Theory and Applications of Graphs

Volume 8 | Issue 1 Article 5

March 2021

The Conditional Strong Matching Preclusion of Augmented Cubes

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Recommended Citation

Abdallah, Mohamad and Cheng, Eddie (2021) "The Conditional Strong Matching Preclusion of Augmented Cubes," *Theory and Applications of Graphs*: Vol. 8 : Iss. 1 , Article 5.
DOI: 10.20429/tag.2021.080105
Available at: https://digitalcommons.georgiasouthern.edu/tag/vol8/iss1/5

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Cover Page Footnote

We would like to thank the two anonymous referees for a number of helpful comments and suggestions.

Abstract

The strong matching preclusion is a measure for the robustness of interconnection networks in the presence of node and/or link failures. However, in the case of random link and/or node failures, it is unlikely to find all the faults incident and/or adjacent to the same vertex. This motivates Park et al. to introduce the conditional strong matching preclusion of a graph. In this paper we consider the conditional strong matching preclusion problem of the augmented cube AQ_n, which is a variation of the hypercube Q_n that possesses favorable properties.

1 Introduction

A *matching* in a graph $G = (V, E)$ is a set M of pairwise nonadjacent edges. A *perfect matching* M in G is a matching such that every vertex in G is incident to exactly one edge in M. An *almost-perfect matching* M in G is a set of edges such that every vertex in G, except one, is incident with exactly one edge in M, and the exceptional vertex is incident to none. If G has a perfect matching, then G has an even number of vertices; if G has an almost-perfect matching, then G has an odd number of vertices. We say that the graph G is *matchable* if it has a perfect matching or an almost-perfect matching. Otherwise, it is called *unmatchable*.

Parallel processing uses computers made up of many separate processor to overcome the limitation of computers with a single processor. When parallel processing is used, one processor may need output generated by another processor. Therefore, these processors must be interconnected. The interconnection network of these processors is usually modeled by graphs. Brigham et al. [3] introduced the concept of matching preclusion as a measure of robustness in the event of link failure in interconnection networks. A *matching preclusion set* of G is a set of edges whose deletion results in an unmatchable graph [3]. The matching preclusion number of G, denoted by $mp(G)$, is the minimum size of all possible matching preclusion sets of G. Any such set is called an *optimal matching preclusion set*. If $mp(G)$ is large, the network will robust in the event of link failures. If G is unmatchable, then $mp(G) = 0$.

Throughout this paper our graphs will always have an even number of vertices. A trivial case of matching preclusion occurs when all edges in G incident to a single vertex are deleted. This case occurs when all faulty edges are incident to a single vertex. In case of random link failure, it is unlikely to have such situation. For this reason, Cheng et al.[6] introduced the *conditional matching preclusion* which removes from consideration the case when the matching preclusion set produces a graph with an isolated vertex after the edge deletion. The *conditional matching preclusion number*, denoted $mp_1(G)$, is the minimum size of all conditional matching preclusion sets of G.

Park and Ihm [16] introduced the concept of strong matching preclusion where the matching preclusion set contains vertices and/or edges. This concept corresponds to the situation when the failure of network occurs through nodes and communication lines. The *strong matching preclusion set* of G is a set of vertices and/or edges whose deletion leads to an unmatchable graph. The *strong matching preclusion number* of G, denoted $smp(G)$, is the minimum size of strong matching preclusion sets in G. Motivated by the same reason Cheng et al. introduced the conditional matching preclusion, Park and Ihm [17] introduced the concept of *conditional strong matching preclusion* where the matching preclusion set contains vertices and/or edges and no isolated vertices are produced after the deletion of vertices and/or edges. The conditional strong matching preclusion of several graphs and interconnection networks has been studied in [17, 1].

The hypercube was first proposed as an interconnection network in 1977 [18]. It is a powerful network for parallel computation that nearly contains all arrays, binary trees, and meshes of trees as subgraphs [12]. The augmented cube, introduced by S. Choudum and V. Sunitha [9], is a variation of the hypercube that maintains all the favorable properties of the hypercube. In addition, the augmented cube has a smaller diameter than the hypercube and possesses embedding properties that the hypercube does not carry. The augmented cube has been studied widely by researchers [8, 14, 5, 4, 10, 11, 13, 19, 20]. The conditional matching preclusion and the strong matching preclusion of the augmented cube has been studied by Cheng et al. [5, 8].

In this paper, we consider the conditional strong matching preclusion problem of the augmented cube. In Section 2, we list some necessary properties about the conditional strong matching preclusion and we define the augmented cube and then present some of its structural properties. In Section 3, we prove our main result. In Section 4, we conclude the paper.

2 Preliminaries

A trivial case of matching preclusion occurs when all edges in G incident to a single vertex are deleted. If a trivial case is an optimal solution, then we call it *trivial optimal matching preclusion set*. Let F be an optimal strong matching preclusion set of a graph $G = (V, E)$, and let $F = F^V \cup F^E$ where F^V consists of vertices in F and F^E consists of edges in F. We may assume that no element in F^E is incident to an element of F^V since F is optimal. In fact, if $f \in F^E$ is incident to $u \in F^V$, then $G - F = G - (F - \{f\})$. If F is an optimal strong matching preclusion set of G and $G - F$ has an isolated vertex, then F is a *basic optimal strong matching preclusion set*. Based on this definition, it is possible to have a basic optimal matching preclusion set F with $G - F$ odd and without almost-perfect matchings. We can further restrict this class by requiring that, in addition, $G - F$ must be even. Then F is called *trivial optimal strong matching preclusion set*.

The following proposition considers the relationship between basic strong matching preclusion sets and trivial strong matching preclusion sets.

Proposition 2.1. *[2] Let G be a r-regular even graph with $r \geq 2$. Suppose that $smp(G) = r$. Then every basic optimal strong matching preclusion set is trivial.*

A conditional fault set $F \subseteq V(G) \cup E(G)$ is called *conditional strong matching preclusion set* of G if $G - F$ has neither a perfect matching nor an almost-perfect matching and no isolated vertices. The minimum cardinality of all such sets is denoted by $smp_1(G)$, and called the *conditional strong matching preclusion number* of G. If G is unmatchable, then $smp_1(G) = 0$. The following propositions follow directly from the fact that a matching preclusion set is a special case of a strong matching preclusion set consisting of edges only.

Proposition 2.2. *[2] Let G be a graph with an even number of vertices. Then $smp(G) \leq mp(G) \leq \delta(G)$, where $\delta(G)$ is the minimum degree of G.*

Proposition 2.3. *[17] For every graph G for which all the four numbers, $mp(G)$, $mp_1(G)$, $smp(G)$, and $smp_1(G)$ are well defined, $smp(G) \leq smp_1(G) \leq mp_1(G)$ and $smp(G) \leq mp(G) \leq mp_1(G)$.*

Under the condition of no isolated vertices allowed after the deletion, an easy way to build a conditional strong matching preclusion set in G is to try a fault set F that leaves after deletion a path (u, z, v) made of the three vertices u, z and v, where $\deg_{G-F}(u) = \deg_{G-F}(v) = 1$. If $G - F$ is even, then the resulting graph becomes unmatchable. Therefore we can build a candidate conditional strong matching preclusion set as follows. Let $N_G(\cdot)$ represents the set of neighboring vertices in G. Given a path (u, z, v) in a graph $G = (V, E)$, build a fault set, denoted F_{uzv}, in such a way that

1. F_{uzv} contains every vertex $w \in (N_G(u) \cap N_G(v)) - \{z\}$,
2. F_{uzv} contains the edge uv if $uv \in E(G)$,
3. for every vertex $w \in N_G(u) - N_G(v)$, F_{uzv} contains exactly one of w and uw,
4. for every vertex $w \in N_G(v) - N_G(u)$, F_{uzv} contains exactly one of w and vw.

The next fundamental proposition provides sufficient conditions to make F_{uzv} a conditional strong matching preclusion set.

Proposition 2.4. *[17] For an arbitrary path (u, z, v) in a graph G, F_{uzv} is a conditional strong matching preclusion set of G if*

1. *there is no isolated vertex in $G - F_{uzv}$, and*
2. *$G - F_{uzv}$ has an even number of vertices.*

The conditional strong matching preclusion set described in Proposition 2.4 is called *trivial* as it is one of the simplest ways of building a conditional strong matching preclusion set. The following proposition provides an upper bound for $smp_1(G)$.

Proposition 2.5. *[17] If there exists a trivial conditional strong matching preclusion set F_{uzv} for some path (u, z, v) in a graph G, then $smp_1(G) \leq deg_G(u) + deg_G(v) - 2 - g_G(u, v)$, where $g_G(u, v)$ is $|N(u) \cap N(v)|$ if $(u, v) \in E(G)$ or $|N(u) \cap N(v)| - 1$ otherwise.*

The augmented cube AQ_n, introduced in [9], is a variation of the hypercube and possesses many superior properties. The n-dimensional augmented cube AQ_n is defined recursively as follows. Let $n \geq 1$, the graph AQ_n has 2^n vertices, each labeled by n-bit binary string $u_1u_2 \ldots u_n$ such that $u_i \in \{0, 1\}$ for all i. AQ_1 is isomorphic to the complete graph K_2 where one vertex is labeled by the digit 0 and the other by 1. For $n \geq 2$, AQ_n is obtained by taking two copies of AQ_{n-1}, denoted by AQ^0_{n-1} and AQ^1_{n-1}, where $V(AQ^0_{n-1}) = \{0u_1u_2 \ldots u_{n-1}; u_i = 0 \text{ or } 1\}$ and $V(AQ^1_{n-1}) = \{1u_1u_2 \ldots u_{n-1}; u_i = 0 \text{ or } 1\}$, and adding $2 \times 2^{n-1}$ edges between the two as follows: $u = 0u_1u_2 \ldots u_{n-1}$ and $v = 1v_1v_2 \ldots v_{n-1}$ are adjacent if and only if one of the following conditions holds:

1. $u_i = v_i$ for all $i \geq 1$. In this case we call the edge uv a *cross edge* and say $u = v^x$ and $v = u^x$.

