Information Sciences Letters

Volume 4
Issue 1 *Jan. 2015*

Article 3

2015

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Aslıhan Sezgin Sezer
Department of Mathematics, Amasya University, 05100 Amasya, Turkey, aslihan.sezgin@amasya.edu.tr

Naim Cağman
Department of Mathematics, Gaziosmanpas¸a University, Tokat, Turkey, aslihan.sezgin@amasya.edu.tr

Akın Osman Atagün
Department of Mathematics, Bozok University, Yozgat, Turkey, aslihan.sezgin@amasya.edu.tr

Information Sciences Letters
An International Journal

http://dx.doi.org/10.12785/isl/040103

A Novel Characterization for Certain Semigroups by Soft Union Ideals

Aslıhan Sezgin Sezer[1,*], *Naim Çağman*[2] *and Akın Osman Atagün*[3]

[1] Department of Mathematics, Amasya University, 05100 Amasya, Turkey
[2] Department of Mathematics, Gaziosmanpaşa University, Tokat, Turkey
[3] Department of Mathematics, Bozok University, Yozgat, Turkey

Received: 3 Aug. 2014, Revised: 16 Sep. 2014, Accepted: 18 Sep. 2014
Published online: 1 Jan. 2015

Abstract: In this paper, we characterize semisimple semigroups, duo semigroups, right (left) zero semigroups, right (left) simple semigroups, semilattice of left (right) simple semigroups, semilattice of left (right) groups and semilattice of groups in terms of soft union semigroups, soft union ideals of semigroups. Moreover, we define soft normal semigroups and give some characterizations of semigroups with soft normality.

Keywords: Soft set, soft union semigroup, soft union (interior, quasi, bi, generalized bi) ideal, semisimple semigroups, duo semigroups, semilattice of groups

1 Introduction

Soft sets was introduced by Molodtsov [19] for modeling vagueness and uncertainty in 1999. Many related concepts with soft sets, especially soft set operations [18, 4, 24, 5] have recently undergone tremendous studies. Soft set theory have found its wide-ranging applications in the mean of algebraic structures such as groups [3, 25], semirings [10], rings [1, 26], BCK/BCI-algebras [13, 14, 15], BL-algebras [31], near-rings [23], and soft substructures and union soft substructures [6, 27].

In [28], Sezgin et al. defined soft union semigroups, soft union left (right, two-sided) ideals and bi-ideals and soft semiprime ideals of semigroups and obtained their basic properties. And in [28], Sezgin et al. [29] defined soft union interior ideals, quasi-ideals, generalized bi-ideals and investigate the interrelations of them. Moreover, they characterized regular, intra-regular, completely regular, weakly regular and quasi-regular semigroups by the properties of these ideals in [28, 29]. Thus, they made a new approach to the classical semigroup theory via soft set theory with these concepts.

In this paper, we characterize certain classes of semigroups, such as semisimple semigroups, duo semigroups, right (left) zero semigroups, right (left) simple semigroups, semilattice of left (right) simple semigroups, semilattice of left (right) groups and semilattice of groups in terms of soft union ideals, bi-ideals, interior ideals, quasi-ideals, generalized bi-ideals. Furthermore, we define soft union normal semigroups and discuss on the relation of this concept with semigroups.

2 Preliminaries

In this section, we recall some notions relevant to semigroups and soft sets. A *semigroup* S is a nonempty set with an associative binary operation. Throughout this paper, S denotes a semigroup. A nonempty subset A of S is called a *right ideal* of S if $AS \subseteq A$ and is called a *left ideal* of S if $SA \subseteq A$. By *two-sided ideal* (or simply *ideal*), we mean a subset of S, which is both a left and right ideal of S. A subsemigroup X of S is called a *bi-ideal* of S if $XSX \subseteq X$. A nonempty subset A of S is called an *interior ideal* of S if $SAS \subseteq A$. A nonempty subset Q of S is called a *quasi-ideal* of S if $QS \cap SQ \subseteq Q$.

We denote by $L[a] (R[a], J[a], B[a] Q[a], I[a])$, the principal left (right, two-sided, bi-ideal, quasi-ideal, interior ideal) of a semigroup S generated by $a \in S$, that

* Corresponding author e-mail: aslihan.sezgin@amasya.edu.tr

is,

$$\begin{aligned} L[a] &= \{a\} \cup Sa, \\ R[a] &= \{a\} \cup aS, \\ J[a] &= \{a\} \cup Sa \cup aS \cup SaS \\ Q[a] &= \{a\} \cup (aS \cap Sa) \\ I[a] &= \{a\} \cup \{a^2\} \cup SaS. \end{aligned}$$

A semigroup S is called *regular* if for every element a of S, there exists an element x in S such that $a = axa$ or equivalently $a \in aSa$. An element a of S is called a *completely regular* if there exists an element $x \in S$ such that $a = axa$ and $ax = xa$. A semigroup S is called *completely regular* if every element of S is completely regular. A semigroup S is called *left (right) regular* if for each element a of S, there exists an element $x \in S$ such that $a = xa^2$ $(a = a^2x)$. A semigroup is called *left (right) regular* if for each element a of S, there exists an element $x \in S$ such that

$$a = xa^2 \ (a = a^2x).$$

A *semilattice* is a structure $S = (S, .)$, where "." is an infix binary operation, called the *semilattice operation*, such that "." is associative, commutative and idempotent. For all undefined concepts and notions about semigroups, we refer to [11,21].

Definition 1.([7,19]) *A soft set f_A over U is a set defined by*

$$f_A : E \to P(U) \text{ such that } f_A(x) = \emptyset \text{ if } x \notin A.$$

Here f_A is also called an approximate function. A soft set over U can be represented by the set of ordered pairs

$$f_A = \{(x, f_A(x)) : x \in E, f_A(x) \in P(U)\}.$$

Definition 2.[7] *Let f_A, $f_B \in S(U)$. Then, f_A is called a soft subset of f_B and denoted by $f_A \tilde{\subseteq} f_B$, if $f_A(x) \subseteq f_B(x)$ for all $x \in E$.*

Definition 3.[7] *Let f_A, $f_B \in S(U)$. Then, union of f_A and f_B, denoted by $f_A \tilde{\cup} f_B$, is defined as $f_A \tilde{\cup} f_B = f_{A\tilde{\cup}B}$, where $f_{A\tilde{\cup}B}(x) = f_A(x) \cup f_B(x)$ for all $x \in E$.*

Definition 4.[7] *Let f_A, $f_B \in S(U)$. Then, intersection of f_A and f_B, denoted by $f_A \tilde{\cap} f_B$, is defined as $f_A \tilde{\cap} f_B = f_{A\tilde{\cap}B}$, where $f_{A\tilde{\cap}B}(x) = f_A(x) \cap f_B(x)$ for all $x \in E$.*

Definition 5.*Let S be a semigroup and f_S and g_S be soft sets over the common universe U. Then, soft union product $f_S * g_S$ is defined by*

$$(f_S * g_S)(x) = \begin{cases} \bigcap_{x=yz} \{f_S(y) \cup g_S(z)\}, & \text{if } \exists y,z \in S \text{ such that } x = yz, \\ \emptyset, & \text{otherwise} \end{cases}$$

for all $x \in S$.

Theorem 1.[28] *Let $f_S, g_S, h_S \in S(U)$. Then,*

i) $(f_S * g_S) * h_S = f_S * (g_S * h_S)$.
ii) $f_S * (g_S \tilde{\cup} h_S) = (f_S * g_S) \tilde{\cup} (f_S * h_S)$ *and* $(f_S \tilde{\cup} g_S) * h_S = (f_S * h_S) \tilde{\cup} (g_S * h_S)$.
iii) $f_S * (g_S \tilde{\cap} h_S) = (f_S * g_S) \tilde{\cap} (f_S * h_S)$ *and* $(f_S \tilde{\cap} g_S) * h_S = (f_S * h_S) \tilde{\cap} (g_S * h_S)$.
iv) If $f_S \tilde{\subseteq} g_S$, *then* $f_S * h_S \tilde{\subseteq} g_S * h_S$ *and* $h_S * f_S \tilde{\subseteq} h_S * g_S$.
v) If $t_S, l_S \in S(U)$ *such that* $t_S \tilde{\subseteq} f_S$ *and* $l_S \tilde{\subseteq} g_S$, *then* $t_S * l_S \tilde{\subseteq} f_S * g_S$.

Definition 6.[28] *Let X be a subset of S. We denote by $\mathscr{S}_{X^c}$ the soft characteristic function of the complement X and define as*

$$\mathscr{S}_{X^c}(x) = \begin{cases} \emptyset, & \text{if } x \in X, \\ U, & \text{if } x \in S \setminus X \end{cases}$$

Definition 7.[28] *Let S be a semigroup and f_S be a soft set over U. Then, f_S is called a soft union semigroup of S, if*

$$f_S(xy) \subseteq f_S(x) \cup f_S(y)$$

for all $x, y \in S$.

Definition 8.[28] *A soft set over U is called a soft union left (right) ideal of S over U if*

$$f_S(ab) \subseteq f_S(b) \ (f_S(ab) \subseteq f_S(a))$$

for all $a, b \in S$. A soft set over U is called a soft union two-sided ideal (soft union ideal) of S if it is both soft union left and soft union right ideal of S over U.

Definition 9.[28] *A soft union semigroup f_S over U is called a soft union bi-ideal of S over U if*

$$f_S(xyz) \subseteq f_S(x) \cup f_S(z)$$

for all $x, y, z \in S$.

Definition 10.[29] *A soft set over U is called a soft union interior of S over U if*

$$f_S(xyz) \subseteq f_S(y)$$

for all $x, y, z \in S$.

Definition 11.[29] *A soft set over U is called a soft union quasi-ideal of S over U if*

$$(f_S * \tilde{\theta}) \tilde{\cup} (\tilde{\theta} * f_S) \tilde{\supseteq} f_S.$$

Definition 12.[29] *A soft set over U is called a soft union generalized bi-ideal of S over U if*

$$f_S(xyz) \subseteq f_S(x) \cup f_S(z)$$

for all $x, y, z \in S$.

For the sake of brevity, soft union semigroup, soft union right (left, two-sided, interior, quasi, generalized bi-) ideal are abbreviated by SU-semigroup, SU-right (left, two-sided, interior, quasi, generalized bi-) ideal, respectively.

It is easy to see that if $f_S(x) = \emptyset$ for all $x \in S$, then f_S is an SU-semigroup (right ideal, left ideal, ideal, bi-ideal, interior ideal, quasi-ideal, generalized bi-ideal) of S over U. We denote such a kind of SU-semigroup (right ideal, left ideal, ideal, bi-ideal) by $\tilde{\theta}$ [28].

Lemma 1. *Let f_S be any SU-semigroup over U. Then, we have the followings:*

i) $\widetilde{\theta} * \widetilde{\theta} \widetilde{\supseteq} \widetilde{\theta}$. *(If S is regular,* $\widetilde{\theta} * \widetilde{\theta} = \widetilde{\theta}$.*)*
ii) $f_S * \widetilde{\theta} \widetilde{\supseteq} \widetilde{\theta}$ *and* $\widetilde{\theta} * f_S \widetilde{\supseteq} \widetilde{\theta}$.
iii) $f_S \widetilde{\cap} \widetilde{\theta} = \widetilde{\theta}$ *and* $f_S \widetilde{\cup} \widetilde{\theta} = f_S$.

Definition 13. *[28] A soft set f_S over U is called soft union semiprime if for all $a \in S$,*

$$f_S(a) \subseteq f_S(a^2).$$

Theorem 2. *[28,29] Let X be a nonempty subset of a semigroup S. Then, X is a subsemigroup (left, right, two-sided ideal, bi-ideal, interior ideal, quasi-ideal, generalized bi-ideal) of S if and only if $\mathscr{S}_{X^c}$ is an SU-semigroup (left, right, two-sided ideal, bi-ideal, interior ideal, quasi-ideal, generalized bi-ideal) of S.*

Proposition 1. *[28,29] Let f_S be a soft set over U. Then,*

i) f_S *is an SU-semigroup over U if and only if* $f_S * f_S \widetilde{\supseteq} f_S$.
ii) f_S *is an SU-left (right) ideal of S over U if and only if* $\widetilde{\theta} * f_S \widetilde{\supseteq} f_S$ $(f_S * \widetilde{\theta} \widetilde{\supseteq} f_S)$.
iii) f_S *is an SU-bi-ideal of S over U if and only if* $f_S * f_S \widetilde{\subseteq} f_S$ *and* $f_S * \widetilde{\theta} * f_S \widetilde{\supseteq} f_S$.
iv) f_S *is an SU-interior ideal of S over U if and only if* $\widetilde{\theta} * f_S * \widetilde{\theta} \widetilde{\supseteq} f_S$.
v) f_S *is an SU-generalized bi-ideal of S over U if and only if* $f_S * \widetilde{\theta} * f_S \widetilde{\supseteq} f_S$.

