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Mathematical Writing: A Descriptive Study to Explore the Relationship Between Analytical / Creative Writing and the Understanding of Mathematical Problem Solving in a Seventh-Grade Classroom

Shiela Ann Puffe

MATHEMATICAL WRITING:

A DESCRIPTIVE STUDY TO EXPLORE

THE RELATIONSHIP BETWEEN ANALYTICAL / CREATIVE WRITING

AND THE UNDERSTANDING OF MATHEMATICAL PROBLEM SOLVING

IN A SEVENTH-GRADE CLASSROOM

by

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This thesis submitted by Shiela Ann Puffe in partial fulfillment of the requirements for the Degree of Master of Science from the University of North Dakota has been read by the Faculty Advisory Committee under whom the work has been done, and is hereby approved.

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This thesis meets the standards for appearance and conforms to the style and format requirements of the Graduate School of the University of North Dakota, and is hereby approved.

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Signature Shiela Puffe

Date 28 November 91

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To the glory of God

and for the benefit of humankind.

ABSTRACT

This naturalistic inquiry explored the relationship between writing and mathematical problem solving. The use of analytical writing and creative writing was examined to see if it was a beneficial heuristic for teaching understanding of mathematical problem solving in a seventh-grade class. The teacher acted as the principal researcher.

The treatment consisted of two distinct segments. Students wrote narrative, analytical accounts of their problem-solving process. They used a teacher-constructed study guide based on Polya's problem-solving framework (Understand the Problem, Devise a Plan, Carry Out the Plan, and Look Back / Evaluate) with metacognitive questioning as a guide for this writing. Students also wrote original mathematical problems and generated stories with mathematical constructs. Throughout this study (1) teacher modeling, (2) guided practice, (3) cooperative small groups, (4) individual work, and (5) journal reflections were used.

Assessment was based on teacher observations, student writing, daily written mathematical work, and the pre- and post-test results. It was determined that analytical/creative writing can be successfully employed to teach for better comprehension of mathematical problem solving.

CHAPTER I

INTRODUCTION

Rationale for the Study

Few people would debate the findings of The National Assessment of Educational Progress which reports that students are extremely deficient in "Moderately Complex Procedures and Reasoning" as well as "Multi-step Problem Solving and Algebra" (Dossey, Mullis, Lidquist, & Chambers, 1988). Kameenui and Griffin (1989) explain students' difficulty with solving multi-step problems as a result of their instruction during their early mathematical experiences. Children have unfortunately gleaned from their instruction by teachers and textbooks that all problems can be solved in a short amount of time by applying one arithmetical operation. While this is certainly false, never-the-less, it has been engrained in learners that once they determine this ONE operation, they have to search no farther in contemplating and comprehending the nature and extent of the problem.

Students also fail to differentiate between necessary and extraneous data. They have difficulty recognizing whether or not the problem makes sense and can be solved. They have a difficult time choosing an appropriate strategy to use in the solving of the problem (Schoenfeld, 1980). Learners also fail to generalize their solutions and apply them later to novel problem-solving experiences.

Lester (1985) stated that students have gleaned the following four practices from their mathematical problem-solving experiences:

> 1. Problem difficulty is determined by the size of the numbers and how many numbers there are.
> 2. All mathematics problems can be solved by direct application of one or more arithmetic operations.
> 3. Which operation to use is determined by the key words in the problem (these key words usually appear in the last sentence or question).
> 4. Whether or not to check computations depends upon the availability of time. For story problems, only computations need to be checked. (p. 42).

In the same publication he contended that

> the ultimate goal of instruction in mathematical problem solving is to enable students to think for themselves. It is my view that most problem-solving instruction not only does not enable students to use

> their heads, but in fact it does more harm than good. (p. 41).

Montague (1988) also asserted that classroom practices and reliance on textbooks "do little to enhance the development of cognitive and metacognitive strategic behavior among students and often inhibit the development of higher order thinking that is the basis of independent problem solving" (p. 277) in her research among learning disabled students.

It seems evident that our students are not able to use mathematical problem-solving skills and strategies because they lack the understanding of WHEN and WHY these skills and strategies work even though they may know HOW to "do" the necessary computation. Is it any wonder that students have a difficult time understanding problem solving, when some educators have the same trouble? (Schroeder & Lester, 1989). In order to benefit our students, something must change. Willoughby (1990) maintained that

> change in teaching methods is needed, . . . because the world is changing. The people who are going to solve the problems of the present and future -- or even understand and evaluate those problems and solutions -- must have a far better grasp of mathematics than most

people have at present, or have ever had in the past. (p. 4).

In this study I seek to explore one such change in teaching methods, the use of writing as a vehicle in understanding mathematical problem solving. When writing has been used in mathematics classes, research has shown that it has been quite successful (Mett, 1989; Tobias, 1989; Powell and Lopez, 1989; Keith, 1989, Lesnak, 1989; Birken, 1989). However, writing has seldom been used as an instructional tool in mathematical classrooms (Kenyon, 1989; Pearce and Davison, 1988). "It becomes evident when the definition for problem solving is examined and compared to the writing stages in WAC [Writing Across the Curriculum] that writing is problem solving" (Kenyon, 1989, p. 76).

A second problem deals with the available research on teaching problem solving. According to Thompson (1988)

> reports of instructional studies in problem solving have generally lacked good descriptions of what actually happened in the classroom (except for those in which programmed instructional booklets were used) and have failed to assess the direct effectiveness of instruction. Rather than assessing whether or not students exhibited thinking and behaviors modeled in instruction, instead they have assessed the number of

problems correctly solved on a post test (Silver, 1987). As a result, our knowledge of desirable instructional practices in problem solving is mostly of folklore rather than research evidence. (p. 232).

Grouws and Good (1988) would agree with this contention. They stated "that most [problem solving] research has focused on individual students, usually in laboratory settings, rather than on actual classroom teaching and learning" (p. 2). Lester (1985) believed that since mathematical problem solving is such a large field with so many variables, it should be studied holistically, or in its entirety. This would necessitate the use of naturalistic study. In an attempt to add to the body of knowledge of actual classroom experience, this study is designed to be naturalistic, or qualitative, in nature rather than quantitative.

Statement of Purpose

This naturalistic inquiry explored the relationship between writing and mathematical problem solving. The use of analytical writing and creative writing was examined to see if it was a beneficial heuristic for teaching understanding of mathematical problem solving in a seventh-grade class.

Definitions

In order to ensure that the reader is on common ground with the author of this study, it is necessary to highlight some definitions and special cases in which terminology may or may not be used synonymously.

Exercise -- computational drill, involve basic arithmetical operations.

Framework -- a basic structure for solving problems. The term is synonymous with model.

Heuristic -- a general technique used during problem solving in order to arrive at a conclusion. The term is used synonymously with strategy in this study.

Metacognition -- the ability to choose and plan what to do and the ability to modify and evaluate one's performance (Garofalo and Lester, 1985).

Problem -- a situation involving mathematical construct in which one is motivated to attempt to solve for an unknown. For purposes of this study (1) routine multi-step textbook problems, (2) nonroutine process problems, and (3) nonroutine real-world problems were included in this definition. It did not include "exercises".

Problem solving -- "a process of applying previously acquired knowledge to new and unfamiliar situations" (Sovchik, 1989, p. 256).

Solution -- the entire process involved in obtaining an answer to the problem. It includes the answer, but it is not synonymous with the "right answer".

CHAPTER II

REVIEW OF LITERATURE

Importance of Problem Solving

Problem solving can be interpreted as a goal, a process, or a basic skill. (Branca, 1980). Problem solving is the primary goal in mathematics education. It is the reason for learning all of the other mathematical skills and concepts. (Enright and Beattle, 1989; Branca, 1980). Dossey (1988) agreed that learning mathematics is "performing an action that calls for reflection and perseverance" (p. 21).

Problem solving is an important and effective means in teaching students to think. In reference to his earlier study which was conducted with Phipps, Osborne (1988) stated that "problem-solving teaching stimulates interest, develops thinking ability, and helps students make decisions. It makes instruction meaningful and relevant and develops attitudes of questioning, comparison, and doubt" (p. 10). Problem solving is an avenue for thinking to occur. Situations arise in which different paths need to be explored and choices made regarding the best direction to take.

Polya (1980) believed that the purpose of education was to develop intelligence which is analogous with solving problems. Dossey (1988) referred to George Polya's goal of education as one that teaches "purposeful thinking" (p. 20). Problem solving has also been equated to Dewey's reflective thinking as well as the scientific method. Osborne (1988) used Beyer's list of "major thinking operations [which] include evaluation and analysis, critical thinking, problem solving, synthesis, application, and decision making" (p. 2). Listed among these one finds the higher order thinking skills of Bloom's Taxonomy of Educational Objectives, application, analysis, synthesis, and evaluation.

Osborne (1988) discussed Beyer's educational goal which is for students to "learn and act responsibly and effectively on their own. This goal implies that the process of learning is much more lasting than the content of learning" (p. 10). Problem solving is a process whereby one can bridge the gap between a student's in-school and out-of-school experiences. It will give students an opportunity to transfer what they have learned in school to novel situations that arise in their daily lives (Branca, 1980).

Problem solving is also a basic skill that we need in order to function effectively in this world (Branca, 1980).

As our society becomes more technologically revolutionized, there will exist a growing need for people to be able to understand the mathematics involved and not just manipulate numbers. "Recent advances in technology ... and a world that is becoming continually more complex and quantitative, show us that mathematical thinking is becoming ever more important" (Willoughby, 1990, p. 3). Much, if not all, of the computation can be done faster and more accurately on calculators and computers. This frees up one's time to spend contemplating more serious issues involved.

Too often mathematics has been taught as computation or a procedure and has not been taught for understanding (Thiessen, Wild, Paige, & Baum, 1989; Kameenui & Griffin, 1989). This lack of comprehension has created difficulties for people to apply their mathematical knowledge in problem solving. Willoughby (1990) stated

> the problem is that we are not doing nearly as good a job as possible to help all of our children learn and understand enough mathematics to lead productive and fulfilling lives in a modern society. We have always failed to teach mathematics so that people would be willing and able to use it effectively. (p. 2).

Students have very few opportunities to make sense of mathematics in school. They are hindered by a reliance on syntax (symbols and rules) rather than semantics (meaning). They do not use common sense and mental computation as often as they do in a similar situation out of school. Lester (1989) reported that students are more likely to make sense of mathematical situations out of school because the problem is set in a familiar place, students are forced to make a decision, everyday problem solving is goal directed, students use their natural language, and they serve as an apprentice by watching other more knowledgeable people in similar situations.

The ability to solve problems also has the benefit of increasing one's self-concept. There is personal joy and satisfaction in knowing that you have met and overcome a challenge.

Problem Solving and Problems

When people think of the term "problem solving", many different ideas come to mind, much of which depends upon how you would define what a problem is. A basic working definition of problem solving is supplied by the National Council of Supervisors of Mathematics. It is "a process of applying previously acquired knowledge to new and unfamiliar situations" (Sovchik, 1989, p. 256).

Problem Categorizations

There are several ways in which one could categorize problems. Thiessen et al. (1989) in their textbook, Elementary Mathematical Methods, arranged them in the following six categories:

1. Computational or Drill Exercises. i.e. 16% of 56.
2. Simple Translation Problems -- one-step story problems. i.e. Keith had 3/4 of the job completed. He assumed the entire project would take him six and a half hours. How many hours has he worked?
3. Complex Translation Problems -- multi-step story problems. i.e. After Susan finished packing ten books into each of five boxes, she weighed one of the boxes to determine postal rates. She determined that it would cost her $2.45 per package. How much change will she get from a $20 bill?
4. Applied Problems. i.e. How much water is used in your school over a period of a year? Could some of this be conserved? How much money could be saved?
5. Process Problems -- no previously learned procedure or algorithm that can be applied for a quick solution. i.e. Find the sum of the numbers one through one hundred.
6. Puzzle Problems -- do not necessarily involve mathematical strategies. i.e. Without lifting your

. . . pencil, draw four straight line segments

. . . that pass through all nine dots. (pp.

. . . 3-4).

Duckett (1990) classified the problems that she used with her students into the following three categories: (1) "classroom quickies" in which a twist or trick is needed, (2) "recreational problems" in which you use one or more strategies to solve, and (3) "real-world problems" such as finding the best value for your money. Troutman and Lichtenberg (1987) in their textbook, Mathematics: A Good Beginning. Strategies for Teaching Children, used just two groups, routine problems and unusual ones.

Criteria for Problem Situations

"Problem solving is what you do when you don't know what to do" (based on Wheatley, cited in Frank, 1988, p. 33). Therefore, a mathematical situation may pose a problem for some people but not for everyone. What is a problem for a younger student may be just an exercise or rote memorization for an older person (Kameenui & Griffin, 1989). What may be a problem for a student today may not be in the future, if he/she remembers the trick to the puzzle or has mastered some appropriate strategies.

Souviney (1981) listed the following comprehensive criteria for an ideal problem situation:

1. be readily understandable, no apparent solution yet,
2. be intrinsically motivating and intellectually stimulating,
3. have more than one solution "path",
4. require only previously learned arithmetic operations and concepts,
5. be solved over a period of time (not a one-step computation),
6. be somewhat open-ended (suggest new problems),
7. integrate various subjects,
8. be well-enough defined to know when it is solved. (p. 5).

Another suggestion as to what constitutes a problem in need of solving is offered by Thiessen et al. (1989). A person

1. has a need or desire to solve the problem.
2. has no established or easily accessible procedure for solving the problem.
3. tries to solve the problem. (p. 4).

Kenyon (1989) stated similar characteristics of a problem. He said

1. A person must be aware of a situation.

2. The person must recognize that the situation requires action.

3. The person must either want or need to act upon the situation and must actually take some action.

4. The resolution of the situation must not be immediately obvious to the person acting on it. (p. 75).

When using problem solving in a classroom, it is necessary to find problems that will challenge all of the students. For well-rounded instruction, it is also important that you use and practice solving both routine problems such as are in textbooks and nonroutine or unusual problems (Kameenui & Griffin, 1989; Troutman & Lichtenberg, 1987).

Characteristics of Good Problem Solvers

Research shows that the best problem solvers have some common characteristics. According to Sovchik (1989) some of the most common are

1. Ability to understand mathematical concepts and terms.

2. Ability to note likenesses, differences, and analogies.

3. Ability to identify critical elements and to select correct procedures and data.

4. Ability to note irrelevant detail.

5. Ability to visualize and interpret quantitative or spatial facts and relationships.

6. Ability to estimate and analyze.

7. Ability to generalize on the basis of a few examples.

8. Ability to switch methods readily.

9. Higher scores for self-esteem and confidence.

10. Lower scores for text anxiety and less impulsive (p. 258).

Montague (1988) found that perseverance was a key common characteristic among the gifted problem solving students with whom she worked. Lester (1985) added that good problem solvers suspend judgment while they explore all of the options. They tend to spend significantly more time in understanding the problem and developing a meaningful representation of it than do inexperienced or ineffective problem solvers. Bloom discovered that successful problem solvers comprehended the problem, activated prior knowledge, used active and verbal problem-solving behavior, and were confident about solving the problem (Ornstein, 1989). In addition to these skills, successful problem solvers also possess metacognitive skills that are discussed later in this chapter.

