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On the Qualitative Analysis of Volterra IDDEs with Infinite Delay

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Abstract

This investigation deals with a nonlinear Volterra integro-differential equation with infinite retardation (IDDE). We will prove three new results on the stability, uniformly stability (US) and square integrability (SI) of solutions of that IDDE. The proofs of theorems rely on the use of an appropriate Lyapunov-Krasovskii functional (LKF). By the outcomes of this paper, we generalize and obtain some former results in mathematical literature under weaker conditions.

Keywords: Nonlinear; Lyapunov-Krasovskii functional; Volterra IDDE; stability; Uniformly stability; Square integrability

MSC 2020 No.: 3K20, 45J05

1. Introduction

In the last 50 years, the studies on various qualitative properties of solutions of integral equations, integro-differential equations without and with retardations, impulsive differential equations and so on have attracted attention from numerous mathematicians, physicists and engineers (for instance, see the books or the papers of Becker (2006), Becker (2007), Becker (2009), Burton (1993), Burton (2005), Burton (2010), Burton and Haddock (2009), Burton and Mahfoud (1983), Burton and Mahfoud (1984), Burton and Mahfoud (1985), Chang and Wang (2011), Chen et al. (2017), Graef and Tunç (2015), Graef et al. (2016), Raffoul and Rai (2016), Slyn'ko and Tunç (2019), Tunç (2016a), Tunç (2016b), Tunç (2016c), Tunç (2017a), Tunç (2017b), Tunç (2017c), Tunç (2018), Tunç and Akbulut (2018), Tunç and Tunç (2018a), Tunç and Tunç (2018b), Tunç and Tunç (2018b), Tunç and Tunç (2018d), Tunç and Tunç (2019), Tunç (2020), Tunç and Golmankhaneh (2020) and the available bibliography in these sources).

It should be insisted that IDEs and IDDEs have proved to be valuable tools in modeling of many physical phenomena in various fields of science, medicine, engineering and new different branches raised under these areas. When we check the available mentioned literature on the qualitative structures of solutions of IDEs and IDDEs, it can be followed that different mathematical models of IDEs and IDDEs are considered and stability, uniformly stability, asymptotic stability, globally asymptotic stability, boundedness, square integrability of solutions of that IDEs and IDDEs are discussed by researchers, without solving that IDEs and IDDEs. Through the available bibliography of this paper, it is seen that the techniques or methods used in the proofs depend on the Lyapunov's second method, the Lyapunov-Krasovskii functional approach, fixed point method, inequality techniques and so on. All of these techniques determine and make available a proper decision related to the stability, uniformly stability, and asymptotic stability, globally asymptotic stability, boundedness, square integrability of solutions, etc., without analytically solving IDEs and IDDEs under discussion.

This fact raises a significant advantage in the course of researches on the mentioned concepts. The motivation of this paper has been inspired from the mentioned bibliography. The target of this work is to generalize and to get some former results in bibliography of this paper under weaker conditions. In fact, we generalize and obtain the results of Raffoul and Rai (2016) under weaker condition. Next, we try to do a contribution to do bibliography of this paper. Thus, it is worthwhile to investigate qualitative behaviors of solutions of nonlinear IDDEs with infinite retardation.

2. Stability

Raffoul and Rai (2016) considered the following nonlinear IDDE with unbounded delay:

$$\frac{dx}{dt} = Px(t) + \int_{-\infty}^{t} g(x(s))C(t,s)ds. \tag{1}$$

They presented some criteria for the US and SI of solutions of IDDE (1) by using a LKF.

In this paper, motivated by the results of Raffoul and Rai (2016), we have the following nonlinear IDDE with unbounded retardation:

$$\frac{dx}{dt} = -a(t)x + \int_{-\infty}^{t} C(t,s)g(s,x(s))ds, \tag{2}$$

where $t, x \in \Re$, $\Re = (-\infty, \infty)$, $a \in C(\Re, [0, \infty)$, $C \in C(\Re^2, \Re)$ and $g \in C(\Re, \Re)$ with $-\infty < s \leq t < \infty$, $g(s, 0) = 0$, are continuous functions. So IDDE (2) includes the zero solution $x(t) \equiv 0$.