2. $u_i \neq v_i$ for all $i \geq 1$. In this case we call the edge uv a *complement edge* and say $u = v^c$ and $v = u^c$.

See Figure 1 for examples of AQ_n when $n = 2, 3$ and 4.

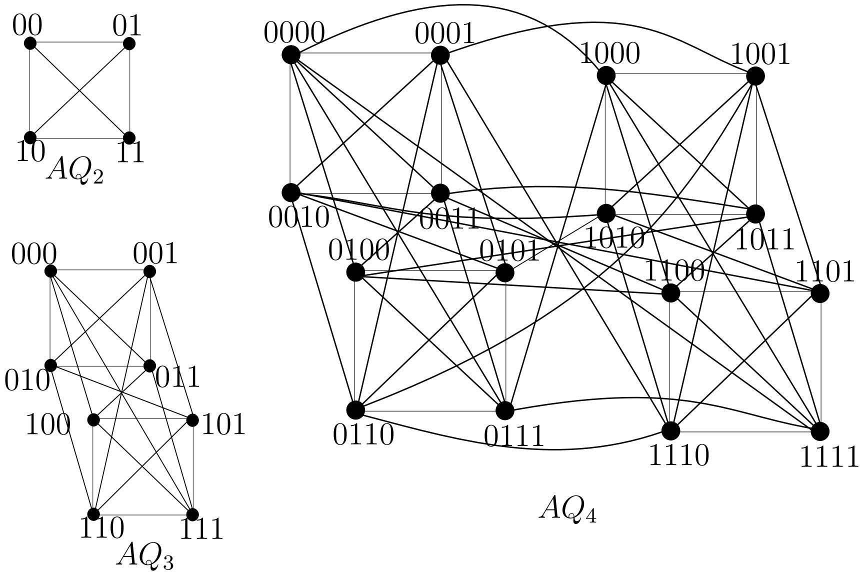

Figure 1: Augmented cube AQ_n for $n = 2, 3$ and 4

Throughout this paper, we denote the set of cross edges in AQ_n by X_n and the set of complement edges in AQ_n by C_n. It is easy to see that $|X_n| = |C_n| = 2^{n-1}$, and the edges in each of X_n and C_n are independent.

The augmented cubes family can be identified as a family of Cayley graphs. Let Γ be a finite group, and let Δ be a set of elements of Γ such that the identity of the group does not belong to Δ. The *Cayley graph* $\Gamma(\Delta)$ is the directed graph with vertex set Γ with an arc directed from u to v if and only if there is an $s \in \Delta$ such that $u = vs$. If whenever $u \in \Delta$, we also have its inverse $u^{-1} \in \Delta$, then for every arc, the reverse arc is also in the graph. So we can treat this Cayley graph as an undirected graph by replacing each pair of arcs by an edge. We denote this simple undirected graph by $G(\Gamma, \Delta)$. Let $\mathbb{Z}_2^n$ denotes the cartesian product of the group $(\mathbb{Z}_2, +)$, where the "+" denotes the sum modulo 2. In [9], the authors showed that $AQ_n \cong G(\mathbb{Z}_n^2, S)$, where $S = \{e_1 = 10\ldots0, e_2 = 010\ldots0, \ldots, e_n = 0\ldots01, e_{n+1} = 1\ldots1, e_{n+2} = 011\ldots1, \ldots, e_{2n-2} = 0\ldots0111, e_{2n-1} = 0\ldots011\}$.

3 Main Result

Theorem 3.1. *[8] Let $n \geq 4$. Then $smp(AQ_n) = 2n - 1$. Moreover, every optimal matching preclusion set is trivial.*

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DOI: 10.20429/tag.2021.080105

Lemma 3.2. *[14] Let $n \geq 3$. Edges of the form $u\overline{u_i}$, where $u = u_n u_{n-1} \dots u_1$ and $\overline{u_i} = u_n \dots \overline{u_i} \dots \overline{u_1}$, have four common neighbors.*

Lemma 3.3. *[14] Let $n \geq 3$. Any two vertices in AQ_n have at most four common neighbors.*

Lemma 3.4. *Let u and v be two vertices in AQ_n, for $n \geq 4$. If $N(u) \cap N(v) \neq \emptyset$, then $|N(u) + N(v)| \geq 2$.*

Proof. Let $z \in N(u) \cap N(v)$. Since $AQ_n \cong G(\mathbb{Z}_n^2, S)$, then $z = u + e_i = v + e_j$, where $e_i, e_j \in S$ for some $i, j \in \{1, \dots, 2n-1\}$ and $i \neq j$. $u+e_i = v+e_j$ implies $u+e_i+e_i+e_j = v+e_j+e_i+e_j$. Since $e_i + e_i = e_j + e_j = 0$, then $u + e_j = v + e_i$. Therefore, the vertex $z' = u + e_j = v + e_i$ is in $N(u) \cap N(v)$ and $z' \neq z$.

□

Lemma 3.5. $smp_1(AQ_4) = 8$.

Proof. By Theorem 3.1, $smp(AQ_4) = 7$, then by Proposition [17] $smp_1(AQ_4) \geq 7$. Since every optimal strong matching preclusion set is trivial, then if $|F| = 7$ and $AQ_4 - F$ has no isolated vertex, the graph $AQ_4 - F$ possesses a perfect or an almost-perfect matching. Therefore $smp_1(AQ_n) \geq 8$. By Proposition 2.5 and Lemma 3.2, we can build a trivial conditional strong matching preclusion set of size less than or equal to $7 + 7 - 2 - 4 = 8$. Thus $smp_1(AQ_4) = 8$. □

Proposition 3.1. *[8] Let $n \geq 3$. Let u be a vertex of AQ_n. Then u^x is adjacent to u^c. Moreover, there is a unique vertex v such that u and v are adjacent, $v^c = u^x$ and $v^x = u^c$. In other words, u, v, u^x, u^c form a complete graph on four vertices.*

Lemma 3.6. *Let $\{x_1, x_2, \dots, x_k\} \subseteq V(AQ_{n-1}^0)$ and suppose that there is at most one faulty edge $f \in X_n \cup C_n$. Then there exists $\{y_1, y_2, \dots, y_k\} \subseteq V(AQ_{n-1}^1)$ such that $x_i y_i \in E(AQ_n - f)$ for $i = 1, \dots, k$.*

Proof. The result is satisfied as the sets X_n and C_n are two independent perfect matchings in AQ_n.

□

Lemma 3.7. *Let $n \geq 4$. Let $F_0 \subseteq V(AQ_{n-1}^0) \cup E(AQ_{n-1}^0)$ such that $|F_0| < 4n - 9$. If $u, v \in V(AQ_{n-1}^0 - F_0)$ and u and v share a common neighbor $z \in V(AQ_{n-1}^1)$, then $uv \in E(AQ_{n-1}^0 - F_0)$ or there exists a vertex $w \in V(AQ_{n-1}^0 - F_0)$ such that uw or vw are in $AQ_{n-1}^0 - F_0$.*

Proof. By Proposition 3.1, $z = u^x = v^c$ or $z = u^c = v^x$. Without loss of generality, assume that $z = u^x = v^c$ and let $t = u^c = v^x$, then u, v, z and t induce the subgraph K_4. By Lemma 3.3, u and v can have at most four common neighbors, then u and v have at most two common neighbors in AQ_{n-1}^0. If $uv \in E(AQ_{n-1}^0 - F_0)$, then we are done. Suppose that $uv \notin E(AQ_{n-1}^0 - F_0)$, then $uv \in F_0$. we claim that there exists a vertex $w \in V(AQ_{n-1}^0 - F_0)$ such that at least one of uw or vw is an edge in the subgraph induced by $AQ_{n-1}^0 - F_0$. If not, then u and v are isolated in the subgraph induced by $AQ_{n-1}^0 - F_0$, hence $|F_0| \geq (2n-3) + (2n-3) - 3 = 4n - 9$, but $|F_0| < 4n - 9$. □

Before we give the proof of our main result, we would like to point out that given the recursive nature of this class of networks, induction is the natural method of proof. The proof considers cases based on the distribution of faults. One may feel that taking cases is not an elegant method. However, all the papers that we are familiar with in this area essentially use this method. One may point out that in [7, 15], results were given to show networks having certain matching preclusion properties by applying sufficient conditions. However, showing the networks satisfying such sufficient conditions typically involve induction with case analysis. We further note that organizing the cases is not an easy task as they have to be organized in such a way that the induction hypothesis and other conditions can apply appropriately (neither too strong or too weak).