Frameworks and Heuristics

Frameworks

Many plans have been developed in an effort to label the steps involved in problem solving. For instance, Dewey's procedure which was introduced in 1910 is

1. Becoming aware of difficulty.
2. Identifying the problem.
3. Assembling and classifying data and formulating hypotheses.
4. Accepting or rejecting tentative hypotheses.
5. Formulating conclusions and evaluating them (Ornstein, 1989, p. 113).

George Polya's model first introduced in 1945 is the most well-known among mathematical educators. It entails four steps which are

1. Understand the Problem -- clarify and identify information.
2. Devise a Plan -- identify operations and procedures.

3. Carry Out the Plan -- perform operations; complete procedures.

4. Look Back -- discuss, refine, modify. (Stiff, 1988, p. 667).

A simplified version of Polya's framework appears in many current grade school mathematical textbooks. This version is (1) read, (2) plan -- make a table, think backward, apply logic, draw a diagram, work a simpler problem, choose the operation, guess and test, and so on, (3) solve, and (4) check (Talton, 1988).

Another variation of Polya's model for solving problems is offered by Beyer and cited by Osborne (1988). It differs from Polya's model in that it separates the first phase (understanding the problem) into two steps -- recognizing the problem and representing or clarifying it. The remaining three stages are devising a solution plan, executing the plan, and evaluating the solution.

IDEAL is an acronym for a similar problem-solving framework. The steps are Identify the problem, Define it, Explore possible strategies, Act on the strategies, and Look at the effects of your efforts (based on Bradford and Stein, cited in Ornstein, 1989).

Yet another variation of a five-step model used by Enright and Beattie (1989) is comprised of the following actions: (1) study the problem, (2) organize the facts,

(3) line up a plan, (4) verify the plan/computation, and (5) examine your answer.

Still another model advanced by Newell and Simon involves only two steps which are

1. Construct a representation of the problem, called the "problem space".
2. Work out a solution that involves a search through the problem space (Ornstein, 1989, p. 113).

Troutman and Lichtenberg (1987) used two different models to generate a solution depending on the type of problem. Their method for nonroutine or unusual problems was (1) become familiar with the problem, (2) collect information related to the problem, (3) devise strategies for solving the problem and evaluate the strategies, and (4) select a strategy and carry it out to find solutions. Evaluate the solutions.

Their model for solving routine or familiar problems was (1) select established procedures, (2) carry out procedures and find solutions, and (3) evaluate the solutions.

As you read through these frameworks of steps engaged in while solving mathematical problems, you will note that they are quite similar. All involve much thinking prior to attempting to calculate a numerical answer. All encourage the consideration of several possible heuristics for solving the problem and then choosing among the alternative strategies for the best approach. All require an evaluation of the solution process as well as the answer in order to generalize the solution process for future use. This last step is vital for problem solvers to transfer their learning to new problems.

Heuristics

In order to teach problem solving effectively, it is necessary to do more than give the students a great quantity of problems to solve (Thompson, 1988). Sowder (1988) pointed to the following strategies (all except the last one are unfortunate) which students have devised for their own use

1. Find the numbers and add (or multiply or subtract...; the choice may be dictated by what has taken place in class recently or by what operation the student feels most competent at doing).
2. Guess the operation to be used.
3. Look at the numbers; they will "tell" you which operation to use.
4. Try all the operations and choose the most reasonable answer.

5. Look for isolated "key" words or phrases to tell which operations to use.

6. Decide whether the answer should be larger or smaller than the given numbers. If larger, try both addition and multiplication and choose the more reasonable answer. If smaller, try both subtraction and division and choose the more reasonable answer.

7. Choose the operation whose meaning fits the story. (p. 2).

Students need to be taught heuristics, general strategies, that may aid them in finding a solution. These are "rule of thumb" procedures and are based on common sense (Silver and Smith, 1980). Lester (1985) grouped most research about problem-solving teaching into the following four main groups:

1. Instruction to develop master thinking strategies (e.g. originality and creativity training)

2. Instruction in the use of specific "tool skills" (e.g. making a table, organizing data, writing an equation)

3. Instruction in the use of specific heuristics (e.g. looking for a pattern, working backward)

4. Instruction in the use of general heuristics (e.g. means-end analysis, planning). (p. 45).

He advocated that a better approach to teaching problem solving would include a combination of all of these strategy categories and training in metacognition. Lester also believed such training should take place over a period of time with many different types of problems used.

Many problem-solving heuristics can be used during this mathematical process. Some problems may be solved by applying one strategy, but many problems will need a combination of strategies for solution. Some of the strategies can be used at various stages throughout the problem; others are more conducive for use in one particular phase (Sovchik, 1989).

Listed are several heuristics that are used when solving mathematical problems, but this is by no means an exhaustive list. There is some variety in how authors choose to group the same heuristics into different steps of Polya's solution model (understand, plan, carry out, look back). Because of the flexibility and uniqueness involved in the art of problem solving, no attempt is made by this author to lock these strategies into a specific step of the solution process.

Souviney (1981) listed several techniques that are useful toward the beginning of the process.

1. Restate the problem in your own words.
2. List the given information.

3. List the given conditions.

4. Write the stated goal in your own words.

5. List the related relevant facts.

6. List implicit conditions.

7. Describe related known problems (p. 22).

Troutman and Lichtenberg (1987) included several abilities that are helpful throughout the process.

1. Recognize attributes that an object or mathematical concept may have.
2. Restate a problem in a variety of ways -- in your own words, making diagrams, tables, charts, graphs, and deriving number sentences.
3. Find similarities and differences.
4. Classify objects and mathematical ideas.
5. Determine when information is sufficient and eliminate irrelevant information.
6. Find relationships or patterns.
7. Systematically determine cases or alternatives -- consider the problem under different circumstances.
8. Approximate.
9. Extend given information -- generate more information or draw conclusions.

10. Compare objects or ideas with a set of criteria -- use, construct, and learn certain mathematical definitions. (pp. 298-301).

Musser and Shaughnessy (1980) supply the following list of problem-solving strategies:

1. Trial and error
    a. Systematic trial and error
    b. Inferential trial and error -- differs from systematic trial and error in that it takes into account relevant knowledge and uses that knowledge to narrow the search.
2. Patterns -- looks at selected instances of the problem. Then a solution is found by generalizing from these specific solutions.
3. Simpler problem -- solving a "special case" of the problem or ... a shortened version. In the latter case, the simpler-problem strategy is often accompanied by a pattern search.
4. Working backward.
5. Simulation. (pp. 137-145).

Ornstein (1989) included the following heuristics in his list of search strategies:

1. Means-ends strategies -- eliminates the differences between the current and the goal condition.

2. Analogies -- make the unfamiliar problem more familiar.

3. Matching -- select a plan of action and carry it out accordingly; the process of matching plans and actions continues until the problem is solved.

4. Sequencing -- step by step behaviors.

5. Examining the problems which one has already worked to find general strategies for a similar group of problems; it does not prepare the problem solver for the atypical problem or a new set of problems (p. 120).

Sovchik (1989) included several of the fore-mentioned techniques for use when solving mathematical problems. In addition to these, he suggested the following strategies:

1. Make a table or graph -- helpful in systematizing information. It is often used in conjunction with the strategy of making a simpler problem.

2. Make a diagram -- essential strategy for understanding a problem as well as designing and carrying out your plan.

3. Make a model -- using objects or simulating the problem's actions in some way, sometimes by drawing a diagram as an aid.

4. Offer various perspectives -- open-ended problems -- those with more than one answer -- can be a valuable vehicle for developing novel, imaginative solutions. Changing your point of view requires a complete and imaginative understanding of the problem.

5. Write a number sentence.

6. Estimate -- determining the reasonableness of an answer is a helpful strategy. (pp. 269-278).

These additional problem-solving heuristics were among those suggested by Stiff (1988)

1. Select appropriate notation.
2. Draw from known information.
3. Exhaust all possibilities.
4. Generalize.
5. Check the solution.
6. Find another way to solve it.
7. Find another result.
8. Study the solution. (p. 667).

Silver and Smith (1980) discussed the drawbacks of teachers asking students to use the related problem strategy. In order for the related problem to be effective in guiding the student toward a solution, the problems need to be similar in mathematical structure (what needs to be done to solve the problem). Students are not always able to get beyond the related context, or "cover story" of the problem to its mathematical structure. The superficial setting for the mathematical problem is peripheral to the solution and is often a stumbling block for students. The second hindrance to analyzing and synthesizing similar mathematical structure is the form of the question. The wording of the question can and often does sidetrack a problem solver from the essential mathematical structure of the problem.

Metacognition

Flavell (1976) stated a widely accepted definition of metcognition. He defined it as follows:

> "Metacognition" refers to one's knowledge concerning one's own cognitive processes and products or anything related to them, e.g., the learning-relevant properties of information or data.... Metacognition refers, among other things, to the active monitoring and consequent regulation and orchestration of these processes in relation to the cognitive objects on which they bear, usually in the service of some concrete goal or objective. (p. 232).

Simply stated, cognition is what one does while metacognition is how one chooses and plans what to do and the ability to evaluate and modify one's performance (Garofalo & Lester, 1985; Ornstein, 1989). Metacognition provides the problem solver with information that determines whether or not he/she is progressing toward a solution (Lester, 1985). In regards to problem solving, Kameenui and Griffin (1989) described both aspects of metacognition. They included one's knowledge of the problem-solving skills and the ability to monitor and check one's problem-solving performance. Bondy (1984) referred to cognition as an automatic and subconscious process, whereas metacognition is a conscious process.

It is important to note that errors in students' solutions are often a result of inappropriately applying a strategy and not due to a lack of ability or a lack of effort on the part of the learner (Ornstein, 1989). At the very least students should have the ability to determine the reasonableness of a given problem or solution (Marshall, 1986).

The following metacognitive skills are possessed by expert problem solvers:

1. Comprehension monitoring.
2. Understanding decisions.
3. Planning.

4. Estimating task difficulty.

5. Task presentation.

6. Coping strategies.

7. Internal cues.

8. Retracking.

9. Noting and correcting.

10. Flexible approaches. (Ornstein, 1989, pp. 114-115).

<u>Teaching Metacognition</u>

Most problem-solving teaching has involved "blind training" where students are only taught what to do. In "informed training" students are taught what, why, and when the heuristics will achieve the result. "Self-regulation training" supplements the informed training with metacognitive processes.

The failure to improve students' performances on problem-solving tasks is caused by concentrating on the teaching of heuristics to the exclusion of teaching managerial skills to monitor and control their actions (Garofalo and Lester, 1985). Vobejda (1987) agreed that instruction in metacognitive techniques rarely occurred in mathematical classrooms. To remedy this situation Garofalo and Lester (1985) offered a framework loosely based on Polya's model with the advantage of offering explicit metacognitive training. These stages involved are

1. Orientation.
2. Organization.
3. Execution.
4. Verification.

Bondy (1984) also made several recommendations for teaching metacognitive aspects in the classroom. They included

To Promote a General Awareness of Metacognitive Activity:

1. Have students keep a daily "learning log".
2. Demonstrate and discuss appropriate metacognitive activity.

To Facilitate Conscious Monitoring of Comprehension:

3. Provide opportunities for feedback.
4. Provide instruction in self-questioning techniques.
5. Teach students to summarize material.
6. Teach students to rate their comprehension. (pp. 235-236).

When teaching metacognitive skills, teachers need to ask questions that encourage students to reflect on their solution attempts. Teachers need to point out potential stumbling blocks to students such as extraneous information and multiple solution processes or answers. They also need to demonstrate actual managerial decisions during problem-solving sessions and not just polished solutions (Garofalo, 1987).

Person, Task, and Strategy Variables

Instruction needs to take into account the metacognitive realm of person variables, task variables, and strategy variables (Garofalo and Lester, 1985). Person variables include what people think about themselves and their capabilities for solving problems. It also includes a self-assessment of strengths and weaknesses as well as the affective variables of motivation, perseverance, and anxiety.

The characteristics of each problem, or task variables, embrace its content, context, structure, and syntax. Task variables also include the belief system a student holds about the field of mathematics and mathematical problems. Garofalo and Lester (1985) reported a profound effect on students' problem-solving successes if they believed such rudimentary principles as a problem may have more than one solution process.

Strategy variables include an awareness of the utility of algorithms and heuristics for solving problems. It is especially important to generalize when and how to apply these strategies for understanding the problem, organizing the collected data, planning and executing heuristics, and monitoring and refining problem-solving attempts (Garofalo and Lester, 1985).

Garofalo (1987) indicated that metacognition causes students to "become watchers, analyzers, assessors, and evaluators of their own mathematical knowledge and behavior" (p. 22).

Writing in the Mathematics Classroom

Writing in Math? Why?

Writing can lead to a deeper understanding and improved mastery of a topic. That is because writing is a mode of language that involves the active manipulation of knowledge. Creating an original piece of writing requires students to analyze and synthesize information, focus their thoughts, and discover new relationships between bits of knowledge. Writing about something involves many of the thought processes teachers would like to foster in their students. Consequently, writing can be an instructional tool to promote learning in areas not usually associated with writing. (based on Emig and Haley-James, cited in Pearce and Davison, 1988, p. 6).

Mathematics is one such academic subject that is not usually associated with writing. Mathematics is considered to be a language in and of itself. It has its own characters or symbols and the meanings that are associated with them. It can also be a "foreign language" to many people, students as well as adults, who do not comprehend it. It is imperative that one learns the language in order to be an effective mathematical problem solver (Rothman & Cohen, 1989). Birken (1989) stated, "Writing allows students to explore constructs of a foreign language (mathematics) using a language in which most are fluent" (p. 41). Lester (1989) added that children using natural language derive more meaning from problem-solving experiences.

Since calculators are able to perform many of the arithmetical operations on which we used to spend the vast majority of mathematical instruction, this time can be put to better use by teaching for the comprehension of mathematics and the communication of mathematics (Keith, 1989). Some instructors (Mett, 1989; Tobias, 1989; Powell and Lopez, 1989; Keith, 1989; Lesnak, 1989; Birken, 1989) have sought to incorporate various forms of writing into their classrooms in an effort to deepen and broaden their students' personal understanding of mathematical concepts and their applications. Unfortunately, these are not typical classrooms.

In a study of the use of writing in junior high mathematics classrooms, Pearce and Davison (1988) categorized all evident forms of writing. These categories were

1. Direct use (copying and transcribing).
2. Linguistic translation (changing symbols to words).
3. Summarizing and interpreting.
4. Applied use (writing story problems or test questions).
5. Creative writing (reports on math projects).

They found that students' writing was infrequently employed as a tool for learning mathematics and when it was observed, it was predominately the non-original, low-level activities that were utilized. Subsequent observations in classrooms that did not participate in this study led them to generalize that much of the same lack of writing activities as instructional tools is prevalent in our mathematics classrooms. Talton and Francis (1987) supported this contention. They noted that writing has rarely "been included as a portion of the instructional procedure to be utilized in teaching mathematical word problem solving" (p. 10).

Writing forces one to slow down the thought processes and to reflect on what one is thinking. In that respect writing may be a very valuable heuristic when applied to mathematical problem solving. It incorporates metacognition which can help the student to spend enough time to fully understand the problem before him/her and choose an appropriate strategy with which to begin the solution process. It also encourages students to generalize and appropriately chunk and store the solution processes in superordinate categories for retrieval during future problem-solving sessions. Birken (1989) reflected on her students who had

> never before thought about what they did or why they did it while problem solving. Since writing requires the ability to communicate (even if only to oneself) a process or idea, most students comment that they have a deeper understanding, further clarity, and better retention of concepts after writing. (p. 41).