When we compare IDDE (2) with IDDE (1) it follows that IDDE (2) includes IDDE (1). In fact, let $a(t) = P$, $P \in \Re$, and $g(t, x)$ be depend only on x. Then IDDE (2) reduces to IDDE (1).

Further, $\varphi \in C(\Re, \Re)$ is a continuous function. Let us show the norm of this function by

$$\sup_{s \in \Re} |\varphi(s)| = \|\varphi\|.$$

We have the initial segment $E_{t_0} = (-\infty, t_0]$. Then, it is assumed that $\psi : E_{t_0} \to \Re$ is an initial function, which is bounded and continuous.

Suppose that $x(t, t_0, \phi)$ is a solution of IDDE (2) having the initial function $\phi \in C((-\infty, t_0], \Re)$, $t_0 \geq 0$.

Then, IDDE (2) can be expressed as the following:

$$\frac{dx}{dt} = -a(t)x - A(t,t)g(t, x(t)) + \frac{d}{dt} \int_{-\infty}^{t} g(s, x(s))A(t, s)ds,$$

where the function $A(.)$ is defined by

$$A(t, s) = \int_{-\infty}^{t-s} C(u + s, s)du, \ t - s \geq 0.$$

2.1. Assumptions

We have the assumptions below through the paper.

(A1) Let $\lambda_1, \lambda_2 \in \Re$, $\lambda_1 > 0, \lambda_2 > 0$. Further, it is assumed that

$$g(t, 0) = 0, \ \lambda_2 x^2 \leq x g(t, x), x \neq 0, \forall t, x \in \Re,$$

$$|g(t, x)| \leq \lambda_1 |x|, \forall t, x \in \Re,$$

and

$$A(t, t) > 0, \forall t \in [0, \infty).$$

(A2) Let $\gamma, \rho \in \Re$, $\gamma > 0, \rho > 0$ with

$$a(t) + 2A(t,t)\lambda_2 - A^2(t,t)\lambda_1^2 - \gamma\lambda_1^2 \int_{t}^{\infty} |A(u, t)|du \geq \rho,$$

$$(a(t) + 1) \int_{-\infty}^{t} |A(t, s)|ds - \gamma \leq 0,$$

$$1-\lambda_1\int_{-\infty}^{t}|A(t,s)|ds>0,$$

and

$$\int_{0}^{\infty}|A(u,t)|du<\infty.$$

Theorem 2.1.

The null solution of IDDE (2) is stable if assumptions (A1) and (A2) hold:

Proof:

We construct a LKF $V(t)=V(t,x)$ by

$$V(t)=(x-\int_{-\infty}^{t}g(s,x(s))A(t,s)ds)^2+\gamma\int_{-\infty}^{t}\int_{t}^{\infty}|A(u,z)|g^2(z,x(z))dudz. \tag{3}$$

Note that $V(t,0)=0$ and $V(t,x)>0$ if $x\neq 0$. Now, the calculation of the derivative of the LKF (3) along IDDE (2) gives

$$\begin{aligned}V'(t)=&-2a(t)x^2-2A(t,t)xg(t,x(t))+2a(t)x\int_{-\infty}^{t}A(t,s)g(s,x(s))ds\\&+2A(t,t)xg(t,x(t))\int_{-\infty}^{t}A(t,s)g(s,x(s))ds+\gamma\int_{t}^{\infty}|A(u,t)|g^2(t,x(t))du\\&-\gamma\int_{-\infty}^{t}|A(t,z)|g^2(z,x(z))dz.\end{aligned}\tag{4}$$

By using the Schwarz inequality, we can observe

$$\begin{aligned}2a(t)x\int_{-\infty}^{t}A(t,s)g(s,x(s))ds&\le a(t)x^2+a(t)[\int_{-\infty}^{t}A(t,s)g(s,x(s))ds]^2\\&=a(t)x^2+a(t)[\int_{-\infty}^{t}|A(t,s)|^{\frac{1}{2}}|A(t,s)|^{\frac{1}{2}}g(s,x(s))ds]^2\\&\le a(t)x^2+a(t)\int_{-\infty}^{t}|A(t,s)|ds\int_{-\infty}^{t}|A(t,s)|g^2(s,x(s))ds.\end{aligned}\tag{5}$$