Theorem 3. *[28] Every SU-left (right, two sided) ideal of a semigroup S over U is an SU-bi-ideal of S over U.*

Proposition 2. *[29] For a semigroup S, the following conditions are equivalent:*

1)Every SU-ideal of a semigroup S over U is an SU-interior ideal of S over U.
2)Every SU-quasi ideal of S is an SU-semigroup of S.
3)Every one-sided SU-ideal of S is an SU-quasi-ideal of S.
4)Every SU-quasi-ideal of S is an SU-bi-ideal of S.

Theorem 4. *[28] For a semigroup S the following conditions are equivalent:*

1)S is regular.
2) $f_S * g_S = f_S \widetilde{\cup} g_S$ *for every SU-right ideal* f_S *of S over U and SU-left ideal* g_S *of S over U.*

Theorem 5. *[28] For a semigroup S the following conditions are equivalent:*

1)S is regular.
2)For every SU-quasi-ideal of S, $f_S = f_S * \widetilde{\theta} * f_S$.

Theorem 6. *[29] Let f_S be a soft set over U, where S is a regular semigroup. Then, the following conditions are equivalent:*

1) f_S *is an SU-ideal of S over U.*
2) f_S *is an SU-interior ideal of S over U.*

Theorem 7. *[28] For a left regular semigroup S, the following conditions are equivalent:*

1)Every left ideal of S is a two-sided ideal of S.
2)Every SU-left ideal of S is an SU-ideal of S.

3 Semisimple semigroups

In this section, we characterize semisimple semigroups with respect to SU-ideals of semigroups. A semigroup S is called *semisimple* if $J^2 = J$ holds for every ideal J of S, that is, every ideal of S is idempotent.

Proposition 3. *[30] For a semigroup S, the following conditions are equivalent:*

1)S is semisimple.
2) $a \in (SaS)(SaS)$ *for every element a of S, that is, there exist elements* $x, y, z \in S$ *such that* $a = xayaz$.

Proposition 4. *Every SU-interior ideal of a semisimple semigroup S is an SU-ideal of S.*

Proof. Let f_S be an SU-interior ideal of S. Let a and b be any elements of S. Then, since S is semisimple, there exist elements $x, y, z \in S$ such that

$$a = xayaz.$$

Thus, we have

$$f_S(ab) = f_S((xayaz)b) = f_S(xay)a(zb)) \subseteq f_S(a)$$

Hence, f_S is an SU-right ideal of S. Similarly, one can prove that f_S is an SU-left ideal of S. Thus, f_S is an SU-ideal of S.

Now we shall give a characterization of a semisimple semigroup by SU-ideals.

Theorem 8. *For a semigroup S, the following conditions are equivalent:*

1)S is semisimple.
2) $f_S * f_S = f_S$ *for every SU-ideal* f_S *of S. (That is, every SU-ideal is idempotent).*
3) $f_S * f_S = f_S$ *for every SU-interior* f_S *of S. (That is, every SU-interior ideal is idempotent).*
4) $f_S \widetilde{\cup} g_S = f_S * g_S$ *for every SU-ideals* f_S *and* g_S *of S.*
5) $f_S \widetilde{\cup} g_S = f_S * g_S$ *for every SU-ideal* f_S *and every SU-interior ideal* g_S *of S.*
6) $f_S \widetilde{\cup} g_S = f_S * g_S$ *for every SU-interior ideal* f_S *and every SU-ideal* g_S *of S.*
7) $f_S \widetilde{\cup} g_S = f_S * g_S$ *for every SU-interior ideals* f_S *and* g_S *of S.*
8)The set of all SU-ideals of a semisimple semigroup S is a semilattice under the soft union product, that is, $f_S * (g_S * h_S) = f_S * (g_S * h_S)$, $f_S * g_S = g_S * f_S$ *and* $f_S * f_S = f_S$ *for all SU-ideals* f_S *and* g_S *of S.*
9)The set of all SU-interior ideals of a semisimple semigroup S is a semilattice under the soft union product.

Proof. First assume that (1) holds. Let f_S and g_S be any SU-interior ideals of S. Since, $\widetilde{\theta}$ itself is an SU-interior ideal of S and since f_S is an SU-ideal of S by Proposition 4, we have:

$$f_S * g_S \widetilde{\supseteq} f_S * \widetilde{\theta} \widetilde{\supseteq} f_S \quad \text{and} \quad f_S * g_S \widetilde{\supseteq} \widetilde{\theta} * g_S \widetilde{\supseteq} g_S.$$

Thus, $f_S * g_S \tilde{\supseteq} f_S \tilde{\cup} g_S$.

Now, let a be any element of S. Since there exist elements $x,y,z,w \in S$ such that

$$a = (xay)(zaw),$$

and since f_S and g_S are SU-interior ideals of S, we have

$$\begin{aligned}(f_S * g_S)(a) &= \bigcap_{a=pq} (f_S(p) \cup g_S(q)) \\ &\subseteq f_S(xay) \cup g_S(zaw) \\ &\subseteq f_S(a) \cup g_S(a) \\ &= (f_S \tilde{\cup} g_S)(a)\end{aligned}$$

and so we have $f_S * g_S \tilde{\subseteq} f_S \tilde{\cup} g_S$. Hence,

$$f_S * g_S = f_S \tilde{\cup} g_S.$$

So, (1) implies (7). (7) implies (6), (6) implies (4), (7) implies (5), (5) implies (4), (4) implies (2), (7) implies (3), (3) implies (2) and (7) implies (9), (9) implies (8), (8) implies (2).

Assume that (2) holds. Let a be any element of S. Since the soft characteristic function $\mathscr{S}_{(J[a])^c}$ of S is an SU-ideal of S and since $a \in J[a]$, we have

$$\mathscr{S}_{(J[a])^c}(a) = \emptyset$$

Now, let $a \notin J[a]J[a]$. Thus, there do not exist $b,c \in J[a]$ such that $a = J[a]J[a]$. Hence,

$$(\mathscr{S}_{(J[a])^c} * \mathscr{S}_{(J[a])^c})(a) = \bigcap_{a=bc} (\mathscr{S}_{(J[a])^c}(b) \cup \mathscr{S}_{(J[a])^c}(c)) = U$$

But this is a contradiction and thus,

$$a \in J[a]J[a] = (\{a\} \cup aS \cup Sa \cup SaS)(\{a\} \cup aS \cup Sa \cup SaS) = \{a^2\} \cup a^2S \cup aSa \cup aSaS \cup aSa \cup aSaS \cup aSSa \cup aSSaS \cup Sa^2 \cup Sa^2S \cup SaSa \cup SaSaS \cup SaSa \cup SaSaS \cup SaSSa \cup SaSSaS \subseteq (SaS)(SaS)$$

Hence, S is semisimple and so, (2) implies (1).

4 Regular duo semigroups

In this section, we characterize a left (right) duo semigroup in terms of SU-ideals. A semigroup S is called *left (right) duo* if every left (right) ideal of S is a two-sided ideal of S. A semigroup S is *duo* if it is both left and right duo.

Definition 14. *A semigroup S is called soft union left (right) duo if every SU-left (right) ideal of S is an SU-ideal of S and is called soft union duo, if it is both soft left and soft right duo.*

Theorem 9. *For a regular semigroup S, the following conditions are equivalent:*

1)S is left (right) duo.
2)S is soft union left (right) duo.

Proof. First assume that S is left duo. Let f_S be any SU-left ideal of S and a and b be any elements of S. It is known that Sa is a left-ideal of S. And so, by hypothesis, it is a two-sided ideal of S. Since S is regular, we have

$$ab \in (aSa)b \subseteq (Sa)S \subseteq Sa$$

This implies that there exists an element $x \in S$ such that

$$ab = xa.$$

Thus, since f_S is an SU-left ideal of S, we have

$$f_S(ab) = f_S(xa) \subseteq f_S(a)$$

This means that f_S is an SU-right ideal of S and so f_S is an SU-ideal of S. Thus, S is soft union left duo and (1) implies (2).

Conversely, assume that S is soft union left duo. Let A be any left ideal of S. Then, the soft characteristic function $\mathscr{S}_{A^c}$ of A is an SU-left ideal of S. By assumption, $\mathscr{S}_{A^c}$ is an SU-ideal of S and so A is a two-sided ideal of S. Thus, S is left duo and (2) implies (1). The right dual of the proof can be seen similarly. So, the proof is completed.

Theorem 10. *For a regular semigroup S, the following conditions are equivalent:*

1)S is duo.
2)S is soft union duo.

Every SU-right (left) ideal of S is an SU-bi-ideal of S ([28]). Moreover, we have the following:

Theorem 11. *Let S be a regular duo semigroup. Then, every SU-bi-ideal of S is an SU-ideal of S.*

Proof. Let f_S be any SU-bi-ideal of S and a,b be any elements of S. It is known that Sa is a left ideal of S. Since S is a duo semigroup, Sa is a right ideal of S. And since S is regular, we have

$$ab \in (aSa)b \subseteq a((Sa)S) \subseteq aSa$$

This implies that there exists an element $x \in S$ such that

$$ab = axa.$$

Then, since f_S is an SU-bi-ideal of S, we have

$$f_S(ab) = f_S(axa) \subseteq f_S(a) \cup f_S(a) = f_S(a).$$

This means that f_S is an SU-right ideal of S. It can be seen in a similar way that f_S is an SU-left ideal of S. Therefore, f_S is an SU-ideal of S. This completes the proof.

Theorem 12. *[9,20] For a semigroup S, the following conditions are equivalent:*

1)S is a regular duo semigroup.
2)$A \cap B = AB$ for every left ideal A and every right ideal B of S.

3) $Q^2 = Q$ for every quasi-ideal of S. (That is, every quasi-ideal is idempotent.)
4) $EQE = E \cap Q \cap E$ for every ideal E and every quasi-ideal Q of S.

Theorem 13. *For a semigroup S, the following conditions are equivalent:*

1) S is a regular duo semigroup.
2) S is a regular soft union duo semigroup.
3) $f_S * g_S = f_S \tilde{\cup} g_S$ for all SU-bi-ideals f_S and g_S of S.
4) $f_S * g_S = f_S \tilde{\cup} g_S$ for all SU-bi-ideal f_S and for all SU-quasi-ideal g_S of S.
5) $f_S * g_S = f_S \tilde{\cup} g_S$ for all SU-bi-ideal f_S and and for all SU-right ideal g_S of S
6) $f_S * g_S = f_S \tilde{\cup} g_S$ for all SU-quasi-ideal f_S and for all SU-bi-ideal g_S of S.
7) $f_S * g_S = f_S \tilde{\cup} g_S$ for all SU-quasi-ideals f_S and g_S of S.
8) $f_S * g_S = f_S \tilde{\cup} g_S$ for all SU-quasi-ideal f_S and for all SU-right ideal g_S of S.
9) $f_S * g_S = f_S \tilde{\cup} g_S$ for all SU-left ideal f_S and for all SU-bi-ideal g_S of S.
10) $f_S * g_S = f_S \tilde{\cup} g_S$ for all SU-left ideal f_S and for all SU-right ideal g_S of S.
11) $f_S * g_S = f_S \tilde{\cup} g_S$ and $h_S * k_S = h_S \tilde{\cup} k_S$ for all SU-right ideals f_S and g_S of S and for all SU-left ideal h_S and k_S of S.
12) Every SU-quasi-ideal of S is idempotent.

Proof. The equivalence of (1) and (2) follows from Theorem 10. Assume that (2) holds. Let f_S and g_S be any SU-bi-ideals of S. Then, by Theorem 11, f_S is an SU-right ideal of S and g_S is an SU-left ideal of S. Since S is regular, it follows by Theorem 4 that

$$f_S * g_S = f_S \tilde{\cup} g_S$$

Thus, (2) implies (3). It is clear that (3) implies (4), (4) implies (5), (5) implies (8), (8) implies (11), (11) implies (3), (3) implies (6), (6) implies (7), (7) implies (8) and (6) implies (9), (9) implies (10), (10) implies (11).

