FACTA UNIVERSITATIS
Series: **Mechanical Engineering**
https://doi.org/10.22190/FUME210125033H

Original scientific paper

THE ENHANCED HOMOTOPY PERTURBATION METHOD FOR AXIAL VIBRATION OF STRINGS

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Abstract. *A governing equation is established for string axial vibrations with temporal and spatial damping forces by the Hamilton principle. It is an extension of the well-known Klein-Gordon equation. The classical homotopy perturbation method (HPM) fails to analyze this equation, and a modification with an exponential decay parameter is suggested. The analysis shows that the amplitude behaves as an exponential decay by the damping parameter. Furthermore, the frequency equation is established and the stability condition is performed. The modified homotopy perturbation method yields a more effective result for the nonlinear oscillators and helps to overcome the shortcoming of the classical approach. The comparison between the analytical solution and the numerical solution shows an excellent agreement.*

Key Words: *Homotopy Perturbations Method, Exponential Decay Parameter, Damping Duffing Equation, Damping Nonlinear Klein-Gordon Equation*

1. INTRODUCTION

String axial vibrations arise in various fields, e.g., suspension bridge, drill string, and ring spinning; the vibration property will significantly affect the safety and the life span, and it can be described by nonlinear differential equations with possible two-fractal derivatives, which are difficult to solve exactly. The scientists have made great efforts to provide information about the approximate solutions of these equations [1-6]. Of the nonlinear

Received January 25, 2021 / Accepted March 11, 2021
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vibration systems that have been fascinating to engineers and scientists, by far the most perplexing is the frequency-amplitude relationship. A simple but accurate approach to the relationship is much needed in practical applications [7-9]. It was revealed that nanofiber's surface morphology can be controlled by the vibration's frequency-amplitude relationship [10], and a successful finding of the Fangzhu's vibration property has revealed the hidden nanotechnology in ancient China [11-13]. The variational principle for nonlinear differential equations is another effective analytical method [14,15].

The string axial vibration can be modeled by the Klein-Gordon equation (KG), which is a significant group of partial differential equations; it is also found in relativistic quantum mechanics and field theory. El-Dib [16] introduced the multiple-scale homotopy perturbation method (He's multiple-scale method) as an outer perturbation of the nonlinear KG equation. A highly accurate periodic unsteady solution was derived from three order perturbations. Furthermore, a complete approximate solution consisting of a temporal solution, which functions as an outer expansion along with the spatial solution serving as an inner expansion of the nonlinear KG equation, was achieved. El-Dib et al. [17,18] recently proposed a new approach to the study of the nonlinear instability analysis. Their analysis has revealed a nonlinear PDE that controls the surface deflection of the interface. They have transferred the characteristic equation to a KG equation. Through a traveling–wave solution, they have examined the stability profile.

It is important to note that the temporal damped KG equation is not naturally dissipative. So, the introduction of the dissipative mechanisms is necessary to force the energy to decay to zero when time goes to infinity. Indeed, the appearance of the damping terms given in the damped KG equation is not required by itself to gauge the energy E(t) associated with the problem, but it is a non-increasing function of the parameter t. Uniform decay rate estimates to the problem have been considered by Cavalcanti et al. [19-21]. The maximum principles of the optimal control problems, governed by a damped KG equation with state constraints, were examined by Parka and Jeong [22]. Lin and Cui [23] investigated a damped nonlinear KG equation, taking into consideration a kernel space. They assumed a new technique in solving this equation to emphasize the feasibility and the rigor of the method. The local solution of the initial-boundary value problem for a class of fourth-order wave equations with a high damping term was obtained by using fixed point theory [24]. He and El-Dib [25] used the reducing rank method to solve a third-order damped nonlinear equation produced from the strong damped nonlinear KG equation

One of the most traditional techniques to find an approximate solution is the homotopy perturbation method (HPM). The major property of the HPM is its ability and flexibility to examine a wide class of nonlinear differential equations conveniently and accurately [26]. It has been developed and improved by scientists and engineers. The HPM with two expanding parameters was suggested by El-Dib [27]. A combination of the HPM and the Laplace transforms were used by El-Dib and Moatimid [25]. El-Dib [29] suggested a modified version of the HPM by the multiple scales technique. This new modification works well, particularly for nonlinear oscillators. Ren et al. [30] developed the multiple scale by combining it with the HPM to improve it by incorporating some known technologies. It provides solutions to nonlinear equations for which the classical perturbation method proved unsuccessful. A modification of the HPM with the so-called reducing-rank technique has been introduced by Shen and El-Dib [31] and El-Dib and Elgazery [32].

In this paper, we present an efficient method for the approximation of the damped nonlinear Klein–Gordon equation, with the aim to solve and study the stability of the periodic solution. The key features of our methods are based on the coupling of the exponential decay parameter to the homotopy perturbation method to overcome the shortcomings of the classical method.

