Arc-disjoint Strong Spanning Subdigraphs of Semicomplete Compositions*

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Abstract

A **strong arc decomposition** of a digraph $D = (V, A)$ is a decomposition of its arc set A into two disjoint subsets A_1 and A_2 such that both of the spanning subdigraphs $D_1 = (V, A_1)$ and $D_2 = (V, A_2)$ are strong. Let T be a digraph with t vertices $u_1, \dots, u_t$ and let $H_1, \dots, H_t$ be digraphs such that H_i has vertices u_{i,j_i}, $1 \le j_i \le n_i$. Then the composition $Q = T[H_1, \dots, H_t]$ is a digraph with vertex set $\bigcup_{i=1}^t V(H_i) = \{u_{i,j_i} \mid 1 \le i \le t, 1 \le j_i \le n_i\}$ and arc set

$$\left(\bigcup_{i=1}^{t} A(H_i)\right) \bigcup \left(\bigcup_{u_i u_p \in A(T)} \{u_{ij_i} u_{pq_p} \mid 1 \le j_i \le n_i, 1 \le q_p \le n_p\}\right).$$

We obtain a characterization of digraph compositions $Q = T[H_1, \dots, H_t]$ which have a strong arc decomposition when T is a semicomplete digraph and each H_i is an arbitrary digraph. Our characterization generalizes a characterization by Bang-Jensen and Yeo (2003) of semicomplete digraphs with a strong arc decomposition and solves an open problem by Sun, Gutin and Ai (2019) on strong arc decompositions of digraph compositions $Q = T[H_1, \dots, H_t]$ in which T is semicomplete and each H_i is arbitrary. Our proofs are constructive and imply the existence of a polynomial algorithm for constructing a strong arc decomposition of a digraph $Q = T[H_1, \dots, H_t]$, with T semicomplete, whenever such a decomposition exists.

Keywords: strong spanning subdigraph; decomposition into strong spanning subdigraphs; semicomplete digraph; digraph composition.

AMS subject classification (2010): 05C20, 05C70, 05C76, 05C85.

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1 Introduction

We refer the reader to [1,2] for graph theoretical notation and terminology not given here. A **digraph** is not allowed to have parallel arcs or loops. A **directed multigraph** $D = (V, A)$ can have parallel arcs, i.e. A is a multiset. A directed multigraph $D = (V, A)$ is **strongly connected** (or **strong**) if there exists a path from x to y and a path from y to x in D for every pair of distinct vertices x, y of D. A directed multigraph D is k**-arc-strong** if $D - X$ is strong for every subset $X \subseteq A$ of size at most $k - 1$.

A directed multigraph $D = (V, A)$ has a **strong arc decomposition** if A can be partitioned into disjoint subsets A_1 and A_2 such that both (V, A_1) and (V, A_2) are strong [6]. A directed multigraph D is **semicomplete** if there is at least one arc between any pair of distinct vertices in D. In particular, a tournament is semicomplete digraph with just one arc between any pair of distinct vertices. (A semicomplete digraph can have two arcs between a pair x, y of distinct vertices: xy and yx.)

Bang-Jensen and Yeo [7] proved that it is NP-complete to decide whether a digraph has a strong arc decomposition. They also characterized semicomplete digraphs with a strong arc decomposition. Note that every digraph with a strong arc decomposition must be 2-arc-strong.

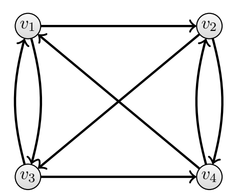


Figure 1: Digraph S_4

Theorem 1.1 *[7] A 2-arc-strong semicomplete digraph D has a strong arc decomposition if and only if D is not isomorphic to S_4, where S_4 is obtained from the complete digraph with four vertices by deleting the arcs of a cycle of length four (see Figure 1). Furthermore, a strong arc decomposition of D can be obtained in polynomial time when it exists.*

The following result by Bang-Jensen and Huang extends Theorem 1.1 to locally semicomplete digraphs. A digraph is **locally semicomplete** if every two vertices with a common out- or in-neighbour have an arc between them. Clearly, the class of locally semicomplete digraphs is a generalization of semicomplete digraphs. The **square** of a directed cycle $v_1 v_2 \dots v_n v_1$ is obtained by adding an arc from v_i to v_{i+2} for every $i \in [n]$, where $v_{n+1} = v_1$ and $v_{n+2} = v_2$.

Theorem 1.2 *[6] A 2-arc-strong locally semicomplete digraph D has a strong arc decomposition if and only if D is not the square of an even cycle.*

Let T be a digraph with t vertices $u_1, \dots, u_t$ and let $H_1, \dots, H_t$ be digraphs such that H_i has vertex set $\{u_{i,j_i} | 1 \leq j_i \leq n_i\}$ Then the **composition** $Q = T[H_1, \dots, H_t]$ is a digraph with vertex set $\cup_{i=1}^{t} V(H_i)$ and arc set

$$\left(\bigcup_{i=1}^{t} A(H_i)\right) \bigcup \left(\bigcup_{u_i u_p \in A(T)} \{u_{ij_i} u_{pq_p} \mid 1 \le j_i \le n_i, 1 \le q_p \le n_p\}\right).$$

We say that a composition $Q = T[H_1, \dots, H_t]$ is a **semicomplete composition** if T is semicomplete. In the important special case when each H_i has no arc we say that Q is an **extension** of T. In particular the class of **extended semicomplete digraphs** consists of all digraphs that are extensions of a semicomplete digraph, that is, of the form $Q = T[\overline{K}_{n_1}, \dots, \overline{K}_{n_t}]$ where T is a semicomplete digraph and $\overline{K}_r$ is a digraph on r vertices and no arcs.

Recently, Sun, Gutin and Ai [12] proved the following charaterization of a subset of semicomplete compositions with a strong arc decomposition, where $\overrightarrow{C}_3$ is a directed cycle on three vertices and $\overrightarrow{P}_2$ is a directed path on two vertices (that is, it is just an arc).

Theorem 1.3 *[12] Let T be a strong semicomplete digraph on $t \ge 2$ vertices and let $H_1, \dots, H_t$ be arbitrary digraphs, each with at least two vertices. Then $Q = T[H_1, \dots, H_t]$ has a strong arc decomposition if and only if Q is not isomorphic to one of the following three digraphs: $\overrightarrow{C}_3[\overline{K}_2, \overline{K}_2, \overline{K}_2]$, $\overrightarrow{C}_3[\overline{K}_2, \overline{K}_2, \overrightarrow{P}_2]$, $\overrightarrow{C}_3[\overline{K}_2, \overline{K}_2, \overline{K}_3]$.*

Remark 1.1 *Note that all three exceptions in Theorem 1.3 are extended semicomplete digraphs (the middle one is an extension of the unique strong tournament T_4^s on four vertices, see Figure 2).*

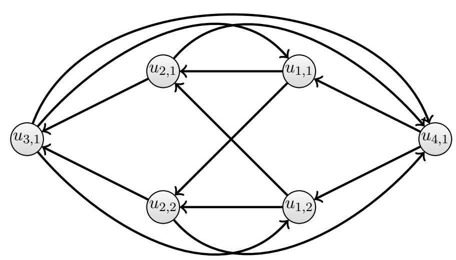


Figure 2: $T_4^s[\overline{K}_2, \overline{K}_2, \overline{K}_1, \overline{K}_1]$

In this paper, solving an open problem in [12], we obtain a characterization of **all** semicomplete compositions with a strong arc decomposition. Note that a digraph with a strong arc decomposition is 2-arc-strong. Our characterization is as follows:

Theorem 1.4 *Let T be a strong semicomplete digraph on $t \ge 2$ vertices and let $H_1, \dots, H_t$ be arbitrary digraphs. Then $D = T[H_1, \dots, H_t]$ has a strong arc decomposition if and only if D is 2-arc-strong and is not isomorphic to one of the following four digraphs: S_4, $\overrightarrow{C}_3[\overline{K}_2, \overline{K}_2, \overline{K}_2]$, $\overrightarrow{C}_3[\overline{K}_2, \overline{K}_2, \overrightarrow{P}_2]$, $\overrightarrow{C}_3[\overline{K}_2, \overline{K}_2, \overline{K}_3]$.*

It is remarkable that all the four exemptions in this theorem are simply the union of the exemptions in Theorems 1.1 and 1.3. However, we see no simple way to prove our theorem by a direct reduction to Theorems 1.1 and 1.3 and do not believe that such a reduction exists. Note that the digraphs covered by Theorem 1.4 but not by Theorem 1.3 are all semicomplete decompositions in which at least one H_i has just one vertex. Having just one vertex in some H_i's makes the strong arc decomposition problem on semicomplete digraphs much more complicated than the case when all H_i's have at least two vertices since in the latter case the semicomplete composition has more symmetries (i.e., authomorphisms) that can be exploited in the proofs. Theorem 1.1 covers just a special subcase of the former case and its proof in [7] is not easier than that of Theorem 1.3 in [12].

Apart from Theorems 1.1 and 1.3 used in our proof of Theorem 1.4, we apply several other results including Edmonds' branching theorem, the existence of nice vertex decompositions proved in [4] (for details see the next section) and an extension of Theorem 1.1 to directed multigraphs (Theorem 3.3) proved in this paper. Interestingly, the extension of Theorem 1.1 has three further exceptions.

Note that the class of strong semicomplete compositions is a generalization of strong quasi-transitive digraphs by the following recursive characterization of quasi-transitive digraphs by Bang-Jensen and Huang [5]. A digraph $D = (V, A)$ is **quasi-transitive** (**transitive**) if for any triple x, y, z of distinct vertices of D, $xy, yz \in A$ implies that there is an arc between x and z (from x to z). Clearly, the class of quasi-transitive digraphs is a generalization of semicomplete digraphs. For a recent overview of quasi-transitive digraphs and their generalization, see [10].

Theorem 1.5 *[5] Let D be a quasi-transitive digraph.*

(a) *If D is strong, then there exists a strong semicomplete digraph S with s vertices and quasi-transitive digraphs $Q_1, Q_2, \ldots, Q_s$ such that Q_i is either a vertex or is non-strong and $D = S[Q_1, Q_2, \ldots, Q_s]$.*

(b) *If D is not strong, then there exist a transitive oriented graph T with t vertices and strong quasi-transitive digraphs $H_1, H_2, \ldots, H_t$ such that $D = T[H_1, H_2, \ldots, H_t]$.*

Theorem 1.4 implies a characterization of quasi-transitive digraphs with a strong arc decomposition (this solves another open question in [12]). In fact the following follows immediately from Theorems 1.5 and 1.4 (observe also that all four exceptions in Theorem 1.4 are quasi-transitive digraphs).