Similarly, we also get

$$\begin{aligned}2A(t,t)g(t,x(t))\int_{-\infty}^{t}A(t,s)g(s,x(s))ds&\le A^2(t,t)g^2(t,x(t))+\left[\int_{-\infty}^{t}A(t,s)g(s,x(s)ds\right]^2\\&\le\lambda_1^2A^2(t,t)x^2+\int_{-\infty}^{t}|A(t,s)|ds\int_{-\infty}^{t}|A(t,s)|g^2(s,x(s))ds.\end{aligned}\tag{6}$$

By putting inequalities (5) and (6) into (4) and using assumption (A2), we derive

$$\begin{aligned} V'(t) \le & -[a(t)+2A(t,t)\lambda_1^2 - A^2(t,t)\lambda_2 - \gamma\lambda_1^2 \int_t^\infty |A(u,t)|du]x^2 \\ & + \left[(a(t)+1)\int_{-\infty}^t |A(t,s)|ds - \gamma\right]\left[\int_{-\infty}^t |A(t,s)|g^2(s,x(s))ds\right] \\ \le & -\rho|x|^2. \end{aligned} \tag{7}$$

Let $\varepsilon > 0$. Then, by the definition of the stability, it is obvious that we can find a constant $\delta > 0$ so that $|x(t,t_0,\psi| < \varepsilon$, when $[\psi \in E_{t_0} \to \Re, \|\psi\| < \delta]$. Since $V'(t,x) \le 0$ in (7), then, this inequality implies that the functional V is decreasing for $t \ge t_0$. If we integrate $V'(t,x) \le 0$ and consider the LKF V given by (3), we observe

$$\begin{aligned} V(t,x(t)) &\le V(t_0,\psi(t_0)) \\ &= [\psi(t_0) - \int_{-\infty}^{t_0} A(t,s)g(s,\psi(s)]^2 + \gamma\int_{-\infty}^{t_0}\int_{t_0}^\infty A(u,z)g^2(z,\psi(z))dudz \\ &\le [|\psi(t_0)| + \lambda_1 \int_{-\infty}^t |A(t,s)|\psi(s)ds]^2 + \gamma\lambda_1^2 \int_{-\infty}^{t_0}\int_{t_0}^\infty |A(u,z)|\psi^2(z)dudz \\ &\le \delta^2([1+\lambda_1\int_{-\infty}^t |A(t,s)|ds]^2 + \gamma\lambda_1^2\int_{-\infty}^{t_0}\int_{t_0}^\infty |A(u,z)|dudz). \end{aligned}$$

Let

$$L^2 = (1+\lambda_1^2\int_{-\infty}^{t_0}|A(t_0,s)|ds)^2 + \lambda_1^2\gamma\int_{-\infty}^{t_0}\int_{t_0}^\infty |A(u,z)|dudz.$$

Then, we can get

$$V(t,x) \le \delta^2 L^2. \tag{8}$$

By the functional (3), we can observe

$$V(t) \ge (|x| - |\int_{-\infty}^t g(s,x(s))A(t,s)|ds)^2.$$

By these last two inequalities, we get

$$|x(t)| \le \delta L + \int_{-\infty}^t |g(s,x(s))||A(t,s)|ds.$$

Since $|x(t)| < \varepsilon$, then, by assumption (A1), we can derive that

$$|x(t)| < \delta L + \varepsilon\lambda_1 \int_{-\infty}^t |A(t,s)|ds, \forall t \ge t_0.$$

Hence, if we choose $\delta < \frac{\varepsilon}{L}(1-\lambda_1\int_{-\infty}^t |A(t,s)|ds)$, then, we can reach that

$$|x(t)| < \varepsilon.$$

Note that by assumption (A2), we get $1 - \lambda_1 \int_{-\infty}^{t} |A(t,s)|ds > 0$. Thus, the last inequality with respect to δ is valid. This finishes the proof of Theorem 2.1. ■

The second theorem presents sufficient conditions for the square integrability of solutions.

Theorem 2.2.