2. MATHEMATICAL MODEL OF THE DAMPING NONLINEAR KLEIN-GORDON EQUATION

The governing equation can be obtained from the following variational principle [14,15]:

$$\int_{t_1}^{t_2}\{\delta K-\delta P+\delta W\}dt=0 \tag{1}$$

where *K*, *P*, and *W* represent, respectively, the kinetic energy, the potential energy, and the work done by the external force; δK. δP and δW can be expressed, respectively, as

$$\delta K=\delta\int_0^L\frac{1}{2}\rho(\frac{\partial y}{\partial t})^2dx-\int_0^L\bar{\eta}\frac{\partial y}{\partial t}\delta ydx \tag{2}$$

$$\delta P=\delta\int_0^L T\varepsilon dx-\int_0^L\bar{\mu}\frac{\partial y}{\partial x}\delta ydx=\int_0^L\frac{1}{2}T(\frac{\partial y}{\partial x})^2dx-\int_0^L\bar{\mu}\frac{\partial y}{\partial x}\delta ydx \tag{3}$$

$$\delta w=\int_0^L f\delta udx \tag{4}$$

Here y is the axial displacement, ρ is the density, T is the surface tension of the string, f is the external force, η represents the temporal damped coefficient, μ stands for the spatial damped coefficient,

$$\rho y_{tt}+\bar{\eta}y_t-Ty_{xx}+\bar{\mu}y_x-f=0 \tag{5}$$

Eq. (5) can be re-written as

$$y_{tt}-Py_{xx}+2\eta y_t+2\mu y_x-F=0 \tag{6}$$

where $P=T/\rho$, $F=f/\rho$, $\eta=\bar{\eta}/\rho$ and $\mu=\bar{\mu}/\rho$, while $y(x,t)$ is the displacement at location x and time t.

When $\eta=\mu=0$, Eq. (6), the Klein-Gordon equation becomes:

$$y_{tt}-Py_{xx}-F=0 \tag{7}$$

When

$$F=-\sigma y \tag{8}$$

we have a linear damped Klein-Gordon equation

$$y_{tt}-Py_{xx}+2\eta y_t+2\mu y_x+\sigma y=0 \tag{9}$$

In this paper, we consider a cubic nonlinear case

$$F = -\sigma y + Qy^3 \tag{10}$$

We have the following one-dimensional nonlinear wave equation with linear damping parameters:

$$y_{tt} - Py_{xx} + 2\eta y_t + 2\mu\, y_x + \sigma y = Qy^3; \quad y = y(x,t). \tag{11}$$

where σ refers to the natural frequency, and Q stands for the cubic-stiffness parameter.

Eq. (11) becomes the classical nonlinear Klein–Gordon equation that appears in various branches of science, including quantum mechanics, nonlinear optics fluid mechanics, and solid physics. In the case of η>0 and μ=0, with some specific nonlinearities, Eq. (11) becomes a one-dimensional temporal damped nonlinear Klein-Gordon equation [33]. El-Dib et al. [34] discussed the time fraction of the damped KG equation to overcome the presence of the non-zero damping coefficients. Besides, they applied the modified multiple scales method for the same aims.

If the cubic term in Eq. (11) becomes vanishing because it is small compared to the linear term, then the right-hand side term contributes to the order of y^3 and thus can be neglected. In the linear limit, Eq. (11) has the well-known exact temporal solution for the damped system given by:

$$y(t) = Ae^{-\eta t - \frac{\mu}{P}x}\cos(\omega t + \theta), \tag{12}$$

where A and θ are real constants determined by the initial conditions. Linear frequency ω is given by

$$\varpi^2 = \sigma - \eta^2 - \frac{3\mu^2}{P}. \tag{13}$$

This procedure cannot be successfully used when the nonlinear terms are involved. The solution of the damping nonlinear Klein-Gordon Eq. (11), by applying the classical analysis of the homotopy perturbation method [26], leads to the disappearance of the damping coefficient.

In what follows the homotopy perturbation with the exponential decay of the slow variable is elaborated [3,35]:

2.1 Solution with the modified HPM by exponential decay description

To build the homotopy equation, we select the linear partial differential operator and the nonlinear operator as:

$$L_t y = y_{tt} + \sigma y \text{ and } Ny = -Py_{xx} + 2\eta y_t + 2\mu y_x - Qy^3. \tag{14}$$

The corresponding temporal homotopy equation may be constructed as

$$H(y;q) = (y_{tt} + \sigma y) + q(-Py_{xx} + 2\eta y_t + 2\mu\, y_x - Qy^3) = 0; \quad q \in [0,1], \tag{15}$$

where function $y(x,t)$ becomes $y(x,t;q) = u(x,t;q)e^{-q\eta t}$. $H(y;q)$ is the homotopy function.

As the primary solution can be obtained by making parameter q tend to zero, we find

$$y_0(x,t) = \lim_{q\to 0} u(x,t;q). \tag{16}$$

The final form of the approximate solution of the original Eq. (11) has the form

$$y(x,t) = e^{-\eta t} \lim_{q\to 1} u(x,t;q). \tag{17}$$

Assuming that Eq. (15) admits the solution [3]

$$y(x,t;q) = u(x,t,q)e^{-q\eta t}. \tag{18}$$

Function $u(x,t,q)$ is expanded as

$$u(x,t,q) = u_0(x,t) + qu_1(x,t) + q^2 u_2(x,t) + \ldots \tag{19}$$

Also, one can use the expanded frequency as

$$\Omega^2 = \sigma + q\Omega_1 + q^2\Omega_2 + \ldots, \tag{20}$$

where Ω_i is unknown in total frequency Ω^2 and determined by the disappearance of the secular term. Employing Eq. (18) with Eq. (15), and using expansions (19) and (20), we obtain homotopy function $H(u;q)$ arranged in the form

$$H(u;q) = H_0(u_0) + qH_1(u_1,u_0) + q^2 H_2(u_2,u_1,u_0) + \ldots = 0. \tag{21}$$