Marshall (1986) argued the need for diagnostic "information about what students do and do not know, including details of student misconceptions and misunderstandings" (p. 1) to be given to teachers in order that they can facilitate the students' correction of the erroneous thinking. Marshall further emphasized the need for such diagnosis of students' higher-level thinking skills of processing, analyzing, synthesizing, and evaluating which are necessary to arrive at solutions to mathematical problems. Writing encompasses these skills and allows for their transfer to problem solving. When students write problems, it improves their ability to solve them (based on Malina, cited in Suydam, 1987).

Beyer (1990) in a regional meeting of the National Council of Teachers of Mathematics provided an additional six-fold rationale for using writing in a mathematics classroom. She stated that writing

> 1. Provides a more supportive learning climate.
> 2. Improves ability to communicate mathematically.
> 3. Facilitates development of conceptual understanding.
> 4. Extends and refines thinking on a concept.
> 5. Develops higher-order thinking skills.
> 6. Facilitates monitoring and adjusting learning/instruction. (p. 4).

Probably, the most important reason why students should be writing in the mathematics classroom is that they become active learners and doers instead of passive sponges soaking in knowledge from a sage. It transfers the responsibility for learning to where it belongs, the student. The student is able to construct his/her own knowledge which builds on the base of his/her existing knowledge (Kenyon, 1989).

Writing in Math? What?

James Britton, a leading theorist in the field of language and learning and advocate of writing-across-the-curriculum, has classified writing activities as transactional, expressive, or poetic. Transactional writing "is the language we use to inform, persuade, or instruct and is directed toward an audience" (Rose, 1989, p. 16). Rose also saw this as an opportunity for students to develop and clarify their own understanding of mathematical concepts, processes, and applications. On the other hand, Powell and Lopez (1989) considered writing to be a product approach for the purpose of diagnosis or evaluation focusing on what the students "know at the moment, not for the evolution of their understanding of mathematical concepts" (p. 159).

The second category is on the opposite end of the product versus process continuum. Expressive writing is a process-product approach to writing with the process being at least as important as the product. It is an exploratory form of writing with oneself as the main audience. It allows for the learner to think independently in order to generate knowledge through reflection of mathematical ideas and to create "new knowledge by making connections between new information and what is already known" (Rose, 1989, p. 17). However, the distinction between transactional and expressive writing is not always clear and both are very beneficial for classroom use to aid student learning (Birken, 1989; Rose, 1989). Transactional writing as well as expressive writing can be used to clarify and build one's understanding of mathematics if prewriting, drafting, and revision is encouraged (Pearce & Davison, 1988).

Britton's third category, poetic writing, is an art and not bound by any limitations. It is characterized by a variety of styles and imagination. It may take the form of fiction, poetry, or the dramatic or musical arts (Birken, 1989).

Transactional writing activities. Students may be provided with a situation involving mathematics, for instance a grocery store advertisement, and asked to generate a question which could be solved. The production of many different questions will allow students to see the myriad of possible directions one could take. It also gives them an opportunity to recognize what are relevant data and what are extraneous data for their individual problems.

Perhaps one of the most widely-used forms of transactional writing is to have students rewrite the mathematical problem in his/her own words. Such action requires a student to focus on what is stated and transcribe it into his/her own language (Rose, 1989).

Another popular transactional form of writing is notetaking. Students can be responsible for taking notes over whole-class or small-cooperative group discussions. These could also be used for the following two activities.

Students may be asked to summarize the given problem or solution process focusing in on what is relevant. As Stiff (1988) claimed

> student solutions to problems should also be written in narrative form by students for use with other students. Usually this means that sketchy student solutions must be replaced by detailed ones. Student narratives that are enhanced by greater detail provide an excellent way to sharpen students' understanding of problem-solving heuristics. (p. 668).

This articulation makes it more personal and facilitates the retention of the concepts involved.

Similar to summaries are explanations. A learner may need to explain in precise terminology how to solve a problem or how to avoid pitfalls when solving a particular type of problem (Rose, 1989). A student will need to use metacognitive reflection and higher-level thinking skills in order to effectively communicate such information to classmates.

Students could extend the current problems they are working by changing the parameters. They could create word problems using a specific structure, content, or question. They could also be given free reign and not limited to any special restraints. These original problems could be shared with classmates for the purpose of solving in later problem-solving sessions (Rose, 1989).

Reports on mathematical problem solving could be written by students. They could choose a heuristic or strategy of interest and explain how, when, and why to use it along with examples. Another topic for reports could be how problem solving is incorporated into various careers (Rose, 1989).

Another transactional writing activity may include characters dialoguing about selecting from the variety of problem-solving strategies available to begin solving a problem or what to do when one has reached a "math block" (Lesnak, 1989, p. 152).

Essays that ask students to "investigate nonstandard contexts and ways of doing mathematics" (Rose, 1989, p. 20) would give learners an opportunity to delve into the unusual, and perhaps even the ridiculous, to motivate and encourage understanding of why it works.

Expressive writing activities. Expressive writing gives the student the opportunity to "explore what he or she thinks, feels, or knows" (based on Britton, Burgress, Martin, McLeod, & Rosen, cited by Birken, 1989, p. 36). It is "a personalized method of understanding the topic" (Birken, 1989, p. 38) being studied. When discussing expressive writing, Mett (1989) stated

> writing can be used at the beginning of class to get students interested and involved in a new idea. Writing can be used at the end of class to help students reflect and summarize before they get distracted by other interests. And writing can be used at an intermediate time to make a transition, to allow absorption of a new idea, or to personalize a theory by creating examples and applications. (p. 295).

It encourages students to reflect on previous studies and make connections with the present material as well as sets the stage for associations with future learning.

Freewriting, also referred to as process writing, tends to be of a stream-of-consciousness nature. Powell and Lopez (1989) discovered that it served a four-fold purpose. It can be used (1) to focus students, (2) to clear their minds of preoccupations and anxieties, (3) to reflect on mathematical processes, and (4) to serve as a source of topics for journal writing. Rose (1989) proposed that freewriting be used to suggest a solution for a given problem, explore errors in an individual's own problem-solving steps, brainstorm discussion questions, and generate problems.

Journals, which are also known by a myriad of other names such as "logs, notebooks, diaries, and thinkbooks" (Rose, 1989, p. 17) can take many forms. They can range from highly structured (answering the predetermined questions of the teacher) to unstructured (choosing own topics). The content may include, as it does for Mett (1989), "a summary of new material learned in class", "a discussion of individual work outside of class", and "an analysis of connections, difficulties, and open questions" (p. 293). As guidelines to his students, Powell "suggested that journal entries focus on one's own learning, feelings, insights, discoveries, and so on" (Powell and Lopez, 1989, p. 167). Tobias (1989) recommended a "divided-page exercise" (p. 51) in which the left side of the page contained affective comments and the right side is reserved for cognitive ones.

Rose (1989) noted that journals benefit students by slowing down the thinking process, so that they become aware of their thought processes. This metacognitive action will enable students to reach solutions to mathematical situations and discover errors in their problem-solving methods when they are stuck.

Letters to uninformed mathematical problem solvers could be written to relay both cognitive and affective elements of problem solving in general or in relation to a specific problem. "Dear Teacher" letters invite the students to write about what they comprehend and don't comprehend with examples of both and any questions that they have about the subject matter, in this case problem solving (Rose, 1989).

"Minute papers" could be written in which students reflect about the most important or most confusing concepts that were studied during mathematics that day (Tobias, 1989).

Poetic writing activities. The creative use of poetic writing in mathematical problem solving may include writing short fictional stories (perhaps mysteries) embedded with math concepts and presenting some question for the reader (sleuth) to solve. It seems possible that this could overlap with transactional writing. These fictional accounts could also be transformed into dramatic presentations. Poems and songs that contain mathematical ideas and problems could also be written and performed.

Writing in Math? How?

Talton and Francis (1987) declared that

> during the writing process each student must be led to understand the essential steps in the problem-solving process, the mathematical language within the stated relationships and the implied language. Thus, when they use the writing process to create their own mathematics word problems, they have the same obligation to their reader.

Pearce and Davison (1988) affirm that a prewriting, drafting, and revision process is necessary to enhance student learning. They noted that "prewriting assistance generates ideas, helps focus thought, and gives some idea of what is expected and how to go about it" (p. 7). They also asserted that "rewriting the content forces students to consider and change what has been written, thereby promoting new learning" (p. 8).

Talton and Francis (1987) recommended a similar process for incorporating writing to benefit in comprehending mathematical problem solving. The teacher begins by modeling the solution of a problem using concrete, pictoral, and symbolic representations and the appropriate mathematical language. Students are allowed to ask questions during this phase. Next students create their own oral problems that follow the theme of the teacher's example.

This is followed by an extensive discussion of these problems between the instructor and EVERY child. Students must have daily opportunities to analyze the nature of the problem and the selection of problem-solving strategies.

Students are then given a chance to practice creating under the supervision of the teacher before they begin drafting their own problems and solutions. These student works are edited by peers and/or the teacher and are revised by the original author. Completed works are then published for display in the classroom and for use by others.

While this process of writing instruction is intended for student-generated mathematical problems, it seems appropriate for use with any of the writing tasks mentioned in this paper.

No matter which form of writing is utilized, Birken (1989) cautioned, "Writing exercises are no more beneficial than tests unless they allow the student to explore, think, test, take risks, and learn through the process" (p. 34). These actions parallel the nature of mathematical problem solving.

A diversity of transactional, expressive, and poetic writing suggestions have been presented in this paper for implementation in mathematics classrooms. If students are taught through a process that combines prewriting (modeling, discussion, and practice), drafting, and revision (editing, revising, and publishing), it "should promote a more reflective, analytical attitude towards the study of mathematics" (Pearce and Davison, 1988, p. 14). Writing can be used to provide students with sorely-needed opportunities to engage in creative and reflective mathematical problem solving. This will permit students to understand not only what to do but why and when it works. It encourages students to think and take responsibility for their learning.

Teaching for Problem Solving

Mathematical Misconceptions

The most awesome yet necessary task that a teacher has to accomplish is to change the students' misconceptions about mathematics. Frank (1988) summarized the beliefs of many students.

1. Mathematics is computation.
2. Mathematics problems should be quickly solvable in just a few steps.
3. The goal of mathematics is to obtain "right answers".

4. The role of the mathematics student is to retrieve mathematical knowledge and to demonstrate that it has been received.

5. The role of the mathematics teacher is to transmit mathematical knowledge and to verify that students have received this knowledge. (p. 33).

Two additional myths cited by Vobejda (1987) from an interview with Schoenfeld are (1) mathematical procedures used in school do not entail discovery and (2) comprehension of mathematical concepts and strategies and their creation are beyond the scope of all persons except the intellectually gifted.

It is indeed unfortunate that students have gleaned such false notions from their mathematical instruction. Student beliefs about mathematics and other affective concerns can influence a students' problem-solving ability. Only after these beliefs are changed, will students have a much better concept of problem solving as the focus of mathematics. It is imperative that in order for mathematics educators to attempt to change these unwholesome views of students toward mathematics and problem solving, that they themselves must be knowledgeable and comfortable in solving problems.

Choosing Good Problems

In order to be most effective in teaching problem solving, teachers need to spend time planning their instructional content and strategies (Burns and Lash, 1988; Thompson, 1988). Teachers should not use their textbooks as the sole source of problems or heuristics. They often need to make instructional decisions which may differ from the methods of a basal series (Kameenui and Griffin, 1989). Such decisions take into account the uniqueness of a classroom and its teacher and the differences in abilities as well as prior knowledge among the students.

A key ingredient is to actively engage students in the learning process (Osborne, 1988). One of the major tasks involved in teaching problem solving is to choose good problems. Students need to perceive the problems as interesting or useful in order to motivate them to attack the challenge. The nature of the problems may be "whimsical" (Butts, 1980, p. 30) or real world and should vary to maintain the enthusiasm of the students. Each of the chosen problems should be in accordance with the goals of the course of instruction (Thompson, 1988). Polya (1980) suggested that the best problems were those that are "not too difficult and not too easy, natural and interesting, challenging their curiosity, proportionate to their knowledge" (p. 2). He also emphasized taking a sufficient amount of time to present the problem to the students.

Sovchik (1989) advised that teachers should use problems that contain too much information, problems that do not have enough information to arrive at a solution, and problems that do not contain any questions. These are more life-like situations than most textbook exercises. Students are unable to find a solution by manipulating the given numbers. Instead they must be able to discern the necessary information and create ways to find the missing information or questions. Sovchik also expressed the desirability for using student-invented problems, teaching reading with mathematics, and using calculators. The use of calculators would put less emphasis on computation, a move which is also supported by Frank (1988).

Teacher Modeling

Stiff (1988) discussed the importance of teachers modeling problem-solving heuristics for the students. This allows the students to serve as an apprentice as they observe the master teacher. Silver and Smith (1980) stated that a teacher should point out the use of the strategies that he/she suggests the students use. They further recommended that the teacher models the more difficult problem and then assigns an easier problem to the students in order for the students to transfer the use of the strategies. This is just the opposite practice employed in textbooks and most classrooms.

Stiff suggested that teachers write out a narrative of their own processes for solving a particular problem and allow students to ask questions or guess why the instructor chose certain strategies. He advised that teachers "anticipate where students will have the most questions and weave responses to them into the narrative" (p. 668). This gives students an opportunity to evaluate the teacher's solution (Garofalo & Mtetwa, 1990). Students can then be asked to write their own accounts of problem-solving attempts. With instruction and guidance from the teacher, these narratives can be quite beneficial (Pearce and Davison, 1988). Garofalo and Mtetwa (1990) also asserted that written work should be given in class, for homework, and on tests.

Teachers should also solve problems from scratch so that students observe the actual process and not just a reflection of it. Students can witness that it takes time and perseverance to solve a problem. They will see that it is permissible, and many times necessary, to retrace one's steps and abandon an ineffective strategy for a better one.

Effective Questions and Discussions

Another aspect of teaching students the art of mathematical problem solving is posing effective questions. According to Cemen (1989) thought-provoking questions should be used to stimulate childrens' understanding of the problem at hand, to move them to proceed with the task, or to extend students' application of the problem-solving processes. Schoenfeld, in the article written by Vobejda (1987), reported that by the end of the school term in which his students frequently had to answer questions regarding their understanding of the problem and justify their solution attempts, they regularly practiced analyzing their own actions and changing their tactics if necessary.

Marshall (1986) designed questions to develop her students' abilities to verbalize what they did in their solutions as well as why they did it. Duckett (1990) developed and refined the abilities of her students in using "specific mathematical terms and precise language" (p. 63). She employed the strategy of wait time, allowed students to respond to the correctness of solutions instead of herself, and did not accept an "I don't know" from a student when asked for his/her thoughts.

Cobb, Yackel, Wood, Wheatley, and Merkel (1988) stressed that teachers should be non-judgmental when accepting children's explanations. Teachers are to be more interested in the children's thinking processes involved in the solution rather than the answer. Marshall (1986) noted that a teacher is able to discern much more about the students' understanding of a problem from the errors than the correct answer.

