \mathrm{PO}(H) \to \mathrm{PO}(G)$ is not injective which is a contradiction with our assumption. □

Corollary 7.27. *Assume N is a normal subgroup of G. Then $\mathrm{Ind}_N^G\colon \mathrm{PO}(N) \to \mathrm{PO}(G)$ is a monomorphism if and only if $(n)_G^{\pm} = (n)_N^{\pm}$ for any $n \in N$ not of prime power order.*

Corollary 7.28. $\mathrm{Ind}_{N_{pq^2}}^{G_{p,q}}\colon \mathrm{PO}(N_{pq^2}) \to \mathrm{PO}(G_{p,q})$ *is a monomorphism.*

Proof. By Corollary 7.27, it suffices to show that for any $n \in N_{pq^2}$ not of prime power order, we have $(n)_{G_{p,q}}^{\pm} = (n)_{N_{pq^2}}^{\pm}$. We know by Table 7.4 that n has to be of order pq and $n = (a^l, 1)$ for some l not divisible by p and q. From the proof of Lemma 7.12, we know that $n^{-1} = (a^{-l}, 1)$ and n are conjugate in $G_{p,q}$. Thus $(n)_{G_{p,q}}^{\pm} = (n)_{G_{p,q}}$. On the other hand n and n^{-1} are not conjugate in N_{pq^2}. Otherwise, there would exists $(a^{l'}, c^{m'})$ such that

$$(a^{l'}, c^{m'})(a^l, 1)(a^{l'}, c^{m'})^{-1} = (a^{-l}, 1).$$

Thus $(a^{li^{m'}}, 1) = (a^{-l}, 1)$ which cannot be true since $li^{m'} \equiv l \not\equiv -l \pmod{q}$ for $q \nmid l$. Therefore $(n^{-1})_{N_{pq^2}} \neq (n)_{N_{pq^2}}$ and it follows by Table 7.3 that $|(n^{\pm})_{N_{pq^2}}| = 2q$. On the other hand, $|(n)_{G_{p,q}}^{\pm}| = |(n)_{G_{p,q}}| = 2q$, and the assertion follows. □

7.4 Proof of Theorem 7.3

Let $H \leq G$ be a subgroup of a group G. Assume that U and V are two $\mathbb{R}H$-modules satisfying the weak gap condition and such that $\dim W^L = 0$ for any $L \in \mathcal{L}(H)$ and $W = U, V$. Then, by [40, Lemma 1.2, Lemma 1.7], the $\mathbb{R}G$-modules $\mathrm{Ind}_H^G(U)$ and $\mathrm{Ind}_H^G(V)$ satisfy the weak gap condition and $\dim W'^L = 0$ for any $L \in \mathcal{L}(G)$ and $W' = \mathrm{Ind}_H^G(U), \mathrm{Ind}_H^G(V)$. In particular, we have

$$\mathrm{Ind}_H^G(\mathrm{PO}_{\mathrm{w}}^{\mathcal{L}}(H)) \subseteq \mathrm{PO}_{\mathrm{w}}^{\mathcal{L}}(G).$$

Let us recall the following result which is a corollary from [39, Theorem 1.6].

Theorem 7.29. *Let G be an Oliver group and U and V two $\mathcal{P}$-matched $\mathbb{R}G$-modules satisfying the weak gap condition and such that $\dim U^L = \dim V^L = 0$ for any $L \in \mathcal{L}(G)$. Then there exists an $\mathbb{R}G$-module W such that $U \oplus W$ and $V \oplus W$ are Smith equivalent.*

Hence, as a corollary from the discussion above, we get the following theorem.

