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# Reformulating the Disjunctive Cut Generating Linear Program

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# Reformulating the Disjunctive Cut Generating Linear Program

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**Abstract** Lift-and-project cuts can be obtained by defining an elegant optimization problem over the space of valid inequalities, the Cut Generating Linear Program (CGLP). A CGLP has two main ingredients: (i) an objective function, which invariably maximizes the violation with respect to a fractional solution $\bar{x}$ to be separated; and (ii) a normalization constraint, which limits the scale in which cuts are represented. One would expect that CGLP optima entail the best cuts, but the normalization may distort how cuts are compared, and the cutting plane may not be a supporting hyperplane with respect to the closure of valid inequalities from the CGLP. This work proposes the Reverse Polar CGLP (RP-CGLP), which switches the roles conventionally played by objective and normalization: violation with respect to $\bar{x}$ is fixed to a positive constant, whereas we minimize the slack for a point $p$ that cannot be separated by the valid inequalities. Cuts from RP-CGLP optima define supporting hyperplanes of the immediate closure. When that closure is full-dimensional, the face defined by the cut lays on facets first intersected by a ray from $\bar{x}$ to $p$, all of which corresponding to cutting planes from RP-CGLP optima if $p$ is an interior point. In fact, these are the cuts minimizing a ratio between the slack for $p$ and the violation for $\bar{x}$. We show how to derive such cuts directly from the simplex tableau in the case of split disjunctions and report experiments on adapting the CglLandP cut generator library for the RP-CGLP formulation.

**Keywords** cutting planes · lift-and-project · integer programming

## 1 Introduction

Many optimization problems can be formulated as a Mixed Integer Linear Program (MILP) of the form $\min\{c^T x : Ax \geq b, x \in \{0,1\}^q \times \mathbb{R}_+^{n-q}\}$. Some are found more frequently and have been studied in more detail, such as the classic traveling

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salesman problem, for which families of valid inequalities are known (Applegate et al., 2006). One can tackle an MILP problem by solving its Linear Program (LP) relaxation $\min\{c^T x : Ax \geq b, x \in [0,1]^q \times \mathbb{R}_+^{n-q}\}$ and then iteratively branch to restrict the domains of integer variables or add inequalities of the form $\alpha^T x \geq \beta$ to separate solutions in which those variables are fractional. These inequalities are denoted as cuts with respect to the fractional solutions that they separate, and in many cases the cuts from different methods are equivalent (Balas and Serra, 2019).

In general, there is a greater and justified interest for facet-defining cuts, which are those essential to characterize a full-dimensional convex hull of feasible solutions. There is also a secondary interest in the broader family of cuts defining supporting hyperplanes, which cannot be strengthened by merely increasing the right-hand side $\beta$. When considering a general MILP instead of a special case, however, it is challenging to guarantee that a cut defines even a supporting hyperplane with respect to the convex hull of the feasible solutions. Hence, we aim for a compromise: which cuts would define facets or supporting hyperplanes of the immediate closure? In other words, among the many cuts that can be obtained in the first round of a method, can we select the essential ones or else those that are as close as possible to the set of valid solutions?

We investigate this question from the perspective of lift-and-project (Balas et al., 1993), which is a method to generate cuts by defining a tighter relaxation of the MILP using a Disjunctive Program (DP) (Balas, 1998). Such DP usually consists of a union of disjoint polyhedra covering the feasible set of the MILP. In a typical example, if a solution $\bar{x}$ of the LP is such that $0 < \bar{x}_k < 1$ for some $1 \leq k \leq q$, we can intersect the split disjunction $\{x : x_k \leq 0\} \cup \{x : x_k \geq 1\}$ with the LP feasible set $\{x : Ax \geq b, x \in [0,1]^q \times \mathbb{R}_+^{n-q}\} := \{x : \tilde{A}x \geq \tilde{b}\}$ to define a system on the inequalities that are valid for each term and thus for the disjunctive hull $\mathrm{P^k} = conv\left(\{x \in \mathbb{R}^n : \tilde{A}x \geq \tilde{b}, -x_k \geq 0\} \cup \{x \in \mathbb{R}^n : \tilde{A}x \geq \tilde{b}, x_k \geq 1\}\right)$. In fact, we can define a restricted system without loss of non-dominated inequalities as follows:

$$\begin{array}{llll} \alpha & -u^T\tilde{A} & +u_0 e_k & = 0 \\ \alpha & -v^T\tilde{A} & -v_0 e_k & = 0 \\ \beta & -u^T\tilde{b} & & = 0 \\ \beta & -v^T\tilde{b} & -v_0 & = 0 \\ & u, v, & u_0, v_0 & \geq 0 \end{array} \tag{C$_k$}$$

where $e_k \in \mathbb{R}^n$ is a vector with value 1 at position $k$ and 0 elsewhere.

Among these inequalities, we obtain one that separates $\bar{x}$ by solving a Cut Generating Linear Program (CGLP) (Balas et al., 1993, 1996) such as

$$\begin{array}{ll} \min & \alpha^T\bar{x} - \beta \\ \text{s.t.} & \mathrm{(C)_k} \\ & u^T e + v^T e + u_0 + v_0 = 1 \end{array} \tag{CGLP$_k$}$$

These formulations have invariably aimed at maximizing the cut violation for $\bar{x}$, i.e., making $\alpha^T\bar{x} - \beta$ as negative as possible. However, cuts from $(\mathrm{CGLP})_k$ optima may neither define a facet nor a supporting hyperplane of the immediate closure (Fischetti et al., 2011). This paradox is due to how cuts are ranked by the CGLP, which relates to how these formulations restrict the algebraic representation of the valid inequalities. Since $(\lambda\alpha)^T x \geq (\lambda\beta)$ is the same cut for any $\lambda > 0$, one has to further limit the feasible set defined by the cone $(\mathrm{C})_\mathrm{k}$ to guarantee the existence of an optimal solution. This is usually done by adding a so-called normalization constraint, such as $u^T e + v^T e + u_0 + v_0 = 1$ in the case above. These constraints are as important to the CGLP outcome as the objective function: each normalization defines a different infeasibility certificate for $\bar{x}$ from solving the CGLP dual (Ceria and Soares, 1997), which in turn validates cuts from CGLP optima. Aiming for cuts with better guarantees, our goal is to understand which normalization to use and, more broadly, how to define the CGLP.

The name for these constraints comes from early approaches fixing the norm of cut coefficients through linear constraints, including $\|\alpha\|_\ell = 1$ for $\ell \in \{1, \infty\}$ and $\beta = \kappa$ for $\kappa \in \{-1, 0, +1\}$. Fixing an $\ell$-norm of $\alpha$ implies that cuts from CGLP maximize the distance to $\bar{x}$ in that norm. While $\ell \in \{1, \infty\}$ can be defined with multiple constraints, Cadoux (2010) explored a nonlinear formulation to maximize Euclidean distance. However, cuts maximizing distance are not necessarily facet-defining. More generally, since facet-defining cuts correspond to extreme rays of $(\mathrm{C})_\mathrm{k}$, it is preferable to apply a normalization consisting of a single linear constraint that intersects all rays to ensure that facet-defining cuts correspond to extreme points of the CGLP. In the case of fixing $\beta$, we are partitioning the cuts into three CGLPs and some of those might remain unbounded. Hence, more recent approaches have focused on constraining the Farkas multipliers instead. Fischetti et al. (2011) shows that the so-called Standard Normalization Constraint (SNC) $u^T e + v^T e + u_0 + v_0 = 1$ tends to generate sparser and lower-rank cuts, but also that the solutions depend on the scaling of the constraints and that they might not define supporting hyperplanes. Variants and a generalization of SNC are discussed by Fischetti et al. (2011) and Balas and Bonami (2009), respectively. Notably, is has been shown that the so-called trivial normalization $u_0 + v_0 = 1$ yields the Gomory fractional cut when $\bar{x}$ is a basic solution of the LP (for example, in Fischetti et al. (2011)).

Finally, we note that explicitly solving a CGLP is considered challenging in practice because the CGLP is at least twice as large as the LP. For a system such as $(\mathrm{C})_k$, there are two rows for each column of the LP and two columns for each row. Hence, generating each cut is more computationally expensive than solving the linear relaxation. However, it is possible to find cuts from CGLP optima through formulations with same size as the LP. In the case of split disjunctions, Balas and Perregaard (2003) have shown that there is a correspondence between cuts from CGLP optima and Gomory fractional cuts from basic solutions of the LP, which may or may not be feasible. Hence, one may pivot among LP basic solutions to find a cut deemed as optimal by the CGLP formulation (Balas and Perregaard, 2003; Balas and Bonami, 2009). In the case of the trivial normalization on 2-term disjunctions, Bonami (2012) has shown how to project out the Farkas multipliers and obtain a dual problem known as the Membership Linear Program (MLP), which only differs from the LP by the objective and right-hand side.

### 1.1 Contribution

We propose a CGLP reformulation that we name the Reverse Polar CGLP (RP-CGLP), which switches the roles played by the objective function and the normalization constraint. Cuts are normalized by fixing their violation with respect to $\bar{x}$ and the objective function evaluates the cut at a point $p \in \mathrm{P^k}$. To the best of our knowledge, this is the first CGLP formulation that uses a different objective, which can be leveraged to generate diverse cuts.

More interestingly, cuts derived from optimal solutions of RP-CGLP define supporting hyperplanes of the immediate closure. When the disjunctive hull $\mathrm{P^k}$ is full-dimensional, there is always a facet-defining cut from some RP-CGLP optimum. In fact, cutting planes from RP-CGLP optima are those exposed when a ray from $\bar{x}$ toward $p$ first intersects $\mathrm{P^k}$. If the point at which that ray first intersects $\mathrm{P^k}$ is at the interior of a facet, then that facet is the unique cutting plane from RP-CGLP optima. More generally, each cutting plane from RP-CGLP optima is a combination of facets that separate or are active at $\bar{x}$. If $p$ is at the interior of $\mathrm{P^k}$, then the cutting plane is a combination of facets separating $\bar{x}$ only.

We note that related work by Balas and Perregaard (2002), Cadoux and Lemaréchal (2013), and Conforti and Wolsey (2016) can be framed as proposing CGLP variants yielding the same cuts. Compared to those, the main advantage of the approach described here is that the feasible set of RP-CGLP does not depend on $p$, which facilitates generating multiple cuts by just reoptimizing the CGLP with a new objective function. This equivalence is shown by unveiling the true objective function of these CGLPs after normalization. To the best of our knowledge, a precise and meaningful objective function has only been previously reported for CGLPs fixing a norm of $\alpha$.

Finally, we show that the solution of RP-CGLP in the case of split disjunctions can be mimicked over the tableau of the LP relaxation, hence requiring little adaptation to be incorporated in solvers generating lift-and-project cuts that way. We report computational results on the implementation.

