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Twin edge coloring of total graph and graphs with twin chromatic index $\Delta + 2$

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Abstract

A twin edge coloring of a graph G is meant a proper edge coloring of G whose colors come from the integers modulo k that induce a proper vertex coloring in which the color of a vertex is the sum of the colors of its incident edges. The minimum k for which G has a twin edge coloring is the twin chromatic index of G. In this paper, I compute twin chromatic index of total graph of path and cycle also construct some special graphs with twin chromatic index is maximum degree plus two.

Keywords: Edge coloring; Vertex coloring; Twin chromatic index; Path; Cycle; Total graph; Regular graph

MSC 2010 No.: 05C15,05C38

1. Introduction

Let G be an undirected finite simple graph. Edge colorings have appeared in a variety of contexts in graph theory. For more than a quarter century, edge colorings have been studied that induce vertex colorings in some manner. Andrews et al. (2014, 2015) obtained the twin chromatic indexes of path, complete graph, cycle, complete bipartite graph, Petersen graph, grids, prisms, trees with small maximum degree and propose twin edge coloring conjecture. Johnston et al. (2014) obtained an upper bound for the twin chromatic index of a graph. Rajarajachozhan et al. (2016) obtained twin edge colorings of certain square graphs and product graphs.

A *proper vertex-coloring* of G is an assignment from a given set of colors to the set of vertices of $G,$ where adjacent vertices are colored differently. The minimum number of colors needed in a proper vertex-coloring of G is the *chromatic number* of $G.$ A *proper edge-coloring* of G is an assignment from a given set of colors to the set of edges of $G,$ where adjacent edges are colored differently. The minimum number of colors needed in a proper edge-coloring of G is the *chromatic index* of G.

For a connected graph G of order at least 3, let c be a function from edges of G to integers modulo k be a proper edge k-coloring of G. A vertex k-coloring be a function from vertex of G to integers modulo k is defined by the set of edges of G incident with a vertex v and the indicated sum is computed in integers modulo k. If the induced vertex k-coloring is proper, then, c is called a *twin edge k-coloring* of G. The minimum k for which G has a twin edge k-coloring is called the *twin chromatic index* of G. Since a twin edge coloring is not only a proper edge-coloring of G but induces a proper vertex-coloring of G.

2. Preliminaries

In this paper, the following notations will be used:

The vertex set of G is denoted by V(G).
The edge set of G is denoted by E(G).
The total graph of G is denoted by $T(G)$.
The maximum degree of G is denoted by $\Delta(G)$ or Δ.
The neighborhood of u is denoted by $N_G(u)$ or $N(u)$.
The chromatic number of G and it is denoted by $\chi(G)$.
The chromatic index of G and it is denoted by $\chi\ '(G)$.
The twin chromatic index of G and it is denoted by $\chi\ '_t(G)$ or $\chi\ '_t$.

Conjecture 2.1. [Andrews et al. (2014)]

If G is a connected graph of order at least 3 that is not a 5-cycle, then, $\chi\ '_t(G) \leq 2 + \Delta(G)$.

The *total graph* of a graph G, is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices are adjacent whenever they are either adjacent or incident in G.

Let

$$V\big(T(P_m)\big) = \{v_1, v_2, \dots, v_m\} \cup \{u_1, u_2, \dots, u_{m-1}\}.$$
$$E(T(P_m)) = \{v_i v_{i+1}: i \in \{1,2,\dots, m-1\}\} \cup \{u_i u_{i+1}: i \in \{1,2,\dots, m-2\}\} \cup \{v_i u_i: i \in \{1,2,\dots, m-1\}\} \cup \{u_i v_{i+1}: i \in \{1,2,\dots, m-1\}\}.$$
$$V\big(T(C_m)\big) = \{v_1, v_2, \dots, v_m\} \cup \{u_1, u_2, \dots, u_m\}.$$
$$E(T(C_m)) = \{v_i v_{i+1}: i \in \{1,2,\dots, m\}\} \cup \{u_i u_{i+1}: i \in \{1,2,\dots, m\}\} \cup \big\{v_i u_i: i \in \{1,2,\dots, m\}\big\} \cup \big\{u_i v_{i+1}: i \in \{1,2,\dots, m\}\big\},$$

where

$$v_{m+1} = v_1 \text{ and } u_{m+1} = u_1.$$

3. Twin chromatic index of total graph

Proposition 3.1.

$$\chi\ '_t(T(P_2)) = 3.$$

Proof:

Since $T(P_2) \cong K_3$ and $\chi\ '_t(K_3) = 3$. Hence, $\chi\ '_t(T(P_2)) = 3$. ■

Proposition 3.2.

$$\chi\ '_t(T(P_3)) = 4.$$

Proof:

Define $c: E(T(P_3)) \to \mathbb{Z}_4$ as follows:

$$c(v_1v_2) = 0,\ c(v_2v_3) = 1,\ c(v_1u_1) = 1,$$
$$c(u_1v_2) = 2,\ c(v_2u_2) = 3,$$
$$c(u_2v_3) = 2, \text{ and } c(u_1u_2) = 0.$$

The induced vertex-coloring are: $\sigma_c(v_1) = 1$, $\sigma_c(v_2) = 2$, $\sigma_c(v_3) = 3$, $\sigma_c(u_1) = 3$, and $\sigma_c(u_1) = 1$. Since σ_c is a proper vertex-coloring. Thus, c is a twin edge 4-coloring of $T(P_3)$. Hence, $\chi\ '_t(T(P_3)) = 4$. ■

Theorem 3.1.

If $n \geq 4$, then, $\chi\ '_t(T(P_n)) = 5$.

Proof:

Case 1. $n \equiv 0 \pmod 6$.

Define $c: E(T(P_n)) \to \mathbb{Z}_5$ as follows:

For $i \in \{1,2,\dots,n-2\}$,

$$c(v_iv_{i+1}) = \begin{cases} 0, & \text{if } i \text{ is odd,} \\ 1, & \text{if } i \text{ is even.} \end{cases} \tag{3.1}$$

$$c(v_{n-1}v_n) = 4. \tag{3.2}$$

For $i \in \{1,2,\dots,n-2\}$,

$$c(u_iu_{i+1}) = \begin{cases} 0, & \text{if } i \text{ is odd,} \\ 1, & \text{if } i \text{ is even.} \end{cases} \tag{3.3}$$

For $i \in \{1,2,\dots,n-2\}$,

$$c(v_iu_i) = \begin{cases} 3, & \text{if } i \equiv 0 \pmod 3, \\ 2, & \text{if } i \equiv 1 \pmod 3, \\ 4, & \text{if } i \equiv 2 \pmod 3. \end{cases} \tag{3.4}$$

$$c(v_{n-1}u_{n-1}) = 2. \tag{3.5}$$

For $i \in \{1,2,\dots,n-2\}$,

$$c(u_i v_{i+1}) = \begin{cases} 4, & \text{if } i \equiv 0 \pmod{3}, \\ 3, & \text{if } i \equiv 1 \pmod{3}, \\ 2, & \text{if } i \equiv 2 \pmod{3}. \end{cases} \tag{3.6}$$

$$c(u_{n-1} v_n) = 0. \tag{3.7}$$

From equation (3.1) to (3.7), by construction, c is a proper edge-coloring.