If assumptions (A1) and (A2) of Theorem 2.1 are satisfied, then, all of solutions of IDDE (2) are square integrable $|x(t)|^2 \in L_[t_0, \infty)$, $t_0 \in E_k$.

Proof:

We know that assumptions (A1) and (A2) imply that zero solution of IDDE (2) is stable. Then, from the definition of stability, we can choose the constant δ such that $|x(t, t_0, \psi)| < 1$. Since LKF V is decreasing, by means of (7) and (3), we can derive

$$\rho \int_{t_0}^{t} |x(s)|^2 ds \leq \rho \int_{t_0}^{t} |x(s)|^2 ds + V(t,x) \leq K.$$

Then, we can derive that

$$\int_{t_0}^{\infty} |x(s)|^2 ds \leq \rho^{-1} K < \infty \text{ as } t \to \infty,$$

where K is positive constant.

Then, the proof of Theorem 2.2 is completed. ■

Let $a(t) = 0$.

2.2. Assumption

We suppose the following assumptions hold through the paper.

(A3)

$$2A(t,t)\lambda_2 - A^2(t,t)\lambda_1^2 - \gamma\lambda_1^2 \int_{t}^{\infty} |A(u,t)|du \geq \rho,$$

$$\int_{-\infty}^{t} |A(t,s)|ds - \gamma \leq 0,$$

$$1 - \lambda_1 \int_{-\infty}^{t} |A(t,s)|ds > 0,$$

and

$$\int_0^\infty |A(u,t)|du < \infty.$$

Theorem 2.3.

If assumptions (A1) and (A3) are satisfied, then, the trivial solution of IDDE (2) is stable and $|x(t)|^2 \in L_[t_0, \infty)$, $t_0 \in E_k$,

Proof:

By using assumptions (A1) and (A3), we can easily finish the proof of Theorem 2.3. Therefore, we omit the details.

For simplicity, let

$$J = \int_{-\infty}^t |A(t,s)|ds. \tag{9}$$

■

Theorem 2.4.

In addition to assumptions (A1) and (A2), let

$$\int_{-\infty}^t \int_t^\infty |A(u,z)|dudz \le R, (R > 0, R \in \Re). \tag{10}$$

Then, the trivial solution of IDDE (2) is US.

Proof:

We benefit from functional V, which is defined in Theorem 2.1. Then, when we consider the conditions of Theorem 2.3 and use (9), it is followed that

$$\begin{aligned} V(t) =& x^2(t) + \int_{-\infty}^t A(t,s)g(s,x(s)ds)^2 \\ & - 2x(t)\int_{-\infty}^t A(t,s)g(s,x(s))ds \\ & + \gamma \int_{-\infty}^t \int_t^\infty |A(u,z)|g^2(z,x(z))dudz \\ \le& 2x^2(t) + 2\lambda_1^2 J \int_{-\infty}^t |A(t,s)|x^2(s)ds + \gamma\lambda_1^2 \int_{-\infty}^t \int_t^\infty |A(u,z)|x^2(z)dudz, \end{aligned} \tag{11}$$

by assumptions (A1) and (A2).

We know that the time derivative of V satisfies $\frac{dV}{dt} \le 0$. Since $V(t)$ is a decreasing functional, we can derive

$$V(t) \le V(t_0), \forall t \ge t_0.$$

Hence, since $x(t) = \phi(t)$ on $E_{t_0}(-\infty, t_0)$, it can be derived that

$$\begin{aligned}\left(x-\int_{-\infty}^{t_0} A(t,s)g(s,x(s))ds\right)^2 &\leq V(t) \leq V(t_0)\\ &= \phi^2(t) + \left(\int_{-\infty}^{t_0} A(t,s)g(s,\phi(s))ds\right)^2 - 2\phi(t)\int_{-\infty}^{t_0} A(t,s)g(s,\phi(s))ds\\ &+ \gamma\int_{-\infty}^{t_0}\int_{t_0}^{\infty} |A(u,z)|g^2(z,\phi(z))dudz.\end{aligned}$$