### 1.2 Organization

First, we present the RP-CGLP and its properties in Section 2 and prove its equivalence to other recently proposed CGLP formulations in Section 3. In the sequence, we show how to solve the RP-CGLP for split disjunctions using the LP tableau in Section 4, some methods to parameterize RP-CGLP in Section 5, and report experiments comparing RP-CGLP to a conventional CGLP in Section 6. We draw some conclusions in Section 7.

## 2 The Reverse Polar Reformulation

We propose the Reverse Polar Cut Generating Linear Program (RP-CGLP) to generate a cut $\alpha^T x \geq \beta$ separating $\bar{x}$ with the orientation of some point $p \in \mathrm{P^k}$:

$$\begin{aligned} \min\ & \alpha^T p - \beta \\ \text{s.t.}\ & (\mathrm{C})_\mathrm{k} \\ & \beta - \alpha^T \bar{x} = 1 \end{aligned} \qquad (\text{RP-CGLP})_k^p$$

Similarly to other CGLP formulations, $(\text{RP-CGLP})_k^p$ contains a single normalization constraint, $\beta - \alpha^T \bar{x} = 1$, which fixes the violation conventionally maximized by a CGLP. Moreover, the interplay between objective and normalization changes.

While most normalizations bound the feasible set to guarantee that there is an optimum, we explain in the next paragraph that the feasible set remains unbounded, whereas the normalization prevents the root $(\alpha, \beta, u, v, u_0, v_0) = \mathbf{0}$ of the cone defined by $(\mathrm{C})_\mathrm{k}$ from being an optimal solution. That is due to $\alpha^T p \geq \beta$ for any valid inequality, which implies that the objective function value of RP-CGLP is always nonnegative for $p \in \mathrm{P}^\mathrm{k}$ while $\alpha^T \mathbf{0} - 0 = 0$. Finally, the separability of $\bar{x}$ is guaranteed by CGLP feasibility instead of optimality.

By reformulating the MILP that we want to solve on coordinates centered at $\bar{x}$, say $x' = x - \bar{x}$, the corresponding RP-CGLP defines a cut of the form $\alpha'^T x' \geq \beta'$ in those coordinates, where $\alpha' = \alpha$ and $\beta' = 1$. Hence, cuts from RP-CGLP optima can be characterized by their left-hand sides, which define a subset of the reverse polar set $\left(\mathrm{P}^\mathrm{k} - \bar{x}\right)^- := \left\{y : y^T(x - \bar{x}) \geq 1 \ \forall x \in \mathrm{P}^\mathrm{k}\right\}$. For any $p \in \mathrm{P}^\mathrm{k}$, $(\text{RP-CGLP})_k^p$ yields a cut of the form $\alpha^T x \geq \beta$ for some $y \in \left(\mathrm{P}^\mathrm{k} - \bar{x}\right)^-$, where $\alpha = y$ and $\beta = 1 + y^T \bar{x}$. In fact, the superset of $(\mathrm{C})_\mathrm{k}$ obtained from replacing the equalities on the terms with $\beta$ by $\leq$ along with normalization $\beta - \alpha^T \bar{x} = 1$ defines an extended formulation of the reverse polar set.

When convenient to elucidate proofs, cuts from $(\text{RP-CGLP})_k^p$ will be denoted in the form $y^T(x - \bar{x}) \geq 1$. Whenever we refer to a cut from $(\text{RP-CGLP})_k^p$ or from any other formulation, we assume that these cuts come from optimal solutions of the corresponding CGLP formulation.

**Lemma 1** *Cuts from $(\text{RP-CGLP})_k^p$ define supporting hyperplanes of* $\mathrm{P}^\mathrm{k}$.

*Proof* Let us suppose, for contradiction, that there is a cut from $(\text{RP-CGLP})_k^p$ where that does not hold. Since the distance from $\bar{x}$ to the hyperplane defined by $\alpha^T \bar{x} = \beta$ is given by $\text{dist}(\alpha^T x = \beta, \bar{x}) = \frac{|\alpha^T \bar{x} - \beta|}{\|\alpha\|}$ with respect to any norm, the norm of parallel cutting planes get smaller as they move away from $\bar{x}$. Hence, if we put that cut in the form $\bar{y}^T(x - \bar{x}) \geq 1$, then $\exists \varepsilon \in (0, 1)$ for which $\varepsilon \bar{y}^T(x - \bar{x}) \geq 1$ is valid for $\mathrm{P}^\mathrm{k}$. However, the former cut would not be optimal since the objective function value of the latter is smaller: $\bar{y}^T(p - \bar{x}) - 1 > \varepsilon \bar{y}^T(p - \bar{x}) - 1$. □

From this point on, let us assume that $\mathrm{P}^\mathrm{k}$ is full-dimensional and then characterize which of its facets characterize cutting planes from $(\text{RP-CGLP})_k^p$ optima. Let $\mathcal{F} = \left\{(\gamma^i)^T(x - \bar{x}) \geq \delta_i\right\}_{i \in F}$ be the set of facet-defining inequalities of $\mathrm{P}^\mathrm{k}$, with $F$ finite for $A, b$ rational. Without loss of generality, we partition $F = F^+ \cup F^0 \cup F^-$, where $\delta_i = 1$ if $i \in F^+$, $\delta_i = 0$ if $i \in F^0$, and $\delta_i = -1$ if $i \in F^-$ . Hence, a face-defining cut $\bar{y}^T(x - \bar{x}) \geq 1$ from an optimal solution of $(\text{RP-CGLP})_k^p$ can be described by some nonnegative combination of multipliers $\{\bar{\lambda}_i\}_{i \in F}$ in which $\bar{y} = \sum_{i \in F} \bar{\lambda}_i \gamma^i$, $\bar{\lambda} \geq 0$, and $\sum_{i \in F^+} \bar{\lambda}_i - \sum_{i \in F^-} \bar{\lambda}_i = 1$ since $\sum_{i \in F} \bar{\lambda}_i \delta_i = 1$.

Now we can characterize the cuts from $(\text{RP-CGLP})_k^p$ through a result that, interestingly, resembles complementary slackness for the facets not separating $\bar{x}$:

**Theorem 1** *For a cut $\bar{y}^T(x - \bar{x}) \geq 1$ from $(\text{RP-CGLP})_k^p$ for $p \in \mathrm{P}^\mathrm{k}$, any combination $\bar{\lambda}$ is such that $\bar{\lambda}_i = 0$ or $(\gamma^i)^T(p - \bar{x}) - \delta_i = 0 \ \forall i \in F^0 \cup F^-$.*

*Proof* Suppose not for a cut of the form $\bar{y}^T(x - \bar{x}) \geq 1$ with a combination $\bar{\lambda}$. Let $G^0 := \{i \in F^0 : \bar{\lambda}_i > 0 \text{ and } (\gamma^i)^T(p - \bar{x}) > 0\}$ and $G^- := \{i \in F^- : \bar{\lambda}_i > 0$ and $(\gamma^i)^T(p - \bar{x}) + 1 > 0\}$, where $G^0 \cup G^- \neq \emptyset$. If $G^0 \neq \emptyset$, then starting with $\lambda' \leftarrow \bar{\lambda}$ and setting $\lambda'_i \leftarrow 0 \ \forall i \in G^0$ would yield a valid cut with objective function value smaller by $\sum_{i \in G^0} \bar{\lambda}_i [(\gamma^i)^T (p - \bar{x})] > 0$. If $G^- \neq \emptyset$, we similarly could start with $\lambda' \leftarrow \bar{\lambda}$, set $\lambda'_i \leftarrow 0 \ \forall i \in G^-$ and accordingly set $\lambda'_j \leftarrow \bar{\lambda}_j / \left(1 + \sum_{i \in G^-} \bar{\lambda}_i\right) \ \forall j \in F^+$ to keep $\sum_{i \in F^+} \lambda'_i - \sum_{i \in F^-} \lambda'_i = 1$. That would decrease the objective function value by $\sum_{i \in G^-} \bar{\lambda}_i [(\gamma^i)^T (p - \bar{x}) + 1] > 0$ and $\sum_{j \in F^+} (\bar{\lambda}_j - \bar{\lambda}'_j)[(\gamma^j)^T (p - \bar{x}) - 1] \geq 0$, respectively. Hence, $\bar{y}^T (x - \bar{x}) \geq 1$ is not obtained from an (RP-CGLP)$_k^p$ optimum, a contradiction. $\square$

**Corollary 1** *For a cut $\bar{y}^T (x - \bar{x}) \geq 1$ with combination $\bar{\lambda}$ derived from (RP-CGLP)$_k^p$ for $p \in \mathrm{P}^\mathrm{k}$, we have $\bar{y}^T (p - \bar{x}) = (\gamma^i)^T (p - \bar{x}) \ \forall i \in F^+ : \bar{\lambda}_i > 0$.*

*Proof* From Theorem 1, we have $\bar{y}^T (p - \bar{x}) = \sum_{i \in F^+} \bar{\lambda}_i [(\gamma^i)^T (p - \bar{x})]$. Furthermore, either $\bar{y}^T (p - \bar{x}) - 1 = 0$ or $\sum_{i \in F^+} \bar{\lambda}_i = 1$, since otherwise $\sum_{i \in F^-} \bar{\lambda}_i > 0$ and we could find a cut with better objective with multipliers $\lambda'$ by starting with $\lambda' \leftarrow \bar{\lambda}$, setting $\lambda'_i \leftarrow 0 \ \forall i \in F^-$, and then scaling down with $\lambda'_j \leftarrow \bar{\lambda}_j / \left(1 + \sum_{i \in F^-} \bar{\lambda}_i\right) \ \forall j \in F^+$. Thus, if $\bar{y}^T (p - \bar{x}) > (\gamma^i)^T (p - \bar{x})$ for some $i \in F^+ : \bar{\lambda}_i > 0$, then $\exists j \in F^+ : \bar{\lambda}_j > 0$ such that $\bar{y}^T (p - \bar{x}) < (\gamma^j)^T (p - \bar{x})$ and vice-versa. If that was possible, however, then there would be a cut with strictly better objective by increasing $\bar{\lambda}_i$ while decreasing $\bar{\lambda}_j$ accordingly. $\square$

**Corollary 2** *For $p \in int(\mathrm{P}^\mathrm{k})$, a cut from (RP-CGLP)$_k^p$ is a combination $\bar{\lambda}$ of facets separating $\bar{x}$ that each correspond to some (RP-CGLP)$_k^p$ optimum.*

*Proof* If $p$ is not on any facet, Theorem 1 implies that $\bar{\lambda}_i = 0 \ \forall i \in F^0 \cup F^-$, and by Corollary 1 the cut from each facet $i$ with $\bar{\lambda}_i > 0$ defines an optimal solution of (RP-CGLP)$_k^p$. $\square$

Thus, cuts from (RP-CGLP)$_k^p$ are a combination of (i) inequalities of $\mathcal{F}$ that are active at $p$, and (ii) inequalities of $\mathcal{F}$ that separate $\bar{x}$. The only set with non-zero objective value is the latter, which comprises inequalities indexed by $F^+$ with same evaluation for $p$ if normalized by the same right-hand side, hence each associated with some (RP-CGLP)$_k^p$ optimum. We are now left with one question — what makes the cuts defining each of such facets optimal?