Theorem 7.30. *Let H be a subgroup of an Oliver group G. Any element of $\mathrm{PO}_{\mathrm{w}}^{\mathcal{L}}(H)$ is representable as the difference $U - V$ in $\mathrm{RO}(H)$ of $\mathcal{P}$-matched $\mathbb{R}H$-modules U and V satisfying the weak gap condition and the condition that $\dim U^L = \dim V^L = 0$ for any $L \in \mathcal{L}(H)$. For these $\mathbb{R}H$-modules U and V, there exists an $\mathbb{R}G$-module W such that the $\mathbb{R}G$-modules $\mathrm{Ind}_H^G(U) \oplus W$ and $\mathrm{Ind}_H^G(V) \oplus W$ are Smith equivalent.*

We can prove now the main theorem of this section.

Proof of Theorem 7.3. Lemmas 7.12 and 7.23 tell us that $G_{p,q}$ is a special Oliver group with primary number at least 2. By Corollary 7.11, there exist non-isomorphic $\mathbb{R}G_{p,q}$-modules U and V with $U - V \in \mathrm{PO}_{\mathrm{w}}^{\mathcal{L}}(N_{pq^2})$. Thus, by Corollary 7.28, $\mathrm{Ind}_{N_{pq^2}}^{G_{p,q}}(U)$ and $\mathrm{Ind}_{N_{pq^2}}^{G_{p,q}}(V)$ are not isomorphic $\mathbb{R}G_{p,q}$-modules. Clearly, U and V can be chosen to satisfy the weak gap condition and such that $\dim U^L = \dim V^L = 0$ for any $L \in \mathcal{L}(N_{pq^2})$. Therefore, by Theorem 7.30, it follows that there exists an $\mathbb{R}G_{p,q}$-module W such that $\mathrm{Ind}_{N_{pq^2}}^{G_{p,q}}(U) \oplus W$ and $\mathrm{Ind}_{N_{pq^2}}^{G_{p,q}}(V) \oplus W$ are Smith equivalent. Obviously, these modules are not isomorphic. □

Notation

- Algebraic topology:
  - ∂M – the boundary of a manifold M p.11
  - $H_k(X, A; R)$,$H^k(X, A; R)$ – homology and cohomology of the pair (X, A) with coefficients in R, when $A = \emptyset$, we omit A and if $R = \mathbb{Z}$, we omit R p.11
  - $[M]$,$[M, \partial M]$ – fundamental class of a manifold M – without and with boundary respectively p.11
  - $\alpha \cup \beta$ – the cup product of cohomology classes α and β p.11
  - $\langle \cdot, \cdot \rangle$ – the Kronecker pairing p.11
  - $\overline{\lambda}$ – the oriented intersection product p.11
  - λ – the unoriented intersection product p.12
  - $\mathrm{Int}(X)$ – the interior of the topological space X p.14
- Geometry and group actions:
  - X^G – the fixed point set of the G-action on X p.9
  - T_xM – the tangent space at point x to the manifold M p.11
  - $\kappa(x)$ p.11
  - $\overline{A \cdot B}$ – oriented intersection number p.11
  - $A \cdot B$ – unoriented intersection number p.12
  - $D(E)$ – disk bundle of the bundle E p.13
  - $G \curvearrowright M$ – the action of a group G on a manifold M p.19
  - $D(\theta)$ – differential of the transformation θ p.20
  - $\exp_p$ – exponential map at point p p.22
  - $C(H)$ p.33
- Abstract and linear algebra:
  - mW – m-fold direct sum of W p.8
  - $\mathrm{char}(F)$ – characteristic of the field F p.8
  - $\dim V^H$ – the dimension of the fixed point space V^H p.10
  - $\mathrm{span}(A)$ – the linear span of A p.11
  - $\mathrm{Tor}(A, B)$ – the Tor functor applied to he pair (A, B) p.33
  - $D_L(G)$ p.37
  - D_G p.38

