THESIS

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by

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High frequency oscillations in bounded elastic media

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Oscillations haute fréquence en milieux élastiques bornés

Résumé

Cette thèse est consacrée à l'étude haute fréquence de problèmes de Dirichlet et Neumann pour le système de l'élasticité. On y étudie le phénomène de réflexion au bord au moyen de deux techniques : la sommation de faisceaux gaussiens et les mesures de Wigner.

Dans les chapitres 1 et 2, on commence par étudier le problème plus simple de l'équation des ondes scalaire à une vitesse. Sous certaines hypothèses sur les conditions initiales, on construit des solutions approchées par superposition de faisceaux gaussiens. La justification de l'asymptotique se fonde sur une estimation de normes de certains opérateurs intégraux à phases complexes. Pour des conditions initiales plus générales, on utilise les mesures de Wigner pour calculer la densité d'énergie microlocale. On calcule explicitement les transformées de Wigner d'intégrales de faisceaux gaussiens. Le comportement de la densité d'énergie microlocale de la solution exacte se déduit de celui établi pour la solution approchée.

Dans le chapitre 3, on utilise les résultats établis pour les sommes infinies de faisceaux gaussiens pour construire une solution approchée pour les équations de l'élasticité et calculer sa densité d'énergie microlocale. L'existence de deux vitesses différentes dans le système de l'élasticité introduit de nouvelles difficultés qui sont traitées dans ce chapitre.

Mots-clefs : élasticité, équation des ondes, conditions de bord, réflexion, faisceaux gaussiens, mesures de Wigner.

Abstract

This thesis is devoted to the study of the high frequency Dirichlet and Neumann problems for the elasticity system. We study the reflection phenomenon at the boundary by means of two techniques: Gaussian beams summation and Wigner measures.

In chapters 1 and 2, we start by studying the simpler problem of the scalar wave equation with one speed. Under some hypotheses on the initial conditions, we build an approximate solution by a Gaussian beams superposition. Justification of the asymptotics is based on norms estimate of some integral operators with complex phases. For more general initial conditions, we use Wigner measures to compute the microlocal energy density. We compute Wigner transforms of Gaussian beams integrals in an explicit way. The behaviour of the microlocal energy density for the exact solution is deduced from the one for the approximate solution.

In chapter 3, we use the established results on infinite sums of Gaussian beams to build an approximate solution for the elasticity equations and to compute its microlocal energy density. We treat new difficulties arising from the existence of two different speeds in the elasticity system.

Keywords: elasticity, wave equation, boundary conditions, reflection, Gaussian beams, Wigner measures.

Contents

Introduction (french)

Position du problème

De nombreux phénomènes physiques sont modélisés par des équations d'onde. Dans un milieu élastique, les équations du mouvement linéarisées dans le cas de petites perturbations sont :

$$\rho\partial_t^2 u = \mathrm{div}\sigma(u), \tag{1}$$

où u est le déplacement autour d'une configuration d'équilibre statique considérée comme état de référence, ρ la densité, $\sigma(u)$ le tenseur des contraintes et $\mathrm{div}\sigma(u)$ le vecteur de composantes $(\mathrm{div}\sigma(u))_j = \sum_{k=1}^{3} \partial_{x_k}\sigma_{jk}(u)$ pour $1 \leq j \leq 3$. La surface de la terre en sismologie, les organes humains en imagerie médicale, ainsi que de nombreuses structures en mécanique industrielle peuvent être considérés comme des milieux élastiques. Si les propriétés du milieu sont les mêmes dans toutes les directions, le milieu est dit isotrope, et le tenseur des contraintes est donné par

$$\sigma(u) = \lambda \mathrm{div} u Id + \mu\left(\partial_x u + \partial_x u^T\right),$$

où $(\partial_x u)_{jk} = \partial_{x_k} u_j$ et $\lambda(x)$, $\mu(x)$ sont les coefficients de Lamé qui vérifient $\mu(x) > 0$ et $\lambda(x) + 2\mu(x) > 0$. Les équations (1) s'écrivent alors

$$\rho\partial_t^2 u - \partial_x(\lambda \mathrm{div} u) - \mathrm{div}\left(\mu\partial_x u + \mu\partial_x u^T\right) = 0. \tag{2a}$$

On supposera $\mu \neq \lambda + 2\mu$ et la densité et les coefficients du milieu réguliers. Dans le cas d'un milieu homogène, la densité et les coefficients de Lamé ne dépendent pas de la position x. La solution u de (2a) se décompose alors en une somme de deux termes u_L et u_T de rotationnel et de divergence nuls respectivement. Chacun de ces termes est solution d'une équation d'onde à une vitesse

$$u = u_L + u_T,\ \partial_t^2 u_L - c_L^2 \triangle u_L = 0,\ \partial_t^2 u_T - c_T^2 \triangle u_T = 0,$$

avec

$$c_L^2 = \frac{\lambda + 2\mu}{\rho} \text{ et } c_T^2 = \frac{\mu}{\rho}.$$

Dans cette thèse, on s'intéresse aux problèmes haute fréquence. Ce type de problème apparaît dans plusieurs applications. L'étude des vibrations de structures industrielles, quand la fréquence d'excitation est importante (chocs) est un exemple d'oscillation à haute fréquence en milieu élastique. On peut également rencontrer ce type de problème dans la propagation d'ondes sismiques quand la longueur d'onde est petite.

On complète ainsi l'équation (2a) par des conditions initiales $(u_\varepsilon^I, v_\varepsilon^I)$

$$u|_{t=0} = u_\varepsilon^I, \partial_t u|_{t=0} = v_\varepsilon^I, \tag{2b}$$

qui dépendent d'un paramètre haute fréquence $\varepsilon \ll 1$. La forme exacte des données initiales n'a pas d'importance dans cette étude. Un exemple typique serait $u_\varepsilon^I = \varepsilon a e^{i\phi_0/\varepsilon}$ et $v_\varepsilon^I = b e^{i\psi_0/\varepsilon}$. La solution du système de l'élasticité avec ces conditions initiales hautement oscillantes dépend désormais de ε (et sera désignée par u_ε). On s'intéresse au comportement de cette solution quand ε est très petit.

Pour les structures industrielles (aéronautiques, automobiles, ferroviaires, génie civil, etc) comme en sismologie, les corps élastiques considérés occupent un domaine Ω avec bord et les relations (2a)-(2b) doivent être complétées par des conditions aux limites sur $\partial\Omega$. La condition d'encastrement du bord ou condition de Dirichlet s'écrit

$$u_\varepsilon|_{\partial\Omega} = 0, \tag{2c}$$

alors qu'un bord libre se traduit par une condition de type Neumann

$$\sigma(u_\varepsilon)\nu|_{\partial\Omega} = 0, \tag{2c'}$$

où ν est la normale extérieure au bord. On peut bien entendu imposer d'autres conditions aux limites (mixtes, dérivées obliques, etc). Résoudre une équation aux dérivées partielles hyperbolique ou un système d'équations avec des conditions initiales et des conditions aux limites données s'appelle problème mixte hyperbolique.

La structure de la solution exacte d'un problème mixte pour les équations d'onde dépend de la géométrie du bord. En effet l'énergie se propage à l'intérieur du domaine le long des bicaractéristiques qui sont des courbes du fibré cotangent $T^*(\mathbb{R} \times \Omega)$. Les projections de ces courbes sur $\mathbb{R} \times \Omega$ sont les rayons optiques. Pour l'opérateur d'onde à une vitesse constante c, ces rayons sont des courbes (t, x^t) de $\mathbb{R}^{n+1}$ qui se déplacent de manière rectiligne à la vitesse c à l'intérieur de Ω. Lors d'un contact transverse avec le bord, les rayons optiques se réfléchissent selon les lois de l'optique géométrique. S'ils rencontrent le bord tangentiellement, ils peuvent donner naissance à des rayons diffractifs qui frôlent le bord sans être déviés. Il peuvent aussi donner lieu à des rayons glissants qui restent dans le bord $\partial\Omega$, et qui sont les limites de rayons se rapprochant du bord et se réfléchissant un grand nombre de fois. L'opérateur de l'élasticité lui possède deux familles de rayons associées à chacune des deux vitesses c_L et c_T. Au contact avec le bord, les rayons associés à la vitesse c_L peuvent donner naissance à des rayons associés à la vitesse c_T et inversement.

On peut étudier les problèmes haute fréquence en construisant des développements asymptotiques de la solution, valides quand ε est très petit. On approche alors la solution au sens d'une norme bien choisie avec une précision qui augmente avec la fréquence. Il existe également d'autres approches qui s'intéressent uniquement à la limite quand $\varepsilon \to 0$ de certaines quantités associées à la solution comme la densité d'énergie locale $|u_\varepsilon|^2$, il s'agit des approches type mesures de Wigner.

Dans cette thèse, nous nous intéressons au comportement des solutions haute fréquence du problème mixte pour l'équation des ondes scalaire

$$\begin{aligned} &\partial_t^2 u_\varepsilon - \partial_x \cdot (c^2(x)\partial_x u_\varepsilon) = 0,\ u_\varepsilon|_{t=0} = u_\varepsilon^I,\ \partial_t u_\varepsilon|_{t=0} = v_\varepsilon^I, \\ &u_\varepsilon|_{\partial\Omega} = 0 \text{ ou } \partial_\nu u_\varepsilon|_{\partial\Omega} = 0, \end{aligned} \tag{3}$$

et le système de l'élasticité (2). Le bord est supposé régulier et seul le phénomène de réflexion est étudié. La démarche adoptée est la suivante. On commence par étudier le problème plus simple de l'équation des ondes scalaire à une vitesse, puis les techniques utilisées sont adaptées à l'élasticité. Sous certaines hypothèses sur les conditions initiales, on approche la solution à $O(\varepsilon^N)$ près pour tout $N \in \mathbb{N}$ en construisant une famille de solutions asymptotiques. La construction est fondée sur une méthode performante dont l'utilisation est bien maîtrisée dans le cas du phénomène de réflexion au bord : la sommation de faisceaux gaussiens. Pour des conditions initiales plus générales, on utilise les mesures de Wigner pour calculer la densité d'énergie par vecteur d'onde. Le comportement de cette quantité se déduit de celui établi pour la solution approchée par des calculs explicites sur les transformées de Wigner.

Sommation de faisceaux gaussiens

Il existe plusieurs modèles mathématiques de solutions approchées des équations d'onde quand la fréquence tend vers l'infini. Les solutions exactes pour ces équations sont connues pour certaines configurations appelées problèmes canoniques. Dans un milieu homogène à une vitesse, les solutions canoniques sont les ondes planes. Dans un milieu où la longueur d'onde est petite par rapport à ses hétérogénéités et aux distances de propagation, cette forme des solutions exactes est valide à haute fréquence localement. On peut alors intuiter les formes des solutions. C'est ce qu'on appelle un ansatz. L'étape suivante est alors de trouver les conditions nécessaires pour que l'ansatz trouvé vérifie effectivement l'équation d'onde considérée.

Un ansatz très simple est de la forme

$$u_\varepsilon \simeq a_0(t,x)e^{i\psi(t,x)/\varepsilon}, \tag{4}$$

où a_0 est une amplitude scalaire ou vectorielle selon le problème considéré, et ψ une phase scalaire. On appelle cette méthode la méthode de l'optique géométrique, ou encore la méthode WKB ou WKBJ [52], du nom des scientifiques Wentzel, Kramers, Brillouin et Jeffreys qui l'ont indépendemment utilisée dans les années 1920. Pour décrire la réflexion en présence d'un bord, des termes similaires avec des amplitudes et phases réfléchies sont rajoutés dans l'ansatz précédent.

Dans le cas de l'équation des ondes à une vitesse $c(x)$, on obtient en appliquant l'opérateur $\partial_t^2 - \partial_x \cdot (c^2\partial_x)$ à cet ansatz les termes suivants organisés selon les puissances de ε

$$\varepsilon^{-2}\left[c^2|\partial_x\psi|^2 - (\partial_t\psi)^2\right]a_0 + i\varepsilon^{-1}\left[2\partial_t\psi\partial_t a_0 - 2c^2\partial_x\psi\partial_x a_0 + (\partial_t^2\psi - \partial_x\cdot(c^2\partial_x\psi))a_0\right] + ...$$

En annulant le premier terme, on obtient une équation eikonale sur la phase ψ

$$c^2|\partial_x\psi|^2 - (\partial_t\psi)^2 = 0.$$

Pour les équations de l'élasticité, ce type de calculs mène à la même équation eikonale avec l'une des deux vitesses c_L ou c_T, couplée avec une information sur la direction de l'amplitude vectorielle a_0.

Dans le cas d'une phase ψ réelle, cette équation de type Hamilton-Jacobi possède deux solutions locales qui vérifient $c|\partial_x\psi| \pm \partial_t\psi = 0$, pour une même phase initiale donnée ψ_0. La méthode traditionnelle pour calculer ces solutions est la méthode des caractéristiques. Il s'agit, pour trouver par exemple la solution de $c|\partial_x\psi| + \partial_t\psi = 0$, de résoudre le système Hamiltonien associé au symbole $h_+(x,\xi) = c(x)|\xi|$

$$\frac{dx^t}{dt} = \partial_\xi h_+(x^t,\xi^t) = c(x^t)\frac{\xi^t}{|\xi^t|}, \frac{d\xi^t}{dt} = -\partial_x h_+(x^t,\xi^t) = -\partial_x c(x^t)|\xi^t|$$

avec comme direction initiale $\xi^0 = \partial_x\psi_0(x^0)$, puis d'intégrer l'équation $\frac{d\psi}{dt} = \partial_t\psi + \partial_x\psi \cdot \frac{dx^t}{dt}$ avec la condition initiale $\psi(0,x) = \psi_0(x)$, le long des courbes x^t. Cependant la phase ψ trouvée n'est en général pas globale en temps. En effet l'application $x^0 \mapsto x^t$ n'est pas toujours bijective, et plusieurs rayons différents peuvent se croiser, formant ainsi ce qu'on appelle une caustique. Il en résulte des fonctions WKB qui ne sont pas valables aux caustiques. D'autre part, la formation de caustiques est une situation récurrente même dans les modèles et structures les plus simples [17].

Si la phase ψ a une partie imaginaire non nulle, la méthode des caractéristiques n'est plus applicable. Cependant les propriétés de $\mathrm{Im}\,\psi$ contrôlent l'enveloppe de la solution asymptotique. En effet, si ψ est réelle sur un rayon (t, x^t) et que la partie imaginaire de sa matrice Hessienne sur ce rayon $\partial_x^2\psi(t, x^t)$ est définie positive, alors à tout instant t la principale partie de la densité d'énergie de $a_0 e^{i\psi/\varepsilon}$ est concentrée au voisinage du point $x = x^t$ pour ε petit. Il n'est plus nécessaire de vérifier l'équation eikonale de façon exacte mais seulement d'annuler la série de Taylor de $c^2|\partial_x\psi|^2 - (\partial_t\psi)^2$ jusqu'à un certain ordre $R \geq 2$ sur les rayons. On est alors ramené à la résolution de systèmes différentiels qui ont des solutions globales.

On obtient ainsi ce qu'on appelle des faisceaux gaussiens, ce nom provenant du fait que leur densité d'énergie à un instant donné est une fonction gaussienne. Ces solutions approchées, qui font partie de l'optique géométrique complexe (voir [56] pour une comparaison entre les différentes méthodes d'optique géométrique complexe), apparaissent aussi sous le nom de "quasiphotons" car à chaque instant t ils sont concentrés au voisinage d'un point qui se déplace selon une certaine géodésique avec une vitesse unitaire et possède plusieurs propriétés des particules (loi de conservation d'énergie, réflexion au bord, etc). Certains auteurs distinguent ces faisceaux gaussiens dépendant du temps et de la variable de l'espace de ceux qui ne dépendent que de la variable de l'espace en les appelant faisceaux gaussiens en temps et en espace, faisceaux gaussiens non-stationnaires ou encore paquets gaussiens et paquets d'onde gaussiens.

Historiquement, les faisceaux gaussiens apparaissent dans les travaux de V.M. Babich dans les années 1960 [5] et sont généralisés dans les années 1980 par J. Ralston [84], V.M. Babich et V.V. Ulin [8]. Ces solutions approchées ont été largement utilisées en élasticité [6, 18, 53, 78], et pour les résonateurs optiques [7]. Les faisceaux gaussiens peuvent être adaptés naturellement à d'autres équations, comme les équations de Helmholtz et de Schrödinger. Tout comme les différentes méthodes d'optique géométrique complexe, ils constituent une alternative à l'optique géométrique traditionnelle pour décrire les solutions au delà des caustiques, et ce de manière globale en temps. Ils peuvent aussi être vus comme une base de solutions élémentaires pour la propagation d'ondes et permettre ainsi d'étudier les solutions générales d'équations aux dérivées partielles [80, 84]. La précision de ces solutions peut être améliorée en rajoutant à l'amplitude a_0 des termes supplémentaires de puissances de ε supérieures $\varepsilon a_1 + \varepsilon^2 a_2 + \ldots$ et en augmentant l'ordre R jusqu'auquel l'équation eikonale est vérifiée sur le rayon.

Pour décrire un champ qui n'a pas de profil gaussien, on utilise la méthode de sommation de faisceaux gaussiens [18, 51, 54, 82]. Le champ initial est décomposé en une somme de gaussiennes. Chaque faisceau gaussien individuel est calculé en résolvant les systèmes différentiels associés. Le champ est alors obtenu en un point d'observation en superposant une sélection de faisceaux gaussiens. Les stratégies de sommation sont nombreuses. La somme peut être discrète ou continue, la sélection des faisceaux gaussiens à superposer peut se faire selon plusieurs critères. On peut citer quelques orientations récentes :

- la sélection des rayons de direction initiale $\partial_x\psi_0$ pour décrire une donnée initiale WKB avec une phase ψ_0 [63, 96].

- l'utilisation de la transformée de Fourier [41, 97]
- l'utilisation de la transformée FBI (de Fourier-Bros-Iagolnitzer) [88] définie de $L^2(\mathbb{R}^n)$ dans $L^2(\mathbb{R}^{2n})$ par

$$T_\varepsilon(f)(x,\xi) = c_n \varepsilon^{-\frac{3n}{4}} \int_{\mathbb{R}^n} f(z) e^{i\xi\cdot(x-z)/\varepsilon-(x-z)^2/(2\varepsilon)} dz, \; c_n = 2^{-\frac{n}{2}}\pi^{-\frac{3n}{4}} \text{ pour } f \in L^2(\mathbb{R}^n).$$

Les deuxième et troisième méthodes permettent de se ramener à des données de la forme amplitude multipliée par l'exponentielle d'une phase.

Quelle que soit la méthode utilisée, il est important d'évaluer ses performances en estimant l'erreur entre le champ théorique et le champ obtenu par la sommation de faisceaux gaussiens. L'erreur de discrétisation d'une intégrale de faisceaux gaussiens pour l'élasticité a été analysée dans [55]. Récemment, la précision d'une superposition continue de faisceaux gaussiens pour approcher la solution exacte d'une équation d'onde acoustique a été étudiée dans [96, 63]. L'erreur relative à l'utilisation d'une série de Taylor pour les phases et les amplitudes des faisceaux gaussiens a été quantifiée par [76] pour l'équation de Helmholtz. Des études similaires ont été réalisées pour l'équation de Schrödinger dans [58, 64].

Dans le chapitre 1 on utilise la transformée FBI pour contruire une famille de solutions du problème mixte (3) comme une intégrale de faisceaux gaussiens. On prouve l'estimation d'erreur suivante :

Théorème 1. [théorème 1.1 du chapitre 1] *Supposons vérifiées les hypothèses nécessaires sur le domaine (B1-B3 p.33), notamment la transversalité au bord de tous les rayons provenant de Ω. Supposons que les conditions initiales vérifient les hypothèses suivantes*

A1. u^I_ε et v^I_ε sont uniformément bornées dans $H^1(\Omega)$ et $L^2(\Omega)$ respectivement,

A2. u^I_ε et v^I_ε sont nulles en dehors d'un compact fixe de Ω,

A3. $T_\varepsilon u^I_\varepsilon(x,\xi)$ et $T_\varepsilon v^I_\varepsilon(x,\xi)$ sont négligeables pour les ξ grands et les ξ proches de zéro (voir p.28).

Alors on peut construire pour $R \in \mathbb{N}$, $R \geq 2$, une solution approchée u^R_ε du problème de Dirichlet ou de Neumann pour l'équation des ondes scalaire comme une intégrale de faisceaux gaussiens. Cette solution vérifie pour tout $T > 0$

$$\underset{t\in[0,T]}{Sup} \|u^R_\varepsilon(t,.) - u_\varepsilon(t,.)\|_{H^1(\Omega)} = O(\varepsilon^{\frac{R-1}{2}}),$$

$$et \; \underset{t\in[0,T]}{Sup} \|\partial_t u^R_\varepsilon(t,.) - \partial_t u_\varepsilon(t,.)\|_{L^2(\Omega)} = O(\varepsilon^{\frac{R-1}{2}}).$$

La démarche est la suivante. On commence par décomposer les conditions initiales en un point z sur la famille des fonctions $\left(e^{i\xi\cdot(x-z)/\varepsilon-(x-z)^2/(2\varepsilon)}\right)_{(x,\xi)\in\mathbb{R}^{2n}}$. A un coefficient de normalisation près, ceci est le noyau de l'adjoint de la transformée FBI qui est une isométrie. Les conditions initiales s'écrivent alors comme une intégrale de faisceaux gaussiens pondérés par leurs transformées FBI (à un coefficient près). On construit les faisceaux gaussiens individuels en suivant le formalisme de [84]. La superposition de faisceaux dont les phases vérifient l'équation eikonale à l'ordre R donne une solution approchée u_ε^R. On estime alors les erreurs dans l'équation à l'intérieur $(\partial_t^2 - \partial_x \cdot (c^2(x)\partial_x))\, u_\varepsilon^R$, la condition au bord et les conditions initiales $u_\varepsilon^R|_{t=0} - u_\varepsilon^I$ et $\partial_t u_\varepsilon^R|_{t=0} - v_\varepsilon^I$. Tous ces termes sont les résultats d'une famille d'opérateurs intégraux à phase complexe appliqués aux transformées FBI des données initiales. Les normes de ces opérateurs de $L^2(\mathbb{R}^{2n})$ dans $H^s(\mathbb{R}^n)$ sont calculées en utilisant la régularité des phases et amplitudes des faisceaux gaussiens ainsi que les propriétés des phases. Une fois toutes les erreurs estimées, la différence entre la solution approchée u_ε^R et la solution exacte est contrôlée par l'estimation d'énergie du problème mixte. On obtient immédiatement l'ordre $\varepsilon^{\frac{R-1}{2}}$ pour la condition de bord de type Dirichlet. Pour prouver le même ordre pour le problème de Neumann, on a recours à la solution approchée u_ε^{R+1} qu'on compare à la solution exacte et à u_ε^R.

Ces idées s'adaptent naturellement au problème de l'élasticité, en généralisant la notion de transformée FBI aux fonctions vectorielles. On a alors l'estimation suivante en élasticité tridimentionnelle :

Théorème 2. [théorème 1.1 du chapitre 3] *Supposons vérifiées les hypothèses nécessaires sur les conditions initiales et le domaine (voir p.100-101), excluant notamment les rayons provenant de* Ω *qui touchent le bord tangentiellement ou à une incidence supérieure ou égale à l'angle critique. On peut construire pour* $R \in \mathbb{N}$, $R \geq 2$, *une solution approchée* u_ε^R *du problème mixte pour l'élasticité comme une intégrale de faisceaux gaussiens. Cette solution vérifie pour tout* $T > 0$

$$\underset{t\in[0,T]}{Sup} \|u_\varepsilon^R(t,.) - u_\varepsilon(t,.)\|_{H^1(\Omega)^3} = O(\varepsilon^{\frac{R-1}{2}}),$$

$$et \underset{t\in[0,T]}{Sup} \|\partial_t u_\varepsilon^R(t,.) - \partial_t u_\varepsilon(t,.)\|_{L^2(\Omega)^3} = O(\varepsilon^{\frac{R-1}{2}}).$$

Mesures de Wigner

Les mesures de Wigner sont des mesures dans l'espace des phases qui permettent de décrire le comportement asymptotique de quantités quadratiques telles que la densité d'énergie locale. La fonction de Wigner a été utilisée en 1932 par E. Wigner [100] en mécanique quantique. Depuis, elle a été appliquée dans divers autres domaines comme l'optique et l'analyse du signal. Dans les années 90, plusieurs mathématiciens s'intéressent aux mesures de Wigner, tels P.-L. Lions, T. Paul [62] et P. Gérard [35] (voir aussi les articles [10, 28, 38] et l'exposé [12]). Les mesures de Wigner sont à rapprocher des H-mesures et mesures de défaut microlocales, introduites par L. Tartar [98] et P. Gérard [36] (voir aussi [33]).

A l'O.N.E.R.A.[1] des travaux récents ont recours aux mesures de Wigner pour déduire le comportement de l'énergie vibratoire à haute fréquence dans un milieu élastique [93] ou visco-élastique [2, 3]. Ces travaux rejoignent les "approches ingénieur" [40, 99] qui constituent une alternative aux techniques habituellement utilisées pour étudier les vibrations des structures à haute fréquence : l'analyse statistique énergétique (SEA) [65, 66] et les modèles de diffusion d'énergie vibratoire [77, 90].

La SEA constitue une approche globale dans la mesure où elle ne fournit que des estimations des énergies vibratoires moyennes par sous-systèmes mécaniques. La difficulté

[1] Office National d'Etudes et de Recherches Aérospatiales

principale de la méthode, encore très heuristique, est la détermination des paramètres physiques qui interviennent : facteurs de perte par couplage entre sous-systèmes, densités modales, puissances injectées.

Les modèles de diffusion d'énergie vibratoire peuvent être qualifiés de locaux car ils fournissent des estimations des densités d'énergie et d'intensité vibratoire. Néanmoins ils n'ont jusqu'à présent été mis en oeuvre que pour des structures simples (poutres, plaques) car ils reposent sur des hypothèses fortes difficilement vérifiables - voire fausses - pour des structures plus complexes. Ces modèles conduisent à une équation de diffusion pour la densité d'énergie vibratoire. Or l'utilisation des solutions WKB traditionnelles montre cependant que l'équation vérifiée par la densité d'énergie est une équation de transport.

Le recours aux mesures de Wigner constitue une alternative rigoureuse pour parer à ces difficultés. De plus cette méthode fournit la direction de propagation de l'énergie.

Une mesure de Wigner $w[f_\varepsilon]$ associée à la suite (f_ε) uniformément bornée dans $L^2(R^n)^p$ est une limite faible de la suite des transformées de Wigner associées à f_ε (quitte à extraire une sous-suite)

$$w_\varepsilon[f_\varepsilon](x,\xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-iv\cdot\xi} f_\varepsilon(x+\frac{\varepsilon}{2}v) f_\varepsilon^*(x-\frac{\varepsilon}{2}v)dv.$$

Moyennant certaines hypothèses, la limite (au sens des mesures) quand $\varepsilon \to 0$ de la densité d'énergie pour les solutions d'équations d'onde peut s'exprimer en terme de mesures de Wigner. Ainsi, pour l'équation des ondes scalaire, la densité d'énergie à l'instant t converge vers

$$\frac{1}{2}\int_{\mathbb{R}^n} w[\partial_t u_\varepsilon(t,.)](x,d\xi) + \frac{1}{2}\int_{\mathbb{R}^n} \mathrm{Tr} w[c\partial_x u_\varepsilon(t,.)](x,d\xi).$$

En élasticité, elle converge vers

$$\frac{\rho}{2}\int_{\mathbb{R}^n} \mathrm{Tr} w[\partial_t u_\varepsilon(t,.)](x,d\xi) + \frac{\mu}{4}\sum_{j=1}^{3}\int_{\mathbb{R}^n} \mathrm{Tr} w[\partial_{x_j} u_\varepsilon(t,.) + \partial_x (u_\varepsilon)_j(t,.)](x,d\xi)$$
$$+ \frac{\lambda}{2}\int_{\mathbb{R}^n} w[\mathrm{div} u_\varepsilon(t,.)](x,d\xi).$$

Ces quantités ont été complètement caractérisées pour les équations d'onde dans tout l'espace [38, 79]. En présence d'un bord, l'étude des mesures de Wigner devient techniquement plus difficile. La notion de mesure de Wigner a été utilisée dans le cas de domaines bornés pour l'analyse des propriétés ergodiques des fonctions propres pour les problème de Dirichlet dans [37, 102], de Neumann et de Robin dans [13]. D'autres études se sont intéressées aux mesures de Wigner dans un domaine borné ou avec une interface, comme dans les articles [11, 74, 91] et les thèses [25, 30]. Tous ces travaux se fondent sur l'utilisation du calcul pseudo-différentiel semi-classique.

Dans le chapitre 2, le comportement de la densité d'énergie microlocale pour la solution du problème (3) est décrit en utilisant une autre approche similaire à [15, 88] fondée sur les faisceaux gaussiens. On prouve le théorème suivant :

Théorème 3. [théorème 1.1 du chapitre 2] *Supposons vérifiées les hypothèses nécessaires sur le domaine (B1-B3 p.61), notamment la transversalité au bord de tous les rayons provenant de Ω. Supposons que les données initiales satisfont A1, A2 et également les conditions suivantes (après extension par* 0 *en dehors de Ω)*

C1. Les mesures de Wigner de v_ε^I *et* $\partial_{x_b} u_\varepsilon^I$ *(*$b = 1, \dots, n$*) sont uniques,*

C2. v_ε^I *et* $\partial_{x_b} u_\varepsilon^I$ *(*$b = 1, \dots, n$*) sont* ε*-oscillantes (voir les équations* (53)*,Chapitre 2),*

C3. Les mesures de Wigner de v_ε^I *et* $\partial_{x_b} u_\varepsilon^I$ *(*$b = 1, \dots, n$*) ne chargent pas l'ensemble* $\mathbb{R}^n \times \{\xi = 0\}$.

Alors la densité d'énergie par vecteur d'onde $\frac{1}{2} w[\partial_t u_\varepsilon(t,.)] + \frac{1}{2} \mathrm{Tr} w[c \partial_x u_\varepsilon(t,.)]$ *s'écrit dans* $\Omega \times (\mathbb{R}^n \backslash \{0\})$ *comme la somme de deux mesures de Wigner initiales transportées le long du flot bicaractéristique brisé obtenu par réflexions successives des rayons au bord.*

La démonstration se divise en deux étapes : on prouve d'abord le théorème pour des conditions initiales qui vérifient l'hypothèse A3 puis on l'étend à des conditions initiales plus générales. Sous l'hypothèse A3, les mesures de Wigner associées aux dérivées de la solution exacte u_ε et aux dérivées d'une solution approchée u_ε^R sont les mêmes. On commence donc par calculer explicitement les transformées de Wigner associées aux dérivées de u_ε^R dans le cas le plus simple $R = 2$. Pour cela on suit la démarche de Robinson [88], qui a calculé des quantités similaires pour l'équation de Schrödinger dans tout l'espace. Il a étudié la transformée de Wigner d'une superposition de faisceaux gaussiens pondérés par une transformée FBI et l'a approchée par une intégrale faisant apparaître une quantité proche du carré du module de la transformée FBI transportée. On calcule la limite de cette intégrale en utilisant le théorème de convergence dominée. On prouve ainsi le théorème 3 pour la solution approchée u_ε^R et par conséquent pour la solution exacte du problème (3) avec des conditions initiales qui vérifient les hypothèses A1-A3 et C1. On veut ensuite s'affranchir de l'hypothèse A3 qui est une hypothèse nécessaire à la sommation des faisceaux gaussiens et non au calcul des transformées de Wigner, et la remplacer par les hypothèses classiques C2, C3 d'ε-oscillation et de non chargement de l'ensemble $\mathbb{R}^n \times \{\xi = 0\}$. Pour cela, on construit une suite de données initiales qui vérifient A3 et telles que les mesures de Wigner associées approchent celles de u_ε^I et v_ε^I.

Pour le système de l'élasticité les calculs sont au départ similaires mais il faut tenir compte des changements de modes à la réflexion : les ondes qui se propagent à la vitesse c_L donnent naissance à des ondes se propageant à la vitesse c_T et inversement. La décomposition de Helmholtz des conditions initiales

$$u_\varepsilon^I = f_\varepsilon + \Psi_\varepsilon, v_\varepsilon^I = g_\varepsilon + \Theta_\varepsilon \text{ avec } \mathrm{rot} f_\varepsilon = \mathrm{rot} g_\varepsilon = 0 \text{ et } \mathrm{div}\Psi_\varepsilon = \mathrm{div}\Theta_\varepsilon = 0,$$

permet d'identifier les quantités transportées selon les flots associés à chacune des vitesses : les termes de rotationnel nul se propagent à la vitesse c_L et les termes de divergence nulle à la vitesse c_T. Cependant des termes supplémentaires apparaissent dans la transformée de Wigner. Il s'agit de termes croisés entre des quantités qui se transportent selon les flots réfléchis associés à des vitesses différentes. On a alors besoin d'une hypothèse supplémentaire pour annuler la contribution de ces termes croisés. On prouve le résultat suivant :

Théorème 4. [théorème 4.1 du chapitre 3] *Supposons vérifiées les hypothèses nécessaires sur les conditions initiales (voir p.100 et p.122) et le domaine (voir p.101), excluant notamment les rayons provenant de* Ω *qui touchent le bord tangentiellement ou à une incidence supérieure ou égale à l'angle critique. Supposons également que*

D1. Les mesures de Wigner associées à f_ε et Ψ_ε sont singulières,

D2. Les mesures de Wigner associées à g_ε et Θ_ε sont singulières.

Alors on peut calculer la densité d'énergie par vecteur d'onde $\frac{\rho}{2}\mathrm{Tr}w[\partial_t u_\varepsilon(t,.)](x,\xi) + \frac{\mu}{4}\sum_{j=1}^{3}\mathrm{Tr}w[\partial_{x_j} u_\varepsilon(t,.) + \partial_x (u_\varepsilon)_j(t,.)](x,\xi) + \frac{\lambda}{2}w[\mathrm{div} u_\varepsilon(t,.)](x,\xi)$ *pour le problème mixte* (2) *en fonction des mesures de Wigner des conditions initiales.*

Ce manuscrit comprend trois chapitres. Les chapitres 1 et 2 sont sous forme d'articles[2][3]. Tous les deux traitent le problème mixte de l'équation des ondes scalaire. Au chapitre 1, une solution approchée est construite par sommation de faisceaux gaussiens. Au chapitre 2, la densité d'énergie microlocale de la solution exacte est calculée au moyen des mesures de Wigner. Le chapitre 3 utilise les mêmes techniques pour l'élasticité.

[2]Chapitre 1 : à paraître dans Comm. Math. Sci.
[3]Chapitre 2 : Prépublication.

Introduction

Statement of the problem

Many physical phenomena are modelled by wave equations. In an elastic medium, the equations of linearized motion in the case of small disturbances are:

$$\rho\partial_t^2 u = \mathrm{div}\sigma(u), \tag{1}$$

where u is the displacement around a static equilibrium configuration considered as a reference state, ρ is the density, $\sigma(u)$ is the stress tensor and $\mathrm{div}\sigma(u)$ is the vector of components $(\mathrm{div}\sigma(u))_j = \sum_{k=1}^{3} \partial_{x_k}\sigma_{jk}(u)$ for $1 \leq j \leq 3$. Earth surface in seismology, human organs in medical imagery, as well as many structures in industrial mechanics can be considered as elastic media. If the properties of the medium are the same in all the directions, the medium is called isotropic, and the stress tensor is given by

$$\sigma(u) = \lambda \mathrm{div} u Id + \mu\left(\partial_x u + \partial_x u^T\right),$$

where $(\partial_x u)_{jk} = \partial_{x_k} u_j$ and $\lambda(x)$, $\mu(x)$ are Lamé coefficients satisfying $\mu(x) > 0$ et $\lambda(x) + 2\mu(x) > 0$. Equations (1) read

$$\rho\partial_t^2 u - \partial_x(\lambda \mathrm{div} u) - \mathrm{div}\left(\mu\partial_x u + \mu\partial_x u^T\right) = 0. \tag{2a}$$

We assume $\mu \neq \lambda + 2\mu$ and the density and the Lamé coefficients are smooth. In a homogeneous medium, the density and the Lamé coefficients do not depend on the position x. The solution u of (2a) can be written in this case as the sum of two terms u_L and u_T, which are curl-free and divergence-free respectively. Each one of these terms is a solution of a wave equation

$$u = u_L + u_T,\ \partial_t^2 u_L - c_L^2 \triangle u_L = 0,\ \partial_t^2 u_T - c_T^2 \triangle u_T = 0,$$

with

$$c_L^2 = \frac{\lambda + 2\mu}{\rho} \text{ and } c_T^2 = \frac{\mu}{\rho}.$$

In this thesis, we are interested in high frequency problems which arise in several applications. The study of industrial structures vibrations, when the frequency of excitation is important (shocks) is an example of high frequency oscillations in elastic media. One can also encounter this kind of problem in seismic waves propagation when the wavelength is small.

We thus complete equations (2a) with initial conditions $(u_\varepsilon^I, v_\varepsilon^I)$

$$u|_{t=0} = u_\varepsilon^I, \partial_t u|_{t=0} = v_\varepsilon^I, \tag{2b}$$

depending on a high frequency parameter $\varepsilon \ll 1$. The exact form of the initial data is not important here. A typical example is $u_\varepsilon^I = \varepsilon a e^{i\phi_0/\varepsilon}$ and $v_\varepsilon^I = b e^{i\psi_0/\varepsilon}$, then the solution of the elasticity system with these highly oscillating initial conditions depends on ε (and will be denoted by u_ε). We are interested in the behavior of this solution when ε is very small.

For industrial structures (aerospace, automotive, railways, civil engineering, etc.) as well as in seismology, the elastic bodies considered occupy a domain Ω with a boundary and the relations (2a)-(2b) must be supplemented by boundary conditions on $\partial\Omega$. The clamped boundary condition or Dirichlet condition reads

$$u_\varepsilon|_{\partial\Omega} = 0, \tag{2c}$$

whereas a free boundary results in a condition of the Neumann type

$$\sigma(u_\varepsilon)\nu|_{\partial\Omega} = 0, \tag{2c'}$$

where ν is the normal exterior to the boundary. One can of course impose other boundary conditions (mixed, oblique derivatives, etc). The problem of solving a hyperbolic partial differential equation or system of equations with given initial conditions and boundary conditions is called a hyperbolic mixed problem.

The structure of the exact solution of a mixed problem for the wave equation depends on the geometry of the domain. Indeed, in the interior of the domain, the energy is propagated along bicharacteristics which are curves of the cotangent bundle $T^*(\mathbb{R} \times \Omega)$. Projections of these curves on $\mathbb{R} \times \Omega$ are the optical rays. For the wave operator with a constant speed c, these rays are curves (t, x^t) of $\mathbb{R}^{n+1}$ moving in a rectilinear way at the speed c inside Ω. When striking the boundary transversally, the optical rays are reflected according to geometrical optics laws. If they meet the boundary tangentially, they may give rise to diffractive rays which hit the boundary without being deviated. They can also give rise to gliding rays which remain on the boundary $\partial\Omega$, and are limits of rays approaching the boundary and reflected a large number of times. As regards the operator of elasticity, it has two families of rays associated with each one of the speeds c_L and c_T. When striking the boundary, the rays associated to the speed c_L can give rise to rays associated to the speed c_T and conversely.

One can study high frequency problems by building asymptotic developments of the solution, valid when ε is very small. The solution is thus approximated for a suitable norm with an accuracy increasing with the frequency. There exist also other approaches which are focused only on the limit when $\varepsilon \to 0$ of some quantities associated with the solution such as the local energy density $|u_\varepsilon|^2$, for example the Wigner measures method.

In this thesis, we are interested in the high frequency solutions of mixed problems for the scalar wave equation

$$\begin{aligned} &\partial_t^2 u_\varepsilon - \partial_x \cdot (c^2(x)\partial_x u_\varepsilon) = 0,\ u_\varepsilon|_{t=0} = u_\varepsilon^I,\ \partial_t u_\varepsilon|_{t=0} = v_\varepsilon^I, \\ &u_\varepsilon|_{\partial\Omega} = 0 \text{ or } \partial_\nu u_\varepsilon|_{\partial\Omega} = 0, \end{aligned} \tag{3}$$

and the system of elasticity (2). The boundary is assumed to be smooth and only the reflection phenomenon is studied. The adopted strategy is the following. We start by studying the simpler problem of the scalar wave equation, then the techniques used are adapted to the elasticity system. Under some hypotheses on the initial conditions, we approach the solution close to $O(\varepsilon^N)$ for all $N \in \mathbb{N}$ by building a family of asymptotic solutions. The construction is based on a powerful method well controlled for the phenomenon of reflection at the boundary: the Gaussian beams summation method. For more general initial conditions, we use Wigner measures to compute the microlocal energy density. This quantity is characterized by analyzing the microlocal energy density of the approximate solution by means of explicit computations on the Wigner transforms.

Gaussian beams summation

There exist several mathematical models of approximate solutions for wave equations when the frequency grows to infinity. The exact solutions for these equations are known for certain configurations called canonical problems. In a homogeneous medium with one wave speed, the canonical solutions are plane waves. In a medium where the wavelength is small compared to its heterogeneities and to the propagation distances, this form of exact solutions is valid for high frequencies locally. One can then guess the shapes of the solutions. This is what is called an ansatz. The following step is then to find some conditions for the ansatz in order to satisfy effectively the considered wave equations.

A very simple ansatz is

$$u_\varepsilon \simeq a_0(t,x)e^{i\psi(t,x)/\varepsilon}, \tag{4}$$

where a_0 is a scalar or vector amplitude according to the considered problem, and ψ a scalar phase. This method is called the geometrical optics method, or the WKB or WKBJ method [52], for the scientists Wentzel, Kramers, Brillouin and Jeffreys who independently introduced it in the 1920's. To describe reflection in the case of a domain with boundary, similar terms with reflected amplitudes and phases are added in the previous ansatz. For the wave equation with speed $c(x)$, one obtains by applying the operator $\partial_t^2 - \partial_x \cdot (c^2\partial_x)$ to this ansatz the following terms organized according to the powers of ε

$$\varepsilon^{-2}\left[c^2|\partial_x\psi|^2 - (\partial_t\psi)^2\right]a_0 + i\varepsilon^{-1}\left[2\partial_t\psi\partial_t a_0 - 2c^2\partial_x\psi\partial_x a_0 + (\partial_t^2\psi - \partial_x\cdot(c^2\partial_x\psi))a_0\right] + ...$$

Making the first term vanish, one gets an eikonal equation for the phase ψ

$$c^2|\partial_x\psi|^2 - (\partial_t\psi)^2 = 0.$$

For the elasticity system, similar computations lead to the same eikonal equation with one of the two speeds c_L or c_T, coupled with an information on the direction of the vectorial amplitude a_0.

In the case of a real phase ψ, this Hamilton-Jacobi type equation has two local solutions which satisfy $c|\partial_x\psi| \pm \partial_t\psi = 0$, for one given initial phase ψ_0. The traditional method to compute these solutions is the method of characteristics. In order to find for example the solution of $c|\partial_x\psi| + \partial_t\psi = 0$, it consists in solving the Hamiltonian system associated with the symbol $h_+(x,\xi) = c(x)|\xi|$

$$\frac{dx^t}{dt} = \partial_\xi h_+(x^t,\xi^t) = c(x^t)\frac{\xi^t}{|\xi^t|}, \frac{d\xi^t}{dt} = -\partial_x h_+(x^t,\xi^t) = -\partial_x c(x^t)|\xi^t|$$

with initial direction $\xi^0 = \partial_x\psi_0(x^0)$, and then integrating the equation $\frac{d\psi}{dt} = \partial_t\psi + \partial_x\psi \cdot \frac{dx^t}{dt}$ with the initial condition $\psi(0,x) = \psi_0(x)$, along the curves x^t. However the phase ψ is generally not global in time. Indeed the map $x^0 \mapsto x^t$ is not always one-to-one, and several different rays can cross, yielding what is called a caustic. It results in WKB solutions which are not valid at the caustics. On the other hand, the formation of caustics is a recurring situation even in the simplest models and structures [17].

If the phase ψ has a non zero imaginary part, the method of characteristics is no more applicable. However the properties of $\operatorname{Im}\psi$ control the envelope of the asymptotic solution. Indeed, if ψ is real on a ray (t, x^t) and if the imaginary part of its Hessian matrix on this ray $\partial_x^2\psi(t, x^t)$ is positive definite, then at any instant t the principal part of the energy density of $a_0 e^{i\psi/\varepsilon}$ is concentrated in the vicinity of the point $x = x^t$ for small ε. It is not necessary any more to satisfy the eikonal equation exactly but only to make the Taylor series of $c^2|\partial_x\psi|^2 - (\partial_t\psi)^2$ vanish up to a certain order $R \geq 2$ on the ray. One has then to solve differential systems which have global solutions. This is what is called Gaussian beams, which owe their name to the fact that their energy density at a given instant is a Gaussian function. These approximate solutions, which belong to complex geometrical optics (see [56] for a comparison between the various methods of complex geometrical optics), also appear under the name of "quasiphotons" because at every instant t they are concentrated in the vicinity of a point which moves along some geodesic line with a unit speed and has several properties of the particles (energy conservation law, reflection at the boundary, etc.). Some authors distinguish these Gaussian beams depending on the time and the space variable from those which depend only on the space variable by calling them space-time Gaussian beams, nonstationary Gaussian beams, or Gaussian packets and Gaussian wave packets.

Historically, Gaussian beams appear in the work of V.M. Babich in the 1960's [5] and are generalized in the 1980's by J. Ralston [84], V.M. Babich and V.V. Ulin [8]. These approximate solutions were widely used in elasticity [6, 18, 53, 78], and for optical resonators [7]. Gaussian beams can be adapted naturally to other equations, such as the Helmholtz and Schrödinger equations. Just like the various methods of complex geometrical optics, they constitute an alternative to traditional geometrical optics to describe the solutions beyond the caustics, globally in time. They can also be seen as a basis of elementary solutions for wave propagation, thus allowing to study the general solutions of partial differential equations [80, 84]. The accuracy of these solutions can be improved by adding to the amplitude a_0 further terms of higher powers of ε of the form $\varepsilon a_1 + \varepsilon^2 a_2 + \dots$ and by increasing the order R up to which the eikonal equation is satisfied on the ray.

To describe a field with non Gaussian profile, one uses the Gaussian beams summation method [18, 51, 54, 82]. The initial field is expanded as a sum of Gaussian beams. Each individual Gaussian beam is computed by solving the associated differential systems. The field is then obtained at an observation point by superposing a selection of Gaussian beams. The summation strategies are numerous. The sum can be discrete or continuous, the selection of the Gaussian beams to be superposed can be done according to several criteria. One can quote some recent orientations:

- selection of rays of initial direction $\partial_x\psi_0$ to describe WKB initial data with a phase ψ_0 [63, 96];
- use of the Fourier transform [41, 97];
- use of the FBI (Fourier-Bros-Iagolnitzer) transform [88] defined from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^{2n})$ by

$$T_\varepsilon(f)(x,\xi) = c_n \varepsilon^{-\frac{3n}{4}} \int_{\mathbb{R}^n} f(z) e^{i\xi\cdot(x-z)/\varepsilon - (x-z)^2/(2\varepsilon)} dz,\ c_n = 2^{-\frac{n}{2}} \pi^{-\frac{3n}{4}} \text{ for } f \in L^2(\mathbb{R}^n).$$

The second and third methods allow to get data of the form of an amplitude multiplied by the exponential of a phase.

Whatever the method is, it is important to evaluate its performances by estimating the error between the theoretical field and the field obtained by the Gaussian beams summation. The discretization error of an integral of Gaussian beams for elasticity was analyzed in [55]. Recently, the accuracy of a continuous superposition of Gaussian beams to approach the exact solution of the acoustic wave equation was studied in [96, 63]. The error related to the use of Taylor series for the phases and the amplitudes of the Gaussian beams was quantified by [76] for the Helmholtz equation. Similar studies were carried out for the Schrödinger equation in [58, 64]. In chapter 1 we use the FBI transform to construct a family of solutions of the mixed problem (3) as an integral of Gaussian beams. The following error estimate is proved:

Theorem 1. [theorem 1.1, chapter 1] *Suppose fulfilled the required hypotheses on the domain (B1-B3 p. 33), in particular transversality at the boundary of all rays originating from Ω. Assume that the initial conditions satisfy the following assumptions*

A1. u_ε^I *and* v_ε^I *are uniformly bounded in* $H^1(\Omega)$ *and* $L^2(\Omega)$ *respectively,*

A2. u_ε^I *and* v_ε^I *vanish outside a fixed compact of* Ω*,*

A3. $T_\varepsilon u_\varepsilon^I(x,\xi)$ *and* $T_\varepsilon v_\varepsilon^I(x,\xi)$ *are negligible for large* ξ *and* ξ *close to zero (see p. 28).*

Then we can construct for $R \in \mathbb{N}$, $R \geq 2$*, an approximate solution* u_ε^R *of the Dirichlet or Neumann problem of the scalar wave equation as an integral of Gaussian beams. This solution satisfies for all* $T > 0$

$$\underset{t\in[0,T]}{Sup} \|u_\varepsilon^R(t,.) - u_\varepsilon(t,.)\|_{H^1(\Omega)} = O(\varepsilon^{\frac{R-1}{2}}),$$

$$\text{and } \underset{t\in[0,T]}{Sup} \|\partial_t u_\varepsilon^R(t,.) - \partial_t u_\varepsilon(t,.)\|_{L^2(\Omega)} = O(\varepsilon^{\frac{R-1}{2}}).$$

The strategy is the following. One starts by decomposing the initial conditions at point z over the family of functions $\left(e^{i\xi\cdot(x-z)/\varepsilon-(x-z)^2/(2\varepsilon)}\right)_{(x,\xi)\in\mathbb{R}^{2n}}$. Up to a scaling coefficient, this is the kernel of the adjoint of the FBI transform, which is an isometry. The initial conditions are then written as an integral of Gaussian beams weighted by their FBI transforms (modulo a scaling coefficient). One builds the individual Gaussian beams following the formalism of [84]. Superposition of beams of which phases satisfy the eikonal equation up to order R gives an approximate solution u_ε^R. One then estimates the errors in the interior equation $(\partial_t^2 - \partial_x \cdot (c^2(x)\partial_x))\, u_\varepsilon^R$, the boundary condition and the initial conditions $u_\varepsilon^R|_{t=0} - u_\varepsilon^I$ and $\partial_t u_\varepsilon^R|_{t=0} - v_\varepsilon^I$. All these terms are the results of a family of integral operators with a complex phase applied to the initial data' FBI transforms. The norms of these operators from $L^2(\mathbb{R}^{2n})$ to $H^s(\mathbb{R}^n)$ are computed by using the smoothness of the Gaussian beams phases and amplitudes and the properties of these phases. Once all the errors are estimated, the difference between the approximate solution u_ε^R and the exact solution is controlled by the mixed problem energy estimate. One obtains immediately the order $\varepsilon^{\frac{R-1}{2}}$ for the Dirichlet boundary condition. To prove the same order for the Neumann problem, one resorts to the approximate solution u_ε^{R+1} and compares it to the exact solution and to u_ε^R.

These ideas can be naturally adapted to the elasticity problem, by generalizing the concept of FBI transform to vector functions. We obtain the following estimate for tridimentional elasticity:

Theorem 2. [theorem 1.1, chapter 3] *Suppose fulfilled the required assumptions on the initial conditions and the boundary (see p.100-101), in particular excluding rays originating from Ω which hit the boundary tangentially or with an incidence equal or larger than the critical angle. One can build for $R \in \mathbb{N}$, $R \geq 2$, an approximate solution u_ε^R of the mixed problem for elasticity as an integral of Gaussian beams. This solution satisfies for all $T > 0$*

$$\underset{t\in[0,T]}{Sup} \|u_\varepsilon^R(t,.) - u_\varepsilon(t,.)\|_{H^1(\Omega)^3} = O(\varepsilon^{\frac{R-1}{2}}),$$

$$\text{and } \underset{t\in[0,T]}{Sup} \|\partial_t u_\varepsilon^R(t,.) - \partial_t u_\varepsilon(t,.)\|_{L^2(\Omega)^3} = O(\varepsilon^{\frac{R-1}{2}}).$$

Wigner measures

Wigner measures are phase space measures which allow to describe the asymptotic behavior of quadratic quantities such as the local energy density. The Wigner function was introduced in 1932 by E. Wigner [100] in quantum mechanics. Since then, it has been applied in various other fields like optics and signal analysis. In the nineties, many mathematicians became interested in Wigner measures, such as P.- L. Lions, T. Paul [62] and P. Gérard [35] (see also the papers [10, 28, 38] and the talk [12]). Wigner measures are related to H-measures and microlocal defect measures, introduced by L. Tartar [98] and P. Gérard [36] (see also [33]). At Onera[4], recent works resort to Wigner measures to deduce the behavior of high frequency vibratory energy in an elastic [93] or viscoelastic [2, 3] medium. These works agree with the engineering approaches [40, 99] which constitute an alternative for the techniques usually used to study the high frequency vibrations of structures: the statistical energy analysis (SEA) [65, 66] and the power flow analysis [77, 90]. The SEA is a global approach insofar as it provides only estimates of average vibratory energies for mechanical subsystems. The main difficulty of the method, which is still very heuristic, is the derivation of the involved physical parameters: subsystems coupling loss factors, modal densities, injected powers. The power flow analysis is a local approach because it provides estimates of the energy densities and the vibratory intensity. Nevertheless, it relies on strong assumptions that can not easily be checked, or are even false for complex structures. That is why it is used only for simple structures (beams, plates). This method leads to a diffusion equation for the vibratory energy density. However the use of traditional WKB solutions shows that the equation satisfied by the energy density is a transport equation. The use of Wigner measures is a rigorous alternative to tackle these difficulties. Moreover this method provides the energy propagation directions and paths.

A Wigner measure $w[f_\varepsilon]$ for a sequence (f_ε) uniformly bounded in $L^2(R^n)^p$ is a weak limit of the sequence of the Wigner transforms associated with f_ε (upon extracting a subsequence)

$$w_\varepsilon[f_\varepsilon](x,\xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-iv\cdot\xi} f_\varepsilon(x + \frac{\varepsilon}{2}v) f_\varepsilon^*(x - \frac{\varepsilon}{2}v) dv.$$

Under some assumptions, the limit (in the sense of measures) when $\varepsilon \to 0$ of the energy density for wave equation solutions can be expressed in term of Wigner measures. For

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the scalar wave equation, the energy density at the instant t converges to

$$\frac{1}{2}\int_{\mathbb{R}^n} w[\partial_t u_\varepsilon(t,.)](x,d\xi) + \frac{1}{2}\int_{\mathbb{R}^n} \mathrm{Tr}w[c\partial_x u_\varepsilon(t,.)](x,d\xi).$$

For the elasticity system, it converges to

$$\frac{\rho}{2}\int_{\mathbb{R}^n} \mathrm{Tr}w[\partial_t u_\varepsilon(t,.)](x,d\xi) + \frac{\mu}{4}\sum_{j=1}^{3}\int_{\mathbb{R}^n} \mathrm{Tr}w[\partial_{x_j} u_\varepsilon(t,.) + \partial_x (u_\varepsilon)_j(t,.)](x,d\xi) + \frac{\lambda}{2}\int_{\mathbb{R}^n} w[\mathrm{div} u_\varepsilon(t,.)](x,d\xi).$$

These quantities were fully characterized for wave equations in the whole space domain [38, 79]. In the presence of a boundary, the study of Wigner measures becomes technically more difficult. The concept of Wigner measures was used in bounded domains for the analysis of the ergodic properties of the eigenfunctions for the Dirichlet problem in [37, 102], or the Neumann and Robin problems in [13]. Other studies have been focused on the Wigner measures in a bounded domain or with an interface, such as the papers [11, 74, 91] and the theses [25, 30]. All these works are based on the use of semi-classical pseudo-differential calculus.

In chapter 2, the microlocal energy density for the solution of the problem (3) is described by using another approach similar to [15, 88] based on Gaussian beams. The following theorem is proved:

Theorem 3. [theorem 1.1, chapter 2] *Suppose fulfilled the required hypotheses on the domain (B1-B3 p.61), in particular transversality at the boundary of all rays originating from Ω. Assume that the initial conditions satisfy A1, A2 together with the following assumptions (after extension by zero outside Ω)*

- *C1. The Wigner measures of v_ε^I and $\partial_{x_b} u_\varepsilon^I$ $(b = 1, \ldots, n)$ are unique,*
- *C2. v_ε^I and $\partial_{x_b} u_\varepsilon^I$ $(b = 1, \ldots, n)$ are ε-oscillatory (see equations (53),chapter 2),*
- *C3. The Wigner measures of v_ε^I and $\partial_{x_b} u_\varepsilon^I$ $(b = 1, \ldots, n)$ do not load the set $\mathbb{R}^n \times \{\xi = 0\}$.*

Then the microlocal energy density $\frac{1}{2}w[\partial_t u_\varepsilon(t,.)] + \frac{1}{2}\mathrm{Tr}w[c\partial_x u_\varepsilon(t,.)]$ is equal in $\Omega \times (\mathbb{R}^n \backslash \{0\})$ to the sum of two initial Wigner measures transported along the broken bicharacteristic flow obtained by successively reflecting the rays at the boundary.

The proof is divided into two steps: the theorem is firstly proved for initial conditions which satisfy the assumption A3 and then extended to more general initial conditions. Under the assumption A3, Wigner measures associated with the derivatives of the exact solution u_ε and with the derivative of an approximate solution u_ε^R are the same. One thus starts by computing explicitly the Wigner transforms associated with the derivatives of u_ε^R in the simple case $R = 2$. To do so one follows the ideas of Robinson [88], who computed similar quantities for the Schrödinger equation in the whole space domain. He analyzed the Wigner transform of a superposition of Gaussian beams weighted by a FBI transform and approached it by an integral involving a quantity close to the square modulus of the transported FBI transform. We compute the limit of this integral by using the dominated convergence theorem. Theorem 3 is thus proved for the approximate solution u_ε^R and consequently for the exact solution of the problem (3) with initial conditions satisfying the assumptions A1-A3 and C1. One wants then to remove the assumption A3 which is needed for the Gaussian beams summation but not for the computation of the Wigner transforms, and to replace it by the traditional assumptions C2, C3 of ε-oscillation and unloading of the set $\mathbb{R}^n \times \{\xi = 0\}$. In order to do that, we build a sequence of initial data which fulfill A3 and such that their associated Wigner measures approach those of u_ε^I and v_ε^I.

For the system of elasticity computations are similar at the beginning but one has to take into account the phenomenon of mode conversion at the reflections: the waves propagating at the speed c_L give rise to waves propagating at the speed c_T and conversely. Helmholtz decomposition of the initial conditions

$$u_\varepsilon^I = f_\varepsilon + \Psi_\varepsilon, v_\varepsilon^I = g_\varepsilon + \Theta_\varepsilon \text{ with } \mathrm{rot} f_\varepsilon = \mathrm{rot} g_\varepsilon = 0 \text{ and } \mathrm{div}\Psi_\varepsilon = \mathrm{div}\Theta_\varepsilon = 0,$$

allows to identify the quantities transported along the flows associated with each speed: the curl-free terms propagate at the speed c_L and the divergence-free terms at the speed c_T. However additional cross terms between quantities transported along different flows appear in the Wigner transform. One then needs a further assumption to cancel the contribution of these cross terms. The following result is proved:

Theorem 4. [theorem 4.1, chapter 3] *Suppose fulfilled the required assumptions on the initial conditions (see p.100 and p.122) and the boundary (see p.101), in particular excluding rays originating from Ω which hit the boundary tangentially or with an incidence equal or larger than the critical angle. Suppose furthermore that*

D1. The Wigner measures associated with f_ε and Ψ_ε are singular,

D2. The Wigner measures associated with g_ε and Θ_ε are singular.

Then one can compute the microlocal energy density

$$\frac{\rho}{2}\mathrm{Tr}w[\partial_t u_\varepsilon(t,.)](x,\xi) + \frac{\mu}{4}\sum_{j=1}^{3}\mathrm{Tr}w[\partial_{x_j} u_\varepsilon(t,.) + \partial_x (u_\varepsilon)_j(t,.)](x,\xi) + \frac{\lambda}{2}w[\mathrm{div}u_\varepsilon(t,.)](x,\xi)$$

for the mixed problem (2) *by using the Wigner measures of the initial conditions.*

This thesis contains three chapters. Chapters 1 and 2 are included in paper forms[5][6]. Both of them deal with the mixed problem of the scalar wave equation. In chapter 1, an approximate solution is constructed by Gaussian beams summation. In chapter 2, the microlocal energy density of the exact solution is computed by means of Wigner measures. Chapter 3 uses the same techniques for elasticity.

[5]Chapter 1: to appear in Comm. Math. Sci.
[6]Chapter 2: Preprint

Chapter I

Gaussian beams summation for the wave equation in a convex domain

Contents

1 Introduction

In this paper, our aim is to provide asymptotic solutions, in a sense to be precised later, to the following initial-boundary value problem (IBVP) for the wave equation

$$\begin{cases} Pu_\varepsilon = \partial_t^2 u_\varepsilon - \partial_x \cdot (c^2(x)\partial_x u_\varepsilon) = 0 \text{ in } [0,T] \times \Omega, \\ u_\varepsilon|_{t=0} = u_\varepsilon^I,\ \partial_t u_\varepsilon|_{t=0} = v_\varepsilon^I \text{ in } \Omega, \\ Bu_\varepsilon = 0 \text{ in } [0,T] \times \partial\Omega, \end{cases} \tag{1}$$

where B is a Dirichlet or Neumann type boundary operator.

Above, $T > 0$ is fixed, and Ω is a bounded domain of $\mathbb{R}^n$, with $n = 2$ or $n = 3$ for important applications to acoustics or elastodynamics problems.

We assume the boundary $\partial\Omega$ is $\mathcal{C}^\infty$ and the domain convex for the bicharacteristic curves of P, see more precisely assumption B1 p.33 below. Furthermore, the coefficient c is assumed to be in $\mathcal{C}^\infty(\bar{\Omega})$, though this assumption may be substantially relaxed.

Our initial data will depend on a small parameter $\varepsilon > 0$, playing the role of a small wavelength, and our main objective is to study the high frequency limit, corresponding to $\varepsilon \to 0$, that is the construction of high frequency solutions. Moreover, we shall assume that u_ε^I, v_ε^I are

A1. uniformly bounded respectively in $H^1(\Omega)$ and $L^2(\Omega)$,

A2. uniformly supported in a fixed compact set $K \subset \Omega$.