**Lemma 2** *A cut $\alpha^T x \geq \beta$ from (RP-CGLP)$_k^p$ with objective value $\zeta$ is active at the point $p' := \bar{x} + \frac{1}{\zeta+1}(p - \bar{x})$, which lies on the ray from $\bar{x}$ to $p$.*

*Proof* It suffices to check that $\alpha^T p' - \beta = 0$: $\alpha^T \left(\bar{x} + \frac{1}{\zeta+1}(p - \bar{x})\right) - \beta = (\alpha^T \bar{x} - \beta) + \left(\frac{1}{\zeta+1}(\alpha^T p - \alpha^T \bar{x})\right) = -1 + \left(\frac{1}{\zeta+1}(\alpha^T p - (\beta - 1))\right) = -1 + \left(\frac{1}{\zeta+1}(\zeta + 1)\right) = 0$. $\square$

**Theorem 2** *A cut from (RP-CGLP)$_k^p$ is active at the first intersection of $\mathrm{P}^\mathrm{k}$ with the ray from $\bar{x}$ to $p$, which corresponds to point $p'$ from Lemma 2.*

*Proof* A cut $\alpha^T x \geq \beta$ from (RP-CGLP)$_k^p$ is such that $\alpha^T \bar{x} - \beta = -1$ and $\alpha^T p - \beta \geq 0$, hence defining a monotonically increasing function for the slack along the ray. Since that slack is negative for any point before $p'$, those are all separated by the cut and $p'$ is the first intersection of the ray with $\mathrm{P}^\mathrm{k}$. $\square$

In other words, cutting planes from $(\text{RP-CGLP})_k^p$ optima are combinations of facets of $\mathrm{P^k}$ that are first intersected by a ray from $\bar{x}$ toward $p$. Note that the ray from $\bar{x}$ to $p$ may intersect other facets separating $\bar{x}$ prior to $p'$, but none of those at a point in $\mathrm{P^k}$. In fact, a cutting plane from $(\text{RP-CGLP})_k^p$ only combines facets indexed by $F_+$ that are last intersected by the ray from $\bar{x}$ to $p$.

We can observe that in a different way. If we replace $\beta$ according to the normalization, the objective can be restated as $\min \alpha^T(p-\bar{x})-1$. Consequently, all points defining the same ray with $\bar{x}$ as $p$ yield the same cuts from $(\text{RP-CGLP})_k^p$. If we choose a point $p'$ along that ray for which the objective value is 0, it becomes clear that facets separating $\bar{x}$ should be active at $p'$. If $p'$ is at the interior of a facet of $\mathrm{P^k}$, then that facet is the unique cutting plane from $(\text{RP-CGLP})_k^p$ optima, as implied by Corollary 1.

## 3 Equivalent Formulations and Related Work

For any CGLP formulation, the imposed normalization and its interplay with the objective function may result in a different ranking of the cuts. In order to compare $(\text{RP-CGLP})_k^p$ with other recent formulations, we need to understand how cuts are truly evaluated.

**Lemma 3** *Valid inequalities of* $\mathrm{P^k}$ *that separate* $\bar{x}$ *are compared by* $(\text{RP-CGLP})_k^p$ *using*

$$\min \frac{\alpha^T p-\beta}{\beta-\alpha^T\bar{x}} \tag{1}$$

*Proof* Let us denote cuts in the form $\mu^T x \geq \nu$, where $\|\mu\|=1$. Hence, for a cut of the form $\alpha^T x \geq \beta$, we consider a correspondence of the form $(\alpha,\beta)=\theta(\mu,\nu)$ for some $\theta>0$.

Normalization $\beta-\alpha^T\bar{x}=1$ implies that $\theta=\frac{1}{\nu-\mu^T\bar{x}}$. The objective function is restated as $\min \alpha^T p-\beta=\min \theta(\mu^T p-\nu)=\min \frac{\mu^T p-\nu}{\nu-\mu^T\bar{x}}$. Therefore, the cuts separating $\bar{x}$ obtained with $(\text{RP-CGLP})_k^p$ are those minimizing the ratio between the slack for $p$ and the violation for $\bar{x}$. Note that the ratio does not depend on the algebraic representation of the cut. □

### 3.1 Equivalent Formulations

A similar reformulation is proposed by Balas and Perregaard (2002), as follows:

$$\begin{aligned} \min\ & \alpha^T\bar{x}-\beta \\ \text{s.t.}\ & (\mathrm{C})_\mathrm{k} \\ & \alpha^T(p-\bar{x})=1 \end{aligned} \qquad (\text{BP-CGLP})_k^p$$

Balas and Perregaard (2002) have proven that $(\text{BP-CGLP})_k^p$ has an optimum if, and only if, the line defined by $\bar{x}$ and $p$ ever intersects $\mathrm{P^k}$. If so, the resulting cut defines a supporting hyperplane, which contains the point of $\mathrm{P^k}$ that is the closest to $\bar{x}$ on the line between $\bar{x}$ and $p$. This normalization is used for multi-row cuts by Louveaux et al. (2015). Formulation $(\text{BP-CGLP})_k^p$ resembles $(\text{RP-CGLP})_k^p$

when the objective of the latter is restated as $\min \alpha^T(p - \bar{x}) - 1$ by substituting $\beta$ according to the normalization, except that switching the expressions for the objective function and the normalization constraint.

However, a key difference between $(\text{RP-CGLP})_k^p$ and $(\text{BP-CGLP})_k^p$ is that the feasible set of the former does not depend on $p$. That allows to generate different cuts by just changing the objective function and reoptimizing, hence entailing more variability if we choose the next point in $\mathrm{P^k}$ to entirely change the set of CGLP optima. In a sense, that generalizes the approach by Balas (1997) of using alternate CGLP optima to generate multiple cuts.

**Corollary 3** *$(BP\text{-}CGLP)_k^p$ and $(RP\text{-}CGLP)_k^p$ derive the same cuts for $p \in \mathrm{P^k}$.*

*Proof* The normalization of $(\text{BP-CGLP})_k^p$ implies $\theta = \frac{1}{\mu^T(p-\bar{x})}$, and the objective becomes $\min \alpha\bar{x} - \beta = \max \beta - \alpha\bar{x} = \max \theta[\nu - \mu^T\bar{x}] = \max \frac{\nu-\mu^T\bar{x}}{\mu^T(p-\bar{x})} = \max \frac{\nu-\mu^T\bar{x}}{(\mu^T p-\nu)+(\nu-\mu^T\bar{x})}$. Since $(\text{BP-CGLP})_k^p$ optima yield cuts separating $\bar{x}$, we can divide numerator and denominator by the violation. The objective becomes $\max \frac{1}{\frac{\mu^T p-\nu}{\nu-\mu^T\bar{x}}+1}$, which is equivalent to $\min \frac{\mu^T p-\nu}{\nu-\mu^T\bar{x}} + 1$, which in turn matches that of $(\text{RP-CGLP})_k^p$ except for a constant term. □

Cadoux and Lemaréchal (2013) praise the boundedness from normalizing by an interior point, hence motivating what we denote as the Polar CGLP (P-CGLP):

$$\begin{aligned} \min\ & \alpha^T\bar{x} - \beta \\ \text{s.t.}\ & (\mathrm{C})_\mathrm{k} \\ & \alpha^T p - \beta = 1 \end{aligned} \qquad (\text{P-CGLP})_k^p$$

We can similarly state $(\text{P-CGLP})_k^p$ as a CGLP for a problem on coordinates centered at $p$ with right-hand side of $-1$, hence characterizing the $\alpha$-projection as a subset of the polar set $\left(\mathrm{P^k} - p\right)^\circ := \{y : y^T(x-p) \leq 1\ \forall x \in \mathrm{P^k}\}$ when $p$ is an interior point. Moreover, we can easily adapt the proof of Lemma 1 to show that cuts from $(\text{P-CGLP})_k^p$ define supporting hyperplanes. However, while normalization $\alpha^T p - \beta = 1$ defines a bounded feasible set if $p \in \mathrm{int}(\mathrm{P^k})$, a valid inequality might be active at $p$ while separating $\bar{x}$ if $p \in \mathrm{bd}(\mathrm{P^k})$, in which case $(\text{P-CGLP})_k^p$ has no optimum. Note that, if $\mathrm{P^k}$ is not full-dimensional, then there are no interior points.

While giving less importance to separating $\bar{x}$, Cadoux and Lemaréchal (2013) nevertheless regard that as a possible role for the objective function in the polar formulation. If separating $\bar{x}$ is of central importance, however, we show below that $(\text{RP-CGLP})_k^p$ is equivalent but more general than $(\text{P-CGLP})_k^p$ because it works with any $p \in \mathrm{bd}(\mathrm{P^k})$. Such distinction is particularly relevant if $\mathrm{P^k}$ is not full-dimensional, and therefore any $p \in \mathrm{P^k}$ is such that $p \in \mathrm{bd}(\mathrm{P^k})$. When $(\text{P-CGLP})_k^p$ has no optimum, then the corresponding cuts have to be derived from rays of the unbounded problem. Cadoux and Lemaréchal (2013) also present a concern with unbounded sets such as the reverse polar, which we address with the objective function evaluating a point in $\mathrm{P^k}$.

**Corollary 4** *If $(P\text{-}CGLP)_k^p$ has an optimum, then $(RP\text{-}CGLP)_k^p$ and $(P\text{-}CGLP)_k^p$ derive the same cuts for $p \in \mathrm{P^k}$.*

*Proof* For $(\text{P-CGLP})_k^p$, the normalization implies $\theta = \frac{1}{\mu^T p - \nu}$ and the objective becomes $\min \alpha\bar{x} - \beta = \max \beta - \alpha\bar{x} = \max \theta[\nu - \mu^T \bar{x}] = \max \frac{\nu - \mu^T \bar{x}}{\mu^T p - \nu}$. Hence, unless $p$ is active at some facet-defining inequality separating $\bar{x}$, we are maximizing a well-defined ratio. □

Another related work is a parallel development by Conforti and Wolsey (2016)[1]. Their approach consists of generating cuts through a coordinate system in which a point $p$ from the relative interior of $\mathrm{P^k}$ is centered at the origin. The LP formulation used by their cut generator maximizes violation subject to the RHS normalization $-1 \leq \beta \leq 1$, which they show to almost surely produce a facet-defining inequality if $\mathrm{P^k}$ is full-dimensional and an improper face otherwise. The resulting approach is in essence very similar to $(\text{P-CGLP})_k^p$. Consequently, generating a second inequality separating $\bar{x}$ may require another formulation due to the change of coordinate systems instead of a change of objective function as in $(\text{RP-CGLP})_k^p$.