Theorem 3.8. *Let n be a positive integer with $n \geq 4$. Then $smp_1(AQ_n) = 4n - 8$.*

Proof. We use proof by mathematical induction. The basis step is satisfied by Lemma 3.5. We assume that $smp_1(AQ_{n-1}) = 4n - 12$, and we want to show that $smp_1(AQ_n) = 4n - 8$. By Proposition 2.5 and Lemma 3.4, we can build a trivial conditional matching preclusion set of size $4n - 8$, thus $smp_1(AQ_n) \leq 4n - 8$. To show that $smp_1(AQ_n) \geq 4n - 8$, we let $F \subseteq V(AQ_n) \cup E(AQ_n)$, such that $|F| \leq 4n - 9$ and $AQ_n - F$ contains no isolated vertex, and we prove that $AQ_n - F$ contains a perfect or an almost-perfect matching. Let $F = F_X \cup F_C \cup F_0 \cup F_1$ where F_0 and F_1 denote the fault sets in AQ^0_{n-1} and AQ^1_{n-1} respectively, F_X is the set of faulty cross edges and F_C is the set of faulty complement edges. We may assume that $|F_0| \geq |F_1|$. We now divide the proof into cases depending on the distribution of faults.

Case 1 $|F_0| = 4n - 9$. Then all the faults are inside AQ^0_{n-1}. We can assume that F_0 contains vertices, since if not the problem becomes the same as the conditional matching preclusion problem of AQ_n discussed in [5]. Let $A \subseteq F_0$, such that $|A| = 4$ and the subgraph induced by the vertices of $AQ^0_{n-1} - (F_0 - A)$ has no isolated vertex. Such set can always be found because there are at most two isolated vertices in the subgraph induced by $AQ^0_{n-1} - F_0$, but these two vertices must be adjacent and/or share common neighbors in AQ^0_{n-1} because the degree of each vertex in AQ^0_{n-1} is $2n - 3$, and by Lemma 3.4 these two vertices must share at least two vertices in AQ^0_{n-1}, so in this case F_0 contains at least two vertices adjacent to these vertices. Let F^V_0 be the set of vertices in F_0 and F^E_0 be the set of edges in F_0. We consider two cases depending on the parity of $|F^V_0|$.

Case 1.1 $|F^V_0|$ *is even.* Then F_0 contains at least two vertices. We choose the set A such that the subgraph induced by $AQ^0_{n-1} - (F_0 - A)$ contains even number of vertices. It is possible to choose the set A such that A consists of two vertices and two edges or exactly four vertices. In fact, by Lemma 3.4, if we have two isolated vertices then they must share two faulty neighbors, and if we have at most one isolated vertex then we include one of its faulty adjacent vertices or faulty incident edges, thereafter we should be able to choose the other elements of A as needed. Let $F'_0 = F_0 - A$.

Case 1.1.1 *Assume that* $A = \{u, v, u_1v_1, u_2v_2\}$. By the induction hypothesis, there exists a perfect matching M_0 in $AQ^0_{n-1} - F'_0$. Let ux and vy be edges in M_0 saturating u and v, and assume that the edges u_1v_1, u_2v_2 are in M_0 as well. Note that this is the worst case scenario. We want to construct a perfect or an almost-perfect matching in $AQ_n - F$ that does not contain edges from A and does not saturate vertices in A. By Lemma 3.6, we can find $u'_1, u'_2, v'_1, v'_2, x', y' \in V(AQ^1_{n-1})$ adjacent to u_1, u_2, v_1, v_2, x, y respectively. Let $A' = \{u'_1, u'_2, v'_1, v'_2, x', y'\}$. $AQ^1_{n-1} - A'$ has no isolated vertex since the degree of every vertex in AQ^1_{n-1} is greater than 6, and $|A'| = 6 \leq 4n - 13$, for $n \geq 5$. By the induction hypothesis, there exists a perfect matching M_1 in the subgraph induced by $AQ^1_{n-1} - A'$. Then the set $(M_0 - \{u_1v_1, u_2v_2, ux, vy\}) \cup M_1 \cup \{u_1u'_1, v_1v'_1, u_2u'_2, v_2v'_2, xx', yy'\}$ is a perfect matching in $AQ_n - F$.

Case 1.1.2 *Assume that* $A = \{u, v, w, z\}$. By the induction hypothesis, there exists a perfect matching M_0 in $AQ^0_{n-1} - F'_0$. It is possible to have some vertices of A adjacent through edges of M_0. However, we will consider the worst case scenario where u, v, w and z are saturated by M_0 through the edges uu_0, vv_0, ww_0, and zz_0. Since there are no faults outside AQ^0_{n-1}, then by Lemma 3.6 can find neighbors for the vertices u_0, v_0, w_0 and z_0 in AQ^1_{n-1}. Let $u_1, v_1, w_1, z_1 \in V(AQ^1_n)$ be the neighbors of u_0, v_0, w_0 and z_0 respectively. Let $A' = \{u_1, v_1, w_1, z_1\}$, then by the induction hypothesis, the subgraph induced by $AQ^1_n - A'$ possesses a perfect matching M_1. Let $M = (M_0 - \{uu_0, vv_0, ww_0, zz_0\}) \cup M_1 \cup \{u_0u_1, v_0v_1, w_0w_1, z_0z_1\}$, then M is a perfect matching in $AQ_n - F$.

Case 1.2 $|F^V_0|$ *is odd.* We want to choose the set A such that the subgraph induced by $AQ^0_{n-1} - (F_0 - A)$ contains even number of vertices. A can consist of one vertex and three edges or three vertices and one edge. Let $F' = F_0 - A$, then $|F'^V|$ is even, $AQ^0_{n-1} - F'$ has no isolated vertices, and $|F'| = 2n - 13$. By the induction hypothesis, there exists a perfect matching M_0 in the subgraph induced by $AQ_n - F'_0$.

Case 1.2.1 *Assume that* $A = \{u, v, w, xy\}$, then u, v, w, x and y are saturated by M_0. Let $uu_0, vv_0, ww_0, xx_0, yy_0 \in M_0$. By Lemma 3.6, there exist vertices u_1, v_1, w_1, x_1, y_1 in AQ^1_{n-1} that are outside neighbors of u_0, v_0, w_0, x_0 and y_0 respectively. Let $A' = \{u_1, v_1, w_1, x_1, y_1\}$, the subgraph induced by $AQ^1_{n-1} - A'$ has no isolated vertices and contains an odd number of vertices, then by the induction hypothesis, there exists an almost-perfect matching M_1 in $AQ^1_{n-1} - A'$. Let $M = (M_0 - \{uu_0, vv_0, ww_0, xx_0, yy_0\}) \cup M_1 \cup \{u_0u_1, v_0v_1, w_0w_1, x_0x_1, y_0y_1\}$, then M is an almost-perfect matching in $AQ_n - F$.

Case 1.2.2 *Assume that* $A = \{u, vw, xy, zt\}$. Again, we assume the worst case scenario where the edges vw, xy, and zt are in M_0. Let uu_0 be the edge in M_0 that saturates u. By Lemma 3.6, there exist vertices $u_1, v_1, w_1, x_1, y_1, z_1, t_1 \in V(AQ^1_{n-1})$, such that $u_0u_1, vv_1, ww_1, xx_1, yy_1, zz_1$, and tt_1 are edges in AQ_n. Let $A' = \{u_1, v_1, w_1, x_1, y_1, z_1, t_1\}$. Note that $|A'| = 7$, which is equal to the degree of the subgraph induced by the vertices of AQ^1_n. However, following the proof of Lemma 3.6, the bipartite graph G constructed will have partitions A and $N(A)$ where $|A| = 7$ and $|N(A)| \geq 8$, so we can always choose the vertices of A' such that the subgraph induced by $AQ^1_{n-1} - A'$ has no isolated vertices. By the induction hypothesis, there exists an almost-perfect matching M_1 in $AQ^1_{n-1} - A'$. Then the set $M = (M_0 - \{uu_0, vw, xy, zt\}) \cup M_1 \cup \{u_0u_1, vv_1, ww_1, xx_1, yy_1, zz_1, tt_1\}$ is an almost-perfect matching in $AQ_n - F$.

Case 2 $|F_0| = 4n-10$. Then $|F_1 \cup F_C \cup F_X| \leq 1$. If $|F_1 \cup F_C \cup F_X| = 0$, then the argument in Case 1 applies.

Case 2.1 $F_1 \cup F_C \cup F_X$ *consists of an edge.* We consider cases depending on the parity of $|F^V_0|$.

Case 2.1.1 $|F^V_0|$ *is even.* Since we are assuming that F contains at least one vertex and $|F_0|$ is even so F_0 contains at least two vertices.

If F_0 contains edges, then let $A \subseteq F_0$ such that A consists of two vertices and one edge and $AQ^0_{n-1} - (F_0 - A)$ has no isolated vertices. Let $A = \{u, v, xy\}$, then by the induction hypothesis there exists a perfect matching M_0 in $AQ^0_{n-1} - (F_0 - A)$. Let uu_0, vv_0, and xy be edges of M_0. Note that we are considering the worst case scenario where the edge xy is in M_0 and the vertices u and v are saturated by M_0 through the edges uu_0 and vv_0 respectively. By Lemma 3.6, there exist outside vertices u_1, v_1, x_1 and y_1 in $AQ_n - F$ of u_0, v_0, x and y respectively, such that $u_0u_1, v_0v_1, xx_1, yy_1$ are edges in $AQ_n - F$. Let $A' = \{u_1, v_1, x_1, y_1, f\}$ where f is the faulty edge in $F_1 \cup F_C \cup F_X$. By the induction hypothesis, there exists a perfect matching M_1 in $AQ^1_{n-1} - A'$. Then the set $M = (M_0 - \{uu_0, vv_0, xy\}) \cup M_1 \cup \{u_0u_1, v_0v_1, xx_1, yy_1\}$ is a perfect matching in $AQ_n - F$.