Assume that (11) holds. Let A and B be any left ideal and right ideal of S, respectively. Let a be any element of $A \cap B$ and $a \notin AB$. Then, $a \in A$ and $a \in B$ and there do not exist $x \in A$ and $y \in B$ such that $a = xy$. Since $\mathscr{S}_{A^c}$ and $\mathscr{S}_{B^c}$ is an SU-left ideal and SU-right ideal of S, respectively, we have

$$\mathscr{S}_{A^c}(a) = \mathscr{S}_{B^c}(a) = \emptyset.$$

and

$$(\mathscr{S}_{A^c} * \mathscr{S}_{B^c})(a) = U$$

But this is a contradiction, so $a \in AB$. Thus, $A \cap B \subseteq AB$. For the converse inclusion, let a be any element of AB and $a \notin A \cap B$. Then, there exist $y \in A$ and $z \in B$ such that $a = yz$. Thus,

$$(\mathscr{S}_{A^c} \tilde{\cup} \mathscr{S}_{B^c})(a) = U$$

and

$$(\mathscr{S}_{A^c} * \mathscr{S}_{B^c})(a) = \bigcap_{a=mn} (\mathscr{S}_{A^c}(m) \cup \mathscr{S}_{B^c}(n)) \subseteq (\mathscr{S}_{A^c}(y) \cup \mathscr{S}_{B^c}(z)) = \emptyset$$

Hence, $(\mathscr{S}_{A^c} * \mathscr{S}_{B^c})(a) = \emptyset$. But this is a contradiction. This implies that $a \in A \cap B$ and that $AB \subseteq A \cap B$. Thus, we have $AB = A \cap B$. It follows by Theorem 12 that S is a regular duo semigroup. Thus (11) implies (1). It is clear that (7) implies (12) by taking $g_S = f_S$.

Conversely, assume that (12) holds. Let Q be any quasi-ideal of S and a be any element of Q and $a \notin QQ$. Then, $\mathscr{S}_{Q^c}$ is an SU-quasi-ideal of S. Thus, we have $\mathscr{S}_{Q^c}(a) = \emptyset$ and since there do not exist $y, z \in Q$ such that $a = yz$,

$$(\mathscr{S}_{Q^c} * \mathscr{S}_{Q^c})(a) = U$$

But this is a contradiction. Hence, we have $a \in Q^2$ and $Q \subseteq Q^2$. Since the converse inclusion always holds, $Q = Q^2$. It follows by Theorem 12 that S is a regular duo semigroup and that (12) implies (1). This completes the proof.

Theorem 14. *For a semigroup S, the following conditions are equivalent:*

1) S is a regular duo semigroup.
2) $f_S * g_S * f_S = f_S \tilde{\cup} g_S$ for every SU-ideal f_S and every SU-bi-ideal g_S of S.
3) $f_S * g_S * f_S = f_S \tilde{\cup} g_S$ for every SU-ideal f_S and every SU-quasi-ideal g_S of S.

Proof. First assume that (1) holds. Let f_S and g_S be any SU-bi-ideal and any SU-ideal of S, respectively. Then, we have

$$f_S * g_S * f_S \tilde{\supseteq} (f_S * \tilde{\theta}) * \tilde{\theta} = f_S * (\tilde{\theta} * \tilde{\theta}) \tilde{\supseteq} f_S * \tilde{\theta} \tilde{\supseteq} f_S$$

On the other hand, since S is regular and duo, f_S is an SU-ideal of S by Theorem 11. Hence, we have

$$f_S * g_S * f_S \tilde{\supseteq} (\tilde{\theta} * g_S) * \tilde{\theta} \tilde{\supseteq} g_S * \tilde{\theta} \tilde{\supseteq} g_S$$

and so

$$(f_S * g_S * f_S) \tilde{\supseteq} f_S \tilde{\cup} g_S$$

In order to show the converse inclusion, let a be any element of S. Then, since S is regular, there exists an element x in S such that

$$a = axa = (axa)xa$$

Thus, we have

$$\begin{aligned}
(f_S * g_S * f_S)(a) &= [f_S * (g_S * f_S)](a) \\
&= \bigcap_{a=pq} [f_S(a) \cup (g_S * f_S)(q)] \\
&\subseteq f_S(ax) \cup (g_S * f_S)(axa) \\
&= f_S(ax) \cup \{ \bigcap_{axa=bc} [g_S(b) \cup f_S(c)] \} \\
&\subseteq f_S(ax) \cup (g_S(a) \cup f_S(xa)) \\
&\subseteq f_S(a) \cup (g_S(a) \cup f_S(a)) \\
&= f_S(a) \cup g_S(a) \\
&= (f_S \tilde{\cup} g_S)(a)
\end{aligned}$$

and so $f_S * g_S * f_S \widetilde{\subseteq} f_S \widetilde{\cup} g_S$ Thus, we obtain that

$$f_S * g_S * f_S = f_S \widetilde{\cup} g_S.$$

Hence, (1) implies (2). It is clear that (2) implies (3).

Assume that (3) holds. Let E and Q any two-sided ideal and quasi-ideal of S, respectively and a be any element of $E \cap Q$ and $a \notin EQE$. Then, $a \in E$ and $a \in Q$ and there do not exist $x, z \in E$ and $y \in Q$ such that $a = xyz$. Since $\mathscr{S}_{E^c}$ and $\mathscr{S}_{Q^c}$ is an SU-ideal and SU-quasi-ideal of S, respectively, we have

$$\mathscr{S}_{E^c}(a) = \mathscr{S}_{Q^c}(a) = \emptyset.$$

and

$$(\mathscr{S}_{E^c} * \mathscr{S}_{Q^c} * \mathscr{S}_{E^c})(a) = U$$

But, this is a contradiction and so $a \in EQE$. Thus, $E \cap Q \subseteq EQE$. For the converse inclusion, let a be any element of EQE and $a \notin E \cap Q$. Then, there exist $x, z \in E$ and $y \in Q$ such that $a = xyz$. Thus,

$$(\mathscr{S}_{E^c} \widetilde{\cup} \mathscr{S}_{Q^c})(a) = U$$

and

$$(\mathscr{S}_{E^c} * \mathscr{S}_{Q^c} * \mathscr{S}_{E^c})(a) = \emptyset$$

But this is a contradiction and so $a \in E \cap Q$. Thus, $EQE \subseteq E \cap Q$ and so $EQE = E \cap Q$. It follows from Proposition 12 that S is regular duo. Hence, (3) implies (1). This completes the proof.

5 Right (left) zero semigroup

In this section, we characterize right (left) zero semigroups in terms of SU-ideals of S. A semigroup S is called *right (left) zero* if $xy = y$ $(xy = x)$ for all $x, y \in S$.

Proposition 5. *For a semigroup S, the following conditions are equivalent:*

1)The set of all idempotent elements of S forms a left (right) zero subsemigroup of S.
2)For every SU-left (right) ideal f_S of S, $f_S(e) = f_S(f)$ for all idempotent elements e and f of S.

Proof. First assume that the set I_S of all idempotent elements of S is a left zero subsemigroup of S. Let $e, f \in I_S$ and f_S be an SU-left ideal of S. Then, since

$$ef = e \text{ and } fe = f$$

we have

$$f_S(e) = f_S(ef) \subseteq f_S(f) = f_S(fe) \subseteq f_S(e)$$

and so

$$f_S(e) = f_S(f).$$

Thus, (1) implies (2).

Conversely, assume that (2) holds. Since S is regular, it is obvious that $I_S \neq \emptyset$. Moreover, the soft characteristic function $\mathscr{S}_{(L[f])^c}$ is an SU-left ideal of S. Thus, by assumption, we have

$$\mathscr{S}_{(L[f])^c}(e) = \mathscr{S}_{(L[f])^c}(f) = \emptyset$$

and so $e \in L[f] = Sf$. (Here note that, if S is a regular semigroup, $L[a] = Sa$ for every $a \in S$ ([9]). Thus, for some $x \in S$, we have

$$e = xf = x(ff) = (xf)f = ef$$

This means that I_S is a left zero semigroup. Thus (2) implies (1). The case when S is right zero, the proof can be seen similarly. This completes the proof.

Corollary 1. *For an idempotent semigroup S, the following conditions are equivalent:*

1)S is left (right) zero.
2)For every SU-left (right) ideal f_S of S, $f_S(e) = f_S(f)$ for all elements $e, f \in S$.

Proposition 6. *Let S be a group. Then, every SU-bi-ideal of S is a constant function.*

Proof. Let S be a group with identity e and f_S be any SU-bi-ideal of S and a be any element of S. Then,

$$f_S(a) = f_S(eae) \subseteq f_S(e) \cup f_S(e) = f_S(e) = f_S(ee) = f_S((aa^{-1})(a^{-1}a)) = f_S(a(a^{-1}a^{-1})a) \subseteq f_S(a) \cup f_S(a) = f_S(a)$$

and so $f_S(e) = f_S(a)$. This implies that f_S is a constant function.

Proposition 7. *For a regular semigroup S, the following conditions are equivalent:*

1)S is a group.
2)For every SU-bi-ideal f_S of S, $f_S(e) = f_S(f)$ for all idempotent elements $e, f \in S$.

Proof. Assume that (1) holds. Let f_S be any SU-bi-ideal of S. Then, it follows from Proposition 6 that f_S is a constant function. This implies that

$$f_S(e) = f_S(f)$$

for all idempotent elements $e, f \in S$. Thus (1) implies (2).

Conversely, assume that (2) holds. Let e and f be any idempotent elements of S. As is well-known, if S is a regular semigroup, $B[x]$, the principal ideal of S generated by $x \in S$ is $B[x] = xSx$ ([9]). Moreover, since the soft characteristic function $\mathscr{S}_{(B[f])^c}$ is an SU-bi-ideal of S and since $f \in B[f]$, we have

$$\mathscr{S}_{(B[f])^c}(e) = \mathscr{S}_{(B[f])^c}(f) = \emptyset$$

and so $e \in B[f] = fsf$, which means that $e = fxf$ for some $x \in S$. One can similarly obtain that $f = eye$ for some $y \in S$. Thus, we have

$$e = fxf = fx(ff) = (fxf)f = ef = e(eye) = (ee)ye = eye = f$$

Since S is regular, $I_S \neq \emptyset$ and S contains exactly one idempotent. Thus, it follows from ([9], p.33) that S is a group. Thus (2) implies (1). This completes the proof.

6 Right (left) simple semigroups

In this section, we define soft union simple semigroup and give the relation of soft union simple semigroup with simple semigroup. A semigroup S is called *left (right) simple* if it contains no proper left (right) ideal of S and is called *simple* if it contains no proper ideal.

Definition 15. *A semigroup S is called soft union left (right) simple if every SU-left (right) ideal of S is a constant function and is called soft union simple if every SU-ideal of S is a constant function.*

Theorem 15. *For a semigroup S, the following conditions are equivalent:*

1) S is left (right) simple.
2) S is soft union left (right) simple.

Proof. First assume that S is left simple. Let f_S be any SU-left ideal of S and a and b be any element of S. Then, it follows from ([9], p. 6) that there exist elements $x, y \in S$ such that $b = xa$ and $a = yb$. Hence, since S is an SU-left ideal of S,

$$f_S(a) = f_S(yb) \subseteq f_S(b) = f_S(xa) \subseteq f_S(a)$$

and so $f_S(a) = f_S(b)$. Since a and b be any elements of S, this means that f_S is a constant function. Thus, we obtain that S is soft union left simple and (1) implies (2).

Conversely, assume that (2) holds. Let A be any left ideal of S. Then, $\mathscr{S}_{A^c}$ is an SU-left ideal of S. By assumption, $\mathscr{S}_{A^c}$ is a constant function. Let x be any element of S. Then, since $A \neq \emptyset$,

$$\mathscr{S}_{A^c}(x) = \emptyset$$

and so $x \in A$. This implies that $S \subseteq A$, and so $S = A$. Hence, S is left simple and (2) implies (1). In the case, when S is soft union right simple, the proof follows similarly.

Theorem 16. *For a semigroup S, the following conditions are equivalent:*

1) S is simple.
2) S is soft union simple.

