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Generalized optical theorem for an arbitrary incident field

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Abstract: A formalism of optical theorem extended for an arbitrarily shaped wave field is presented. The formalism concerns only time-harmonic scattering in free space. The theorem relates the extinction cross section to the imaginary part of the total scattering amplitude at the forward direction of the individual plane wave components multiplied by the corresponding plane wave amplitude in the angular spectrum of the incident wave. A differential extinction cross section is defined from the theorem. An alternative formalism relating the total cross section to plane wave scattering is also presented. Physical interpretation is provided and applications are discussed.

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1. Introduction

In illumination of waves on objects, the energy extinction from incident wave field is either by scattering or absorption. The total extinction cross section is a sum of scattering cross section and absorption cross section. A fundamental relation in the scattering of plane waves, called the optical theorem or extinction theorem,[1] relates the total extinction cross section due to a scatterer to the complex scattering amplitude in the forward direction.

To be explicit, let an incident plane wave $\psi_i^p = \exp(i\mathbf{k}_0 \cdot \mathbf{r})$ with the wave vector $\mathbf{k}_0 = k\mathbf{n}_0$ at the direction $\mathbf{n}_0$, and the scattered far field at the direction $\mathbf{n}$ in terms of a scattering amplitude $A(\mathbf{n}, \mathbf{n}_0)$ is

$$\psi_s^p = \frac{e^{ikr}}{r} A(\mathbf{n}, \mathbf{n}_0), \tag{1}$$

where $\mathbf{n}$ is the radially outward normal. Then the optical theorem states that the total extinction cross section is

$$\sigma_{\text{ext}}^p = \frac{4\pi}{k} \text{Im}\left[A(\mathbf{n} = \mathbf{n}_0, \mathbf{n}_0)\right], \tag{2}$$

where Im represents the imaginary part of the complex forward scattering $A(\mathbf{n} = \mathbf{n}_0, \mathbf{n}_0)$. The extinction theorem can be applied to measure the extinction cross section (or scattering cross section for nonabsorbing scattering) from the forward scattering amplitude.

The optical theorem for plane waves is not applicable to non-plane wave incidence, as one can recognize from formulas of extinction cross section for some specific types of beams, such as fields emitted by a point source,[2] Gaussian beams,[3] Bessel beams,[4,5] or general non-diffracting beams.[6] An extended optical theorem was presented for non-diffracting acoustic beams[6] and later extended to electromagnetic scattering.[7]

In this letter, a generalized optical theorem applicable to an arbitrarily shaped incident wave is derived and presented for the time-harmonic (single-frequency) scattering. The theorem relates the extinction cross section to the angular function of the incident field (Sec. 3). A differential extinction cross section is introduced (Sec. 4). The extinction is also related to the plane wave scattering (Sec. 5). Even though the presentation starts in the context of acoustic waves, the formalism of the extinction cross section is applicable to any scalar waves.

2. Formulas of acoustic extinction power and cross section

Consider the total field as a sum of the incident and scattered field for both the acoustic pressure and velocity fields,

$$p_t = p_i + p_s, \quad \mathbf{u}_t = \mathbf{u}_i + \mathbf{u}_s. \tag{3}$$

Let S be a spherical surface of radius r centered on the scatterer (see Fig. 1). The scattered power is calculated by integrating the outward-going component of the energy flux of the scattered field

$$P_{\text{sca}} = \int \langle p_s \mathbf{u}_s \rangle \cdot d\mathbf{S}, \tag{4}$$

where $\langle\ \rangle$ denotes the time average over a wave circle. The absorbed power can be calculated by integrating the inward-going component of the energy flux of the total field

$$P_{\text{abs}} = -\int \langle p_t \mathbf{u}_t \rangle \cdot d\mathbf{S} = -\int \langle p_i \mathbf{u}_i + p_s \mathbf{u}_i + p_i \mathbf{u}_s + p_s \mathbf{u}_s \rangle \cdot d\mathbf{S}, \tag{5}$$

where Eq. (3) was used. P_{abs} is non-negative for a passive object.[8] In Eq. (5), the term involved with the incident field only is zero, following from the conservation of energy when there is no scatterer. Then the sum of Eqs. (4) and (5) gives the power extinguished from the beams when scattering from the object (i.e., the extinction power),

$$P_{\text{ext}} = P_{\text{abs}} + P_{\text{sca}} = -\int \langle p_i \mathbf{u}_s + p_s \mathbf{u}_i \rangle \cdot d\mathbf{S}, \tag{6}$$

which involves the interference of the incident and scattered fields.[9]