Closely connected to the art of questioning is that of discussing. Discussion is another valuable tool for teachers to use. It allows students to use their natural language which makes it more meaningful for them (Troutman and Lichtenberg, 1987). Discussions should be held in both large group and small group settings. They should revolve around the heuristics invoked in solving the problem and the generalizability of the solution to future problem situations. They should be centered around the students' thoughts rather than the teacher's ideas (Thompson, 1988).

Cooperative Learning

Cooperative learning is a teaching strategy used in more and more classrooms (Duckett, 1990; Frank, 1988; Theissen et al., 1989). Cooperative learning is a practice engaged in frequently in the real world (Vobejda, 1987). Burns (1988) and Cobb et al. (1988) are convinced that this will provide the students with the opportunity to be responsible for both their learning and their conduct.

Thiessen et al. (1989) offered the following assessment of cooperative learning:

> Small group work combines the best elements of both individual and whole-class activities. Small groups provide an opportunity for students to be more actively involved in the process of problem solving. Due to the

sharing of ideas, the students are more likely to be successful and to develop positive attitudes. It has been found that students who work in small groups solve more problems than if they had been working alone. It is also easier for the teacher to provide appropriate guidance for small groups than for students working individually. (p. 38).

Frank (1988) stressed the importance of moving away from a teacher-centered approach to learning toward a student-centered approach. It is necessary that students are weaned from dependence on a teacher and rely on themselves and each other as mathematical authorities.

Wick (1990) listed the following four outcomes of cooperative learning.

1. Increased achievement.
2. Improved attitudes towards classmates.
3. Enhanced self-esteem.
4. Improved skills in cooperation, communication, building and maintaining trust, and dealing effectively with controversy. (p. 2).

Students must work together and help each other to master the mathematical concepts and strategies. Only after each person in the group has been consulted and everyone is stuck can the group ask the teacher for help (Burns, 1988; Wick, 1990).

Depending on the task, groups may range in size of up to six students (Burns, 1988). All students must be participating members of their groups if they want to be successful problem solvers. Wick (1990) gave each of the members of her groups one of the following four specific roles: checker, observer, encourager, or recorder. She also rotated her random groups of four every three weeks so they had an opportunity to work with many classmates.

Assessing Students' Progress

Since the focus of mathematical education is changing in terms of content and instructional practices, it is necessary that assessment techniques also change. Stacey (1987) pointed out that it is necessary to know HOW as well as WHEN to perform strategies throughout the problem-solving process. This is in accordance with other educators who espouse the importance of students justifying WHY they used certain techniques (Schoenfeld, 1980; Lester, 1985). This may mean that the evaluation practices currently used may need to be modified or totally changed.

Charles, Lester, and O'Daffer (1987) identified the following goals for mathematical problem-solving education:

1. To develop students' thinking skills.

2. To develop students' abilities to select and to use problem-solving strategies.

3. To develop helpful attitudes and beliefs about problem solving.

4. To develop students' abilities to use related knowledge.

5. To develop students' abilities to monitor and evaluate their thinking and progress while solving problems.

6. To develop students' abilities to solve problems in cooperative learning situations.

7. To develop students' abilities to find correct answers to a variety of types of problems. (p. 7).

There are several means of assessment that can be used to tell us where students are in relation to solving problems.

<u>Observations and Interviews</u>

Observing students solving problems during class will provide data on what a student understands about the problem and what strategies he/she uses proficiently. During this observation period an instructor may want to question the students in order to clarify and extend the data gathered by observation alone. The teacher can record notes on the students' problem-solving strengths, weaknesses, and progress. This can become too time consuming to use on a daily basis so it might be wise to alternate the anecdotal notes with pre-established checklists and/or pre-established rating scales (Charles, Lester, and O'Daffer, 1987).

A formal interview involving a student solving one or more problems could also be used. This observation and questioning format follows established guidelines and seeks either general or specific data. Again anecdotal notes, checklists, or rating scales could be used to record observations (Charles et al., 1987).

Students' Writing

Students' writing can be used to ascertain students' thinking and ability to use heuristics. Mett (1989), Tobias (1989), Powell and Lopez (1989), Keith (1989), Lesnak (1989), and Birken (1989) have successfully used various forms of writing from their students to assess their students' abilities. Student inventories are an excellent means of providing data about the students' beliefs and attitudes toward mathematics (Charles et al., 1987).

Scoring of Written Mathematical Work

Charles et al. (1987) suggested the use of a five-point holistic scoring system in assessing a students' written mathematical work. Points are awarded based on criteria which takes into account strategy selection and implementation. It stresses a holistic view of the process and the product.

Analytical scoring differs from holistic scoring in that it breaks the solution into three distinct categories and assigns a numerical value to each phase. Scores of 0, 1, and 2 are assigned in each of the categories of "Understanding the problem", "Planning a solution", and "Getting an answer". Established guidelines to determine the different point values are also used with analytical scoring.

Tests

Multiple choice tests can be used to measure students' thinking processes and problem-solving skills. Each test question is followed by a choice of several answers, one of which is the best one. "The other possible answers, usually called distractors, often reflect common mistakes or misinterpretations, and are designed to entice students who are unsure about the correct response" (Charles et al., 1987).

Completion tests that ask students to supply necessary data, desired strategies, or appropriate written explanations and justifications of given problem-solving heuristics would be another excellent means of determining students' problem-solving aptitudes.

Mathematical Portfolios

Lester and Kroll (in press) suggested the use of student mathematical portfolios which are similar in nature to an artist's portfolio. This collection of work is a combination of all of the other forms of assessment (teacher observations and interviews, student writing, written mathematical work, and tests). It should span an extended period of time. A mathematical portfolio should demonstrate the use of a vast repertoire of problem-solving tactics at the students' disposal. It may be compiled by the student or by the teacher and the student, but the quantity of entries needs to be limited so that it is feasible for the instructor to assess the students' progress. This limitation of the number of pieces of work forces the student to be reflective about choosing the samples. These portfolios should be evaluated in terms of the quality and variety of work and not in terms of the quantity of entries.

Purposes of Assessment

These assessment techniques can also be used for the purpose of modifying instructional content or teaching methods to enable the educator to better work toward the desired student outcomes or goals. These evaluation techniques can be vital in determining the climate of the classroom. Changes may be necessary in order to create positive student attitudes and beliefs. A third purpose for assessing student problem solving is assigning grades. Evaluation is not synonymous with grading, however, it does provide documentation for a letter grade or percentage grade (Charles et al., 1987).

CHAPTER III

METHODOLOGY

Setting

The setting for this naturalistic study was a self-contained seventh-grade class of twenty-one students. The large classroom had a row of windows along the entire southern side of the room. The other three walls each held a bulletin board of a different size. The north and west walls contained chalkboards. The position of the door was along the east wall in the northeast corner of the room. The top half of the four walls was painted mint green and the bottom half was a grayish-brown. Colorful posters were strung along the east, north, and west walls above the chalkboards and bulletin boards. Students' works were displayed along all available wall space.

The large pieces of classroom furniture consisted of 23 students' desks, the teacher's desk and cabinet, and a long table for small group work. The instructor's desk stayed in the southwest corner of the room along with the enormous cabinet/shelving unit, the file drawer and the plant stand. The work table and 21 of the students' desks were frequently rearranged as the students and/or teacher desired. The globe and the atlases were placed on one of the extra desks near the door. The classroom computer was on the other extra desk which shifted positions as the seating arrangement dictated. The northwest corner was the most popular area of the room. It contained an 8'x11' Oriental rug, two over-stuffed chairs, several large pillows, a footstool, and three mismatched book cases. It was usually used for reading and small-group work, but students were free to work there as they wished.

The seventh-grade classroom was located on the third floor of the parochial school along with the classrooms for the fifth grade and the sixth grade, wash rooms, and a huge meeting room. Coat racks lined both walls of the hallway. Three stairwells led down, two led to the driveway entrances used by the students, and the other led past the gymnasium to the church offices on the first floor and the cafeteria in the basement.

The 28-year-old Christian Day School is located in a multi-cultural neighborhood of a large Midwestern city. The neighborhood is predominantly Hispanic and African American with some Caucasian and Middle Easterners. The racial composition of the school is approximately one-third Caucasian, one-third Hispanic, and one-third others, mainly composed of African Americans and Middle Easterners.

Treatment

This inquiry into the use of writing as a heuristic to teach mathematical problem solving engaged the teacher as the principal researcher. Polya's framework for problem solving, (1) Understand the Problem, (2) Devise a Plan, (3) Carry Out the Plan, and (4) Look Back / Evaluate, was used since it was similar to previous basal instruction that the students had in earlier grades. Metacognitive questioning was incorporated into Polya's model. Routine multi-step problems, nonroutine process problems, and nonroutine real-world problems from a multitude of sources were used.

The forms of writing used in this study were analytical, creative, and reflective. Analytical writing was accomplished with the use of a teacher/researcher constructed study guide based on Polya's framework (See Appendix 1 -- Analytical Study Guide for Problem Solving) to aid the students in the problem-solving process. The Analytical Study Guide contains many general heuristics that can be employed while comprehending and reflecting on a variety of problems. After solving a problem, students wrote a narrative of their solution using the steps of the problem-solving framework as topics for each paragraph. They were to identify what, when, and why they used certain strategies and what they were thinking as they attempted to solve the problem.

Creative writing was in the form of (1) student-generated mathematical problems for which they wrote solutions (See Appendix 2 -- Original Writing of Math Problems), and (2) student-generated mathematical stories. Daily journals were used for the students to reflect on both their thoughts and their feelings and attitudes toward problem solving.

Daily instructional periods were approximately one hour in length. Mathematics/writing was scheduled for just before lunch. However, tests were taken in the afternoon so that students had an ample amount of time.

Initially each class period involved five steps. (1) Class began with the teacher modeling problem-solving heuristics and metacognitive strategies. The students had an opportunity to ask questions to clarify their understanding. (2) This was followed by the teacher guiding the students through the problem-solving process. (3) Students would then work in small, cooperative groups solving or writing problems. Students had written guidelines for both of these activities. Cooperative groups were changed periodically and chosen by various means -- draw, numbered-off seating arrangement, alphabetically, birth dates, student choice, etc. (4) Students would then work as individuals to write a narrative account of their group problem-solving experience or they might solve a problem on their own. (5) Each instructional period would end with journal writing.

After a short period of time, it became apparent that this five-step format was an unrealistic expectation for a one-hour session. The daily lesson plan was modified. A much more-flexible time schedule was used that allowed for a sufficient amount of time during each of the phases (teacher modeling, guided practice, small groups, individual work, and journal reflection). The attempt to accomplish the entire five-step format each and every day was abandoned. We simply picked up where we left off the previous day. There were no time deadlines to infringe on any part of the research which created a need for more instructional time. In order to accommodate this need, the time span of the study was increased from eight to eleven weeks.

It also became obvious within a short period of time that a majority of students in the class were not comfortable with solving problems individually. Some of the students expressed their lack of confidence in their problem-solving abilities. A few others flatly refused to attempt solving a problem on their own. Again instruction was modified to place an emphasis on the guided practice and cooperative working stages. Journals were still used extensively. Modeling was also evident.

Pre- and post-test instruments were teacher/researcher constructed and consisted of ten problems (five routine and five nonroutine). The problems of each assessment were paired as closely as possible. These assessments along with all daily written mathematical solutions were evaluated using the holistic techniques and the analytical techniques of Charles et al. (1987).

Students were also asked to create routine and nonroutine problems prior to and throughout the course.

A combination of teacher observations, students' writing, daily written mathematical work, and the pre- and post tests was used to assess if students' writing would aid in the understanding of mathematical problems and their solutions.

CHAPTER IV

ANALYSIS AND PRESENTATION OF DATA

Choosing the Topic for Research

In the past I have been concerned about the lack of problem-solving ability of my students. Some of my lower-ability students have not felt comfortable with problem solving and were reluctant to attempt to solve textbook word problems. My students who do well with routine problems have not performed as well as expected on the nonroutine, process problems of our state's mathematics contest for junior high students.

In choosing a topic for this research project, I wanted something that would benefit my students and be relevant to others. Problem solving, then, became an obvious choice for subject matter for my thesis.

I was overwhelmed at the variety of paths that were explored in the vast quantity of literature regarding problem solving. I was especially intrigued with the combination of writing and mathematical problem solving for two reasons. First, every year I begin with the resolution to have my students write more than in the previous year.

Second, "connections" is a major tenet of my philosophy of education. Connections within a subject curriculum and between the different subject curricula. The infusion of writing into mathematical problem solving satisfied both of these conditions.

The Study

I was still on a "high" from my first North Central Regional Conference of the National Council of Teachers of Mathematics from one-and-a-half weeks earlier when I began this research project with my students in the fall. Although I was excited about this study, I was even more nervous about it. Since it was a naturalistic inquiry with no pre-conceived ideas about the results but rather it was to unfold naturally in the classroom environment, I was frustrated at not always being able to control the direction of the study.

I told my students that we would be doing some work outside of our mathematics textbooks for the next quarter. The students were very delighted about this. I also told them that we would be connecting our mathematics with our language instruction. The students accepted this as one more new-fangled idea of their seventh-grade teacher.

Up until now, all of their teachers had been very textbook-driven in their approach to instruction. This was the beginning of the year, and the students were still unaccustomed to all of my non-stereotypical methods of teaching. While the students knew that I was working toward a Master's degree, they were unaware that they were the objects of a thesis. To them this was just one more of these nonstandard ideas that I incorporated from my summer school courses.

<u>Analysis of Pretest Results</u>

On the first day of the study, Tuesday, the students took the pretest consisting of five routine problems and five nonroutine ones. They were ecstatic about taking a test that was not going to be recorded in a grade book. There were also a few who were skeptical about such an idea.

It was interesting to note that even without instruction in problem-solving heuristics, some students used them. Michael took his change out of his billfold to help him solve a nonroutine money problem. His solution of incorporating the acting-it-out strategy with the aid of manipulatives led him to one of the correct answers for that problem.

Luke, Megan, Tim, Guy, and Joel used the breaking-into-parts strategy to glean more than the obvious combinations of parts from a nonroutine rectangle problem. However, only Megan followed through with the strategy to find all of the rectangles within the given diagram.

Melanie also found the correct answer although her use of a specific heuristic was not evident.

Angela, Tiffany, Megan, George and Guy all used patterning on a football score problem. Megan found 19 of the answers, Tiffany found 20, Guy found 33, Angela found 40, and George found all 43 answers to the problem. To solve this same problem, Tim, Missy, and Joel used systematic elimination. Joel found 19 of the answers, Missy found 21, and Tim found 40.

There were two problems which gave the students the most trouble. One asked them to find the dimensions of a rectangle when given its perimeter and its area. I think that the vocabulary word, dimensions, may have been a hindrance to many of the students. Most of the students tried to add or multiply the perimeter and the area. This did not surprise me since we had not studied these concepts yet. George was the only one who found the correct answer.

The second troublesome problem gave the students four single digits and asked them to write two whole numbers with the largest product. The digits could only be used once. Again it may be the vocabulary word, product, that had been the problem. Several students added or incorrectly divided instead of multiplying the digits. Most of the students also ignored the condition of writing two whole numbers. Some students gave one number while others listed fractions or mixed numerals. Two students ignored the condition of using the digits only once. Four of my students used digits other than the given ones. One of my girls drew four circles and unequally divided them. However, Tiffany and Christine did solve the problem correctly.