Theorem 1.6 *Let D be a quasi-transitive digraph. D has a strong arc decomposition if and only if D is 2-arc-strong and is not isomorphic to one of the following four digraphs: S_4, $\overrightarrow{C}_3[\overline{K}_2, \overline{K}_2, \overline{K}_2]$, $\overrightarrow{C}_3[\overrightarrow{P}_2, \overline{K}_2, \overline{K}_2]$, $\overrightarrow{C}_3[\overline{K}_2, \overline{K}_2, \overline{K}_3]$.*

To see that strong quasi-transitive digraphs form a relatively small subset of strong semicomplete compositions, note that the Hamiltonicity problem is polynomial-time solvable for quasi-transitive digraphs [11], but NP-complete for strong semicomplete compositions. Indeed, consider a digraph H with vertex set U and an extra vertex x. Construct a new digraph D by adding to H all arcs between U and x. Observe that D is a strong semicomplete composition and it has a Hamilton cycle if and only if H has a Hamilton path.

The paper is organized as follows. The next section provides additional terminology and notation and a number of results used later in the paper. We prove an extension of Theorem 1.1 to directed multigraphs in Section 3. In Section 4 we prove a lemma which simplifies our further proofs: every 2-arc-strong semicomplete composition containing a cut-vertex has a strong arc decomposition. Our main result, Theorem 1.4, is proved in Section 5. However, the proof of Theorem 1.4 uses our main technical result, Theorem 5.1, which is proved in Section 6. We complete the paper in Section 7, where we briefly discuss some open problems.

2 Additional Terminology, Notation and Results

Let $D = (V, A)$ be a directed multigraph. The **multiplicity**, $\mu(x, y)$ of an arc xy in D is the number of copies of xy in D. An arc is **single** (**double**, respectively) if it is of multiplicity 1 (2, respectively). For $S \subset V$ such that $S \neq \emptyset$, let $(S, \overline{S})_D$ be the set of arcs of D with tails in S and heads in $,\overline{S}$. The sets of tails in S and heads in $\overline{S}$ of arcs in $(S, \overline{S})_D$ are denoted by $N^+_D(S)$ and $N^-_D(\overline{S})$, respectively. The cardinalities of $N^-_D(\overline{S})$ and $N^+_D(S)$ are denoted by $d^-_D(\overline{S})$ and $d^+_D(S)$, respectively.

If X and Y are disjoint vertex sets in a digraph, then we use the notation $X \to Y$ to denote that xy is an arc for every choice of $x \in X, y \in Y$. For a non-empty subset X of V, the subdigraph of D induced by X is denoted by $D\langle X\rangle$. Let $P = x_1x_2 \dots x_p$ be a path in D. For $1 \leq i \leq j \leq p$, $P[x_i, x_j] = x_ix_{i+1} \dots x_j$ denotes the subpath of P from x_i to x_j. An arc uv of a digraph $D = (V, A)$ is a **cut-arc** if $D - uv$ is not strongly connected. A vertex $v \in V$ is a **cut-vertex** if $D - v$ is not strongly connected.

A **vertex decomposition of a digraph** D is a partition $(S_1, \dots, S_p)$, $p \geq 1$, of its vertex set. The **index** of vertex v in the decomposition, denoted by $\mathrm{ind}(v)$, is the integer i such that $v \in S_i$. An arc uv is **forward** if $\mathrm{ind}(u) < \mathrm{ind}(v)$, **backward** if $\mathrm{ind}(u) > \mathrm{ind}(v)$. A vertex decomposition $(S_1, \dots, S_p)$ is **strong** if $D\langle S_i\rangle$ is strong for all $1 \leq i \leq p$. A **nice vertex decomposition** of a digraph D is a strong decomposition such that the set of cut-arcs of D is exactly the set of backward arcs.

Theorem 2.1 *[4] Every strong semicomplete digraph S of order at least 4 admits a nice decomposition $(S_1, S_2, \dots, S_p)$ and this decomposition is unique. The sets S_i, $i \in [p]$ are exactly the strong components of the digraph D that we obtain from S by deleting all cut-arcs.*

Proposition 2.2 *[4] Let $(S_1, \dots, S_p)$ be a nice decomposition of a strong semicomplete digraph D. The following properties hold:*

(i) If u_1v_1 and u_2v_2 are two cut-arcs, then $\mathrm{ind}(u_1) \neq \mathrm{ind}(u_2)$ and $\mathrm{ind}(v_1) \neq \mathrm{ind}(v_2)$.

(ii) If $\text{ind}(u_1) < \text{ind}(u_2)$ *then* $\text{ind}(v_1) < \text{ind}(v_2)$.

The following simple lemma sometimes allows one to reduce the number of digraphs under consideration in proofs of results on strong arc decompositions.

Lemma 2.3 *[12] Let $D = Q[H_1, \dots, H_t]$, where Q is an arbitrary digraph and every H_i has no arcs. If an induced subdigraph D' of D with at least one vertex in each H_i has a strong arc decomposition, then so has D.*

Lemma 2.4 *Let $D = R[\overline{K}_{n_1}, \dots, \overline{K}_{n_r}]$ be an extension of a digraph R. If D is 2-arc-strong and some n_i is larger than 2, then the digraph D' obtained from D by deleting a vertex from $\overline{K}_{n_i}$ is also 2-arc-strong.*

Proof: Let x be the vertex that we deleted from $H_i = \overline{K}_{n_i}$ and let y, z be two other vertices of H_i. Suppose that D' is not 2-arc-strong. Then there exists a vertex partition $(X, \overline{X})$ of $V(D')$ so that there is at most one arc from X to $\overline{X}$ in D'. As D is 2-arc-strong this implies that x has an out-neighbour w^+ in $\overline{X}$ and an in-neighbour w^- in X. However now w^-yw^+ and w^-zw^+ are two arc-disjoint paths from X to $\overline{X}$ in D', contradicting the fact that there is at most one arc from X to $\overline{X}$ in D'. Hence D' is 2-arc-strong. $\square$

An **out-branching** (**in-branching**, resp.) B **rooted** at vertex z in a directed multigraph D is a spanning subdigraph, which is an oriented tree such that only z has in-degree (out-degree, resp.) zero. A vertex of an out-branching (in-branching, resp.) is called a **leaf** if its out-degree (in-degree, resp.) equals zero. We will use the following result called Edmonds' branching theorem.

Theorem 2.5 *[9] A directed multigraph $D = (V, A)$ with a vertex z, has k arc-disjoint out-branchings rooted at z if and only if $d^-(X) \geq k$ for all non-empty $X \subseteq V \setminus \{z\}$.*

Note that, by Menger's theorem, the condition of Theorem 2.5 is equivalent to the existence of k arc-disjoint paths from z to any vertex $x \in V \setminus \{z\}$.

3 Extending Theorem 1.1 to Semicomplete Directed Multigraphs

If the arc xy in a directed multigraph D has multiplicity $\mu(x, y) \geq 3$, we may delete $\mu(x, y) - 2$ copies xy of D and the resulting directed multigraph has a strong arc decomposition if and only if so has D. Thus, we may assume that all directed multigraphs considered in this paper have no arcs of multiplicity 3 or more.

Recall the semicomplete digraph S_4 from Theorem 1.1. Without loss of generality, we may assume that $V(S_4) = \{v_1, v_2, v_3, v_4\}$ and $A(S_4) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_1, v_1v_3, v_3v_1, v_2v_4, v_4v_2\}$. We call the cycle $v_1v_2v_3v_4v_1$ the **Hamilton cycle** of S_4 and cycles $v_1v_3v_1$ and $v_2v_4v_2$ **2-cycles** of S_4.

To prove Theorem 3.3, we will use the following lemma.

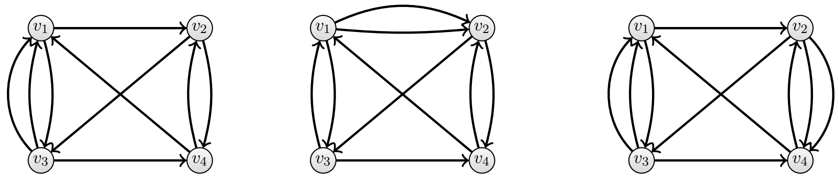


Figure 3: The digraphs $S_{4,1}$, $S_{4,2}$, $S_{4,3}$

Lemma 3.1 *Let D be a directed multigraph with no arcs of multiplicity more than 2 and let D contain S_4 as a spanning subdigraph. Then D has no strong arc decomposition if and only if D is isomorphic to one of the following* **exceptional digraphs***:*

- *S_4.*
- *A directed multigraph obtained from S_4 by adding a copy of an arc in S_4 (isomorphic to $S_{4,1}$ or $S_{4,2}$, see Figure 3).*
- *A directed multigraph obtained from S_4 by adding a copy of one arc in each of the two 2-cycles of S_4 (isomorphic to $S_{4,3}$, see Figure 3).*

Proof: Observe that the Hamilton cycle $v_1v_2v_3v_4v_1$ is the only Hamilton cycle in S_4 and $|A(S_4)| = 8$. Let $D' \in \{S_4, S_{4,1}, S_{4,2}, S_{4,3}\}$ be arbitrary and for the sake of contradiction assume that D' has a strong arc decomposition consisting of strong subdigraphs D_1, D_2 with arc sets A_1 and A_2.

First consider the case when $|A_1| = 4$, which implies that A_1 contains the arcs of the unique (up to copies of the same arc) Hamilton cycle $v_1v_2v_3v_4v_1$. However $D' - \{v_1v_2, v_2v_3, v_3v_4, v_4v_1\}$ is not strong for any digraph in $\{S_4, S_{4,1}, S_{4,2}, S_{4,3}\}$, implying that $|A_1| \geq 5$. Analogously $|A_2| \geq 5$, which implies that $D' = S_{4,3}$ and $|A_1| = |A_2| = 5$.

In $D' = S_{4,3}$ we note that $d^+(v_4) = 2$, so we may without loss of generality assume that $v_4v_2 \in A_1$ and $v_4v_1 \in A_2$ and obtain the following:

- $d^-(v_2) = 2$ implies that $v_1v_2 \in A_2$ (as $v_4v_2 \in A_1$).
- $d^+(v_1) = 2$ implies that $v_1v_3 \in A_1$ (as $v_1v_2 \in A_2$).
- $d^-(v_3) = 2$ implies that $v_2v_3 \in A_2$ (as $v_1v_3 \in A_1$).

Thus, all arcs from $\{v_2, v_4\}$ to $\{v_1, v_3\}$ belong to A_2, a contradiction with the assumption that D_1 is strong.

Assume now that D is not isomorphic to any directed multigraph described in the statement of the lemma. We will show that D has a strong arc decomposition with subdigraphs with disjoint arc sets A_1 and A_2. We have four cases, which cover all possibilities subject to isomorphism. It is not hard to check that the subdigraphs induced by both A_1 and A_2 given below are strong (see Figure 4).

Case 1: $\mu(v_2, v_4) = \mu(v_4, v_2) = 2$. Then let $A_1 = \{v_1v_2, v_2v_3, v_3v_1, v_2v_4, v_4v_2\}$ and $A_2 = \{v_1v_3, v_3v_4, v_4v_1, v_2v_4, v_4v_2\}$.

Case 2: $\mu(v_1, v_2) = \mu(v_2, v_3) = 2$. Then let A_1 contain the arcs of the Hamilton cycle of S_4 and A_2 the rest of the arcs of D.