The search for such approximate solutions and related notions of parametrices for the wave equation and similar equations has been an intensive field of activities. A widely used technique to produce such high frequency solutions is given by geometric optics, also called WKB method [72]. This technique is well known in the Physics literature [52]. Then, and in the full space case, approximate solutions are constructed under the form

$$\sum_{j=0}^{N} \varepsilon^j a_j e^{i\psi/\varepsilon}, \tag{2}$$

with a real phase function ψ and complex amplitudes functions a_j. The presence of a boundary may lead to further terms with reflected phases and amplitudes.

Typically, initial data should have the same form as in (2), but solutions for more general initial conditions can be obtained by summing an infinite number of WKB solutions. Mathematically, this technique relies on the well known theory of Fourier Integral Operators (FIOs), see for instance [43], see also the earlier works of Maslov and Fedoruk [72] and the recent lecture notes by Rauch and Markus [87]. In general, the global construction of a FIO breaks down at some time, due to generic existence of caustics, see [24].

The caustics problem is also linked to the local solvability of the eikonal equation for the phase, which is derived by substituting the WKB ansatz in the partial differential equation. Indeed, the eikonal equation is solved using the method of characteristics and the phase therefore cannot be defined near every point of the domain, at the exception of some very particular cases.

To overcome this difficulty, one either uses a collection of local FIOs or, more generally, constructs a global FIO. This is the way chosen by Chazarain to produce a parametrix for the mixed problem of the wave equation in [20]. Though this method is quite satisfying for the mathematical analysis of propagation of singularities, it does not give approximate solutions directly. A computationally oriented alternative to this mathematical elaborate method is the use of Gaussian beams summation.

Gaussian beams are high frequency asymptotic solutions to linear partial differential equations that are concentrated on a single ray. In the mathematical literature, their first use dates back to the 1960s, see [5]. Since then, they have been useful in a variety of problems in mathematical physics such as modelling seismic [42] or electromagnetic [27] wave fields. They also have been used in pure mathematics, such as propagation of singularities [44, 84] and semiclassical measures [80], see [46] and [38] for other methods concerning these problems.

One advantage of this method over the WKB precedure is that an individual Gaussian beam has no singularities at caustics. Note that Gaussian beams summation is naturally linked to FIOs with complex phases [43] (see [14, 57, 58, 95] for recent contributions).

In a bounded domain of general geometry, both of the WKB and the Gaussian beams ansatzs are inadequate to produce asymptotic solutions. Other models are needed to describe the diffraction phenomena or the gliding of rays along the boundary, such as the Fourier-Airy Integral Operators [73] or the gliding beams [86]. However, in our precise setting of a convex domain with compactly supported initial data, only the reflection effects at the boundary must be considered.

Dirichlet or Neumann boundary conditions can be taken into account by combining a finite sum of successively reflected Gaussian beams [50, 67]. Using an infinite sum of Gaussian beams, one can then match quite general initial conditions. This summation can be achieved in different ways, see [18, 51, 54] and the recent [42, 49, 60, 63, 64, 76, 96]. In [63] and [96], superpositions of Gaussian beams are used to solve wave equations with initial data of WKB form. In fact, see Theorem 1.1 below, more general initial conditions are allowed through the use of their FBI transforms, which is also naturally linked with the concept of a Gaussian beam.

The FBI or Fourier-Bros-Iagolnitzer transform (see [23, 71, 94]) is, for a given scale ε, the operator $T_\varepsilon : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^{2n})$ defined by

$$T_\varepsilon(a)(y,\eta) = c_n \varepsilon^{-\frac{3n}{4}} \int_{\mathbb{R}^n} a(w) e^{i\eta.(y-w)/\varepsilon - (y-w)^2/(2\varepsilon)} dw, \; c_n = 2^{-\frac{n}{2}} \pi^{-\frac{3n}{4}}, \; a \in L^2(\mathbb{R}^n). \quad (3)$$

Its adjoint is the operator

$$T_\varepsilon^*(f)(x) = c_n \varepsilon^{-\frac{3n}{4}} \int_{\mathbb{R}^{2n}} f(y,\eta) e^{i\eta.(x-y)/\varepsilon - (x-y)^2/(2\varepsilon)} dy d\eta, \; f \in L^2(\mathbb{R}^{2n}). \quad (4)$$

As the Fourier Transform, the FBI transform is an isometry, satisfying $T_\varepsilon^* T_\varepsilon = Id$. Its main property is to decompose an L^2_x function over the family of functions $(e^{i\eta.(x-y)/\varepsilon - (x-y)^2/(2\varepsilon)})_{(y,\eta)\in\mathbb{R}^{2n}}$. For instance, FBI transformation was the method used in [88] to construct an approximate solution for the Schrödinger equation with WKB initial conditions. The FBI transform is of course again connected with FIOs with complex phases and an interesting result on their global L^2 boundedness has been proven recently in [95], regarding the Hermann Kluck propagator.

In this paper, our approach to find asymptotic solutions to the problem (1) is to achieve a superposition of incident and reflected Gaussian beams weighted by the FBI transforms of the initial data, satisfying both the condition at the boundary and the initial conditions. Our main result is given by

Theorem 1.1. *Under assumptions A1 and A2, suppose the FBI transforms of the initial data is infinitely small on the complement of some ring*

$$R_\eta = \{\eta \in \mathbb{R}^n, r_0 \leq |\eta| \leq r_\infty\},\ 0 < r_0 < r_\infty,$$

in the sense that

A3. $\|T_\varepsilon u^I_\varepsilon\|_{L^2(\mathbb{R}^n \times R^c_\eta)} = O(\varepsilon^s)$ *and* $\|T_\varepsilon v^I_\varepsilon\|_{L^2(\mathbb{R}^n \times R^c_\eta)} = O(\varepsilon^s)$, $\forall s \geq 0$.

Then for any integer $R \geq 2$, *there is an asymptotic solution to* (1) *of the form*

$$u^R_\varepsilon(t,x) = \sum_k \int_{\mathbb{R}^{2n}} a^k_\varepsilon(t,x,y,\eta,R) e^{i\psi_k(t,x,y,\eta,R)/\varepsilon} dy d\eta,$$

where $a^k_\varepsilon e^{i\psi_k/\varepsilon}$ *are Gaussian beams and the summation over* k *is finite.*
u^R_ε *is asymptotic to the exact solution of the IBVP* (1) *in the following sense*

$$\underset{t\in[0,T]}{Sup} \|u^R_\varepsilon(t,.) - u_\varepsilon(t,.)\|_{H^1(\Omega)} = O(\varepsilon^{\frac{R-1}{2}}),$$

$$\text{and } \underset{t\in[0,T]}{Sup} \|\partial_t u^R_\varepsilon(t,.) - \partial_t u_\varepsilon(t,.)\|_{L^2(\Omega)} = O(\varepsilon^{\frac{R-1}{2}}).$$

Let us note that construction of asymptotic solutions such as a summation of Gaussian beams is certainly not new, but rigorous justification is the main point of our work, together with precise estimates.

This paper is organized as follows. In section 2 we recall the construction of Gaussian beams for a strictly hyperbolic differential operator as achieved in [84]. Then, we study the case of the wave equation and construct the incident and reflected beams, and in a final step, we construct approximate solutions for (1) by a Gaussian beams summation. Justification of the asymptotics is given in section 3. Therein, we introduce approximation operators acting from $L^2(\mathbb{R}^{2n})$ to $L^2(\mathbb{R}^n)$ with a complex phase and compute their norms. We apply these operators on FBI transforms of initial data, and estimate the error of the constructed asymptotic solutions near the boundary, thus taking into account the precise boundary condition, and in the interior set. These estimates are combined with the errors in the initial conditions and yield the justification of the asymptotics by means of energy type estimates.

We close this introduction by a short discussion on the notations. Throughout this paper, we will use standard multiindex notations. The inner product of two vectors $a, b \in \mathbb{R}^d$ will be denoted by $a \cdot b$. The transpose of a matrix A will be noted A^T. If E is a subset of $\mathbb{R}^d$, we denote $\mathbf{1}_E$ its characteristic function. For a smooth function

$f \in \mathcal{C}^\infty(\mathbb{R}^d_x, \mathbb{C})$, we will use the notation $\partial_x f$ to denote its gradient vector $(\partial_{x_b} f)_{1\le b\le d}$, $\partial_x^2 f$ to denote its Hessian matrix $(\partial_{x_b}\partial_{x_c} f)_{1\le b,c\le d}$ and $\partial_x^r f$, $r > 2$ to denote the family $(\partial_{x_{b_1}} \dots \partial_{x_{b_r}} f)_{1\le b_1,\dots,b_r\le d}$. For a vector function $F \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{C}^p)$, we denote its Jacobian matrix by DF with $(DF)_{j,k} = \partial_k F_j$ and its second derivatives by D^2F with $(D^2F)_{j,k,l} = \partial_j\partial_k F_l$. For $y_\varepsilon, z_\varepsilon \in \mathbb{R}_+$, we use the notation $y_\varepsilon \lesssim z_\varepsilon$ if there exists a constant $c > 0$ independent of ε such that $y_\varepsilon \le c z_\varepsilon$. We write $y_\varepsilon \lesssim \varepsilon^\infty$ or $y_\varepsilon = O(\varepsilon^\infty)$ if $\forall s \ge 0$ there exists $c_s > 0$ s.t. $y_\varepsilon \le c_s \varepsilon^s$ for ε small enough . Finally, the word cons denotes a positive constant (different each time it appears).

2 Construction of the asymptotic solutions

In this section we first introduce the notion of Gaussian beams for strictly hyperbolic differential operators, following the presentation of [84]. Then the construction of incident and reflected Gaussian beams in the particular case of the wave equation is explained. Finally, the approximate solution for the IBVP (1) is given in the last section as an infinite sum of Gaussian beams.

2.1 Gaussian beams for stricly hyperbolic operators

This section follows basically the presentation of [84].

Let $P(t, x, \partial_t, \partial_x)$ be a strictly hyperbolic differential operator of order m_P and of principal symbol p. That is, we suppose that the roots τ of $p(t, x, \tau, \xi) = 0$ are simple and real for all (t, x) and $\xi \ne 0$. The symbol p is assumed real. A Gaussian beam for P is a function of the form

$$w_\varepsilon(t, x) = \sum_{j=0}^{N} \varepsilon^j a_j(t, x) e^{i\psi(t,x)/\varepsilon},\ N \in \mathbb{N}, \tag{5}$$

satisfying

$$\exists m > 0 \text{ s.t. } \|P w_\varepsilon\|_{L^2_{t,x}} = O(\varepsilon^m).$$

Note that the above expansion is similar to the usual WKB expansion, but it is required here that:

(i) the beam w_ε is concentrated on some fixed ray $(t(s), x(s))$ associated to p. Here s is the "time" parameter of this curve.

(ii) the phase ψ is a complex-valued function, but real-valued on the ray $(t(s), x(s))$.

The exact definition of a ray $(t(s), x(s))$ is as follows. First of all, we introduce the so-called null bicharacteristics, which are the curves, solutions of the Hamiltonian equations

$$\begin{aligned} \dot t(s) &= \partial_\tau p(t(s), x(s), \tau(s), \xi(s)), & \dot\tau(s) &= -\partial_t p(t(s), x(s), \tau(s), \xi(s)), \\ \dot x(s) &= \partial_\xi p(t(s), x(s), \tau(s), \xi(s)), & \dot\xi(s) &= -\partial_x p(t(s), x(s), \tau(s), \xi(s)), \end{aligned} \tag{6}$$

with initial conditions satisfying $p(t(0), x(0), \tau(0), \xi(0)) = 0$. Note that it follows that $p(t(s), x(s), \tau(s), \xi(s)) = 0$, for all s. Then by definition, the projection on $\mathbb{R}^{n+1}_{t,x}$ of such a curve $(t(s), x(s), \tau(s), \xi(s))$, that is $(t(s), x(s))$, is called a ray. We suppose fulfilled the conditions for local existence, uniqueness and smoothness with respect to initial conditions of solutions to the Hamiltonian system (6), see [39].

The construction of a Gaussian beam w_ε is achieved by making Pw_ε vanish to a certain order on a fixed and given ray $(t(s), x(s))$. For this purpose, applying P to the form (5) of a Gaussian beam, we obtain a similar form

$$Pw_\varepsilon = \sum_{j=0}^{N+m_P} \varepsilon^{j-m_P} c_j e^{i\psi/\varepsilon}, \tag{7}$$

where

$$\begin{aligned} c_0 &= p(t, x, \partial_t\psi, \partial_x\psi)a_0, \\ c_j &= La_{j-1} + p(t, x, \partial_t\psi, \partial_x\psi)a_j + g_j,\ j \geq 1. \end{aligned} \tag{8}$$

Above, $a_j = 0$ for $j > N$, $g_1 = 0$ and g_j is a function of $\psi, a_0, \ldots, a_{j-2}$ for $j \geq 2$. Furthermore, L is a linear differential operator with coefficients depending on ψ. Using p', the symbol of the terms of order $m_P - 1$ of P, L can be written in an explicit way as

$$L = \frac{1}{i}\partial_{\tau,\xi}p(t, x, \partial_t\psi, \partial_x\psi)\cdot\partial_{t,x} + \frac{1}{2i}Tr(\partial^2_{\tau,\xi}p(t, x, \partial_t\psi, \partial_x\psi)\partial^2_{t,x}\psi) + p'(t, x, \partial_t\psi, \partial_x\psi). \tag{9}$$

For the construction of a Gaussian beam adapted to P, the first step, and by far the most important one, is to build a phase ψ satisfying the eikonal equation

$$p(t, x, \partial_t\psi(t, x), \partial_x\psi(t, x)) = 0 \text{ on } (t, x) = (t(s), x(s)) \text{ up to order } R \text{ only}, \tag{10}$$

with $R \geq 2$, which means

$$\partial^\alpha_{t,x}[p(t, x, \partial_t\psi(t, x), \partial_x\psi(t, x))]|_{(t(s),x(s))} = 0 \text{ for } |\alpha| \leq R.$$

Compare this with the usual eikonal equation $p(t, x, \partial_t\psi(t, x), \partial_x\psi(t, x)) = 0$ required by the WKB method in full space.
Order 0 of eikonal (10)

$$p\,(t(s), x(s), \partial_t\psi(t(s), x(s)), \partial_x\psi(t(s), x(s))) = 0,$$

is fulfilled by setting

$$\Big(\partial_t\psi, \partial_x\psi\Big)|_{(t(s),x(s))} = \Big(\tau(s), \xi(s)\Big). \tag{P.a}$$

This constraint insures that $\frac{d}{ds}\psi(t(s), x(s))$ is real, which leads by choosing

$$\psi(t(0), x(0)) \text{ a real quantity},$$

to the required property

$$\psi(t(s), x(s)) \text{ is real}. \tag{P.b}$$

Replacing $\partial_{\tau,\xi}p|_{(t(s),x(s),\tau(s),\xi(s))}$ by $(\dot{t}(s), \dot{x}(s))$ yields in the differentiation of (10) to the compatibility condition

$$\partial^2_{t,x}\psi|_{(t(s),x(s))}\begin{pmatrix} \dot{t}(s) \\ \dot{x}(s) \end{pmatrix} = -\begin{pmatrix} \partial_t p \\ \partial_x p \end{pmatrix}|_{(t(s),x(s),\tau(s),\xi(s))} = \begin{pmatrix} \dot{\tau}(s) \\ \dot{\xi}(s) \end{pmatrix}. \tag{11}$$

It also gives for every function $f \in \mathcal{C}^\infty(\mathbb{R}_t \times \mathbb{R}^n_x, \mathbb{C})$

$$\partial_{\tau,\xi} p|_{(t(s),x(s),\tau(s),\xi(s))} \cdot \partial_{t,x} f|_{(t(s),x(s))} = \frac{d}{ds} f|_{(t(s),x(s))}. \tag{12}$$

Using this relation on $\partial^\alpha_{t,x}\psi$, $|\alpha| = 2$, we may write order 2 of eikonal (10) as

$$\frac{d}{ds}\partial^2_{t,x}\psi|_{(t(s),x(s))} + H_{12}(s)^T \partial^2_{t,x}\psi|_{(t(s),x(s))} + \partial^2_{t,x}\psi|_{(t(s),x(s))} H_{12}(s) + \partial^2_{t,x}\psi|_{(t(s),x(s))} H_{22}(s) \partial^2_{t,x}\psi|_{(t(s),x(s))} + H_{11}(s) = 0,$$

where $H_{11}(s) = \partial^2_{t,x} p|_{(t(s),x(s),\tau(s),\xi(s))}$, $(H_{12})_{bc}(s) = (\partial_{\tau,\xi})_b(\partial_{t,x})_c p|_{(t(s),x(s),\tau(s),\xi(s))}$ and $H_{22}(s) = \partial^2_{\tau,\xi} p|_{(t(s),x(s),\tau(s),\xi(s))}$. One can substitute for $\partial_t\partial_x\psi|_{(t(s),x(s))}$ and $\partial^2_t\psi|_{(t(s),x(s))}$ from the compatibility condition (11), since $\dot{t}(s) \neq 0$ by the strict hyperbolicity of P. The previous Riccati equation yields then a similar Riccati equation on $\partial^2_x\psi|_{(t(s),x(s))}$. Although non-linear, this equation has a unique global symmetric solution which satisfies the fundamental property

$$\operatorname{Im} \partial^2_x\psi|_{(t(s),x(s))} \text{ is positive definite,} \tag{P.c}$$

given an initial symmetric matrix $\partial^2_x\psi|_{(t(0),x(0))}$ with a positive definite imaginary part (see the proof of Lemma 2.56 p.101 in [50]).
Higher order derivatives of the phase on the ray are determined recursively. For $3 \leq r \leq R$, order r of the eikonal equation (10) combined with the relation (12) leads to linear inhomogeneous ordinary differential equations (ODEs) on $\partial^r_x\psi|_{(t(s),x(s))}$. They have a unique solution for a fixed initial condition $\partial^r_x\psi|_{(t(0),x(0))}$.

The second step of the construction is to make c_j, for $1 \leq j \leq N+1$, vanish on the ray up to the order $R-2j$. The choice of the order $R-2j$ is related to the quadratic imaginary part in the phase and the study of estimates in Sobolev spaces. This will appear clearly in the justification of the approximation in Lemma 2.2. In any case, the equations on the amplitudes $c_j = 0$ can be solved on the ray at most up to the order $R-2$, due to the term $\partial^2_{t,x}\psi$ in the operator L (9).
Taking into account the eikonal equation (10), one gets the following evolution equations on a_j, $0 \leq j \leq N$

$$\frac{1}{i}\partial_{\tau,\xi} p(t,x,\partial_t\psi,\partial_x\psi) \cdot \partial_{t,x} a_j + \Big[\frac{1}{2i} Tr(\partial^2_{\tau,\xi} p(t,x,\partial_t\psi,\partial_x\psi)\partial^2_{t,x}\psi) + p'(t,x,\partial_t\psi,\partial_x\psi)\Big] a_j + g_{j+1} = 0$$

$$\text{on } (t,x) = (t(s),x(s)) \text{ up to order } R-2j-2. \tag{13}$$

This equation *uniquely* determines the Taylor series of a_j on $(t(s),x(s))$ up to the order $R-2j-2$, given the values of their spatial derivatives at $(t(0),x(0))$ up to the same order.

Remark 2.1. *The number N of amplitudes in the ansatz* (5) *and the order R up to which the eikonal equation* (10) *is solved are not independent. Indeed, the computations of the amplitudes derivatives require*

$$R - 2N - 2 \geq 0.$$

Another condition ([84] p.219) is assumed to insure that the remainder terms c_j, $N+2 \leq j \leq N+m_P$, contribute with the right power of ε (see [97] for an alternative justification)

$$R - 2N - 3 \leq 0. \tag{14}$$

An essential point for the use of Gaussian beams is the smoothness of the phase and the amplitudes with respect to (w.r.t.) $(t(0), x(0))$. To this aim, the needed initial values of the derivatives of the phase $\partial_x^r\psi|_{(t(0),x(0))}$, $2 \leq r \leq R$, and of the amplitudes $\partial_x^r a_j|_{(t(0),x(0))}$, $0 \leq r \leq R-2j-2$, are chosen smooth w.r.t. $(t(0), x(0))$. The phase and the amplitudes are then prescribed to be equal to their Taylor developments (truncated up to fixed orders) on the ray.

The final step of the construction is to multiply the amplitudes by a cutoff equal to 1 near the ray.

2.2 Incident and reflected beams for the wave equation

The preceeding results will now be applied and detailed for the particular case of the wave equation and the construction of reflected beams. The computations rely on the results of [67] and [84].

We extend c in a smooth way outside $\bar{\Omega}$. Let $p(x,\tau,\xi) = c^2(x)|\xi|^2 - \tau^2$ be the principal symbol of the wave operator $P = \partial_t^2 - \partial_x \cdot (c^2\partial_x)$. Then $\tau(s) = \tau(0)$ from the Hamiltonian equations (6). Writing

$$p = -p_+p_- \text{ with } p_+(x,\tau,\xi) = c(x)|\xi| + \tau \text{ and } p_-(x,\tau,\xi) = -c(x)|\xi| + \tau,$$

shows that null bicharacteristics $s \mapsto (t(s), x(s), \tau(0), \xi(s))$ for p s.t. $\tau(0) \neq 0$ are either null bicharacteristics for p_+ if $\tau(0) < 0$ or for p_- if $\tau(0) > 0$, by using the parametrization $s' = -2\tau s$.
Denote $h_+(x,\xi) = c(x)|\xi|$ and let $(x_0^t(y,\eta), \xi_0^t(y,\eta))$ (or simply (x_0^t, ξ_0^t)) be the Hamiltonian flow for h_+ starting from the point (y,η), that is

$$\begin{aligned} &\frac{dx_0^t}{dt} = \partial_\xi h_+(x_0^t,\xi_0^t) = c(x_0^t)\frac{\xi_0^t}{|\xi_0^t|}, \quad &&\frac{d\xi_0^t}{dt} = -\partial_x h_+(x_0^t,\xi_0^t) = -\partial_x c(x_0^t)|\xi_0^t|, \\ &x_0^t|_{t=0} = y, &&\xi_0^t|_{t=0} = \eta, \eta \neq 0. \end{aligned} \tag{15}$$

Then the null bicharacteristic curve $(t(s), x(s), \tau(s), \xi(s))$ for p starting at $s = 0$ from $(0, y, \mp c(y)|\eta|, \eta)$ is exactly $(t, x_0^{\pm t}(y,\eta), \mp c(y)|\eta|, \xi_0^{\pm t}(y,\eta))$ the null bicharacteristic curve for $p_\pm$.

As in [92], one can prove that the Hamiltonian system (15) associated to h_+ has a unique solution global in time (by Cauchy-Lipschitz theorem), which depends smoothly on $(t,y,\eta) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \backslash \{0\}$.

The remainder of this section is organised as follows. In section 2.2.1, one explains the construction of incident and reflected beams associated to p_+, then section 2.2.2 is a simple repetition for p_- and finally in section 2.2.3 we give error estimates for the individual beams gathered in (22).

2.2.1 Construction of beams associated to p_+

For the ray $(t, x_0^t(y,\eta))$ associated with p_+, denote by $w_\varepsilon^0(t,x,y,\eta)$ a Gaussian beam concentrated on that ray, by $\psi_0(t,x,y,\eta)$ and $a_j^0(t,x,y,\eta)$ its associated phase and amplitudes. If no confusion is possible, symbols y, η and even t, x, y, η in the notations above will be dropped.

The phase ψ_0 is determined by solving the eikonal equation (10) on the ray (t, x_0^t) together with the conditions

$$\partial_t \psi_0(t, x_0^t) = -h_+(x_0^t, \xi_0^t),\ \partial_x \psi_0(t, x_0^t) = \xi_0^t, \tag{P_0.a}$$

and the choice of

$\psi_0(0, y)$ a real function ,
$\partial_x^2 \psi_0(0, y)$ a symmetric matrix with a positive definite imaginary part,
$\partial_x^r \psi_0(0, y)$, $3 \le r \le R$, permutable families.

In particular ψ_0 satisfies the important properties

$$\psi_0(t, x_0^t) \text{ is real,} \tag{P_0.b}$$

and

$$\operatorname{Im} \partial_x^2 \psi_0(t, x_0^t) \text{ is positive definite.} \tag{P_0.c}$$

The phase ψ_0 is assumed to be equal to its Taylor series up to the order R on $x = x_0^t$

$$\psi_0(t, x) = \sum_{|\alpha| \le R} \frac{1}{\alpha!}(x - x_0^t)^\alpha \partial_x^\alpha \psi_0(t, x_0^t). \tag{16}$$

The amplitudes of $w_\varepsilon^0(t, x)$ are also determined by the requirement that the c_j, $1 \le j \le N+1$ in (8) are null up to orders $R - 2j$ on the ray (t, x_0^t), given their initial spatial derivatives on the ray $\partial_x^r a_j^0(0, y)$, $r = 0, \dots, R - 2j - 2$. We choose them as

$$a_j^0(t, x) = \chi_d(x - x_0^t) \sum_{|\alpha| \le R-2j-2} \frac{1}{\alpha!}(x - x_0^t)^\alpha \partial_x^\alpha a_j^0(t, x_0^t),\ j = 0, \dots, N, \tag{17}$$

where $d > 0$ and χ_d is a cut-off of $\mathcal{C}_0^\infty(\mathbb{R}^n, [0, 1])$ satisfying

$$\chi_d(x) = 1 \text{ if } |x| \le d/2 \text{ and } \chi_d(x) = 0 \text{ if } |x| \ge d.$$

Throughout the paper, the parameter d will be adjusted to obtain requested estimates.

This construction leads to a beam $w_\varepsilon^0(t, x, y, \eta)$ called an incident beam for p_+, satisfying

$$\sup_{t \in [0,T]} \|P w_\varepsilon^0(t, .)\|_{L^2(\Omega)} = O(\varepsilon^m) \text{ for some } m > 0.$$

Let $\overset{o}{T^*}\Omega = T^*\Omega \backslash \{\eta = 0\}$. To study the reflection on the boundary, we make the following assumptions

B1. The domain Ω is convex for the bicharacteristic curves of P, that is for every $(y, \eta) \in \overset{o}{T^*}\Omega$, $x_0^t(y, \eta)$ cuts the boundary at only two times of opposite signs and transversally, B2. For every $(y,\eta) \in \overset{o}{T^*}\Omega$, $x_0^t(y,\eta)$ does not remain in a compact of $\mathbb{R}^n$ when t varies in $\mathbb{R}$,

B3. The boundary has no dead-end trajectories, that is infinite number of successive reflections cannot occur in a finite time.

For $(y,\eta) \in \overset{o}{T^*}\Omega$, let $T_1(y,\eta)$ be the instant (that is the exit time) s.t.

$$x_0^{T_1(y,\eta)}(y,\eta) \in \partial\Omega \text{ and } T_1(y,\eta) > 0.$$

Note that $\overset{o}{T^*}\Omega$ is an open set, and thanks to B1, the function $(y,\eta) \in \overset{o}{T^*}\Omega \mapsto T_1(y,\eta)$ is well-defined and C^∞, as follows from the implicit functions theorem. The reflection involution associated to the considered symbol p is the map

$$\begin{aligned} \mathcal{R}: \ \overset{o}{T^*}\mathbb{R}^n|_{\partial\Omega} &\to \overset{o}{T^*}\mathbb{R}^n|_{\partial\Omega} \\ (X,\Xi) &\mapsto (X, (Id - 2\nu(X)\nu(X)^T)\Xi). \end{aligned}$$

Above ν denotes the exterior normal field to $\partial\Omega$. Let $\varphi_0^t = (x_0^t, \xi_0^t)$ denote the incident Hamiltonian flow solution of (15). We define the first reflected flow φ_1^t by the condition

$$\varphi_1^{T_1} = \mathcal{R}\, o \varphi_0^{T_1},$$

that is the Hamiltonian flow for h_+ having at $t = T_1$, position $x_0^{T_1}$, the direction being given by the reflected vector of $\xi_0^{T_1}$.

Then the broken flow is defined recursively after a finite number of successive reflections as follows (see fig.1): for $k > 1$, T_k and $\varphi_k^t = (x_k^t, \xi_k^t)$ are determined by:

$T_k(y,\eta)$ is the instant s.t. $x_{k-1}^{T_k(y,\eta)}(y,\eta) \in \partial\Omega$ and $T_k(y,\eta) > T_{k-1}(y,\eta)$,
$\varphi_k^{T_k} = \mathcal{R}\, o \varphi_{k-1}^{T_k}$.

The convexity of the boundary B1 implies the non-grazing hypothesis

$$\forall (y,\eta) \in \overset{o}{T^*}\Omega \text{ and } k \geq 1,\ \dot{x}_{k-1}^{T_k(y,\eta)}(y,\eta) \cdot \nu(x_{k-1}^{T_k(y,\eta)}(y,\eta)) > 0,$$

where $\dot{x}_{k-1}^t$ denotes $\frac{d}{dt}x_{k-1}^t$. Assumption B3 leads to

$$\forall (y,\eta) \in \overset{o}{T^*}\Omega,\ T_k(y,\eta) \underset{k\to+\infty}{\to} +\infty. \tag{18}$$

It insures that for a fixed point (y,η) in $\overset{o}{T^*}\Omega$, there is a finite number $q_+(y,\eta)$ of reflections in $[0,T]$.

Following the method of Ralston in [84] p.220, we shall construct reflected beams $w_\varepsilon^1, \ldots, w_\varepsilon^{q_+}$ which satisfy the boundary estimate

$$\exists m' > 0 \text{ and } s \geq 0 \text{ s.t. } \|B(w_\varepsilon^0 + \cdots + w_\varepsilon^{q_+})\|_{H^s([0,T]\times\partial\Omega)} = O(\varepsilon^{m'}),$$

together with the interior estimates

$$\sup_{t\in[0,T]} \|Pw_\varepsilon^k(t,.)\|_{L^2(\Omega)} = O(\varepsilon^m),\ 1 \leq k \leq q_+.$$

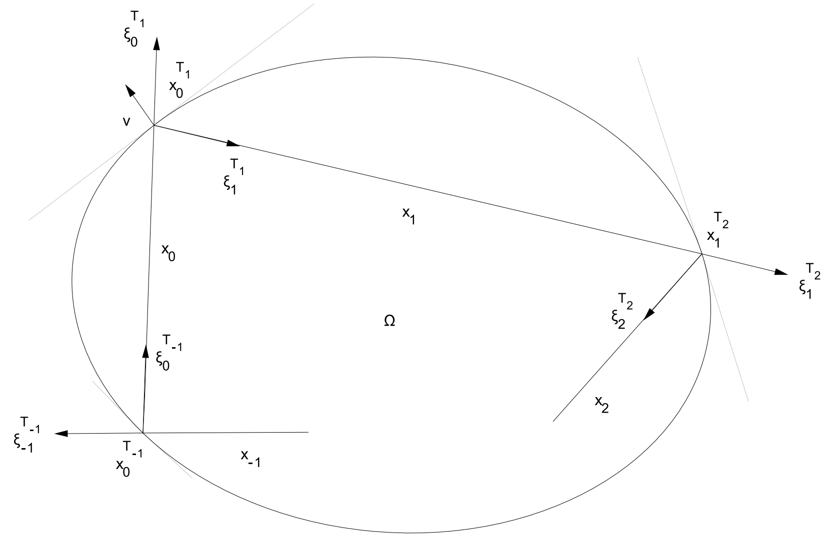

Figure 1: successive reflections.

For each $1 \leq k \leq q_+$, the reflected beam w_ε^k will be written as

$$w_\varepsilon^k = e^{i\psi_k/\varepsilon}(a_0^k + \cdots + \varepsilon^N a_N^k).$$

To insure the interior estimates, each phase ψ_k and amplitudes a_j^k $(0 \leq j \leq N)$ must satisfy equations (10) and (13) on the reflected ray (t, x_k^t).
As the beams vanish away from their associated rays, the contribution to the boundary norm of $w_\varepsilon^0 + \cdots + w_\varepsilon^{q+}$ occurs when t is close to some T_k and then from the beams w_ε^{k-1} and w_ε^k. The construction of the reflected beams is completed recursively. Assume that the beam w_ε^{k-1} has been constructed and that its associated phase satisfies

$$\partial_t \psi_{k-1}(t, x_{k-1}^t) = -h_+(x_{k-1}^t, \xi_{k-1}^t),\ \partial_x \psi_{k-1}(t, x_{k-1}^t) = \xi_{k-1}^t, \qquad (P_{k-1}.\text{a})$$
$$\psi_{k-1}(t, x_{k-1}^t) \text{ is real,} \qquad (P_{k-1}.\text{b})$$
$$\operatorname{Im} \partial_x^2 \psi_{k-1}(t, x_{k-1}^t) \text{ is positive definite.} \qquad (P_{k-1}.\text{c})$$

One may write on the boundary $\partial\Omega$

$$B\Big(w_\varepsilon^{k-1} + w_\varepsilon^k\Big) = \Big(\varepsilon^{-m_B} d_{-m_B}^{k-1} + \cdots + \varepsilon^N d_N^{k-1}\Big) e^{i\psi_{k-1}/\varepsilon} + \Big(\varepsilon^{-m_B} d_{-m_B}^{k} + \cdots + \varepsilon^N d_N^{k}\Big) e^{i\psi_k/\varepsilon},$$

m_B being the order of B ($m_B = 0$ for Dirichlet and $m_B = 1$ for Neumann).

In order to satisfy the boundary estimate, the first step is to impose on ψ_k to have the same time and tangential derivatives as ψ_{k-1} at $(T_k, x_{k-1}^{T_k})$, up to the order R. More precisely, let us introduce boundary coordinates near $x_{k-1}^{T_k} = x_k^{T_k}$ as follows. We partition $\partial\Omega$ with a finite number of small open subsets $\mathcal{U}_1, \ldots, \mathcal{U}_L$ s.t. there exist $\mathcal{C}^\infty$ parametrizations

$$\sigma_l : \mathcal{N}_l \to \mathbb{R}^n,\ l = 1, \ldots, L,$$

where $\mathcal{N}_l$ are open subsets of $\mathbb{R}^{n-1}$, $\sigma_l(\mathcal{N}_l) = \mathcal{U}_l$ and σ_l a diffeomorphism from $\mathcal{N}_l$ to $\mathcal{U}_l$. Suppose that $x_{k-1}^{T_k}$ belongs to $\mathcal{U}_{l_0}$ and denote $x_{k-1}^{T_k} = \sigma_{l_0}(\hat{z}_k)$. For $x \in \mathbb{R}^n$ close to $x_{k-1}^{T_k}$, we may write

$$x = \sigma_{l_0}(\hat{v}) + v_n \nu(\sigma_{l_0}(\hat{v})),$$

with $\hat{v} \in \mathcal{N}_{l_0}$ and $v_n \in \mathbb{R}$. If we use the notation

$$^{\sigma}f(t, \hat{v}, v_n) = f(t, x),$$

then we impose

$$\partial^{\alpha}_{t,\hat{v}}\, {}^{\sigma}\psi_k(T_k, \hat{z}_k, 0) = \partial^{\alpha}_{t,\hat{v}}\, {}^{\sigma}\psi_{k-1}(T_k, \hat{z}_k, 0),\ |\alpha| \leq R. \tag{19}$$

Order 0 of (19) gives a real value for $\psi_k(T_k, x^{T_k}_{k-1})$. Order 1 of this same constraint and order 0 of the eikonal equation (10) on ψ_k are both satisfied by setting

$$\partial_t \psi_k(t, x^t_k) = -h_+(x^t_k, \xi^t_k),\ \partial_x \psi_k(t, x^t_k) = \xi^t_k. \tag{P_k.a}$$

It follows that

$$\psi_k(t, x^t_k) \text{ is real.} \tag{P_k.b}$$

Due to the non-grazing hypothesis, (19) and the compatibility condition resulting from order 1 of the eikonal equation (10) provide with $\partial^2_x \psi_k(T_k, x^{T_k}_{k-1})$. To solve the Riccati equation on $\partial^2_x \psi_k(t, x^t_k)$ with its given value at $t = T_k$, we need to study the imaginary part of $\partial^2_x \psi_k(T_k, x^{T_k}_{k-1})$. For $k' = k-1, k$, one has

$$\partial_t \partial_{\hat{v}}\, {}^{\sigma}\psi_{k'}(t, \hat{v}, 0) = D\sigma_{l_0}(\hat{v})^T \partial_t \partial_x \psi_{k'}(t, x^t_{k'}),$$

and

$$\begin{aligned}\partial^2_{\hat{v}}\, {}^{\sigma}\psi_{k'}(t, \hat{v}, 0) = D^2\sigma_{l_0}(\hat{v}) \left(\partial_x \psi_{k'}(t, x^t_{k'})\right) \\ + D\sigma_{l_0}(\hat{v})^T \partial^2_x \psi_{k'}(t, x^t_{k'}) D\sigma_{l_0}(\hat{v}).\end{aligned}$$

Differentiating (P_{k-1}.a) and (P_k.a) yields

$$\operatorname{Im} \partial_t \partial_x \psi_{k'}(t, x^t_{k'}) = -\operatorname{Im} \partial^2_x \psi_{k'}(t, x^t_{k'})\, \dot{x}^t_{k'}$$

and

$$\operatorname{Im} \partial^2_t \psi_{k'}(t, x^t_{k'}) = \dot{x}^t_{k'} \cdot \operatorname{Im} \partial^2_x \psi_{k'}(t, x^t_{k'})\, \dot{x}^t_{k'}.$$

Denote

$$M_k = \partial^2_{t,\hat{v}}\, {}^{\sigma}\psi_{k-1}(T_k, \hat{z}_k, 0) = \partial^2_{t,\hat{v}}\, {}^{\sigma}\psi_k(T_k, \hat{z}_k, 0). \tag{20}$$

One has therefore

$$\operatorname{Im} M_k = \begin{pmatrix} -\dot{x}^{T_k}_{k'}, & D\sigma_{l_0}(\hat{z}_k) \end{pmatrix}^T \operatorname{Im} \partial^2_x \psi_{k'}(T_k, x^{T_k}_{k-1}) \begin{pmatrix} -\dot{x}^{T_k}_{k'}, & D\sigma_{l_0}(\hat{z}_k) \end{pmatrix}.$$

The non-grazing hypothesis insures that the matrices $\begin{pmatrix} -\dot{x}^{T_k}_{k'}, & D\sigma_{l_0}(\hat{z}_k) \end{pmatrix}$ are non singular. Since $\operatorname{Im} \partial^2_x \psi_{k-1}(T_k, x^{T_k}_{k-1})$ is positive definite by (P_{k-1}.c), it follows that the same property holds true for $\operatorname{Im} M_k$ and consequently for $\operatorname{Im} \partial^2_x \psi_k(T_k, x^{T_k}_{k-1})$. Hence, the matrix $\partial^2_x \psi_k(t, x^t_k)$ solution of a Riccati equation with its given value at $t = T_k$ satisfies

$$\operatorname{Im} \partial^2_x \psi_k(t, x^t_k) \text{ is positive definite.} \tag{P_k.c}$$

Higher order derivatives of the reflected phase on the associated ray are determined recursively. For $3 \leq r \leq R$, $\partial^r_x \psi_k(t, x^t_k)$ satisfies linear ODEs with a given value at $t = T_k$.

The second step is to prescribe that $d^{k-1}_{-m_B+j}+d^{k}_{-m_B+j}$ vanish up to the order $R-2j-2$ at $(T_k, x^{T_k}_{k-1})$. These requirements provide with the derivatives of a^k_j up to the order $R-2j-2$ at $(T_k, x^{T_k}_{k-1})$. Hence, for $0 \le r \le R-2j-2$, $\partial^r_x a^k_j(t, x^t_k)$ satisfy linear systems of ODEs with initial conditions given at $t = T_k$.

It follows from this construction that the choice of the (truncated up to fixed orders) Taylor series of the phase and the amplitudes of the incident beam on the starting point of the ray determines recursively the (truncated up to fixed orders) Taylor series of successively reflected beams' phases and amplitudes.

Finally, the amplitudes a^k_j are multiplied by a cutoff equal to 1 near x^t_k. The reflected phases and amplitudes have the same forms as the incident ones

$$\psi_k(t,x) = \sum_{|\alpha|\le R} \frac{1}{\alpha!}(x - x^t_k)^\alpha \partial^\alpha_x \psi_k(t, x^t_k),$$

and

$$a^k_j(t,x) = \chi_d(x - x^t_k) \sum_{|\alpha|\le R-2j-2} \frac{1}{\alpha!}(x - x^t_k)^\alpha \partial^\alpha_x a^k_j(t, x^t_k),\ j = 1, \dots, N.$$

2.2.2 Construction of beams associated to p_-

For the symbol p_-, the same construction applies for the associated incident and reflected beams.

An incident beam for p_- is a beam concentrated on the ray (t, x_0^{-t}), so it is simply $w^0_\varepsilon(-t, x)$. In fact, denoting $Pw^0_\varepsilon = \sum\limits_{j=0}^{N+2} \varepsilon^{j-2} c^0_j e^{i\psi_0/\varepsilon}$, one can notice that $P[w^0_\varepsilon(-t,x)] = [Pw^0_\varepsilon](-t,x)$, and the amplitudes $c^0_j(-t,x)$ vanish on $x = x_0^{-t}$ up to the required orders.

Reflected beams for p_- are obtained by reflecting φ^t_0 backwards. For $(y,\eta) \in \overset{o}{T^*}\Omega$, let $T_{-1}(y,\eta) < 0$ be the instant s.t. $x_0^{T_{-1}(y,\eta)}(y,\eta)$ strikes the boundary $\partial\Omega$. Denote by φ^t_{-1} the Hamiltonian flow for h_+ determined by the condition (see fig.1)

$$\varphi^{T_{-1}}_{-1} = \mathcal{R}\, o \varphi^{T_{-1}}_0.$$

For $k > 1$, one can define recursively the instants of reflections T_{-k} and the Hamiltonians flows φ^t_{-k} for h_+ as follows:

$T_{-k}(y,\eta)$ is the instant s.t. $x^{T_{-k}(y,\eta)}_{-k+1}(y,\eta) \in \partial\Omega$ and $T_{-k}(y,\eta) < T_{-k+1}(y,\eta)$,
$\varphi^{T_{-k}}_{-k} = \mathcal{R}\, o \varphi^{T_{-k}}_{-k+1}$.

Assumption B3 implies that $T_k(y,\eta) \to -\infty$ when k goes to $-\infty$, and thus insures a finite number $q_-(y,\eta)$ of reflections in $[-T, 0]$.

Then we build Gaussian beams w^{-k}_ε for p_- after $1 \le k \le q_-$ backwards reflections, by imposing $\|B(w^0_\varepsilon + \cdots + w^{-q_-}_\varepsilon)\|_{H^s([-T,0]\times\partial\Omega)} = O(\varepsilon^{m'})$ for some $m' > 0$ and $s \ge 0$. We write these beams as

$$w^{-k}_\varepsilon = e^{i\psi_{-k}/\varepsilon}(a^{-k}_0 + \cdots + \varepsilon^N a^{-k}_N).$$

In particular, for $1 \leq k \leq q_-$, the phase ψ_{-k} satisfies the following properties

$$\partial_t \psi_{-k}(t, x^t_{-k}) = -h_+(x^t_{-k}, \xi^t_{-k}),\ \partial_x \psi_{-k}(t, x^t_{-k}) = \xi^t_{-k}, \tag{P_{-k}.a}$$
$$\psi_{-k}(t, x^t_{-k}) \text{ is real}, \tag{P_{-k}.b}$$
$$\operatorname{Im} \partial_x^2 \psi_{-k}(t, x^t_{-k}) \text{ is positive definite}. \tag{P_{-k}.c}$$

Noting that (t, x^t_{-k}), $k = 1, \dots, q_-$, are successively reflected rays for p_-, the reflected beam of p_- after k reflections is simply $w_\varepsilon^{-k}(-t, x)$.

2.2.3 Error estimates for individual Gaussian beams

We fix $(y, \eta) \in \overset{o}{T^*\Omega}$ and choose d sufficiently small s.t. for $k = 0, \dots, q_\pm$, $t \in [0, T]$ and $|x - x^{\pm t}_{\pm k}| \leq d$,

$$\operatorname{Im} \psi_{\pm k}(\pm t, x) \geq \operatorname{cons}(x - x^{\pm t}_{\pm k})^2. \tag{21}$$

One can see that this choice is always possible by the properties (P_k.a)-(P_k.b)-(P_k.c) of each phase ψ_k, $-q_- \leq k \leq q_+$.

For $t \in [0, T]$ and $x \in \mathbb{R}^n$, let

$$\underline{w}_\varepsilon^+(t, x) = \sum_{k=0}^{q_+} w_\varepsilon^k(t, x) \text{ and } \underline{w}_\varepsilon^-(t, x) = \sum_{k=0}^{q_-} w_\varepsilon^{-k}(-t, x). \tag{22}$$

Then we have the following estimates on these constructed beams

Lemma 2.2. *1.* $\|\varepsilon^{-\frac{n}{4}+1} \underline{w}_\varepsilon^\pm(t, .)\|_{H^1(\Omega)} \lesssim 1$ *and* $\|\varepsilon^{-\frac{n}{4}+1} \partial_t \underline{w}_\varepsilon^\pm(t, .)\|_{L^2(\Omega)} \lesssim 1$ *uniformly w.r.t.* $t \in [0, T]$,

2. $\|P\left(\varepsilon^{-\frac{n}{4}+1} \underline{w}_\varepsilon^\pm\right)(t, .)\|_{L^2(\Omega)} \lesssim \varepsilon^{\frac{R-1}{2}}$ *uniformly w.r.t.* $t \in [0, T]$,

3. $\|B\left(\varepsilon^{-\frac{n}{4}+1} \underline{w}_\varepsilon^\pm\right)\|_{H^s([0,T] \times \partial\Omega)} \lesssim \varepsilon^{-m_B - s + \frac{R+1}{2}}$, $s \geq 0$.

The proof of this Lemma and other results rely on this standard estimate for $p \in \mathbb{N}$

$$|x|^p e^{-x^2/\varepsilon} dx \lesssim \varepsilon^{\frac{p}{2}} e^{-x^2/(2\varepsilon)},\ \forall x \in \mathbb{R}^n. \tag{23}$$

For more details, we refer the interested reader to [84] or [67].

2.3 Gaussian beams summation

The constructed functions $\varepsilon^{-\frac{n}{4}+1} \underline{w}_\varepsilon^\pm$ are approximate solutions for the IBVP of the wave equation with initial data

$$\varepsilon^{-\frac{n}{4}+1} \underline{w}_\varepsilon^\pm|_{t=0} = \varepsilon^{-\frac{n}{4}+1} \sum_{j=0}^{N} \varepsilon^j a_j^0|_{t=0} e^{i\psi_0|_{t=0}/\varepsilon} + \varepsilon^{-\frac{n}{4}+1} \sum_{k=1}^{q_\pm} w_\varepsilon^{\pm k}|_{t=0},$$

and

$$\partial_t \left(\varepsilon^{-\frac{n}{4}+1} \underline{w}_\varepsilon^\pm\right)|_{t=0} = \pm \varepsilon^{-\frac{n}{4}} \sum_{j=0}^{N+1} \varepsilon^j f_j^0 e^{i\psi_0|_{t=0}/\varepsilon} \pm \varepsilon^{-\frac{n}{4}+1} \sum_{k=1}^{q_\pm} \partial_t w_\varepsilon^{\pm k}|_{t=0},$$

where the f_j^0 are related to the phase and amplitudes of w_ε^0. One can show that the assumptions B1-B2 imply that $x_k^0 \notin \bar{\Omega}$ for $k \neq 0$. The exponential decrease of the phases away from their associated rays leads to

$$\|w_\varepsilon^k|_{t=0}\|_{H^1(\Omega)} \lesssim \varepsilon^\infty \text{ and } \|\partial_t w_\varepsilon^k|_{t=0}\|_{L^2(\Omega)} \lesssim \varepsilon^\infty,\ k \neq 0.$$

Modulo infinitely small remainders, the initial conditions of $\varepsilon^{-\frac{n}{4}+1}\underline{w}_\varepsilon^\pm$ are then

$$\left(\varepsilon^{-\frac{n}{4}+1}\sum_{j=0}^{N}\varepsilon^j a_j^0|_{t=0}e^{i\psi_0|_{t=0}/\varepsilon}, \pm\varepsilon^{-\frac{n}{4}}\sum_{j=0}^{N+1}\varepsilon^j f_j^0 e^{i\psi_0|_{t=0}/\varepsilon}\right).$$

We wish to consider the IBVP (1) with general initial conditions $(u_\varepsilon^I, v_\varepsilon^I)$ in $H^1(\Omega)\times L^2(\Omega)$. Note that $\psi_0|_{t=0}$ has properties similar to ϕ_0, where $c_n\varepsilon^{-\frac{3n}{4}}e^{i\phi_0(x,y,\eta)/\varepsilon}$ denotes the kernel of T_ε^*, see formula (4) in the introduction. The first step is to build, for a fixed point $(y,\eta)\in \overset{o}{T^*}\Omega$, asymptotic solutions with initial conditions close to $(\varepsilon^{-\frac{n}{4}+1}e^{i\phi_0(.,y,\eta)/\varepsilon}, 0)$ and $(0, \varepsilon^{-\frac{n}{4}}e^{i\phi_0(.,y,\eta)/\varepsilon})$ in $H^1(\Omega)\times L^2(\Omega)$. Then one expects to fulfill more general initial data $(u_\varepsilon^I, v_\varepsilon^I)$ by decomposing u_ε^I on the family $(\varepsilon^{-\frac{n}{4}+1}e^{i\phi_0/\varepsilon})_{(y,\eta)\in \overset{o}{T^*}\Omega}$ and v_ε^I on the family $(\varepsilon^{-\frac{n}{4}}e^{i\phi_0/\varepsilon})_{(y,\eta)\in \overset{o}{T^*}\Omega}$, indexed by (y,η).

Let us recover the notation of the beams referring to the starting points of the incident flow. We fix $(y,\eta)\in \overset{o}{T^*}\Omega$ and consider the incident beam $w_\varepsilon^0(t,x,y,\eta)$ associated to the ray $(t, x_0^t(y,\eta))$ and the reflected beams $w_\varepsilon^{\pm k}(t,x,y,\eta)$, $k = 1,\ldots,q_\pm$. Taylor formulae (16) yields at $t = 0$

$$\psi_0(0,x,y,\eta) = \sum_{|\alpha|\leq R}\frac{1}{\alpha!}(x-y)^\alpha\partial_x^\alpha\psi_0(0,y,y,\eta).$$

If one chooses the following initial spatial derivatives on the ray for the incident beam's phase

$$\psi_0(0,y,y,\eta) = 0, \partial_x^2\psi_0(0,y,y,\eta) = iId \text{ and } \partial_x^\alpha\psi_0(0,y,y,\eta) = 0,\ 3\leq|\alpha|\leq R,$$

then $(P_0.\text{a})$ implies

$$\psi_0(0,x,y,\eta) = \eta\cdot(x-y) + i(x-y)^2/2 = \phi_0(x,y,\eta). \tag{24}$$

We assume henceforth that the incident beam's phase satisfies (24). Consider an approximate solution

$$\frac{1}{2}\varepsilon^{-\frac{n}{4}+1}(\underline{w}_\varepsilon^+ + \underline{w}_\varepsilon^-).$$

Its initial data are

$$\left(\varepsilon^{-\frac{n}{4}+1}\sum_{j=0}^{N}\varepsilon^j a_j^0|_{t=0}e^{i\phi_0/\varepsilon}, 0\right),$$

with a redidue of order ε^∞ in $H^1(\Omega)\times L^2(\Omega)$. To get the form $(\varepsilon^{-\frac{n}{4}+1}e^{i\phi_0/\varepsilon}, 0)$, one has to make a suitable choice for the amplitudes. The expansion (17) at $t = 0$ yields

$$a_j^0(0,x,y,\eta) = \chi_d(x-y)\sum_{|\alpha|\leq R-2j-2}\frac{1}{\alpha!}(x-y)^\alpha\partial_x^\alpha a_j^0(0,y,y,\eta),\ j = 0,\ldots,N,$$

and one has full choice for the initial spatial derivatives of a_j^0 on the ray up to the order $R-2j-2$. Under the assumptions

$$a_0^0(0,y,y,\eta)=1,\ \partial_x^\alpha a_0^0(0,y,y,\eta)=0 \text{ for } 1\leq|\alpha|\leq R-2,$$
$$\partial_x^\alpha a_j^0(0,y,y,\eta)=0 \text{ for } |\alpha|\leq R-2j-2,\ 1\leq j\leq N,$$

one gets

$$\sum_{j=0}^{N}\varepsilon^j a_j^0(0,x,y,\eta)=\chi_d(x-y). \tag{25}$$

Taking advantage of the exponential decrease of $e^{i\phi_0(x,y,\eta)/\varepsilon}$ for $|x-y|\geq d/2$, one deduces that

$$\|\varepsilon^{-\frac{n}{4}+1}\sum_{j=0}^{N}\varepsilon^j a_j^0(0,.,y,\eta)e^{i\phi_0(.,y,\eta)/\varepsilon}-\varepsilon^{-\frac{n}{4}+1}e^{i\phi_0(.,y,\eta)/\varepsilon}\|_{H^1(\Omega)}\lesssim\varepsilon^\infty.$$

We keep the notations a_j^0 and w_ε^0 to denote the amplitudes satisfying (25) and the associated incident beam. For $1\leq k\leq q_\pm$, we denote by $w_\varepsilon^{\pm k}$ the corresponding reflected beams and by $\underline{w}_\varepsilon^\pm$ the sum of the incident and reflected beams for $p_\pm$.

Next, we shift to the initial condition on the time derivative, for which we construct a new incident beam $w_\varepsilon^{0\prime}$ with amplitudes $a_j^{0\prime}$. Indeed, an approximate solution

$$\frac{1}{2}\varepsilon^{-\frac{n}{4}+1}(\underline{w}_\varepsilon^{+\prime}-\underline{w}_\varepsilon^{-\prime}),$$

has initial data

$$\left(0,\varepsilon^{-\frac{n}{4}}\sum_{j=0}^{N+1}\varepsilon^j\left(i\partial_t\psi_0 a_j^{0\prime}+\partial_t a_{j-1}^{0\ \prime}\right)|_{t=0}e^{i\phi_0/\varepsilon}\right),$$

modulo a remainder of order ε^∞ in $H^1(\Omega)\times L^2(\Omega)$, with $a_{-1}^{0\ \prime}=a_{N+1}^{0\ \prime}=0$. In order to approach the form $(0,\varepsilon^{-\frac{n}{4}}e^{i\phi_0/\varepsilon})$, we derive new initial Taylor series for the incident beam's amplitudes. As $\partial_t\psi_0(0,y,y,\eta)=-c(y)|\eta|$, we impose

$$a_0^{0\prime}(0,y,y,\eta)=i\left(c(y)|\eta|\right)^{-1},\ \partial_x^\alpha\left(\partial_t\psi_0 a_0^{0\prime}\right)(0,y,y,\eta)=0 \text{ for } 1\leq|\alpha|\leq R-2,$$
$$\partial_x^\alpha\left(i\partial_t\psi_0 a_j^{0\prime}+\partial_t a_{j-1}^{0\ \prime}\right)(0,y,y,\eta)=0 \text{ for } |\alpha|\leq R-2j-2,\ 1\leq j\leq N.$$

One gets

$$\begin{aligned}\sum_{j=0}^{N+1}\varepsilon^j\left(i\partial_t\psi_0 a_j^{0\prime}+\partial_t a_{j-1}^{0\ \prime}\right)(0,x,y,\eta)&=1+\sum_{j=0}^{N}\varepsilon^j\sum_{|\alpha|=R-2j-1}(x-y)^\alpha z_\alpha(x,y,\eta)\\&\quad+\varepsilon^{N+1}\partial_t a_N^{0\ \prime}(0,x,y,\eta),\end{aligned} \tag{26}$$

where z_α are smooth remainders that vanish for $|x-y|\geq d$. Making use of (14) and (23), one can show that

$$\|\varepsilon^{-\frac{n}{4}}\sum_{j=0}^{N+1}\varepsilon^j\left(i\partial_t\psi_0 a_j^{0\prime}+\partial_t a_{j-1}^{0\ \prime}\right)(0,.,y,\eta)e^{i\phi_0(.,y,\eta)/\varepsilon}-\varepsilon^{-\frac{n}{4}}e^{i\phi_0(.,y,\eta)/\varepsilon}\|_{L^2(\Omega)}\lesssim\varepsilon^{\frac{R-1}{2}}.$$

Let $w_\varepsilon^{\pm k\prime}$, $1\leq k\leq q_\pm$, be the reflected beams associated to $w_\varepsilon^{0\prime}$ and denote by $\underline{w}_\varepsilon^{\pm\prime}$ the sum of the so obtained incident and reflected beams for $p_\pm$. Hence, the approximate solutions

$$\frac{1}{2}\varepsilon^{-\frac{n}{4}+1}(\underline{w}_\varepsilon^{+}+\underline{w}_\varepsilon^{-})(t,x,y,\eta) \text{ and } \frac{1}{2}\varepsilon^{-\frac{n}{4}+1}(\underline{w}_\varepsilon^{+\prime}-\underline{w}_\varepsilon^{-\prime})(t,x,y,\eta),$$

have the required initial data

$$\left(\varepsilon^{-\frac{n}{4}+1}e^{i\phi_0(x,y,\eta)/\varepsilon},0\right) \text{ and } \left(0,\varepsilon^{-\frac{n}{4}}e^{i\phi_0(x,y,\eta)/\varepsilon}\right),$$

modulo remainders of respective orders ε^∞ and $\varepsilon^{\frac{R-1}{2}}$ in $H^1(\Omega)\times L^2(\Omega)$.

To fulfill general initial conditions $(u_\varepsilon^I, v_\varepsilon^I)$, the previous computations together with the identity $T_\varepsilon^* T_\varepsilon = Id$, suggest that we look for an approximate solution such as

$$\frac{c_n}{2}\varepsilon^{-\frac{3n}{4}}\int_{\overset{o}{T^*\Omega}} T_\varepsilon u_\varepsilon^I(y,\eta)\left(\underline{w}_\varepsilon^+(t,x,y,\eta)+\underline{w}_\varepsilon^-(t,x,y,\eta)\right)dyd\eta \\ +\frac{c_n}{2}\varepsilon^{-\frac{3n}{4}}\int_{\overset{o}{T^*\Omega}}\varepsilon T_\varepsilon v_\varepsilon^I\left(\underline{w}_\varepsilon^{+\prime}(t,x,y,\eta)-\underline{w}_\varepsilon^{-\prime}(t,x,y,\eta)\right)dyd\eta.$$

Let us notice that it is not clear that the previous integral is well defined.

Firstly, the construction of $\underline{w}_\varepsilon^{\pm(\prime)}(t,x,y,\eta)$ breaks down when y approaches the boundary $\partial\Omega$ because the numbers of reflections in $[0,\pm T]$ become infinitely large. Next we need to tackle the problem of integration for large η.

One way to overcome these two problems is to require that the initial FBI transforms are compactly supported modulo small remainders. This requirement is in the spirit of considering only compactly supported symbols in the study of the FIOs of [58]. Nevertheless, this restriction was removed recently by Rousse and Swart in [89]. In particular, a general boundedness result of FIOs with complex phases for subquadratic Hamiltonians was established therein. The proof is rather subtle and relies in particular on Cotlar-Stein estimate. The same arguments can be used for the constant coefficient wave equation but seem not to work for the general wave equation. In fact, in this case, the second derivatives of the Hamiltonian are not bounded and thus the proof of [89] needs to be adapted.

A last problem related to the wave equation is the integration for small η.

In view of all these difficulties, this explains why we have made in the introduction the assumptions A2 and A3 on the initial data, which we recall

$$u_\varepsilon^I \text{ and } v_\varepsilon^I \text{ are supported in a fixed compact } K\subset\Omega, \\ \|T_\varepsilon u_\varepsilon^I\|_{L^2(\mathbb{R}^n\times R_\eta^c)} = O(\varepsilon^\infty) \text{ and } \|T_\varepsilon v_\varepsilon^I\|_{L^2(\mathbb{R}^n\times R_\eta^c)} = O(\varepsilon^\infty),$$

where $R_\eta = \{\eta\in\mathbb{R}^n, r_0\le|\eta|\le r_\infty\}$, $0<r_0<r_\infty$. These assumptions are satisfied for instance by a large class of WKB functions $ae^{i\Phi/\varepsilon}$, $a\in\mathcal{C}_0^\infty(\Omega)$. Indeed the non-stationary phase lemma implies that the FBI transform of such a function is of order $O(\varepsilon^\infty)$ outside the compact set

$$\mathcal{A}\times\mathcal{B} = \{y\in\mathbb{R}^n, \operatorname{dist}(y,\operatorname{supp}a)\le c\}\times\{\eta\in\mathbb{R}^n, \operatorname{dist}(\eta,\partial_x\Phi(\mathcal{A}))\le c\},\ c>0,$$

see Lemmas 4.2 and 4.3 of [88]. Thus $ae^{i\Phi/\varepsilon}$ satisfies assumption A3, provided that $\partial_x\Phi$ does not vanish on supp$a$.

Remark 2.3. *Another strategy can be used to match initial conditions of WKB form in a Gaussian beams summation [63, 96]. It consists of integrating the beams associated to rays that start from $y\in$ supp$a$ with the direction $\eta=\partial_x\Phi(y)$. The accuracy of such obtained solutions faces a damage caused by caustics, namely an extra factor*

$\varepsilon^{\frac{1-n}{4}}$ *appears in the error estimate. This loss originates from the restriction to rays* $x^{\pm t}_{\pm k}(y, \partial_x \Phi(y))$ *(*$k = 0, \dots, N_\pm$*), which technically leads to considering the deformation matrices* $\partial_y[x^{\pm t}_{\pm k}(y, \partial_x \Phi(y))]$ *singular at caustics (see Lemma 5.1 of [63]). The summation over rays starting with general directions* η *independent of* y *uses the symplectic maps* $\varphi^{\pm t}_{\pm k}$ *and thus provides a phase space description of the solution that ŤunfoldsŤ the caustics.*

Let ρ be a cut-off of $\mathcal{C}^\infty_0(\mathbb{R}^n, [0,1])$ supported in a compact $K_y \subset \Omega$ and satisfying

$$\rho(y) = 1 \text{ if } \operatorname{dist}(y, K) < \Delta \text{ for a small } \Delta > 0, \tag{27}$$

and ϕ a cut-off of $\mathcal{C}^\infty_0(\mathbb{R}^n, [0,1])$ supported in a compact $K_\eta \subset \mathbb{R}^n \backslash \{0\}$ s.t. $\phi = 1$ on R_η.