Students' reflections of the pretest ranged from easy (Joel) to "dumb and too hard" (Christine), from fun (Tiffany) to boring (Scott), from liking it (Guy) to hating it (Angela). Fifteen students commended though that at least part of the test was hard or that they performed poorly on it. Angela and Tiffany wrote two of the more interesting journal entries for the day. Angela exclaimed

> I am mad that I did bad because I wasn't sure about things. Then I got mad at Miss Puffe for giving us the pretest. I hated it! I am stupid because I don't know this stuff!

Tiffany's journal entry for the day read

> I think that taking these pretests before taking the actual test is better then we are prepared and we know what we should expect and what to look forward to on an actual test. It was pretty fun because we never do this and it is different.

The following day the students created two routine problems and two nonroutine problems. To make sure that they knew what I wanted, I asked them which of the problems from the previous day's test fell into each of these categories. After class members quickly and correctly classified the problems, they wrote their original ones. With only examples from the test and previous textbooks, I was pleased to note that the students' routine problems were all correctly categorized. There was variety among the "cover stories" and the mathematical structure of the routine problems. Here is a sample of their original problems with their grammatical and mechanical errors.

> Nancy is a dress maker, she needs 16 yards of material for one certain dress. The store only sells the material in boxes of 3 yards. How many boxes does Nancy have to buy? (Missy)
>
> Gail had 80 stickers. She had to split her stickers between 4 of her friends and herself. How many stickers does each person get? (Melanie)
>
> Nancy has $5. She buys a cassette tape of "Teenge Mutant Ninja Turtles" and also buys a pack of gum. The gum costs 46¢ tax included. The tape costs $4.19 with

tax included. She then found $10 in her pocket. How much does she have left? (Scott)

Mr. and Mrs. Asonavage were on a vacation their van had a 30 gallon tank he was going to drive for two straight days the van needed 30 gallons for 120 miles Mr. Asonavage was going to go 480 miles how many times did he have to fill the tank? (Damon)

Lisa has 1 quarter, 3 dimes, 2 nickles and 9 pennies. If Lisa bought 2 suckers for 10¢ each how much money does she have left? (Megan)

Martha liked going shopping she went to the mall with $25.00 She bought a blouse for $10.00 She bought the pants for the blouse The pants were $12.00 She bought a barette for $3.89. did Martha have change could she have bought something else or did she have to leave something she bought. (Frances)

I was also surprised that many of them were longer and mathematically more difficult than I expected. Missy's problem requires that you round off your answer to the next highest number. You cannot "crunch" the numbers to solve the problems written by Melanie, Damon, and Megan. Damon also includes the extraneous data of two days. Multiple steps are involved in Scott and Frances' problems. Frances also does not ask for a numerical answer. From this I gleaned that the students were excited about this new approach to mathematical instruction.

Of course, many of the problems mirrored those that one could find in a typical mathematics textbook. As with every class, there is a range of ability. This class has fewer average ability students. They tend to be clustered at the two extremes. There was one student who wrote a particularly interesting "problem".

> One day me and Justin went to the store to get 9 gallons of milk. Last week we got 72 packs of paper. How many packs of paper are left? (Jordan)

This problem contains both extraneous information and insufficient information. The latter case precludes the problem from having a solution. For this reason, it would be a good problem to discuss in class. However, doing so would greatly embarrass this student.

Having extremely limited exposure to nonroutine mathematical problems, the vast majority of the students wrote routine problems. Of those that were nonroutine process problems, most of them were spin-offs of the rectangle problem on the pretest. The students did choose a variety of sizes and shapes for their original problems.

There were three other original nonroutine problems that were non-geometrical and are included here. They are also similar in mathematical structure to the football score and coin problems from the pretest. I am sure that the girls did not realize when they wrote these problems that they have more than one answer. These would be excellent problems to use to change students' misconceptions that mathematical problems have only one right answer.

> The basketball game lasted for 2 hours. The Celtics won and the Suns lost. The Celtics score was 87 points. Baskets are 3 points and freethrows are 1 point. How many of each was scored? (Melanie)

> Kim was at her cash register and the customer wanted change for $2. She wanted quarters, nickels, and dimes but she only wanted 15 coins. What would Kim give her? (Frances)

> There was a basketball game. Each basket was worth 5 points. Freethrows were worth 6 points. How many baskets and freethrows would Jordan have gotten if his score was 136? (Tiffany -- Believe it or not, Tiffany was an excellent basketball player and was a starting guard for our girls' team.)

Step One -- Understand the Problem

We continued our mathematics/writing class that day with the modeling and discussing of two nonroutine, process problems. Because I wanted students to realize how important it is to take time to understand a problem instead of just rushing through calculation, we spent several days working with only the first step, "Understand the Problem", with the aid of the Analytical Study Guide. I also chose to start with nonroutine problems because they would peak my students' interest.

The first example problem involved buying exactly 100 farm animals for $100. Each of the cows, pigs, and chickens had set prices and all of the different animals had to be purchased. The second one asked for the time that three race cars traveling at different rates would be at the same place on a circular race track.

After today's lesson, only Frances, Veronica, and Christine reflected that they still thought that problem solving was hard or confusing. All of the other students commented to some extent that they were understanding it, or as Donna, Melanie, and George wrote "getting the hang of it". Even Angela's self-confidence completely turned around from the previous day. She wrote

> I am impressed with myself because I understand more about solving story problems and how to work them

> slowly and carefully. I now understand what to do and how to work a story problem step-by-step. Now I know I understand how to do this.

I found that this held true for most students throughout the study. There was a direct relationship between the success that students experienced in class and their perceptions of themselves as competent problem solvers.

Guy wrote that he had learned a lot, and Donna said that it was fun and educational. Joel and Tim agreed that problem solving was fun. Damon and Missy commented that the whole class discussion was excellent. Megan said that she felt that writing out the study guides would "help us solve them easier. I knew how to solve the problems but writing them up on paper was different [sic]."

The next day I guided my students through the understanding stage of two more problems. The first involved a peasant who had to get his farm products safely across a river in order to take them to market. The other problem involved finding the highest unobtainable score on a pictured dart board.

Most of the students actively participated in the discussion. Those who did not participate wrote down the comments of their classmates on the Analytical Study Guide. My favorite part of both discussions was searching for "insights" into the problems. The students were amazed that the first problem asked for a list of steps and not a numerical answer. In the second problem Melanie realized that the highest unobtainable score could not be a multiple of four or nine which were the numerical values on the dart board.

With more positive experiences in problem solving, yesterday's students who lacked confidence in their abilities were becoming more optimistic. Veronica knew that "if we keep doing the problems we will learn how to do the problems better". Frances stated, "It was a little complicated but I will understand as we go on." Christine expressed that although the work was hard, it was "o.k." and that she understood "a little bit". Joel wrote, "I think it is a lot of fun because we really start to think about the problem." Megan added

> I feel that the sheets that we are writing on make the problems easier and make me see how I am solving it not just speeding through them without looking and checking them. I know how to solve the problems but the sheets [Analytical Study Guide] help me understand more and see how to do it.

On Friday we tackled two routine problems. One involved determining if a basketball player had enough playing time to merit a letter for the sport. The other involved finding the cost of putting new baseboards around a rec room.

The problems were not difficult for the students. Nine of my boys and at least eight of my girls play and/or follow the sport of basketball, so they quickly worked through "Understand the Problem", step one of their study guide. The second problem was also easy for them, perhaps too easy. Some of the students would have preferred to solve it rather than just understand it. Scott, Guy, and Joel calculated the number of baseboards and the cost in the margin of their study guide.

When I suggested that the students now try the understanding step of problem solving in small groups, they became verbal about their lack of confidence and ability. The students were used to small group work where everything is laid out for them such as following the steps in a science laboratory experience or reading and answering questions for a class. They are not used to having to generate information on their own. Since I wanted the students to be as successful as possible, I allowed them to "convince" me to spend the remainder of the day working as a whole class with the provision that on Monday we would begin cooperative-learning problem solving.

I chose nonroutine problems to work on for the remainder of the class period. The process problem was to find the sum of the first 100 odd numbers. The real-world problem was to locate a dripping faucet, determine the amount of money lost in a year, and the length of time it would take for a new washer to pay for itself by the water it saves.

Students readily participated in discussion. I felt that they were gaining confidence as we worked together. Joel mentioned in his journal that it was "fun because it gets us to think". George "was impressed of [sic] myself because I had an idea of solving each of the problems. I am gaining progress in these problems. I knew more what to do." Tiffany and Melanie reflected on this novel focus and novel approach to mathematical instruction. Tiffany wrote, "I think this was weird. I don't think any other class does anything like this." She suggested that the class get a good grade for the day since everyone did a good job during discussion. Melanie said, "To me this is not as bad as it was at first. It's not better than regular math but it's o.k."

Time was a key element mentioned by two students. One thought that we were moving slowly so that everyone could absorb the skills, the other too fast which inhibited learning. Angela's daily entry said, "I know I understand this better -- Miss Puffe is teaching us slowly so we could understand it better." On the other hand, Josh reflected, "We think this is getting to our heads a little. We need more time."

Monday came and the students were ready for small group work. The students were asked to give the number of pages that 381 digits would fill. The other problem asked if it was possible for a bank robber to carry a bag containing $1 million in small bills. The unfinished work was their individual work for that evening.

Journal comments centered around the use of small groups. Some students liked it while others did not. Luke thought, "This is o.k. because it is educational now I may be able to do this by myself." Tim preferred whole class discussion to small groups. Guy liked the cooperative groups because they worked as a team. He "learned more about teams and more about problem solving". Missy wrote, "This was pretty interesting again, but I didn't like the groups."

Donald was not specific in his referent; he simply stated, "I feel good about it. I know I like it." George discussed his learning as well as the importance of active participation. He wrote

> Today we finished the first step in the math solving. I understand how to do it. I learned more by reading the problem better and looking back at it. In the groups I learned alot [sic] more, probably the other

kids in the group too. We all gave ideas to our problems in the questions.

Analytical Writing in Problem Solving

On the following day we began to look at the entire problem-solving process using the Analytical Study Guide for Problem Solving. Again I chose nonstandard problems with which to begin. The first problem which I modeled dealt with two numbered cubes which could make all of the combinations for the 31 days of a month.

During the first step of the process I thought of some important insights for solving this problem. I knew that there must be doubles of some of the numbers and that the cubes needed to be interchangeable in order to make all 31 days. In step two, "Devise a Plan", I thought about the strategies of trial and error (commonly referred to as guess and check -- I prefer to call it guess, check, and refine guess) and systematic elimination (or in this case systematic inclusion). I decided to use the latter heuristic to begin phase three. I was able to narrow the possibilities to 13 choices for the 12 spots on the two dice. I shared with the students that this problem had taken me quite a while to find the "twist" needed to solve it. During the "Look Back / Evaluate" step I showed how we could interchange the numbers of the two cubes to get different answers and generalized the problem to other situations in which flexibility of mind is necessary.

The other problem which I presented asked for the number of diagonals in an octagon. I began my solution attempt with "Understand the Problem", step one of the Analytical Study Guide. After drawing a diagram of an octagon with a few of its diagonals, I realized that it had the same mathematical structure as the familiar "Handshake" problem. I considered a few strategies for step two. I chose to begin step three, "Carry Out the Plan", with the heuristic of a diagram. But I lost count of the number of diagonals and had to switch strategies. I next decided to use a "simpler problem" (triangle, quadrilateral, pentagon, etc.). I found a pattern and used it to solve a problem. In step four, "Look Back / Evaluate", I derived a formula to solve this problem. We had a good discussion of the study guides and written narratives. In fact, we lost track of time and were ten minutes late for lunch.

We took a few minutes after lunch to write in our journals. I knew that it was a great quantity of information to assimilate in a short amount of time, and I expected that the students would be overwhelmed at first. They reflected that the wealth of information about all of the new steps involved in the problem-solving process had indeed overwhelmed them. There was some discomfort gnawing at their levels of confidence. They were all less comfortable with their ability to succeed; some more than others. Missy stated

> I felt that today's cubing problem was frustrating and confusing but at the end I understood it. The diagonals were very challenging and it is still a little foggy. All in all it was a fun process. I know how to do both of these problems. The whole process is simpler.

Donna also admitted that she was lost at the beginning of the geometric diagonals but felt comfortable with it by the time we finished it. She found it interesting to learn that you could solve problems with the strategies of formulas and patterns. Josh again spoke of the need for more time to comprehend everything.

Donald expressed the extreme toll that had been taken on him. He wrote, "I feel like I don't understand anything new or old. I got very confused. I know that I am lost with the new things and I am lost with the old things." What a turn around from the previous day! In my written comment to his entry I asked if he understood step one and suggested that it would make more sense as we do more problems. He responded the following day with "no, they [step one] did not make sense. Yes, I think it [more problems] will help me and the whole class. I feel like I don't understand the last questions. I know how to do most of it but I need help."

I was thrilled that Melanie understands "the tiniest bit. I don't know what it is but I kind of understand it when it is taught. Understanding the problem seems kind of easy now. Everything else is lost in my mind."

Wednesday was spent with the class as a whole working through a geometric problem to find the number of angles given a diagram of ten rays with a common endpoint. Scott realized that the angles would be different sizes, a key insight to unlock the solution of this problem. Several students also recognized this problem as similar to the rectangle problem from the pretest.

The class decided to count the number of each of the various-sized angles and to use a chart to keep track of the sizes of the angles and the quantity of each size. After counting the number of angles of the first three sizes, many of the students recognized an obvious pattern. They abandoned the counting strategy and switched to the pattern. They shouted out the partial answers to the quantity for each of the angle sizes faster than I could write them on the overhead projector. We later used the counting strategy as an alternative strategy to verify our result. It was very interesting to note that the students generalized the patterning heuristic to a triangle containing many smaller triangles, which unbeknownst to them was tomorrow's problem.

Journal entries were much more positive today. As students worked through the problem with my guidance, they realized that they could do it. Joel wrote, "I now think it is easier to catch the pattern after doing Angle Tangle." Frances commented

> I feel that I am understanding better today everything started to make more sense but when we got to the part where we are counting the angels [sic] they went too fast but I will get the hang of it. I knew how to answer the questions.

Veronica wrote extensively in her journal. She said

> I think that the problems now are making more sense because we are going threw [sic] them a lot of times. I feel that they are getting easier but some are still hard. The one we had today was o.k. but I couldn't write so fast so a lot of it I just didn't understand. Because while you were explaining one thing I was trying to finish writing the last problem somebody had just said. I hope the problems we have later are easy if there [sic] hard I hope I can understand. I learned angels [sic] can be made with two rays.

Melanie mentioned the specific phases of the problem-solving process when she wrote

> This is much better now. I kind of understand evaluating [step 4] but I need to know more about that. Devising plans [step 2] is kind of easy, but not really. I understand it though. This PS [problem solving] is getting better. I'm not as lost as I was yesterday, figuring the problem was a bit easier. I'm understanding more about carrying out the plan [step 3].

Megan chose to commend about the Analytical Study Guide. She wrote

> My feelings on these sheets are that these sheets make the work easy and almost like first grade math work, by explaining every step to the detail. I also feel these sheets will help us learn more by not doing all the steps but letting us do the steps. My knowledge on this problem was already known.