The induced vertex-coloring are:

$$\sigma_c(v_1) = 2. \tag{3.8}$$

For $i \in \{2,3,4, \dots, n-2\}$,

$$\sigma_c(v_i) = \begin{cases} 1, & \text{if } i \equiv 0 \pmod{3}, \\ 2, & \text{if } i \equiv 1 \pmod{3}, \\ 3, & \text{if } i \equiv 2 \pmod{3}. \end{cases} \tag{3.9}$$

$$\sigma_c(v_{n-1}) = 0, \text{ and } \sigma_c(v_n) = 4. \tag{3.10}$$

$$\sigma_c(u_1) = 0. \tag{3.11}$$

For $i \in \{2,3,4, \dots, n-2\}$,

$$\sigma_c(u_i) = \begin{cases} 3, & \text{if } i \equiv 0 \pmod{3}, \\ 1, & \text{if } i \equiv 1 \pmod{3}, \\ 2, & \text{if } i \equiv 2 \pmod{3}. \end{cases} \tag{3.12}$$

$$\sigma_c(u_{n-1}) = 3. \tag{3.13}$$

Observe equation (3.8) to (3.13), σ_c is a proper vertex-coloring. Thus, c is a twin edge 5-coloring of $T(P_n)$.

Case 2. $n \equiv 1 \pmod{6}$.

Define $c: E(T(P_n)) \to \mathbb{Z}_5$ as follows:

For $i \in \{1,2, \dots, n-1\}$,

$$c(v_i v_{i+1}) = \begin{cases} 0, & \text{if } i \text{ is odd}, \\ 1, & \text{if } i \text{ is even}. \end{cases} \tag{3.14}$$

For $i \in \{1,2, \dots, n-2\}$,

$$c(u_i u_{i+1}) = \begin{cases} 0, & \text{if } i \text{ is odd}, \\ 1, & \text{if } i \text{ is even}. \end{cases} \tag{3.15}$$

For $i \in \{1,2, \dots, n-1\}$,

$$c(v_i u_i) = \begin{cases} 3, & \text{if } i \equiv 0 \pmod{3}, \\ 2, & \text{if } i \equiv 1 \pmod{3}, \\ 4, & \text{if } i \equiv 2 \pmod{3}. \end{cases} \tag{3.16}$$

For $i \in \{1,2, \dots, n-2\}$,

$$c(u_iv_{i+1}) = \begin{cases} 4, & \text{if } i \equiv 0 \pmod 3), \\ 3, & \text{if } i \equiv 1 \pmod 3), \\ 2, & \text{if } i \equiv 2 \pmod 3). \end{cases} \tag{3.17}$$

$$c(u_{n-1}v_n) = 2. \tag{3.18}$$

From equation (3.14) to (3.18), by construction, c is a proper edge-coloring.

The induced vertex-coloring are:

$$\sigma_c(v_1) = 2. \tag{3.19}$$

For $i \in \{2,3,4,\dots,n-1\}$,

$$\sigma_c(v_i) = \begin{cases} 1, & \text{if } i \equiv 0 \pmod 3), \\ 2, & \text{if } i \equiv 1 \pmod 3), \\ 3, & \text{if } i \equiv 2 \pmod 3). \end{cases} \tag{3.20}$$

$$\sigma_c(v_n) = 3. \tag{3.21}$$

$$\sigma_c(u_1) = 0. \tag{3.22}$$

For $i \in \{2,3,4,\dots,n-2\}$,

$$\sigma_c(u_i) = \begin{cases} 3, & \text{if } i \equiv 0 \pmod 3), \\ 1, & \text{if } i \equiv 1 \pmod 3), \\ 2, & \text{if } i \equiv 2 \pmod 3). \end{cases} \tag{3.23}$$

$$\sigma_c(u_{n-1}) = 0. \tag{3.24}$$

Observe equation (3.19) to (3.24), σ_c is a proper vertex-coloring. Thus, c is a twin edge 5-coloring of $T(P_n)$.

Case 3. $n \equiv 2, 4, 5 \pmod 6)$.

Define $c: E(T(P_n)) \to \mathbb{Z}_5$ as follows:

For $i \in \{1,2,\dots,n-1\}$,

$$c(v_iv_{i+1}) = \begin{cases} 0, & \text{if } i \text{ is odd}, \\ 1, & \text{if } i \text{ is even}. \end{cases} \tag{3.25}$$

For $i \in \{1,2,\dots,n-2\}$,

$$c(u_iu_{i+1}) = \begin{cases} 0, & \text{if } i \text{ is odd}, \\ 1, & \text{if } i \text{ is even}. \end{cases} \tag{3.26}$$

For $i \in \{1,2,\dots,n-1\}$,

$$c(v_iu_i) = \begin{cases} 3, & \text{if } i \equiv 0 \pmod 3), \\ 2, & \text{if } i \equiv 1 \pmod 3), \\ 4, & \text{if } i \equiv 2 \pmod 3). \end{cases} \tag{3.27}$$

For $i \in \{1,2,\dots,n-1\}$,

$$c(u_i v_{i+1}) = \begin{cases} 4, & \text{if } i \equiv 0 \pmod{3}, \\ 3, & \text{if } i \equiv 1 \pmod{3}, \\ 2, & \text{if } i \equiv 2 \pmod{3}. \end{cases} \quad (3.28)$$

See equation (3.25) to (3.28), by construction, c is a proper edge-coloring.

The induced vertex-coloring are:

$$\sigma_c(v_1) = 2. \quad (3.29)$$

For $i \in \{2,3,4,\dots,n-1\}$,

$$\sigma_c(v_i) = \begin{cases} 1, & \text{if } i \equiv 0 \pmod{3}, \\ 2, & \text{if } i \equiv 1 \pmod{3}, \\ 3, & \text{if } i \equiv 2 \pmod{3}. \end{cases} \quad (3.30)$$

$$\sigma_c(v_n) = \begin{cases} 3, & \text{if } n \equiv 2 \pmod{6}, \\ 4, & \text{if } n \equiv 4 \pmod{6} \text{ or } n \equiv 5 \pmod{6}. \end{cases} \quad (3.31)$$

$$\sigma_c(u_1) = 0. \quad (3.32)$$

For $i \in \{2,3,4,\dots,n-2\}$,

$$\sigma_c(u_i) = \begin{cases} 3, & \text{if } i \equiv 0 \pmod{3}, \\ 1, & \text{if } i \equiv 1 \pmod{3}, \\ 2, & \text{if } i \equiv 2 \pmod{3}; \end{cases} \quad (3.33)$$

$$\sigma_c(u_{n-1}) = \begin{cases} 1, & \text{if } n \equiv 1 \pmod{6}, \\ 3, & \text{if } n \equiv 3 \pmod{6}, \\ 0, & \text{if } n \equiv 4 \pmod{6}. \end{cases} \quad (3.34)$$

Observe equation (3.29) to (3.34), σ_c is a proper vertex-coloring. Thus, c is a twin edge 5-coloring of $T(P_n)$.

Case 4. $n \equiv 3 \pmod{6}$.