Case 3: $\mu(v_1, v_2) = \mu(v_3, v_4) = 2$. Then let $A_1 = \{v_1v_2, v_2v_3, v_3v_4, v_4v_2, v_3v_1\}$ and A_2 the rest of the arcs of D.

Case 4: $\mu(v_1, v_2) = \mu(v_3, v_1) = 2$. Then let $A_1 = \{v_1v_2, v_2v_4, v_4v_1, v_1v_3, v_3v_1\}$ and A_2 the rest of the arcs of D.

It is not hard to check that up to isomorphism all the exception digraphs are depicted in Figures 1 and 3. □

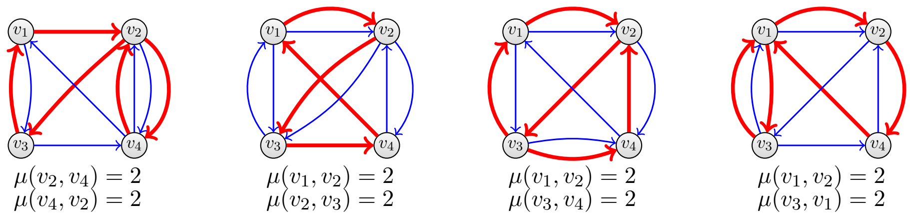


Figure 4: The strong arc decompositions given in the proof of Lemma 3.1.

The following theorem was used in [7] to prove Theorem 1.1. We will use it to prove Theorem 3.3. Note that while in [7] Theorem 3.2 was stated only for semicomplete digraphs, its proof in [7] shows that it holds also for semicomplete directed multigraphs.

Theorem 3.2 *[7] Let $k \geq 1$ and let $D = (V, A)$ be a k-arc-strong semicomplete directed multigraph such that there is a set $S \subset V$, with $2 \leq |S| \leq |V|-2$ and $|(S, V - S)_D| = k$. Then there exist k arc-disjoint strong spanning subgraphs of D except if $D = S_4$.*

Now we are ready to prove the main result of this section.

Theorem 3.3 *A 2-arc-strong semicomplete directed multigraph $D = (V, A)$ has a strong arc decomposition if and only if it is not isomorphic to one of the exceptional digraphs depicted in Figures 1 and 3. Furthermore, a strong arc decomposition of D can be obtained in polynomial time when it exists.*

Proof: We are going to prove the first part of the statement by induction over $n = |V|$ and then over the number of double arcs. The second part of the statement then follows as our proof is constructive. If there are no double arcs, then the claim follows from Theorem 1.1. If $n = 2$ then D is a directed multigraph consisting of two vertices u, v with $\mu(u, v) = \mu(v, u) = 2$. Clearly this has a strong arc decomposition.

Let D be a semicomplete directed multigraph on at least 3 vertices with a double arc uv. If we can delete one copy of uv and still have a 2-arc-strong semicomplete directed multigraph D', then the claim follows by the induction hypothesis, so we may assume that D' is not 2-arc-strong. Hence there is a partition $(X, V-X)$ of V with $u \in X$ and $v \in V-X$ so that the two copies of uv are the only arcs from X to $V-X$. If $\min\{|X|, |V-X|\} \geq 2$, then it follows from Theorem 3.2 that D has a strong arc decomposition. Hence, we may assume without loss of generality that $X = \{u\}$.

Let D^* be the digraph obtained from D by contracting $\{u, v\}$ into one vertex, say w (that is remove $\{u, v\}$ and add w such that for all $x \in V(D) \setminus \{u, v\}$ we have $\mu_{D^*}(w, x) = \mu_D(u, x) + \mu_D(v, x)$ and $\mu_{D^*}(x, w) = \mu_D(x, u) + \mu_D(x, v)$). Clearly D^* is 2-arc-strong since any cut $(X, V-X)$ in D^* gives a cut in D (by replacing w by $\{u, v\}$) with equaly many arcs across. By the induction hypothesis, D^* has a strong arc decomposition unless it has four vertices and is one of the exceptions from Lemma 3.1. However this is not the case as no vertex in these exceptions is an out-neighbour of all the other three vertices (note that $\mu(x, w) \geq 1$ for all $x \in V(D^*) - w$ as u is dominated by $V(D) - \{u, v\}$). Hence D^* has a strong arc decomposition D_1, D_2 and it remains to show that this can be modified to a strong arc decomposition of D.

Start by replacing w by u, v in each of D_1, D_2 and then add a copy of uv to each of the new versions of D_1 and D_2 and for the $\mu_{D^*}(x, w)$ arcs from every $x \in V \setminus \{u, v\}$ into w in D^* let $\mu_D(x, u)$ of these go to u and $\mu_D(x, v)$ of these go to v (keeping them in the same D_i as they were before). Note that all arcs out of w in D^* correspond to arcs out of v in D. If we can do the above procedure such that u receives an arc into it in both D_1 and D_2, then we obtain a strong arc decomposition of D. Furthermore this is always possible if there exist two vertices z_1, z_2 so that z_i is an in-neighbour of u in D_i (in D^*), $i \in \{1, 2\}$ and either $z_1 \neq z_2$ or $z_1 = z_2$ and this vertex has a double arc to u in D. So we may assume that this is not the case, which implies that $n = 3$ and $V = \{u, v, z\}$. By the arguments above and the fact that D is 2-arc-strong we get that vz is a double arc and there is at least one arc from v to u since zu is not a double arc. Now we get the desired strong arc decomposition by taking the two sets of arcs $\{uv, vz, zu\}$ and $\{uv, vu, vz, zv\}$.

□

4 Semicomplete compositions containing a cut-vertex

In this section we will prove the following lemma, which will turn out to be very useful in the proofs below, and is of interest in its own right.

Lemma 4.1 *Let $D = T[H_1, \ldots, H_t]$ be 2-arc-strong ($t \geq 2$), where T is a semicomplete digraph and every H_i is an arbitrary digraph. If D contains a cut-vertex then D has a strong arc decomposition.*

Proof: Let u be a cut-vertex in D and let $D' = D - u$. Let (X, Y) be a cut in D' such that there is no arc from $Y = V(D') - X$ to X in D'. Let $D^* = D\langle X \cup \{u\}\rangle$ and let $D^{**} = D\langle Y \cup \{u\}\rangle$. Note that $|V(D^*)| \geq 2$ and $|V(D^{**})| \geq 2$. We now prove the following claims.

Claim 1. We may assume that D' contains vertices from more than one H_i.

Proof of Claim 1: Suppose that $V(D') \subseteq V(H_1)$. We will prove that then D has a strong arc decomposition, which will complete the proof of Claim 1.

Since $D = T[H_1, \ldots, H_t]$ and $t \geq 2$, this implies that $t = 2$, T is a 2-cycle, $V(H_2) = \{u\}$ and for every $v \in V(H_1)$ we have $vu, uv \in A(D)$. Let $Q_1, Q_2, \ldots, Q_l$ be strong components in D', such that there is no arc from Q_i to Q_j when $i > j$. We will now construct a strong arc decomposition, (G_1, G_2) of D as follows.

For all Q_i with $|V(Q_i)| \geq 2$, do the following. Let $x_i y_i$ be any arc in Q_i. Add the arcs $(A(Q_i) \setminus \{x_i y_i\}) \cup \{x_i u, u y_i\}$ to G_1 and add all arcs $\{u x_i, x_i y_i, y_i u\}$ and all arcs $\{uw, wu\}$ for all $w \in V(Q_i) \setminus \{x_i, y_i\}$ to G_2. Note that this implies that $G_a\langle V(Q_i) \cup \{u\}\rangle$ is strong for $a = 1, 2$.

Now add all arcs between different Q_i's to G_1. Furthermore for all Q_i with $|V(Q_i)| = 1$ assume that $V(Q_i) = \{x_i\}$ and add $\{x_i u, u x_i\}$ to G_2. As $d^+(x_i) \geq 2$ and $d^-(x_i) \geq 2$ in D we note that $2 \leq i \leq l - 1$ and x_i has an arc into it from a Q_j with $j < i$ and an arc out of it to a Q_k with $k > i$ and these arcs belong to G_1. It follows that x_i therefore belongs to a path from a Q_a to a Q_b in G_1, where $a < b$ and $|V(Q_a)| \geq 2$ and $|V(Q_b)| \geq 2$. Therefore (G_1, G_2) is a strong arc decomposition in D.

Claim 2. We may assume that $X \cap V(H_i) = \emptyset$ or $Y \cap V(H_i) = \emptyset$ for all $i \in \{1, 2, \ldots, t\}$.

Proof of Claim 2: Assume without loss of generality that $X \cap V(H_1) \neq \emptyset$ and $Y \cap V(H_1) \neq \emptyset$. By Claim 1 either X or Y contains a vertex not in H_1. Assume without loss of generality that $Y \setminus V(H_1) \neq \emptyset$. Let $X' = X \cup V(H_1) \setminus \{u\}$ and let $Y' = Y \setminus (V(H_1) \cup \{u\})$. Note that (X', Y') is a cut in D' with no arc from Y' to X' as otherwise there are arcs from Y' to H_1, contradicting that there is no arc from Y to X. The process of moving from (X, Y) to (X', Y') decreased the number of H_i with vertices in both X and Y, so continuing this process we will obtain that $X \cap V(H_i) = \emptyset$ or $Y \cap V(H_i) = \emptyset$ for all $i \in \{1, 2, \ldots, t\}$. This completes the proof of Claim 2.

Claim 3. There exists two arc-disjoint out-branchings in D^* both rooted at u and there exists two arc-disjoint in-branchings in D^{**} both rooted at u.

Proof of Claim 3: Let $w \in X$ be arbitrary. As D is 2-arc-strong there are 2 arc-disjoint paths, P_1 and P_2, from u to w in D. We note that $V(P_1) \subseteq V(D^*)$ and $V(P_2) \subseteq V(D^*)$, as any path from a vertex not in X to w goes through u. Therefore there exists two arc-disjoint paths from u to w in D^*. By Edmonds' branching theorem, Theorem 2.5, two arc-disjoint out-branchings exist in D^*, both rooted at u.

Analogously if $w \in Y$ then there exist two arc-disjoint paths from w to u in D and therefore also in D^{**}. Again, by Theorem 2.5, there exists two arc-disjoint in-branchings in D^{**} both rooted at u. This completes the proof of Claim 3.

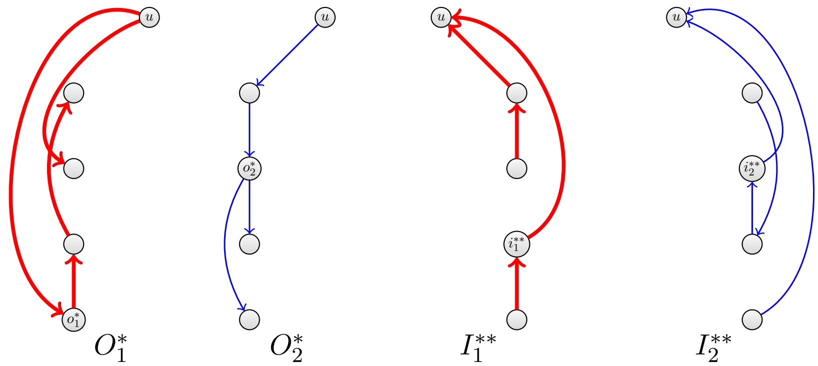


Figure 5: An example of O_1^*, O_2^*, I_1^{**} and I_2^{**} and o_1^*, o_2^*, i_1^{**} and i_2^{**}.

