THÈSE

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Présentée par

Paul Sauvy

Étude de quelques problèmes elliptiques et paraboliques quasi-linéaires avec singularités

Directeur de thèse : **M. Jacques Giacomoni**

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Devant la Commission d'Examen

JURY

M. Gilles Carbou	Professeur à l'U.P.P.A.	Examinateur
M. Jacques Giacomoni	Professeur à l'U.P.P.A.	Directeur de Thèse
M. Jean Pierre Gossez	Professeur à l'Université Libre de Bruxelles	Président
M. Jean Michel Rakotoson	Professeur à l'Université de Poitiers	Rapporteur
M. Peter Takáč	Professeur à l'Université de Rostock	Rapporteur
M. Guillaume Warnault	Maître de Conférences à l'U.P.P.A.	Examinateur

Thèse préparée au sein du Laboratoire de Mathématiques Appliquées de Pau (L.M.A.P.), dans l'école doctorale des sciences exactes et leurs applications de l' U.P.P.A.

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Paul.

Table des matières

Notations

Notations générales

$N \geq 2$ (ou d)	Entier naturel, dimension de l'espace de travail
$\mathbb{R}^N$	Espace euclidien muni de sa norme usuelle notée $\lvert \cdot \rvert$
Ω	Domaine borné, de classe $\mathscr{C}^2$ de $\mathbb{R}^N$
$(\Omega_k)_{k\in\mathbb{N}^*}$	Suite croissante de sous-domaines de Ω, convergeant vers Ω au sens de Hausdorff
$\partial\Omega$	Frontière de Ω
$D = \mathrm{diam}(\Omega)$	Diamètre de Ω
ν	Normale unitaire extérieure de Ω
$x = (x_1, \cdots, x_N)$	Elément de Ω
$d(x)$	Distance du point x à $\partial\Omega$
$r > 1$ (ou p ou q)	Exposant de Lebesgue
$r' > 1$	Exposant conjugé de r vérifiant $\frac{1}{r} + \frac{1}{r'} = 1$
$\mathbb{1}_A$	Fonction indicatrice de l'ensemble A
int(A)	Intérieur de l'ensemble A
$\mathrm{supp}_A\, f$	Support de de la fonction f sur A
$f(x) \sim g(x)$ dans A	Fonctions positives de $\mathrm{L}^1_{\mathrm{loc}}(A)$ dites équivalentes sur A, c'est à dire pour lesquelles il existe $C_1, C_2 > 0$, telles que pour tout $x \in A$, $C_1 f(x) \leq g(x) \leq C_2 f(x)$
∇v	Gradient de v défini par $\nabla v \overset{\mathrm{def}}{=} \left(\frac{\partial v}{\partial x_1}, \cdots, \frac{\partial v}{\partial x_N}\right)$
$\Delta_r v$	r-Laplacien de v défini par $\Delta_r v \overset{\mathrm{def}}{=} \mathrm{div}(\lvert\nabla v\rvert^{r-2}\nabla u)$
λ_1 (ou $\lambda_{1,r}$)	Première valeur propre du r-Laplacien sur Ω
φ_1 (ou $\varphi_{1,r}$)	Fonction propre strictement positive et L^r-renormalisée sur Ω, associée à λ_1
$\mathscr{C}(\overline{\Omega})$	Ensemble des fonctions continues sur $\overline{\Omega}$

$\mathscr{C}_0(\overline{\Omega})$	Ensemble des fonctions continues sur $\overline{\Omega}$ s'annulant sur $\partial\Omega$
$\mathscr{C}^{0,\alpha}(\overline{\Omega})$	Ensemble des fonctions de $\mathscr{C}(\overline{\Omega})$ α-Hölderiennes, avec $0<\alpha<1$; c'est à dire, $\mathscr{C}^{0,\alpha}(\overline{\Omega}) \stackrel{\text{def}}{=} \left\{v \in \mathscr{C}(\overline{\Omega}) \;\middle|\; \exists C>0, \quad \forall x,y \in \overline{\Omega}, \quad \|v(x)-v(y)\| \leq C\|x-y\|^\alpha\right\}$
$\mathscr{C}^{1,\alpha}(\overline{\Omega})$	Ensemble des fonctions $\mathscr{C}^1(\overline{\Omega})$ α-Hölderiennes, avec $0<\alpha<1$; c'est à dire, $\mathscr{C}^{1,\alpha}(\overline{\Omega}) \stackrel{\text{def}}{=} \left\{v \in \mathscr{C}^1(\overline{\Omega}) \;\middle|\; \forall i \in \{1,\cdots,N\}, \quad \frac{\partial v}{\partial x_i} \in \mathscr{C}^{0,\alpha}(\overline{\Omega})\right\}$
$\mathscr{D}(\Omega)$	Ensemble des fonctions $\mathscr{C}^\infty$ à support compact sur Ω
$\mathscr{D}'(\Omega)$	Espace des distributions su Ω
$\mathrm{L}^r(\Omega)$, $\mathrm{L}^\infty(\Omega)$	Espaces de Lebesgue standards sur Ω d'exposants r et ∞
$\mathrm{L}^1_{\mathrm{loc}}(\Omega)$	Ensemble des fonctions localement intégrables sur Ω défini par $\mathrm{L}^1_{\mathrm{loc}}(\Omega) \stackrel{\text{def}}{=} \{v \text{ mesurable sur } \Omega \mid \forall \Omega' \subset\subset \Omega, \quad v \in \mathrm{L}^1(\Omega')\}$
$\mathrm{W}^{1,r}(\Omega)$	Espace de Sobolev standard sur Ω d'exposant r
$\mathrm{W}^{1,r}_{\mathrm{loc}}(\Omega)$	Espace défini par $\mathrm{W}^{1,r}_{\mathrm{loc}}(\Omega) \stackrel{\text{def}}{=} \left\{v \in \mathrm{L}^1_{\mathrm{loc}}(\Omega) \;\middle|\; \forall i \in \{1,\cdots,N\}, \quad \frac{\partial v}{\partial x_i} \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)\right\}$
$\mathrm{W}^{1,r}_0(\Omega)$	Adhérence de $\mathscr{D}(\Omega)$ dans $\mathrm{W}^{1,r}(\Omega)$ pour la norme $\|\cdot\|_{\mathrm{W}^{1,r}(\Omega)}$
$\mathrm{W}^{-1,r'}(\Omega)$	Dual topologique de $\mathrm{W}^{1,r}_0(\Omega)$
$\langle\cdot,\cdot\rangle_{\mathrm{W}^{-1,r'}(\Omega)\times\mathrm{W}^{1,r}_0(\Omega)}$	Produit de dualité entre $\mathrm{W}^{-1,r'}(\Omega)$ et $\mathrm{W}^{1,r}_0(\Omega)$
$\|v\|_{\mathrm{L}^r(\Omega)}$	Norme de v sur $\mathrm{L}^r(\Omega)$ définie par $\|v\|_{\mathrm{L}^r(\Omega)} \stackrel{\text{def}}{=} \left(\int_\Omega \|v\|^r\,dx\right)^{\frac{1}{r}}$
$\|v\|_{\mathrm{L}^\infty(\Omega)}$	Norme de v dans $\mathrm{L}^\infty(\Omega)$ définie par $\|v\|_{\mathrm{L}^\infty(\Omega)} \stackrel{\text{def}}{=} \operatorname*{ess\,sup}_{x\in\Omega} \|v(x)\|$
$\|v\|_{\mathrm{W}^{1,r}(\Omega)}$	Norme de v sur $\mathrm{W}^{1,r}(\Omega)$ définie par $\|v\|_{\mathrm{W}^{1,r}(\Omega)} \stackrel{\text{def}}{=} \left(\|v\|_{\mathrm{L}^r(\Omega)} + \|\nabla v\|_{\mathrm{L}^r(\Omega)}\right)^{\frac{1}{r}}$
$\|v\|_{\mathrm{W}^{1,r}_0(\Omega)}$	Norme de v sur $\mathrm{W}^{1,r}_0(\Omega)$ définie par $\|v\|_{\mathrm{W}^{1,r}_0(\Omega)} \stackrel{\text{def}}{=} \|\nabla v\|_{\mathrm{L}^r(\Omega)}$, équivalente à $\|v\|_{\mathrm{W}^{1,r}(\Omega)}$
$\|f\|_{\mathrm{W}^{-1,r'}(\Omega)}$	Norme duale de f sur $\mathrm{W}^{-1,r'}(\Omega)$

Chapitre I

$p>1$	Paramètre relatif à Δ_p
$\lambda>0$	Paramètre du problème (P_λ)
q, r	Exposants du second membre dans (P_λ), vérifiant $-1<r<q<p-1$
K	Poids singulier du second membre dans (P_λ)
L	Perturbation singulière dite de Karamata dans l'expression de K
k	Taux d'explosion de la singularité K tel que $0 \leq k < p$
Λ_1	Valeur critique du paramètre λ lorsque $k<1+r$
Λ_2	Valeur critique du paramètre λ lorsque $k \geq 1+r$

Ω_ε	Ensemble défini pour $\varepsilon > 0$ par $\Omega_\varepsilon \overset{\text{def}}{=} \{x \in \Omega \mid d(x) < \varepsilon\}$
$\underline{u}_\lambda$, $\overline{u}_\lambda$	Sous- et sur-solution de (P_λ) dans Ω
$\mathrm{H}^1_0(\Omega)$	Espace de Hilbert standard représentant $\mathrm{W}^{1,2}_0(\Omega)$
$\mathscr{H}$	Espace des fonctions de $\mathrm{H}^1_0(\Omega)$, L^2-renormalisées défini par $\mathscr{H} \overset{\text{def}}{=} \{v \in \mathrm{H}^1_0(\Omega) \mid \|v\|_{\mathrm{L}^2(\Omega)} = 1\}$
$\mathrm{H}^{-1}(\Omega)$	Dual topologique de $\mathrm{H}^1_0(\Omega)$ représentant $\mathrm{W}^{-1,2}(\Omega)$

Chapitre II

p, $q > 1$	Paramètres relatifs à Δ_p et Δ_q
$(\underline{u}, \underline{v})$, $(\overline{u}, \overline{v})$	Paire de sous- et sur-solutions de (P)
$[\underline{w}, \overline{w}]$	Ensemble convexe défini par $[\underline{w}, \overline{w}] \overset{\text{def}}{=} \left\{w \in \mathscr{C}(\overline{\Omega}) \mid \underline{w} \le w \le \overline{w}\right\}$
$\mathcal{C}$	Cône de paires de sous- et sur-solutions de (P) défini par $\mathcal{C} \overset{\text{def}}{=} [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$
$\mathscr{C}^{1,\alpha}_{\mathrm{loc}}(\Omega)$	Ensemble des fonctions $\mathscr{C}^1(\Omega)$ localement α- Hölderiennes, avec $0 < \alpha < 1$, c'est à dire, $\mathscr{C}^{1,\alpha}_{\mathrm{loc}}(\Omega) \overset{\text{def}}{=} \left\{v \in \mathscr{C}^1(\Omega) \mid \forall \Omega' \subset\subset \Omega, \quad v \in \mathscr{C}^{1,\alpha}(\overline{\Omega'})\right\}$
$\mathrm{W}^{1/r',r}(\partial\Omega)$	Espace des traces de $\mathrm{W}^{1,r}(\Omega)$ sur $\partial\Omega$
$v\|_{\partial\Omega}$	Trace de $v \in \mathrm{W}^{1,r}(\Omega)$ sur $\partial\Omega$
$\mathrm{sign}(\sigma)$	Signe de $\sigma \in \mathbb{R}$, à savoir $\mathrm{sign}(\sigma) = \begin{cases} + & \text{si } \sigma \ge 0, \\ - & \text{si } \sigma < 0 \end{cases}$
a_1, b_1, a_2, b_2, α_1, α_2, β_1, β_2	Exposants des seconds membres des équations de (P) apparaissant dans les Exemples 1 à 5
K_1, K_2	Poids singulier des second membres des équations de (P)
L_1,L_2	Perturbations singulières dite de Karamata dans les expressions de K_1 et K_2
k_1, k_2	Taux d'explosion des singularités K_1 et K_2 tels que $0 \le k_1 < p$ et $0 \le k_2 < q$

Chapitre III

$d \ge 2$	Dimension de l'espace de travail
$p > 1$	Paramètre relatif à Δ_p
β	Exposant du terme d'absorption singulier du problème (P) vérifiant $0 < \beta < 1$
$T > 0$	Temps maximum de l'étude
Q	Domaine d'étude défini par $Q \overset{\text{def}}{=} (0, T) \times \Omega$
Γ	Frontière de Q définie par $\Gamma \overset{\text{def}}{=} (0, T) \times \partial\Omega$

$\partial_t u$	Dérivée partielle de u par rapport au temps
u_0	Condition initiale positive choisie dans $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ ou seulement dans $L^\infty(\Omega)$
$\varepsilon > 0$	Paramètre du problème régularisé (P_ε)
$N >> 1$	Entier suffisamment grand
$0 = t_0 < \cdots < t_N = T$	Subdivision régulière de l'intervalle $[0, T]$
Δ_t	Pas de semi-discrétisation en temps égal à $\frac{T}{N}$
y	Fonction barrière supérieure de (P) et (P_ε) indépendante de Δ_t et ε
$\mathscr{D}(Q)$	Ensemble des fonctions $\mathscr{C}^\infty$ à support compact sur Q
$\mathscr{D}'(Q)$	Espace des distributions de Q
$\mathscr{C}\left([0,T], L^2(\Omega)\right)$	Ensemble des fonctions continues de $[0,T]$ dans $L^2(\Omega)$
$\mathscr{C}\left(I, W_0^{1,p}(\Omega)\right)$	Ensemble des fonctions continues de I (intervalle de $[0,T]$) dans $W_0^{1,p}(\Omega)$
$L^2(Q)$, $L^\infty(Q)$	Espaces de Lebesgue standard sur Q d'exposants 2 et ∞
$L^r\left(0,T; W_0^{1,p}(\Omega)\right)$	Espace de Sobolev à valeurs vectorielles avec $r = p$ ou $r = \infty$
$L^{p'}\left(0,T; W^{-1,p'}(\Omega)\right)$	Dual topologique de $L^p\left(0,T; W_0^{1,p}(\Omega)\right)$
$\langle \cdot, \cdot \rangle$	Produit de dalité entre $W^{-1,p'}(\Omega)$ et $W_0^{1,p}(\Omega)$
$\langle\langle \cdot, \cdot \rangle\rangle$	Produit de dalité entre $L^{p'}\left(0,T; W^{-1,p'}(\Omega)\right)$ et $L^p\left(0,T; W_0^{1,p}(\Omega)\right)$
$\|v\|_{L^p\left(0,T;W_0^{1,p}(\Omega)\right)}$	Norme de v dans $L^p\left(0,T; W_0^{1,p}(\Omega)\right)$ définie par $\|v\|_{L^p\left(0,T;W_0^{1,p}(\Omega)\right)} \stackrel{\text{def}}{=} \left(\int_0^T \|v(t,\cdot)\|^p_{W_0^{1,p}(\Omega)} ds\right)^{\frac{1}{p}}$
$\|v\|_{L^\infty\left(0,T;W_0^{1,p}(\Omega)\right)}$	Norme de v dans $L^\infty\left(0,T; W_0^{1,p}(\Omega)\right)$ définie par $\|v\|_{L^\infty\left(0,T;W_0^{1,p}(\Omega)\right)} \stackrel{\text{def}}{=} \operatorname*{ess\,sup}_{t\in(0,T)} \|v(t,\cdot)\|^p_{W_0^{1,p}(\Omega)}$
$\mathscr{U}$	Espace dans lequel on cherche les solutions de (P), défini par $\mathscr{U} \stackrel{\text{def}}{=} \left\{v \in L^\infty\left(0,T; W_0^{1,p}(\Omega) \cap L^\infty(\Omega)\right) \;\middle|\; \partial_t v \in L^2(Q)\right\}$
$\mathscr{V}$	Espace dans lequel on cherche les solutions de (P_ε), défini par $\mathscr{V} \stackrel{\text{def}}{=} \left\{v \in L^p\left(0,T; W_0^{1,p}(\Omega)\right) \cap L^\infty(Q) \;\middle|\; \partial_t v \in L^{p'}\left(0,T; W^{-1,p'}(\Omega)\right)\right\}$
z	Fonction représentant la norme $L^2(\Omega)$ de $u \in \mathscr{U}$, solution de (P), sur $[0,T]$

Introduction générale

Cette thèse concerne l'étude de certains problèmes elliptiques et paraboliques, quasi-linéaires singuliers. Dans les problèmes-modèles considérés au cours de ce travail, nous nous plaçons sur un domaine borné Ω de $\mathbb{R}^N$, $N \geq 2$, avec des conditions aux limites de type Dirichlet homogène. Le caractère singulier des différents problèmes rencontrés, se traduit alors par la présence dans l'équation d'un terme non-linéaire de la forme $u^{-\delta}$, avec $\delta > 0$, qui tend vers l'infini au bord du domaine Ω. Ceci pose un certain nombre de difficultés, liées aux manque de régularité et donc de compacité des solutions, qui ne nous permettent pas d'utiliser directement les méthodes classiques de l'analyse non-linéaire. A travers les Chapitres I à III, nous avons montré comment ces difficultés peuvent être surmontées et démontré de nouveaux résultats concernant l'existence, l'unicité, la régularité et le comportement asymptotique des solution faibles. Un des éléments essentiels permettant de pallier ces obstacles réside dans la détermination du comportement précis des solutions au bord du domaine d'étude. Il est alors possible d'adapter ou de généraliser certaines méthodes classiques comme les méthodes de monotonie, qui s'appuient sur le principe de comparaison, les méthodes variationnelles ou encore certains arguments de convexité. Un des outils dont nous nous servons également est l'inégalité de Hardy qui permet de tirer profit de l'étude du comportement des solutions au bord du domaine.

