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Computational analysis of a nonlinear fractional emerging telecommunication model with higher–order dispersive cubic–quintic

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Abstract: In this paper, the modified simplest equation method as one of the computational schemes is applied to a nonlinear fractional emerging telecommunication model with higher-orderer dispersive cubic - quintic for constructing the exact traveling and solitary wave solutions. This model is also known with higher - order dispersive cubic - quintic nonlinear complex fractional Schrödinger ($NLCFS$) equation. Moreover, it is used to explain the physical nature of the waves spread, especially in the dispersive medium. The disadvantages of the B - spline schemes are investigated based on the obtained analytical solutions. Some obtained computational solutions are sketched to more illustration of the wave dynamics in the dispersive medium.

Keywords: Emerging telecommunication model; NLCFS model; Extended simplest equation method; B-spline schemes; Exact traveling and solitary waves solutions.

1 Introduction

Partial differential equations (PDEs) have been playing an essential role in the emerging technologies where many nonlinear evolution equations have been derived to describe the dynamical behavior of several phenomena in several fields, for example, nonlinear optics, fluid dynamics, Bose–Einstein condensates, quantum mechanics and several other areas [1,2,3,4,5,6,7]. However, the inadequate of the PDEs with an–integer order have been clarified because of the nonlocal property where this kind of equation does not explain that kind of features [8,9,10]. Therefore, several natural phenomena have been formulated with nonlinear PDEs with fractional order [11]. Thus, PDEs have been playing an important role in the emerging technologies where many nonlinear evolution equations have been derived to describe the dynamical behaviour of several phenomenon in several fields, for example, nonlinear optics, fluid dynamics, Bose–Einstein condensates, quantum mechanics and several other areas [12,13,14,15]. However, the inadequate of the PDEs with an–integer order have been clarified because of the nonlocal property where this kind of equation do not explain that kind of properties [16,17,18]. Therefore, several nature phenomena have been formulated with nonlinear PDEs with fractional order [19,20,21]. Thus, many fractional operators have been derived such as conformable fractional derivative, fractional Riemann–Liouville derivatives, Caputo, Caputo–Fabrizio definition, and so on [22,23,24,25,26].

These definitions have been being employed to convert the fractional nonlinear partial differential equations to a nonlinear integer–order ordinary differential equation. Then the computational and numerical schemes can be applied to get various types of solutions for these models and the examples of these schemes [27,28,29,30,31,32,33,34,35,36,37,38].

This paper studies the analytical and numerical solutions of the $NLCFS$ with higher–order dispersive

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cubic–quintic arising in the emerging telecommunication [39]. This fractional model describes the wave function or state function of a quantum–mechanical system [40]. Moreover, it is also used in the optical fiber where it occurs in the Manakov system. The NLCFS equation is given by [41,42]

$$\imath\, D_x^{\alpha}\,\mathcal{U} - \frac{s_1}{2}\,\mathcal{U}_{t\,t} + r_1\,\mathcal{U}\,|\mathcal{U}|^2 - \imath\,\frac{s_2}{6}\,D^{3\,\alpha}_{t\,t\,t}\mathcal{U} - \frac{s_3}{24}\,D^{4\,\alpha}_{t\,t\,t\,t}\mathcal{U} + r_2\,\mathcal{U}\,|\mathcal{U}|^4 = 0, \tag{1}$$

where $(0 < \alpha < 1)$, $\mathcal{U}$ describes the propagation of the wave through a nonlinear medium. Additionally, the s_1, s_2 and s_3 are dispersions of order 2^{nd} , 3^{nd} and 4^{nd} respectively, while the r_1 and r_2 are the coefficients of two nonlinearities of the medium, the function q is the gradually varying envelope of the electromagnetic material, the variables t and x are the retarded time and the distance along the direction of propagation respectively. The second and third terms in the above equations are revealed from the velocity dispersion and the Kerr effects. Using the next wave transformation $[\mathcal{U} = \mathcal{U}(x,t) = \mathcal{U}(\mathfrak{z})\,e^{\imath\,(d_1\,x + d_2\,t + d_3)},\ \mathfrak{z} = \frac{1}{\alpha}\,(l_1\,x^{\alpha} + l_2\,t^{\alpha})]$ to Eq. (1), yields,

$$\begin{cases} l_2^2\,s_3\,\mathcal{U}^{(4)} + \left(12\,s_1\,l_2^2 - 6\,s_3\,d_2^2\,l_2^2 - 12\,s_2\,d_2\,l_2^2\right)\mathcal{U}'' + \left(s_3\,d_2^4 + 4\,s_2\,d_1^3 - 12\,s_1\,d_2 + 24\,d_1\right)\mathcal{U} - 24\,r_2\,\mathcal{U}^5 - 24\,r_1\,\mathcal{U}^3 = 0, \\ \left(-4\,s_3\,d_2^3\,l_2 - 12\,s_2\,d_2^2\,l_2 + 24\,s_1\,d_2\,l_2 - 24\,k_1\right)\mathcal{U}' + 4\,l_2^3\,\left(s_3\,d_2 + s_2\right)\mathcal{U}''' = 0, \end{cases} \tag{2}$$

where $[d_i,\, l_j,\, (i = 1,2,3),\, (j = 1,2)]$ are arbitrary constants. Differentiating second equation of the system (2) and then substitute the result into the first equation of the same equation, gives

$$k_1\,\mathcal{U}'' + k_2\,\mathcal{U}^5 + k_3\,\mathcal{U}^3 + k_4\,\mathcal{U} = 0, \tag{3}$$

where

$$\begin{aligned} k_1 &= l_2^2\left(12 s_1 - 6 s_3\,d_2^2 - 12\,s_2\,d_2\right) \\ &\quad \times \left(\frac{4\,s_3\,d_2^3\,l_2 + 12\,s_3\,d_2^2\,l_2 - 24\,s_1\,d_2\,l_2 + 24\,k_1}{4\,l_2^3\,\left(s_3\,d_2 + s_2\right)}\right), \\ k_2 &= -24\,r_2,\ k_3 = -24\,r_1, \\ k_4 &= \left(s_3\,d_2^4 + 4\,s_2\,d_2^3 - 12\,s_1\,d_2^2 + 24\,d_1\right) \end{aligned}$$

Applying the homogeneous balance principle to Eq. (3), leads to $(m = \frac{1}{2})$. Thus, we use the next transformation $[\mathcal{U} = \mathcal{Q}^{\frac{1}{2}}]$ to the Eq. (3), yields

$$\frac{-k_1}{4}\,\mathcal{Q}'^2 + \frac{k_1}{2}\,\mathcal{Q}\,\mathcal{Q}'' + k_2\,\mathcal{Q}^4 + k_3\,\mathcal{Q}^3 + k_4\,\mathcal{Q}^2 = 0. \tag{4}$$

Applying the homogeneous balance principle to Eq. (4), obtains $(m = 1)$.