Define $c: E(T(P_n)) \to \mathbb{Z}_5$ as follows:

For $i \in \{1,2,\dots,n-1\}$,

$$c(v_i v_{i+1}) = \begin{cases} 0, & \text{if } i \text{ is odd}, \\ 1, & \text{if } i \text{ is even}. \end{cases} \quad (3.35)$$

For $i \in \{1,2,\dots,n-2\}$,

$$c(u_i u_{i+1}) = \begin{cases} 0, & \text{if } i \text{ is odd}, \\ 1, & \text{if } i \text{ is even}. \end{cases} \quad (3.36)$$

For $i \in \{1,2,\dots,n-1\}$,

$$c(v_i u_i) = \begin{cases} 3, & \text{if } i \equiv 0 \pmod{3}, \\ 2, & \text{if } i \equiv 1 \pmod{3}, \\ 4, & \text{if } i \equiv 2 \pmod{3}. \end{cases} \quad (3.37)$$

For $i \in \{1,2,\dots,n-2\}$,

$$c(u_i v_{i+1}) = \begin{cases} 4, & \text{if } i \equiv 0 \pmod 3, \\ 3, & \text{if } i \equiv 1 \pmod 3, \\ 2, & \text{if } i \equiv 2 \pmod 3. \end{cases} \quad (3.38)$$

$$c(u_{n-1} v_n) = 3. \quad (3.39)$$

See equation (3.35) to (3.39), by construction, c is a proper edge-coloring.

The induced vertex-coloring are:

$$\sigma_c(v_1) = 2. \quad (3.40)$$

For $i \in \{2,3,4,\dots,n-1\}$,

$$\sigma_c(v_i) = \begin{cases} 1, & \text{if } i \equiv 0 \pmod 3, \\ 2, & \text{if } i \equiv 1 \pmod 3, \\ 3, & \text{if } i \equiv 2 \pmod 3. \end{cases} \quad (3.41)$$

$$\sigma_c(v_n) = 4. \quad (3.42)$$

$$\sigma_c(u_1) = 0. \quad (3.43)$$

For $i \in \{2,3,4,\dots,n-2\}$,

$$\sigma_c(u_i) = \begin{cases} 3, & \text{if } i \equiv 0 \pmod 3, \\ 1, & \text{if } i \equiv 1 \pmod 3, \\ 2, & \text{if } i \equiv 2 \pmod 3. \end{cases} \quad (3.44)$$

$$\sigma_c(u_{n-1}) = 2. \quad (3.45)$$

Observe equation (3.40) to (3.45), σ_c is a proper vertex-coloring. Thus, c is a twin edge 5-coloring of $T(P_n)$. Hence, $\chi\ '_t(T(P_n)) = 5$.
■

Theorem 3.2.

If $n \geq 1$, *then,* $\chi\ '_t(T(C_{3n})) = 5$.

Proof:

Define $c: E(T(C_{3n})) \to \mathbb{Z}_5$ as follows.

For $i \in \{1,2,\dots,3n\}$,

$$c(v_i v_{i+1}) = \begin{cases} 2, & \text{if } i \equiv 0 \pmod 3, \\ 0, & \text{if } i \equiv 1 \pmod 3, \\ 1, & \text{if } i \equiv 2 \pmod 3. \end{cases} \quad (3.46)$$

$$c(u_i u_{i+1}) = \begin{cases} 1, & \text{if } i \equiv 0 \pmod 3, \\ 2, & \text{if } i \equiv 1 \pmod 3, \\ 0, & \text{if } i \equiv 2 \pmod 3. \end{cases} \tag{3.47}$$

$$c(v_i u_i) = 3, \text{ and } c(u_i v_{i+1}) = 4. \tag{3.48}$$

See equation (3.46) to (3.48), by construction, c is a proper edge-coloring.

The induced vertex-coloring are.

For $i \in \{1,2,\dots,3n\}$,

$$\sigma_c(v_i) = \begin{cases} 0, & \text{if } i \equiv 0 \pmod 3, \\ 4, & \text{if } i \equiv 1 \pmod 3, \\ 3, & \text{if } i \equiv 2 \pmod 3. \end{cases} \tag{3.49}$$

$$\sigma_c(u_i) = \begin{cases} 3, & \text{if } i \equiv 0 \pmod 3, \\ 0, & \text{if } i \equiv 1 \pmod 3, \\ 4, & \text{if } i \equiv 2 \pmod 3. \end{cases} \tag{3.50}$$

Observe equation (3.49) and (3.50), σ_c is a proper vertex-coloring. Thus, c is a twin edge 5-coloring of $T(C_{3n})$. Hence, $\chi\ '_t(T(C_{3n})) = 5$. ■

Theorem 3.3.

If $n \geq 1$, *then,* $\chi\ '_t(T(C_{4n})) = 5$.

Proof:

Define $c: E(T(C_{4n})) \to \mathbb{Z}_5$ as follows:

For $i \in \{1,2,\dots,4n\}$,

$$c(v_i v_{i+1}) = \begin{cases} 3, & \text{if } i \equiv 0 \pmod 4, \\ 0, & \text{if } i \equiv 1 \pmod 4, \\ 1, & \text{if } i \equiv 2 \pmod 4, \\ 2, & \text{if } i \equiv 3 \pmod 4. \end{cases} \tag{3.51}$$

$$c(u_i u_{i+1}) = \begin{cases} 2, & \text{if } i \equiv 0 \pmod 4, \\ 3, & \text{if } i \equiv 1 \pmod 4, \\ 0, & \text{if } i \equiv 2 \pmod 4, \\ 1, & \text{if } i \equiv 3 \pmod 4. \end{cases} \tag{3.52}$$

$$c(v_i u_i) = \begin{cases} 0, & \text{if } i \equiv 0 \pmod 4, \\ 1, & \text{if } i \equiv 1 \pmod 4, \\ 2, & \text{if } i \equiv 2 \pmod 4, \\ 3, & \text{if } i \equiv 3 \pmod 4. \end{cases} \tag{3.53}$$

$$c(u_i v_{i+1}) = 4. \tag{3.54}$$

See equation (3.51) to (3.54), by construction, c is a proper edge-coloring.

The induced vertex-coloring are.

For $i \in \{1,2,\dots,4n\}$,

$$\sigma_c(v_i) = \begin{cases} 4, & \text{if } i \equiv 0 \pmod 4, \\ 3, & \text{if } i \equiv 1 \pmod 4, \\ 2, & \text{if } i \equiv 2 \pmod 4, \\ 0, & \text{if } i \equiv 3 \pmod 4. \end{cases} \tag{3.55}$$

$$\sigma_c(u_i) = \begin{cases} 2, & \text{if } i \equiv 0 \pmod 4, \\ 0, & \text{if } i \equiv 1 \pmod 4, \\ 4, & \text{if } i \equiv 2 \pmod 4, \\ 3, & \text{if } i \equiv 3 \pmod 4. \end{cases} \tag{3.56}$$

Observe equation (3.55) and (3.56), σ_c is a proper vertex-coloring. Thus, c is a twin edge 5-coloring of $T(C_{4n})$. Hence, $\chi\ '_t(T(C_{4n})) = 5$. ■

Theorem 3.4.