The estimation of $H_n; n = 0,1,2,\ldots$ is as follows:

$$H_0(u_0) = \lim_{q\to 0} H(y;q), \tag{22}$$

$$H_1(u_1,u_0) = \lim_{q\to 0} \frac{\partial}{\partial q} H(y;q), \tag{23}$$

$$H_r(,u_r,\ldots,u_1,u_0) = \frac{1}{r!}\lim_{q\to 0} \frac{\partial^r}{\partial q^r} H(y;q). \tag{24}$$

According to Eqs. (12-14) and equating each H_i to zero, we obtain

$$H_0(u_0) = \left(\partial_{tt} + \Omega^2\right)u_0(x,t) = 0, \tag{25}$$

$$H_1(u_1,u_0) = \left(\partial_{tt} + \Omega^2\right)u_1(x,t) - \eta t\left(\partial_{tt} + \Omega^2\right)u_0 + \left(-P\partial_{xx} + 2\mu\partial_x - \Omega_1 - Qu_0^2\right)u_0 = 0, \tag{26}$$

$$\begin{aligned} H_2(u_2,u_1,u_0) &= \left(\partial_{tt} + \Omega^2\right)u_2(x,t) - \eta t\left(\partial_{tt} + \Omega^2\right)u_1 - P\partial_{xx}u_1 + \left(2\mu\partial_x - \Omega_1 - 3Qu_0^2\right)u_1 \\ &+ \tfrac{1}{2}\eta^2 t^2\left(\partial_{tt} + \Omega^2\right)u_0 + P\partial_{xx}u_0 - \Omega_2 u_0 - \eta t\left(2\mu\partial_x - \Omega_1 - 3Qu_0^2\right)u_0 - \eta^2 u_0 = 0. \end{aligned} \tag{27}$$

Handling Eq. (25) with Eq. (26) yields

$$\left(\partial_{tt}+\Omega^2\right)u_1(x,t)=P\partial_{xx}u_0-2\mu\partial_x u_0+\Omega_1 u_0+Qu_0^3. \tag{28}$$

Eq. (27) will be simplified by using Eqs. (25) and (28), to become

$$\left(\partial_{tt}+\Omega^2\right)u_2(x,t)=\left(P\partial_{xx}-2\mu\partial_x+\Omega_1+3Qu_0^2\right)u_1+\left(\Omega_2+\eta^2+P\partial_{xx}u_0\right)u_0-2\eta tQu_0^3. \tag{29}$$

Suppose the solution of Eq. (25) has the form

$$u_0(x,t)=Ae^{-\frac{\mu}{P}x}\cos(\Omega t+\theta), \tag{30}$$

where unknown arbitrary function $B(x)$ is chosen to be $B(x)=Ae^{-\frac{\mu}{P}x}$, with constant A.

By making use of (20) into (18), we transformed the right-hand side into a polynomial in $\exp(-\frac{\mu}{P}x)$.

$$\begin{aligned}\left(\partial_{tt}+\Omega^2\right)u_1(x,t)=&\left(\Omega_1+\frac{3\mu^2}{P}+\frac{3}{4}A^2Qe^{-\frac{2\mu}{P}x}\right)e^{-\frac{\mu}{P}x}A\cos\left(\Omega t+\theta\right)\\&+\frac{1}{4}e^{-\frac{3\mu}{P}x}A^3Q\cos\left(3\Omega t+3\theta\right)\end{aligned} \tag{31}$$

Avoiding the secular term that appeared in Eq. (31) gives

$$\Omega_1=-\frac{3\mu^2}{P}-\frac{3}{4}A^2Qe^{-\frac{2\mu}{P}x}. \tag{32}$$

Consequently, the total solution of Eq. (31) becomes

$$u_1(x,t)=-\frac{QA^3}{32\Omega^2}e^{-\frac{3\mu}{P}x}\cos(3\Omega t+3\theta). \tag{33}$$

Employing Eqs. (30) and (33) into Eq. (29), and using Eq. (32), the result reduces to

$$\begin{aligned}\left(\partial_{tt}+\Omega^2\right)u_2(x,t)=&\left(\Omega_2+\eta^2-\frac{3}{2}\eta tQA^2e^{-2\frac{\mu}{P}x}-\frac{3Q^2A^4}{128\Omega^2}e^{-\frac{4\mu}{P}x}\right)Ae^{-\frac{\mu}{P}x}\cos\left(\Omega t+\theta\right)\\&+QA^3\left(-\frac{1}{2}\eta t-\frac{3\mu^2}{8P\Omega^2}-\frac{3}{128\Omega^2}A^2Qe^{-\frac{2\mu}{P}x}\right)e^{-3\frac{\mu}{P}x}\cos\left(3\Omega t+3\theta\right)\\&-\frac{3Q^2}{128\Omega^2}A^5e^{-\frac{5\mu}{P}x}\cos\left(5\Omega t+5\theta\right).\end{aligned} \tag{34}$$

Avoiding the secular terms from Eq. (34) requires that

$$\Omega_2 + \eta^2 - \frac{3}{2}\eta t\, QA^2 e^{-2\frac{\mu}{P}x} - \frac{3Q^2A^4}{128\Omega^2} e^{-\frac{4\mu}{P}x} = 0. \tag{35}$$