  - $\mathrm{tr}(A)$ – the trace of the matrix A p.69

- Representation theory:
  - $\overline{\chi}$ – the conjugate of the character χ p.8
  - $\mathrm{Re}(\chi)$ – the real part of the character χ p.8
  - $\iota(\chi)$ – the Frobenius-Schur indicator of the character χ p.9
  - $\mathrm{Ind}_H^G(V)$ – the induced module of V from H to G p.10
  - $\mathrm{Ker}(V)$ – the kernel of module V p.39
  - $\mathrm{RO}(G)$ – the representation group of G p.55
  - Res_H^G – the restriction homomorphism from G to H p.55
  - $\mathrm{PO}(G)$ – the primary group of G p.55
  - $\widetilde{\mathrm{PO}}(G)$ – the reduced primary group of G p.55
  - $\mathrm{PO}_{\mathrm{w}}^{\mathcal{L}}(G)$ p.59
  - $\zeta_n = e^{2\pi i/n}$ p.60
  - $\sigma_{t,x}$ p.60
  - $\psi_{s,t}$ p.60

- Group theory:
  - $H \leq G$ – H is a subgroup of the group G p.9
  - $[G : H]$ – the index of the subgroup H in G p.10
  - $C_K(x)$ – the centralizer in K of x p.10
  - $H \trianglelefteq G$ – H is a normal subgroup of the group G p.27
  - G^{sol} – the smallest normal subgroup of G such that G/G^{sol} is solvable p.29
  - $\mathcal{G}_p^q, \mathcal{G}^q, \mathcal{G}, \mathcal{G}_p^q(G), \mathcal{G}^q(G), \mathcal{G}(G)$ pp.31,32
  - $\langle H_1, H_2 \rangle$ – the subgroup generated by the subgroups H_1 and H_2 .... p.32
  - r_G – the number of real irreducible representations of G diminished by 1 .... p. 38
  - $\mathcal{A}(G), \mathcal{B}(G)$ – good subgroup triples of G of type A and B respectively p.41
  - $I_2(G)$ – the intersection subgroups of G of index at most 2 .... p. 41
  - $\mathrm{SmallGroup}(n, k)$ – the group of order n and id k from the GAP SmallGroup library p.46
  - $(g)^{\pm} = (g) \cup (g^{-1})$ – the real conjugacy class of g p.56
  - $\lambda(G)$ – the Laitinen number of G p.56
  - G^{nil} – the smallest normal subgroup of G such that G/G^{nil} is nilpotent p.57
  - $\mathcal{O}^p(G)$ – the smallest normal subgroup of G such that $G/\mathcal{O}^p(G)$ is a p-group ... p.57
  - $\mathcal{L}(G)$ – the family of large subgroups of G p.57
  - $N \rtimes_\varphi H$ – the semidirect product of N and H defined by the homomorphism $\varphi\colon H \to \mathrm{Aut}(N)$ p.58
  - $G_{p,q}$ p.58

- Algorithms:
  - MN_{odd} p.39

- MN_{one} p.39
- MNF_{one} p.40
- MT_{one} p.40
- $M(n, G)$ p.40

- Smith problem:
  - $\mathrm{Sm}(G)$ – the Smith set of the group G p.56
  - $\mathrm{PSm}(G)$ – the primary Smith set of the group G p.56
  - $\mathrm{Br}(k)$ – the Bredon number of k p.54

Bibliography

[1] M. F. Atiyah and R. Bott. A Lefschetz fixed point formula for elliptic complexes. II. Applications. *Ann. of Math. (2)*, 88:451–491, 1968.

[2] M. Audin. *Torus actions on symplectic manifolds*, volume 93 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, revised edition, 2004.

[3] A. Bak and M. Morimoto. Equivariant surgery and applications. In *Topology Hawaii (Honolulu, HI, 1990)*, pages 13–25. World Sci. Publ., River Edge, NJ, 1992.

[4] A. Bak and M. Morimoto. The dimension of spheres with smooth one fixed point actions. *Forum Math.*, 17(2):199–216, 2005.