Notons que les problèmes que nous avons étudiés dans ce travail de thèse apparaissent dans de nombreux modèles cinétiques de réactions chimiques de type catalyse (voir les ouvrages d'Aris-Cho-Carr [6] et Banks [8]), de dynamique des populations, de physique des plasmas, ainsi que dans certains modèles d'écoulement de fluides non-newtoniens. On pourra consulter à ce sujet l'article-revue de Hernández-Mancebo [55], ainsi que l'ouvrage de Ghergu-Rădulescu [43], où une présentation de ces modèles ainsi qu'une bibliographie fournie relative aux problèmes singuliers y sont données.

Cette thèse est organisée de la façon suivante :

Dans le premier chapitre, nous étudions un problème quasi-linéaire elliptique singulier, dépendant d'un paramètre $\lambda > 0$, qui modélise un phénomène d'absorption. Le second membre de cette équation se compose d'une non-linéarité sous-homogène à dominante négative du type concave-convexe, multipliée par un poids singulier apparaissant comme une certaine puissance négative de la distance au bord, appelée "taux d'explosion". Cette singularité amplifie le phénomène d'absorption au voisinage de $\partial\Omega$. Nous démontrons alors l'existence d'une valeur critique du taux d'explosion séparant deux comportements bien distincts des solutions d'un tel problème : positivité stricte d'une part et à support compact d'autre part. Dans chacun de ces deux cas, nous mettons en évidence l'existence d'une valeur critique du paramètre λ séparant existence et non existence de solutions non-triviales d'un tel problème. Nous démontrons également la régularité $\mathscr{C}^{1,\alpha}(\overline{\Omega})$ des solutions, pour un certain $0 < \alpha < 1$. Enfin, dans la dernière section de ce chapitre nous étudions plus en détail le problème dans le cas particulier du Laplacien et lorsque le second membre est concave. Dans cette partie, nous démontrons la stabilité des solutions du problème, ainsi que l'existence et l'unicité de la solution du problème extrémal, c'est à dire lorsque le paramètre λ atteint la valeur critique séparant existence et non-existence de solution.

Dans le Chapitre II, nous étudions une classe générale de systèmes quasi-linéaires singuliers pour la quelle nous démontrons l'existence de solutions faibles. Dans cette étude, nous ne tenons pas compte de la structure des systèmes étudiés (coopératifs, compétitifs ou autres). Les seules conditions requises concernent l'existence de sur- et sous-solutions, dans un sens que nous définirons, et la connaissance précise du comportement des seconds membres des équations du système par rapport à la distance au bord. Dans le cas d'un système coopératif, nous démontrons également un résultat d'existence de solution dans un sens plus faible mais ne requérant que peu d'hypothèses sur les seconds membres. Dans une deuxième partie, nous appliquons nos deux résultats généraux d'existence à des systèmes du type GIERER-MEINHARDT [51], modélisant des phénomènes biologiques (compétition, symbiose, proies/prédateurs), les singularités exprimant que les espèces évoluent en milieu hostile. L'application de ces deux théorèmes, est rendue possible par la connaissance de sur- et sous-solutions adaptées. Pour se faire, nous utilisons ici quelques résultats d'existence établis dans la Chapitre I.

Dans le dernier chapitre de cette thèse, nous étudions un problème parabolique quasi-linéaire singulier. La singularité de ce problème se concentre sur un terme d'absorption de la forme $-\mathbb{1}_{\{u>0\}}u^{-\beta}$, avec $0 < \beta < 1$. Sous la condition de régularité $\mathrm{W}_0^{1,p}(\Omega)\cap\mathrm{L}^\infty(\Omega)$ de la donnée initiale, nous démontrons l'existence d'une solutions faible d'un tel problème. Pour cela, nous sommes tout d'abord amenés à étudier un problème régularisé pour lequel nous faisons une étude détaillée. Pour ce problème, nous démontrons par une méthode de semi-discrétisation en temps non standard, l'existence, l'unicité et la régularité $\mathscr{C}\left(]0,T],\mathrm{W}_0^{1,p}(\Omega)\right)$ d'une solution faible vérifiant de surcroit une certaine égalité d'énergie, avec seulement une donnée initiale dans $\mathrm{L}^\infty(\Omega)$. Dans un deuxième temps, nous analysons le comportement asymptotique de cette solution dans le cas particulier où le second membre ne se compose que du terme d'absorption cité précédemment. Dans ce cas, lorsque $p \geq \frac{2N}{N+2}$, nous établissons l'extinction en temps fini des solutions sur tout le domaine Ω. Ceci se généralise à la situation où le second membre contient un terme de réaction vérifiant un certain comportement asymptotique.

Dans les annexes de ce manuscrit, nous donnons quelques résultats techniques utilisés dans les chapitres de cette thèse. Dans l'Annexe A nous améliorons quelque peu un résultat de régularité Höl-derienne, dû à GIACOMONI-SCHINDLER-TAKÁČ [48], que nous utilisons pour l'étude de nos problèmes. Ce résultat permet en particulier d'établir la régularité des solutions dans le Chapitre I, et donne des bornes dans $\mathscr{C}^{0,\alpha}(\overline{\Omega})$, pour un certain $0 < \alpha < 1$, desquelles découle la compacité nécessaire pour appliquer un argument de point fixe dans le Chapitre II. Dans l'Annexe B, nous démontrons d'autres résultats techniques du Chapitre I, en particulier le résultat d'existence dont nous servons dans les exemples du Chapitre II pour construire les sur- et sous- solutions des systèmes étudiés.

Avant de faire une présentation plus détaillée des résultats que nous avons obtenus, faisons brièvement l'état de l'art sur les problèmes elliptiques et paraboliques singuliers.

Un des tout premiers travaux sur les problèmes elliptiques singuliers est dû à STUART [81], où l'existence de solutions classiques dans $\mathscr{C}^2(\Omega)\cap\mathscr{C}(\overline{\Omega})$ est démontrée *via* l'utilisation d'un schéma itératif et du principe du maximum. Dans CRANDALL-RABINOWITZ-TARTAR [18], qui peut être considéré à bien des égards comme le travail pionnier sur ces problèmes et qui a motivé par la suite une longue série de travaux pour cette classe de problèmes, une version plus générale du problème suivant est étudiée :

$$\begin{cases} L(u) = K(x)u^{-\delta} \quad \text{dans } \Omega, \\ u = 0 \quad \text{sur } \partial\Omega, \quad u > 0 \quad \text{dans } \Omega. \end{cases} \tag{1}$$

Dans cette étude, Ω est un domaine borné régulier (au moins de classe $\mathscr{C}^2$), L désigne un opérateur linéaire uniformément elliptique, $K \in \mathscr{C}(\overline{\Omega})$ est un potentiel strictement positif sur Ω et $\delta > 0$ (bien sûr !). L'existence et l'unicité d'une solution classique y est démontrée par une méthode de sur- et sous-solutions. Cette approche permet également d'établir le comportement précis de cette solution au voisinage du bord de Ω. Le cas où le potentiel K est singulier a été ensuite étudié par GOMES [52] qui a démontré moyennant une restriction sur le paramètre δ l'existence d'une solution dans $\mathscr{C}^1(\overline{\Omega})$ *via* un argument de point fixe de Schauder sur la formulation intégrale équivalente. La compacité requise est alors assurée par des estimations près du bord de la fonction de Green et de son gradient. Dans DEL PINO [23] l'auteur considère une classe plus large de potentiels, ne présentant pas de condition de stricte positivité, et démontre l'existence d'une solution à variation bornée. Dans le cadre de non-linéarités plus singulières, moyennant une régularité $\mathscr{C}^{0,\alpha}(\overline{\Omega})$ et la stricte positivité sur $\overline{\Omega}$ du potentiel K, des résultats sont apportés par LAZER-MCKENNA [61] sur la régularité des solutions de (1) dans l'espace d'énergie. Les auteurs démontrent que la solution qui appartient à $\mathscr{C}^{2,\delta}(\Omega) \cap \mathscr{C}(\overline{\Omega})$ est dans $\mathrm{H}_0^1(\Omega)$ si et seulement si $\delta < 3$ et dans $\mathscr{C}^1(\overline{\Omega})$ si et seulement si $\delta < 1$. La régularité Hölderienne des solutions, dans $C^{0,\alpha}(\overline{\Omega})$ et $C^{1,\alpha}(\overline{\Omega})$, est établie de manière plus générale dans GUI-LIN [53] par des estimations sur les représentations intégrales *via* la fonction de Green. Concernant la régularité dans les espaces de Sobolev, DÍAZ-HERNÁNDEZ-RAKOTOSON [29] ont récemment montré que les solutions appartiennent à $\mathrm{W}^{1,q}(\Omega)$, avec $q > 1$ dépendant de δ. La preuve de ce résultat fait appel à de récentes contributions de DÍAZ-RAKOTOSON [31, 32] se situant dans la continuité de travaux entrepris par BRÉZIS- CAZENAVE-MARTEL-RAMIANDRISOA [13] sur l'existence et l'unicité de solutions faibles pour des problèmes semi-linéaires elliptiques avec données dans les espaces à poids $\mathrm{L}^q\left(\Omega, d(x)^\beta\right)$, avec $0 < \beta < 1$ (voir aussi l'extension dans le cas parabolique dans RAKOTOSON [76]). Lorsque le potentiel $K \in \mathrm{L}^m(\Omega)$, l'existence d'une solution u telle que $u^\gamma \in \mathrm{H}_0^1(\Omega)$ est établie dans BOCCARDO-ORSINA [11], en reprenant une approche similaire à celle de CRANDALL-RABINOWITZ-TARTAR [18] et reprenant des méthodes de troncatures de Stampachia (voir aussi STUART). Des résultats de non-existence sont également démontrés dans ce travail lorsque K est une mesure de Radon (une masse de Dirac, par exemple).

Avant d'aborder la bibliographie concernant plus particulièrement les problèmes quasi-linéaires singuliers, rappelons ici deux résultats classique concernant la classe de problèmes quasi-linéaires (non-singuliers) suivante faisant intervenir l'opérateur p-Laplacien :

$$\begin{cases} -\Delta_p u = f & \text{dans } \Omega, \\ u = 0 & \text{sur } \partial\Omega, \end{cases} \tag{2}$$

avec $1 < p < \infty$ et $f \in \mathrm{W}^{-1,p'}(\Omega)$. Le premier concerne la régularité des solutions du problème (2). DiBenedetto [34] et Tolksdorf [82] ont obtenu indépendamment et à la même période des résultats de régularité locale des solutions de ce problème. Lieberman [64] obtient un résultat de régularité globale (*i.e.* jusqu'au bord). Ce dernier démontre que sous l'hypothèse que $f \in \mathrm{L}^\infty(\Omega)$, l'unique solution faible de (2) admet en fait une régularité $\mathscr{C}^{1,\alpha}(\overline{\Omega})$, pour un certain $0 < \alpha < 1$. Le deuxième résultat classique, dû à Vázquez [83], généralise le lemme de Hopf (voir par exemple Evans [37, p. 330]) aux opérateurs quasi-linéaires et montre la validité d'un principe du maximum fort pour le problème (2) lorsque les solutions de ce dernier sont dans $\mathscr{C}^1(\overline{\Omega})$. Ce résultat permet entre autres d'obtenir un comportement précis de φ_1, l'unique fonction propre strictement positive et L^p-renormalisée sur Ω associée à la première valeur propre de Δ_p, sur Ω en fonction de la distance au bord. Cette estimation sera fort utile pour décrire le comportement des solutions obtenues. Dans ce travail , Vázquez donne de façon plus générale et presque optimale un critère de stricte positivité pour les solutions d'un problème de la forme

$$-\Delta_p u + \beta(u) = f \quad \text{dans } \Omega, \tag{3}$$

où β est une fonction croissante telle $\beta(0) = 0$. Il démontre que sous la condition de non-intégrabilité

$$\int_0^1 (s\beta(s))^{-\frac{1}{p}}\, ds = +\infty, \tag{4}$$

les solutions de (2) sont strictement positives sur Ω.

