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Christian Kasumo
Nelson Mandela African Institution of Science and Technology

Juma Kasozi
Makerere University

Dmitry Kuznetsov
Nelson Mandela African Institution of Science and Technology

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Dividend Maximization Under a Set Ruin Probability Target in the Presence of Proportional and Excess-of-loss Reinsurance

[1*]Christian Kasumo, [2]Juma Kasozi and [3]Dmitry Kuznetsov

[1,3]Department of Applied Mathematics and Computational Science
Nelson Mandela African Institution of Science and Technology
P.O. Box 447
Arusha, Tanzania

[2]Department of Mathematics
Makerere University
P.O. Box 7062
Kampala, Uganda
[1]kasumoc@nm-aist.ac.tz; [2]kasozi@cns.mak.ac.ug; [3]dmitry.kuznetsov@nm-aist.ac.tz

*Corresponding Author

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Abstract

We study dividend maximization with set ruin probability targets for an insurance company whose surplus is modelled by a diffusion perturbed classical risk process. The company is permitted to enter into proportional or excess-of-loss reinsurance arrangements. By applying stochastic control theory, we derive Volterra integral equations and solve numerically using block-by-block methods. In each of the models, we have established the optimal barrier to use for paying dividends provided the ruin probability does not exceed a predetermined target. Numerical examples involving the use of both light- and heavy-tailed distributions are given. The results show that ruin probability targets result in an improvement in the optimal barrier to be used for dividend payouts. This is the case for light- and heavy-tailed distributions and applies regardless of the risk model used.

Keywords: Hamilton-Jacobi-Bellman equation; Volterra equation; Block-by-block method; Reinsurance; Dividends; Ruin probability; Ruin probability target

MSC 2010 No.: 45D05, 49L20, 62P05

1. Introduction

The dividend maximization problem has preoccupied researchers for several decades now. Several studies have investigated the optimal dividend problem for maximizing equity value by applying stochastic control techniques, among them Choulli et al. (2003), Kasozi and Mahera (2013) and Nansubuga et al. (2016). Kasozi et al. (2011) used homotopy analysis method (HAM) to maximize dividend payments in the Cramér-Lundberg model under a barrier strategy but found that the HAM was not convergent when applied to a model with stochastic return on investments. Kasozi and Paulsen (2005a) studied the problem of dividend maximization in the classical risk model for a company that has invested some of the surplus in a risk-free asset. They obtained the optimal barrier strategy that maximizes the dividends to be paid out prior to ruin. Marciniak and Palmowski (2016) focused on the optimal dividend problem for insurance risk models with surplus-dependent premiums, using as their basic model a piece-wise deterministic Markov process (PDMP).

But many other studies have emerged in the actuarial literature focusing on dividend maximization under solvency constraints. Paulsen (2003) solved the dividend optimization problem of a firm under solvency constraints and showed that the optimal policy is of barrier type. Dickson and Drekic (2006) considered dividend optimization under a ruin probability constraint but for models different from ours. He et al. (2008) studied the optimal control problem for an insurance company that adopts a proportional reinsurance policy under solvency constraints. They gave a rigorous probability proof on the bankrupt probability decreasing with respect to some dividend barrier. Nansubuga et al. (2016) considered maximization of dividend payouts under infinite ruin probability constraints. They derived Volterra integral equations which they solved using the block-by-block method and established the optimal barrier to use to pay dividends provided the ruin probability is no larger than the predetermined tolerance.

Hernández and Junca (2015) studied the classical optimal dividend problem in the Cramér-Lundberg model with exponential claim sizes subject to a constraint on the ruin time and obtained the value function as a point-wise infimum of auxiliary value functions indexed by Lagrange multipliers. Using the fundamental tool of scale functions and fluctuation theory, Hernández et al. (2018) extended the results of Hernández and Junca (2015) for spectrally one-sided Lévy risk processes by introducing a longevity feature in the classical dividend problem through addition of a constraint on the time of ruin of the firm. Hipp (2016) studied control for minimizing ruin probability as well as maximizing dividend payments. In particular, he considered an optimal control problem concerned with maximizing the total expected discounted dividend payments with a ruin constraint and found that a ruin constraint is cheaper when an appropriate reinsurance cover is available.

In this paper, we consider dividend maximization under a set ruin probability target in a jump-diffusion model compounded by proportional and excess-of-loss (XL) reinsurance. In addition to managing the company's risk through reinsurance, management is allowed to pay dividends to the shareholders provided they are paid continuously according to a barrier level b and only until ruin. No dividends are paid when the surplus falls below b, and the ruin probability target is taken into account. The term 'dividend' refers to taxable payments declared by the insurer's board of directors and given to shareholders out of the company's current or retained earnings (Kasozi and Paulsen

(2005a)). The 'ruin probability' is given by $\psi(u) = \mathbb{P}(\tau_u < \infty)$, where $\tau_u = \{t > 0 | U_t < 0\}$, called the 'ruin time', is the first time the surplus process U becomes negative, with $\tau_u = \infty$ if U always stays positive. At an initial surplus u, the probability of ruin occurring before time horizon T is $\psi(t, u) = \mathbb{P}(\tau_u < T)$. If the surplus process is $U_t^{\overline{D},\overline{R}}$, where $(\overline{D}, \overline{R}) \in \Pi_u^{D,R}$ is an admissible dividend and reinsurance strategy, and if $\overline{D}$ incorporates a dividend barrier b, then the ruin probability *at barrier level* b is defined as $\psi(u) = \mathbb{P}(\tau_b^{\overline{D},\overline{R}} < T)$. At time horizon T and a ruin tolerance $\epsilon > 0$, the ruin probability at barrier level b is defined as $\psi_b(T, b) := \psi_b(T, u)|_{u=b} = \mathbb{P}(\tau_b^{\overline{D},\overline{R}} < T) = \epsilon$. The 'ultimate ruin probability' is the probability that the surplus process ever falls below zero, represented mathematically as $\psi(u) = \mathbb{P}(\tau_u^{\overline{D},\overline{R}} < \infty | U_0 = u) = 1 - \phi(u)$, where $\phi(u)$ is the survival probability.

The main contribution of this paper is that it seeks to extend the work of Nansubuga et al. (2016) by allowing the company to enter into reinsurance arrangements involving a combination of proportional and excess-of-loss reinsurance while at the same time distributing a portion of the surplus in the form of dividends to the shareholders. But there has to be a delicate trade-off between stability and profitability. Maximizing dividend payments leads to certain ruin (which is unacceptable for the policyholders), while maximizing survival probability results in a reduction in solvency capital, thus making dividend distribution impossible (which is unacceptable for the shareholders). To strike a balance, we seek to maximize dividend payments under a ruin probability constraint or target (Hipp (2003)). The idea in this paper is to find the optimal reinsurance strategies and then use them to determine the dividend value functions under a set ruin probability target.

The models studied in this paper result in Volterra integral equations (VIEs) of the second kind. As Press et al. (1992) have pointed out, there is general consensus that the block-by-block method, first proposed by Young (1954), is the best of the higher order methods for solving VIEs of the second kind. The block-by-block methods are essentially extrapolation procedures which produce a block of values at a time. They are advantageous over linear multistep and step-by-step methods in that they can be of higher order and still be self-starting. In addition to not requiring special starting procedures or values, block-by-block methods have a simple structure, allow for easy switching of step-size and have the ability to compute several values of the unknown function at the same time (Linz (1985); Katani and Shahmorad (2012)).

Furthermore, the block-by-block method is chosen in this paper over such methods as saddlepoint approximation, importance sampling simulation, upper and lower bounds, Fast-Fourier Transform (FFT) and diagonally implicit multistep block (Gatto and Mosimann (2012); Gatto and Baumgartner (2016); Baharum et al. (2018)) because it is a fourth-order method while most of the other methods are of order less than four. In fact, some of the methods mentioned above are used for directly

Other methods have been used to solve integrodifferential equations arising in engineering. These include the local Galerkin integral equation and thin plate spline collocation methods for solving second-order Volterra integrodifferential equations (VIDEs) with time-periodic coefficients (Assari and Dehghan (2018); Assari (2018)). Being meshless, both of these methods do not require any background interpolation. The collocation method proposed by Cardone et al. (2018) has the advantages of variable step-size implementation, high order of convergence, strong stability and a high degree of flexibility. However, it suffers from the order-reduction phenomenon when applied to stiff problems since it does not have a uniform order of convergence. Saeedi et al. (2013) solved a class of nonlinear VIEs of the first kind by converting them into linear VIEs of the second kind and then applying the Tau method which they found to be highly accurate.