Definition of O_1^*, O_2^*, I_1^{**} and I_2^{**}: Let O_1^* and O_2^* be two arc-disjoint out-branchings in D^* rooted at u and let I_1^{**} and I_2^{**} be two arc-disjoint in-branchings in D^{**} rooted at u found in Claim 3.

Claim 4. Each branching O_1^*, O_2^*, I_1^{**} and I_2^{**} contains a vertex that is neither a leaf nor the root u.

Denote these vertices by o_1^*, o_2^*, i_1^{**} and i_2^{**}, respectively (see Figure 5).

Proof of Claim 4: Consider O_1^*. As $|V(D^*)| \geq 2$ we note that there is an arc $uv \in O_2^*$. This implies that $uv \notin O_1^*$ (as we do not have parallel arcs in D). However some arc, say o_1^*v, enters v in O_1^*. Now o_1^* is not the root of O_1^* and is also not a leaf of O_1^*. The cases of O_2^*, I_1^{**} and I_2^{**} can be proved analogously, which completes the proof of Claim 4.

Definition of G_1 and G_2: We now define an arc decomposition of D as follows (see Figures 5 and 6 for an illustration). Let G_1 contain all arcs of O_1^* and I_1^{**} and let G_2 contain all arcs of O_2^* and I_2^{**}. Note that all arcs between o_1^* and $V(D) \setminus V(D^*)$ exist (by Claim 2 and the fact that T is semicomplete) and go out of o_1^*. Add all arcs from o_1^* to $V(D) \setminus (V(D^*) \cup \{i_2^{**}\})$ to G_2. Analogously add all arcs from o_2^* to $V(D) \setminus (V(D^*) \cup \{i_1^{**}\})$ to G_1. Also, add all arcs from $V(D) \setminus (V(D^{**} \cup \{o_2^*\})$ to i_1^{**} to G_2 and add all arcs from $V(D) \setminus (V(D^{**} \cup \{o_1^*\})$ to i_2^{**} to G_1. Any remaining arcs from D which have not been added to G_1 or G_2 yet can be added arbitrarily. This completes the definition of G_1 and G_2.

Claim 5: (G_1, G_2) is a strong arc decomposition of D.

Proof of Claim 5: First let $v \in X$ be arbitrary. We will now show that there exists a (v, u)-path in G_1. As $O_1^* \subseteq G_1$, there is a path, P_1, from v to a leaf l^* in O_1^*. By construction the arc $l^*i_2^{**}$ belongs to G_1 (as $l^* \neq o_1^*$, since l^* is a leaf in O_1^* and o_1^* is not a leaf). As $I_1^{**} \subseteq G_1$, there is a path, P_2, from i_2^{**} to u in I_1^{**}. The path P_1P_2 is now the desired (v, u)-path in G_1.

Analogously we can show that there exists a (v, u)-path in G_2. Furthermore as O_1^* and O_2^* are out-branchings in G_1 and G_2, respectively we can also find a (u, v)-path in both G_1 and G_2.

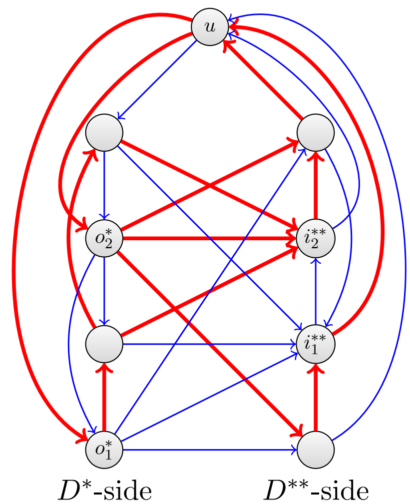


Figure 6: An illustration of G_1 and G_2 obtained from O_1^*, O_2^*, I_1^{**} and I_2^{**} seen in Figure 5. The thick arcs give us G_1 and the thin arcs G_2. Note that both G_1 and G_2 induce strong spanning subdigraphs.

Let $w \in Y$ be arbitrary. Analogously we can find a (u, w)-path in both G_1 and G_2, by considering a path from a leaf l_r^{**} in G_r ($r \in \{1, 2\}$) such that there exists a (l_r^{**}, w)-path in I_r^{**} and noting that $o_{3-r}^* l_r^{**}$ is an arc in G_r and there exists a (u, o_{3-r}^*)-path in O_r^*. As I_1^{**} and I_2^{**} are in-branchings in G_1 and G_2, respectively we can also find a (w, u)-path in both G_1 and G_2.

This implies that every vertex in D' has a path to u and a path from u in G_1 and in G_2, showing that G_1 and G_2 is a strong arc decomposition, thereby proving Claim 5 and the lemma. □

5 Main Results

Our main technical result is the following theorem. Recall that by Remark 1.1, $\overrightarrow{C}_3[\overline{K}_2, \overline{K}_2, \overrightarrow{P}_2] = T_4^s[\overline{K}_2, \overline{K}_2, \overline{K}_1, \overline{K}_1]$, where T_4^s is the unique strong tournament on four vertices, see Figure 2.

Theorem 5.1 *Let $D = T[\overline{K}_{n_1}, \dots, \overline{K}_{n_t}]$ be a 2-arc-strong extended semicomplete digraph where $n_i \le 2$ for $i \in [t]$. If D is not isomorphic to one of the following three digraphs: S_4, $\overrightarrow{C}_3[\overline{K}_2, \overline{K}_2, \overline{K}_2]$, $\overrightarrow{C}_3[\overline{K}_2, \overline{K}_2, \overrightarrow{P}_2] = T_4^s[\overline{K}_2, \overline{K}_2, \overline{K}_1, \overline{K}_1]$ then D has a strong arc decomposition.*

Before proving this theorem in the next section, we use it to prove Theorem 5.2, which is the special case of our main result, Theorem 1.4, for extended semicomplete digraphs. Theorem 5.2 and Lemma 5.3 will imply Theorem 1.4.

Theorem 5.2 *Let $D = T[\overline{K}_{n_1}, \ldots, \overline{K}_{n_t}]$ be a 2-arc-strong extended semicomplete digraph. If D is not isomorphic to one of the following four digraphs: S_4, $\overrightarrow{C}_3[\overline{K}_2, \overline{K}_2, \overline{K}_2]$, $\overrightarrow{C}_3[\overline{K}_2, \overline{K}_2, \overrightarrow{P}_2]$, $\overrightarrow{C}_3[\overline{K}_2, \overline{K}_2, \overline{K}_3]$ then D has a strong arc decomposition.*

Proof: Let $\mathcal{C} = \{\overrightarrow{C}_3[\overline{K}_2, \overline{K}_2, \overline{K}_2], \overrightarrow{C}_3[\overline{K}_2, \overline{K}_2, \overrightarrow{P}_2], \overrightarrow{C}_3[\overline{K}_2, \overline{K}_2, \overline{K}_3]\}$. We will prove the theorem by induction over $|V(D)|$. If $|V(D)| \leq 3$, then the theorem clearly holds, so the base case holds. If $n_i \leq 2$ for all $i = 1, 2, \ldots, t$ then we are done by Theorem 5.1, so we may assume that $n_j \geq 3$ for some j. Let $D' = D - u_{j,n_j}$. Since $n_j - 1 \geq 2$, $D' \neq S_4$.

It follows from Lemma 2.4 that D' is 2-arc-strong and hence it fulfils the statement of the theorem by induction. If $D' \notin \mathcal{C}$, then, by induction, it has a strong arc decomposition and hence D also has a strong arc decomposition by Lemma 2.3. Hence we may assume that $D' \in \mathcal{C}$ and consider the corresponding three cases.

Case 1: $D' = \overrightarrow{C}_3[\overline{K}_2, \overline{K}_2, \overline{K}_2]$**.** This implies that $D = \overrightarrow{C}_3[\overline{K}_2, \overline{K}_2, \overline{K}_3]$, and therefore $D \in \mathcal{C}$, which completes this case.

Case 2: $D' = \overrightarrow{C}_3[\overline{K}_2, \overline{K}_2, \overline{K}_3]$**.** In this case $D = \overrightarrow{C}_3[\overline{K}_2, \overline{K}_3, \overline{K}_3]$ or $D = \overrightarrow{C}_3[\overline{K}_2, \overline{K}_2, \overline{K}_4]$. These digraphs have strong arc decompositions by Theorem 1.3.

Case 3: $D' = \overrightarrow{C}_3[\overline{K}_2, \overline{K}_2, \overrightarrow{P}_2]$**.** As in Case 2, D has a strong arc decomposition by Theorem 1.3.

This completes the proof of the theorem. □

Now Theorem 5.2 and the following lemma imply our main result, Theorem 1.4.

Lemma 5.3 *Let $D = T[H_1, \ldots, H_t]$ be 2-arc-strong ($t \geq 2$), where T is a semicomplete digraph and every H_i is an arbitrary digraph. Then at least one of the following cases holds.*

(a) *D has a strong arc decomposition.*

(b) *D is an extended semicomplete digraph.*

(c) *For every $i \in [t]$ and every arc e of H_i, $D - e$ is 2-arc-strong.*

Proof: For the sake of contradiction assume that none of (a)-(c) hold and let $D = (V, A)$. As D is not an extended semicomplete digraph (otherwise (b) holds) there exists an arc in some H_i and since (c) does not hold we can choose $i \in [t]$ and an arc $e = uv$ of H_i such that the digraph $D' = D - e$ is strong, but not 2-arc-strong. Let $(X, V - X)$ be a cut in D' such that there is only one arc, xy, from X to $V - X$ and note that $u \in X$ and $v \in V - X$ as D is 2-arc-strong. Note that either $x \neq u$ or $y \neq v$, and we assume without loss of generality that $y \neq v$. This implies that v has no arc into it from X in D'.

First consider the case when $X \backslash V(H_i) \neq \emptyset$. In this case let $Y = X \backslash V(H_i)$ and note that $(Y, V - Y)$ is a cut in D with at most one arc from Y to $V - Y$

(the only possible arc is the arc xy since if $z \in Y$ has an arc to a vertex in $V(H_i) \cap X$ then zv is an arc from X to $V - X$), a contradiction to D being 2-arc-strong. We may therefore assume that $X \setminus H_i = \emptyset$, which is equivalent to $X \subseteq V(H_i)$.

If $y \in V(H_i)$ then $N^+_{D'}(x) \subseteq V(H_i)$ which implies that there is no arc leaving $V(H_i)$ in D', contradicting that D' is strong (and $t \geq 2$). Therefore $y \notin V(H_i)$ and $N^+_{D'}(V(H_i)) = \{y\}$. If $V(D) \neq H_i \cup \{y\}$ then we note that y is a cut-vertex in D', separating H_i from $V \setminus (H_i \cup \{y\})$ in D'. In this case there is a strong arc decomposition in D' by Lemma 4.1, a contradiction to (a) not holding as clearly D has a strong arc decomposition, too).