[5] A. Borowiecka. $\mathrm{SL}(2,5)$ has no smooth effective one-fixed-point action on S^8. *Bull. Pol. Acad. Sci. Math.*, 64(1):85–94, 2016.

[6] A. Borowiecka and P. Mizerka. Nonexistence of smooth effective one fixed point actions of finite Oliver groups on low-dimensional spheres. *Bull. Pol. Acad. Sci. Math.*, 66(2):167–177, 2018.

[7] G. E. Bredon. Representations at fixed points of smooth actions of compact groups. *Ann. of Math. (2)*, 89:515–532, 1969.

[8] G. E. Bredon. *Introduction to compact transformation groups*. Academic Press, New York-London, 1972. Pure and Applied Mathematics, Vol. 46.

[9] N. P. Buchdahl, S. a. Kwasik, and R. Schultz. One fixed point actions on low-dimensional spheres. *Invent. Math.*, 102(3):633–662, 1990.

[10] S. E. Cappell and J. L. Shaneson. Fixed points of periodic maps. *Proc. Nat. Acad. Sci. U.S.A.*, 77(9, part 1):5052–5054, 1980.

[11] S. E. Cappell and J. L. Shaneson. Fixed points of periodic differentiable maps. *Invent. Math.*, 68(1):1–19, 1982.

[12] E. C. Cho. Smith equivalent representations of generalized quaternion groups. In *Group actions on manifolds (Boulder, Colo., 1983)*, volume 36 of *Contemp. Math.*, pages 317–322. Amer. Math. Soc., Providence, RI, 1985.

[13] E. C. Cho. s-Smith equivalent representations of dihedral groups. *Pacific J. Math.*, 135(1):17–28, 1988.

[14] E. C. Cho and D. Y. Suh. Induction in equivariant K-theory and s-Smith equivalence of representations. In *Group actions on manifolds (Boulder, Colo., 1983)*, volume 36 of *Contemp. Math.*, pages 311–315. Amer. Math. Soc., Providence, RI, 1985.

[15] C. W. Curtis and I. Reiner. *Representation theory of finite groups and associative algebras.* AMS Chelsea Publishing, Providence, RI, 2006. Reprint of the 1962 original.

[16] J. F. Davis and P. Kirk. *Lecture notes in algebraic topology*, volume 35 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI, 2001.

[17] S. De Michelis. The fixed point set of a finite group action on a homology four sphere. *Enseign. Math. (2)*, 35(1-2):107–116, 1989.

[18] L. Dornhoff. *Group representation theory. Part A: Ordinary representation theory.* Marcel Dekker, Inc., New York, 1971. Pure and Applied Mathematics, 7.

[19] K. H. Dovermann. Even-dimensional s-Smith equivalent representations. In *Algebraic topology, Aarhus 1982 (Aarhus, 1982)*, volume 1051 of *Lecture Notes in Math.*, pages 587–602. Springer, Berlin, 1984.

[20] K. H. Dovermann and T. Petrie. G surgery. II. *Mem. Amer. Math. Soc.*, 37(260):xxiii+118, 1982.

[21] K. H. Dovermann and T. Petrie. Smith equivalence of representations for odd order cyclic groups. *Topology*, 24(3):283–305, 1985.

[22] M. Furuta. A remark on a fixed point of finite group action on S^4. *Topology*, 28(1):35–38, 1989.

[23] The GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.10.2*, 2019.

[24] M. W. Hirsch. *Differential topology.* Springer-Verlag, New York-Heidelberg, 1976. Graduate Texts in Mathematics, No. 33.

[25] W.-c. Hsiang and W.-y. Hsiang. Some problems in differentiable transformation groups. In *Proc. Conf. on Transformation Groups (New Orleans, La., 1967)*, pages 223–234. Springer, New York, 1968.

[26] S. Illman. Representations at fixed points of actions of finite groups on spheres. In *Current trends in algebraic topology, Part 2 (London, Ont., 1981)*, volume 2 of *CMS Conf. Proc.*, pages 135–155. Amer. Math. Soc., Providence, R.I., 1982.