In the literature, two-, three- and four-block block-by-block methods have been used to solve Volterra integral equations of the second kind (e.g., Linz (1969) for non-linear VIEs; Saify (2005) for a system of linear VIEs). More recently, Kasozi and Paulsen (2005a) used the two-block block-by-block method to study the flow of dividends under a constant interest force. They derived a linear VIE and applied the fourth-order block by-block method of Paulsen et al. (2005) in conjunction with Simpson's rule to solve the Volterra integral equation for the optimal dividend barrier. In another study, Kasozi and Paulsen (2005b) applied a fourth-order block-by-block method to the numerical solution of the VIE for ultimate ruin in the Cramér–Lundberg model compounded by a constant force of interest. More pertinent literature on the block-by-block method is available, for example, in Paulsen (2003) and Paulsen and Gjessing (1997).

The outline for the rest of the paper is as follows. We formulate the model in Section 2 and derive the relevant Hamilton-Jacobi-Bellman (HJB), integrodifferential and integral equations corresponding to the problem in Section 3. Section 4 outlines the numerical method and presents numerical results for validation of the method. In Section 5, we present a few conclusions and suggest possible extensions to this work.

2. Model formulation

To make a rigorous mathematical formulation of the problem, we assume throughout this paper that all random variables and stochastic quantities are defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}^+}, \mathbb{P})$ satisfying the usual conditions, that is, the filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}^+}$ is right-continuous and $\mathbb{P}$-complete. In the absence of dividend payouts and reinsurance, the surplus of an insurance company is governed by the diffusion-perturbed classical risk process:

$$U_t = u + ct + \sigma W_t - \sum_{i=1}^{N_t} X_i,\ t \geq 0, \tag{1}$$

where $u = U_0 \geq 0$ is the initial reserve, $c = (1 + \theta)\lambda\mu > 0$ is the premium rate assumed to be computed by the expected value premium principle, $\{N_t\}$ is a homogeneous Poisson process with intensity $\lambda > 0$ and $\{X_i\}$ is an i.i.d. sequence of strictly positive random variables with distribution function $F(x)$ and probability density function $f(x)$. The quantity θ is called the *safety loading* and represents the additional premium received by the insurer on account of uncertainty. The claim arrival process $\{N_t\}$ and claim sizes $\{X_i\}$ are assumed to be independent. Here $\{W_t\}$ is a standard Brownian motion independent of the compound Poisson process $S_t = \sum_{i=1}^{N_t} X_i$. We assume that $\mathbb{E}[X_i] = \mu < \infty$ and $F(0) = 0$. The Brownian term σW_t represents random variations or fluctuations in the surplus process. However, when there is no volatility in the surplus and claim amounts (that is, when $\sigma = 0$), Equation (1) reduces to the Cramér-Lundberg model (CLM) or the classical risk process.

We assume that the insurer takes a combination of quota-share and XL reinsurance as proposed by Centeno (1985). Let the quota-share retention level be $k \in [0,1]$. Then the insurer's aggregate claims, net of quota-share reinsurance, are kX. Also, let the XL reinsurance retention level be $a \in [0,\infty)$. Then the insurer's aggregate claims, net of quota-share and XL reinsurance, are $kX \wedge a$. When the retention limit a of the XL reinsurance is infinite, then the treaty becomes a *pure quota-share* reinsurance, while a quota-share level $k = 1$ makes it a *pure excess-of-loss* reinsurance treaty. Though these two scenarios are somewhat extreme, they are still real possibilities for an insurance company. The premium income of the insurance company is non-negative if $c \geq (1+\theta)\mathbb{E}[(kX - a)^+]$. Therefore, we will let $\underline{a}$ be the XL retention level at which equality $c = (1+\theta)\mathbb{E}[(kX - \underline{a})^+]$ holds.

Thus, given a reinsurance strategy $\overline{R} = (R_t)_{t\in\mathbb{R}^+}$ combining quota-share and XL reinsurance, the controlled surplus process becomes

$$U_t^{\overline{R}} = u + c^{\overline{R}}t + \sigma W_t - \sum_{i=1}^{N_t}(kX_i \wedge a),\ t \geq 0, \tag{2}$$

where $U_0^{\overline{R}} = u$ is the initial surplus of the company and $c^{\overline{R}}$ is the premium rate net of the reinsurance premium. By Itô's formula, the generator for the process $U_t^{\overline{R}}$ is given by

$$\mathcal{L}g(u) = \frac{1}{2}\sigma^2 g''(u) + c^{\overline{R}}g'(u) + \lambda\int_0^\infty [g(u - kx \wedge a) - g(u)]dF(x). \tag{3}$$

Paulsen and Gjessing (1997) have shown that if the equation $\mathcal{L}(\psi)(u) = 0$, where $\mathcal{L}$ is the infinitesimal generator defined in (3), has a solution satisfying the boundary conditions

$$\begin{aligned} \psi(u) &= 1 \text{ on } u < 0, \\ \psi(0) &= 1 \text{ if } \sigma^2 > 0, \\ \lim_{u\to\infty}\psi(u) &= 0, \end{aligned} \tag{4}$$

then that solution is the probability of ruin. Minimizing the ultimate ruin probability $\psi(u)$ is the same as maximizing the ultimate survival probability $\phi(u)$ such that $\mathcal{L}(\phi)(u) = \mathcal{L}(1 - \psi(u)) = 0$ with the boundary conditions

$$\begin{aligned} \phi(u) &= 0 \text{ on } u < 0, \\ \phi(0) &= 0 \text{ if } \sigma^2 > 0, \\ \lim_{u\to\infty}\phi(u) &= 1. \end{aligned} \tag{5}$$

3. Model analysis

Definition 3.1.

The set of all reinsurance strategies, denoted by $\mathcal{R}$, is the collection of all possible proportional and excess-of-loss reinsurance strategies for which $k \in [0,1]$ and $a \in [0,\infty)$, respectively. In other words, $\mathcal{R}$ contains all *admissible* reinsurance strategies.

3.1. HJB equation

The following theorem presents the Hamilton-Jacobi-Bellman (HJB) equation for the survival probability maximization problem. It is this HJB equation that forms the basis for the derivation of the integrodifferential and integral equations.

Theorem 3.2.

Assume that the survival probability $\phi(u)$ is twice continuously differentiable on $(0, \infty)$. Then $\phi(u)$ satisfies the HJB equation

$$\sup_{R \in \mathcal{R}} \mathcal{L}\phi(u) = 0, \; u > 0, \tag{6}$$

where $\mathcal{R}$ is the set of all reinsurance policies and $\mathcal{L}$ is the infinitesimal generator defined in Equation (3) for $0 < u \leq \infty$.

Proof:

The proof is motivated by Schmidli (2008). Let $(0, h]$ be a small interval, and suppose that for each surplus $u(h) > 0$ at time h we have a reinsurance strategy $\overline{R}_\varepsilon$ such that $\phi^{\overline{R}_\varepsilon}(u(h)) > \phi(u(h)) - \varepsilon$. Knowing that $\overline{R} = (R_t)$, with $R_t = kX \wedge a$, where k and a are, respectively, the quota-share and excess-of-loss retention levels, we let $k(t) = k \in [0, 1]$ and $a(t) = a \in [0, \infty)$ for $t \leq h$. Then

$$\phi(u) \geq \phi^{\overline{R}}(u) = \mathbb{E}\left[\phi^{\overline{R}_\varepsilon}\left(U^{\overline{R}}(h)\right)\mathbf{1}_{\{\tau^{\overline{R}} > h\}}\right] = \mathbb{E}\left[\phi^{\overline{R}_\varepsilon}\left(U^{\overline{R}}(\tau^{\overline{R}} \wedge h)\right)\right] \geq \mathbb{E}\left[\phi\left(U^{\overline{R}}(\tau^{\overline{R}} \wedge h)\right)\right] - \varepsilon.$$