The rest of research paper is organized as follows: Section (2), applies the extended simplest equation method and B - spline schemes to the suggested model to get exact traveling and solitary wave solutions of it [43, 44, 45, 46, 47, 48, 49]. Section (4), explains the conclusion of all the steps of our paper is detailed.

2 Application

Here in this section, the extended simplest equation method and B - spline schemes are applied to the NLCFS equation t;o explain the restricted electromagnetic wave which stretches in media of nonlinear dispersive. Due to stability among nonlinearity and dispersion effects, the intensity of optical solitons are unchanged, and such categories of solitary waves are more significant because of their suppleness in optical of long distance.

2.1 Computational solutions

Applying the extended simplest equation method to Eq. (4), leads to formulate the general solution of this model in the following formula

$$\mathcal{Q}(\mathfrak{z}) = \sum_{i=-n}^{n} a_i\,\mathcal{F}^i(\mathfrak{z}) = \frac{a_{-1}}{\mathcal{F}(\mathfrak{z})} + a_1\mathcal{F}(\mathfrak{z}) + a_0, \tag{5}$$

where $[a_{-1},\, a_0,\, a_1]$ are arbitrary constants to be determined later. Additionally, $\mathcal{F}(\mathfrak{z})$ is the solution function of the following ordinary differential equation

$$\mathcal{F}'(\mathfrak{z}) = \alpha + \lambda\,\mathcal{F}(\mathfrak{z}) + \mu\,\mathcal{F}(\mathfrak{z})^2, \tag{6}$$

where $[\alpha,\, \lambda,\, \mu]$ are arbitrary constant. Substituting Eq. (5) along (6) into Eq. (4) and collecting all terms with the same power of $[\mathcal{F}^i(\mathfrak{z}),\, i = -5, -4, ..., 4, 5]$, give a system of algebraic equation. Using the Mathematica 12 program for solving this system, yields

Family I:
$\left[a_{-1}\to\frac{\alpha a_0}{\lambda}, a_1\to\frac{a_0\mu}{\lambda}, k_2\to-\frac{3\lambda^2 k_1}{4a_0^2}, k_3\to\frac{\lambda^2 k_1}{a_0}, k_4\to-\frac{1}{4}k_1\left(\lambda^2-4\alpha\mu\right)\right]$
Thus, the explicit wave solutions of Eq. (1) are formulated in the following formulas
For $[\alpha=0]$ when $(\lambda>0)$

$$\mathcal{U}_1=e^{id_2t+id_1x+id_3}\sqrt{\frac{a_0}{1-\mu e^{\lambda(\xi+\vartheta)}}}. \tag{7}$$

When $(\lambda<0)$

$$\mathcal{U}_2=e^{id_2t+id_1x+id_3}\sqrt{a_0\left(1-\frac{\mu-\frac{\mu}{\mu e^{\lambda(\xi+\vartheta)}+1}}{\lambda}\right)}. \tag{8}$$

For $[4\,\alpha\,\mu>\lambda^2]$,

$$\mathcal{U}_3=e^{id_2t+id_1x+id_3}\sqrt{\frac{a_0\left(\lambda^2-4\alpha\mu\right)}{\lambda\left(-\sqrt{4\alpha\mu-\lambda^2}\sin\left((\xi+\vartheta)\sqrt{4\alpha\mu-\lambda^2}\right)+\lambda\cos\left((\xi+\vartheta)\sqrt{4\alpha\mu-\lambda^2}\right)+\lambda\right)}}, \tag{9}$$

$$\mathcal{U}_4=e^{id_2t+id_1x+id_3}\sqrt{-\frac{a_0\left(\lambda^2-4\alpha\mu\right)}{\lambda\left(\sqrt{4\alpha\mu-\lambda^2}\sin\left((\xi+\vartheta)\sqrt{4\alpha\mu-\lambda^2}\right)+\lambda\cos\left((\xi+\vartheta)\sqrt{4\alpha\mu-\lambda^2}\right)-\lambda\right)}}. \tag{10}$$

Family II:
$\left[a_{-1}\to\frac{\sqrt{\alpha}a_0}{2\sqrt{\mu}}, a_1\to\frac{a_0\sqrt{\mu}}{2\sqrt{\alpha}}, k_2\to-\frac{3\alpha k_1\mu}{a_0^2}, k_3\to\frac{k_1\left(8\alpha\mu-2\sqrt{\alpha}\lambda\sqrt{\mu}\right)}{a_0}, k_4\to\frac{1}{4}\left(12\sqrt{\alpha}k_1\lambda\sqrt{\mu}-20\alpha k_1\mu-\lambda^2 k_1\right)\right]$
Thus, the explicit wave solutions of Eq. (1) are formulated in the following formulas
For $[\lambda=0]$, when $(\alpha\,\mu>0)$

$$\mathcal{U}_5=e^{id_2t+id_1x+id_3}\sqrt{a_0\left(\csc\left(2\sqrt{\alpha}\sqrt{\mu}(\xi+\vartheta)\right)+1\right)}. \tag{11}$$

When $(\alpha\,\mu<0)$

$$\mathcal{U}_6=e^{id_2t+id_1x+id_3}\sqrt{a_0\left(\frac{\sqrt{\alpha}\sqrt{\mu}\operatorname{csch}\left(2\left(\xi\sqrt{-\alpha\mu}\mp\frac{\log(\vartheta)}{2}\right)\right)}{\sqrt{-\alpha\mu}}+1\right)}. \tag{12}$$