One can establish that the assumptions A2 and A3 imply

$$\| (1 - \rho(y)\phi(\eta))\, T_\varepsilon u^I_\varepsilon \|_{L^2_{y,\eta}} \lesssim \varepsilon^\infty \text{ and } \| (1 - \rho(y)\phi(\eta))\, T_\varepsilon v^I_\varepsilon \|_{L^2_{y,\eta}} \lesssim \varepsilon^\infty.$$

In fact, viewing the FBI transform as the Fourier Transform of some auxiliary function yields by Parseval equality the following result

Lemma 2.4. *Let* a *be a positive real and* G *a measurable subset of* $\mathbb{R}^n$ *s.t.* $\operatorname{dist}(G, K) \geq a$*. If* $u \in L^2(\mathbb{R}^n_w)$ *is supported in* K *then*

$$\|\mathbf{1}_G(y) T_\varepsilon u\|_{L^2_{y,\eta}} = c_n \varepsilon^{-\frac{n}{4}} \|\mathbf{1}_G(y) u(w) e^{-(w-y)^2/(2\varepsilon)}\|_{L^2_{w,y}} \lesssim e^{-a^2/(4\varepsilon)} \|u\|_{L^2_w}.$$

On the other hand, if (y, η) varies in $K_y \times K_\eta$, then $q_+(y, \eta)$ is uniformly bounded. In fact, for $j \geq 1$, the T_j are positive, depend continuously on (y, η) and the property (18) insures that $T_j \nearrow +\infty$ when $j \to +\infty$. Thus they uniformly go to $+\infty$ on the compact $K_y \times K_\eta$, by Dini's theorem on the sequence $(1/T_j)_{j \geq 1}$. As $T_{q_+} \leq T$, it follows that $\sup\limits_{K_y \times K_\eta} q_+ < +\infty$. The same result holds true for q_-. Denote $N_\pm = \sup\limits_{K_y \times K_\eta} q_\pm$. The construction of the reflected beams in section 2.2 may be continued up to $N_\pm$ reflections.

The final result of the discussion above is an approximate solution proposed as

$$\begin{aligned} u^R_\varepsilon(t,x) = \frac{1}{2} \varepsilon^{-\frac{3n}{4}} c_n \int_{\mathbb{R}^{2n}} \rho(y)\phi(\eta) \Big[& \varepsilon T_\varepsilon v^I_\varepsilon(y,\eta) (\sum_{k=0}^{N_+} w^{k'}_\varepsilon(t,x,y,\eta) - \sum_{k=0}^{N_-} w^{-k'}_\varepsilon(-t,x,y,\eta)) \\ & + T_\varepsilon u^I_\varepsilon(y,\eta) (\sum_{k=0}^{N_+} w^k_\varepsilon(t,x,y,\eta) + \sum_{k=0}^{N_-} w^{-k}_\varepsilon(-t,x,y,\eta)) \Big] dy d\eta. \end{aligned} \tag{28}$$

This approximate solution is indexed by R, the order of vanishing of the eikonal equation (10) on the ray. The incident beams' phase fulfills the initial conditions (24) and their amplitudes satisfy respectively (25) for w^0_ε and (26) for $w^{0'}_\varepsilon$ for every $(y, \eta) \in \operatorname{supp}\rho \otimes \phi$,. The size $d \in]0,1]$ of the support of the cut-offs multiplying the amplitudes no longer depends on (y, η) and would be chosen sufficiently small to satisfy various constraints we set out in the following section.

In the sequel, we prove that this family of functions (u^R_ε) indeed allows to approach the exact solution of the IBVP problem (1) to any arbitrary power of ε by choosing the order R. The difference between the asymptotic solutions and the exact one is investigated in $\mathcal{C}([0,T], H^1(\Omega)) \times \mathcal{C}^1([0,T], L^2(\Omega))$ by means of error estimates in the interior equation, the boundary condition and the initial conditions. The only assumptions needed on the initial data are A1, A2 and A3.

3 Justification of the asymptotics

We aim to estimate $\|u_\varepsilon^R(t,.)-u_\varepsilon(t,.)\|_{H^1(\Omega)}$ and $\|\partial_t u_\varepsilon^R(t,.)-\partial_t u_\varepsilon(t,.)\|_{L^2(\Omega)}$ for $t\in[0,T]$.

It follows from standard results [21] that the IBVP for the wave equation is well-posed, and furthermore one has the energy estimate (as a consequence of [59] p.185 for the Dirichlet problem and of [9] p.224 for the Neumann problem)

$$\begin{aligned}\underset{t\in[0,T]}{Sup}\|u_\varepsilon^R(t,.)-u_\varepsilon(t,.)\|_{H^1(\Omega)}+\underset{t\in[0,T]}{Sup}\|\partial_t u_\varepsilon^R(t,.)-\partial_t u_\varepsilon(t,.)\|_{L^2(\Omega)}\lesssim\\ \underset{t\in[0,T]}{Sup}\|Pu_\varepsilon^R\|_{L^2(\Omega)}+\|Bu_\varepsilon^R\|_{H^s([0,T]\times\partial\Omega)}\\ +\|u_\varepsilon^R(0,.)-u_\varepsilon^I\|_{H^1(\Omega)}+\|\partial_t u_\varepsilon^R(0,.)-v_\varepsilon^I\|_{L^2(\Omega)},\end{aligned}\tag{29}$$

where $s=1$ for Dirichlet and $s=\frac{1}{2}$ for Neumann.

The asymptotics will be proven by estimating each term of the r.h.s. of this energy estimate.

Since the error estimates in the interior and near the boundary use similar computations, a unified framework will be used by considering the more general problem of estimates linked with a suitable family of approximation operators O^α in section 3.1. Then in section 3.2 we use these estimates for the interior term $\|Pu_\varepsilon^R\|_{L^2(\Omega)}$ in 3.2.1, the boundary term $\|Bu_\varepsilon^R\|_{H^s([0,T]\times\partial\Omega)}$ in 3.2.2 and the initial conditions errors in 3.2.3. All these estimates are gathered in section 3.3 to prove Theorem 1.1.

3.1 Approximation operators

Let $K_{z,\theta}$ be a compact of $\mathbb{R}^{2n}$ and

$$E_r=\{(x,z,\theta)\in\mathbb{R}^n\times K_{z,\theta},\ |x-z|\leq r\},\ r>0.$$

Consider a complex phase function Φ smooth on an open set containing E_{r_0} for some $r_0\in]0,1]$. We assume, for $(z,\theta)\in K_{z,\theta}$,that

$$\begin{aligned}&\partial_x\Phi(z,z,\theta)=\theta,\\ &\Phi(z,z,\theta)\text{ is real,}\\ &\partial_x^2\Phi(z,z,\theta)\text{ has a positive definite imaginary part.}\end{aligned}\tag{Q1}$$

Taylor expansion of Φ together with assumptions (Q1) imply the existence of some constant $r[\Phi]\in]0,r_0]$ s.t. for $(x,z,\theta)\in E_{r[\phi]}$

$$\operatorname{Im}\Phi(x,z,\theta)\geq\operatorname{cons}(x-z)^2.$$

Consider a sequence $l_\varepsilon\in\mathcal{C}^\infty(\mathbb{R}^n_x\times\mathbb{R}^{2n}_{z,\theta},\mathbb{C})$. We assume that

$$\begin{aligned}&l_\varepsilon(x,z,\theta)=0\text{ if }(x,z,\theta)\notin E_{r[\Phi]},\\ &l_\varepsilon\text{ is uniformly bounded in }L^\infty(\mathbb{R}^{3n}).\end{aligned}\tag{Q2}$$

For a given multi-index α, let the operators $O^0\left(l_\varepsilon,\Phi/\varepsilon\right)$ and $O^\alpha\left(l_\varepsilon,\Phi/\varepsilon\right)$ be given by

$$\left[O^0\left(l_\varepsilon,\Phi/\varepsilon\right)h\right](x)=\int_{\mathbb{R}^{2n}}h(z,\theta)l_\varepsilon(x,z,\theta)e^{i\Phi(x,z,\theta)/\varepsilon}dzd\theta,\ h\in L^2(\mathbb{R}^{2n}),$$

and

$$[O^\alpha(l_\varepsilon,\Phi/\varepsilon)h](x)=\int_{\mathbb{R}^{2n}}h(z,\theta)l_\varepsilon(x,z,\theta)(x-z)^\alpha e^{i\Phi(x,z,\theta)/\varepsilon}dzd\theta,\ h\in L^2(\mathbb{R}^{2n}),$$

with $x\in\mathbb{R}^n$.

Let us show that these are operators from $L^2(\mathbb{R}^{2n})$ to $L^2(\mathbb{R}^n)$. For $x\in\mathbb{R}^n$ we have

$$\int|l_\varepsilon e^{i\Phi/\varepsilon}|dzd\theta\lesssim\int_{(z,\theta)\in K_{z,\theta}}e^{-\mathrm{cons}(x-z)^2/\varepsilon}dzd\theta,$$

and thus

$$\int|l_\varepsilon e^{i\Phi/\varepsilon}|dzd\theta\lesssim\varepsilon^{\frac{n}{2}}.$$

Similarly, for $(z,\theta)\in K_{z,\theta}$

$$\int|l_\varepsilon e^{i\Phi/\varepsilon}|dx\lesssim\varepsilon^{\frac{n}{2}}.$$

It is then immediate by Schur's lemma, that

$$\|O^0(l_\varepsilon,\Phi/\varepsilon)\|_{L^2(\mathbb{R}^{2n})\to L^2(\mathbb{R}^n)}\lesssim\varepsilon^{\frac{n}{2}}.$$

Similar arguments lead to the estimate

$$\|O^\alpha(l_\varepsilon,\Phi/\varepsilon)\|_{L^2(\mathbb{R}^{2n})\to L^2(\mathbb{R}^n)}\lesssim\varepsilon^{\frac{n}{2}+\frac{|\alpha|}{2}}.$$

However, the use of the module inside the previous integrals makes one lose the highly oscillatory character of $e^{i\Phi/\varepsilon}$, that is the contribution of $e^{i\theta\cdot(x-z)/\varepsilon}$. In fact, a better estimate on the norms of these operators is available if a precise control on l_ε is assumed. This is stated in the following lemma

Lemma 3.1. *Assume that $\varepsilon^{\frac{k}{2}}\partial^k_{x_b}l_\varepsilon$ ($b=1,\dots,n$) is uniformly bounded in $L^\infty(\mathbb{R}^{3n})$, at any order $k\in\mathbb{N}$. Then, one has*

1. $\|O^0(l_\varepsilon,\Phi/\varepsilon)\|_{L^2(\mathbb{R}^{2n})\to L^2(\mathbb{R}^n)}\lesssim\varepsilon^{\frac{3n}{4}}$,
2. $\|O^\alpha(l_\varepsilon,\Phi/\varepsilon)\|_{L^2(\mathbb{R}^{2n})\to L^2(\mathbb{R}^n)}\lesssim\varepsilon^{\frac{3n}{4}+\frac{|\alpha|}{2}}$.

Proof. 1. Let $h\in L^2(\mathbb{R}^{2n})$. We shall use the notations $f(x)$ for $f(x,z,\theta)$ and $f'(x)$ for $f(x,z',\theta')$. First of all, we explicit the L^2 norm of $O^0(l_\varepsilon,\Phi/\varepsilon)h$ as

$$\begin{aligned}\|O^0(l_\varepsilon,\Phi/\varepsilon)h\|^2_{L^2}=&\int_{\mathbb{R}^{4n}}h\overline{h}'e^{i\Phi(z)/\varepsilon-i\Phi'(z')/\varepsilon}e^{i(\theta'.z'-\theta.z)/\varepsilon}\\&\Big[\int_{\mathbb{R}^n}l_\varepsilon(x)\overline{l}'_\varepsilon(x)e^{i(\theta-\theta').x/\varepsilon}e^{i\Theta(x,z,\theta,z',\theta')/\varepsilon}dx\Big]dzdz'd\theta d\theta',\end{aligned}\tag{30}$$

where

$$\begin{aligned}\Theta(x,z,\theta,z',\theta')=&\sum_{|\alpha|=2}(x-z)^\alpha\int_0^1\tfrac{2}{\alpha!}(1-s)\partial^\alpha_x\Phi(z+s(x-z),z,\theta)ds\\&-\sum_{|\alpha|=2}(x-z')^\alpha\int_0^1\tfrac{2}{\alpha!}(1-s)\partial^\alpha_x\overline{\Phi}(z'+s(x-z'),z',\theta')ds.\end{aligned}$$

Let I_ε denote the integral inside the brackets, that we begin to estimate. For $1 \le b \le n$ and $K \in \mathbb{N}$, successive integrations by parts give

$$I_\varepsilon(z,z',\theta,\theta')i^K\varepsilon^{-K}(\theta_b-\theta'_b)^K = (-1)^K \sum_{N+N'=K} \begin{pmatrix} K \\ N \end{pmatrix} \int_{\mathbb{R}^n} e^{i(\theta-\theta').x/\varepsilon}\partial_{x_b}^N[e^{i\Theta/\varepsilon}]\partial_{x_b}^{N'}[l_\varepsilon \bar{l}'_\varepsilon]dx,$$

where $\begin{pmatrix} K \\ N \end{pmatrix}$ denotes the standard binomial coefficient. To estimate $\partial_{x_b}^N[e^{i\Theta/\varepsilon}]$, $N \in \mathbb{N}$, we use the following result, of which proof is postponed to the end of this section

Lemma 3.2. *Let $p \in \mathbb{N}^*$ and consider a complex phase function F_p of the form*

$$F_p(x,z) = \sum_{|\alpha|=p} (x-z)^\alpha f_\alpha(x,z),$$

with f_α smooth on some open set of $\mathbb{R}^{2n}$ containing a subset S and $\partial_x^k f_\alpha$ bounded on S for any $k \ge 0$.
Then for $(x,z) \in S$, $|x-z| \le 1$, small ε, $N \in \mathbb{N}$ and $b = 1,\dots,n$, one has

$$|\partial_{x_b}^N[e^{iF_p/\varepsilon}]| \le \max_{\substack{|\alpha|=p \\ 0\le s\le N \\ 1\le k\le N}} \Big(\sup_S|\partial_{x_b}^s f_\alpha|\Big)^k \Big(\sum_{\frac{N}{p}\le k\le N} \varepsilon^{-k}|x-z|^{kp-N} + \sum_{1\le k<\frac{N}{p}} \varepsilon^{-N/p}\Big)|e^{iF_p/\varepsilon}|.$$

We write $\Theta = F_2 - \bar{F}'_2$ with

$$F_2(x,z,\theta) = \sum_{|\alpha|=2} (x-z)^\alpha \int_0^1 \frac{2}{\alpha!}(1-s)\partial_x^\alpha \Phi(z+s(x-z),z,\theta)ds,$$

for $(x,z,\theta) \in E_{r[\Phi]}$. By Leibnitz formula, $\partial_{x_b}^N[e^{i\Theta/\varepsilon}]$ is a sum of terms of the form

$$\partial_{x_b}^{N_1}[e^{iF_2/\varepsilon}]\partial_{x_b}^{N_2}[e^{-i\bar{F}'_2/\varepsilon}],\ 0 \le N_1, N_2 \le N,\ N_1+N_2 = N.$$

Note that $\operatorname{Im} F_2 = \operatorname{Im}\Phi$. Lemma 3.2 yields for $N_1 \in \mathbb{N}$ and $(x,z,\theta) \in E_{r[\Phi]}$

$$|\partial_{x_b}^{N_1}[e^{iF_2/\varepsilon}]| \lesssim \left(\sum_{\frac{N_1}{2}\le k\le N_1} \varepsilon^{-k}|x-z|^{2k-N_1} + \varepsilon^{-N_1/2} \right) e^{-\mathrm{cons}(x-z)^2/\varepsilon}.$$

Hence

$$|\partial_{x_b}^{N_1}[e^{iF_2/\varepsilon}]| \lesssim \varepsilon^{-\frac{N_1}{2}} e^{-\mathrm{cons}(x-z)^2/\varepsilon}.$$

A similar estimate may be obtained for $|\partial_{x_b}^{N_2}[e^{i\bar{F}'_2/\varepsilon}]|$ when $(x,z',\theta') \in E_{r[\Phi]}$. It follows, for $(x,z,\theta),(x,z',\theta') \in E_{r[\Phi]}$, that

$$|\partial_{x_b}^{N_1}[e^{iF_2/\varepsilon}]\partial_{x_b}^{N_2}[e^{-i\bar{F}'_2/\varepsilon}]| \lesssim \varepsilon^{-\frac{N_1+N_2}{2}} e^{-\mathrm{cons}(x-z)^2/\varepsilon}e^{-\mathrm{cons}(x-z')^2/\varepsilon},$$

and thus

$$|\partial_{x_b}^{N}[e^{i\Theta/\varepsilon}]| \lesssim \varepsilon^{-\frac{N}{2}} e^{-\mathrm{cons}(2x-z-z')^2/\varepsilon}e^{-\mathrm{cons}(z-z')^2/\varepsilon},\ N \in \mathbb{N}.$$

Since $\varepsilon^{\frac{N'}{2}}\partial_{x_b}^{N'}[l_\varepsilon \bar{l}'_\varepsilon]$, $N' \in \mathbb{N}$, is uniformly bounded

$$|\partial_{x_b}^{N}[e^{i\Theta/\varepsilon}]\partial_{x_b}^{N'}[l_\varepsilon \bar{l}'_\varepsilon]| \lesssim \varepsilon^{-\frac{N+N'}{2}} e^{-\text{cons}(2x-z-z')^2/\varepsilon} e^{-\text{cons}(z-z')^2/\varepsilon},$$

and we deduce that

$$|I_\varepsilon(z,z',\theta,\theta')\left(\frac{\theta_b-\theta'_b}{\sqrt{\varepsilon}}\right)^K| \lesssim \varepsilon^{\frac{n}{2}} e^{-\text{cons}(z-z')^2/\varepsilon},$$

for $b = 1, \dots, n$ and $K \in \mathbb{N}$.

Choosing $K > n$ and coming back to (30) gives

$$\|O^0\left(l_\varepsilon, \Phi/\varepsilon\right)h\|_{L^2}^2 \lesssim \varepsilon^{\frac{n}{2}} \int_{\mathbb{R}^{4n}} |h||h'|e^{-\text{cons}(z-z')^2/\varepsilon} dz dz' (1+(\theta-\theta')^2/\varepsilon)^{-\frac{K}{2}} d\theta d\theta'.$$

Upon using the change of variables:

$$(z,z') = (u+\sqrt{\varepsilon}v, u-\sqrt{\varepsilon}v) \text{ and } (\theta,\theta') = (\sigma+\sqrt{\varepsilon}\delta, \sigma-\sqrt{\varepsilon}\delta),$$

we have

$$\begin{aligned}\|O^0\left(l_\varepsilon, \Phi/\varepsilon\right)h\|_{L^2}^2 \quad \lesssim \quad & \varepsilon^{\frac{3n}{2}} \int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} |h(u+\sqrt{\varepsilon}v, \sigma+\sqrt{\varepsilon}\delta)||h(u-\sqrt{\varepsilon}v, \sigma-\sqrt{\varepsilon}\delta)| du d\sigma \\ & e^{-\text{cons} v^2}(1+4\delta^2)^{-\frac{K}{2}} dv d\delta,\end{aligned}$$

from which, using Cauchy-Schwartz inequality for the first integral, we get

$$\|O^0\left(l_\varepsilon, \Phi/\varepsilon\right)h\|_{L^2}^2 \lesssim \varepsilon^{\frac{3n}{2}} \|h\|_{L^2}^2.$$

2. Arguments are similar to the previous case. For a multi-index α, we have

$$\begin{aligned}\|O^\alpha\left(l_\varepsilon, \Phi/\varepsilon\right)h\|_{L^2}^2 = \int_{\mathbb{R}^{4n}} h\bar{h}' e^{i\Phi(z)/\varepsilon - i\Phi'(z')/\varepsilon} e^{i(\theta'.z'-\theta.z)/\varepsilon} \\ I_\varepsilon^\alpha(z,z',\theta,\theta') dz dz' d\theta d\theta',\end{aligned}$$

where, for $b = 1, \dots, n$ and $K \in \mathbb{N}$

$$\begin{aligned}I_\varepsilon^\alpha(z,z',\theta,\theta') i^K \varepsilon^{-K}(\theta_b-\theta'_b)^K = (-1)^K \sum_{N+N'=K} \begin{pmatrix} K \\ N \end{pmatrix} \int_{\mathbb{R}^n} e^{i(\theta-\theta').x/\varepsilon} \\ \partial_{x_b}^N[(x-z)^\alpha (x-z')^\alpha e^{i\Theta/\varepsilon}]\partial_{x_b}^{N'}[l_\varepsilon \bar{l}'_\varepsilon] dx.\end{aligned}$$

We note that $\partial_{x_b}^N[(x-z)^\alpha(x-z')^\alpha e^{i\Theta/\varepsilon}]$ is a finite sum of terms of the form

$$(x-z)^{\alpha-ke^b}(x-z')^{\alpha-le^b}\partial_{x_b}^m[e^{i\Theta/\varepsilon}],$$

where $k, l \leq \alpha_b$, $k+l+m = N$ and e^b denotes the vector of $\mathbb{R}^n$ s.t. $e_a^b = \delta_{ab}$.
For $(x,z,\theta),(x,z',\theta') \in E_{r[\Phi]}$, it follows that

$$|\partial_{x_b}^N[(x-z)^\alpha(x-z')^\alpha e^{i\Theta/\varepsilon}]| \lesssim \varepsilon^{|\alpha|-\frac{N}{2}} e^{-\text{cons}(2x-z-z')^2/\varepsilon} e^{-\text{cons}(z-z')^2/\varepsilon}.$$

Since $\varepsilon^{\frac{N'}{2}}\partial_{x_b}^{N'}[l_\varepsilon l'_\varepsilon]$ is uniformly bounded

$$\begin{aligned}|\partial_{x_b}^N[(x-z)^\alpha(x-z')^\alpha e^{i\Theta/\varepsilon}]\partial_{x_b}^{N'}[l_\varepsilon \bar{l}'_\varepsilon]| \lesssim \\ \varepsilon^{|\alpha|-\frac{N+N'}{2}} e^{-\text{cons}(2x-z-z')^2/\varepsilon} e^{-\text{cons}(z-z')^2/\varepsilon},\end{aligned}$$

and thus

$$|I_\varepsilon^\alpha(z,z',\theta,\theta')\left(\frac{\theta_b-\theta_b'}{\sqrt{\varepsilon}}\right)^K| \lesssim \varepsilon^{\frac{n}{2}+|\alpha|}e^{-\text{cons}(z-z')^2/\varepsilon},$$

and finally

$$\|O^\alpha\left(l_\varepsilon,\Phi/\varepsilon\right)h\|_{L^2}^2 \lesssim \varepsilon^{\frac{3n}{2}+|\alpha|}\|h\|_{L^2}^2.$$

□

Similar computations can be carried out for a phase Φ and a sequence of amplitudes l_ε that depend on a parameter $m\in[0,M]$. In this case, we consider for $m\in[0,M]$ a compact $K_{z,\theta}(m)\subset\mathbb{R}^{2n}$ and denote for $r>0$

$$E_r=\{(m,x,z,\theta)\in[0,M]\times\mathbb{R}^{3n},\ (z,\theta)\in K_{z,\theta}(m),\ |x-z|\leq r\}.$$

We are interested in a phase function Φ smooth on an open set containing E_{r_0} for some $r_0\in]0,1]$. We make the further assumption

$$E_{r_0}\text{ is compact},$$

which is obviously fulfilled when no parameter m interferes. Assuming, for $m\in[0,M]$ and $(z,\theta)\in K_{z,\theta}(m)$, that

$$\begin{array}{l}\partial_x\Phi(m,z,z,\theta)=\theta,\\ \Phi(m,z,z,\theta)\text{ is real,}\\ \partial_x^2\Phi(m,z,z,\theta)\text{ has a positive definite imaginary part,}\end{array}\tag{Q1'}$$

one can find $r[\Phi]\in]0,r_0]$ s.t. for $(m,x,z,\theta)\in E_{r[\phi]}$

$$\operatorname{Im}\Phi(m,x,z,\theta)\geq\text{cons}(x-z)^2.$$

Similarly, the sequence l_ε will be assumed to belong to $\mathcal{C}^\infty([0,M]\times\mathbb{R}_x^n\times\mathbb{R}_{z,\theta}^{2n},\mathbb{C})$ and to satisfy

$$\begin{array}{l}\text{for }m\in[0,M],\ l_\varepsilon(m,x,z,\theta)=0\text{ if }(m,x,z,\theta)\notin E_{r[\Phi]},\\ l_\varepsilon\text{ is uniformly bounded in }L^\infty([0,M]\times\mathbb{R}^{3n}).\end{array}\tag{Q2'}$$

One can then define, for every given $m\in[0,M]$ and α multiindex ($|\alpha|\geq 0$), the operators $O^\alpha\left(l_\varepsilon(m,.),\Phi(m,.)/\varepsilon\right)$, for which the following estimate may be established

Lemma 3.3. *Assume that $\varepsilon^{\frac{k}{2}}\partial_{x_b}^k l_\varepsilon$ ($b=1,\ldots,n$) is uniformly bounded in $L^\infty([0,M]\times\mathbb{R}^{3n})$, at any order $k\in\mathbb{N}$. Then, one has*

$$\|O^\alpha\left(l_\varepsilon(m,.),\Phi(m,.)/\varepsilon\right)\|_{L^2(\mathbb{R}^{2n})\to L^2(\mathbb{R}^n)}\lesssim\varepsilon^{\frac{3n}{4}+\frac{|\alpha|}{2}},\ \textit{uniformly w.r.t. } m\in[0,M].$$

In fact, all the estimates used in the proof of Lemma 3.1 hold true with a parameter $m\in[0,M]$, since $E_{r[\phi]}$ is still compact, owing to the compactness of E_{r_0}.

We now give the proof of Lemma 3.2. Using the formula of composite functions' high derivatives (see, e.g., [32] p.161), the N^{th} partial derivative of $e^{iF_p/\varepsilon}$ is

$$\partial_{x_b}^N[e^{iF_p/\varepsilon}]=\sum_{k=1}^N\left(\frac{i}{\varepsilon}\right)^k\prod_{\substack{j^1+\cdots+j^k=N\\ j^1,\ldots,j^k\geq 1}}\frac{N!}{k!j^1!\ldots j^k!}\,\partial_{x_b}^{j^1}F_p\ldots\partial_{x_b}^{j^k}F_p\,e^{iF_p/\varepsilon},\ N\in\mathbb{N}^*.$$

Each derivative $\partial_{x_b}^j F_p$ is a linear combination of

$$(x-z)^{\alpha+(s-j)e^b}\partial_{x_b}^s f_\alpha,\ |\alpha|=p, 0\le s\le j \text{ and } \alpha_b\ge j-s.$$

The product $\partial_{x_b}^{j^1}F_p\dots\partial_{x_b}^{j^k}F_p$ is then a linear combination of

$$(x-z)^{\alpha^1+(s^1-j^1)e^b+\dots+\alpha^k+(s^k-j^k)e^b}\partial_{x_b}^{s^1}f_{\alpha^1}\dots\partial_{x_b}^{s^k}f_{\alpha^k},$$

where for $i=1,\dots,k$, $|\alpha^i|=p$, $0\le s^i\le j^i$ and $\alpha_b^i\ge j^i-s^i$. As $j^1+\dots+j^k=N$, then for $N/p\le k\le N$ and $|x-z|\le 1$ one has

$$|(x-z)^{\alpha^1+(s^1-j^1)e^b+\dots+\alpha^k+(s^k-j^k)e^b}|\le|x-z|^{kp-N}.$$

Thus for $N\in\mathbb{N}^*$, $(x,z)\in S$, $|x-z|\le 1$ and small ε

$$|\partial_{x_b}^N[e^{iF_p/\varepsilon}]|\le\max_{\substack{|\alpha|=p\\0\le s\le N\\1\le k\le N}}\Big(\sup_S|\partial_{x_b}^s f_\alpha|\Big)^k\Big(\sum_{\frac{N}{p}\le k\le N}\varepsilon^{-k}|x-z|^{kp-N}+\sum_{1\le k<\frac{N}{p}}\varepsilon^{-N/p}\Big)|e^{iF_p/\varepsilon}|,$$

which of course is also valid for $N=0$.

3.2 Error estimates

The different terms of the energy estimate (29) will be estimated separately. Our main interest is to prove that the interior and boundary errors given for individual beams in Lemma 2.2 hold true for an infinite sum of beams, when the starting points of the incident flow vary in the compact $K_y\times K_\eta$. The control we have is that we can make the Gaussian beams vanish outside the very neighbourhood of their associated rays, by making the parameter d as small as needed.

3.2.1 The interior estimate of Pu_ε^R

In this section, we will prove that

$$\underset{t\in[0,T]}{Sup}\,\|Pu_\varepsilon^R(t,.)\|_{L^2(\Omega)}\lesssim\varepsilon^{\frac{R-1}{2}}.$$

For $0\le k\le N_+$, one has by construction

$$Pw_\varepsilon^k=\sum_{j=0}^{N+2}\varepsilon^{j-2}c_j^k e^{i\psi_k/\varepsilon},$$

where c_j^k is null on (t,x_k^t), up to the order $R-2j$, for $j=0,\dots,N+1$. One may write

$$Pw_\varepsilon^k(t,x)=\sum_{j=0}^{N+1}\varepsilon^{j-2}\Big(\sum_{|\alpha|=R-2j+1}(x-x_k^t)^\alpha r_\alpha^k(t,x)e^{i\psi_k(t,x)/\varepsilon}\Big)+\varepsilon^N c_{N+2}^k(t,x)e^{i\psi_k(t,x)/\varepsilon},$$

where r_α^k denotes the remainder in the Taylor formulae of c_j^k near x_k^t. Applying P to (28) gives then terms of the form

$$p_\varepsilon^{j,k}(t,x) = \varepsilon^{-\frac{3n}{4}-1+j} \sum_{|\alpha|=R-2j+1} \int_{\mathbb{R}^{2n}} \rho(y)\phi(\eta)h_\varepsilon(y,\eta) \\ (x-x_k^t)^\alpha r_\alpha^k(t,x,y,\eta)e^{i\psi_k(t,x,y,\eta)/\varepsilon}dyd\eta,$$

with $j = 0,\dots,N+1$, and

$$p_\varepsilon^{N+2,k}(t,x) = \varepsilon^{-\frac{3n}{4}+N+1} \int_{\mathbb{R}^{2n}} \rho(y)\phi(\eta)h_\varepsilon(y,\eta)c_{N+2}^k(t,x,y,\eta)e^{i\psi_k(t,x,y,\eta)/\varepsilon}dyd\eta,$$

where h_ε is either $\varepsilon^{-1}T_\varepsilon u_\varepsilon^I$ or $T_\varepsilon v_\varepsilon^I$ and $0 \le k \le N_+$. Other terms of the same form come from $Pw_\varepsilon^{k'}$, $0 \le k \le N_+$, and $P[w_\varepsilon^{-k(')}(-t,.)]$, $0 \le k \le N_-$.

Let $\tilde f(t,x,z,\theta) = f(t,x,\{\varphi_k^t\}^{-1}(z,\theta))$. Using the volume preserving change of variables $(z,\theta) = \varphi_k^t(y,\eta)$ in the definition of $p_\varepsilon^{j,k}(t,x)$, $0 \le j \le N+1$, writes it as a sum of terms of the form

$$\varepsilon^{-\frac{3n}{4}-1+j} \int_{\mathbb{R}^{2n}} \widetilde{\rho\otimes\phi}(t,z,\theta)\tilde h_\varepsilon(t,z,\theta)(x-z)^\alpha \tilde r_\alpha^k(t,x,z,\theta)e^{i\tilde\psi_k(t,x,z,\theta)/\varepsilon}dzd\theta,$$

with $|\alpha| = R-2j+1$. Similarly, $p_\varepsilon^{N+2,k}(t,x)$ is a sum of terms of the form

$$\varepsilon^{-\frac{3n}{4}+N+1} \int_{\mathbb{R}^{2n}} \widetilde{\rho\otimes\phi}(t,z,\theta)\tilde h_\varepsilon(t,z,\theta)\tilde c_{N+2}^k(t,x,z,\theta)e^{i\tilde\psi_k(t,x,z,\theta)/\varepsilon}dzd\theta.$$

We want to estimate these integrals with the help of the operators O^α applied to $\mathbf{1}_{\mathrm{supp}\widetilde{\rho\otimes\phi}(t,.)}\tilde h_\varepsilon$. Clearly $\mathbf{1}_{\mathrm{supp}\widetilde{\rho\otimes\phi}(t,.)}\widetilde{T_\varepsilon v_\varepsilon^I}(t,.)$ is uniformly bounded (w.r.t. ε and t) in $L^2(\mathbb{R}^{2n})$. But more work is needed for estimating $\varepsilon^{-1}\mathbf{1}_{\mathrm{supp}\widetilde{\rho\otimes\phi}(t,.)}\widetilde{T_\varepsilon u_\varepsilon^I}(t,.)$, which is given in the following result

Lemma 3.4. $\|\varepsilon^{-1}T_\varepsilon u_\varepsilon^I\|_{L^2(\mathbb{R}^{2n})} \lesssim 1$.

Proof. Differentiating (3) w.r.t. y_b, $1 \le b \le n$, yields

$$\varepsilon^{\frac12}\partial_{y_b}(T_\varepsilon u_\varepsilon^I) = i\eta_b\varepsilon^{-\frac12}T_\varepsilon u_\varepsilon^I - c_n\varepsilon^{-\frac{3n}{4}} \int_{\mathbb{R}^n} u_\varepsilon^I(w)\varepsilon^{-\frac12}(y_b-w_b)\, e^{i\eta.(y-w)/\varepsilon-(y-w)^2/(2\varepsilon)}dw.$$

The l.h.s. is bounded in $L^2_{y,\eta}$ because $\partial_{y_b}(T_\varepsilon u_\varepsilon^I) = T_\varepsilon(\partial_{w_b}u_\varepsilon^I)$. The second term of the r.h.s. is the Fourier transform of a bounded function in L^2_w, thus it can be estimated using Parseval equality. One gets

$$\|\varepsilon^{-\frac{3n}{4}} \int_{\mathbb{R}^n} u_\varepsilon^I(w)\varepsilon^{-\frac12}(y_b-w_b)\, e^{i\eta.(y-w)/\varepsilon-(y-w)^2/(2\varepsilon)}dw\|_{L^2_{y,\eta}} \lesssim \|u_\varepsilon^I\|_{L^2_w}.$$

Thus $\|\varepsilon^{-\frac12}\eta_b T_\varepsilon u_\varepsilon^I\|_{L^2_{y,\eta}} \lesssim 1$ and consequently $\|\varepsilon^{-\frac12}\phi(\eta)T_\varepsilon u_\varepsilon^I\|_{L^2_{y,\eta}} \lesssim 1$. Assumption $A3$ yields

$$\|\varepsilon^{-\frac12}T_\varepsilon u_\varepsilon^I\|_{L^2_{y,\eta}} \lesssim 1.$$

Hence $\|u_\varepsilon^I\|_{L^2} \lesssim \sqrt{\varepsilon}$. Reproducing the same arguments on the following equality

$$\partial_{y_b}(T_\varepsilon u_\varepsilon^I) = i\eta_b\varepsilon^{-1}T_\varepsilon u_\varepsilon^I - c_n\varepsilon^{-\frac{3n}{4}} \int_{\mathbb{R}^n} \left(\varepsilon^{-\frac12}u_\varepsilon^I\right)(w)\varepsilon^{-\frac12}(y_b-w_b)\, e^{\frac{i}{\varepsilon}\eta.(y-w)-\frac{1}{2\varepsilon}(y-w)^2}dw,$$

leads to $\|u_\varepsilon^I\|_{L^2} \lesssim \varepsilon$. □

Let us now check if a family of operators O^α may be used. First, each phase $\tilde{\psi}_k$ is smooth on an open set containing

$$E_1 = \{(t,x,z,\theta) \in [0,T] \times \mathbb{R}^{3n}, (z,\theta) \in \varphi_k^t(K_y \times K_\eta), |x-z| \leq 1\}.$$

E_1 is compact, since the map $(t,y,\eta) \mapsto (t, \varphi_k^t(y,\eta))$ is continuous. For $t \in [0,T]$ and $(z,\theta) \in \varphi_k^t(K_y \times K_\eta)$, one has by ($P_k$.a), ($P_k$.b) and ($P_k$.c)

$$\begin{aligned}
&\partial_x \tilde{\psi}_k(t,z,z,\theta) = \tilde{\xi}_k^t(z,\theta) = \theta,\\
&\tilde{\psi}_k(t,z,z,\theta) \text{ is real},\\
&\partial_x^2 \tilde{\psi}_k(t,z,z,\theta) \text{ has a positive definite imaginary part.}
\end{aligned}$$

Hence $\tilde{\psi}_k$ satisfies properties (Q1'). We fix some $r[\tilde{\psi}_k] \in]0,1]$ so that

$$\operatorname{Im} \tilde{\psi}_k(t,x,z,\theta) \geq \operatorname{cons}(x-z)^2 \text{ for every } (t,x,z,\theta) \in E_{r[\tilde{\psi}_k]}. \tag{31}$$

Next, for $R-2N-1 \leq |\alpha| \leq R+1$, let

$$l^{\alpha,k}(t,x,z,\theta) = \widetilde{\rho \otimes \phi}(t,z,\theta) \tilde{r}_\alpha^k(t,x,z,\theta),\ t \in [0,T],$$

and

$$l^{0,k}(t,x,z,\theta) = \widetilde{\rho \otimes \phi}(t,z,\theta) \tilde{c}_{N+2}^k(t,x,z,\theta),\ t \in [0,T].$$

Then the $l^{\alpha,k}$, $|\alpha| = R-2N-1, \ldots, R+1$, and $l^{0,k}$ are smooth w.r.t. all their variables. Assume that

$$d \leq r[\tilde{\psi}_k],\ k = 0, \ldots, N_+. \tag{32}$$

Because of the cut-offs χ_d in the beams' amplitudes, it follows that $\tilde{c}_{N+2}^k(t,x,z,\theta) = \tilde{r}_\alpha^k(t,x,z,\theta) = 0$ if $|x-z| \geq r[\tilde{\psi}_k]$. Furthermore, $\widetilde{\rho \otimes \phi}(t,z,\theta) = 0$ for $(z,\theta) \notin \varphi_k^t(K_y \times K_\eta)$. The $l^{\alpha,k}$ satisfy therefore assumptions (Q2').

It follows that the operators O^α can be used to obtain for $t \in [0,T]$ and $x \in \mathbb{R}^n$

$$p_\varepsilon^{j,k}(t,x) = \varepsilon^{-\frac{3n}{4}-1+j} \sum_{|\alpha|=R-2j+1} \left[O^\alpha \left(l^{\alpha,k}(t,.), \tilde{\psi}_k(t,.)/\varepsilon \right) \mathbf{1}_{\operatorname{supp} \widetilde{\rho \otimes \phi}(t,.)} \tilde{h}_\varepsilon(t,.) \right](x),$$

with $j = 0, \ldots N+1$, and

$$p_\varepsilon^{N+2,k}(t,x) = \varepsilon^{-\frac{3n}{4}+N+1} \left[O^0 \left(l^{0,k}(t,.), \tilde{\psi}_k(t,.)/\varepsilon \right) \mathbf{1}_{\operatorname{supp} \widetilde{\rho \otimes \phi}(t,.)} \tilde{h}_\varepsilon(t,.) \right](x).$$

Applying Lemma 3.3 and making use of (14) yields

$$\|p_\varepsilon^{j,k}(t,.)\|_{L^2(\Omega)} \lesssim \varepsilon^{\frac{R-1}{2}},\ \text{uniformly w.r.t. } t \in [0,T],\ \text{for } j = 0, \ldots, N+2.$$

3.2.2 The boundary estimate of Bu_ε^R

We will now estimate $Bu_\varepsilon^R|_{\partial\Omega}$, $B = \mathbf{D}$ or $\mathbf{N}$ standing for Dirichlet and Neumann operators respectively. We shall prove that

$$\|\mathbf{D}u_\varepsilon^R\|_{H^1([0,T]\times\partial\Omega)} \lesssim \varepsilon^{\frac{R-1}{2}} \text{ and } \|\mathbf{N}u_\varepsilon^R\|_{H^{1/2}([0,T]\times\partial\Omega)} \lesssim \varepsilon^{\frac{R-2}{2}}. \tag{33}$$

To this end, we note that the boundary operator B applied to (28) is a sum of terms arising from Bw_ε^k, $0 \leq k \leq N_+$ such as

$$b_\varepsilon^j(t,x) = \varepsilon^{-\frac{3n}{4}+1-m_B+j} \int_{\mathbb{R}^{2n}} \rho(y)\phi(\eta)h_\varepsilon(y,\eta) \sum_{k=0}^{N_+} d^k_{-m_B+j}(t,x',y,\eta)e^{i\psi_k(t,x',y,\eta)/\varepsilon}dyd\eta, \tag{34}$$

with $j = 0,\dots,N+m_B$, and others with the same form arising from $Bw_\varepsilon^{k'}$, $0 \leq k \leq N_+$, and $B[w_\varepsilon^{-k(')}(-t,.)]$, $0 \leq k \leq N_-$.

Above and as in the previous section, h_ε is either $\varepsilon^{-1}T_\varepsilon u^I_\varepsilon$ or $T_\varepsilon v^I_\varepsilon$ and thus is uniformly bounded in L^2.

We first study the support of the amplitudes. Next we use local boundary coordinates to expand the boundary phases and introduce a change of variables on (y,η) that makes the obtained phases satisfy properties (Q1). The previous results on the approximation operators O^α are then used to estimate the boundary norms.

Support of the amplitudes Due to assumptions B1-B2-B3, the rays stay away from the boundary except for times near the instants of reflections. For $(y,\eta) \in K_y \times K_\eta$ and $t \in [0,T]$ near some $T_k(y,\eta)$, $0 \leq k \leq N_+$, only $x_k^t(y,\eta)$ and, if $k \neq 0$, $x^t_{k-1}(y,\eta)$ approach the boundary. This suggests that the meaningful contributions to the boundary norm of b^j_ε are the quantities $d^{k-1}_{-m_B+j}(.,y,\eta)e^{i\psi_{k-1}(.,y,\eta)/\varepsilon} + d^k_{-m_B+j}(.,y,\eta)e^{i\psi_k(.,y,\eta)/\varepsilon}$ near $T_k(y,\eta)$, $k = 1,\dots,N_+$. Furthermore, for t in the neighbourhood of $T_k(y,\eta)$ and $x' \in \partial\Omega$, one expects $d^{k-1}_{-m_B+j}(t,x',y,\eta)$ and $d^k_{-m_B+j}(t,x',y,\eta)$ to vanish away from $x^{T_k(y,\eta)}_{k-1}(y,\eta)$, because of the cut-offs in the amplitudes. In the remainder, we show that these two intuitive points are true. The key argument is that (t,y,η) vary in a compact set.

The first point is rather easy to see. For $(y,\eta) \in K_y \times K_\eta$, let us consider a period smaller than any lapse of time between two successive reflections, say $\beta(y,\eta) = \min_{0\leq k\leq N_+} (T_k(y,\eta) - T_{k-1}(y,\eta))/3$, $(T_0 = 0)$, and define the intervals

$$I_0(y,\eta) = \emptyset,\ I_k(y,\eta) = [T_k(y,\eta) - \beta(y,\eta), T_k(y,\eta) + \beta(y,\eta)] \text{ for } k = 1,\dots,N_+, \text{ and } I_{N_++1}(y,\eta) = \emptyset.$$

For each $k = 0,\dots,N_+$, let

$$A_k = \{(t,y,\eta) \in [0,T] \times K_y \times K_\eta,\ t \notin \overset{o}{I}_k(y,\eta) \cup \overset{o}{I}_{k+1}(y,\eta)\}.$$

For $(t,y,\eta) \in A_k$, $\operatorname{dist}(x^t_k(y,\eta),\partial\Omega) > 0$ and has then a positive lower bound by continuity on the compact A_k. One has by (31) and (32)

$$\psi_k(t,x,y,\eta) \geq \operatorname{cons}(x - x^t_k(y,\eta))^2,$$

for $(t,x,y,\eta) \in [0,T] \times \mathbb{R}^n \times K_y \times K_\eta$ s.t. $|x - x^t_k(y,\eta)| \leq d$. Thus

$$|d^k_{-m_B+j}(t,x',y,\eta)e^{i\psi_k(t,x',y,\eta)/\varepsilon}| \leq e^{-\operatorname{cons}/\varepsilon} \text{ for } (t,y,\eta) \in A_k \text{ and } x' \in \partial\Omega.$$

All we have to care about is then the contribution to the norm at the boundary of $d^{k-1}_{-m_B+j}e^{i\psi_{k-1}/\varepsilon}$ and $d^k_{-m_B+j}e^{i\psi_k/\varepsilon}$ at times in the interval I_k, $k = 1,\dots,N_+$. Let

$$q^{j,k}_\varepsilon = \varepsilon^{-\frac{3n}{4}+1-m_B+j} \int_{\mathbb{R}^{2n}} \rho\otimes\phi h_\varepsilon \mathbf{1}_{I_k}(t)(d^{k-1}_{-m_B+j}e^{i\psi_{k-1}/\varepsilon} + d^k_{-m_B+j}e^{i\psi_k/\varepsilon})dyd\eta.$$

Summing over $k=1,\dots,N_+$ yields

$$\|b_\varepsilon^j\|_{L^2([0,T]\times\partial\Omega)} \lesssim \sum_{k=1}^{N_+} \|q_\varepsilon^{j,k}\|_{L^2([0,T]\times\partial\Omega)} + \varepsilon^\infty. \tag{35}$$

For the second point, we partition the set of starting points (y,η) according to the part of the boundary the rays $x_{k-1}^t(y,\eta)$ reach at $t=T_k(y,\eta)$. Let (u_l) be a $\mathcal{C}^\infty$ partition of unity associated to the covering $(\mathcal{U}_l)$ introduced in subsection 2.2.1 and $\pi_l(y,\eta)=\rho(y)\phi(\eta)u_l(x_{k-1}^{T_k}(y,\eta))$. Then

$$\|q_\varepsilon^{j,k}\|_{L^2([0,T]\times\partial\Omega)} \lesssim \sum_{l=1}^{L} \|m_\varepsilon^{j,k,l}\|_{L^2([0,T]\times\partial\Omega)},$$

where

$$\begin{aligned} m_\varepsilon^{j,k,l} = \varepsilon^{-\frac{3n}{4}+1-m_B+j} \int_{\mathbb{R}^{2n}} h_\varepsilon \pi_l \mathbf{1}_{I_k}(t)(d_{-m_b+j}^{k-1} e^{i\psi_{k-1}/\varepsilon} \\ + d_{-m_b+j}^{k} e^{i\psi_k/\varepsilon}) dy d\eta. \end{aligned} \tag{36}$$

We fix $1\le l\le L$ and $1\le k\le N_+$. For $0<\delta<\min\limits_{K_y\times K_\eta}\beta$, let

$$B_\delta = \{(t,y,\eta)\in[0,T]\times \mathrm{supp}\pi_l,\ t\in I_k(y,\eta)\backslash]T_k(y,\eta)-\delta, T_k(y,\eta)+\delta[\}.$$

If (t,y,η) is in the compact set B_δ, then $\mathrm{dist}(x_k^t(y,\eta),\partial\Omega)>0$. Let $d(\delta)\in]0,\delta]$ s.t. $d(\delta)<\min\limits_{(t,y,\eta)\in B_\delta}\mathrm{dist}(x_k^t(y,\eta),\partial\Omega)$ and consider the set

$$S_\delta = \{(t,x',y,\eta)\in[0,T]\times\partial\Omega\times\mathrm{supp}\pi_l, t\in I_k(y,\eta) \text{ and } |x'-x_k^t(y,\eta)|\le d(\delta)\}.$$

If $(t,x',y,\eta)\in S_\delta$ then $t\in]T_k(y,\eta)-\delta,T_k(y,\eta)+\delta[$ and consequently

$$\begin{aligned} |x'-x_{k-1}^{T_k(y,\eta)}(y,\eta)| &\le |x'-x_{k-1}^t(y,\eta)| + |t-T_k(y,\eta)| \sup_{s\in[t,T_k(y,\eta)]} |\dot{x}_{k-1}^s(y,\eta)| \\ &\le (1+\|c\|_\infty)\delta, \end{aligned}$$

which implies that $x'\in\mathcal{U}_l$ for sufficiently small δ, since $x_{k-1}^{T_k(y,\eta)}(y,\eta)$ varies in a compact set of $\mathcal{U}_l$. Assume that $d\le d(\delta)$. Thus, $\mathrm{supp}\left(\pi_l(y,\eta)\mathbf{1}_{I_k(y,\eta)}(t)d_{-m_B+j}^k(t,x',y,\eta)\right)$ is included in S_δ. On the other hand, as σ_l is a diffeomorphism between $\mathcal{N}_l$ and $\mathcal{U}_l$, one has

$$|\sigma_l(\hat{v})-\sigma_l(\hat{v}')|\ge \mathrm{cons}|\hat{v}-\hat{v}'| \text{ for every } \hat{v},\hat{v}'\in\mathcal{N}_l.$$

Therefore, there exists $\kappa>0$ s.t.

$$\pi_l(y,\eta)\mathbf{1}_{I_k(y,\eta)}(t)d_{-m_B+j}^k(t,\sigma_l(\hat{v}),y,\eta)=0 \text{ if } |t-T_k(y,\eta)|\ge\delta \text{ or } |\hat{v}-\hat{z}_k(y,\eta)|\ge\kappa\delta,$$

where $\sigma_l(\hat{z}_k(y,\eta))=x_{k-1}^{T_k(y,\eta)}(y,\eta)$.
The same result holds true for $\pi_l(y,\eta)\mathbf{1}_{I_k(y,\eta)}(t)d_{-m_B+j}^{k-1}(t,\sigma_l(\hat{v}),y,\eta)$, assuming that $d\le d'(\delta)$ with $d'(\delta)\in]0,\delta]$ and $d'(\delta)<\min\limits_{(t,y,\eta)\in B_\delta}\mathrm{dist}(x_{k-1}^t(y,\eta),\partial\Omega)$. Furthermore

$$m_\varepsilon^{j,k,l}(t,x')=0 \text{ if } x'\notin\mathcal{U}_l.$$

Expansion of the boundary phases For simplicity of notation, we shall drop the exponents and indexes l. We expand the phase ${}^{\sigma}\psi_{k-1}$ on $[0,T]\times\mathcal{N}\times\{0\}$ near $(T_k,\hat{z}_k)$

$$\begin{aligned}{}^{\sigma}\psi_{k-1}(t,\hat{v},0) &= \psi_{k-1}(t,\sigma(\hat{v})+v_n\nu(\sigma(\hat{v})))|_{v_n=0}\\ &= {}^{\sigma}\psi_{k-1}(T_k,\hat{z}_k,0)+(t-T_k,\hat{v}-\hat{z}_k)\cdot(\tau,\hat{\theta}_k)\\ &+\frac{1}{2}(t-T_k,\hat{v}-\hat{z}_k)\cdot M_k(t-T_k,\hat{v}-\hat{z}_k)\\ &+\sum_{|\alpha|=3}(t-T_k,\hat{v}-\hat{z}_k)^{\alpha}\\ &\qquad\int_0^1\frac{3}{\alpha!}(1-s)^2\partial^{\alpha}_{t,\hat{v}}{}^{\sigma}\psi_{k-1}(T_k+s(t-T_k),\hat{z}_k+s(\hat{v}-\hat{z}_k),0)ds,\end{aligned}$$

where $\hat{\theta}_k = D\sigma(\hat{z}_k)^T\xi^{T_k}_{k-1}$ and the matrix M_k defined in (20) has a positive definite imaginary part. Remember that all the quantities of the previous formulae depend on $(y,\eta)\in(x^{T_k}_{k-1})^{<-1>}(\mathcal{U})$. For the purpose of obtaining a phase satistfying (Q1), the form of ${}^{\sigma}\psi_{k-1}|_{v_n=0}$ suggests the change of variables $(C):(z,\theta)=\vartheta(y,\eta)$, with

$$\vartheta:(y,\eta)\in(x^{T_k}_{k-1})^{<-1>}(\mathcal{U})\mapsto(T_k,\hat{z}_k,\tau,\hat{\theta}_k).$$

Because tangential rays are avoided, the function $T_k\in\mathcal{C}^{\infty}((x^{T_k}_{k-1})^{<-1>}(\mathcal{U}))$ so ϑ is $\mathcal{C}^{\infty}$. Note that $\xi^{T_k}_{k-1}=\Sigma(\hat{z}_k)\hat{\theta}_k+(\nu(\sigma(\hat{z}_k))\cdot\xi^{T_k}_{k-1})\nu(\sigma(\hat{z}_k))$ with $\Sigma=D\sigma\left(D\sigma^TD\sigma\right)^{-1}$. Hence ϑ is bijective and its inverse is given by

$$\begin{aligned}\vartheta^{-1}:(T_k,\hat{z}_k,\tau,\ \hat{\theta}_k)\ &\in\vartheta((x^{T_k}_{k-1})^{<-1>}(\mathcal{U}))\\ \mapsto\ &\{\varphi^{T_k}_{k-1}\}^{-1}(\sigma(\hat{z}_k),\Sigma(\hat{z}_k)\hat{\theta}_k+(\tau^2/c^2(\sigma(\hat{z}_k))-|\Sigma(\hat{z}_k)\hat{\theta}_k|^2)^{\frac{1}{2}}\nu(\sigma(\hat{z}_k))).\end{aligned}$$

ϑ^{-1} is $\mathcal{C}^{\infty}$ on $\vartheta\left((x^{T_k}_{k-1})^{<-1>}(\mathcal{U})\right)$ because the square root in the previous expression never vanish. Consequently, ϑ is a $\mathcal{C}^{\infty}$ diffeomorphism.

Let $v=(t,\hat{v})$, $z=(T_k,\hat{z}_k)$ and $\theta=(\tau,\hat{\theta}_k)$ and denote $\tilde{f}(v,z,\theta)=f(v,\vartheta^{-1}(z,\theta))$. We may write ${}^{\sigma}\tilde{\psi}_{k-1}|_{v_n=0}$ as

$$\begin{aligned}{}^{\sigma}\tilde{\psi}_{k-1}(v,0,z,\theta) &= {}^{\sigma}\tilde{\psi}_{k-1}(z,0,z,\theta)+\theta\cdot(v-z)+\tfrac{1}{2}(v-z)\tilde{M}_k(z,\theta)(v-z)\\ &+\sum_{3\le|\alpha|\le R}\tfrac{1}{\alpha!}(v-z)^{\alpha}\partial^{\alpha}_{t,\hat{v}}{}^{\sigma}\tilde{\psi}_{k-1}(z,0,z,\theta)+\tilde{r}_{k-1}(v,z,\theta)\\ &:= \tilde{\lambda}(v,z,\theta)+\tilde{r}_{k-1}(v,z,\theta).\end{aligned}$$

Since ${}^{\sigma}\psi_k$ and ${}^{\sigma}\psi_{k-1}$ have by construction the same derivatives w.r.t. v up to the order R at $(z,0)$, the expansion of ${}^{\sigma}\tilde{\psi}_k|_{v_n=0}$ involves the same derivatives up to the order R and a remainder $\tilde{r}_k$

$${}^{\sigma}\tilde{\psi}_k(v,0,z,\theta)=\tilde{\lambda}(v,z,\theta)+\tilde{r}_k(v,z,\theta).$$

With the change of variables (C), ${}^{\sigma}m^{j,k}_{\varepsilon}$ may be written on $[0,T]\times\mathcal{N}\times\{0\}$ as

$$\begin{aligned}{}^{\sigma}m^{j,k}_{\varepsilon}=\varepsilon^{-\frac{3n}{4}+1-m_B+j}\int_{\mathbb{R}^{2n}}\tilde{h}_{\varepsilon}\tilde{\pi}\mathbf{1}_{\tilde{I}_k}(t)({}^{\sigma}\tilde{d}^{k-1}_{-m_B+j}e^{i(\tilde{\lambda}+\tilde{r}_{k-1})/\varepsilon}\\ +{}^{\sigma}\tilde{d}^{k}_{-m_B+j}e^{i(\tilde{\lambda}+\tilde{r}_k)/\varepsilon})|\det\vartheta|dzd\theta,\end{aligned}$$

where $\tilde{I}_k$ denotes $[\tilde{T}_k-\tilde{\beta},\tilde{T}_k+\tilde{\beta}]$. We split the previous integral into two integrals which can be estimated using the operators O^{α}

$$\varepsilon^{-\frac{3n}{4}+1-m_B+j}\int_{\mathbb{R}^{2n}}\tilde{h}_{\varepsilon}\tilde{\pi}\mathbf{1}_{\tilde{I}_k}(t)({}^{\sigma}\tilde{d}^{k-1}_{-m_B+j}+{}^{\sigma}\tilde{d}^{k}_{-m_B+j})e^{i(\tilde{\lambda}+\tilde{r}_{k-1})/\varepsilon}|\det\vartheta|dzd\theta:=①,$$

$$\varepsilon^{-\frac{3n}{4}+1-m_B+j}\int_{\mathbb{R}^{2n}}\tilde{h}_{\varepsilon}\tilde{\pi}\mathbf{1}_{\tilde{I}_k}(t){}^{\sigma}\tilde{d}^{k}_{-m_B+j}e^{i\tilde{\lambda}/\varepsilon}(e^{i\tilde{r}_k/\varepsilon}-e^{i\tilde{r}_{k-1}/\varepsilon})|\det\vartheta|dzd\theta:=②.$$

Estimate of ①: The phase $\tilde\lambda + \tilde r_{k-1}$ is smooth on an open set containing $E_{r_0} = \{(v,z,\theta) \in \mathbb{R}^n \times \mathrm{supp}\tilde\pi,\ |v-z| \le r_0\}$ for some $r_0 \in]0,1]$. Furthermore, $\tilde\lambda + \tilde r_{k-1}$ satisfies the required properties (Q1). We fix $r[\tilde\lambda + \tilde r_{k-1}] \in]0, r_0]$.

Since ${}^\sigma d^{k-1}_{-m_B+j} + {}^\sigma d^{k}_{-m_B+j}$ is zero at $v=z$ up to the order $R-2j-2$ by construction, one has

$$\left({}^\sigma\tilde d^{k-1}_{-m_B+j} + {}^\sigma\tilde d^{k}_{-m_B+j}\right)(v,z,\theta) = \sum_{|\alpha|=R-2j-1} (v-z)^\alpha \tilde s^k_\alpha(v,z,\theta),$$

where $\tilde s^k_\alpha$ are smooth remainders. Let

$$a^{\alpha,k}(v,z,\theta) = \tilde\pi(z,\theta)\mathbf{1}_{\tilde I_k}(t)\tilde s^k_\alpha(v,z,\theta)|\det\vartheta(z,\theta)|.$$

The $a^{\alpha,k}$ are smooth and $a^{\alpha,k}(t,\hat v, T_k, \hat z_k,\theta) = 0$ if $|t-T_k| \ge \delta$ or $|\hat v - \hat z_k| \ge \kappa\delta$ or $(z,\theta) \notin \mathrm{supp}(\tilde\pi)$. Then the $a^{\alpha,k}$ satisfy the properties (Q2), assuming δ small enough to insure $|(\delta,\kappa\delta)| \le r[\tilde\lambda + \tilde r_{k-1}]$.

Therefore

$$① = \varepsilon^{-\frac{3n}{4}+1-m_B+j} \sum_{|\alpha|=R-2j-1} O^\alpha\left(a^{\alpha,k}, (\tilde\lambda + \tilde r_{k-1})/\varepsilon\right)\mathbf{1}_{\mathrm{supp}\tilde\pi}\tilde h_\varepsilon.$$

One deduces

$$\|①\|_{L^2([0,T]\times\mathcal{N})} \lesssim \varepsilon^{\frac{R+1}{2}-m_B}\|h_\varepsilon\|_{L^2}. \tag{37}$$

Estimate of ②: This is the term for which Lemma 3.1 is fully used. We write $\tilde\lambda$ as $\tilde\lambda = \beta + 2\gamma$ where

$$\gamma = \frac{1}{4}(v-z)\tilde M_k(z,\theta)(v-z) \text{ and } \beta = \tilde\lambda - \frac{1}{2}(v-z)\tilde M_k(z,\theta)(v-z).$$

The part $\beta+\gamma$ will play the role of the phase for the operators O^α, while $e^{i\gamma/\varepsilon}$ will be enclosed in the amplitude to give it a good behavior. The phase $\beta+\gamma$ is smooth on an open set containing E_{r_0} and satisfies the properties (Q1). We associate to this phase some constant $r[\beta+\gamma]$ and impose on δ to satisfy $|(\delta,\kappa\delta)| \le r[\beta+\gamma]$.