Since ε is arbitrary, we can choose $\varepsilon = 0$, so that we have

$$\phi(u) \geq \mathbb{E}\left[\phi\left(U^{\overline{R}}(\tau^{\overline{R}} \wedge h)\right)\right]. \tag{7}$$

By Itô's formula, and substituting into the expectation (7), provided that $\phi(u)$ is twice continuously differentiable, we have

$$\mathbb{E}\left[\int_0^{\tau^{\overline{R}} \wedge h}\left\{\frac{1}{2}\sigma^2\phi^{''}\left(U^{\overline{R}}(x)\right) + c^{\overline{R}}\phi^{'}\left(U^{\overline{R}}(x)\right) + \lambda\left[\int_0^u \phi\left(U^{\overline{R}}(x) - kx \wedge a\right)dF(x) - \phi\left(U^{\overline{R}}(x)\right)\right]\right\}dx\right] \leq 0,$$

where $kx \wedge a$ denotes the retained loss function (i.e., the part of the claim X_i paid by the cedent). Dividing through by h and letting $h \to 0$ yields, provided that the limit and expectation are interchangeable,

$$\frac{1}{2}\sigma^2\phi^{''}\left(U^{\overline{R}}(x)\right) + c^{\overline{R}}\phi^{'}\left(U^{\overline{R}}(x)\right) + \lambda\left[\int_0^u \phi\left(U^{\overline{R}}(x) - kx \wedge a\right)dF(x) - \phi\left(U^{\overline{R}}(x)\right)\right] \leq 0.$$

This equation must hold $\forall\; k \in [0, 1]$ and $a \in [0, \infty)$, i.e.,

$$\sup_{R \in \mathcal{R}}\left\{\frac{1}{2}\sigma^2\phi^{''}(u) + c^{\overline{R}}\phi^{'}(u) + \lambda\int_0^u \phi\left(u - kx \wedge a\right)dF(x) - \lambda\phi(u)\right\} \leq 0.$$

Suppose there is an optimal reinsurance strategy $\overline{R}$ with $k \in [0, 1]$, $a \in [0, \infty)$ such that $\lim_{t \downarrow 0} k(t) = k(0) = k_0$ and $\lim_{t \downarrow 0} a(t) = a(0) = a_0$. Then, as above,

$$\mathbb{E}\left[\int_0^{\tau^{\overline{R}} \wedge h}\left\{\frac{1}{2}\sigma^2\phi^{''}\left(U^{\overline{R}}(x)\right) + c^{\overline{R}}\phi^{'}\left(U^{\overline{R}}(x)\right) + \lambda\left[\int_0^u \phi\left(U^{\overline{R}}(x) - kx \wedge a\right)dF(x) - \phi\left(U^{\overline{R}}(x)\right)\right]\right\}dx\right] = 0.$$

Dividing by h and letting $h \to 0$ yields

$$\frac{1}{2}\sigma^2\phi^{''}(u) + c^{\overline{R}}\phi^{'}(u) + \lambda \int_0^u \phi(u - k_0 x \wedge a_0)dF(x) - \lambda\phi(u) = 0,$$

which motivates the HJB equation

$$\sup_{R \in \mathcal{R}} \mathcal{L}\phi(u) = 0,$$

where

$$\mathcal{L}\phi(u) = \frac{1}{2}\sigma^2\phi^{''}(u) + c^{\overline{R}}\phi^{'}(u) + \lambda \int_0^u \phi(u - kx \wedge a)dF(x) - \lambda\phi(u),$$

with boundary conditions $\phi(u) = 0$ on $u < 0$ and $\lim_{u\to\infty} \phi(u) = 1$. ■

3.2. Integral equations

It follows from Theorem 3.2 that

$$\frac{1}{2}\sigma^2\phi''(u) + c^{\overline{R}}\phi'(u) + \lambda \int_0^u \phi(u - kx \wedge a)dF(x) - \lambda\phi(u) = 0, \tag{8}$$

which is a second-order Volterra integrodifferential equation (VIDE). This equation is transformed into an ordinary Volterra integral equation (VIE) using successive integration by parts as stated in the following theorem.

Theorem 3.3.

The Volterra integrodifferential equation (8) can be represented as a Volterra integral equation of the second kind

$$\phi(u) + \int_0^u K(u,x)\phi(x)dx = \alpha(u), \tag{9}$$

where

(i) For $u \leq \underline{a} < a$, we have

$$\begin{aligned} K(u,x) &= -\frac{\lambda\overline{F}(u-kx)}{c^{\overline{R}}}, \\ \alpha(u) &= \phi(0), \end{aligned} \tag{10}$$

with $\overline{F}(x) = 1 - F(x)$, when there is no diffusion (that is, when $\sigma^2 = 0$), and

$$\begin{aligned} K(u,x) &= \frac{2\left(c^{\overline{R}} + \lambda G(x,u) - \lambda(u-kx)\right)}{\sigma^2}, \\ \alpha(u) &= u\phi'(0) \text{ if } \sigma^2 > 0, \end{aligned} \tag{11}$$

when there is diffusion.

(ii) For $\underline{a} < a < u$, we have

$$K(u,x) = -\frac{\lambda H_1(x,u)}{c^{\overline{R}}}, \quad \alpha(u) = \phi(0), \tag{12}$$

with

$$H_1(x,u) = \begin{cases} \overline{F}(u-kx), & kx < a, \\ 1-(F(kx+a)-F(a)), & kx \geq a, \end{cases}$$

when there is no diffusion, and

$$K(u,x) = \frac{2\left(c^{\overline{R}} + \lambda H_2(x,u) - \lambda(u-kx)\right)}{\sigma^2}, \quad \alpha(u) = u\phi'(0) \text{ if } \sigma^2 > 0, \tag{13}$$

with

$$H_2(x,u) = \begin{cases} G(u-kx), & kx < a, \\ (F(kx+a)-F(a))(u-kx), & kx \geq a, \end{cases}$$

and $G(x) = \int_0^x F(v)dv$ when there is diffusion.

Proof:

The proof for the case $u \leq \underline{a} < a$ is similar to the proof of Theorem 2.2 in Paulsen et al. (2005) but with $r = \sigma_R^2 = 0$, $k = 1$ and $p = c^{\overline{R}}$. The case $\underline{a} < a < u$ has been proved in Kasumo et al. (2018a). ∎

Assume that the company pays out dividends D_t^b up to time t under a ruin probability target, say $\psi(u) = \epsilon$. The dividend process $\overline{D} = (D_t^b)_{t \in \mathbb{R}^+}$ is non-negative, non-decreasing, right-continuous with left limits (or càdlàg) and $\mathcal{F}_t$-adapted. The dynamics of the company's wealth is therefore given by

$$dU_t^{\overline{D},\overline{R}} = dU_t^{\overline{R}} - dD_t^b, \tag{14}$$

where $dU_t^{\overline{R}} = c^{\overline{R}}dt + \sigma dW_t - d\left(\sum_{i=1}^{N_t} kX_i \wedge a\right)$ and the superscript b is a dividend barrier level. The insurance premium rate under these conditions is $c^{\overline{R}} = c - c^R$, where $c^R = (1+\theta)\lambda\mathbb{E}[(kX_i - a)^+]$ is the reinsurance premium, and $dW_t = \xi_t dt$, ξ_t being a white noise process. The time of ruin, when dividends and reinsurance are taken into account, is defined as $\tau_b^{\overline{D},\overline{R}} = \inf\{t \geq 0 | U_t^{\overline{D},\overline{R}} < 0\}$ and the probability of ultimate ruin is defined as $\psi^{\overline{D},\overline{R}} = \mathbb{P}(U_t^{\overline{D},\overline{R}} < 0 \text{ for some } t > 0)$.