According to the solvability condition of Eq. (35), the solution of Eq. (34) becomes

$$\begin{aligned} u_2(x,t) = {} & QA^3\left[\frac{1}{16\Omega^2}\eta t + \frac{1}{256\Omega^4}\left(\frac{3\mu^2}{P} + \frac{3}{16}A^2Qe^{-\frac{2\mu}{P}x}\right)\right]e^{-3\frac{\mu}{P}x}\cos(3\Omega t + 3\theta) \\ & + \frac{3\eta QA^3}{32\Omega^2}e^{-3\frac{\mu}{P}x}\sin(3\Omega t + 3\theta) + \frac{Q^2}{1024\Omega^4}A^5 e^{-\frac{5\mu}{P}x}\cos(5\Omega t + 5\theta). \end{aligned} \tag{36}$$

The approximate second-order solution can be performed by inserting Eqs. (30), (33), and (36) into Eq. (18), and taking $q = 1$ yields:

$$\begin{aligned} y(x,t) = {} & Ae^{-\left(\eta t + \frac{\mu}{P}x\right)}\cos(\Omega t + \theta) - \frac{QA^3}{32\Omega^2}e^{-3\left(\eta t + \frac{\mu}{P}x\right)}\cos(3\Omega t + 3\theta) \\ & + \frac{3\eta QA^3}{32\Omega^2}e^{-\left(\eta t + \frac{3\mu}{P}x\right)}\sin(3\Omega t + 3\theta) \\ & + \frac{QA^3}{256\Omega^4}\left(\frac{3\mu^2}{P} + \frac{3}{16}A^2Qe^{-\frac{2\mu}{P}x}\right)e^{-\left(\eta t + \frac{3\mu}{P}x\right)}\cos(3\Omega t + 3\theta) \\ & + \frac{Q^2}{1024\Omega^4}A^5 e^{-\left(\eta t + \frac{5\mu}{P}x\right)}\cos(5\Omega t + 5\theta). \end{aligned} \tag{37}$$

If the higher orders of the perturbation are done, the compact form of $(1 - 2\eta t + \ldots)$ becomes $e^{-2\eta t}$ [3]. This solution still contains unknown frequency Ω. Its determination is the subject of the following section.

2.2 The nonlinear frequency in the perturbed form to get stability condition

The two solvability conditions (32) and (35) can be used to establish the frequency formula by inserting them into the expansion (20), and making $q = 1$ yields:

$$\Omega^2 = \varpi^2 - \frac{3}{4}A^2Qe^{-2\left(\eta t + \frac{\mu}{P}x\right)} - \frac{3Q^2A^4}{128\Omega^2}e^{-\frac{4\mu}{P}x}. \tag{38}$$

Without upgrading the power of the frequency, we need to obtain an approximate solution of the above frequency equation. To accomplish this solution, we put Eq. (38) in the perturbed form by introducing an artificial small parameter $\varepsilon \in [0, 1]$. Thus, we have

$$\Omega^2(\varepsilon) = \varpi^2 - \frac{3}{4}\varepsilon A^2Qe^{-2\varepsilon\left(\eta t + \frac{\mu}{P}x\right)} - \varepsilon\frac{3Q^2A^4}{128\Omega^2(\varepsilon)}e^{-\varepsilon\frac{4\mu}{P}x}. \tag{39}$$

Consider the following expansion:

$$\Omega^2 = \varpi^2 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \dots \tag{40}$$

Inserting Eq. (30) into Eq. (29) and equating the identical power of ε to zero, we obtain

$$\omega_1 = -\frac{3}{4}A^2Q - \frac{3Q^2A^4}{128\varpi^2}. \tag{41}$$

To perform the first-order approximate solution of Eq. (39), we employ Eqs. (13) and (41) in Eq. (40), which yields

$$\Omega^2 = \varpi^2 - \frac{3}{4}A^2Q - \frac{3Q^2A^4}{128\varpi^2}. \tag{42}$$

The stability requires Ω to be real, which needs the following condition:

$$\varpi^2 - \frac{3}{4}A^2Q - \frac{3Q^2A^4}{128\varpi^2} > 0. \tag{43}$$

Employing the linear frequency as defined by Eq. (13), we obtain

$$\left(\sigma - \eta^2 - \frac{3\mu^2}{P}\right)^2 - \frac{3}{4}A^2Q\left(\sigma - \eta^2 - \frac{3\mu^2}{P}\right) - \frac{3Q^2A^4}{128} > 0. \tag{44}$$

2.3 Numerical illustrations

The numerical solution of the damped nonlinear KG-equation (11) is obtained by using a mathematical software for the system of P=1, σ=3, Q=0.3, A=1, η=0.1, μ=0.1. The results are displayed in Fig. 1. It is noted that the small values in damping coefficients η and μ lead to the gradual damping in the periodic wave solution. The analytical solution as given by Eq. (37) is also calculated numerically for the same system as used in the numerical solution, assuming that θ=0. The results are located in Fig. 2.

