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Robust Filtering for Discrete Systems with Unknown Inputs and Jump Parameters

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Abstract—The paper deals with robust filtering algorithms for discrete systems with unknown inputs (disturbances) and Markovian jump parameter. The proposed filtering algorithm is based on the separation principle, minimization of a quadratic criterion and the use of Kalman filters with unknown input and smoothing procedures. Solving a non-stationary problem is represented solving a two-point boundary value problem in kind of difference matrix equations. In the stationary case problem is represented matrix algebraic equations. Robustness ensures the stability of the filter dynamics when errors occur in identifying the jump parameter. An example is provided to illustrate the proposed approach, which showed that the use of smoothing procedures for estimating an unknown input improves the accuracy of estimates.

Keywords: robust filtering, jump parameter, unknown inputs, smoothing estimators

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1. INTRODUCTION

Systems with Markovian jump parameters are a special class of switching systems, and they are modeled by a set of systems with the transitions between the models determined by a Markov chain taking values in a finite set. There are many real applications of these systems, for example, economic systems [1–3], power systems [4, 5], flight systems [6], communication systems [7].

The problems of estimating the states of systems with continuous time and with random jump parameters described by a Markov chain with a finite number of states were considered in [8–11]. Similar problems were studied for discrete systems (see, e.g. [12–15]). There is currently an interest in the literature for the Robust filtering problem (see, e.g. [16–19]). In [20–25] filtering problems for discrete systems with unknown inputs were considered.

In this paper, we construct a solution to the problem of synthesizing a robust filter for discrete systems with random jump parameters, a finite number of states and unknown inputs. The solution was obtained using the separation principle, Kalman filtering and smoothing algorithms. A filter transfer matrix is proposed to choose based on minimizing the sum of quadratic forms of estimation errors with averaging over the probabilities of the state of the jump process. A numerical example for a two-mode Markovian jump linear system, to show the advantage of using robust techniques with smoothing algorithms to filter, is provided.

2. PROBLEM FORMULATION

Let the mathematical model of the linear discrete-time stochastic system with an unknown input and a jump process is described by equation

$$x(k+1) = A_{\gamma(k)}x(k) + f(k) + q_{\gamma(k)}(k), \quad x(0) = x_0, \tag{1}$$

where $x(k) \in \mathrm{R}^n$ is the state of the system, $f(k)$ is an unknown input, $\gamma(k)$ is Markov chain with r states $(\gamma_1, \gamma_2, \ldots, \gamma_r)$; x_0 is random vector with known math expected value and covariance $\bar{x}_0 = \mathrm{E}\{x_0\}$, $N_{0,i} = \mathrm{E}\{(x_0 - \bar{x}_0)(x_0 - \bar{x}_0)^{\mathrm{T}} | \gamma = \gamma_i\}$, $i = \overline{1, r}$; $A_{\gamma(k)}$ are matrices of the appropriate dimensions; $q_{\gamma(k)}(k)$ – random perturbations with characteristics: $\mathrm{E}\{q_{\gamma(k)}(k)\}=0$, $\mathrm{E}\{q_{\gamma(k)}(k)q^{\mathrm{T}}_{\gamma(k)}(j)\mid\gamma(\xi)=\gamma(k),\ k\le\xi\le j\}=Q_{\gamma(k)}\delta_{kj}$ ($\mathrm{E}\{\cdot\}$ is the math expectation of a random variable, $^{\mathrm{T}}$ is denotes matrix transposition, δ_{kj} is Kronecker symbol).

An observation vector is:

$$y(k)=S_{\gamma(k)}x(k)+v_{\gamma(k)}(k), \tag{2}$$

where $v_{\gamma(k)}(k)$ is the Gaussian random sequence with characteristics: $\mathrm{E}\{v_{\gamma(k)}(k)\}=0$; $\mathrm{E}\{v_{\gamma(k)}(k)v^{\mathrm{T}}_{\gamma(k)}(j)\mid\gamma(\xi)=\gamma(k),\ k\le\xi\le j\}=V_{\gamma(k)}\ \delta_{kj}$.