Let

$$c^{j,k}_\varepsilon = \varepsilon^{-\frac{R-1}{2}}\tilde\pi\mathbf{1}_{\tilde I_k}(t)\,{}^\sigma\tilde d^k_{-m_B+j} e^{i\gamma/\varepsilon}(e^{i\tilde r_k/\varepsilon} - e^{i\tilde r_{k-1}/\varepsilon})|\det\vartheta|.$$

One has

$$|c^{j,k}_\varepsilon| \lesssim \varepsilon^{-\frac{R-1}{2}} e^{-\mathrm{cons}(v-z)^2/\varepsilon}|e^{i\tilde r_k/\varepsilon} - e^{i\tilde r_{k-1}/\varepsilon}|.$$

If δ is small enough,

$$e^{-\mathrm{cons}(v-z)^2/\varepsilon}|e^{i\tilde r_k/\varepsilon} - e^{i\tilde r_{k-1}/\varepsilon}| \lesssim \varepsilon^{-1}|v-z|^{R+1}e^{-\mathrm{cons}(v-z)^2/(2\varepsilon)},$$

so that

$$|c^{j,k}_\varepsilon| \lesssim 1.$$

Hence $c^{j,k}_\varepsilon$ is smooth and satisfies the properties (Q2):

$$\begin{aligned}&c^{j,k}_\varepsilon(v,z,\theta) = 0 \text{ if } |v-z| \ge r[\beta+\gamma] \text{ or } (z,\theta) \notin \mathrm{supp}(\tilde\pi),\\ &c^{j,k}_\varepsilon \text{ is uniformly bounded in } L^\infty(\mathbb{R}^{3n}).\end{aligned}$$

To make use of the estimates of Lemma 3.1, we aim to show that for $N \in \mathbb{N}$, $\varepsilon^{\frac{N}{2}} \partial_{v_b}^N c_\varepsilon^{j,k}$ $(b = 1, \dots, n)$ is uniformly bounded in $L^\infty(\mathbb{R}^{3n})$. For this purpose, we write $\partial_{v_b}^N [e^{i\gamma/\varepsilon}(e^{i\tilde{r}_k/\varepsilon} - e^{i\tilde{r}_{k-1}/\varepsilon})]$ as a sum of terms of the form

$$\partial_{v_b}^{N_1}[e^{i\gamma/\varepsilon}]\partial_{v_b}^{N_2}[e^{i\tilde{r}_k/\varepsilon} - e^{i\tilde{r}_{k-1}/\varepsilon}],\ 0 \le N_1, N_2 \le N, N_1 + N_2 = N.$$

As the remainders $\tilde{r}_k$ and $\tilde{r}_{k-1}$ are of order $R+1$, Lemma 3.2 yields for $N_1, N_2 \in \mathbb{N}$, $(z,\theta) \in \mathrm{supp}\tilde{\pi}$, $|v - z| \le |(\delta, \kappa\delta)|$ and δ sufficiently small

$$|\partial_{v_b}^{N_1}[e^{i\gamma/\varepsilon}]| \lesssim \varepsilon^{-\frac{N_1}{2}} e^{-\mathrm{cons}(v-z)^2/\varepsilon},$$

$$|\partial_{v_b}^{N_2}[e^{i\tilde{r}_k/\varepsilon} - e^{i\tilde{r}_{k-1}/\varepsilon}]| \lesssim \Bigg(\sum_{\frac{N_2}{R+1} \le k \le N_2} \varepsilon^{-k}|v-z|^{k(R+1)-N_2} + \sum_{1 \le k < \frac{N_2}{R+1}} \varepsilon^{-\frac{N_2}{R+1}} \Bigg) \Big(|e^{i\tilde{r}_k/\varepsilon}| + |e^{i\tilde{r}_{k-1}/\varepsilon}|\Big).$$

The second sum in the last inequality is zero when $N_2/(R+1) \le 1$. Remember that $R \ge 2$. If $N_2/(R+1) > 1$ then $N_2(R-1)/(2(R+1)) > (R-1)/2$ and consequently $-N_2/(R+1) > -N_2/2 + (R-1)/2$. Thus

$$|\partial_{v_b}^{N_2}[e^{i\tilde{r}_k/\varepsilon} - e^{i\tilde{r}_{k-1}/\varepsilon}]| \lesssim \Bigg(\sum_{\frac{N_2}{R+1} \le k \le N_2} \varepsilon^{-k}|v-z|^{k(R+1)-N_2} + \varepsilon^{-\frac{N_2}{2}+\frac{R-1}{2}} \Bigg) \Big(|e^{i\tilde{r}_k/\varepsilon}| + |e^{i\tilde{r}_{k-1}/\varepsilon}|\Big).$$

Hence, for $(z,\theta) \in \mathrm{supp}\tilde{\pi}$ and $|v - z| \le |(\delta, \kappa\delta)|$

$$|\partial_{v_b}^{N_1}[e^{i\gamma/\varepsilon}]\partial_{v_b}^{N_2}[e^{i\tilde{r}_k/\varepsilon} - e^{i\tilde{r}_{k-1}/\varepsilon}]| \lesssim \varepsilon^{-\frac{N_1}{2}-\frac{N_2}{2}+\frac{R-1}{2}}.$$

It follows that

$$|\partial_{v_b}^N c_\varepsilon^{j,k}| \lesssim \varepsilon^{-\frac{N}{2}}.$$

One can use the operator O^0 to write

$$② = \varepsilon^{-\frac{3n}{4}+1-m_B+j} \varepsilon^{\frac{R-1}{2}} O^0\left(c_\varepsilon^{j,k}, (\beta+\gamma)/\varepsilon\right) \mathbf{1}_{\mathrm{supp}\tilde{\pi}} \tilde{h}_\varepsilon,$$

and thus

$$\|②\|_{L^2([0,T]\times\mathcal{N})} \lesssim \varepsilon^{\frac{R+1}{2}-m_B+j}\|h_\varepsilon\|_{L^2}. \tag{38}$$

Using (37) and (38) yields

$$\|m_\varepsilon^{j,k,l}\|_{L^2([0,T]\times\partial\Omega)} \lesssim \varepsilon^{\frac{R+1}{2}-m_B}\|h_\varepsilon\|_{L^2}.$$

One has a similar bound for $q_\varepsilon^{j,k}$ by summing over $l = 1, \dots, L$,

$$\|q_\varepsilon^{j,k}\|_{L^2([0,T]\times\partial\Omega)} \lesssim \varepsilon^{\frac{R+1}{2}-m_B}\|h_\varepsilon\|_{L^2}.$$

Plugging this into (35) gives

$$\|b_\varepsilon^j\|_{L^2([0,T]\times\partial\Omega)} \lesssim \varepsilon^{\frac{R+1}{2}-m_B}.$$

All in all, we have shown that

$$\|Bu_\varepsilon^R\|_{L^2([0,T]\times\partial\Omega)} \lesssim \varepsilon^{\frac{R+1}{2}-m_B}.$$

This result can be adapted to the integer Sobolev spaces as follows

$$\|Bu_\varepsilon^R\|_{H^s([0,T]\times\partial\Omega)} \lesssim \varepsilon^{\frac{R+1}{2}-m_B-s},\ s \in \mathbb{N}.$$

An interpolation argument ([61], p.49) enables the same estimate for non integer Sobolev spaces $H^s([0,T]\times\partial\Omega)$, $s > 0$. This proves (33).

3.2.3 The initial conditions

In this section we estimate the difference between $(u_\varepsilon^R|_{t=0}, \partial_t u_\varepsilon^R|_{t=0})$ and $(u_\varepsilon^I, v_\varepsilon^I)$ in $H^1(\Omega)\times L^2(\Omega)$.

By construction,

$$\begin{aligned} u_\varepsilon^R(0,x) &= \frac{1}{2}\varepsilon^{-\frac{3n}{4}} c_n \int_{\mathbb{R}^{2n}} \rho(y)\phi(\eta)\varepsilon T_\varepsilon v_\varepsilon^I(y,\eta)\Big[\sum_{k=0}^{N_+} w_\varepsilon^{k'}(0,x,y,\eta) - \sum_{k=0}^{N_-} w_\varepsilon^{-k'}(0,x,y,\eta)\Big] \\ &\quad + \rho(y)\phi(\eta) T_\varepsilon u_\varepsilon^I(y,\eta)\Big[\sum_{k=0}^{N_+} w_\varepsilon^{k}(0,x,y,\eta) + \sum_{k=0}^{N_-} w_\varepsilon^{-k}(0,x,y,\eta)\Big] dy d\eta. \end{aligned}$$

As $\mathrm{dist}(x^0_{\pm k}(y,\eta),\bar{\Omega}) > 0$ for $(y,\eta)\in K_y\times K_\eta$, $k=1,\dots,N_\pm$, $w_\varepsilon^{\pm k^{(\prime)}}(0,x,y,\eta)$ are uniformly exponentially decreasing for $x\in\Omega$ and $(y,\eta)\in K_y\times K_\eta$. Thus, only the incident beams contribute to $u_\varepsilon^R(0,x)$ in Ω and

$$u_\varepsilon^R(0,x) = \varepsilon^{-\frac{3n}{4}} c_n \int_{\mathbb{R}^{2n}} \rho(y)\phi(\eta) T_\varepsilon u_\varepsilon^I(y,\eta) w_\varepsilon^0(0,x,y,\eta) dy d\eta + O(\varepsilon^\infty),$$
$$\text{uniformly w.r.t. } x\in\Omega.$$

The initial values for the phase and the amplitudes of w_ε^0 have been fixed in (24) and (25). Hence

$$u_\varepsilon^R(0,x) = \varepsilon^{-\frac{3n}{4}} c_n \int_{\mathbb{R}^{2n}} \rho(y)\phi(\eta) T_\varepsilon u_\varepsilon^I(y,\eta)\chi_d(x-y) e^{i\phi_0(x,y,\eta)/\varepsilon} dy d\eta + O(\varepsilon^\infty),$$
$$\text{uniformly w.r.t. } x\in\Omega.$$

It follows, uniformly for $x\in\Omega$, that

$$u_\varepsilon^R(0,x) = T_\varepsilon^*\rho\otimes\phi T_\varepsilon u_\varepsilon^I(x) + \varepsilon^{-\frac{3n}{4}} c_n \int_{\mathbb{R}^{2n}} \rho(y)\phi(\eta) T_\varepsilon u_\varepsilon^I(y,\eta)(\chi_d(x-y)-1)$$
$$e^{i\phi_0(x,y,\eta)/\varepsilon} dy d\eta + O(\varepsilon^\infty).$$

One wants to get rid of the second integral by making use of the exponential decrease of $e^{i\phi_0(x,y,\eta)/\varepsilon}$ for $|x-y|\geq d/2$. The following estimate is immediate by Cauchy-Schwartz inequality:

Lemma 3.5. *Let a be a positive real and $h\in L^2(\mathbb{R}^{2n}_{y,\eta})$. Then*

$$\|\int_{|x-y|\geq a} h\mathbf{1}_{K_y\times K_\eta}(y,\eta) e^{-(x-y)^2/(2\varepsilon)} dy d\eta\|_{L^2_x} \lesssim \|h\|_{L^2_{y,\eta}} e^{-a^2/(4\varepsilon)}.$$

The previous Lemma leads to

$$\|u_\varepsilon^R|_{t=0} - T_\varepsilon^*\rho\otimes\phi T_\varepsilon u_\varepsilon^I\|_{L^2(\Omega)} \lesssim \varepsilon^\infty,$$

by using the boundedness of T_ε^* from $L^2(\mathbb{R}^{2n})$ to $L^2(\mathbb{R}^n)$ (this result follows, e.g., from [71] p.97). On the other hand, $\rho\otimes\phi T_\varepsilon u_\varepsilon^I$ approaches $T_\varepsilon u_\varepsilon^I$ up to a small remainder. In fact, as $\rho(y)=1$ if $\mathrm{dist}(y,K)<\Delta$, one has by Lemma 2.4 and assumption A3

$$\|T_\varepsilon u_\varepsilon^I - \rho\otimes\phi T_\varepsilon u_\varepsilon^I\|_{L^2_{y,\eta}} \lesssim \varepsilon^\infty,$$

and consequently

$$\|u_\varepsilon^R|_{t=0} - u_\varepsilon^I\|_{L^2(\Omega)} \lesssim \varepsilon^\infty.$$

Moving to the spatial derivatives of u_ε^R, one has

$$\partial_{x_b} u_\varepsilon^R(0,x) = \varepsilon^{-\frac{3n}{4}} c_n \int_{\mathbb{R}^{2n}} \rho(y)\phi(\eta) T_\varepsilon u_\varepsilon^I(y,\eta) \sum_{j=0}^{N} \varepsilon^j \partial_{x_b} \left[a_j^0(0,x,y,\eta) e^{i\phi_0(x,y,\eta)/\varepsilon} \right] dy d\eta$$
$$+O(\varepsilon^\infty), \text{ uniformly w.r.t. } x \in \Omega.$$

Plugging the initial condition (25) for the incident amplitudes into the previous equation yields a simpler expression

$$\partial_{x_b} u_\varepsilon^R(0,x) = \varepsilon^{-\frac{3n}{4}} c_n \int_{\mathbb{R}^{2n}} \rho(y)\phi(\eta) T_\varepsilon u_\varepsilon^I(y,\eta) \partial_{x_b} \Big(\chi_d(x-y) e^{i\phi_0(x,y,\eta)/\varepsilon} \Big) dy d\eta + O(\varepsilon^\infty),$$
$$\text{uniformly w.r.t. } x \in \Omega.$$

Since $\partial_{x_b}\Big(\chi(x-y)e^{i\phi_0/\varepsilon}\Big) = -\partial_{y_b}\Big(\chi(x-y)e^{i\phi_0/\varepsilon}\Big)$, integration by parts leads to

$$\partial_{x_b} u_\varepsilon^R(0,x) = \varepsilon^{-\frac{3n}{4}} c_n \int_{\mathbb{R}^{2n}} \partial_{y_b} \Big(\rho T_\varepsilon u_\varepsilon^I \Big) \phi \chi_d(x-y) e^{i\phi_0(x,y,\eta)/\varepsilon} dy d\eta + O(\varepsilon^\infty),$$
$$\text{uniformly w.r.t. } x \in \Omega.$$

Application of Lemma 3.5 and then Lemma 2.4 shows that the term involving $\partial_{y_b}\rho$ has an exponentially decreasing contribution in $L^2(\Omega)$. On the other hand, the y derivative of the FBI transform is the FBI transform of the derivative. Thus

$$\|\partial_{x_b} u_\varepsilon^R|_{t=0} - \varepsilon^{-\frac{3n}{4}} c_n \int_{\mathbb{R}^{2n}} \rho \otimes \phi T_\varepsilon \left(\partial_{x_b} u_\varepsilon^I \right) \chi_d(x-y) e^{i\phi_0(x,y,\eta)/\varepsilon} dy d\eta \|_{L^2(\Omega)} \lesssim \varepsilon^\infty.$$

Again, Lemmas 3.5-2.4 and assumption A3 imply

$$\|\partial_{x_b} u_\varepsilon^R|_{t=0} - \partial_{x_b} u_\varepsilon^I\|_{L^2(\Omega)} \lesssim \varepsilon^\infty.$$

Time differentiation of u_ε^R is somewhat different. The contribution of reflected beams is still uniformly exponentially decreasing for $x \in \Omega$

$$\partial_t u_\varepsilon^R|_{t=0}(x) = \varepsilon^{-\frac{3n}{4}} c_n \int_{\mathbb{R}^{2n}} \rho(y)\phi(\eta) T_\varepsilon v_\varepsilon^I(y,\eta) \varepsilon \partial_t w_\varepsilon^{0\prime}(0,x,y,\eta) dy d\eta + O(\varepsilon^\infty),$$

with

$$\varepsilon \partial_t w_\varepsilon^{0\prime} = \sum_{j=0}^{N+1} \varepsilon^j \left(i \partial_t \psi_0 a_j^{0\prime} + \partial_t a_{j-1}^{0\;\prime} \right) e^{i\psi_0/\varepsilon}.$$

The initial values (24) and (26) for the phase and amplitudes of $w_\varepsilon^{0\prime}$ yield

$$\varepsilon \partial_t w_\varepsilon^{0\prime}(0,x,y,\eta) = e^{i\phi_0(x,y,\eta)/\varepsilon} + \sum_{j=0}^{N} \varepsilon^j \sum_{|\alpha|=R-2j-1} (x-y)^\alpha z_\alpha(x,y,\eta) e^{i\phi_0(x,y,\eta)/\varepsilon}$$
$$+ \varepsilon^{N+1} \partial_t a_N^{0\;\prime}(0,x,y,\eta) e^{i\phi_0(x,y,\eta)/\varepsilon},$$

where z_α are smooth remainders that vanish for $|x-y| \geq d$. We can use the operators O^α to estimate the contribution of the terms $(x-y)^\alpha z_\alpha$ to the norm of $u_\varepsilon^R|_{t=0}$

$$\begin{aligned} \|\varepsilon^{-\frac{3n}{4}} \int_{\mathbb{R}^{2n}} \rho \otimes \phi T_\varepsilon v_\varepsilon^I \varepsilon^j (x-y)^\alpha z_\alpha e^{i\phi_0/\varepsilon} dy d\eta \|_{L_x^2} &= \varepsilon^{-\frac{3n}{4}+j} \|O^\alpha \left(\rho \otimes \phi z_\alpha, \phi_0/\varepsilon \right) T_\varepsilon v_\varepsilon^I \|_{L_x^2} \\ &\lesssim \varepsilon^{\frac{R-1}{2}}, \text{ for } j = 0, \dots, N. \end{aligned}$$

We also have

$$\begin{aligned}\|\varepsilon^{-\frac{3n}{4}}\int_{\mathbb{R}^{2n}}\rho\otimes\phi T_\varepsilon v_\varepsilon^I\varepsilon^{N+1}\partial_t a_N^0{}'|_{t=0}e^{i\phi_0/\varepsilon}dyd\eta\|_{L_x^2}\\ &= \varepsilon^{-\frac{3n}{4}+N+1}\|O^0\left(\rho\otimes\phi\partial_t a_N^0{}'|_{t=0},\phi_0/\varepsilon\right)T_\varepsilon v_\varepsilon^I\|_{L_x^2}\\ &\lesssim \varepsilon^{N+1}.\end{aligned}$$

It follows, with the help of (14), that

$$\|\partial_t u_\varepsilon^R|_{t=0}-T_\varepsilon^*\rho\otimes\phi T_\varepsilon v_\varepsilon^I\|_{L^2(\Omega)} \lesssim \varepsilon^{\frac{R-1}{2}},$$

and finally, from Lemma 2.4 and assumption A3,

$$\|\partial_t u_\varepsilon^R|_{t=0}-v_\varepsilon^I\|_{L^2(\Omega)}\lesssim\varepsilon^{\frac{R-1}{2}}.$$

Hence

$$\|\partial_t u_\varepsilon^R|_{t=0}-v_\varepsilon^I\|_{L^2(\Omega)}+\|u_\varepsilon^R|_{t=0}-u_\varepsilon^I\|_{H^1(\Omega)}\lesssim\varepsilon^{\frac{R-1}{2}}.$$

3.3 Proof of the main theorem

Now we may collect the previous estimates in order to bound the difference between u_ε the exact solution for (1) and u_ε^R the approximate solution of order R.

For the Dirichlet case, the errors in the interior, at the boundary and in the initial conditions exhibit the same scale of ε, and the energy estimate leads to

$$\underset{t\in[0,T]}{Sup}\|u_\varepsilon(t,.)-u_\varepsilon^R(t,.)\|_{H^1(\Omega)}\lesssim\varepsilon^{\frac{R-1}{2}},\ \underset{t\in[0,T]}{Sup}\|\partial_t u_\varepsilon(t,.)-\partial_t u_\varepsilon^R(t,.)\|_{L^2(\Omega)}\lesssim\varepsilon^{\frac{R-1}{2}}. \quad (39)$$

For the Neumann case, one looses an order $\sqrt{\varepsilon}$ in the boundary estimate, and thus the energy estimate yields

$$\underset{t\in[0,T]}{Sup}\|u_\varepsilon(t,.)-u_\varepsilon^R(t,.)\|_{H^1(\Omega)}\lesssim\varepsilon^{\frac{R-2}{2}},\ \underset{t\in[0,T]}{Sup}\|\partial_t u_\varepsilon(t,.)-\partial_t u_\varepsilon^R(t,.)\|_{L^2(\Omega)}\lesssim\varepsilon^{\frac{R-2}{2}}.$$

However, when comparing the ansatz at order R and $R+1$ in the difference between u_ε^{R+1} and u_ε^R, we can make use of further powers of $((x-x_k^t)^\alpha)_{|\alpha|=R+1}$ between the phases and $((x-x_k^t)^\alpha)_{|\alpha|=R-2j-1}$ in the amplitudes. Using the approximation operators yields uniformly in time

$$\|u_\varepsilon^{R+1}(t,.)-u_\varepsilon^R(t,.)\|_{H^1(\Omega)}\lesssim\varepsilon^{\frac{R-1}{2}},\ \|\partial_t u_\varepsilon^{R+1}(t,.)-\partial_t u_\varepsilon^R(t,.)\|_{L^2(\Omega)}\lesssim\varepsilon^{\frac{R-1}{2}}.$$

Hence one may improve the estimate for the Neumann case by using the approximate solution at the next order $R+1$

$$\begin{aligned}\|u_\varepsilon(t,.)-u_\varepsilon^R(t,.)\|_{H^1(\Omega)} &\lesssim \|u_\varepsilon(t,.)-u_\varepsilon^{R+1}(t,.)\|_{H^1(\Omega)}+\|u_\varepsilon^{R+1}(t,.)-u_\varepsilon^R(t,.)\|_{H^1(\Omega)}\\ &\lesssim \varepsilon^{\frac{R-1}{2}}.\end{aligned}$$

This leads to the same estimate (39) for the Neumann case.

Remark 3.6. *The FBI transforms of u_ε^I and v_ε^I are uniformly locally infinitely small outside the frequency sets $Fs(u_\varepsilon^I)$ and $Fs(v_\varepsilon^I)$ respectively, as ε tends to 0 (see [71] p.98). These sets may be phase space submanifolds of lower dimensions. For instance, for WKB initial data, $Fs(ae^{i\Phi/\varepsilon}) = \{(y, \partial_x \Phi(y)), y \in \mathrm{supp} a\}$. For numerical computations, one has therefore to discretize neighbourhoods of $(K_y \times K_\eta) \cap Fs(u_\varepsilon^I)$ and $(K_y \times K_\eta) \cap Fs(v_\varepsilon^I)$. Studying numerically the behaviour of FBI transforms in the associated computational domains could lead to interesting results on the optimal mesh size. Details on numerical FBI transforms are given in [60].*

Chapter II

Wigner measures for the wave equation in a convex domain

Contents

1 Introduction

In this article, we are interested in the high frequency limit of the initial-boundary value problem (IBVP) for the wave equation

$$Pu_\varepsilon = \partial_t^2 u_\varepsilon - \partial_x.(c^2(x)\partial_x u_\varepsilon) = 0 \text{ in } [0,T]\times\Omega, \tag{1a}$$

$$Bu_\varepsilon = 0 \text{ in } [0,T]\times\partial\Omega, \tag{1b}$$

$$u_\varepsilon|_{t=0} = u_\varepsilon^I,\ \partial_t u_\varepsilon|_{t=0} = v_\varepsilon^I \text{ in } \Omega, \tag{1c}$$

where B stands for a Dirichlet or Neumann type boundary operator.

Above, $T > 0$ is fixed, and Ω is a bounded domain of $\mathbb{R}^n$ with a $\mathcal{C}^\infty$ boundary. The coefficient c is assumed to be in $\mathcal{C}^\infty(\bar{\Omega})$, though this assumption may be relaxed.

Herein, the initial data depend on a small parameter $\varepsilon > 0$, playing the role of a small wavelength, the high frequency limit corresponding to $\varepsilon \to 0$. In any case, we shall assume that u_ε^I, v_ε^I are

A1. uniformly bounded respectively in $H^1(\Omega)$ and $L^2(\Omega)$,

A2. uniformly supported in a fixed compact set of Ω.

We shall assume that the following hypotheses holds on the domain Ω:

B1. Ω is convex with respect to the bicharacteristics of the wave operator, that is every ray originating from Ω hits the boundary twice and transversally,

B2. No ray remains in a compact of $\mathbb{R}^n$ for increasing times,

B3. The boundary has no dead-end trajectories, that is infinite number of successive reflections cannot occur in a finite time.

These geometric hypotheses insure that the rays starting from the compact support of the initial data do not face diffraction on the boundary, neither do they glide along $\partial\Omega$. The only phenomena occuring at the boundary is reflection according to geometrical optics laws.

We investigate the high frequency limit in terms of Wigner measures. The Wigner function is a phase space distribution introduced by E. Wigner [100] in 1932 to study quantum corrections to classical statistical mechanics. In the 90's, mathematicians became increasingly interested by the Wigner transforms and related measures. In [62, 68, 69, 70], those transforms are applied to the semiclassical limit of Schrödinger equations. A general theory for their use in the homogenization of energy densities of dispersive equations was laid out by Gérard et al. in [38], see also [34, 35]. Wigner measures are related to the H-measures and microlocal defect measures introduced in [98] and [36], see also [4, 12]. Whereas there is no notion of scale for the latter measures, Wigner transforms are associated to a parameter $\varepsilon \to 0$. In quantum mechanics, this parameter is the rescaled Planck constant, while it will be the distance between two points of the medium's periodic structure for homogenisation problems.

The Wigner transform, at the scale ε, is defined for a given sequence $(a_\varepsilon, b_\varepsilon)$ in $\mathcal{S}'(\mathbb{R}^n)^p \times \mathcal{S}'(\mathbb{R}^n)^p$ by the duality weak formula

$$w_\varepsilon(a_\varepsilon, b_\varepsilon)(x,\xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-iv\cdot\xi} a_\varepsilon(x + \frac{\varepsilon}{2}v) b_\varepsilon^*(x - \frac{\varepsilon}{2}v) dv.$$

If (a_ε) is uniformly bounded w.r.t. ε in $L^2(\mathbb{R}^n)^p$, then $w_\varepsilon[a_\varepsilon] = w_\varepsilon(a_\varepsilon, a_\varepsilon)$ converges as ε goes to 0 to a positive hermitian matrix measure in $\mathcal{S}'(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ (modulo the extraction of a subsequence). This measure is called a Wigner measure associated to (a_ε) and denoted $w[a_\varepsilon]$. The Wigner measures associated to the solution of the wave equation (and hyperbolic problems in general, see e.g. [38],[79]) are related to the energy density in the high frequency limit. More precisely, under suitable hypotheses, the density of energy converges in the sense of measures to (proposition 1.7 in [38])

$$\frac{1}{2} \int_{\mathbb{R}^n} e^t(x, d\xi),$$

where

$$e^t = w[\underline{\partial_t u}_\varepsilon(t,.)] + \mathrm{Tr} w[c\underline{\partial_x u}_\varepsilon(t,.)].$$

Above, the involved Wigner measures are obtained after extending $\partial_t u_\varepsilon$ and $c\partial_{x_b} u_\varepsilon$, $b = 1, \ldots, n$, to functions of $L^2(\mathbb{R}^n)$ by setting $\underline{\partial_t u}_\varepsilon = \mathbf{1}_\Omega \partial_t u_\varepsilon$, $\underline{\partial_x u}_\varepsilon = \mathbf{1}_\Omega \partial_x u_\varepsilon$ and extending c outside $\bar{\Omega}$ in a smooth way.

Wigner measures for the wave equation have been studied by Miller [74] who proved refraction results for sharp interfaces and Burq [11] who described their support for a Dirichlet boundary condition. Similar results have been established for other problems [26, 31], in particular eigenfunctions for the Dirichlet problem [37, 102] and for the Neumann and Robin problems [13]. All these works are based on pseudo-differential calculus.

In this paper, we shall investigate the Wigner measure by means of direct computations on an approximate solution of the IBVP for the wave equation. The approximate or asymptotic solutions used here are obtained by superposition (or mixing) of Gaussian beams, and more precisely by a weighted integral of Gaussian beams suitably designed to fit initial data as in chapter 1.

Gaussian beams are waves with a Gaussian shape at any instant, localized near a single ray [5, 84]. The summation of different beams allows to approach non localized wave fields, see e.g. [18, 51, 54] and the recent [42, 64, 76, 96]. Gaussian beams (or the related coherent states) can be treated as a basis of fundamental solutions of wave motion and used to study general solutions of partial differential equations. They hence allowed amongst others to describe propagation of singularities [84], to prove lack of observability [67] and to study semiclassical measures [80] and trace formulaes [22, 101].

This feature seems to be very well suited for the study of Wigner measures. Indeed, the Wigner transform of two different beams vanishes when ε goes to zero. Even better, the Wigner measure of one individual Gaussian beam associated to the wave equation is a Dirac mass localized on the corresponding bicharacteristic. Thus Gaussian beams form a sort of an orthogonal family for the Wigner measure. The appealing to these elementary solutions for studying Wigner measures is not new; they have been used in the whole space domain by Robinson [88] for the Schrödinger equation and more recently by Castella [15] who used a coherent states approach for the Helmholtz equation.

In view of known results, one expects that the Wigner measure of a summation of Gaussian beams would give easily that the associated weights are transported along the broken bicharacteristic flow (see p.66 for the construction of reflected flows and p.90 for the definition of the broken flow). Unfortunately this result is not immediate as even different beams become infinitely close to each other.

We show however by elementary computations that this intuition is indeed true and that the Wigner measure of the considered approximate solution is transported along the broken bicharacterisitic flow. Since the asymptotic solution is close to the exact one, we may deduce the same outcome for the Wigner measure e^t. In particular, we shall prove the following theorem

Theorem 1.1. *Set* $\underline{v}^I_\varepsilon = \mathbf{1}_\Omega v^I_\varepsilon$ *and* $\underline{u}^I_\varepsilon = \mathbf{1}_\Omega u^I_\varepsilon$*. Assume the conditions A1 and A2 fulfilled, and furthermore that:*

C1. The Wigner measures of $\underline{v}^I_\varepsilon$ *and* $\partial_{x_b}\underline{u}^I_\varepsilon$*,* $b = 1, \ldots, n$*, are unique,*

C2. $\underline{v}^I_\varepsilon$ *and* $\partial_{x_b}\underline{u}^I_\varepsilon$*,* $b = 1, \ldots, n$ *are* ε*-oscillatory (see equation* (53)*),*

C3. The Wigner measures of $\underline{v}^I_\varepsilon$ *and* $\partial_{x_b}\underline{u}^I_\varepsilon$*,* $b = 1, \ldots, n$ *do not charge the set* $\mathbb{R}^n \times \{\xi = 0\}$*.*

Let $e^\pm = w[\underline{v}^I_\varepsilon \pm ic|\mathbf{D}|\underline{u}^I_\varepsilon]$*, and denote by* φ^t_b *the broken bicharacteristic flow associated to* $-i\partial_t - c|\mathbf{D}|$ *obtained after successives reflections on the boundary* $\partial\Omega$*. Then*

$$e^t = \frac{1}{2}\Big(e^+ o(\varphi_b^{-t})^{-1} + e^- o(\varphi_b^t)^{-1}\Big) \text{ in } \Omega \times (\mathbb{R}^n \backslash \{0\}).$$

For general properties of Wigner measures and transforms, we will refer to the usual framework [38]. The rest of the paper is organized as follows. In the first section, we recall the construction of first order Gaussian beams and the structure of the asymptotic solutions obtained as an infinite sum of such beams. The derivatives of the asymptotic solutions are then expressed using what we call Gaussian integrals. We simplify the expression of the Wigner transform of such integrals in section 3, following initial computations of [88] in the Schrödinger case. We then compute the scalar Wigner measure for the asymptotic solution by exploiting the expressions of the beams' phases and amplitudes and using the dominated convergence theorem. Finally, we prove the propagation of the Wigner measure along the broken flow for the exact solution of the IBVP (1) with the help of assumptions C2 and C3 on the initial data.

A few useful notations will be used hereafter. The inner product of two vectors $a, b \in \mathbb{R}^d$ will be denoted by $a \cdot b$. The transpose of a matrix A will be noted A^T. If E is a subset of $\mathbb{R}^d$, we denote E^c its complementary and $\mathbf{1}_E$ its characteristic function. For a function $f \in L^2(\Omega)$, we denote $\underline{f} = \mathbf{1}_\Omega f$. For $r > 0$, χ_r denotes a cut-off of $\mathcal{C}^\infty_0(\mathbb{R}^n, [0, 1])$ satisfying

$$\chi_r(x) = 1 \text{ if } |x| \leq r/2 \text{ and } \chi_r(x) = 0 \text{ if } |x| \geq r.$$

We use the following definition of the Fourier transform

$$\mathcal{F}_x u(\xi) = \int_{\mathbb{R}^d} u(x) e^{-ix\cdot\xi} dx \text{ for } u \in L^2(\mathbb{R}^d).$$

If no confusion is possible, we shall omit the reference to the lower index x.
For a smooth function $f \in \mathcal{C}^\infty(\mathbb{R}^d_x, \mathbb{C})$, we will use the notation $\partial_x f$ to denote its gradient vector $(\partial_{x_b} f)_{1\le b\le d}$ and $\partial_x^2 f$ to denote its Hessian matrix $(\partial_{x_b}\partial_{x_c} f)_{1\le b,c\le d}$. For a function $F \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{C}^p)$, the notation DF is used for its Jacobian matrix. We use the letter C to denote a positive constant (different each time it appears). We specify the parameters some constants depend on by denoting them $C(V)$, where V may be a variable or a set of $\mathbb{R}^d$.
For y_ε and z_ε sequences of $\mathbb{R}_+$ with $\varepsilon \in]0, \varepsilon_0]$, we use the notation $y_\varepsilon \lesssim z_\varepsilon$ if there exists a constant $C > 0$ independent of ε such that $y_\varepsilon \le C z_\varepsilon$ for ε small enough. We write $y_\varepsilon \lesssim \varepsilon^\infty$ or $y_\varepsilon = O(\varepsilon^\infty)$ if for any $s \ge 0$ there exists $c_s > 0$ s.t. for ε small enough $y_\varepsilon \le c_s \varepsilon^s$.
Finally, if E is in an open subset of $\mathbb{R}^{2n}$ and ν_ε, ν'_ε are two distributions s.t.

$$\lim_{\varepsilon\to 0}(\nu_\varepsilon - \nu'_\varepsilon) = 0 \text{ in } E,$$

we shall write

$$\nu_\varepsilon \approx \nu'_\varepsilon \text{ in } E.$$

2 Asymptotic solution

In this section, we explain the notion of Gaussian beam for the wave equation focusing on first order beams. We then construct the asymptotic solution as a superposition of these beams and express its time and spatial derivatives with the help of Gaussian integrals.

2.1 First order Gaussian beams

We recall the construction of individual first order Gaussian beams in section 2.1.1, and apply it to describe the incident beam and the reflected beams in section 2.1.2. A useful general relation linking reflected beams' phases to the phase of an incident beam is given for first order and higher order beams.

2.1.1 Beams in the whole space

Denote $h_+(x,\xi) = c(x)|\xi|$ and let (x^t, ξ^t) be a Hamiltonian flow for h_+, that is a solution of the system

$$\frac{dx^t}{dt} = \partial_\xi h_+(x^t,\xi^t) = c(x^t)\frac{\xi^t}{|\xi^t|}, \quad \frac{d\xi^t}{dt} = -\partial_x h_+(x^t,\xi^t) = -\partial_x c(x^t)|\xi^t|.$$

The curves $(t, x^{\pm t})$ of $\mathbb{R}^{n+1}$ are called the rays of P.

An individual first order (Gaussian) beam for the wave equation associated to a ray (t, x^t) has the following form

$$\omega_\varepsilon(t,x) = a_0(t,x)e^{i\psi(t,x)/\varepsilon}, \tag{2}$$

with a complex phase function ψ real-valued on (t, x^t) and an amplitude function a_0 null outside a neighbourhood of (t, x^t). It satisfies

$$\sup_{t\in[0,T]} \|P\omega_\varepsilon(t,.)\|_{L^2(\Omega)} = O(\varepsilon^m),$$

for some $m > 0$.

The construction is achieved by making the amplitudes of $P\omega_\varepsilon$ vanish on the ray up to fixed suitable orders [50, 67, 84]

$$P\omega_\varepsilon = \Big(\varepsilon^{-2}p(x, \partial_t\psi, \partial_x\psi)a_0 + \varepsilon^{-1}i[2\partial_t\psi\partial_t a_0 - 2c^2\partial_x\psi\partial_x a_0 + P\psi a_0] + h.o.t.\Big)e^{i\psi/\varepsilon}, \quad (3)$$

where $p(x, \tau, \xi) = c^2(x)|\xi|^2 - \tau^2$ is the principal symbol of P. The first equation is then the eikonal equation

$$p(x, \partial_t\psi(t, x), \partial_x\psi(t, x)) = 0, \quad (4)$$

on $x = x^t$ up to order 2 (see Remark 2.1 in chapter 1 for an explanation of the choice of this specific order), which means

$$\partial_x^\alpha[p(x, \partial_t\psi(t, x), \partial_x\psi(t, x))]|_{x=x^t} = 0 \text{ for } |\alpha| \le 2.$$

Orders 0 and 1 of the previous equation are fulfilled on the ray by setting

$$\partial_t\psi(t, x^t) = -h_+(x^t, \xi^t) \text{ and } \partial_x\psi(t, x^t) = \xi^t. \quad \text{(P.a)}$$

It follows, by choosing

$$\psi(0, x^0) \text{ a real quantity,}$$

that

$$\psi(t, x^t) \text{ is real.} \quad \text{(P.b)}$$

Order 2 of eikonal (4) on the ray may be written as a Riccati equation

$$\frac{d}{dt}\partial_x^2\psi(t, x^t) + H_{21}(x^t, \xi^t)\partial_x^2\psi(t, x^t) + \partial_x^2\psi(t, x^t)H_{12}(x^t, \xi^t) + \partial_x^2\psi(t, x^t)H_{22}(x^t, \xi^t)\partial_x^2\psi(t, x^t) + H_{11}(x^t, \xi^t) = 0, \quad (5)$$

where $H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}$ is the Hessian matrix of h_+. Although non-linear, this Riccati equation has a unique global symmetric solution which satisfies the fundamental property

$$\operatorname{Im} \partial_x^2\psi\left(t, x^t\right) \text{ is positive definite,} \quad \text{(P.c)}$$

given an initial symmetric matrix $\partial_x^2\psi\,(0, x^0)$ with a positive definite imaginary part (see the proof of Lemma 2.56 p.101 in [50]).

The phase is defined beyond the ray as a polynomial of order 2 with respect to (w.r.t.) $(x - x^t)$ [97]

$$\psi(t, x) = \psi(t, x^t) + \xi^t\cdot(x - x^t) + \frac{1}{2}(x - x^t)\cdot\partial_x^2\psi(t, x^t)(x - x^t). \quad (6)$$

Next, we make the term associated to the power ε^{-1} in the expansion (3) vanish on (t, x^t)

$$2\partial_t\psi\partial_t a_0 - 2c^2\partial_x\psi\partial_x a_0 + P\psi a_0 = 0 \text{ on } (t, x^t), \quad (7)$$

which leads to a linear ordinary differential equation (ODE) on $a_0(t, x^t)$. The amplitude is then chosen under the form

$$a_0(t, x) = \chi_d(x - x^t) a_0(t, x^t),$$

where $d > 0$ will be fixed later. The constructed beams are thus defined for all $(t, x) \in \mathbb{R}^{n+1}$ and they satisfy the estimate

$$\|\varepsilon^{-\frac{n}{4}+1} P\omega_\varepsilon(t, .)\|_{L^2(\Omega)} = O(\sqrt{\varepsilon}) \text{ uniformly w.r.t. } t \in [0, T].$$

Gaussian beams for P associated to the ray (t, x^{-t}) are $\omega_\varepsilon(-t, x)$.

2.1.2 Incident and reflected beams in a convex domain

Construction of flows and beams We suppose $c(x)$ constant for $\operatorname{dist}(x, \bar{\Omega})$ larger than some $D > 0$. Given a point (y, η) in the phase space $\overset{o}{T^*}\mathbb{R}^n$ where

$$\overset{o}{T^*}U \text{ denotes } U \times (\mathbb{R}^n \backslash \{0\}) \text{ if } U \text{ is an open set of } \mathbb{R}^n,$$

an incident beam is a beam associated to the ray $(t, x_0^t(y, \eta))$ satisfying:

$$\frac{dx_0^t}{dt} = c(x_0^t)\frac{\xi_0^t}{|\xi_0^t|}, \quad \frac{d\xi_0^t}{dt} = -\partial_x c(x_0^t)|\xi_0^t|,$$

$$x_0^t|_{t=0} = y, \qquad \xi_0^t|_{t=0} = \eta, \eta \neq 0.$$

The Hamiltonian flow $\varphi_0^t = (x_0^t, \xi_0^t)$ for h_+ is called an incident flow. The associated beam is denoted ω_ε^0 and called an incident beam.
Since we have dependence w.r.t. the initial conditions (y, η), we shall write the incident beam as

$$\omega_\varepsilon^0(t, x, y, \eta) = a_0(t, x, y, \eta) e^{i\psi_0(t,x,y,\eta)/\varepsilon}.$$

Let $\mathcal{R}$ be the reflection involution

$$\begin{aligned} \mathcal{R}: \ \overset{o}{T^*}\mathbb{R}^n|_{\partial\Omega} &\to \overset{o}{T^*}\mathbb{R}^n|_{\partial\Omega} \\ (X, \Xi) &\mapsto (X, (Id - 2\nu(X)\nu(X)^T)\Xi). \end{aligned}$$

Above ν denotes the exterior normal field to $\partial\Omega$. We restrain the study to starting points $(y, \eta) \in \mathcal{B} = \cup_{t \in \mathbb{R}} \varphi_0^t(\overset{o}{T^*}\Omega)$. Each associated flow $\varphi_0^t(y, \eta)$ strikes the boundary twice. Reflection of $\varphi_0^t(y, \eta)$ at the exit time $t = T_1(y, \eta)$ s.t.

$$x_0^{T_1(y,\eta)}(y, \eta) \in \partial\Omega \text{ and } \dot{x}_0^{T_1(y,\eta)}(y, \eta) \cdot \nu(x_0^{T_1(y,\eta)}(y, \eta)) > 0,$$

gives birth to the reflected flow $\varphi_1^t(y, \eta) = (x_1^t(y, \eta), \xi_1^t(y, \eta))$ defined by the condition

$$\varphi_1^{T_1(y,\eta)}(y, \eta) = \mathcal{R}\, o \varphi_0^{T_1(y,\eta)}(y, \eta).$$

Similarly, we also define the reflection time $T_{-1}(y, \eta)$ and the flow $\varphi_{-1}^t(y, \eta)$ by reflecting $\varphi_0^t(y, \eta)$ as follows

$$\begin{aligned} &x_0^{T_{-1}(y,\eta)}(y, \eta) \in \partial\Omega \text{ and } \dot{x}_0^{T_{-1}(y,\eta)}(y, \eta) \cdot \nu(x_0^{T_{-1}(y,\eta)}(y, \eta)) < 0, \\ &\varphi_{-1}^{T_{-1}(y,\eta)}(y, \eta) = \mathcal{R}\, o \varphi_0^{T_{-1}(y,\eta)}(y, \eta). \end{aligned}$$

We denote, for $k = \pm 1$, the reflected beams by

$$\omega_\varepsilon^k(t,x,y,\eta) = a_0^k(t,x,y,\eta)e^{i\psi_k(t,x,y,\eta)/\varepsilon}.$$

These beams are associated to the reflected bicharacteristics φ_k^t. Let us introduce, for $k = 0, \pm 1$, the boundary amplitudes $d_{-m_B+j}^k$ s.t.

$$B\omega_\varepsilon^k = \sum_{j=0}^{m_B} \varepsilon^{-m_B+j} d_{-m_B+j}^k e^{i\psi_k/\varepsilon}.$$

Above, m_B denotes the order of B ($m_B = 0$ for Dirichlet and $m_B = 1$ for Neumann). The construction of the reflected phases and amplitudes is achieved by imposing that the time and tangential derivatives of

$$\psi_k \text{ equal at } (T_k, x_0^{T_k}) \text{ those of } \psi_0 \text{ up to order 2,} \tag{8}$$
$$d_{-m_B}^0 + d_{-m_B}^k \text{ vanish at } (T_k, x_0^{T_k}), \tag{9}$$

for $k = \pm 1$. These constraints *uniquely* determine the reflected phases and amplitudes, once the incident ones are fixed [84]. Moreover, if T is sufficiently small, the reflected rays is in the interior of the domain at the instant T ($x_{\pm 1}^{\pm T}(y,\eta) \in \Omega$), and the following boundary estimates are satisfied [84]

$$\|B(\varepsilon^{-\frac{n}{4}+1}\omega_\varepsilon^0(.,y,\eta) + \varepsilon^{-\frac{n}{4}+1}\omega_\varepsilon^1(.,y,\eta))\|_{H^s([0,T]\times\partial\Omega)} = O(\varepsilon^{-m_B-s+\frac{3}{2}}),$$
$$\text{and } \|B(\varepsilon^{-\frac{n}{4}+1}\omega_\varepsilon^0(.,y,\eta) + \varepsilon^{-\frac{n}{4}+1}\omega_\varepsilon^{-1}(.,y,\eta))\|_{H^s([-T,0]\times\partial\Omega)} = O(\varepsilon^{-m_B-s+\frac{3}{2}}) \text{ for } s \geq 0.$$

Let us point out that the construction of the reflected beams is also valid for other boundary conditions if the IBVP is well-posed ([84] p.221).

Remark 2.1. *Higher order beams, possibly with more than one amplitude, can be constructed to satisfy better interior and boundary estimates. In this case, the eikonal equation* (4) *must be satisfied up to order larger that* 2 *on the rays. If* $r \geq 3$, *the equations* $\partial_x^\alpha \left[p(x,\partial_t\psi,\partial_x\psi)\right](t,x^t) = 0$ *for* $|\alpha| = r$ *give systems of linear ODEs on* $(\partial_x^\alpha\psi(t,x^t))_{|\alpha|=r}$ *with a second member involving lower order derivatives of the phase. The key observation to prove this statement is to replace each term*

$$\partial_\tau p(\varphi^t)\partial_x^\alpha\partial_t\psi(t,x^t) + \partial_\xi p(\varphi^t)\cdot\partial_x^\alpha\partial_x\psi(t,x^t),\ |\alpha| = r,$$

by $2c(x^t)|\xi^t|\frac{d}{dt}\left(\partial_x^\alpha\psi(t,x^t)\right)$. *We refer to [84] for further details.*

2.1.3 General relation between incident and reflected beams' phases

In this paragraph, we give a useful relation between an incident phase ψ_{inc} and a reflected phase ψ_{ref} for beams of any order. This relation provides with the derivatives of the reflected phase up to order R, which may be useful in other applications of Gaussian beams. Here we will apply the obtained results for first order beams to compute the Hessian matrices of $\psi_{\pm 1}$ on the rays. The matrices $\psi_{\pm 1}(t, x_{\pm 1}^{\pm t})$ can also be computed by solving the Riccati equations with the proper values at the instants of reflections $t = T_{\pm 1}$ (see eg. [78]).

Consider the following auxiliary function linking φ_1^t to φ_0^t for any fixed time t

$$\begin{array}{llll} s_1: & \mathcal{B} & \rightarrow & \mathcal{B} \\ & (y,\eta) & \mapsto & \varphi_0^{-T_1(y,\eta)}\, o\, \mathcal{R}\, o\, \varphi_0^{T_1(y,\eta)}(y,\eta). \end{array}$$

For a given point $(y,\eta) \in \mathcal{B}$, $s_1(y,\eta)$ is its "image by the mirror" $\partial\Omega$. For instance, Chazarain used this type of auxiliary functions in [20] to show propagation of regularity for wave type equations in a convex domain.

By the Implicit functions theorem, T_1 is $\mathcal{C}^\infty$ on the open set $\mathcal{B}$ and so is s_1. Since $\varphi_0^t\ o\ s_1$ satisfies the same Hamiltonian equations as φ_1^t and $\varphi_1^{T_1(y,\eta)}(y,\eta) = \varphi_0^{T_1(y,\eta)}\ o\ s_1(y,\eta)$ for $(y,\eta) \in \mathcal{B}$, one has

$$\varphi_1^t = \varphi_0^t\ o\ s_1.$$

Besides, noting that $T_1(\varphi_0^t) = T_1 - t$, one also has

$$\varphi_1^t = s_1\ o\varphi_0^t. \tag{10}$$

φ_0^t and φ_1^t are symplectic $\mathcal{C}^\infty$ diffeomorphisms from $\mathcal{B}$ to $\mathcal{B}$ [48], and so is s_1. One can define a similar auxiliary function $s_{-1} : \mathcal{B} \rightarrow \mathcal{B}$ s.t. $\varphi_{-1}^t = \varphi_0^t\ o\ s_{-1}$ and $\varphi_{-1}^t = s_{-1}\ o\varphi_0^t$ for $t \in \mathbb{R}$.

Let us introduce the components of s_1 as

$$s_1 = (r, \lambda).$$

For every functions $f, g \in \mathcal{C}^\infty\left(\mathbb{R}_u^n \times (\mathbb{R}_\xi^n \backslash \{0\}), \mathbb{C}^p\right)$ and phase function $V \in \mathcal{C}^\infty(\mathbb{R}_u^n, \mathbb{C}_\xi^n)$ s.t. $V(u_0) \in \mathbb{R}_\xi^n \backslash \{0\}$, we introduce the notation

$$f(u, V(u)) \underset{u=u_0}{\overset{m}{\asymp}} g(u, V(u)),$$

to denote that the formal derivatives of $f(u,V(u))$ and $g(u,V(u))$ up to order m coincide on u_0. The derivation here is viewed formally, since V may be complex valued out of u_0, which makes $f(u,V(u))$ and $g(u,V(u))$ not defined for $u \neq u_0$. However, on the exact point u_0, one can always use the formulae of composite functions' derivatives to get a formal expression of the derivatives. We will use the same notation

$$f(t, x, V(t,x)) \underset{x=x^t}{\overset{m}{\asymp}} g(t, x, V(t,x)),$$

for functions $f, g \in \mathcal{C}^\infty\left(\mathbb{R}_t \times \mathbb{R}_x^n \times (\mathbb{R}_\xi^n \backslash \{0\}), \mathbb{C}^p\right)$ and phase function $V \in \mathcal{C}^\infty(\mathbb{R}_t \times \mathbb{R}_x^n, \mathbb{C}_\xi^n)$ s.t. for $t \in \mathbb{R}$, $V(t, x^t) \in \mathbb{R}_\xi^n \backslash \{0\}$ to denote that the formal derivatives of $f(t,x,V(t,x))$ and $g(t,x,V(t,x))$ w.r.t. x up to order m coincide on (t, x^t) for $t \in \mathbb{R}$.

In the following, we will be sloppy with respect to the notation of the dependence of $\partial_t\psi_0$ and $\partial_x\psi_0$ on their variables (t,x).

Since the reflection $\mathcal{R}$ conserves $|\Xi|$, one has for every $(x,\xi) \in \mathcal{B}$ and $\tau \in \mathbb{R}^*$

$$p(r(x,\xi), \tau, \lambda(x,\xi)) = p(x, \tau, \xi). \tag{11}$$

Thus

$$p(r(x, \partial_x\psi_0), \partial_t\psi_0, \lambda(x, \partial_x\psi_0)) \underset{x=x_0^t}{\overset{\infty}{\asymp}} p(x, \partial_t\psi_0, \partial_x\psi_0),$$

which implies, by construction of ψ_0

$$p(r(x,\partial_x\psi_0),\partial_t\psi_0,\lambda(x,\partial_x\psi_0)) \underset{x=x_0^t}{\overset{2}{\asymp}} 0.$$

Compare this with the equation

$$p(r(x,\partial_x\psi_0),\partial_t\psi_1(t,r(x,\partial_x\psi_0)),\partial_x\psi_1(t,r(x,\partial_x\psi_0))) \underset{x=x_0^t}{\overset{2}{\asymp}} 0,$$

resulting from the construction of ψ_1 and (10). This suggests the following Lemma

Lemma 2.2. *Let R be an integer larger than 1 and ψ_{inc} and ψ_{ref} an incident and a reflected phase of $\mathcal{C}^\infty(\mathbb{R}_t\times\mathbb{R}^n_x,\mathbb{C})$ satisfying*

$$\partial_t\psi_{inc}(t,x_0^t) = -c(x_0^t)|\xi_0^t| \text{ and } \partial_t\psi_{ref}(t,x_1^t) = -c(x_1^t)|\xi_1^t|,$$
$$\partial_x\psi_{inc}(t,x_0^t) = \xi_0^t \text{ and } \partial_x\psi_{ref}(t,x_1^t) = \xi_1^t,$$
$$p(x,\partial_t\psi_{inc},\partial_x\psi_{inc}) \underset{x=x_0^t}{\overset{R}{\asymp}} 0 \text{ and } p(x,\partial_t\psi_{ref},\partial_x\psi_{ref}) \underset{x=x_1^t}{\overset{R}{\asymp}} 0,$$

and having the same time and tangential derivatives at the instant and the point of reflection $(T_1,x_0^{T_1})$ up to the order R. Then $\partial_t\psi_{ref}(t,r(x,\partial_x\psi_{inc})) \underset{x=x_0^t}{\overset{R-1}{\asymp}} \partial_t\psi_{inc}$ and $\partial_x\psi_{ref}(t,r(x,\partial_x\psi_{inc})) \underset{x=x_0^t}{\overset{R-1}{\asymp}} \lambda(x,\partial_x\psi_{inc})$.

The proof is postponed to Appendix A. A similar result linking the reflected phase associated to the ray (t,x_{-1}^{-t}) to the incident phase can be established.

2.2 Gaussian beams summation

We begin this section by a reminder of the construction of asymptotic solutions to the IBVP (1a), (1b) with some initial conditions (1c') (see below). These solutions are obtained by a Gaussian beams summation as achieved in chapter 1. We focus on a superposition of first order beams, for which exact expressions of the phases and amplitudes are displayed in 2.2.2. These beams lead to a first order approximate solution, close to the exact one up to $\sqrt{\varepsilon}$. We end the section approaching the derivatives of the first order solution by some Gaussian type integrals we introduce.

2.2.1 Construction of the approximate solution

In chapter 1, we have constructed a family of asymptotic solutions to the IBVP for the wave equation for initial data satisfying A1, A2 and an additionnal hypothesis A3 concerning their FBI transforms.

Let us recall here that the FBI transform (see [71]) is, for a given scale ε, the operator $T_\varepsilon : L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^{2n})$ defined by

$$T_\varepsilon(a)(y,\eta) = c_n\varepsilon^{-\frac{3n}{4}}\int_{\mathbb{R}^n} a(x)e^{\frac{i}{\varepsilon}\eta\cdot(y-x)-\frac{k}{2}(y-x)^2}dx,\ c_n = 2^{-\frac{n}{2}}\pi^{-\frac{3n}{4}},\ a\in L^2(\mathbb{R}^n). \quad (12)$$

Its adjoint is the operator

$$T_\varepsilon^*(f)(x) = c_n \varepsilon^{-\frac{3n}{4}} \int_{\mathbb{R}^{2n}} f(y,\eta) e^{\frac{i}{\varepsilon}\eta\cdot(x-y)-\frac{k}{2}(x-y)^2} dy d\eta,\ f \in L^2(\mathbb{R}^{2n}). \tag{13}$$

As the Fourier Transform, the FBI transform is an isometry, satisfying $T_\varepsilon^* T_\varepsilon = Id$. The extra assumption needed in chapter 1 is

A3. $\|T_\varepsilon \underline{u}_\varepsilon^I\|_{L^2(\mathbb{R}^n\times R^c)} = O(\varepsilon^\infty)$ and $\|T_\varepsilon \underline{v}_\varepsilon^I\|_{L^2(\mathbb{R}^n\times R^c)} = O(\varepsilon^\infty)$,

where R^c denotes the complementary in $\mathbb{R}^n$ of some ring $R = \{\eta \in \mathbb{R}^n, r_0 \le |\eta| \le r_\infty\}$, $0 < r_0 < r_\infty$.

In general, this assumption may be not satisfied. We thus construct a family of initial data $(u_{\varepsilon,\gamma}^I, v_{\varepsilon,\gamma}^I)$ close to $(u_\varepsilon^I, v_\varepsilon^I)$ that satisfy assumptions A1 and A2 and furthermore have FBI transforms small in $L^2(\mathbb{R}^n \times R^c)$. Letting r_0 go to 0 and r_∞ go to $+\infty$ will make these data approach $(u_\varepsilon^I, v_\varepsilon^I)$ in a sense that will be specified in section 3.3. In any case, the needed convergence is weaker than a L^2 convergence since we are interested in the study of Wigner measures.

Let us first truncate $T_\varepsilon \underline{u}_\varepsilon^I$ and $T_\varepsilon \underline{v}_\varepsilon^I$ outside R by multiplying them by a cut-off $\gamma \in \mathcal{C}_0^\infty(\mathbb{R}^n, [0,1])$ supported in the interior of R

$$\gamma = \chi_{r_\infty/2}(1-\chi_{4r_0}). \tag{14}$$

Lemma B.4 yields

$$\|T_\varepsilon T_\varepsilon^* \gamma(\eta) T_\varepsilon \underline{u}_\varepsilon^I\|_{L^2(\mathbb{R}^n\times R^c)} = O(\varepsilon^\infty) \text{ and } \|T_\varepsilon T_\varepsilon^* \gamma(\eta) T_\varepsilon \underline{v}_\varepsilon^I\|_{L^2(\mathbb{R}^n\times R^c)} = O(\varepsilon^\infty). \tag{15}$$

In order to satisfy A2, we multiply $(T_\varepsilon^*\gamma(\eta)T_\varepsilon \underline{u}_\varepsilon^I, T_\varepsilon^*\gamma(\eta)T_\varepsilon \underline{v}_\varepsilon^I)$ by a cut-off $\rho \in \mathcal{C}_0^\infty(\mathbb{R}^n, [0,1])$ supported in Ω, and consider

$$u_{\varepsilon,\gamma}^I = \rho T_\varepsilon^* \gamma(\eta) T_\varepsilon \underline{u}_\varepsilon^I,\ v_{\varepsilon,\gamma}^I = \rho T_\varepsilon^* \gamma(\eta) T_\varepsilon \underline{v}_\varepsilon^I. \tag{1c'}$$

We index this initial data by γ because the parameters r_0 and r_∞ will vary in section 3.3. We suppose that $\rho(v) = 1$ if $\mathrm{dist}(v, \mathrm{supp} u_\varepsilon^I \cup \mathrm{supp} v_\varepsilon^I) \le D$ for some $D > 0$. The required estimate

A3'. $\|T_\varepsilon u_{\varepsilon,\gamma}^I\|_{L^2(\mathbb{R}^n\times R^c)} = O(\varepsilon^\infty)$ and $\|T_\varepsilon v_{\varepsilon,\gamma}^I\|_{L^2(\mathbb{R}^n\times R^c)} = O(\varepsilon^\infty)$,

is fulfilled since Lemma B.3 implies that

$$\|(1-\rho)T_\varepsilon^*\gamma(\eta)T_\varepsilon \underline{u}_\varepsilon^I\|_{L_x^2} \lesssim e^{-C/\varepsilon} \text{ and } \|(1-\rho)T_\varepsilon^*\gamma(\eta)T_\varepsilon \underline{v}_\varepsilon^I\|_{L_x^2} \lesssim e^{-C/\varepsilon}. \tag{16}$$

Using the boundedness of the operator $T_\varepsilon^*\gamma T_\varepsilon$ from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ and the relations

$$\partial_{y_b} T_\varepsilon = T_\varepsilon \partial_{x_b},\ \partial_{x_b} T_\varepsilon^* = T_\varepsilon^* \partial_{y_b}, \tag{17}$$

obtained by integrations by parts in the expressions of T_ε and T_ε^*, one can show that the assumption A1 is also fulfilled by these new initial data. Let ρ' be a cut-off of $\mathcal{C}_0^\infty(\mathbb{R}^n, [0,1])$ supported in a compact $K_y \subset \Omega$ and satisfying

$$\rho'(y) = 1 \text{ if } \mathrm{dist}(y, \mathrm{supp}\rho) < \Delta \text{ for some } \Delta > 0,$$

and γ' a cut-off of $\mathcal{C}_0^\infty(\mathbb{R}^n,[0,1])$ supported in $K_\eta \subset \mathbb{R}^n\backslash\{0\}$ s.t. $\gamma' \equiv 1$ on R. Without loss of generality, we assume that the reflected rays at the instant T $(x_{\pm 1}^{\pm T}(y,\eta))$ remain in the interior of the domain when y varies in K_y and η in $\mathbb{R}^n\backslash\{0\}$. This is always possible upon reducing T because the number of reflections for initial position and speed varying in $K_y \times (\mathbb{R}^n\backslash\{0\})$ is uniformly bounded (see section 2.3 of chapter 1 for similar arguments). Then, the IBVP (1a), (1b) with initial conditions (1c') has a family of approximate solutions $u_{\varepsilon,\gamma}^{appr}$ in $\mathcal{C}^0([0,T],H^1(\Omega))\cap\mathcal{C}^1([0,T],L^2(\Omega))$ obtained as a summation of first order beams. A general result using a superposition of beams of any order was proven in chapter 1, and it reads for first order beams as follows

Proposition 2.3. [Theorem 1.1, chapter 1] *Denote for $t \in [0,T]$ and $x \in \mathbb{R}^n$ the following superposition of Gaussian beams*

$$\begin{aligned} u_{\varepsilon,\gamma}^{appr}&(t,x) \\ &= \frac{1}{2}\varepsilon^{-\frac{3n}{4}+1} c_n \int_{\mathbb{R}^{2n}} \rho'(y)\gamma'(\eta) T_\varepsilon v_{\varepsilon,\gamma}^I(y,\eta) \Big[\sum_{k=0,1} \omega_\varepsilon^{k'}(t,x,y,\eta) - \sum_{k=0,-1} \omega_\varepsilon^{k'}(-t,x,y,\eta)\Big] \\ &+ \rho'(y)\gamma'(\eta)\varepsilon^{-1} T_\varepsilon u_{\varepsilon,\gamma}^I(y,\eta) \Big[\sum_{k=0,1} \omega_\varepsilon^{k}(t,x,y,\eta) + \sum_{k=0,-1} \omega_\varepsilon^{k}(-t,x,y,\eta)\Big] dy d\eta. \end{aligned}$$

Above, ω_ε^0, $\omega_\varepsilon^{0'}$ are incident Gaussian beams with the same phase ψ_0 satisfying at $t=0$

$$\psi_0(0,x,y,\eta) = \eta\cdot(x-y) + \frac{i}{2}(x-y)^2, \tag{18}$$

and different amplitudes a_0^0, $a_0^{0'}$ satisfying

$$a_0^0(0,x,y,\eta) = \chi_d(x-y),\ [i\partial_t\psi_0 a_0^{0'}](0,x,y,\eta) = \chi_d(x-y) + O(|x-y|). \tag{19}$$

$\omega_\varepsilon^{\pm 1}$ and $\omega_\varepsilon^{\pm 1'}$ denote the associated reflected beams. Then $u_{\varepsilon,\gamma}^{appr}$ is asymptotic to $u_{\varepsilon,\gamma}$ the exact solution of (1a)-(1b) *with initial conditions* (1c') *in the sense that*

$$\sup_{t\in[0,T]} \|u_{\varepsilon,\gamma} - u_{\varepsilon,\gamma}^{appr}\|_{H^1(\Omega)} \le C(\gamma,\Omega,T)\sqrt{\varepsilon} \text{ and } \sup_{t\in[0,T]} \|\partial_t u_{\varepsilon,\gamma} - \partial_t u_{\varepsilon,\gamma}^{appr}\|_{L^2(\Omega)} \le C(\gamma,\Omega,T)\sqrt{\varepsilon}.$$

Remark 2.4. *The final error is obtained by summing the errors in the interior equation, the boundary condition and the initial conditions. Note that the asymptotics is of the same order $\sqrt{\varepsilon}$ for both Dirichlet and Neumann boundary conditions. More generally, the construction may be achieved for any boundary condition if the IBVP is well-defined in $\mathcal{C}^0([0,T],H^1(\Omega))\cap\mathcal{C}^1([0,T],L^2(\Omega))$ for second members, initial data and boundary data in some Sobolev spaces $\mathcal{C}^0([0,T],H^{s_1}(\Omega)) \times H^{s_2}([0,T]\times\partial\Omega) \times (H^{s_3}(\Omega)\times H^{s_4}(\Omega))$ (see e.g. [75] for boundary conditions with energy estimates in this type of spaces). It may look surprising, but the asymptotics does not depend on the orders s_i of these Sobolev spaces. Indeed, using higher order beams increases the accuracy of the Gaussian beams integral and for sufficiently high order, the error falls to $O(\sqrt{\varepsilon})$. On the other hand, summation of higher order beams and summation of first order ones are close to order $\sqrt{\varepsilon}$. So the error in the approximate solution obtained by superposing first order beams is at any case $O(\sqrt{\varepsilon})$.*

The proof relies on the use of a family of approximate operators acting from $L^2(\mathbb{R}^{2n})$ to $L^2(\mathbb{R}^n)$ (chapter 1). We recall a simple version of the estimate of the norm of these operators established therein.