For $[4\alpha\mu>\lambda^2]$

$$\mathcal{U}_7=\frac{1}{2}e^{id_2t+id_1x+id_3}\sqrt{-\frac{a_0\left(-2\sqrt{\alpha}\sqrt{\mu}-\sqrt{4\alpha\mu-\lambda^2}\tan\left(\frac{1}{2}(\xi+\vartheta)\sqrt{4\alpha\mu-\lambda^2}\right)+\lambda\right)^2}{\sqrt{\alpha}\sqrt{\mu}\left(\lambda-\sqrt{4\alpha\mu-\lambda^2}\tan\left(\frac{1}{2}(\xi+\vartheta)\sqrt{4\alpha\mu-\lambda^2}\right)\right)}}, \tag{13}$$

$$\mathcal{U}_8=\frac{1}{2}e^{id_2t+id_1x+id_3}\sqrt{-\frac{a_0\left(-2\sqrt{\alpha}\sqrt{\mu}-\sqrt{4\alpha\mu-\lambda^2}\cot\left(\frac{1}{2}(\xi+\vartheta)\sqrt{4\alpha\mu-\lambda^2}\right)+\lambda\right)^2}{\sqrt{\alpha}\sqrt{\mu}\left(\lambda-\sqrt{4\alpha\mu-\lambda^2}\cot\left(\frac{1}{2}(\xi+\vartheta)\sqrt{4\alpha\mu-\lambda^2}\right)\right)}}. \tag{14}$$

Family III:
$[a_{-1} \to \frac{\sqrt{a_0^2(\lambda^2-4\alpha\mu)}+a_0\lambda}{2\mu}, a_1 \to 0, k_2 \to \frac{3k_1\left(\lambda\sqrt{a_0^2(\lambda^2-4\alpha\mu)}-a_0\left(\lambda^2-2\alpha\mu\right)\right)}{8a_0^3},$
$k_3 \to \frac{k_1\left(a_0\left(\lambda^2-4\alpha\mu\right)-\lambda\sqrt{a_0^2(\lambda^2-4\alpha\mu)}\right)}{2a_0^2}, k_4 \to -\frac{1}{4}k_1\left(\lambda^2-4\alpha\mu\right)]$
Thus, the explicit wave solutions of Eq. (1) are formulated in the following formulas.
For $[\lambda = 0]$ when $(\alpha\mu < 0)$

$$\mathcal{U}_9 = e^{id_2t+id_1x+id_3}\sqrt{\frac{\sqrt{-\alpha a_0^2\mu}\coth\left(\xi\sqrt{-\alpha\mu}\mp\frac{\log(\vartheta)}{2}\right)}{\sqrt{-\alpha\mu}}+a_0}, \tag{15}$$

$$\mathcal{U}_{10} = e^{id_2t+id_1x+id_3}\sqrt{\frac{\sqrt{-\alpha a_0^2\mu}\tanh\left(\xi\sqrt{-\alpha\mu}\mp\frac{\log(\vartheta)}{2}\right)}{\sqrt{-\alpha\mu}}+a_0}. \tag{16}$$

For $[\alpha = 0]$ when $(\lambda > 0)$

$$\mathcal{U}_{11} = e^{id_2t+id_1x+id_3}\sqrt{a_0-\frac{\left(\sqrt{a_0^2\lambda^2}+a_0\lambda\right)e^{-\lambda(\xi+\vartheta)}\left(\mu e^{\lambda(\xi+\vartheta)}-1\right)}{2\lambda\mu}}. \tag{17}$$

Family IV:
$\left[a_{-1} \to 0, a_1 \to \frac{a_0\lambda-\sqrt{a_0^2\lambda^2-4\alpha a_0^2\mu}}{2\alpha}, k_2 \to -\frac{3k_1\left(\lambda\sqrt{a_0^2(\lambda^2-4\alpha\mu)}+a_0\left(\lambda^2-2\alpha\mu\right)\right)}{8a_0^3}, k_3 \to \frac{k_1\left(\lambda\sqrt{a_0^2(\lambda^2-4\alpha\mu)}+a_0\left(\lambda^2-4\alpha\mu\right)\right)}{2a_0^2},\right.$
$\left.k_4 \to -\frac{1}{4}k_1\left(\lambda^2-4\alpha\mu\right)\right]$
Thus, the explicit wave solutions of Eq. (1) are formulated in the following formulas
For $[\lambda = 0]$ when $(\alpha\mu < 0)$

$$\mathcal{U}_{12} = e^{id_2t+id_1x+id_3}\sqrt{\frac{\sqrt{-\alpha a_0^2\mu}\tanh\left(\xi\sqrt{-\alpha\mu}\mp\frac{\log(\vartheta)}{2}\right)}{\sqrt{-\alpha\mu}}+a_0}, \tag{18}$$

$$\mathcal{U}_{13} = e^{id_2t+id_1x+id_3}\sqrt{\frac{\sqrt{-\alpha a_0^2\mu}\coth\left(\xi\sqrt{-\alpha\mu}\mp\frac{\log(\vartheta)}{2}\right)}{\sqrt{-\alpha\mu}}+a_0}. \tag{19}$$

2.2 Numerical solutions

Here, we try to find the numerical solutions of the *NLCFS* equation via the B - spline schemes as following

2.2.1 Cubic B–Spline

Employing the cubic spline technique to Eq. (4) with the above conditions, yields elicit its numerical solutions as following

$$\mathfrak{P}(\mathfrak{Z}) = \sum_{\mathfrak{T}=-1}^{\mathfrak{M}+1}\mathfrak{C}_{\mathfrak{T}}\,\mathcal{E}_{\mathfrak{T}}, \tag{20}$$

where $\mathfrak{C}_{\mathfrak{T}}$, $\mathcal{E}_{\mathfrak{T}}$ follow the next conditions, respectively:

$$\mathfrak{L}\,\mathfrak{B}(\mathfrak{Z}) = \mathcal{F}(\mathfrak{Z}_{\mathfrak{T}}, \mathfrak{B}(\mathfrak{Z}_{\mathfrak{T}}))\ \text{where}\ (\mathfrak{T} = 0, 1, ..., n) \tag{21}$$