It is assumed that the sequences $q_\gamma(k),v_\gamma(k)$ are independent and a pair of matrices A_γ,S_γ is detectable (here and below, the variable k is not indicated if this does not cause uncertainty).

The probability of states of the jump process $p_j(k)=P\{\gamma(k)=j\},\ j=\overline{1,r}$ satisfies the equation

$$p_j(k+1)=\sum_{i=1}^{r}p_i(k)p_{i,j},\quad p_j(0)=p_{j,0},\ j=\overline{1,r}, \tag{3}$$

where $p_{i,j}$ is the probability of transition from state i to state j for one step, $p_{j,0}$ is the initial probability of the j-th state.

According to the information received at the moment k, it is required to find an estimate of a state vector based on the Kalman filtering algorithm with a transfer matrix independent of the jump process $\gamma(k)$.

3. SYNTHESIS OF NONSTATIONARY FILTER

The synthesis of the robust filtering algorithm will be carried out on the basis of the separation principle. This means that we first construct estimate of the vector $\hat{x}(k)$ under the assumption that the vector $f(k)$ is known and then the vector of the estimate $\hat{f}(k)$ can be constructed under the assumption that the estimate of the state vector $\hat{x}(k)$ is known.

We construct an estimate of the state vector under the assumption that the vector is known, minimizing the following criterion for $k\in[0,T]$:

$$J[0;T_f]=\mathrm{E}\left\{\left[\sum_{k=0}^{T-1}\sum_{i=1}^{r}p_i(k)e^{\mathrm{T}}(k)R_ie(k)+\sum_{i=1}^{r}p_i(T)e^{\mathrm{T}}(T)M_ie(T)\right]\mid\gamma(0)=\gamma_0\right\}, \tag{4}$$

where $e(k)=x(k)-\hat{x}(k)$, $R_i>0$ and $M_i>0$ are weight matrices, γ_0 is initial value of variable γ.

We define the estimate $\hat{x}(k)$ using the Kalman filter:

$$\hat{x}(k+1)=A_\gamma\hat{x}(k)+f(k)+K(k)[y(k+1)-S_\gamma(A_\gamma\hat{x}(k)+f(k))],\quad \hat{x}(0)=\overline{x}_0, \tag{5}$$

where $K(k)$ is transfer matrix of the filter that does not depend on process $\gamma(k)$.

Subtracting (5) from (1) obtain the equation for the error vector:

$$e(k+1)=x(k+1)-\hat{x}(k+1)=(I-K(k)S_\gamma)A_\gamma e(k)+(I-K(k)S_\gamma)q_\gamma(k)-K(k)v_\gamma(k+1). \tag{6}$$

We introduce notations for matrices $Q_\gamma,V_\gamma,\ R_\gamma,\ N_\gamma,L_\gamma,\ A_\gamma,\ S_\gamma$, for $\gamma=\gamma_i$: $Q_i,\ V_i,\ R_i,\ N_i,L_i,\ A_i,\ S_i$, respectively ($i=\overline{1,r}$).

Theorem 1. Let matrices $N_i(k)>0$ and $L_i(k)>0$ satisfies of a two-point boundary value problem:

$$\begin{gathered}N_i(k+1)=(A_i-K(k)S_iA_i)\left(\sum_{j=1}^{r}p_{i,j}N_j(k)\right)(A_i-K(k)S_iA_i)^{\mathrm{T}}\\+(I-K(k)S_i)Q_i(I-K(k)S_i)^{\mathrm{T}}+K(k)V_iK(k)^{\mathrm{T}},\ \ N_i(0)=N_0,\end{gathered} \tag{7}$$