The objective is to maximize the total expected discounted dividends paid out to the shareholders until ruin

$$V^{\overline{D},\overline{R}}(u) = \mathbb{E}_u\left[\int_0^{\tau_b^{\overline{D},\overline{R}}} e^{-\delta t} dD_t^b\right], \tag{15}$$

under a set ruin probability target

$$\psi^{\overline{D},\overline{R}}(u) = \mathbb{P}(\tau_b^{\overline{D},\overline{R}} < \infty) \leq \epsilon. \tag{16}$$

The quantity $\delta > 0$ is the constant rate at which dividends are discounted and $\mathbb{E}_u$ denotes expectation with respect to $\mathbb{P}_u$, the probability measure conditioned on the initial capital $U_0^{\overline{D},\overline{R}} = u$. Thus, the optimal value function of this problem becomes

$$V_b(u) = V^{\overline{D},\overline{R}}(u,\epsilon) = \sup_{(\overline{D},\overline{R})\in\Pi_u^{D,R}} \left\{ \mathbb{E}_u \left[\int_0^{\tau_b^{\overline{D},\overline{R}}} e^{-\delta t} dD_t^b \right] : \psi^{\overline{D},\overline{R}}(u) \leq \epsilon \right\}, \tag{17}$$

where $0 < \epsilon \leq 1$ is the permitted ruin probability and $\psi^{\overline{D},\overline{R}}(u)$ is the with-dividend-and-reinsurance ruin probability.

Paulsen and Gjessing (1997) have shown that if $V_b(u)$ solves $\mathcal{L}V_b(u) = \delta V_b(u)$ on $0 < u < b$, subject to the conditions

$$\begin{aligned} V_b(u) &= 0 \text{ on } u < 0, \\ V_b(0) &= 0 \text{ if } \sigma^2 > 0, \\ V_b^{'}(b) &= 1, \\ V_b(u) &= V_b(b) + u - b \text{ on } u > b, \end{aligned} \tag{18}$$

then $V_b(u)$ is given by Equation (17). For $0 \leq u \leq b$, the integrodifferential equation for V_b is

$$\frac{1}{2}\sigma^2 V_b^{''}(u) + c^{\overline{R}} V_b^{'}(u) + \lambda \int_0^u V_b(u - kx \wedge a) dF(x) - (\lambda + \delta) V_b(u) = 0. \tag{19}$$

Equation (19) is a Volterra integrodifferential equation (VIDE) which can be transformed into a VIE of the second kind using successive integration by parts, as shown by the following theorem.

Theorem 3.4.

The Volterra integrodifferential equation (19) can be represented as a Volterra integral equation of the second kind

$$V_b(u) + \int_0^u K(u,x) V_b(x) dx = \alpha(u), \tag{20}$$

where

(i) For $u \leq \underline{a} < a$, we have

$$\begin{aligned} K(u,x) &= -\frac{\delta + \lambda \overline{F}(u - kx)}{c^{\overline{R}}}, \\ \alpha(u) &= V_b(0), \end{aligned} \tag{21}$$

with $\overline{F}(x) = 1 - F(x)$, when there is no diffusion (that is, when $\sigma^2 = 0$), and

$$\begin{aligned} K(u,x) &= \frac{2\left(c^{\overline{R}} + \lambda G(x,u) - (\lambda + \delta)(u - kx)\right)}{\sigma^2}, \\ \alpha(u) &= u V_b^{'}(0) \text{ if } \sigma^2 > 0, \end{aligned} \tag{22}$$

when there is diffusion.

(ii) For $\underline{a} < a < u$, we have

$$K(u,x) = -\frac{\delta + \lambda H_1(x,u)}{c^{\overline{R}}}, \quad \alpha(u) = V_b(0), \tag{23}$$

with

$$H_1(x,u) = \begin{cases} \overline{F}(u-kx), & kx < a, \\ 1-(F(kx+a)-F(a)), & kx \geq a, \end{cases}$$

and $V_b(u) = V_b(b) + u - b$, for $u > b$ when there is no diffusion, and

$$K(u,x) = \frac{2\left(c^{\overline{R}} + \lambda H_2(x,u) - (\lambda+\delta)(u-kx)\right)}{\sigma^2}, \quad \alpha(u) = uV_b^{'}(0) \text{ if } \sigma^2 > 0, \tag{24}$$

with

$$H_2(x,u) = \begin{cases} G(u-kx), & kx < a, \\ (F(kx+a)-F(a))(u-kx), & kx \geq a, \end{cases}$$

and $G(x) = \int_0^x F(v)dv$ when there is diffusion.

Proof:

The proof for the case $u \leq \underline{a} < a$ is similar to the proof of Theorem 2.2 in Paulsen et al. (2005) but with $r = \sigma_R^2 = 0$, $k = 1$ and $p = c^{\overline{R}}$, while the proof for the case $\underline{a} < a < u$ can be found in Kasumo et al. (2018b). ■

3.3. Ruin probability targets

This section presents results relevant to the case involving ruin probability targets.

Theorem 3.5.

At every dividend barrier level b, there exists a unique probability ϵ_b such that $\mathbb{P}(\tau_b^{\overline{D},\overline{R}} < T) = \epsilon_b$ and if $b_1 < b_2$, then $\epsilon_{b_2} < \epsilon_{b_1}$.

Proof:

We note that $\mathbb{P}(\tau_u^{\overline{D},\overline{R}} < T)$ is defined $\forall\ u > 0$. This implies that $\mathbb{P}(\tau_b^{\overline{D},\overline{R}} < T)$ follows by setting $u = b$ and since $\mathbb{P}(\tau_b^{\overline{D},\overline{R}} < T)$ is a decreasing function of u, $b_1 < b_2$ implies that $\epsilon_{b_2} < \epsilon_{b_1}$. ■

Theorem 3.6.

Suppose that the barrier that solves the VIDE (19) is b^* and that a ruin target $\mathbb{P}(\tau_b^{\overline{D},\overline{R}} < T) = \epsilon_b$ is enforced by the insurance company. Then

(a) If $b \leq b^*$, the optimal strategy is to pay dividends using barrier level b^*.
(b) If $b > b^*$, the optimal strategy is to pay dividends using barrier level b.

Proof:

(a) This should be obvious and should always hold since the optimal policy b^* is feasible under the prescribed ruin target.
(b) If $b > b^*$, the management of the insurance company is prohibited from paying dividends at b^* according to the imposed ruin target. Applying the generalized Itô's formula (Theorem 4.2.1 in Øksendal (2003)), we have

$$\begin{aligned}
\mathrm{e}^{-\delta(\tau_b^{\overline{D},\overline{R}}\wedge t)} V\left(U^b_{\tau_b^{\overline{D},\overline{R}}\wedge t}\right) &= V_b(u) + \int_0^{\tau_b^{\overline{D},\overline{R}}\wedge t} \mathrm{e}^{-\delta s}\left[\tilde{\mathcal{L}}(V_b)(U_s^b) - \delta V_b(U_s^b)\right] ds \\
&+ \int_0^{\tau_b^{\overline{D},\overline{R}}\wedge t} \mathrm{e}^{-\delta s}\sigma V_b^{'}(U_s^b) dW_s - \int_0^{\tau_b^{\overline{D},\overline{R}}\wedge t} \mathrm{e}^{-\delta s} V_b^{'}(U_s^b)\mathbf{1}_{\{U^b_{s-}\geq b\}} dD_s^b \\
&+ \sum_{\substack{0\leq s\leq \tau_b^{\overline{D},\overline{R}}\wedge t \\ U_s^b\neq U_{s-}^b}} \mathrm{e}^{-\delta s}\left[V(U_s^b) - V(U_{s-}^b)\right] \\
&= V_b(u) + \int_0^{\tau_b^{\overline{D},\overline{R}}\wedge t} \mathrm{e}^{-\delta s}\left[\tilde{\mathcal{L}}(V_b)(U_s^b) - \delta V_b(U_s^b)\right] ds \\
&+ \int_0^{\tau_b^{\overline{D},\overline{R}}\wedge t} \mathrm{e}^{-\delta s}\sigma V_b^{'}(U_s^b) dW_s - \int_0^{\tau_b^{\overline{D},\overline{R}}\wedge t} \mathrm{e}^{-\delta s} V_b^{'}(U_s^b)\mathbf{1}_{\{U^b_{s-}\geq b\}} dD_s^b \\
&+ \sum_{\substack{0\leq \tau_i\leq \tau_b^{\overline{D},\overline{R}}\wedge t \\ U_s^b\neq U_{s-}^b}} \mathrm{e}^{-\delta \tau_i}\left[V(U_{\tau_i}^b) - V(U_{\tau_i-}^b)\right]. \qquad (25)
\end{aligned}$$