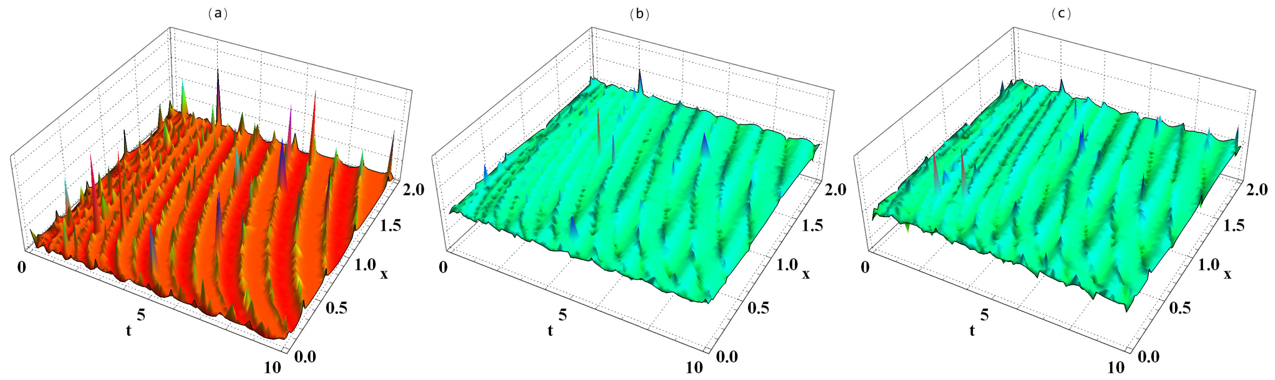

Fig. 1: Numerical simulation of Eq. (4) in three–dimensional sketches.

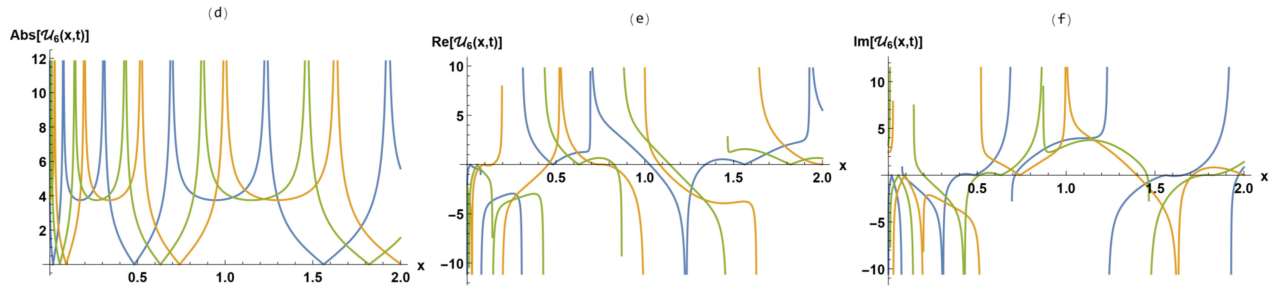

Fig. 2: Numerical simulation of Eq. (4) in two–dimensional sketches.

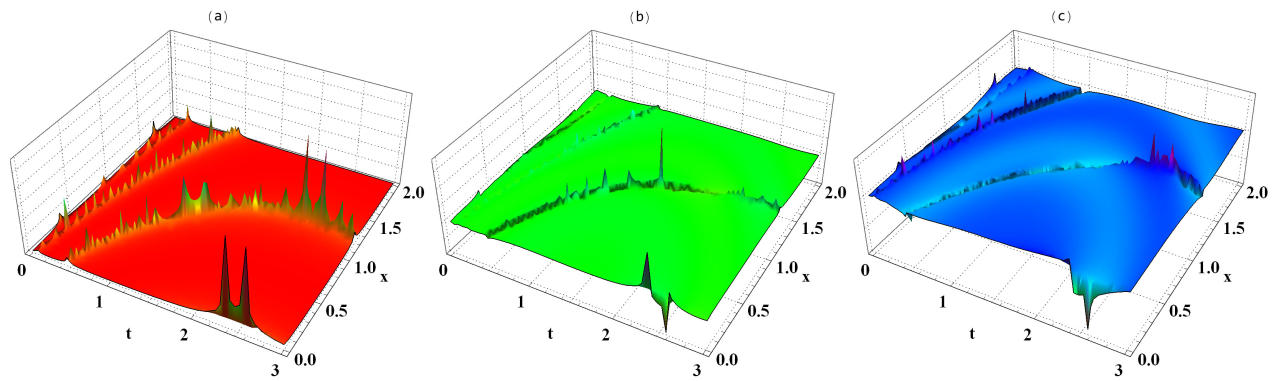

Fig. 3: Numerical simulation of Eq. (4) in three–dimensional sketches.

and

$$\mathcal{E}_{\mathfrak{T}}(\mathfrak{Z}) = \frac{1}{6\,\mathfrak{H}^3}\begin{cases} (\mathfrak{Z}-\mathfrak{Z}_{\mathfrak{T}-2})^3, & \mathfrak{Z}\in[\mathfrak{Z}_{\mathfrak{T}-2},\mathfrak{Z}_{\mathfrak{T}-1}],\\ -3\,(\mathfrak{Z}-\mathfrak{Z}_{\mathfrak{T}-1})^3+3\,\mathfrak{H}\,(\mathfrak{Z}-\mathfrak{Z}_{\mathfrak{T}-1})^2+3\,\mathfrak{H}^2\,(\mathfrak{Z}-\mathfrak{Z}_{\mathfrak{T}-1})+\mathfrak{H}^3, & \mathfrak{Z}\in[\mathfrak{Z}_{\mathfrak{T}-1},\mathfrak{Z}_i],\\ -3\,(\mathfrak{Z}_{\mathfrak{T}+1}-\mathfrak{Z})^3+3\,\mathfrak{H}\,(\mathfrak{Z}_{\mathfrak{T}+1}-\mathfrak{Z})^2+3\,\mathfrak{H}^2\,(\mathfrak{Z}_{\mathfrak{T}+1}-\mathfrak{Z})+\mathfrak{H}^3, & \mathfrak{Z}\in[\mathfrak{Z}_{\mathfrak{T}},\mathfrak{Z}_{\mathfrak{T}+1}],\\ (\mathfrak{Z}_{\mathfrak{T}+2}-\mathfrak{Z})^3, & \mathfrak{Z}\in[\mathfrak{Z}_{\mathfrak{T}+1},\mathfrak{Z}_{\mathfrak{T}+2}],\\ 0, & \text{otherwise}. \end{cases} \tag{22}$$

For $\mathfrak{T}\in[-2,\mathfrak{M}+2]$, we obtain

$$\mathfrak{B}_{\mathfrak{T}}(\mathfrak{Z}) = \mathfrak{C}_{\mathfrak{T}-1} + 4\,\mathfrak{C}_{\mathfrak{T}} + \mathfrak{C}_{\mathfrak{T}+1}. \tag{23}$$

Substituting Eq. (23) into Eq. (4), yields $(\mathfrak{M}+3)$ of equations. Resolving this system gives the results in table 1.