$$L_i(k)=(A_i-K(k+1)S_iA_i)^{\mathrm{T}}\left(\sum_{j=1}^{r}p_{i,j}L_j(k+1)p_{i,1}\right)(A_i-K(k+1)S_iA_i)+R_i,\quad L_i(T)=M_i, \tag{8}$$

then the matrix $K(k)$ expressed from the formula

$$\operatorname{vec}(K(k)) = \left\{\sum_{i=1}^{r} p_i(k+1)\left[L_i(k+1)\otimes S_i A_i\left(\sum_{j=1}^{r} p_{i,j}N_j(k)\right)A_i^{\mathrm T}S_i^{\mathrm T} + L_i(k+1)\otimes(S_i Q_i S_i^{\mathrm T} + V_i)\right]\right\}^{-1}$$
$$\times \operatorname{vec}\left\{\sum_{i=1}^{r} p_i(k+1)\left[L_i(k+1)A_i\left(\sum_{j=1}^{r} p_{i,j}N_j(k)\right)A_i^{\mathrm T}S_i^{\mathrm T} + L_i(k+1)Q_i S_i^{\mathrm T}\right]\right\}, \tag{9}$$

will provide a minimum of criterion (4). In (7) I is identity matrix of appropriate dimension, in (9) $\operatorname{vec}(K(k))$ is a vector, composed of transpose rows of the matrix $K(k)$.

Theorem 1 can be proven with using Lyapunov function. At first present criterion (4) as a sum

$$J[0;T] = \sum_{i=1}^{r} J_i[0;T], \tag{10}$$

where

$$J_i[0;T] = \sum_{k=0}^{T-1} \operatorname{tr} p_i(k)N_i(k)R_i + \operatorname{tr} p_i(T)N_i(T)M_i. \tag{11}$$

In (11) function tr is trace of a quadratic matrix, the matrix $N_i(k) = \mathrm{E}\{e(k)e(k)^T\ |\gamma = \gamma_i\}$ is determined from the equation

$$N_i(k+1) = (A_i - K(k)S_iA_i)\left(\sum_{j=1}^{r} p_{i,j}N_j(k)\right)(A_i - K(k)S_iA_i)^{\mathrm T}$$
$$+ (I - K(k)S_i)Q_i(I - K(k)S_i)^{\mathrm T} + K(k)V_iK(k)^{\mathrm T}, \quad N_i(0) = N_0. \tag{12}$$

We introduce the Lyapunov function of the following form:

$$W(k, N_i(k)) = \operatorname{tr} p_i(k)\,N_i(k)\,L_i(k) + \operatorname{tr}\sum_{t=k}^{T} p_i(t)\,\overline{\Psi}_i(t)\,L_i(t), \tag{13}$$

where

$$\overline{\Psi}_i(t) = (I - KS_i)\,Q_i(I - KS_i)^{\mathrm T} + K(t)V_i\,K(t)^{\mathrm T} + \Psi_i(t), \tag{14}$$

matrix $L_i(t)$ satisfies the equation

$$L_i(k) = (A_i - K(k+1)\,S_iA_i)^{\mathrm T}\left(\sum_{j=1}^{r} p_{i,j}\,L_j(k+1)p_{i,1}\right)(A_i - K(k+1)\,S_iA_i) + C_i, \quad L_i(T) = M_i, \tag{15}$$

and matrix C_i is determined later.

Let's sum up $k = \overline{0, T-1}$ the final differences of the function $W(k, N_i(k))$, taking into account formula (15)

$$\sum_{k=0}^{T-1}\Delta W(k, N_i(k)) = \sum_{k=0}^{T-1}[W(k+1, N_i(k+1)) - W(k, N_i(k))]$$
$$= \sum_{k=0}^{T-1}\operatorname{tr}\left[p_i(k+1)N_i(k+1)L_i(k+1) - p_i(k)N_i(k)L_i(k) - p_i(k)\overline{\Psi}_i(k)L_i(k)\right]. \tag{16}$$