If $n \geq 1$, *then*, $\chi\ '_t(T(C_{5n})) = 5$.

Proof:

Define $c: E(T(C_{5n})) \to \mathbb{Z}_5$ as follows.

For $i \in \{1,2,\dots,5n\}$,

$$c(v_i v_{i+1}) = \begin{cases} 4, & \text{if } i \equiv 0 \pmod 5, \\ 0, & \text{if } i \equiv 1 \pmod 5, \\ 1, & \text{if } i \equiv 2 \pmod 5, \\ 2, & \text{if } i \equiv 3 \pmod 5, \\ 3, & \text{if } i \equiv 4 \pmod 5. \end{cases} \tag{3.57}$$

$$c(u_i u_{i+1}) = \begin{cases} 2, & \text{if } i \equiv 0 \pmod 5, \\ 3, & \text{if } i \equiv 1 \pmod 5, \\ 4, & \text{if } i \equiv 2 \pmod 5, \\ 0, & \text{if } i \equiv 3 \pmod 5, \\ 1, & \text{if } i \equiv 4 \pmod 5. \end{cases} \tag{3.58}$$

$$c(v_i u_i) = \begin{cases} 0, & \text{if } i \equiv 0 \pmod 5, \\ 1, & \text{if } i \equiv 1 \pmod 5, \\ 2, & \text{if } i \equiv 2 \pmod 5, \\ 3, & \text{if } i \equiv 3 \pmod 5, \\ 4, & \text{if } i \equiv 4 \pmod 5. \end{cases} \tag{3.59}$$

$$c(u_i v_{i+1}) = \begin{cases} 3, & \text{if } i \equiv 0 \pmod 5, \\ 4, & \text{if } i \equiv 1 \pmod 5, \\ 0, & \text{if } i \equiv 2 \pmod 5, \\ 1, & \text{if } i \equiv 3 \pmod 5, \\ 2, & \text{if } i \equiv 4 \pmod 5. \end{cases} \tag{3.60}$$

See equation (3.57) to (3.60), by construction, c is a proper edge-coloring.

The induced vertex-coloring are.

For $i \in \{1,2,\dots,5n\}$,

$$\sigma_c(v_i) = \begin{cases} 4, & \text{if } i \equiv 0 \pmod 5, \\ 3, & \text{if } i \equiv 1 \pmod 5, \\ 2, & \text{if } i \equiv 2 \pmod 5, \\ 1, & \text{if } i \equiv 3 \pmod 5, \\ 0, & \text{if } i \equiv 4 \pmod 5. \end{cases} \tag{3.61}$$

$$\sigma_c(u_i) = \begin{cases} 1, & \text{if } i \equiv 0 \pmod 5, \\ 0, & \text{if } i \equiv 1 \pmod 5, \\ 4, & \text{if } i \equiv 2 \pmod 5, \\ 3, & \text{if } i \equiv 3 \pmod 5, \\ 2, & \text{if } i \equiv 4 \pmod 5. \end{cases} \tag{3.62}$$

Observe equation (3.61) and (3.62), σ_c is a proper vertex-coloring. Thus, c is a twin edge 5-coloring of $T(C_{5n})$. Hence, $\chi'_t(T(C_{5n})) = 5$. ■

Conjecture 3.1.

For any $n \geq 3$, $\chi'_t\big(T(C_n)\big) = 5$.

4. Twin chromatic index of total graph with nowhere-zero coloring

A twin edge coloring c of a graph G is a *nowhere-zero coloring* if $c(e) \neq 0$ for each edge e of G.

Proposition 4.1.

The nowhere-zero twin chromatic index of $T(P_2)$ *is 4.*

Proof:

Define $c: E(T(P_2)) \to \mathbb{Z}_4 - \{0\}$ as follows: $c(v_1v_2) = 1$, $c(v_1u_1) = 2$, and $c(u_1v_2) = 3$. The induced vertex-coloring are: $\sigma_c(v_1) = 3$, $\sigma_c(v_2) = 0$, and $\sigma_c(u_1) = 1$. Since σ_c is a proper vertex-coloring. Thus, c is a nowhere-zero twin edge 4-coloring $T(P_2)$. Hence, $\chi'_t(T(P_2)) = 4$.
■

Proposition 4.2.

The nowhere-zero twin chromatic index of $T(P_3)$ is 6.

Proof:

Define $c: E(T(P_3)) \to \mathbb{Z}_6 - \{0\}$ as follows: $c(v_1v_2) = 1$, $c(u_1u_2) = 1$, $c(v_2v_3) = 2$, $c(v_1u_1) = 2$, $c(u_1v_2) = 3$, $c(v_2u_2) = 4$, and $c(u_2v_3) = 5$. The induced vertex-coloring are: $\sigma_c(v_1) = 3$, $\sigma_c(v_2) = 4$, $\sigma_c(v_3) = 1$, $\sigma_c(u_1) = 0$, and $\sigma_c(u_1) = 4$. Since σ_c is a proper vertex-coloring. Thus, c is a nowhere-zero twin edge 6-coloring $T(P_3)$. Therefore, $\chi\ '_t(T(P_3)) \leq 6$.

Since $\Delta(T(P_3)) = 4$, then, we prove that $\chi\ '_t(T(P_3)) \neq 5$. Suppose $\chi\ '_t(T(P_3)) = 5$, then, assume $c(v_1v_2) = 1$, $c(v_2v_3) = 2$, $c(u_1v_2) = 3$, and $c(v_2u_2) = 4$.

Four possibility:

- $c(v_1u_1) = 2$, $c(u_1u_2) = 1$, and $c(u_2v_3) = 3$: then, $\sigma_c(v_2) = \sigma_c(v_3) = 0$.
- $c(v_1u_1) = 4$, $c(u_1u_2) = 1$, and $c(u_2v_3) = 3$:
  Then, $\sigma_c(v_1) = \sigma_c(v_2) = \sigma_c(v_3) = 0$ and $\sigma_c(u_1) = \sigma_c(u_2) = 3$.
- $c(v_1u_1) = 4$, $c(u_1u_2) = 2$, and $c(u_2v_3) = 1$: then, $\sigma_c(v_1) = \sigma_c(v_2) = 0$.
- $c(v_1u_1) = 4$, $c(u_1u_2) = 2$, and $c(u_2v_3) = 3$:
  Then, $\sigma_c(v_1) = \sigma_c(v_2) = \sigma_c(v_3) = 0$ and $\sigma_c(u_1) = \sigma_c(u_2) = 4$.

Therefore, $\chi\ '_t(T(P_3)) \geq 6$. Hence, $\chi\ '_t(T(P_3)) = 6$. ■

Theorem 4.1.

If $n \geq 4$, the nowhere-zero twin chromatic index of $T(P_n)$ is 6.

Proof:

Case 1. $n \equiv 0 \pmod{6}$

Define $c: E(T(P_n)) \to \mathbb{Z}_5 - \{0\}$ as follows.