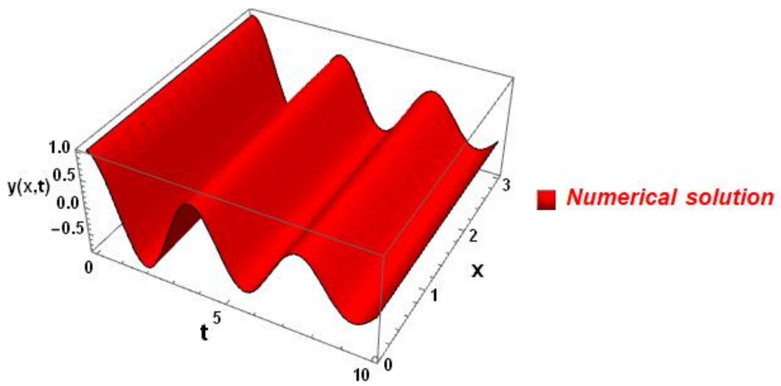

Fig. 1 The numerical solution of Eq. (11) for a system having P=1, σ=3, Q=0.3, A=1, η=0.1, μ=0.1

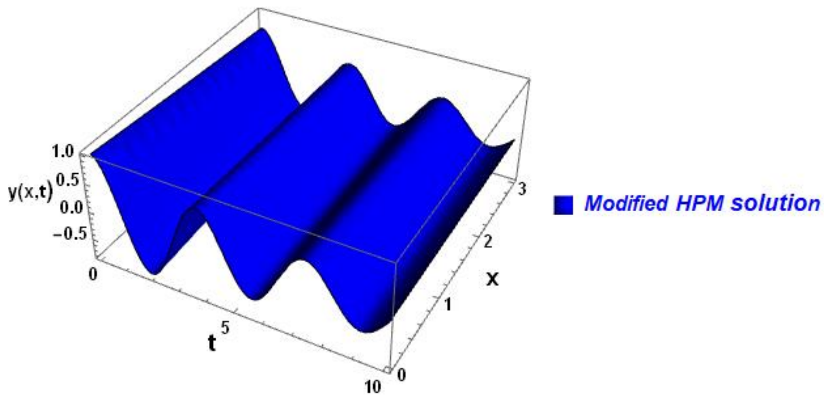

Fig. 2 The analytical approximate solution of Eq. (11) as given by Eq. (37) using the full frequency as given by Eq. (39). The system is considered as given in Fig. (1)

For comparison, the two kinds of solutions of Eq. (11) are collected together in one graph as shown in Fig. 3. It is observed that there is an excellent agreement between the numerical and analytical solutions along the time.

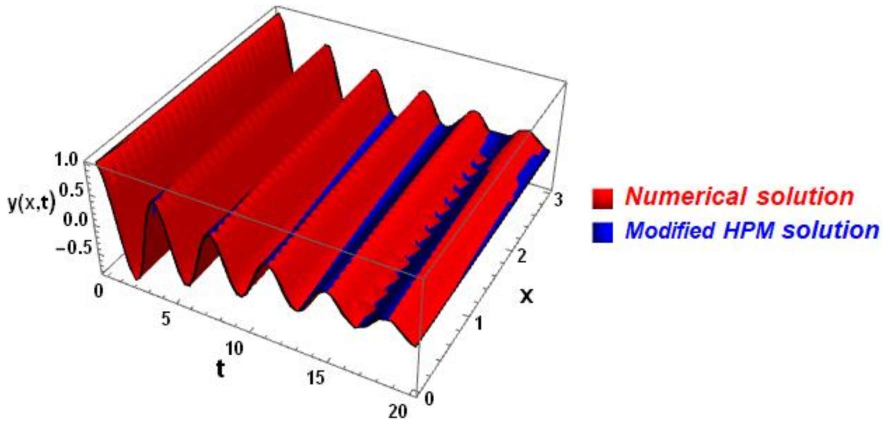

Fig. 3 The comparison between the numerical and analytical solutions of Eq. (11) for the same system as given in Fig. 1

The examination of the impact of temporal damping coefficient η on the picture of the periodic wave solution of the modified analytical solution of Eq. (37) is displayed in Fig. 4. The numerical values are the same as those used in Fig. (2), except that the spatial variable is fixed to value x=1. It is observed that as η=0 the pure periodic solution is found. By a slight change in η from the zero value, the damping influence is observed. The damping influence of the increase of spatial damping coefficient μ is shown in Fig. 4. The calculation is made for the same system as illustrated in Fig. 4 by fixing η at the zero value. It is noted that increase in μ decreases the amplitude of the wave solution.

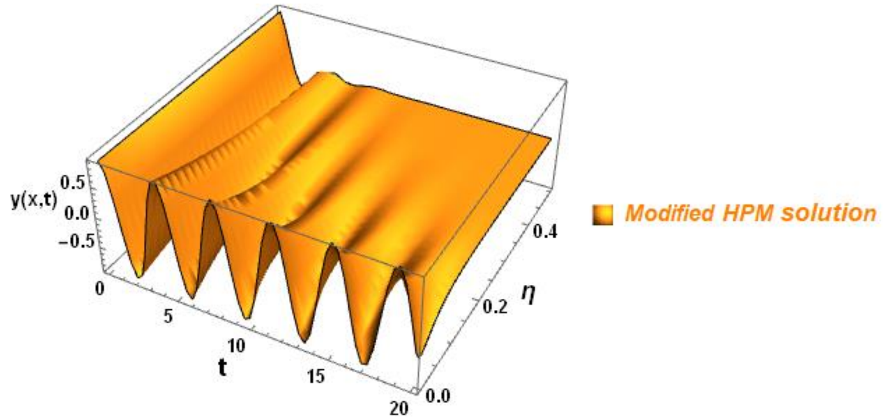

Fig. 4 The variation of temporal damping coefficient η for the same system as given in Fig. 1 except that the spatial variable is considered fixed at the value x=1