On the other hand, this expression can be represented as:

$$\sum_{k=0}^{T-1}\Delta W(k, N_i(k)) = W(k+1, N_i(k+1)) - W(k, N_i(k)) + \ldots + W(T, N_i(T)) - W(T-1, N_i(T-1))$$
$$= \operatorname{tr} p_i(T)N_i(T)L_i(T) - \operatorname{tr} p_i(0)\,N_i(0)\,L_i(0) - \operatorname{tr}\sum_{k=0}^{T-1} p_i(k)\overline{\Psi}_i(k)L_i(k). \tag{17}$$

Add to formula (11) the difference of the right sides of expressions (16) and (17), as a result we get:

$$J_i[0;T_f] = \sum_{k=0}^{T-1} \text{tr}\, p_i(k)N(k)_i R_i + \text{tr}\, p_i(T)N_i(T)L_i(T) + \sum_{k=0}^{T-1} \text{tr}\big[p_i(k+1)N_i(k+1)L_i(k+1) - p_i(k)N_i(k)L_i(k)$$
$$- p_i(k)\overline{\Psi}_i(k)L_i(k)\big] - \left[\text{tr}\, p_i(T)N_i(T)L_i(T) - \text{tr}\, p_i(0)N_i(0)L_i(0) - \text{tr}\sum_{k=0}^{T-1} p_i(k)\overline{\Psi}_i(k)L_i(k)\right] \quad (18)$$
$$= \sum_{k=0}^{T-1} \text{tr}\, p_i(k)N_i(k)R_i + \sum_{k=0}^{T-1} \text{tr}\, p_i(k+1)N_i(k+1)L_i(k+1) - \sum_{k=1}^{T-1} \text{tr}\, p_i(k)N_i(k)L_i(k).$$

In criterion (10) we substitute (18) and (12):

$$J[0;T] = \sum_{i=1}^{r}\left\{\sum_{k=0}^{T-1} \text{tr}\, p_i(k)N_i(k)R_i + \sum_{k=0}^{T-1} \text{tr}\, p_i(k+1)N_i(k+1)L_i(k+1) - \sum_{k=1}^{T-1} \text{tr}\, p_i(k)N_i(k)L_i(k)\right\}$$
$$= \sum_{i=1}^{r}\left\{\sum_{k=0}^{T-1} \text{tr}\, p_i(k)N_i(k)R_i - \sum_{k=1}^{T-1} \text{tr}\, p_i(k)N_i(k)L_i(k) + \sum_{k=0}^{T-1} \text{tr}\, p_i(k+1)L_i(k+1)\Big[(A_i - K(k)S_iA_i)\right. \quad (19)$$
$$\left.\times\left(\sum_{j=1}^{r} p_{i,j}N_j(k)\right)(A_i - K(k)S_iA_i)^{\text{T}} + (I - K(k)S_i)Q_i(I - K(k)S_i)^{\text{T}} + K(k)V_iK^{\text{T}}(k)\Big]\right\}.$$

Using the technique of differentiation of the trace function from the product of matrices (see [26]) technique of, we calculate the derivative of $K(k)$:

$$\frac{\partial J}{\partial K(k)} = \sum_{k=0}^{T-1}\sum_{i=1}^{r}[-L_i(k+1)p_i(k+1)A_i\left(\sum_{j=1}^{r} p_{i,j}N_j(k)\right)A_i^{\text{T}}S_i^{\text{T}} - p_i(k+1)L_i(k+1)A_i\left(\sum_{j=1}^{r} p_{ij}N_j(k)\right)A_i^{\text{T}}S_i^{\text{T}}$$
$$+ p_i(k+1)L_i(k+1)K(k)S_iA_i\left(\sum_{j=1}^{r} p_{i,j}N_j(k)\right)A_i^{\text{T}}S_i^{\text{T}} + L_i(k+1)p_i(k+1)K(k)S_iA_i\left(\sum_{j=1}^{r} p_{i,j}N_j(k)\right)A_i^{\text{T}}S_i^{\text{T}} \quad (20)$$
$$- L_i(k+1)p_i(k+1)Q_iS_i^{\text{T}} - p_i(k+1)L_i(k+1)Q_iS_i^{\text{T}} + p_i(k+1)L_i(k+1)K(k)S_iQ_iS_i^{\text{T}}$$
$$+ L_i(k+1)p_i(k+1)K(k)S_iQ_iS_i^{\text{T}} + p_i(k+1)L_i(k+1)K(k)V_i + L_i(k+1)p_i(k+1)K(k)V_i] = 0.$$

