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NONPARAMETRIC ESTIMATION FOR AN AUTOREGRESSIVE MODEL

The paper deals with the nonparametric estimation problem at a given fixed point for an autoregressive model with unknown distributed noise. Kernel estimate modifications are proposed. Asymptotic minimax and efficiency properties for proposed estimators are shown.

Key words: *asymptotical efficiency, kernel estimates, minimax, nonparametric autoregression.*

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1. Introduction

We consider the following nonparametric autoregressive model

$$y_k = S(x_k) y_{k-1} + \xi_k \, , \; 1 \le k \le n, \tag{1.1}$$

where $S(.)$ is an unknown $\mathbf{R} \to \mathbf{R}$ function, $x_k = k/n, y_0$ is a constant and the noise random variables $(\xi_k)_{1 \le k \le n}$ are i.i.d. with $\mathbf{E}\xi_k = 0$ and $\mathbf{E}\xi_k^2 = 1$.

The model (1.1) is a generalization of autoregressive processes of the first order. In [4] the process (1.1) is considered with the function S having a parametric form. Moreover, the paper [5] studies spectral properties of the stationary process (1.1) with the nonparametric function S.

This paper deals with a nonparametric estimation of the autoregression coefficient function S at a given point z_0, when the smoothness of S is known. For this problem we make use of the following modified kernel estimator

$$\hat{S}_n(z_0) = \frac{1}{A_n} \sum_{k=1}^{n} Q(u_k) y_{k-1} y_k \mathbf{1}_{(A_n \ge d)} \, , \tag{1.2}$$

where $Q(.)$ is a kernel function,

$$A_n = \sum_{k=1}^{n} Q(u_k) y_{k-1}^2 \quad \text{with } u_k = \frac{x_k - z_0}{h} \, ,$$

d and h are some positive parameters.

First we assume that the unknown function S belongs to the *stable local Hölder class* at the point z_0 with a known regularity $1 \le \beta < 2$. This class will be defined below. We find an asymptotical (as $n \to \infty$) positive lower bound for the minimax risk with the normalyzing coefficient

$$\varphi_n = n^{\frac{2\beta}{2\beta+1}} . \tag{1.3}$$

To obtain this convergence rate we set in (1.2)

$$h = n^{-\frac{1}{2\beta+1}} \text{ and } d = \kappa_n n h \, , \tag{1.4}$$

where $\kappa_n \geq 0$,

$$\lim_{n\to\infty} \kappa_n = 0 \text{ and } \lim_{n\to\infty} \frac{h}{\kappa_n^2} = 0 . \tag{1.5}$$

As to the the kernel function we assume that

$$\int_{-1}^{1} Q(z)dz > 0 \quad \text{and} \quad \int_{-1}^{1} zQ(z)dz = 0 . \tag{1.6}$$

In this paper we show that the estimator (1.2) with the parameters (1.4)-(1.6) is asymptotically minimax, i.e. we show that the asymptotical upper bound for the minimax risk with respect to the stable local Holder class is finite.

At the next step we study sharp asymptotic properties for the minimax estimators (1.2).

To this end similarly to [1] we introduce the *weak stable local Hölder class*. In this case we find a positive constant giving the exact asymptotic lower bound for the minimax risk with the normalyzing coefficient (1.3). Moreover, we show that for the estimator (1.2) with the parameters (1.4)-(1.5) and the indicator kernel $Q = \mathbf{1}_{[-1,1]}$ the asymptotic upper bound of the minimax risk coincides with this constant, i.e. in this case such estimators are asymptotically efficient. In [9], Belitser consider the above model with lipshitz condtions.

The autor proposed a recursive estimator , and consider the estimatimation problem in a fixed t. By the quadratic risk, Belitser establish the convergence rate witout showing it's optimality. Moulines at al in [10], show that the convergence rate is optimal for the quadratic risk by using a recursive method for autoregressive model of order d. We note that in our paper we establish an optimal convergence rate but the risk considered is different from the one used in [10], and assymptions are weaker then those of [10].

The paper is organized as follows. In the next section we give the main results. In Section 3 we find asymptotical lowers bounds for the minimax risks. Section 4 is devoted to uppers bounds. Appendix contains some technical results.

2. Main results

Fisrt of all we assume that the noise in the model (1.1), i.e. the i.i.d. random variables $(\xi_k)_{1\leq k\leq n}$ have a density p (with respect to the Lebesgue measure) from the functional class $\mathcal{P}$ defined as

$$\mathcal{P} := \Big\{ p \geq 0 : \int_{-\infty}^{+\infty} p(x)dx = 1 , \int_{-\infty}^{+\infty} xp(x)dx = 0 , \int_{-\infty}^{+\infty} x^2 p(x)dx = 1$$

$$\text{and } \int_{-\infty}^{+\infty} |x|^4 p(x)dx \leq \sigma^* \Big\} \tag{2.1}$$

with $\sigma^* \geq 3$. Note that the (0,1) -gaussian density belongs to $\mathcal{P}$. In the sequel we denote this density by p_0.