La version quasi-linéaire de la classe de problèmes associés à (1) a fait l'objet de contributions plus récentes. En particulier, les problèmes de la forme suivante faisant intervenir l'opérateur p-Laplacien :

$$\begin{cases} -\Delta_p u = K(x)u^{-\delta} + f(x,u) & \text{dans } \Omega, \\ u = 0 \quad \text{sur } \partial\Omega, \quad u > 0 & \text{dans } \Omega, \end{cases} \tag{5}$$

où f est une fonction de Carathéodory satisfaisant des hypothèses adéquates, ont été étudiés dans plusieurs travaux. Dans Aranda-Godoy [5], quelques exemples partiels sur l'existence de solutions faibles sont établis par la théorie de la bifurcation. Dans Giacomoni-Schindler-Takáč [49], l'existence ainsi que la multiplicité de solutions faibles sont démontrées dans le cas où $f(x,u) = u^q$, avec $1 < q \leq \frac{Np}{N-p} - 1$ (incluant donc le cas critique) et $0 < \delta < 1$, par méthodes variationnelles. En adaptant les estimations dans les espaces de Campanato (voir Giaquinta [50]) utilisées dans Lieberman [64], les auteurs ont également démontré dans ce travail la régularité $\mathscr{C}^{1,\alpha}(\overline{\Omega})$ pour un certain $0 < \alpha < 1$, des solutions. Ce résultat étend les résultats de régularité de Gui-Lin [53] au p-Laplacien en suivant une approche différente. Récemment, les mêmes auteurs ont établi dans Giacomoni-Schindler-Takáč [48] un résultat de régularité $\mathscr{C}^{0,\alpha}(\overline{\Omega})$ des solutions de (5) dans le cas où $1 \leq \delta \leq 2 + \frac{1}{p-1}$. Ces résultats de régularité sont utilisés dans le Chapitre II, *via* le théorème de point fixe de Schauder pour étudier l'existence de solutions faibles de systèmes quasi-linéaires singuliers de la forme :

$$\begin{cases} -\Delta_p u = f_1(x,u,v) \quad \text{in } \Omega\,; \quad u|_{\partial\Omega} = 0, \quad u > 0 \quad \text{dans } \Omega, \\ \\ -\Delta_q v = f_2(x,u,v) \quad \text{in } \Omega\,; \quad v|_{\partial\Omega} = 0, \quad v > 0 \quad \text{dans } \Omega. \end{cases} \tag{6}$$

Les systèmes elliptiques, ont été beaucoup étudiés durant ces dernières décennies. Ils s'expriment sous la même forme (6) en considérant différents opérateurs elliptiques à la place de $-\Delta_p$ et $-\Delta_q$. Le lecteur intéressé pourra consulter à ce titre l'article de Hernández-Mancebo-Vega [56], où une revue des travaux réalisés dans ce domaine y est présentée. Dans le cas des systèmes elliptiques singuliers, citons les travaux de Choi-McKenna [15, 16], où les auteurs étudient des systèmes dits de Gierer-Meinhardt [51], modélisant des phénomènes biologiques (interaction entre deux populations de bactéries par exemple), dans un cas classique et dans un cas dégénéré. Dans le premier article cité, $f_1(u,v) = \alpha_1 u + \beta_1 u^r v^{-1}$ et $f_2(u,v) = \alpha_2 v + \beta_2 u^r$, avec $r > 0$. Il y est établi, par un argument de point fixe, un résultat d'existence de solutions classiques radiales sur la boule unité de $\mathbb{R}^2$. Dans le second article, $f_1(u,v) = u - uv^{-1}$ et $f_2(u,v) = \alpha v - uv^{-1}$. Choi et Mc Kenna démontrent alors par une méthode similaire l'existence de solutions positives lorsque $N \geq 2$ et établissent l'unicité d'une telle solution dans le cas mono-dimensionnel. Dans Ghergu [42], l'auteur donne sous des hypothèses de sous-homogénéité sur les non-linéarités, des résultats d'existence de solutions classiques pour des systèmes compétitifs présentant des singularités du type $f_1(u,v) = u^{-a_1} v^{-b_1}$ et $f_2(u,v) = v^{-a_2} u^{-b_2}$, avec $a_1\,, a_2 \geq 0$ et $b_1\,, b_2 > 0$. Ce résultat a été récemment généralisé au cas quasi-linéaire par Giacomoni-Schindler-Takáč [48]. Dans Ni-Wei [68] est étudié un autre type de système de Gierer-Meinhardt. Cette fois ci le système est étudié sur la boule unité avec des conditions de Neumann homogènes, $f_1(u,v) = \lambda \left(u - u^{a_1} v^{-b_1}\right)$ et $f_2(u,v) = v - u^{a_2} v^{-b_2}$, avec $\lambda > 0$, a_1, $a_2 > 1$ et b_1, $b_2 \geq 0$ satisfaisant également une condition de sous-homogénéité. Des résultats similaires d'existence de solutions positives et radiales sont alors démontrés. L'article de Hernández-Mancebo-Vega [56], donne un premier résultat d'existence général, généralisant les précédents résultats cités ici lorsque les solutions se comportent comme la distance au bord. Concernant les systèmes quasi-linéaires singuliers de la forme (6), Giacomoni-Hernández-Mouassaoui [44] établissent un résultat d'existence de solutions positives pour des systèmes coopératifs similaires à ceux de [42] en utilisant un argument de point fixe couplé à un méthode de sur- et sous- solutions. De leurs côtés, Lee-Shivaji-Ye [62] établissent par le même procédé des résultats d'existence pour un autre type de système de Gierer-Meinhardt présentant les mêmes caractéristiques singulières et sous-homogènes. Dans le Chapitre II, nous présentons des résultats issus Giacomoni-Hernández-Sauvy [45] qui généralisent les résultats de [44], [42], [48] et [62].

Dans le premier chapitre de cette thèse, nous étudierons un problème d'absorption avec un terme singulier. Nous démontrons pour certaines valeurs des paramètres l'existence de solutions non-triviales à support compact. Ce type de problème se retrouve notamment dans les équations aux dérivées partielles issues de la modélisation en sciences du vivant, en particulier en dynamique des populations ; nous renvoyons le lecteur au livre de DÍAZ [26] où bon nombre de modèles sont explicités. Dans ce genre de problèmes apparaissent des "zones mortes" où les solutions u s'annulent et donc où la diffusion est nulle. La frontière de l'ensemble $\Omega_0 = \{x \in \Omega \mid u(x) = 0\}$ est *a priori* inconnue et peut évoluer dans le temps pour les problèmes paraboliques ; on parle alors de problèmes à frontière libre. Dans DÁVILA-MONTENEGRO [20], les auteurs étudient un problème à frontière libre elliptique avec un second membre de la forme $\mathbb{1}_{\{u>0\}}\left(\lambda f(x,u) - (1-\beta)u^{-\beta}\right)$, où $\beta \in]0,1[$, $\lambda > 0$ et f est une fonction croissante concave et sous-linéaire. Il est démontré l'existence valeur critique du paramètre λ séparant solutions strictement positives et solutions telles que $\mathcal{L}^N\left(\{x \in \Omega \mid u(x) = 0\}\right) > 0$. HAITAO [54], quant à lui, étudie un problème à frontière libre où la singularité est concentrée sur un potentiel $K(x)$, se comportant comme une puissance négative de la distance au bord, qui accentue l'effet d'absorption au voisinage de $\partial\Omega$. Il est alors démontré l'existence d'une solution à support compact lorsque $K(x)$ est suffisamment singulier. Pour se faire, l'auteur s'appuie sur une méthode développée par BÉNILAN-BRÉZIS-CRANDALL [10] qui, sous la condition d'intégrabilité sur β (satisfaisant (3)) :

$$\int_0^1 \left(s\beta(s)\right)^{-\frac{1}{2}}\, ds < +\infty, \tag{7}$$

garantit que les solutions du problème

$$-\Delta u + \beta(u) = f \qquad \text{dans } \Omega,$$

sont à support compact. Par la suite, cet argument a été généralisé par DÍAZ-HERRERO [30] dans le cas quasi-linéaire. Ces travaux se situent dans le prolongement de ceux de VÁZQUEZ [83] sur la stricte positivité des solutions de (3). Concernant les problèmes elliptiques quasi-linéaires, DÍAZ-HERNÁNDEZ-MANCEBO [28], ont étudié plus en détail le problème mono-dimensionnel suivant :

$$\begin{cases} -\left(|u'|^{p-2}u'\right)' = f(u) & \text{sur }]-L, L[, \\ u(-L) = u(L) = 0, \end{cases}$$

où $f :]0,+\infty[\to \mathbb{R}$ est telle que

$$\limsup_{s\to 0^+} f(s) \leq 0 \ ;$$

en particulier, f peut être singulière en 0. Les auteurs analysent alors, par des méthodes de comparaison locale, la bifurcation de la branche de solutions positives sur $]-L, L[$, où L correspond au paramètre de bifurcation. Le lecteur pourra également trouver dans cet article de nombreuses autres références en lien avec les problèmes de frontière libre.

Dans le Chapitre III, nous étudierons un problème d'absorption quasi-linéaire singulier pour lequel, sous des hypothèses convenables, nous démontrerons l'extinction en temps fini des solutions. Ce type de phénomène, où les solutions s'annulent sur une partie du domaine d'étude, est appelé "quenching" dans la littérature. Dans le cas de l'équation de la chaleur, des résultats ont mis en lumière une grande variété de comportements des solutions suivant la nature de la non-linéarité.

Dans Díaz-Hernández [27], les auteurs étudient un problème elliptique du type suivant :

$$\begin{cases} \partial_t u - \Delta u + f(u) &= 0 \quad \text{dans } Q =]0, +\infty[\times\Omega, \\ u &= h \quad \text{on } \Gamma =]0, +\infty[\times\partial\Omega, \\ u(0, \cdot) &= u_0 \quad \text{in } \Omega. \end{cases} \tag{8}$$

Dans cette étude, la donnée initiale $0 \leq u_0 \in L^\infty(\Omega)$, f est croissante et positive et h est une fonction positive de $L^\infty(\Gamma)$. Sous les hypothèses d'intégrabilité suivantes :

$$\int_0^1 \int_0^s f(t)^{-\frac{1}{2}} dt ds < +\infty \quad \text{et} \quad \int_0^1 f(s)^{-1} ds < +\infty, \tag{9}$$

qui sont à rapprocher de celles données dans (7), les auteurs démontrent l'existence d'un temps critique T^* et d'une constante $L > 0$ ne dépendant que de f, h, u_0, Ω et N tel que pour $t \geq T^*$,

$$\left\{ x \in \Omega \;\middle|\; d\left(x, \bigcup_{\tau \geq 0} \operatorname{supp}_\Gamma h(\tau, \cdot)\right) \geq L \right\} \subset \{x \in \Omega \mid u(t, x) = 0\},$$

avec supp $h(\tau, \cdot)$ le support de la fonction h à l'instant $\tau \geq 0$. Ceci prouve en particulier que les solutions du problème (8) avec conditions de Dirichlet homogènes sont à support compact à partir d'un certain temps. La preuve de ce résultat est basée sur une méthode de comparaison permettant d'établir des estimations locale des solutions. Dans Fila-Kawohl [39], le problème (8) est étudié avec un terme d'absorption singulier de la forme $f(u) = u^{-\beta}$, avec $\beta > 0$. Dans cette étude Ω est convexe $u = 1$ sur Γ et $-\Delta u_0 + u_0 \geq 0$ dans Ω. Par des inégalités différentielles, les auteurs établissent alors des estimations sur le profil d'extinction en temps fini des solutions. Plus précisément, si une solution positive u de (8) s'éteint à partir d'un instant T^*, il existe $C_1, C_2 > 0$ telles que

$$C_1 (T^* - t)^{\frac{1}{1+\beta}} \leq u(t, x) \leq C_2 (T^* - t)^{\frac{1}{1+\beta}} \quad \text{dans } [0, T^*] \times \Omega.$$

Dans le cas où $\Omega = B$ est une boule de $\mathbb{R}^N$ et lorsque $f(u) = \lambda u^{-\beta}$, avec $0 < \beta < 1$ et $\lambda > 0$, Fila-Levine-Vázquez [40] ont montré que pour des petites valeurs de λ, l'ensemble $\{(t, x) \in]0, +\infty[\times B \mid u(t, x) = 0\}$ était borné dans $]0, +\infty[\times B$. Dans Winkler [85], l'extinction en temps fini est démontrée pour une classe similaire de singularités grace à des estimations du type (9), tandis que dans Winkler [84, 86] des résultats de non-unicité et de contraction de support des solutions sont établis. Des résultats d'existence ont également été démontrés dans Phillips [70], pour le même genre de problèmes singuliers définis sur tout l'espace $]0, +\infty[\times\mathbb{R}^N$. Un lien entre le comportement des solutions d'un problème parabolique à frontière libre et celui des solutions du problème stationnaire associé est établi par Davila-Montenegro [21]. Dans ce travail, les auteurs ont étudié les solutions positives du problème d'évolution

$$\partial_t u - \Delta u + \mathbb{1}_{\{u>0\}} u^{-\beta} = f(u) \quad \text{dans } Q, \tag{10}$$

pour une donnée initiale u_0 positive et dans $\mathrm{L}^\infty(\Omega)$, $0 < \beta < 1$ et f sous-linéaire. Un résultat d'existence de solutions globales, basé sur l'étude d'un problème approché, y est démontré. Il est également prouvé que dans l'hypothèse où le problème stationnaire,

$$-\Delta u + \mathbb{1}_{\{u>0\}} \Big(u^{-\beta} - \lambda f(u) \Big) = 0 \quad \text{dans } \Omega \tag{11}$$

avec $\lambda > 0$, admet des solutions strictement positives presque partout dans Ω (ce qui est le cas d'après [20] pour une certaine classe de fonctions f), les solutions de (10) convergent vers les solutions (11) quand $t \to +\infty$. Inversement, si les solutions de (11) s'annulent sur des sous-ensembles de mesure non-nulle (ce qui est le cas toujours d'après [20] pour une certaine classe de fonctions f), alors les solutions du problème (10) s'éteignent en temps fini. A notre connaissance, peu d'études ont été entreprises pour ce genre de problèmes dans les cas quasi-linéaire. C'est l'objet du Chapitre III présentant des travaux récents de Giacomoni-Sauvy-Shmarev [47].

Nous allons maintenant décrire avec plus de précision les principaux résultats obtenus dans les Chapitres I, II et III et les idées principales utilisées dans les preuves de ceux-ci.

Résultats principaux du Chapitre I : Dans ce premier chapitre, nous nous sommes intéressés au problème elliptique, quasi-linéaire et singulier suivant :

$$(\mathrm{P}_\lambda) \begin{cases} -\Delta_p u = \mathbb{1}_{\{u>0\}} K(x)(\lambda u^q - u^r) & \text{dans } \Omega, \\ u = 0 \quad \text{sur } \partial\Omega, \quad u \geq 0 & \text{dans } \Omega. \end{cases}$$

Dans la première équation de (P_λ), $\lambda > 0$ est un paramètre réel, $-1 < r < q < p-1$ et $K \in \mathscr{C}(\Omega)$ est une fonction positive ayant un comportement singulier au voisinage de $\partial\Omega$. Plus précisément,

$$K(x) = d(x)^{-k} L(d(x)) \quad \text{dans } \Omega,$$

où $0 < k < p$ est appelé ***taux d'explosion de K*** et $L \in \mathscr{C}^2((0, D])$, est une perturbation positive d'ordre plus petit, dite de Karamata (voir Karamata [58]). Le lecteur intéressé pourra aussi consulter à ce sujet l'article de Mâagli-Zribi [67] ainsi que ses références, où les classes de Kato et les fonctions de Karamata y sont définies et étudiées en détail à travers la théorie du potentiel.

Dans cette étude, on dira que $u \in \mathrm{W}_0^{1,p}(\Omega) \cap \mathrm{L}^\infty(\Omega)$ est une ***solution faible de (P_λ)*** si l'équation suivante est satisfaite pour toute fonction test $\varphi \in \mathcal{D}(\Omega)$:

$$\int_\Omega |\nabla u|^{p-2} \nabla u . \nabla \varphi \, dx = \int_\Omega \mathbb{1}_{\{u>0\}} K(x) \left(\lambda u^q - u^r\right) \varphi \, dx. \tag{12}$$

Le second membre de l'équation principale de (P_λ) présente une non-linéarité du type à effet absorbant au bord du domaine (l'exposant r étant plus petit que q) et une dépendance par rapport au paramètre $\lambda > 0$. On se pose alors les questions suivantes :

1. Pour quelles valeurs de λ a-t-on l'existence ou la non-existence de solutions faibles (non-triviales) de (P_λ) ?
2. Que peut-on dire du comportement des éventuelles solutions de (P_λ) en fonction de k ? Plus précisément, les solutions de (P_λ) sont-elles strictement positives sur Ω ou à support compact ?
3. Quelle est la régularité des solutions faibles ?

Les théorèmes suivants apportent des réponses aux questions précédentes. De ces deux résultats ci-dessous il découle l'existence d'une valeur critique du taux d'explosion k séparant deux comportements des solutions de (P_λ) : stricte positivité lorsque $k < 1+r$ et solutions à frontière libre lorsque $k \geq 1+r$.