We obtain the equations for determining the elements of the matrix $K(k)$ by equating this derivative to zero. Then an analytical solution of the linear matrix equation (20) for the vector ct$(K(k))$ using properties the Kronecker product operation [27] is represented as:

$$\text{vec}(K(k)) = \left\{\sum_{i=1}^{r} p_i(k+1)\left[L_i(k+1)\otimes S_iA_i\left(\sum_{j=1}^{r} p_{i,j}N_j(k)\right)A_i^{\text{T}}S_i^{\text{T}} + L_i(k+1)\otimes(S_iQ_iS_i^{\text{T}} + V_i)\right]\right\}^{-1}$$
$$\times\, \text{vec}\left\{\sum_{i=1}^{r} p_i(k+1)\left[L_i(k+1)A_i\left(\sum_{j=1}^{r} p_{i,j}N_j(k)\right)A_i^{\text{T}}S_i^{\text{T}} + L_i(k+1)Q_iS_i^{\text{T}}\right]\right\}. \quad (21)$$

We calculate the matrix C_i such that criterion (19):

$$J[0;T_f] = \sum_{i=1}^{r}\left\{\sum_{k=0}^{T-1} \text{tr}\, p_i(k)N_i(k)R_i + \sum_{k=0}^{T-1} \text{tr}\, p_i(k+1)N_i(k+1)L_i(k+1) - \sum_{k=1}^{T-1} \text{tr}\, p_i(k)N_i(k)L_i(k)\right\}$$

is minimal. Substitute (15) in the criterion (19)

$$J[0;T_f] = \sum_{i=1}^{r}\left\{\sum_{k=0}^{T-1} \text{tr}\, p_i(k)N_i(k)R_i + \sum_{k=0}^{T-1} \text{tr}\, p_i(k+1)N_i(k+1)L_i(k+1)\right.$$
$$\left. - \sum_{k=1}^{T-1} \text{tr}\, p_i(k)N_i(k)\left[(A_i - K(k+1)S_iA_i)^{\text{T}}\left(\sum_{j=1}^{r} p_{i,j}L_j(k+1)p_{i,1}\right)(A_i - K(k+1)S_iA_i) + C_i\right]\right\}$$
$$= \sum_{i=1}^{r}\left\{\text{tr}\, p_i(0)N_i(0)R_i + \sum_{k=0}^{T-1} \text{tr}\, p_i(k+1)N_i(k+1)L_i(k+1) + \sum_{k=1}^{T-1} \text{tr}\, p_i(k)N_i(k)[R_i - C_i]\right.$$
$$\left. - \sum_{k=1}^{T-1} \text{tr}\, p_i(k)N_i(k)\left[(A_i - K(k+1)S_iA_i)^{\text{T}}\left(\sum_{j=1}^{r} p_{i,j}L_j(k+1)p_{i,1}\right)(A_i - K(k+1)S_iA_i)\right]\right\},$$

we obtain that the minimum non-negative value of the criterion is achieved if

$$C_i = R_i. \tag{22}$$

Because of the matrices $N_i > 0$ and $L_i > 0$ by the condition of the theorem, and the matrix $\Psi_i(k) > 0$ is given arbitrarily, it is obvious that it can be chosen such that the finite difference (17):