The problem is to estimate the function $S(.)$at a fixed point $z_0 \in (0,1)$, i.e. the value $S(z_0)$. For this problem we make use of the risk proposed in [1]. Namely, for any estimate $\tilde{S} = \tilde{S}_n(z_0)$ (i.e. any mesurable with respect to the observations $(y_k)_{1\leq k\leq n}$ function) we set

$$\Re_n(\tilde{S}_n, S) = \sup_{p\in\mathcal{P}} \mathbf{E}_{S,p} \left|\tilde{S}_n(z_0) - S(z_0)\right| , \tag{2.2}$$

where $\mathbf{E}_{S,p}$, is the expectation taken with respect to the distribution $\mathbf{P}_{S,p}$ of the vector $(y_1,...,y_n)$ in (1.1) corresponding to the function S and the density p from $\mathcal{P}$.

To obtain a stable (uniformly with respect to the function S) model (1.1) we assume (see [4] and [5]) that for some fixed $0 < \varepsilon < 1$ the unknown function S belongs to the *stability set*

$$\Gamma_\varepsilon = \{S \in C_1[0,1] : \|S\| \le 1-\varepsilon\}\,, \tag{2.3}$$

where $\|S\| = \sup_{0\le x\le 1} |S(x)|$. Here $C_1[0,1]$ is the Banach space of continuously differentiable $[0,1] \to \mathbf{R}$ functions.

For fixed constants $K > 0$ and $0 \le \alpha < 1$ we define the corresponding *stable local Hölder class* at the point z_0 as

$$H^{(\beta)}(z_0,K,\varepsilon) = \left\{S \in \Gamma_\varepsilon : \|\dot{S}\| \le K \ \text{ and } \ \Omega^*(z_0,S) \le K\right\} \tag{2.4}$$

with $\beta = 1 + \alpha$ and

$$\Omega^*(z_0,S) = \sup_{x\in[0,1]} \frac{|\dot{S}(x) - \dot{S}(z_0)|}{|x - z_0|^\alpha}.$$

First we show that the sequence (1.3) gives the optimal convergence rate for the functions S from $H^{(\beta)}(z_0,K,\varepsilon)$. We start with a lower bound.

Theorem 2.1. For any $K > 0$ and $0 < \varepsilon < 1$

$$\underline{\lim}_{n\to\infty} \inf_{\tilde{S}} \sup_{S\in H^{(\beta)}(z_0,K,\varepsilon)} \varphi_n \Re_n(\tilde{S}_n, S) > 0\ , \tag{2.5}$$

where the infimum is taken over all estimators.

Now we obtain an upper bound for the kernel estimator (1.2)

Theorem 2.2. For any $K > 0$ and $0 < \varepsilon < 1$ the kernel estimator (1.2) with the parameters (1.4) – (1.6) satisfies the following inequality

$$\overline{\lim_{n\to\infty}} \sup_{S\in H^{(\beta)}(z_0,K,\varepsilon)} \varphi_n \Re_n(\hat{S}_n, S) < \infty\,. \tag{2.6}$$

Theorem 2.1 and Theorem .2.2 imply that the sequence (1.3) is the optimal (minimax) convergence rate for any stable Holder class of regularity β, i.e. the estimator (1.2) with the parameters (1.4) – (1.6) is minimax with respect to the functional class (2.4).

Now we study some efficiency properties for the minimax estimators (1.2). To this end similarly to [1] we make use of the family of the *weak stable local Hölder classes* at the point z_0, i.e. for any $\delta > 0$ we set

$$U^\beta_{\delta,n}(z_0,\varepsilon) = \left\{S \in \Gamma_\varepsilon : \|\dot{S}\| \le \delta^{-1} \ \text{ and } \ |\Omega_h(z_0,S)| \le \delta h^\beta\right\}, \tag{2.7}$$

where

$$\Omega_h(z_0,S) = \int_{-1}^{1} (S(z_0 + uh) - S(z_0))du$$

and h is given in (1.4).

Moreover, we set

$$\tau(S) = 1 - S^2(z_0). \tag{2.8}$$

With the help of this function we describe the sharp lower bound for the minimax risks in this case.