But since $V_b^{'}(b) = 1$, it follows by rearrangement that

$$\begin{aligned}
\int_0^{\tau_b^{\overline{D},\overline{R}}\wedge t} \mathrm{e}^{-\delta s}\mathbf{1}_{\{U^b_{s-}\geq b\}} dD_s^b &= -\mathrm{e}^{-\delta(\tau_b^{\overline{D},\overline{R}}\wedge t)} V\left(U^b_{\tau_b^{\overline{D},\overline{R}}\wedge t}\right) \\
&+ V_b(u) + \int_0^{\tau_b^{\overline{D},\overline{R}}\wedge t} \mathrm{e}^{-\delta s}\left[\tilde{\mathcal{L}}(V_b)(U_s^b) - \delta V_b(U_s^b)\right] ds \\
&+ \int_0^{\tau_b^{\overline{D},\overline{R}}\wedge t} \mathrm{e}^{-\delta s}\sigma V_b^{'}(U_s^b) dW_s \\
&+ \sum_{\substack{0\leq \tau_i\leq \tau_b^{\overline{D},\overline{R}}\wedge t \\ U_s^b\neq U_{s-}^b}} \mathrm{e}^{-\delta \tau_i}\left[V(U_{\tau_i}^b - kX_i\wedge a) - V(U_{\tau_i-}^b)\right].
\end{aligned} \qquad (26)$$

Since

$$\sum_{\substack{0\leq\tau_i\leq\tau_b^{\overline{D},\overline{R}}\wedge t\\ U_s^b\neq U_{s-}^b}} \mathrm{e}^{-\delta\tau_i}\left[V(U_{\tau_i}^b-kX_i\wedge a)-V(U_{\tau_i-}^b)\right]$$

$$-\lambda\int_0^{\tau_b^{\overline{D},\overline{R}}\wedge t}\int_0^{U_s^b}\mathrm{e}^{-\delta s}\left[V(U_{s-}^b-kx\wedge a)-V(U_{s-}^b)\right]dF(x)ds,$$

and

$$\int_0^{\tau_b^{\overline{D},\overline{R}}\wedge t}\mathrm{e}^{-\delta s}\sigma V_b^{'}(U_s^b)dW_s,$$

are martingales with mean zero, taking expectations gives

$$\mathbb{E}\left[\int_0^{\tau_b^{\overline{D},\overline{R}}\wedge t}\mathrm{e}^{-\delta s}dD_s^b\right]=V_b(u)+\mathbb{E}\left[-\mathrm{e}^{-\delta(\tau_b^{\overline{D},\overline{R}}\wedge t)}V\left(U^b_{\tau_b^{\overline{D},\overline{R}}\wedge t}\right)\right]$$
$$+\mathbb{E}\left[\int_0^{\tau_b^{\overline{D},\overline{R}}\wedge t}\mathrm{e}^{-\delta s}\left[\tilde{\mathcal{L}}(V_b)(U_s^b)-\delta V_b(U_s^b)\right]ds\right]. \quad (27)$$

Combining Equation (27) with Lemma 2.4 in Nansubuga (2016), and recalling that $\tilde{\mathcal{L}}(V_b)(U_s^b)=\delta V_b(U_s^b)$, we have

$$\mathbb{E}\left[\int_0^{\tau_b^{\overline{D},\overline{R}}\wedge t}\mathrm{e}^{-\delta s}dD_s^b\right]\leq V_b(u)-\mathbb{E}\left[\mathrm{e}^{-\delta(\tau_b^{\overline{D},\overline{R}}\wedge t)}V\left(U^b_{\tau_b^{\overline{D},\overline{R}}\wedge t}\right)\right]. \quad (28)$$

But by definition of $\tau_b^{\overline{D},\overline{R}}$ and by the fact that $V_b(0)=0$, we have that

$$\lim_{t\to\infty}\mathrm{e}^{-\delta(\tau_b^{\overline{D},\overline{R}}\wedge t)}V_b\left(U^b_{\tau_b^{\overline{D},\overline{R}}\wedge t}\right)=\mathrm{e}^{-\delta\tau_b^{\overline{D},\overline{R}}}V_b(0)\mathbf{1}_{\{\tau_b^{\overline{D},\overline{R}}<\infty\}}+\lim_{t\to\infty}\mathrm{e}^{-\delta\tau_b^{\overline{D},\overline{R}})}V_b(U_t)\mathbf{1}_{\{\tau_b^{\overline{D},\overline{R}}=\infty\}}=0,$$

so that inequality (28) becomes

$$\mathbb{E}\left[\int_0^{\tau_b^{\overline{D},\overline{R}}\wedge t}\mathrm{e}^{-\delta s}dD_s^b\right]\leq V_b(u). \quad (29)$$

Letting $t\to\infty$, we have

$$\mathbb{E}\left[\int_0^{\tau_b^{\overline{D},\overline{R}}}\mathrm{e}^{-\delta s}dD_s^b\right]\leq V_b(u), \quad (30)$$

and the result follows. ■

4. Methods and results

To solve for the survival probability $\phi(u)$ (from which we obtain the ruin probability $\psi(u) = 1-\phi(u)$) and for the dividend value function $V_b(u)$, we use the fourth-order block-by-block method outlined in this section and described fully in Linz (1969, 1985) and Paulsen et al. (2005). In this regard, we use a fixed grid $u = 0, h, 2h, \ldots$ The numerical solution of the general linear Volterra integral equation of the second kind

$$g(u) + \int_0^u K(u,x)g(x)dx = \alpha(u), \tag{31}$$

where the kernel $K(u,x)$ and the forcing function $\alpha(u)$ are known functions and $g(u)$ is the unknown function to be determined, is of the form

$$g_n + h\sum_{i=1}^{n} w_i K_{n,i} g_i = \alpha_n, \tag{32}$$

where g_i is the numerical approximation to $g(ih)$, $K_{n,i} = K(nh, ih)$, $g_n = g(nh)$ and $\alpha_n = \alpha(nh)$. The w_i are the integration weights. The forcing function $g(u)$ refers to the value function which, in this paper, may be the dividend value function $V_b(u)$ or the ultimate survival probability $\phi(u)$. Here, the block-by-block method will be used in conjunction with Simpson's rule of integration to obtain solutions in blocks of two values.

Linz (1969) has shown that the block-by-block method always converges and has an order of convergence of four (see also Huang et al. (2012)). This method reduces the VIE of the second kind into a system of algebraic equations which are solved by matrix methods to obtain the blocks (for details, see Kasumo (2011)).

Definition 4.1.

Convergence: Let $g_0(h), g_1(h), \ldots$ denote the approximation obtained by a given method using step-size h. Then a method is said to be convergent if and only if

$$|g_i(h) - g(u_i)| \to 0, \ for\ \boldsymbol{i} = 0, 1, 2, \ldots, N,$$

as $h \to 0$, $N \to \infty$, such that $Nh = a$.

Definition 4.2.

Order of convergence: A method is said to be of order q if q is the largest number for which there exists a finite constant C such that

$$|g_i(h) - g(u_i)| \leq Ch^q, \quad i = 0, 1, 2, \ldots, \text{ for all } \boldsymbol{h} > 0.$$

Remark 4.3.

By Theorem 3.1 in Paulsen et al. (2005) and from results in Chapter 7 of Linz (1985), it follows that for a fixed u so that $nh = u$, the solution satisfies

$$|g_n - g(u)| = O(h^4), \tag{33}$$

provided that g is four times continuously differentiable as is the case here by Theorem 2.4 in Paulsen et al. (2005). On the other hand, for the block-by-block method $|g_{2m+2} - g_{2m+1}| = O(h^4)$ as well.

To maximize dividends under a ruin probability target for a model with initial capital u and ruin probability tolerance ϵ, the following calculations have been performed:

(a) Using u, we solve the dividend maximization problem to determine the optimal barrier b^*.
(b) For each optimal dividend barrier b^*, we incorporate proportional and XL reinsurance into the CLM and the DPM.
(c) In the ultimate ruin problem, we choose b_0 so that $\psi(b_0) = \epsilon$, which is the ultimate ruin probability at b_0. This means that dividends cannot be paid unless the survival probability $1-\epsilon$ is greater than ϵ.

