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Characterizing 2-Trees Relative to Chordal and Series-Parallel Graphs

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Abstract

The 2-connected 2-tree graphs are defined as being constructible from a single 3-cycle by recursively appending new degree-2 vertices so as to form 3-cycles that have unique edges in common with the existing graph. Such 2-trees can be characterized both as the edge-minimal chordal graphs and also as the edge-maximal series-parallel graphs. These are also precisely the 2-connected graphs that are simultaneously chordal and series-parallel, where these latter two better-known types of graphs have themselves been both characterized and applied in numerous ways that are unmotivated by their interaction with 2-trees and with each other.

Toward providing such motivation, the present paper examines several simple, very closely-related characterizations of chordal graphs and 2-trees and, after that, of series-parallel graphs and 2-trees. This leads to showing a way in which chordal graphs and series-parallel graphs are special—indeed, extremal—in this regard.

1 Introduction to the relevant graph classes

The 2-*dimensional trees*—commonly called 2-*trees*—can be defined recursively by beginning with a single edge and then repeatedly appending new degree-2 vertices, each with two new edges so as to form a new 3-cycle—a K_3 *triangle*—with existing edges of previously constructed 2-trees; see [12]. This definition supports simple inductive arguments to show that, for example, 2-trees of order n have exactly $2n - 3$ edges.

The following two graph classes are fundamental for our purposes, with both thoroughly discussed in [2]. *Chordal graphs* can be defined by every cycle C of length 4 or more having at least one chord, where a *chord* is an edge $ab \notin E(C)$ with $a, b \in V(C)$. *Series-parallel graphs* can be defined by every two edges st and xy in a common 2-connected subgraph being *confluent edges*, where this means that no two cycles that contain both these edges have the vertices in the order s, x, y, t around one of the cycles and in the order s, y, x, t around the other.

These graph classes arose from applications, with chordal graphs repeatedly discovered and independently studied in a variety of applications surveyed in §2.4 of [10]. Series-parallel graphs were primarily motivated by early electric network applications, as surveyed in [3]. In contrast, 2-trees were first defined in the 1960s and yet have been related to applications in geodetic surveying more than a hundred years earlier in [7].

The following two propositions follow from translating [11, 13] into our terminology, and they will be the fundamental characterizations of 2-trees used in this paper.

Proposition 1.1. (Patil [11]) *A* 2*-connected graph is a* 2*-tree if and only if it is an edge-minimal chordal graph,* where this means that deleting any existing edge while maintaining 2-connectedness would always create a graph that is not chordal.

Proposition 1.2. (Wald & Colbourn [13]) *A* 2*-connected graph is a* 2*-tree if and only if it is an edge-maximal series-parallel graph,* where this means that inserting any new edge between existing nonadjacent vertices would always create a graph that is not series-parallel.

Unless stated otherwise, we will restrict our discussion to the *nontrivial* 2*-trees*, meaning the 2-connected 2-trees, which simply means that the recursive construction begins from a single triangle, instead of a single edge. This enables the following characterization to be proved from the two propositions above in [8]: *The nontrivial 2-trees are the 2-connected graphs that are simultaneously chordal and series-parallel*. Note that the restriction to nontrivial 2-trees avoids the fact that all trees are trivially both chordal and series-parallel.

Section 2 will consist of three pairs of characterizations of chordal graphs and of 2-trees, where the chordal characterizations, usually well known, can be strengthened—using Proposition 1.1—into similar sounding, usually new, 2-tree characterizations. Section 3 will be structured similarly, except strengthening—using Proposition 1.2—series-parallel graph characterizations into similarly sounding 2-tree characterizations. Section 4 will consider why the specific properties of being chordal and being series-parallel play such special roles, concluding by citing an application in which these two particular properties play interchangeable roles.

2 2-trees in relation to chordal graphs

Recall the definition of chordal graphs given in §1: *A graph is chordal if and only if every k-cycle with $k \geq 4$ has a chord.* Theorem 2.1 will show the first of three ways in this section to change a word or two in a characterization of chordal graphs so as to characterize nontrivial 2-trees. A chord ab of a cycle C is called an *uncrossed chord* whenever C has no chord cd such that the four vertices a, c, b, d occur in that order around C.

Theorem 2.1. *A 2-connected graph is a 2-tree if and only if every k-cycle with $k \geq 4$ has an* uncrossed *chord.*

Proof. By the recursive definition of nontrivial 2-trees in §1, each newly appended degree-2 vertex x, along with the two new edges xy and xz that form the new triangle xyz, can be inserted into either one of the two faces bordered by the existing edge yz of the plane representation of the previous 2-tree, with yz an uncrossed chord of every new cycle thereby created.