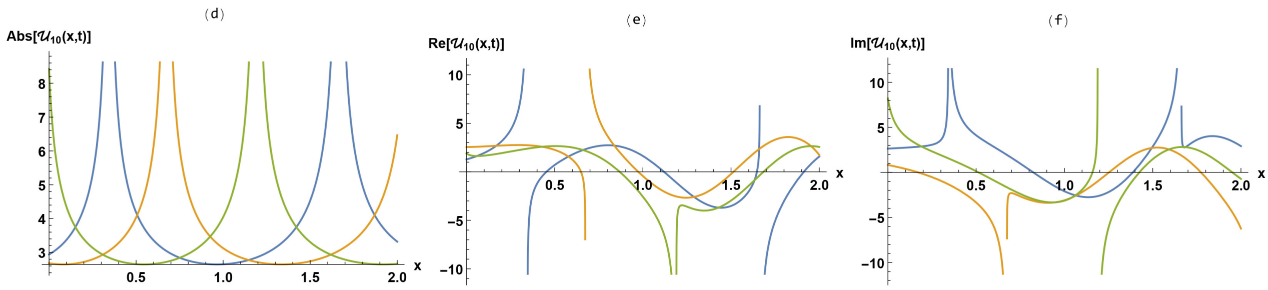

Fig. 4: Numerical simulation of Eq. (4) in two–dimensional sketches.

2.2.2 Quantic B–spline

Employing the cubic spline technique to Eq. (4) with the above conditions gives its numerical solutions in the next formula

$$\mathfrak{B}(\mathfrak{z}) = \sum_{\mathfrak{M}=-1}^{\mathfrak{M}+1} \mathfrak{C}_{\mathfrak{T}}\, \mathcal{E}_{\mathfrak{T}}, \tag{24}$$

where $\mathfrak{C}_{\mathfrak{T}}$, $\mathcal{E}_{\mathfrak{T}}$ follow the next conditions, respectively:

$$\mathfrak{L}\,\mathfrak{B}(\mathfrak{z}) = \mathcal{F}(\mathfrak{z}_{\mathfrak{T}}, \mathfrak{B}(\mathfrak{z}_{\mathfrak{T}})) \text{ where } (\mathfrak{T} = 0, 1, ..., n) \tag{25}$$

and

$$\mathcal{E}_{\mathfrak{T}}(\mathfrak{z}) = \frac{1}{\mathfrak{H}^5} \begin{cases} (\mathfrak{z}-\mathfrak{z}_{\mathfrak{T}-3})^5, & \mathfrak{z} \in [\mathfrak{z}_{\mathfrak{T}-3}, \mathfrak{z}_{\mathfrak{T}-2}], \\ (\mathfrak{z}-\mathfrak{z}_{\mathfrak{T}-3})^5 - 6(\mathfrak{z}-\mathfrak{z}_{\mathfrak{T}-2})^5, & \mathfrak{z} \in [\mathfrak{z}_{\mathfrak{T}-2}, \mathfrak{z}_{\mathfrak{T}-1}], \\ (\mathfrak{z}-\mathfrak{z}_{\mathfrak{T}-3})^5 - 6(\mathfrak{z}-\mathfrak{z}_{\mathfrak{T}-2})^5 + 15(\mathfrak{z}-\mathfrak{z}_{\mathfrak{T}-1})^5, & \mathfrak{z} \in [\mathfrak{z}_{\mathfrak{T}-1}, \mathfrak{z}_{\mathfrak{T}}], \\ (\mathfrak{z}_{\mathfrak{T}+3}-\mathfrak{z})^5 - 6(\mathfrak{z}_{\mathfrak{T}+2}-\mathfrak{z})^5 + 15(\mathfrak{z}_{\mathfrak{T}+1}-\mathfrak{z})^5, & \mathfrak{z} \in [\mathfrak{z}_{\mathfrak{T}}, \mathfrak{z}_{\mathfrak{T}+1}], \\ (\mathfrak{z}_{\mathfrak{T}+3}-\mathfrak{z})^5 - 6(\mathfrak{z}_{\mathfrak{T}+2}-\mathfrak{z})^5, & \mathfrak{z} \in [\mathfrak{z}_{\mathfrak{T}+1}, \mathfrak{z}_{\mathfrak{T}+2}], \\ (\mathfrak{z}_{\mathfrak{T}+3}-\mathfrak{z})^5, & x \in [\mathfrak{z}_{\mathfrak{T}+2}, \mathfrak{z}_{\mathfrak{T}+3}], \\ 0, & \text{otherwise.} \end{cases} \tag{26}$$

For $\mathfrak{T} \in [-2, \mathfrak{M}+2]$, we get

$$\mathfrak{B}_{\mathfrak{T}}(\mathfrak{z}) = \mathfrak{C}_{\mathfrak{T}-2} + 26\,\mathfrak{C}_{\mathfrak{T}-1} + 66\,\mathfrak{C}_{\mathfrak{T}} + 26\,\mathfrak{C}_{\mathfrak{T}+1} + \mathfrak{C}_{\mathfrak{T}+2}. \tag{27}$$

Substituting Eq. (27) into Eq. (4) gives $(\mathfrak{M}+5)$ of equations. Resolving this system, leads to the results in table 2.