Our work contributes to this literature in the following ways. First, we propose a reformulation that permits generating multiple cuts with the intended properties by only changing the objective function. Second, we show the equivalence between the different formulations presented above and how those formulations rank all valid cuts. Third, we complement the analyses of Balas and Perregaard (2002), Cadoux (2010), and Conforti and Wolsey (2016) by characterizing the decomposition of the resulting cuts in terms of facet-defining inequalities of $\mathrm{P^k}$ when the cut is not facet-defining. Finally, we present the first method to generate such cuts using the simplex tableau of the LP relaxation, as described in Section 4.

### 3.2 Duality

The dual of a CGLP is regarded as the lift-and-project primal, where the solutions correspond to a convex combination of points on each term of the disjunction. The effect of each normalization constraint in the CGLP is to relax the lift-and-project in a different way to make $\bar{x}$ feasible, hence defining an infeasibility certificate. Ceria and Soares (1997) explore the interpretation of these duals in conventional CGLP formulations.

Among the formulations that we have proven equivalent above, the dual for $(\text{BP-CGLP})_k^p$ is particularly insightful because it yields the point alluded by Lemma 2 as $p' = x^0 + x^1$:

$$
\begin{array}{lrrl}
\min & \omega & & \\
\text{s.t.} & \tilde{A}x^0 & -\tilde{b}y_0 & \geq 0 \\
& x_j^0 & & \geq 0 \\
& \tilde{A}x^1 & -\tilde{b}y_1 & \geq 0 \\
& x_j^1 & -y_1 & \geq 0 \\
& & y_0 + y_1 & = 1 \\
& (\bar{x} - p)\omega + x^0 + x^1 & & = \bar{x}
\end{array}
\qquad (\text{BP-L\&P})_k^p
$$

[1] A poster with the results that we prove up to this point in the paper was presented on May 2016 at the MIP Workshop (https://sites.google.com/site/mipworkshop2016/posters), which was almost simultaneous with their presentation at the CORE@50 Conference.

### 3.3 Supporting Hyperplane Methods

There is a broader stream of literature on related techniques dating back to Veinott (1967), which are often denoted as supporting hyperplane methods. They are used for network design by Ben-Ameur and Neto (2007) and for mixed-integer nonlinear programs by Kronqvist et al. (2016), both considering a segment between an interior point $p$ and $\bar{x}$ to obtain a boundary point $p'$ and a cutting plane active at $p'$. Using a conventional CGLP, the in-out approach by Fischetti and Salvagnin (2010) separates another exterior point within that segment. For $p$ sufficiently close to $\mathrm{P}^{\mathrm{k}}$, the latter approach is intuitively equivalent to ours, as described next. For any valid inequality derived by the finite set of basic CGLP solutions that does not define a supporting hyperplane of $\mathrm{P}^{\mathrm{k}}$, there is a set of exterior points that are sufficiently close to $\mathrm{P}^{\mathrm{k}}$ in the line segment between $\bar{x}$ and $p$ that would not be separated. Therefore, an exterior point that is sufficiently close to $\mathrm{P}^{\mathrm{k}}$ would imply that all valid inequalities from CGLP optima define supporting hyperplanes of $\mathrm{P}^{\mathrm{k}}$.

The target cuts by Buchheim et al. (2008) exploit a projection where it is possible to enumerate all extreme points. Using a polar normalization and an objective maximizing violation, a linear program prevents inequalities from separating the extreme points and yields a facet-defining cut for the projection. This facet is active at the boundary point between $p$ and $\bar{x}$. The method is later applied to solve robust network design (Buchheim et al., 2011) and quadratic integer programming (Buchheim et al., 2010). The same idea is used by Tjandraatmadja and van Hoeve (2016) to generate facet-defining cuts with respect to the polytope associated with the convex hull of the solutions represented by a decision diagram corresponding to a relaxation of the problem being solved.

## 4 Cut Generation from the Simplex Tableau

When the CGLP is defined on a split disjunction with a single normalization constraint, Balas and Perregaard (2003) have shown that there is a correspondence between basic solutions of the CGLP defining a cut and those of the LP relaxation. Such result assumes the restricted set of inequalities defined by $(\mathrm{C})_{\mathrm{k}}$, where there are $2n+3$ basic variables in any basic CGLP solution. In the case of cuts separating $\bar{x}$, these basic variables consist of $\alpha, \beta, u_0, v_0$, and $n$ multipliers among $u$ and $v$. Furthermore, the basic variables among $u$ and $v$ correspond to linearly independent inequalities of the LP feasible set. From each of those basic multipliers, we infer that the slack of the corresponding inequality is non-basic at a basic LP solution where the same cut can be derived as a Gomory cut. These slacks correspond to the variables in $x$ for the inequalities defining bounds.

In what follows, we show how to generate cuts from $(\text{RP-CGLP})_k^p$ directly from the simplex tableau, hence using some results and proof steps from Balas and Perregaard (2003) when appropriate. Let $\bar{a}_{ij}$ denote the $j$-th column and $\bar{a}_{i0}$ the right-hand side of the $i$-th row of the simplex tableau for basic solution $\bar{x}$. The set $J = M_1 \cup M_2$ defines the index set of nonbasic variables of the LP for a given CGLP solution, where $M_1$ correspond to basic multipliers among $u$ and $M_2$ to basic multipliers among $v$. Finally, let the slacks of the linear relaxation with respect to $\bar{x}$ and $p$ denote $\bar{s} = \tilde{A}\bar{x} - \tilde{b}$ and $\bar{\bar{s}} = \tilde{A}p - \tilde{b}$, respectively.

**Theorem 3** *For a given basic solution of the LP relaxation, the reduced costs of non-basic multipliers $u_i$ and $v_i$ of (RP-CGLP)$_k^p$ for some row $i$ for the corresponding given basic solution $(\bar{\alpha}, \bar{\beta}, \bar{u}, \bar{v})$ are*

$$r_{u_i} = -\sigma\left[\sum_{j\in M_2} \bar{a}_{ij}\bar{s}_j - \bar{a}_{i0}(1-\bar{x}_k)\right] - \left[\sum_{j\in M_2} \bar{a}_{ij}\bar{\bar{s}}_j - \bar{a}_{i0}(1-p_k)\right] \tag{2}$$

*and*

$$r_{v_i} = -\sigma\left[\sum_{j\in M_1} \bar{a}_{ij}\bar{s}_j - \bar{a}_{i0}\bar{x}_k\right] - \left[\sum_{j\in M_1} \bar{a}_{ij}\bar{\bar{s}}_j - \bar{a}_{i0}p_k\right], \tag{3}$$

*where the objective function value of the CGLP solution corresponds to*

$$\sigma = -\frac{\sum_{j\in M_2} \bar{a}_{kj}\bar{\bar{s}}_j - \bar{a}_{k0}(1-p_k)}{\sum_{j\in M_2} \bar{a}_{kj}\bar{s}_j - \bar{a}_{k0}(1-\bar{x}_k)} \tag{4}$$

*or*

$$\sigma = -\frac{\sum_{j\in M_1} \bar{a}_{kj}\bar{\bar{s}}_j + (1-\bar{a}_{k0})p_k}{\sum_{j\in M_1} \bar{a}_{kj}\bar{s}_j + (1-\bar{a}_{k0})\bar{x}_k} \tag{5}$$

*with $\bar{a}_{kj} \le 0\ \forall j \in M_1$ and $\bar{a}_{kj} \ge 0\ \forall j \in M_2$.*

*Proof* By restricting to the basic multipliers in $u$ and $v$ along with one non-basic multiplier for each term, $u_i$ and $v_i$, we have the following expressions for the cut coefficients:

$$\alpha = u_{M_1}^T \tilde{A}_{M_1} + u_i\tilde{A}_i \quad -u_0 e_k = v_{M_2}^T \tilde{A}_{M_2} + v_i\tilde{A}_i \quad +v_0 e_k \tag{6}$$
$$\beta = u_{M_1}^T \tilde{b}_{M_1} + u_i\tilde{b}_i \quad = v_{M_2}^T \tilde{b}_{M_2} + v_i\tilde{b}_i \quad +v_0 \tag{7}$$

After substitutions using Lemma 8 from Balas and Perregaard (2003), we have

$$u_j = -(u_0+v_0)\bar{a}_{kj} + (u_i - v_i)\bar{a}_{ij} \qquad \forall j \in M_1 \tag{8}$$
$$v_j = (u_0+v_0)\bar{a}_{kj} - (u_i - v_i)\bar{a}_{ij} \qquad \forall j \in M_2 \tag{9}$$
$$v_0 = (u_0+v_0)\bar{a}_{k0} - (u_i - v_i)\bar{a}_{i0} \tag{10}$$

Since $u_0, v_0 > 0$ because the inequality separates $\bar{x}$, we show the partitioning among $M_1$ and $M_2$ by setting $u_i, v_i = 0$:

$$u_j = -(u_0+v_0)\bar{a}_{kj} \qquad \to \bar{a}_{kj} \le 0\ \forall j \in M_1 \tag{11}$$
$$v_j = (u_0+v_0)\bar{a}_{kj} \qquad \to \bar{a}_{kj} \ge 0\ \forall j \in M_2 \tag{12}$$