For $i \in \{1,2,\ldots,n-2\}$,

$$c(v_iv_{i+1}) = \begin{cases} 1, & \text{if } i \text{ is odd,} \\ 2, & \text{if } i \text{ is even.} \end{cases} \tag{4.1}$$

$$c(v_{n-1}v_n) = 3. \tag{4.2}$$

For $i \in \{1,2,\ldots,n-3\}$,

$$c(u_iu_{i+1}) = \begin{cases} 1, & \text{if } i \text{ is odd,} \\ 2, & \text{if } i \text{ is even.} \end{cases} \tag{4.3}$$

$$c(u_{n-2}u_{n-1}) = 5. \tag{4.4}$$

For $i \in \{1,2,\ldots,n-2\}$,

$$c(v_i u_i) = \begin{cases} 4, & \text{if } i \equiv 0 \pmod 3, \\ 3, & \text{if } i \equiv 1 \pmod 3, \\ 5, & \text{if } i \equiv 2 \pmod 3. \end{cases} \tag{4.5}$$

$$c(v_{n-1} u_{n-1}) = 1. \tag{4.6}$$

For $i \in \{1,2,\dots,n-2\}$,

$$c(u_i v_{i+1}) = \begin{cases} 5, & \text{if } i \equiv 0 \pmod 3, \\ 4, & \text{if } i \equiv 1 \pmod 3, \\ 3, & \text{if } i \equiv 2 \pmod 3. \end{cases} \tag{4.7}$$

$$c(u_{n-1} v_n) = 2. \tag{4.8}$$

See equation (4.1) to (4.8), by construction, c is a proper edge-coloring.

The induced vertex-coloring are:

$$\sigma_c(v_1) = 4. \tag{4.9}$$

For $i \in \{2,3,4,\dots,n-2\}$,

$$\sigma_c(v_i) = \begin{cases} 4, & \text{if } i \equiv 0 \pmod 3, \\ 5, & \text{if } i \equiv 1 \pmod 3, \\ 0, & \text{if } i \equiv 2 \pmod 3. \end{cases} \tag{4.10}$$

$$\sigma_c(v_{n-1}) = 4, \text{ and } \sigma_c(v_n) = 5. \tag{4.11}$$

$$\sigma_c(u_1) = 2. \tag{4.12}$$

For $i \in \{2,3,4,\dots,n-3\}$,

$$\sigma_c(u_i) = \begin{cases} 0, & \text{if } i \equiv 0 \pmod 3, \\ 4, & \text{if } i \equiv 1 \pmod 3, \\ 5, & \text{if } i \equiv 2 \pmod 3. \end{cases} \tag{4.13}$$

$$\sigma_c(u_{n-2}) = 1, \text{ and } \sigma_c(u_{n-1}) = 2. \tag{4.14}$$

Observe equation (4.9) to (4.14), σ_c is a proper vertex-coloring. Thus, c is a nowhere-zero twin edge 6-coloring of $T(P_n)$.

Case 2. $n \equiv 1 \pmod 6$.

Define $c: E(T(P_n)) \to \mathbb{Z}_5 - \{0\}$ as follows:

For $i \in \{1,2,\dots,n-1\}$,

$$c(v_i v_{i+1}) = \begin{cases} 1, & \text{if } i \text{ is odd}, \\ 2, & \text{if } i \text{ is even}. \end{cases} \tag{4.15}$$

For $i \in \{1,2,\dots,n-2\}$,

$$c(u_iu_{i+1}) = \begin{cases} 1, & \text{if } i \text{ is odd,} \\ 2, & \text{if } i \text{ is even.} \end{cases} \tag{4.16}$$

For $i \in \{1,2,\dots,n-1\}$,

$$c(v_iu_i) = \begin{cases} 4, & \text{if } i \equiv 0 \pmod 3, \\ 3, & \text{if } i \equiv 1 \pmod 3, \\ 5, & \text{if } i \equiv 2 \pmod 3. \end{cases} \tag{4.17}$$

For $i \in \{1,2,\dots,n-2\}$,

$$c(u_iv_{i+1}) = \begin{cases} 5, & \text{if } i \equiv 0 \pmod 3, \\ 4, & \text{if } i \equiv 1 \pmod 3, \\ 3, & \text{if } i \equiv 2 \pmod 3. \end{cases} \tag{4.18}$$

$$c(u_{n-1}v_n) = 3. \tag{4.19}$$

See equation (4.15) to (4.19), by construction, c is a proper edge-coloring.

The induced vertex-coloring are:

$$\sigma_c(v_1) = 4. \tag{4.20}$$

For $i \in \{2,3,4,\dots,n-1\}$,

$$\sigma_c(v_i) = \begin{cases} 4, & \text{if } i \equiv 0 \pmod 3, \\ 5, & \text{if } i \equiv 1 \pmod 3, \\ 0, & \text{if } i \equiv 2 \pmod 3. \end{cases} \tag{4.21}$$

$$\sigma_c(v_n) = 5. \tag{4.22}$$

$$\sigma_c(u_1) = 2. \tag{4.23}$$

For $i \in \{2,3,4,\dots,n-2\}$,

$$\sigma_c(u_i) = \begin{cases} 0, & \text{if } i \equiv 0 \pmod 3, \\ 4, & \text{if } i \equiv 1 \pmod 3, \\ 5, & \text{if } i \equiv 2 \pmod 3. \end{cases} \tag{4.24}$$

$$\sigma_c(u_{n-1}) = 2. \tag{4.25}$$

Observe equation (4.20) to (4.25), σ_c is a proper vertex-coloring.

Thus, c is a nowhere-zero twin edge 6-coloring of $T(P_n)$.

Case 3. $n \equiv 2 \pmod 6$.

Define $c: E(T(P_n)) \to \mathbb{Z}_6 - \{0\}$ as follows.

For $i \in \{1,2,\dots,n-1\}$,

$$c(v_iv_{i+1}) = \begin{cases} 1, & \text{if } i \text{ is odd,} \\ 2, & \text{if } i \text{ is even.} \end{cases} \tag{4.26}$$

For $i \in \{1,2,\dots,n-2\}$,

$$c(u_iu_{i+1}) = \begin{cases} 1, & \text{if } i \text{ is odd,} \\ 2, & \text{if } i \text{ is even.} \end{cases} \tag{4.27}$$

For $i \in \{1,2,\dots,n-1\}$,

$$c(v_iu_i) = \begin{cases} 4, & \text{if } i \equiv 0 \pmod 3, \\ 3, & \text{if } i \equiv 1 \pmod 3, \\ 5, & \text{if } i \equiv 2 \pmod 3. \end{cases} \tag{4.28}$$

For $i \in \{1,2,\dots,n-2\}$,

$$c(u_iv_{i+1}) = \begin{cases} 5, & \text{if } i \equiv 0 \pmod 3, \\ 4, & \text{if } i \equiv 1 \pmod 3, \\ 3, & \text{if } i \equiv 2 \pmod 3. \end{cases} \tag{4.29}$$

$$c(u_{n-1}v_n) = 5. \tag{4.30}$$

See equation (4.26) to (4.30), by construction, c is a proper edge-coloring.