With both b^* and b_0, the decision is based on Theorem 3.6. Some results are now presented based on the exponential and Pareto distributions. The Exp(β) distribution, a special case of the Weibull(α, β) distribution, has density $f(x) = \beta e^{-\beta x}$, with corresponding distribution and tail functions $F(x) = 1 - e^{-\beta x}$ and $\overline{F}(x) = 1 - F(x) = e^{-\beta x}$, respectively. The mean excess function for the exponential distribution is $e_F(x) = \frac{1}{\beta}$ and $G(x) = x - \frac{1}{\beta}F(x)$. The Pareto($\alpha, \kappa$) distribution, which is a special case of the three-parameter Burr(α, κ, τ) distribution, has density $f(x) = \frac{\alpha\kappa^\alpha}{(\kappa+x)^{\alpha+1}}$ for $\alpha > 0$ and $\kappa = \alpha - 1 > 0$, and its distribution function is $F(x) = 1 - \left(\frac{\kappa}{\kappa+x}\right)^\alpha$. The Pareto tail distribution is $\overline{F}(x) = \left(\frac{\kappa}{\kappa+x}\right)^\alpha$ and its mean excess function is $e_F = 1 + \frac{x}{\kappa}$, so that $G(x) = x - \left(1 + \frac{x}{\kappa}\right) F(x)$. Alternatively, for the Pareto distribution $G(x)$ can be written as $x - 1 + \left(\frac{\kappa}{\kappa+x}\right)^\kappa$.

A grid size of $h = 0.01$ was used throughout. The data simulations in this section were performed on a Samsung Series 3 PC with an Intel Celeron 847 processor at 1.10GHz and 6.0GB RAM. To reduce computing time, the numerical method was implemented using the FORTRAN programming language, taking advantage of its DOUBLE PRECISION feature which gives a high degree of accuracy. The graphs were constructed using MATLAB R2016a.

The various cases of dividend payouts with reinsurance can be derived from Equations (20)-(24) (Theorem 3.4), which represent dividend models compounded by proportional and XL reinsurance, with appropriate values of k, a and σ. The results based on the Cramér-Lundberg model (CLM) and diffusion-perturbed model (DPM) are given in the following sections.

4.1. Ruin probability targets: CLM (exponential claims)

Here the kernel and forcing function are given by

$$\begin{aligned} K(u,x) &= -\frac{\lambda\overline{F}(u-x)}{c}, \\ \alpha(u) &= \phi(0), \end{aligned} \tag{34}$$

with $\overline{F}(x) = 1 - F(x)$. Table 1 gives the ultimate ruin probabilities in the CLM with no reinsurance and dividends.

Table 1. Ultimate ruin probabilities in the CLM for Exp(0.5) claims, $c = 6,\ \lambda = 2$

u	0	4	8	10	12	14	16	18	20
$\psi(u)$	0.6667	0.3423	0.1757	0.1259	0.0902	0.0646	0.0463	0.0332	0.0238

As expected, increasing the initial capital u reduces the ruin probability $\psi(u)$. We now set ruin probabilities to obtain different values of initial capital to be used as ruin probability target values in the dividend model for the CLM without reinsurance, that is, with $k = 1$ and $a = \infty$. Let $\epsilon_i \equiv i$-th ruin probability used. Then, choosing arbitrarily from Table 1,

$\psi(b_0^1) = \epsilon_1 = 0.1259$ gives $b_0^1 = 10.00$,
$\psi(b_0^2) = \epsilon_2 = 0.0902$ gives $b_0^2 = 12.00$,
$\psi(b_0^3) = \epsilon_3 = 0.0646$ gives $b_0^3 = 14.00$,
$\psi(b_0^4) = \epsilon_4 = 0.0332$ gives $b_0^4 = 18.00$.

4.2. Dividends: CLM (exponential claims)

The VIE in this case has kernel and forcing function

$$\begin{aligned} K(u,x) &= -\frac{\delta + \lambda \overline{F}(u-x)}{c}, \\ \alpha(u) &= V_b(0). \end{aligned} \tag{35}$$

The exact solution can be found in Kasozi and Paulsen (2005a). The total expected present value of dividends is given by the value function

$$V_b(u) = \begin{cases} \frac{f(u)}{f'(b)}, & u \le b, \\ \frac{f(b)}{f'(b)} + u - b, & u > b, \end{cases} \tag{36}$$

where $f(u) = (\beta + r_1)\mathrm{e}^{r_1 u} - (\beta + r_2)\mathrm{e}^{r_2 u}$ (Kasozi et al. (2011)), where r_1 and r_2 are given by

$$r_{1,2} = \frac{-(c\beta - \lambda - \delta) \pm \sqrt{(c\beta - \lambda - \delta)^2 + 4c\beta\delta}}{2c}. \tag{37}$$

The optimal barrier b^* is obtained by solving the equation $f''(b^*) = 0$, that is, $(\beta + r_1)r_1^2\mathrm{e}^{r_1 b^*} - (\beta + r_2)r_2^2\mathrm{e}^{r_2 b^*} = 0$. For any arbitrary starting point $f(0)$, $f(u)$ is the $O(h^4)$ numerical solution obtained using the block-by-block method. To find $f'(b)$, we use the approximation $f'(b) \approx \lim_{h\to 0} \frac{f(b+h) - f(b-h)}{2h}$, where h is the grid size. We have solved (36) for several values of b. Using a FORTRAN program which, at each run, gives the $O(h^4)$ solution to the Volterra equations of the second kind (Theorem 3.4), we have computed the values of $f'(b)$. The results indicate that for any two barriers b_1 and b_2, with $0 < b_1 < b_2 < \infty$, $f'(b_1) > f'(b_2)$. Eventually, some interval $[b_1, b_2]$ gives $f'(b_1) < f'(b_2)$ for the first time. This interval contains the optimal b^* which gives the optimal value function $V_b(b^*)$. The results are given in Table 2 and Figure 1.

Table 2. Dividends in the CLM with Exp(0.5) claims, $c = 6,\ \lambda = 2,\ \delta = 0.1$

b	$u = 2$	$u = 4$	$u = 6$	$u = 8$	$u = 10$	$u = 12$	$u = 14$
2	5.8278	7.8278	9.8278	11.8278	13.8278	15.8278	17.8278
4	6.9460	9.1224	11.1224	13.1224	15.1224	17.1224	19.1224
6	7.8320	10.2860	12.4009	14.4009	16.4009	18.4009	20.4009
8	8.3886	11.0169	13.2822	15.3439	17.3439	19.3439	21.3439
10	8.5897	11.2810	13.6006	15.7117	17.7294	19.7294	21.7294
12	8.4894	11.1494	13.4419	15.5284	17.5225	19.5058	21.5058
14	8.1872	10.7525	12.9634	14.9757	16.8988	18.8115	20.7718
16	7.7567	10.1871	12.2817	14.1881	16.0101	17.8223	19.6794
18	7.2728	9.5516	11.5155	13.3030	15.0114	16.7105	18.4517
20	6.7561	8.8729	10.6974	12.3579	13.9448	15.5232	17.1408
$b^* = 10.27$	8.5923	11.2844	13.6047	15.7165	17.7348	19.9749	21.9749
$b_0^1 = 10.00$	10.0000	11.2844	13.6047	15.7165	17.7348	19.9749	21.9749
$b_0^2 = 12.00$	12.0000	12.0000	13.6047	15.7165	17.7348	19.9749	21.9749
$b_0^3 = 14.00$	14.0000	14.0000	14.0000	15.7165	17.7348	19.9749	21.9749
$b_0^4 = 18.00$	18.0000	18.0000	18.0000	18.0000	18.0000	19.9749	21.9749

Using Theorem 3.6, we obtain the optimal barriers under ruin probability targets. For example, for initial capital $u = 2$ the optimal dividend barrier is $b^* = 8.5923$ and $b_0^1 = 10.0000$. Since $b_0^1 > b^*$, we take 10.0000 as the optimal barrier. For $u = 6$, $b^* = 13.6047$ and $b_0^1 = 10.0000$. Since $b^* > b_0^1$, we take 13.6047 as the optimal barrier. The optimal barriers for other values of u can be obtained in a similar manner. The results are presented in Table 2 and Figure 1. The company pays out dividends to the shareholders whenever $b^* > b_0^1$ because of the ruin probability target. Figure 1 shows that as the ruin probability reduces, the optimal dividend barrier increases and this is precisely the goal of dividend maximization.

It should be noted that for Pareto claim sizes, $b^* > b_0^i$ $(i = 1, 2, 3, 4)$ for all u. Thus, the company can pay dividends at all barrier levels.