Conversely, suppose G is 2-connected, and each k-cycle with $k \geq 4$ of G has an uncrossed chord. Thus, G is automatically chordal and inserting any additional edge uv into G would make uv a chord of the minimal-length cycle C of the 2-connected graph G that has $u, v \in V(C)$, contradicting that C already had an uncrossed chord. Therefore, G is an edge-minimal chordal graph and so is a 2-tree by Proposition 1.1. □

Define the *sum of a set of cycles* of a graph G to be the subgraph of G formed by the edges that are in precisely an odd number of those cycles. Lemma 2.2 is from [4] and reproved in [9], where two 3-cycles K_3 are called *distinct triangles* whenever their vertex sets are not equal.

Lemma 2.2. *A graph is chordal if and only if every k-cycle is the sum of* at least *one set of $k-2$ distinct triangles.*

Theorem 2.3 first appeared in [8], where it is proved directly from Lemma 2.2 and Proposition 1.1.

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Theorem 2.3. *A 2-connected graph is a 2-tree if and only if every k-cycle is the sum of* exactly *one set of $k-2$ distinct triangles.*

Lemma 2.4 is a simple consequence of the definition of chordal graphs in §1 (arguing by induction on $k \geq 3$).

Lemma 2.4. *A graph is chordal if and only if every k-cycle has* at least *$k-3$ chords.*

Theorem 2.5. *A 2-connected graph is a 2-tree if and only if every k-cycle has* exactly *$k-3$ chords.*

Proof. First, suppose G is a 2-connected 2-tree and argue by induction on $k \geq 3$. The $k = 3$ basis case holds since triangles have exactly $k-3 = 0$ chords. Suppose G is constructed from a 2-tree G' of at least 3 vertices by appending a new vertex u to an edge vw. A k-cycle C of G that does not contain u is also part of G' and has exactly $k-3$ chords by the induction hypothesis. Otherwise, if $u \in V(C)$, then since u has degree 2, both v and w are also in C, with vw a chord of C. The $(k-1)$-cycle that resembles C but takes the shortcut edge vw instead of vu and uw is in G' and, except for vw, shares the same set of $(k-1)-3$ chords with C. Thus, C has exactly $(k-1)-3+1 = k-3$ chords.

Conversely, suppose every k-cycle of a 2-connected graph G has exactly $k-3$ chords, and argue by induction on $n = |V(G)| \geq 3$. Since G is chordal by Lemma 2.4, G has a v such that every two neighbors of v are adjacent in G (see[2, 10], where v is called a "simplicial vertex"). The $n = 3$ basis case holds since $G \cong K_3$ is a 2-tree. Hence, suppose $n \geq 4$. If $\deg_G(v) \geq 3$, then v and three of its neighbors form a complete subgraph, which would constitute a 4-cycle C with two chords, contradicting that C is supposed to have exactly $4-3 \neq 2$ chords. Therefore, $\deg_G(v) = 2$, and so v and its two neighbors form a triangle. Since $G-v$ is a 2-tree by the inductive hypothesis and Lemma 2.4, G is a 2-tree by the recursive construction in §1. □

3 2-trees in relation to series-parallel graphs

For distinct $s, x, y, t \in V(G)$ with $xy \in E(G)$, define G to be an $\{s, xy, t\}$-*confluent graph* if no two s-to-t paths P_1 and P_2 of G have $xy \in E(P_1) \cap E(P_2)$ with vertices in the orders s, x, y, t along P_1 and s, y, x, t along P_2. For example, consider the complete tripartite graph $K_{1,1,2}$, which can be viewed as K_4 with one edge deleted. Thus $K_{1,1,2}$ is a 2-connected 2-tree and, if s and t are its nonadjacent degree-2 vertices and x and y are its degree-3 vertices, then $K_{1,1,2}$ is not $\{s, xy, t\}$-confluent. But if, instead, st and xy are two vertex-disjoint edges of $K_{1,1,2}$, then $K_{1,1,2}$ is $\{s, xy, t\}$-confluent. In fact, every two edges of $K_{1,1,2}$ are confluent edges, as defined in §1. Lemma 3.1 restates the definition of series-parallel graphs from §1 using this new terminology.