Théorème 1 *Lorsque $k < 1 + r$, il existe une constante $\Lambda_1 > 0$ telle que :*

1. *Pour $\lambda > \Lambda_1$, (P_λ) admet une solution faible strictement positive.*
2. *Toute solution faible de (P_λ) est dans $\mathscr{C}^{1,\beta}(\overline{\Omega})$, pour un certain $0 < \beta < 1$.*
3. *Pour $\lambda < \Lambda_1$, (P_λ) n'admet pas de solution non-triviale.*

Théorème 2 *Supposons que $r > 0$ et qu'une des deux hypothèses suivantes soit satisfaite :*

$$1 + r > q \quad \text{et} \quad k \in \left[1 + r, 1 + \frac{(p-1)(r+1)}{p-q+r}\right), \tag{13}$$

$$1 + r \geq q \quad \text{et} \quad k \in [1 + r, 2 + r). \tag{14}$$

Alors, il existe une constante $\Lambda_2 > 0$ telle que :

1. *Pour $\lambda > \Lambda_2$, (P_λ) admet une solution faible à support compact dans Ω.*
2. *Toute solution faible de (P_λ) est dans $\mathscr{C}^{1,\beta}(\overline{\Omega})$, pour un certain $0 < \beta < 1$.*
3. *Pour $\lambda < \Lambda_2$, (P_λ) n'admet pas de solution non-triviale.*

La preuve du Théorème 1, est basée sur une méthode non standard de sur- et sous-solutions. Cette approche repose sur l'argument suivant : lorsque $k < 1 + r$, il est possible de construire une sous-solution de (P_λ) à l'aide d'une puissance positive bien choisie de φ_1, la fonction propre strictement positive et L^p-renormalisée sur Ω associée à λ_1, première valeur propre du p-Laplacien avec conditions de Dirichlet homogènes sur Ω, définie par

$$\lambda_1 \stackrel{\text{def}}{=} \inf \left\{ \int_\Omega |\nabla v|^p \, dx \in \mathbb{R}_+ \quad \middle| \quad v \in \mathrm{W}_0^{1,p}(\Omega), \quad \int_\Omega |v|^p \, dx = 1 \right\}$$

(voir Anane [3], [4] pour plus de détails à ce sujet). La preuve du Théorème 2, quant à elle, se divise en deux parties distinctes. Premièrement, nous commençons par établir l'existence d'une solution faible de (P_λ), pour $\lambda > 0$ suffisamment grand, en utilisant une méthode dite de Perron qui consiste à minimiser la fonctionnelle énergie sur un convexe défini par les sur- et sous- solutions du problème (voir Struwe [80, Theorem 2.4 p.17] pour plus de détails sur la méthode de Perron). Ensuite, nous démontrons que cette solution est alors à support compact. Pour cela, nous établissons une estimation fine du comportement de la solution au voisinage de $\partial\Omega$, qui permet de construire une sur-solution de (P_λ) à support compact, sur un voisinage de $\partial\Omega$. Cette méthode a été développée par Alvarez-Díaz [2], puis adaptée par Haitao [54] dans le cas semi-linéaire.

Dans les preuves des Théorèmes 1 et 2, nous avons été amenés à étudier de façon détaillée le problème quasi-linéaire suivant :

$$(\mathrm{Q}) \begin{cases} -\Delta_p v = K(x) v^q \quad \text{dans } \Omega, \\ \quad v = 0 \quad \text{sur } \partial\Omega, \quad v > 0 \quad \text{dans } \Omega. \end{cases}$$

Par une méthode de sur-et sous-solutions, détaillée dans la Section 1 de l'Annexe B (ou voir Giacomoni-Mâagli-Sauvy [46]), nous avons établi le résultat suivant :

Proposition 1

1. *Lorsque $k \in (0, 1+q)$, le problème* (Q) *admet une unique solution faible $v \in \mathscr{C}^{1,\alpha}(\overline{\Omega})$, pour un certain $0 < \alpha < 1$, satisfaisant la condition de cône :*

$$v(x) \sim d(x) \quad \textit{dans } \Omega.$$

2. *Lorsque $k = 1+q$, le problème* (Q) *admet une unique solution faible $v \in \mathrm{W}_0^{1,p}(\Omega) \cap \mathscr{C}^{0,\alpha}(\overline{\Omega})$, pour un certain $0 < \alpha < 1$, satisfaisant la condition de cône :*

$$v(x) \sim d(x) \left(\int_{d(x)}^{2d} \frac{L(t)}{t}\, dt \right)^{\frac{1}{p-k}} \quad \textit{dans } \Omega.$$

3. *Lorsque $k \in \left(1+q, 1+q+\frac{p-(1+q)}{p}\right)$, le problème* (Q) *admet une unique solution faible $v \in \mathrm{W}_0^{1,p}(\Omega) \cap \mathscr{C}^{0,\alpha}(\overline{\Omega})$, pour un certain $0 < \alpha < 1$, satisfaisant la condition de cône :*

$$v(x) \sim d(x)^{\frac{p-k}{p-(1+q)}} \Big(L\left(d(x)\right) \Big)^{\frac{1}{p-(1+q)}} \quad \textit{dans } \Omega.$$

Enfin, nous donnons dans la dernière section de ce chapitre une série de résultats concernant le cas semi-linéaire. Précisément, dans cette partie le problème (P_λ) est analysé plus en détail dans le cas particulier du Laplacien (*i.e.* $p = 2$) et lorsque le second membre de l'équation principale de (P_λ) est une fonction concave. Les résultats qui y sont établis concernent le comportement précis et la stabilité des solutions de (P_λ), ainsi que l'existence de solution pour le problème extrémal (P_{Λ_1}). Concernant les problèmes de stabilité des solutions, nous référons au livre Dupaigne [36] qui donne un large spectre des résultats établis à sujet et à Sattinger [77] où il est démontré la stabilité des solutions minimales d'un problème elliptique, obtenues par méthode se sur- et sous-solutions.

Résultats principaux du Chapitre II : Dans ce chapitre, nous nous sommes intéressés à un système elliptique quasi-linéaire général, défini de la façon suivante :

$$(\mathrm{P})\begin{cases} -\Delta_p u = f_1(x,u,v) \quad \text{in } \Omega\,; \quad u|_{\partial\Omega} = 0, \quad u > 0 \quad \text{ dans } \Omega, \\ \\ -\Delta_q v = f_2(x,u,v) \quad \text{in } \Omega\,; \quad v|_{\partial\Omega} = 0, \quad v > 0 \quad \text{ dans } \Omega. \end{cases}$$

Dans cette étude, f_1 et f_2 sont deux fonctions de Carathéodory sur $\Omega \times \left(\mathbb{R}_+^* \times \mathbb{R}_+^*\right)$ éventuellement singulier. Plus précisément, pour tout $(t_1, t_2) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ et presque tout $x \in \Omega$, on suppose que

(H_1) $f_1(\cdot, t_1, t_2)$ et $f_2(\cdot, t_1, t_2)$ sont deux fonctions Lebesgue mesurables sur Ω,

(H_2) $f_1(x, \cdot, \cdot)$ et $f_2(x, \cdot, \cdot)$ sont fonctions de $\mathscr{C}^1(\mathbb{R}_+^* \times \mathbb{R}_+^*)$.

Dans les différentes études citées précédemment, à savoir [15, 16, 42, 44, 68], les auteurs établissent des résultats d'existence de solutions faibles, *via* une méthode de sur- et sous-solutions, en se servant de la structure particulière des systèmes étudiés. Principalement, l'hypothèse de ***coopérativité du système***, c'est à dire le fait d'imposer aux seconds membres de (P) la condition

$$\frac{\partial f_1}{\partial v}(x,u,v) > 0 \quad \text{ et } \quad \frac{\partial f_2}{\partial u}(x,u,v) > 0 \quad \text{ dans } \Omega \times \mathbb{R}_+^* \times \mathbb{R}_+^*, \tag{15}$$

permet d'obtenir un principe de comparaison, qui induit de la monotonie dans les schémas itératifs.

Le but de notre étude est de trouver un résultat d'existence de solutions faibles de (P), toujours basé sur une méthode de sur- et sous-solutions, qui traite des cas très généraux. En particulier, nous voudrions résoudre aussi bien les systèmes coopératifs, les systèmes compétitifs que les systèmes de type proies/prédateurs. Pour cela, nous n'imposons aucune condition de croissance du type (15) sur f_1 et f_2. Nous ne requérons pas non plus d'hypothèse concernant le signe des seconds membres f_1 et f_2.

Cette approche plus générale requiert en premier lieu de généraliser la notion de sur- et sous-solutions existant pour les systèmes coopératifs ou compétitifs. Nous commençons donc par rappeler ici les principales définitions liées à cette généralisation : dans les trois points suivants, pour $1 < r < +\infty$, $\mathscr{A}_r(\Omega)$ désignera soit l'espace $\mathrm{W}_0^{1,r}(\Omega)$, soit l'espace $\mathrm{W}_{\mathrm{loc}}^{1,r}(\Omega)$.

1. Soit $\underline{w}, \overline{w} \in \mathscr{A}_r(\Omega)$, deux fonctions strictement positives p.p. sur Ω telles que $\underline{w} \leq \overline{w}$ p.p. dans Ω. On définit l'ensemble convexe et fermé, pour la topologie de $\mathscr{C}(\overline{\Omega})$, suivant :

$$[\underline{w}, \overline{w}] \stackrel{\text{def}}{=} \left\{ w \in \mathscr{C}(\overline{\Omega}), \quad \underline{w} \leq w \leq \overline{w} \quad \text{ p.p. dans } \Omega \right\}.$$

2. Soient $\underline{u}, \overline{u} \in \mathscr{A}_p(\Omega)$ et $\underline{v}, \overline{v} \in \mathscr{A}_q(\Omega)$ quatre fonction strictement positives p.p. dans Ω telles que $\underline{u} \leq \overline{u}$ et $\underline{v} \leq \overline{v}$ p.p. dans Ω. Le couple $(\overline{u}, \overline{v})$ et $(\underline{u}, \underline{v})$ est appelé ***couple de sur- et***

***sous-solutions de* (P)** si les inégalités suivantes sont satisfaites au sens des distributions :

$$-\Delta_p \underline{u} \le f_1(x,\underline{u},v) \quad \text{dans } \Omega, \quad \text{pour tout } v \in [\underline{v},\overline{v}], \tag{16}$$

$$-\Delta_q \underline{v} \le f_2(x,u,\underline{v}) \quad \text{dans } \Omega, \quad \text{pour tout } u \in [\underline{u},\overline{u}], \tag{17}$$

$$-\Delta_p \overline{u} \ge f_1(x,\overline{u},v) \quad \text{dans } \Omega, \quad \text{pour tout } v \in [\underline{v},\overline{v}], \tag{18}$$

$$-\Delta_q \overline{v} \ge f_2(x,u,\overline{v}) \quad \text{dans } \Omega, \quad \text{pour tout } u \in [\underline{u},\overline{u}]. \tag{19}$$

3. Soit $(\overline{u},\overline{v})$, $(\underline{u},\underline{v}) \in \mathscr{A}_p(\Omega)\times\mathscr{A}_q(\Omega)$ un couple de sur- et sous-solutions de (P). Le cône défini par $[\underline{u},\overline{u}]\times[\underline{v},\overline{v}]$ sera alors noté $\mathcal{C}$.

Le théorème général d'existence que nous obtenons alors est le suivant. Il ne s'appuie que sur l'existence d'un couple de sur- et sous-solutions de (P) ainsi que sur la connaissance précise du comportement de f_1 et f_2 sur le cône $\Omega\times\mathcal{C}$ par rapport à la fonction distance.

Théorème 3 *Soient* $(\overline{u},\overline{v})$ *et* $(\underline{u},\underline{v}) \in \mathrm{W}_0^{1,p}(\Omega)\times\mathrm{W}_0^{1,q}(\Omega)$ *un couple de sur- et sous-solutions de* (P) *et supposons que conditions suivantes sont satisfaites :*

1. *il existe* $k_1,k_2>0$ *et* $\delta_1,\delta_2\in\mathbb{R}$ *tels que*

$$|f_1(x,u,v)| \le k_1 d(x)^{\delta_1} \quad et \quad |f_2(x,u,v)| \le k_2 d(x)^{\delta_2} \quad dans\ \ \Omega\times\mathcal{C}, \tag{20}$$

2. *il existe* $C_1,C_2>0$ *et* $b_1,b_2>0$ *tels que*

$$\overline{u} \le C_1 d(x)^{b_1} \quad et \quad \overline{v} \le C_2 d(x)^{b_2} \quad dans\ \ \Omega, \tag{21}$$

3. *et il existe* $\kappa_1,\kappa_2>0$ *et* $\alpha_1,\alpha_2>0$ *tels que*

$$\left|\frac{\partial f_1}{\partial u}(x,u,v)\right| \le \kappa_1 d(x)^{\delta_1-\alpha_1} \quad dans\ \ \Omega\times\mathcal{C}, \tag{22}$$

$$\left|\frac{\partial f_2}{\partial v}(x,u,v)\right| \le \kappa_2 d(x)^{\delta_2-\alpha_2} \quad dans\ \ \Omega\times\mathcal{C}. \tag{23}$$

Si de plus, les exposants mentionnés dans les équations (20) *à* (23) *vérifient les conditions*

$$\delta_1 > -2+\frac{1}{p}+(\alpha_1-b_1)^+, \qquad \delta_2 > -2+\frac{1}{q}+(\alpha_2-b_2)^+. \tag{24}$$

alors il existe une paire de solutions faibles positives $(u,v)\in\mathrm{W}_0^{1,p}(\Omega)\times\mathrm{W}_0^{1,q}(\Omega)$ *au problème* (P) *telles que* $(u,v)\in\mathcal{C}$.

Nous donnons également un deuxième théorème d'existence de solutions dans un sens plus faible que précédemment pour des systèmes coopératifs. La particularité du résultat énoncé ci-dessous réside dans l'affaiblissement des hypothèses imposées sur les seconds membres f_1 et f_2.

Théorème 4 *Supposons que* (P) *soit un système coopératif. Soient* $(\overline{u},\overline{v})$ *et* $(\underline{u},\underline{v}) \in \left[\mathscr{C}(\overline{\Omega}) \cap \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega)\right] \times \left[\mathscr{C}(\overline{\Omega}) \cap \mathrm{W}^{1,q}_{\mathrm{loc}}(\Omega)\right]$ *un couple de sur- et sous-solutions de* (P). *Supposons de plus que les conditions suivante soient vérifiées :*

1. *il existe* $C_1, C_2 > 0$ *et* $b_1, b_2 > 0$ *tels que*

$$\overline{u} \leq C_1 d(x)^{b_1} \quad \text{et} \quad \overline{v} \leq C_2 d(x)^{b_2} \quad \text{dans } \Omega, \tag{25}$$

2. *il existe* $\kappa_1, \kappa_2 > 0$ *et* $\delta_1, \delta_2 \in \mathbb{R}$ *tels que*

$$\left|\frac{\partial f_1}{\partial u}(x,u,v)\right| \leq \kappa_1 d(x)^{\delta_1} \quad \text{et} \quad \left|\frac{\partial f_2}{\partial v}(x,u,v)\right| \leq \kappa_2 d(x)^{\delta_2} \quad \text{dans } \Omega \times \mathcal{C}. \tag{26}$$

Alors, il existe une paire de solutions très faibles $(u,v) \in \left[\mathrm{L}^\infty(\Omega) \cap \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega)\right] \times \left[\mathrm{L}^\infty(\Omega) \cap \mathrm{W}^{1,q}_{\mathrm{loc}}(\Omega)\right]$ *au problème* (P) *telles que* $(u,v) \in \mathcal{C}$.

La démonstration du Théorème 3 repose sur un argument variationnel couplé à une méthode de point fixe. En effet, nous appliquons ici le théorème de point fixe de Schauder à une application $T : \mathcal{C} \to \mathcal{C}$, bien choisie. Cette approche utilise une amélioration d'un résultat récent de régularité Hölderienne du à Giacomoni-Schindler-Takáč [48], présenté dans l'Annexe A, garantissant la compacité de T. La démonstration du Théorème 4, quant à elle, utilise une méthode de sur- et sous-solutions non standard, similaire à celle de la preuve du Théorème 1, où nous nous servons du caractère coopératif de (P) pour comparer deux solutions consécutives du schéma itératif associé à (P).