When expressing the acoustic velocity and pressure of the incident and scattered fields in terms of dimensionless complex velocity potential ψ_i and ψ_s,

$$\mathbf{u}_{i,s} = \text{Re}\left[\psi_0 \nabla \psi_{i,s} \exp(-i\omega t)\right], \quad p_{i,s} = \text{Re}\left[i\omega\rho_0\psi_0\psi_{i,s} \exp(-i\omega t)\right], \tag{7}$$

where Re denotes the real part and ρ_0 is the static density of external medium, and an intensity $I_0 = \rho_0 c_0 k^2 |\psi_0|^2/2$ is introduced to define an extinction cross section $\sigma_{\text{ext}} = P_{\text{ext}}/I_0$.

In Eq. (6), using Eq. (7) and evaluating the time average by using $\langle \text{Re}[Ae^{-i\omega t}] \times \text{Re}[Be^{-i\omega t}] \rangle = \text{Re}[AB^\star]/2$ (the star represents complex conjugate), the extinction cross section reduces to be in terms of complex potentials as

$$\sigma_{\text{ext}} = -\frac{1}{k}\int \text{Im}\left[\left(\psi_i^\star \nabla \psi_s - \psi_s \nabla \psi_i^\star\right) \cdot \mathbf{n}\right] dA, \tag{8}$$

where $d\mathbf{S} = \mathbf{n}dA$ was used. Equation (8) and equations from now on are applicable to any scalar waves, not limited in acoustics. The extinction cross section can also be written in terms of dimensionless extinction efficiency, $Q_{\text{ext}} = \sigma_{\text{ext}}/\pi a^2$, and related to the dimensionless scattering function, $f_s = 2A_s/a$, for an object of size a and with an illuminating area πa^2.

3. Generalized optical theorem

The interest is to relate the extinction power to the scattering amplitude. The scattered far field at large distance $kr \to \infty$ in terms of a scattering amplitude $A_s(\mathbf{n})$ at the direction $\mathbf{n}$ is

$$\psi_s = \frac{e^{ikr}}{r} A_s(\mathbf{n}), \tag{9}$$

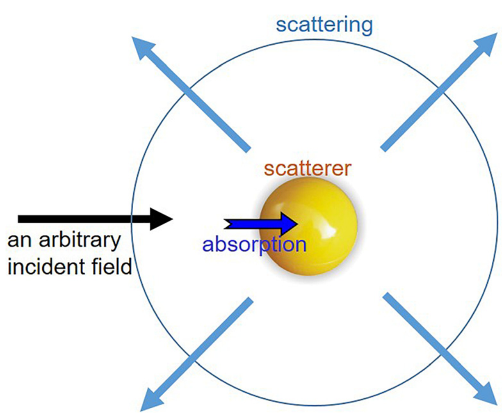

Fig. 1. (Color online) Illustration of energy extinction from the arbitrary shaped field via scattering and absorption. The generalized optical theorem presented in Eqs. (18) and (21) relates the extinction cross section to the scattering amplitude and the angular spectrum of the incident field [see Eq. (12)].

and it has

$$\nabla\psi_s \simeq ik\mathbf{n}\psi_s. \tag{10}$$

Equation (9) is for an object producing a scattered wave with three-dimensional spreading. Then the extinction cross section, Eq. (8), expressed as a function of the scattering amplitude $A_s(\mathbf{n})$ is

$$\sigma_{\text{ext}} = -\frac{1}{r}\int \text{Re}\left[\left(\psi_i^\star - \frac{1}{ik}\mathbf{n}\cdot\nabla\psi_i^\star\right)e^{ikr}A_s(\mathbf{n})\right]dA. \tag{11}$$