Now we compute the slack of $\bar{x}$ with respect to the $M_2$ and also $M_1$:

$$\begin{aligned}
\alpha^T\bar{x} - \beta &= v_{M_2}^T(\tilde{A}_{M_2}\bar{x} - \tilde{b}_{M_2}) + v_i(\tilde{A}_i\bar{x} - \tilde{b}_i) + v_0(e_k^T\bar{x} - 1)\\
&= v_{M_2}^T\bar{s}_{M_2} + v_i\bar{s}_i + v_0(\bar{x}_k - 1)\\
&= (u_0+v_0)\sum_{j\in M_2}\bar{a}_{kj}\bar{s}_j - (u_i - v_i)\sum_{j\in M_2}\bar{a}_{ij}\bar{s}_j + v_i\bar{s}_i + [(u_0+v_0)\bar{a}_{k0} - (u_i-v_i)\bar{a}_{i0}](\bar{x}_k - 1)\\
&= (u_0+v_0)\left[\sum_{j\in M_2}\bar{a}_{kj}\bar{s}_j - \bar{a}_{k0}(1-\bar{x}_k)\right] - (u_i - v_i)\left[\sum_{j\in M_2}\bar{a}_{ij}\bar{s}_j - \bar{a}_{i0}(1-\bar{x}_k)\right] + v_i\bar{s}_i
\end{aligned} \tag{13}$$

$$\begin{aligned}
\alpha^T \bar{x} - \beta = & \quad u_{M_1}^T(\tilde{A}_{M_1}\bar{x} - \tilde{b}_{M_1}) + u_i(\tilde{A}_i \bar{x} - \tilde{b}_i) - u_0 e_k^T \bar{x} \\
= & \quad u_{M_1}^T \bar{s}_{M_1} + u_i \bar{s}_i - u_0 \bar{x}_k \\
= & \quad -(u_0+v_0)\sum_{j \in M_1} \bar{a}_{kj}\bar{s}_j + (u_i - v_i)\sum_{j \in M_1} \bar{a}_{ij}\bar{s}_j + u_i \bar{s}_i - [(u_0+v_0)(1-\bar{a}_{k0}) + (u_i - v_i)\bar{a}_{i0}]\bar{x}_k \\
= & \quad -(u_0+v_0)\left[\sum_{j \in M_1} \bar{a}_{kj}\bar{s}_j + (1-\bar{a}_{k0})\bar{x}_k\right] + (u_i - v_i)\left[\sum_{j \in M_1} \bar{a}_{ij}\bar{s}_j - \bar{a}_{i0}\bar{x}_k\right] + u_i \bar{s}_i
\end{aligned} \tag{14}$$

We can obtain similar expressions with respect to $p$:

$$\alpha^T p - \beta = \quad (u_0+v_0)\left[\sum_{j \in M_2} \bar{a}_{kj}\bar{\bar{s}}_j - \bar{a}_{k0}(1-p_k)\right] - (u_i - v_i)\left[\sum_{j \in M_2} \bar{a}_{ij}\bar{\bar{s}}_j - \bar{a}_{i0}(1-p_k)\right] + v_i \bar{\bar{s}}_i \tag{15}$$

$$\alpha^T p - \beta = \quad -(u_0+v_0)\left[\sum_{j \in M_1} \bar{a}_{kj}\bar{\bar{s}}_j + (1-\bar{a}_{k0})p_k\right] + (u_i - v_i)\left[\sum_{j \in M_1} \bar{a}_{ij}\bar{\bar{s}}_j - \bar{a}_{i0}p_k\right] + u_i \bar{\bar{s}}_i \tag{16}$$

This is the point where our proof differs from Balas and Perregaard (2003). We use our normalization to determine the value of $u_0 + v_0$ and plug that in the objective function. We first use the expressions depending on $M_2$:

$$\alpha^T \bar{x} - \beta = \quad -1 \rightarrow \quad (u_0+v_0) = \quad \frac{-1 + (u_i - v_i)\left[\sum_{j \in M_2} \bar{a}_{ij}\bar{s}_j - \bar{a}_{i0}(1-\bar{x}_k)\right] - v_i \bar{s}_i}{\sum_{j \in M_2} \bar{a}_{kj}\bar{s}_j - \bar{a}_{k0}(1-\bar{x}_k)} \tag{17}$$

$$\begin{aligned}
\alpha^T p - \beta = & \quad \frac{-1 + (u_i - v_i)\left[\sum_{j \in M_2} \bar{a}_{ij}\bar{s}_j - \bar{a}_{i0}(1-\bar{x}_k)\right] - v_i \bar{s}_i}{\sum_{j \in M_2} \bar{a}_{kj}\bar{s}_j - \bar{a}_{k0}(1-\bar{x}_k)}\left[\sum_{j \in M_2} \bar{a}_{kj}\bar{\bar{s}}_j - \bar{a}_{k0}(1-p_k)\right] \\
& -(u_i - v_i)\left[\sum_{j \in M_2} \bar{a}_{ij}\bar{\bar{s}}_j - \bar{a}_{i0}(1-p_k)\right] + v_i \bar{\bar{s}}_i
\end{aligned} \tag{18}$$

Note that fixing $u_i, v_i = 0$ above yields the CGLP objective as in (4), whereas fixing only $v_i = 0$ and subtracting $\sigma$ yields (2). Now we use the expressions depending on $M_1$:

$$\alpha^T \bar{x} - \beta = \quad -1 \rightarrow \quad (u_0+v_0) = \quad \frac{1 + (u_i - v_i)\left[\sum_{j \in M_1} \bar{a}_{ij}\bar{s}_j - \bar{a}_{i0}\bar{x}_k\right] + u_i \bar{s}_i}{\left[\sum_{j \in M_1} \bar{a}_{kj}\bar{s}_j + (1-\bar{a}_{k0})\bar{x}_k\right]} \tag{19}$$

$$\alpha^T p - \beta = \quad \frac{-1-(u_i-v_i)\left[\sum_{j\in M_1}\bar{a}_{ij}\bar{s}_j - \bar{a}_{i0}\bar{x}_k\right] - u_i\bar{s}_i}{\left[\sum_{j\in M_1}\bar{a}_{kj}\bar{s}_j + (1-\bar{a}_{k0})\bar{x}_k\right]}\left[\sum_{j\in M_1}\bar{a}_{kj}\bar{\bar{s}}_j + (1-\bar{a}_{k0})p_k\right] + (u_i-v_i)\left[\sum_{j\in M_1}\bar{a}_{ij}\bar{\bar{s}}_j - \bar{a}_{i0}p_k\right] + u_i\bar{\bar{s}}_i \tag{20}$$

Similarly, fixing $u_i, v_i = 0$ above yields the CGLP objective as in (5), whereas fixing $u_i = 0$ and subtracting $\sigma$ yields (3). □

In fact, it would also be possible to define $r_{v_i}$ in terms of $M_1$ as well as $r_{u_i}$ in terms of $M_2$, but both of these would leave the slack of a nonbasic multiplier in the expression, which does not need to be computed otherwise.

Note that $\sigma$ resembles (1). In contrast, Balas and Perregaard (2003) showed that the tableau expression for $(\text{CGLP})_k$ is $\sigma' = \dfrac{\sum_{j\in M_2}\bar{a}_{kj}\bar{s}_j - \bar{a}_{k0}(1-\bar{x}_k)}{1+\sum_{j\in J}|\bar{a}_{kj}|}$, where the denominator evidences the effect of coefficient scale pointed out by Fischetti et al. (2011).

Finally, note that choosing which variable enters the CGLP basic solution corresponds to choosing which variable leaves the LP basic solution: solving the CGLP through the LP implies dualizing the solution method. However, LP solutions do not need to be feasible. In fact, it is very common for a CGLP solution to be associated with an infeasible LP solution.

If we decide to put $u_i$ or $v_i$ into the CGLP basis, we need to pivot out $x_i$ from the LP basis and replace it by some other variable $x_l$. Consequently, we are changing the coefficients of the non-basic variables in the line defining $x_k$ and thus the cut that we obtain.

To simplify notation, we assume $x_k$ is defined by row $k$, $x_i$ by row $i$, and $s_j$ denotes a nonbasic variable in the LP that is a basic multiplier in the CGLP. Hence, the rows associated with $x_k$ and $x_i$ in the LP relaxation can be denoted as

$$x_k + \sum_{j\in J}\bar{a}_{kj}s_j = \bar{a}_{k0} \tag{21}$$

$$x_i + \sum_{j\in J}\bar{a}_{ij}s_j = \bar{a}_{i0} \tag{22}$$

**Corollary 5** *Given an LP basis where $\lfloor\bar{x}_k\rfloor < x_k < \lceil\bar{x}_k\rceil$ and variable $x_i$ leaves the basis, pivoting a non-basic variable $x_l$ preserves $\lfloor\bar{x}_k\rfloor < x_k < \lceil\bar{x}_k\rceil$ if $\dfrac{\lfloor\bar{x}_k\rfloor - \bar{a}_{k0}}{\bar{a}_{i0}} < \gamma_l < \dfrac{\lceil\bar{x}_k\rceil - \bar{a}_{k0}}{\bar{a}_{i0}}$, where $\gamma_l = -\dfrac{\bar{a}_{kl}}{\bar{a}_{il}}$. If $\gamma_l > 0$, the corresponding improvement in the objective function of (RP-CGLP)$_k^p$ is given by*

$$f^+(\gamma) := \quad -\frac{\sum_{j\in J}\left(\min\{0, \bar{a}_{kj}+\gamma\bar{a}_{ij}\}\right)\bar{\bar{s}}_j + (1-\bar{a}_{k0}-\gamma\bar{a}_{ij})p_k}{\sum_{j\in J}\left(\min\{0, \bar{a}_{kj}+\gamma\bar{a}_{ij}\}\right)\bar{s}_j + (1-\bar{a}_{k0}-\gamma\bar{a}_{ij})\bar{x}_k} - \sigma.$$

*Otherwise, if $\gamma_l < 0$, then it is given by*

$$f^-(\gamma) := \quad -\frac{\sum_{j\in J}\left(\max\{0, \bar{a}_{kj}+\gamma\bar{a}_{ij}\}\right)\bar{\bar{s}}_j - (\bar{a}_{k0}+\gamma\bar{a}_{i0})(1-p_k)}{\sum_{j\in J}\left(\max\{0, \bar{a}_{kj}+\gamma\bar{a}_{ij}\}\right)\bar{s}_j - (\bar{a}_{k0}+\gamma\bar{a}_{i0})(1-\bar{x}_k)} - \sigma.$$

*The pivot operation yields no improvement if $\gamma_l = 0$.*

*Proof* If we add row $i$ multiplied by some $\gamma > 0$ to row $k$, we obtain

$$x_k + \gamma x_i + \sum_{j \in J} (\bar{a}_{kj} + \gamma \bar{a}_{ij}) s_j = \bar{a}_{k0} + \gamma \bar{a}_{i0}$$

and the right-hand side remains in the range $(\lfloor \bar{x}_k \rfloor, \lceil \bar{x}_k \rceil)$ if $\dfrac{\lfloor \bar{x}_k \rfloor - \bar{a}_{k0}}{\bar{a}_{i0}} < \gamma < \dfrac{\lceil \bar{x}_k \rceil - \bar{a}_{k0}}{\bar{a}_{i0}}$.

If $x_i$ is pivoted out and replaced by the variable in the $l$-th column, then setting the coefficient of that variable to 0 in the $k$-th row requires $\gamma = -\dfrac{\bar{a}_{kl}}{\bar{a}_{il}} = \gamma_l$.