For $t \in [0,T]$, let $K_{z,\theta}(t)$ be a compact of $\mathbb{R}^{2n}$ and consider the set

$$E_1 = \{(t,x,z,\theta) \in [0,T] \times \mathbb{R}^{3n},\ (z,\theta) \in K_{z,\theta}(t),\ |x-z| \leq 1\},$$

which we assume compact. Let Φ be a phase function smooth on an open set containing E_1 and satisfying, for $t \in [0,T]$ and $(z,\theta) \in K_{z,\theta}(t)$

$$\begin{array}{l} \partial_x \Phi(t,z,z,\theta) = \theta, \\ \Phi(t,z,z,\theta) \text{ is real,} \\ \partial_x^2 \Phi(t,z,z,\theta) \text{ has a positive definite imaginary part.} \end{array} \tag{20}$$

Then there exists $r[\Phi] \in]0,1]$ s.t.

$$\operatorname{Im} \Phi(t,x,z,\theta) \geq C(x-z)^2 \text{ for } t \in [0,T],\ (z,\theta) \in K_{z,\theta}(t) \text{ and } |x-z| \leq r[\Phi].$$

Let $l_\varepsilon \in \mathcal{C}^\infty([0,T] \times \mathbb{R}^{3n}, \mathbb{C})$ satisfying

$$\begin{array}{l} \text{For } t \in [0,T], l_\varepsilon(t,x,z,\theta) = 0 \text{ if } (z,\theta) \notin K_{z,\theta}(t) \text{ or } |x-z| > r[\Phi], \\ \varepsilon^{\frac{k}{2}} \partial_{x_b}^k l_\varepsilon \text{ is uniformly bounded in } L^\infty([0,T] \times \mathbb{R}^{3n}) \text{ for every } 1 \leq b \leq n \text{ and } k \in \mathbb{N}. \end{array}$$

If $O^\alpha\left(l_\varepsilon(t,.), \Phi(t,.)/\varepsilon\right)$ denotes, for a given multiindex α and $t \in [0,T]$, the operator

$$\begin{aligned} &\left[O^\alpha\left(l_\varepsilon(t,.), \Phi(t,.)/\varepsilon\right) h\right](x) \\ &\qquad = \int_{\mathbb{R}^{2n}} h(z,\theta) l_\varepsilon(t,x,z,\theta)(x-z)^\alpha e^{i\Phi(t,x,z,\theta)/\varepsilon} dz d\theta,\ h \in L^2(\mathbb{R}^{2n}), \end{aligned}$$

then, under the previous hypotheses on Φ and l_ε, we have the following estimate

Proposition 2.5. [Lemma 3.3, chapter 1]
$\left\| O^\alpha\left(l_\varepsilon(t,.), \Phi(t,.)/\varepsilon\right)\right\|_{L^2(\mathbb{R}^{2n}) \to L^2(\mathbb{R}^n)} \lesssim \varepsilon^{\frac{3n}{4} + \frac{|\alpha|}{2}}$ *uniformly w.r.t.* $t \in [0,T]$.

2.2.2 Expression of the phases

Incident beams' phases By the requirement (P.a) for the incident phase, one has $\frac{d}{dt}\psi_0(t,x_0^t) = \partial_t \psi_0(t,x_0^t) + \partial_x \psi_0(t,x_0^t) \cdot \dot{x_0}^t = 0$. Taking into account the initial null value $\psi_0(0,y) = 0$ chosen in (18), one gets a null phase on the ray

$$\psi_0(t,x_0^t) = 0.$$

With the aim of computing $\partial_x^2 \psi_0(t,x_0^t)$, let us examine the Jacobian matrix of the bicharacteristic $F_0^t = D\varphi_0^t$. The matrix F_0^t satisfies the linear ODE

$$\left\{ \begin{array}{l} \frac{d}{dt} F_0^t = JH(x_0^t, \xi_0^t) F_0^t, \\ F_0^0 = Id, \end{array} \right. .$$

Writing F_0^t as

$$F_0^t = \left(\begin{array}{cc} \partial_y x_0^t & \partial_\eta x_0^t \\ \partial_y \xi_0^t & \partial_\eta \xi_0^t \end{array} \right),$$

leads to the following system of ODEs on $(U_0^t, V_0^t) = (\partial_y x_0^t + i\partial_\eta x_0^t, \partial_y \xi_0^t + i\partial_\eta \xi_0^t)$

$$\frac{d}{dt} U_0^t = H_{21}(x_0^t, \xi_0^t) U_0^t + H_{22}(x_0^t, \xi_0^t) V_0^t, \tag{21}$$

$$\frac{d}{dt} V_0^t = -H_{11}(x_0^t, \xi_0^t) U_0^t - H_{12}(x_0^t, \xi_0^t) V_0^t. \tag{22}$$

Moreover, F_0^t is a symplectic matrix in that

$$(F_0^t)^T J F_0^t = J.$$

Due to the symetry of the following matrices

$$(\partial_y x_0^t)^T \partial_y \xi_0^t, (\partial_\eta x_0^t)^T \partial_\eta \xi_0^t, \partial_y x_0^t (\partial_\eta x_0^t)^T, \text{ and } \partial_y \xi_0^t (\partial_\eta \xi_0^t)^T,$$

and the relations

$$(\partial_y x_0^t)^T \partial_\eta \xi_0^t - (\partial_y \xi_0^t)^T \partial_\eta x_0^t = Id \text{ and } \partial_y x_0^t (\partial_\eta \xi_0^t)^T - \partial_\eta x_0^t (\partial_y \xi_0^t)^T = Id,$$

one has

$$\begin{array}{l} (U_0^t)^T V_0^t = (V_0^t)^T U_0^t, \\ (V_0^t)^T \bar{U}_0^t - (U_0^t)^T \bar{V}_0^t = 2iId, \\ U_0^t \text{ is invertible.} \end{array} \tag{23}$$

Putting together (21), (22) and (23) shows that $V_0^t (U_0^t)^{-1}$ is a symmetric matrix with a positive definite imaginary part and fulfills the Riccati equation (5) with initial value iId. Since this is the initial condition for $\partial_x^2 \psi_0(t, x_0^t)$ given by (18), it follows that

$$\partial_x^2 \psi_0(t, x_0^t) = V_0^t (U_0^t)^{-1}.$$

Reflected beams' phases The expression of the reflected phases ψ_k, $k = \pm 1$, is similar. In fact, since $\frac{d}{dt}\psi_k(t, x_k^t) = 0$ and $\psi_k(T_k, x_0^{T_k}) = \psi_0(T_k, x_0^{T_k})$ by (8), we get

$$\psi_k(t, x_k^t) = 0.$$

The relation connecting the incident and the reflected phases stated in Lemma 2.2 gives at order one

$$\partial_x^2 \psi_1(t, x_1^t) \left(\partial_x r(x_0^t, \xi_0^t) + \partial_\xi r(x_0^t, \xi_0^t) \partial_x^2 \psi_0(t, x_0^t) \right) = \partial_x \lambda(x_0^t, \xi_0^t) + \partial_\xi \lambda(x_0^t, \xi_0^t) \partial_x^2 \psi_0(t, x_0^t),$$

and one has a similar relation for $\partial_x^2 \psi_{-1}(t, x_{-1}^t)$. One obtains by plugging the expression of $\partial_x^2 \psi_0(t, x_0^t)$ and using (10)

$$\partial_x^2 \psi_k(t, x_k^t) = V_k^t (U_k^t)^{-1} \text{ where } U_k^t = \partial_y x_k^t + i \partial_\eta x_k^t \text{ and } V_k^t = \partial_y \xi_k^t + i \partial_\eta \xi_k^t.$$

As φ_k^t is symplectic, it follows that (U_k^t, V_k^t) share the same properties (23) as (U_0^t, V_0^t)

$$\begin{array}{l} (U_k^t)^T V_k^t = (V_k^t)^T U_k^t, \\ (V_k^t)^T \bar{U}_k^t - (U_k^t)^T \bar{V}_k^t = 2iId, \\ U_k^t \text{ is invertible.} \end{array} \tag{24}$$

2.2.3 Expression of the amplitudes

Incident beams' amplitudes Using (P.a) and the Hamiltonian system satisfied by (x_0^t, ξ_0^t), the equation (7) at order zero implies the following transport equation for the value of the amplitude on the ray [50]

$$\frac{d}{dt} a_0^{0(')}(t, x_0^t) + \frac{1}{2} Tr(H_{21}(x_0^t, \xi_0^t) + H_{22}(x_0^t, \xi_0^t) \partial_x^2 \psi_0(t, x_0^t)) a_0^{0(')}(t, x_0^t) = 0, \tag{25}$$

which may be written, using the matrices U_0^t and V_0^t, as

$$\frac{d}{dt}a_0^{0(')}(t,x_0^t)+\frac{1}{2}Tr\Big[\Big(H_{21}(x_0^t,\xi_0^t)U_0^t+H_{22}(x_0^t,\xi_0^t)V_0^t\Big)(U_0^t)^{-1}\Big]a_0^{0(')}(t,x_0^t)=0.$$

The time evolution for U_0^t, see (21), combined with the choice of the initial values $a_0^0(0,y)=1$ and $a_0^{0\prime}(0,y)=(-ic(y)|\eta|)^{-1}$ from (19), yields

$$a_0^0(t,x_0^t)=[\det U_0^t]^{-\frac{1}{2}} \text{ and } a_0^{0\prime}(t,x_0^t)=i(c(y)|\eta|)^{-1}[\det U_0^t]^{-\frac{1}{2}}.$$

Above the square root is defined by continuity in t from 1 at $t=0$.

reflected beams' amplitudes The first reflected amplitudes evaluated on the ray satisfy a transport equation similar to (25), which may be written as

$$\frac{d}{dt}a_0^{k(')}(t,x_k^t)+\frac{1}{2}Tr\Big[\Big(H_{21}(x_k^t,\xi_k^t)U_k^t+H_{22}(x_k^t,\xi_k^t)V_k^t\Big)(U_k^t)^{-1}\Big]a_0^{k(')}(t,x_k^t)=0.$$

One can obtain a similar equation to (21) on U_k^t involving $H_{21}(x_k^t,\xi_k^t)$ and $H_{22}(x_k^t,\xi_k^t)$, by using the relation $\varphi_k^t=\varphi_0^t\ o\ s_k$. On the whole

$$a_0^{k(')}(t,x_k^t)=a_0^{k(')}(T_k,x_0^{T_k})\left[\frac{\det U_k^t}{\det U_k^{T_k}}\right]^{-\frac{1}{2}},\ k=\pm1,$$

where the square root is obtained by continuity from 1 at $t=T_k$.
On the other hand, for $k=\pm1$

$$d^0_{-m_B}+d^k_{-m_B}=b(x,\partial_x\psi_0)a_0^0+b(x,\partial_x\psi_k)a_0^k,$$

where b denotes the principal symbol of B. Thus, the condition (9) required for the construction of the reflected amplitudes implies that $a_0^{k(')}(T_k,x_0^{T_k})=sa_0^{0(')}(T_k,x_0^{T_k})$, with $s=-1$ for Dirichlet conditions and $s=1$ for Neumann conditions.

In order to find the relationship between $U_k^{T_k}$ and $U_0^{T_k}$ for $k=\pm1$, we differentiate the equality $x_k^{T_k}=x_0^{T_k}$

$$\partial_y x_k^{T_k}+\dot{x}_k^{T_k}(\partial_y T_k)^T=\partial_y x_0^{T_k}+\dot{x}_0^{T_k}(\partial_y T_k)^T,\ \partial_\eta x_k^{T_k}+\dot{x}_k^{T_k}(\partial_\eta T_k)^T=\partial_\eta x_0^{T_k}+\dot{x}_0^{T_k}(\partial_\eta T_k)^T,$$

and compute the derivatives of T_k from the condition $x_0^{T_k}\in\partial\Omega$

$$\partial_{y,\eta}T_k=-\frac{1}{(\dot{x_0}^{T_k}\cdot\nu(x_0^{T_k}))}(\partial_{y,\eta}x_0^{T_k})^T\nu(x_0^{T_k}),$$

to get after elementary computations

$$U_k^{T_k}=(Id-2\nu(x_0^{T_k})\nu(x_0^{T_k})^T)U_0^{T_k}. \tag{26}$$

Hence

$$a_0^k(t,x_k^t)=-si[\det U_k^t]^{-\frac{1}{2}} \text{ and } a_0^{k\prime}(t,x_k^t)=s(c|\eta|)^{-1}[\det U_k^t]^{-\frac{1}{2}} \text{ for } k=\pm1,$$

where the square root is defined by continuity from $i[\det U_0^{T_k}]^{-\frac{1}{2}}$ at $t=T_k$.

The previous form of the beams is summarized in the following result

Lemma 2.6. *For $k = 0, \pm 1$, the incident and reflected beams ω_ε^k have the form*

$$\omega_\varepsilon^{k^{(\prime)}}(t,x) = \beta_k \chi_d(x - x_k^t) a_k^{(\prime)}(t) e^{i\psi_k/\varepsilon},$$

with

$$\begin{aligned}
&\beta_0 = 1, \beta_1 = \beta_{-1} = -si,\\
&a_k(t) = [\det U_k^t]^{-\frac{1}{2}},\ a_k'(t) = i(c(y)|\eta|)^{-1}[\det U_k^t]^{-\frac{1}{2}},\\
&\psi_k = \xi_k^t \cdot (x - x_k^t) + \frac{i}{2}(x - x_k^t)\cdot \Lambda_k(t)(x - x_k^t),\ \Lambda_k(t) = -iV_k^t (U_k^t)^{-1}.
\end{aligned}$$

2.2.4 Gaussian integrals

It follows that the approximate solution $u_{\varepsilon,\gamma}^{appr}$ has the form (recall the dependence of Gaussian beams w.r.t. variables (y,η))

$$\begin{aligned}
u_{\varepsilon,\gamma}^{appr}(t,x) = \frac{1}{2}\varepsilon^{-\frac{3n}{4}+1} c_n \int_{\mathbb{R}^{2n}} \Big(\rho'(y)\gamma'(\eta) \sum_{k=0,1} \chi_d(x - x_k^t)\beta_k p_{\varepsilon,k}(t,y,\eta) e^{i\psi_k(t,x,y,\eta)/\varepsilon}\\
+ \rho'(y)\gamma'(\eta) \sum_{k=0,-1} \chi_d(x - x_k^{-t})\beta_k q_{\varepsilon,k}(-t,y,\eta) e^{i\psi_k(-t,x,y,\eta)/\varepsilon}\Big) dyd\eta,
\end{aligned}$$

with

$$\begin{aligned}
p_{\varepsilon,k}(t,y,\eta) &= a_k(t,y,\eta)\varepsilon^{-1} T_\varepsilon u_{\varepsilon,\gamma}^I(y,\eta) + a_k'(t,y,\eta) T_\varepsilon v_{\varepsilon,\gamma}^I(y,\eta)\\
\text{and } q_{\varepsilon,k}(t,y,\eta) &= a_k(t,y,\eta)\varepsilon^{-1} T_\varepsilon u_{\varepsilon,\gamma}^I(y,\eta) - a_k'(t,y,\eta) T_\varepsilon v_{\varepsilon,\gamma}^I(y,\eta).
\end{aligned} \tag{27}$$

In the remainder of this section, we write the derivatives of the approximate solution using Gaussian type integrals $I_\varepsilon(h,\Phi)$ that we define by

$$I_\varepsilon(h,\Phi)(t,x) = \varepsilon^{-\frac{3n}{4}} c_n \int_{\mathbb{R}^{2n}} h(t,z,\theta) e^{i\Phi(t,x,z,\theta)/\varepsilon} dz d\theta,$$

for phase functions $\Phi \in \mathcal{C}^\infty(\mathbb{R}_{t,x}^{n+1} \times \mathcal{B}, \mathbb{C})$ satisfying properties (20) and polynomial of order 2 in $x - z$ and amplitude functions $h \in \mathcal{C}^0([0,T], L^2(\mathbb{R}_{z,\theta}^{2n}))$ supported for every fixed $t \in [0,T]$ in a compact of $\mathcal{B}$. By Proposition 2.5, $\| \int_{\mathbb{R}^{2n}} h(t,z,\theta)\chi(x - z) e^{i\Phi(t,x,z,\theta)/\varepsilon} dz d\theta \|_{L_x^2} \lesssim \|h(t,.)\|_{L_{z,\theta}^2}$. Noting that the phase Φ provides with an exponentially decreasing function for $|x - z| \geq 1$, one can use the following crude estimate

$$\| \int_{|x-z|\geq a} h(t,z,\theta) e^{i\Phi(t,x,z,\theta)/\varepsilon} dz d\theta \|_{L_x^2} \lesssim e^{-C/\varepsilon} \|h(t,.)\|_{L_{z\theta}^2} \text{ for } a > 0, \tag{28}$$

to deduce that $I_\varepsilon(h,\Phi)(t,.)$ is uniformly bounded in L_x^2. The same notation I_ε will be also used for vector valued functions h. For a function f depending on $(t,x,z,\theta) \in \mathbb{R}^{n+1} \times \mathcal{B}$ and $k = 0, \pm 1$, denote

$$\widetilde{f}^k(t,x,z,\theta) = f(t,x;\{\varphi_k^t\}^{-1}(z,\theta)).$$

Set $K_{z,\theta}^k(t) = \varphi_k^t(K_y \times K_\eta)$. Let $\Pi_k(t)$ be a cutoff of $\mathcal{C}_c^\infty(\mathbb{R}^{2n},[0,1])$ supported in $\mathcal{B}$ and satisfying $\Pi_k(t) \equiv 1$ on $K_{z,\theta}^k(t)$. We prove the following Lemma

Lemma 2.7. *$\partial_t u^{appr}_{\varepsilon,\gamma}(t,.)$ is uniformly bounded in $L^2(\mathbb{R}^n)$ and satisfies*

$$\partial_t u^{appr}_{\varepsilon,\gamma}(t,x) = \frac{1}{2}(v_t^+(t,x) - v_t^-(-t,x)) + O(\sqrt{\varepsilon}) \text{ in } L^2(\mathbb{R}^n) \text{ uniformly w.r.t. } t \in [0,T],$$

where v_t^+ and v_t^- are uniformly bounded sequences of $L^2(\mathbb{R}^n)$ given by

$$v_t^+ = \sum_{k=0,1} \beta_k I_\varepsilon(-ic(z)|\theta|\Pi_k \widetilde{\rho' \otimes \gamma'}^k \widetilde{p_{\varepsilon,k}}^k, \widetilde{\psi_k}^k),$$
$$v_t^- = \sum_{k=0,-1} \beta_k I_\varepsilon(-ic(z)|\theta|\Pi_k \widetilde{\rho' \otimes \gamma'}^k \widetilde{q_{\varepsilon,k}}^k, \widetilde{\psi_k}^k).$$

Likewise, $\partial_x u^{appr}_{\varepsilon,\gamma}(t,.)$ is uniformly bounded in $L^2(\mathbb{R}^n)^n$ and satisfies

$$\partial_x u^{appr}_{\varepsilon,\gamma}(t,x) = \frac{1}{2}(v_x^+(t,x) + v_x^-(-t,x)) + O(\sqrt{\varepsilon}) \text{ in } L^2(\mathbb{R}^n)^n \text{ uniformly w.r.t. } t \in [0,T],$$

where v_x^+ and v_x^- are uniformly bounded sequences of $L^2(\mathbb{R}^n)^n$ given by

$$v_x^+ = \sum_{k=0,1} \beta_k I_\varepsilon(i\theta\Pi_k \widetilde{\rho' \otimes \gamma'}^k \widetilde{p_{\varepsilon,k}}^k, \widetilde{\psi_k}^k),$$
$$v_x^- = \sum_{k=0,-1} \beta_k I_\varepsilon(i\theta\Pi_k \widetilde{\rho' \otimes \gamma'}^k \widetilde{q_{\varepsilon,k}}^k, \widetilde{\psi_k}^k).$$

Proof. Because of (6), time derivative of $u^{appr}_{\varepsilon,\gamma}$ may be written as a sum of integrals of the form

$$\varepsilon^{-\frac{3n}{4}+j} \int_{\mathbb{R}^{2n}} \rho'(y)\gamma'(\eta)\, f_\varepsilon(y,\eta) r^k_{j,\alpha}(t,x,y,\eta)(x-x^t_k)^\alpha e^{i\psi_k(t,x,y,\eta)/\varepsilon} dy d\eta,$$

with $j,k = 0,1$ and $|\alpha| \leq 2$, arising from $\partial_t \omega^0_\varepsilon(t,.)$ and $\partial_t \omega^1_\varepsilon(t,.)$. Other terms of the same form come from the derivatives of $\omega^0_\varepsilon(-t,.)$ and $\omega^{-1}_\varepsilon(-t,.)$ w.r.t. t. f_ε stands for $\varepsilon^{-1}T_\varepsilon u^I_{\varepsilon,\gamma}$ or $T_\varepsilon v^I_{\varepsilon,\gamma}$ and the $r^k_{j,\alpha}$ are smooth functions vanishing for $|x - x^t_k| \geq d$.
The volume preserving change of variables

$$(C_k) : (z,\theta) = \varphi^t_k(y,\eta),$$

writes the previous integral as

$$\varepsilon^{-\frac{3n}{4}+j} \int_{\mathbb{R}^{2n}} \Pi_k(t) \widetilde{\rho' \otimes \gamma'}^k \widetilde{f_\varepsilon}^k \widetilde{\left(r^k_{j,\alpha}\right)}^k (x-z)^\alpha e^{i\widetilde{\psi_k}^k(t,x)/\varepsilon} dz d\theta. \tag{29}$$

Then, the transported phase $\widetilde{\psi_k}^k$ is smooth and satisfies by (P.a)-(P.b)-(P.c) the following properties, for $t \in [0,T]$ and $(z,\theta) \in K^k_{z,\theta}(t)$

$\partial_x \widetilde{\psi_k}^k(t,z,z,\theta) = \theta$,
$\widetilde{\psi_k}^k(t,z,z,\theta)$ is real,
$\partial^2_x \widetilde{\psi_k}^k(t,z,z,\theta)$ has a positive definite imaginary part.

We fix some $r[\widetilde{\psi_k}^k] \in]0,1]$ so that $\operatorname{Im} \widetilde{\psi_k}^k(t,x,z,\theta) \geq C(x-z)^2$ for $t \in [0,T]$, $(z,\theta) \in K^k_{z,\theta}(t)$ and $|x-z| \leq r[\widetilde{\psi_k}^k]$.

For $t \in [0,T]$, $\Pi_k \widetilde{\rho' \otimes \gamma'}^k (t,z,\theta) \widetilde{\left(r^k_{j,\alpha}\right)}^k (t,x,z,\theta)$ depends smoothly on its variables and vanishes for $|x-z| > d$ or $(z,\theta) \notin K^k_{z,\theta}(t)$. Hence, upon choosing $d \leq \min_{k=0,\pm 1} r[\widetilde{\psi_k}^k]$, every phase $\widetilde{\psi_k}^k$ and amplitude $\Pi_k \widetilde{\rho' \otimes \gamma'}^k \widetilde{\left(r^k_{j,\alpha}\right)}^k$ satisfy the properties formulated in Proposition 2.5. Let us check if $\mathbf{1}_{\mathcal{B}} \widetilde{f_\varepsilon}^k$ is uniformly bounded in $L^2(\mathbb{R}^{2n})$. Clearly $T_\varepsilon v^I_{\varepsilon,\gamma}$ is, and the property holds true for $\varepsilon^{-1} T_\varepsilon v^I_{\varepsilon,\gamma}$ by Lemma B.5. One can then use the approximation operators O^α to write the integral (29) as

$$\varepsilon^{-\frac{3n}{4}+j} \int_{\mathbb{R}^{2n}} \Pi_k \widetilde{\rho' \otimes \gamma'}^k \widetilde{f_\varepsilon}^k \widetilde{(r^k_{j,\alpha})}^k (x-z)^\alpha e^{i\widetilde{\psi_k}^k/\varepsilon} dz d\theta$$
$$= \varepsilon^{-\frac{3n}{4}+j} O^\alpha [\Pi_k \widetilde{\rho' \otimes \gamma'}^k \widetilde{(r^k_{j,\alpha})}^k (t,.), \widetilde{\psi_k}^k (t,.)/\varepsilon] \, \mathbf{1}_{\mathcal{B}} \widetilde{f_\varepsilon}^k,$$

The estimate established in Proposition 2.5 yields

$$\|\varepsilon^{-\frac{3n}{4}+j} \textstyle\int_{\mathbb{R}^{2n}} \Pi_k \widetilde{\rho' \otimes \gamma'}^k \widetilde{f_\varepsilon}^k \widetilde{(r^k_{j,\alpha})}^k (x-z)^\alpha e^{i\widetilde{\psi_k}^k/\varepsilon} dz d\theta \|_{L^2_x} \lesssim \varepsilon^{j+\frac{|\alpha|}{2}}.$$

Hence, only $\widetilde{(r^k_{0,0})}^k$ contributes to $\partial_t u^{appr}_{\varepsilon,\gamma}$, the residue being of order $\sqrt{\varepsilon}$. Since

$$\begin{aligned} r^k_{0,0}(t,x,y,\eta) &= \frac{i}{2} c_n \beta_k \partial_t \psi_k(t,x^t_k) \chi_d(x-x^t_k) a^{(\prime)}_k(t,y,\eta) \\ &= -\frac{i}{2} c_n \beta_k c(x^t_k) |\xi^t_k| \chi_d(x-x^t_k) a^{(\prime)}_k(t,y,\eta), \end{aligned}$$

by (P.a), it follows that

$$\begin{aligned} \partial_t u^{appr}_{\varepsilon,\gamma}(t,x) =& \frac{1}{2}\varepsilon^{-\frac{3n}{4}} c_n \int_{\mathbb{R}^{2n}} \sum_{k=0,1} (-i)\beta_k c(z)|\theta| \chi_d(x-z) \Pi_k(t,z,\theta) \widetilde{\rho' \otimes \gamma'}^k (t,z,\theta) \\ & \widetilde{p_{\varepsilon,k}}^k (t,z,\theta) e^{i\widetilde{\psi_k}^k (t,x,z,\theta)/\varepsilon} dz d\theta \\ & + \frac{1}{2}\varepsilon^{-\frac{3n}{4}} c_n \int_{\mathbb{R}^{2n}} \sum_{k=0,-1} i\beta_k c(z)|\theta| \chi_d(x-z) \Pi_k(-t,z,\theta) \widetilde{\rho' \otimes \gamma'}^k (-t,z,\theta) \\ & \widetilde{q_{\varepsilon,k}}^k (-t,z,\theta) e^{i\widetilde{\psi_k}^k (-t,x,z,\theta)/\varepsilon}) dz d\theta \\ & + O(\sqrt{\varepsilon}), \end{aligned}$$

in $L^2(\mathbb{R}^n)$, uniformly for $t \in [0,T]$. Similar arguments apply to the spatial derivatives of $u^{appr}_{\varepsilon,\gamma}$ and lead to

$$\begin{aligned} \partial_{x_b} u^{appr}_{\varepsilon,\gamma}(t,x) =& \frac{1}{2}\varepsilon^{-\frac{3n}{4}} c_n \int_{\mathbb{R}^{2n}} \sum_{k=0,1} i\beta_k (\theta_k)_b \chi_d(x-z) \Pi_k(t,z,\theta) \widetilde{\rho' \otimes \gamma'}^k (t,z,\theta) \\ & \widetilde{p_{\varepsilon,k}}^k (t,z,\theta) e^{i\widetilde{\psi_k}^k (t,x,z,\theta)/\varepsilon} dz d\theta \\ & + \frac{1}{2}\varepsilon^{-\frac{3n}{4}} c_n \int_{\mathbb{R}^{2n}} \sum_{k=0,-1} i\beta_k (\theta_k)_b \Pi_k(-t,z,\theta) \widetilde{\rho' \otimes \gamma'}^k (-t,z,\theta) \\ & \widetilde{q_{\varepsilon,k}}^k (-t,z,\theta) e^{i\widetilde{\psi_k}^k (-t,x,z,\theta)/\varepsilon}) dz d\theta \\ & + O(\sqrt{\varepsilon}) \end{aligned}$$

in $L^2(\mathbb{R}^n)$, uniformly w.r.t. $t \in [0,T]$.

One can get rid of the cutoff $\chi_d(x-z)$ appearing in $\partial_t u^{appr}_{\varepsilon,\gamma}(t,x)$ and $\partial_{x_b} u^{appr}_{\varepsilon,\gamma}(t,x)$ by using the estimate (28). □

3 Wigner transforms and measures

In this section, we compute the scalar measures associated to $\partial_t u^{appr}_{\varepsilon,\gamma}(t,.)$ and $c\partial_x u^{appr}_{\varepsilon,\gamma}(t,.)$. As $|\beta_k| = 1$, the Wigner transforms associated to $v^{\pm}_t(\pm t,.)$ are finite sums of terms of the form

$$w_\varepsilon(I_\varepsilon(f^k_t,\Phi_k)(\pm t,.),\ I_\varepsilon(g^l_t,\Phi_l)(\pm t,.)),$$

where $k,l = 0,\pm 1$, $f^k_t = c|\theta|\Pi_k\widetilde{\rho'\otimes\gamma'}^k\widetilde{p_{\varepsilon,k}}^k$, $g^l_t = c|\theta|\Pi_l\widetilde{\rho'\otimes\gamma'}^l\widetilde{q_{\varepsilon,l}}^l$, $\Phi_k = \widetilde{\psi_k}^k$ and $\Phi_l = \widetilde{\psi_l}^l$. As regards the Wigner transforms associated to $cv^{\pm}_x(\pm t,.)$, since c is uniformly continuous on $\mathbb{R}^n$, one has by a classical result ([38] p.8)

$$w_\varepsilon(cv^{\pm}_x(\pm t,.),cv^{\pm}_x(\pm t,.)) \approx c^2 w_\varepsilon(v^{\pm}_x(\pm t,.),v^{\pm}_x(\pm t,.)) \text{ in } \mathbb{R}^{2n},$$

so that the involved quantities have the form

$$c^2 w_\varepsilon(I_\varepsilon(f^k_x,\Phi_k)(\pm t,.),I_\varepsilon(g^l_x,\Phi_l)(\pm t,.)),$$

with $f^k_x = \theta\Pi_k\widetilde{\rho'\otimes\gamma'}^k\widetilde{p_{\varepsilon,k}}^k$ and $g^l_x = \theta\Pi_l\widetilde{\rho'\otimes\gamma'}^l\widetilde{q_{\varepsilon,l}}^l$. Forgetting the powers of ε factors, the involved Wigner transform integrals have the form

$$\int_{\mathbb{R}^{6n}} \widetilde{T_\varepsilon\kappa_\varepsilon}^k(z,\theta)\overline{\widetilde{T_\varepsilon\tau_\varepsilon}}^l(z',\theta')b_1(z,\theta,z',\theta',x,v)e^{i\Psi_1(z,\theta,z',\theta',x,v)/\varepsilon}dzd\theta dz'd\theta'dxdv, \tag{30}$$

with $\kappa_\varepsilon,\tau_\varepsilon = \varepsilon^{-1}u^I_{\varepsilon,\gamma},v^I_{\varepsilon,\gamma}$ and $k,l = 0,\pm 1$, or by expanding the FBI transforms

$$\int_{\mathbb{R}^{8n}} \kappa_\varepsilon(w)\tau_\varepsilon(w')b_2(z,\theta,z',\theta',x,v)e^{i\Psi_2(w,w',z,\theta,z',\theta',x,v)/\varepsilon}dwdw'dzd\theta dz'd\theta'dxdv. \tag{31}$$

Traditionally, this type of oscillating integrals is estimated by the stationary phase theorem. This method was successfully used in [15] for the computation of a Wigner measure for smooth data. The involved phase is complex and its Hessian restricted to the stationary set is non-degenerate in normal direction to this set. In our case however, the amplitude is not smooth because no such assumption was made on $\underline{u}^I_\varepsilon$ and $\underline{v}^I_\varepsilon$. So we can not estimate immediately the global integral (31) by the same techniques. One research lead is to resort to the stationary phase theorem with a complex phase depending on parameters for estimating $\int_{\mathbb{R}^{6n}} b_2(z,\theta,z',\theta',x,v)e^{i\Psi_2(w,w',z,\theta,z',\theta',x,v)/\varepsilon}dzd\theta dz'd\theta'dxdv$.

An alternative method was used in [88], where an integral of the form (30) associated to the Wigner transform for the Schrödinger equation with a WKB initial condition was simplified by elementary computations into an integral over $\mathbb{R}^{4n}$. However the method faced difficulties in deducing the exact relation between the Wigner measure of the solution and of the initial data.

We adapt the result of [88] to our problem in section 3.1 and complete the analysis to prove the propagation of the microlocal energy density of $u^{appr}_{\varepsilon,\gamma}$ along the flow in section 3.2. The proof is simple and elementary and the computations are made in an explicit way. Section 3.3 is devoted to the Wigner measures associated to the derivatives of the exact solution u_ε of (1).

3.1 Wigner transform for Gaussian integrals

The functions $f_{t,x}^k$ and $g_{t,x}^l$ depend on ε but they are uniformly bounded in $L^2(\mathbb{R}^{2n})$ and their support is contained in a fixed compact. Slight modifications of the computations of [88] lead to the following more general result

Lemma 3.1. *Let f_ε and g_ε be sequences uniformly bounded in $L^2(\mathbb{R}^{2n})$ and supported in compacts independent of ε. Let F be an open set containing $\text{supp} f_\varepsilon \cup \text{supp} g_\varepsilon$ and Φ, Ψ be phase functions in $\mathcal{C}^\infty(\mathbb{R}^n_x \times F, \mathbb{C})$ satisfying*

$$\Phi(x,z,\theta) = r_\Phi(z,\theta) + \theta\cdot(x-z) + \frac{i}{2}(x-z)\cdot H_\Phi(z,\theta)(x-z) \text{ for } (x,z,\theta) \in \mathbb{R}^n \times F,$$

$$\Psi(x,z',\theta') = r_\Psi(z',\theta') + \theta'\cdot(x-z') + \frac{i}{2}(x-z')\cdot H_\Psi(z',\theta')(x-z') \text{ for } (x,z',\theta') \in \mathbb{R}^n \times F,$$

where $r_\Phi, r_\Psi \in \mathcal{C}^\infty(F,\mathbb{R})$ and the matrices $H_\Phi, H_\Psi \in \mathcal{C}^\infty(F,\mathcal{M}_n(\mathbb{C}))$ with positive definite real parts. Then for $\phi \in \mathcal{C}_c^\infty(F,\mathbb{R})$

$$< w_\varepsilon(I_\varepsilon(f_\varepsilon,\Phi), I_\varepsilon(g_\varepsilon,\Psi)), \phi >= \int_{\mathbb{R}^{4n}} \phi(s,\sigma)A(\Phi,\Psi)(s,\sigma) f_\varepsilon(s+\sqrt{\varepsilon}r, \sigma+\sqrt{\varepsilon}\delta)g_\varepsilon^*(s-\sqrt{\varepsilon}r,\sigma-\sqrt{\varepsilon}\delta)e^{i\Theta_\varepsilon(\Phi,\Psi)(s,\sigma,r,\delta)}drd\delta dsd\sigma + o(1),$$

where

$$A(\Phi,\Psi)(s,\sigma) = c_n^2 2^{\frac{5n}{2}}\pi^{\frac{n}{2}}(\det[H_\Phi(s,\sigma)+\bar{H}_\Psi(s,\sigma)])^{-\frac{1}{2}},$$

and

$$\Theta_\varepsilon(\Phi,\Psi)(s,\sigma,r,\delta) = r_\Phi(s+\sqrt{\varepsilon}r,\sigma+\sqrt{\varepsilon}\delta)/\varepsilon - r_\Psi(s-\sqrt{\varepsilon}r,\sigma-\sqrt{\varepsilon}\delta)/\varepsilon - 2\sigma\cdot r/\sqrt{\varepsilon} + i(r,\delta)\cdot\mathbf{Q}(H_\Phi(s,\sigma),\bar{H}_\Psi(s,\sigma))(r,\delta).$$

The matrix $\mathbf{Q}(H_\Phi(s,\sigma),\bar{H}_\Psi(s,\sigma))$ and the square root are defined in Lemma 3.2.

Proof. The proof is given in two steps. In a first time, the Fourier transform of a Gaussian type function is computed explicitly. Then, in a second time, a Gaussian approximation is used for several smooth functions appearing in the Wigner transform integral.

For simplicity we denote $u(x,z,\theta)$ by u and $u(x,z',\theta')$ by u' when integrating w.r.t. z,θ,z',θ'. We also omit the index ε in the notation of f_ε and g_ε.

Step 1. Fourier transform. Firstly, we note that the Wigner transform at point $(x,\xi) \in \mathbb{R}^{2n}$ may be written as

$$w_\varepsilon(I_\varepsilon(f,\Phi), I_\varepsilon(g,\Psi))(x,\xi) = \pi^{-n}c_n^2\varepsilon^{-\frac{5n}{2}}\int_{\mathbb{R}^{5n}} fg^{*\prime}e^{ir_\Phi/\varepsilon - ir'_\Psi/\varepsilon + ix\cdot(\theta-\theta')/\varepsilon + i(\theta'\cdot z'-\theta\cdot z)/\varepsilon} \mathcal{F}_v[e^{-(v+x-z)\cdot H_\Phi(v+x-z)/(2\varepsilon)-(v-x+z')\cdot \bar{H}'_\Psi(v-x+z')/(2\varepsilon)}]((2\xi-\theta-\theta')/\varepsilon) dvdzdz'd\theta d\theta'.$$

The Fourier transform of Gaussian functions' product is given by the following Lemma, of which proof is postponed to the end of this section

Lemma 3.2. *Let $a, b \in \mathbb{R}^d$ and $M, N \in \mathcal{M}_d(\mathbb{C})$ symmetric matrices with positive definite real parts, then*

$$\mathcal{F}_x[e^{-(x-a)\cdot M(x-a)/2}e^{-(x-b)\cdot N(x-b)/2}](\xi) = (2\pi)^{\frac{d}{2}}(\det[M+N])^{-\frac{1}{2}}e^{-i\xi\cdot(b+a)/2}e^{-(b-a,\xi)\cdot\mathbf{Q}(M,N)(b-a,\xi)/4},$$

where $\mathbf{Q}(M,N)$ is the symmetric symplectic matrix given by

$$\mathbf{Q}(M,N) = \begin{pmatrix} 2M(M+N)^{-1}N & i(N-M)(M+N)^{-1} \\ i(M+N)^{-1}(N-M) & 2(M+N)^{-1} \end{pmatrix},$$

and the square root is defined as explained in section 3.4 of [45]. Moreover, $\mathbf{Q}(M,N)\mathbf{A}(M,N) = \mathbf{B}(M,N)$ with $\mathbf{A}(M,N) = \begin{pmatrix} Id & Id \\ -iN & iM \end{pmatrix}$ and $\mathbf{B}(M,N) = \begin{pmatrix} N & M \\ -iId & iId \end{pmatrix}$, and $\mathbf{Q}(M,N)$ has a positive definite real part

$$\operatorname{Re}\mathbf{Q}(M,N) = 2\mathbf{A}(M,N)^{*-1}\begin{pmatrix} \operatorname{Re}N & 0 \\ 0 & \operatorname{Re}M \end{pmatrix}\mathbf{A}(M,N)^{-1}.$$

Hence

$$w_\varepsilon(I_\varepsilon(f,\Phi), I_\varepsilon(g,\Psi))(x,\xi) = c_n^2 2^{\frac{n}{2}}\pi^{-\frac{n}{2}}\varepsilon^{-2n}\int_{\mathbb{R}^{4n}} fg^{*\prime}(\det[H_\Phi + \bar{H}'_\Psi])^{-\frac{1}{2}}e^{ir_\Phi/\varepsilon - ir'_\Psi/\varepsilon}$$
$$e^{i(\theta+\theta'-2\xi)\cdot(z-z')/(2\varepsilon)+i(\theta-\theta')\cdot x/\varepsilon+i(\theta'\cdot z'-\theta\cdot z)/\varepsilon}$$
$$e^{-(2x-z-z',2\xi-\theta-\theta')\cdot\mathbf{Q}(H_\Phi,\bar{H}'_\Psi)(2x-z-z',2\xi-\theta-\theta')/(4\varepsilon)}dzdz'd\theta d\theta'.$$

Making the change of variables

$$(z,z') = (s+\sqrt{\varepsilon}r, s-\sqrt{\varepsilon}r) \text{ and } (\theta,\theta') = (\sigma+\sqrt{\varepsilon}\delta, \sigma-\sqrt{\varepsilon}\delta),$$

and writing f_+ for $f(s+\sqrt{\varepsilon}r, \sigma+\sqrt{\varepsilon}\delta)$ and g_- for $g(s-\sqrt{\varepsilon}r, \sigma-\sqrt{\varepsilon}\delta)$, leads to

$$w_\varepsilon(I_\varepsilon(f,\Phi), I_\varepsilon(g,\Psi))(x,\xi) = c_n^2 2^{\frac{5n}{2}}\pi^{-\frac{n}{2}}\varepsilon^{-n}\int_{\mathbb{R}^{4n}} f_+g^*{}_-(\det[H_{\Phi+} + \bar{H}_{\Psi\,-}])^{-\frac{1}{2}}$$
$$e^{ir_{\Phi+}/\varepsilon - ir_{\Psi\,-}/\varepsilon + 2i\delta\cdot(x-s)/\sqrt{\varepsilon} - 2i\xi\cdot r/\sqrt{\varepsilon}}$$
$$e^{-(x-s,\xi-\sigma)\cdot\mathbf{Q}(H_{\Phi+},\bar{H}_{\Psi\,-})(x-s,\xi-\sigma)/\varepsilon}drd\delta dsd\sigma.$$

Step 2. Gaussian approximations. Taking the duality product of the Wigner transform with a test function $\phi \in \mathcal{C}_0^\infty(F,\mathbb{R})$, and after setting $(x',\xi') = (x-s,\xi-\sigma)/\sqrt{\varepsilon}$, one has

$$< w_\varepsilon(I_\varepsilon(f,\Phi), I_\varepsilon(g,\Psi)), \phi >= c_n^2 2^{\frac{5n}{2}}\pi^{-\frac{n}{2}}\int_{\mathbb{R}^{6n}} \phi(s+\sqrt{\varepsilon}x', \sigma+\sqrt{\varepsilon}\xi')f_+g^*{}_-$$
$$(\det[H_{\Phi+} + \bar{H}_{\Psi\,-}])^{-\frac{1}{2}}e^{ir_{\Phi+}/\varepsilon - ir_{\Psi\,-}/\varepsilon - 2i\sigma\cdot r/\sqrt{\varepsilon} + 2i(x',\xi')\cdot(\delta,-r)}$$
$$e^{-(x',\xi')\cdot\mathbf{Q}(H_{\Phi+},\bar{H}_{\Psi\,-})(x',\xi')}dx'd\xi'drd\delta dsd\sigma. \tag{32}$$

Let ${\rho'}_f$ and ${\rho'}_g$ be cut-off functions supported in F s.t. ${\rho'}_f \equiv 1$ on a fixed compact containing supp$f$ and ${\rho'}_g \equiv 1$ on a fixed compact containing supp$g$, and consider

$$b_\varepsilon : (x',\xi',s,\sigma,r,\delta) \mapsto \Big[\phi(s+\sqrt{\varepsilon}x', \sigma+\sqrt{\varepsilon}\xi') - \phi(s,\sigma)\Big]{\rho'}_{f+}{\rho'}_{g-}e^{-(x',\xi')\cdot\mathbf{Q}(H_{\Phi+},\bar{H}_{\Psi\,-})(x',\xi')}.$$

The r.h.s. of (32) may then be written as

$$
\begin{aligned}
< w_\varepsilon(I_\varepsilon(f,\Phi), I_\varepsilon(g,\Psi)), \phi >= c_n^2 2^{\frac{5n}{2}}\pi^{-\frac{n}{2}} \int_{\mathbb{R}^{6n}} & \phi(s,\sigma) f_+ g^*{}_-(\det[H_{\Phi+}+\bar{H}_{\Psi}{}_-])^{-\frac{1}{2}} \\
& e^{ir\Phi_+/\varepsilon - ir_{\Psi}{}_-/\varepsilon - 2i\sigma\cdot r/\sqrt{\varepsilon} + 2i(x',\xi')\cdot(\delta,-r)} \\
& e^{-(x',\xi')\cdot \mathbf{Q}(H_{\Phi+},\bar{H}_{\Psi}{}_-)(x',\xi')} dx'd\xi' dr d\delta ds d\sigma \\
+ c_n^2 2^{\frac{5n}{2}}\pi^{-\frac{n}{2}} \int_{\mathbb{R}^{4n}} & (\det[H_{\Phi+}+\bar{H}_{\Psi}{}_-])^{-\frac{1}{2}} f_+ g^*{}_- \\
& e^{ir\Phi_+/\varepsilon - ir_{\Psi}{}_-/\varepsilon - 2i\sigma\cdot r/\sqrt{\varepsilon}} \\
& \mathcal{F}_{(x',\xi')} b_\varepsilon(-2\delta, 2r, s, \sigma, r, \delta) dr d\delta ds d\sigma.
\end{aligned}
\tag{33}
$$

Leibnitz formula yields for α multi-index

$$
\begin{aligned}
&\partial^\alpha_{x',\xi'} b_\varepsilon(x',\xi', s,\sigma,r,\delta) = \\
&\quad \rho'_{f_+}\rho'_{g_-}\Big[\phi(s+\sqrt{\varepsilon}x', \sigma+\sqrt{\varepsilon}\xi') - \phi(s,\sigma)\Big]\partial^\alpha_{x',\xi'}\Big[e^{-(x',\xi')\cdot \mathbf{Q}(H_{\Phi+},\bar{H}_{\Psi}{}_-)(x',\xi')}\Big] \\
&\quad + \rho'_{f_+}\rho'_{g_-} \sum_{\beta+\gamma=\alpha, \beta\neq 0} C_{\beta,\gamma}\varepsilon^{\frac{|\beta|}{2}} \partial^\beta_{x,\xi}\phi(s+\sqrt{\varepsilon}x', \sigma+\sqrt{\varepsilon}\xi')\partial^\gamma_{x',\xi'}\Big[e^{-(x',\xi')\cdot \mathbf{Q}(H_{\Phi+},\bar{H}_{\Psi}{}_-)(x',\xi')}\Big].
\end{aligned}
$$

As $(s+\sqrt{\varepsilon}r, \sigma+\sqrt{\varepsilon}\delta)$ varies in $\mathrm{supp}\rho'_f$ and $(s-\sqrt{\varepsilon}r, \sigma-\sqrt{\varepsilon}\delta)$ varies in $\mathrm{supp}\rho'_g$, one can find by continuity a constant $C>0$ s.t.

$$
\mathrm{Re}\,\mathbf{Q}(H_{\Phi+},\bar{H}_{\Psi}{}_-) \geq C Id \text{ on } \mathrm{supp}(\rho'_{f_+}\rho'_{g_-}).
$$

Since

$$
(s,\sigma) \text{ and } \sqrt{\varepsilon}(r,\delta) \text{ are bounded on } \mathrm{supp}(\rho'_{f_+}\rho'_{g_-}), \tag{34}
$$

it follows that there exists a constant $C'>0$ s.t.

$$
|\partial^\alpha_{x',\xi'} b_\varepsilon(x',\xi',s,\sigma,r,\delta)| \lesssim \sqrt{\varepsilon} e^{-C'(x',\xi')^2} \text{ for all } (x',\xi',s,\sigma,r,\delta),
$$

which leads to

$$
|\mathcal{F}_{(x',\xi')} b_\varepsilon(-2\delta, 2r, s,\sigma,r,\delta)| \lesssim \sqrt{\varepsilon}(1+(r,\delta)^2)^{-n-1} \text{ for all } (s,\sigma,r,\delta).
$$

The second integral on the r.h.s. of (33) is then dominated by

$$
\sqrt{\varepsilon}\int_{\mathbb{R}^{4n}} |f_+||g_-|(1+(\delta,r)^2)^{-n-1} dr d\delta ds d\sigma.
$$

We deduce with Cauchy-Schwartz inequality w.r.t. s,σ that

$$
\begin{aligned}
| < w_\varepsilon(I_\varepsilon(f,\Phi), I_\varepsilon(g,\Psi)), \phi > - c_n^2 2^{\frac{5n}{2}}\pi^{\frac{n}{2}} \int_{\mathbb{R}^{4n}} & \phi(s,\sigma)(\det[H_{\Phi+}+\bar{H}_{\Psi}{}_-])^{-\frac{1}{2}} f_+ g^*{}_- \\
& e^{ir\Phi_+/\varepsilon - ir_{\Psi}{}_-/\varepsilon - 2i\sigma . r/\sqrt{\varepsilon}} e^{-(\delta,-r)\cdot \mathbf{Q}(H_{\Phi+},\bar{H}_{\Psi}{}_-)^{-1}(\delta,-r)} dr ds d\delta d\sigma| \\
\lesssim \sqrt{\varepsilon}\|f\|_{L^2(\mathbb{R}^{2n})}\|g\|_{L^2(\mathbb{R}^{2n})},
\end{aligned}
$$

where we have replaced $\det \mathbf{Q}(H_{\Phi+},\bar{H}_{\Psi}{}_-)$ by 1 since $\mathbf{Q}(H_{\Phi+},\bar{H}_{\Psi}{}_-)$ is symplectic.

To go further, let us extend H_Φ and H_Ψ as $\lambda H_\Phi + (1-\lambda)Id$ and $\lambda H_\Psi + (1-\lambda)Id$ by using a cut-off $\lambda \in \mathcal{C}^\infty_c(\mathbb{R}^{2n},[0,1])$ supported in F s.t. $\lambda \equiv 1$ on the compact set

$\mathrm{supp}\rho'_f \cup \mathrm{supp}\rho'_g \cup \mathrm{supp}\phi$. The extended matrices have positive definite real parts. The smoothness of these matrices implies by the mean value theorem and (34) that

$$|(\det[H_{\Phi+} + \bar{H}_{\Psi\,-}])^{-\frac{1}{2}} - (\det[H_\Phi(s,\sigma) + \bar{H}_\Psi(s,\sigma)])^{-\frac{1}{2}}| \lesssim \sqrt{\varepsilon}|(r,\delta)| \text{ on } \mathrm{supp}(\phi f_+ g^*{}_-).$$

Using the symplecticity and the symmetry of $\mathbf{Q}(H_{\Phi+}, \bar{H}_{\Psi\,-})$, its inverse is $-J\mathbf{Q}(H_{\Phi+}, \bar{H}_{\Psi\,-})J$.
Thus the quantity $|e^{-(\delta,-r)\cdot\mathbf{Q}(H_{\Phi+},\bar{H}_{\Psi\,-})^{-1}(\delta,-r)} - e^{-(r,\delta)\cdot\mathbf{Q}(H_\Phi(s,\sigma),\bar{H}_\Psi(s,\sigma))(r,\delta)}|$ is dominated by

$$|(r,\delta)\cdot\Big[\mathbf{Q}(H_{\Phi+}, \bar{H}_{\Psi\,-}) - \mathbf{Q}(H_\Phi(s,\sigma), \bar{H}_\Psi(s,\sigma))\Big](r,\delta)|$$
$$\sup_{s\in[0,1]}|e^{-s(r,\delta)\cdot\mathbf{Q}(H_{\Phi+},\bar{H}_{\Psi\,-})(r,\delta)-(1-s)(r,\delta)\cdot\mathbf{Q}(H_\Phi(s,\sigma),\bar{H}_\Psi(s,\sigma))(r,\delta)}|.$$

The positivity of $\mathrm{Re}\,\mathbf{Q}(H_{\Phi+}, \bar{H}_{\Psi\,-})$ and $\mathrm{Re}\,\mathbf{Q}(H_\Phi(s,\sigma), \bar{H}_\Psi(s,\sigma))$ and the mean value theorem for the matrix function $\mathbf{Q}(\lambda H_\Phi + (1-\lambda)Id, \lambda\bar{H}_\Psi + (1-\lambda)Id)$ give by (34)

$$|e^{-(\delta,-r)\cdot\mathbf{Q}(H_{\Phi+},\bar{H}_{\Psi\,-})^{-1}(\delta,-r)} - e^{-(r,\delta)\cdot\mathbf{Q}(H_\Phi(s,\sigma),\bar{H}_\Psi(s,\sigma))(r,\delta)}| \lesssim \sqrt{\varepsilon}|(r,\delta)|^3 e^{-C(r,\delta)^2}$$

for $(s,\sigma) \in \mathrm{supp}\phi$, $(s+\sqrt{\varepsilon}r, \sigma+\sqrt{\varepsilon}\delta) \in \mathrm{supp}\rho'_f$ and $(s-\sqrt{\varepsilon}r, \sigma-\sqrt{\varepsilon}\delta) \in \mathrm{supp}\rho'_g$. It follows that

$$\begin{aligned}|< w_\varepsilon(I_\varepsilon(f,\Phi), I_\varepsilon(g,\Psi)), \phi > -c_n^2 2^{\frac{5n}{2}}\pi^{\frac{n}{2}} \int_{\mathbb{R}^{4n}} \phi(s,\sigma)(\det[H_\Phi + \bar{H}_\Psi])^{-\frac{1}{2}}(s,\sigma) f_+ g^*{}_- \\ e^{ir_{\Phi+}/\varepsilon - ir_{\Psi\,-}/\varepsilon - 2i\sigma\cdot r/\sqrt{\varepsilon} - (r,\delta)\cdot\mathbf{Q}(H_\Phi,\bar{H}_\Psi)(s,\sigma)(r,\delta)} dr d\delta ds d\sigma| \\ \lesssim \sqrt{\varepsilon}\|f\|_{L^2(\mathbb{R}^{2n})}\|g\|_{L^2(\mathbb{R}^{2n})}.\end{aligned}$$

□

Now we give the proof of the Lemma 3.2.

Proof. The matrix $M+N$ has a positive definite real part and is then non-singular. By elementary calculus we have

$$\begin{aligned}(x-a)\cdot M(x-a) + (x-b)\cdot N(x-b) = (b-a)\cdot M(M+N)^{-1}N(b-a) \\ + (x-(M+N)^{-1}(Ma+Nb))\cdot(M+N)(x-(M+N)^{-1}(Ma+Nb)).\end{aligned}$$

Thus, using the value of the Fourier transform of a Gaussian function (see Theorem 7.6.1 of [45]), it follows that

$$\begin{aligned}\mathcal{F}_x[e^{-(x-a)\cdot M(x-a)/2}e^{-(x-b)\cdot N(x-b)/2}](\xi) =& (2\pi)^{\frac{d}{2}}(\det[M+N])^{-\frac{1}{2}}e^{-(b-a)\cdot M(M+N)^{-1}N(b-a)/2} \\ & e^{-i\xi.(M+N)^{-1}(Ma+Nb)-\xi\cdot(M+N)^{-1}\xi/2}.\end{aligned}$$

Writing $M = 1/2(M+N) + 1/2(M-N)$ and $N = 1/2(M+N) - 1/2(M-N)$, we get the expression with the matrix $\mathbf{Q}(M,N)$ and the relation

$$\mathbf{Q}(M,N)\mathbf{A}(M,N) = \mathbf{B}(M,N).$$

One can easily show that

$$\mathbf{B}(M,N)^T J\mathbf{B}(M,N) = \begin{pmatrix} 0 & i(M+N) \\ -i(M+N) & 0 \end{pmatrix} = \mathbf{A}(M,N)^T J\mathbf{A}(M,N),$$

from which follows the symplecticity of $\mathbf{Q}(M,N)$. We then write

$$\begin{aligned}\mathbf{Q}(M,N) &+ \overline{\mathbf{Q}}(M,N) \\ &= \mathbf{A}(M,N)^{*-1}\Big[\mathbf{A}(M,N)^{*}\mathbf{B}(M,N) + \mathbf{B}(M,N)^{*}\mathbf{A}(M,N)\Big]\mathbf{A}(M,N)^{-1}, \\ &= \mathbf{A}(M,N)^{*-1}\mathbf{A}(M,N)^{-1},\end{aligned}$$

and obtain the value of $\operatorname{Re}\mathbf{Q}(M,N)$. □

In the remainder of this paper we fix $t \in [0,T]$ and apply Lemma 3.1 with $F = \mathcal{B}$ to (f_t^k, Φ_k) and (g_t^l, Φ_l) for the Wigner transforms associated with $v_t^{\pm}(\pm t, .)$, and (f_x^k, Φ_k) and (g_x^l, Φ_l) for the Wigner transforms associated with $v_x^{\pm}(\pm t, .)$.

3.2 Wigner measure for superposed Gaussian beams

In this section we prove that the cross Wigner transforms $w_\varepsilon(I_\varepsilon(f_{t,x}^k, \Phi^k), I_\varepsilon(g_{t,x}^l, \Phi^l))$ with $k \neq l$, do not contribute to $w[\partial_t u_{\varepsilon,\gamma}^{appr}(t,.)] + c^2 \mathrm{Tr} w[\partial_x u_{\varepsilon,\gamma}^{appr}(t,.)]$ in $\overset{o}{T^*}\Omega$. We compute $\Theta_\varepsilon(\Phi_k, \Phi_k)$ and $A(\Phi_k, \Phi_k)$ and analyze the transported FBI transforms at points $(s \pm \sqrt{\varepsilon} r, \sigma \pm \sqrt{\varepsilon}\delta)$. This enables to complete the study of the Wigner measure for superposed Gaussian beams.