As is well-known, a semigroup S is a group if it is left and right simple. From this, we have the following theorem:

Proposition 8. *For a semigroup S, the following conditions are equivalent:*

1) S is a group.
2) S is both soft union left and soft union right simple.

Proposition 9. *Let S be a left simple semigroup. Then, every SU-bi-ideal of S is an SU-right ideal of S.*

Proof. Let f_S be an SU-bi-ideal of S and a and b be any elements of S. Then, since S is left simple, there exists an element x in S such that

$$b = xa.$$

Then, since f_S is an SU-bi-ideal of S, we have

$$f_S(ab) = f_S(a(xa)) = f_S(a) \cup f_S(a) = f_S(a)$$

which means that f_S is an SU-right ideal of S. This completes the proof.

7 Semilattices of left (right) simple semigroups

In this section, we characterize a semigroup that is a semilattice of left (right) simple semigroups by SU-ideals. A semigroup S is a *semilattice of left simple semigroups* if it is the set-theoretical intersection of the family of left simple semigroups S_i $(i \in M)$ such that,

$$S = \bigcap_{i \in M} S_i$$

such that the products S_iS_j and S_jS_i are both contained in the same S_k $(k \in M)$.

Theorem 17. *[9, 22] For a semigroup S, the following conditions are equivalent:*

1) S is a semilattice of left simple semigroups.
2) S is left regular and every left ideal of S is two-sided.
3) S is left regular and $AB = BA$ for any left ideals A and B of S.

Theorem 18. *[28] For a left regular semigroup S, the following conditions are equivalent:*

1) Every left ideal of S is a two-sided ideal of S.
2) Every SU-left ideal of S is an SU-ideal of S.

Theorem 19. *For a semigroup S, the following conditions are equivalent:*

1) S is a semilattice of left simple semigroups.
2) S is left regular and every SU-left ideal of S is an SU-ideal of S.
3) $f_S * g_S = f_S \widetilde{\cup} g_S$ for every SU-left ideals of S.
4) The set of all SU-left ideals of S is a semilattice under the soft union product.
5) The set of all left ideals of S is a semilattice under the multiplication of subsets.

Proof. The equivalence of (1) and (2) follows from Theorem 17 and Theorem 18. Assume that (2) holds. Let f_S and g_S be any SU-left ideals of S and a be any element of S. Then, since S is left regular, there exists an element

$x \in S$ such that $a = xa^2$. By assumption, f_S is also an SU-right ideal of S. So, we have

$$\begin{aligned}(f_S * g_S)(a) &= \bigcap_{a=yz} (f_S(y) \cup g_S(z)) \\ &\subseteq (f_S(xa) \cup g_S(a)) \\ &\subseteq (f_S(a) \cup g_S(a)) \\ &= (f_S \tilde{\cup} g_S)(a)\end{aligned}$$

Thus, $f_S * g_S \tilde{\subseteq} f_S \tilde{\cup} g_S$. On the other hand, by assumption, g_S is SU-right ideal of S, and so

$$\begin{aligned}(f_S * g_S)(a) &= \bigcap_{a=yz} (f_S(y) \cup g_S(z)) \\ &\supseteq (f_S(yz) \cup g_S(yz)) \\ &= f_S(a) \cup g_S(a) \\ &= (f_S \tilde{\cup} g_S)(a)\end{aligned}$$

Thus, $f_S * g_S \tilde{\supseteq} f_S \tilde{\cup} g_S$. Thus, $f_S * g_S = f_S \tilde{\cup} g_S$ and so (2) implies (3).

(3) implies (4) is clear. Assume that (4) holds. Let A and B be any left ideals of S and a be any element of BA and $a \notin AB$. Then, there exist $y \in B$ and $z \in A$ such that $a = yz$ and there do not exist $m \in A$ and $n \in B$ such that $a = mn$. Then, since the soft characteristic function $\mathscr{S}_{A^c}$ and $\mathscr{S}_{B^c}$ are SU-left ideals of S, we have

$$(\mathscr{S}_{B^c} * \mathscr{S}_{A^c})(a) = \emptyset$$

and

$$(\mathscr{S}_{A^c} * \mathscr{S}_{B^c})(a) = U.$$

But this is a contradiction. Hence, $a \in AB$. Thus, $BA \subseteq AB$. Similarly, we have $AB \subseteq BA$. Thus, $AB = BA$.

In order to see that any left ideal A of S is idempotent, let a be any element of A and $a \notin AA$. Since $\mathscr{S}_{A^c}$ is an SU-left ideal of S, we have

$$(\mathscr{S}_{A^c} * \mathscr{S}_{A^c})(a) = U$$

and

$$\mathscr{S}_{A^c}(a) = \emptyset.$$

But this is a contradiction and so $a \in A^2$. Thus, $A \subseteq A^2$ and so $A = A^2$. Therefore (4) implies (5).

Finally, assume that (5) holds. Let A be any left ideal of S and a be any element of S. Then, since S itself is a left ideal, by assumption we have

$$AS = SA \subseteq A$$

Thus, A is a right ideal of S, and so A is a two-sided ideal of S.

Let a be any element of S. Then, since the left ideal $L[a]$ of S is idempotent by assumption and since $a \in L[a]$, we have

$$\begin{gathered}a \in L[a]L[a] = (\{a\} \cup Sa)(\{a\} \cup Sa) = \\ \{a^2\} \cup aSa \cup Sa^2 \cup SaSa \subseteq \{a^2\} \cup (aS)aSa \cup Sa^2 \cup SaSa \subseteq \\ \{a^2\} \cup SaSa \cup Sa^2 \subseteq \{a^2\} \cup Sa^2\end{gathered}$$

which implies that S is left-regular. Thus, it follows by Theorem 17-(2) that S is a semilattice of left simple groups. That is to say (5) implies (1). This completes the proof.

The left-right dual of Theorem 19 reads as follows:

Theorem 20. *For a semigroup S, the following conditions are equivalent:*

1)S is a semilattice of right simple semigroups.
2)S is right regular and every SU-right ideal of S is an SU-ideal of S.
3)$f_S * g_S = f_S \tilde{\cup} g_S$ for every SU-right ideals of S.
4)The set of all SU-right ideals of S is a semilattice under the soft union product.
5)The set of all right ideals of S is a semilattice under the multiplication of subsets.

Theorem 21. *[28] For a semigroup S, the following conditions are equivalent:*

1)S is left regular.
2)For every SU-left ideal f_S of S, $f_S(a) = f_S(a^2)$ for all $a \in S$.

Theorem 22. *For a semigroup S, the following conditions are equivalent:*

1)S is a semilattice of left simple semigroups.
2)For every SU-left ideal f_S of S, $f_S(a) = f_S(a^2)$ and $f_S(ab) = f_S(ba)$ for all $a, b \in S$.

Proof. Assume that S is a semilattice of left simple semigroups. Let f_S be any SU-left ideal of S. Then, by Theorem 17-(2), S is left regular and f_S is an SU-ideal of S. Let a be any element of S. Thus, by Theorem 21, we have

$$f_S(ab) = f_S((ab)^2) = f_S(a(ba)b) \subseteq f_S(ba).$$

Similarly, we have $f_S(ba) \subseteq f_S(ab)$. Hence, we obtain that

$$f_S(ab) = f_S(ba).$$

Thus, (1) implies (2).

Conversely, assume that (2) holds. Let f_S be any SU-ideal of S. Since $f_S(a) = f_S(a^2)$ for all $a \in S$, it follows from Theorem 21 that S is left regular. Let A and B be any left ideal of S and ab be any element of AB. Since the soft characteristic function $\mathscr{S}_{(L[ba])^c}$ is an SU-left ideal of S and since $ba \in L[ba]$, we have

$$\mathscr{S}_{(L[ba])^c}(ab) = \mathscr{S}_{(L[ba])^c}(ba) = \emptyset$$

This implies that

$$ab \in L[ba] = \{ba\} \cup Sba \subseteq BA \cup SBA \subseteq BA$$

and so we have $AB \subseteq BA$. Similarly, it can be seen that the converse inclusion holds. Thus, we obtain that

$$AB = BA$$

Then, it follows by Theorem 17-(3) that S is a semilattice of left simple semigroups. Therefore (3) implies (1). This completes the proof.

The right dual of Theorem 22 reads as follows:

Theorem 23. *For a semigroup S, the following conditions are equivalent:*

1)S is a semilattice of right simple semigroups.
2)For every SU-right ideal f_S of S, $f_S(a) = f_S(a^2)$ and $f_S(ab) = f_S(ba)$ for all $a, b \in S$.

8 A semilattice of left (right) groups

In this section, we characterize a semigroup that is a semilattice of left (right) simple groups by SU-ideals. An element a of S is said to be *left (right) cancellable* if, for any $x, y \in S$ $ax = ay$ $(xa = ya)$ implies $x = y$. A semigroup S is called *left (right) cancellative* if every element of S is left (right) cancellative. A semigroup S is called a *left group* if it is left simple and right cancellable ([9]), that is, for all $a \in S$, there exists a unique element $x \in S$ such that $xa^2 = a$ ([21]). Dually, a semigroup S is called a *right group* if it is right simple and left cancellable.

Theorem 24. *[21] For a semigroup S, the following conditions are equivalent:*

1)S is a semilattice of left groups.
2)S is regular and $aS \subseteq Sa$ for every $a \in S$.

Theorem 25. *Let S be a semigroup that is a semilattice of left groups. Then, every SU-(generalized) bi-ideal of S is an SU-right ideal of S.*

Proof. Let f_S be any SU-bi-ideal of S, and a and b any elements of S. Then, it follows from Theorem 24 that there exist elements $x, y \in S$ such that

$$a = axa \text{ and } ab = ya.$$

Thus,

$$ab = (axa)b = (ax)(ab) = (ax)(ya) = a(xy)a.$$

Since f_S is an SU-bi-ideal of S,

$$f_S(ab) = f_S(a(xy)a) \subseteq f_S(a) \cup f_S(a) = f_S(a).$$

Hence, f_S is an SU-right ideal of S.

Corollary 2. *Let S be a semigroup that is a semilattice of left groups. Then, every SU-left ideal of S is an SU-right ideal of S, that is to say, S is soft union left duo.*

Theorem 26. *Let S be a semigroup that is a semilattice of left groups. Then, every SU-interior ideal of S is an SU-left ideal of S.*

Proof. Let f_S be any SU-interior ideal of S, and a and b any elements of S. Then, it follows from Theorem 24 that there exist element $z \in S$ such that

$$b = bzb.$$

Thus,

$$ab = (axa)b = (ax)(ab) = (ax)(ya) = a(xy)a.$$

Since f_S is an SU-bi-ideal of S,

$$f_S(ab) = f_S(a(bzb)) = f_S((a)b(zb)) \subseteq f_S(b).$$

Hence, f_S is an SU-left ideal of S.

Theorem 27. *[29] For a semigroup S the following conditions are equivalent:*

1)S is regular.
2)$f_S \tilde{\cup} g_S = f_S * g_S * f_S$ for every SU-quasi-ideal f_S of S and SU-ideal g_S of S over U.

Theorem 28. *For a semigroup S, the following conditions are equivalent:*

1)S is a semilattice of left groups.
2)$f_S \tilde{\cup} g_S = f_S * g_S$ for every SU-quasi-ideal f_S and SU-left ideal g_S of S.
3)$f_S \tilde{\cup} g_S = f_S * g_S$ for every SU-quasi-ideal f_S and SU-ideal g_S of S.
4)$f_S \tilde{\cup} g_S = f_S * g_S$ for every SU-quasi-ideal f_S and SU-interior ideal g_S of S.
5)$f_S \tilde{\cup} g_S = f_S * g_S$ for every SU-bi-ideal f_S and SU-left ideal g_S of S.
6)$f_S \tilde{\cup} g_S = f_S * g_S$ for every SU-bi-ideal f_S and SU-ideal g_S of S.
7)$f_S \tilde{\cup} g_S = f_S * g_S$ for every SU-bi-ideal f_S and SU-interior ideal g_S of S.
8)$f_S \tilde{\cup} g_S = f_S * g_S$ for every SU-bi-ideal f_S and SU-left ideal g_S of S.
9)$f_S \tilde{\cup} g_S = f_S * g_S$ for every SU-generalized bi-ideal f_S and SU-left ideal g_S of S.
10)$f_S \tilde{\cup} g_S = f_S * g_S$ for every SU-generalized bi-ideal f_S and SU-ideal g_S of S.
11)$f_S \tilde{\cup} g_S = f_S * g_S$ for every SU-generalized bi-ideal f_S and SU-interior ideal g_S of S.
12)$f_S \tilde{\cup} g_S = f_S * g_S$ for every SU-one-sided ideal f_S and SU-ideal g_S of S.
13)$f_S \tilde{\cup} g_S = f_S * g_S$ for every SU-one-sided ideal f_S and SU-interior ideal g_S of S.
14)S is regular left duo.