[27] G. James and M. Liebeck. *Representations and characters of groups.* Cambridge University Press, New York, second edition, 2001.

[28] E. Laitinen and M. Morimoto. Finite groups with smooth one fixed point actions on spheres. *Forum Math.*, 10(4):479–520, 1998.

[29] E. Laitinen, M. Morimoto, and K. Pawałowski. Deleting-inserting theorem for smooth actions of finite nonsolvable groups on spheres. *Comment. Math. Helv.*, 70(1):10–38, 1995.

[30] E. Laitinen and K. Pawałowski. Smith equivalence of representations for finite perfect groups. *Proc. Amer. Math. Soc.*, 127(1):297–307, 1999.

[31] J. Milnor. Whitehead torsion. *Bull. Amer. Math. Soc.*, 72:358–426, 1966.

[32] P. Mizerka. A new family of finite Oliver groups satisfying the Laitinen conjecture. *Topology Appl.*, 283:107336, 17, 2020.

[33] P. Mizerka. one_fixed_point_spheres_scripts. `https://github.com/piotrmizerka/one_fixed_point_spheres_scripts`, 2020.

[34] D. Montgomery and H. Samelson. Fiberings with singularities. *Duke Math. J.*, 13:51–56, 1946.

[35] M. Morimoto. On one fixed point actions on spheres. *Proc. Japan Acad. Ser. A Math. Sci.*, 63(4):95–97, 1987.

[36] M. Morimoto. Most of the standard spheres have one fixed point actions of A_5. In *Transformation groups (Osaka, 1987)*, volume 1375 of *Lecture Notes in Math.*, pages 240–258. Springer, Berlin, 1989.

[37] M. Morimoto. Most standard spheres have smooth one fixed point actions of A_5. II. *K-Theory*, 4(3):289–302, 1991.

[38] M. Morimoto. Smith equivalent $\mathrm{Aut}(A_6)$-representations are isomorphic. *Proc. Amer. Math. Soc.*, 136(10):3683–3688, 2008.

[39] M. Morimoto. Deleting and inserting fixed point manifolds under the weak gap condition. *Publ. Res. Inst. Math. Sci.*, 48(3):623–651, 2012.

[40] M. Morimoto, T. Sumi, and M. Yanagihara. Finite groups possessing gap modules. In *Geometry and topology: Aarhus (1998)*, volume 258 of *Contemp. Math.*, pages 329–342. Amer. Math. Soc., Providence, RI, 2000.

[41] M. Morimoto and S. Tamura. Spheres not admitting smooth odd-fixed-point actions of S_5 and $\mathrm{SL}(2,5)$. *Osaka Journal of Mathematics*, 2018.

[42] M. Morimoto and K. Uno. Remarks on one fixed point A_5-actions on homology spheres. In *Algebraic topology Poznań 1989*, volume 1474 of *Lecture Notes in Math.*, pages 337–364. Springer, Berlin, 1991.

[43] B. Oliver. Fixed point sets and tangent bundles of actions on disks and Euclidean spaces. *Topology*, 35(3):583–615, 1996.

[44] R. Oliver. Fixed-point sets of group actions on finite acyclic complexes. *Comment. Math. Helv.*, 50:155–177, 1975.

[45] R. Oliver. Smooth compact Lie group actions on disks. *Math. Z.*, 149(1):79–96, 1976.

[46] R. Oliver. G-actions on disks and permutation representations. II. *Math. Z.*, 157(3):237–263, 1977.

[47] R. Oliver. G-actions on disks and permutation representations. *J. Algebra*, 50(1):44–62, 1978.

[48] R. A. Oliver. *Smooth fixed-point free-actions of compact Lie groups on disks.* PhD thesis, 1974. Princeton University.