Then, in view of $\|\phi(t)\| < \delta$ and assumption (A1), it follows that

$$\begin{aligned}&\left(x-\int_{-\infty}^{t_0} A(t,s)g(s,x(s))ds\right)^2\\ &\qquad\leq \delta^2 + \left(\int_{-\infty}^{t_0} |A(t,s)|ds\right)^2\lambda_1^2\delta^2 + \delta^2 + \left(\int_{-\infty}^{t_0} |A(t,s)|ds\right)^2\lambda_1^2\delta^2\\ &\qquad+ \gamma\lambda_1^2 + \delta^2\int_{-\infty}^{t_0}\int_{t_0}^{\infty} |A(u,z)|dudz\\ &\qquad= 2\delta^2 + 2\lambda_1^2\delta^2\left(\int_{-\infty}^{t_0} |A(t,s)|ds\right)^2 + \gamma\delta^2\lambda_1^2\int_{-\infty}^{t_0}\int_{t_0}^{\infty} |A(u,z)|dudz\\ &\qquad= \delta^2(2 + 2\lambda_1^2 J + \gamma\lambda_1^2 R).\end{aligned}$$

Given an $\varepsilon > 0$ and a fixed $t_0 \in E_k$. Let $\delta > 0$, $0 < \delta < \varepsilon$, such that

$$\sqrt{(2 + 2\lambda_1^2 J + \gamma\lambda_1^2 R)}\,\delta < \varepsilon(1 - \lambda_1 J).$$

Also, we note that

$$|x| - \lambda_1\int_{-\infty}^{t} |A(t,s)||x(s)|ds \leq |x - \int_{-\infty}^{t} A(t,s)g(s,x(s))ds|.$$

Hence, we assert that $|x(t)| < \varepsilon$, $\forall t \geq t_0$. We know that $|x(u)| < \delta < \varepsilon$, $\forall u \in (-\infty, t_0]$. If this claim is not correct, hence, let $t = t^*$ be such that $|x(t^*)| = \varepsilon$ and $|x(s)| < \varepsilon$ for $t_0 \leq s < t^*$. Then, by above information, we can get

$$\begin{aligned}\varepsilon(1 - \lambda_1 J) &= \varepsilon(1 - \lambda_1\int_{-\infty}^{t} |A(t,s)|ds)\\ &\leq |x(t^*) - \lambda_1\int_{-\infty}^{t^*} A(t^*,s)g(s,x(s))ds|\\ &\leq \sqrt{(2 + 2\lambda_1^2 J + \gamma\lambda_1^2 R)}\,\delta.\end{aligned}$$

This result is a contradiction, that is, the above claim is not true. This outcome finishes the proof of Theorem 2.4. ∎

Example 2.5.

We have the following nonlinear IDDE, which is a modified equation of Example 3.1 in Raffoul and Rai (2016):

$$\frac{dx}{dt} = -(4 + \frac{1}{1+t^2})x - \int_{-\infty}^{t} 16^{-1}(t-s+1)^{-4}x(s)\left(\frac{\sin^2 x(s) + \sin^2 s + 1}{6}\right) ds. \tag{12}$$

When we compare this equation with IDDE (2), we can derive the following expressions:

$$a(t) = 4 + \frac{1}{1+t^2},$$

$$g(t,x) = \frac{x(\sin^2 x + \sin^2 t + 1)}{6},$$

$$xg(t,x) > \frac{x^2}{6}, x \neq 0,$$

$$\lambda_1 = \frac{1}{2}, \lambda_2 = \frac{1}{6},$$

$$|g(t,x)| \leq \frac{1}{2}|x|,$$

$$C(t,s) = -16^{-1}(t-s+1)^{-4},$$

$$C(u+s,s) = -16^{-1}(u+1)^{-4},$$

$$A(t,s) = \int_{-\infty}^{t-s} C(u+s,s)ds = 4^{-1}(t-s+1)^{-3}.$$

The last equality satisfies that $A(t,t) = \frac{1}{4}$. It is also clear that

$$\int_{t}^{\infty} |A(u,t)|du = \frac{3}{4}, \int_{-\infty}^{t} |A(t,s)|ds = \frac{3}{4},$$

and

$$\begin{aligned}\int_{-\infty}^{t}\int_{t}^{\infty} |A(u,z)|dudz &= \int_{-\infty}^{t}\int_{t}^{\infty} 4^{-1}(u-z+1)^{-3}dudz \\ &= \frac{3}{4}\int_{-\infty}^{t} (t-z+1)^{-2}dz \\ &= \frac{3}{2}.\end{aligned}$$