$$\sum_{k=0}^{T-1} \Delta W(k, N_i(k)) = \operatorname{tr} p_i(T) N_i(T) L_i(T) - \operatorname{tr} p_i(0) N_i(0) L_i(0)$$
$$- \operatorname{tr} \sum_{k=0}^{T-1} p_i(k) [(I - KS_i) Q_i (E - KS_i)^{\mathrm{T}} + K(k) V_i K(k)^{\mathrm{T}} + \Psi_i(k)] L_i(k),$$

became negative. This condition guarantees Lyapunov stability. Thus, to find the matrix $K(k)$ it is necessary to solve the two-point boundary value problem (7) and (8) taking into account equation (9).

4. SYNTHESIS OF STATIONARY FILTER

In the stationary case, the matrix of transfer coefficients K will be constant, and the criterion will take the form:

$$J[0;\infty] = \sum_{i=1}^{r} \sum_{k=0}^{\infty} \operatorname{tr} \bar{p}_i N_i R_i, \tag{23}$$

where $\bar{p}_i$ — steady-state probabilities ($\bar{p}_i = \lim_{k\to\infty} p_i(k)$).

The two-point boundary value problem is transformed into the following system of matrix algebraic equations:

$$N_i = (A_i - KS_iA_i)\left(\sum_{j=1}^{r} p_{i,j} N_j\right)(A_i - KS_iA_i)^{\mathrm{T}} + (I - KS_i)Q_i(I - KS_i)^{\mathrm{T}} + KV_iK^{\mathrm{T}}, \tag{24}$$

$$L_i = (A_i - KS_iA_i)^{\mathrm{T}}\left(\sum_{j=1}^{r} p_{i,j} L_j\right)(A_i - KS_iA_i) + R_i. \tag{25}$$

The matrix K is determined by the formula

$$\begin{aligned}\operatorname{vec}(K) = &\left\{\sum_{i=1}^{r} \bar{p}_i \left[L_i \otimes S_iA_i\left(\sum_{j=1}^{r} p_{i,j} N_j\right) A_i^{\mathrm{T}} S_i^{\mathrm{T}} + L_i \otimes (S_i Q_i S_i^{\mathrm{T}} + V_i)\right]\right\}^{-1} \\ &\times \operatorname{vec}\left\{\sum_{i=1}^{r} \bar{p}_i \left[L_iA_i\left(\sum_{j=1}^{r} p_{i,j} N_j\right) A_i^{\mathrm{T}} S_i^{\mathrm{T}} + L_iQ_iS_i^{\mathrm{T}}\right]\right\}.\end{aligned} \tag{26}$$

Thus, for the synthesis of stationary filters, it is necessary to solve the system of matrix equations (24)–(26).

Note that if there are positive definite solutions N_i, L_i $(i = \overline{1,n})$ of the matrix equations (24)–(26), then from equation (25) and the condition $R_i > 0$ follows the validity of Theorem 1.6 [9], and this means the fulfillment of the condition of stochastic stability of the filter with matrices of dynamic $(A_i - KS_i A_i)$. The stationary filter has the form:

$$\hat{x}(k+1) = A_i\hat{x}(k) + \hat{f}(k) + K(y(k+1) - S_i(A_i\hat{x}(k) + \hat{f}(k)), \quad \hat{x}(0) = \bar{x}_0. \tag{27}$$

This means that the filter (27) will maintain the stability of the dynamics of the filter at errors in determining the values of the jump process. This fact ensures the robust condition of the filtering.

The problem of calculating the estimate of the unknown input $f(k)$ will be considered in the next section.