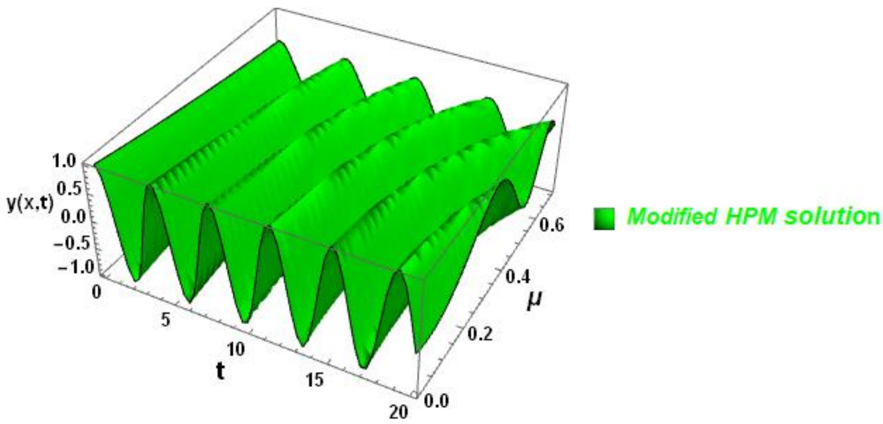

Fig. 5 The variation of spatial damping coefficient μ for the same system as given in Fig. 4 except that temporal damping coefficient η is considered fixed at the value η=0

3. Alternative Process Across the Traveling Wave Description

Traveling waves arise naturally in many physical systems, usually qualified by partial differential equations. Therefore, an alternative homotopy equation can be formulated by introducing traveling wave variable $\zeta(x,t)$ defined as

$$\zeta(x,t) = 2\mu x + 2P\eta t. \tag{45}$$

According to the above new independent variable, we have

$$y_t = 2P\eta y'(\zeta), \quad y_x = 2\mu y'(\zeta), \quad y_{tt} = 4P^2\eta^2 y''(\zeta), \quad y_{xx} = 4\mu^2 y''(\zeta), \tag{46}$$

where the prime denotes the total derivative with respect to variable ζ. Employing Eq. (45) with Eq. (11), it will handle the following damping Duffing equation:

$$y''(\zeta)+\frac{1}{P}y'(\zeta)+\omega_0^2 y(\zeta)=Ry^3(\zeta), \tag{47}$$

where ω_0^2 and R are given by the following notations:

$$\omega_0^2=\frac{\sigma}{4P\left(P\eta^2+\mu^2\right)},\qquad R=\frac{Q}{4P\left(P\eta^2+\mu^2\right)}. \tag{48}$$

Accordingly, the homotopy equation can be built in the form

$$H(y;q)=y''+\omega_0^2 y+q\left(\frac{1}{P}y'-Ry^3\right)=0;\ q\in[0,1] \tag{49}$$

The homotopy expansion is applied associated with the modification by the exponential decay parameter [3] as follows:

$$y(\zeta;q)=e^{-\frac{1}{2P}q\zeta}\left(y_0(\zeta)+qy_1(\zeta)+q^2y_2(\zeta)+\ldots\right), \tag{50}$$

Introducing modified frequency ω which is given by

$$\omega^2=\omega_0^2+q\omega_1+q^2\omega_2+\ldots, \tag{51}$$

where ω_1 is unknown, determined through the perturbation analysis.

Substituting Eqs. (50) and (51) into Eq. (49), and proceeding as requested by the homotopy perturbation method, the above unknowns can be determined and have the following results:

$$y_0(\zeta)=a\cos(\omega\zeta+\phi), \tag{52}$$

where a and ϕ are two arbitrary constants determined by the initial conditions.

Besides, we have

$$y_1(\zeta)=-\frac{Ra^3}{32\Omega^2}\cos(3\omega\zeta+3\varphi),\quad \text{under the condition } \omega_1=-\frac{3}{4}Ra^2. \tag{53}$$

Moreover, we get

$$y_2(\zeta)=\frac{Ra^3}{32\omega^2}\left(\frac{\zeta}{P}+\frac{3Ra^2}{32\omega^2}\right)\cos(3\omega\zeta+3\varphi)+\frac{3Ra^3}{128P\omega^2}\sin(3\omega\zeta+3\varphi)+\frac{R^2a^5}{1024\omega^4}\cos(5\omega\zeta+5\varphi), \tag{54}$$

under the condition:

$$\omega_2 = -\frac{1}{4P^2} + \frac{3\zeta}{4P} Ra^2 + \frac{3R^2a^4}{128\omega^2}. \tag{55}$$

As known, the approximate second-order solution of Eq. (47) is given by

$$y(\zeta) = \lim_{q\to 1}\left(e^{-\frac{1}{2P}q\zeta} \left(y_0(\zeta) + q y_1(\zeta) + q^2 y_2(\zeta) + \dots\right)\right), \tag{56}$$

Also, the approximate frequency is formulated by

$$\omega^2 = \lim_{q\to 1}\left(\omega_0^2 + q\omega_1 + q^2\omega_2 + \dots\right). \tag{57}$$