It was pointed out in A3' that $T_\varepsilon u_{\varepsilon,\gamma}^I, T_\varepsilon v_{\varepsilon,\gamma}^I$ have infinitely small contributions in $L^2(\mathbb{R}^n \times \mathrm{supp}(1-\gamma'))$. Besides, Lemma B.2 shows that $T_\varepsilon u_{\varepsilon,\gamma}^I, T_\varepsilon v_{\varepsilon,\gamma}^I$ have infinitely small contributions in $L^2(\mathrm{supp}(1-\rho') \times \mathbb{R}^n)$ because $\rho' \equiv 1$ on $\mathrm{supp} u_{\varepsilon,\gamma}^I$ and $\mathrm{supp} v_{\varepsilon,\gamma}^I$. Therefore

$$\begin{aligned}w_\varepsilon(I_\varepsilon(f_t^k, \Phi_k), I_\varepsilon(g_t^l, \Phi_l)) \\ \approx A(\Phi_k, \Phi_l) \int \left(c|\sigma|\Pi_k \widetilde{p_{\varepsilon,k}}^k\right)_+ \left(c|\sigma|\Pi_l \widetilde{\bar{p}_{\varepsilon,l}}^l\right)_- e^{i\Theta_\varepsilon(\Phi_k,\Phi_l)} dr d\delta \text{ in } \overset{o}{T^*}\Omega,\end{aligned}$$

and a similar relation holds true for $w_\varepsilon(I_\varepsilon(f_x^k, \Phi_k), I_\varepsilon(g_x^l, \Phi_l))$.
We start by approaching $(c(s)|\sigma|)_+ (c(s)|\sigma|)_-$ by $c(s)^2|\sigma|^2$ in $w_\varepsilon(I_\varepsilon(f_t^k, \Phi_k), I_\varepsilon(g_t^l, \Phi_l))$

$$\begin{aligned}w_\varepsilon(I_\varepsilon(f_t^k, \Phi_k), I_\varepsilon(g_t^l, \Phi_l)) \approx A(\Phi_k, \Phi_l) c(s)^2 |\sigma|^2 \int \left(\Pi_k \widetilde{p_{\varepsilon,k}}^k\right)_+ \left(\Pi_l \widetilde{\bar{p}_{\varepsilon,l}}^l\right)_- \\ e^{i\Theta_\varepsilon(\Phi_k,\Phi_l)} dr d\delta \text{ in } \overset{o}{T^*}\Omega,\end{aligned} \tag{35}$$

and $\sigma_+ \sigma_-^*$ by $\sigma\sigma^*$ in $w_\varepsilon(I_\varepsilon(f_x^k, \Phi_k), I_\varepsilon(g_x^l, \Phi_l))$

$$\begin{aligned}w_\varepsilon(I_\varepsilon(f_x^k, \Phi_k), I_\varepsilon(g_x^l, \Phi_l)) \\ \approx A(\Phi_k, \Phi_l)\sigma\sigma^* \int \left(\Pi_k \widetilde{p_{\varepsilon,k}}^k\right)_+ \left(\Pi_l \widetilde{\bar{p}_{\varepsilon,l}}^l\right)_- e^{i\Theta_\varepsilon(\Phi_k,\Phi_l)} dr d\delta \text{ in } \overset{o}{T^*}\Omega.\end{aligned} \tag{36}$$

These approximations result from the following Lemma

Lemma 3.3. *Let Φ, Ψ and $f_\varepsilon, g_\varepsilon$ satisfy the hypotheses of Lemma 3.1. If α and β are in $\mathcal{C}^1(F, \mathbb{C})$ then*

$$w_\varepsilon(I_\varepsilon(\alpha f_\varepsilon, \Phi), I_\varepsilon(\beta g_\varepsilon, \Psi)) \approx \alpha\bar{\beta} w_\varepsilon(I_\varepsilon(f_\varepsilon, \Phi), I_\varepsilon(g_\varepsilon, \Psi)) \text{ in } F.$$

Proof. The proof relies on the use of Taylor's formula on ${\rho'}_f\alpha$ and ${\rho'}_g\bar{\beta}$, where ${\rho'}_f$ and ${\rho'}_g$ are the cut-offs used in the proof of Lemma 3.1 (supported in F and equal to 1 on $\mathrm{supp} f_\varepsilon$ and $\mathrm{supp} g_\varepsilon$ respectively). □

It follows by (35) and (36) that

$$c^2 \mathrm{Tr}\, w_\varepsilon(I_\varepsilon(f_x^k,\Phi_k), I_\varepsilon(g_x^l,\Phi_l)) \approx w_\varepsilon(I_\varepsilon(f_t^k,\Phi_k), I_\varepsilon(g_t^l,\Phi_l)) \text{ in } \overset{o}{T^*}\Omega,$$

which leads to

$$\begin{aligned} w_\varepsilon(v_t^\pm(\pm t,.),&v_t^\pm(\pm t,.)) \approx c^2 \mathrm{Tr}\, w_\varepsilon(v_x^\pm(\pm t,.), v_x^\pm(\pm t,.)), \\ &\text{and } w_\varepsilon(v_t^\pm(\pm t,.), v_t^\mp(\mp t,.)) \approx c^2 \mathrm{Tr}\, w_\varepsilon(v_x^\pm(\pm t,.), v_x^\mp(\mp t,.)) \text{ in } \overset{o}{T^*}\Omega. \end{aligned} \tag{37}$$

The standard estimate (see Proposition 1.1 in [38])

$$| < w_\varepsilon(a_\varepsilon, b_\varepsilon), \phi > | \lesssim \|a_\varepsilon\|_{L^2(\mathbb{R}^n)} \|b_\varepsilon\|_{L^2(\mathbb{R}^n)}, \text{ for } a_\varepsilon, b_\varepsilon \in L^2(\mathbb{R}^n) \text{ and } \phi \in \mathcal{C}_c^\infty(\mathbb{R}^{2n},\mathbb{R}), \tag{38}$$

leads by using the approximations of the derivatives of $u_{\varepsilon,\gamma}^{appr}$ given in Lemma 2.7 to

$$\begin{aligned} 4(w_\varepsilon[\partial_t u_{\varepsilon,\gamma}^{appr}(t,.)]+&c^2 \mathrm{Tr} w_\varepsilon[\partial_x u_{\varepsilon,\gamma}^{appr}(t,.)]) \\ &\approx w_\varepsilon[v_t^+(t,.)] + c^2 \mathrm{Tr} w_\varepsilon[v_x^+(t,.)] + w_\varepsilon[v_t^-(-t,.)] + c^2 \mathrm{Tr} w_\varepsilon[v_x^-(-t,.)] \\ &- w_\varepsilon(v_t^+(t,.), v_t^-(-t,.)) + c^2 \mathrm{Tr}\, w_\varepsilon(v_x^+(t,.), v_x^-(-t,.)) \\ &- w_\varepsilon(v_t^-(-t,.), v_t^+(t,.)) + c^2 \mathrm{Tr}\, w_\varepsilon(v_x^-(-t,.), v_x^+(t,.)) \text{ in } \mathbb{R}^{2n}. \end{aligned}$$

The cross terms between v^+ and v^- cancel in $\overset{o}{T^*}\Omega$ from (37), leading to

$$w_\varepsilon[\partial_t u_{\varepsilon,\gamma}^{appr}(t,.)] + c^2 \mathrm{Tr} w_\varepsilon[\partial_x u_{\varepsilon,\gamma}^{appr}(t,.)] \approx \frac{1}{2} w_\varepsilon[v_t^+(t,.)] + \frac{1}{2} w_\varepsilon[v_t^-(-t,.)] \text{ in } \overset{o}{T^*}\Omega. \tag{39}$$

Thus, we are left with the computation of the Wigner measure associated with v_t^+, computations being also similar for v_t^-. One has

$$\begin{aligned} w_\varepsilon[v_t^+] \approx \sum_{k,l=0,1} c(s)^2 |\sigma|^2 A(\Phi_k,\Phi_l) \int_{\mathbb{R}^{2n}} \left(\Pi_k\, \widetilde{p_{\varepsilon,k}}^k\right)_+ \\ \left(\Pi_l\, \widetilde{\overline{p_{\varepsilon,l}}}^l\right)_- e^{i\Theta_\varepsilon(\Phi_k,\Phi_l)} dr d\delta \text{ in } \overset{o}{T^*}\Omega. \end{aligned} \tag{40}$$

Moreover the inverse of the reflected/incident flow in $\overset{o}{T^*}\Omega$ is a reflected/incident flow

$$\{\varphi_k^t\}^{-1} = \varphi_{-k}^{-t},\ k = 0,1.$$

Thus for $(s,\sigma) \in \overset{o}{T^*}\Omega$, at most one of rays $x_{-k}^{-t}(s,\sigma)$ and $x_{-l}^{-t}(s,\sigma)$ is in Ω. Consequently, the contribution of cross terms between different Gaussian beams in (40) vanishes in $\overset{o}{T^*}\Omega$, and we are left with the computation when ε goes to zero of each of the following two distributions

$$\mu_{\varepsilon,k}^t = c(s)^2 |\sigma|^2 w_\varepsilon(I_\varepsilon(\Pi_k \widetilde{p_{\varepsilon,k}}^k,\Phi_k), I_\varepsilon(\Pi_k \widetilde{p_{\varepsilon,k}}^k,\Phi_k)),\ k = 0,1. \tag{41}$$

Remember that $\widetilde{p_{\varepsilon,k}}^k = \widetilde{a_k}^k \varepsilon^{-1}\widetilde{T_\varepsilon u^I_{\varepsilon,\gamma}}^k + \widetilde{a'_k}^k \widetilde{T_\varepsilon v^I_{\varepsilon,\gamma}}^k$, so $\mu^t_{\varepsilon,k}$ may be written as

$$\begin{aligned}\mu^t_{\varepsilon,k} &= c^2(s)|\sigma|^2 w_\varepsilon(I_\varepsilon(\Pi_k\widetilde{a_k}^k\varepsilon^{-1}T_\varepsilon u^I_{\varepsilon,\gamma},\Phi_k), I_\varepsilon(\Pi_k\widetilde{a_k}^k\varepsilon^{-1}T_\varepsilon u^I_{\varepsilon,\gamma},\Phi_k))\\ &+ w_\varepsilon(I_\varepsilon(\Pi_k\widetilde{a_k}^k T_\varepsilon v^I_{\varepsilon,\gamma},\Phi_k), I_\varepsilon(\Pi_k\widetilde{a_k}^k T_\varepsilon v^I_{\varepsilon,\gamma},\Phi_k))\\ &- ic(s)|\sigma| w_\varepsilon(I_\varepsilon(\Pi_k\widetilde{a_k}^k\varepsilon^{-1}T_\varepsilon u^I_{\varepsilon,\gamma},\Phi_k), I_\varepsilon(\Pi_k\widetilde{a_k}^k T_\varepsilon v^I_{\varepsilon,\gamma},\Phi_k))\\ &+ ic(s)|\sigma| w_\varepsilon(I_\varepsilon(\Pi_k\widetilde{a_k}^k T_\varepsilon v^I_{\varepsilon,\gamma},\Phi_k), I_\varepsilon(\Pi_k\widetilde{a_k}^k\varepsilon^{-1}T_\varepsilon u^I_{\varepsilon,\gamma},\Phi_k)) \text{ in } \overset{o}{T^*}\Omega, \qquad (42)\end{aligned}$$

In the remainder of this section we prove the following result

Proposition 3.4. *Let $\kappa_\varepsilon, \tau_\varepsilon$ be unifromly bounded sequences in $L^2(\mathbb{R}^n)$. Then*

$$w_\varepsilon(I_\varepsilon(\Pi_k\widetilde{a_k}^k T_\varepsilon\kappa_\varepsilon,\Phi_k), I_\varepsilon(\Pi_k\widetilde{a_k}^k T_\varepsilon\tau_\varepsilon,\Phi_k)) \approx \Pi_k^2 w_\varepsilon(\kappa_\varepsilon,\tau_\varepsilon)\, o\{\varphi^t_k\}^{-1} \text{ in } \overset{o}{T^*}\Omega.$$

Above φ^t_k is extended outside $\mathcal{B}$ as the identity.

Proof. **Computation of the phase $\Theta_\varepsilon(\Phi_k,\Phi_k)$ and the amplitude $A(\Phi_k,\Phi_k)$.**

We consider $(s,\sigma) \in \overset{o}{T^*}\Omega$ and start from

$$\Theta_\varepsilon(\Phi_k,\Phi_k)(s,\sigma,r,\delta) = -2\sigma\cdot r/\sqrt{\varepsilon} + i(r,\delta)\cdot \mathbf{Q}\left(\widetilde{\Lambda_k}^k(t,s,\sigma), \widetilde{\bar{\Lambda}_k}^k(t,s,\sigma)\right)(r,\delta).$$

The particular form of $\Lambda_k(t) = -iV^t_k(U^t_k)^{-1}$, see Lemma 2.6, induces a similar form for the matrix $\mathbf{Q}\left(\widetilde{\Lambda_k}^k(t), \widetilde{\bar{\Lambda}_k}^k(t)\right)$

$$\mathbf{Q}\left(\widetilde{\Lambda_k}^k(t), \widetilde{\bar{\Lambda}_k}^k(t)\right)\mathbf{U}^t_k = -i\mathbf{V}^t_k,$$

where $\mathbf{U}^t_k$ and $\mathbf{V}^t_k$ are the $2n\times 2n$ matrices

$$\mathbf{U}^t_k = \begin{pmatrix} \widetilde{\bar{U}^t_k}^k & \widetilde{U^t_k}^k \\ \widetilde{\bar{V}^t_k}^k & \widetilde{V^t_k}^k \end{pmatrix} \text{ and } \mathbf{V}^t_k = \begin{pmatrix} -\widetilde{\bar{V}^t_k}^k & \widetilde{V^t_k}^k \\ \widetilde{\bar{U}^t_k}^k & -\widetilde{U^t_k}^k \end{pmatrix}.$$

Replacing U^t_k and V^t_k by their definitions links $\mathbf{U}^t_k$ and $\mathbf{V}^t_k$ to the Jacobian matrix F^t_k

$$\mathbf{V}^t_k = J\widetilde{F^t_k}^k \begin{pmatrix} -Id & Id \\ iId & iId \end{pmatrix} \text{ and } \mathbf{U}^t_k = -i\widetilde{F^t_k}^k J \begin{pmatrix} -Id & Id \\ iId & iId \end{pmatrix},$$

so that

$$\mathbf{Q}\left(\widetilde{\Lambda_k}^k(t), \widetilde{\bar{\Lambda}_k}^k(t)\right) = -J\widetilde{F^t_k}^k J\left(\widetilde{F^t_k}^k\right)^{-1}.$$

As $\varphi^t_k \ o\ \varphi^{-t}_{-k} = Id$, one has

$$\widetilde{F^t_k}^k F^{-t}_{-k} = Id.$$

Combining this relation with the symplecticity of F^t_k, one gets the following relation for the matrix $\mathbf{Q}\left(\widetilde{\Lambda_k}^k(t), \widetilde{\bar{\Lambda}_k}^k(t)\right)$

$$\mathbf{Q}\left(\widetilde{\Lambda_k}^k(t), \widetilde{\bar{\Lambda}_k}^k(t)\right) = (F^{-t}_{-k})^T F^{-t}_{-k}.$$

Therefore

$$\Theta_\varepsilon(\Phi_k,\Phi_k) = -2\sigma\cdot r/\sqrt{\varepsilon} + i[F_{-k}^{-t}(r,\delta)]^2. \tag{43}$$

Moving to the amplitude $A(\Phi_k,\Phi_k) = c_n^2 2^{\frac{5n}{2}}\pi^{\frac{n}{2}}\left(\det[\widetilde{\Lambda_k}^k + \widetilde{\bar{\Lambda}_k}^k]\right)^{-\frac{1}{2}}$, one gets by using (23) and (24)

$$\Lambda_k(t) + \bar{\Lambda}_k(t) = 2(\bar{U}_k^t)^{-1T}(U_k^t)^{-1}.$$

Hence

$$A(\Phi_k,\Phi_k) = c_n^2 2^{2n}\pi^{\frac{n}{2}}\left|\det \widetilde{U_k^t}^k\right|.$$

Plugging the form of the indident and reflected amplitudes in Lemma 2.6 and using the $\mathcal{C}^1$ smoothness of $a_k^{(\prime)}$ on $\mathcal{B}$ yields by Lemmas 3.1 and 3.3

$$\begin{aligned}&w_\varepsilon(I_\varepsilon(\Pi_k\widetilde{a_k}^k T_\varepsilon\kappa_\varepsilon\Phi_k), I_\varepsilon(\Pi_k\widetilde{a_k}^k T_\varepsilon\tau_\varepsilon,\Phi_k))\\ &\qquad\approx c_n^2 2^{2n}\pi^{\frac{n}{2}}\int\left(\Pi_k\widetilde{T_\varepsilon\kappa_\varepsilon}^k\right)_+\left(\Pi_k\widetilde{\overline{T_\varepsilon\tau_\varepsilon}}^k\right)_- e^{-i2\sigma\cdot r/\sqrt{\varepsilon}-[F_{-k}^{-t}(r,\delta)]^2}drd\delta =: J_{\varepsilon,k}^t(\kappa_\varepsilon,\tau_\varepsilon).\end{aligned}$$

It remains to analyze the most difficult terms in the amplitude, which involve transported FBI transforms

$$\left(\Pi_k\widetilde{T_\varepsilon\kappa_\varepsilon}^k\right)_+ = [\Pi_k\, T_\varepsilon\kappa_\varepsilon\, o\varphi_{-k}^{-t}](s+\sqrt{\varepsilon}r,\sigma+\sqrt{\varepsilon}\delta)$$

$$\left(\Pi_k\widetilde{\overline{T_\varepsilon\tau_\varepsilon}}^k\right)_- = [\Pi_k\, T_\varepsilon\kappa_\varepsilon\, o\varphi_{-k}^{-t}](s-\sqrt{\varepsilon}r,\sigma-\sqrt{\varepsilon}\delta).$$

Analysis of the transported FBI transforms

Let ϑ_{-k}^{-t} be a map of $\mathcal{C}_c^\infty(\mathbb{R}^{2n},\mathbb{R}^{2n})$ that coincides with φ_{-k}^{-t} on $K_{z,\theta}^k(t)\cup \mathrm{supp}\phi$ (see Theorem 1.4.1 of [45]) and use Taylor's formula for this map to get for $(s\pm\sqrt{\varepsilon}r,\sigma\pm\sqrt{\varepsilon}\delta)\in K_{z,\theta}^k(t)$ and $(s,\sigma)\in\mathrm{supp}\phi$

$$\begin{aligned}\left(x_{-k}^{-t}\right)_\pm &= x_{-k}^{-t}\pm\sqrt{\varepsilon}\,\partial_y x_{-k}^{-t}\, r\pm\sqrt{\varepsilon}\,\partial_\eta x_{-k}^{-t}\,\delta+\varepsilon r_\varepsilon^{x\pm},\\ \left(\xi_{-k}^{-t}\right)_\pm &= \xi_{-k}^{-t}\pm\sqrt{\varepsilon}\,\partial_y \xi_{-k}^{-t}\, r\pm\sqrt{\varepsilon}\,\partial_\eta \xi_{-k}^{-t}\,\delta+\varepsilon r_\varepsilon^{\xi\pm},\end{aligned}$$

with

$$(r_\varepsilon^{x\pm}, r_\varepsilon^{\xi\pm})(s,\sigma,r,\delta) = \sum_{|\alpha|=2}\frac{2}{\alpha!}(r,\delta)^\alpha\int_0^1(1-u)\partial_{y,\eta}^\alpha\vartheta_{-k}^{-t}\left((s,\sigma)\pm u\sqrt{\varepsilon}\,(r,\delta)\right)du.$$

The change of variables $\begin{pmatrix} r'\\ \delta'\end{pmatrix} = F_{-k}^{-t}(s,\sigma)\begin{pmatrix} r\\ \delta\end{pmatrix}$ in $J_{\varepsilon,k}^t(\kappa_\varepsilon,\tau_\varepsilon)(s,\sigma)$ is appropriate. Notice that for $(s,\sigma)\in \overset{o}{T^*\Omega}$ one has the following relations [58]

$$x_{-k_y}^{-u}(s,\sigma)\cdot\xi_{-k}^{-u}(s,\sigma)-\sigma = 0 \text{ and } x_{-k_\eta}^{-u}(s,\sigma)\cdot\xi_{-k}^{-u}(s,\sigma) = 0 \text{ for } u\in\mathbb{R}.$$

In fact, one can show that the derivatives of the previous equations w.r.t. u are zero. Besides, the equalities clearly hold true at $u=0$ for $k=0$ and at $u=T_k(s,\sigma)$ for $k=\pm1$, as a consequence of (26). Hence, it follows that

$$\sigma\cdot r = \xi_{-k}^{-t}(s,\sigma)\cdot(\partial_y x_{-k}^{-t}(s,\sigma)r+\partial_\eta x_{-k}^{-t}(s,\sigma)\delta) = \xi_{-k}^{-t}(s,\sigma)\cdot r',$$

which leads in $\overset{o}{T^*\Omega}$ to

$$\begin{aligned}J^t_{\varepsilon,k}(\kappa_\varepsilon,\tau_\varepsilon) =& c_n^2 2^{2n}\pi^{\frac{n}{2}}\int(\Pi_k)_+T_\varepsilon\kappa_\varepsilon(x^{-t}_{-k}+\sqrt{\varepsilon}r'+\varepsilon r^{x+'}_\varepsilon,\xi^{-t}_{-k}+\sqrt{\varepsilon}\delta'+\varepsilon r^{\xi+'}_\varepsilon)\\ &(\Pi_k)_-\overline{T_\varepsilon\tau_\varepsilon}(x^{-t}_{-k}-\sqrt{\varepsilon}r'+\varepsilon r^{x-'}_\varepsilon,\xi^{-t}_{-k}-\sqrt{\varepsilon}\delta'+\varepsilon r^{\xi-'}_\varepsilon)\\ &e^{-2i\xi^{-t}_{-k}\cdot r'/\sqrt{\varepsilon}-r'^2-\delta'^2}dr'd\delta',\end{aligned}$$

where

$$(r^{x\prime}_\varepsilon,r^{\xi\prime}_\varepsilon)(s,\sigma,r',\delta')=(r^x_\varepsilon,r^\xi_\varepsilon)(s,\sigma,r,\delta).$$

Let ϕ be a test function of $\mathcal{C}^\infty_c(\mathbb{R}^{2n},\mathbb{R})$ supported in $\overset{o}{T^*\Omega}$. We want to use the change of variables $(s,\sigma)=\varphi^t_k(y,\eta)$ when computing $< J^t_{\varepsilon,k}(\kappa_\varepsilon,\tau_\varepsilon),\phi >$, so we extend φ^t_k outside $\mathcal{B}$ by the identity and still denote it φ^t_k, so that φ^t_k is now a one to one map from $\mathbb{R}^{2n}$ to $\varphi^t_k(\mathbb{R}^{2n})$. Then $\Pi_k\, o\varphi^t_k$ and $\phi\, o\varphi^t_k$ belong to $\mathcal{C}^\infty_c(\mathbb{R}^{2n},\mathbb{R})$ and are supported in $\mathcal{B}$. Expanding the FBI transforms gives

$$\begin{aligned}< J^t_{\varepsilon,k}(\kappa_\varepsilon,\tau_\varepsilon),\phi > = c_n^4 2^{2n}\pi^{\frac{n}{2}}\varepsilon^{-\frac{3n}{2}}\int_{\mathbb{R}^{6n}}&\phi\, o\varphi^t_k(y,\eta)\\ &(\Pi_k\, o\varphi^t_k)(y+\sqrt{\varepsilon}r'+\varepsilon R^{x+}_\varepsilon,\eta+\sqrt{\varepsilon}\delta'+\varepsilon R^{\xi+}_\varepsilon)\\ &(\Pi_k\, o\varphi^t_k)(y-\sqrt{\varepsilon}r'+\varepsilon R^{x-}_\varepsilon,\eta-\sqrt{\varepsilon}\delta'+\varepsilon R^{\xi-}_\varepsilon)\\ &\kappa_\varepsilon(z)\bar{\tau}_\varepsilon(z')e^{i\eta\cdot(2\sqrt{\varepsilon}r'+\varepsilon R^{x+}_\varepsilon-\varepsilon R^{x-}_\varepsilon-z+z')/\varepsilon+i\delta'\cdot(2y-z-z'+\varepsilon R^{x+}_\varepsilon+\varepsilon R^{x-}_\varepsilon)/\sqrt{\varepsilon}}\\ &e^{iR^{\xi+}_\varepsilon\cdot(y+\sqrt{\varepsilon}r'+\varepsilon R^{x+}_\varepsilon-z)-iR^{\xi-}_\varepsilon\cdot(y-\sqrt{\varepsilon}r'+\varepsilon R^{x-}_\varepsilon-z')-(y+\sqrt{\varepsilon}r'+\varepsilon R^{x+}_\varepsilon-z)^2/(2\varepsilon)}\\ &e^{-(y-\sqrt{\varepsilon}r'+\varepsilon R^{x-}_\varepsilon-z')^2/(2\varepsilon)-2i\eta\cdot r'/\sqrt{\varepsilon}-r'^2-\delta'^2}dr'd\delta'dzdz'dyd\eta,\end{aligned}$$

where

$$(R^x_\varepsilon,R^\xi_\varepsilon)(y,\eta,r',\delta')=(r^{x\prime}_\varepsilon,r^{\xi\prime}_\varepsilon)(s,\sigma,r',\delta').$$

We perform the following changes of variables

$$(x,u)=(\frac{z+z'}{2},\frac{z-z'}{\varepsilon})\text{ and }y'=(y-\frac{z+z'}{2})/\sqrt{\varepsilon}$$

to obtain

$$< J^t_{\varepsilon,k}(\kappa_\varepsilon,\tau_\varepsilon),\phi > = c_n^4 2^{2n}\pi^{\frac{n}{2}}\int_{\mathbb{R}^{6n}}\kappa_\varepsilon(x+\frac{\varepsilon}{2}u)\bar{\tau}_\varepsilon(x-\frac{\varepsilon}{2}u)d_\varepsilon e^{i\gamma_\varepsilon-i\eta\cdot u}dr'd\delta'dxdudy'd\eta,$$

where

$$\begin{aligned}&d_\varepsilon(x,y',\eta,r',\delta')\\ &\quad=\phi\, o\varphi^t_k(x+\sqrt{\varepsilon}y',\eta)(\Pi_k\, o\varphi_k)(x+\sqrt{\varepsilon}y'+\sqrt{\varepsilon}r'+\varepsilon R^{x+'}_\varepsilon,\eta+\sqrt{\varepsilon}\delta'+\varepsilon R^{\xi+'}_\varepsilon)\\ &\quad(\Pi_k\, o\varphi_k)(x+\sqrt{\varepsilon}y'-\sqrt{\varepsilon}r'+\varepsilon R^{x-'}_\varepsilon,\eta-\sqrt{\varepsilon}\delta'+\varepsilon R^{\xi-'}_\varepsilon),\end{aligned}$$

$$\begin{aligned}&\gamma_\varepsilon(x,y',\eta,r',\delta',u)\\ &\quad=\eta\cdot(R^{x+'}_\varepsilon-R^{x-'}_\varepsilon)+\delta'\cdot(2y'+\sqrt{\varepsilon}R^{x+'}_\varepsilon+\sqrt{\varepsilon}R^{x-'}_\varepsilon)\\ &\quad+\sqrt{\varepsilon}R^{\xi+'}_\varepsilon\cdot(y'+r'+\sqrt{\varepsilon}R^{x+'}_\varepsilon-\sqrt{\varepsilon}\frac{u}{2})\\ &\quad-\sqrt{\varepsilon}R^{\xi-'}_\varepsilon\cdot(y'-r'+\sqrt{\varepsilon}R^{x-'}_\varepsilon+\sqrt{\varepsilon}\frac{u}{2})+ir'^2+i\delta'^2\\ &\quad+i(y'+r'+\sqrt{\varepsilon}R^{x+'}_\varepsilon-\sqrt{\varepsilon}u/2)^2/2+i(y'-r'+\sqrt{\varepsilon}R^{x-'}_\varepsilon+\sqrt{\varepsilon}u/2)^2/2,\end{aligned}$$

and

$$(R^{x\prime}_{\varepsilon}, R^{\xi\prime}_{\varepsilon})(x, y', \eta, r', \delta') = (R^x_{\varepsilon}, R^{\xi}_{\varepsilon})(x + \sqrt{\varepsilon}y', \eta, r', \delta').$$

Notice that $d_\varepsilon(x, y', \eta, r', \delta')$ converges when $\varepsilon \to 0$ to

$$d_0(x, \eta) = \phi\, o\varphi^t_k(x, \eta)(\Pi_k\, o\varphi_k)^2(x, \eta).$$

On the other hand, since $\varepsilon r^{\pm}_x$ are the remainder terms in the Taylor expansions of $x^{-t}_{-k}(s \pm \sqrt{\varepsilon}r, \sigma \pm \sqrt{\varepsilon}\delta)$ at order 2, $r^{x+}_{\varepsilon} - r^{x-}_{\varepsilon}$ is of order $\sqrt{\varepsilon}$ and so is $R^{x+\prime}_{\varepsilon} - R^{x-\prime}_{\varepsilon}$, leading to

$$\gamma_\varepsilon(x, y', \eta, r', \delta', u) \underset{\varepsilon\to 0}{\to} \gamma_0(y', r', \delta') = 2\delta' \cdot y' + iy'^2 + 2ir'^2 + i\delta'^2.$$

One has

$$|< J^t_{\varepsilon,k}(\kappa_\varepsilon, \tau_\varepsilon), \phi > -c_n^4 2^{2n}\pi^{\frac{n}{2}} \int \kappa_\varepsilon(x + \frac{\varepsilon}{2}u)\bar{\tau}_\varepsilon(x - \frac{\varepsilon}{2}u) d_0 e^{i\gamma_0} e^{-i\eta\cdot u} dr' d\delta' du dy' dx d\eta| \lesssim$$

$$\int \left[\int_{\mathbb{R}^n} |\kappa_\varepsilon|(x + \frac{\varepsilon}{2}u)|\tau_\varepsilon|(x - \frac{\varepsilon}{2}u) dx\right]$$

$$\sup_x \left|\mathcal{F}_\eta[d_\varepsilon e^{i\gamma_\varepsilon} - d_0 e^{i\gamma_0}](x, y', u, r', \delta', u)\right| dr' d\delta' du dy'. \quad (44)$$

Cauchy-Schwartz inequality w.r.t. dx insures that the bracket integral is less than $\|\kappa_\varepsilon\|_{L^2}\|\tau_\varepsilon\|_{L^2}$. Let us examin the term

$$\int \sup_x \left|\mathcal{F}_\eta[d_\varepsilon e^{i\gamma_\varepsilon} - d_0 e^{i\gamma_0}](x, y', u, r', \delta', u)\right| dr' d\delta' du dy'.$$

For fixed y', u, r', δ', the functions d_ε and d_0 are compactly supported w.r.t. (x, η) so

$$\sup_x |\mathcal{F}_\eta[d_\varepsilon e^{i\gamma_\varepsilon} - d_0 e^{i\gamma_0}](x, y', u, r', \delta', u)| \lesssim \sup_{(x,\eta)} |[d_\varepsilon e^{i\gamma_\varepsilon} - d_0 e^{i\gamma_0}](x, y', \eta, r', \delta', u)|.$$

Note that $|d_\varepsilon e^{i\gamma_\varepsilon} - d_0 e^{i\gamma_0}|$ is dominated by $|d_\varepsilon - d_0| + |d_0||e^{i\gamma_\varepsilon - i\gamma_0} - 1|$. The convergence of d_ε when $\varepsilon \to 0$ to its limit d_0 is uniform w.r.t. (x, η) and so is the convergence of γ_ε to γ_0 on the support of d_0. Thus $d_\varepsilon e^{i\gamma_\varepsilon}$ converges to $d_0 e^{i\gamma_0}$ uniformly w.r.t. (x, η). It follows that

$$\sup_x |\mathcal{F}_\eta[d_\varepsilon e^{i\gamma_\varepsilon} - d_0 e^{i\gamma_0}](x, y', u, r', \delta', u)| \underset{\varepsilon\to 0}{\to} 0 \text{ for every } y', u, r', \delta'.$$

On the other hand, successive integrations by parts give

$$\int_{\mathbb{R}^n} d_\varepsilon e^{i\gamma_\varepsilon} e^{-i\eta\cdot u} d\eta = (1 + u^2)^{-n} \int_{\mathbb{R}^n} L\left(d_\varepsilon e^{i\gamma_\varepsilon}\right) e^{-i\eta\cdot u} d\eta,$$

with L a differential operator w.r.t. η, of order $2n$. Thus,

$$\sup_x |\mathcal{F}_\eta[d_\varepsilon e^{i\gamma_\varepsilon}](x, y', u, r', \delta', u)| \lesssim (1 + u^2)^{-n} \sup_{(x,\eta)} \max_{|\alpha|\le 2n} |\partial^\alpha_\eta \left(d_\varepsilon e^{i\gamma_\varepsilon}\right)(x, y', \eta, r', \delta', u)|, \quad (45)$$

for every y', u, r', δ'. The quantities $(x + \sqrt{\varepsilon}y', \eta)$ and $\sqrt{\varepsilon}(r', \delta')$ are bounded on the support of d_ε, so $R^{x\pm\prime}_{\varepsilon}$, $R^{\xi\pm\prime}_{\varepsilon}$ and their derivatives w.r.t. η are dominated by $(r', \delta')^2$ and for every multiindex α, there exists $C > 0$ s.t.

$$|\partial^\alpha_\eta d_\varepsilon| \le C, \quad |\partial^\alpha_\eta \gamma_\varepsilon| \le C(r', \delta')^2(1 + |\eta| + |\delta'| + (r', \delta')^2$$
$$+ |y' + r' + \sqrt{\varepsilon}R^{x+\prime}_{\varepsilon} - \sqrt{\varepsilon}u/2| + |y' - r' + \sqrt{\varepsilon}R^{x-\prime}_{\varepsilon} - \sqrt{\varepsilon}u/2|)$$

for all $(x, y', \eta, r', \delta') \in \text{supp} d_\varepsilon$ and $u \in \mathbb{R}^n$. Thus, there exists $C, C' > 0$ s.t.

$$\begin{aligned}|\partial_\eta^\alpha[d_\varepsilon e^{i\gamma_\varepsilon}]| &\leq Ce^{-C'(y'+r'+\sqrt{\varepsilon}R_\varepsilon^{x+'}-\sqrt{\varepsilon}u/2)^2-C'(y'-r'+\sqrt{\varepsilon}R_\varepsilon^{x-'}+\sqrt{\varepsilon}u/2)^2-C'r'^2-C'\delta'^2}\\ &\leq Ce^{-C'(2y'+\sqrt{\varepsilon}R_\varepsilon^{x+'}+\sqrt{\varepsilon}R_\varepsilon^{x-'})^2-C'r'^2-C'\delta'^2}\end{aligned}$$

for all $(x, y', \eta, r', \delta') \in \text{supp} d_\varepsilon$ and $u \in \mathbb{R}^n$. On the support of d_ε, $\sqrt{\varepsilon}R_\varepsilon^{x\pm'}$ are dominated by $|(r', \delta')|$, which implies for some $C_0 > 0$ that

$$(2y' + \sqrt{\varepsilon}R_\varepsilon^{x+'} + \sqrt{\varepsilon}R_\varepsilon^{x-'})^2 \geq 4y'^2 - C_0|(r', \delta')||y'|.$$

Hence, if $|y'| \geq C_0|(r', \delta')|$, $e^{-C'(2y'+\sqrt{\varepsilon}R_\varepsilon^{x+'}+\sqrt{\varepsilon}R_\varepsilon^{x-'})^2} \leq e^{-C''y'^2}$. Otherwise, $e^{-C'r'^2-C'\delta'^2} \leq e^{-C''y'^2-C''r'^2-C''\delta'^2}$. In all cases, there exists $C', C'' > 0$ s.t.

$$|\partial_\eta^\alpha[d_\varepsilon e^{i\gamma_\varepsilon}]| \leq C'e^{-C''y'^2-C''r'^2-C''\delta'^2}$$

for every $x, y', \eta, r', \delta', u$ and $\varepsilon \in]0, \varepsilon_0]$ with some $\varepsilon_0 > 0$. Using this in (45) leads to

$$\sup_x |\mathcal{F}_\eta[d_\varepsilon e^{i\gamma_\varepsilon}](x, y', u, r', \delta', u)| \lesssim (1 + u^2)^{-n} e^{-Cy'^2-Cr'^2-C\delta'^2},$$

and repeating the same arguments for $\sup_x |\mathcal{F}_\eta[d_0 e^{i\gamma_0}]|$ gives

$$\sup_x |\mathcal{F}_\eta[d_\varepsilon e^{i\gamma_\varepsilon} - d_0 e^{i\gamma_0}](x, y', u, r', \delta', u)| \lesssim (1 + u^2)^{-n} e^{-Cy'^2-Cr'^2-C\delta'^2}$$

for every $x, y', \eta, r', \delta', u$ and $\varepsilon \in]0, \varepsilon_0]$. By the dominated convergence theorem, one obtains

$$\int \sup_x |\mathcal{F}_\eta[d_\varepsilon e^{i\gamma_\varepsilon} - d_0 e^{i\gamma_0}](x, y', u, r', \delta', u)| dy' du dr' d\delta' \underset{\varepsilon \to 0}{\longrightarrow} 0.$$

From the inequality (44) concerning the distribution $J_{\varepsilon,k}^t(\kappa_\varepsilon, \tau_\varepsilon)$, one finally has by plugging the expressions of d_0 and γ_0

$$\begin{aligned}< J_{\varepsilon,k}^t(\kappa_\varepsilon, \tau_\varepsilon), \phi >= c_n^4 2^{2n} \pi^{\frac{n}{2}} \int_{\mathbb{R}^{6n}} d_0 \kappa_\varepsilon(x + \frac{\varepsilon}{2}u)\bar{\tau}_\varepsilon(x - \frac{\varepsilon}{2}u)\\ e^{2i\delta'\cdot y' - y'^2 - 2r'^2 - \delta'^2} e^{-i\eta\cdot u} dr' d\delta' dx du dy' d\eta + o(1).\end{aligned}$$

Integration w.r.t. r', δ', y' yields

$$\begin{aligned}&< J_{\varepsilon,k}^t(\kappa_\varepsilon,\tau_\varepsilon), \phi >\\ &\quad= (2\pi)^{-n} \int_{\mathbb{R}^{2n}} \mathcal{F}_\eta \left[\Pi_k^2\, o\varphi_k^t \phi o\varphi_k^t\right](x, u)\kappa_\varepsilon(x + \frac{\varepsilon}{2}u)\bar{\tau}_\varepsilon(x - \frac{\varepsilon}{2}u)dxdu + o(1).\end{aligned}$$

The integral in the l.h.s. is exactly the Wigner transform of $(\kappa_\varepsilon, \tau_\varepsilon)$ tested on $\Pi_k^2\, o\varphi_k^t \phi o\varphi_k^t$. □

One gets by using Proposition 3.4, Lemma B.6 and the expression (42) of $\mu_{\varepsilon,k}^t$

$$\mu_{\varepsilon,k}^t \approx \Pi_k^2 \left(w_\varepsilon[v_{\varepsilon,\gamma}^I - ic|D|u_{\varepsilon,\gamma}^I]\right) o\{\varphi_k^t\}^{-1} \text{ in } \overset{o}{T^*}\Omega.$$

Recalling the relation between the Wigner measure and the FBI transform (see Proposition 1.4 of [37])

$$\int |T_\varepsilon a_\varepsilon|^2 \theta dy d\eta \underset{\varepsilon \to 0}{\longrightarrow} < w[a_\varepsilon], \theta >$$
$$\text{for } \theta \in \mathcal{C}_c^\infty(\mathbb{R}^{2n}, \mathbb{R}) \text{ and } (a_\varepsilon) \text{ uniformly bounded in } L^2(\mathbb{R}^n), \quad (46)$$

it follows that $w_\varepsilon[v^I_{\varepsilon,\gamma} - ic|D|u^I_{\varepsilon,\gamma}] \approx 0$ in $(K_y \times K_\eta)^c$ or equivalently

$$w_\varepsilon[v^I_{\varepsilon,\gamma} - ic|D|u^I_{\varepsilon,\gamma}]\, o\{\varphi^t_k\}^{-1} \approx 0 \text{ in } (K^k_{z,\theta}(t))^c.$$

Since $\Pi_k \equiv 1$ on $K^k_{z,\theta}(t)$, one deduces

$$\mu^t_{\varepsilon,k} \approx w_\varepsilon[v^I_{\varepsilon,\gamma} - ic|D|u^I_{\varepsilon,\gamma}]\, o\{\varphi^t_k\}^{-1} \text{ in } \overset{o}{T^*}\Omega.$$

By summing over $k = 0, 1$ and letting $\varepsilon \to 0$, we get the transport on the incident and the reflected flows of $\mu[v^I_{\varepsilon,\gamma} - ic|D|u^I_{\varepsilon,\gamma}]$

$$w[v^+_t(t,.)] \;=\; \sum_{k=0,1} \mu[v^I_{\varepsilon,\gamma} - ic|D|u^I_{\varepsilon,\gamma}]\, o\{\varphi^t_k\}^{-1} \text{ in } \overset{o}{T^*}\Omega,$$

For $u \in [-T, T]$ and $(y, \eta) \in K_y \times (\mathbb{R}^n\backslash\{0\})$, the incident and reflected flows are related to the broken bicharacteristic flow associated to $-i\partial_t + c|D|$ as follows

$$\varphi^u_b(y,\eta) = \begin{cases} \varphi^u_{-1}(y,\eta) \text{ if } u < T_{-1}(y,\eta) \\ \varphi^u_0(y,\eta) \text{ if } T_{-1}(y,\eta) < u < T_1(y,\eta) \\ \varphi^u_1(y,\eta) \text{ if } u > T_1(y,\eta). \end{cases}$$

We define φ^u_b in $(\Omega\backslash K_y) \times (\mathbb{R}^n\backslash\{0\})$ by successively reflecting the rays at the boundary. We extend φ^u_b at times of reflections arbitrary. As only one incident/reflected ray can be in the interior of the domain at a fixed time

$$\phi\, o\varphi^t_b = \sum_{k=0,1} \phi\, o\varphi^t_k \text{ in } K_y \times \mathbb{R}^n\backslash\{0\}.$$

It follows that

$$w[v^+_t(t,.)] \;=\; \mu[v^I_{\varepsilon,\gamma} - ic|D|u^I_{\varepsilon,\gamma}]\, o \left(\varphi^t_b\right)^{-1} \text{ in } \overset{o}{T^*}\Omega.$$

The computations for v^-_t are similar. One has just to replace the index $k = 1$ by $k = -1$ and $\widetilde{p_{\varepsilon,k}}^k$ by $\widetilde{q_{\varepsilon,k}}^k$ in equations (40), (41). Set

$$\Upsilon^\pm_\varepsilon = \underline{v}^I_\varepsilon \pm ic|D|\underline{u}^I_\varepsilon \text{ and } \Upsilon^\pm_{\varepsilon,\gamma} = v^I_{\varepsilon,\gamma} \pm ic|D|u^I_{\varepsilon,\gamma}.$$

One gets

$$w[v^-_t(-t,.)] \;=\; w[\Upsilon^+_{\varepsilon,\gamma}]\, o \left(\varphi^{-t}_b\right)^{-1} \text{ in } \overset{o}{T^*}\Omega.$$

Plugging these results in the expression (39) of the scalar Wigner measure associated to $u^{appr}_{\varepsilon,\gamma}$ leads to

$$\begin{aligned} w[\partial_t u^{appr}_{\varepsilon,\gamma}(t,.)] &+ \mathrm{Tr} w[c\partial_x u^{appr}_{\varepsilon,\gamma}(t,.)] \\ &= \frac{1}{2} w[\Upsilon^+_{\varepsilon,\gamma}]\, o \left(\varphi^{-t}_b\right)^{-1} + \frac{1}{2} w[\Upsilon^-_{\varepsilon,\gamma}]\, o \left(\varphi^t_b\right)^{-1} \text{ in } \overset{o}{T^*}\Omega. \end{aligned} \tag{47}$$

3.3 Proof of the main theorem

A consequence of the estimate (38) is

$$| < w(a_\varepsilon, b_\varepsilon), \theta > | \lesssim \overline{\lim_{\varepsilon\to 0}} \|a_\varepsilon\|_{L^2(\Omega)} \overline{\lim_{\varepsilon\to 0}} \|b_\varepsilon\|_{L^2(\Omega)}, \tag{48}$$

for $a_\varepsilon, b_\varepsilon$ uniformly bounded in $L^2(\mathbb{R}^n)$ and $\theta \in \mathcal{C}_c^\infty(T^*\overset{o}{\Omega}, \mathbb{R})$.

Using this estimate (38) on the difference between the derivatives of the exact and approximate solutions of the IBVP (1a)- (1b) with initial conditions (1c'), one deduces the measures associated to $\underline{\partial_t u}_{\varepsilon,\gamma}$ and $\underline{\partial_x u}_{\varepsilon,\gamma}$ and gets by (47)

$$w[\underline{\partial_t u}_{\varepsilon,\gamma}(t,.)] + \mathrm{Tr} w[c\underline{\partial_x u}_{\varepsilon,\gamma}(t,.)] = \frac{1}{2} w[\Upsilon^+_{\varepsilon,\gamma}]\, o \left(\varphi_b^{-t}\right)^{-1} + \frac{1}{2} w[\Upsilon^-_{\varepsilon,\gamma}]\, o \left(\varphi_b^{t}\right)^{-1} \text{ in } T^*\overset{o}{\Omega}.$$

Remark 3.5. *Gaussian beams summation of first order beams allows to compute the Wigner measure for the solution of the IBVP* (1)*, under hypothesis* (A1)-(A3) *on initial conditions. Summation of higher order beams may imply asymptotic formulas for the Wigner transform. Higher order terms in the Wigner transform's expansion were studied for instance in [83] and [29] for WKB initial data.*

Let us now study the scalar Wigner measure for the problem (1), by making the data $(u^I_{\varepsilon,\gamma}, v^I_{\varepsilon,\gamma})$ approach $(u^I_\varepsilon, v^I_\varepsilon)$. The contribution of the sets $\{\eta \in \mathbb{R}^n, |\eta| \geq r_\infty/4\}$ and $\{\eta \in \mathbb{R}^n, |\eta| \leq 4r_0\}$ where $\gamma \not\equiv 1$ (remember the definition of γ in (14)) is controlled asymptotically by assumptions C2 and C3 respectively.
Denote $\phi^t = \phi o \varphi_b^t$, then $\phi^t \in \mathcal{C}_c^\infty(\mathbb{R}^{2n}, \mathbb{R})$. One has

$$\begin{aligned}
&\Big| < w[\underline{\partial_t u}_\varepsilon(t,.)] + \mathrm{Tr} w[c\underline{\partial_x u}_\varepsilon(t,.)], \phi > -\frac{1}{2} < w[\Upsilon^+_\varepsilon], \phi^{-t} > -\frac{1}{2} < w[\Upsilon^-_\varepsilon], \phi^t > \Big| \leq \\
&| < w[\underline{\partial_t u}_\varepsilon(t,.)] - w[\underline{\partial_t u}_{\varepsilon,\gamma}(t,.)], \phi > | + \sum_{b=1}^n | < w[c\underline{\partial_{x_b} u}_\varepsilon(t,.)] - w[c\underline{\partial_{x_b} u}_{\varepsilon,\gamma}(t,.)], \phi > | \\
&+| < w[\underline{\partial_t u}_{\varepsilon,\gamma}(t,.)] + \mathrm{Tr} w[c\underline{\partial_x u}_{\varepsilon,\gamma}(t,.)], \phi > -\frac{1}{2} < w[\Upsilon^+_{\varepsilon,\gamma}], \phi^{-t} > -\frac{1}{2} < w[\Upsilon^-_{\varepsilon,\gamma}], \phi^t > | \\
&+\frac{1}{2}| < w[\Upsilon^+_{\varepsilon,\gamma}] - w[\Upsilon^+_\varepsilon], \phi^{-t} > | + \frac{1}{2}| < w[\Upsilon^-_{\varepsilon,\gamma}] - w[\Upsilon^-_\varepsilon], \phi^t > |.
\end{aligned} \tag{49}$$

We use (38) to get

$$\begin{aligned}
| < w[\Upsilon^+_{\varepsilon,\gamma}] - w[\Upsilon^+_\varepsilon], \phi^{-t} > | &\lesssim \overline{\lim_{\varepsilon\to 0}} \|\Upsilon^+_{\varepsilon,\gamma} - \Upsilon^+_\varepsilon\|_{L^2(\mathbb{R}^n)} \overline{\lim_{\varepsilon\to 0}} \left(\|\Upsilon^+_{\varepsilon,\gamma}\|_{L^2(\mathbb{R}^n)} + \|\Upsilon^+_\varepsilon\|_{L^2(\mathbb{R}^n)} \right) \\
&\lesssim \overline{\lim_{\varepsilon\to 0}} \|v^I_\varepsilon - v^I_{\varepsilon,\gamma}\|_{L^2(\Omega)} + \overline{\lim_{\varepsilon\to 0}} \|u^I_\varepsilon - u^I_{\varepsilon,\gamma}\|_{H^1(\Omega)}.
\end{aligned}$$

Similarly, by (48)

$$\begin{aligned}
&| < w[\underline{\partial_t u}_\varepsilon(t,.)] - w[\underline{\partial_t u}_{\varepsilon,\gamma}(t,.)], \phi > | \lesssim \\
&\quad \overline{\lim_{\varepsilon\to 0}} \|\partial_t u_\varepsilon(t,.) - \partial_t u_{\varepsilon,\gamma}(t,.)\|_{L^2(\Omega)} (\overline{\lim_{\varepsilon\to 0}} \|\partial_t u_\varepsilon(t,.)\|_{L^2(\Omega)} + \overline{\lim_{\varepsilon\to 0}} \|\partial_t u_{\varepsilon,\gamma}(t,.)\|_{L^2(\Omega)}),
\end{aligned}$$

and for $b = 1, \ldots, n$

$$\begin{aligned}
&| < w[\underline{\partial_{x_b} u}_\varepsilon(t,.)] - w[\underline{\partial_{x_b} u}_{\varepsilon,\gamma}(t,.)], \phi > | \lesssim \\
&\quad \overline{\lim_{\varepsilon\to 0}} \|\partial_{x_b} u_\varepsilon(t,.) - \partial_{x_b} u_{\varepsilon,\gamma}(t,.)\|_{L^2(\Omega)} \left(\overline{\lim_{\varepsilon\to 0}} \|\partial_{x_b} u_\varepsilon(t,.)\|_{L^2(\Omega)} + \overline{\lim_{\varepsilon\to 0}} \|\partial_{x_b} u_{\varepsilon,\gamma}(t,.)\|_{L^2(\Omega)} \right).
\end{aligned}$$

The solution of the IBVP for the wave equation is given by a continuous unitary evolution group on the space $H^1(\Omega, dx) \times L^2(\Omega, dx)$. Hence

$$\|\partial_t u_\varepsilon(t,.) - \partial_t u_{\varepsilon,\gamma}(t,.)\|_{L^2(\Omega)} \lesssim \|v_\varepsilon^I - v_{\varepsilon,\gamma}^I\|_{L^2(\Omega)} + \|u_\varepsilon^I - u_{\varepsilon,\gamma}^I\|_{H^1(\Omega)},$$
$$\|\partial_{x_b} u_\varepsilon(t,.) - \partial_{x_b} u_{\varepsilon,\gamma}(t,.)\|_{L^2(\Omega)} \lesssim \|v_\varepsilon^I - v_{\varepsilon,\gamma}^I\|_{L^2(\Omega)} + \|u_\varepsilon^I - u_{\varepsilon,\gamma}^I\|_{H^1(\Omega)},\ b = 1,\dots,n.$$

We then have by (47)

$$| < w[\underline{\partial_t u_\varepsilon}] + \mathrm{Tr} w[c\underline{\partial_x u}_\varepsilon], \phi > - \frac{1}{2} < w[\Upsilon_\varepsilon^+], \phi^{-t} > - \frac{1}{2} < w[\Upsilon_\varepsilon^-], \phi^t > | \lesssim \overline{\lim_{\varepsilon\to 0}} \|v_\varepsilon^I - v_{\varepsilon,\gamma}^I\|_{L^2(\Omega)} + \overline{\lim_{\varepsilon\to 0}} \|u_\varepsilon^I - u_{\varepsilon,\gamma}^I\|_{H^1(\Omega)}. \tag{50}$$

We therefore need to estimate the difference between initial data (1c) and (1c'). We start by the initial speed. By the exponential decrease of $T_\varepsilon^* \gamma T_\varepsilon \underline{v}_\varepsilon^I$ on the support of $1-\rho$ (16), one has

$$\|v_\varepsilon^I - v_{\varepsilon,\gamma}^I\|_{L^2(\Omega)} \lesssim \varepsilon^\infty + \|\underline{v}_\varepsilon^I - T_\varepsilon^* \gamma T_\varepsilon \underline{v}_\varepsilon^I\|_{L^2(\mathbb{R}^n)}.$$

Because T_ε^* is bounded on $L^2(\mathbb{R}^{2n}) \to L^2(\mathbb{R}^n)$ and $T_\varepsilon^* T_\varepsilon = Id$

$$\|\underline{v}_\varepsilon^I - T_\varepsilon^* \gamma T_\varepsilon \underline{v}_\varepsilon^I\|_{L^2(\mathbb{R}^n)} \le \underset{①}{\|(1-\chi_{r_\infty/2}) T_\varepsilon \underline{v}_\varepsilon^I\|_{L^2(\mathbb{R}^{2n})}} + \underset{②}{\|\chi_{r_\infty/2}\chi_{4r_0} T_\varepsilon \underline{v}_\varepsilon^I\|_{L^2(\mathbb{R}^{2n})}}.$$

Firstly, writing the expression of the FBI transform given in Lemma B.1 as the Fourier transform of some auxiliary function, it follows by Parseval equality that

$$\|\varepsilon^{-\frac{n}{4}}(1-\chi_{r_\infty/2}(\eta)) \int \mathcal{F}\underline{v}_\varepsilon^I(\xi) e^{i\xi\cdot y - (\eta - \varepsilon\xi)^2/(2\varepsilon)} d\xi\|_{L^2(\mathbb{R}^{2n})}^2 =$$
$$(2\pi)^n \varepsilon^{-\frac{n}{2}} \int_{|\varepsilon\xi| \le r_\infty/4} (1-\chi_{r_\infty/2}(\eta))^2 |\mathcal{F}\underline{v}_\varepsilon^I(\xi)|^2 e^{-(\eta-\varepsilon\xi)^2/\varepsilon} d\xi d\eta$$
$$+ (2\pi)^n \varepsilon^{-\frac{n}{2}} \int_{|\varepsilon\xi| \ge r_\infty/4} (1-\chi_{r_\infty/2}(\eta))^2 |\mathcal{F}\underline{v}_\varepsilon^I(\xi)|^2 e^{-(\eta-\varepsilon\xi)^2/\varepsilon} d\xi d\eta.$$

The first integral in the r.h.s. is exponentially decreasing, which leads to

$$\overline{\lim_{\varepsilon\to 0}}\ ① \lesssim \overline{\lim_{\varepsilon\to 0}} \Big(\int_{|\varepsilon\xi| \ge r_\infty/4} |\mathcal{F}\underline{v}_\varepsilon^I(\xi)|^2 d\xi \Big)^{\frac{1}{2}}.$$

Secondly, as $\mathrm{dist}(\mathrm{supp} v_\varepsilon^I, \mathrm{supp}(1-\rho)) > 0$, one gets $\|(1-\rho) T_\varepsilon \underline{v}_\varepsilon^I\|_{L^2(\mathbb{R}^{2n})} \le e^{-C/\varepsilon}$ by Lemma B.2 and thus

$$\|\chi_{r_\infty/2}(\eta)\chi_{4r_0}(\eta) T_\varepsilon \underline{v}_\varepsilon^I\|_{L^2(\mathbb{R}^{2n})} \lesssim \varepsilon^\infty + \|\rho(y)\chi_{r_\infty/2}(\eta)\chi_{4r_0}(\eta) T_\varepsilon \underline{v}_\varepsilon^I\|_{L^2(\mathbb{R}^{2n})}.$$

It results from the relation (46) applied to $a_\varepsilon = v_\varepsilon^I$ that

$$(②)^2 \underset{\varepsilon\to 0}{\to} < w[v_\varepsilon^I], \rho^2 \otimes \chi_{4r_0}^2 \chi_{r_\infty/2}^2 > .$$

Because $w[\underline{v}_\varepsilon^I]$ is a regular measure, assumption C3 yields

$$\forall \alpha > 0, \exists l_0(\alpha) > 0 \text{ s.t. } w[\underline{v}_\varepsilon^I](\{|\xi| \le l_0(\alpha)\}) \le \alpha. \tag{51}$$

One deduces, for $4r_0 \le l_0(\alpha)$, that

$$\overline{\lim_{\varepsilon\to 0}}\ ② \lesssim \sqrt{\alpha},$$

which leads to

$$\overline{\lim_{\varepsilon\to 0}}\|v_\varepsilon^I - v_{\varepsilon,\gamma}^I\|_{L^2(\Omega)} \lesssim \overline{\lim_{\varepsilon\to 0}}\left(\int_{|\varepsilon\xi|\geq r_\infty/4}|\mathcal{F}v_\varepsilon^I(\xi)|^2 d\xi\right)^{\frac{1}{2}} + \sqrt{\alpha}.$$

Moving to the difference between u_ε^I and $u_{\varepsilon,\gamma}^I$ in $H^1(\Omega)$, we begin by estimating the spatial derivatives of the difference. It follows, by the formula of the inverse FBI transform's derivative given in (17), that

$$\partial_{x_b}u_\varepsilon^I - \partial_{x_b}u_{\varepsilon,\gamma}^I = \partial_{x_b}u_\varepsilon^I - (\partial_{x_b}\rho)T_\varepsilon^*\gamma T_\varepsilon \underline{u}_\varepsilon^I - \rho T_\varepsilon^*\gamma\partial_{y_b}T_\varepsilon \underline{u}_\varepsilon^I.$$

The term involving the derivative of ρ is exponentially decreasing by Lemma B.3. Since the FBI transform of a derivative is a derivative of the FBI transform by (17), one has to estimate $\|\partial_{x_b}u_\varepsilon^I - \rho T_\varepsilon^*\gamma T_\varepsilon\partial_{x_b}\underline{u}_\varepsilon^I\|_{L^2(\Omega)}$. Employing the same previous techniques yields for $b = 1,\dots,n$

$$\overline{\lim_{\varepsilon\to 0}}\|\partial_{x_b}u_\varepsilon^I - \partial_{x_b}u_{\varepsilon,\gamma}^I\|_{L^2(\Omega)} \lesssim \overline{\lim_{\varepsilon\to 0}}\left(\int_{|\varepsilon\xi|\geq r_\infty/4}|\mathcal{F}\partial_{x_b}\underline{u}_\varepsilon^I(\xi)|^2 d\xi\right)^{\frac{1}{2}} + \sqrt{\alpha},$$

if $4r_0 \leq l_b(\alpha)$ and $w[\partial_{x_b}\underline{u}_\varepsilon^I](\{|\xi| \leq l_b(\alpha)\}) \leq \alpha$. Set $r_0 = \frac{1}{4}\min_{0\leq b\leq n} l_b(\alpha)$, then the Poincaré inequality yields the same bound for $\|u_\varepsilon^I - u_{\varepsilon,\gamma}^I\|_{L^2(\Omega)}$.

Coming back to (50) we deduce that

$$\begin{aligned}|< w[\underline{\partial_t u}_\varepsilon(t,.)] + \mathrm{Tr}w[c\underline{\partial_x u}_\varepsilon(t,.)],\phi > -\frac{1}{2} < w[\Upsilon_\varepsilon^+],\phi^{-t} > -\frac{1}{2} < w[\Upsilon_\varepsilon^-],\phi^t > | \\ \lesssim \sqrt{\alpha} + \Big(\overline{\lim_{\varepsilon\to 0}}\int_{\varepsilon|\xi|\geq r_\infty/4}|\mathcal{F}\Big[\underline{v_\varepsilon^I}\Big](\xi)|^2 d\xi\Big)^{\frac{1}{2}} \\ + \sum_{b=1}^n\Big(\overline{\lim_{\varepsilon\to 0}}\int_{\varepsilon|\xi|\geq r_\infty/4}|\mathcal{F}\Big[\partial_{x_b}\underline{u_\varepsilon^I}\Big](\xi)|^2 d\xi\Big)^{\frac{1}{2}}.\end{aligned} \tag{52}$$

The assumption C2 of $\varepsilon-$oscillation means by definition that

$$\begin{aligned}\overline{\lim_{\varepsilon\to 0}}\int_{\varepsilon|\xi|\geq r_\infty/4}|\mathcal{F}\Big[\underline{v_\varepsilon^I}\Big](\xi)|^2 d\xi \underset{r_\infty\to+\infty}{\longrightarrow} 0, \\ \overline{\lim_{\varepsilon\to 0}}\int_{\varepsilon|\xi|\geq r_\infty/4}|\mathcal{F}\Big[\partial_{x_b}\underline{u_\varepsilon^I}\Big](\xi)|^2 d\xi \underset{r_\infty\to+\infty}{\longrightarrow} 0 \text{ for } b = 1,\dots,n.\end{aligned} \tag{53}$$

Since the l.h.s. of (52) does not depend on α nor r_∞, one deduces by passing to the limits $\alpha \to 0$ and $r_\infty \to \infty$ that

$$w[\underline{\partial_t u}_\varepsilon(t,.)] + \mathrm{Tr}w[c\underline{\partial_x u}_\varepsilon(t,.)] = \frac{1}{2}w[\Upsilon_\varepsilon^+]\,o\left(\varphi_b^{-t}\right)^{-1} + \frac{1}{2}w[\Upsilon_\varepsilon^-]\,o\left(\varphi_b^{t}\right)^{-1} \text{ in } \overset{o}{T^*}\Omega.$$

A Proof of the relation between incident and reflected beams' phases

Let $A(t,x,\xi) = \partial_x r(x,\xi) + \partial_\xi r(x,\xi)\partial_x^2\psi_{\text{inc}}(t,x)$ and $B(t,x,\xi) = \partial_x\lambda(x,\xi) + \partial_\xi\lambda(x,\xi)\partial_x^2\psi_{\text{inc}}(t,x)$. One can dispose of a phase function $\theta \in \mathcal{C}^\infty(\mathbb{R}_t \times \mathbb{R}_x^n, \mathbb{C})$ s.t.

$$\partial_x\theta(t, r(x, \partial_x\psi_{\text{inc}})) \underset{x=x_0^t}{\overset{R-1}{\asymp}} \lambda(x, \partial_x\psi_{\text{inc}}), \tag{A.1}$$

if $A(t, x_0^t, \xi_0^t)$ is non singular and

$$B(t,x,\partial_x\psi_{\text{inc}})A(t,x,\partial_x\psi_{\text{inc}})^{-1} \underset{x=x_0^t}{\overset{R-2}{\asymp}} A(t,x,\partial_x\psi_{\text{inc}})^{T-1}B(t,x,\partial_x\psi_{\text{inc}})^T. \tag{A.2}$$

From (10) one gets

$$A(t,x_0^t,\xi_0^t)(\partial_y x_0^t + i\partial_\eta x_0^t) = \partial_y x_1^t + i\partial_\eta x_1^t.$$

Since φ_1^t is symplectic, the matrix $\begin{pmatrix} \partial_y x_1^t & \partial_\eta x_1^t \\ \partial_y \xi_1^t & \partial_\eta \xi_1^t \end{pmatrix}$ is symplectic. This implies in particular the relation

$$\partial_\eta\xi_1^t(\partial_y x_1^t)^T - \partial_y\xi_1^t(\partial_\eta x_1^t)^T = Id,$$

and the symmetry of $\partial_y x_1^t(\partial_\eta x_1^t)^T$. Thus, $\ker(\partial_\eta x_1^t)^T \cap \ker(\partial_y x_1^t)^T = \{0\}$ and at the same time,

$$(\partial_y x_1^t + i\partial_\eta x_1^t)(\partial_y x_1^t + i\partial_\eta x_1^t)^* = \partial_y x_1^t(\partial_y x_1^t)^T + \partial_\eta x_1^t(\partial_\eta x_1^t)^T.$$

This proves that $\partial_y x_1^t + i\partial_\eta x_1^t$ is invertible and so is $A(t,x_0^t,\xi_0^t)$. On the other hand,

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \partial_x r & \partial_\xi r \\ \partial_x\lambda & \partial_\xi\lambda \end{pmatrix}\begin{pmatrix} Id \\ \partial_x^2\psi_{\text{inc}} \end{pmatrix}.$$

Let $\mathbf{M}(x,\xi) = \begin{pmatrix} \partial_x r(x,\xi) & \partial_\xi r(x,\xi) \\ \partial_x\lambda(x,\xi) & \partial_\xi\lambda(x,\xi) \end{pmatrix}$. Then

$$[A^TB - B^TA] = \begin{pmatrix} Id \\ \partial_x^2\psi_{\text{inc}} \end{pmatrix}^T \mathbf{M}^TJ\mathbf{M}\begin{pmatrix} Id \\ \partial_x^2\psi_{\text{inc}} \end{pmatrix},$$

where is $J = \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix}$ is the standard symplectic matrix. Since $\mathbf{M}^TJ\mathbf{M} = Ds_1^TJDs_1$, the symplecticity of s_1 leads to

$$\mathbf{M}^TJ\mathbf{M} = J.$$

Hence

$$[A^TB - B^TA] = \begin{pmatrix} Id \\ \partial_x^2\psi_{\text{inc}} \end{pmatrix}^T J\begin{pmatrix} Id \\ \partial_x^2\psi_{\text{inc}} \end{pmatrix} = 0,$$

and the requirement (A.2) is fulfilled.