5. ESTIMATING AN UNKNOWN INPUT

As an algorithm for estimating an unknown input, we will use LSM-estimates; in this case, an estimate can be constructed on the basis of minimizing the additional criterion [10, 11] under the assumption that the value of the jump process is known:

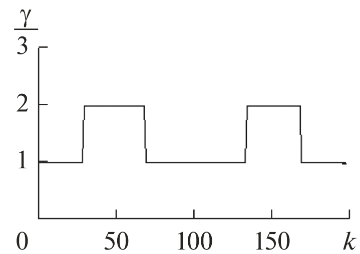

Fig. 1. Plot of the jump parameter γ.

$$J(f(k)) = \sum_{t=1}^{k+1} \left\{ \left\| y(t) - S_i(A_i\hat{x}(t-1) + f(t-1)) \right\|_{\overline{W}}^2 + \left\| f(t-1) \right\|_{\overline{\overline{W}}}^2 \right\}, \tag{28}$$

where $\overline{W}$, $\overline{\overline{W}}$ are positive definite weight matrices.

Minimizing (28) we obtain estimates of the unknown input:

$$\hat{f}^{\langle \mathrm{LMS} \rangle}(k) = [S_i^{\mathrm{T}} \overline{W} S_i + \overline{\overline{W}}]^{-1} S_i^{\mathrm{T}} \overline{W} \{y(k+1) - S_i A_i \hat{x}(k)\}. \tag{29}$$

Consider also another estimate of the unknown input using smoothing for the innovation process $y(k+1) - S_i A_i \hat{x}(k)$:

$$\hat{f}^{\langle \mathrm{SM} \rangle}(k) = [S_i^{\mathrm{T}} \overline{W} S_i + \overline{\overline{W}}]^{-1} S_i^{\mathrm{T}} \overline{W} \bar{\Omega}_i(k+1), \tag{30}$$

where $\bar{\Omega}_i(k+1)$ is calculated using the moving average algorithm:

$$\bar{\Omega}_i(k) = \sum_{\tau=k-l}^{k} \frac{[y(\tau+1) - S_i A_i \hat{x}(\tau)]_j}{l+1},$$

where $l + 1$ is averaging interval.

A numerical comparison of the two methods will be presented in the next section.

6. SIMULATION RESULTS

Consider the problem of modeling a robust filter for discrete-time stochastic system with two-dimensional state vector, unknown input $f(k)$, 2-mode Markovian jump parameter $\gamma(k)$ ($\gamma_1 = 1, \gamma_2 = 2$) and with transition probability matrix $P = [p_{i,j}] = \begin{pmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{pmatrix}$ on Fig. 1 represented by process $\gamma(k)$.

The simulation was performed on a time interval $k \in [0, 200]$.

Consider system (1) with the following data:

$$A_1 = \begin{pmatrix} 0.85 & 0.1 \\ -0.05 & 0.94 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0.89 & 0.05 \\ -0.02 & 0.45 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} 0.03 & 0 \\ 0 & 0.04 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0.05 & 0 \\ 0 & 0.03 \end{pmatrix},$$

$$f(k) = \begin{cases} \begin{pmatrix} -0.3 & -0.2 \end{pmatrix}^{\mathrm{T}} & \text{if } k \in [1;30], \\ \begin{pmatrix} 0.4 & 0.3 \end{pmatrix}^{\mathrm{T}} & \text{if } k \in [31;75], \\ \begin{pmatrix} 0 & -0.2 \end{pmatrix}^{\mathrm{T}} & \text{if } k \in [76;126], \\ \begin{pmatrix} 0.5 & 0.1 \end{pmatrix}^{\mathrm{T}} & \text{if } k \in [127;184], \\ \begin{pmatrix} 0.1 & 0.4 \end{pmatrix}^{\mathrm{T}} & \text{if } k \in [185;200]. \end{cases}$$

The matrices describing observation vector (2) are as follows:

$$S_1 = S_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad V_1 = \begin{pmatrix} 0.06 & 0 \\ 0 & 0.02 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0.07 & 0 \\ 0 & 0.03 \end{pmatrix}.$$