Theorem 2.3. For any $\delta > 0$ and $0 < \varepsilon < 1$

$$\underline{\lim}_{n\to\infty} \inf_{\tilde{S}} \sup_{S\in U^{(\beta)}_{\delta,n}(z_0,\varepsilon)} \tau^{-1/2}(S)\varphi_n \Re_n(\tilde{S}_n, S) \ge \mathbf{E}|\eta|, \tag{2.9}$$

where η is a gaussian random variable with the parameters (0,1/2).

Theorem 2.4. The estimator (1.2) with the parameters (1.4) – (1.5) and $Q(z) = 1_{[-1,1]}$ satisfies the following inequality

$$\overline{\lim_{\delta\to 0}}\, \overline{\lim_{n\to\infty}} \sup_{S\in U^{(\beta)}_{\delta,n}(z_0,\varepsilon)} \tau^{-1/2}(S)\varphi_n \Re_n(S_n, S) \le \mathbf{E}|\eta|,$$

where η is a gaussian random variable with the parameters (0,1/2).

Theorems 2.3 and 2.4 imply that the estimator (1.2), (1.4) – (1.5) with the indicator kernel is asymptotically efficient.

Remark 2.1. One can show (see [1]) that for any $0 < \delta < 1$ and $n \ge 1$

$$H^{(\beta)}(z_0,\delta,\varepsilon) \subset U^{(\beta)}_{\delta,n}(z_0,\varepsilon).$$

This means that the «natural» normalyzing coefficient for the functional class (2.7) is the sequence (1.3). Theorem 2.3 and Theorem 2.4 extend usual the Hölder approach for the point estimation by keeping the minimax convergence rate (1.3).

3. Lower bounds

3.1. Proof of Theorem 2.1

Note that to prove (2.5) it suffices to show that

$$\underline{\lim}_{n\to\infty} \inf_{\tilde{S}} \sup_{S\in H^{(\beta)}(z_0,K,\varepsilon)} \mathbf{E}_{S,p_0} \psi_n(\tilde{S}_n, S) > 0\,, \tag{3.1}$$

where

$$\psi_n(\tilde{S}_n, S) = \varphi_n \left|\tilde{S}_n(z_0) - S(z_0)\right|.$$

We make use of the similar method proposed by Ibragimov and Hasminskii to obtain a lower bound for the density estimation problem in [7]. First we chose the corresponding parametric family in $H^{(\beta)}(z_0,K,\varepsilon)$. Let V be a two times continuously differentiable function such that $\int_{-1}^{1} V(z)dz > 0$ and $V(z) = 0$ for any $|z| \ge 1$ We set

$$S_u(x) = \frac{u}{\varphi_n} V\left(\frac{x - z_0}{h}\right), \tag{3.2}$$

where φ_n and h are defined in (1.3) and (1.4).

It is easy to see that for any $z_0 - h \le x \le z_0 + h$

$$\left|\dot{S}_u(x) - \dot{S}_u(z_0)\right| = \frac{|u|}{h\varphi_n}\left|\dot{V}\left(\frac{x-z_0}{h}\right) - \dot{V}(0)\right| \le \frac{|u|}{h\varphi_n} V_*^{''} \left|\frac{x-z_0}{h}\right| \le |u| V_*^{''} \left|x - z_0\right|^{\alpha},$$

where $V_*^{''} = \max_{|z|\le 1}\left|\ddot{V}(z)\right|$. Therefore, for all $0 < u \le u^* = K/V_*^{''}$ we obtain that

$$\sup_{z_0-h\le x\le z_0+h} \frac{\left|\dot{S}_u(x) - \dot{S}_u(z_0)\right|}{\left|x - z_0\right|^{\alpha}} \le K.$$

Moreover, by the definition (3.2) for all $x > z_0 + h$

$$\dot{S}_u(x) = \dot{S}_u(z_0 + h) = 0 \text{ and } \dot{S}_u(x) = \dot{S}_u(z_0 - h) = 0$$

for all $x < z_0 - h$ respectively. Therefore, the last inequality implies that

$$\sup_{|u| \le u^*} \Omega^*(z_0, S_u) \le K,$$

where the function $\Omega^*(z_0, S)$ is defined in (2.4).

This means that there exists $n_{K,\varepsilon} > 0$ such that $S_u \in H^{(\beta)}(z_0, K, \varepsilon)$ for all $|u| \le u^*$ and $n \ge n_{K,\varepsilon}$. Therefore, for all $n \ge n_{K,\varepsilon}$ and for any estimator $\tilde{S}_n$ we estimate with below the supremum in (3.1) as

$$\sup_{S \in H^{(\beta)}(z_0,K,\varepsilon)} \mathbf{E}_{S,p} \psi_n(\tilde{S}_n, S) \ge \sup_{|u| \le u^*} \mathbf{E}_{S_u,p_0} \psi_n(\tilde{S}_n, S_u) \ge \frac{1}{2b} \int_{-b}^{b} \mathbf{E}_{S_u,p_0} \psi_n(\tilde{S}_n, S_u) du \quad (3.3)$$

for any $0 < b \le u^*$.