Lemma 3.1. *A graph G is series-parallel if and only if, for* all *adjacent vertices $s, t \in V(G)$ and every $xy \in E(G - \{s, t\})$, the graph G is $\{s, xy, t\}$-confluent.*

Theorem 3.2. *A 2-connected graph G is a 2-tree if and only if, for* exactly the *adjacent vertices $s, t \in V(G)$ and every $xy \in E(G - \{s, t\})$, the graph G is $\{s, xy, t\}$-confluent.*

Proof. First, suppose G is a 2-connected 2-tree, and so is series-parallel by Proposition 1.1, with two arbitrary $s, t \in V(G)$ and arbitrary $xy \in E(G - \{s, t\})$. If s and t are adjacent, then G is $\{s, xy, t\}$-confluent by Lemma 3.1.

Now assume s and t are nonadjacent and yet G is $\{s, xy, t\}$-confluent, arguing by contradiction. By Proposition 1.2, inserting a new edge st into G will create a 2-connected graph G^+ that is not series-parallel. Since s and t are adjacent in G^+ and G is series-parallel, but G^+ is *not* series-parallel, the contrapositive of Lemma 3.1 assures that, for *some* $xy \in E(G^+ - \{s, t\})$, the graph G^+ is *not* $\{s, xy, t\}$-confluent. Thus, there are two s-to-t paths P_1 and P_2 of G^+ that have $xy \in E(P_1) \cap E(P_2)$ with vertices in the orders s, x, y, t along P_1 and s, y, x, t along P_2. But since $st \notin E(P_1) \cup E(P_2)$, both P_1 and P_2 are paths of G, and so G would not be $\{s, xy, t\}$-confluent, contradicting the assumption above.

Conversely, suppose G is 2-connected and, for exactly the adjacent vertices $s, t \in V(G)$ and every $xy \in E(G - \{s, t\})$, the graph G is $\{s, xy, t\}$-confluent. Thus, for all adjacent vertices $s, t \in V(G)$ and $xy \in E(G - \{s, t\})$, the graph G is $\{s, xy, t\}$-confluent, and so G is series-parallel by Lemma 3.1.

Because of the "for exactly the adjacent vertices $s, t \in V(G)$" modifier, if $s', t' \in V(G)$ are *non*adjacent, then for some $xy \in E(G - \{s', t'\})$, G will not be $\{s, xy, t\}$-confluent. In other words, inserting any new edge $s't'$ between existing nonadjacent vertices s' and t' of G would always produce a graph G' that has an edge $xy \in E(G' - \{s', t'\})$ such that G' is not be $\{s, xy, t\}$-confluent; thus, G' would not be series-parallel by Lemma 3.1.

By the preceding two paragraphs, G is an edge-maximal series-parallel graph, and so G is a 2-tree by Proposition 1.2. □

For $s, t \in V(G)$, a *minimal* s, t*-cutset* of G is an inclusion-minimal $S \subset E(G)$ such that s and t are in different components of $G - S$ (and so $st \in S$ if $st \in E(G)$). Call $S \subset E(G)$ a *mincut* of G if S is a *minimal* s, t*-cutset* for some $s, t \in V(G)$. Three edges $st, wx, yz \in E(G)$ are in a common cycle if and only if wx and yz are in a common s-to-t path, and those three edges are in a common mincut if and only if wx and yz are in a common minimal s, t-cutset. It is proved in [6] that a graph is series-parallel if and only if no three edges are in both a common cycle and a common mincut. Lemma 3.3 restates this in our current terminology.

Lemma 3.3. *A graph G is series-parallel if and only if, for* all *adjacent vertices $s, t \in V(G)$, no two $wx, yz \in E(G) \setminus \{st\}$ are in both a common s-to-t path and a common minimal s, t-cutset of G.*

Theorem 3.4. *A 2-connected graph G is a 2-tree if and only if, for* exactly the *adjacent vertices $s, t \in V(G)$, no two $wx, yz \in E(G) \setminus \{st\}$ are in both a common s-to-t path and a common minimal s, t-cutset of G.*

Proof. First, suppose G is a 2-connected 2-tree, and so is series-parallel by Proposition 1.1, with two arbitrary vertices s and t. If s and t are adjacent, then (by Lemma 3.3) no two $wx, yz \in E(G) \setminus \{st\}$ are in both a common s-to-t path and a common minimal s, t-cutset of G.

Now assume s and t are nonadjacent and yet no two $wx, yz \in E(G) \setminus \{st\}$ are in both a common s-to-t path and a common minimal s, t-cutset of G, arguing by contradiction. By Proposition 1.2, inserting a new edge st into G will create a 2-connected graph G^+

that is not series-parallel. Since s and t are adjacent in G^+ and G is series-parallel, but G^+ is *not* series-parallel, the contrapositive of Lemma 3.3 assures that *some* two edges $wx, yz \in E(G^+) \setminus \{st\}$ are in both a common s-to-t path π and a common minimal s,t-cutset S of G^+. Since $st \notin E(\pi)$ and $st \in E(S)$, this in turn implies that $wx, yz \in E(G) \setminus \{st\}$ are in both a common s-to-t path π and a common minimal s,t-cutset S of G, contradicting the assumption above.