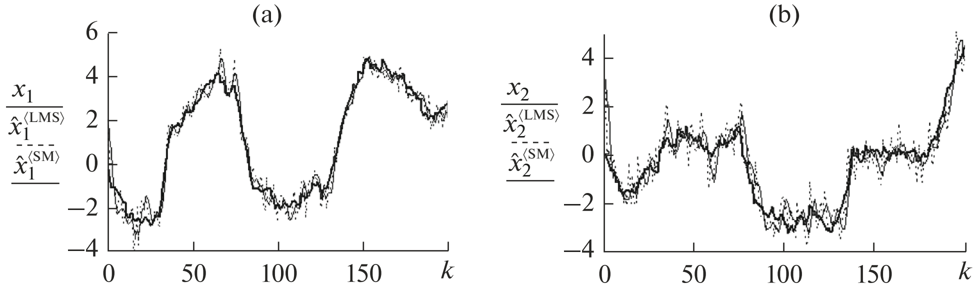

Fig. 2. Plots of the $x_1(k)$, $x_2(k)$ and its estimates.

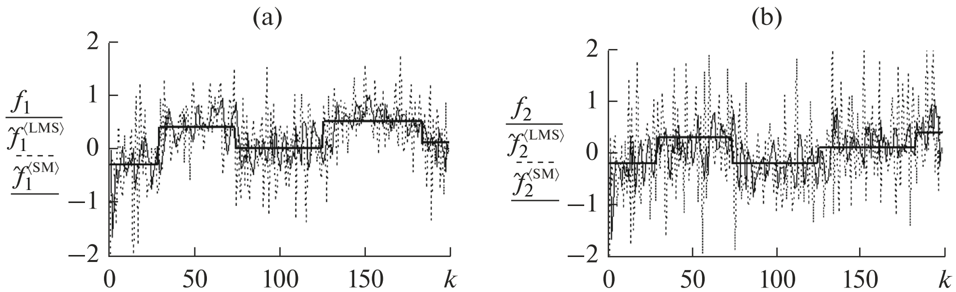

Fig. 3. Plots of the unknown component $f_1(k), f_2(k)$ and its estimates.

Weight matrices of criterion (4) and (29) are:

$$R_1 = \begin{pmatrix} 0.09 & 0 \\ 0 & 0.15 \end{pmatrix},\ R_2 = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.17 \end{pmatrix},\ \overline{W} = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix},\ \overline{\overline{W}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

At the first stage we calculated the matrix K from equations (24)–(26). Next, a filter (27) is realized, which uses the observations $y(k)$ and an unknown input estimate calculated by the algorithms: LSM (29) or smoothing method (SM) for the innovation process (30).

The simulation results are presented in Figs. 2 and 3, Tables 1 and 2. These results illustrate the quality of estimation using algorithms (29) and (30).

Figures 2 and 3 shows the implementations of the processes $(x_1(k), x_2(k))$, $(f_1(k), f_2(k))$, and their estimates, calculated using LSM and SM.

Tables 1 and 2 show the standard error values of the state and the unknown input for two methods (29) and (30). Formulas for calculating standard errors are

$$\sigma_{\hat{x},i} = \sqrt{\frac{\sum_{k=1}^{T} (x_i(k) - \hat{x}_i(k))^2}{T-1}},\quad \sigma_{f,i} = \sqrt{\frac{\sum_{k=1}^{T} (f_i(k) - \hat{f}_i(k))^2}{T-1}},\quad i = \overline{1, 2}.$$

Results presented in the Tables are an averaging over 100 implementations.

Table 1. Standard errors of the state vector

Components	LMS-estimates	SM-estimates
1	0.507	0.438
2	0.661	0.584

Table 2. Standard errors of the unknown input

Components	LMS-estimates	SM-estimates
1	0.626	0.275
2	0.811	0.323

7. CONCLUSIONS

This paper provides algorithms of the robust filtering for discrete systems with a random jump process and an unknown input. The proposed method was verified by simulations. The simulation results show that filtering procedures using smoothing for the innovation process have advantages in accuracy over algorithms using LSM-estimates.

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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