If F_0 does not contain edges, then we let $A = \{u, v, w\}$. By the induction hypothesis, there exists an almost-perfect matching M_0 in the subgraph induced by $AQ^0_{n-1} - (F_0 - A)$. We consider the worst case scenario where the unsaturated vertex by M_0 is $x \notin A$. Let uu_0, vv_0 and ww_0 be the edges of M_0 saturating u, v, and w respectively. By Lemma 3.6 there exist $u_1, v_1, w_1, x_1 \in V(AQ^1_{n-1})$ such that u_0u_1, v_0v_1, w_0w_1, and xx_1 are edges in $AQ_n - F$. Let $A' = \{u_1, v_1, w_1, x_1\}$, then by the induction hypothesis there exists a perfect matching M_1 in $AQ^1_{n-1} - A'$. Then the set $M = (M_0 - \{uu_0, vv_0, ww_0\}) \cup M_1 \cup \{u_0u_1, v_0v_1, w_0w_1, xx_1\}$ is a perfect matching in $AQ_n - F$.

Case 2.1.2 $|F^V_0|$ *is odd.* Then $|F^E_0| \geq 1$ and $|F^V_0| \geq 1$, so we can choose A to be a set of two vertices and one edge or two edges and one vertex.

Assume that $A = \{u, v, xy\}$. By the induction hypothesis, there exists an almost-perfect matching M_0 in $AQ^0_{n-1} - (F - A)$. Consider the worst case scenario where $xy, uu_0, vv_0 \in M_0$ and u_0 and v_0 are vertices in the subgraph induced by $AQ^0_{n-1} - (F - A)$. By Lemma 3.6, there exist vertices u_1, v_1, x_1 and y_1 in AQ^1_{n-1} such that $u_0u_1, v_0v_1, xx_1, yy_1$ are edges in $AQ_n - F$. Let $A' = \{u_1, v_1, x_1, y_1\}$, then by the induction hypothesis, there exists a perfect matching M_1 in the subgraph induced by $AQ^1_{n-1} - A'$. Then the set $M = (M_0 - \{uu_0, vv_0, xy\}) \cup M_1 \cup \{u_0u_1, v_0v_1, xx_1, yy_1\}$ is an almost-perfect matching in $AQ_n - F$.

Assume that $A = \{u, vw, xy\}$. By the induction hypothesis, there exists a perfect matching M_0 in $AQ^0_{n-1} - (F - A)$. Consider the worst case scenario where $xy, vw, uu_0 \in M_0$ and u_0 is a vertex in the subgraph induced by $AQ^0_{n-1} - (F - A)$. By Lemma 3.6, there exist vertices u_1, v_1, w_1, x_1 and y_1 in AQ^1_{n-1} such that $u_0u_1, vv_1, ww_1, xx_1, yy_1$ are edges in $AQ_n - F$. Let $A' = \{u_1, v_1, w_1, x_1, y_1\}$, then by the induction hypothesis, there exists an almost-perfect matching M_1 in the subgraph induced by $AQ^1_{n-1} - A'$. Then the set $M = (M_0 - \{uu_0, vw, xy\}) \cup M_1 \cup \{u_0u_1, vv_1, ww_1, xx_1, yy_1\}$ is an almost-perfect matching in $AQ_n - F$.

Case 2.2 *$F_1 \cup F_C \cup F_X$ consists of a vertex.* Let z be this vertex, then z is in AQ^1_{n-1}. We want to choose the set A so that $|A| = 3$, and the subgraph induced by $AQ^0_{n-1} - (F - A)$ has no isolated vertices.

Case 2.2.1 *$|F_0^V|$ is even.* F_0 contains no vertices or at least two vertices, then we choose A such that A contains two vertices and one edge, three edges or three vertices.

Assume that $A = \{u, v, xy\}$. By the induction hypothesis, there exists a perfect matching M_0 in $AQ^0_{n-1} - (F - A)$. Suppose that $xy, uu_0, vv_0 \in M_0$. If we can find outside neighbors for x, y, u_0 and v_0 in $AQ^1_{n-1} - \{z\}$, then we find the desired matching M as we did in the previous cases. Suppose that we can not find such neighbors. This means that two of the four vertices x, y, u_0 and v_0 are adjacent to z. Without loss of generality, assume that u_0 and v_0 are these two vertices. By Lemma 3.2, u_0 and v_0 must be adjacent in AQ^0_{n-1}. If $u_0v_0 \notin F_0$, then we can add u_0v_0 to the matching we are looking for, if not then by Lemma 3.7 there exists a vertex a in $AQ^0_{n-1} - F$ that is adjacent to u_0 or v_0. Suppose that u_0 is adjacent to a and a is saturated by the matching M_0. Let $ab \in M_0$, then b has outside neighbor $b_1 \in AQ^1_{n-1} - \{z\}$. Let x_1, y_1, v_1 be the outside neighbors of x, y, and v respectively in $AQ^1_{n-1} - \{z, b_1\}$, and let $A' = \{x_1, y_1, v_1, b_1, z\}$. By the induction hypothesis, there exists an almost-perfect matching M_1 in the subgraph induced by $AQ^1_{n-1} - A'$. Let $M'_0 = (M_0 - \{ab\}) \cup \{u_0a\}$, then $M = M'_0 \cup M_1 \cup \{v_0v_1, xx_1, yy_1\}$ is an almost-perfect matching in $AQ_n - F$.

Assume that $A = \{ab, cd, ef\}$. By the induction hypothesis, there exists a perfect matching M_0 in $AQ^0_{n-1} - (F - A)$. Suppose that $ab, cd, ef \in M_0$, note that this is the worst case scenario. Suppose that we can find vertices $a_1, b_1, c_1, d_1, e_1, f_1$ in $AQ^1_{n-1} - \{z\}$ such that $aa_1, bb_1, cc_1, dd_1, ee_1, ff_1 \in E(AQ_n - F)$. Let $A' = \{a_1, b_1, c_1, d_1, e_1, f_1, z\}$, then by the induction hypothesis, there exists an almost-perfect matching M_1 in the subgraph induced by $AQ^1_{n-1} - A'$. Let $M'_0 = M_0 - \{ab, cd, ef\}$, then $M = M'_0 \cup M_1 \cup \{aa_1, bb_1, cc_1, dd_1, ee_1, ff_1\}$ is an almost perfect matching in $AQ_n - F$. If we can not find the vertices $a_1, b_1, c_1, d_1, e_1, f_1$ as defined above, then two of these vertices will be adjacent to z, but in this case these two vertices must be adjacent and if the edge connecting them is not in $AQ_n - F$, then by Lemma 3.7 we can find a vertex adjacent to one of them and we repeat the construction done above.

Assume that $A = \{u, v, w\}$. By the induction hypothesis, there exists an almost-perfect matching M_0 in $AQ^0_{n-1} - (F - A)$. We consider the worst case scenario where the unsaturated vertex is a vertex different than u, v and w. Let $s \in V(AQ^0_{n-1} - F)$ be the unsaturated vertex by M_0 and suppose that $uu_0, vv_0, ww_0 \in M_0$. If we can find outside neighbors u_1, v_1 and w_1 for u_0, v_0 and w_0 in $AQ^1_{n-1} - \{z\}$, then we can find the desired matching M as we did in the previous cases. Suppose that we can not find such neighbors. This means that two of the three vertices u_0, v_0 and w_0 are adjacent to z. Without loss of generality, assume that u_0 and v_0 are these two vertices. In addition u_0 and v_0 are adjacent to another vertex $t \in V(AQ^1_{n-1} - \{z\})$. By Lemma 3.2, u_0 and v_0 must be adjacent in AQ^0_{n-1}. If $u_0v_0 \notin F_0$, then we can add u_0v_0 to the matching we are looking for. If not then by Lemma 3.7 there exists a vertex a in $AQ^0_{n-1} - F$ that is adjacent to u_0 or v_0. Without loss of generality, suppose that $u_0a \in E(AQ^0_{n-1} - F)$. *Assume that* $a \neq s$, then a is saturated by M_0. Let $ab \in M_0$, so b has outside neighbor b_1 in $AQ^1_{n-1} - \{z\}$. Let $A' = \{z, t, w_1, b_1\}$, then by the induction hypothesis, there exists a perfect matching M_1 in $AQ^1_{n-1} - A'$. Let $M'_0 = M_0 - \{uu_0, vv0, ww_0, ab\}$, then $M = M'_0 \cup M_1 \cup \{u_0a, v_0t, bb_1, w_0w_1\}$ is an almost-perfect matching in $AQ_n - F$. *Assume that* $a = s$. Let $A'' = \{z, t, w_1\}$, then by the induction hypothesis, there exists an almost-perfect matching M_1 in $AQ^1_{n-1} - A''$. Let $M'_0 = M_0 - \{uu_0, vv_0, ww_0\}$, then $M = M'_0 \cup M_1 \cup \{u_0s, v_0t, w_0w_1\}$ is an almost-perfect matching in $AQ_n - F$.

Case 2.2.2 $|F^V_0|$ *is odd.* Note that F_0 can not contain $4n - 10$ edges nor $4n - 10$ vertices, then we can choose A such that A contains a vertex and two edges or two vertices and an edge.