Cooperative groups were used the following day to begin solving the triangle problem to which they had generalized the pattern strategy from the day before. It was interesting to note that none of the groups found the pattern. Instead they used the heuristic, breaking-it-into-simpler-parts. These parts were the similar-shaped triangles which the students then counted. The students wrote up their solutions for homework using the Analytical Study Guide sections as topics for the paragraphs.

While most students proclaimed in their journals that they understood the problem which was easy, their total number of triangles varied between the different groups. Christine's group found 56 triangles, Angela's 65, Luke's 42, Megan's 78, and Melanie's 83. It seemed that arriving at an answer constituted an easy problem.

On Friday I collected these first write-ups. I asked the students to journal at the beginning of class today instead of at the end. I suggested that they write about their mathematical write-ups. As they thought and wrote about their weekend writing assignment on triangles, I scanned through the pile of papers. Most of the write-ups were good, solid explanations of their understanding, planning, working, and reflecting. A few were poorly written with an abundance of run-on sentences. These also tended to be the ones with unclear explanations.

An example of a very sophisticated write-up was submitted by George who recognized that his group was not thorough in finding all of the shapes of triangles possible. George reworked the problem and had an extensive and complete page of drawings of the various-shaped triangles to accompany his narrative which gave some details of his thought process as he struggled with the problem. Interestingly, his answer of 103 triangles was two fewer than mine. As I checked his work, it appeared to be correct, but then so did mine. Several of George's friends wanted to see whose answer was correct. Other students wanted to know how there could be so many triangles in the one triangle. I thought it would be fun to rework the problem that day.

As it turned out George's answer was correct. Joel hypothesized that I had counted the two little triangles at the top twice as a part of two different shapes. As I went back through my work, I discovered that that was exactly what I had done.

I also guided the students in the difficult task of finding the relatively simple pattern for the problem. Later Joel reflected, "I thought I knew most of it about carrying out my plan. I should look at the problem from more than one angle."

Angela commented that the write-up was easy. Most of the other students talked more about their perceived success with finding patterns. Melanie, who at first thought problem solving was so complicated, now thinks that "it seems kind of easy". Reworking problems a second time with the same strategy was heralded as ensuring success. Donna commented that this problem "was easier than some problems but confusing and I had to rest a little because I got mad". I was really pleased when I read Melanie's entry. She wrote, "Now I understand PS [problem solving] is important and we need to know about it."

We still had twelve minutes before lunch so we finished class that day with the presentation of an exponential problem -- the grass population of a pond doubles daily. If we begin with one blade of grass on day one and if the pond's capacity is one million blades of grass, how many days will it take for the pond to be filled? half-filled? Students worked through the "Understand the Problem" step over the weekend.

The next three weeks were shortened to three days each on our school calendar. The first week had no school on Thursday and Friday due to our school's parent-teacher conferences. The following week the teachers were at the district teachers' convention on Thursday and Friday. The next week was Thanksgiving vacation. The students had almost as many days out of school as in school during November.

The Monday of the first shortened week we finished solving the grass in the pond problem. This was the first exposure for most of the students to exponential growth. Only a few of the students had ever heard of the "Penny-a-Day" problem in which the amount is double that of the previous day for a month. We set up a chart to keep track of the days and the grass growth. The students used calculators to find the growth. They called out the answers faster than I could write them on the chalkboard. It was easy for them to come up with the pattern when I asked for it. But they were lost when I derived a formula for the pattern since they had done little work with exponential numbers in the past.

George reflected on his solving of the triangle problem. He acknowledged the need for time to solve problems, the benefit of cooperative learning, and an increased level of self-confidence in his problem-solving capabilities. George stated

> I was very, very impressed with myself during the past week. I came from nowhere and got the hang of the problems. I thought the problem was hard so I was sort of mad but I took time and figured out the problem. I knew how to do the first two steps by the group help we have had. I think this gave me more to think about. The problems seem easy but when you get into it, it gets pretty hard.

Damon believed that he is now able to generalize the patterning strategy to a former problem. He wrote, "It was confusing and I had a headache during it. It was o.k.

because with the pretest I know I got the rectangle part wrong but now I can fix it easily."

During the next three class periods, the students voted to choose from among the problems that had already been introduced for the three which they wanted to solve and write about. It was interesting that all of the problems which they chose were nonroutine, process problems.

They worked cooperatively in groups of four on Tuesday to solve the problem of the peasant getting his goods to market. All of the groups chose to act it out. Five groups of students were scattered about the room. I also participated in one of the groups due to absentees. (It was advantageous to use groups of four since there were four characters in the problem.)

One of the groups of girls was very clever. They labeled the group members with their respective characters. This quickly caught on with the other groups. One of the small groups also chose to diagram it in their work space for step three of the Analytical Study Guide.

I was not quite as pleased with this set of individual write-ups as a whole. As a class they tended to be very sketchy about what happened during the group experience.

Again much of the comments were about cooperative learning. Melanie said it was better than doing it on your own. Luke agreed that "two heads are better than one". Tiffany wrote, "I think it was pretty fun. We should do more things like that. It took a lot of thought." Guy created a new vocabulary word. He said

> I liked this because of a team effort, but 2 people did not participate, but I understand a lot more. I learned how to devise a plan to find the answer. I understand more about understanding problems and "stratezing" [quotation marks added for emphasis] them. I learned how to use diagrams to finish a problem.

Guy did not include a diagram on his study guide or his write-up, but fellow group member, Tim, drew a pictorial representation for step three, "Carry Out the Plan", of the Analytical Study guide. His sketch of the peasant's series of river crossings was sequential and clearly labeled.

On Wednesday, the last day of school for that week, we discussed how we could improve our written narratives. After about ten minutes of sharing ideas, we split into groups. The students wanted to choose their own groups for the day and were allowed to do so. Three of the boys with the lowest academic ability were seen as undesirable group members and were left to make up their own group. Knowing that they would not succeed if left to themselves, I decided that it was important for me to participate in their group for the day. I wanted to be part of their group so that they would succeed and so that their classmates would see them as worthy and contributing members of the class. The day's problem required that we purchase 100 farm animals for \$100. We had to buy at least one cow for \$10, one pig for \$3, and one chicken for \$0.50.

Jordan, Donald, Josh, and I reviewed the "Understand the Problem" for this problem which I had modeled and the class had discussed three weeks earlier. The boys stated the given information and what was wanted. They could put the problem with all of its conditions in their own words. We talked about having to buy an even number of chickens to arrive at a whole dollar amount. We decided it would be necessary to use the guess, check, and refine guess strategy. The group felt confident to begin the actual solving of the problem.

Jordan's first guess included 120 chickens. I kept my mouth closed and let the boys calculate the total number of animals and the money spent. Looking at their totals, they realized that they had ignored the parameter of 100 animals. Donald guessed this time. He chose an odd number of chickens. At this point I was really wondering where these boys' minds were during our two earlier discussions of understanding this problem. I wanted to tear my hair out, but instead I calmly let the boys discover their error. It did not take long for Donald to realize that this also was an unsuitable guess. At this point the proverbial light bulb clicked on for Donald. With a little help from me, they were able to refine their guesses. From here on, Donald was able to lead Jordan and Josh through the solution process with very little direction from me.

We, then, examined the solution and decided that the strategy of guess, check, and refine guess could be generalized to other problems which had two conditions that had to be satisfied. During this reflection time Jordan exclaimed, "Donald did an awesome job! He really knew what he was doing." Donald beamed with pride at the compliment from his classmate. The boys were impressed with themselves that they had solved the problem and that they had done so before all of their classmates who did not want them as group members. They literally strutted back to their desks to write up their problem-solving experience. It was fascinating to see other students asking these three "dummies" for hints. This boost in self-esteem of these three boys was one of the most rewarding moments in my entire teaching career.

All three of these boys reflected an improved measure of self-esteem. Donald wrote, "I feel good about the problem. I know how to do a lot of it." Josh said that he learned a lot and felt good. Jordan was more enthusiastic when he wrote, "I feel great it is exciting. The understanding is getting to me."

After a long weekend the students were ready for school to begin on Monday. The problem was to number the pages of a book with 381 digits. All of the groups attempted to use the heuristic of breaking-the-problem-into-simpler-parts, (i.e. single digit numbers, double digit numbers, and triple digit numbers). However, only one group used this strategy throughout their solution. The others changed to counting the digits of pages in a book. Angela specifically mentioned in her write-up that her group used the Bible for counting.

Students wrote positive and negative comments about the interaction of small group members. Damon "learned that if we work together we have 5 different answers and we can put it together to have more fun with it". Luke also thought that his group work was beneficial. He said, "I think this is super because I can help others and not get mad at all."

Unfortunately, another one of his group members did not feel the same way toward him. Angela explained, "I am still mad because Luke always acts like the boss. He won't let me reason with him. He is not the boss or group leader! I try to reason and explain but he won't listen!"

Creative Writing of Mathematical Problems

The two remaining days of the second abbreviated week in November and the following week of Thanksgiving vacation were used for small cooperative groups to create 12 problems and solve them. The students were given guidelines for their original problems (See Appendix 2). In order to fulfill the different criteria, the students had to undergo problem solving. The first three days of the last week in November were used for peer revision, peer editing, and publication of the group problems. During the last two days of the week the students enjoyed spending the mathematics class time solving the problems which were composed by other groups.

Since Luke and Angela had problems working with each other on the last problem, I strategically arranged the "random grouping" so that they would again be together. Over the next five days the improvement in their working relationship was evidenced in Angela's journal when she mentioned that Luke liked the problems which she had created. Unfortunately, Angela and Megan developed a personal conflict outside of math class and this made it difficult for them to work together. Angela wrote

> I understand how to add and devise a problem but it gets hard making up problems in my head when Megan is complaining. If she would understand what I feel and what I mean she might change her attitude toward me. This is really hard.

Megan also expressed her lack of enjoyment with her groupmates although she does not mention specific names. She wrote

> I did not really have as much fun as I always have working in a group because one person in our group worked by themself [sic] and made the rest of us copy and one person just fell back and stayed behind us while we worked.

The small groups tended to gravitate to the same area of the room throughout this creative writing segment of the study. Jordan, Angela, Megan, Tiffany, and Luke rearranged desks near the side chalkboard along the north wall as their workspace. Melanie, Donald, Guy, and Josh huddled at the table in the back of the room. Michael, Suzanne, Donna, and Tim basked in the sun as they worked along the southern windows. Christine, George, Joel, and Missy sprawled out in the overstuffed chairs and carpet. Damon, Scott, Frances, and Veronica clustered desks together in the center of the room for their writing area. I circulated among the groups observing them and offered suggestions when asked.

The individual groups began discussing how to attack the assignment. Interestingly, all of the groups chose to divide the assignment equally among its members. Each member of the group was assigned to write three problems for the group with the exception of the five-member group.

During the first two days students busily chattered with fellow group members about their creations as I watched. However, when I wandered off to observe another group, more than one conversation changed to the students' prospective plans for Thanksgiving break.

As I read through the students' journals, I realized that work was accomplished on those days. Joel reflected, "The thinking of the problems were [sic] fun and easy our group really works well together. I can't wait to do this again tomorrow." Guy declared

> I love this work because I use my creativity not someone else's. I made 3 problems today. I got help from Melanie, but Donald and Josh were [sic] daydreaming. I learned that certain numbers can go into [other] numbers. So I will try that in the future.

After the long weekend we began Writer's Workshop for the written solutions. The students were not used to revising and editing. In their previous years of education the first draft was the only draft. I struggled all year to have my students rework pieces for clarity and interest. These three days were no different.

My students were very lethargic. Perhaps this was because it was the first full week of school in a month's time. No one was particularly looking forward to it. It was difficult to keep their attention on task.

Students wrote solutions for their own problems. They did not take full advantage of their groupmates. There was little communication among themselves. In some groups much of the limited interactions had little to do with mathematics or writing. Donna, in her journal, wrote, "Our group is cooperating very well. I think our group is quiet, but we talk when we need help. Everyone lets them share and try to solve there [sic] problem but we get it wrong sometimes." The students were very possessive of their problems. Therefore, they would not share them with other groups to verify their written solution processes.

Frances reflected on writing the problems. She said, "After writing a problem it was kind of hard to make your own problems but you think of the problems we did before and you get different ideas." Donna, in another entry later that week, wrote, "I know how to figure out geometry problems but not two-step problems. I know how to solve non-numerical problems. I can create lots of geometry and non-numerical problems."

The students also commented on the benefit of the task of creating problems. Missy noted that writing original problems "is a good experience for the class. I think it will help the class to increase the understanding of why and how we are solving problems." Christine thought it was easy and fun. Tiffany said, "It was neat to make up our own problems. We had a lot of discussion. I know that lots of people learned from this."

When I stared in disbelief at the problems that were turned in, I was not as certain that this generative process led to much learning. I was extremely disappointed with the finished problems and written solutions of four of the five groups.

I really wondered how some of the groups had spent their time while engaged in Writer's Workshop. Two of the groups failed to meet the problem criteria. The vast majority of the problems were short, simple, and unimaginative. Some of the problems included the solution process in the question. Others were unrealistic, even for seventh-grade students. There were too many grammatical and spelling errors. One group failed to use question marks after five problems that ended with questions. As a whole they were of much poorer quality than the individual problems submitted prior to instruction. Most of the nonroutine problems were patterned after the angle and triangle problems. They involved a diagram containing rectangles, diamonds, triangles, or line segments and rays.

A sample of the cooperatively-generated routine problems is provided.

A cop went to buy some donuts. If they cost $3.60 for a dozen, how many donuts could he buy with $3.

Elisa went to the store for school supplies. She bought a pack of pencils that costs $1.39, a pack of pens for $2.69 and 4 notebooks for $1.89 each. How much did it cost? How much did she get back from $20?

Judy had to save forty-five dollars to go to California in one week her parents are paying for the flight and Judy is saving for spending money she gets 3.50 a day will she have enough spending money? How much will she have if she doesn't have enough.

If Shirley makes $5 an hour. How much will she make in 8 hours? 10 hours? Then take $2 out of every hour for tax.

How many miles from Chicago to New York City? kilometers? Use Social Studies Book, pp. 492-3.

John needs to know the area of his garden. The length is 25.63 m and the width is 30.8 m. What is the area of his garden?

There is a heavy storm. A ditch could contain 1 gallon of water. 300 drops of water equals 1/2 pint. How many raindrops will it take to fill the ditch?

The fifth group, composed of Josh, Guy, Melanie, and Donald turned in finished problems that had showed substance, originality, and variety. Neither the problems nor their written solutions were perfect, but they were worth five days of valuable class time. Two examples are given.

Alex works 25 hours, 2040 minutes, and 1980 seconds. How many hours and minutes did he work this week? The normal work week is 40 hours. How much over time did he work?

The Smith family has many members in it. Judy is the oldest. She has 3 brothers and 2 sisters. Judy got married and had two sons and 1 daughter. Her sons all got married and had three kids each. One son later got divorced. Her daughters got married also and had two kids. Judy's brother Charlie, got married, had 4 sons, but 1 died. All of his sons got married. Judy's other brother, Johnny, was married, had 3 daughters, then got divorced. One daughter got married and had 1 child. Angela, Judy's youngest sister, is married with 1 son and 1 daughter. They are both married. Judy has 2 other siblings, both single. How many people are in the Smith family? How many people don't have the last name Smith?