Conversely, suppose G is 2-connected and, for exactly the adjacent vertices $s, t \in V(G)$, no two $wx, yz \in E(G) \setminus \{st\}$ are in both a common s-to-t path and a common minimal s,t-cutset. Thus, for all adjacent vertices $s, t \in V(G)$, no two $wx, yz \in E(G) \setminus \{st\}$ are in both a common s-to-t path and a common minimal s,t-cutset, and so G is series-parallel by Lemma 3.3.

Because of the "for exactly the adjacent vertices $s, t \in V(G)$" modifier, if $s', t' \in V(G)$ are *non*adjacent, then some two $wx, yz \in E(G)$ will be in both a common s'-to-t' path and a common minimal s', t'-cutset of G. In other words, inserting any new edge between existing nonadjacent vertices s' and t' of G would always produce a graph G' that would not be series-parallel by Lemma 3.3.

By the preceding two paragraphs, G is an edge-maximal series-parallel graph, and so G is a 2-tree by Proposition 1.2. □

4 What makes the chordal and series-parallel graphs special here?

Having seen how Propositions 1.1 and 1.2 can be exploited in §2 and §3, an intriguing question remains. This section will consider why "chordal" and "series-parallel" seem to play special roles in these two propositions—especially since certain related properties could be substituted for them. In Proposition 1.1, for instance, "chordal" could be replaced with the stronger property of "chordal planar," as follows from every series-parallel graph being planar by the first paragraph of the proof of Theorem 2.1, and from planarity being preserved under edge deletion. In contrast to such modifications, Theorem 4.1 will first suggest a sense in which "chordal" and "series-parallel" are special—in fact, are extremal—in Propositions 1.1 and 1.2. As in [2, 10], a graph-theoretic property is a *hereditary property* if it is preserved under taking induced subgraphs.

Theorem 4.1. *Proposition* 1.1 *cannot be modified by replacing "chordal" with a weaker hereditary property, and Proposition* 1.2 *cannot be modified by replacing "series-parallel" with a stronger hereditary property.*

Proof. First, suppose "*chordal*$^-$" is any hereditary property that is strictly weaker than "chordal" among 2-connected graphs; thus, some 2-connected chordal$^-$ graph is not chordal and so, by definition, contains an induced subgraph $G \cong C_n$ with $n \geq 4$. Since G would then also be chordal$^-$, but not a 2-tree, and since no edge can be deleted from G while maintaining 2-connectedness, the attempted modification of Proposition 1.1 would fail.

Similarly, suppose "*series-parallel*$^+$" is a hereditary property that is strictly stronger than being 2-connected and series-parallel; thus, some 2-connected series-parallel graph G is not series-parallel$^+$. Assume further that such a G is chosen for which inserting any additional edge between existing nonadjacent vertices of G would create a subgraph homeomorphic to K_4, which would make the larger graph non-series-parallel and so not non-series-parallel$^+$. Thus G would be a 2-connected, edge-maximal series-parallel graph, and so G is a 2-tree by Proposition 1.2. Since this G is not an edge-maximal series-parallel$^+$ graph, the attempted modification of Proposition 1.2 would fail. □

As further support for the roles of "chordal" and "series-parallel" in the characterization of nontrivial 2-trees mentioned in §1 as being simultaneously chordal and series-parallel, there is a bipartite analogy of 2-trees that are called 2∗-*trees* in [8]. These 2∗-trees are defined recursively, now starting from any 2-connected complete bipartite graph instead of from a triangle, and they are characterized in [8] by being the bipartite graphs that are both series-parallel and chordal.

Since §1 mentions a number of unrelated applications of chordal and of series-parallel graphs, it seems interesting that there is another application of these two properties—this time simultaneously—to a matrix completion problem that is explained in detail in [1]. The resulting so-called *cycle completable graphs* are characterized there as those graphs in which new edges can be inserted so as to produce a chordal graph without forming any new K_4 subgraphs. This blend of chordal and series-parallel graphs is shown in [5] to be equivalent to the graphs being constructible by sequentially identifying $m-1$ edge pairs from $m \geq 1$ subgraphs that are 2-connected with each of those subgraphs *either* chordal *or* series-parallel. This contrasts with the characterization of nontrivial 2-trees as being similarly constructible from 2-connected subgraphs, each of which is a triangle—in other words, with each of those subgraphs *both* chordal *and* series-parallel.

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