Dans une deuxième partie, nous appliquons les Théorèmes 3 et 4 à plusieurs exemples de systèmes particuliers. Nous nous sommes particulièrement intéressés au système quasi-linéaire suivant :

$$(\mathrm{P}) \begin{cases} -\Delta_p u = K_1(x) u^{a_1} v^{b_1} & \text{dans } \Omega\,; \quad u|_{\partial\Omega} = 0, \quad u > 0 \quad \text{dans } \Omega, \\ -\Delta_q v = K_2(x) v^{a_2} u^{b_2} & \text{dans } \Omega\,; \quad v|_{\partial\Omega} = 0, \quad v > 0 \quad \text{dans } \Omega. \end{cases}$$

Dans ce problème,

1. Les exposents $a_1 < p-1$, $a_2 < q-1$ et $b_1, b_2 \neq 0$ satisfont la condition de sous-homogénéité suivante :

$$(p-1-a_1)(q-1-a_2) - |b_1 b_2| > 0, \tag{27}$$

2. K_1, K_2 son deux fonctions positives sur Ω vérifiant

$$K_1(x) = d(x)^{-k_1} L_1(d(x)) \quad \text{and} \quad K_2(x) = d(x)^{-k_2} L_2(d(x)) \quad \text{in } \Omega, \tag{28}$$

avec $0 \leq k_1 < p$, $0 \leq k_2 < q$ et pour $i = 1, 2$, L_i sont des fonctions de Karamata.

Comme dans les travaux de GHERGU [42] , GIACOMONI-HERNÁNDEZ-MOUASSAOUI [44] et GIACOMONI-SCHINDLER-TAKÁČ [48], l'existence de solutions faibles de (P) repose sur la détermination d'un couple de sur- et sous-solutions adéquat. Nous utilisons pour cela la Proposition 1 et l'étude du problème (Q) que nous avons entreprise dans le Chapitre I. Nous obtenons alors le résultat suivant :

Théorème 5 *Supposons que les exposants* $a_1 < p-1$, $a_2 < q-1$ *et* $b_1, b_2 \neq 0$ *du problème* (P) *vérifient l'hypothèse* (II.32).

1. *Notons*

$$\alpha_1 = \frac{q-1-a_2}{(p-1-a_1)(q-1-a_2)-b_1b_2}, \quad \alpha_2 = \frac{p-1-a_1}{(p-1-a_1)(q-1-a_2)-b_1b_2},$$

$$\beta_1 = \frac{b_1}{(p-1-a_1)(q-1-a_2)-b_1b_2}, \quad \beta_2 = \frac{b_2}{(p-1-a_1)(q-1-a_2)-b_1b_2},$$

$$\gamma_1 = \frac{(p-k_1)(q-1-a_2)+(q-k_2)b_1}{(p-1-a_1)(q-1-a_2)-b_1b_2}, \quad \gamma_2 = \frac{(q-k_2)(p-1-a_1)+(p-k_1)b_2}{(p-1-a_1)(q-1-a_2)-b_1b_2}$$

et supposons que

$$1-\frac{1}{p} < \gamma_1 < 1 \quad et \quad 1-\frac{1}{q} < \gamma_2 < 1. \tag{29}$$

Alors (P) *admet un paire de solutions positives* $(u,v) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ *satisfaisant les estimations :*

$$u(x) \sim d(x)^{\gamma_1} L_1(d(x))^{\alpha_1} L_2(d(x))^{\beta_1} \quad dans\ \Omega, \tag{30}$$

$$v(x) \sim d(x)^{\gamma_2} L_2(d(x))^{\alpha_2} L_1(d(x))^{\beta_2} \quad dans\ \Omega. \tag{31}$$

De plus, $(u,v) \in \mathscr{C}^{0,\alpha}(\overline{\Omega}) \times \mathscr{C}^{0,\alpha}(\overline{\Omega})$, *pour un certain* $0 < \alpha < 1$.

2. *Maintenant supposons que*

$$k_1 - 1 < a_1 + b_1 < p-1 \quad et \quad k_2 - 1 < a_2 + b_2 < q-1. \tag{32}$$

Alors (P) *admet un paire de solutions positives* $(u,v) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ *satisfaisant les estimations :*

$$u(x) \sim d(x) \quad et \quad v(x) \sim d(x) \quad dans\ \Omega. \tag{33}$$

De plus, $(u,v) \in \mathscr{C}^{1,\alpha}(\overline{\Omega}) \times \mathscr{C}^{1,\alpha}(\overline{\Omega})$, *pour un certain* $0 < \alpha < 1$.

3. *Notons ici,*

$$\gamma = \frac{p-k_1+b_1}{p-1-a_1}$$

et supposons que

$$1-\frac{1}{p} < \gamma < 1 \quad et \quad k_2 - 1 < a_2 + b_2\gamma < q-1. \tag{34}$$

Alors (P) *admet un paire de solutions positives* $(u,v) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ *satisfaisant les* *estimations :*

$$u(x) \sim d(x)^{\gamma} L_1(d(x))^{\frac{1}{p-1-a_1}} \quad \textit{et} \quad v(x) \sim d(x) \quad \textit{dans } \Omega. \tag{35}$$

De plus, $(u,v) \in \mathscr{C}^{0,\alpha}(\overline{\Omega}) \times \mathscr{C}^{1,\alpha}(\overline{\Omega})$, *pour un certain* $0 < \alpha < 1$.

4. *Symétriquement, notons*

$$\gamma = \frac{q - k_2 + b_2}{q - 1 - a_2}$$

et supposons que

$$k_1 - 1 < a_1 + b_1\gamma < p - 1 \quad \textit{and} \quad 1 - \frac{1}{q} < \gamma < 1. \tag{36}$$

Alors (P) *admet un paire de solutions positives* $(u,v) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ *satisfaisant les estimations :*

$$u(x) \sim d(x) \quad \textit{et} \quad v(x) \sim d(x)^{\gamma} L_2(d(x))^{\frac{1}{q-1-a_2}} \quad \textit{dans } \Omega. \tag{37}$$

De plus, $(u,v) \in \mathscr{C}^{1,\alpha}(\overline{\Omega}) \times \mathscr{C}^{0,\alpha}(\overline{\Omega})$, *pour un certain* $0 < \alpha < 1$.

La démonstration de ce théorème provient de l'application du Théorème 3 et de la construction de couples bien choisis de sur- et sous-solutions *via* la Proposition 1 énoncée au paragraphe précédent.

Par une méthode similaire, nous établissons également des résultats d'existence de solutions plus faibles au moyen du Théorème 4. Lorsque le système présente un caractère soit coopératif soit compétitif, nous démontrons également l'unicité des solutions dans les cônes définis au Théorème 5 :

Théorème 6 *Soient* $a_1 < p-1$, $a_2 < q-1$ *et* $b_1, b_2 \neq 0$ *vérifiant la condition de sous-homogénéité* (II.32). *Supposons de plus que le système* (P) *est soit coopératif soit compétitif, i.e.* $b_1 b_2 > 0$. *Alors chaque paire de solutions positives explicitée au Théorème 5 est unique.*

La preuve de ce résultat repose sur la validité (seulement lorsque $b_1 b_2 > 0$) d'un principe de comparaison permettant d'appliquer un argument classique due à Kransnoselskii [60].

Dans les exemples 2 à 5 sont également étudiés des systèmes quasi-linéaires présentant des termes négatifs dans les seconds membres des équations de (P). Nous nous sommes particulièrement intéressés aux systèmes de la forme

$$(\mathrm{P}) \begin{cases} -\Delta_p u = \lambda_1 u^{\alpha_1} - u^{\beta_1} - \mu_1 u^{a_1} v^{b_1} \quad \text{dans } \Omega\,; \quad u|_{\partial\Omega} = 0, \quad u > 0 \quad \text{dans } \Omega, \\ \\ -\Delta_q v = \lambda_2 v^{\alpha_2} - v^{\beta_2} - \mu_2 v^{a_2} u^{b_2} \quad \text{dans } \Omega\,; \quad v|_{\partial\Omega} = 0, \quad v > 0 \quad \text{dans } \Omega, \end{cases}$$

et avons établi des résultats d'existence similaires à ceux énoncés pour l'exemple précédent (voir Exemples 3-5, p. 87-93).

Résultats principaux du Chapitre III : Dans ce dernier chapitre, nous nous sommes intéressés au problème parabolique, quasi-linéaire et singulier suivant :

$$(\mathrm{P})\begin{cases} \partial_t u - \Delta_p u + \mathbb{1}_{\{u>0\}}u^{-\beta} &= f(x,u) \quad \text{dans } Q, \\ u &= 0 \quad \text{sur } \Gamma, \\ u(0,\cdot) &= u_0 \quad \text{dans } \Omega. \end{cases}$$

Dans cette étude, $T > 0$, $Q \stackrel{\text{def}}{=}]0,T[\times\Omega$ est le domaine sur lequel est posé le problème et $\Gamma \stackrel{\text{def}}{=} (0,T)\times\partial\Omega$ la frontière de Q. L'exposant β vérifie $0 < \beta < 1$ et la donnée initiale u_0, la condition suivante :

$$u_0 \in \mathrm{W}_0^{1,p}(\Omega) \cap \mathrm{L}^\infty(\Omega) \quad \text{et} \quad u_0 \geq 0 \quad \text{p.p. dans } \Omega. \tag{38}$$

Dans le membre de droite de la première équation de (P), f vérifie les conditions suivantes :

1. $f : \Omega \times [0,+\infty) \to [0,+\infty)$ est une fonction de Carathéodory, localement lipschitzienne par rapport à la seconde variable telle que

$$\forall \text{ p.p.} x \in \Omega, \qquad f(x,0) = 0.$$

2. f possède le comportement asymptotique suivant : il existe $q > 0$ et deux constantes $\alpha \geq 0$ et $C_\alpha \geq 0$ telles que

$$\forall \text{ p.p} \, x \in \Omega, \ \forall w \in [0,+\infty), \qquad f(x,w) \leq \alpha w^q + C_\alpha. \tag{39}$$

On dénotera $\mathbb{1}_{\{u>0\}}$, la fonction caractéristique de l'ensemble $\{(t,x) \in Q \mid u(t,x) > 0\}$ et on fera l'hypothèse naturelle que $\mathbb{1}_{\{u>0\}}u^{-\beta} = 0$ dès que $u = 0$. Ce problème a été, à notre connaissance, peu étudié dans le cadre du p-Laplacien. Le cas de l'équation de la chaleur est présenté dans Dávila-Montenegro [21], où on pourra consulter une bibliographie assez fournie sur les problèmes d'extinction en temps fini.

Définition 1 *Définissons tout d'abord l'espace dans lequel on cherchera les solutions de* (P) *:*

$$\mathscr{U} \stackrel{\text{def}}{=} \left\{ v \in \mathrm{L}^\infty\left(0,T; \mathrm{W}_0^{1,p}(\Omega) \cap \mathrm{L}^\infty(\Omega)\right) \quad \Big| \quad \partial_t v \in \mathrm{L}^2(Q) \right\}.$$

Définition 2 *Une fonction $u \in \mathscr{U}$ sera appelée* ***solution faible du problème* (P)** *si :*

1. $u \geq 0$ p.p. *dans Q.*
2. *Pour toute fonction test $\varphi \in \mathscr{D}(Q)$,*

$$\int_Q \partial_t u \varphi \, dz + \int_Q |\nabla u|^{p-2}\nabla u.\nabla\varphi \, dz + \int_Q \mathbb{1}_{\{u>0\}}u^{-\beta}\varphi \, dz = \int_Q f(x,u)\varphi \, dz, \tag{40}$$

où $dz \stackrel{\text{def}}{=} dx\,ds$.

3. *$u_\varepsilon(0,\cdot) = u_0$ p.p dans Ω.*

Remarque 1 *Le point 3. de cette définition a bien un sens car on peut montrer que $\mathscr{U} \hookrightarrow \mathscr{C}\left([0,T], \mathrm{L}^2(\Omega)\right)$.*

Aux vues du caractère singulier du problème (P), les questions naturelles sur lesquelles nous nous sommes penchées sont les suivantes :

1. Existe-t-il une solution faible d'un tel problème ?
2. Que peut-on dire du comportement asymptotique des éventuelles solutions ?
3. Que peut-on dire sur la régularité des éventuelles solutions ?

Les deux théorèmes suivants donnent quelques éléments de réponse à ces interrogations. En particulier, lorsque le second membre f de l'équation principale de (P) est nul, le Théorème 8 établit l'extinction en temps fini de la solution sur tout le domaine Ω.

Théorème 7 *Supposons que u_0 satisfasse la condition* (38)*. Alors, il existe $T^* > 0$ tel que pour tout $T < T^*$, le problème* (P) *admet au moins une solutions faible $u \in \mathscr{U}$. Cette solution vérifie l'identité d'énergie suivante : $\forall t \in [0, T]$,*

$$\frac{1}{2}\|u(t)\|^2_{\mathrm{L}^2(\Omega)} - \frac{1}{2}\|u_0\|^2_{\mathrm{L}^2(\Omega)} + \int_0^t \int_\Omega |\nabla u|^p \, dz + \int_0^t \int_\Omega u^{1-\beta} \, dz = \int_0^t \int_\Omega f(x,u) u \, dz. \tag{41}$$

De plus, lorsque $q \leq 1$, on a $T^* = +\infty$ et u est une solution globale.

Théorème 8 *Supposons que u_0 satisfasse la condition* (38)*. Supposons également que $f \equiv 0$ dans $\Omega \times [0, +\infty[$ et que $p \geq \frac{2d}{d+2}$. Alors, pour tout $T > 0$,* (P) *admet une solution faible $u \in \mathscr{U}$. De plus, il existe un instant critique $t_* > 0$ dépendant uniquement des paramètres p, d, Ω, $\|u\|_{\mathrm{L}^\infty(Q)}$ et $\|u_0\|_{\mathrm{L}^2(\Omega)}$ tel que*

$$\forall t > t_*, \qquad \|u(t)\|_{\mathrm{L}^2(\Omega)} = 0,$$

avec la notation naturelle $u(t) \overset{\text{def}}{=} u(t, \cdot)$ p.p. *dans Ω.*

La démonstration du Théorème 7 s'appuie sur l'étude d'un problème régularisé (P_ε), avec $\varepsilon > 0$, où on approxime le terme singulier $\mathbb{1}_{\{u>0\}} u^{-\beta}$ par

$$g_\varepsilon(w) = \begin{cases} 0 & \text{si } w = 0, \\ \varepsilon^{-\beta} & \text{si } w \in (0, \varepsilon), \\ w^{-\beta} & \text{si } w \geq \varepsilon. \end{cases} \tag{42}$$

On obtient alors le résultat suivant :

Théorème 9 *Supposons que u_0 satisfasse la condition* (38)*. Alors, il existe $T^* > 0$ tel que pour tout $T < T^*$, le problème* (P_ε) *admet au moins une solutions faible $u_\varepsilon \in \mathscr{U}$ (au sens de la Définition 2). Cette solution vérifie l'identité d'énergie suivante : $\forall t \in [0, T]$,*

$$\frac{1}{2}\|u_\varepsilon(t)\|^2_{\mathrm{L}^2(\Omega)} - \frac{1}{2}\|u_0\|^2_{\mathrm{L}^2(\Omega)} + \int_0^t \int_\Omega |\nabla u_\varepsilon|^p \, dz + \int_0^t \int_\Omega g_\varepsilon(u_\varepsilon) u_\varepsilon \, dz = \int_0^t \int_\Omega f(x, u_\varepsilon) u_\varepsilon \, dz, \tag{43}$$

De plus, lorsque $q \leq 1$, on a $T^* = +\infty$ et u est une solution globale.

Le problème (P_ε) représente la situation limite d'un nouveau problème régularisé ($\mathrm{P}_{\varepsilon,\eta}$), avec $\varepsilon > \eta > 0$, où le terme g_ε est approché par

$$g_{\varepsilon,\eta}(w) = \begin{cases} 2\varepsilon^{-\beta}\eta^{-1}w & \text{si } w \in [0,\eta)\,, \\ \varepsilon^{-\beta} & \text{si } w \in [\eta,\varepsilon)\,, \\ w^{-\beta} & \text{si } w \geq \varepsilon. \end{cases} \tag{44}$$

Une solution de (P) est alors démontrée par passage à la limite en $\eta \to 0$ dans le problème ($\mathrm{P}_{\varepsilon,\eta}$), puis en $\varepsilon \to 0$ dans le problème (P_ε). Ce dernier passage à la limite est rendu possible par la monotonie en ε de g_ε sur $]0,+\infty[$. En effet, la présence du terme singulier $\mathbb{1}_{\{u>0\}}u^{-\beta}$, ne permet pas directement d'obtenir une régularité $\mathrm{L}^{p'}\left(0,T;\mathrm{W}^{-1,p'}(\Omega)\right)$ pour le terme $\partial_t u$ et de fait, de pouvoir considérer des solutions faibles de (P) au sens de la Définition 4. La preuve du Théorème 8, quant à elle, repose sur l'estimation d'énergie (41) du Théorème 7 et sur l'inégalité de Gagliardo-Nirenberg. Une inégalité différentielle découle alors pour $\|u(t)\|_{\mathrm{L}^2(\Omega)}$, où $u \in \mathscr{U}$ est une solution faible de (P).