Assume that $A = \{u, v, xy\}$. By the induction hypothesis, there exists an almost-perfect matching M_0 in $AQ^0_{n-1} - (F - A)$. Suppose that $xy, uu_0, vv_0 \in M_0$. Let $a_0a_1 \in E(AQ_n - F)$ such that $a_0 \in V(AQ^0_{n-1})$, $a_1 \in V(AQ^1_{n-1})$ and a_1 is not adjacent to any of u_0, v_0, x and y. Suppose that we can find u_1, v_1, x_1 and y_1 in $AQ^1_{n-1} - \{z, a_1\}$, such that u_0u_1, v_0v_1, xx_1 and yy_1 are edges in $AQ_n - F$. Let $A' = \{u_1, v_1, x_1, y_1, a_1, z\}$, then by the induction hypothesis, there exists a perfect matching M_1 in the subgraph induced by $AQ^1_{n-1} - A'$. The set $M = (M_0 - \{xy, uu_0, vv_0\}) \cup M_1 \cup \{u_0u_1, v_0v_1, xx_1, yy_1, a_0a_1\}$ is a perfect matching in $AQ_n - F$. If we can not find u_1, v_1, x_1 and y_1 in $AQ^1_{n-1} - \{z, a_1\}$ as above, and since we chose a_1 such that it is not adjacent to any of these four vertices, then two of the four vertices, u_0, v_0, x, y, must be adjacent to z. Without loss of generality, assume that u_0 and v_0 are adjacent to z. Then by Lemma 3.7, u_0v_0 is an edge in $AQ^0_{n-1} - F_0$ and we can add it to the matching we are looking to construct, or at least one of u_0 and v_0 is adjacent to some vertex w in $AQ^0_{n-1} - F_0$. Without loss of generality, suppose that u_0w is an edge in $AQ^0_{n-1} - F_0$. The vertex w is saturated by M_0, let $ww_0 \in M_0$ where $w_0 \in V(AQ^0_{n-1} - F_0)$. w_0 has two outside neighbors, and at least one of them, say w_1, is different that z and a_1. Let $A' = \{w_1, v_1, x_1, y_1, a_1, z\}$, then by the induction hypothesis, then there exists a perfect matching M_1 in the subgraph induced by $AQ^1_{n-1} - A'$. Let $M'_0 = (M_0 - \{ww_0\}) \cup \{u_0w\}$. Then the set $M = M_0' \cup M_1 \cup \{w_0w_1, v_0v_1, xx_1, yy_1, a_0a_1\}$ is a perfect matching in $AQ_n - F$.

Assume that $A = \{u, vw, xy\}$. By the induction hypothesis, there exists a perfect matching M_0 in $AQ_{n-1}^0 - (F - A)$. Consider the worst case scenario where $xy, vw, uu_0 \in M_0$ and u_0 is a vertex in the subgraph induced by $AQ_{n-1}^0 - (F - A)$. Suppose that we can find u_1, v_1, w_1, x_1 and y_1 in $AQ_{n-1}^1 - \{z\}$, such that $u_0u_1, v_0v_1, w_0w_1, xx_1$ and yy_1 are edges in $AQ_n - F$. Let $A' = \{u_1, v_1, w_1x_1, y_1, z\}$. Then by the induction hypothesis, there exists a perfect matching M_1 in the subgraph induced by $AQ_{n-1}^1 - A'$. The set $M = (M_0 - \{xy, vw, uu_0\}) \cup M_1 \cup \{u_0u_1, vv_1, ww_1, xx_1, yy_1\}$ is a perfect matching in $AQ_n - F$. If we can not find the vertices u_1, v_1, w_1, x_1 and y_1 in $AQ_{n-1}^1 - \{z\}$ as described above, then two of the five vertices u_0, v, w, x and y are adjacent to z. Without loss of generality, suppose that v and x are adjacent to z, then by Lemma 3.1 $vx \in E(AQ_{n-1}^0)$. If $vx \notin F$, then we can add it to the matching. Assume that $vx \in F$, then by Lemma 3.7, there exists a vertex $a \in V(AQ_{n-1}^0 - F)$ such that at least one of the vertices, v and x, is adjacent to a. Without loss of generality, assume that va is an edge in the subgraph induced by $AQ_{n-1}^0 - F$. The vertex a is saturated by M_0. Suppose that $ab \in M_0$, then b has two outside neighbors in $AQ_{n-1}^1 - \{z\}$. Hence, we can find vertices u_1, w_1, x_1, y_1 and b_1 in $AQ_{n-1}^1 - \{z\}$, such that $u_0u_1, ww_1, xx_1, yy_1, bb_1$ are independent edges in $AQ_n - F$. Let $A_1 = \{u_1, w_1, x_1, y_1, b_1, z\}$, then by the induction hypothesis, there exists a perfect matching M_1 in the subgraph induced by $AQ_{n-1}^1 - A_1$. Then the set $M = (M_0 - \{xy, vw, uu_0, ab\}) \cup \{va, bb_1, u_0u_1, ww_1, xx_1, yy_1\} \cup M_1$ is a perfect matching in $AQ_n - F$.

Case 3 $|F_0| = 4n - 11$. Then $|F_1 \cup F_C \cup F_X| \leq 2$.

Case 3.1 $|F_0^V|$ *is even.* We want to choose a set $A \subseteq F_0$ such that the subgraph induced by $AQ_{n-1}^0 - (F_0 - A)$ has no isolated vertices and even number of vertices. The choice of such set A is always possible since we can choose A to be a set of two vertices or a set of two edges.

Case 3.1.1 *If* $A = \{u, v\}$, then by the induction hypothesis, there exists a perfect matching M_0 in $AQ_{n-1}^0 - (F_0 - A)$ saturating u and v. Let uu_0 and vv_0 be in M_0. If there exist two vertices u_1 and v_1 in $AQ_{n-1}^1 - F$ such that u_0u_1 and v_0v_1 are edges in $AQ_n - F$, then we let $A' = \{u_1, v_1, f_1, f_2\}$ where f_1 and f_2 are the two faults outside F_0. By the induction hypothesis, there exists a perfect or an almost-perfect matching M_1 in the subgraph induced by $AQ_{n-1}^1 - A'$. Then the set $M = (M_0 - \{uu_0, vv_0\}) \cup M_1 \cup \{u_0u_1, v_0v_1\}$ is a perfect or an almost-perfect matching in $AQ_n - F$.

Assume that one of u_0 and v_0 is adjacent (incident) to two faulty vertices (edges) outside AQ_{n-1}^0. Without loss of generality, suppose that u_0 is such vertex. Since u_0 can not be isolated in $AQ_n - F$, then u_0 must have a neighbor w in $AQ_{n-1}^0 - F$. w is saturated by M_0. Let $ww_0 \in M_0$, where $w_0 \in V(AQ_n - F)$. If f_1 and f_2 are two edges incident to u_0, then w_0 has two neighbors in $AQ^1_{n-1} - F$. Then we can find two edges w_0w_1 and v_0v_1 in $AQ_n - F$, such that w_1 and v_1 are in $AQ^1_{n-1} - F$. Now we choose the set A' to be $A' = \{w_1, v_1, f_1, f_2\}$, and by the induction hypothesis there exists a perfect or an almost-perfect matching M_1 in the subgraph induced by $AQ^1_{n-1} - A'$. Then the set $M = (M_0 - \{ww_0, uu_0, vv_0\}) \cup M_1 \cup \{u_0w, w_0w_1, v_0v_1\}$ is a perfect or an almost-perfect matching in $AQ_n - F$.

We consider the last possibility where f_1 and f_2 are two vertices, and u_0 and v_0 are both adjacent to f_1 and f_2. By Lemma 3.1, u_0 and v_0 are adjacent. If $u_0v_0 \in E(AQ_n - F)$, then we add this edge to the matching. By the induction hypothesis, we can find a perfect or an almost-perfect matching M_1 in $AQ^1_{n-1} - F$. Therefore, $M = (M_0 - \{uu_0, vv_0\}) \cup M_1 \cup \{u_0v_0\}$ is a perfect or an almost-perfect matching in $AQ_n - F$. If $u_0v_0 \in F$, then we can claim that u_0 is adjacent to t and v_0 is adjacent to z where $z, t \in V(AQ^0_{n-1} - F)$. In fact, if this is not true, then u_0 and v_0 are adjacent to exactly one vertex, then F_0 contains at least $4n - 9$ faults and this is not possible. The vertices z and t are saturated by M_0. Let zz_0 and tt_0 be in M_0. The vertices z_0 and t_0 have their outside neighbors in $AQ^1_{n-1} - F$. Let z_1 and t_1 be outside neighbors of z_0 and t_0 respectively. Let $A' = \{z_1, t_1, f_1, f_2\}$ By the induction hypothesis, there exists a perfect matching in the subgraph induced by $AQ_n - A'$. Then $M = (M_0 - \{uu_0, vv_0, zz_0, tt_0\}) \cup M_1 \cup \{z_0z_1, t_0t_1\}$ is a perfect matching in $AQ_n - F$.

Case 3.1.2 *If* $A = \{uv, xy\}$. In this case all the faults inside AQ^0_{n-1} are edges, then we can always choose the edges uv and xy such that at least one endpoint of each edge has two outside neighbors in $AQ^1_{n-1} - F$. Suppose that v and y are the endpoints adjacent to v' and y' in $AQ^1_{n-1} - F$ and that vv' and yy' are edges in $AQ_n - F$, then we can include the edges vv' and yy' in the matching M to saturate the vertices v and y. The only problem we may have is when we fail to find independent cross/complement edges in $AQ_n - F$ incident to u and x respectively. This occurs when both or one of them is incident to two faulty cross/complement edges or adjacent to two faulty vertices in AQ^1_{n-1}. This situation is similar to what we had in Case 3.1.1. We can proceed as in Case 3.1.1, but the set A' we choose here will be of size 6 instead of 4, and the induction hypothesis will be applied as well.