Notice that for any S the measure $\mathbf{P}_{S,p_0}$ is equivalent to the measure $\mathbf{P}_{0,p_O}$, where $\mathbf{P}_{0,p_O}$ is the distribution of the vector $(y_1, \cdots, y_n)$ in (1.1) corresponding to the function $S = 0$ and the gaussian (0,1) noise density p_0, i.e. the random variables $(y_1, \cdots, y_n)$ are i.i.d. $N(0,1)$ with respect to the measure $\mathbf{P}_{0,p_0}$. In the sequel we denote $\mathbf{P}_{0,p_0}$ by $\mathbf{P}$. It is easy to see that in this case the Radon-Nikodym derivative can be written as

$$\rho_n(u) = \frac{d\mathbf{P}_{S_u,p_0}}{d\mathbf{P}} = e^{u\varsigma_n \eta_n - \frac{u^2}{2}\varsigma_n^2}$$

with
$$\varsigma_n^2 = \frac{1}{\varphi_n^2} \sum_{k=1}^{n} V^2(u_k) \xi_{k-1}^2 \text{ and } \eta_n = \frac{1}{\varphi_n \varsigma_n} \sum_{k=1}^{n} V(u_k) \xi_{k-1} \xi_k .$$

Through the large numbers law we obtain

$$\mathbf{P} - \lim_{n \to \infty} \varsigma_n^2 = \lim_{n \to \infty} \frac{1}{nh} \sum_{k=k_*}^{k^*} V^2(u_k) \xi_{k-1}^2 = \int_{-1}^{1} V^2(u) du = \sigma^2 ,$$

where

$$k_* = [nz_0 - nh] + 1 \text{ and } k^* = [nz_0 + nh] . \quad (3.4)$$

Here $[a]$ is the integer part of a.

Moreover, by the central limit theorem for martingales (see [2] and [3]), it is easy to see that under the measure **P**

$$\eta_n \Rightarrow N(0,1) \quad \text{as } n \to \infty.$$

Therefore we represent the Radon-Nykodim density in the following asymptotic form

$$\rho_n(u) = e^{u\sigma\eta_n - \frac{u^2\sigma^2}{2} + r_n},$$

where
$$\mathbf{P} - \lim_{n \to \infty} r_n = 0.$$

This means that in this case the Radon-Nikodym density $(\rho_n(u))_{n\geq 1}$ satisfies the L.A.N. property and we can make use the method from theorem 12.1 of [7] to obtain the following inequality

$$\underline{\lim}_{n\to\infty} \inf_{\tilde{S}} \frac{1}{2b}\int_{-b}^{b} \mathbf{E}_{S_u,p_0} \psi_n(\tilde{S}_n, S_u)du \geq I(b,\sigma), \tag{3.5}$$

where $$I(b,\sigma) = \frac{\max(1, b-\sqrt{b})}{b}\frac{\sigma}{\sqrt{2\pi}}\int_{-\sqrt{b}}^{\sqrt{b}} e^{-\sigma^2\frac{u^2}{2}}du$$

and $0 < b < u^*$. Therefore, inequalities (3.3) and (3.4) imply (3.1). Hence Theorem 2.1. ■

3.2 Proof of Theorem 2.3

First, similarly to the proof of Theorem 2.1 we choose the corresponding parametric functional family $S_{u,\nu}(.)$ in the form (3.2) with the function $V = V_\nu$ defined as

$$V_\nu(x) = \nu^{-1}\int_{-\infty}^{+\infty}\tilde{Q}_\nu(u)g\left(\frac{u-x}{\nu}\right)du\ .$$