The induced vertex-coloring are:

$$\sigma_c(v_1) = 4. \tag{4.31}$$

For $i \in \{2,3,4,\dots,n-1\}$,

$$\sigma_c(v_i) = \begin{cases} 4, & \text{if } i \equiv 0 \pmod 3, \\ 5, & \text{if } i \equiv 1 \pmod 3, \\ 0, & \text{if } i \equiv 2 \pmod 3. \end{cases} \tag{4.32}$$

$$\sigma_c(v_n) = 0. \tag{4.33}$$

$$\sigma_c(u_1) = 2. \tag{4.34}$$

For $i \in \{2,3,4,\dots,n-2\}$,

$$\sigma_c(u_i) = \begin{cases} 0, & \text{if } i \equiv 0 \pmod 3, \\ 4, & \text{if } i \equiv 1 \pmod 3, \\ 5, & \text{if } i \equiv 2 \pmod 3. \end{cases} \tag{4.35}$$

$$\sigma_c(u_{n-1}) = 4. \tag{4.36}$$

Observe equation (4.31) to (4.36), σ_c is a proper vertex-coloring. Thus, c is a nowhere-zero twin edge 6-coloring of $T(P_n)$.

Case 4. $n \equiv 3, 5 \pmod 6$.

Define $c: E(T(P_n)) \to \mathbb{Z}_6 - \{0\}$ as follows.

For $i \in \{1,2,\ldots,n-1\}$,

$$c(v_iv_{i+1}) = \begin{cases} 1, & \text{if } i \text{ is odd}, \\ 2, & \text{if } i \text{ is even}. \end{cases} \tag{4.37}$$

For $i \in \{1,2,\ldots,n-2\}$,

$$c(u_iu_{i+1}) = \begin{cases} 1, & \text{if } i \text{ is odd}, \\ 2, & \text{if } i \text{ is even}. \end{cases} \tag{4.38}$$

For $i \in \{1,2,\ldots,n-1\}$,

$$c(v_iu_i) = \begin{cases} 4, & \text{if } i \equiv 0 \pmod 3, \\ 3, & \text{if } i \equiv 1 \pmod 3, \\ 5, & \text{if } i \equiv 2 \pmod 3. \end{cases} \tag{4.39}$$

For $i \in \{1,2,\ldots,n-1\}$,

$$c(u_iv_{i+1}) = \begin{cases} 5, & \text{if } i \equiv 0 \pmod 3, \\ 4, & \text{if } i \equiv 1 \pmod 3, \\ 3, & \text{if } i \equiv 2 \pmod 3. \end{cases} \tag{4.40}$$

See equation (4.37) to (4.40), by construction, c is a proper edge-coloring.

The induced vertex-coloring are:

$$\sigma_c(v_1) = 4. \tag{4.41}$$

For $i \in \{2,3,4,\ldots,n-1\}$,

$$\sigma_c(v_i) = \begin{cases} 4, & \text{if } i \equiv 0 \pmod 3, \\ 5, & \text{if } i \equiv 1 \pmod 3, \\ 0, & \text{if } i \equiv 2 \pmod 3. \end{cases} \tag{4.42}$$

$$\sigma_c(v_n) = \begin{cases} 0, & \text{if } n \equiv 5 \pmod 6, \\ 5, & \text{if } n \equiv 3 \pmod 6. \end{cases} \tag{4.43}$$

$$\sigma_c(u_1) = 2. \tag{4.44}$$

For $i \in \{2,3,4,\ldots,n-2\}$,

$$\sigma_c(u_i) = \begin{cases} 0, & \text{if } i \equiv 0 \pmod 3, \\ 4, & \text{if } i \equiv 1 \pmod 3, \\ 5, & \text{if } i \equiv 2 \pmod 3. \end{cases} \tag{4.45}$$

$$\sigma_c(u_{n-1}) = \begin{cases} 2, & \text{if } n \equiv 5 \pmod 6, \\ 3, & \text{if } n \equiv 3 \pmod 6. \end{cases} \tag{4.46}$$

Observe equation (4.41) to (4.46), σ_c is a proper vertex-coloring. Thus, c is a nowhere-zero twin edge 6-coloring of $T(P_n)$.

Case 5. $n \equiv 4 \pmod 6$

Define $c: E(T(P_n)) \to \mathbb{Z}_6 - \{0\}$ as follows:

For $i \in \{1,2,\dots,n-2\}$,

$$c(v_i v_{i+1}) = \begin{cases} 1, & \text{if } i \text{ is odd}, \\ 2, & \text{if } i \text{ is even}. \end{cases} \quad (4.47)$$

$$c(v_{n-1}v_n) = 4. (4.48)$$

For $i \in \{1,2,\dots,n-2\}$,

$$c(u_i u_{i+1}) = \begin{cases} 1, & \text{if } i \text{ is odd}, \\ 2, & \text{if } i \text{ is even}. \end{cases} \quad (4.49)$$

For $i \in \{1,2,\dots,n-2\}$,

$$c(v_i u_i) = \begin{cases} 4, & \text{if } i \equiv 0 \pmod 3, \\ 3, & \text{if } i \equiv 1 \pmod 3, \\ 5, & \text{if } i \equiv 2 \pmod 3. \end{cases} \quad (4.50)$$

$$c(v_{n-1}u_{n-1}) = 1. \quad (4.51)$$

For $i \in \{1,2,\dots,n-1\}$,

$$c(u_i v_{i+1}) = \begin{cases} 5, & \text{if } i \equiv 0 \pmod 3, \\ 4, & \text{if } i \equiv 1 \pmod 3, \\ 3, & \text{if } i \equiv 2 \pmod 3. \end{cases} \quad (4.52)$$

See equation (4.47) to (4.52), by construction, c is a proper edge-coloring.

The induced vertex-coloring are:

$$\sigma_c(v_1) = 4. \quad (4.53)$$

For $i \in \{2,3,4,\dots,n-1\}$,

$$\sigma_c(v_i) = \begin{cases} 4, & \text{if } i \equiv 0 \pmod 3, \\ 5, & \text{if } i \equiv 1 \pmod 3, \\ 0, & \text{if } i \equiv 2 \pmod 3. \end{cases} \quad (4.54)$$

$$\sigma_c(v_n) = 3. \quad (4.55)$$

$$\sigma_c(u_1) = 2. \quad (4.56)$$

For $i \in \{2,3,4,\dots,n-2\}$,

$$\sigma_c(u_i) = \begin{cases} 0, & \text{if } i \equiv 0 \pmod 3, \\ 4, & \text{if } i \equiv 1 \pmod 3, \\ 5, & \text{if } i \equiv 2 \pmod 3. \end{cases} \quad (4.57)$$

$$\sigma_c(u_{n-1}) = 2. \quad (4.58)$$

Observe equation (4.53) to (4.58), σ_c is a proper vertex-coloring. Thus, c is a nowhere-zero twin edge 6-coloring of $T(P_n)$. Hence, $\chi\ '_t(T(P_n)) = 6$. ■

Theorem 4.2.

If $n \geq 1$, *the nowhere-zero twin chromatic index of* $T(C_{3n})$ *is* 6.