The written solutions of the groups of students were as depressing as the problems they attempted to explain. One group only showed calculations for all of their problems and offered absolutely no written explanations whatsoever. Others gave sketchy accounts of the solution process for a problem. Some students ventured to write more detailed paragraphs which usually ended up filled with run-on sentences that created a lack of clarity in the mind of the reader. Still others chose to write lists of steps, unfortunately several of these contained a mixed-up sequence of procedures. Not one of the groups used the Analytical Study Guide as an aid for writing their solutions.

The students knew that I was upset with the overall quality of their original problems and accompanying written solutions. In order to appease me they very wisely used their time the remaining two days of the week solving problems belonging to each other.

I gave the students the option of improving their problems and solutions for extra credit which is generally a very motivating force with this class. Not one of my 21 students opted to do so. Apparently, in this instance the incentive of extra credit was not worth the time or thought it would take to redo their work. C'est la vie!

<u>More Problem Solving from an Analytical Approach</u>

On Monday of the following week I guided my students through a routine problem which required that we calculate the cost to paint a given room. I was surprised to find out that Joel was the only one of my students who had helped his parents to paint rooms of their house.

The students participated energetically throughout the discussion. It only took a few minutes of discussion until Tim realized that you could make a 3-dimensional model of the room with paper. Michael passed out extra scratch paper that I keep on hand for just such an occasion. The students busily labeled and tore, folded, or cut the paper to model the four walls of the room. Joel immediately mentioned that if we took into account the doors and windows, we would need less paint. His classmates thought this would be a good consideration for the question, "What if __________? Could we use the same strategy? Explain.", included in the "Look Back / Evaluate" stage of problem solving.

At this point the students individually calculated the area of the room. Scott had already calculated the answer while we were discussing the understanding and planning steps of the problem-solving framework. I asked Scott, Christine, Michael, and Tiffany to put their work on the chalkboard. The class was amazed to discover that they had used four different means of finding the area of the room. We found the cost to paint the room with and without the consideration of the door and windows. We used the sizes of the classroom door and one of the windows for our extension of the given problem.

The students seemed to enjoy the change back and forth between the analytical writing emphasis and the creative writing emphasis. Joel expressed his appreciation of the day's task which allowed him to make connections with real-life situations. Christine also enjoyed it. She reflected, "This problem was fun and easy but had to be careful on each step."

Suzanne offered an interesting but confusing comment. She declared, "I knew how to [do] the problem but except for 3 [Carry Out the Plan] because I really didn't understand how to do it. I think that this problem was pretty easy." At first Suzanne thought that she understood at least part of the solution, but as she continued to write, she was no longer certain that she did. Frances' journal entry displayed the opposite thoughts of Suzanne. Frances stated, "When I first wrote the problem down I thought it was going to be hard but when we solved it it was not easy but I understood it."

We did not have mathematics/writing on Tuesday because we spent half of the day making Christmas ornaments for our classroom Christmas tree.

The following two days were spent on another redecorating problem with a different mathematical structure. Wednesday the students worked in cooperative groups working on finding the number of strips of molding for a given ceiling and the cost involved. Most of the students did not finish the problem in class so they completed the solution including its evaluation for homework. On Thursday we discussed the problem from the previous day and reviewed the elements of the narrative write-ups. Then the students wrote their personal accounts of their small group solutions.

The students' Analytical Study Guides and their write-ups showed that they still were not spending enough time in step one, "Understand the Problem". One group had a difficult time drawing and labeling a picture containing only the necessary information. One group of girls gave the exact cost to the nearest penny as an estimate. Another group stated that there was no intermediate step, yet they did find the perimeter in order to determine the number of strips of molding and the cost.

I was pleased with their evaluations of their solutions. All of the students used an alternative computation as a second strategy. All of the students extended the problem with either a different length of molding or different dimension for the room. They all generalized the problem. However, it was evident that some of the students did not realize that perimeter problems have a different mathematical structure than area ones. Donna's journal further documented this. She wrote, "This problem was very similar from the problem we had yesterday."

Many of the students described how they solved the molding problem in their journal entries. Up until now, students had not generally talked about specific solution strategies. Jordan stated, "That the promble [sic] was easy and my diagram worked for the promble. I got the promble by working it out in ft [feet] and adding it out." George's procedure, adding all the sides then dividing, was similar to Jordan's and made it easy for him also. Angela wrote, "We added 12 ft + 15 ft and multiplied by 2. Then we divided and got 7 pieces $26.60. We checked it twice on the calculator we are sure we are right!"

Cooperative group work was also seen as a means of producing correct solutions and answers. Veronica wrote, "I know that we have a reasonable answer because we worked as a group." Missy echoed this thought. She reflected, "I know that this is a good way to work out problems, by working in groups, because some people can help others with things they don't understand. I think that this problem was easy to figure out."

That Friday we again broke into small groups and worked on wallpapering a room. The students were to find the number of rolls of wallpaper needed and the appropriate cost. The post problem-solving discussion was held the following Monday with the write-ups.

The students' written works were well done. Four of the groups of students realized that this problem was similar to the painting problem. Only one group stated that it was like the molding problem. The students' "insights" and "intermediate steps" reflected an understanding of and transfer from our previous work. Most of the students rounded off the fractions to find an estimate before beginning to find the actual cost.

Some of the groups used an alternative solution of converting the fractions into decimals. One of the groups stated that a chart would be another heuristic that would aid in solving this problem. Problem extensions and generalizations were fairly limited to the scope of our recent work. Christine, Michael, and Guy were more reflective when they generalized the use of the break-into-parts strategy to the nonroutine problem dealing with angles that we had worked through as a class weeks ago.

There was confidence exhibited in their daily reflections. Melanie thought, "PS [problem solving] is much easier. The only hard thing is explaining our strategy and generalizing." Josh realized his need "to learn more about measurement". He also felt more confident. He added that he was "getting good" at solving problems. Like Josh, Damon also praised his own work and offered a suggestion for improvement. Damon wrote, "The problem was easy and I'm proud that when I thought, it all worked out. I know that I have to concentrate and not goof around."

Tiffany talked about cooperative learning. She said, "I feel that this was easy because we all discussed it and compromised. I know that we have all of this write [sic] because we all did it together so it made it easier."

The school held extended, all-school Christmas program rehearsals on Tuesday, Thursday, and Friday which curtailed much of our studies including seventh-grade mathematics/writing for those days.

<u>Creative Writing of Mathematical Stories</u>

On Wednesday the students were given a problem situation of a girl who bought a bicycle and then sold it, and then she bought it and sold it again. The question embedded in the story was "How much money does she make or lose in the process?" The stories were to include a plot, setting, and characters. They spent the next week which was the remaining three days before Christmas vacation in Writers' Workshop with their stories.

Students were more enthusiastic about sharing their stories than their group problems from last month. They worked well with partners or in small groups. Talk was centered around their creative writings. Students praised each other's work. A few offered suggestions for improvement to their classmates. Tim, George, Melanie, and

Angela were busy proofreading their friends' papers for punctuation and spelling errors. I collected them before vacation and read them while I was home for Christmas. They were wonderful!

Most of the stories had well-developed plots. Common reasons for buying and selling the bike so often were written in the individual stories. Donna, Joel, Guy, Christine, George, and Tim all had the main character fix up the bike then sell it to make more money. Melanie, Donald, and Megan's main characters needed the bike for transportation. Jordan, Tiffany, Angela, and Scott all used the bike as a means for their character to get other things.

Not everyone was totally serious about the writing assignment. Donald was a little silly choosing names for his characters. The fourteen siblings in Tim's story all had the same birthday. Jordan's family of 21 people shared one bathroom until the bicycle was sold to pay for a second bathroom. Other than these minor details, the boys' plots were fairly well developed.

Scott and Tim added so much extraneous numerical data to their stories that it would have been easy to generate a long list of additional problems needing solutions.

Melanie and Megan developed extensive conversations between characters in their stories. Veronica, Josh, Melanie, and George portrayed accurate accounts of many, real-life family relationships. Scott, Melanie, and Tiffany relied heavily on friendships in developing their plots.

The majority of students gave good descriptions of the main character and the bicycle. Megan used a simile in her lengthy description of her main character. Guy gave an extremely detailed description of the bicycle. The descriptions of minor characters used by the students were not nearly as detailed as those of the main character.

All of the students included a setting in their stories. George wrote a particularly detailed one -- down to the exact minute.

Two of the students were confused about the mathematical problem and/or the writing assignment. One turned all the costs around. The other student used the dollar amounts for items other than the bicycle.

After a two week holiday, we returned to school and to mathematical problem solving. The first day back from vacation was a Thursday in which we reviewed what we had done so far. I complimented my students on their excellent bicycle stories. We gathered on the carpeting for the rest of Thursday and all of Friday sharing and discussing the students' stories. Most of the students chose to have a friend read their story rather than read it themselves. I also shared my story. My students and I really enjoyed sharing the stories. It was a relaxed way to ease back into our studies after vacation.

The following Monday, Wednesday, and Thursday we again wrote stories. That Tuesday we did not have mathematics as we went to a local museum for the day. Friday was set aside to share the stories.

This time they were about the changing summer and winter rates of a motel over a period of four years. I had not even finished writing the problem on the overhead projector for the students to copy when Joel, Megan, Scott, Tim, and Guy had already discovered the pattern and correctly predicted this year's summer motel rate. The students were to include all of the story elements as they had done previously. They were allowed class time and encouraged to use peer revision and editing.

The stories were good, but not as terrific as their first ones. The stories tended to be limited to explanations of the motel rate increases. Remodeling and the number of motel guests were popular reasons.

Many of the students chose to write about families on vacation. Luke, George, Angela, and Donna had characters who took year-long vacations and then told friends who also took year-long vacations. Must be nice! Tim, Joel, Missy, and Jordan had characters who returned to the same motel twice a year for vacations.

Two of the students used unique plots. Veronica had her main character doing research on motel rates for a school project. Tiffany wrote about a travel agent helping a customer select a motel. Her main character used a travel agent's computer printout of four data tables. Each table contained the seasonal rates of seven popular motels for a specific year. Interestingly, Tiffany used the same pattern as needed to find the given motel's rates in order to generate all 48 of the other rates of her additional six motels. The characterizations were not very thorough. Two of the students used well-known characters. Melanie included former President and Mrs. Reagan as motel patrons and Donna wrote about the Simpsons.

George described a picturesque mountain lake setting for his motel. Then he very ingeniously revolved the plot around the setting of his story. Luke chose to set his story in the late 1760's. Apparently, he never thought that they may not have had $4 million lotteries back then.

There was also a particularly unique occurrence. Above each word on the first three lines of Scott's story, he wrote its literal translation into Spanish.

Real-World Problems

During our last week of problem solving instruction we worked through two real-world problems. The students and I spent Monday's entire class period discussing steps one and two, "Understand the Problem" and "Devise a Plan", for a problem which called for creating a plan to cut electrical costs at school. Because we were so heavily involved in discussion, we lost track of time and were once again late for lunch.

On Tuesday the students brought boxes of breakfast cereal to school. They spent one-and-a-half days figuring out how they would determine which was the best cereal. Small groups of students were scattered throughout the room busily passing cereal boxes among themselves. The students recorded the nutritional information in charts and compared the number of servings in a box with its price.

Michael, Donna, Guy, Suzanne, and Luke decided that the order of their criteria for determining the best cereal would be taste, nutrition, and cost. Damon, Jordan, Christine, Angela, and Donald decided that the cereal with the most calcium and vitamins and the least calories would be the best. Another group, Suzanne, Scott, Melanie, George, Josh, and Veronica also used nutritional content for determining the best cereal. The fourth group, Missy, Frances, Tim, Megan, and Joel decided to survey 300 - 500 people on the best cereal. The cereal with the most votes would win.

Student reflections on solving real-world problems touched both ends of the continuum. Guy revealed, "It makes me think more than word problems and number problems. I get more 'stirred' up." Jordan thought it was fun to solve the cereal problem even though it was "one hard problem". On the other hand, Tim wrote that real-world problems got him confused. He was frustrated because there was not a cut-and-dried answer. Joel thought, "This is a very complicated problem. It's hard to compare two totally different things. I did not know how to figure out the problem."

Analysis of Post-test Results

The last half hour of class on Wednesday was spent writing individual problems. These individual problems were of better quality than their group efforts.

> If you buy two shirts for $50 and two pairs of pants for $30 each and then you buy a pair of Air Jordan's for $110. How much money did you spend on clothes? (Tim)

> Brandon and his sister Brenda were going to the movies Brandon invited one of his friends, Brenda invited two of her friends. If they get to the show before 12:00 p.m. they would only have to pay $3.75 if they get to there after 12:00 p.m. they would have to pay $6.50 so, they all left at 11:20 p.m. [sic] It takes them 15 minutes to get to the movies 5 minutes to find parking 15 minutes to buy their tickets. How much did they have to pay? (Frances)

Rick drinks 3 cups of Coke. The bottle contains 10 cups. Joey drinks 13 ounces, Alex drinks 1 1/4 cups, and Jeff drinks 350 ml of Coke. How much Coke is left? (Melanie)

Jim is going to the supermarket to get food. Jim needs to get corn, meat, apples and pears. The corn costs 59¢ for one can and the meat costs $4.58 for one pound of meat. The cost of the apples and pears is 39¢ for 2 each. Jim has $5 and is short on money. How much more money does Jim need? (Jordan)

The recycle center pays 7¢ per can. Last week they [the center] got $210.42. This week they want more. Monday they got 200 cans Tuesday they got 179 cans. On Wednesday they got 804 cans. On Thursday they got 193 less than on Wednesday. Friday they were closed but Saturday they got three times what they got on Tuesday. On Sunday they got 1.5 times what they got on Wednesday. Did they beat last week's amount? How much were they short or over? (Megan)

The Mayan worship three gods each month for the first five months. The next four months they worship four gods per month. The last three months they worship twelve gods per month. How many gods will they worship in two years? (George)

Nancy has the following number grades: 67, 82, 97, 100, 10, and 73. What is her average? If she needs a percentage of 88 to be on Honorable Mention and a percentage of 95 to be on Honor Roll. Will she make it? If so, which one? If not, how many points did she miss both Honor Roll and Honorable Mention by? (Angela)

If you buy a 6 pack of pop a day. You drink them all. How many pops will you drink in 543 1/2 days? (Donald)

Just as with the original problems created for the pretest, these problems contained a variety of "cover stories" and mathematical structures. As before these problems contained the difficulties of multiple steps and inability to "crunch" numbers. Unlike the pre-analysis, these created problems did not contain any extraneous data although students included extraneous numerical information in their recent stories. Donald and Melanie added difficulty to their problems by using fractions and a variety of units of measurement, respectively.

Most students wrote problems that were about the same length as before. A few students wrote problems which were considerably longer than their pretest ones. This has both positive and negative points. While longer problems written by Frances, Megan, Jordan, and Josh contained significantly more mathematical content than their shorter ones, they also contained more writing errors. The longer the girls' problems were, the more run-on sentences they had. Both Josh and Jordan lacked sufficient information to solve their problems.

Students took the post problem-solving test on Thursday afternoon so the students would not be rushed for time. It was evident by many of their faces that they were struggling with this test which I had constructed even though I had tried to pair the questions with those of the pretest. It seems to be difficult for me to write an average test (in any subject matter) for my students. I asked my students how they felt about this test. All of the students with the exception of Guy replied that it was much harder than the pretest. I told the students not to worry that I would not grade it.