Using the compatibility conditions

$$\begin{aligned}\frac{d}{dt}&\Big\{\partial_{t,x}^\alpha\Big[f(t,x,\partial_x\psi_{\text{inc}}(t,x))\Big]\Big(t,x_0^t\Big)\Big\} \\ &= \partial_t\partial_{t,x}^\alpha[f(t,x,\partial_x\psi_{\text{inc}}(t,x))]\Big(t,x_0^t\Big) + \partial_x\partial_{t,x}^\alpha[f(t,x,\partial_x\psi_{\text{inc}}(t,x))]\Big(t,x_0^t\Big)\,\dot{x}_0^t\end{aligned}$$

on the maps $(t,x,\xi)\mapsto \partial_x\theta(t,r(x,\xi))$ and $(x,\xi)\mapsto\lambda(x,\xi)$ yields recursively on $|\alpha|\leq R-1$

$$\partial_t\partial_x\theta(t,r(x,\partial_x\psi_{\mathrm{inc}}))+\partial_x^2\theta(t,r(x,\partial_x\psi_{\mathrm{inc}}))\partial_\xi r(x,\partial_x\psi_{\mathrm{inc}})\partial_t\partial_x\psi_{\mathrm{inc}} \underset{x=x_0^t}{\overset{R-2}{\asymp}} \partial_\xi\lambda(x,\partial_x\psi_{\mathrm{inc}})\partial_t\partial_x\psi_{\mathrm{inc}}.$$

On the other hand

$$\partial_x^2\theta(t,r(x,\partial_x\psi_{\mathrm{inc}})) \underset{x=x_0^t}{\overset{R-2}{\asymp}} [BA^{-1}](t,r(x,\partial_x\psi_{\mathrm{inc}})).$$

Using (A.2), one gets

$$\begin{aligned}\partial_t\partial_x\theta(t,r(x,&\partial_x\psi_{\mathrm{inc}}))\\ &\underset{x=x_0^t}{\overset{R-2}{\asymp}} [\partial_\xi\lambda(x,\partial_x\psi_{\mathrm{inc}})-(A^{-1T}B^T)(t,r(x,\partial_x\psi_{\mathrm{inc}}))\partial_\xi r(x,\partial_x\psi_{\mathrm{inc}})]\partial_t\partial_x\psi_{\mathrm{inc}}.\end{aligned}$$

Since

$$A^T\partial_\xi\lambda - B^T\partial_\xi r = \begin{pmatrix} Id \\ \partial_x^2\psi_{\mathrm{inc}}\end{pmatrix}^T \mathbf{M}^T J\mathbf{M}\begin{pmatrix} 0 \\ Id\end{pmatrix} = Id,$$

it follows that

$$\partial_t\partial_x\theta(t,r(x,\partial_x\psi_{\mathrm{inc}}))A(t,x,\partial_x\psi_{\mathrm{inc}}) \underset{x=x_0^t}{\overset{R-2}{\asymp}} \partial_t\partial_x\psi_{\mathrm{inc}}.$$

Setting $\partial_t\theta(t,r(x_0^t,\xi_0^t))=\partial_t\psi_{\mathrm{inc}}(t,x_0^t)$ implies then that

$$\partial_t\theta(t,r(x,\partial_x\psi_{\mathrm{inc}})) \underset{x=x_0^t}{\overset{R-1}{\asymp}} \partial_t\psi_{\mathrm{inc}}. \tag{A.3}$$

Putting together (11),(A.1),(A.3) and the eikonal equation satisfied by ψ_{inc}, the phase θ satisfies

$$p(r(x,\partial_x\psi_{\mathrm{inc}}),\partial_t\theta(t,r(x,\partial_x\psi_{\mathrm{inc}})),\partial_x\theta(t,r(x,\partial_x\psi_{\mathrm{inc}}))) \underset{x=x_0^t}{\overset{R-1}{\asymp}} 0.$$

In fact, the relation holds true also at order R. To see this, let $u(t,x)=p(x,\partial_t\theta(t,x),\partial_x\theta(t,x))$. The formula of composite functions' high derivatives yields for $|\alpha|=R$

$$\partial_x^\alpha\Big[u(t,r(x,\partial_x\psi_{\mathrm{inc}}(t,x)))\Big]=\sum_{|\beta|=R}\partial_x^\beta u(t,r(x,\partial_x\psi_{\mathrm{inc}}(t,x)))v_\beta(t,x)+z_\alpha(t,x),$$

where z_α depends on derivatives of u of order lower than R. By Remark 2.1, the terms $\partial_x^\beta u(t,x_1^t)$ involve partial derivatives of θ of order at most R, so one can substitute for them from (A.1) and (A.3) to obtain

$$p(r(x,\partial_x\psi_{\mathrm{inc}}),\partial_t\theta(t,r(x,\partial_x\psi_{\mathrm{inc}})),\partial_x\theta(t,r(x,\partial_x\psi_{\mathrm{inc}}))) \underset{x=x_0^t}{\overset{R}{\asymp}} 0. \tag{A.4}$$

To compare time and tangential derivatives of θ and ψ_{inc} at $(T_1,x_0^{T_1})$, let us introduce a $\mathcal{C}^\infty$ parametrization of a neighbourhood $\mathcal{U}$ of $x_0^{T_1}$ in $\partial\Omega$

$$\sigma:\mathcal{N}\to\mathbb{R}^n,$$

where $\mathcal{N}$ is an open subset $\mathbb{R}^{n-1}$, $\sigma(\mathcal{N}) = \mathcal{U}$ and σ is a diffeomorphism from $\mathcal{N}$ to $\mathcal{U}$. For $x \in \mathbb{R}^n$ close to $x_0^{T_1}$, we may write $x = \sigma(\hat{v}) + v_n \nu(\sigma(\hat{v}))$, with $\hat{v} \in \mathcal{N}$ and $v_n \in \mathbb{R}$. Denote $\sigma(\hat{v}_1) = x_0^{T_1}$ and set $\theta_b(t,\hat{v}) = \theta(t,\sigma(\hat{v}))$ and $\psi_{\text{inc}b}(t,\hat{v}) = \psi_{\text{inc}}(t,\sigma(\hat{v}))$ the phases at the boundary near $x_0^{T_1}$. Since $r(X,\Xi) = X$ for $(X,\Xi) \in \overset{o}{T^*\mathbb{R}^n}|_{\partial\Omega}$, it follows that

$$r(\sigma(\hat{v}), \partial_x \psi_{\text{inc}}(t,\sigma(\hat{v}))) \underset{(t,\hat{v})=(T_1,\hat{v}_1)}{\overset{\infty}{\asymp}} \sigma(\hat{v}),$$

which implies by (A.3) that

$$\partial_t \theta_b \underset{(t,\hat{v})=(T_1,\hat{v}_1)}{\overset{R-1}{\asymp}} \partial_t \psi_{\text{inc}b}.$$

Similarly $\lambda(X,\Xi) = \Xi - 2(\Xi\cdot\nu(X))\nu(X)$ for $(X,\Xi) \in \overset{o}{T^*\mathbb{R}^n}|_{\partial\Omega}$, leading to

$$D\sigma(\hat{v})\cdot\lambda(\sigma(\hat{v}), \partial_x\psi_{\text{inc}}(t,\sigma(\hat{v}))) \underset{(t,\hat{v})=(T_1,\hat{v}_1)}{\overset{\infty}{\asymp}} D\sigma(\hat{v})\cdot\partial_x\psi_{\text{inc}}(t,\sigma(\hat{v})).$$

Since $\partial_{\hat{v}}\theta_b(t,\hat{v}) = D\sigma(\hat{v})\cdot\partial_x\theta(t,\sigma(\hat{v}))$ and a similar relation holds true for $\partial_{\hat{v}}\psi_{\text{inc}b}$, one gets from (A.1)

$$\partial_{\hat{v}}\theta_b \underset{(t,\hat{v})=(T_1,\hat{v}_1)}{\overset{R-1}{\asymp}} \partial_{\hat{v}}\psi_{\text{inc}b}.$$

Hence θ_b and $\psi_{\text{inc}b}$ have the same time and tangential derivatives at $(T_1,\hat{v}_1)$ from order 1 to order $R-1$.

If we add to the relation (A.1) the condition

$$\theta(T_1, x_0^{T_1}) = \psi_{\text{inc}}(T_1, x_0^{T_1}),$$

then the phase θ satisfies the same requirements that uniquely determine the reflected phase ψ_{ref} for a fixed phase ψ_{inc} (see Remark 2.1 for the uniqueness the spatial derivatives involving a normal derivation being deduced from time and tangential derivatives). The two phases are thus equal on (t, x_1^t) up to the order R.

B Results related to the FBI and the Wigner transforms

Lemma B.1. *For u in $L^2(\mathbb{R}^n)$*

$$T_\varepsilon u(y,\eta) = \varepsilon^{-\frac{n}{4}} c_n (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \mathcal{F}u(\xi) e^{i\xi\cdot y - (\eta-\varepsilon\xi)^2/(2\varepsilon)} d\xi.$$

Proof. The equality is proven by Parseval formula. □

Lemma B.2. Lemma 2.4 of chapter 1 *Let a be a positive real and G a measurable subset of $\mathbb{R}^n$ s.t.* $\operatorname{dist}(G,K) \geq a$. *If $u \in L^2(\mathbb{R}^n_x)$ is supported in K then*

$$\|\mathbf{1}_G(y) T_\varepsilon u\|_{L^2_{y,\eta}} = c_n \varepsilon^{-\frac{n}{4}} \|\mathbf{1}_G(y) u(x) e^{-(x-y)^2/(2\varepsilon)}\|_{L^2_{xy}} \lesssim e^{-a^2/(4\varepsilon)} \|u\|_{L^2_x}.$$

Proof. The proof consists of writing the FBI transform as the Fourier Transform of some auxiliary function and using Parseval equality. □

Lemma B.3. *Let θ be a cut-off of $\mathbb{R}^n$ and $u \in L^2(\mathbb{R}^n)$ compactly supported. If E is a measurable subset of $\mathbb{R}^n$ s.t.* $\operatorname{dist}(E, \operatorname{supp} u) > 0$, *then*

$$\|T_\varepsilon^* \theta(\eta) T_\varepsilon u\|_{L^2(E)} \lesssim e^{-C/\varepsilon} \|u\|_{L^2(\mathbb{R}^n)}.$$

Proof. The kernel of $T_\varepsilon^* \theta T_\varepsilon$ is

$$\begin{aligned} K_\varepsilon(x, x') &= \varepsilon^{-\frac{3n}{2}} c_n^2 \int \theta(\eta) e^{i\eta\cdot(x-x')/\varepsilon-(y-x')^2/(2\varepsilon)-(x-y)^2/(2\varepsilon)} dy d\eta \\ &= \varepsilon^{-n} (2\pi)^{-n} \mathcal{F}\theta(\frac{x'-x}{\varepsilon}) e^{-(x'-x)^2/(4\varepsilon)}. \end{aligned}$$

Cauchy-Schwartz inequality yields

$$\|T_\varepsilon^* \theta T_\varepsilon u\|^2_{L^2(E)} \leq (2\pi)^{-n} \|\mathcal{F}\theta\|^2_{L^2(\mathbb{R}^n)} \int |u(x')|^2 \mathbf{1}_E(x) e^{-(x'-x)^2/(2\varepsilon)} dx dx' \lesssim e^{-C/\varepsilon} \|u\|^2_{L^2(\mathbb{R}^n)}.$$

□

Lemma B.4. *Let θ be a cut-off of $\mathbb{R}^n$ and $u \in L^2(\mathbb{R}^n)$. If F is a measurable subset of $\mathbb{R}^n$ s.t.* $\operatorname{dist}(F, \operatorname{supp}\theta) > 0$, *then*

$$\|T_\varepsilon T_\varepsilon^* \theta(\eta) T_\varepsilon u\|_{L^2(\mathbb{R}^n \times F)} \lesssim e^{-C/\varepsilon} \|u\|_{L^2(\mathbb{R}^n)}.$$

Proof. $T_\varepsilon^* \theta T_\varepsilon u$ may be written as acting on $\mathcal{F}u$, using the expression of the FBI transform given in Lemma B.1

$$T_\varepsilon^* \theta T_\varepsilon u(x) = \varepsilon^{-\frac{n}{2}} c_n^2 \int \mathcal{F}u(\xi) e^{i\xi\cdot x} \mathcal{G}_\varepsilon \theta(\varepsilon\xi) d\xi,$$

where $\mathcal{G}_\varepsilon$ denotes the operator defined by

$$\mathcal{G}_\varepsilon a(\xi) = \int_{\mathbb{R}^n} a(\xi') e^{-(\xi-\xi')^2/\varepsilon} d\xi', \text{ for } a \in L^2(\mathbb{R}^n).$$

It follows that

$$T_\varepsilon T_\varepsilon^* \theta T_\varepsilon u(y, \eta) = \varepsilon^{-\frac{3n}{4}} c_n^3 (2\pi)^{\frac{n}{2}} \int \mathcal{F}u(\xi) e^{i\xi\cdot y-(\eta-\varepsilon\xi)^2/(2\varepsilon)} \mathcal{G}_\varepsilon \theta(\varepsilon\xi) d\xi,$$

By Parseval equality, one has

$$\|T_\varepsilon T_\varepsilon^* \theta(\eta) T_\varepsilon u\|^2_{L^2(\mathbb{R}^n \times F)} = \varepsilon^{-\frac{3n}{2}} c_n^6 (2\pi)^{2n} \int |\mathcal{F}u(\xi)|^2 \mathbf{1}_F(\eta) e^{-(\eta-\varepsilon\xi)^2/\varepsilon} |\mathcal{G}_\varepsilon \theta(\varepsilon\xi)|^2 d\xi d\eta.$$

Let $d = \operatorname{dist}(F, \operatorname{supp}\theta)$. If $\operatorname{dist}(\varepsilon\xi, F) \geq d/2$ then $|\mathbf{1}_F(\eta) e^{-(\eta-\varepsilon\xi)^2/\varepsilon}| \leq e^{-C(\eta-\varepsilon\xi)^2/\varepsilon} e^{-C/\varepsilon}$. If $\operatorname{dist}(\varepsilon\xi, F) \leq d/2$ then $\operatorname{dist}(\varepsilon\xi, \operatorname{supp}\theta) \geq d/2$, and $|\mathcal{G}_\varepsilon \theta(\varepsilon\xi)|^2 \leq e^{-C/\varepsilon}$. Since $\mathcal{G}_\varepsilon \theta$ is bounded, integrating w.r.t. ξ and η ends the proof. □

Lemma B.5. Lemma 3.4 of chapter 1 $\|\varepsilon^{-1} T_\varepsilon u^I_{\varepsilon,\gamma}\|_{L^2(\mathbb{R}^{2n})} \lesssim 1$.

Proof. Derivating (12) w.r.t. y_b, $0 \leq b \leq n$, yields

$$\varepsilon^{\frac{1}{2}} \partial_{y_b} (T_\varepsilon u^I_{\varepsilon,\gamma}) = i\eta_b \varepsilon^{-\frac{1}{2}} T_\varepsilon u^I_{\varepsilon,\gamma} - c_n \varepsilon^{-\frac{3n}{4}} \int_{\mathbb{R}^n} u^I_{\varepsilon,\gamma}(w) \varepsilon^{-\frac{1}{2}} (y_b - w_b)\, e^{i\eta.(y-w)/\varepsilon-(y-w)^2/(2\varepsilon)} dw.$$

The l.h.s. is bounded in $L^2_{y,\eta}$ because $\partial_{y_b}(T_\varepsilon u^I_{\varepsilon,\gamma}) = T_\varepsilon(\partial_{w_b} u^I_{\varepsilon,\gamma})$. The second term of the r.h.s. is the Fourier transform of a bounded function in L^2_w, thus it can be estimated using Parseval equality. One gets

$$\|\varepsilon^{-\frac{3n}{4}} \int_{\mathbb{R}^n} u^I_{\varepsilon,\gamma}(w)\varepsilon^{-\frac{1}{2}}(y_b - w_b)\, e^{i\eta\cdot(y-w)/\varepsilon-(y-w)^2/(2\varepsilon)}dw\|_{L^2_{y,\eta}} \lesssim \|u^I_{\varepsilon,\gamma}\|_{L^2_w}.$$

Thus $\|\varepsilon^{-\frac{1}{2}}\eta_b T_\varepsilon u^I_{\varepsilon,\gamma}\|_{L^2_{y,\eta}} \lesssim 1$ and consequently $\|\varepsilon^{-\frac{1}{2}}\phi(\eta) T_\varepsilon u^I_{\varepsilon,\gamma}\|_{L^2_{y,\eta}} \lesssim 1$. $A3'$ yields

$$\|\varepsilon^{-\frac{1}{2}} T_\varepsilon u^I_{\varepsilon,\gamma}\|_{L^2_{y,\eta}} \lesssim 1.$$

Hence $\|u^I_{\varepsilon,\gamma}\|_{L^2} \lesssim \sqrt{\varepsilon}$. Reproducing the same arguments on the following equality

$$\partial_{y_b}(T_\varepsilon u^I_{\varepsilon,\gamma}) = i\eta_b\varepsilon^{-1}T_\varepsilon u^I_{\varepsilon,\gamma} - c_n\varepsilon^{-\frac{3n}{4}} \int_{\mathbb{R}^n} \left(\varepsilon^{-\frac{1}{2}} u^I_{\varepsilon,\gamma}\right)(w)\varepsilon^{-\frac{1}{2}}(y_b - w_b)\, e^{\frac{i}{\varepsilon}\eta.(y-w)-\frac{1}{2}(y-w)^2}dw,$$

leads to $\|u^I_{\varepsilon,\gamma}\|_{L^2(\mathbb{R}^n)} \lesssim \varepsilon$. □

Lemma B.6. *Let a_ε and b_ε two sequences uniformly bounded in $L^2(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$ respectively. If $\varepsilon^{-1}b_\varepsilon$ is uniformly bounded in $L^2(\mathbb{R}^n)$, then*

$$w_\varepsilon(a_\varepsilon, |D|b_\varepsilon) \approx |\xi| w_\varepsilon(a_\varepsilon, \varepsilon^{-1}b_\varepsilon) \text{ on } \mathbb{R}^n \times (\mathbb{R}^n \backslash \{0\}).$$

Proof. We use another expression of the Wigner transform using the Fourier transform. Let ϕ be a test function of $\mathcal{C}^\infty_c(\mathbb{R}^n \times (\mathbb{R}^n\backslash\{0\}), \mathbb{R})$ and denote $c_\varepsilon = |D|b_\varepsilon$. Then

$$< w_\varepsilon(a_\varepsilon, c_\varepsilon), \phi >= (2\pi)^{-n} \int_{\mathbb{R}^{2n}} \mathcal{F}_\xi\phi(x - \frac{\varepsilon}{2}v, v) a_\varepsilon(x)\bar{c}_\varepsilon(x - \varepsilon v) dv dx.$$

Since $\mathcal{F}_\xi\phi$ is rapidly decreasing

$$\sup_x |\mathcal{F}_\xi\phi(x - \frac{\varepsilon}{2}v, v) - \mathcal{F}_\xi\phi(x, v)| \lesssim \varepsilon(1 + v^2)^{-n-1}.$$

By Cauchy-Schwartz inequality

$$\int_{\mathbb{R}^{2n}} | \left(\mathcal{F}_\xi\phi(x - \frac{\varepsilon}{2}v, v) - \mathcal{F}_\xi\phi(x, v)\right) a_\varepsilon(x)\bar{c}_\varepsilon(x - \varepsilon v)| \lesssim \varepsilon\|a_\varepsilon\|_{L^2}\|c_\varepsilon\|_{L^2}.$$

It follows that

$$< w_\varepsilon(a_\varepsilon, c_\varepsilon), \phi >= (2\pi)^{-n} \int_{\mathbb{R}^{3n}} \phi(x, \xi)e^{-iv\cdot\xi} a_\varepsilon(x)\bar{c}_\varepsilon(x - \varepsilon v) dv dx d\xi + o(1).$$

Integrating w.r.t. v leads to

$$< w_\varepsilon(a_\varepsilon, c_\varepsilon), \phi >= (2\pi)^{-n}\varepsilon^{-n} \int_{\mathbb{R}^{2n}} \phi(x, \xi)e^{-ix\cdot\xi/\varepsilon} a_\varepsilon(x)\mathcal{F}\bar{c}_\varepsilon(-\xi/\varepsilon) dx d\xi + o(1),$$

and replacing $\mathcal{F}\bar{c}_\varepsilon(-\xi/\varepsilon)$ by $\varepsilon^{-1}|\xi|\mathcal{F}\bar{b}_\varepsilon(-\xi/\varepsilon)$ ends the proof. □

Chapter III

Elasticity system : asymptotic solutions and Wigner measures

Contents

1 Introduction

In this chapter, we focus on the elasticity equations for $x \in \Omega$ a three dimensional bounded domain and $t \in [0,T]$

$$Eu_\varepsilon = \rho\partial_t^2 u_\varepsilon - \partial_x(\lambda \mathrm{div} u_\varepsilon) - \sum_{b=1}^{3} \partial_{x_b}(\mu\partial_x(u_\varepsilon)_b + \mu\partial x_b u_\varepsilon) = 0, \tag{1a}$$

subject to the initial conditions

$$u_\varepsilon|_{t=0} = u_\varepsilon^I,\ \partial_t u_\varepsilon|_{t=0} = v_\varepsilon^I, \tag{1b}$$

and the boundary conditions

$$Bu_\varepsilon|_{\partial\Omega} = 0, \tag{1c}$$

where B is Dirichlet or Neumann operator. We construct asymptotic solutions for the problem (1) and compute the associated scalar Wigner measure under suitable assumptions on the geometry of the domain and the initial data.

We again use the Gaussian beams summation technique to produce our approximate solutions. Gaussian beams for the elasticity system have the form

$$\sum_{j=0}^{N} \varepsilon^j a_j e^{i\psi/\varepsilon},$$

with vector amplitudes a_j and a complex phase ψ. Individual beams in the whole space domain have been constructed by Ralston in [85], and reflection of these beams has been studied in [19, 78, 81]. Summation of Gaussian beams for the elasticity system has been investigated in [16]. As for the scalar wave equations, different techniques may be used to superpose an infinite number of beams. Here we use the same strategy we applied for the scalar wave equation in chapter 1, appealing to the FBI transforms to fulfill general initial data. Hypotheses similar to those we imposed on initial data for the scalar wave equation are assumed

A1. u_ε^I and v_ε^I are uniformly bounded respectively in $H^1(\Omega)^3$ and $L^2(\Omega)^3$,

A2. u_ε^I and v_ε^I are uniformly supported in a fixed compact set $K \subset \Omega$,

A3. $\|T_\varepsilon u_\varepsilon^I\|_{L^2(\mathbb{R}^3\times R_\eta^c)^3} = O(\varepsilon^\infty)$ and $\|T_\varepsilon v_\varepsilon^I\|_{L^2(\mathbb{R}^3\times R_\eta^c)^3} = O(\varepsilon^\infty)$,

where $R_\eta = \{\eta \in \mathbb{R}^3, r_0 \le |\eta| \le r_\infty\}$, $0 < r_0 < r_\infty$.

For our construction to work, we must avoid tangential rays. Since the elasticity equations have two different wave speeds, a ray that emanates from the domain and strikes the boundary transversally may give birth to a tangential ray. So we add a further assumption

A4. $\|T_\varepsilon u_\varepsilon^I\|_{L^2(Tg^c)^3} = O(\varepsilon^\infty)$ and $\|T_\varepsilon v_\varepsilon^I\|_{L^2(Tg^c)^3} = O(\varepsilon^\infty)$,

where $Tg \subset \mathbb{R}^{2n}$ denotes the set of points giving birth to rays having a transversal contact with the boundary after one or more reflections on $[0,T]$. As regards the domain, we suppose that

B2. No ray remains in a compact of $\mathbb{R}^3$ for growing times,

B3. The boundary has no dead-end trajectories, that is infinite number of successive reflections cannot occur in a finite time.

Under these assumptions on initial data and the domain Ω, we prove the following theorem

Theorem 1.1. *For any integer $R \geq 2$, there is an asymptotic solution to (1) of the form*

$$u_\varepsilon^R(t,x) = \sum_k \int_{\mathbb{R}^{2n}} a_\varepsilon^k(t,x,y,\eta,R)e^{i\psi_k(t,x,y,\eta,R)/\varepsilon}dyd\eta,$$

where $a_\varepsilon^k e^{i\psi_k/\varepsilon}$ are Gaussian beams and the summation over k is finite.
u_ε^R is asymptotic to the exact solution of the IBVP (1) in the following sense

$$\underset{t\in[0,T]}{Sup} \|u_\varepsilon^R(t,.) - u_\varepsilon(t,.)\|_{H^1(\Omega)^3} = O(\varepsilon^{\frac{R-1}{2}}),$$

$$\text{and } \underset{t\in[0,T]}{Sup} \|\partial_t u_\varepsilon^R(t,.) - \partial_t u_\varepsilon(t,.)\|_{L^2(\Omega)^3} = O(\varepsilon^{\frac{R-1}{2}}).$$

As an application of this construction we compute the microlocal energy density associated to the elasticity system [79]

$$e = \frac{\rho}{2}\mathrm{Tr}w[\partial_t u_\varepsilon] + \frac{\mu}{4}\sum_{b=1}^{3}\mathrm{Tr}w[\partial_{x_b}u_\varepsilon + \partial_x(u_\varepsilon)_b] + \frac{\lambda}{2}w[divu_\varepsilon]. \tag{2}$$

The techniques used to describe the Wigner measures are an adaptation of chapter 2. However a new difficulty arises due to the existence of two different families of rays.

This chapter is organised as follows. Section 2 is devoted to the construction of individual beams for the elasticity equations and their reflection at the boundary. In section 3 we superpose an inifinite number of these beams to construct asymptotic solutions to the problem (1) and prove theorem 1.1. Computation of the Wigner measure is achieved in section 4 through the first order asymptotic solution.

2 Gaussian beams for the elasticity equations

We search for a high frequency solution of (1a) under the form

$$\omega_\varepsilon = \sum_{j=0}^{N} \varepsilon^j a_j e^{i\psi/\varepsilon}.$$

Applying the equations of elasticity to ω_ε gives a function of the same form

$$E\omega_\varepsilon = \sum_{j=0}^{N+2} \varepsilon^{j-2} c_j e^{i\psi/\varepsilon},$$

with

$$c_j = i^j(Ja_j + Ma_{j-1} + Na_{j-2}),$$

$$J = \rho[c_L^2 \partial_x\psi\partial_x\psi \cdot + c_T^2(|\partial_x\psi|^2 Id - \partial_x\psi\partial_x\psi\cdot) - (\partial_t\psi)^2],$$

$$\begin{aligned} M = & -2\rho\partial_t\psi\partial_t + (\lambda+\mu)\partial_x(\partial_x\psi\cdot) + (\lambda+\mu)\partial_x\psi(\partial_x\cdot) + 2\mu(\partial_x\psi\cdot\partial_x) \\ & -\rho\partial_t^2\psi + \mu\triangle\psi + \partial_x\lambda(\partial_x\psi\cdot) + (\partial_x\mu\cdot)\partial_x\psi + (\partial_x\mu\cdot\partial_x\psi), \end{aligned} \tag{3}$$

and N a matrix differential operator of order 2 that we won't specify (see 2.4 in [78] for explicit expression). Above, c_T and c_L denote the speeds associated to the elasticity system

$$c_T^2 = \frac{\mu}{\rho},\ c_L^2 = \frac{\lambda+2\mu}{\rho}, \tag{4}$$

and $a_{-1} = a_{-2} = a_{N+1} = a_{N+2} = 0$. As in [78, 85], the construction of the Gaussian beams consists in making the terms c_j vanish up to the order $R-2j$ on some fixed ray. The elasticity operator has two families of rays : the rays assocaited to the speed c_T and those associated to the speed c_L. In fact, the principal symbol of E is

$$\rho^3(c_L^2(x)|\xi|^2 - \tau^2)(c_T^2(x)|\xi|^2 - \tau^2)^2.$$

Each ray is a curve $(t, x^{\pm t})$ where x^t is the projection of a Hamiltonian flow (x^t, ξ^t) either associated to the symbol $c_L(x)|\xi|$ or $c_T(x)|\xi|$

$$\begin{cases} \dfrac{dx_L^t}{dt} = c_L(x_L^t)\dfrac{\xi_L^t}{|\xi_L^t|}, \\ \\ \dfrac{d\xi_L^t}{dt} = -\partial_x c_L(x_L^t)|\xi_L^t|, \end{cases} \quad \text{or} \quad \begin{cases} \dfrac{dx_T^t}{dt} = c_T(x_T^t)\dfrac{\xi_T^t}{|\xi_T^t|}, \\ \\ \dfrac{d\xi_T^t}{dt} = -\partial_x c_T(x_T^t)|\xi_T^t|. \end{cases} \tag{5}$$

In the remainder, we build Gaussian beams associated to rays propagating in the positive sense, that is $(t, x_{L,T}^t)$.

One can make $c_0 = Ja_0$ vanish on some ray of one family of rays or the other by imposing one of the following conditions

$$\begin{aligned} & c_L^2|\partial_x\psi|^2 - (\partial_t\psi)^2 \text{ and } a_0 \wedge \partial_x\psi \text{ vanish up to the order } R \text{ on } (t,x) = (t, x_L^t), \\ & \text{or} \\ & c_T^2|\partial_x\psi|^2 - (\partial_t\psi)^2 \text{ and } a_0 \cdot \partial_x\psi \text{ vanish up to the order } R \text{ on } (t,x) = (t, x_T^t), \end{aligned} \tag{6}$$

where R is an integer larger than 1. The previous two constraints lead respectively to Gaussian beams of type L or longitudinal beams and Gaussian beams of type T or transversal beams.

The associated phases ψ_L and ψ_T are thus constructed the same way the phase has been in section 2 of chapter 1, that is by solving the eikonal equations

$$c_L^2|\partial_x\psi_L|^2 - (\partial_t\psi_L)^2 \underset{x=x_L^t}{\overset{R}{\asymp}} 0 \text{ and } c_T^2|\partial_x\psi_T|^2 - (\partial_t\psi_T)^2 \underset{x=x_T^t}{\overset{R}{\asymp}} 0.$$

The notation $\underset{x=x^t}{\overset{k}{\asymp}}$ is used to denote that the spatial derivatives of the quantities at its left and at its right match up to the order k on (t, x^t). We omit the indexes L and T for relations that hold true for both of the two phases. From each eikonal equation results systems of ODEs on the spatial derivatives of the associated phase up to the order R on the ray. As usual, we then set

$$\psi(t,x) = \sum_{|\alpha|\le R} \frac{1}{\alpha!}(x-x^t)^\alpha \partial_x^\alpha \psi(t,x^t).$$

The equations on the amplitudes in (6) may be written

$$a_0^L \underset{x=x_L^t}{\overset{R}{\asymp}} s_0^L p_L \text{ for a beam L}, a_0^T \underset{x=x_T^t}{\overset{R}{\asymp}} b_0^T \text{ for a beam T},$$

where p is a fixed complex unit vector (that is $p\cdot p = 1$) satisfying $p \wedge \partial_x\psi = 0$ and oriented like $\partial_x\psi$, and b_0 orthogonal to p. The indexes/exponents L and T in p and b_0 indicate the associated phase (ψ_L or ψ_T).

To make the terms c_j for $j > 0$ vanish, let us analyse the action of the operator M on vectors colinear and orthogonal to $\partial_x\psi$. We start by computing $M(ap)$, for a scalar $a \in \mathbb{C}$. Focusing on the second and third operators in the sum (3), we note that

$$\partial_x(\partial_x\psi \cdot ap) = \partial_x(|\partial_x\psi|)a + |\partial_x\psi|\partial_x a,\ \partial_x\psi\partial_x\cdot(ap) = a\partial_x\psi\partial_x\cdot p + \partial_x\psi(\partial_x a \cdot p).$$

$$p\cdot\partial_x(|\partial_x\psi|) + |\partial_x\psi|\partial_x\cdot p = \triangle\psi,$$

and thus

$$(\lambda+\mu)p\cdot(\partial_x|\partial_x\psi| + \partial_x\cdot p\partial_x\psi) = (\lambda+\mu)\triangle\psi.$$

Therefore, the dot product of p with the second and third terms of $M(ap)$ is

$$(\lambda+\mu)p\cdot[\partial_x(\partial_x\psi\cdot ap) + \partial_x\psi\partial_x\cdot(ap)] = (\lambda+\mu)\triangle\psi a + 2(\lambda+\mu)\partial_x\psi\cdot\partial_x a. \tag{7}$$

On the other hand, since $p\cdot\partial_x p = 0$ and

$$\partial_x\psi\cdot\partial_x(ap) = (\partial_x\psi\cdot\partial_x a)p + \partial_x p\partial_x\psi a,$$

one gets

$$2\mu p\cdot[\partial_x\psi\cdot\partial_x(ap)] = 2\mu(\partial_x\psi\cdot\partial_x a), \tag{8}$$

which is the dot product of p with the forth term in $M(ap)$. Let

$$L_{T,L}(\psi) = -2\rho\partial_t\psi\partial_t + 2\rho c_{T,L}^2\partial_x\psi\partial_x,$$
$$P_{T,L} = \partial_t^2 - \partial_x\cdot(c_{T,L}^2\partial_x) \text{ and } \beta_{T,L}(\psi) = -\rho P_{T,L}\psi + c_{T,L}^2(\partial_x\rho\cdot\partial_x\psi).$$

Taking the dot product of $M(ap)$ with p, (7) and (8) give

$$M(ap) = (L_L(\psi)a + \beta_L(\psi)a)p + T_\perp a,$$

with $T_\perp$ a differential operator of order 1 satisfying $p\cdot T_\perp = 0$.

Moving to vectorial amplitudes b s.t $b\cdot\partial_x\psi = 0$, one has

$$Mb = L_T(\psi)b + (\lambda+\mu)\partial_x\psi\partial_x\cdot b + (\partial_x\mu\cdot b)\partial_x\psi + \beta_T(\psi)b,$$

which may be written

$$Mb = L_T(\psi)b + q_\| b + \beta_T(\psi)b,$$

with $q_\|$ a differential operator of order 1, colinear to p.

Let us decompose each amplitude tangentially and orthogonally to p as

$$a_j = s_j p + b_j.$$

To make c_1 vanish up to the order $R-2$, one plugs one of the forms of the amplitude a_0 chosen in (6).

2.1 Longitudinal beams

For a beam L, one obtains (equations (2.10), (3.3) in [78])

$$\rho(c_T^2 - c_L^2)|\partial_x \psi_L|^2 (Id - p_L p_L \cdot) a_1^L + [L_L(\psi_L) + \beta_L(\psi_L)] s_0^L p_L + T_\perp s_0^L \underset{x=x_L^t}{\overset{R-2}{\asymp}} 0. \tag{9}$$

This equation determines s_0^L and $(Id - p_L p_L \cdot) a_1^L$ up to order $R-2$ at $x = x_L^t$. Indeed, starting by taking the dot product of (9) with p_L, one gets an evolution equation on s_0^L

$$(L_L(\psi_L) + \beta_L(\psi_L)) s_0^L \underset{x=x_L^t}{\overset{R-2}{\asymp}} 0. \tag{10}$$

Compare this equation with the one satisfied by the amplitude of a first order beam for the scalar wave equation (see (7) p.65). For a non constant density, it has an extra term $c_L^2(\partial_x \rho \cdot \partial_x \psi_L)$. By writing $\partial_x \psi_L(t, x_L^t) = \xi_L^t$ and $\partial_t \psi_L(t, x_L^t) = -c_L(x_L^t)|\xi_L|$, one gets

$$(L_L(\psi_L) f)\,|_{(t,x_L^t)} = -2\rho(x_L^t)\partial_t \psi_L(t, x_L^t) \frac{d}{dt} f(t, x_L^t) \text{ for } f \in \mathcal{C}^\infty(\mathbb{R}_t \times \mathbb{R}_x^3, \mathbb{C}), \tag{11}$$

so order 0 of (10) gives the transport equation

$$\frac{d}{dt} s_0^L(t, x_L^t) = \frac{1}{2}\left[\frac{P_L \psi_L(t, x_L^t)}{c_L(x_L^t)|\xi_L^t|} - \frac{d}{dt}\ln(\rho(x_L^t))\right] s_0^L(t, x_L^t).$$

The amplitude of a first order beam for the scalar wave equation is $\det({x^t}_y + i{x^t}_\eta)]^{-\frac{1}{2}}$ up to a coefficient (see (25) p.73). Thus

$$s_0^L(t, x_L^t) = s_0^L(0, x_L^0)\left[\frac{\rho(x_L^t)}{\rho(x_L^0)} \det(x_{Ly}^t + i x_{L\eta}^t)\right]^{-\frac{1}{2}}.$$

Above, the square root is obtained by continuity from 1 at $t = 0$. This result is similar to the formulae (3.28) in [78].

For $|\alpha| \geq 1$,

$$\partial_x^\alpha L_L(\psi_L) = L_L(\psi_L)\partial_x^\alpha + R_L^\alpha \tag{12}$$

with R_L^α a differential operator of order less than $|\alpha|$. By (11), it follows that

$$[\partial_x^\alpha (L_L(\psi_L) f)]\,|_{(t,x_L^t)} = -2\rho(x_L^t)\partial_t \psi_L(t, x_L^t)\frac{d}{dt}\partial_x^\alpha f(t, x_L^t) + (R_L^\alpha f)|_{(t,x_L^t)}$$

Thus, equation (10) gives at order $0 < k \leq R$ a non-homogeneous transport equation on $\left(\partial_x^\alpha s_0^L(t, x_L^t)\right)_{(|\alpha|=k)}$. To summarize, the spatial derivatives of s_0^L up to order $R-2$ on (t, x_L^t) are uniquely determined by (10) given their values on $(0, x_L^0)$. Now we choose

$\partial_x^\alpha s_0^L(0, x_L^0)$ for $|\alpha| \leq R-2$ arbitrary permutable families,
$\partial_x^\alpha s_0^L(t, x_L^t)$ for $t \in \mathbb{R}, R-2 < |\alpha| \leq R$ arbitrary permutable families,

and set

$$a_0^L(t, x) = \chi_d(x - x_L^t)\sum_{|\alpha| \leq R} \frac{1}{\alpha!}(x - x_L^t)^\alpha \partial_x^\alpha \left(s_0^L p_L\right)(t, x_L^t). \tag{13}$$

Next, using the evolution equations (10), equation (9) becomes

$$\rho(c_T^2 - c_L^2)|\partial_x\psi_L|^2(Id - p_L p_L\cdot)a_1^L + T_\perp s_0^L \underset{x=x_L^t}{\overset{R-2}{\asymp}} 0.$$

One obtains then $b_1^L = (Id - p_L p_L\cdot)a_1^L$ up to order $R-2$ on (t, x_L^t) by plugging the value of s_0^L's derivatives on the ray. Similar results were obtained by [78] (see equations (6.6), (6.7)).

For $1 \leq j \leq N$, the amplitudes a_j^L are computed recursively as follows. Let us assume that $a_0^L, \ldots, a_{j-1}^L$ and b_j^L have been determined by the vanishing of $c_0^L, \ldots, c_j^L$ on the ray up to orders $R, \ldots, R-2j$ under the choice of

$\partial_x^\alpha s_k^L(0, x_L^0)$ for $|\alpha| \leq R-2k-2$ arbitrary permutable families,
$\partial_x^\alpha s_k^L(t, x_L^t)$ for $t \in \mathbb{R}, R-2k-2 < |\alpha| \leq R-2k$ arbitrary permutable families,

and the setting of

$$a_k^L(t,x) = \chi_d(x - x_L^t) \sum_{|\alpha| \leq R-2k} \frac{1}{\alpha!}(x - x_L^t)^\alpha \partial_x^\alpha \left(s_k^L p_L + b_k^L\right)(t, x_L^t),$$

for $k = 0, \ldots, j-1$. By using the eikonal equation on ψ_L

$$\begin{aligned} c_{j+1}^L \underset{x=x_L^t}{\overset{R}{\asymp}} \; & \rho(c_T^2 - c_L^2)|\partial_x\psi_L|^2 b_{j+1}^L \\ & + \left[(L_L(\psi_L) + \beta_L(\psi_L))s_j^L\right] p_L + T_\perp s_j^L + (L_T(\psi_L) + \beta_T(\psi_L))b_j^L + q_\| b_j^L \\ & + N a_{j-1}^L. \end{aligned}$$

Making the dot product of c_{j+1}^L with p_L vanish up to the order $R-2j-2$ gives a non-homogeneous evolution equation on s_j^L

$$(L_L(\psi_L) + \beta_L(\psi_L))s_j^L + [q_\| b_j^L + (L_T(\psi_L) + \beta_T(\psi_L))b_j^L + N a_{j-1}^L].p_L \underset{x=x_L^t}{\overset{R-2j-2}{\asymp}} 0. \qquad (14)$$

This equation determines $\partial_x^\alpha s_j^L(t, x_L^t)$ given $\partial_x^\alpha s_j^L(0, x_L^0)$ for $|\alpha| = 0, \ldots, R-2j-2$. Once chosen

$\partial_x^\alpha s_j^L(0, x_L^0)$ for $|\alpha| \leq R-2j-2$ arbitrary permutable families,
$\partial_x^\alpha s_j^L(t, x_L^t)$ for $t \in \mathbb{R}, R-2j-2 < |\alpha| \leq R-2j$ arbitrary permutable families,

we set

$$a_j^L(t,x) = \chi_d(x - x_L^t) \sum_{|\alpha| \leq R-2j} \frac{1}{\alpha!}(x - x_L^t)^\alpha \partial_x^\alpha \left(s_j^L p_L + b_j^L\right)(t, x_L^t),$$

b_{j+1}^L is then fully determined by the equation $c_{j+1}^L \underset{x=x_L^t}{\overset{R-2j-2}{\asymp}} 0$. The system is closed, since $b_{N+1} = 0$ and the vanishing of c_{N+1}^L determines the spatial derivatives of s_N^L on the ray.

2.2 Transversal beams

The analysis for beams T is similar. One has

$$c_1^T \underset{x=x_T^t}{\overset{R}{\asymp}} \rho(c_L^2 - c_T^2)|\partial_x \psi_T|^2 s_1^T p_T + q_\| b_0^T + (L_T(\psi_T) + \beta_T(\psi_T)) b_0^T. \tag{15}$$

Since $p_T \cdot b_0^T = 0$, it follows that

$$p_T \cdot \partial_t b_0^T = -\partial_t p_T \cdot b_0^T \text{ and } p_T \cdot \partial_x b_0^T = -\partial_x p_T \cdot b_0^T.$$

Thus

$$p_T \cdot L_T(\psi_T) b_0^T = -(L_T(\psi_T) p_T) \cdot b_0^T, \tag{16}$$

which leads, by using $(Id - p_T p_T \cdot) c_1^T \underset{x=x_T^t}{\overset{R-2}{\asymp}} 0$, to the following equation

$$(L_T(\psi_T) + \beta_T(\psi_T))\, b_0^T + p_T (L_T(\psi_T) p_T) \cdot b_0^T \underset{x=x_T^t}{\overset{R-2}{\asymp}} 0. \tag{17}$$

Let us write, for $j = 0, \dots, N$, $b_j^T = r_j^T q_j^T$, with q_j^T a unit vector. We first consider the projection of (17) on q_0^T and then on $p_T \wedge q_0^T$. Since $(L_T(\psi_T) q_0^T) \cdot q_0^T = 0$, the dot product of (17) with q_0^T may be written

$$(L_T(\psi_T) + \beta_T(\psi_T)) r_0^T \underset{x=x_T^t}{\overset{R-2}{\asymp}} 0. \tag{18}$$

Thus we dispose of an evolution equation on r_0^T that determines its spatial derivatives up to order $R-2$ on the ray. In particular, replacing $\partial_x \psi_T(t, x_T^t)$ by ξ_T^t and $\partial_t \psi_T(t, x_T^t)$ by $-c_T(x_T^t)|\xi_T|$ and using a similar relation to (11) with indexes T give at order 0 the following transport equation

$$\frac{d}{dt} r_0^T(t, x_T^t) = \frac{1}{2} \left[\frac{P_T \psi_T(t, x_T^t)}{c_T(x_T^t)|\xi_T^t|} - \frac{d}{dt} \ln(\rho(x_T^t)) \right] r_0^T(t, x_T^t).$$

Thus

$$r_0^T(t, x_T^t) = \left[\frac{\rho(x_T^t)}{\rho(x_T^0)} \det(x_{Ty}^t + i x_{T\eta}^t) \right]^{-\frac{1}{2}},$$

where the square root is defined by continuity from 1 at $t = 0$. r_0^T is computed up to the order $R-2$ on the ray as s_0^L and its spatial derivatives on the ray up to this order are fully determined by their initial values.

On the other hand, taking the dot product of (17) with $p_T \wedge q_0^T$ leads to

$$(L_T(\psi_T) q_0^T) \cdot (p_T \wedge q_0^T) \underset{x=x_T^t}{\overset{R-2}{\asymp}} 0. \tag{19}$$

As q_0^T is unit and orthogonal to p_T, one can prove that this equation fully determines the spatial derivatives of q_0^T up to order $R-2$ on the ray given their initial values. We show this recursively on $k = 0, \dots, R-2$. At order 0, equation (19) reads

$$\frac{d}{dt} q_0^T(t, x_T^t) = \left[\frac{d}{dt} q_0^T(t, x_T^t) \cdot p_T(t, x_T^t) \right] p_T(t, x_T^t),$$

which is equivalent to the system of ODEs

$$\frac{d}{dt}q_0^T(t,x_T^t) = -\left[p_T(t,x_T^t)\frac{d}{dt}p_T(t,x_T^t)\cdot\right] q_0^T(t,x_T^t).$$

$q_0^T(t,x_T^t)$ is thus determined modulo its initial value. Now suppose $\partial_x^\beta q_0^T(t,x_T^t)$ determined for $|\beta| \le k$. Since q_0^T is unit and orthogonal to p_T, $\partial_x^\alpha q_0^T(t,x_T^t)\cdot q_0^T(t,x_T^t)$ and $\partial_x^\alpha q_0^T(t,x_T^t)\cdot p_T$ can be determined for $|\alpha| \le k+1$. The remaining unknowns are thus $\partial_x^\alpha q_0^T(t,x_T^t)\cdot(p_T(t,x_T^t)\wedge q_0^T(t,x_T^t))$ for $|\alpha| = k+1$. Using (11) and (12), the equation (19) gives at order $k+1$

$$\frac{d}{dt}\left[\partial_x^\alpha q_0^T(t,x_T^t)\cdot(p_T(t,x_T^t)\wedge q_0^T(t,x_T^t))\right] + \left(Z_\alpha q_0^T\right)|_{(t,x_T^t)} = 0,\ |\alpha| = k+1,$$

with Z_α a differential operator of order α. The latter equations provide the projections of $\partial_x^\alpha q_0^T(t,x_T^t)$ on $p_T(t,x_T^t)\wedge q_0^T(t,x_T^t)$.

Remark 2.1. *A useful property of q_0^T is determined by using the transverse vectors e_1^t, e_2^t defined by*

$$e_1^t = \cos\theta^t n^t + \sin\theta^t b^t,\ e_2^t = -\sin\theta^t n^t + \cos\theta^t b^t,$$

where n^t and b^t are the ray x_T^t normal and binormal, $\theta^t = \int_{s_0}^s \tau(s)ds + \theta^0$ and $\tau(s)$ the ray torsion. Indeed $\frac{de_1^t}{dt}$ and $\frac{de_2^t}{dt}$ have nonzero projections only on the vector $p_T(t,x_T^t)$. It follows that $\frac{d}{dt}\left[q_0^T(t,x_T^t)\cdot e_1^t\right] = \frac{d}{dt}q_0^T(t,x_T^t)\cdot e_1^t + q_0^T(t,x_T^t)\cdot\frac{d}{dt}e_1^t$ is zero as well as $\frac{d}{dt}\left[q_0^T(t,x_T^t)\cdot e_2^t\right]$ and $q_0^T(t,x_T^t)$ remains constant in the basis (e_1^t,e_2^t) of the hyperplane $p_T(t,x_T^t)^\perp$ (see [78], section 4).

We choose

$q_0^T(0,x_T^0)$ arbitrary unit vector orthogonal to ξ_T^0,

$\partial_x^\alpha r_0(0,x_T^0)$ for $|\alpha| \le R-2$ and $\partial_x^\beta q_0^T(0,x_T^0)\cdot(\frac{\xi_T^0}{|\xi_T^0|}\wedge q_0^T(0,x_T^0))$ for $1 \le |\beta| \le R-2$ arbitrary permutable families,

$\partial_x^\alpha r_0(t,x_T^t)$ and $\partial_x^\alpha r_0(t,x_T^t)\cdot(\frac{\xi_T^t}{|\xi_T^t|}\wedge q_0^T(t,x_T^t))$ for $t\in\mathbb{R}, R-2 < |\alpha| \le R$ arbitrary permutable families, (20)

and set

$$a_0^T(t,x) = \chi_d(x-x_T^t)\sum_{|\alpha|\le R}\frac{1}{\alpha!}(x-x_T^t)^\alpha\partial_x^\alpha\left(r_0^T q_0^T\right)(t,x_T^t). \tag{21}$$

Finally, the spatial derivatives of s_1^T on the ray are obtained up to order $R-2$ by plugging the values of r_0^T and q_0^T in (15).

For $0 \le j \le N$, the eikonal equation on ψ_T implies

$$\begin{aligned} c_{j+1}^T \ \overset{R}{\underset{x=x_T^t}{\asymp}}\ & (c_L^2 - c_T^2)|\partial_x\psi_T|^2 s_{j+1}^T p_T \\ +\ & [L_L(\psi_T)+\beta_L(\psi_T)]s_j^T p_T + q_\parallel b_j^T + (L_T(\psi_T)+\beta_T(\psi_T))b_j^T + T_\perp s_j^T \\ +\ & N a_{j-1}^T. \end{aligned} \tag{22}$$

Assume that $a_0^T, \dots, a_{j-1}^T$ and s_j^T have been determined by the vanishing of $c_0^T, \dots, c_j^T$ on the ray up to the orders $R, \dots, R-2j$, similar choices to (20) and the setting of

$$a_k^T(t,x) = \chi_d(x - x_T^t) \sum_{|\alpha| \leq R-2k} \frac{1}{\alpha!}(x - x_T^t)^\alpha \partial_x^\alpha \left(s_k^T p_T + b_k^T\right)(t, x_T^t),$$

for $k \leq j-1$.

Since $p_T \cdot b_{j-1}^T = 0$, a similar equation to (16) gives, by using $(Id - p_T p_T \cdot) c_j^T \underset{x=x_T^t}{\overset{R-2j}{\asymp}} 0$

$$\left(L_T(\psi_T) + \beta_T(\psi_T)\right) b_{j-1}^T + [(L_T(\psi_T) p_T) \cdot b_{j-1}^T] p_T + T_\perp s_{j-1}^T + (Id - p_T p_T \cdot) N a_{j-2}^T \underset{x=x_T^t}{\overset{R-2j}{\asymp}} 0.$$

Writing $b_{j-1}^T = r_{j-1}^T q_{j-1}^T$ with q_{j-1}^T a unit vector leads to ODEs on the spatial derivatives of r_{j-1}^T and q_{j-1}^T up to order $R-2j$ by taking the dot product of the previous equation with q_{j-1}^T and $p_T \wedge q_{j-1}^T$. We set

$$a_{j-1}^T(t,x) = \chi_d(x - x_T^t) \sum_{|\alpha| \leq R-2j+2} \frac{1}{\alpha!}(x - x_T^t)^\alpha \partial_x^\alpha \left(s_{j-1}^T p_T + b_{j-1}^T\right)(t, x_T^t).$$

Next, s_j^T is obtained by plugging the value of b_{j-1}^T in (22) and making c_j^T vanish up to order $R-2j$ on (t, x_T^t). Hence, b_{j-1}^T and s_j^T are determined up to the orders $R-2j$ on the ray (t, x_T^t) under the knowledge of b_{j-1}^T up to the order $R-2j$ on $(0, x_T^0)$.

We end the construction of the beams' amplitudes by expanding them for all $(t,x) \in \mathbb{R}^{n+1}$ as follows

$$a_j = \chi_d(x - x^t) \sum_{|\alpha| \leq p-2j-2} \frac{1}{\alpha!}(x - x^t)^\alpha \partial_x^\alpha a_j(t, x^t).$$

2.3 Reflection of a beam L

The reflection of a beam L gives birth to two reflected beams : a beam L and a beam T. After one reflection, we search for a solution of the form

$$\omega_\varepsilon^L + \mathrm{Ref}\, \omega_\varepsilon^L = \sum_{j=0}^{j} \varepsilon^j a_j^L e^{i\psi_L/\varepsilon} + \varepsilon^j a_j^{LL} e^{i\psi_{LL}/\varepsilon} + \varepsilon^j a_j^{LT} e^{i\psi_{LT}/\varepsilon},$$

where the index LL denotes the reflected beam L and the index LT the reflected beam T.

To fulfill the boundary condition, the reflected phases must have the same time and tangential derivatives as the incident phase at the instant t_L and the point $x_L^{t_L}$ of reflection, and this up to the order R. The phase ψ_{LL} is thus constructed like the reflected phase for the scalar wave equation associated to the symbol $\tau + c_L|\xi|$ (chapter 1).

The reflected bicharacteristic φ_{LT} is given by the following conditions at $t = t_L$

$$\begin{cases} x_{LT}^{t_L} = x_L^{t_L}, \\ c_T(x_L^{t_L})|\xi_{LT}^{t_L}| = c_L(x_L^{t_L})|\xi_L^{t_L}|, \\ \xi_{LT}^{t_L} - (\xi_{LT}^{t_L} \cdot \nu(x_L^{t_L}))\nu(x_L^{t_L}) = \xi_L', \\ \xi_{LT}^{t_L} \cdot \nu(x_L^{t_L}) \leq 0, \end{cases}$$

with $\xi'_L = \xi_L^{t_L} - (\xi_L^{t_L} \cdot \nu(x_L^{t_L}))\nu(x_L^{t_L})$. Carrying out elementary computations yields

$$\begin{cases} x_{LT}^{t_L} = x_L^{t_L}, \\ \Big(\xi_{LT}^{t_L} \cdot \nu(x_L^{t_L})\Big)^2 = \dfrac{c_L^2(x_L^{t_L})}{c_T^2(x_L^{t_L})}\Big(\xi_{LT}^{t_L} \cdot \nu(x_L^{t_L})\Big)^2 + (\dfrac{c_L^2(x_L^{t_L})}{c_T^2(x_L^{t_L})} - 1)|\xi'_L|^2, \\ \xi_{LT}^{t_L} = \xi'_L + \Big(\xi_{LT}^{t_L} \cdot \nu(x_L^{t_L})\Big)\nu(x_L^{t_L}) \\ \xi_{LT}^{t_L} \cdot \nu(x_L^{t_L}) \leq 0. \end{cases}$$

Since $c_T < c_L$

$$\xi_{LT}^{t_L} = \xi'_L - \Big[\frac{c_L^2(x_L^{t_L})}{c_T^2(x_L^{t_L})}|\xi'_L|^2 + (\frac{c_L^2(x_L^{t_L})}{c_T^2(x_L^{t_L})} - 1)|\xi'_L|^2\Big]^{\frac{1}{2}}\nu(x_L^{t_L}).$$

The reflected amplitudes are determined by the boundary condition. Let m_B be the order of the boundary operator, that is $m_B = 0$ or 1 for Dirichlet ou Neumann problem. We adapt the method [84] p. 224 for a vector beam. To do this we introduce vector amplitudes d_{-m_B+j} defined on the boundary $\partial\Omega$ by

$$\begin{aligned} B(\omega_\varepsilon^L + \mathrm{Ref}\,\omega_\varepsilon^L) &= \Big(\varepsilon^{-m_B} d_{-m_B}^L + \cdots + \varepsilon^N d_N^L\Big)e^{i\psi_L/\varepsilon} \\ &+ \Big(\varepsilon^{-m_B} d_{-m_B}^{LL} + \cdots + \varepsilon^N d_N^{LL}\Big)e^{i\psi_{LL}/\varepsilon} \\ &+ \Big(\varepsilon^{-m_B} d_{-m_B}^{LT} + \cdots + \varepsilon^N d_N^{LT}\Big)e^{i\psi_{LT}/\varepsilon}. \end{aligned}$$

We impose on $d_{-m_B+j}^L + d_{-m_B+j}^{LL} + d_{-m_B+j}^{LT}$ to vanish on $(t_L, x_L^{t_L})$ up to the order $R-2j-2$ for $j = 0, \ldots, N + m_B$. For $j = 0$, we plug the form of the reflected amplitudes a_0^{LL} and a_0^{LT} to get

$$d_{-m_B}^L + d_{-m_B}^{LL} + d_{-m_B}^{LT} = b(x, \partial_x\psi_L)a_0^L + b(x, \partial_x\psi_{LL})s_0^{LL}p_{LL} + b(x, \partial_x\psi_{LT})b_0^{LT}. \tag{23}$$

Taking the dot product of $d_{-m_B}^L + d_{-m_B}^{LL} + d_{-m_B}^{LT}$ with the vector $p_{LT}(t_L, x_L^{t_L})$, one obtains

$$b(x, \partial_x\psi_{LL})(p_{LL} \cdot p_{LT})s_0^{LL} \underset{(t,x')=(t_L,x_L^{t_L})}{\overset{R-2}{\asymp}} -b(x, \partial_x\psi_L)(a_0^L \cdot p_{LT}). \tag{24}$$

Here, we use the notation $\underset{(t,x')=(t_a,x_a^{t_a})}{\overset{k}{\asymp}}$ to denote that both sides have the same time and tangential derivatives at the instant t_a and the boundary point $x_a^{t_a}$ up to order k. If the contact with the boundary is transversal, $b(x_L^{t_L}, \xi_{LL}^{t_L}) \neq 0$ and

$$p_{LL}(t_L, x_L^{t_L}) \cdot p_{LT}(t_L, x_L^{t_L})|\xi_{LL}^{t_L}||\xi_{LT}^{t_L}| = |\xi'_L|^2 + (\nu(x_L^{t_L}) \cdot {\xi_{LL}}^{t_L})(\nu(x_L^{t_L}) \cdot {\xi_{LT}}^{t_L}) > 0.$$

Therefore, (24) determines time and tangential derivatives of s_0^{LL} at $(t_L, x_L^{t_L})$ up to the order $R-2$. Once those values computed, we use them in (23) to get b_0^{LT}

$$b(x, \partial_x\psi_{LT})b_0^{LT} \underset{(t,x')=(t_L,x_L^{t_L})}{\overset{R-2}{\asymp}} b(x, \partial_x\psi_L)\Big[-a_0^L + \frac{(a_0^L \cdot p_{LT})}{(p_{LL} \cdot p_{LT})}p_{LL}\Big]. \tag{25}$$

Since $b(x_L^{t_L}, \xi_{LT}^{t_L}) \neq 0$ when the contact is non tangential, (25) determines time and tangential derivatives of b_0^{LT} at $(t_L, x_L^{t_L})$ up to the order $R-2$.

For $j > 0$, we write the reflected amplitudes as $a_j^{L\alpha} = s_j^{L\alpha} p_{L\alpha} + b_j^{L\alpha}$ with $b_j^{L\alpha} \cdot p_{L\alpha} = 0$ for $\alpha = L, T$. It follows that

$$d^L_{-m_B+j} + d^{LL}_{-m_B+j} + d^{LT}_{-m_B+j} = b(x, \partial_x \psi_L) a_j^L + b(x, \partial_x \psi_{LL})[s_j^{LL} p_{LL} + b_j^{LL}] + b(x, \partial_x \psi_{LT})[s_j^{LT} p_{LT} + b_j^{LT}] + g_j^L,$$

where g_j^L depends only on the amplitudes $r < j$. The constraint

$$d^L_{-m_B+j} + d^{LL}_{-m_B+j} + d^{LT}_{-m_B+j} \underset{(t,x')=(t_L, x_L^{t_L})}{\overset{R-2j-2}{\asymp}} 0$$

may be written as

$$b(x, \partial_x \psi_{LL}) s_j^{LL} p_{LL} + b(x, \partial_x \psi_{LT}) b_j^{LT} \underset{(t,x')=(t_L, x_L^{t_L})}{\overset{R-2j-2}{\asymp}} -b(x, \partial_x \psi_L) a_j^L - b(x, \partial_x \psi_{LL}) b_j^{LL} - b(x, \partial_x \psi_{LT}) s_j^{LT} p_{LT} - g_j^L. \tag{26}$$

As pointed out in the construction of the amplitudes of beams L, b_j^{LL} is fully determined by the knowledge of $a_0^{LL}, \dots, a_{j-1}^{LL}$. Likewise, for beams T, s_j^{LT} is fully determined once $a_0^{LT}, \dots, a_{j-1}^{LT}$ have been computed. Thus, taking the dot product of (26) with $p_{LT}(t_L, x_L^{t_L})$, one can compute time and tangential derivatives of s_j^{LL} up to the order $R - 2j - 2$ at $(t_L, x_L^{t_L})$. We then use these values to determine b_j^{LT} .

2.4 Reflection of a beam T

Reflection of a beam T gives birth in general to a beam L and a beam T. We search for a solution of the form

$$\omega_\varepsilon^T + \mathrm{Ref}\, \omega_\varepsilon^T = \sum_{j=0}^{j} \varepsilon^j a_j^T e^{i\psi_T/\varepsilon} + \varepsilon^j a_j^{TL} e^{i\psi_{TL}/\varepsilon} + \varepsilon^j a_j^{TT} e^{i\psi_{TT}/\varepsilon},$$

where the index TL denotes a reflected beam L and the index TT a reflected beam T. The construction is similar to the reflection of a beam L. The reflected phase ψ_{TT} is built as the reflected phase for the scalar wave equation associated to the symbol $\tau + c_T(x)|\xi|$.