Case 3.2 $|F^V_0|$ *is odd.* We can always choose the set A so that $AQ^0_{n-1} - (F_0 - A)$ has no isolated vertices and A contains two vertices or a vertex and an edge.

Case 3.2.1 *If* $A = \{u, v\}$, then by the induction hypothesis, there exists an almost-perfect matching M_0 in $AQ^0_{n-1} - (F_0 - A)$. We consider the worst case scenario where u and v are saturated by M_0. Let uu_0 and vv_0 be in M_0, and let $z \in V(AQ^0_{n-1} - (F_0 - A))$ be the unsaturated vertex by M_0. Assume that we can find $u_1, v_1, z_1 \in V(AQ^1_{n-1} - F)$ such that $u_0u_1, v_0v_1, zz_1 \in E(AQ_n - F)$, and let $A' = \{f_1, f_2, u_1, z_1, v_1\}$ where f_1 and f_2 are the faults outside AQ^0_{n-1}. By the induction hypothesis, there exists a perfect or an almost-perfect matching M_1 in the subgraph induced by $AQ^1_{n-1} - A'$. The set $M = (M_0 - \{uu_0, vv_0\}) \cup M_1 \cup \{u_0u_1, v_0v_1, zz_1\}$ is a perfect or an almost-perfect

DOI: 10.20429/tag.2021.080105

matching in $AQ_n - F$. Assume that we can not find the vertices u_1, v_1, z_1 as described above, then either one or two of the vertices from $\{u_0, v_0, z\}$ is adjacent (or incident) to two faulty vertices (or faulty edges). This situation is similar to the one we had in Case 3.1.1. Using a similar construction we can find the perfect or the almost-perfect matching we are trying to find.

Case 3.2.2 *If* $A = \{u, xy\}$, then by the induction hypothesis, there exists a perfect matching M_0. We consider the worst case scenario where $xy \in M_0$. Let $M_0' = \{xy, uu_0\}$, so M_0' is a matching in $AQ^0_{n-1} - F$ missing three vertices. This situation is similar to the previous case when M_0 was missing three vertices, namely u_0, v_0, w. Hence we can proceed as in Case 3.1.1 to find a perfect or an almost-perfect matching in $AQ_n - F$.

Case 4 $|F_0| = 4n - 12$. Then $|F_1 \cup F_C \cup F_X| \leq 3$. We consider two cases depending on the parity of $|F_0^V|$.

Case 4.1 *Assume* $|F_0^V|$ *is even.* Let $A \subseteq F$ such that $|A| = 1$. We can always choose A so that $AQ^0_{n-1} - (F - A)$ has no isolated vertices. In fact, if we have an isolated vertex, then we choose A to be a vertex (edge) adjacent (incident) to that vertex, and if we have two isolated vertices, then they should share at least one vertex and/or adjacent, so we can choose A to be the edge connecting them or a common neighbor for both vertices. We consider two cases depending on whether A is a set containing a vertex or an edge.

Case 4.1.1 *If* $A = \{u\}$, then by the induction hypothesis, there exists an almost-perfect matching M_0 in $AQ^0_{n-1} - (F - A)$. Assume the worst case scenario, which is the case when u is saturated by M_0. Let $uu_0 \in M_0$ and let z be the vertex in $AQ^0_{n-1} - (F - A)$ missed by M_0. Suppose that we can find vertices u_1 and z_1, outside neighbors of u_0 and z respectively, in $AQ^1_{n-1} - F$. By The induction hypothesis, there exists a perfect or an almost perfect-matching M_1 in the subgraph induced by $AQ^1_{n-1} - (F \cup \{u_1, z_1\})$. The set $M = (M_0 - \{uu_0\}) \cup M_1 \cup \{u_0u_1, zz_1\}$ is a perfect or an almost-perfect matching in $AQ_n - F$. If we can not find such u_1 and z_1, then either one of the vertices u_0 or z is adjacent to two faulty vertices in AQ^1_{n-1}, or one of them is incident to two faulty edges in $F_C \cup F_X$, or both vertices are adjacent to two faulty vertices in AQ^1_{n-1}.

Case 4.1.1(a) *Suppose that one of the vertices* u_0 *and* z*, say* u_0*, is adjacent to two faulty vertices or incident to two faulty edges.* Then u_0 must be adjacent to some vertex in $AQ^0_{n-1} - F$. If z is such vertex then we include the edge u_0z in the matching we are looking to find. If z is not adjacent to u_0, then u_0 is adjacent to some vertex u_0' that is saturated by M_0. Let tu_0' be the edge of M_0 saturating u_0'. If we can find an outside neighbor for t in $AQ^1_{n-1} - (F \cup \{z_1\})$ where z_1 is an outside neighbor of z in $AQ^1_{n-1} - F$, then we can proceed as above. If this is not the case, then t is either adjacent to the two vertices in AQ^1_{n-1} that u_0 is adjacent to, or t and z share their two outside neighbors in AQ^1_{n-1} and one of them is a vertex in F_1.

- *If t and u_0 are both adjacent to two faulty vertices in AQ^1_{n-1}*, then by Lemma 3.1 u_0 and t are adjacent in AQ_n. If the edge u_0t is not in F, then we can add this edge to the matching and proceed as above. If not, then we claim that the vertices u_0 and t must be adjacent to at least three vertices in $AQ^0_{n-1}-F$. In fact, by Lemma 3.3 u_0 and t can have at most four common neighbors in AQ_n, two of them are in AQ^1_{n-1}, then they have at most two common neighbors in AQ^0_{n-1}, and if u_0 and t are adjacent to less than three vertices in $AQ^0_{n-1}-F$, then $|F_0|$ must contain at least $2(2n-6)+1 = 4n-11$ elements which is not possible. Therefore, there are two vertices a and b adjacent to u_0 and/or t other than u'_0. Let aa' and bb' be the edges of M_0 saturating a and b. At least one of the vertices a' and b' has two outside neighbors in $AQ^1_{n-1} - F$. Without loss of generality, we can assume that a' has two outside neighbors in $AQ^1_{n-1} - F$. Let a_1 be one of these neighbors. By the induction hypothesis, there exists a perfect or an almost-perfect matching M_1 in the subgraph induced by $AQ^1_{n-1} - (F_1 \cup \{a_1, z_1\})$. If $u_0a \in E(AQ^0_{n-1} - F)$, then the set $M = (M_0-\{uu_0, aa'\})\cup M_1\cup\{u_0a, a'a_1, zz_1\}$ is a perfect or an almost-perfect matching in AQ_n-F. If $ta \in E(AQ^0_{n-1}-F)$, then set $M = (M_0 - \{uu_0, aa', tu'_0\}) \cup M_1 \cup \{ta, u_0u'_0, a'a_1, zz_1\}$ is a perfect or an almost-perfect matching in $AQ_n - F$.
- *If t and z are both adjacent to a faulty vertex in AQ^1_{n-1}*, then by Lemma 3.1 z and t are adjacent in AQ_n. If $zt \notin F_0$, then $M'_0 = (M_0 - \{uu_0\}) \cup \{uu'_0, zt\}$ is a perfect matching in $AQ^0_{n-1} - F$. By the induction hypotheses, there exists a perfect or an almost-perfect matching M_1 in $AQ^1_{n-1} - F$, then $M'_0 \cup M_1$ is a perfect or an almost perfect matching in $AQ_n - F$. If $tz \in F_0$. As we did above, we can find three vertices in $AQ^0_{n-1} - F$ that are adjacent to t or z. If not, then t and z will be adjacent/incident to at least $(2n-6)+(2n-7)+3 = 4n-10$ faults in AQ^0_{n-1} and this is not possible because $|F_0| = 4n-12$. So we can proceed as above to find the desired matching.

Case 4.1.1(b) *Suppose that the two vertices u_0 and z have two common outside neighbors in F_1.* Since there is no isolated vertex in $AQ_n - F$, then z has neighbors in $AQ^0_{n-1} - F$. If $u_0z \in E(AQ_n - F)$, then $M'_0 = (M_0 - \{uu_0\}) \cup \{u_0z\}$ is a perfect matching in AQ^0_{n-1}. By The induction there is a perfect or an almost-perfect matching in $AQ^1_{n-1} - F$. Therefore $M'_0 \cup M_1$ is a perfect or an almost-perfect matching in $AQ_n - F$. If $zu_0 \in F_0$, then we can claim, as we did previously, that z and/or u_0 are adjacent to at least three vertices in $AQ^0_{n-1} - F$. Note that since there is no isolated vertex in $AQ_n - F$, then each u_0 and z has a neighbor in $AQ^0_{n-1} - F$, and in this case there must be an additional neighbor for u_0 and z, or else the size of F_0 will be $4n-11$. In case u_0 and z share a common non-faulty neighbor, then we can find at least two additional neighbors for u_0 and z, or else the size of F_0 will be $4n-10$. Then we can proceed as in the previous case to find the desired matching.

Case 4.1.2 *If* $A = \{xy\}$, then by the induction hypothesis, there exists a perfect matching M_0 in $AQ^0_{n-1} - (F - A)$. Assume the worst case scenario, which is the case where $xy \in M_0$. We should note that the only case we are forced to choose A to be an edge is when all faults in AQ^0_{n-1} are edges or when there is one vertex incident to $(2n-3)$ faulty edges in AQ^0_{n-1}. In either cases, we can always find an edge xy such that at least one endpoint has no faulty outside neighbors. Let y be this endpoint.