2.2.3 Septic B–Spline

Employing the septic spline technique to Eq. (4) with the above conditions gives its numerical solutions in the next form

$$\mathfrak{B}(\mathfrak{z}) = \sum_{\mathfrak{T}=-1}^{\mathfrak{M}+1} \mathfrak{C}_{\mathfrak{M}}\, \mathcal{E}_{\mathfrak{M}}, \tag{28}$$

where $c_{\mathfrak{M}}$, $\mathcal{E}_{\mathfrak{M}}$ follow the next conditions, respectively:

$$\mathfrak{L}\,\mathfrak{B}(\mathfrak{z}) = \mathcal{F}(\mathfrak{z}_{\mathfrak{M}}, \mathfrak{B}(\mathfrak{z}_{\mathfrak{M}})) \text{ where } (\mathfrak{M} = 0, 1, ..., n) \tag{29}$$

Table 1: Value of exact solutions, and numerical obtained solutions by cubic spline scheme of (4) under the following conditions $[a_0 = 2,\ \alpha = -1,\ \lambda = 0,\ k_1 = 3,\ k_2 = -\frac{9}{4},\ k_3 = 12,\ k_4 = -12,\ \mu = 4,\ \vartheta = 1]$ for Eq. (16)

Value of $\mathfrak{z}$	Exact value	Numerical Value	Absolute error
0	2	2	0
0.1	2.39475	0.994884	1.39987
0.2	2.7599	0.335965	2.42393
0.3	3.0741	0.0302715	3.04383
0.4	3.32807	-0.00460121	3.33267
0.5	3.52319	-0.000604126	3.52379
0.6	3.66731	-0.0137043	3.68101
0.7	3.7707	0.0684398	3.70226
0.8	3.84334	0.707378	3.13596
0.9	3.89361	2.02339	1.87023
1	3.92806	3.92806	0

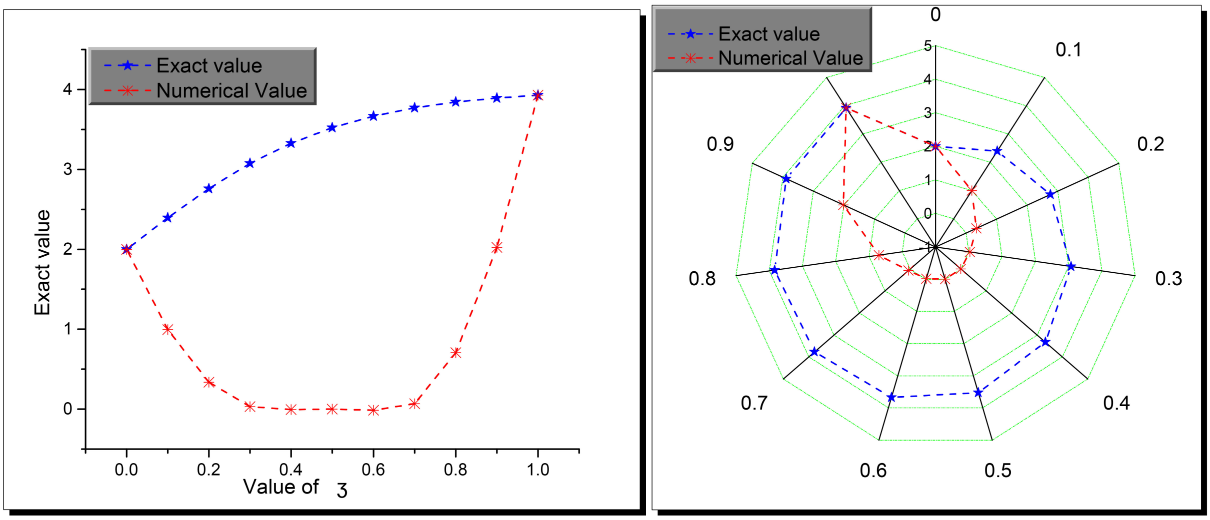

Fig. 5: Exact and numerical values by using cubic B–spline scheme for the three analytical schemes

and

$$\mathcal{E}_{\mathfrak{T}}(\mathfrak{z}) = \frac{1}{\mathfrak{H}^5}\begin{cases} (\mathfrak{z}-\mathfrak{z}_{\mathfrak{T}-4})^7, & \mathfrak{z}\in[\mathfrak{z}_{\mathfrak{T}-4},\mathfrak{z}_{\mathfrak{T}-3}],\\ (\mathfrak{z}-\mathfrak{z}_{\mathfrak{T}-4})^7-8(\mathfrak{z}-\mathfrak{z}_{\mathfrak{T}-3})^7, & \mathfrak{z}\in[\mathfrak{z}_{\mathfrak{T}-3},\mathfrak{z}_{\mathfrak{T}-2}],\\ (\mathfrak{z}-\mathfrak{z}_{\mathfrak{T}-4})^7-8(\mathfrak{z}-\mathfrak{z}_{\mathfrak{T}-3})^7+28(\mathfrak{z}-\mathfrak{z}_{\mathfrak{T}-2})^7, & \mathfrak{z}\in[\mathfrak{z}_{\mathfrak{T}-2},\mathfrak{z}_{\mathfrak{T}-1}],\\ (\mathfrak{z}-\mathfrak{z}_{\mathfrak{T}-4})^7-8(\mathfrak{z}-\mathfrak{z}_{\mathfrak{T}-3})^7+28(\mathfrak{z}-\mathfrak{z}_{\mathfrak{T}-2})^7+56(\mathfrak{z}-\mathfrak{z}_{\mathfrak{T}-1})^7, & \mathfrak{z}\in[\mathfrak{z}_{\mathfrak{T}-1},\mathfrak{z}_{\mathfrak{T}}],\\ (\mathfrak{z}_{\mathfrak{T}+4}-\mathfrak{z})^7-8(\mathfrak{z}_{\mathfrak{T}+3}-\mathfrak{z})^7+28(\mathfrak{z}_{\mathfrak{T}+2}-\mathfrak{z})^7+56(\mathfrak{z}_{\mathfrak{T}+1}-\mathfrak{z})^7, & \mathfrak{z}\in[\mathfrak{z}_{\mathfrak{T}},\mathfrak{z}_{\mathfrak{T}+1}],\\ (\mathfrak{z}_{\mathfrak{T}+4}-\mathfrak{z})^7-8(\mathfrak{z}_{\mathfrak{T}+3}-\mathfrak{z})^7+28(\mathfrak{z}_{\mathfrak{T}+2}-\mathfrak{z})^7, & \mathfrak{z}\in[\mathfrak{z}_{\mathfrak{T}+1},\mathfrak{z}_{\mathfrak{T}+2}],\\ (\mathfrak{z}_{\mathfrak{T}+4}-\mathfrak{z})^7-8(\mathfrak{z}_{\mathfrak{T}+3}-\mathfrak{z})^7, & \mathfrak{z}\in[\mathfrak{z}_{\mathfrak{T}+2},\mathfrak{z}_{\mathfrak{T}+3}],\\ (\mathfrak{z}_{\mathfrak{T}+4}-\mathfrak{z})^7, & \mathfrak{z}\in[\mathfrak{z}_{\mathfrak{T}+3},\mathfrak{z}_{\mathfrak{T}+4}],\\ 0, & \text{otherwise.} \end{cases} \tag{30}$$