Comme nous l'avons signalé précédemment, pour démontrer le Théorème 7, nous avons commencé par faire l'étude détaillée d'un problème régularisé. Plus précisément pour $\varepsilon > \eta > 0$, nous nous sommes tout d'abord intéressés au problème suivant :

$$(\mathrm{P}_{\varepsilon,\eta}) \begin{cases} \partial_t u_{\varepsilon,\eta} - \Delta_p u_{\varepsilon,\eta} &= h_{\varepsilon\eta}(x,u_{\varepsilon,\eta}) \quad \text{dans } Q, \\ u_{\varepsilon,\eta} &= 0 \quad \text{sur } \Gamma, \\ u_{\varepsilon,\eta}(0,\cdot) &= u_0 \quad \text{dans } \Omega, \end{cases}$$

où $h_{\varepsilon,\eta} : \Omega \times [0,+\infty) \to \mathbb{R}$ est définie par $h_{\varepsilon,\eta}(x,w) \overset{\text{def}}{=} f(x,w) - g_{\varepsilon,\eta}(w)$, avec $g_{\varepsilon,\eta}$ l'approximation du terme singulier défini dans (44). Pour le problème ($\mathrm{P}_{\varepsilon,\eta}$), nous considérons une donnée initiale plus générale ; à savoir,

$$u_0 \in \mathrm{L}^\infty(\Omega) \quad \text{et} \quad u_0 \geq 0 \quad \text{p.p. dans } \Omega. \tag{45}$$

Les solutions faibles obtenues pour le problème (P_ε) sont définies comme suit :

Définition 3 *Définissons l'espace dans lequel on cherchera les solutions de* (P_ε) *:*

$$\mathscr{V} \overset{\text{def}}{=} \left\{ v \in \mathrm{L}^p\left(0,T;\mathrm{W}_0^{1,p}(\Omega)\right) \cap \mathrm{L}^\infty(Q) \;\middle|\; \partial_t v \in \mathrm{L}^{p'}\left(0,T;\mathrm{W}^{-1,p'}(\Omega)\right) \right\}.$$

Définition 4 *Une fonction $u_{\varepsilon,\eta} \in \mathscr{V}$ sera appelée* ***solution faible du problème* (P_ε)** *si :*

1. $u_{\varepsilon,\eta} \geq 0$ p.p. *dans* Q.
2. *Pour toute fonction test* $\varphi \in \mathrm{L}^p\left(0,T;\mathrm{W}_0^{1,p}(\Omega)\right)$,

$$\int_0^T \langle \partial_t u_{\varepsilon,\eta}(s), \varphi(s)\rangle\, ds + \int_Q |\nabla u_{\varepsilon,\eta}|^{p-2}\nabla u_\varepsilon.\nabla\varphi\, dz + = \int_Q h_{\varepsilon,\eta}(x,u_{\varepsilon,\eta})\varphi\, dz, \tag{46}$$

où $\langle\cdot,\cdot\rangle$ *désigne le produit de dualité entre* $\mathrm{W}^{-1,p'}(\Omega)$ *et* $\mathrm{W}_0^{1,p}(\Omega)$.

3. $u_{\varepsilon,\eta}(0,.) = u_0$ *p.p dans* Ω.

Remarque 2 *On a bien l'injection* $\mathscr{V} \hookrightarrow \mathscr{C}\left([0,T],\mathrm{L}^2(\Omega)\right)$. *Ainsi, le point 3. de la définition précédente a bien un sens.*

Nous obtenons pour le problème $(\mathrm{P}_{\varepsilon,\eta})$:

Théorème 10 *Supposons que* u_0 *satisfasse la condition* (45). *Alors, il existe* $T^* > 0$ *tel que pour tout* $T < T^*$ *et pour tout* $\varepsilon > \eta > 0$, *le problème* $(\mathrm{P}_{\varepsilon,\eta})$ *admet une unique solution faible* $u_\varepsilon \in \mathscr{V}$. *De plus,* $u_{\varepsilon,\eta} \in \mathscr{C}\Big((0,T],\mathrm{W}_0^{1,p}(\Omega)\Big)$, $\sqrt{t}\partial_t u_{\varepsilon,\eta} \in \mathrm{L}^2(Q)$ *et vérifient l'identité d'énergie suivante :* $\forall t \in [0,T]$,

$$\int_0^t \int_\Omega s\left(\partial_t u_{\varepsilon,\eta}\right)^2 \, dz + \frac{t}{p}\int_\Omega |\nabla u_{\varepsilon,\eta}(t)|^p \, dx - \frac{1}{p}\int_0^t\int_\Omega |\nabla u_{\varepsilon,\eta}|^p \, dz$$
$$= t\int_\Omega H_{\varepsilon,\eta}\left(x, u_{\varepsilon,\eta}(t)\right) \, dx - \int_0^t \int_\Omega H_{\varepsilon,\eta}(x, u_{\varepsilon,\eta}) \, dz, \quad (47)$$

avec $\forall$ p.p.$x \in \Omega$, $\forall w \in [0,+\infty)$,

$$H_{\varepsilon,\eta}(x,w) \overset{\text{def}}{=} \int_0^w h_{\varepsilon,\eta}(x,v)\, dv.$$

Enfin, lorsque $q \leq 1$, *on a* $T^* = +\infty$ *et* u_ε *est une solution globale.*

Dans le cas où la donnée initiale $u_0 \in \mathrm{W}_0^{1,p}(\Omega) \cap \mathrm{L}^\infty(\Omega)$, nous établissons le résultat plus fort suivant :

Théorème 11 *Supposons que* u_0 *satisfasse la condition* (38). *Alors il existe* $T^* > 0$ *tel que pour tout* $T < T^*$ *et pour tout* $\varepsilon > \eta > 0$, *le problème* $(\mathrm{P}_{\varepsilon,\eta})$ *admet une unique solution faible* $u_{\varepsilon,\eta} \in \mathscr{V}$. *De plus,* $u_{\varepsilon,\eta} \in \mathscr{C}\Big([0,T],\mathrm{W}_0^{1,p}(\Omega)\Big)$, $\partial_t u_{\varepsilon,\eta} \in \mathrm{L}^2(Q)$ *et vérifient l'identité d'énergie suivante :* $\forall t \in [0,T]$,

$$\int_0^t \int_\Omega \left(\partial_t u_{\varepsilon,\eta}\right)^2 \, dz + \frac{1}{p}\int_\Omega |\nabla u_{\varepsilon,\eta}(t)|^p \, dz - \frac{1}{p}\int_\Omega |\nabla u_0|^p \, dx$$
$$= \int_\Omega H_{\varepsilon,\eta}\left(x, u_{\varepsilon,\eta}(t)\right) \, dx - \int_\Omega H_{\varepsilon,\eta}\left(x, u_0\right) \, dx. \quad (48)$$

Enfin, lorsque $q \leq 1$, *on a* $T^* = +\infty$ *et* u_ε *est une solution globale.*

Les démonstrations de ces deux théorèmes sont basées sur une méthode de semi-discrétisation en temps (non-standard pour le Théorème 10), permettant d'obtenir des estimations *a priori* des solutions approchées dans des espaces de Sobolev à valeurs vectorielles appropriés ; ainsi que sur la construction d'une borne dans $\mathrm{L}^\infty(Q)$ des solutions approchées, obtenue grâce à l'existence d'une fonction barrière supérieure pour le problème (P) indépendante de η et de ε. A partir du Théorème 11, par passage à la limite quand $\eta \to 0^+$, on démontre le Théorème 9 ; puis par passage à la limite quand $\varepsilon \to 0^+$, on démontre le Théorème 7 sur l'existence locale d'une solution faible.

N.B. La lecture des Chapitres I, II et III peut se faire de façon indépendante. De ce fait, les mêmes notations, notions ou résultats pourront occasionnellement apparaitre à plusieurs endroits dans ce manuscrit.

Chapitre I

Study of a quasilinear and singular absorption elliptic problem

Nous présentons ici les résultats issus de Giacomoni-Mâagli-Sauvy [46], travail réalisé en collaboration avec Jacques Giacomoni et Habib Mâagli, Professeur à l'Unversité de Tunis.

1 Introduction

Let Ω be a $\mathscr{C}^2$ bounded domain of $\mathbb{R}^N$, $N \geq 2$. We discuss the existence of weak solutions in $\mathrm{W}_0^{1,p}(\Omega) \cap \mathrm{L}^\infty(\Omega)$ to

$$(\mathrm{P}_\lambda)\begin{cases} -\Delta_p u = \mathbb{1}_{\{u>0\}} K(x)(\lambda u^q - u^r) & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \quad u \geq 0 & \text{in } \Omega. \end{cases}$$

$u \in \mathrm{W}_0^{1,p}(\Omega) \cap \mathrm{L}^\infty(\Omega)$ is a weak solution to (P_λ) if for all test functions $\varphi \in \mathcal{D}(\Omega)$,

$$\int_\Omega |\nabla u|^{p-2} \nabla u . \nabla \varphi \, dx = \int_\Omega \mathbb{1}_{\{u>0\}} K(x) \left(\lambda u^q - u^r\right) \varphi \, dx. \tag{I.1}$$

In the equation in (P_λ), $\lambda > 0$ is a positive parameter, $-1 < r < q < p-1$ and $K \in \mathscr{C}(\Omega)$ is a positive function having a singular behaviour near the boundary $\partial\Omega$. Precisely, $K(x) = d(x)^{-k} L(d(x))$ in Ω, with $0 < k < p$ and $L \in \mathscr{C}^2((0, D])$ a positive function, with $D \stackrel{\text{def}}{=} \operatorname{diam}(\Omega)$, defined as follows :

$$L(t) = \exp\left(\int_t^D \frac{z(s)}{s} ds\right), \tag{I.2}$$

with $z \in \mathscr{C}([0, D]) \cap \mathscr{C}^1((0, D])$ and $z(0) = 0$. Let us note that (I.2) implies that

$$\lim_{t \to 0^+} \frac{t L'(t)}{L(t)} = 0 \tag{I.3}$$

and for all $\varepsilon > 0$,

$$\lim_{t \to 0^+} t^\varepsilon L(t) = 0 \tag{I.4}$$

and

$$\lim_{t\to 0^+} t^{-\varepsilon}L(t) = +\infty. \tag{I.5}$$

The above asymptotics of L force

$$\forall\xi > 0\,, \quad \lim_{t\to 0^+} \frac{L(\xi t)}{L(t)} = 1.$$

Then, L belongs to the Karamata class (see KARAMATA [58]). Setting $\mathcal{K}$ the class of functions satisfying (I.2), we get the following properties : if $L_1, L_2 \in \mathcal{K}$ and if $p \in \mathbb{R}$, then $L_1.L_2 \in \mathcal{K}$ and ${L_1}^p \in \mathcal{K}$.

Example 1.1 *Let $m \in \mathbb{N}^*$ and $A >> 0$ large enough. Let us define*

$$L(t) = \prod_{n=1}^{m} \left(\log_n\left(\frac{A}{t}\right)\right)^{\mu_n}, \quad t \in (0, D]$$

with $\log_n \stackrel{\text{def}}{=} \log\circ\cdots\circ\log$ (n times) and $\mu_n > 0$. Then $L \in \mathcal{K}$.

In this chapter, we investigate first the following issues for the problem (P_λ) : existence of non-trivial weak solutions according to $\lambda > 0$, Hölder regularity of weak solutions. Next, we study further the properties of non-trivial solutions. Since the non-linearity in the right-hand side is a singular absorption term near the boundary, a non-trivial weak solution may not be positive everywhere in Ω and compact support (non-trivial) weak solutions or compactons (solutions with zero normal derivative at the boundary) may exist for stronger singularities, that is for large $k > 0$ whereas for small $k > 0$ any non-trivial weak solution is positive. Then, the natural question is to determine the borderline condition on the parameter k, which gives the strength of the singular potential K, between existence of positive weak solutions and existence of free boundary weak solutions. The existence of compact support solutions is important in the applications, in particular in biology models (population dynamics and epidemiology models for instance) and was investigated quite intensely for non-linear reaction diffusion equations with absorption in the last decades. In particular, concerning the case where the equation involves a quasilinear and degenerate operator, we can refer to the result in VÁZQUEZ [83] where under a suitable condition about the behaviour of the non-linearity near the origin, a strong maximum principle is proved and thereby the positivity of solutions. The given condition is sharp in the sense that for different situations where this condition is not satisfied, the existence of free boundary solutions is shown. In ALVAREZ-DÍAZ [2] (see also DÍAZ [26] for related results on the subject), the authors consider a class of non-homogeneous reaction-diffusion equations with strong absorption and study the behaviour of the solution near the free boundary. In particular, a non degeneracy property (the solution grows faster than some function of the distance to the free boundary) is obtained when the growth of the reaction term near the boundary satisfies some estimate by below. In IL'YASOV-EGOROV [57], the authors consider a semi-linear equation with a similar (and non singular) conflicting non-linearity as in the equation in (P_λ) and the existence of compactons is proved using the fibering method. An interesting feature of this result is that the Hopf lemma is violated for such kind of equations. In the present work, we consider, further the case where the equation involves a p-Laplace operator and a singular potential in the right-hand side and show that a more complex situation occurs in respect to the non singular case.

In the next section, we give the main results proved in this chapter. These results extend a previous work due to HAITAO [54] in the semi-linear case ($p = 2$) and which involves a smaller class of non-linearities.

2 Main results

The main results of this chapter concerning the problem (P_λ) are stated below :

Theorem 2.1 *When $k < 1 + r$, there exists a constant $\Lambda_1 > 0$ such that :*

1. *For $\lambda > \Lambda_1$, (P_λ) admits a positive weak solution.*
2. *Any weak solution of (P_λ) is $\mathscr{C}^{1,\beta}(\overline{\Omega})$, for some $0 < \beta < 1$.*
3. *For $\lambda < \Lambda_1$, (P_λ) has no positive solution.*

Theorem 2.2 *Let $r > 0$ and one of the two following conditions be satisfied :*

$$1 + r > q \quad \text{and} \quad k \in \left[1 + r, 1 + \frac{(p-1)(r+1)}{p-q+r}\right), \tag{I.6}$$

$$1 + r \geq q \quad \text{and} \quad k \in [1 + r, 2 + r). \tag{I.7}$$

Then, there exists $\Lambda_2 > 0$ such that :

1. *For $\lambda > \Lambda_2$, (P_λ) has a compact support weak solution u_λ.*
2. *Any weak solution of (P_λ) is $\mathscr{C}^{1,\beta}(\overline{\Omega})$, for some $0 < \beta < 1$.*
3. *For $\lambda < \Lambda_2$, (P_λ) has no non-trivial solution.*

The outline of this Chapter is as follows. Before giving the proofs of those theorems, we establish some useful preliminary results in the next section. The proof of Theorem 2.1 is given in Section 4 and the proof of Theorem 2.2 is given in Section 5. The technical results stated in Section 3 are finally proved in Sections 1 and 2 of Appendix B. The related regularity results are a consequence of the general regularity results stated in Appendix A.