If x is incident to a cross or a complement edge in $AQ_n - F$, then there is a vertex $x_1 \in AQ^1_{n-1} - F_1$ such that $xx_1 \in E(AQ_n - F)$. Let y_1 be an outside neighbor of y such that $x_1 \neq y_1$. By the induction hypothesis, there exists a perfect or an almost-perfect matching M_1 in the subgraph induced by $AQ^1_{n-1} - (F_1 \cup \{x_1, y_1\})$. Then the set $(M_0 - \{xy\}) \cup M_1 \cup \{xx_1, yy_1\}$ is a perfect or an almost-perfect matching in $AQ_n - F$.

If x is incident and/or adjacent to two faults outside AQ^0_{n-1}, then x must be adjacent to some vertex $a \in AQ^0_{n-1} - F$. The vertex a is saturated by M_0 through an edge ab. If b has an outside neighbor b_1 in $AQ^1_{n-1} - F$, then by the induction hypothesis, there exists a perfect or an almost-perfect matching M_1 in the subgraph induced by $AQ^1_{n-1} - (F_1 \cup \{b_1, y_1\})$. The set $(M_0 - \{xy\}) \cup M_1 \cup \{xa, bb_1, yy_1\}$ is a perfect or an almost-perfect matching in $AQ_n - F$. If we can not find such vertex b_1, then b and x are both adjacent to two faulty vertices in F_1. By Lemma 3.1, bx is an edge of AQ^1_{n-1}. Assume that $bx \notin F$, the vertex a has at least one outside neighbor a_1 in $AQ^1_{n-1} - F$ such that $aa_1 \in E(AQ_n - F)$. By the induction hypothesis, there exists a perfect or an almost-perfect matching M_1 in the subgraph induced by $AQ^1_{n-1} - (F_1 \cup \{a_1, y_1\})$. The set $(M_0 - \{xy\}) \cup M_1 \cup \{xb, aa_1, yy_1\}$ is a perfect or an almost-perfect matching in $AQ_n - F$. Finally, assume that $bx \in F$, then we claim that b and/or x have at least three neighbors (including a) in AQ^0_{n-1}. This is very similar to Case 4.1.1; we follow the same construction to find the matching in $AQ_n - F$.

Case 4.2 *Assume* $|F^V_0|$ *is odd.*

Case 4.2.1 *If* $A = \{u\}$, then by the induction hypothesis, there exists a perfect matching M_0 in $AQ^0_{n-1} - (F - A)$, then u is saturated by M_0. Let $uu_0 \in M_0$. The vertex u_0 has a neighbor $a \in V(AQ^0_{n-1} - F)$ which is also saturated by M_0 through the edge ab. From here, we can proceed with the construction of the Matching as we did in Case 4.1, by considering cases depending on whether b is incident to complement/cross edges in $E(AQ_n - F)$.

Case 4.2.2 *If* $A = \{xy\}$, then by the induction hypothesis, there exists an almost-perfect matching M_0 in $AQ^0_{n-1} - (F - A)$. Assume the worst case scenario, which is the case when xy is in M_0. Let $z \in V(AQ^0_{n-1} - F)$ be the missing vertex by M_0. If we can find outside neighbors x_1, y_1 and z_1 in $AQ^1_{n-1} - F$ for x, y and z, then by the induction hypothesis we can find a perfect or almost-perfect matching M_1 in $AQ^1_{n-1} - (F_1 \cup \{x_1, y_1, z_1\})$. Hence, $M = (M_0 - \{uu_0\}) \cup M_1 \cup \{xx_1, yy_1, zz_1\}$. If we can not find such vertices, then two of the vertices x, y and z have two common neighbors in F_1.

Since we can always choose the edge xy such that one endpoint, say y, is incident to two non-faulty cross and complement edges in $AQ_n - F$, then we can assume that x and z are both adjacent to faulty vertices in AQ^1_{n-1}. By Lemma 3.1, $xz \in E(AQ_n)$. If $xz \notin F$, then as we did in Case 4.1, we can find at least three vertices a, b and c that are adjacent to x or z in $AQ^0_{n-1} - F$. Note that at least one of them is adjacent to x and one is adjacent to y. These vertices a, b, and c are saturated by M_0. Let aa_0, bb_0, cc_0, be the edges saturating a, b and c respectively. It is possible that two of the vertices a and b are adjacent and the edge $ab \in M_0$, but this will not affect the proof. We can choose two vertices from $\{a, b, c\}$, say a and b, such that $xa, zb \in E(AQ^0_{n-1} - F)$ and a_0 and b_0 have outside neighbors a_1 and b_1, respectively, in $AQ^1_{n-1} - F$. Let y_1 be one of the outside neighbors of y. By the induction hypothesis, there exists a perfect or an almost-perfect matching M_1 in the subgraph induced by $AQ^1_{n-1} - (F_1 \cup \{a_1, b_1, y_1\})$. The set $M = (M_0 - \{xy, aa_1, bb_1\}) \cup M_1 \cup \{xa, zb, a_0a_1, b_0b_1, yy_1\}$ is a perfect or an almost-perfect matching in $AQ_n - F$.

Case 5 $2n - 2 \leq |F_0| \leq 4n - 13$. We consider two cases depending on whether $AQ^0_{n-1} - F$ contains an isolated vertex or not. Note that when $|F_0| \leq 4n - 13$ then $AQ^0_{n-1} - F$ can have at most one isolated vertex.

Case 5.1 *Assume that $AQ^0_{n-1} - F$ contains an isolated vertex u.* Since $AQ_n - F$ has no isolated vertices, then u has an outside neighbor u_1 in $AQ^1_{n-1} - F$.

Case 5.1.1 *Suppose that u is adjacent to a faulty vertex v in F_0.* Let $F'_0 = F_0 - \{v\}$, then $|F'_0| \leq 4n - 14$ and the subgraph induced by $AQ^0_{n-1} - F'_0$ has no isolated vertices. By the induction hypothesis, there exists a perfect or an almost-perfect matching M_0 in $AQ^0_{n-1} - F'_0$.

If $|F^V_0|$ is odd, then M_0 is a perfect matching and $uv \in M_0$. Let $F'_1 = F_1 \cup \{u_1\}$, then $|F'_1| \leq 2n - 7 \leq 4n - 13$ for $n \geq 4$. In addition, $AQ^1_{n-1} - F'_1$ has no isolated vertices, then by the induction hypothesis, there exists a perfect or an almost-perfect matching M_1 in the subgraph induced by $AQ^1_{n-1} - F'_1$. Let $M = (M_0 - \{uv\}) \cup M_1 \cup \{uu_1\}$, then M is a perfect or an almost-perfect matching in $AQ_n - F$.

If $|F^V_0|$ is even, let z_0 be a vertex in $AQ^0_{n-1} - F$ such that z_0 has an outside neighbor $z_1 \neq u_1$ in $AQ^1_{n-1} - F$ and the subgraph induced by $AQ^0_{n-1} - (F'_0 \cup \{z_0\})$ has no isolated vertices. Let $F''_0 = F'_0 \cup \{z_0\}$, then $|F''_0| \leq 4n - 13$. Hence, by the induction hypothesis, there exists a perfect matching M'_0 in the subgraph induced by $AQ^0_{n-1} - F'_0$. Let $F'_1 = F_1 \cup \{u_1, z_1\}$, then $|F'_1| \leq 2n - 6 \leq 4n - 13$ for $n \geq 4$. In addition, $AQ^1_{n-1} - F'_1$ has no isolated vertices, then by the induction hypothesis, there exists a perfect or an almost-perfect matching M_1 in the subgraph induced by $AQ^1_{n-1} - F'_1$. Let $M = (M'_0 - \{uv\}) \cup M_1 \cup \{uu_1, z_0z_1\}$, then M is a perfect or an almost-perfect matching in $AQ_n - F$.

Case 5.1.2 *Suppose that u is not adjacent to a faulty vertex*, then u is incident to $2n - 3$ faulty edges in AQ^0_{n-1}. Let $f_1, f_2, \ldots, f_{2n-3}$ be these edges, and let

$F_0' = \{u\} \cup (F_0 - \{f_1, \ldots, f_{2n-3}\})$. $|F_0'| \leq 2n - 9$.
Assume F_0' has even number of vertices. By Theorem 3.1, $AQ^0_{n-1} - F_0'$ has a perfect matching M_0'. Let $F_1' = F_1 \cup \{u_1\}$, then $|F_1'| \leq 2n - 7 \leq 4n - 13$ for $n \geq 4$. In addition, $AQ^1_{n-1} - F_1'$ has no isolated vertices, then by the induction hypothesis, there exists a perfect or an almost-perfect matching M_1 in the subgraph induced by $AQ^1_{n-1} - F_1'$. Let $M = M_0 \cup M_1 \cup \{uu_1\}$, then M is a perfect or an almost-perfect matching in $AQ_n - F$.
Assume that F_0' has odd number of vertices. We can find a vertex z_0 in $AQ^0_{n-1} - F_0'$ such that z_0 has an outside neighbor $z_1 \neq u_1$ in $AQ^1_{n-1} - F$ and the subgraph induced by $AQ^0_{n-1} - (F_0' \cup \{z_0\})$ has no isolated vertices. Let $F_0'' = F_0 \cup \{z_0\}$, $|F_0''| \leq 2n - 8$ and has an even number of vertices. By Theorem 3.1, there exists a perfect matching M_0' in the subgraph induced by $AQ^0_{n-1} - F_0''$. Let $F_1' = F_1 \cup \{u_1, z_1\}$, then $|F_1'| \leq 2n - 6 \leq 4n - 13$ for $n \geq 4$. In addition, $AQ^1_{n-1} - F_1'$ has no isolated vertices, then by the induction hypothesis, there exists a perfect or an almost-perfect matching M_1 in the subgraph induced by $AQ^1_{n-1} - F_1'$. Let $M = M_0' \cup M_1 \cup \{uu_1, z_0z_1\}$, then M is a perfect or an almost-perfect matching in $AQ_n - F$.