For $\mathfrak{T}\in[-3,\mathfrak{M}+3]$, we get

$$\begin{aligned}\mathfrak{B}_{\mathfrak{T}}(\mathfrak{z}) = {}& \mathfrak{C}_{\mathfrak{T}-3}+120\,\mathfrak{C}_{\mathfrak{T}-2}+1191\,\mathfrak{C}_{\mathfrak{T}-1}+2416\,\mathfrak{C}_{\mathfrak{T}}\\ &+1191\,\mathfrak{C}_{\mathfrak{T}+1}+120\,\mathfrak{C}_{\mathfrak{T}+2}+\mathfrak{C}_{\mathfrak{T}+3}.\end{aligned} \tag{31}$$

Substituting Eq. (31) into Eq. (4) gives $(\mathfrak{M}+7)$ of equations. Resolving this system leads to the results in table (3).

Table 2: Value of exact solutions, and numerical obtained solutions by quantic spline scheme of (4) under the following conditions $[a_0 = 2,\ \alpha = -1,\ \lambda = 0,\ k_1 = 3,\ k_2 = -\frac{9}{4},\ k_3 = 12,\ k_4 = -12,\ \mu = 4,\ \vartheta = 1]$ for Eq. (16)

Value of $\mathfrak{z}$	Exact value	Numerical Value	Absolute error
0	2	1.92032	0.0796816
0.1	2.39475	0.92752	1.46723
0.2	2.7599	0.256607	2.50329
0.3	3.0741	0.0981479	2.97595
0.4	3.32807	0.0251699	3.3029
0.5	3.52319	0.0226624	3.50053
0.6	3.66731	-0.00303086	3.67034
0.7	3.7707	0.152334	3.61837
0.8	3.84334	0.825615	3.01772
0.9	3.89361	2.28397	1.60964
1	3.92806	3.77156	0.156497

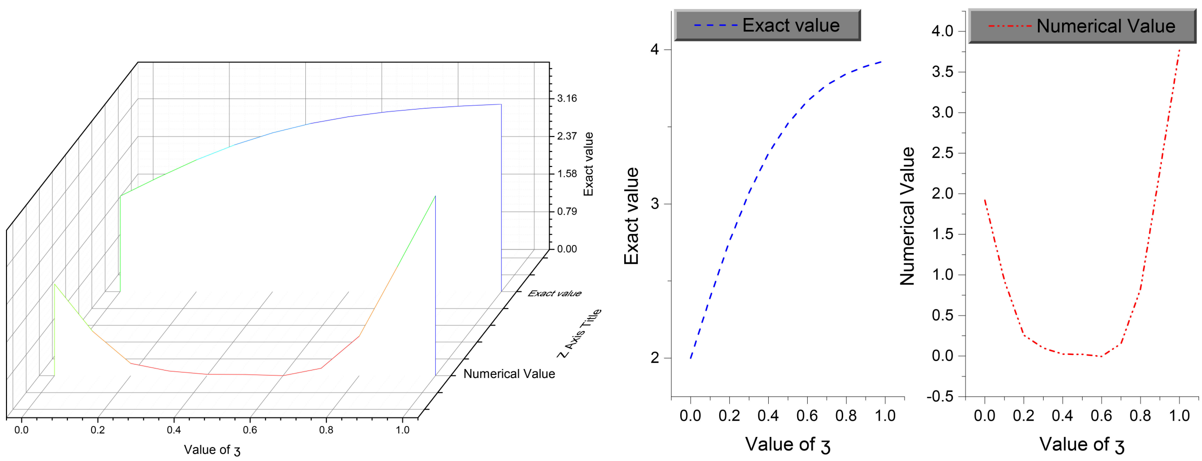

Fig. 6: Exact and numerical values by using quantic B–spline scheme for the three analytical schemes

3 Results and discussion

All computational obtained solutions are considered as optical soliton wave solutions that are used to explain the dynamical behavior of the particles in the optical waves where the optical soliton is restricted electromagnetic wave which stretches in media of nonlinear dispersive. The physical illustration of the presented figures is given as follows

1.Figs. (1, 2) show the soliton shape of the absolute, real, and imaginary solution (12) in the three–dimensional plot (a, b, c) to explain the perspective view of the solution and the absolute, real, and imaginary sketches in two–dimensional plot (d, e, f) to explain the wave propagation pattern of the wave along x axis when $[a_0 = 7,\ \alpha = -1,\ \alpha = 0.5,\ d_1 = 5,\ d_2 = 6,\ d_3 = 9,\ \mu = 4,\ l_1 = 2,\ l_2 = 3,\ \vartheta = 1]$.
2.Figs. (3, 4) show the soliton of the absolute, real, and imaginary solution (18) in the three–dimensional plot (a, b, c) to explain the perspective view of the solution and the absolute, real, and imaginary sketches in two– dimensional plot (d, e, f) to explain the wave propagation pattern of the wave along x axis when $[a_0 = 7,\ \alpha = -1,\ \alpha = 0.5,\ d_1 = 5,\ d_2 = 6,\ d_3 = 9,\ \mu = 4,\ l_1 = 2,\ l_2 = 3,\ \vartheta = 1]$.
3.Fig. 5 shows the exact and numerical solutions of Eq. (4) via cubic B - spline scheme under the following conditions $[a_0 = 2,\ \alpha = -1,\ \lambda = 0,\ k_1 = 3,\ k_2 = -\frac{9}{4},\ k_3 = 12,\ k_4 = -12,\ \mu = 4,\ \vartheta = 1]$ for Eq. (16).
4.Fig. 6 shows the exact and numerical solutions of Eq. (4) via quantic B - spline scheme under the following conditions $[a_0 = 2,\ \alpha = -1,\ \lambda = 0,\ k_1 = 3,\ k_2 = -\frac{9}{4},\ k_3 = 12,\ k_4 = -12,\ \mu = 4,\ \vartheta = 1]$ for Eq. (16).
5.Fig. 7 shows the exact and numerical solutions of Eq. (4) via septic B - spline scheme under the following conditions $[a_0 = 2,\ \alpha = -1,\ \lambda = 0,\ k_1 = 3,\ k_2 = -\frac{9}{4},\ k_3 = 12,\ k_4 = -12,\ \mu = 4,\ \vartheta = 1]$ for Eq. (16).