The impact of such pivot on $(\text{RP-CGLP})_k^p$ depends on $\gamma_l$ being positive or negative. If $\gamma_l > 0$, column $i$ joins $M_2$. In such case, $M_1$ remains a subset of the non-basic variables from the previous basis, which corresponds to those variables with nonpositive coefficients in the $k$-th row. Hence, the objective function of $(\text{RP-CGLP})_k^p$ after the pivot is given by

$$-\frac{\sum_{j \in J} \left(\min\{0, \bar{a}_{kj} + \gamma \bar{a}_{ij}\}\right) \bar{\bar{s}}_j + (1 - \bar{a}_{k0} - \gamma \bar{a}_{ij}) p_k}{\sum_{j \in J} \left(\min\{0, \bar{a}_{kj} + \gamma \bar{a}_{ij}\}\right) \bar{s}_j + (1 - \bar{a}_{k0} - \gamma \bar{a}_{ij}) \bar{x}_k}$$

Otherwise, if $\gamma_l < 0$, column $i$ joins $M_1$ and the objective with respect to $M_2$ becomes

$$-\frac{\sum_{j \in J} \left(\max\{0, \bar{a}_{kj} + \gamma \bar{a}_{ij}\}\right) \bar{\bar{s}}_j - (\bar{a}_{k0} + \gamma \bar{a}_{i0})(1 - p_k)}{\sum_{j \in J} \left(\max\{0, \bar{a}_{kj} + \gamma \bar{a}_{ij}\}\right) \bar{s}_j - (\bar{a}_{k0} + \gamma \bar{a}_{i0})(1 - \bar{x}_k)}$$

Note that there is no change in $\sigma$ if $\gamma_l = 0$ because the $k$-th row remains the same. □

Finally, we observe that there are LP basic solutions corresponding to solutions with negative objective for $(\text{RP-CGLP})_k^p$, which are those yielding inequalities that do not separate $\bar{x}$. These solutions have no correspondence in the CGLP because they are removed by the normalization constraint. When using the LP relaxation to generate the cut, one should not pivot to such bases. If we keep using $p \in \mathrm{P^k}$, they are easily spotted through the objective function of the CGLP.

## 5 Parameterizing the Cut Generator

We are now left to discuss the choice of a point $p \in \mathrm{P^k}$ to parameterize the cut generator. From a theoretical perspective, we want to preferably choose a point $p \in \mathrm{int}(\mathrm{P^k})$ to obtain a cutting plane that is a combination of facets separating $\bar{x}$. From a practical perspective, we want to keep the computational cost closer to that of competing alternatives. As mentioned previously, using the simplex tableau associated with the LP relaxation instead of an explicit CGLP formulation halves the size of the LP formulation with which the cuts are generated. Therefore, any method to choose $p$ must preferably not require a computational effort that fairly exceeds that of solving the LP relaxation or of using it as a surrogate.

One immediate option is to use the line defined by the gradient of the objective function. Namely, choose a point $p$ such that $(p - \bar{x}) = \lambda c$ for some $\lambda \neq 0$. Alternately, or if it is not straightforward to check if such line intersects $\mathrm{P}^{\mathrm{k}}$, we can compute $p$ as the convex combination of points of the LP relaxation in each of the terms of the split disjunction $\{x : x_k \leq 0\} \cup \{x : x_k \geq 1\}$, say $p_0$ and $p_1$, hence entailing the independent solution of two LP formulations of similar size as the LP relaxation. When pursuing the latter option, we augment these formulations with a single decision variable $s$ that captures a lower bound on the slacks of the linear inequalities. Subsequently, those points are combined in proportion to their corresponding slack. If we denote the solutions of each LP as $(p_\gamma, s_\gamma)$ for $\gamma \in \{0, 1\}$, then we choose $p := \frac{s_0}{s_0+s_1} p_0 + \frac{s_1}{s_0+s_1} p_1$.

Our first method combines both ideas. First, we check if extending the objective gradient yields a point in the LP relaxation on either side of the split disjunction. If not, we calculate a pair of points on both sides with a formulation that prioritizes moving away from the boundary of $\mathrm{P}^{\mathrm{k}}$ by maximizing $s$. We calculate those points as $p_\gamma := \arg_x \max_{x,s}\{s : Ax + s \geq b, x_k = \gamma, x \in [0, 1]^q \times \mathbb{R}_+^{n-q}, s \in \mathbb{R}_+\}$ for $\gamma \in \{0, 1\}$, and we denote the method as Gradient or Maximize Minimum Slack (**GMMS**). If it is possible to increase $s$ indefinitely, we use $\gamma = 1$ to truncate the resulting unbounded solution $(x_v, s_v) + \gamma(x_r, s_r)$, $\gamma \geq 0$, which is defined by the extreme point $(x_v, s_v)$ and the extreme ray $(x_r, s_r)$.

Our second method also leverages the objective function of the MILP while ensuring a lower bound on the slacks. We calculate those points as $p_\gamma := \arg_x \max\{c^T x : Ax + s \geq b, s \geq \varepsilon, x_k = \gamma, x \in [0, 1]^q \times \mathbb{R}_+^{n-q}, s \in \mathbb{R}_+\}$ for $\gamma \in \{0, 1\}$ and some $\varepsilon > 0$, and we denote the method as Required Minimum Slack (**RMS**). We can choose $\varepsilon = 1$ and keep halving its value up to a threshold if the linear system is infeasible. If the formulation is feasible, the change of objective function in contrast to the MMS formulation ensures that an optimal solution exists since the same is true for the MILP, whereas the objective function induces some proximity to $\bar{x}$.

## 6 Computational Experiments

This section compares the cuts generated by solving $(\mathrm{CGLP})_k$ and $(\mathrm{RP\text{-}CGLP})_k^p$ in the case of split disjunctions through the tableau of the linear relaxation. We use the implementation for $(\mathrm{CGLP})_k$ described in Balas and Bonami (2009) and adapt it to also generate cuts using $(\mathrm{RP\text{-}CGLP})_k^p$ with methods GMMS and RMS.

Our primary focus is solving MILP formulations faster, which can be at odds with metrics evaluating the local impact of a cut or collection on cuts. For example, a cut maximizing the distance between the separating hyperplane and the fractional solution may not necessarily be facet-defining. Furthermore, the optimality gap closed by a collection of cuts may not capture the complementary of such cuts beyond such immediate impact. While average distance and optimality gap closed are reported for the cuts generated, the development of the cut generator has focused on parameterizations that improve the final outcome of the solver. Nevertheless, we disable all cuts that are automatically generated by the solver in order to avoid generating redundant cuts with either approach, since that could potentially distort the comparison.

In the experiments, we first solve the linear relaxation of instances from the MIPLIB benchmarks[2] and then generate a lift-and-project cut for each fractional variable $x_k$ using either $(\text{CGLP})_k$ or $(\text{RP-CGLP})_k^p$, which is then strengthened, up to a limit of 300 distinct cuts. We circumvent most of the numerical issues that arise when the objective function gets too close to zero or RMS does not yield a feasible solution by reverting to the conventional formulation in those cases. As we generate the cuts for each instance using each formulation, we measure the total gap closed when adding all the cuts generated and resolving the linear relaxation as well as the average Euclidean distance of each cutting plane to $\bar{x}$. The gap closed here refers to the optimal value of the linear relaxation after adding the cuts in comparison with the optimal value of the linear relaxation with no cuts and the known optimal value of each instance. Finally, we try solving the resulting MILP formulations on a single thread using the CPLEX solver 12.9 with the cuts from each method added and automatic cut generation disabled and a time limit of ten minutes, from which we report the runtime or the remaining gap after the time limit is reached. All code is written in C++ and the CPLEX experiments ran on a Windows 10 machine with an Intel(R) Core(TM) i5-6200U CPU @ 2.30GHz processor and 16 GB of RAM.

Tables 1 to 4 describe the results per instance. Table 5 summarizes them. Table 5 also aggregates the results in a different form with a column for best virtual method, which counts the cases in which using $(\text{RP-CGLP})_k^p$ with either GMMS or RMS has a better result than $(\text{CGLP})_k$ and the cases in which using $(\text{CGLP})_k$ has a worse result than both.

[2] miplib.zib.de

**Table 1** Comparison of default solver behavior with total gap closed (%), average Euclidean distance of the cuts, and runtime in seconds to solve the problem or gap by time limit after adding the generated cuts.

| | | $(\mathrm{CGLP})_k$ | | | $(\mathrm{RP\text{-}CGLP})_k^p$ with GMMS | | | $(\mathrm{RP\text{-}CGLP})_k^p$ with RMS | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Instance | Default | Gap | Avg. dist. | CPLEX | Gap | Avg. dist. | CPLEX | Gap | Avg. dist. | CPLEX |
| 10teams | 3.8 s | 0.6 | 0.024 | 9.3 s | 0.6 | 0.025 | 10.6 s | 0.6 | 0.025 | 12.7 s |
| a1c1s1 | 0.0 % | 0.2 | 0.91 | 0.3 % | 0.2 | 0.851 | 0.3 % | 0.2 | 0.903 | 0.3 % |
| aflow30a | 6.2 s | 0.2 | 0.055 | 27.8 s | 0.2 | 0.055 | 27.5 s | 0.2 | 0.042 | 27.1 s |
| aflow40b | 139.0 s | 0.1 | 0.021 | 0.0 % | 0.1 | 0.021 | 0.0 % | 0.1 | 0.016 | 0.0 % |
| air01 | 0.1 s | 1.0 | 0.025 | 0.1 s | 1.0 | 0.025 | 0.1 s | 1.0 | 0.025 | 0.1 s |
| air02 | 0.5 s | 0.8 | 0.005 | 0.7 s | 0.8 | 0.005 | 0.5 s | 0.3 | 0.006 | 0.5 s |
| air03 | 0.8 s | 1.0 | 0.006 | 1.4 s | 1.0 | 0.006 | 0.9 s | 1.0 | 0.006 | 1.3 s |
| air04 | 6.5 s | 0.1 | 0.004 | 279.5 s | 0.1 | 0.004 | 291.6 s | 0.1 | 0.004 | 152.3 s |
| air05 | 16.8 s | 0.1 | 0.005 | 143.6 s | 0.0 | 0.005 | 88.7 s | 0.1 | 0.005 | 117.5 s |
| air06 | 0.9 s | 0.5 | 0.005 | 14.0 s | 0.5 | 0.005 | 14.6 s | 0.5 | 0.005 | 12.8 s |
| arki001 | 22.8 s | 0.3 | 0.168 | 9.8 s | 0.3 | 0.168 | 5.6 s | 0.3 | 0.168 | 6.9 s |
| atlanta-ip | 0.1 % | 0.0 | 0.016 | 0.1 % | 0.0 | 0.016 | 0.1 % | 0.0 | 0.016 | 0.1 % |
| bell3a | 0.2 s | 0.5 | 0.537 | 1.6 s | 0.5 | 0.582 | 0.9 s | 0.5 | 0.549 | 1.2 s |
| bell3b | 0.9 s | 0.5 | 0.55 | 0.5 s | 0.5 | 0.544 | 1.0 s | 0.5 | 0.55 | 0.9 s |
| bell4 | 0.5 s | 0.3 | 0.562 | 0.9 s | 0.3 | 0.561 | 0.5 s | 0.3 | 0.561 | 0.6 s |
| bell5 | 0.1 s | 0.9 | 0.479 | 0.1 s | 0.9 | 0.479 | 0.1 s | 0.9 | 0.479 | 0.1 s |
| blend2 | 1.1 s | 0.2 | 0.11 | 0.5 s | 0.2 | 0.11 | 0.6 s | 0.2 | 0.11 | 0.7 s |
| bm23 | 0.1 s | 0.2 | 0.121 | 0.0 s | 0.1 | 0.128 | 0.0 s | 0.1 | 0.083 | 0.0 s |
| cap6000 | 0.3 s | 0.3 | 0.006 | 0.8 s | 0.4 | 0.006 | 0.4 s | 0.3 | 0.006 | 0.4 s |
| cracpb1 | 0.1 s | -inf | 0.046 | 0.0 s | -inf | 0.023 | 0.0 s | -inf | 0.004 | 0.1 s |
| danoint | 0.0 % | 0.0 | 0.113 | 0.0 % | 0.0 | 0.066 | 0.0 % | 0.0 | 0.117 | 0.0 % |
| dcmulti | 0.4 s | 0.4 | 1.52 | 0.1 s | 0.4 | 1.239 | 0.1 s | 0.4 | 1.073 | 0.2 s |
| ds | 0.8 % | 0.0 | 0.001 | 0.9 % | 0.0 | 0.001 | 0.8 % | 0.0 | 0.001 | 0.8 % |
| dsbmip | 0.3 s | 0.0 | 0.476 | 0.3 s | 0.0 | 0.477 | 0.3 s | 0.0 | 0.48 | 0.3 s |
| egout | 0.1 s | 0.6 | 0.88 | 0.0 s | 0.6 | 0.903 | 0.0 s | 0.6 | 0.903 | 0.0 s |
| fast0507 | 64.5 s | 0.0 | 0.002 | 0.0 % | 0.0 | 0.002 | 0.0 % | 0.0 | 0.002 | 0.0 % |
| fiber | 0.8 s | 0.4 | 0.052 | 1.4 s | 0.4 | 0.052 | 1.2 s | 0.5 | 0.045 | 2.7 s |
| fixnet3 | 0.2 s | 0.7 | 0.729 | 0.1 s | 0.7 | 0.729 | 0.1 s | 0.7 | 0.729 | 0.1 s |
| fixnet4 | 0.3 s | 0.3 | 0.732 | 0.1 s | 0.3 | 0.732 | 0.1 s | 0.3 | 0.732 | 0.1 s |