4.3. Dividends: CLM with proportional reinsurance (exponential claims)

The VIE for the CLM compounded by proportional reinsurance has kernel and forcing function

$$\begin{aligned} K(u,x) &= -\frac{\delta + \lambda\overline{F}(u-kx)}{kc}, \\ \alpha(u) &= V_b(0), \end{aligned} \tag{38}$$

where $k \in [0,1]$ is the retention level for quota-share reinsurance. It has been shown by Kasumo et al. (2018b) that for dividend maximization it is optimal not to take proportional reinsurance in the small claim case involving the CLM. Therefore, the ruin probability targets and optimal barriers under proportional reinsurance are the same as shown in the immediately preceding sections (see Tables 1 and 2).

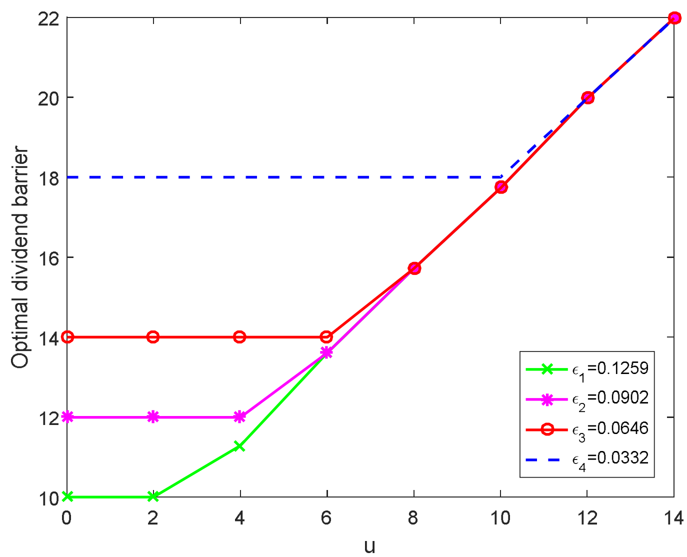

Figure 1. Numerical optimal barriers in CLM for Exp(0.5) claims, $c = 6,\ \lambda = 2,\ \delta = 0.1$

4.4. Dividends: CLM with XL reinsurance (exponential claims)

The kernel and forcing function for this case are given by

$$K(u,x) = -\frac{\delta + \lambda H_1(x,u)}{c^{\overline{R}}}, \quad \alpha(u) = V_b(0), \tag{39}$$

with

$$H_1(x,u) = \begin{cases} \overline{F}(u-x), & x < a, \\ 1 - (F(x+a) - F(a)), & x \geq a, \end{cases}$$

where $c^{\overline{R}} = c - (1+\theta)\lambda\mathbb{E}[(X_i - a)^+]$ is the insurance premium rate. Kasumo et al. (2018b) have shown that it is optimal not to take XL reinsurance in the CLM for exponential claims. This means that the ruin probability targets and optimal dividend barriers for XL reinsurance are the same as those for QS reinsurance as shown in Tables 1 and 2.

4.5. Ruin probability targets: CLM with XL reinsurance (Pareto claims)

Since in the large claim case involving the CLM the optimal policy is to take XL reinsurance with $a^* = 10$ (see Kasumo et al. (2018b)), we have to compute ruin probabilities for $a = 10$. These are shown in Table 3.

The ruin probability $\psi(u)$ reduces quite slowly as the initial capital u increases. We now set ruin probabilities to obtain different values of initial capital to be used as ruin probability target values

Table 3. Ultimate ruin probabilities in the CLM for Par(3,2) claims, $c = 6,\ \lambda = 2$

u	0	4	8	10	10.5	11	11.5	12	12.5
$\psi(u)$	0.6667	0.5363	0.5169	0.5132	0.4888	0.4318	0.3512	0.2492	0.1251

in the dividend model for the CLM with optimal XL reinsurance retention $a^* = 10$. Let $\epsilon_i \equiv i$-th ruin probability used. Then, choosing arbitrarily from Table 3,

$\psi(b_0^1) = \epsilon_1 = 0.5169$ gives $b_0^1 = 8.00$,
$\psi(b_0^2) = \epsilon_2 = 0.5132$ gives $b_0^2 = 10.00$,
$\psi(b_0^3) = \epsilon_3 = 0.4318$ gives $b_0^3 = 11.00$,
$\psi(b_0^4) = \epsilon_4 = 0.2492$ gives $b_0^4 = 12.00$.

4.6. Dividends: CLM with XL reinsurance (Pareto claims)

The optimal dividend barrier for this model is found to be $b^* \approx 9.75$ and the results are given in Table 4. It turns out that for Pareto(3,2) claims in the CLM compounded by XL reinsurance $b^* > b_0^i\ (i = 1, 2, 3, 4)\ \forall\ u > 0$. Therefore, the company can pay dividends at all barrier levels.

Table 4. Dividends in the CLM with XL reinsurance: Par(3,2) claims, $c = 6,\ \lambda = 2,\ \delta = 0.1$

b	$u = 2$	$u = 4$	$u = 6$	$u = 8$	$u = 10$	$u = 12$	$u = 14$
2	13.7872	15.7872	17.7872	19.7872	21.7872	23.7872	25.7872
4	22.2092	24.6831	26.6831	28.6831	30.6831	32.6831	34.6831
6	26.8729	29.8662	32.0364	34.0364	36.0364	38.0364	40.0364
8	28.6866	31.8819	34.1985	36.2515	38.2515	40.2515	42.2515
10	27.6877	30.7718	33.0077	34.9893	36.9052	38.9052	40.9052
$b^* = 9.75$	28.9684	32.1951	34.5345	36.6077	38.3522	40.3522	42.3522

4.7. Ruin probability targets: DPM (exponential claims)

The ruin probabilities for the DPM with $\sigma = 1$ are given in Table 5.

Table 5. Ultimate ruin probabilities in the DPM for Exp(0.5) claims, $c = 6,\ \lambda = 2, \sigma = 1$

u	0	4	8	10	12	14	16	18	20
$\psi(u)$	1.0000	0.3581	0.1872	0.1354	0.0979	0.0708	0.0512	0.0370	0.0268

Since the optimal reinsurance policy for the dividend maximization problem is $(k^*, a^*) = (1, \infty)$, that is, do not reinsure, we have used only the ruin probabilities $\psi_{k=1}(u)$ which are the same as $\psi_{a=\infty}(u)$ to choose ruin probability targets and set optimal dividend barriers under set ruin probability targets. Thus, from Table 5, we arbitrarily choose

$\psi(b_0^1) = \epsilon_1 = 0.1354$, giving $b_0^1 = 10.00$,
$\psi(b_0^2) = \epsilon_2 = 0.0708$, giving $b_0^2 = 14.00$,
$\psi(b_0^3) = \epsilon_3 = 0.0512$, giving $b_0^3 = 16.00$,
$\psi(b_0^4) = \epsilon_4 = 0.0268$, giving $b_0^4 = 20.00$.

4.8. Dividends: DPM with proportional reinsurance (exponential claims)

The kernel and forcing function for this case are, respectively,

$$\begin{aligned} K(u,x) &= \frac{2\left(kc + \lambda G(u-kx) - (\lambda+\delta)(u-kx)\right)}{\sigma^2}, \\ \alpha(u) &= uV_b^{'}(0) \text{ if } \sigma^2 > 0, \end{aligned} \tag{40}$$

with $G(x) = \int_0^x F(v)dv$. The optimal barriers for varying initial surplus values are shown in Figure 2. The optimal barrier without a ruin probability target in this case was found as $b^* = 12.35$.