Proof. First assume that (1) holds. Let f_S and g_S be any SU-generalize bi-ideal of S and SU-interior ideal of S, respectively and a be any element of S. Then, since S is regular by Theorem 24, there exists an element $x \in S$ such that

$$a = axa (= axaxa).$$

Since g_S is an SU-interior ideal of S, $g_S((x)a(xa)) \subseteq g_S(a)$. Thus, we have

$$\begin{aligned}(f_S * g_S)(a) &= \bigcap_{a=pq} (f_S(p) \cup g_S(q)) \\ &\subseteq f_S(a) \cup g_S((x)a(xa)) \\ &\subseteq f_S(a) \cup g_S(a) \\ &= (f_S \tilde{\cup} g_S)(a)\end{aligned}$$

and so $f_S * g_S \widetilde{\subseteq} f_S \widetilde{\cup} g_S$. Moreover, it follows by Theorem 25 that f_S is an SU-right ideal of S. Thus, we have

$$\begin{aligned}(f_S * g_S)(a) &= \bigcap_{a=pq} (f_S(p) \cup g_S(q)) \\ &\supseteq \bigcap_{a=pq} (f_S(pq) \cup g_S(pq)) \\ &= \bigcap_{a=pq} (f_S(a) \cup g_S(a)) \\ &= f_S(a) \cup g_S(a) \\ &= (f_S \widetilde{\cup} g_S)(a)\end{aligned}$$

and so $f_S * g_S \widetilde{\supseteq} f_S \widetilde{\cup} g_S$. Therefore, we obtain that $f_S * g_S = f_S \widetilde{\cup} g_S$ and that (1) implies (10). It is clear that (10) implies (9), (9) implies (8), (8) implies (5), (5) implies (2), (10) implies (7), (7) implies (6), (6) implies (5), (5) implies (2), (7) implies (4), (4) implies (3), (3) implies (2) and (4) implies (12), (12) implies (11).

Assume that (2) holds. Then, it follows by Theorem 27 that S is regular. Let Q be any quasi-ideal of S. Then, the soft characteristic function $\mathscr{S}_{Q^c}$ is an SU-quasi-ideal of S. Since $\widetilde{\theta}$ itself is an SU-left ideal of S and so by assumption, we have

$$\mathscr{S}_{Q^c} = \mathscr{S}_{Q^c} \widetilde{\cup} \widetilde{\theta} = \mathscr{S}_{Q^c} * \widetilde{\theta}.$$

Thus, $\mathscr{S}_Q$ is an SU-right ideal of S, and so Q is a right ideal of S. Thus, any quasi-ideal of S is a right ideal of S. Let $a \in S$. Then, the quasi-ideal Sa is a right ideal of S. Since S is regular, we have

$$aS \subseteq (aSa)S = ((aS)a)S \subseteq (Sa)S \subseteq Sa.$$

Thus, $aS \subseteq Sa$ and since S is regular, S is a semilattice of left groups by Theorem 24. Thus, (2) implies (1).

Assume that (11) holds. Let f_S and g_S be any SU-right ideal and any SU-left ideal of S, respectively. Then, since $\widetilde{\theta}$ itself is an SU-ideal of S and so by assumption, we have

$$g_S = g_S \widetilde{\cup} \widetilde{\theta} = g_S * \widetilde{\theta}$$

Thus, g_S is an SU-right ideal of S, that is, g_S is an SU-ideal of S. Thus, by assumption, $f_S * g_S = f_S \widetilde{\cup} g_S$ for every SU-right ideal f_S of S over U and SU-left ideal g_S of S over U. It follows by Theorem 4 that S is regular. As is proved in (2) implies (1), we have $aS \subseteq Sa$. Thus, S is a semilattice of left groups, so (11) implies (1).

Assume that (1) holds. Then, it follows by Theorem 24 that S is regular. Moreover, it follows by Corollary 2 that S is soft union left duo and so by Theorem 9, S is left duo. Thus (1) implies (13).

Conversely assume that (13) holds. Then, it follows by Theorem 9 that S is left duo, that is, every left ideal of S is a right ideal of S. In order to prove that S is semilattice of left groups, by Theorem 24, it suffices to show that $aS \subseteq Sa$ for all $a \in S$. As is proved in (2) implies (1), we have $aS \subseteq Sa$. Thus, S is a semilattice of left groups, so (13) implies (1). This completes the proof.

The left-right dual of Theorem 29 reads as follows:

Theorem 29. *For a semigroup S, the following conditions are equivalent:*

1) S is a semilattice of right groups.
2) $f_S \widetilde{\cup} g_S = f_S * g_S$ for every SU-quasi-ideal f_S and SU-right ideal g_S of S.
3) $f_S \widetilde{\cup} g_S = f_S * g_S$ for every SU-quasi-ideal f_S and SU-ideal g_S of S.
4) $f_S \widetilde{\cup} g_S = f_S * g_S$ for every SU-quasi-ideal f_S and SU-interior ideal g_S of S.
5) $f_S \widetilde{\cup} g_S = f_S * g_S$ for every SU-bi-ideal f_S and SU-right ideal g_S of S.
6) $f_S \widetilde{\cup} g_S = f_S * g_S$ for every SU-bi-ideal f_S and SU-ideal g_S of S.
7) $f_S \widetilde{\cup} g_S = f_S * g_S$ for every SU-bi-ideal f_S and SU-interior ideal g_S of S.
8) $f_S \widetilde{\cup} g_S = f_S * g_S$ for every SU-bi-ideal f_S and SU-right ideal g_S of S.
9) $f_S \widetilde{\cup} g_S = f_S * g_S$ for every SU-generalized bi-ideal f_S and SU-right ideal g_S of S.
10) $f_S \widetilde{\cup} g_S = f_S * g_S$ for every SU-generalized bi-ideal f_S and SU-ideal g_S of S.
11) $f_S \widetilde{\cup} g_S = f_S * g_S$ for every SU-generalized bi-ideal f_S and SU-interior ideal g_S of S.
12) $f_S \widetilde{\cup} g_S = f_S * g_S$ for every SU-one-sided ideal f_S and SU-ideal g_S of S.
13) $f_S \widetilde{\cup} g_S = f_S * g_S$ for every SU-one-sided ideal f_S and SU-interior ideal g_S of S.
14) S is regular right duo.

Theorem 30. *For a semigroup S, the following conditions are equivalent:*

1) S is a semilattice of left groups.
2) $f_S \widetilde{\cup} g_S = f_S * g_S * f_S$ for every SU-quasi-ideal f_S and SU-left ideal g_S of S.
3) $f_S \widetilde{\cup} g_S = f_S * g_S * f_S$ for every SU-bi-ideal f_S and SU-left ideal g_S of S.
4) $f_S \widetilde{\cup} g_S = f_S * g_S * f_S$ for every SU-generalized bi-ideal f_S and SU-left ideal g_S of S.

Proof. First assume that (1) holds. Let f_S and g_S be any SU-generalized bi-ideal of S. Then, we have

$$f_S * g_S * f_S \widetilde{\supseteq} f_S * \widetilde{\theta} * f_S \widetilde{\supseteq} f_S$$

On the other hand, since the SU-left ideal g_S is an SU-bi-ideal of S, we have

$$f_S * g_S * f_S \widetilde{\supseteq} (\widetilde{\theta} * g_S) * \widetilde{\theta} \widetilde{\supseteq} g_S * \widetilde{\theta} \widetilde{\supseteq} g_S$$

Therefore, we have

$$f_S * g_S * f_S \widetilde{\supseteq} f_S \widetilde{\cup} g_S.$$

Let a be any element of S. Then, it follows by Theorem 24 that there exist elements $x, y \in S$ such that $a = axa$ and $ax = ya$. Hence,

$$ax = axaxax = axax(ya) = (axa)(xya).$$

Thus,

$$\begin{aligned}(f_S * g_S * f_S)(a) &= [(f_S * g_S) * f_S](a)\\ &= \bigcap_{a=pq} [(f_S * g_S)(p) * f_S(q)]\\ &\subseteq (f_S * g_S)(ax) \cup f_S(a)\\ &= \{ \bigcap_{ax=pq} (f_S(p) \cup g_S(q)) \cup f_S(a)\\ &\subseteq (f_S(axa) \cup g_S(xya)) \cup f_S(a)\\ &\subseteq (f_S(a) \cup g_S(a)) \cup f_S(a)\\ &= (f_S \widetilde{\cup} g_S)(a)\end{aligned}$$

and so, $f_S * g_S * f_S \widetilde{\subseteq} f_S \widetilde{\cup} g_S$. Thus, $f_S * g_S * f_S = f_S \widetilde{\cup} g_S$ and (1) implies (4). It is clear that (4) implies (3) and (3) implies (2).

Assume that (2) holds. Let f_S be any SU-quasi ideal of S. Then, $\widetilde{\theta}$ is an SU-left ideal of S and so by assumption,

$$f_S = f_S \widetilde{\cup} \widetilde{\theta} = f_S * \widetilde{\theta} * f_S$$

Thus,it follows by Theorem 5 that S is regular. On the other hand, let g_S be any SU-left ideal of S. then, by assumption,

$$g_S = \widetilde{\theta} \widetilde{\cup} g_S = \widetilde{\theta} * g_S * \widetilde{\theta}$$

Thus, g_S is an SU-interior ideal of S. Since S is regular, g_S is an SU-ideal of S by Theorem 6. Therefore, we obtain that every SU-left ideal of S is an ideal of S. It follows by Theorem 7 that every SU-left ideal of S is an SU-ideal of S. Let $a \in S$. Since S is regular, the left ideal Sa is an ideal of S. Thus, we have

$$aS \subseteq (aSa)S \subseteq a((Sa)S) \subseteq a(Sa) = (aS)a \subseteq Sa.$$

Thus, $aS \subseteq Sa$ and since S is regular, S is a semilattice of left groups by Theorem 24. Thus (2) implies (1).

The left-right dual of Theorem 30 reads as follows:

Theorem 31. *For a semigroup S, the following conditions are equivalent:*

1) S is a semilattice of right groups.
2) $f_S \widetilde{\cup} g_S = f_S * g_S * f_S$ for every SU-quasi-ideal f_S and SU-right ideal g_S of S.
3) $f_S \widetilde{\cup} g_S = f_S * g_S * f_S$ for every SU-bi-ideal f_S and SU-right ideal g_S of S.
4) $f_S \widetilde{\cup} g_S = f_S * g_S * f_S$ for every SU-generalized bi-ideal f_S and SU-right ideal g_S of S.

Theorem 32. *For a semigroup S, the following conditions are equivalent:*

1) S is a semilattice of left groups.
2) $f_S \widetilde{\cup} g_S = f_S * \widetilde{\theta} * g_S$ for every SU-quasi-ideal f_S and SU-left ideal g_S of S.
3) $f_S \widetilde{\cup} g_S = f_S * \widetilde{\theta} * g_S$ for every SU-bi-ideal f_S and SU-left ideal g_S of S.
4) $f_S \widetilde{\cup} g_S = f_S * \widetilde{\theta} * g_S$ for every SU-generalized bi-ideal f_S and SU-left ideal g_S of S.

Proof. First assume that (1) holds. Let f_S and g_S be any SU-generalized bi-ideal and SU-left ideal of S, respectively. Then, we have

$$f_S * \widetilde{\theta} * g_S = f_S * (\widetilde{\theta} * g_S) \widetilde{\supseteq} f_S * g_S \widetilde{\supseteq} \widetilde{\theta} * g_S \widetilde{\supseteq} g_S.$$

Moreover, by Theorem 25 that f_S is an SU-right ideal of S. Thus,

$$f_S * \widetilde{\theta} * g_S = (f_S * \widetilde{\theta}) * g_S \widetilde{\supseteq} f_S * g_S \widetilde{\supseteq} f_S * \widetilde{\theta} \widetilde{\supseteq} f_S.$$

Thus, we have $f_S * \widetilde{\theta} * g_S \widetilde{\supseteq} f_S \widetilde{\cup} g_S$.