**Table 2** Continuation of Table 1

| Instance | Default | $(\text{CGLP})_k$ | | | $(\text{RP-CGLP})_k^p$ with GMMS | | | $(\text{RP-CGLP})_k^p$ with RMS | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | Gap | Avg. dist. | CPLEX | Gap | Avg. dist. | CPLEX | Gap | Avg. dist. | CPLEX |
| fixnet6 | 1.0 s | 0.1 | 0.728 | 0.2 s | 0.1 | 0.727 | 0.2 s | 0.1 | 0.727 | 0.3 s |
| flugpl | 0.0 s | 0.1 | 0.352 | 0.0 s | 0.1 | 0.355 | 0.0 s | 0.1 | 0.355 | 0.0 s |
| gen | 0.1 s | 0.7 | 0.051 | 0.1 s | 0.7 | 0.051 | 0.1 s | 0.7 | 0.052 | 0.1 s |
| gesa2 | 0.6 s | 0.3 | 0.156 | 3.3 s | 0.3 | 0.224 | 5.3 s | 0.2 | 0.234 | 5.1 s |
| gesa2_o | 2.7 s | 0.3 | 0.212 | 4.3 s | 0.3 | 0.201 | 4.7 s | 0.3 | 0.194 | 5.4 s |
| gesa3 | 1.0 s | 0.6 | 0.091 | 0.7 s | 0.3 | 0.079 | 0.2 s | 0.5 | 0.101 | 0.2 s |
| gesa3_o | 0.9 s | 0.7 | 0.126 | 0.2 s | 0.5 | 0.12 | 0.2 s | 0.6 | 0.12 | 0.2 s |
| glass4 | 148.9 s | -0.0 | 0.964 | 232.6 s | -0.0 | 0.964 | 238.7 s | -0.0 | 0.964 | 234.4 s |
| gt2 | 0.2 s | 0.9 | 0.582 | 0.0 s | 0.9 | 0.582 | 0.1 s | 0.1 | 0.194 | 0.0 s |
| harp2 | 7.3 s | 0.3 | 0.025 | 28.2 s | 0.2 | 0.017 | 33.4 s | 0.2 | 0.017 | 43.9 s |
| khb05250 | 0.2 s | 0.7 | 0.638 | 0.5 s | 0.7 | 0.638 | 0.1 s | 0.7 | 0.638 | 0.1 s |
| l152lav | 1.5 s | 0.0 | 0.014 | 2.9 s | 0.0 | 0.007 | 3.4 s | 0.0 | 0.008 | 4.1 s |
| lp4l | 0.1 s | 0.5 | 0.02 | 0.2 s | 0.9 | 0.024 | 0.1 s | 0.9 | 0.024 | 0.1 s |
| lseu | 0.1 s | 0.6 | 0.107 | 0.1 s | 0.6 | 0.071 | 0.2 s | 0.5 | 0.069 | 0.3 s |
| manna81 | 0.6 s | 1.0 | 0.289 | 0.3 s | 1.0 | 0.289 | 0.2 s | 1.0 | 0.289 | 0.2 s |
| markshare1 | 1.0 % | 0.0 | 0.028 | 1.0 % | 0.0 | 0.028 | 1.0 % | 0.0 | 0.028 | 1.0 % |
| markshare2 | 1.0 % | 0.0 | 0.027 | 1.0 % | 0.0 | 0.027 | 1.0 % | 0.0 | 0.027 | 1.0 % |
| mas74 | 291.0 s | 0.1 | 0.009 | 206.3 s | 0.1 | 0.013 | 209.2 s | 0.0 | 0.011 | 266.7 s |
| mas76 | 15.5 s | 0.1 | 0.009 | 17.1 s | 0.0 | 0.017 | 17.6 s | 0.0 | 0.012 | 16.6 s |
| misc01 | 0.3 s | 0.0 | 0.138 | 0.1 s | 0.0 | 0.14 | 0.1 s | 0.0 | 0.133 | 0.0 s |
| misc02 | 0.1 s | 0.1 | 0.137 | 0.0 s | 0.1 | 0.134 | 0.0 s | 0.1 | 0.137 | 0.0 s |
| misc03 | 0.1 s | 0.1 | 0.064 | 0.2 s | 0.1 | 0.064 | 0.2 s | 0.1 | 0.066 | 0.1 s |
| misc04 | 0.2 s | 0.2 | 0.118 | 0.1 s | 0.2 | 0.081 | 0.2 s | 0.2 | 0.081 | 0.1 s |
| misc05 | 0.2 s | 0.2 | 0.114 | 0.1 s | 0.2 | 0.081 | 0.1 s | 0.2 | 0.114 | 0.1 s |
| misc06 | 0.2 s | 0.3 | 0.035 | 0.2 s | 0.3 | 0.035 | 0.1 s | 0.3 | 0.035 | 0.1 s |
| misc07 | 1.3 s | 0.0 | 0.044 | 6.2 s | 0.0 | 0.047 | 8.5 s | -0.0 | 0.048 | 10.9 s |
| mitre | 0.7 s | 0.8 | 0.04 | 0.7 s | 0.8 | 0.04 | 0.4 s | 0.6 | 0.037 | 0.5 s |
| mkc | 0.0 % | 0.2 | 0.076 | 0.0 % | 0.1 | 0.055 | 0.0 % | 0.1 | 0.064 | 0.0 % |
| mod008 | 0.3 s | 0.2 | 0.016 | 0.3 s | 0.0 | 0.016 | 0.3 s | 0.0 | 0.016 | 0.2 s |
| mod010 | 0.3 s | 0.1 | 0.018 | 0.6 s | 0.7 | 0.009 | 0.3 s | 0.7 | 0.009 | 0.3 s |