The extinction cross section, Eq. (11), exhibits a dependence on the scattering amplitude at all directions but can be simplified by optical theorem. Here we derive a generalized optical theorem for relating the extinction power to scattering at the forward direction of the plane wave components of the incident field by using the angular representation of the incident field. The incident field can be represented as the superposition of plane wave components in angular spectrum as

$$\psi_i = \int g(\mathbf{n}_k)\exp(i\mathbf{k}\cdot\mathbf{r})d\Omega_k, \tag{12}$$

where $\mathbf{r}$ is the field point, the wave-vectors $\mathbf{k} = k\mathbf{n}_k$ with $\mathbf{n}_k$ being the incident direction of the individual plane wave components, $g(\mathbf{n}_k)$ is the angular function, and the integral is over the solid angle element $d\Omega_k = \sin\theta_k d\theta_k d\phi_k$ with θ_k and ϕ_k being the polar and azimuthal angles at the direction of $\mathbf{n}_k$. The angular function $g(\mathbf{n}_k)$ carries the information of both the amplitude and phase of the wave components. It follows

$$\nabla\psi_i = \int i\mathbf{k}g(\mathbf{n}_k)\exp(i\mathbf{k}\cdot\mathbf{r})d\Omega_k. \tag{13}$$

Substituting Eqs. (12) and (13) into Eq. (11) leads to express the extinction power as a function of the scattering amplitude $A_s(\mathbf{n})$ and the angular function $g(\mathbf{n}_k)$. The expression contains double integrals: one over dA and the other over $d\Omega_k$. The interest is to express the extinction power in the angular space, like the incident wave in Eq. (12). For this purpose, exchanging the sequence of the two integrals over dA and $d\Omega_k$ gives

$$\sigma_{\text{ext}} = \int \text{Re}\left[g^\star(\mathbf{n}_k)e^{ikr}\left(\frac{1}{r}\int(1+\mathbf{n}_k\cdot\mathbf{n})A_s(\mathbf{n})e^{-ik\mathbf{n}_k\cdot\mathbf{n}r}dA\right)\right]d\Omega_k. \tag{14}$$

The next step is to simplify Eq. (14) by evaluating the inner integral over dA. Using the Jones' lemma for $kr \to \infty$:[10,11]

$$\frac{1}{r}\int G(\mathbf{n})e^{-ik\mathbf{n}_k\cdot\mathbf{n}r}dA = \frac{2\pi i}{k}\left[G(\mathbf{n}_k)e^{-ikr} - G(-\mathbf{n}_k)e^{ikr}\right], \tag{15}$$

with $G(\mathbf{n})$ being an arbitrary function of $\mathbf{n}$, and letting

$$G(\mathbf{n}) = (1+\mathbf{n}_k\cdot\mathbf{n})A_s(\mathbf{n}), \tag{16}$$

leads to

$$\frac{1}{r}\int(1+\mathbf{n}_k\cdot\mathbf{n})A_s(\mathbf{n})e^{-ik\mathbf{n}_k\cdot\mathbf{n}r}dA = \frac{4\pi i}{k}A_s(\mathbf{n}_k)e^{-ikr}. \tag{17}$$

Using Eq. (17) for Eq. (14) results in

$$\sigma_{\text{ext}} = \frac{4\pi}{k}\text{Im}\left[\int g^\star(\mathbf{n}_k)A_s(\mathbf{n}_k)d\Omega_k\right], \tag{18}$$

which is our formalism of generalized optical theorem relating the extinction cross section for an arbitrary field incidence to the *total* scattering amplitude at the forward direction of the plane wave component at the direction of $\mathbf{n}_k$ and to the angular function $g(\mathbf{n}_k)$ at that direction. When there is no absorption, the extinction cross section σ_{ext} should equal to the scattering cross section $\sigma_{\text{sca}} = \int|A_s(\mathbf{n}_k)|^2 d\Omega_k$.

4. Differential cross section

The formalism, Eq. (18), is termed as generalized optical theorem in the manner that, when the angular function $g(\mathbf{n}_k)$ survives at a particular direction of $\mathbf{n}_0$, it reduces to the optical theorem for a plane wave incidence in Eq. (2). In Eq. (18), the extinction of an arbitrary incident field can be regarded as an angular integral of extinction from individual plane wave components in the angular spectrum. The individual extinction replies on interference of the angular function of the incident plane wave component, $g(\mathbf{n}_k)$, with the *total* scattering amplitude in the forward direction of the plane wave component, $A_s(\mathbf{n}_k)$.