The reflected bicharacteristic φ_{TL} is given by the following conditions at the instant of reflection t_T

$$\begin{cases} x_{TL}^{t_T} = x_T^{t_T}, \\ c_L(x_T^{t_T})|\xi_{TL}^{t_T}| = c_T(x_T^{t_T})|\xi_T^{t_T}|, \\ \xi_{TL}^{t_T} - (\xi_{TL}^{t_T} \cdot \nu(x_T^{t_T}))\nu(x_T^{t_T}) = \xi'_T, \\ \xi_{TL}^{t_T} \cdot \nu(x_T^{t_T}) \le 0, \end{cases} \tag{27}$$

where $\xi'_T = \xi_T^{t_T} - (\xi_T^{t_T} \cdot \nu(x_T^{t_T}))\nu(x_T^{t_T})$. Computations give

$$\begin{cases} x_{TL}^{t_T} = x_T^{t_T}, \\ \Big(\xi_{TL}^{t_T} \cdot \nu(x_T^{t_T})\Big)^2 = \Big(\frac{c_T^2(x_T^{t_T})}{c_L^2(x_T^{t_T})} - 1\Big)|\xi'_T|^2 + \frac{c_T^2(x_T^{t_T})}{c_L^2(x_T^{t_T})}\Big(\xi_T^{t_T} \cdot \nu(x_T^{t_T})\Big)^2, \\ \xi_{TL}^{t_T} = \xi'_T + \Big(\xi_{TL}^{t_T} \cdot \nu(x_T^{t_T})\Big)\nu(x_T^{t_T}), \\ \xi_{TL}^{t_T} \cdot \nu(x_T^{t_T})) \le 0. \end{cases}$$

If the assumption

$$\xi_T^{t_T} \cdot \nu(x_T^{t_T}) \geq \Big(\frac{c_L^2(x_T^{t_T})}{c_T^2(x_T^{t_T})} - 1\Big)^{\frac{1}{2}} |\xi_T'|,$$

is satisfied, one has

$$\xi_{TL}^{t_T} = \xi_T' - \frac{c_T(x_T^{t_T})}{c_L(x_T^{t_T})}\Big[\Big(\xi_T^{t_T} \cdot \nu(x_T^{t_T})\Big)^2 - \Big(\frac{c_L(x_T^{t_T})^2}{c_T(x_T^{t_T})^2} - 1\Big)|\xi_T'|^2\Big]^{\frac{1}{2}} \nu(x_T^{t_T}). \tag{28}$$

The needed hypothesis on $\xi_T^{t_T} \cdot \nu(x_T^{t_T})$ shows that there is a critical angle θ_c between $\xi_T^{t_T}$ and the hyperplane tangent to $\partial\Omega$ at $x_T^{t_T}$ defined by

$$\sin\theta_c = \Big(\frac{c_L^2(x_T^{t_T})}{c_T^2(x_T^{t_T})} - 1\Big)^{\frac{1}{2}} \frac{|\xi_T'|}{|\xi_T|}, \tag{29}$$

for which $\xi_{TL}^{t_T}$ is tangential to the boundary [1]. In the sequel we assume that this critical angle is not reached. The phase ψ_{TL} is constructed similarly to ψ_{LT}.
Let us compute the reflected amplitudes. We replace in the computations of the previous section, the indexes L, LL and LT by T, TL and TT respectively. One gets

$$b(x, \partial_x\psi_{TL}) s_j^{TL} p_{TL} + b(x,\partial_x\psi_{TT}) b_j^{TT} \underset{(t,x')=(t_T,x_L^{t_T})}{\overset{R-2j-2}{\asymp}} - b(x, \partial_x\psi_T) a_j^T - b(x, \partial_x\psi_{TL}) b_j^{TL} - b(x, \partial_x\psi_{TT}) s_j^{TT} p_{TT} - g_j^T,$$

with g_j^T depending only on the amplitudes a_r, $r < n$ and $g_j^0 = 0$.

Taking the dot product with $p_{TT}^{t_T}$ gives time and tangential derivatives of s_j^{TL} up to the order $R - 2j - 2$ at $(t_T, x_T^{t_T})$. We then use these values to compute b_j^{TT}. For $j = 0$ one obtains

$$b(x, \partial_x\psi_{TL})(p_{TL} \cdot p_{TT}) s_0^{TL} \underset{(t,x')=(t_T,x_T^{t_T})}{\overset{R-2}{\asymp}} -b(x, \partial_x\psi_T)(a_0^T \cdot p_{TT}),$$

and

$$b(x, \partial_x\psi_{TT}) b_0^{TT} \underset{(t,x')=(t_T,x_T^{t_T})}{\overset{R-2}{\asymp}} b(x, \partial_x\psi_T)\Big[- a_0^T + \frac{(a_0^T \cdot p_{TT})}{(p_{TL} \cdot p_{TT})} p_{TL}\Big].$$

3 Construction of the approximate solution

In this section, we build u_ε^R, the approximate solution up to order $O(\varepsilon^{\frac{R-1}{2}})$ and justify the asymptotics.

3.1 Gaussian beams summation

The summation process based on the FBI technique consists of integrating over the phase space domain Gaussian beams microlocalized at $t = 0$ near (y, η), after weighting them by some quantities related to the initial data's FBI transforms at point (y, η). The very first step is then to generalize the notion of a FBI transform to vector functions. This is done naturally by setting for $u \in L^2(\mathbb{R}^3)^3$

$$T_\varepsilon u(y,\eta) = c_n \varepsilon^{-\frac{3n}{4}} \int_{\mathbb{R}^3} u(x) e^{i\eta\cdot(y-x)/\varepsilon-(y-x)^2/(2\varepsilon)} dx. \tag{30}$$

One gets properties similar to the scalar case. In particular, T_ε is an isometry from $L^2(\mathbb{R}^3)^3 \to L^2(\mathbb{R}^3 \times \mathbb{R}^3)^3$ and

$$T_\varepsilon^* T_\varepsilon = Id, \tag{31}$$

where T_ε^* is the adjoint of T_ε and satisfies

$$T_\varepsilon^* f(x) = c_n \varepsilon^{-\frac{3n}{4}} \int_{\mathbb{R}^6} f(y,\eta) e^{i\eta\cdot(x-y)/\varepsilon-(x-y)^2/(2\varepsilon)} dy d\eta =: c_n \varepsilon^{-\frac{3n}{4}} \int_{\mathbb{R}^6} f(y,\eta) e^{i\phi_0(x,y,\eta)/\varepsilon} dy d\eta.$$

Next we select beams with phases that coincide at $t = 0$ with ϕ_0

$$\psi(0,x) = \eta \cdot (x-y) + \frac{i}{2}(x-y)^2. \tag{32}$$

We refer to the dependence of such beams on the starting point and direction of their associated rays. We thus denote $\omega_\varepsilon^L(t,x,y,\eta)$ an L-beam associated with the bicharacteristic (x_L^t, ξ_L^t) s.t. $(x_L^0, \xi_L^0) = (y,\eta)$, and use a similar notation for a T-beam. Hence, we consider only bicharacteristics satisfying

$$x_L^0 = x_T^0 = y \text{ and } \xi_L^0 = \xi_T^0 = \eta.$$

Let us now examine the amplitudes of these beams at $t = 0$. The first term in the Taylor series of $a_0^L(0,x)$ is $s_0^L(0,y)\frac{\eta}{|\eta|}$. As regards $a_0^T(0,x)$, it is a unit vector orthogonal to $\frac{\eta}{|\eta|}$ multiplied by $r_0^T(0,y)$. We shall thus decompose the FBI transforms on the basis $(\frac{\eta}{|\eta|}, e_1^0, e_2^0)$ and fit each component by using three appropriate elementary solutions based on Gaussian beams. So we aim to construct solutions ι_ε^1, ι_ε^2, ι_ε^3 that equal at $t = 0$ respectively $\frac{\eta}{|\eta|} e^{i\phi_0/\varepsilon}$, $e_1^0 e^{i\phi_0/\varepsilon}$ and $e_2^0 e^{i\phi_0/\varepsilon}$ modulo residues of order $O(\varepsilon^p)$ with p sufficiently large. For any two beams $\omega_\varepsilon^L(.,y,\eta)$ and $\omega_\varepsilon^T(.,y,\eta)$ we may write

$$\omega_\varepsilon^L(0,x,y,\eta) + \omega_\varepsilon^T(0,x,y,\eta) = \sum_{j=0}^N \varepsilon^j \left[a_j^L(0,x,y,\eta) + a_j^T(0,x,y,\eta)\right] e^{i\phi_0(x,y,\eta)/\varepsilon}. \tag{33}$$

We write $r_j^T q_j^T =: r_j^S q_S + r_j^Z q_Z$ for $j = 0,\dots,N$ where (p_T, q_S, q_Z) is a fixed orthonormal basis that coincides on (t, x_T^t) with the basis $(\frac{\xi_T^t}{|\xi_T^t|}, e_1^t, e_2^t)$. One has by (13) and (21)

$$a_0^L(0,x) + a_0^T(0,x) = \chi_d(x-y) \sum_{|\alpha|\le R} \frac{1}{\alpha!}(x-y)^\alpha \partial_x^\alpha \left[s_0^L p_L + r_0^S q_S + r_0^Z q_Z\right](0,y).$$

We set $(s_0^L, r_0^S, r_0^Z)(0,y)$ equal to $(1,0,0)$ for the construction of ι_ε^1, $(0,1,0)$ for the construction of ι_ε^2 or $(0,0,1)$ for the construction of ι_ε^3. We then impose that

$$\partial_x^\alpha \left[M \begin{pmatrix} s_0^L \\ r_0^S \\ r_0^Z \end{pmatrix} \right](0,y) = 0 \text{ for } 1 \le |\alpha| \le R-2, \tag{34}$$

where M is the matrix (p_L, q_S, q_Z). Since the vectors p_L, q_S and q_Z form at point $(0, y)$ an orthonormal basis, $M(0, y)$ is invertible and the previous equations determine the spatial derivatives of s_0^L, r_0^S and r_0^Z at $(0, y)$ up to order $R-2$. One gets

$$a_0^L(0,x) + a_0^T(0,x) = \chi_d(x-y)\left[s_0^L(0,y)\frac{\eta}{|\eta|} + r_0^S(0,y)e_1^0 + r_0^Z(0,y)e_2^0\right] + \chi_d(x-y)\sum_{|\alpha|=R-1}^{R}\frac{1}{\alpha!}(x-y)^\alpha\partial_x^\alpha\left[s_0^L p_L + r_0^S q_S + r_0^Z q_Z\right](0,y).$$

Recursively, for $j = 1, \dots, N$, we impose that

$$\partial_x^\alpha\left[s_j^L p_L + r_j^S q_S + r_j^Z q_Z\right](0,y) = -\partial_x^\alpha\left[b_j^L + s_j^T p_L\right](0,y) \text{ for } |\alpha| \le R-2j-2. \tag{35}$$

The r.h.s. is known by the construction of the amplitudes a_k^L and a_k^T for $k \le j-1$. Therefore, hypothesis (34) and (35) lead to

$$\omega_\varepsilon^L(0,x) + \omega_\varepsilon^T(0,x) = \Big[\chi_d(x-y)\Big(s_0^L(0,y)\frac{\eta}{|\eta|} + r_0^S(0,y)e_1^0 + r_0^Z(0,y)e_2^0\Big) + \chi_d(x-y)\sum_{j=0}^{N}\varepsilon^j\sum_{|\alpha|=R-2j-1}^{R-2j}(x-y)^\alpha \mathrm{res}_j^\alpha\Big]e^{i\phi_0(x,y,\eta)/\varepsilon}.$$

We thus succeeded in building the elementary solutions ι_ε^1, ι_ε^2 and ι_ε^3 as a sum of two suitable beams of type L and T. Similar ideas give elementary solutions $\iota_\varepsilon^{j'}$, $j = 1, 2, 3$, s.t. their time derivatives at $t = 0$ approach respectively $\varepsilon^{-1}\frac{\eta}{|\eta|}e^{i\phi_0/\varepsilon}$, $\varepsilon^{-1}e_1^0 e^{i\phi_0/\varepsilon}$ and $\varepsilon^{-1}e_2^0 e^{i\phi_0/\varepsilon}$ modulo small residues.

Without loss of generality, one can choose T sufficiently small so that at most one reflection occurs for rays originating from some compact $K_y \times K_\eta \subset \mathbb{R}^{2n}$. Let ρ' be a cut-off of $\mathcal{C}_0^\infty(\mathbb{R}^3, [0,1])$ supported in a compact $K_y \subset \Omega$ and satisfying

$$\rho'(y) = 1 \text{ if } \mathrm{dist}(y, K) < \Delta \text{ for a small } \Delta > 0, \tag{36}$$

and γ' a cut-off of $\mathcal{C}_0^\infty(\mathbb{R}^3, [0,1])$ supported in a compact $K_\eta \subset \mathbb{R}^3\backslash\{0\}$ s.t. $\gamma' \equiv 1$ on R_η.

We search for an approximate solution such as

$$\begin{aligned} u_\varepsilon^R(t,x) = \frac{c_n}{2}\varepsilon^{-\frac{3n}{4}+1}\int\Bigg[&\Big(\sum_{k=0,1}\mathrm{Ref}^k\iota_\varepsilon^1(t,x) + \sum_{k=0,-1}\mathrm{Ref}^k(\iota_\varepsilon^1(-t,x))\Big)\frac{\eta}{|\eta|} \\ &+ \Big(\sum_{k=0,1}\mathrm{Ref}^k\iota_\varepsilon^2(t,x) + \sum_{k=0,-1}\mathrm{Ref}^k(\iota_\varepsilon^2(-t,x))\Big)e_1^0 \\ &+ \Big(\sum_{k=0,1}\mathrm{Ref}^k\iota_\varepsilon^3(t,x) + \sum_{k=0,-1}\mathrm{Ref}^k(\iota_\varepsilon^3(-t,x))\Big)e_2^0\Bigg]\cdot\Big(\varepsilon^{-1}T_\varepsilon u_\varepsilon^I\Big)\rho'\otimes\gamma' \\ &+ \Bigg[\Big(\sum_{k=0,1}\mathrm{Ref}^k\iota_\varepsilon^{1'}(t,x) - \sum_{k=0,-1}\mathrm{Ref}^k(\iota_\varepsilon^{1'}(-t,x))\Big)\frac{1}{c_L}\frac{\eta}{|\eta|} \\ &+ \Big(\sum_{k=0,1}\mathrm{Ref}^k\iota_\varepsilon^{2'}(t,x) - \sum_{k=0,-1}\mathrm{Ref}^k(\iota_\varepsilon^{2'}(-t,x))\Big)\frac{1}{c_T}e_1^0 \\ &+ \Big(\sum_{k=0,1}\mathrm{Ref}^k\iota_\varepsilon^{3'}(t,x) - \sum_{k=0,-1}\mathrm{Ref}^k(\iota_\varepsilon^{3'}(-t,x))\Big)\frac{1}{c_T}e_2^0\Bigg]\cdot\left(i\frac{1}{|\eta|}T_\varepsilon v_\varepsilon^I\right)\rho'\otimes\gamma' dy d\eta, \end{aligned}$$

where ι_ε^j and $\iota_\varepsilon^{j'}$ are the elemntary solutions constructed previously as sums of two beams and Ref denotes the reflections of each one of these beams.

3.2 Justification of the asymptotics

In this section we prove theorem 1.1. The initial boundary value problem (1) is well posed and moreover we dispose of the following energy estimate as a consequence of e.g. [47]

$$\begin{aligned}\sup_{[0,T]}\|u_\varepsilon^R(t,.)-u_\varepsilon(t,.)\|_{H^1(\Omega)^3}+\sup_{[0,T]}\|\partial_t u_\varepsilon^R(t,.)-\partial_t u_\varepsilon(t,.)\|_{L^2(\Omega)^3}\lesssim\\ \|\partial_t u_\varepsilon^R(0,.)-v_\varepsilon^I\|_{L^2(\Omega)^3}+\|u_\varepsilon^R(0,.)-u_\varepsilon^I\|_{H^1(\Omega)^3}\\ +\sup_{[0,T]}\|Eu_\varepsilon^R(t,.)\|_{L^2(\Omega)^3}+\|Bu_\varepsilon^R\|_{H^s([0,T]\times\partial\Omega)^3},\end{aligned}$$

where $s=\frac{3}{2}$ for Dirichlet boundary condition and 1/2 for Neumann. We estimate each term of the r.h.s. of the previous inequality. The computations are based on the results established in section 3 of chapter 1.

We start by estimating the error in the initial conditions. The phases ψ_L and ψ_T satisfy by construction the fundamental estimate

$$\operatorname{Im}\psi(t,x)\gtrsim|x-x^t|^2\text{ for }|x-x^t|\le d.$$

At $t=0$, all of the rays created by reflection at the boundary are in the exterior of the domain and their contribution to u_ε^R is then exponentially decreasing in L^2 as explained in chapter 1 p.39.

Only elementary solutions weighted by $T_\varepsilon u_\varepsilon^I$ have a non zero contribution initially. Moreover, at $t=0$, the contribution of $x-y\notin supp\chi_d$ is then exponentielly decreasing. Hence,

$$\begin{aligned}u_\varepsilon^R(0,x)=c_n\varepsilon^{-\frac{3n}{4}}\int\Big[\frac{\eta}{|\eta|}\frac{\eta}{|\eta|}\cdot+e_1^0e_1^0\cdot+e_2^0e_2^0\cdot\Big]T_\varepsilon u_\varepsilon^I e^{i\eta\cdot(x-y)/\varepsilon-(x-y)^2/(2\varepsilon)}dyd\eta\\ +c_n\varepsilon^{-\frac{3n}{4}}\sum_{j=0}^{N}\varepsilon^j\sum_{|\alpha|=R-2j-1}^{R-2j}\int(x-y)^\alpha\operatorname{res}_\alpha T_\varepsilon u_\varepsilon^I e^{i\eta\cdot(x-y)/\varepsilon-(x-y)^2/(2\varepsilon)}dyd\eta+O(\varepsilon^\infty).\end{aligned}$$

The estimates established in chapter 1 for similar quantities yield

$$u_\varepsilon^R|_{t=0}=u_\varepsilon^I+O(\varepsilon^{\frac{R-1}{2}})\text{ in }L^2(\Omega)^3.$$

Similar arguments show that

$$\partial_{x_b}u_\varepsilon^R|_{t=0}=\partial_{x_b}u_\varepsilon^I+O(\varepsilon^{\frac{R-1}{2}})\text{ in }L^2(\Omega)^3,\ b=1,2,3,$$

and

$$\partial_t u_\varepsilon^R|_{t=0}=v_\varepsilon^I+O(\varepsilon^{\frac{R-1}{2}})\text{ in }L^2(\Omega)^3.$$

To estimate the interior equation, we note that by construction, Eu_ε^R is a sum of terms of the form

$$\varepsilon^{-\frac{3n}{4}-1+j}\int c_j e^{i\psi/\varepsilon}f_\varepsilon dyd\eta,$$

where f_ε denotes some projection of $\varepsilon^{-1}T_\varepsilon u_\varepsilon^I$ or $i/(c|\eta|)T_\varepsilon v_\varepsilon^I$ and $c_j\underset{x=x^t}{\overset{R-2j}{\asymp}}0$, vanishing for $|x-x^t|\ge d$. For $k=1,2,3$, one may then write $\{Eu_\varepsilon^R\}_k$ as a sum of terms of the form

$$\varepsilon^{-\frac{3n}{4}+j-1}\int e_j^\alpha(x-x^t)^\alpha e^{i\psi/\varepsilon}f_\varepsilon dyd\eta,$$

with $|\alpha| = R - 2j + 1$ and e_j^α vanish for $|x - x^t| \geq d$. Since f_ε is uniformly bounded in $L^2(\mathbb{R}^3)$, the operators O^α introduced in chapter 1 give the following estimate

$$\| \int e_j^\alpha (x - x^t)^\alpha e^{i\psi/\varepsilon} f_\varepsilon dy d\eta \|_{L^2(\Omega)} \lesssim \varepsilon^{\frac{3n}{4} + \frac{|\alpha|}{2}},$$

which leads to

$$\sup_{[0,T]} \| E u_\varepsilon^R(t,.) \|_{L^2(\Omega)^3} \lesssim \varepsilon^{\frac{R-1}{2}}.$$

The estimate of the boundary conditions is similar to the scalar case. One obtains

$$\| B u_\varepsilon^R \|_{H^s([0,T]\times\partial\Omega)^3} \lesssim \varepsilon^{\frac{R+1}{2} - m_B - s}.$$

4 Wigner transforms and measures

In this section we compute the scalar Wigner measure e defined in (2) for initial data satisfying A1-A4 by using the first order approximate solution obtained as a summation of first order beams. We denote henceforth this asymptotic solution by u_ε^{appr}.

4.1 First order beams

We give explicit expressions for the beams' phases and the first term in the amplitudes when $R = 2$. In this case, the phases are quadratic on $(x - x^t)$ and may be written as

$$\psi(t,x) = \xi^t \cdot (x - x^t) + \frac{1}{2}(x - x^t) \cdot \left(\xi_y^t + i\xi_\eta^t\right) \left(x_y^t + i x_\eta^t\right)^{-1} (x - x^t).$$

Above the considered phases and flows may be incident or reflected ones.

Only the first terms in the Taylor expansion of the first amplitudes a_0 near the ray contribute to u_ε^{appr}. The constraints on the beams used in the construction of the elementary solutions lead to

$$a_0^L(t, x_L^t) = a_L(t) \frac{\xi_L^t}{|\xi_L^t|},$$
$$a_0^S(t, x_T^t) = a_T(t) e_1^t, a_0^Z(t, x_T^t) = a_T(t) e_2^t,$$

where a_L and a_T denote the quantities associated to the flows φ_L and φ_T defined by

$$a(t) = \left[\frac{\rho(x^t)}{\rho(y)} \det(x_y^t + i x_\eta^t) \right]^{-\frac{1}{2}}.$$

After reflection, the beams of type T are projected on e_1^t and e_2^t. We then obtain the scalar amplitudes $r_0^{\grave{\alpha}S}$ and $r_0^{\grave{\alpha}Z}$ defined as

$$b_0^{LT}(t, x_{LT}^t) = r_0^{LS}(t, x_{LT}^t) {e_1}_{LT}^t + r_0^{LZ}(t, x_{LT}^t) {e_2}_{LT}^t,$$
$$b_0^{\grave{\alpha}T}(t, x_{\grave{\alpha}T}^t) = r_0^{\grave{\alpha}S}(t, x_{\grave{\alpha}T}^t) {e_1}_{TT}^t + r_0^{\grave{\alpha}Z}(t, x_{\grave{\alpha}T}^t) {e_2}_{TT}^t, \ \grave{\alpha} = S, Z,$$

We next define the coefficients of reflection linking the scalar amplitudes of the reflected beams to the one of the incident beam at the instant of reflection

$$\begin{pmatrix} R^+_{LL} & R^+_{SL} & R^+_{ZL} \\ R^+_{LS} & R^+_{SS} & R^+_{ZS} \\ R^+_{LZ} & R^+_{SZ} & R^+_{ZZ} \end{pmatrix} \begin{pmatrix} s^L_0(t_L, x^{t_L}_L) & 0 & 0 \\ 0 & r^S_0(t_T, x^{t_T}_T) & 0 \\ 0 & 0 & r^Z_0(t_T, x^{t_T}_T) \end{pmatrix} = \begin{pmatrix} s^{LL}_0(t_L, x^{t_L}_L) & s^{SL}_0(t_T, x^{t_T}_T) & s^{ZL}_0(t_T, x^{t_T}_T) \\ r^{LS}_0(t_L, x^{t_L}_L) & r^{SS}_0(t_T, x^{t_T}_T) & r^{ZS}_0(t_T, x^{t_T}_T) \\ r^{LZ}_0(t_L, x^{t_L}_L) & r^{SZ}_0(t_T, x^{t_T}_T) & r^{ZZ}_0(t_T, x^{t_T}_T) \end{pmatrix}$$

Thus, the scalar reflected amplitudes satisfy at any time t

$$\begin{aligned} s^{LL}_0(t, x^t_{LL}) = R^+_{LL} a_{LL}(t),\ s^{\grave{\alpha}L}_0(t, x^t_{\grave{\alpha}L}) = R^+_{\grave{\alpha}L} a_{TL}(t), \\ r^{L\acute{\alpha}}_0(t, x^t_{L\acute{\alpha}}) = R^+_{LS} a_{LT}(t),\ r^{\grave{\alpha}\acute{\alpha}}_0(t, x^t_{\grave{\alpha}\acute{\alpha}}) = R^+_{\grave{\alpha}\acute{\alpha}} a_{TT}(t), \end{aligned}$$

where a_{LL}, a_{LT}, a_{TL} and a_{TT} denote the scalar amplitudes associated to the flows φ_{LL}, φ_{LT}, φ_{TL} and φ_{TT} respectively and $\grave{\alpha}, \acute{\alpha} = S, Z$. We also define $R^+_L = R^+_S = R^+_Z = 1$ to get similar expressions for the incident amplitudes. For beams assocaited to flows that propagate in the positive sens, we associate reflection coefficients with an exponent $-$.

4.2 Wigner measures for the asymptotic solution

We denote the Gaussian beams used in u^{appr}_ε as $\omega^{\alpha,\pm}_\varepsilon$ with

$$\alpha \in \mathcal{I} = \{L, S, Z\},$$

for the incident beams and

$$\alpha \in \mathcal{R} = \{LL, LS, LZ, SL, SS, SZ, ZL, ZS, ZZ\},$$

for the reflected beams. For $\alpha \in \mathcal{I} \cup \mathcal{R}$, we use the following notations

$\grave{\alpha} := \alpha$, $\acute{\alpha} := \alpha$ if $\alpha \in \mathcal{I}$ and $\alpha = \grave{\alpha}\acute{\alpha}$ if $\alpha \in \mathcal{R}$,

$[\alpha]$ denotes the index of the associated flow, that is $[L] = L, [S] = [Z] = T$ and for $\grave{\alpha}\acute{\alpha} \in \mathcal{R}, [\grave{\alpha}\acute{\alpha}] = [\grave{\alpha}][\acute{\alpha}]$,

$\vec{\alpha}$ is the initial unit vector associated with $\omega^{\alpha,\pm}_\varepsilon$. It may be $\eta/|\eta|$, e^0_1 or e^0_2.

For example, for a beam ω^{LS}_ε, that is $\alpha = LS$, we have $\grave{\alpha} = L$, $\acute{\alpha} = S$, $[\alpha] = LT$ and $\vec{\alpha} = e^0_1$. Time and spatial derivatives of the asymptotic solution u^{appr}_ε can then be written using the integrals

$$\begin{aligned} I^\pm_\alpha(\varepsilon, f_\varepsilon, v)(t, x) \quad = \quad & c_n \varepsilon^{-\frac{3n}{4}+1} \int_{\mathbb{R}^6} (\rho' \otimes \gamma')\, o\{\varphi^t_{[\alpha],\pm}\}^{-1}(z, \theta) f_\varepsilon\, o\{\varphi^t_{\alpha,\pm}\}^{-1}(z, \theta) \\ & (R^\pm_\alpha a^\alpha)(t, \{\varphi^t_{[\alpha],\pm}\}^{-1}(z, \theta)) e^{i\psi_\alpha(t,x,\{\varphi^t_{[\alpha],\pm}(z,\theta)\}^{-1})/\varepsilon} v(z, \theta) dz d\theta, \end{aligned}$$

where v is a vector smooth on $\{\varphi^t_{\alpha,\pm}\}^{-1}(K_y \times K_\eta)$ and f_ε is uniformly bounded in $L^2(\mathbb{R}^6)^3$, and the functions

$$\gamma^L_\varepsilon = (T_\varepsilon u^I_\varepsilon + \frac{i}{c_L|\eta|} T_\varepsilon v^I_\varepsilon) \cdot \frac{\eta}{|\eta|},\ (\gamma^S_\varepsilon, \gamma^Z_\varepsilon) = (T_\varepsilon u^I_\varepsilon + \frac{i}{c_T|\eta|} T_\varepsilon v^I_\varepsilon) \cdot (e^0_1, e^0_2),$$

$$\kappa_\varepsilon^L = (T_\varepsilon u_\varepsilon^I - \frac{i}{c_L|\eta|}T_\varepsilon v_\varepsilon^I)\cdot\frac{\eta}{|\eta|},\ (\kappa_\varepsilon^S,\kappa_\varepsilon^Z) = (T_\varepsilon u_\varepsilon^I - \frac{i}{c_T|\eta|}T_\varepsilon v_\varepsilon^I)\cdot(e_1^0,e_2^0).$$

In fact, as proven for the scalar wave equation in chapter 2, one has in $L^2(\Omega)^3$, uniformly for $t\in[0,T]$

$$2\partial_t u_\varepsilon^{appr} = v_t^+ + v_t^- + O(\sqrt{\varepsilon}),$$

with

$$\begin{aligned} v_t^+ &= -iI_L^+(\varepsilon,\gamma_\varepsilon^L,c_L\eta) - iI_S^+(\varepsilon,\gamma_\varepsilon^S,c_L|\eta|e_1^0) - iI_Z^+(\varepsilon,\gamma_\varepsilon^Z,c_L|\eta|e_2^0)\\ &\quad -iI_{LL}^+(\varepsilon,\gamma_\varepsilon^L,c_L\eta) - iI_{SL}^+(\varepsilon,\gamma_\varepsilon^S,c_L|\eta|e_1^0) - iI_{ZL}^+(\varepsilon,\gamma_\varepsilon^Z,c_L|\eta|e_2^0)\\ &\quad -iI_{LS}^+(\varepsilon,\gamma_\varepsilon^L,c_T\eta) - iI_{SS}^+(\varepsilon,\gamma_\varepsilon^S,c_T|\eta|e_1^0) - iI_{ZS}^+(\varepsilon,\gamma_\varepsilon^Z,c_T|\eta|e_2^0)\\ &\quad -iI_{LZ}^+(\varepsilon,\gamma_\varepsilon^L,c_T\eta) - iI_{SZ}^+(\varepsilon,\gamma_\varepsilon^S,c_T|\eta|e_1^0) - iI_{ZZ}^+(\varepsilon,\gamma_\varepsilon^Z,c_T|\eta|e_2^0)\\ &= -i\sum_{\alpha\in\mathcal{I}\cup\mathcal{R}} I_\alpha^+(\varepsilon,\gamma_\varepsilon^{\grave{\alpha}},c_{[\acute{\alpha}]}|\eta|\vec{\alpha}), \end{aligned} \tag{37}$$

and

$$v_t^- = i\sum_{\alpha\in\mathcal{I}\cup\mathcal{R}} I_\alpha^-(\varepsilon,\kappa_\varepsilon^{\grave{\alpha}},c_{[\acute{\alpha}]}|\eta|\vec{\alpha}). \tag{38}$$

Likewise, for $b=1,2,3$

$$2\partial_{x_b} u_\varepsilon^{appr} = v_b^+ + v_b^- + O(\sqrt{\varepsilon}),$$

with

$$\begin{aligned} v_b^+ &= I_L^+(\varepsilon,\gamma_\varepsilon^L,\eta_b\tfrac{\eta}{|\eta|}) + I_S^+(\varepsilon,\gamma_\varepsilon^S,\eta_b e_1^0) + I_Z^+(\varepsilon,\gamma_\varepsilon^Z,\eta_b e_2^0)\\ &\quad + I_{LL}^+(\varepsilon,\gamma_\varepsilon^L,\eta_b\tfrac{\eta}{|\eta|}) + I_{SL}^+(\varepsilon,\gamma_\varepsilon^S,\eta_b e_1^0) + I_{ZL}^+(\varepsilon,\gamma_\varepsilon^Z,\eta_b e_2^0)\\ &\quad + I_{LS}^+(\varepsilon,\gamma_\varepsilon^L,\eta_b\tfrac{\eta}{|\eta|}) + I_{SS}^+(\varepsilon,\gamma_\varepsilon^S,\eta_b e_1^0) + I_{ZS}^+(\varepsilon,\gamma_\varepsilon^Z,\eta_b e_2^0)\\ &\quad + I_{LZ}^+(\varepsilon,\gamma_\varepsilon^L,\eta_b\tfrac{\eta}{|\eta|}) + I_{SZ}^+(\varepsilon,\gamma_\varepsilon^S,\eta_b e_1^0) + I_{ZZ}^+(\varepsilon,\gamma_\varepsilon^Z,\eta_b e_2^0)\\ &= \sum_{\alpha\in\mathcal{I}\cup\mathcal{R}} iI_\alpha^+(\varepsilon,\gamma_\varepsilon^{\grave{\alpha}},\eta_b\vec{\alpha}), \end{aligned} \tag{39}$$

and

$$v_b^- = \sum_{\alpha\in\mathcal{I},\mathcal{R}} iI_\alpha^-(\varepsilon,\kappa_\varepsilon^{\grave{\alpha}},\eta_b\vec{\alpha}).$$

We denote V_x^+ the 3×3 matrix (v_1^+,v_2^+,v_3^+) and compute the Wigner measures associated to $\mathrm{Tr}w_\varepsilon[v_t^\pm]$, $\sum_{b=1}^{3}\mathrm{Tr}w_\varepsilon[v_b^\pm + V_x^\pm\cdot k_b]$, $w_\varepsilon[\sum_{b=1}^{3} v_b^\pm\cdot k_b]$, as well as the cross measures involving (v_t^+,v_t^-) and (v_b^+,v_b^-) for $b=1,2,3$. Here k_b denotes the vector of $\mathbb{R}^3$ s.t. $(k_b)_j=\delta_{bj}$. One needs then to estimate

$$w_\varepsilon(I_\alpha^p(\varepsilon,\gamma_\varepsilon^{\grave{\alpha}},b),I_\beta^q(\varepsilon,\gamma_\varepsilon^{\grave{\beta}},d)),$$

for $p,q=\pm$, $\alpha,\beta\in\mathcal{I}\cup\mathcal{R}$ and b,d vector functions smooth respectively on $\{\varphi_{[\alpha],\pm}^t\}^{-1}(K_y\times K_\eta)$ and $\{\varphi_{[\beta],\pm}^t\}^{-1}(K_y\times K_\eta)$. The previous analysis carried in chapter 2 for similar quantities associated to the scalar wave equation leads to

$$\begin{aligned} < w_\varepsilon(I_\alpha^p(\varepsilon,\gamma_\varepsilon^{\grave{\alpha}},b),I_\alpha^q(\varepsilon,\gamma_\varepsilon^{\grave{\beta}},d)),\psi >&= \textstyle\int_{\mathbb{R}^6}\psi(s,\sigma)b(s,\sigma)d^*(s,\sigma)\\ &(\rho'\otimes\gamma')\,o\{\varphi_{[\alpha],p}^t\}^{-1}(s,\sigma)\rho^{-1}(s,\sigma)\Big(R_\alpha^p\rho^{\frac12}\gamma_\varepsilon^{\grave{\alpha}}\Big)\,o\{\varphi_{[\alpha],p}^t\}^{-1}(s,\sigma)\\ &(\rho'\otimes\gamma')\,o\{\varphi_{[\beta],q}^t\}^{-1}(s,\sigma)\Big(R_\beta^q\rho^{\frac12}\bar{\gamma}_\varepsilon^{\grave{\beta}}\Big)\,o\{\varphi_{[\beta],q}^t\}^{-1}(s,\sigma)dsd\sigma + o(1). \end{aligned}$$

Let

$$E_\alpha^p = (\rho'\otimes\gamma')\,o\{\varphi_{[\alpha],p}^t\}^{-1}\rho^{-\frac12}\Big(R_\alpha^p\rho^{\frac12}\gamma_\varepsilon^{\grave{\alpha}}\Big)\,o\{\varphi_{[\alpha],p}^t\}^{-1}.$$

We may write the previous equation in the sense of distributions as

$$w_\varepsilon(I_\alpha^p(\varepsilon,\gamma_\varepsilon^{\grave{\alpha}},b),I_\beta^q(\varepsilon,\gamma_\varepsilon^{\grave{\beta}},d)) \approx E_\alpha^p\bar{E}_\beta^q bd^*,$$

and the trace measure satisfies then

$$\mathrm{Tr}w_\varepsilon(I^p_\alpha(\varepsilon,\gamma^{\grave\alpha}_\varepsilon,b),I^q_\beta(\varepsilon,\gamma^{\grave\beta}_\varepsilon,d))\approx E^p_\alpha\bar E^q_\beta d^*b. \tag{40}$$

We get

$$\mathrm{Tr}w_\varepsilon[v^+_t]\approx\sum_{\alpha,\beta\in\mathcal I\cup\mathcal R}c_{[\acute\alpha]}c_{[\acute\beta]}|\eta|^2E^+_\alpha\bar E^+_\beta(\vec\alpha.\vec\beta).$$

The terms coming from the cross Wigner measure between beams with different directions vanish and only the terms satisfying $\vec\alpha=\vec\beta$ contribute to $\mathrm{Tr}w[v^+_t]$. On the other hand, for $s\in\Omega$, $\sigma\in\mathbb R^n\backslash\{0\}$, $t\in[0,T]$ and $\alpha\in\mathcal R$, only one of the points $x^t_{\grave\alpha}(s,\sigma)$ and $x^t_\alpha(s,\sigma)$ may be in Ω. It follows that

$$E^+_{\grave\alpha}\bar E^+_\alpha=E^+_\alpha\bar E^+_{\grave\alpha}=0.$$

All in all, the cross terms that have a non-zero contribution to $\mathrm{Tr}w[v^+_t]$ are associted to reflected beams having the same direction. One has therefore

$$\rho\mathrm{Tr}w_\varepsilon[v^+_t]\approx|\eta|^2\Big(c^2_T|E^+_S|^2+c^2_T|E^+_Z|^2+c^2_T\sum_{\substack{\alpha,\beta\in\mathcal R\\ \vec\alpha=\vec\beta=e^0_1,e^0_2}}E^+_\alpha\bar E^+_\beta+c^2_L|E^+_L|^2+c^2_L\sum_{\substack{\alpha,\beta\in\mathcal R\\ \vec\alpha=\vec\beta=\frac{\eta}{|\eta|}}}E^+_\alpha\bar E^+_\beta\Big).$$

From (39), one gets

$$v^+_b+V^+_x\cdot k_b=\sum_{\alpha\in\mathcal I\cup\mathcal R}iI^+_\alpha(\varepsilon,\gamma^{\grave\alpha}_\varepsilon,\eta_b\vec\alpha+\vec\alpha_b\eta). \tag{41}$$

It follows, by using approximation (40), that the Wigner transform $\mathrm{Tr}w_\varepsilon[v^+_b+V^+_x\cdot k_b]$ is a sum of terms of the form

$$E^+_\alpha\bar E^+_\beta(\eta_b\vec\alpha+\vec\alpha_b\eta)(\eta_b\vec\beta^T+\vec\beta_b\eta^T),\ \alpha,\beta\in\mathcal I\cup\mathcal R,$$

modulo a vanishing residue. Since

$$\begin{aligned}\sum_{b=1}^3\mathrm{Tr}[(\eta_b\vec\alpha+\vec\alpha_b\eta)(\eta_b\vec\beta^T+\vec\beta_b\eta^T)]&=\sum_{b=1}^3(\eta_b^2\delta_{\alpha\beta}+\eta_b\vec\beta_b\delta_{\alpha L}+\eta_b\vec\alpha_b\delta_{\beta L}+\vec\alpha_b\vec\beta_b|\eta|^2)\\&=2|\eta|^2(\delta_{\alpha\beta}+\delta_{L\alpha}\delta_{L\beta}),\end{aligned}$$

the cross terms between beams of different directions do not contribute to $\sum\limits_{b=1}^3\mathrm{Tr}w_\varepsilon[v^+_b+V^+_x\cdot k_b]$ and one obtains

$$\begin{aligned}&\sum_{b=1}^3\mathrm{Tr}w_\varepsilon[v^+_b+V^+_x\cdot k_b]\\&\qquad\approx2|\eta|^2\left(\sum_{\alpha=S,Z}|E^+_\alpha|^2+2|E^+_L|^2\right)+2|\eta|^2\Big(\sum_{\substack{\alpha,\beta\in\mathcal R\\ \vec\alpha=\vec\beta=e^0_1,e^0_2}}E^+_\alpha\bar E^+_\beta+2\sum_{\substack{\alpha,\beta\in\mathcal R\\ \vec\alpha=\vec\beta=\frac{\eta}{|\eta|}}}E^+_\alpha\bar E^+_\beta\Big).\end{aligned}$$

Finally,

$$\sum_{b=1}^3v^+_b\cdot k_b=\sum_{\alpha\in\mathcal I\cup\mathcal R}I^+_\alpha(\varepsilon,\gamma^{\grave\alpha}_\varepsilon,i\eta^T\vec\alpha)=\sum_{\vec\alpha=\frac{\eta}{|\eta|}}I^+_\alpha(\varepsilon,\gamma^{\grave\alpha}_\varepsilon,i|\eta|), \tag{42}$$

which implies, using approximation (40), that

$$w_\varepsilon[\sum_{b=1}^{3} v_b^+ \cdot k_b] \approx |\eta|^2 |E_L^+|^2 + |\eta|^2 \sum_{\substack{\alpha,\beta\in\mathcal{R} \\ \vec{\alpha}=\vec{\beta}=\frac{\eta}{|\eta|}}} E_\alpha^+ \bar{E}_\beta^+. \tag{43}$$

By replacing the speeds c_T and c_L by their values, it follows that

$$\begin{aligned} &\rho \mathrm{Tr} w_\varepsilon[v_t^+] \\ &\quad \approx |\eta|^2 \Big[c_T^2 |E_S^+|^2 + c_T^2 |E_Z^+|^2 + c_T^2 \sum_{\substack{\alpha,\beta\in\mathcal{R} \\ \vec{\alpha}=\vec{\beta}=e_1^0,e_2^0}} E_\alpha^+ \bar{E}_\beta^+ + c_L^2 |E_L^+|^2 + c_L^2 \sum_{\substack{\alpha,\beta\in\mathcal{R} \\ \vec{\alpha}=\vec{\beta}=\frac{\eta}{|\eta|}}} E_\alpha^+ \bar{E}_\beta^+ \Big] \\ &\quad \approx \frac{\mu}{2} \sum_{b=1}^{3} \mathrm{Tr} w_\varepsilon[v_b^+ + V_x^+ \cdot k_b] + \lambda w_\varepsilon[\sum_{b=1}^{3} v_b^+ \cdot k_b]. \end{aligned} \tag{44}$$

The analysis for v_t^- and v_b^-, $b = 1, 2, 3$, is similar. The only differences are that the Hamiltonian flows propagate in the negative sense and that $\gamma_\varepsilon^{\dot{\alpha}}$ is replaced by $\kappa_\varepsilon^{\dot{\alpha}}$. One has

$$v_b^- + V_x^- \cdot k_b \approx \sum_{\alpha\in\mathcal{I}\cup\mathcal{R}} i I_\alpha^-(\varepsilon, \kappa_\varepsilon^{\dot{\alpha}}, \eta_b \vec{\alpha} + \vec{\alpha}_b \eta), \tag{45}$$

and

$$\sum_{b=1}^{3} v_b^- \cdot k_b \approx \sum_{\alpha\in\mathcal{I}\cup\mathcal{R}} I_\alpha^-(\varepsilon, \kappa_\varepsilon^{\dot{\alpha}}, i\eta^T \vec{\alpha}) = \sum_{\vec{\alpha}=\frac{\eta}{|\eta|}} I_\alpha^-(\varepsilon, \gamma_\varepsilon^{\dot{\alpha}}, i|\eta|). \tag{46}$$

Let

$$E_\alpha^- = (\rho' \otimes \gamma')\, o\{\varphi_{[\alpha],-}^t\}^{-1} \rho^{-\frac{1}{2}} \Big(R_\alpha^- \rho^{\frac{1}{2}} \kappa_\varepsilon^{\dot{\alpha}} \Big)\, o\{\varphi_{[\alpha],-}^t\}^{-1}.$$

One may write a similar equation to (44)

$$\begin{aligned} \rho \mathrm{Tr} w_\varepsilon[v_t^-] &\approx |\eta|^2 \Big(c_T^2 |E_S^-|^2 + c_T^2 |E_Z^-|^2 + c_T^2 \sum_{\substack{\alpha,\beta\in\mathcal{R} \\ \vec{\alpha}=\vec{\beta}=e_1^0,e_2^0}} E_\alpha^- \bar{E}_\beta^- + c_L^2 |E_L^-|^2 + c_L^2 \sum_{\substack{\alpha,\beta\in\mathcal{R} \\ \vec{\alpha}=\vec{\beta}=\frac{\eta}{|\eta|}}} E_\alpha^- \bar{E}_\beta^- \Big) \\ &\approx \frac{\mu}{2} \sum_{b=1}^{3} \mathrm{Tr} w_\varepsilon[v_b^- + V_x^- \cdot k_b] + \lambda w_\varepsilon[\sum_{b=1}^{3} v_b^- \cdot k_b]. \end{aligned} \tag{47}$$

It remains to estimate the cross terms $\mathrm{Tr} w_\varepsilon(v_t^+, v_t^-)$, $\sum_{b=1}^{3} \mathrm{Tr} w_\varepsilon(v_b^- + V_x^- \cdot k_b, v_b^- + V_x^- \cdot k_b)$ and $w_\varepsilon(\sum_{b=1}^{3} v_b^+ \cdot k_b, \sum_{b=1}^{3} v_b^- \cdot k_b)$. By (37) and (38) and approximation (40), one gets

$$\begin{aligned} &\rho \mathrm{Tr} w_\varepsilon(v_t^+, v_t^-) \\ &\quad \approx -|\eta|^2 \Big(c_T^2 E_S^+ \bar{E}_S^- + c_T^2 E_Z^+ \bar{E}_Z^- + c_T^2 \sum_{\substack{\alpha,\beta\in\mathcal{R} \\ \vec{\alpha}=\vec{\beta}=e_1^0,e_2^0}} E_\alpha^+ \bar{E}_\beta^- + c_L^2 E_L^+ \bar{E}_L^- + c_L^2 \sum_{\substack{\alpha,\beta\in\mathcal{R} \\ \vec{\alpha}=\vec{\beta}=\frac{\eta}{|\eta|}}} E_\alpha^+ \bar{E}_\beta^- \Big). \end{aligned}$$

On the other hand, equations (41) and (45) lead to

$$\begin{aligned} &\sum_{b=1}^{3} \mathrm{Tr} w_\varepsilon(v_b^+ + V_x^+ \cdot k_b, v_b^- + V_x^- \cdot k_b) \\ &\quad \approx 2|\eta|^2 \Big(\sum_{\alpha=S,Z} E_\alpha^+ \bar{E}_\alpha^- + 2 E_L^+ \bar{E}_L^- \Big) + 2|\eta|^2 \Big[\sum_{\substack{\alpha,\beta\in\mathcal{R} \\ \vec{\alpha}=\vec{\beta}=e_1^0,e_2^0}} E_\alpha^+ \bar{E}_\beta^- + 2 \sum_{\substack{\alpha,\beta\in\mathcal{R} \\ \vec{\alpha}=\vec{\beta}=\frac{\eta}{|\eta|}}} E_\alpha^+ \bar{E}_\beta^- \Big], \end{aligned}$$

and (42) and (46) imply

$$w_\varepsilon(\sum_{b=1}^{3} v_b^+ \cdot k_b, \sum_{b=1}^{3} v_b^- \cdot k_b) \approx |\eta|^2 E_L^+ \bar{E}_L^- + |\eta|^2 \sum_{\substack{\alpha,\beta\in\mathcal{R} \\ \vec{\alpha}=\vec{\beta}=\frac{\eta}{|\eta|}}} E_\alpha^+ \bar{E}_\beta^- .$$

Hence, the cross terms have a zero contribution to the measure e and one has

$$e = \frac{1}{4}(e_+ + e_-),$$

where

$$\begin{aligned} e_+ = |\eta|^2 c_T^2 \Big(& \sum_{\alpha=S,Z} \Big(\rho|\gamma_\varepsilon^\alpha|^2\Big)\, o\{\varphi_{\alpha,+}^t\}^{-1} \\ & + \sum_{\substack{\vec{\alpha}=\vec{\beta}=e_1^0,e_2^0 \\ \alpha,\beta\in\mathcal{R}}} \Big(\rho^{\frac12} R_\alpha^+ \gamma_\varepsilon^{\grave{\alpha}}\Big)\, o\{\varphi_{\alpha,+}^t\}^{-1} \Big(\rho^{\frac12} R_\beta^+ \bar{\gamma}_\varepsilon^{\grave{\beta}}\Big)\, o\{\varphi_{\beta,+}^t\}^{-1}\Big) \\ + |\eta|^2 c_L^2 \Big(\Big(\rho|\gamma_\varepsilon^L|^2\Big)\, o\{\varphi_{L,+}^t\}^{-1} & + \sum_{\substack{\vec{\alpha}=\vec{\beta}=\frac{\eta}{|\eta|} \\ \alpha,\beta\in\mathcal{R}}} \Big(\rho^{\frac12} R_\alpha^+ \gamma_\varepsilon^{\grave{\alpha}}\Big)\, o\{\varphi_{\alpha,+}^t\}^{-1} \Big(\rho^{\frac12} R_\beta^+ \bar{\gamma}_\varepsilon^{\grave{\beta}}\Big)\, o\{\varphi_{\beta,+}^t\}^{-1}\Big), \end{aligned}$$

and

$$\begin{aligned} e_- = |\eta|^2 c_T^2 \Big(& \sum_{\alpha=S,Z} \Big(\rho|\kappa_\varepsilon^\alpha|^2\Big)\, o\{\varphi_{\alpha,-}^t\}^{-1} \\ & + \sum_{\substack{\vec{\alpha}=\vec{\beta}=e_1^0,e_2^0 \\ \alpha,\beta\in\mathcal{R}}} \Big(\rho^{\frac12} R_\alpha^- \kappa_\varepsilon^{\grave{\alpha}}\Big)\, o\{\varphi_{\alpha,-}^t\}^{-1} \Big(\rho^{\frac12} R_\beta^- \bar{\kappa}_\varepsilon^{\grave{\beta}}\Big)\, o\{\varphi_{\beta,-}^t\}^{-1}\Big) \\ + |\eta|^2 c_L^2 \Big(\Big(\rho|\gamma_\varepsilon^L|^2\Big)\, o\{\varphi_{L,-}^t\}^{-1} & + \sum_{\substack{\vec{\alpha}=\vec{\beta}=\frac{\eta}{|\eta|} \\ \alpha,\beta\in\mathcal{R}}} \Big(\rho^{\frac12} R_\alpha^- \gamma_\varepsilon^{\grave{\alpha}}\Big)\, o\{\varphi_{\alpha,-}^t\}^{-1} \Big(\rho^{\frac12} R_\beta^- \bar{\gamma}_\varepsilon^{\grave{\beta}}\Big)\, o\{\varphi_{\beta,-}^t\}^{-1}\Big). \end{aligned}$$

For $\alpha \in \mathcal{I}$, $\{\varphi_{\alpha,\pm}^t\}^{-1} = \varphi_{\alpha,\mp}^t$. For $\grave{\alpha}\acute{\alpha} \in \mathcal{R}$, the inverse of $\varphi_{\grave{\alpha}\acute{\alpha},\pm}^t$ is $\varphi_{\acute{\alpha}\grave{\alpha},\mp}^t$. Moreover, the flows φ_T and φ_{TT} keep $c_T|\eta|$ invariant and the flows φ_L and φ_{LL} keep $c_L|\eta|$ invariant, while for φ_{TL} and φ_{LT} one has

$$c_L(x_{TL}(y,\eta))|\xi_{TL}(y,\eta)| = c_T(y)|\eta| \text{ and } c_T(x_{LT}(y,\eta))|\xi_{LT}(y,\eta)| = c_L(y)|\eta|.$$

It follows that

$$\begin{aligned} e_+ \;&=\; \left|\sqrt{\mu}|\eta| \begin{pmatrix} \gamma_\varepsilon^S \\ \gamma_\varepsilon^Z \end{pmatrix}\right|^2 o\varphi_{T,-}^t + |\sqrt{\lambda+2\mu}|\eta|\gamma_\varepsilon^L|^2\, o\varphi_{L,-}^t \\ &+ \left|\left[\sqrt{\mu}|\eta| \begin{pmatrix} R_{SS}^+ & R_{ZS}^+ \\ R_{SZ}^+ & R_{ZZ}^+ \end{pmatrix} \begin{pmatrix} \gamma_\varepsilon^S \\ \gamma_\varepsilon^Z \end{pmatrix}\right] o\varphi_{TT,-}^t + \left[\sqrt{\lambda+2\mu}|\eta| \begin{pmatrix} R_{LS}^+ \\ R_{LZ}^+ \end{pmatrix} \gamma_\varepsilon^L\right] o\varphi_{TL,-}^t\right|^2 \\ &+ \left|\left[\sqrt{\mu}|\eta| \begin{pmatrix} R_{SL}^+ & R_{ZL}^+ \end{pmatrix} \begin{pmatrix} \gamma_\varepsilon^S \\ \gamma_\varepsilon^Z \end{pmatrix}\right] o\varphi_{LT,-}^t + \left[\sqrt{\lambda+2\mu}|\eta| R_{LL}^+ \gamma_\varepsilon^L\right] o\varphi_{LL,-}^t\right|^2 . \end{aligned} \tag{48}$$

A similar result can be established for e_-. In order to understand the transported terms, let us write the Helmholtz decomposition of the initial conditions as

$$u_\varepsilon^I = f_\varepsilon + \Psi_\varepsilon, v_\varepsilon^I = g_\varepsilon + \Theta_\varepsilon \text{ with } f_\varepsilon = \partial_x a_\varepsilon,\ g_\varepsilon = \partial_x b_\varepsilon,\ \mathrm{div}\Psi_\varepsilon = 0 \text{ and } \mathrm{div}\Theta_\varepsilon = 0.$$

Since $\|T_\varepsilon \partial_x u - i\eta T_\varepsilon u\|_{L^2(\mathbb{R}^{2n})} \lesssim \sqrt{\varepsilon}\|u\|_{L^2(\mathbb{R}^n)}$ for $u \in H^1(\mathbb{R}^n)$, one has

$$\|(Id - \frac{\eta\eta^t}{|\eta|^2})T_\varepsilon g_\varepsilon\|_{L^2(\mathbb{R}^6)^3} \lesssim \sqrt{\varepsilon}. \tag{49}$$

On the other hand

$$-i\varepsilon T_\varepsilon \text{div}\Theta_\varepsilon = \eta \cdot T_\varepsilon \Theta_\varepsilon + \int (x-y)\cdot \Theta_\varepsilon(x) e^{\frac{i}{\varepsilon}\eta\cdot(y-x) - \frac{1}{2\varepsilon}(y-x)^2} dx.$$

Since $\Theta_\varepsilon \in L^2(\Omega)^3$, the integral term is of order $\sqrt{\varepsilon}$ in $L^2(\mathbb{R}^6)$. Thus

$$\|\eta \cdot T_\varepsilon \Theta_\varepsilon\|_{L^2(\mathbb{R}^6)} \lesssim \sqrt{\varepsilon}.$$

The same arguments hold true for $\varepsilon^{-1}T_\varepsilon u^I_\varepsilon$ because $\varepsilon^{-1}u^I_\varepsilon \in L^2(\Omega)^3$, leading to

$$\|(Id - \frac{\eta\eta^t}{|\eta|^2})T_\varepsilon f_\varepsilon\|_{L^2(\mathbb{R}^6)^3} \lesssim \sqrt{\varepsilon} \text{ and } \|\eta \cdot T_\varepsilon \Psi_\varepsilon\|_{L^2(\mathbb{R}^6)} \lesssim \sqrt{\varepsilon}.$$

The Helmholtz decomposition of the initial data implies then a decomposition of their FBI transforms tangentially and orthogonally to η as follows

$$\begin{aligned} T_\varepsilon\left(\varepsilon^{-1}f_\varepsilon + i(c_L|\eta|)^{-1}g_\varepsilon\right) =& \gamma^L_\varepsilon \frac{\eta}{|\eta|} + O(\sqrt{\varepsilon}), \\ &\text{and } T_\varepsilon\left(\varepsilon^{-1}\Psi_\varepsilon + i(c_T|\eta|)^{-1}\Theta_\varepsilon\right) = \gamma^S_\varepsilon e^0_1 + \gamma^Z_\varepsilon e^0_2 + O(\sqrt{\varepsilon}), \end{aligned}$$

in $L^2(\mathbb{R}^3)^3$. We deduce in the sens of measures that

$$\begin{aligned} |\gamma^L_\varepsilon|^2 &\approx |T_\varepsilon\left(\varepsilon^{-1}f_\varepsilon + i(c_L|\eta|)^{-1}g_\varepsilon\right)|^2, \\ |\gamma^S_\varepsilon|^2 + |\gamma^Z_\varepsilon|^2 &\approx |T_\varepsilon\left(\varepsilon^{-1}\Psi_\varepsilon + i(c_T|\eta|)^{-1}\Theta_\varepsilon\right)|^2. \end{aligned}$$

By Lemma B.1 in chapter 2, $T_\varepsilon|D| \approx |\eta|T_\varepsilon$ and crude computations show that $T_\varepsilon\vartheta \approx \vartheta T_\varepsilon$ for functions $\vartheta \in \mathcal{C}^\infty$. One obtains the following approximations in the sens of measures

$$\begin{aligned} (\lambda+2\mu)|\eta|^2|\gamma^L_\varepsilon|^2 &\approx |T_\varepsilon\left(\sqrt{\rho}g_\varepsilon - i\sqrt{\lambda+2\mu}\varepsilon^{-1}|D|f_\varepsilon\right)|^2, \\ \mu|\eta|^2\left(|\gamma^S_\varepsilon|^2 + |\gamma^Z_\varepsilon|^2\right) &\approx |T_\varepsilon\left(\sqrt{\rho}\Theta_\varepsilon - i\sqrt{\mu}\varepsilon^{-1}|D|\Psi_\varepsilon\right)|^2. \end{aligned}$$

Similar results can be established for κ_ε with a change of signs. Since the FBI transform is related to the Wigner measure (see Lemma 1.2 of [37]) in that

$$|T_\varepsilon z_\varepsilon|^2 \approx w[z_\varepsilon] \text{ for } z_\varepsilon \text{ uniformly bounded in } L^2(\mathbb{R}^n),$$

the first two terms of (48) may be written as transported Wigner measures of $\sqrt{\rho}g_\varepsilon + ip\sqrt{\lambda+2\mu}|D|f_\varepsilon$ and $\sqrt{\rho}\Theta_\varepsilon + ip\sqrt{\mu}|D|\Psi_\varepsilon$. The four remaing terms in (48) are however troublesome. Indeed, they exhibit cross quantities $\gamma^{\dot{\alpha}}_\varepsilon\, o\{\varphi^t_{\alpha,+}\}^{-1}\bar{\gamma}^{\dot{\beta}}_\varepsilon\, o\{\varphi^t_{\beta,+}\}^{-1}$ with $\alpha \neq \beta$, which quantities can not be interpreted as Wigner measures. However, if one assumes the additionnal hypotheses

D1. The Wigner measures associated to f_ε and Ψ_ε are singular,

D2. The Wigner measures associated to g_ε and Θ_ε are singular, which are commonly assumed hypotheses when studying the Wigner measure for a system with two wave speeds at the boundary (or equivalently the problem of waves refraction on an interface [74]), then the cross terms between γ_ε^L and $(\gamma_\varepsilon^S, \gamma_\varepsilon^Z)$ vanish. In this case, (48) may be written in terms of transported initial Wigner measures as

$$\begin{aligned} e_+ &= w[\sqrt{\rho}g_\varepsilon - i\sqrt{\lambda+2\mu}|D|f_\varepsilon]\, o\, \varphi_{L,+} + w[\sqrt{\rho}\Theta_\varepsilon - i\sqrt{\mu}|D|\Psi_\varepsilon]\, o\, \varphi_{T,-} \\ &+ w[R_{LL}^+\left(\sqrt{\rho}g_\varepsilon - i\sqrt{\lambda+2\mu}|D|f_\varepsilon\right)]\, o\varphi_{LL,-} + w[Y_{LT}^+\left(\sqrt{\rho}g_\varepsilon - i\sqrt{\lambda+2\mu}|D|f_\varepsilon\right)]\, o\varphi_{TL,-} \\ &+ w[M_{TT}^+\left(\sqrt{\rho}\Theta_\varepsilon - i\sqrt{\mu}|D|\Psi_\varepsilon\right)]\, o\varphi_{TT,-} + w[Y_{TL}^+\left(\sqrt{\rho}\Theta_\varepsilon - i\sqrt{\mu}|D|\Psi_\varepsilon\right)]\, o\varphi_{LT,-}, \end{aligned} \tag{50}$$

with

$$Y_{LT}^+ = (R_{LS}^+ e_1^0 + R_{LZ}^+ e_2^0)(\frac{\eta}{|\eta|})^T,\ Y_{TL}^+ = (R_{SL}^+ e_1^0 + R_{ZL}^+ e_2^0)(\frac{\eta}{|\eta|})^T,$$
$$\text{and } M_{TT}^+ = (R_{SS}^+ e_1^0 + R_{SZ}^+ e_2^0)(e_1^0)^T + (R_{ZS}^+ e_1^0 + R_{ZZ}^+ e_2^0)(e_2^0)^T.$$

Above, the exponent T denotes transposition. A similar expression can be obtained for e_-

$$\begin{aligned} e_- &= w[\sqrt{\rho}g_\varepsilon + i\sqrt{\lambda+2\mu}|D|f_\varepsilon]\, o\, \varphi_{L,-} + w[\sqrt{\rho}\Theta_\varepsilon + i\sqrt{\mu}|D|\Psi_\varepsilon]\, o\, \varphi_{T,+} \\ &+ w[R_{LL}^-\left(\sqrt{\rho}g_\varepsilon + i\sqrt{\lambda+2\mu}|D|f_\varepsilon\right)]\, o\varphi_{LL,+} + w[Y_{LT}^-\left(\sqrt{\rho}g_\varepsilon + i\sqrt{\lambda+2\mu}|D|f_\varepsilon\right)]\, o\varphi_{TL,+} \\ &+ w[M_{TT}^-\left(\sqrt{\rho}\Theta_\varepsilon + i\sqrt{\mu}|D|\Psi_\varepsilon\right)]\, o\varphi_{TT,+} + w[Y_{TL}^-\left(\sqrt{\rho}\Theta_\varepsilon + i\sqrt{\mu}|D|\Psi_\varepsilon\right)]\, o\varphi_{LT,+}. \end{aligned} \tag{51}$$

This leads to our final theorem

Theorem 4.1. *Suppose that the initial conditions satisfy assumtions A1-A2 and D1-D2. Assume the following further assumptions*

- *C1. The Wigner measures of* $\underline{v}_\varepsilon^I$ *and* $\partial_{x_b}\underline{u}_\varepsilon^I$, $b = 1, 2, 3$, *are unique,*
- *C2.* $\underline{v}_\varepsilon^I$ *and* $\partial_{x_b}\underline{u}_\varepsilon^I$, $b = 1, 2, 3$ *are* ε*-oscillatory (see equation* (53)*, chapter 2),*
- *C3. The Wigner measures of* $\underline{v}_\varepsilon^I$ *and* $\partial_{x_b}\underline{u}_\varepsilon^I$, $b = 1, 2, 3$ *do not charge the set* $\mathbb{R}^3 \times \{\xi = 0\}$,
- *C4. The Wigner measures of* $\underline{v}_\varepsilon^I$ *and* $\partial_{x_b}\underline{u}_\varepsilon^I$, $b = 1, 2, 3$ *do not charge* Tg.

Then the scalar Wigner measure e *equals* $\frac{1}{4}(e_+ + e_-)$ *where* e_+ *and* e_- *depend on the Wigner measures of the initial data and the reflection coefficients and are given by formulas* (50) *and* (51).

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