Where $\tilde{Q}_\nu(u) = \mathbf{1}_{\{|u|\leq 1-2\nu\}} + 2\mathbf{1}_{\{1-2\nu\leq|u|\leq 1-\nu\}}$ with $0<\nu<1/4$ and g is some even nonnegative infinitely differentiable function such that $g(z)=0$ for $|z|=1$ and $\int_{-1}^{1} g(z)dz = 1$. One can show (see [1]) that for any $b>0$, $0<\delta<1$ and $0<\nu<1/4$ there exists $n_* = n_*(b,\delta,\nu) > 0$ such that for all $|u|\leq b$ and $n\geq n_*$

$$S_{u,\nu} \in U^{(\beta)}_{\delta,n}(z_0,\varepsilon).$$

Therefore, in this case for any $n \geq n_*$

$$\varphi_n \sup_{S\in U^{(\beta)}_{\delta,n}(z_0,\varepsilon)} \tau^{-1/2}(S)\Re_n(\tilde{S}_n,S) \geq \sup_{S\in U^{(\beta)}_{\delta,n}(z_0,\varepsilon)} \tau^{-1/2}(S)\mathbf{E}_{S,p_0}\psi_n(\tilde{S}_n,S)$$

$$\geq \tau_*(n,b)\frac{1}{2b}\int_{-b}^{b}\mathbf{E}_{u,\nu,p_0}\psi_n(\tilde{S}_n,S_{u,\nu})du\ ,$$

where $$\tau_*(n,b) = \inf_{|u|\leq b}\left|\tau^{-1/2}\right|(S_{u,\nu}).$$

The definition (2.8) and (3.2) imply that for any $b > 0$

$$\lim_{n\to\infty}\sup_{|u|\leq b}\left|\tau(S_{u,\nu})-1\right| = 0.$$

Therefore, by the same way as in the proof of Theorem 2.1 we obtain that for any $b>0$ and $0<\nu<1/4$

$$\underline{\lim}_{n\to\infty}\inf_{\tilde{S}}\sup_{S\in U^{(\beta)}_{\delta,n}(z_0,\varepsilon)} \tau^{-1/2}(S)\varphi_n\Re_n(\tilde{S}_n,S) \geq I(b,\sigma_\nu), \tag{3.6}$$

where the function $I(b,\sigma_\nu)$ is defined in (3.5) with $\sigma^2_\nu = \int_{-1}^{1} V^2_\nu(u)du$. It is easy to check that $\sigma^2_\nu \to 2$ as $\nu\to 0$. Limiting $b\to\infty$ and $\nu\to 0$ in (3.6) yield the inequality (2.9). Hence Theorem 2.3. ■

4. Upper bounds

4.1. Proof of Theorem 2.2

First of all we set

$$\tilde{A}_n = \frac{A_n}{\varphi_n^2} \quad \text{and} \quad \hat{A}_n = \frac{1}{\tilde{A}_n} \mathbf{1}_{(\tilde{A}_n > \kappa_n)}. \tag{4.1}$$

Now from (1.2) we represent the estimate error as

$$\hat{S}_n(z_0) - S(z_0) = -S(z_0)\mathbf{1}_{(\tilde{A}_n \le \kappa_n)} + \frac{1}{\varphi_n} \hat{A}_n \zeta_n + \frac{1}{\varphi_n} \hat{A}_n B_n, \tag{4.2}$$

with

$$\zeta_n = \frac{\sum_{k=1}^n Q(u_k) y_{k-1} \xi_k}{\varphi_n} \quad \text{and} \quad B_n = \frac{\sum_{k=1}^n Q(u_k)(S(x_k) - S(z_0)) y_{k-1}^2}{\varphi_n}.$$

Note that, the first term in the right hand of (4.2) is studied in Lemma A.3. To estimate the second term we make use of Lemma A.2 which implies directly

$$\overline{\lim_{n\to\infty}} \sup_{S \in H^{(\beta)}(z_0, K, \varepsilon)} \sup_{p \in \mathcal{P}} \mathbf{E}_{S,p} \zeta_n^2 < \infty$$

and, therefore, by A.8 we obtain

$$\overline{\lim_{n\to\infty}} \sup_{S \in H^{(\beta)}(z_0, K, \varepsilon)} \sup_{p \in \mathcal{P}} \mathbf{E}_{S,p} \left|\hat{A}_n\right| |\zeta_n| < \infty.$$

Let us estimate now the last term in the right hand of (4.2). To this end we need to show that

$$\overline{\lim_{n\to\infty}} \sup_{S \in H^{(\beta)}(z_0, K, \varepsilon)} \sup_{p \in \mathcal{P}} \mathbf{E}_{S,p} B_n^2 < \infty. \tag{4.3}$$

Indeed, putting $r_k = S(x_k) - S(z_0) - \dot{S}(z_0)(x_k - z_0)$ by the Taylor Formula we represent B_n as

$$B_n = \frac{h}{\varphi_n} \dot{S}(z_0) \tilde{B}_n + \frac{1}{\varphi_n} \hat{B}_n,$$

where $\tilde{B}_n = \sum_{k=1}^n Q(u_k) u_k y_{k-1}^2$ and $\hat{B}_n = \sum_{k=1}^n Q(u_k) r_k y_{k-1}^2$. We remind that by the condition (1.6) $\int_{-1}^{1} u Q(u) du = 0.$ Therefore through Lemma A.2 we obtain

$$\lim_{n\to\infty} \frac{h^2}{\varphi_n^2} \sup_{S \in H^{(\beta)}(z_0, K, \varepsilon)} \sup_{p \in \mathcal{P}} \mathbf{E}_{S,p} \tilde{B}_n^2 = 0.$$