The generalized optical theorem, Eq. (18), then allows us to define a differential cross section of *extinction*,

$$\frac{d\sigma_{\text{ext}}}{d\Omega_k} = \frac{4\pi}{k}\,\text{Im}\left[g^\star(\mathbf{n}_k)A_s(\mathbf{n}_k)\right], \tag{19}$$

which is an analogue of the differential cross section of scattering, $d\sigma_{\text{sca}}/d\Omega_k = |A_s(\mathbf{n}_k)|^2$, that characterizes the scattering strength at the direction $\mathbf{n}_k$. The differential cross section of extinction Eq. (19) is likewise regarded as the *extinction* strength at the direction $\mathbf{n}_k$. The strength also relies on the phase between the complex conjugate of the angular function, $g^\star(\mathbf{n}_k)$, and the total scattering at that direction, $A_s(\mathbf{n}_k)$. While the scattering strength $d\sigma_{\text{sca}}/d\Omega_k$ is non-negative, it would be interesting to examine if the extinction strength $d\sigma_{\text{ext}}/d\Omega_k$ could be negative at certain directions in the three-dimensional spreading. Note that the differential extinction cross section is defined in the spectrum domain, but the differential scattering cross section is defined in the spatial domain.

5. Extinction and plane wave scattering

The aforementioned generalized optical theorem relates the extinction cross section to the *total* scattering at the forward direction of the individual plane wave component. One can further relate the extinction to the scattering of plane wave incidence.

Since the incident arbitrary field was expressed as the angular superposition of the plane wave [see Eq. (12)], correspondingly the scattering amplitude $A_s(\mathbf{n_k})$ at the direction $\mathbf{n}_k$ is an angular superposition of the plane wave scattering amplitude $A(\mathbf{n}_k, \mathbf{n}_0)$ incident from all directions, denoted by $\mathbf{n}_0$ [see Eq. (1)], to have

$$A_s(\mathbf{n}_k) = \int g(\mathbf{n}_0)A(\mathbf{n}_k,\mathbf{n}_0)d\Omega_0, \tag{20}$$

where the integral is over the solid angle element $d\Omega_0 = \sin\theta_0 d\theta_0 d\phi_0$ with θ_0 and ϕ_0 being the polar and azimuthal angles at the direction of $\mathbf{n}_0$. Using Eq. (20), we reform the optical theorem, Eq. (18), for relating to the plane wave scattering amplitude $A(\mathbf{n}_k, \mathbf{n}_0)$ as

$$\sigma_{\text{ext}} = \frac{4\pi}{k}\,\text{Im}\left[\int\int g^\star(\mathbf{n}_k)g(\mathbf{n}_0)A(\mathbf{n}_k,\mathbf{n}_0)d\Omega_k d\Omega_0\right]. \tag{21}$$

The theorem in the form of Eq. (21) illustrates the total extinction cross section for an arbitrarily shaped field, determined by the angular function $g(\mathbf{n})$ [see Eq. (12)], relies on the scattering at all directions in the plane wave incidence [see Eq. (1)]. This form of theorem can be useful to directly evaluate the extinction from plane wave scattering.

6. Summary

The extended extinction theorem presented herein relates the extinction cross section to the total scattering amplitude at the forward direction $\mathbf{n}_k$ of the plane wave components of the incident fields [Eq. (18)], where the amplitude is weighted by the complex conjugate of the angular function [Eq. (12)] which determines the phase and amplitude of the plane wave component. Importantly, a differential extinction cross section, as an analogue of the differential scattering cross section, is introduced [Eq. (19)]. The differential extinction cross section relies on the interference of the angular function and the total scattering amplitude in the forward direction of the plane wave components in the angular spectrum of the incident wave. An alternative form of optical theorem relates the extinction to the scattering amplitude of plane wave incidence [Eq. (21)]. These formalisms are applicable to any scalar waves including quantum fields. The theorem can reduce to the specific types of optical theorem for specific beams, for instance, non-diffracting Bessel beams.[4–6] The theorem can potentially be extended for situations of inhomogeneous media,[12,13] objects with particular symmetry,[14] integrated extinction over multiple frequencies,[15] or in the presence of boundaries in space or extra objects.[16]

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