We may therefore assume that $V(D) = V(H_i) \cup \{y\}$. By Lemma 4.1 we may assume that H_i is strongly connected, as y otherwise would be a cut-vertex. Let us consider the following pair G_1, G_2 of disjoint spanning subdigraphs of D. The arcs of G_1 are $(A(H_i) \setminus \{uv\}) \cup \{uy, yv\}$ and the arcs of G_2 are $\{yu, uv, vy\}$ and all arcs $\{yw, wy\}$ for all $w \in V(H_i) \setminus \{u, v\}$. Since both G_1 and G_2 are strong, D has a strong arc decomposition, contradicting the assumption that (a) does not hold, and thereby completing the proof. □

Recall the statement of Theorem 1.4.

Theorem 1.4 *Let T be a strong semicomplete digraph on $t \geq 2$ vertices and let $H_1, \ldots, H_t$ be arbitrary digraphs. Then $D = T[H_1, \ldots, H_t]$ has a strong arc decomposition if and only if D is 2-arc-strong and is not isomorphic to one of the following four digraphs: S_4, $\overrightarrow{C}_3[\overline{K}_2, \overline{K}_2, \overline{K}_2]$, $\overrightarrow{C}_3[\overline{K}_2, \overline{K}_2, \overrightarrow{P}_2]$, $\overrightarrow{C}_3[\overline{K}_2, \overline{K}_2, \overline{K}_3]$.*

Proof: By Theorems 1.1 and 1.3, S_4, $\overrightarrow{C}_3[\overline{K}_2, \overline{K}_2, \overline{K}_2]$, $\overrightarrow{C}_3[\overline{K}_2, \overline{K}_2, \overrightarrow{P}_2]$, $\overrightarrow{C}_3[\overline{K}_2, \overline{K}_2, \overline{K}_3]$ have no strong arc decompositions. Suppose that D satisfies the conditions of the theorem and yet has no strong arc decomposition. Then D satisfies either Case (b) or (c) of Lemma 5.3. However, Case (c) of Lemma 5.3 can be reduced to Case (b). Thus, Theorem 5.2 implies Theorem 1.4. □

6 Proof of Theorem 5.1

Before giving the proof of Theorem 5.1 we will prove the following lemma, which is needed in the proof of Theorem 5.1.

Lemma 6.1 *Let $D = T[H_1, \ldots, H_t]$ be a 2-arc-strong extended semicomplete digraph which has no cut-vertex, $V(T) = \{u_1, \ldots, u_t\}$, $|V(H_i)| \leq 2$ for $i \in [t]$ and $V(H_r) = \{x, y\}$. Suppose $D' = D - y$ is not 2-arc strong. Then there exists an index $q \neq r$ such that one of the following holds,*

(i) *$u_r u_q$ is a cut-arc of T and $N^-(V(H_q)) = V(H_r)$, or*

(ii) *$u_q u_r$ is a cut-arc of T and $N^+(V(H_q)) = V(H_r)$.*

Proof: As D' is strong (since D contains no cut-vertex) but not 2-arc-strong, there is a proper subset S of $V(D')$ such that there is exactly one arc uv from S to $\bar{S} = V(D) - S$ in D'. Suppose first that $x \in S$. If $x \neq u$ then $(S + y, \bar{S})$ is a vertex partition of $V(D)$ with only one arc from $S + y$ to $\bar{S}$, contradicting that D is 2-arc-strong (here we used the fact that x and y have the same out-neighbours). Thus we must have $x = u$ which implies that $u_r u_q$ is a cut-arc of T, where $v \in V(H_q)$, as $u_r u_q$ is the only arc from S' to $V(T) - S'$ where $S' \subset V(T)$ is the set of vertices in T that we obtain by taking u_j in S' precisely when $V(H_j) \cap S \neq \emptyset$.

If $|V(H_q)| = 2$, then we may assume that $H_q = \{v, w\}$ for some $w \neq v$. In this case we must have $w \in S$ as $xw \in A(D)$. Since v is not a cut-vertex we must have $\bar{S} = \{v\}$. Similarly if $|V(H_q)| = 1$ then $\bar{S} = \{v\}$, as otherwise v is a cut-vertex in D. As $\bar{S} = \{v\}$, we note that $N^-(V(H_q)) = V(H_r)$, implying that Part (i) of the lemma holds in this case.

It is easy to see that case when $x \in \bar{S}$ leads to Part (ii) of the lemma. □

Recall the statement of Theorem 5.1.

Theorem 5.1 *Let $D = T[\overline{K}_{n_1}, \ldots, \overline{K}_{n_t}]$ be a 2-arc-strong extended semicomplete digraph where $n_i \leq 2$ for $i \in [t]$. If D is not isomorphic to one of the following three digraphs: S_4, $\overrightarrow{C}_3[\overline{K}_2, \overline{K}_2, \overline{K}_2]$, $\overrightarrow{C}_3[\overline{K}_2, \overline{K}_2, \overrightarrow{P}_2] = T_4^s[\overline{K}_2, \overline{K}_2, \overline{K}_1, \overline{K}_1]$ then D has a strong arc decomposition.*

Proof: Let $\mathcal{D}_2 = \{S_4, \overrightarrow{C}_3[\overline{K}_2, \overline{K}_2, \overline{K}_2], \overrightarrow{C}_3[\overline{K}_2, \overline{K}_2, \overrightarrow{P}_2]\}$, D and T defined as in the theorem, and $V(T) = \{u_1, \ldots, u_t\}$. For all $i \in [t]$ let H_i denote the i'th subdigraph in the decomposition, i.e. $H_i = \overline{K}_{n_i}$ and denote the vertices of H_i by u_{i,j_i}, $1 \leq j_i \leq n_i$. By the assumption of the theorem $n_i \in \{1, 2\}$ for all $i \in [t]$. We consider the following cases.

Case 1. $|V(T)| \leq 2$.

As D is 2-arc-strong we must have $|V(T)| = t = 2$, $A(T) = \{u_1u_2, u_2u_1\}$ and $|H_1| = |H_2| = 2$. Then $u_{1,1}u_{2,1}u_{1,2}u_{2,2}u_{1,1}$ and $u_{1,1}u_{2,2}u_{1,2}u_{2,1}u_{1,1}$ form arc-disjoint Hamilton cycles (see Figure 7(a)), thereby proving that D has a strong arc decomposition.

Case 2. $|V(T)| = 3$.

If $|H_1| = |H_2| = |H_3| = 2$ then we are done by Theorem 1.3, so we may without loss of generality assume that $|H_1| = 1$. As T is a strong semicomplete digraph, it contains a Hamilton cycle, by Camion's theorem (see [8] or [3, Theorem 2.2.6]), so we may assume that $u_1u_2, u_2u_3, u_3u_1 \in A(T)$. As $|H_1| = 1$ and the vertices in H_2 have in-degree at least two we must then have $u_3u_2 \in A(T)$.

If $|H_2| = |H_3| = 2$, then we note that $D\langle H_2 \cup H_3\rangle$ has a strong arc decomposition which can easily be extended to a strong arc decomposition of D as can be seen in Figure 7(b). We may therefore assume without loss of generality that $|H_1| = |H_2| = 1$. As the vertices in H_3 have in-degree at least two we must then have $u_1u_3 \in A(T)$.

If $|H_3| = 2$ then note that $u_{1,1}u_{3,1}u_{2,1}u_{3,2}u_{1,1}$ and $u_{1,1}u_{3,2}u_{2,1}u_{3,1}u_{1,1}$ form arc-disjoint Hamilton cycles, thereby proving that D has a strong arc

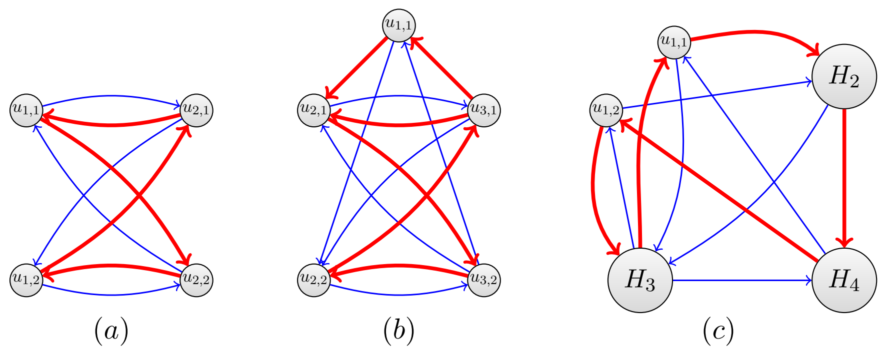


Figure 7: Strong arc decompositions of different digraphs. The red arcs form one strong spanning subdigraph and the blue arcs form the other strong spanning subdigraph. The arcs between $\{u_{1,1}, u_{1,2}\}$ and the H_i's in (c) indicate the direction of all arcs between these sets.

decomposition.

The only remaining case is when $|H_1| = |H_2| = |H_3| = 1$ and $u_2u_1 \in A(T)$. However in this case D has a strong arc decomposition as it consists of two 3-cycles in the opposite directions. This completes the proof of Case 2.

Case 3. T is 2-arc-strong and $|V(T)| \geq 4$.

If T is not isomorphic to S_4, then we are done by Theorem 1.1 and Lemma 2.3. Now assume that T is isomorphic to S_4. If $|H_1| = |H_2| = |H_3| = |H_4| = 1$, then D is isomorphic to S_4 and $D \in \mathcal{D}_2$. By symmetry we may therefore without loss of generality assume that $|H_1| = 2$ and a strong arc decomposition of D can be seen in Figure 7(c) (the arcs shown between all $u_{1,i}$'s and H_j's and between all H_j's and H_k's show the direction of all arcs between them), completing the proof of Case 3.

Case 4. T is not 2-arc-strong and $|V(T)| \geq 4$.

We will now prove that D has a strong arc decomposition. Suppose this is not true and consider a minimum counterexample D.

Suppose first that D has a vertex $z \in H_i$, where $|H_i| = 2$, so that $\hat{D} = D - z$ is 2-arc-strong. By minimality of D, $\hat{D}$ has a strong arc decomposition, unless it is one of the exceptions in the theorem. As we have assumed that $|V(H_i)| \leq 2$ for $i \in [t]$ we note that $\hat{D}$ in this case is either S_4 or $T_4^s[\overline{K}_2, \overline{K}_2, \overline{K}_1, \overline{K}_1]$. If $\hat{D}$ is isomorphic to S_4, then $T = \hat{D}$ and T is 2-arc-strong, a contradiction by the statement of Case 4. Thus, we may assume that $\hat{D}$ is isomorphic to $T_4^s[\overline{K}_2, \overline{K}_2, \overline{K}_1, \overline{K}_1]$. However in this case we can find a strong arc decomposition of $D = T_4^s[\overline{K}_2, \overline{K}_2, \overline{K}_2, \overline{K}_1]$ as seen in Figure 8. Hence $\hat{D}$ has a strong arc decomposition and by Lemma 2.3 so has D, contradiction our assumption.