[49] K. Pawałowski and R. Solomon. Smith equivalence and finite Oliver groups with Laitinen number 0 or 1. *Algebr. Geom. Topol.*, 2:843–895, 2002.

[50] K. Pawałowski and T. Sumi. The Laitinen conjecture for finite solvable Oliver groups. *Proc. Amer. Math. Soc.*, 137(6):2147–2156, 2009.

[51] K. M. Pawałowski. The Smith equivalence problem and the Laitinen conjecture. In *Handbook of group actions. Vol. III*, volume 40 of *Adv. Lect. Math. (ALM)*, pages 485–537. Int. Press, Somerville, MA, 2018.

[52] P. Petersen. *Riemannian geometry*, volume 171 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2006.

[53] T. Petrie. Pseudoequivalences of G-manifolds. In *Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 1*, Proc. Sympos. Pure Math., XXXII, pages 169–210. Amer. Math. Soc., Providence, R.I., 1978.

[54] T. Petrie. Three theorems in transformation groups. In *Algebraic topology, Aarhus 1978 (Proc. Sympos., Univ. Aarhus, Aarhus, 1978)*, volume 763 of *Lecture Notes in Math.*, pages 549–572. Springer, Berlin, 1979.

[55] T. Petrie. The equivariant J homomorphism and Smith equivalence of representations. In *Current trends in algebraic topology, Part 2 (London, Ont., 1981)*, volume 2 of *CMS Conf. Proc.*, pages 223–233. Amer. Math. Soc., Providence, R.I., 1982.

[56] T. Petrie. One fixed point actions on spheres. I, II. *Adv. in Math.*, 46(1):3–14, 15–70, 1982.

[57] T. Petrie. Smith equivalence of representations. *Math. Proc. Cambridge Philos. Soc.*, 94(1):61–99, 1983.

[58] T. Petrie and J. Randall. Spherical isotropy representations. *Inst. Hautes Études Sci. Publ. Math.*, (62):221–256, 1985.

[59] T. Petrie and J. D. Randall. *Transformation groups on manifolds*, volume 82 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker, Inc., New York, 1984.

[60] C. U. Sanchez. Actions of groups of odd order on compact, orientable manifolds. *Proc. Amer. Math. Soc.*, 54:445–448, 1976.

[61] J.-P. Serre. *Linear representations of finite groups*. Springer-Verlag, New York-Heidelberg, 1977. Translated from the second French edition by Leonard L. Scott, Graduate Texts in Mathematics, Vol. 42.

[62] P. Smith. Fixed-point theorems for periodic transformations. *Amer. J. Math.*, 63:1–8, 1941.

[63] P. A. Smith. Transformations of finite period. *Ann. of Math. (2)*, 39(1):127–164, 1938.

[64] P. A. Smith. Transformations of finite period. II. *Ann. of Math. (2)*, 40:690–711, 1939.

[65] P. A. Smith. New results and old problems in finite transformation groups. *Bull. Amer. Math. Soc.*, 66:401–415, 1960.

[66] E. Stein. Surgery on products with finite fundamental group. *Topology*, 16(4):473–493, 1977.

[67] D. Y. Suh. s-Smith equivalent representations of finite abelian groups. In *Group actions on manifolds (Boulder, Colo., 1983)*, volume 36 of *Contemp. Math.*, pages 323–329. Amer. Math. Soc., Providence, RI, 1985.

[68] D. Y. Suh. Isotropy representations of cyclic group actions on homotopy spheres. *Bull. Korean Math. Soc.*, 25(2):175–178, 1988.

[69] T. Sumi. Gap modules for direct product groups. *J. Math. Soc. Japan*, 53(4):975–990, 2001.

[70] T. Sumi. Gap modules for semidirect product groups. *Kyushu J. Math.*, 58(1):33–58, 2004.

[71] T. Sumi. The gap hypothesis for finite groups which have an abelian quotient group not of order a power of 2. *J. Math. Soc. Japan*, 64(1):91–106, 2012.