Table 3: Value of exact solutions, and numerical obtained solutions by septic spline scheme of (4) under the following conditions $[a_0 = 2,\ \alpha = -1,\ \lambda = 0,\ k_1 = 3,\ k_2 = -\frac{9}{4},\ k_3 = 12,\ k_4 = -12,\ \mu = 4,\ \vartheta = 1]$ for Eq. (16)

Value of $\mathfrak{z}$	Exact value	Numerical Value	Absolute error
0	2	0.496895	1.5031
0.1	2.39475	0.347676	2.04707
0.2	2.7599	0.203475	2.55642
0.3	3.0741	0.156736	2.91736
0.4	3.32807	0.0408894	3.28718
0.5	3.52319	-0.00272479	3.52591
0.6	3.66731	0.0759522	3.59136
0.7	3.7707	0.231326	3.53938
0.8	3.84334	0.378552	3.46479
0.9	3.89361	0.650581	3.24303
1	3.92806	0.975916	2.95214

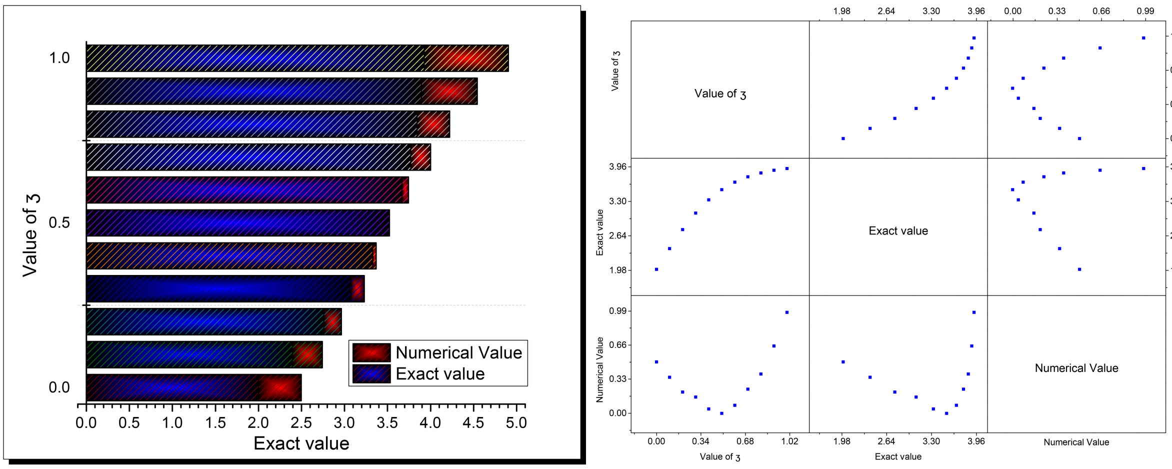

Fig. 7: Exact and numerical values by using septic B–spline scheme for the three analytical schemes

Here, the obtained computational and numerical solutions are investigated to show the novelty of our solutions as following:

1. **Computational schemes:**
The extended simplest equation method is employed to find the exact traveling and solitary wave solutions of the higher - order dispersive cubic - quintic NLCFS equation. The employed method is considered one of the most recent derived analytical schemes. It depends on an auxiliary equation (6) that has the following general solution,

$$\mathcal{F}(\mathfrak{z}) = \frac{\sqrt{4\alpha\mu - \lambda^2}\tan\left(\frac{1}{2}\left(\mathfrak{z}\sqrt{4\alpha\mu - \lambda^2} + c_1\sqrt{4\alpha\mu - \lambda^2}\right)\right) - \lambda}{2\mu}, \tag{32}$$

where c_1 is arbitrary constant. Consequently, all other obtained solutions via this method are just special case of this general solution under specific conditions.

2. **Obtained computational wave solutions:**

   – Eq. (16) is equal to Eq. (29) [42] when $[\vartheta = 0,\ -\alpha\,\mu = \chi^2 - 4\,\delta\,\varrho,\ k_1 = a_0]$.
   – All other obtained solutions are different from that obtained in [42] where the modified Khater method was employed to construct the exact traveling and solitary wave solutions.

3. **Obtained Numerical solutions:**
Here, we explain the obtained numerical results via B - spline schemes (cubic & quantic & septic). These solutions illustrate the disadvantage of this kind of numerical schemes where the value of absolute error is relativity not small. The reason of this value is the higher and degree in Eq. (1).

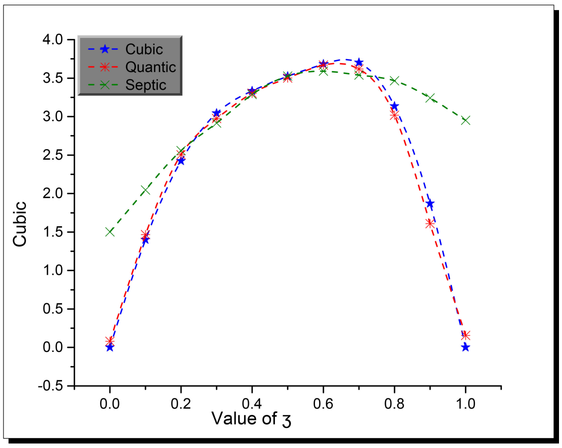

Fig. 8: The obtained absolute value of error via cubic, quantic, and septic spline schemes

4 Conclusion

This research investigated the exact traveling and solitary wave solutions of the higher-order dispersive cubic - quintic NLCFS equation via the extended simplest equation method through the conformable fractional derivative. Novel solitary wave solutions were obtained and some of them were explained by plotting them in two, three-dimensional in absolute, real, and imaginary values of these solutions. The novelty of our paper was shown by making the comparison between our obtained solutions and that were purchased in previously published articles.

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