Hence we may assume below that

$$\begin{array}{l}\text{for every } H_i \text{ with } |H_i| = 2 \text{ and every } z \in V(H_i) \\ \text{the digraph } D - z \text{ is not 2-arc-strong.}\end{array} \quad (1)$$

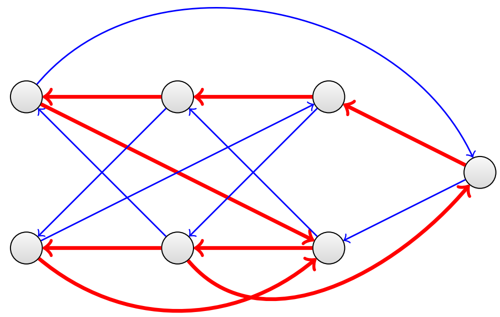


Figure 8: A strong arc decomposition of $D = T_4^s[\overline{K}_2, \overline{K}_2, \overline{K}_2, \overline{K}_1]$

By Theorem 2.1, T has a nice vertex decomposition $(T_1, \ldots, T_p)$. Let $(D_1, D_2, \ldots, D_p)$ be the vertex decompostion of D obtained by replacing each vertex u_i of T by the corresponding independent set H_i (so $V(D_j) = \bigcup_{u_i \in V(T_j)} V(H_i)$). By our assumption on D, it has no strong arc decomposition. We now prove the following claims and use these to obtain a contradiction to the existence of D.

Claim A. If u_iu_j is a cut-arc in T and $|H_j| = 1$, then $V(T_1) = \{u_j\}$.

Proof of Claim A: Assume that u_iu_j is a cut-arc in T and $|H_j| = 1$. Define r such that $u_j \in T_r$. Let $X = V(D_1) \cup V(D_2) \cup \cdots \cup V(D_r) \setminus \{u_{j,1}\}$ and let $Y = V(D_{r+1}) \cup \cdots \cup V(D_p)$. Note that $Y \neq \emptyset$ as $V(H_i) \subseteq Y$. Also note that there is no arc from Y to X in D (as the only arcs out of Y go to $H_j = \{u_{j,1}\}$, because u_iu_j is a cut-arc of T). If $X \neq \emptyset$ then u_j is a cut-vertex in D and we are done by Lemma 4.1. So, $X = \emptyset$ which implies that $V(T_1) = \{u_j\}$, completing the proof of Claim A. ∎

Claim B. If u_iu_j is a cut-arc in T and $|H_i| = 1$, then $T_p = \{u_i\}$.

Proof of Claim B: This can be proved analogously to Claim A. ∎

Claim C. If $|H_r| = 2$, for some $r \in [t]$, then u_r is incident with a cut-arc into T_1 or a cut-arc out of T_p (or both).

This implies that for every $u_q \in V(T)$ that is not incident to a cut-arc we have $|V(H_q)| = 1$.

Proof of Claim C: Let $|H_r| = 2$, for some $r \in [t]$. By Lemma 6.1 we note that there exists a $q \neq r$ such that one of the following holds.

(i) u_ru_q is a cut-arc of T and $N^-(V(H_q)) = V(H_r)$, or

(ii) u_qu_r is a cut-arc of T and $N^+(V(H_q)) = V(H_r)$.

Assume without loss of generality that (i) above holds. Therefore u_r is incident with the cut-arc u_ru_q in T. As $N^-(V(H_q)) = V(H_r)$, we note that $d_T^-(u_q) = 1$. If $u_q \in T_1$, then we are done, so assume that this is not the case. However, as u_ru_q is a cut-arc in T observe that $u_r \notin T_1$, which implies that u_q dominates T_1 (as (i) holds). This implies that $|T_1| = 1$ since there is only one arc entering T_1 in T, and assuming that $T_1 = \{u_1\}$, we note that $u_q u_1$ is the cut-arc into T_1 in T. Therefore $N_T^-(u_1) = \{u_q\}$. As $u_r u_q$ is a cut-arc in T we note that $u_q \not\in T_p$, which by Claims A and B implies that $|H_q| = 2$ (as $u_q u_1$ is a cut-arc in T).

We will now show that $|H_1| = 1$. For the sake of contradiction assume that $|H_1| = 2$, which by Lemma 6.1(ii), implies that $N^+(H_q) = H_1$. However this implies that $d_T^+(u_q) = 1$ and from above $d_T^-(u_q) = 1$, implying that $|V(T)| = 3$, contradicting the fact that $|V(T)| \geq 4$. Therefore $|H_1| = 1$. As $T_1 = \{u_1\}$ we note that $d_T^-(u_1) = 1$.

To summarize, we have now shown the following.

(I) $H_1 = \{u_{1,1}\}$, $H_q = \{u_{q,1}, u_{q,2}\}$ and $H_r = \{u_{r,1}, u_{r,2}\}$ in D.

(II) $u_r u_q$ and $u_q u_1$ are cut-arcs in T.

(III) $N_T^-(u_1) = \{u_q\}$ and $N_T^-(u_q) = \{u_r\}$.

We now consider $D' = D - u_{1,1}$. By Lemma 4.1 we note that D contains no cut-vertices and therefore D' is strongly connected. If D' has a strong arc decomposition then this can easily be extended to a strong arc decomposition of D, as $u_{1,1}$ has at least two arcs into D' and at least two arcs out of D'. We may therefore assume that D' has no strong arc decomposition.

If D' is 2-arc-strong, this implies that D' is one of our exceptions. In this case D' is either $\overrightarrow{C}_3[\overline{K}_2, \overline{K}_2, \overline{K}_2]$ or $\overrightarrow{C}_3[\overline{K}_2, \overline{K}_2, \overrightarrow{P}_2]$, since there is an H_i with at least two vertices. If $D' = \overrightarrow{C}_3[\overline{K}_2, \overline{K}_2, \overline{K}_2]$, then it follows from (III) that $D = T_4^s[\overline{K}_1, \overline{K}_2, \overline{K}_2, \overline{K}_2]$ and we obtain a good decomposition of D using the decomposition in Figure 8 and then reversing all arcs. This contradicts our assumption that D has no strong arc decomposition. Suppose now that $D' = \overrightarrow{C}_3[\overline{K}_2, \overline{K}_2, \overrightarrow{P}_2]$. Then D contains $T_4^s[\overline{K}_1, \overline{K}_2, \overline{K}_2, \overline{K}_2]$ as a spanning subdigraph and hence it has a strong arc decomposition , contradiction.

So we may now assume that D' is not 2-arc-strong and we will let $(S, \overline{S})$ be a partition of $V(D')$ with exactly one arc from S to $\overline{S}$. Since T is strong, $d_{D'}^-(u_r) \geq 1$. We will now show that $d_{D'}^-(u_{r,1}) = 1$ (and $d_{D'}^-(u_{r,2}) = 1$). In order to do this we consider the three possible placements of the vertices $u_{q,1}$ and $u_{q,2}$ in the partition $(S, \overline{S})$.

Case C.1: $\{u_{q,1}, u_{q,2}\} \subseteq S$. If $\overline{S}$ contains any vertex not in $V(H_r)$ then there are at least two arcs from S to $\overline{S}$ (comming from $u_{q,1}$ and $u_{q,2}$), a contradiction. Therefore, $\overline{S} \subseteq H_r$. Without loss of generality $u_{r,1} \in \overline{S}$ and $xu_{r,1}$ is the arc from S to $\overline{S}$. Considering $u_{r,1}$ we note that in D' it only has one arc into it (from x), implying that $d_{D'}^-(u_{r,1}) = 1$ as desired.

Case C.2: $\{u_{q,1}, u_{q,2}\} \subseteq \overline{S}$. Adding $u_{1,1}$ to $\overline{S}$ we note that there is still only one arc from S to $\overline{S}$ (as $u_{1,1}$ only has arcs into it from $u_{q,1}$ and $u_{q,2}$), a contradiction to D being 2-arc-strong.

Case C.3: $|\{u_{q,1}, u_{q,2}\} \cap S| = 1$. Without loss of generality assume that $u_{q,1} \in S$ and $u_{q,2} \in \overline{S}$. As there is only one arc from S to $\overline{S}$ we note that

$u_{r,1}$ or $u_{r,2}$ must belong to $\overline{S}$. Without loss of generality assume that $u_{r,2} \in \overline{S}$. For the sake of contradiction assume that $d^-_{D'}(u_{r,1}) \geq 2$ (and therefore $d^-_{D'}(u_{r,2}) \geq 2$) and let $z_1, z_2 \in N^-_{D'}(u_{r,1})$ (and therefore $z_1, z_2 \in N^-_{D'}(u_{r,2})$) be arbitrary.

If $z_i \in S$, then we note that $z_i u_{r,2}$ is an arc from S to $\overline{S}$ and if $z_i \in \overline{S}$ then either $z_i = u_{q,2}$ or $u_{q,1} z_i$ is an arc from S to $\overline{S}$ for $i = 1, 2$. As there is only one arc from S to $\overline{S}$ we note that $z_1 = u_{q,2}$ or $z_2 = u_{q,2}$. Without loss of generality we may assume that $z_1 = u_{q,1}$ and $z_2 = u_{q,2}$ (as if $u_{q,2}$ dominates $u_{r,1}$ then so does $u_{q,1}$). However if $u_{r,1} \in S$ then both $u_{r,1}u_{q,2}$ and $u_{q,1}u_{r,2}$ go from S to $\overline{S}$ and if $u_{r,1} \in \overline{S}$ then both $u_{q,1}u_{r,1}$ and $u_{q,1}u_{r,2}$ go from S to $\overline{S}$, a contradiction. This completes Case C.3.

We have now shown that $d^-_{D'}(u_{r,1}) = 1$, so we may define z, such that $N^-_{D'}(u_{r,1}) = \{u_{z,1}\}$. Note that $|H_z| = 1$. Let $Y = H_1 \cup H_q \cup H_r \cup H_z$. We will show that $V(D) = Y$, so assume for the sake of contradiction that there exists a vertex $y \in V(D) \setminus Y$. Then there is no path from y to $Y \setminus \{u_{z,1}\}$ in $D - u_{z,1}$, as $N^-_T(u_1) = \{u_q\}$ and $N^-_T(u_q) = \{u_r\}$ and $N^-_T(u_r) = \{u_1, u_z\}$ (so all arcs into $Y \setminus \{u_{z,1}\}$ come from $u_{z,1}$). This implies that $u_{z,1}$ is a cut-vertex in D, a contradiction by Lemma 4.1. Therefore we must have $V(D) = Y$.

As $|V(T)| \geq 4$, we note that H_z, H_1, H_q and H_r are distinct. However in this case D is the exception $\overrightarrow{C}_3[\overline{K}_2, \overline{K}_2, \overrightarrow{P}_2]$. This completes the proof of Claim C. ∎

Claim D. T has at most three cut-arcs.