Let a be any element of S. Then, it follows by Theorem 24 that there exist elements $x, y \in S$ such that $a = axa$ and $ax = ya$. Hence,

$$ax = axaxax = axax(ya) = (axa)(xya).$$

Thus, we have

$$\begin{aligned}(f_S * \widetilde{\theta} * g_S)(a) &= [(f_S * \widetilde{\theta}) * g_S](a)\\ &= [\bigcap_{a=pq} (f_S * \widetilde{\theta})(p)] * g_S(q)\\ &\subseteq (f_S * \widetilde{\theta})(ax) \cup g_S(a)\\ &= \{ \bigcap_{ax=pq} (f_S(p) \cup \widetilde{\theta}(q))\} \cup g_S(a)\\ &\subseteq (f_S(axa) \cup \widetilde{\theta}(aya)) \cup g_S(a)\\ &= (f_S(a) \cup \emptyset) \cup g_S(a)\\ &\subseteq f_S(a) \cup g_S(a)\\ &= (f_S \widetilde{\cup} g_S)(a)\end{aligned}$$

and so, $f_S * \widetilde{\theta} * g_S \widetilde{\subseteq} f_S \widetilde{\cup} g_S$. And so, $f_S * \widetilde{\theta} * g_S = f_S \widetilde{\cup} g_S$. Thus, (1) implies (4).

It is clear that (4) implies (3) and (3) implies (2).

Assume that (2) holds. Let f_S and g_S be any SU-quasi-ideal and SU-left ideal of S, respectively. Then, by assumption, we have

$$f_S \widetilde{\cup} g_S = f_S * \widetilde{\theta} * g_S = f_S * (\widetilde{\theta} * g_S) \widetilde{\supseteq} f_S \widetilde{*} g_S.$$

Hence, it follows by Theorem 4 that S is regular. Let g_S be any SU-left ideal of S. Then, since g_S is an SU-quasi-ideal of S and since $\widetilde{\theta}$ itself is an SU-left ideal of S, we have

$$g_S = g_S \widetilde{\cup} \widetilde{\theta} = g_S * \widetilde{\theta} * \widetilde{\theta}.$$

Let L be any left ideal of S, $a \in L$ and $a \notin LSS$. Then, the soft characteristic function $\mathscr{S}_{L^c}$ is an SU-left ideal of S. Thus,

$$\mathscr{S}_{L^c}(a) = \emptyset$$

and

$$(\mathscr{S}_{L^c} * \mathscr{S}_{S^c} * \mathscr{S}_{S^c})(a) = U$$

which is a contradiction, and so $a \in LSS$. Thus, $L \subseteq LSS$. Moreover, let $a \in LSS$ and $a \notin L$. Then,

$$\mathscr{S}_{L^c}(a) = U$$

and

$$(\mathscr{S}_{L^c} * \mathscr{S}_{S^c} * \mathscr{S}_{S^c})(a) = \emptyset$$

which is a contradiction, and so $a \in L$. Thus, $LSS \subseteq L$, and so $LSS = L$. Since Sa is a left ideal of S, we have $(Sa)SS = Sa$ and so,

$$aS \subseteq (aSa)S = a(Sa)S = a((Sa)SS)S \subseteq a((Sa)SS) \subseteq a(Sa) = (aS)a \subseteq Sa.$$

It follows by Theorem 24 that S is a semilattice of left groups and so (2) implies (1).

The left-right dual of Theorem 32 reads as follows:

Theorem 33. *For a semigroup S, the following conditions are equivalent:*

1) S is a semilattice of right groups.
2) $f_S \widetilde{\cup} g_S = f_S * \widetilde{\theta} * g_S$ for every SU-quasi-ideal f_S and SU-right ideal g_S of S.
3) $f_S \widetilde{\cup} g_S = f_S * \widetilde{\theta} * g_S$ for every SU-bi-ideal f_S and SU-right ideal g_S of S.
4) $f_S \widetilde{\cup} g_S = f_S * \widetilde{\theta} * g_S$ for every SU-generalized bi-ideal f_S and SU-right ideal g_S of S.

Theorem 34. *For a semigroup S, the following conditions are equivalent:*

1) S is a semilattice of left groups.
2) $f_S \widetilde{\cup} h_S \widetilde{\cup} g_S = f_S * h_S * g_S$ for every SU-quasi-ideal f_S, for every SU-ideal h_S and every SU-left ideal g_S of S.
3) $f_S \widetilde{\cup} h_S \widetilde{\cup} g_S = f_S * h_S * g_S$ for every SU-bi-ideal f_S, for every SU-ideal h_S and every SU-left ideal g_S of S.
4) $f_S \widetilde{\cup} h_S \widetilde{\cup} g_S = f_S * h_S * g_S$ for every SU-generalized bi-ideal f_S, for every SU-ideal h_S and every SU-left ideal g_S of S.

Proof. First assume that (1) holds. Let f_S be any SU-generalized bi-ideal of S, h_S be any SU-ideal of S and g_S be any SU-left ideal of S. Then, we have

$$f_S * h_S * g_S \widetilde{\supseteq} \widetilde{\theta} * (\widetilde{\theta} * g_S) \widetilde{\supseteq} \widetilde{\theta} * g_S \widetilde{\supseteq} g_S$$

and

$$f_S * h_S * g_S \widetilde{\supseteq} \widetilde{\theta} * h_S * \widetilde{\theta} \widetilde{\supseteq} h_S.$$

Moreover, by Theorem 25, since SU-generalized bi-ideal f_S of S is an SU-right ideal of S, we have

$$f_S * h_S * g_S \widetilde{\supseteq} (f_S * \widetilde{\theta}) * \widetilde{\theta} \widetilde{\supseteq} f_S * \widetilde{\theta} \widetilde{\supseteq} f_S.$$

Hence, we have

$$f_S * h_S * g_S \widetilde{\supseteq} f_S \widetilde{\cup} h_S \widetilde{\cup} g_S.$$

Let $a \in S$. Then, by Theorem 24, $a = axa$ and $ax = ya$ for some $x, y \in S$. Then,

$$ax = axaxax = axax(ya) = (axa)(xya).$$

Hence, we have

$$\begin{aligned}
(f_S * h_S * g_S)(a) &= [(f_S * h_S) * g_S](a) \\
&= [\bigcap_{a=pq} (f_S * h_S)(p)] * g_S(q) \\
&\subseteq (f_S * h_S)(ax) \cup g_S(a) \\
&= \{\bigcap_{ax=pq} (f_S(p) \cup h_S(q))\} \cup g_S(a) \\
&\subseteq (f_S(axa) \cup h_S(xya)) \cup g_S(a) \\
&\subseteq (f_S(a) \cup h_S(a)) \cup g_S(a) \\
&= (f_S \widetilde{\cup} h_S \widetilde{\cup} g_S)(a)
\end{aligned}$$

and so, $f_S * h_S * g_S \widetilde{\subseteq} f_S \widetilde{\cup} h_S \widetilde{\cup} g_S$. Thus, $f_S * h_S * g_S = f_S \widetilde{\cup} h_S \widetilde{\cup} g_S$ and (1) implies (4).

It is clear that (4) implies (3) and (3) implies (2).

Conversely, assume that (2) holds. Let f_S be any SU-quasi-ideal and g_S be any SU-left ideal of S. Then, since $\widetilde{\theta}$ itself is an SU-ideal of S, we have by assumption that

$$f_S \widetilde{\cup} g_S = f_S \widetilde{\cup} \widetilde{\theta} \widetilde{\cup} g_S = f_S * \widetilde{\theta} * g_S = f_S * (\widetilde{\theta} * g_S) \widetilde{\supseteq} f_S * g_S.$$

It follows by Theorem 4 that S is regular. As in the above Theorem, one can easily show that $aS \subseteq Sa$. Thus, S is a semilattice of left groups. Thus, (2) implies (1). This completes the proof.

The left-right dual of Theorem 34 reads as follows:

Theorem 35. *For a semigroup S, the following conditions are equivalent:*

1) S is a semilattice of right groups.
2) $f_S \widetilde{\cup} h_S \widetilde{\cup} g_S = f_S * h_S * g_S$ for every SU-quasi-ideal f_S, for every SU-ideal h_S and every SU-right ideal g_S of S.
3) $f_S \widetilde{\cup} h_S \widetilde{\cup} g_S = f_S * h_S * g_S$ for every SU-bi-ideal f_S, for every SU-ideal h_S and every SU-right ideal g_S of S.
4) $f_S \widetilde{\cup} h_S \widetilde{\cup} g_S = f_S * h_S * g_S$ for every SU-generalized bi-ideal f_S, for every SU-ideal h_S and every SU-right ideal g_S of S.

9 A semilattice of groups

Let S be a semigroup. We shall say that S is a *semilattice of groups* if it is the set-theoretical union of a family of mutually disjoint subgroups G_i ($i \in M$) such that, for any pair i, j in M, the products G_iG_j and G_jG_i are both contained in the same subgroups G_k ($k \in M$). The following is due to [9,17,21].

Proposition 10. *[9,17,21] For a semigroup S, the following conditions are equivalent:*

1) S is a semilattice of groups.
2) S is regular and $aS = Sa$ for all $a \in S$.
3) $LR = L \cap R$ for every left ideal L and every right ideal R of S.
4) $LB = L \cap B$ for every left ideal L and every bi-ideal B of S.

5)$BR = B \cap R$ for every bi-ideal B and every right ideal R of S.
6)S is regular and every one-sided ideal of S is two-sided.

Proposition 11.*Let S be a semigroup that is a semilattice of groups. Then, every SU-(generalized) bi-ideal of S is an SU-ideal of S.*

*Proof.*Let f_S be any SU-bi-ideal of S and a and b be any elements of S. Then, it follows by Proposition 10 that

$$ab \in (aSa)S = (aS)(aS) = (aS)(Sa) = a(SS)a \subseteq aSa$$

Thus, there exists an element $x \in S$ such that $ab = axa$. Hence,

$$f_S(ab) = f_S(axa) \subseteq f_S(a) \cup f_S(a) = f_S(a).$$

Hence, f_S is an SU-right ideal of S. Similarly,

$$ab \in S(bSb) = (Sb)(Sb) = (bS)(Sb) = b(SS)b \subseteq bSb$$

Thus, there exists an element $x \in S$ such that $ab = bxb$. Hence,

$$f_S(ab) = f_S(bxb) \subseteq f_S(b) \cup f_S(a) = f_S(b).$$

Therefore, f_S is an SU-left ideal of S. That is to say, f_S is an SU-ideal of S.

Proposition 12.*[17] For a semigroup S, the following conditions are equivalent:*

1)S is a semilattice of groups.
2)The set of all (generalized) bi-ideals of S is a semilattice under the multiplication of subsets.

Now, we shall give a characterization of a semigroup which s a semilattice of groups in terms of SU-ideals of semigroups.