Moreover, for any function $S \in H^{(\beta)}(z_0, K, \varepsilon)$ and for $k_* \le k \le k^*$ (k_* and k^* are given in (3.4))

$$|r_k| = \left| \int_{z_0}^{x_k} \left((\dot{S}(u) - \dot{S}(z_0)) \right) du \right| \le K |x_k - z_0|^{\beta} \le K h^{\beta} = K \varphi_n^{-1},$$

i.e. $\hat{B}_n \le \varphi_n \tilde{A}_n$. Therefore, by Lemma A.2

$$\overline{\lim_{n\to\infty}} \sup_{S\in H^{(\beta)}(z_0,K,\varepsilon)} \sup_{p\in\mathcal{P}} \frac{1}{\varphi_n^2} \mathbf{E}_{S,p} \hat{B}_n^2 < \infty.$$

This implies (4.3). Hence Theorem 2.2. ■

4.2. Proof of Theorem 2.4

Similarly to Lemma A.2 from [1] by making use of Lemma A.1 and Lemma A.2 we can show that

$$\sqrt{\frac{\tau(S)}{2}}\zeta_n \Rightarrow N(0,1) \text{ as } n\to\infty$$

uniformly in $S\in\Gamma_\varepsilon$ and $p\in\mathcal{P}$. Therefore, by Lemma A.2 we obtain that uniformly in $S\in\Gamma_\varepsilon$ and $p\in\mathcal{P}$

$$\tau^{-1/2}(S)\hat{A}_n\zeta_n \Rightarrow N(0,1/2) \text{ as } n\to\infty$$

Moreover, by applying the Burkhölder inequality and Lemma A.2 to the martingale ζ_n we deduce that

$$\overline{\lim_{n\to\infty}} \sup_{S\in H^{(\beta)}(z_0,K,\varepsilon)} \sup_{p\in\mathcal{P}} \mathbf{E}_{S,p}\zeta_n^4 < \infty.$$

Therefore, inequality A.8 implies that the sequence $(\hat{A}_n\zeta_n)_{n\geq 1}$ is uniformly integrable. This means that

$$\lim_{n\to\infty} \sup_{S\in H^{(\beta)}(z_0,K,\varepsilon)} \sup_{p\in\mathcal{P}} \left|\tau^{-1/2}(S)\mathbf{E}_{S,p}\left|\hat{A}_n\zeta_n\right| - \mathbf{E}|\eta|\right| = 0,$$

where η is a gaussian random variable with the parameters (0,1/2). Now to finish this proof we have to show that

$$\lim_{\delta\to 0}\overline{\lim_{n\to\infty}} \sup_{S\in H^{(\beta)}_{\delta,n}(z_0,\varepsilon)} \sup_{p\in\mathcal{P}} \mathbf{E}_{S,p}B_n^2 = 0. \quad (4.4)$$

Indeed, by setting $f_S(u) = S(z_0+hu) - S(z_0)$ we rewrite B_n as

$$B_n = \frac{1}{\varphi_n}\sum_{k=k_*}^{k^*} f_S(u_k)y_{k-1}^2 = \varphi_n\; G_n\;(f_S,S) + \frac{\varphi_n}{\tau(S)}\Omega_h(z_0,S), \quad (4.5)$$

where
$$G_n(f,S) = \frac{\sum_{k=1}^n f(u_k)y_{k-1}^2}{\varphi_n^2} - \frac{1}{\tau(S)}\int_{-1}^1 f(u)du$$

and $\Omega_h(z_0,S)$ is defined in (2.7). The definition (2.8) implies that for any $S\in\Gamma_\varepsilon$

$$\varepsilon^2 \leq \tau(S) \leq 1. \quad (4.6)$$

From here by the definition (2.7) we obtain that

$$|B_n| \leq \varphi_n|\quad G_n(f_S,S)\;\;| + \frac{\delta}{\varepsilon^2}.$$

Moreover, for any $S\in U^\beta_{\delta,n}(z_0,\varepsilon)$ the function f_S satisfies the following inequality

$$\|f_S\| + \|\dot{f}_S\| \leq \delta^{-1}h.$$

We note also that $\varphi_n h^2 \to 0$ as $n \to \infty$. Therefore, by making use of Lemma A.2 with $R = h/\delta$ we obtain (4.4). Hence Theorem 2.4. ■

5. Appendix

In this section we study distribution properties of the stationary process (1.1).