***Proof*:**

Define $c: E(T(C_{3n})) \to \mathbb{Z}_6 - \{0\}$ as follows:

For $i \in \{1,2, \dots, 3n\}$,

$$c(v_i v_{i+1}) = \begin{cases} 3, & \text{if } i \equiv 0 \pmod 3, \\ 1, & \text{if } i \equiv 1 \pmod 3, \\ 2, & \text{if } i \equiv 2 \pmod 3. \end{cases} \tag{4.59}$$

$$c(u_i u_{i+1}) = \begin{cases} 2, & \text{if } i \equiv 0 \pmod 3, \\ 3, & \text{if } i \equiv 1 \pmod 3, \\ 1, & \text{if } i \equiv 2 \pmod 3. \end{cases} \tag{4.60}$$

$$c(v_i u_i) = 4, \tag{4.61}$$

$$c(u_i v_{i+1}) = 5. \tag{4.62}$$

See equation (4.59) to (4.62), by construction, c is a proper edge-coloring.

The induced vertex-coloring are.

For $i \in \{1,2, \dots, 3n\}$,

$$\sigma_c(v_i) = \begin{cases} 2, & \text{if } i \equiv 0 \pmod 3, \\ 1, & \text{if } i \equiv 1 \pmod 3, \\ 0, & \text{if } i \equiv 2 \pmod 3. \end{cases} \tag{4.63}$$

$$\sigma_c(u_i) = \begin{cases} 0, & \text{if } i \equiv 0 \pmod 3, \\ 2, & \text{if } i \equiv 1 \pmod 3, \\ 1, & \text{if } i \equiv 2 \pmod 3. \end{cases} \tag{4.64}$$

Observe equation (4.63) and (4.64), σ_c is a proper vertex-coloring. Thus, c is a nowhere-zero twin edge 6-coloring of $T(C_{3n})$. Hence, $\chi\ '_t(T(C_{3n})) = 6$. ■

Theorem 4.3.

If $n \geq 1$, *the nowhere-zero twin chromatic index of* $T(C_{4n})$ *is* 6.

Proof:

Define $c: E(T(C_{4n})) \to \mathbb{Z}_6 - \{0\}$ as follows.

For $i \in \{1,2, \dots, 4n\}$,

$$c(v_iv_{i+1}) = \begin{cases} 4, & \text{if } i \equiv 0 \pmod 4, \\ 1, & \text{if } i \equiv 1 \pmod 4, \\ 2, & \text{if } i \equiv 2 \pmod 4, \\ 3, & \text{if } i \equiv 3 \pmod 4. \end{cases} \tag{4.65}$$

$$c(u_iu_{i+1}) = \begin{cases} 3, & \text{if } i \equiv 0 \pmod 4, \\ 4, & \text{if } i \equiv 1 \pmod 4, \\ 1, & \text{if } i \equiv 2 \pmod 4, \\ 2, & \text{if } i \equiv 3 \pmod 4. \end{cases} \tag{4.66}$$

$$c(v_iu_i) = \begin{cases} 5, & \text{if } i \equiv 0 \pmod 4, \\ 2, & \text{if } i \equiv 1 \pmod 4, \\ 3, & \text{if } i \equiv 2 \pmod 4, \\ 4, & \text{if } i \equiv 3 \pmod 4; \end{cases} \tag{4.67}$$

$$c(u_iv_{i+1}) = 5. \tag{4.68}$$

See equation (4.65) to (4.68), by construction, c is a proper edge-coloring.

The induced vertex-coloring are.

For $i \in \{1,2,\dots,4n\}$,

$$\sigma_c(v_i) = \begin{cases} 1, & \text{if } i \equiv 0 \pmod 4, \\ 0, & \text{if } i \equiv 1 \pmod 4, \\ 5, & \text{if } i \equiv 2 \pmod 4, \\ 2, & \text{if } i \equiv 3 \pmod 4. \end{cases} \tag{4.69}$$

$$\sigma_c(u_i) = \begin{cases} 5, & \text{if } i \equiv 0 \pmod 4, \\ 2, & \text{if } i \equiv 1 \pmod 4, \\ 1, & \text{if } i \equiv 2 \pmod 4, \\ 0, & \text{if } i \equiv 3 \pmod 4. \end{cases} \tag{4.70}$$

Observe equation (4.69) and (4.70), σ_c is a proper vertex-coloring. Thus, c is a nowhere-zero twin edge 6-coloring of $T(C_{4n})$. Hence, $\chi\ '_t(T(C_{4n})) = 6$. ■

Theorem 4.4.

If $n \geq 1$, the nowhere-zero twin chromatic index of $T(C_{5n})$ is 6.

Proof:

Define $c: E(T(C_{5n})) \to \mathbb{Z}_6 - \{0\}$ as follows.

For $i \in \{1,2,\dots,5n\}$,

$$c(v_iv_{i+1}) = \begin{cases} 5, & \text{if } i \equiv 0 \pmod 5, \\ 1, & \text{if } i \equiv 1 \pmod 5, \\ 2, & \text{if } i \equiv 2 \pmod 5, \\ 3, & \text{if } i \equiv 3 \pmod 5, \\ 4, & \text{if } i \equiv 4 \pmod 5. \end{cases} \quad (4.71)$$

$$c(u_iu_{i+1}) = \begin{cases} 3, & \text{if } i \equiv 0 \pmod 5, \\ 4, & \text{if } i \equiv 1 \pmod 5, \\ 5, & \text{if } i \equiv 2 \pmod 5, \\ 1, & \text{if } i \equiv 3 \pmod 5, \\ 2, & \text{if } i \equiv 4 \pmod 5. \end{cases} \quad (4.72)$$

$$c(v_iu_i) = \begin{cases} 1, & \text{if } i \equiv 0 \pmod 5, \\ 2, & \text{if } i \equiv 1 \pmod 5, \\ 3, & \text{if } i \equiv 2 \pmod 5, \\ 4, & \text{if } i \equiv 3 \pmod 5, \\ 5, & \text{if } i \equiv 4 \pmod 5. \end{cases} \quad (4.73)$$

$$c(u_iv_{i+1}) = \begin{cases} 4, & \text{if } i \equiv 0 \pmod 5, \\ 5, & \text{if } i \equiv 1 \pmod 5, \\ 1, & \text{if } i \equiv 2 \pmod 5, \\ 2, & \text{if } i \equiv 3 \pmod 5, \\ 3, & \text{if } i \equiv 4 \pmod 5. \end{cases} \quad (4.74)$$

See equation (4.71) to (4.74), by construction, c is a proper edge-coloring.

The induced vertex-coloring are.

For $i \in \{1,2,\dots,5n\}$,

$$\sigma_c(v_i) = \begin{cases} 1, & \text{if } i \equiv 0 \pmod 5, \\ 0, & \text{if } i \equiv 1 \pmod 5, \\ 5, & \text{if } i \equiv 2 \pmod 5, \\ 4, & \text{if } i \equiv 3 \pmod 5, \\ 2, & \text{if } i \equiv 4 \pmod 5. \end{cases} \quad (4.75)$$

$$\sigma_c(u_i) = \begin{cases} 4, & \text{if } i \equiv 0 \pmod 5, \\ 2, & \text{if } i \equiv 1 \pmod 5, \\ 1, & \text{if } i \equiv 2 \pmod 5, \\ 0, & \text{if } i \equiv 3 \pmod 5, \\ 5, & \text{if } i \equiv 4 \pmod 5. \end{cases} \quad (4.76)$$

Observe equation (4.75) and (4.76), σ_c is a proper vertex-coloring.