Inserting $y_0(\zeta)$,$y_1(\zeta)$ and $y_2(\zeta)$ into Eq. (50) yields

$$\frac{y(\zeta)}{ae^{-\frac{1}{2P}\zeta}} = \cos(\omega\zeta + \varphi) + \frac{Ra^2}{32\omega^2}\left(-e^{-\frac{1}{P}\zeta} + \frac{3Ra^2}{32\omega^2}\right)\cos(3\omega\zeta + 3\varphi)$$
$$+ \frac{3Ra^2}{128P\omega^2}\sin(3\omega\zeta + 3\varphi) + \frac{R^2a^4}{1024\omega^4}\cos(5\omega\zeta + 5\varphi). \tag{58}$$

where the frequency equation is determined [35] as

$$\omega^2 = \omega_0^2 - \frac{1}{4P^2} - \frac{3}{4}Ra^2 e^{-\frac{\zeta}{P}} + \frac{3R^2a^4}{128\omega^2}. \tag{59}$$

It is worth observing that through performing the final solution (58), we replace the terms (1-ζ/P+…) which refer to the first two terms of exponential exp(-ζ/P); therefore, this compact form is used in Eqs. (58) and (59).

Without upgrading the power of frequency ω^2, we proceed to obtain an approximate solution of Eq. (49). This approximate solution is formulated in the form

$$\omega^2 = \omega_0^2 - \frac{1}{4P^2} - \frac{3}{4}Ra^2 + \frac{3R^2a^4}{128\left(\omega_0^2 - \frac{1}{4P^2} - \frac{3}{4}Ra^2\right)}. \tag{60}$$

The necessary stability requirements need to be positive ω^2. This requires

$$\omega_0^2 - \frac{1}{4P^2} - \frac{3}{4}Ra^2 + \frac{3R^2a^4}{128\left(\omega_0^2 - \frac{1}{4P^2} - \frac{3}{4}Ra^2\right)} > 0. \tag{61}$$

The above stability is arranged in terms of the definition given in Eq. (48), which leads to the same stability condition as in Eq. (44).

3.1 Numerical estimations

The comparison between the numerical solution of the Duffing Eq. (47) and the modified HPM solution given by (58), with the full frequency as given by (59), is shown in Fig. 6. The numerical solution is made by the use of mathematical software. The calculation is made for the same system as given in Fig. 1. The numerical solution is located in red, while the analytical solution is plotted in blue. This graph indicates that there is an excellent agreement between the numerical solution and the analytical modified HPM solution.

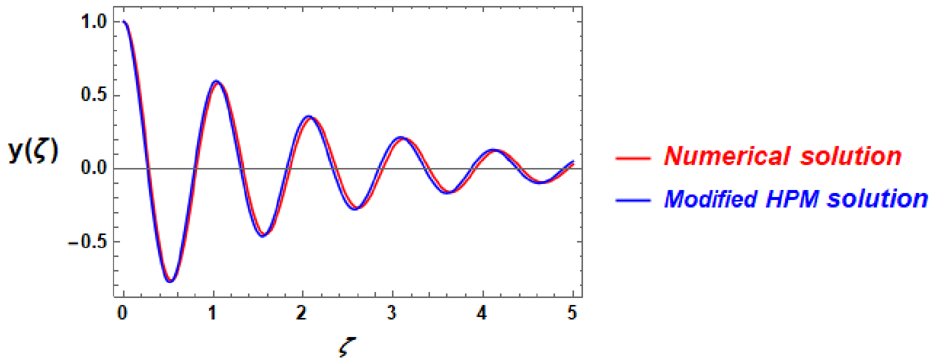

Fig. 6 The comparison between the numerical solution and the analytical solution to the Duffing Eq. (47), for the same system of Fig. 1

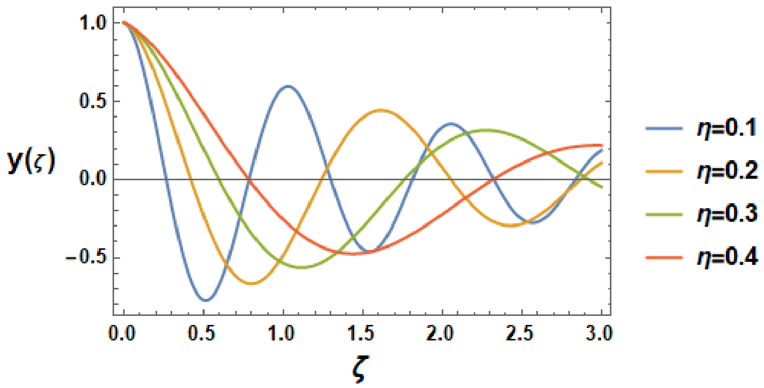

Fig. 7 The modified HPM solution in Eq. (58) for the same system as given in Fig. 2 with the variation of parameter η

In Fig. 7, we plot the modified HPM solution given by Eq. (58) with the full frequency as given by Eq. (59) for variation of temporal damping coefficient η=0.1, 0.2, 0.3, 0.4. Inspection of the graph shows that as η increases, the amplitude of the wave solution decreases gradually until vanishing. This indicates the loss of energy as η increases. The variation of spatial damping coefficient μ with fixation of η=0.1 is displayed in Fig. (8). The conclusion turns out to be similar to that of η. The examination of the impact of coefficient P on the picture of the wave solution is indicated in Fig. 9. It is observed that as P increases, so does the amplitude of the wave solution. This observation is expected because parameter P represents the inverse of the damping coefficient of the Duffing Eq. (47).