Proof of Claim D: Let $u_i u_j$ be a cut-arc in T with $1 < j < i < p$. By Claims A and B we note that $|H_j| = |H_i| = 2$. By Claim C we note that u_j is incident with a cut-arc into T_1 and u_i is incident with a cut-arc from T_p. This implies that there are only these three cut-arcs in this case. Furthermore, if there is no cut-arc, $u_i u_j$, in T with $1 < j < i < p$ then there are at most two cut-arcs in T (one into T_1 and one out of T_p). ∎

The remaining part of the proof is split into three cases, covering the number of possible cut-arcs in T according to Claim D.

Case 4.1. T has exactly one cut-arc $u_p u_1$.

Let T' be the semicomplete multigraph that we obtain by adding an extra copy of the arc $u_p u_1$ to T (so T' has exactly one pair of parallel arcs). As $u_p u_1$ was the only cut-arc in T, we note that T' is 2-arc strong. By the statement of Case 4 we note that T is not 2-arc-strong, and therefore not isomorphic to S_4, implying that T' is not one of the exceptions in Theorem 3.3. Therefore T' contains a strong arc decomposition (R_1, R_2).

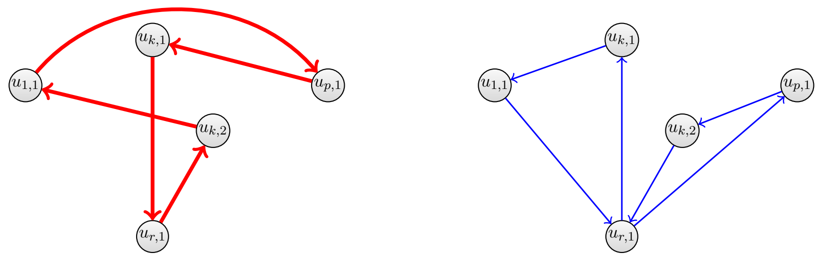
Let R'_1, R'_2 be the arc-disjoint spanning subdigraphs of D that we obtain by replacing the vertex u_p by $\{u_{p,1}, u_{p,2}\}$ and the vertex u_1 by $\{u_{1,1}, u_{1,2}\}$. That is, if $x u_p$ ($u_p y$) is an arc of R_i, then R'_i contains the arcs $x u_{p,1}, x u_{p,2}$ ($u_{p,1} y, u_{p,2} y$) and analogously for arcs entering and leaving u_1. Note that R'_1, R'_2 well-defined for all arcs apart from the ones from $\{u_{p,1}, u_{p,2}\}$ to $\{u_{1,1}, u_{1,2}\}$, for these we let $u_{p,1}u_{1,1}$ and $u_{p,2}u_{1,2}$ belong to R'_1 and $u_{p,1}u_{1,2}$ and $u_{p,2}u_{1,1}$ belong to R'_2. We will now show that (R'_1, R'_2) is a strong arc decomposition of D , contradiction the assumption on D.

First consider the case when $|H_p| = |H_1| = 2$. As there is a path from u_1 to u_p in R_i ($i \in [2]$), we note that for $j \in [2]$ the vertex $u_{p,j}$ can reach every vertex in $\{u_{1,1}, u_{1,2}\}$ in R'_i, either by a direct arc, or by an arc from H_p to H_1 followed by the equivalent of a (u_1, u_p)-path in R_i followed by another arc from H_p to H_1. (For example, in R'_1 the vertex $u_{p,1}$ can reach $u_{1,2}$ via arc $u_{p,1}u_{1,1}$ followed by a $(u_{1,1}, u_{p,2})$-path in R_1 and finally the arc $u_{p,2}u_{1,2}$). Therefore we have all the same connections in R'_1 and R'_2 as in R_1 and R_2, completing the proof of the case when $|H_p| = |H_1| = 2$.

We may therefore without loss of generality assume that $|H_1| = 1$. As D is 2-arc-strong and $u_{1,1}$ is not a cut-vertex we note that $|H_p| = 2$, $V(D_1) = \{u_{1,1}\}$ and there exists a vertex $u_{y,1} \in N^+_D(u_{p,1}) \setminus \{u_{1,1}\}$. Without loss of generality assume that $u_p u_y \in A(R_1)$. As $D - u_{p,2} = T$ we can assign every arc of $D - u_{p,2}$ to R'_i if and only if it was assigned to R_i in T', except the arc from $u_{p,1}$ to $u_{1,1}$ which gets assigned to R'_2. Let $u_x u_p$ be the last arc on a path from u_y to u_p in R_1. Now let the arcs $u_{x,1}u_{p,2}$ and $u_{p,2}u_{1,1}$ belong to R'_1. Now we note that R'_1 is a strong spanning subdigraph of D, as there exists a path from both $u_{p,1}$ and $u_{p,2}$ to $u_{1,1}$ (and therefore all paths in R_1 also work for R'_1). Adding any arc into $u_{p,2}$, different from $u_{x,1}u_{p,2}$ and any arc out of $u_{p,2}$, different from $u_{p,2}u_{1,1}$ to R'_2 makes R'_2 into a strong spanning subdigraph of D' (as it was already strong in $D - u_{p,2}$). Therefore (R'_1, R'_2) is a strong arc decomposition of D.

Case 4.2. T has exactly two cut-arcs $u_p u_h$ and $u_k u_1$. In Claim C we proved that $|H_1| = 1$. We can show similarly that $|H_p| = 1$. Since D is 2-arc-strong we note that $|H_h| = |H_k| = 2$. By Claim A and B we note that $|T_1| = |T_p| = 1$. This and the fact that D has no cut-vertex implies that $|V(D_1)| = |V(D_p)| = 1$. There are 3 subcases to consider: $u_k = u_h$, u_k, u_h are distinct but $u_k, u_h \in V(T_i)$ for some $i \in [p]$ and finally the case where $u_k \in V(T_j), u_h \in V(T_i)$ where $i < j$ (as otherwise T is not strong).

Case 4.2.1. $u_k = u_h$. Let the index i be chosen so that $u_k \in V(T_i)$. As D has no cut-vertex we note that $i = 2$ and $p = 3$. As $|T_1| = |T_3| = 1$ and $|V(T)| \geq 4$ we must have $|T_2| \geq 2$. Therefore the set $W = V(D_2) - \{u_{k,1}, u_{k,2}\}$ contains at least one vertex. As $V(T_2)$ is strong the digraph $D'_2 = D_2 - \{u_{k,1}\}$ is strong. If $V(T_2) = \{u_k, u_r\}$ for some r (that is, $V(T) = \{u_1, u_k, u_r, u_p\}$) then the hamiltonian cycle $u_{1,1}u_{p,1}u_{k,1}u_{r,1}u_{k,2}u_{1,1}$ (see (a) below) is arc-disjoint from the strong spanning subdigraph whose arc set is the arcs of the two paths $u_{p,1}u_{k,2}u_{r,1}u_{k,1}u_{1,1}$ and $u_{1,1}u_{r,1}u_{p,1}$ (see (b) below), showing that D has a strong arc decomposition , contradicting our assumption.



(a) (b)

Suppose now that $|V(T_2)| > 2$. Let v be an in-neighbour of $u_{k,1}$ in $V(D_2)$ and let $w \neq v$ be an out-neighbour of $u_{k,1}$ in $V(D_2)$. Let $D_2' = D_2 - u_{k,1}$ and note that D_2' is strong. Now let the two spanning digraphs $G_1 = (V, A_1), G_2 = (V, A_2)$ contain the following arcs (see Figure 9).

- $A_1 = \{u_{p,1}u_{k,1}, u_{k,1}u_{1,1}, u_{1,1}w, vu_{p,1}\} \cup A(D_2')$
- $A_2 = \{u_{p,1}u_{k,2}, u_{k,2}u_{1,1}, u_{1,1}v, vu_{k,1}, u_{k,1}w, wu_{p,1}\} \cup \{u_{1,1}z, zu_{p,1} | z \in V(D_2') - \{u_{k,2}, v, w\}$

It is easy to verify that G_1, G_2 are arc-disjoint strong spanning subdigraphs of D , contradiction the assumption on D.

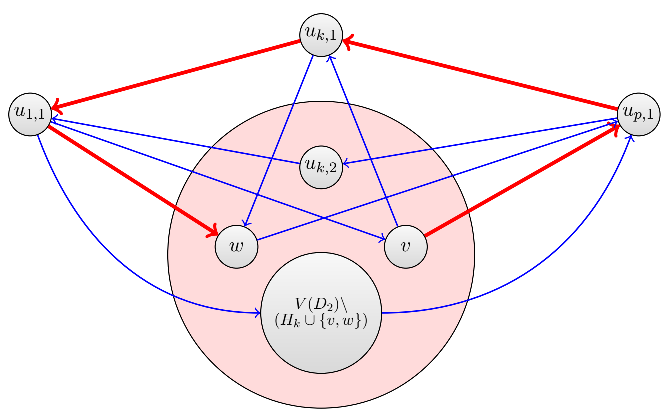


Figure 9: The strong arc decomposition used in Case 4.2.1, where all arcs within the big red circle are red arcs.

Case 4.2.2 u_k and u_h are distinct but belong to the same T_i.

As in the proof of Case 4.2.1, we note that $i = 2$ and $p = 3$ (as D has no cut-vertex) and $|V(D_1)| = |V(D_3)| = 1$.

As u_pu_h and u_ku_1 are the only cut-arcs of T it follows from Menger's theorem that there are two arc-disjoint (u_h, u_k)-paths P_1, P_2 in T_2. For $i \in [2]$ let A_i' be the arcs of D_2 that correspond to $A(P_i)$, that is, we replace the first arc u_hv (last arc $v'u_k$) of P_i by the two arcs $u_{h,1}v, u_{h,2}v$ (respectively, $v'u_{k,1}, v'u_{k,2}$). Recall that, by Claim C, we have $|H_g| = 1$ when u_g is not incident to a cut-arc of T so $A(P_i)$ corresponds exactly to A_i' in D_2.

We will now construct F_1 and F_2 as follows. Let $X = V(D_2) \setminus (H_k \cup H_h)$. Initially let F_1 and F_2 consist of the following arcs (see Figure 10):

- $A(F_1)$ initially consists of the arcs $\{u_{p,1}u_{h,1}, u_{k,2}u_{1,1}, u_{1,1}u_{h,2}, u_{k,1}u_{p,1}\}$ and all arcs of A'_1.
- $A(F_2)$ initially consists of the arcs $\{u_{p,1}u_{h,2}, u_{k,2}u_{p,1}, u_{1,1}u_{h,1}, u_{k,1}u_{1,1}\}$ and all arcs of A'_2.