Lemma A.1 For any $0 < \varepsilon < 1$ the random variables (1.1) satisfy the following moment inequality

$$m^* = \sup_{n\ge 1} \sup_{0\le k\le n} \sup_{S\in\Gamma_\varepsilon} \sup_{p\in\mathcal{P}} \mathbf{E}_{S,p} y_k^4 < \infty. \qquad \text{(A.1)}$$

Proof. One can deduce from (1.1) with $S \in \Gamma_\varepsilon$ that for all $1 \le k \le n$

$$y_k^4 \le \left((1-\varepsilon)^k |y_0| + \sum_{j=1}^{k} (1-\varepsilon)^{k-j} \left|\xi_j\right| \right)^4 \le 8 y_0^4 + 8 \left(\sum_{j=1}^{k} (1-\varepsilon)^{k-j} \left|\xi_j\right| \right)^4 .$$

Moreover, by the Hölder inequality with $q = 4/3$ and $p = 4$

$$y_k^4 \le 8|y_0|^4 + \frac{8}{\varepsilon^3} \sum_{j=1}^{k} (1-\varepsilon)^{k-j} \xi_j^4 .$$

Therefore, for any $p \in \mathcal{P}$

$$\mathbf{E}_{S,p} y_k^4 \le 8|y_0|^4 + \frac{8}{\varepsilon^4} \sigma_* .$$

Hence Lemma A.1. ■

Now for any $K > 0$ and $0 < \varepsilon < 1$ we set

$$\Theta_{K,\varepsilon} = \left\{ S \in \Gamma_\varepsilon : \|\dot{S}\| \le K \right\}. \qquad \text{(A.2)}$$

Lemma A.2. Let the function f is two times continuously differentiable in [-1,1], such that $f(u) = 0$ for $|u| \ge 1$. Then

$$\overline{\lim_{n\to\infty}} \sup_{R>0} \frac{1}{(Rh)^2} \sup_{\|f\|_1 \le R} \sup_{S\in\Theta_{K,\varepsilon}} \sup_{p\in\mathcal{P}} \mathbf{E}_{S,p}\, G_n^2(f,S) < \infty, \qquad \text{(A.3)}$$

where $\|f\|_1 = \|f\| + \|\dot{f}\|$ and $G_n(f,S)$ is defined in (4.5).

Proof. First of all, note that

$$\sum_{k=1}^{n} f(u_k) y_{k-1}^2 = T_n + a_n , \qquad \text{(A.4)}$$

where

$$T_n = \sum_{k=k_*}^{k^*} f(u_k) y_k^2 \quad \text{and} \quad a_n = \sum_{k=k_*}^{k^*} (f(u_k) - f(u_{k-1})) y_{k-1}^2 - f(u_{k^*}) y_{k^*}^2$$

with k^* and k_* defined in (3.4). Moreover, from the model (1.1) we find

$$T_n = I_n(f) + \sum_{k=k_*}^{k^*} f(u_k) S^2(x_k) y_{k-1}^2 + M_n ,$$

where $\quad I_n(f) = \sum_{k=k_*}^{k^*} f(u_k)$ and $M_n = \sum_{k=k_*}^{k^*} f(u_k)(2S(x_k) y_{k-1}\xi_k + \eta_k)$

with $\eta_k = \xi_k^2 - 1$. By setting

$$C_n = \sum_{k=k_*}^{k^*} (S^2(x_k) - S^2(z_0)) f(u_k) y_{k-1}^2 \text{ and } D_n = \sum_{k=k_*}^{k^*} f(u_k)(y_{k-1}^2 - y_k^2)$$

we get
$$\frac{1}{\varphi_n^2} T_n = \frac{1}{\tau(S)} \frac{I_n(f)}{\varphi_n^2} + \frac{1}{\tau(S)} \frac{\Delta_n}{\varphi_n^2} \tag{A.5}$$

with $\Delta_n = M_n + C_n + S^2(z_0) D_n$. Moreover, taking into account that $\varphi_n^2 = nh$ we obtain

$$\frac{I_n(f)}{\varphi_n^2} = \int_{-1}^{1} f(t)dt + \sum_{k=k_*}^{k^*} \int_{u_{k-1}}^{u_k} f(u_k)dt - \int_{-1}^{1} f(t)dt =$$

$$= \int_{-1}^{1} f(t)dt + \sum_{k=k_*}^{k^*} \int_{u_{k-1}}^{u_k} (f(u_k) - f(t))dt + \int_{u_{k_*-1}}^{u_{k^*}} f(t)dt - \int_{-1}^{1} f(t)dt.$$