Le us recall the main notations we will use throughout this chapter :

1. To $p > 1$ we associate $p' \stackrel{\text{def}}{=} \frac{p}{p-1}$.
2. For $x \in \Omega$, $d(x) \stackrel{\text{def}}{=} \mathrm{dist}(x, \Omega) = \inf_{y \in \Omega} d(x, y)$.
3. $D \stackrel{\text{def}}{=} \mathrm{diam}(\Omega) = \sup_{x,y \in \Omega} d(x, y)$.
4. Let ω be a non-empty set of Ω and $f, g : \omega \longrightarrow [0, +\infty]$. We write

$$f(x) \sim g(x) \quad \text{in } \omega$$

if there exist two positive constants C_1 and C_2 such that

$$\forall x \in \omega, \quad C_1 f(x) \leq g(x) \leq C_2 f(x).$$

5. Let $\omega \subset \mathbb{R}^N$, $\mathcal{L}^N(\omega)$ denotes the $N-$dimensional Lebegue's mesure of ω.
6. Let $\varepsilon > 0$, we define $\Omega_\varepsilon \overset{\text{def}}{=} \{x \in \Omega \mid d(x) < \varepsilon\}$.
7. $\nu : \partial\Omega \longrightarrow \mathbb{R}^N$ denotes the outward normal associated to Ω.
8. For $v \in \mathrm{W}_0^{1,p}(\Omega)$, we write $\|v\|_{\mathrm{W}_0^{1,p}(\Omega)} \overset{\text{def}}{=} \|\nabla v\|_{\mathrm{L}^p(\Omega)} = \left(\int_\Omega |\nabla v|^p dx\right)^{\frac{1}{p}}$.
9. The function $\varphi_1 \in \mathrm{W}_0^{1,p}(\Omega)$ denotes the positive and L^p-renormalized ($\|\varphi_1\|_{\mathrm{L}^p(\Omega)} = 1$) eigenfunction corresponding to the first eigenvalue of $-\Delta_p$,

$$\lambda_1 \overset{\text{def}}{=} \inf \left\{ \int_\Omega |\nabla v|^p dx \;\middle|\; v \in \mathrm{W}_0^{1,p}(\Omega),\ \int_\Omega |v|^p dx = 1 \right\}.$$

It is a weak solution of the following eigenvalue problem :

$$\begin{cases} -\Delta_p u = \lambda_1 u^{p-1} \quad \text{in } \Omega, \\ \quad u = 0 \quad \text{on } \partial\Omega, \quad u \geq 0 \quad \text{in } \Omega. \end{cases}$$

Using Moser iterations and the regularity result in Lieberman [64], $\varphi_1 \in \mathscr{C}^{1,\alpha}(\overline{\Omega})$, for some $0 < \alpha < 1$. Moreover the strong maximum and boundary principles from Vázquez [83], guarantee that φ_1 satisfies those two properties :

(a) There exist two positive constants K_1 and K_2 only depending on p, Ω and on the dimension N such that :

$$\forall x \in \Omega, \qquad K_1 d(x) \leq \varphi_1(x) \leq K_2 d(x). \tag{I.8}$$

(b) There exist $\varepsilon^* > 0$ and $\delta^* > 0$ only depending on p, Ω and on the dimension N such that :

$$\forall x \in \Omega_{\delta^*}, \qquad |\nabla \varphi_1(x)| > \varepsilon^*. \tag{I.9}$$

3 Preliminary results

3.1 A non-existence lemma

Lemma 3.1 *When $k < 1 + r$, there exists $\lambda_* > 0$ such that* (P_λ) *has no non-trivial solution for $\lambda \leq \lambda_*$.*

Proof. Let us define

$$\lambda_{1,K} \overset{\text{def}}{=} \inf_{\substack{v \in \mathrm{W}_0^{1,p}(\Omega) \\ v \neq 0}} \frac{\displaystyle\int_\Omega |\nabla v|^p dx}{\displaystyle\int_\Omega K(x)|v|^p dx}.$$

From the Hardy's inequality, there exists a constant $C > 0$ only depending on Ω and p such that for all $v \in \mathrm{W}_0^{1,p}(\Omega)$,

$$\int_\Omega \frac{|v|^p}{d(x)^p} dx \leq C \int_\Omega |\nabla v|^p dx.$$

Since $k < p$, $\lambda_{1,K} > 0$. Let $u \in \mathrm{W}_0^{1,p}(\Omega) \cap \mathrm{L}^\infty(\Omega)$ be a non-trivial solution of (P_λ), then from (I.1) taking $u \in \mathrm{W}_0^{1,p}(\Omega)$ as a test function we get,

$$0 < \lambda_{1,K} \int_\Omega K(x) u^p dx \leq \int_\Omega |\nabla u|^p dx = \int_\Omega K(x) \left(\lambda u^{q+1} - u^{r+1} \right) dx. \tag{I.10}$$

This inequality can not hold for $\lambda \leq \lambda_* \stackrel{\text{def}}{=} \min\{1, \lambda_{1,K}\}$. Indeed,

1. if $u(x) \leq 1$, $\lambda u^{q+1} - u^{r+1} \leq 0$ as soon as $\lambda \leq 1$,
2. if $u(x) > 1$, $K(x) \left(\lambda u^{q+1} - u^{r+1} \right) < \lambda_{1,K} K(x) u^{q+1}$ as soon as $\lambda \leq \lambda_{1,K}$. Therefore, either $\mathcal{L}^N \left(\{x \in \Omega \mid u(x) > 1\} \right) = 0$ and we get

$$0 < \lambda_{1,K} \int_\Omega K(x) u^p dx \leq 0,$$

or

$$\lambda_{1,K} \int_\Omega \mathbb{1}_{\{u>1\}} K(x) u^p dx \leq \lambda_{1,K} \int_\Omega \mathbb{1}_{\{u>1\}} K(x) u^{q+1} dx,$$

which contradicts $q < p - 1$. □

3.2 Construction of a sub-solution for (P_λ)

Lemma 3.2 *When $k < 1 + r$, there exist $M > 0$, $\lambda^* > 0$ and $\tau > 1$ such that $\underline{u}_\lambda \stackrel{\text{def}}{=} M {\varphi_1}^\tau$ is a sub-solution of* (P_λ) *in Ω, provided that $\lambda \geq \lambda^*$.*

Proof. Let $M > 0$ and $\tau > 1$, then we define $\underline{u}_\lambda = M {\varphi_1}^\tau$ in Ω. A straightforward computation yields

$$-\Delta_p \underline{u}_\lambda = -\left(M\tau\right)^{p-1} \left[(p-1)(\tau-1) |\nabla \varphi_1|^p {\varphi_1}^{(\tau-1)(p-1)-1} - \lambda_1 {\varphi_1}^{\tau(p-1)} \right]$$

and

$$K(x) \left(\lambda {\underline{u}_\lambda}^q - {\underline{u}_\lambda}^r \right) = -K(x) \left(M^r {\varphi_1}^{\tau r} - \lambda M^q {\varphi_1}^{\tau q} \right).$$

By properties (I.8) and (I.9) of the function φ_1, if we let

$$\delta_0 \stackrel{\text{def}}{=} \min \left\{ \delta^*, \frac{\varepsilon^*}{K_2} \left(\frac{(\tau-1)(p-1)}{2\lambda_1} \right)^{\frac{1}{p}}, \frac{1}{K_2} \left(\frac{1}{2\lambda M^{q-r}} \right)^{\frac{1}{\tau(q-r)}} \right\},$$

both of the above expressions are negative on Ω_{δ_0}. Moreover,

$$\Delta_p \underline{u}_\lambda(x) \sim (M\tau)^{p-1} (\tau-1) d(x)^{(\tau-1)(p-1)-1} \quad \text{in } \Omega_{\delta_0}$$

and

$$K(x)\left(\underline{u}_\lambda{}^r - \lambda \underline{u}_\lambda{}^q\right) \sim M^r L(d(x)) d(x)^{\tau r - k} \quad \text{in } \Omega_{\delta_0}.$$

Since $k < 1 + r$, we can choose a constant $\tau > 1$ satisfying $(\tau - 1)(p - 1) - 1 < \tau r - k$. Hence, for $M > 0$ large enough we get

$$-\Delta_p \underline{u}_\lambda \leq K(x)\left(\lambda \underline{u}_\lambda^q - \underline{u}_\lambda^r\right) \quad \text{in } \Omega_{\delta_0}.$$

In $\Omega \setminus \Omega_{\delta_0}$, K, φ_1 and $|\nabla \varphi_1|$ are bounded, therefore there exists $\lambda^* > 0$ such that for $\lambda \geq \lambda^*$,

$$-\Delta_p \underline{u}_\lambda \leq K(x)\left(\lambda \underline{u}_\lambda^q - \underline{u}_\lambda^r\right) \quad \text{in } \Omega \setminus \Omega_{\delta_0}.$$

Thus, $\underline{u}_\lambda$ is a sub-solution of (P_λ) in Ω for M large enough and $\lambda \geq \lambda^*$. □

3.3 Construction of a super-solution for $(\mathbf{P}_\lambda)$

We consider the following problem :

$$(\mathrm{Q}) \begin{cases} -\Delta_p v = K(x) v^q & \text{in } \Omega, \\ \quad v = 0 \quad \text{on } \partial\Omega, \quad v > 0 & \text{in } \Omega, \end{cases}$$

with q, p and K satisfying the above assumptions.

Proposition 3.1

1. *If $k \in (0, 1+q)$, problem* (Q) *has a unique solution $v \in \mathscr{C}^{1,\alpha}(\overline{\Omega})$, for some $0 < \alpha < 1$, satisfying*

$$v(x) \sim d(x) \quad \text{in } \Omega.$$

2. *If $k = 1 + q$, problem* (Q) *has a unique solution $v \in \mathrm{W}_0^{1,p}(\Omega) \cap \mathscr{C}^{0,\alpha}(\overline{\Omega})$, for some $0 < \alpha < 1$, satisfying*

$$v(x) \sim d(x) \left(\int_{d(x)}^{D} L(t) t^{-1}\, dt \right)^{\frac{1}{p-k}} \quad \text{in } \Omega.$$

3. *If $k \in \left(1+q, 1+q+\frac{p-(1+q)}{p}\right)$, problem* (Q) *has a unique solution $v \in \mathrm{W}_0^{1,p}(\Omega) \cap \mathscr{C}^{0,\alpha}(\overline{\Omega})$, for some $0 < \alpha < 1$, satisfying*

$$v(x) \sim d(x)^{\frac{p-k}{p-(1+q)}} \Big(L\left(d(x)\right) \Big)^{\frac{1}{p-(1+q)}} \quad \text{in } \Omega.$$

4. *If $k \in \left[1+q+\frac{p-(1+q)}{p}, p\right)$,* (Q) *has a unique solution $v \in \mathrm{W}_{\mathrm{loc}}^{1,p}(\Omega) \cap \mathscr{C}_0(\overline{\Omega})$ satisfying*

$$v(x) \sim d(x)^{\frac{p-k}{p-(1+q)}} \Big(L\left(d(x)\right) \Big)^{\frac{1}{p-(1+q)}} \quad \text{in } \Omega.$$

5. *If $k = p$ and if L satisfies the following condition :*

$$\int_0^D t^{-1} L(t)^{\frac{1}{p-1}} dt < +\infty, \tag{I.11}$$

problem (Q) *has a unique solution* $v \in W^{1,p}_{loc}(\Omega) \cap \mathscr{C}_0(\overline{\Omega})$ *satisfying*

$$v(x) \sim \left(\int_0^{d(x)} t^{-1} L(t)^{\frac{1}{p-1}}\, dt \right)^{\frac{p-1}{p-(1+q)}} \qquad in\ \Omega.$$

Proof. See Section 1 in Appendix B. □

From a solution of (Q), we can easily construct a super-solution of (P_λ). Indeed, let us consider $v \in W^{1,p}_{loc}(\Omega) \cap \mathscr{C}_0(\overline{\Omega})$ the solution of (Q) given by Proposition 3.1. Then, $\overline{u}_\lambda \stackrel{\text{def}}{=} \overline{M} v$ is a super-solution of (P_λ) in Ω as soon as $\overline{M} \geq \lambda^{\frac{1}{p-(1+q)}}$. Particularly when $k < 1+r$ and $\lambda \geq \lambda^*$, choosing $\overline{M}$ sufficiently large $\overline{u}_\lambda \in W^{1,p}_0(\Omega) \cap \mathscr{C}(\overline{\Omega})$ and is a super-solution of (P_λ) in Ω satisfying

$$\underline{u}_\lambda \leq \overline{u}_\lambda \quad \text{and} \quad \overline{u}_\lambda(x) \sim d(x) \quad \text{in } \overline{\Omega}.$$

Now let us state a non-existence result for the problem (Q).

Proposition 3.2 *Let* $\underline{v} \in W^{1,p}_0(\Omega) \cap \mathscr{C}(\overline{\Omega})$ *be a positive sub-solution of* (Q) *in* Ω *and assume that there exists* $\varepsilon > 0$ *such that*

$$\int_\Omega K(x) {\varphi_1}^{p-1+\varepsilon} dx = +\infty. \tag{I.12}$$

Then, for any $\eta > 0$, (Q) *has no weak solution* $v \in W^{1,p}_{loc}(\Omega) \cap \mathscr{C}_0(\overline{\Omega})$ *such that* $v \geq \eta \underline{v}$ *in* Ω.

Proof. See Section 2 in Appendix B. □

Corollary 3.1 *If* $k > p$, *there is no non-trivial weak solution of* (Q).

4 Proof of Theorem 2.1

4.1 Existence of a $\mathscr{C}^{1,\beta}$ positive solution when $\boldsymbol{\lambda \geq \lambda^*}$

Proposition 4.1 *When* $k < 1+r$, *provided* $\lambda \geq \lambda^*$, (P_λ) *has a weak solution* $u_\lambda \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$. *Furthermore,* $u_\lambda \in \mathscr{C}^{1,\beta}(\overline{\Omega})$, *for some* $0 < \beta < 1$.

Proof. In the equation of (P_λ), the expression $h_\lambda(x,v) \stackrel{\text{def}}{=} K(x)(\lambda v^q - v^r)$ involves a singular term $K(x)$ which blows up as $d(x) \to 0$, that prevents the direct application of the sub- and super-solution method. To overcome this difficulty, we apply this method in a sequence of sub-domains of Ω. Precisely, let us introduce $(\Omega_k)_{k \in \mathbb{N}^*} \subset \Omega$ an increasing sequence of smooth sub-domains of Ω such that $\Omega_k \xrightarrow[k \to +\infty]{} \Omega$ in the Hausdorff topology with

$$\forall k \in \mathbb{N}^*, \qquad \frac{1}{k+1} < \operatorname{dist}(\partial\Omega, \partial\Omega_k) < \frac{1}{k}.$$

Then, for all $k \in \mathbb{N}^*$ we consider the following problem :

$$(P_k) \begin{cases} -\Delta_p u_k = K(x)(\lambda {u_k}^q - {u_k}^r) & \text{in } \Omega_k, \\ u_k = \overline{u}_\lambda \quad \text{on } \partial\Omega_k, \quad u_k > 0 & \text{in } \Omega_k. \end{cases}$$

By definition of Ω_k, there exists $C_k > 0$ such that

$$\forall v \in I_k \overset{\text{def}}{=} \left[\min_{\Omega_k} \underline{u}_\lambda \, , \, \max_{\Omega_k} \overline{u}_\lambda\right], \quad \sup_{x\in\Omega_k} \left|\frac{\partial h_\lambda}{\partial v}(x,v)\right| \leq C_k.$$

As a consequence, there exists $\mu_k > 0$ such that for all $x \in \Omega_k$, the function $v \longmapsto h_\lambda(x,v) + \mu_k v^{p-1}$ is increasing on I_k. Therefore by sub- and super-solution method, (P_k) has a solution $u_k \in \mathrm{W}^{1,p}(\Omega_k)$. Indeed, we can construct the following iterative monotone scheme : for all $n \in \mathbb{N}^*$, let $u_{k,n} \in \mathrm{W}^{1,p}(\Omega_k)$ be the weak solution of

$$(\mathrm{P}_{k,n}) \begin{cases} -\Delta_p u_{k,n} + \mu_k \left(u_{k,n}\right)^{p-1} = h_\lambda(x, u_{k,n-1}) + \mu_k \left(u_{k,n-1}\right)^{p-1} & \text{in } \Omega_k, \\ u_{k,n} = \overline{u}_\lambda \quad \text{on } \partial\Omega_k, \quad u_{k,n} > 0 \quad \text{in } \Omega_k \end{cases}$$

with $u_{k,0} = \overline{u}_\lambda$ in Ω_k. By induction on $n \in \mathbb{N}$, $(\mathrm{P}_{k,n})$ has a unique solution $u_{k,n} \in \mathrm{W}^{1,p}(\Omega_k)$. From the weak comparison principle, $\left(u_{k,n}\right)_{n\in\mathbb{N}}$ satisfies

$$\underline{u}_\lambda \leq u_{k,n+1} \leq u_{k,n} \leq \overline{u}_\lambda \quad \text{in } \Omega_k.$$

Consequently, for all $n \in \mathbb{N}^*$,

$$\left|h_\lambda(x, u_{k,n-1}) + \mu_k \left(\left(u_{k,n-1}\right)^{p-1} - \left(u_{k,n}\right)^{p-1}\right)\right| \in \mathrm{L}^\infty(\Omega_k)$$

and since $\underline{u}_\lambda$ is smooth in Ω, we can state by a regularity result due to LIEBERMAN [64, Theorem 1] that $\left(u_{k,n}\right)_{n\in\mathbb{N}}$ is bounded in $\mathscr{C}^{1,\gamma}(\overline{\Omega_k})$, for some $0 < \gamma < 1$. Precisely, there exists a constant $C > 0$ only depending on γ, Ω_k, $\|\overline{u}_\lambda\|_{\mathrm{L}^\infty(\Omega_k)}$ and $\|\underline{u}_\lambda\|_{\mathrm{L}^\infty(\Omega_k)}$ such that $\|u_{k,n}\|_{\mathscr{C}^{1,\gamma}(\overline{\Omega_k})} \leq C$. From Ascoli-Arzelà theorem, there exist $u_k \in \mathscr{C}^1(\overline{\Omega_k})$ and a subsequence $\left(u_{k,m}\right)_{m\in\mathbb{N}}$ such that $u_{k,m} \xrightarrow[m\to+\infty]{} u_k$ in $\mathscr{C}^1(\overline{\Omega_k})$. Passing to the limit when $n \to +\infty$ in $(\mathrm{P}_{k,n})$, u_k is a weak solution of (P_k).