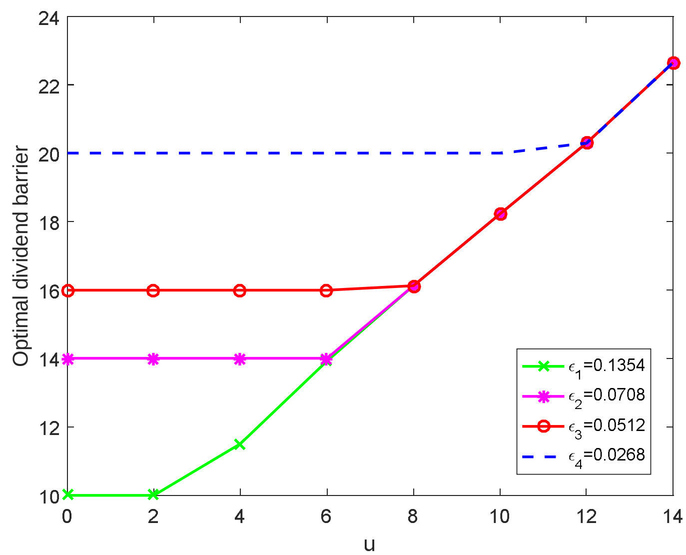

Figure 2. Numerical optimal barriers in DPM for Exp(0.5) claims, $c = 6,\ \lambda = 2,\ \delta = 0.1,\ \sigma = 1$

At $u = 2$, for example, the optimal barrier without a ruin probability target is 12.35. Thus, the insurer only pays out dividends to the shareholders when the surplus exceeds 12.35. However, with a ruin probability target, say $\epsilon_3 = 0.0512$, the optimal barrier is 16.00 instead of 12.35. This increases the optimal barrier, thereby reducing the chances of the company undergoing ruin. This applies to all other models considered in this section (see Figures 1 and 2).

4.9. Ruin probability targets: DPM with XL reinsurance (Pareto claims)

The kernel and forcing function for this case are given by Equation (13) (Theorem 3.3) when $k = 1$ and $c^{\overline{R}}$ is as defined in Section 4.4. The infinite ruin probabilities for the DPM for Pareto(3,2) claim sizes are given in Table 6.

Table 6. Ultimate ruin probabilities in the DPM for Par(3,2) claims, $c = 6,\ \lambda = 2,\ \sigma = 1$

u	0	8	16	24	32	34	36	38	40
$\psi(u)$	1.0000	0.0273	0.0080	0.0036	0.0020	0.0018	0.0016	0.0014	0.0012

Choosing arbitrarily from Table 6, we have

$\psi(b_0^1) = \epsilon_1 = 0.0020$, giving $b_0^1 = 32.00$,
$\psi(b_0^2) = \epsilon_2 = 0.0016$, giving $b_0^2 = 36.00$,
$\psi(b_0^3) = \epsilon_3 = 0.0014$, giving $b_0^3 = 38.00$,
$\psi(b_0^4) = \epsilon_4 = 0.0012$, giving $b_0^4 = 40.00$.

4.10. Dividends: DPM with XL reinsurance (Pareto claims)

By Theorem 3.4, the kernel and forcing function are given by Equation (24) for $k = 1$ and $c^{\overline{R}}$ is as defined in Section 4.4. We use the same analysis and discussion of results as in Sections 4.1 and 4.2. We find the optimal barrier without a ruin probability target to be $b^* = 11.50$. However, under ruin probability targets, the optimal barriers for Pareto claims are obtained using Theorem 3.6 and given in Figure 3 for selected ruin probabilities.

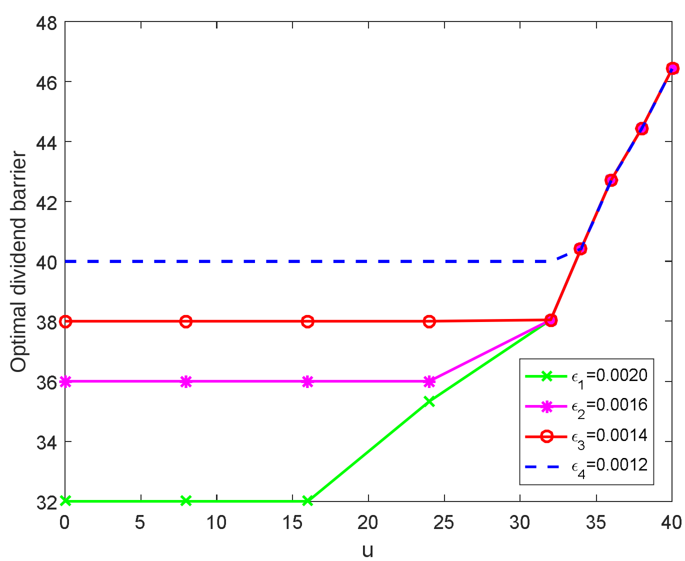

Figure 3. Numerical optimal barriers in DPM for Par(3,2) claims, $c = 6,\ \lambda = 2,\ \delta = 0.1,\ \sigma = 1$

4.11. Convergence results

In this section, convergence results of the block-by-block method are presented in the form of comparisons of the exact and approximate solutions based on two VIEs of the second kind.

Example 4.4.

Consider the VIE:

$$u(x) = 1 - x\sin x + x\cos x + \int_0^x tu(t)dt. \tag{41}$$

This equation has exact solution

$$u(x) = \sin x + \cos x.$$

The forcing function is

$$1 - x\sin x + x\cos x.$$

For the purpose of the numerical algorithm used here, the kernel is the entire function between the integral and dt, which is

$$tu(t),$$

and the derivative of the kernel with respect to u is t. Substituting these into the program for implementing the 2-block block-by-block method yields the results in Figure 4(a).

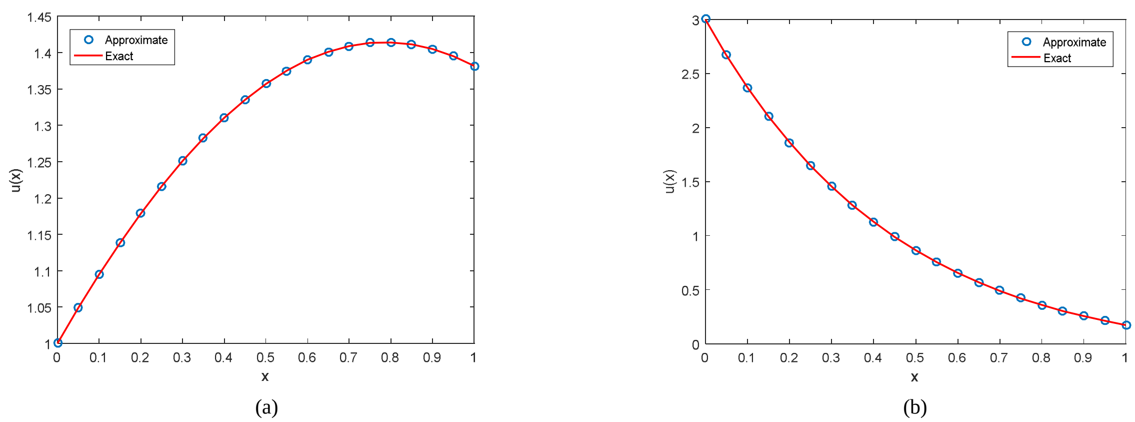

Figure 4. Fourth-order approximate solution for (a) Equation (41) (b) Equation (42)

Example 4.5.

Consider the VIE

$$u(x) = 3 + 2x - \int_0^x [2(x-t) + 3]u(t)dt. \tag{42}$$

The exact solution of Equation (42) is

$$u(x) = 4\mathrm{e}^{-2x} - \mathrm{e}^{-x}.$$

The forcing function is

$$3 + 2x,$$

the kernel is

$$-[2(x-t)+3]u(t)$$

and the derivative of the kernel with respect to u is $-[2(x-t)+3]$. The numerical results are compared with the exact solution in Figure 4(b) and show that the block-by-block method performs very well.

5. Conclusion

The study has shown that as the ruin probability reduces the optimal dividend barrier to use for payment of dividends increases, and vice versa. Therefore, the use of ruin probability targets by the insurance company is highly desirable from the shareholders' point of view. This is because a lower ruin probability makes it possible for the insurance company to pay more in dividends to the shareholders (that is, to use a higher dividend value function or optimal barrier).

The study has established that the reinsurance strategies are no different than before imposing ruin probability targets but that the optimal dividend barriers to use for dividend maximization increase as the ruin probability reduces. Insurance companies should therefore work towards reducing their ruin probabilities using some risk measures as this has a desirable effect on the optimal dividend barriers to be used for dividend payouts.

This work could be extended by: (1) including investments; (2) incorporating transaction costs when paying dividends; (3) exploring optimality of other dividend strategies (e.g., threshold or band); and (4) replacing the claim number process N with a general renewal process so that the surplus process becomes a Sparre-Andersen model.

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