We remind that $\|f\| + \|\dot{f}\| \le R$. Therefore

$$\left| \frac{1}{nh} \sum_{k=k_*}^{k^*} f(u_k) - \int_{-1}^{1} f(t)dt \right| \le \frac{2R}{nh}.$$

Taking this into account in (A.5) and the lower bound for $\tau(S)$ given in (4.6) we find that

$$\left| \frac{T_n}{\varphi_n^2} - \frac{1}{\tau(S)} \int_{-1}^{1} f(t)dt \right| \le \frac{1}{\varepsilon^2} \left(\frac{2R}{nh} + \frac{M_n}{nh} + \frac{C_n}{nh} + \frac{D_n}{nh} \right). \tag{A.6}$$

Note that the sequence $(M_n)_{n\ge1}$ is a square integrable martingale. Therefore,

$$\mathbf{E}_{S,p} \left(\frac{1}{nh} M_n \right)^2 = \frac{1}{(nh)^2} \mathbf{E}_{S,p} \sum_{k=k_*}^{k^*} f^2(u_k)(2S(x_k) y_{k-1} \xi_k + \eta_k)^2 \le \frac{4R^2(4\sqrt{m^*} + \sigma^*)}{nh},$$

where m^* is given in (A.1). Moreover, taking into account that $|S(x_k) - S(z_0)| \le L|x_k - z_0|$ for any $S \in \Theta_{L,\varepsilon}$ and that $k^* - k_* \le 2nh$ we obtain that

$$\frac{1}{(nh)^2} \mathbf{E}_{S,p} C_n^2 \le \frac{2}{nh} \sum_{k=k_*}^{k^*} \left| (S^2(x_k) - S^2(z_0)) \right|^2 f^2(u_k) \mathbf{E}_{S,p} y_{k-1}^4 \le 16 R^2 L^2 m^* h^2.$$

Let us consider now the last term in the right hand of the inequality (A.6). To this end we make use of the integration by parts formula, i.e. we represent D_n as

$$D_n = \sum_{k=k_*}^{k^*} ((f(u_k) - f(u_{k-1})) y_{k-1}^2 + f(u_{k_*-1}) y_{k_*-1}^2 - f(u_{k^*}) y_{k^*}^2.$$

Therefore, taking into account that $\|f\| + \|\dot{f}\| \le R$ we obtain that

$$\mathbf{E}_{S,p} D_n^2 \le 3R^2 \mathbf{E}_{S,p} \left(\frac{2}{nh} \sum_{k=k_*}^{k^*} y_{k-1}^4 + y_{k^*}^4 + y_{k_*-1}^4 \right) \le 18 R^2 m^*.$$

By the same way we estimate the second term in the right hand of (A.4). Hence Lemma A.2. ■

Lemma A.3. The sequences $(\tilde{A}_n)_{n\ge1}$ and $(\hat{A}_n)_{n\ge1}$ defined in (4.1) satisfy the following properties

$$\overline{\lim_{n\to\infty}} \frac{1}{h^2} \sup_{S\in\Theta_{K,\varepsilon}} \sup_{p\in\mathcal{P}} \mathbf{P}_{S,p}(\tilde{A}_n \le \kappa_n) < \infty \tag{A.7}$$

and

$$\overline{\lim_{n\to\infty}} \sup_{S\in\Theta_{K,\varepsilon}} \sup_{p\in\mathcal{P}} \mathbf{E}_{S,p} \hat{A}_n^4 < \infty. \tag{A.8}$$

Proof. It is easy to see that the inequality (A.7) follows directly from Lemma A.2. We check now the inequality (A.8). By setting $\gamma_* = \varepsilon^{-2}\int_{-1}^{1} Q(u)du$ we get

$$\mathbf{E}_{S,p}\hat{A}_n^4 = 4\int_0^\infty t^3 \quad {}_{S,p}(\tilde{A}_n \le t^{-1}, \tilde{A}_n > \kappa_n)dt \le 4\int_0^{\kappa_n^{-1}} t^3 \quad {}_{S,p}(\quad G_n(Q,S) + \gamma_* \le t^{-1})dt$$

$$\le \left(\frac{2}{\gamma_*}\right)^4 + \frac{1}{\kappa_n^4} \quad {}_{S,p}(| \quad G_n(Q,S) \quad | \ge \gamma_*/2).$$

By making use of Lemma A.2 with the condition (1.5) we obtain the inequality (A.8). ■

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