Before the day was over, I told them that we would take another test on Monday. (School was dismissed on Friday at noon because many of the teachers and upper-grade students were traveling out of state for a seventh and eighth grade basketball tournament.)

I was disappointed that none of the students were able to understand, let alone solve, a problem involving area and perimeter of a farmyard pasture. We had dealt with these concepts in our redecorating problems. It may have been the lack of knowledge of rural settings and its specific vocabulary that hindered students from visualizing the problem. Or it may have been the differing "cover stories" that prevented them from understanding it.

Students used problem-solving heuristics on many of the test questions. Guy set up a very sophisticated chart which enabled him to find the correct answer to a nonroutine problem. Suzanne, Guy, Donna, Angela, Megan, and Francis drew at least one diagram to aid in solving problems. Joel, Missy, Michael, Megan, Tim, and Guy worked a problem backwards. Almost all of the students used guess, check, and refine guess on at least one of the problems. Most of the students looked for patterns on some of the problems. Luke underlined key information in one of the problems.

Although the students thought that this was a much more difficult test than the pretest, 11 of them scored better on this test than the pretest. Angela, Tiffany, Megan, Scott, Tim, George, Christine, Donald, Melanie, Guy, and Joel had higher holistic scores for their solutions on this difficult test after receiving problem-solving instruction. Three students, Jordan, Damon, and Suzanne, earned the same score.

I did not have much time over the weekend to write a second post test so I decided to use the pretest as the other post test. Although the students had been exposed to the problems before, it had been three months since they had attempted to solve them. None of the problems had been worked or even discussed in class. I was curious to see if the students were better able to transfer what they had learned to a problem which they had already seen.

Monday afternoon the class members attempted to solve the same five routine and five nonroutine problems which they had not seen for three months. Students used their problem-solving strategies that we had been practicing daily for the past three months. Jordan drew pictures for four of the ten problems. The majority of others drew at least one diagram. Joel, Missy, Guy, Melanie, Tim, Megan, Luke, Tiffany, and Angela broke down problems into simpler parts and found patterns. Luke demonstrated his prowess at mental math during the test. He wrote, "I know that we can use a calculator but for most of it I used my head to figure it out."

The two problems which caused the most trouble during the first testing were still difficult for the students to solve. Josh, Megan, Joel, Tiffany, and Michael correctly answered the problem asking for two whole numbers with the largest possible product when given four specific digits which can only be used once. Tiffany, Frances, Donna, Suzanne, George, Josh, Melanie, and Joel demonstrated that they had some understanding of the problem requesting the dimensions of a rectangle when given its area and its perimeter. Only Missy, though, was able to solve it.

Thirteen students performed better on the pre/post test the second time than they did the first time. Frances, Jordan, Angela, Veronica, Luke, Scott, Joel, George, Suzanne, Missy, Christine, Donald, and Megan earned higher holistic scores the second time they took the test. Tiffany's scores were identical. Guy, Josh, Michael, Megan, and Tim only scored one point less than their original scores.

Megan and George both found educational benefit in using the same test. Megan wrote, "I think it was a good idea to give us the same test that we had in the beginning so you can see what we have learned." George stated, "It showed if you remembered and thought right."

Reflections about the test itself ranged from "very easy" (Tim and Joel) or "too easy" (Scott) to "very hard" (Christine and Frances), "pretty difficult" (Suzanne), or "still a challenge" (George). Melanie changed her mind about which test was more difficult as she wrote in her journal. She reflected

> Since we already took this test it seemed easy, but I don't know. When I first looked at it I thought it was harder. It's weird now I feel the other test was better. Parts of this test was [sic] easy like the

tape thing [problem] but I didn't understand the last one.

With the possible exception of Melanie, every student readily agreed that this test was easier than last Thursday's test. Students' perceptions of how they did on the two tests were not necessarily correct. It was interesting to discover that nine of the students, Tiffany, Scott, Guy, Tim, Megan, George, Angela, Donald, and Melanie, had higher holistic scores on the original, "hard" post test than the repeated version of the "easier" pretest. Generally speaking, these students were the better problem solvers. Joel's holistic scores for both post tests were the same. Eleven students, Veronica, Michael, Christine, Missy, Luke, Suzanne, Donna, Jordan, Frances, Josh, and Damon performed better on the second post test than on the first post test.

It was evident on several of the problem solvers' second post tests that they tried to recall answers or partial steps from three months ago without thinking through or writing out the solution as George's comment on remembering indicates. Even though the answers may have been right, there was no evidence of the thought processes or the degree of understanding, so under holistic guidelines, they only received two of the possible five points. This definitely had a negative impact on their holistic assessment scores for those problems. It was the reason Megan, Michael, and Josh scored higher on the first administration of the pretest than on its second one. It also explained why Megan and Melanie's scores of the first post test with the novel problems were higher than those of the second post test.

In their final journal entries, some students perceived benefits of their problem-solving instruction. Damon "learned to go over the problem". Missy felt, "The problems helped everyone, mainly me." Veronica wrote, "I think the math things are getting harder but we are getting smarter." Angela was the only student who was "so happy this boring stuff is over".

On the following day, I handed out large sheets of lime-green construction paper to each student. The students made folders out of the paper and transferred all of their work involving mathematical problem solving and writing.

It was at this point that I told students that they were part of my research project. Joel looked at me with a gleam in his eye as if he had suspected that something other than regular class instruction was involved.

I asked my students to choose a pseudonym to protect their anonymity in the event that I made a specific reference to them or quoted them. After a few questions from students, I stipulated that they were neither allowed to choose a first name or last name of a classmate nor the name of a real person. So Michael Jordan became Jordan Michael. Problem solving at work!

CHAPTER V

DISCUSSION AND CONCLUSIONS

In this naturalistic inquiry, I explored the relationship between writing and my students' understanding of mathematical problem solving. I used both analytical writing and creative writing to teach problem solving.

Prior to the study the students did not generally talk about mathematical concepts and strategies. Students were very comfortable with a "recipe" approach to mathematical situations. In the past mathematical exercises had been emphasized over routine textbook problems. Students had limited exposure to nonroutine problems. Instruction had been limited to a few heuristics. Students had never been encouraged to generalize mathematical structures or a solution process.

They had never before had to analyze a mathematical problem so extensively. Previously, students had not been concerned with why or when a heuristic worked; they just wanted the answer. They had never been exposed to metacognition until now. In the past students had not learned how to engage in actively thinking about what they were doing and where it was leading them. They simply did it.

Themes

Misunderstanding of "Understand"

One of the most prevalent themes that I discovered as I read through and reflected on my students' writing was their misunderstanding of the word, "understand". After only one day of modeling and discussing just the first step of the problem-solving framework, Angela "understood". Donald used a qualified statement. He said, "I feel like I understood a little bit." At this point, they had absolutely no idea of what "understanding" entailed.

By the end of the first week, six people had expressed that they had varying degrees of "understanding" for at least one of the problems that we discussed. Twelve journals contained the synonym, "know". These students perceived that they "knew" how to do the problem to some extent. They did not realize that observing someone else model the problem-solving process or being guided through it was far different from actually solving a problem.

Furthermore, they did not know that "doing" is not the same as "understanding". These students believed that if you found an answer, then you understood how to do it. It is possible to find the answer to a problem without knowing why or under which conditions the strategy worked. One cannot truly say that they "understand" problem solving unless they can grasp the concepts and can apply or generalize the strategies to other problems.

The students' use of the phrase, "getting the hang of it", was probably a more accurate description of the cognitive ability of the students who, at this point, claimed to "understand".

There were also students who claimed to not understand how to solve problems. Their later work indicated that they did not know what we were doing. These students had a much better grasp of what it means to "understand".

Students' affective nature played a large part in their "understanding" and their perception of themselves as problem solvers. In general, if the students perceived problem solving as fun or felt "proud", "happy", "impressed", or "great" about themselves, then they also tended to say that they "understood". The converse also held true. If the students felt "mad" or "confused" about problem solving, then they would usually write that they did "not understand". A few students were fickle with their use of the word. Yesterday they understood what we were doing, but today they do not and tomorrow they will again.

Learning Problem Solving Takes Time

Another theme that appeared was learning to solve problems takes time. In every one of Josh's entries for the first two weeks, he stressed the need for more time to absorb and assimilate all of the things which we were doing. Angela appreciated the slow speed of the lessons. Megan reflected on the use of the Analytical Study Guide as a tool to help her in slowing down and "not just speeding through them without looking and checking them".

Problem Solving Requires Thought

Another major theme that was embedded in the students' remarks was the necessity to think when solving problems. Almost all of the students specifically referred to or alluded to their need to think. Students experienced that deeper thinking led them to a greater understanding of the problem and its solution.

I was very pleased to observe that Tim, Guy, Missy, and Joel who are seldom challenged by class work were stimulated or "stirred up" by the challenging problems. They found it very motivating and rewarding to use their higher-level thought processes.

Writing Aids Problem Solving

Several times Megan reflected on the benefit of writing while problem solving. The Analytical Study Guides and the accompanying write-ups of their solutions helped the students to engage in metacognitive thinking while problem solving. The analytical writing required the students to think about (1) what they knew about the problem, and (2) how they should go about solving it. The study guide and write-up gave the students an opportunity to be thoughtful and reflective throughout the process, especially before and after an answer was found. Evaluation of the solution process and the answer is vital for true understanding of the problem and its application and generalization to other problem situations.

The students also saw creative writing as beneficial for learning and stimulating creativity. It gave them an opportunity to express mathematical ideas in their own words. This use of natural language aided in student learning and understanding.

"Two Heads are Better Than One"

Cooperative learning also helped students to understand problem solving. Students reflected on the advantages of working with other students. They mentioned that their group discussions profited them. Students were usually able to help someone who was having trouble.

Students were forced to be more actively involved in small groups than in whole-class settings. This increased their learning. Talking about the heuristics broadened their repertoire of them, and using natural language to discuss the problems increased their knowledge.

Concluding Remarks

In retrospect I think that the broad nature of this study placed an enormous and difficult expectation on my seventh-grade students which they handled well. They were faced with many new concepts which were not in line with their preconceptions of mathematics. Early in the study Melanie referred to problem solving as not "regular math". Tiffany called it "weird". As time went on, the students' comments reflected that they saw the need for and benefits of learning to solve problems. At first the use of analytical writing was foreign to them, nor had they ever before been exposed to creative writing assignments in mathematics.

This research experiment in which they participated was indeed an experience that was filled with many new and strange methods and ideas. Some students struggled more than others with all that I asked or expected them to do, but they all weathered the study in good shape. All of the students except for one ended their reflections of problem solving on a positive note.

Recommendations

An intention of the author of this study was to provide data on actual classroom experiences of using writing to teach mathematical problem solving. Six recommendations are now provided for future research in established classroom settings.

As a cautionary note for teachers who intend to research within their own classrooms, I found at times a tension existed between the roles of the writer as teacher and the writer as researcher. There were times when I wondered if my purpose was to benefit my students' academic growth or to collect data. It is my belief that these goals do not have to be mutually exclusive, but I highly recommend that the teacher/researcher be aware of the possibility that this tension may exist and be prepared to deal with it.

Another recommendation of this author is to study the nature of metacognition in more depth. It is suggested that future researchers probe the relationship of the students' levels of metacognition as related to their levels of understanding of mathematical problems. As students grow in developing metacognitive abilities, do they also mature as mathematical problem solvers? Which strategies for teaching metacognition are most effective in developing junior high / middle school students' mathematical problem-solving abilities?

I suggest the exploration of how students develop an "understanding" of complex mathematical concepts and strategies be explored in order to effect future instruction. How do junior high / middle school students develop conceptual mathematical understanding? How does this research determine the choice of methods and teaching strategies used in the classroom?

Another recommendation is to research the benefits of individualized instruction for the various levels of problem-solving abilities within the class. This author believes that individualizing instruction by providing students with problems and strategies for their specific mathematical problem solving strengths and weaknesses would provide greater benefit in learning to students as individuals. Will students become better problem solvers as a result of individualized instruction compared with instruction geared toward the entire class?

The study of the use of prewriting activities to improve mathematical problem-solving abilities is encouraged by this author. The use of prewriting has been shown to improve students' comprehension of reading. Does this same relationship exist within the mathematics curriculum? Would the use of prewriting activities aid in understanding mathematical problems and their solution processes?

A final recommendation for further study is to develop students' thinking skills so that students can generalize the mathematical structure of a problem and its solution process to new and different problems. It may also be appropriate to study the ability of students to transfer higher-level thinking to other topics within the mathematics curriculum or other subject areas beyond mathematics. Which strategies for teaching problem solving will most effectively enable students to develop higher-level thinking skills? Are there thinking skills at this level of junior high / middle school students which are generalizable within mathematical problem solving that are not content specific and can be transferred to other mathematical topics and/or to other subject curricula?

APPENDICES

APPENDIX 1

ANALYTICAL STUDY GUIDE FOR PROBLEM SOLVING

PROBLEM TITLE____

1. UNDERSTAND THE PROBLEM

What is given?

What is wanted?

Draw a diagram.

What intermediate steps will you have to solve?

Is all of the data that you need there? If not, how are you going to solve the problem?

What information do you need to solve the problem that is not given to you? (conversion factors, data that you have to collect)

What extraneous data, if any, is given?

Restate the problem in your own words.

What insights do you have about this problem?

How will your answer be labeled?

Make a reasonable guess (including units) of your answer.

2. DEVISE A PLAN

What similar problem have you seen before?

What strategies might you use to solve the problem?

Which one do you think is the best one with which to begin? Why?

3. CARRY OUT THE PLAN

(Where is this strategy leading you? Is it helping you to solve the problem, or do you need to change strategies?)

4. LOOK BACK / EVALUATE

Write the answer in a sentence. Is it reasonable?

What stimulated a particularly useful idea?

Did you take any "detours" that you now recognize as unnecessary? What were they?

Find another answer, if possible.

Use another strategy to solve the problem.

When does this strategy, ______________, work?

Why did this strategy, ______________, work for this problem?

What if ______________? Could you use the same strategy? Explain.

Generalize to another kind of problem in which this strategy, ______________, might work.

APPENDIX 2

ORIGINAL WRITING OF MATH PROBLEMS

1. Working cooperatively, student groups will create and submit twelve word problems for a class book. The twelve problems must meet the following criteria:

-- The problems must be evenly divided into easy, medium, and difficult categories.

-- Each category must contain routine and nonroutine problems.

-- Half of the problems must involve whole-number operations.

-- One-fourth of the problems must involve fractions or decimals.

-- One-sixth of the problems must be related to geometry.

-- One of the problems must not contain any numerical information.

-- Half of the problems must be two-step problems.

-- Appropriate graphics (pictures) must accompany at least one-third of the problems.

2. All final drafts of problems must be neatly written on loose-leaf paper, with step-by-step solutions on the back of the paper. The solutions must be neat as well as a clear and accurate description of the method used to arrive at an answer.

3. All problems will be shared with students in another group in order to verify the correctness of the mathematical solution and for proposing editorial changes to be made by the authors of the problem. The teacher may be consulted in the event of a disagreement among the group members.

4. All students will demonstrate that they are able to solve the problems submitted by their group on an individual test at the end of this unit.

5. Group grades will be given for positive participation and meeting daily deadlines.

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