**Table 3** Continuation of Table 1

| Instance | Default | $(\mathrm{CGLP})_k$ Gap | $(\mathrm{CGLP})_k$ Avg. dist. | $(\mathrm{CGLP})_k$ CPLEX | $(\mathrm{RP\text{-}CGLP})_k^p$ with GMMS Gap | $(\mathrm{RP\text{-}CGLP})_k^p$ with GMMS Avg. dist. | $(\mathrm{RP\text{-}CGLP})_k^p$ with GMMS CPLEX | $(\mathrm{RP\text{-}CGLP})_k^p$ with RMS Gap | $(\mathrm{RP\text{-}CGLP})_k^p$ with RMS Avg. dist. | $(\mathrm{RP\text{-}CGLP})_k^p$ with RMS CPLEX |
|---|---|---|---|---|---|---|---|---|---|---|
| mod011 | 40.5 s | 0.2 | 0.588 | 32.1 s | 0.3 | 0.318 | 25.3 s | 0.3 | 0.318 | 25.2 s |
| mod013 | 0.2 s | 0.0 | 0.485 | 0.1 s | 0.0 | 0.496 | 0.1 s | 0.0 | 0.485 | 0.1 s |
| modglob | 0.2 s | 0.2 | 0.542 | 20.1 s | 0.2 | 0.555 | 44.9 s | 0.1 | 0.504 | 72.0 s |
| momentum1 | 0.2 % | 0.4 | 0.147 | 0.1 % | 0.4 | 0.152 | 0.2 % | 0.4 | 0.153 | 0.4 % |
| momentum2 | 0.2 % | 0.4 | 0.178 | 0.0 % | 0.4 | 0.179 | 0.1 % | 0.4 | 0.174 | 0.1 % |
| msc98-ip | 0.0 % | 0.1 | 0.059 | 0.1 % | 0.1 | 0.059 | 0.0 % | 0.1 | 0.064 | 0.0 % |
| mzzv11 | 21.8 s | 0.3 | 0.027 | 292.2 s | 0.3 | 0.027 | 189.9 s | 0.3 | 0.027 | 201.8 s |
| mzzv42z | 16.4 s | 0.2 | 0.021 | 71.4 s | 0.2 | 0.021 | 69.7 s | 0.2 | 0.021 | 59.5 s |
| net12 | 114.1 s | 0.1 | 0.135 | 143.9 s | 0.1 | 0.136 | 97.2 s | 0.1 | 0.137 | 108.5 s |
| nsrand-ipx | 86.5 s | 0.4 | 0.008 | 0.0 % | 0.3 | 0.004 | 0.0 % | 0.3 | 0.005 | 0.0 % |
| nw04 | 5.1 s | 0.6 | 0.001 | 4.6 s | 0.6 | 0.001 | 6.3 s | 0.6 | 0.001 | 6.0 s |
| opt1217 | 0.5 s | 0.1 | 0.072 | 0.2 % | -0.0 | 0.072 | 0.2 % | 0.1 | 0.067 | 0.2 % |
| p0033 | 0.0 s | 0.6 | 0.368 | 0.0 s | 0.6 | 0.368 | 0.0 s | 0.6 | 0.368 | 0.0 s |
| p0040 | 0.0 s | 1.0 | 0.096 | 0.0 s | 1.0 | 0.096 | 0.0 s | 1.0 | 0.096 | 0.0 s |
| p0201 | 0.2 s | 0.4 | 0.038 | 0.1 s | 0.4 | 0.038 | 0.1 s | 0.4 | 0.042 | 0.1 s |
| p0282 | 0.2 s | 0.0 | 0.149 | 0.1 s | 0.0 | 0.148 | 0.1 s | 0.0 | 0.147 | 0.1 s |
| p0291 | 0.0 s | 0.4 | 0.235 | 0.1 s | 0.3 | 0.18 | 0.1 s | 0.4 | 0.235 | 0.0 s |
| p0548 | 0.1 s | 0.4 | 0.598 | 0.6 s | 0.4 | 0.598 | 0.5 s | 0.4 | 0.598 | 0.6 s |
| p2756 | 0.3 s | 0.0 | 0.427 | 44.4 s | 0.0 | 0.427 | 51.3 s | 0.0 | 0.416 | 36.6 s |
| p6000 | 0.3 s | 0.3 | 0.006 | 0.4 s | 0.4 | 0.006 | 0.4 s | 0.3 | 0.006 | 0.4 s |
| pipex | 0.1 s | 0.3 | 0.153 | 0.1 s | 0.3 | 0.153 | 0.1 s | 0.3 | 0.103 | 0.0 s |
| pk1 | 19.3 s | 0.0 | 0.013 | 22.9 s | 0.0 | 0.013 | 22.3 s | 0.0 | 0.013 | 22.5 s |
| pp08a | 0.5 s | 0.5 | 0.812 | 388.5 s | 0.5 | 0.812 | 420.9 s | 0.5 | 0.812 | 404.0 s |
| pp08aCUTS | 0.7 s | 0.4 | 0.283 | 12.1 s | 0.3 | 0.268 | 15.0 s | 0.4 | 0.252 | 9.8 s |
| protfold | 0.6 % | 0.3 | 0.047 | 0.9 % | 0.1 | 0.025 | 0.9 % | 0.1 | 0.024 | 0.8 % |
| qnet1 | 0.8 s | 0.2 | 0.043 | 0.9 s | 0.1 | 0.012 | 1.5 s | 0.2 | 0.017 | 1.2 s |
| qnet1_o | 0.5 s | 0.5 | 0.194 | 0.7 s | 0.4 | 0.137 | 1.0 s | 0.5 | 0.141 | 0.8 s |
| rd-rplusc-21 | 1.0 % | 0.0 | 0.169 | 505.4 s | -0.0 | 0.167 | 1.0 % | 0.0 | 0.173 | 1.0 % |
| rentacar | 0.6 s | 0.2 | 0.968 | 2.0 s | 0.2 | 0.93 | 2.0 s | 0.2 | 0.93 | 2.3 s |
| rgn | 0.5 s | 0.1 | 0.087 | 0.1 s | 0.1 | 0.117 | 0.2 s | 0.1 | 0.12 | 0.1 s |

**Table 4** Continuation of Table 1

| | | $(\mathrm{CGLP})_k$ | | | $(\mathrm{RP\text{-}CGLP})_k^p$ with GMMS | | | $(\mathrm{RP\text{-}CGLP})_k^p$ with RMS | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Instance | Default | Gap | Avg. dist. | CPLEX | Gap | Avg. dist. | CPLEX | Gap | Avg. dist. | CPLEX |
| roll3000 | 19.8 s | 0.3 | 0.079 | 0.0 % | 0.3 | 0.081 | 0.0 % | 0.3 | 0.081 | 0.0 % |
| rout | 5.7 s | 0.0 | 0.077 | 33.7 s | 0.0 | 0.073 | 27.6 s | 0.0 | 0.075 | 22.3 s |
| sample2 | 0.0 s | 0.1 | 0.59 | 0.0 s | 0.1 | 0.591 | 0.0 s | 0.1 | 0.59 | 0.0 s |
| sentoy | 0.1 s | 0.2 | 0.029 | 0.0 s | 0.2 | 0.051 | 0.0 s | 0.2 | 0.031 | 0.0 s |
| set1ch | 0.6 s | 0.4 | 0.82 | 0.2 % | 0.4 | 0.814 | 0.2 % | 0.4 | 0.814 | 0.1 % |
| seymour | 0.0 % | 0.1 | 0.074 | 0.0 % | 0.1 | 0.058 | 0.0 % | 0.1 | 0.075 | 0.0 % |
| sp97ar | 0.0 % | 0.2 | 0.003 | 0.0 % | 0.2 | 0.003 | 0.0 % | 0.2 | 0.003 | 0.0 % |
| stein15 | 0.0 s | 0.0 | 0.255 | 0.0 s | 0.0 | 0.241 | 0.0 s | 0.0 | 0.241 | 0.0 s |
| stein27 | 0.0 s | -0.0 | 0.239 | 0.3 s | -0.0 | 0.231 | 0.2 s | -0.0 | 0.239 | 0.2 s |
| stein45 | 5.7 s | 0.0 | 0.294 | 6.4 s | 0.0 | 0.174 | 6.1 s | 0.0 | 0.182 | 5.9 s |
| stein9 | 0.0 s | 0.0 | 0.214 | 0.0 s | -0.0 | 0.186 | 0.0 s | 0.0 | 0.207 | 0.0 s |
| swath | 0.1 % | 0.3 | 0.071 | 0.2 % | 0.1 | 0.026 | 0.2 % | 0.1 | 0.022 | 0.2 % |
| timtab1 | 112.8 s | 0.2 | 0.298 | 0.2 % | 0.2 | 0.298 | 0.2 % | 0.2 | 0.298 | 0.2 % |
| timtab2 | 0.2 % | 0.2 | 0.225 | 0.6 % | 0.2 | 0.225 | 0.6 % | 0.2 | 0.225 | 0.6 % |
| vpm1 | 0.1 s | 0.1 | 0.139 | 4.7 s | 0.1 | 0.135 | 7.6 s | 0.1 | 0.139 | 3.8 s |
| vpm2 | 0.6 s | 0.2 | 0.069 | 2.2 s | 0.2 | 0.059 | 2.2 s | 0.2 | 0.069 | 1.8 s |

**Table 5** Performance of $(\text{RP-CGLP})_k^p$ against $(\text{CGLP})_k$ for each method generating $p$ and a virtual best method consisting of the best result from either method.

| | GMMS | | RMS | | Virtual best method | |
|---|---|---|---|---|---|---|
| | Better | Worse | Better | Worse | Better | Worse |
| Total gap closed | 21 | 48 | 23 | 44 | 32 | 31 |
| Average distance | 31 | 55 | 28 | 56 | 41 | 39 |
| CPLEX performance | 57 | 47 | 63 | 41 | 75 | 29 |

We observe that using $(\text{RP-CGLP})_k^p$ with both GMMS or RMS often yields a better performance solving the problem than $(\text{CGLP})_k$. However, the total gap closed and the average distance of the cutting planes tends to be smaller. An intuitive explanation is that these cutting planes do not maximize violation for some scale of the constraints. Furthermore, the aggregated results in the last columns indicate that a more calibrated choice of $p$ could potentially improve the results.

## 7 Conclusion

This paper introduced the Reverse Polar Cut Generating Linear Program (RP-CGLP), which is parameterized by a point $\bar{x}$ that we want to separate and a point $p$ that we cannot. We have shown that these lift-and-project cuts define supporting hyperplanes of the immediate closure. When that closure if full-dimensional, the cutting plane is a combination of facets that are active at the point that a ray from $\bar{x}$ to $p$ first intersects the closure, with each facet separating $\bar{x}$ also corresponding to an optimal solution to RP-CGLP. We also adapt this formulation to generate cuts from the tableau of the LP relaxation in the case of split disjunctions. Note, however, that all other results remain valid for arbitrary disjunctions.

While we switch the roles of normalization and objective in comparison to other CGLP formulations, we nevertheless observe that a distortion in how the cuts are compared with respect to the objective is unavoidable. We fix violation to guarantee separability and then choose to minimize the slack for $p$ to ensure boundedness. That intuitively favors cuts that are farther away from $\bar{x}$ and closer to $p$. In fact, we show that RP-CGLP actually minimizes the ratio between slack for $p$ and violation for $\bar{x}$ across all valid cuts, consequently proving the equivalence between RP-CGLP and other recent CGLP formulations. Previously, the CGLPs for which an explicit representation of the objective was known were those fixing a norm of $\alpha$. Moreover, in comparison to the equivalent CGLP formulations that have been recently proposed, RP-CGLP has the benefit of preserving the feasible set for different choices of $p$, whereas changing $p$ affects the left-hand side of those formulations. Hence, RP-CGLP may facilitate generating multiple cuts because we only need to reoptimize after changing $p$. In fact, any feasible solution of RP-CGLP yields a cut separating $\bar{x}$, while that depends on the objective function value for other formulations. In addition, we can potentially use sensitivity analysis on the objective of RP-CGLP to look for points in the disjunctive hull yielding a disjoint set of cuts.

The experimental results have shown that there is some potential for RP-CGLP, but also that finding a good choice for $p$ deserves further study. There is a noticeable difference in the results according to the method used to generate

$p$, especially for the second experiment of testing the performance on CPLEX with those cuts and automatic cut generation disabled. In fact, one could argue that there is a way to choose $p$ in each case for which the same cuts from the conventional formulation are obtained, or else strictly better ones are found in the case that they do not define supporting hyperplanes.

Ultimately, one could argue that the reformulation shifts where the numerical issues are. While the concept of a most violated cut depends on an adequate scale of the constraints in the conventional formulation, the family of equivalent in-out formulations to which RP-CGLP belongs depends on a careful choice of point or ray parameterizing the direction of separation. Therefore, an important milestone for these approaches is finding points in the disjunctive hull that yield strictly better cuts. One would expect the ideal point to define a CGLP optimum with unique $(\alpha, \beta)$-projection, which in turn derives a cut $\alpha^T x \geq \beta$ defining a facet of the disjunctive hull. When the disjunctive hull is not full-dimensional and the point is inevitably at the boundary, further restricting the CGLP by facial reduction (Borwein and Wolkowicz, 1981) could be a possibility for better results.

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