Theorem 36.*For a semigroup S, the following conditions are equivalent:*

1)S is a semilattice of groups.
2)$f_S * g_S = f_S \widetilde{\cup} g_S$ for every SU-left ideal f_S and every SU-right ideal g_S of S.
3)$f_S * g_S = f_S \widetilde{\cup} g_S$ for every SU-left ideal f_S and every SU-quasi ideal g_S of S.
4)$f_S * g_S = f_S \widetilde{\cup} g_S$ for every SU-left ideal f_S and every SU-bi-ideal g_S of S.
5)$f_S * g_S = f_S \widetilde{\cup} g_S$ for every SU-left ideal f_S and every SU-generalized bi-ideal g_S of S.
6)$f_S * g_S = f_S \widetilde{\cup} g_S$ for every SU-quasi-ideal f_S and every SU-right ideal g_S of S.
7)$f_S * g_S = f_S \widetilde{\cup} g_S$ for all SU-quasi-ideals f_S and g_S of S.
8)$f_S * g_S = f_S \widetilde{\cup} g_S$ for every SU-quasi-ideal f_S and every SU-bi-ideal g_S of S.
9)$f_S * g_S = f_S \widetilde{\cup} g_S$ for every SU-quasi-ideal f_S and every SU-generalized bi-ideal g_S of S.
10)$f_S * g_S = f_S \widetilde{\cup} g_S$ for every SU-bi-ideal f_S and every SU-right ideal g_S of S.
11)$f_S * g_S = f_S \widetilde{\cup} g_S$ for every SU-bi-ideal f_S and every SU-quasi-ideal g_S of S.
12)$f_S * g_S = f_S \widetilde{\cup} g_S$ for all SU-bi-ideals f_S and g_S of S.
13)$f_S * g_S = f_S \widetilde{\cup} g_S$ for every SU-bi-ideal f_S and every SU-generalized bi-ideal g_S of S.
14)$f_S * g_S = f_S \widetilde{\cup} g_S$ for every SU-generalized bi-ideal f_S and every SU-right ideal g_S of S.
15)$f_S * g_S = f_S \widetilde{\cup} g_S$ for every SU-generalized bi-ideal f_S and every SU-quasi-ideal g_S of S.
16)$f_S * g_S = f_S \widetilde{\cup} g_S$ for every SU-generalized bi-ideal f_S and every SU-bi-ideal g_S of S.
17)$f_S * g_S = f_S \widetilde{\cup} g_S$ for all SU-generalized bi-ideals f_S and g_S of S.
18)S is regular and every SU-one-sided ideal of S is an SU-ideal of S.
19)The set of all SU-quasi-ideals of S is a semilattice under the multiplication of soft union product.
20)The set of all SU-bi-ideals of S is a semilattice under the multiplication of soft union product.
21)The set of all SU-generalized-bi-ideals of S is a semilattice under the multiplication of soft union product.

*Proof.*First assume that (1) holds. In order to prove that (17) holds, let f_S and g_S be any SU-generalized bi-ideals of S. Then, it follows by Proposition 11 that f_S and g_S are SU-ideals of S. Since S is regular by Proposition 10, it follows from Theorem 4 that $f_S * g_S = f_S \widetilde{\cup} g_S$. Hence, we obtain that (1) implies (17). It is clear that (17) implies (16), (16) implies (15), (15) implies (14), (14) implies (10), (10) implies (6), (6) implies (2), (17) implies (13), (13) implies (12), (12) implies (11), (11) implies (10), (13) implies (9), (9) implies (8), (8) implies (7), (7) implies (6) and (9) implies (5), (5) implies (4), (4) implies (3) and (3) implies (2).

Assume that (2) holds. Let L and R be any left and right ideal of S, respectively. Let a be any element of $L \cap R$ and $a \notin LR$. Then, $a \in L$ and $a \in R$ and there do not exist $x \in L$ and $y \in R$ such that $a = xy$. Since $\mathscr{S}_{L^c}$ and $\mathscr{S}_{R^c}$ is an SU-left ideal and SU-right ideal of S, respectively, we have

$$\mathscr{S}_{L^c}(a) = \mathscr{S}_{R^c}(a) = \emptyset.$$

and

$$(\mathscr{S}_{L^c} * \mathscr{S}_{R^c})(a) = U$$

But this is a contradiction, so $a \in LR$. Thus, $L \cap R \subseteq LR$. For the converse inclusion, let a be any element of LR and $a \notin L \cap R$. Then, there exist $y \in L$ and $z \in R$ such that $a = yz$. Thus,

$$(\mathscr{S}_{L^c} \widetilde{\cup} \mathscr{S}_{L^c})(a) = U$$

and

$$(\mathscr{S}_{L^c} * \mathscr{S}_{R^c})(a) = \bigcap_{a=mn} (\mathscr{S}_{L^c}(m) \cup \mathscr{S}_{R^c}(n)) \subseteq (\mathscr{S}_{L^c}(y) \cup \mathscr{S}_{R^c}(z)) = \emptyset$$

Hence, $(\mathscr{S}_{L^c} * \mathscr{S}_{R^c})(a) = \emptyset$. But this is a contradiction. This implies that $a \in L \cap R$ and that $LR \subseteq L \cap R$. Thus, we have $LR = L \cap R$. It follows by Proposition 10 that S is a semilattice of groups and so (2) implies (1).

Assume that (1) holds. Then, as shown above, (17) holds and (21) holds. It is obvious that (21) implies (20) and (20) implies (19). Assume that (19) holds. Then, since every SU-quasi-ideal of S is idempotent, it follows that S is regular ([29]. Let L and R be any left and right ideal of S, respectively. Then, since L and R are quasi-ideal of S, soft characteristic functions $\mathscr{S}_{L^c}$ and $\mathscr{S}_{R^c}$ are SU-quasi-ideal of S. Thus, by assumption $\mathscr{S}_{L^c} * \mathscr{S}_{R^c} = \mathscr{S}_{R^c} * \mathscr{S}_{L^c}$. Let $a \in LR$ and $a \notin RL$. Then,

$$(\mathscr{S}_{L^c} * \mathscr{S}_{R^c})(a) = \emptyset$$

and

$$(\mathscr{S}_{R^c} * \mathscr{S}_{L^c})(a) = U.$$

But this is a contradiction, hence $LR \subseteq RL$. One can similarly show that $RL \subseteq LR$ and so $LR = L \cup R$. Then, since S is regular, we have

$$R \cap L = RL = LR.$$

It follows by Proposition 12 that S is a semilattice of groups. Thus (19) implies (1).

Now assume that (2) holds. To see that (18) holds, let f_S be any SU-left ideal of S. Since $\widetilde{\theta}$ is an SU-right ideal of S, we have

$$f_S = f_S \widetilde{\cup} \widetilde{\theta} = f_S * \widetilde{\theta}$$

Thus, f_S is an SU-right ideal of S. One can similarly show that every SU-right ideal of S is an SU-left ideal of S. As shown above, S is regular. Thus, (2) implies (18). Assume that (17) holds. In order to show that (1) holds, let A and B be any generalized bi-ideals of S and a be any element of AB and $a \notin BA$. Then, the soft characteristic functions $\mathscr{S}_{A^c}$ and $\mathscr{S}_{B^c}$ are SU-generalized bi-ideals of S. Thus,

$$(\mathscr{S}_{B^c} * \mathscr{S}_{A^c})(a) = U$$

and

$$\mathscr{S}_{A^c} * \mathscr{S}_{B^c})(a) = \emptyset.$$

But this is a contradiction and so $a \in BA$. Thus, $AB \subseteq BA$. It can be seen in a similar way that the converse inclusion holds. Thus, we obtain that $AB = BA$. Now, we shall prove that any generalized bi-ideal of S is idempotent. Let B be any generalized bi-ideal of S and $a \in B$ and $a \notin BB$. Then, since the soft characteristic function $\mathscr{S}_{B^c}$ is an SU-generalized bi-ideal of S, we have

$$(\mathscr{S}_{B^c} * \mathscr{S}_{B^c})(a) = U$$

and

$$\mathscr{S}_{B^c}(a) = \emptyset$$

which is a contradiction and so $a \in BB$. Thus, $B \subseteq BB$. Similarly, one can show that $BB \subseteq B$. Hence, $B = BB$. This means that the set of all generalized bi-ideals of S is a semilattice under the multiplication of subsets. It follows by Proposition 12 that S is a semilattice of groups. Thus (2) implies (1). This completes the proof.

Theorem 37.[28] *For a semigroup S the following conditions are equivalent:*

1)S is completely regular.
2)Every bi-ideal of S is semiprime.
3)Every SU-bi-ideal of S is soft union semiprime.
4)$f_S(a) = f_S(a^2)$ for every SU-bi-ideal f_S of S and for all $a \in S$.

Theorem 38.*For a semigroup S, the following conditions are equivalent:*

1)S is a semilattice of groups.
2)For every SU-quasi-ideal f_S of S, $f_S(a) = f_S(a^2)$ and $f_S(ab) = f_S(ba)$ for all $a, b \in S$.
3)For every SU-bi-ideal f_S of S, $f_S(a) = f_S(a^2)$ and $f_S(ab) = f_S(ba)$ for all $a, b \in S$.
4)For every SU-generalized bi-ideal f_S of S, $f_S(a) = f_S(a^2)$ and $f_S(ab) = f_S(ba)$ for all $a, b \in S$.

*Proof.*First assume that (1) holds. Let f_S be any SU-generalized bi-ideal of S and a and b be any elements of S. Then, since S is regular by Proposition 10, there exists an element x in S such that $a = axa = axaxaxa$. Since $aS \subseteq Sa$ by Proposition 10, there exist elements $y, z \in S$ such that $xa = ya$ and $ax = za$. Thus, we have

$$a = axa = a(xaxaxa) = a(xa)x(ax)a = a(ya)x(za)a = a^2(yxz)a^2.$$

Hence, since f_S is an SU-generalized bi-ideal of S, we have

$$f_S(a) = f_S(a^2(yxz)a^2) \subseteq f_S(a^2) \cup f_S(a^2) = f_S(a^2) = f_S(a(axa)) = f_S(a(ax)a) \subseteq f_S(a) \cup f_S(a) = f_S(a)$$

and so $f_S(a) = f_S(a^2)$. Moreover, by Proposition 10, we have

$$(ab)^4 = a(ba)ba(ba)b \in (Sba)S(baS) = (baS)S(Sba).$$

Hence, there exists an element $u \in S$ such that $(ab)^4 = bauba$. Thus,

$$f_S(ab) = f_S((ab)^2) = f_S((ab)^4) = f_S((ba)u(ba)) \subseteq f_S(ba) \cup f_S(ba) = f_S(ba).$$

Similarly, we have $f_S(ba) \subseteq f_S(ab)$ and so $f_S(ab) = f_S(ba)$. Thus, (1) implies (4).

It is clear that (4) implies (3) and (3) implies (2).

Conversely, assume that (2) holds. Then, it follows by Theorem 37 that S is completely regular and so regular. Let a be any element of S. To see that $aS = Sa$, let ax be any element of aS. Since the soft characteristic function $\mathscr{S}_{(B[xa])^c}$ is an SU-bi-ideal of S, by assumption, we have

$$\mathscr{S}_{(B[xa])^c}(ax) = \mathscr{S}_{(B[xa])^c}(xa) = \emptyset$$

and so $ax \in B[xa] = \{xa\} \cup (xa)^2 \cup (xa)S(xa)$. If $ax = xa$, then $ax = xa \in Sa$, and so $aS \subseteq Sa$. If $ax = (xa)^2$, then $ax = (xax)a \in Sa$. Hence, $aS \subseteq Sa$. If $ax \in (xa)S(xa)$, then

$$ax \in (xa)S(xa) = (xaSx)a \in Sa$$

and so $aS \subseteq Sa$. In any case, $aS \subseteq Sa$. Similarly, $Sa \subseteq aS$. Thus, $aS = Sa$. Hence, it follows by Proposition 10 that S is a semilattice of groups. Thus, (2) implies (1). This completes the proof.

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Theorem 39. *For a semigroup S, the following conditions are equivalent:*

1)S is a semilattice of groups.
2)$f_S \tilde{\cup} g_S = g_S * f_S * g_S$ for every SU-quasi-ideal f_S of S and for all SU-ideal g_S of S.
3)$f_S \tilde{\cup} g_S = g_S * f_S * g_S$ for every SU-quasi-ideal f_S of S and for all SU-interior ideal g_S of S.
4)$f_S \tilde{\cup} g_S = g_S * f_S * g_S$ for every SU-bi-ideal f_S of S and for all SU-ideal g_S of S.
5)$f_S \tilde{\cup} g_S = g_S * f_S * g_S$ for every SU-bi-ideal f_S of S and for all SU-interior ideal g_S of S.
6)$f_S \tilde{\cup} g_S = g_S * f_S * g_S$ for every SU-generalized bi-ideal f_S of S and for all SU-ideal g_S of S.
7)$f_S \tilde{\cup} g_S = g_S * f_S * g_S$ for every SU-generalized bi-ideal f_S of S and for all SU-interior ideal g_S of S.