Case 5.2 *Assume that $AQ^0_{n-1} - F$ has no isolated vertex.* Then by the induction hypothesis, there exists a perfect or an almost-perfect M_0 in $AQ^0_{n-1} - F$ and a perfect or an almost-perfect matching M_1 in $AQ^1_{n-1} - F$. If both M_0 and M_1 are perfect matchings, then $M = M_0 \cup M_1$ is a perfect matching in $AQ_n - F$. If exactly one of them is a perfect matching, then $M = M_1 \cup M_2$ is an almost-perfect matching in $AQ_n - F$. Suppose that M_1 and M_2 are almost-perfect matchings. Let z be the vertex in $AQ^0_{n-1} - F$ missed by M_0. If z is adjacent to some vertex $z_1 \in V(AQ^1_{n-1} - F)$ in $AQ_n - F$ then we can find a perfect matching M_1' in $AQ^1_{n-1} - (F_1 \cup \{z_1\})$ and the matching $M = M_0 \cup M_1' \cup \{zz_1\}$ is a perfect matching in $AQ_n - F$. Suppose that z is adjacent to two faulty outside edges or adjacent to two faulty outside vertices. Then, z must be adjacent to some vertex $u \in V(AQ^0_{n-1} - F)$ since $AQ_n - F$ contains no isolated vertex. The vertex u is saturated by M_0, let $uw \in M_0$. If w has an outside neighbor w_1 in $AQ^1_{n-1} - F$ such that $ww_1 \in E(AQ_n - F)$, then $M_0' = (M_0 - \{uw\}) \cup \{uz\}$ is an almost-perfect matching in $AQ^0_{n-1} - F$ missing w. By the induction hypothesis, there exists a perfect or almost-perfect matching M_1 in the subgraph induced by $AQ^1_{n-1} - (F_1 \cup \{w_1\})$. Then $M = M_0' \cup M_1 \cup \{ww_1\}$ is a perfect or an almost-perfect matching in $AQ_n - F$.

Suppose that z and w have no outside neighbors in $AQ^1_{n-1} - F$. We want to show that it is possible to have a vertex x in $AQ^0_{n-1} - (F \cup \{u\})$ such that $xy \in M_0$ and y has an outside neighbor in $AQ^1_{n-1} - F$.

If z and w share a common neighbor in AQ^1_{n-1}, then by Lemma 3.1, z and w share two common faulty neighbors in AQ^1_{n-1}, and in this case the vertices z and w can have at most one common neighbor in $AQ^0_{n-1} - F$ other than u. So z and w are adjacent to at least $(2n - 4) + (2n - 4) - 1 - |F_0| = 4n - 9 - |F_0| \geq |F_1|$ vertices in $AQ^0_{n-1} - F$ other than u. This means that z and w are adjacent to at least m vertices in $AQ^0_{n-1} - F$, where $m \geq |F_1|$. Let $x_1, \ldots, x_m$ be these vertices, and since all of them are saturated by M_0, we let $x_1y_1, \ldots, x_my_m \in M_0$. There are at least m neighbors for $\{y_1, \ldots, y_m\}$ in AQ^1_{n-1}. If all those neighbors are faulty, then $m = |F_1|$ and we know that there are already two faults in AQ^1_{n-1} other than these m faulty vertices, namely the faulty vertices adjacent to z and w. Then $|F_1| > |F_1| + 2$, contradiction. Note that z and w can be incident to faulty cross/complement edges, but we are considering the worst case scenario where we get the most number of faults in AQ^1_{n-1}. Therefore we can always find a vertex from the set $\{y_1, \ldots, y_m\}$ having a neighbor in $AQ^1_{n-1} - F$. Let y_1 be such vertex and let $y_1 y_1' \in E(AQ_n - F)$, where $y_1' \in V(AQ^1_{n-1} - F)$. By the induction hypothesis, there exists a perfect matching M_1'' in the subgraph induced by $AQ^1_{n-1} - (F \cup \{y_1'\})$. If $zx_1 \in E(AQ^0_{n-1} - F)$, then $M = (M_0 - \{x_1y_1\}) \cup M_1'' \cup \{y_1y_1', zx_1\}$ is a perfect matching in $AQ_n - F$. If $wx_1 \in E(AQ^0_{n-1} - F)$, then $M = (M_0 - \{uw, x_1y_1\}) \cup M_1'' \cup \{y_1y_1', zu, wx_1\}$ is a perfect matching in $AQ_n - F$. *If z and w do not share common neighbors in AQ^1_{n-1}*, then z and w are incident/adjacent to four faults. In this case z and w are adjacent to at least $(2n-4)+(2n-4)-3-|F_0| = 4n-11-|F_0| \geq |F_1|-2$ vertices in $AQ^0_{n-1} - F$ other than u. Then z and w are adjacent to at least $m-2$ vertices in $AQ^0_{n-1} - F$, where $m = |F_1|$. Let $x_1, \ldots, x_{m-2}$ be these vertices, and since all of them are saturated by M_0, we let $x_1y_1, \ldots, x_{m-2}y_{m-2} \in M_0$. Let $N(\{y_1, \ldots, y_{m-2}, w, z\})$ be the set of the outside neighbors of $\{y_1, \ldots, y_{m-2}, w, z\}$, then $|N(\{y_1, \ldots, y_{m-2}, w, z\})| \geq m$. If $|N(\{y_1, \ldots, y_{m-2}, w, z\})| > m$, then we can find y_i, such that y_i has an outside neighbor y_i' in $AQ^1_{n-1} - F$, and we proceed as we did above. If $|N(\{y_1, \ldots, y_{m-2}, w, z\})| = m$, then m is even, so $|F_1| = |F_1^V|$ is even and this contradicts the assumption that M_1 is an almost-perfect matching.

Case 6 $|F_0| = 2n-3$. This is can be treated the same as Case 5. The only difference is when $AQ^0_{n-1} - F_0$ has an isolated vertex, then by Proposition 2.1 $|F_0^V|$ is even.

Case 7 $|F_0| < 2n-3$ *and* $|F_1| < 2n-3$. Then by the induction hypothesis (or by Theorem 3.1), there exists a perfect or an almost-perfect M_0 in $AQ^0_{n-1} - F$ and a perfect or an almost-perfect matching M_1 in $AQ^1_{n-1} - F$. If both M_0 and M_1 are perfect matchings, then $M = M_0 \cup M_1$ is a perfect matching in $AQ_n - F$. If exactly one of them is a perfect matching, then $M = M_1 \cup M_2$ is an almost perfect matching in $AQ_n - F$. Suppose that M_1 and M_2 are almost-perfect matchings. Let x_0x_1 be an edge in $AQ_n - F$, such that $x_0 \in V(AQ^0_{n-1} - F)$, $x_1 \in V(AQ^1_{n-1} - F)$ and the subgraphs induced by $AQ^0_{n-1} - (F \cup \{x_0\})$ and by $AQ^1_{n-1} - (F \cup \{x_1\})$ has no isolated vertex. We can always find such edge. In fact, there are 2^n edges between AQ^0_{n-1} and AQ^1_{n-1}, and there is at most one vertex $x_0 \in V(AQ^0_{n-1} - F)$ whose deletion results in having an isolated vertex in $AQ^0_{n-1} - F$, and at most one vertex $x_1 \in V(AQ^1_{n-1} - F)$ whose deletion results in having an isolated vertex in $AQ^1_{n-1} - F$. Moreover, each faulty-vertex is incident to two cross/complement edges, so there are at most $2(4n-9+2) = 8n-14$ edges that can not be chosen, and this number is less than 2^n, for $n \geq 5$. By the induction hypothesis, there exist two perfect matchings M_0 and M_1 in the subgraphs induced by $AQ^0_{n-1} - (F \cup \{x_0\})$ and by $AQ^1_{n-1} - (F \cup \{x_1\})$ respectively. Therefore, $M = M_0 \cup M_1 \cup \{x_0x_1\}$ is a perfect matching in $AQ_n - F$.

□

DOI: 10.20429/tag.2021.080105

4 Conclusion

In this paper, we have studied the strong matching preclusion problem of augmented cubes under the condition that no isolated vertex is created in the presence of faulty edges and/or vertices. We proved that the conditional strong matching preclusion number of AQ_n is $4n-8$. We note that in the proof of our main theorem, we only make use of certain properties of the augmented cubes. So one can consider generalizing the result to a class of networks by starting with two copies of K_4 and add two sets of perfect matchings between them so that certain properties are satisfied, that is, we restrict the two sets of perfect matchings. One may wonder why we did not present this paper under this more general class. This is because we used Theorem 3.1, which was only proved for augmented cubes. We remark that if one examines the proof of this theorem in [8], it only relies on specific properties of the two added sets of perfect matchings. So one can generalize Theorem 3.1 to a larger class of graphs by allowing any two sets of perfect matchings with these properties. While this is interesting from a graph theory perspective, it is less important from an interconnection networks perspective that these graphs were designed for, as while the two specific sets of perfect matchings is less important regarding matching preclusion, they are very important in terms of designing nice distributed routing algorithms.

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