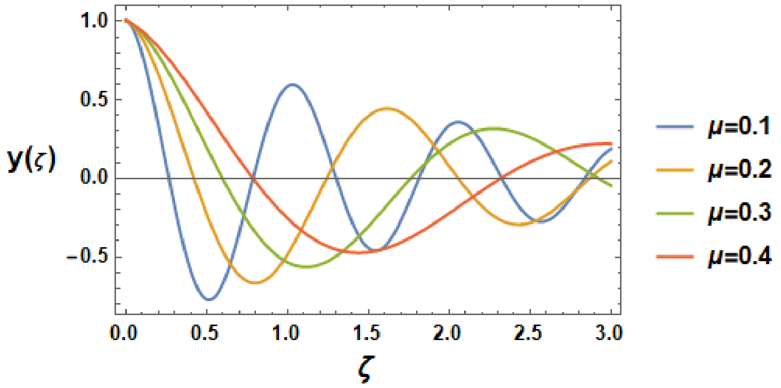

Fig. 8 The same system of Fig. 2 with the variation of μ

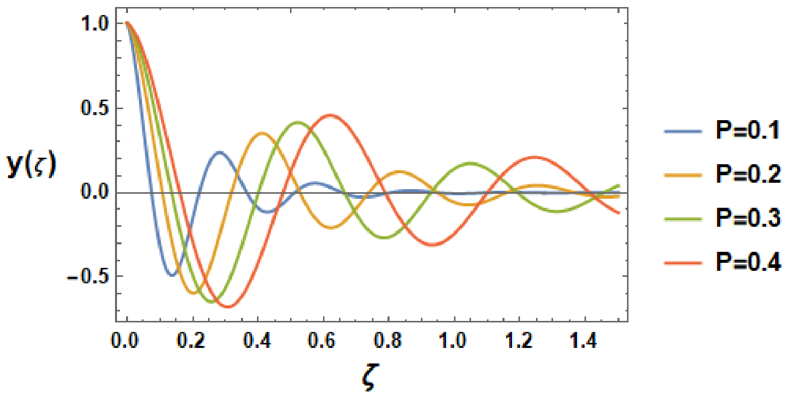

Fig. 9 The same system of Fig. 2 with the variation of P

4. RESULTS AND DISCUSSION

If damped parameters μ and η tend to zero in Eq. (11), the result is the classical Klein-Gordon equation:

$$y_{tt} + Py_{xx} + \sigma y = Qy^3; \quad y = y(x,t). \tag{62}$$

Further, let $\mu \to 0$ and $\eta \to 0$ in the frequency formula (38), it reduces to

$$\Omega^2 = \sigma - \frac{3}{4} A^2 Q - \frac{3Q^2 A^4}{128\Omega^2}. \tag{63}$$

This is equivalent to those obtained before in [16] by using the classical HPM. The classical HPM has successfully solved this equation. El-Dib [16] applied the He's-multiple scale technique to find the periodic solution and to obtain the stability conditions. Further, El-Dib [27] investigated Eq. (62) by applying the method of variable separation to variable $y(x,t)$, and investigated the periodic solution using the technique of the Multi-homotopy perturbations. These previous studies failed to analyze the nonlinear damped Klein-Gordon Eq. (11). In Ref. [35], the authors tried to overcome the shortcomings of the classical HPM. They converted the integer second-time partial derivative to the fractional nonlinear partial differential of order (α+1; $0<\alpha\leq 1$), and applied the properties of the Riemann–Liouville fraction derivatives, through the HPM. This technique allowed the authors to obtain the periodic solution of Eq. (11) and the stability conditions in terms of parameter α and letting $\alpha \to 1$ in the final results. The new modification to HPM discussed in the present study depends on the idea of the normal form technique. It is known that this technique is successfully used in the linear damped equation. In the current work, we succeeded to adapt the technique of the normal form when coupled with the HPM, in which the exponential decay parameters are established. The analysis shows that the amplitude behaves as an exponential decay by the damping parameter. An excellent agreement with the numerical solution is observed in these calculations. This scheme yielded a more effective result for the nonlinear oscillators and helped to overcome the shortcoming of the classical approach.

5. CONCLUSION

The periodic solution of nonlinear models has much significance and has drawn a great deal of interest. In the current work, we have investigated the solution of the damping nonlinear wave equation. The normal form transformation is used to solve the damping linear Klein-Gordon equation. Further, the frequency equation is estimated as displayed in Eqs. (12) and (13). Although the classical HPM is not successful in analyzing the damping nonlinear oscillator when it is associated with the exponential decay parameter as defined in Eq. (18), the solution is obtained. This modification has been successful in suppressing the shortcoming of the classical HPM. The first-order approximate solution is derived in Eq. (37), the frequency-amplitude equation is established in Eq. (42) and the stability condition is performed in Eq. (44). Furthermore, the traveling-wave description is used to transform the Klein-Gordon equation in the damping Duffing equation into a new dependent variable. The same procedure of homotopy perturbation with the exponential decay parameter is used, and the solution and the frequency formula are obtained. Numerical calculations are done and some graphs are displayed. The periodic damping analytical solution is found to have an excellent agreement with the numerical solution. This conclusion is reached in the case of the partial differential Eq. (11), and, also, in the case of the ordinary nonlinear Duffing Eq. (37).

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