Proof. First assume that (1) holds. Let f_S be any SU-generalized bi-ideal and g_S be any SU-interior ideal of S. It follows by Proposition 11 that f_S is an SU-ideal of S. Thus,

$$g_S * f_S * g_S \tilde{\supseteq} \tilde{\theta} * f_S * \tilde{\theta} \tilde{\supseteq} f_S.$$

Moreover, $g_S * f_S * g_S \tilde{\supseteq} g_S * (\tilde{\theta} * g_S) \tilde{\supseteq} g_S * g_S \tilde{\supseteq} g_S * \tilde{\theta} \tilde{\supseteq} g_S$. Therefore, we have

$$g_S * f_S * g_S \tilde{\supseteq} f_S \tilde{\cup} g_S.$$

Now, let a be any element of S. Since S is regular by Proposition 10, there exists an element $x \in S$ such that $a = axa$. Hence

$$\begin{aligned}(g_S * f_S * g_S)(a) &= [(g_S * f_S) * g_S](a) \\ &= [\bigcap_{a=pq} (g_S * f_S)(p)] * g_S(q) \\ &\subseteq (g_S * f_S)(a) \cup g_S(xa) \\ &= \{\bigcap_{a=uv} (g_S(u) \cup f_S(v))\} \cup g_S(a) \\ &\subseteq (g_S(ax) \cup f_S(a)) \cup g_S(a) \\ &\subseteq f_S(a) \cup g_S(a) \\ &= (f_S \tilde{\cup} g_S)(a)\end{aligned}$$

and so, $g_S * f_S * g_S \tilde{\subseteq} f_S \tilde{\cup} g_S$. Thus, $g_S * f_S * g_S = f_S \tilde{\cup} g_S$, so, (1) implies (7). It is clear that (7) implies (6), (6) implies (4), (4) implies (2) and (7) implies (5), (5) implies (3) and (3) implies (2).

Assume that (2) holds. Let Q and J be any quasi-ideal and ideal of S, respectively. Let $a \in JQJ$ and $a \notin J \cap Q$. Since the soft characteristic function $\mathscr{S}_{Q^c}$ and $\mathscr{S}_{J^c}$ are SU-quasi-ideal and SU-ideal of S, respectively, we have

$$(\mathscr{S}_{J^c} \tilde{\cup} \mathscr{S}_{Q^c})(a) = U$$

and

$$(\mathscr{S}_{J^c} * \mathscr{S}_{Q^c} * \mathscr{S}_{J^c})(a) = \emptyset$$

which is a contradiction and so $a \in JQJ$. Thus, $J \cap Q \subseteq JQJ$. Similarly, one can show that $JQJ \subseteq J \cup Q$. Therefore, we have that $JQJ = J \cap Q$ for every quasi-ideal Q and ideal J of S, which implies that S is regular and (2) implies (1). This completes the proof.

10 Soft normal semigroups

In this section, we introduce the concepts of soft normality in a semigroup. It is known that a semigroup S is called *normal* if $aS = Sa$ for all $a \in S$.

Definition 16. *An SU-quasi-ideal f_S of S is called union $Q-normal$ if $f_S(ab) = f_S(ba)$ for all $a, b \in S$.*

Definition 17. *An SU-bi-ideal f_S of S is called union $B-normal$ if $f_S(ab) = f_S(ba)$ for all $a, b \in S$.*

Definition 18. *A semigroup S is called soft union $B^\star - normal$ if every SU-bi ideal of S is union$B - normal$.*

Definition 19. *A semigroup S is called soft union $Q^\star - normal$ if every SU-quasi-ideal of S is union$Q - normal$.*

Theorem 40. *Any soft union $Q^\star - normal$ semigroup is normal.*

Proof. Let f_S be an SU-quasi-ideal of a soft union $Q^\star - normal$ semigroup of S. Let a be any element of S. To see that $aS = Sa$, let ax be any element of aS. Since the soft characteristic function $\mathscr{S}_{(Q[xa])^c}$ is an SU-quasi-ideal of S, by assumption, we have

$$\mathscr{S}_{(Q[xa])^c}(ax) = \mathscr{S}_{(Q[xa])^c}(xa) = \emptyset$$

which implies that

$$ax \in Q[xa] = \{xa\} \cup (xaS \cup Sxa) \subseteq Sa$$

Thus, we have $aS \subseteq Sa$. Similarly, $Sa \subseteq aS$ holds. Thus, $aS = Sa$ and S is normal. This completes the proof.

The following theorem shows that the converse of Theorem 40 holds for a regular semigroup.

Theorem 41. *For a regular semigroup S, the following conditions are equivalent:*

1)S is soft union $Q^\star - normal$.
2)S is normal.

Proof. It suffices to prove that (2) implies (1). Assume that (2) holds. Let f_S be any SU-quasi-ideal of S and a and b be any elements of S. Since S is regular and normal, we have

$$ab \in (aSa)(bSb) = (aS)(ab)(Sb) \subseteq (aS)(abSab)(Sb) = (aSa)b(Sa)(bSb) \subseteq (Sb)(Sa)S = (Sb)(aS)S = S(ba)SS = (ba)SSS \subseteq baS$$

This implies that there exists an element $x \in S$ such that $ab = bax$. Thus, since f_S is an SU-bi-ideal of S, we have

$$(f_S * \tilde{\theta})(ab) = \bigcap_{ab=pq} \{(f_S(p) \cup \tilde{\theta}(q)\} \subseteq f_S(ba) \cup \tilde{\theta}(x) = f_S(ba).$$

One can similarly show that

$$(\tilde{\theta} * f_S)(ab) \subseteq f_S(ba)$$

Since, f_S is an SU-quasi-ideal of S,

$$f_S(ab) \subseteq ((f_S * \widetilde{\theta})\widetilde{\cup}(\widetilde{\theta} * f_S))(ab) =$$
$$(f_S * \widetilde{\theta})(ab) \cup (\widetilde{\theta} * f_S)(ab) \subseteq f_S(ba) \cup f_S(ba) = f_S(ba)$$

Similarly, it can be proved that $f_S(ba) \subseteq f_S(ab)$. Thus, $f_S(ba) = f_S(ab)$, and so S is soft union $Q^\star - normal$ and that (2) implies (1). This completes the proof.

Theorem 42. *Any soft union $B^\star - normal$ semigroup is normal.*

Proof. Let f_S be an SU-bi-ideal of a soft $B^\star - normal$ semigroup of S. Let a be any element of S and ax be any element of aS. Since the soft characteristic function $\mathscr{S}_{(B[xa])^c}$ is an SU-bi-ideal of S, by assumption, we have

$$\mathscr{S}_{(B[xa])^c}(ax) = \mathscr{S}_{(B[xa])^c}(xa) = \emptyset$$

which implies that

$$ax \in B[xa] = \{xa\} \cup \{xaxa\} \cup (xa)S(xa) \subseteq Sa$$

Thus, we have $aS \subseteq Sa$. Similarly, $Sa \subseteq aS$ holds. Thus, $aS = Sa$ and S is normal. This completes the proof.

The following theorem shows that the converse of Theorem 42 holds for a regular semigroup.

Theorem 43. *For a regular semigroup S, the following conditions are equivalent:*

1)S is soft $B^\star - normal$.
2)S is normal.

Proof. It suffices to prove that (2) implies (1). Assume that (2) holds. Let f_S be any SU-bi-ideal of S and a and b be any elements of S. Since S is regular, we have

$$ab \in (aSa)(bSb) = (aS)(ab)(Sb) \subseteq (aS)(abSab)(Sb) =$$
$$(aSa)b(Sa)(bSb) \subseteq (Sb)(aS)S = S(ba)SS = (ba)SSS \subseteq$$
$$baS = (baSba)S = (baS)(Sba) = ba(SS)ba \subseteq baSba.$$

This implies that there exists an element $x \in S$ such that $a = baxba$. Thus, since f_S is an SU-bi-ideal of S, we have

$$f_S(ab) = f_S((ba)x(ba)) \subseteq f_S(ba) \cup f_S(ba) = f_S(ba).$$

One can similarly show that $f_S(ba) \subseteq f_S(ab)$. Hence $f_S(ab) = f_S(ba)$ which implies that S is soft union $B^\star - normal$ and that (2) implies (1). This completes the proof.

Proposition 13. *For an idempotent semigroup S, the following conditions are equivalent:*

1)S is commutative.
2)S is soft union $Q^\star - normal$.
3)S is soft union $B^\star - normal$.

Proof. (1) implies (3) and (3) implies (2) is obvious. Assume that (2) holds. Then, S is normal. Let $a, b \in S$. Then, $ab \in Sb = bS$. Thus, there exists an element x in S such that $ab = bx$. Similarly, we have $ba = yb$ for some $b \in S$. Hence, since S is idempotent, we have

$$ab = bx = (bb)x = b(bx) = b(ab) = (ba)b = (yb)b = yb = ba$$

which implies that S is commutative. Hence (2) implies (1).

Definition 20. [21] *A semigroup S is called archimedean if for all $a, b \in S$, there exists a positive integer n such that $a^n \in SbS$.*

Definition 21. [21] *A semigroup S is called weakly commutative if for all $a, b \in S$, there exists a positive integer n such that $(ab)^n \in bSa$.*

Proposition 14. [21] *Every weakly commutative semigroup is a semilattice of archimedean semigroups.*

Proposition 15. *Any soft union $B^\star - normal$ semigroup is a semilattice of archimedean semigroups.*

Proof. Let S be any soft union $B^\star - normal$ semigroup. Let a and b be any elements of S, and f_S be any SU-bi-ideal of S. Since the soft characteristic function $\mathscr{S}_{(B[ba])c}$ is an SU-bi-ideal of S, by assumption, we have

$$\mathscr{S}_{(B[ba])c}(ab) = \mathscr{S}_{(B[ba])c}(ba) = \emptyset$$

and so

$$ab \in B[ba] = \{ba\} \cup \{baba\} \cup (baSba) \subseteq Sa$$

Thus, we have $(ab)^2 \in baSba \subseteq bSa$. Therefore, S is weakly commutative. Hence by Proposition 14, S is a semilattice of archimedean semigroups.

One can similarly prove the following proposition.

Proposition 16. *Any soft union $Q^\star - normal$ semigroup is a semilattice of archimedean semigroups.*

Theorem 44. *For a completely regular semigroup S, the following conditions are equivalent:*

1)S is soft union $Q^\star - normal$.
2)S is soft union $B^\star - normal$.
3)For each elements a and b of S, there exists a positive integer n such that $(ab)^n \in baSba$.

Proof. It is obvious that (2) implies (1). Assume that (1) holds. Then, S is normal. Let a and b be any elements of S. Thus, we have

$$(ab)^3 = ababab = a(ba)bab \subseteq (Sba)(baS) = (baS)(Sba) = (ba)SS(ba) \subseteq baSba$$

which shows that (1) implies (3).

Conversely, assume that (3) holds. To see that (2) holds, l et f_S be any SU-bi-ideal of S and a and b be any elements of S. Then, by assumption, there exists a positive integer n such that $(ab)^n = baxba$. Since S is completely regular, for this positive integer, there exists an element $y \in S$ such that $ab = (ab)^n y (ab)^n$. Then, since f_S is an SU-bi-ideal of S, we have

$$f_S(ab) = f_S((ab)^n y (ab)^n) \subseteq f_S((ab)^n) \cup f_S((ab)^n) =$$
$$f_S((ab)^n)) = f_S(baxba) \subseteq f_S(ba) \cup f_S(ba) = f_S(ba).$$

One can similarly show that $f_S(ba) \subseteq f_S(ab)$. Hence, $f_S(ab) = f_S(ba)$ which implies that f_S is soft $B^\star - normal$. Thus, (3) implies (2).

11 Conclusion

In this paper, we characterize certain classes of semigroups such as semisimple semigroups, duo semigroups, right (left) zero semigroups, right (left) simple semigroups, semilattice of left (right) simple semigroups, semilattice of left (right) groups and semilattice of groups via different soft union ideals of semigroups. Also, we define soft union normal semigroups and study on the relation of this concept with semigroups. In order to be useful to characterize the classical semigroups, some further work can be done on the properties of soft union semigroups and different classes of soft union ideals.

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Aslıhan Sezgin Sezer is Assistant Professor Doctor of Mathematics at Amasya University. She received the PhD degree in Mathematics at Gaziosmanpaşa University (Turkey). Her research interests are in the areas of soft set and its algebraic applications and general near-ring theory.

Naim Çağman is Associate Professor of Mathematics at Gaziosmanpaşa University. He received the PhD degree in Mathematics at University of Leeds (UK). His main research interests are: Mathematical Logic, Fuzzy Logic, Soft Set Theory, Game Development, Mathematics Education.

Akın Osman Atagün is Associate Professor of Mathematics at Bozok University. He received the PhD degree in Mathematics at Erciyes University (Turkey). His main research interests are: Near-ring, Prime ideals of near-rings and near-ring modules, Soft set and its algebraic applications.