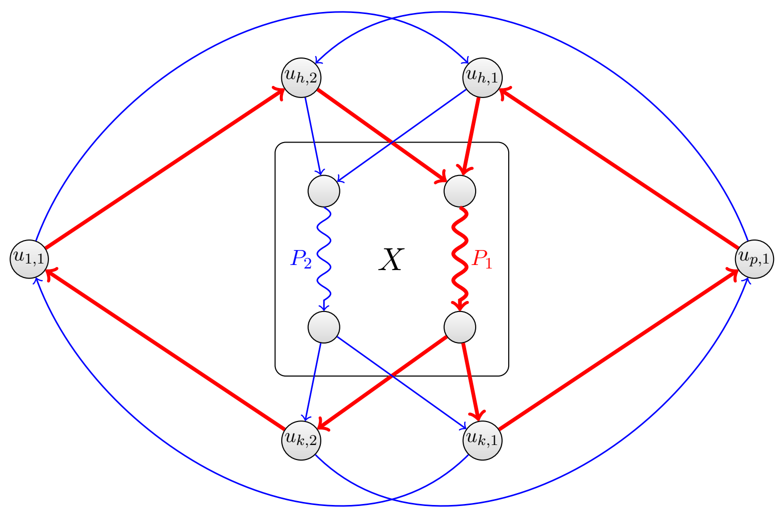


Figure 10: The initial assignments of arcs to F_1 and F_2 in Case 4.2.2.

Now for every vertex $x \in X$ we add the following arcs to F_1 and F_2.

- If $x \notin V(P_1)$ then add the arcs $u_{1,1}x$ and $xu_{p,1}$ to $A(F_1)$.
- If $x \in V(P_1)$ then add the arcs $u_{1,1}x$ and $xu_{p,1}$ to $A(F_2)$.

Finally we add all arcs not assigned to any F_i yet to $A(F_2)$. It is easy to check that F_1 is a strong spanning subdigraph of D. In order to show that F_2 is also a strong spanning subdigraph of D we consider any $x \in X$ and will show that x has a path to and from $\{u_{p,1}, u_{1,1}\}$ in F_2. If $x \in V(P_1) \cup V(P_2)$, then this is clearly the case by the construction above (as either the arcs $u_{1,1}x$ and $xu_{p,1}$ belong to F_2 or $x \in V(P_2)$). So assume that $x \notin V(P_1) \cup V(P_2)$. In this case any path from x to $V(P_1) \cup V(P_2)$ in D_2, and any path to x from $V(P_1) \cup V(P_2)$, belongs to F_2, so we are done as all vertices in $V(P_1) \cup V(P_2)$ have a path to and from $\{u_{p,1}, u_{1,1}\}$ in F_2. Therefore (F_1, F_2) is a strong arc decomposition of D , contradiction our assumption on D. This completes the proof of Case 4.2.2.

Case 4.2.3 There are indices $1 < i < j < p$ such that $u_h \in V(T_i)$ and $u_k \in V(T_j)$.

Recall that $|V(H_1)| = |V(H_p)| = 1$ and $|T_1| = |T_p| = 1$, implying that we must have $V(D_1) = \{u_{1,1}\}$ and $V(D_p) = \{u_{p,1}\}$. As D has no cut vertex we must furthermore have $i = 2$ (or $u_{1,1}$ would be a cut-vertex of D) and $j = p - 1$ (or $u_{p,1}$ would be a cut-vertex of D). If $V(D) = \{u_{1,1}, u_{h,1}, u_{h,2}, u_{k,1}, u_{k,2}, u_{p,1}\}$, then T has another cut-arc, namely u_2u_3, contradicting that we are in Case 4.2. Thus we can choose a vertex $z \in V(D) - \{u_{1,1}, u_{h,1}, u_{h,2}, u_{k,1}, u_{k,2}, u_{p,1}\}$ so that z is an out-neighbour of $u_{h,1}, u_{h,2}$ and an in-neighbour of $u_{k,1}, u_{k,2}$. Let $U = \{z, u_{1,1}, u_{h,1}, u_{h,2}, u_{k,1}, u_{k,2}, u_{p,1}\}$ and note that every vertex of $V(D)-U$ has at least two in-neighbours and at least two out-neighbours in U so it suffices to give a strong arc decomposition for $D[U]$.

Observe that D has a pair arc-disjoint strong spanning subdigraphs: one induced by the arcs of the 6-cycle $u_{p,1}u_{h,1}u_{k,1}u_{1,1}u_{h,2}u_{k,2}u_{p,1}$ and the two arcs $u_{1,1}z, zu_{p,1}$ and the other by the arcs of the 5-cycle $u_{k,2}u_{1,1}u_{p,1}u_{h,2}zu_{k,2}$ and the arcs of the paths $u_{1,1}u_{h,1}u_{k,2}$ and $u_{h,2}u_{k,1}u_{p,1}$ (see Figure 11). Thus, D has a strong arc decomposition , contradiction our assumption on D.

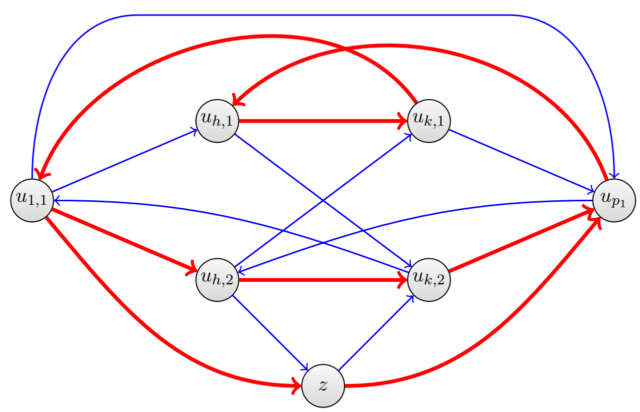


Figure 11: A pair of arc-disjoint strong spanning subgraphs of D.

Case 4.3. T has three cut-arcs u_pu_h, u_hu_k, u_ku_1.

Recall that $|T_1| = |T_p| = 1$ and $|H_1| = |H_p| = 1$, which implies that $V(D_1) = \{u_{1,1}\}$ and $V(D_p) = \{u_{p,1}\}$. Suppose first that $|V(D)| = 6$. Then $V(D) = \{u_{1,1}, u_{k,1}, u_{k,2}, u_{h,1}, u_{h,2}, u_{p,1}\}$ and either D is isomorphic to $T_4^s[\overline{K}_2, \overline{K}_2, \overline{K}_1, \overline{K}_1] = \overrightarrow{C}_3[\overline{K}_2, \overline{K}_2, \overrightarrow{P}_2]$ or at least one of the three cut-arcs are part of a 2-cycle in T. In the later case we obtain the contradiction that D has a strong arc decomposition as shown in Figure 12. Thus, we may assume that $|V(D)| \geq 7$. Now we can choose a vertex w which is an out-neighbour of $u_{k,1}, u_{k,2}$ and an in-neighbour of $u_{h,1}, u_{h,2}$. As above it suffices to show that the subdigraph D^* induced by the vertices in $\{w, u_{1,1}, u_{k,1}, u_{k,2}, u_{h,1}, u_{h,2}, u_{p,1}\}$ has a strong arc decomposition. This follows from the fact that D^* has a pair of arc-disjoint strong spanning subdigraphs: the first is induced by the arcs of the 5-cycle $u_{1,1}wu_{p,1}u_{h,2}u_{k,2}u_{1,1}$ and the path $u_{1,1}u_{h,1}u_{k,1}u_{p,1}$ and the second is induced by the 7-cycle $u_{1,1}u_{p,1}u_{h,1}u_{k,2}wu_{h,2}u_{k,1}u_{1,1}$ (see Figure 13).

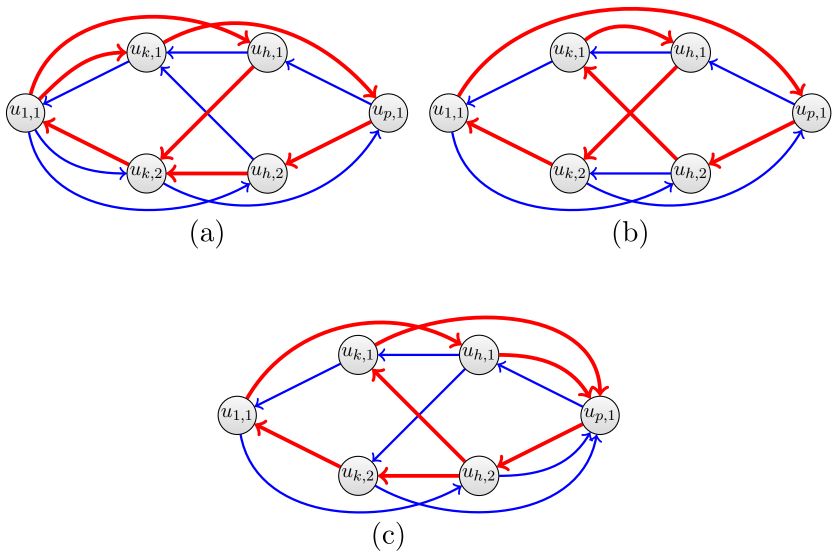


Figure 12: Arc disjoint spanning subdigraphs of D when $|V(D)| = 6$ and one of the cut-arcs of T is in a 2-cycle. The case when $u_k u_1$ is in a 2-cycle of T is shown in (a), the one where $u_h u_k$ is in a 2-cycle in T is shown in (b) and finally (c) shows the case when $u_p u_h$ is in a 2-cycle in T.

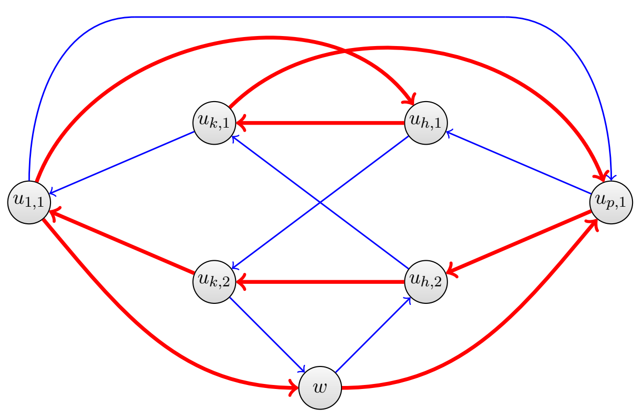


Figure 13: A pair of arc-disjoint strong spanning subgraphs of D^*.

Thus we have shown that the counterexample D does not exist and the proof of the theorem is complete. □

7 Concluding remarks

All proofs in this paper are constructive and can be turned into polynomial algorithms for finding strong arc decompositions. Thus, the problem of finding a strong arc decomposition in a semicomplete composition, which has one, admits a polynomial time algorithm.

Recall that strong semicomplete compositions generalize both strong semicomplete digraphs and strong quasi-transitive digraphs. However, they do not generalize locally semicomplete digraphs and their generalizations in- and out-locally semicomplete digraphs. A digraph D is **in-locally semicomplete** (**out-locally semicomplete**, respectively) if the in-neighbourhood (out-neighbourhood, respectively) of every vertex of D indices a semicomplete digraph. (For information on in- and out-locally semicomplete digraphs, see e.g. [1] and [2, Chapter 6].)

While there is a characterization of locally semicomplete digraphs having a strong arc decomposition (see Theorem 1.2), no such a characterization is known for in-locally semicomplete digraphs[1] and it would be interesting to obtain such a characterization or at least establish the complexity of deciding whether an in-locally semicomplete digraph has a strong arc decomposition. Similar questions are of interest for other generalizations of semicomplete digraphs such as generalizations of quasi-transitive digraphs overviewed in [10].

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