We can also conclude that

$$\begin{aligned}\mu(t) &= a(t) + 2A(t,t)\lambda_2 - A^2(t,t)\lambda_1^2 - \gamma\lambda_1^2 \int_t^{\infty} |A(u,t)|du \\ &= 4 + \frac{1}{1+t^2} + \frac{1}{12} - \frac{1}{64} - \gamma\frac{3}{16} \\ &\geq 4 + \frac{1}{12} - \frac{1}{64} - \gamma\frac{3}{16} \\ &= \frac{781-36\gamma}{192}.\end{aligned}$$

Let $\gamma = 3$. Then, we get

$$\mu(t) = \frac{781-108}{192} = \frac{673}{192} = \rho.$$

Thus, we can verify that the trivial solution IDDE (12) is US.

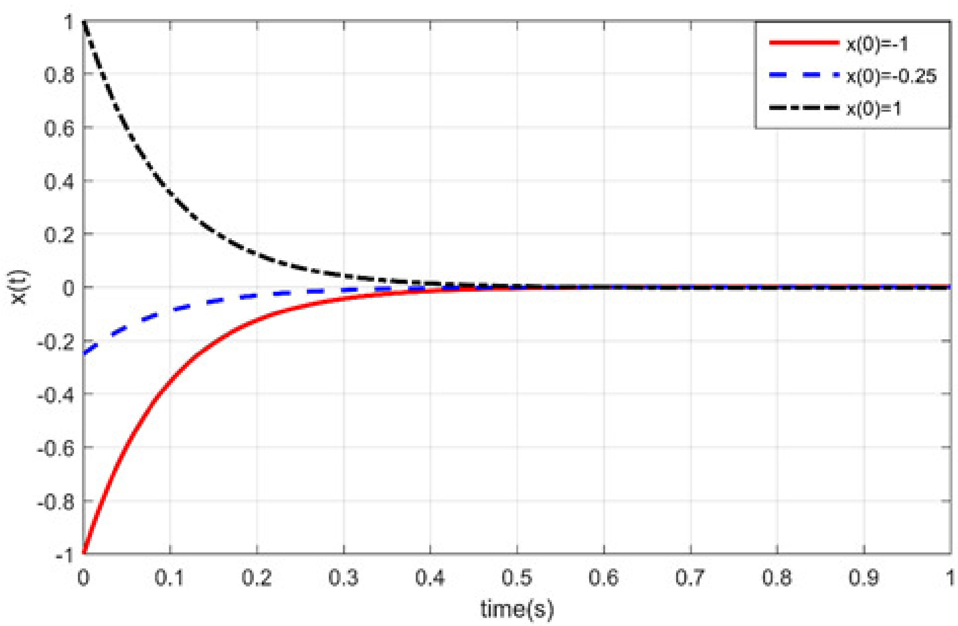

Figure 1. Trajectory of solution $x(t)$ of Equation (12) in Example 1

3. Conclusion

In this paper, the authors investigate a nonlinear Volterra integro-differential equation with infinite retardation. They discuss certain qualitative aspects of solutions of that nonlinear Volterra IDDE such as stability, square integrability and uniform stability of solutions. Three new theorems are presented on stability, square integrability and uniform stability of solutions. The constructed hypotheses through the theorems are recognized as sufficient conditions and they guarantee the mentioned qualitative properties of solutions of that Volterra IDDE with infinite delay. The technique used to proceed the proofs of the main results is known as the Lyapunov-Krasovskii functional approach. For this approach, a new Lyapunov-Krasovskii functional is constructed. In particular case, an example of IDDEs with the plots of the paths of solutions is given to verify and show the applicability of the given results. It can be observed that the obtained results extend, include and improve some results can be found in the bibliography of this paper. Finally, our findings have contributions to the qualitative theory of integral and integro-differential equations.

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