Thus, c is a nowhere-zero twin edge 6-coloring of $T(C_{5n})$. Hence, $\chi'_t(T(C_{5n})) = 6$. ■

Conjecture 4.1.

For any $n \geq 3$, the nowhere-zero twin chromatic index of $T(C_n)$ is 6.

5. Graphs with twin chromatic index $\Delta + 2$

Problem 4.1. [Rajarajachozhan et al. (2016)]

If possible to find a twin edge $(2 + \Delta)$-coloring of $K_{2n+1} - E(H)$, where H is a triangle-free 2-factor of K_{2n+1}.

Tables 1, 2 3 and 4 below yield, respectively, twin edge 10-coloring of $K_{11} - E(C_{11})$, twin edge 12-coloring of $K_{13} - E(C_{13})$, twin edge 14-coloring of $K_{15} - E(C_{15})$, and twin edge 16-coloring of $K_{17} - E(C_{17})$.

Table 1. A twin edge 10-coloring of $K_{11} - E(C_{11})$, where $V(K_{11}) = \{v_0, v_1, \dots, v_{10}\}$ and $C_{11} = v_0 v_1 v_2 \dots v_{10} v_0$

	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}
v_0	-	-	1	6	0	2	3	5	7	9	-
v_1	-	-	-	9	8	0	5	7	3	1	2
v_2	1	-	-	-	7	8	9	0	4	2	5
v_3	6	9	-	-	-	7	1	2	5	3	4
v_4	0	8	7	-	-	-	2	3	6	4	9
v_5	2	0	8	7	-	-	-	4	9	5	6
v_6	3	5	9	1	2	-	-	-	8	6	7
v_7	5	7	0	2	3	4	-	-	-	8	1
v_8	7	3	4	5	6	9	8	-	-	-	0
v_9	9	1	2	3	4	5	6	8	-	-	-
v_{10}	-	2	5	4	9	6	7	1	0	-	-

Table 2. A twin edge 12-coloring of $K_{13} - E(C_{13})$, where $V(K_{13}) = \{v_0, v_1, \dots, v_{12}\}$ and $C_{13} = v_0 v_1 v_2 \dots v_{12} v_0$

	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}
v_0	-	-	2	3	10	6	8	5	4	0	9	11	-
v_1	-	-	-	4	5	8	3	2	11	9	10	7	6
v_2	2	-	-	-	3	4	11	0	5	8	1	6	10
v_3	3	4	-	-	-	7	5	1	8	6	11	10	2
v_4	10	5	3	-	-	-	0	4	6	2	8	1	9
v_5	6	8	4	7	-	-	-	10	3	5	0	2	11
v_6	8	3	11	5	0	-	-	-	9	7	6	4	1
v_7	5	2	0	1	4	10	-	-	-	11	7	8	3
v_8	4	11	5	8	6	3	9	-	-	-	2	0	7
v_9	0	9	8	6	2	5	7	11	-	-	-	3	4
v_{10}	9	10	1	11	8	0	6	7	2	-	-	-	5

v_{11}	11	7	6	10	1	2	4	8	0	3	-	-	-
v_{12}	-	6	10	2	9	11	1	3	7	4	5	-	-

Table 3. A twin edge 14-coloring of $K_{15} - E(C_{15})$, where $V(K_{15}) = \{v_0, v_1, \dots, v_{14}\}$ and $C_{15} = v_0 v_1 v_2 \dots v_{14} v_0$

	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}
v_0	-	-	13	12	11	8	9	10	7	6	5	4	3	2	-
v_1	-	-	-	13	12	11	10	9	8	7	6	5	4	3	2
v_2	13	-	-	-	10	9	8	7	6	5	4	3	2	1	12
v_3	12	13	-	-	-	10	7	6	5	4	3	2	1	9	8
v_4	11	12	10	-	-	-	6	5	2	3	9	8	0	7	4
v_5	8	11	9	10	-	-	-	4	3	2	1	7	6	12	0
v_6	9	10	8	7	6	-	-	-	4	1	2	12	11	0	5
v_7	10	9	7	6	5	4	-	-	-	12	11	0	8	13	1
v_8	7	8	6	5	2	3	4	-	-	-	0	1	12	11	10
v_9	6	7	5	4	3	2	1	12	-	-	-	9	10	8	11
v_{10}	5	6	4	3	9	1	2	11	0	-	-	-	13	10	7
v_{11}	4	5	3	2	8	7	12	0	1	9	-	-	-	6	13
v_{12}	3	4	2	1	0	6	11	8	12	10	13	-	-	-	9
v_{13}	2	3	1	9	7	12	0	13	11	8	10	6	-	-	-
v_{14}	-	2	12	8	4	0	5	1	10	11	7	13	9	-	-

Table 4. A twin edge 16-coloring of $K_{17} - E(C_{17})$, where $V(K_{17}) = \{v_0, v_1, \dots, v_{16}\}$ and $C_{17} = v_0 v_1 v_2 \dots v_{16} v_0$

	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}	v_{15}	v_{16}
v_0	-	-	13	12	11	10	9	8	7	0	1	2	3	4	5	6	-
v_1	-	-	-	1	0	2	3	4	5	6	13	12	11	10	9	8	7
v_2	13	-	-	-	2	3	14	5	6	1	12	11	10	9	8	7	0
v_3	12	1	-	-	-	0	5	6	13	11	10	9	8	7	14	2	3
v_4	11	0	2	-	-	-	15	1	12	10	5	8	7	13	3	4	9
v_5	10	2	3	0	-	-	-	11	1	12	8	7	9	5	4	15	13
v_6	9	3	14	5	15	-	-	-	0	2	7	10	13	8	11	1	12
v_7	8	4	5	6	1	11	-	-	-	9	2	0	14	12	13	10	15
v_8	7	5	6	13	12	1	0	-	-	-	4	3	2	11	15	9	14
v_9	0	6	1	11	10	12	2	9	-	-	-	15	5	3	7	13	8

v_{10}	1	13	12	10	5	8	7	2	4	-	-	-	15	6	0	3	11
v_{11}	2	12	11	9	8	7	10	0	3	15	-	-	-	14	1	5	6
v_{12}	3	11	10	8	7	9	13	14	2	5	15	-	-	-	6	12	4
v_{13}	4	10	9	7	13	5	8	12	11	3	6	14	-	-	-	0	2
v_{14}	5	9	8	14	3	4	11	13	15	7	0	1	6	-	-	-	10
v_{15}	6	8	7	2	4	15	1	10	9	13	3	5	12	0	-	-	-
v_{16}	-	7	0	3	9	13	12	15	14	8	11	6	4	2	10	-	-

I strongly feel that finding graphs with $\chi'_t = 2 + \Delta$ or obtaining special classes of graphs with $\chi'_t \geq 2 + \Delta$ is difficult.

6. Conclusion

The study of twin edge-coloring of total graph of path and cycle are important due to its applications in many real life problems. In this paper, I investigated the twin edge-chromatic index of total graph of path and cycle only, also two problems in section 3 and 4 complete solution are perfect upper bound is not easy one. The investigation of analogous results for different graphs and different operation of above families of graph are still open.

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