For all $k \in \mathbb{N}$, we define $\tilde{u}_k \overset{\text{def}}{=} \mathbb{1}_{\Omega_k}.u_k$ in order to extend u_k on Ω by zero. We prove that $(\tilde{u}_k)_{k\in\mathbb{N}}$ is a non-increasing sequence in Ω. Indeed, since $\Omega_k \subset \Omega_{k+1}$, if we compare u_{k+1} with every term of $\left(u_{k,n}\right)_{n\in\mathbb{N}}$ in Ω_k, using the weak comparison principle we get

$$\forall n \in \mathbb{N}, \quad \tilde{u}_{k+1} \leq u_{k,n} \quad \text{in } \Omega_k.$$

Passing to the limit in the above inequality, $\left(\tilde{u}_k(x)\right)_{k\in\mathbb{N}}$ is non-increasing for all $x \in \Omega$ and satisfies for any $k \in \mathbb{N}^*$:

$$\underline{u}_\lambda \leq \tilde{u}_k \leq \overline{u}_\lambda \quad \text{in } \Omega_k. \tag{I.13}$$

Therefore, there exists $u_\lambda \in \mathrm{L}^\infty(\Omega)$ such that $\tilde{u}_k \xrightarrow[k\to+\infty]{} u_\lambda$ a.e. in Ω and

$$\underline{u}_\lambda \leq u_\lambda \leq \overline{u}_\lambda \quad \text{in } \Omega. \tag{I.14}$$

Let $\varphi \in \mathscr{D}(\Omega)$. By definition of the sequence $(\Omega_k)_{k\in\mathbb{N}^*}$, there exists $k_0 \in \mathbb{N}^*$ such that for any $k \geq k_0$, $K \overset{\text{def}}{=} \operatorname{supp}_\Omega \varphi \subset \Omega_k$. Then for any $k \geq k_0$, $\tilde{u}_k \in \mathrm{W}^{1,p}(\Omega_{k_0}) \cap \mathrm{L}^\infty(\Omega_{k_0})$ and satisfies

$$-\Delta_p \tilde{u}_k = K(x)\left[\lambda\left(\tilde{u}_k\right)^q - \left(\tilde{u}_k\right)^r\right] \quad \text{ in } \Omega_{k_0}.$$

Since by estimate (I.13), $\tilde{u}_k$ is bounded in $\mathrm{L}^\infty(\Omega_{k_0})$, using the local regularity result of DiBenedetto [34, Theorem 2], $(\tilde{u}_k)_{k\geq k_0}$ is bounded in $\mathscr{C}^{1,\gamma}(K)$, for some $0<\gamma<1$, independently of k. Therefore, arguing similarly as above and passing to the limit as $k \to +\infty$, $u_\lambda \in \mathscr{C}_0(\overline{\Omega}) \cap \mathscr{C}^1(\Omega)$ and satisfies equation (I.1). Using inequality (I.14) and Hardy's inequality, $K(x)\left[\lambda(u_\lambda)^q - (u_\lambda)^r\right] \in \mathrm{W}^{-1,p'}(\Omega)$, from which it follows that $u_\lambda \in \mathrm{W}_0^{1,p}(\Omega)$. Finally applying Theorem 0.1 of Appendix A, we get the $\mathscr{C}^{1,\beta}(\overline{\Omega})$ regularity of u_λ. □

4.2 Existence of Λ_1

Let us define

$$\Lambda_1 \overset{\text{def}}{=} \inf\left\{\lambda > 0,\ (\mathrm{P}_\lambda) \text{ has a positive solution}\right\}.$$

By Lemma 3.1 and Proposition 4.1, $\lambda_* \leq \Lambda_1 \leq \lambda^* < +\infty$. By definition of Λ_1, for any $\lambda > \Lambda_1$ there exists $\mu \in (\Lambda_1, \lambda)$ such that (P_μ) has a positive solution $u_\mu \in \mathrm{W}_0^{1,p}(\Omega) \cap \mathrm{L}^\infty(\Omega)$. Moreover using Theorem 0.1 of Appendix A, $u_\mu \in \mathscr{C}^{1,\beta}(\overline{\Omega})$. Since u_μ is a sub-solution to (P_λ), we prove that $u_\mu \leq \overline{u}_\lambda$ in Ω. Indeed, $K(x) > 0$ in Ω, so there exists $\delta_0 > 0$ such that

$$-\Delta_p u_\mu \leq 0 \leq -\Delta_p\left(C_0\varphi_1\right) \quad \text{ in } \Omega_{\delta_0},$$

with $C_0 > 0$ large enough to satisfy

$$u_\mu \leq C_0\varphi_1 \quad \text{ on } \partial\Omega_{\delta_0}.$$

By the weak comparison principle, $u_\mu \leq C_0\varphi_1$ in Ω_{δ_0}. Moreover u_μ and φ_1 are bounded in $\Omega \setminus \Omega_{\delta_0}$, thus $u_\mu \leq C\varphi_1$ in Ω, for some constant $C > 0$. Therefore choosing $\overline{M}$ sufficiently large in the definition of $\overline{u}_\lambda$, we get $u_\mu \leq \overline{u}_\lambda$ in Ω. Finally, applying again sub- and super-solution technique as in Subsection 4.1, we get a solution $u_\lambda \in \mathscr{C}^{1,\beta}(\overline{\Omega})$ of (P_λ).

Proof. (of Theorem 2.1). The proof follows from Proposition 4.1 and Subsection 4.2 □

5 Proof of Theorem 2.2

5.1 Existence of a solution under condition (I.6) or (I.7).

Proposition 5.1 *Let $k \in \left[1+r, 1+q+\frac{p-(1+q)}{p}\right)$. Then, under condition*

$$\int_\Omega K(x)\left(\overline{u}_\lambda\right)^{r+1} dx < +\infty, \tag{I.15}$$

there exists $\lambda^{**} > 0$ *such that the problem* (P_λ) *has a non-trivial weak solution* $u_\lambda \in \mathrm{W}_0^{1,p}(\Omega) \cap \mathrm{L}^\infty(\Omega)$ *for* $\lambda > \lambda^{**}$.

Remark 5.1 *Since* $\overline{u}_\lambda \in \mathrm{W}_0^{1,p}(\Omega)$, *by Hardy's inequality* $\frac{\overline{u}_\lambda}{d(x)} \in \mathrm{L}^p(\Omega)$ *. So using Hölder's inequality, assumption* (I.15) *in Theorem 5.1 is satisfied if*

$$L(d(x))d(x)^{r+1-k} \in \mathrm{L}^{\alpha'}(\Omega),$$

where $\alpha = \frac{p}{r+1} > 1$. *And this last condition is satisfied if*

$$k < 1 + r + \frac{p-(1+r)}{p}. \tag{I.16}$$

So (I.16) *implies* (I.15) *; but this condition is not sharp and can be weakened by using the precise behaviour of* u_λ *given in Proposition 3.1. Indeed,*

1. *if* $k \in [1+r, 1+q)$, $\overline{u}_\lambda(x) \sim d(x)$ *in* Ω. *Therefore condition* (I.15) *is satisfied if*

$$k < 2 + r. \tag{I.17}$$

2. *if* $k = 1+q$, *condition* (I.15) *is also satisfied if* $k < 2+r$.
3. *if* $k \in \left(1+q, 1+q+\frac{p-(1+q)}{p}\right)$, *then*

$$\overline{u}_\lambda \sim d(x)^{\frac{p-k}{p-(1+q)}} \Big(L\left(d(x)\right)\Big)^{\frac{1}{p-(1+q)}} \quad \text{in } \Omega.$$

Therefore, condition (I.15) *is satisfied if*

$$k < 1 + \frac{(p-1)(r+1)}{p-q+r}. \tag{I.18}$$

Remark that if $1+r > q$, (I.17) *is always true for* $k \in [1+r, 1+q]$ *and since*

$$1+q < 1 + \frac{(p-1)(r+1)}{p-q+r} \iff r+1 > q, \tag{I.19}$$

condition (I.6) *implies* (I.15). *Similarly if* $1+r \le q$, *by equivalence* (I.19), (I.18) *is never satisfied for* $k \in \left(1+q, 1+q+\frac{p-(1+q)}{p}\right)$ *and condition* (I.7) *implies* (I.15). *We can easily check that both conditions* (I.6) *and* (I.7) *are weaker than* (I.16). *Moreover, if one of the following conditions holds :*

$$1+r > q \quad \text{and} \quad k \in \left(1 + \frac{(p-1)(r+1)}{p-q+r}, 1+q+\frac{p-(1+q)}{p}\right),$$

$$1+r \ge q \quad \text{and} \quad k \in \left(2+r, 1+q+\frac{p-(1+q)}{p}\right),$$

then, using Proposition 3.1 again, condition (I.15) *is not satisfied, which shows the "sharpness" of conditions* (I.6) *and* (I.7).

In the proof of Proposition 5.1, we will need the following well known lemma.

Lemma 5.1 *There exists a constant $C_p > 0$ such that, for all $x, y \in \mathbb{R}^N$*

$$(|x|^{p-2}x - |y|^{p-2}y).(x-y) \geq \begin{cases} C_p|x-y|^p & if \quad p \geq 2, \\ C_p \dfrac{|x-y|^2}{(|x|+|y|)^{2-p}} & if \quad 1 < p < 2. \end{cases}$$

Proof. See Lemma 4.2 in LINDQVIST [65]. □

Proof. (OF PROPOSITION 5.1). Let us introduce the functional

$$I_\lambda(v) = \frac{1}{p}\int_\Omega |\nabla v|^p \, dx + \frac{1}{r+1}\int_\Omega K(x)|v|^{r+1} \, dx - \frac{\lambda}{q+1}\int_\Omega K(x)|v|^{q+1} \, dx,$$

with $v \in \mathrm{W}_0^{1,p}(\Omega)$. Let $\varphi_0 \neq 0 \in \mathcal{D}(\Omega)$ be a non-negative function. Therefore there exists $\lambda^{**} > 0$ such that $I_\lambda(\varphi_0) < 0$ for $\lambda > \lambda^{**}$. Let us fix a constant $M > 1$ such that $M\overline{u}_\lambda \geq \varphi_0$ in Ω and introduce the cut-off function f_λ defined in $\Omega \times \mathbb{R}$ by :

$$f_\lambda(x,v) = \begin{cases} K(x)\left[\lambda(M\overline{u}_\lambda)^q - (M\overline{u}_\lambda)^r\right] & \text{if } \ v > M\overline{u}_\lambda(x), \\ K(x)\left[\lambda|v|^q - |v|^r\right] & \text{if } \ v \in [0, M\overline{u}_\lambda(x)], \\ 0 & \text{if } \ v < 0. \end{cases}$$

It is easy to see that the function $v \longmapsto f_\lambda(x,v)$ is a Carathéodory function. For $(x,v) \in \Omega \times \mathbb{R}$, let us set $F_\lambda(x,v) = \displaystyle\int_0^v f_\lambda(x,t)\,dt$ and consider the functional E_λ defined as follows :

$$\forall v \in \mathrm{W}_0^{1,p}(\Omega), \qquad E_\lambda(v) = \frac{1}{p}\int_\Omega |\nabla v|^p \, dx - \int_\Omega F_\lambda(x, v(x)) \, dx.$$

A straightforward computation yields

$$\begin{aligned} E_\lambda(v) = \frac{1}{p}\int_\Omega |\nabla v|^p \, dx - \frac{\lambda}{q+1}A(v,q) + \frac{1}{r+1}A(v,r) \\ - \lambda B(v,q) + B(v,r) - \frac{r}{r+1}C(r) + \lambda\frac{q}{q+1}C(q), \quad \text{(I.20)} \end{aligned}$$

with

$$A(v,s) \stackrel{\text{def}}{=} \int_\Omega \mathbb{1}_{\{0 \leq v \leq M\overline{u}_\lambda\}} K(x)|v|^{s+1} dx, \ B(v,s) \stackrel{\text{def}}{=} \int_\Omega \mathbb{1}_{\{v \geq M\overline{u}_\lambda\}} K(x)\left(M\overline{u}_\lambda\right)^s v dx$$

and

$$C(s) \stackrel{\text{def}}{=} \int_\Omega \mathbb{1}_{\{v \geq M\overline{u}_\lambda\}} K(x)(M\overline{u}_\lambda)^{s+1} dx.$$

Let $\varepsilon > 0$ and $v \in \mathrm{W}_0^{1,p}(\Omega)$, then we split the integral $A(v,q)$ in $\Omega \setminus \Omega_\varepsilon$ and Ω_ε :

$$\begin{aligned} A(v,q) &= \int_{\Omega\setminus\Omega_\varepsilon} \mathbb{1}_{\{0 \leq v \leq M\overline{u}_\lambda\}} K(x)|v|^{q+1} dx + \int_{\Omega_\varepsilon} \mathbb{1}_{\{0 \leq v \leq M\overline{u}_\lambda\}} K(x)|v|^{q+1} dx \\ &\stackrel{\text{def}}{=} A_{\Omega\setminus\Omega_\varepsilon}(v,q) + A_{\Omega_\varepsilon}(v,q). \end{aligned}$$

Since in $\Omega \setminus \Omega_\varepsilon$, K is bounded, from the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{q+1}(\Omega)$, there exists a constant C_1 such that

$$A_{\Omega\setminus\Omega_\varepsilon}(v,q) \le C_1 \|v\|^{q+1}_{W_0^{1,p}(\Omega)}. \tag{I.21}$$

Furthermore, by the Hölder's inequality we have,

$$A_{\Omega_\varepsilon}(v,q) \le A_{\Omega_\varepsilon}(v,r)^\tau \left(\int_{\Omega_\varepsilon} \mathbb{1}_{\{0\le v\le M\overline{u}_\lambda\}} K(x)|v|^p dx \right)^{1-\tau},$$

with $\tau = \frac{p-(1+q)}{p-(1+r)} < 1$. Using inequality (I.4) and Hardy's Inequality, we finally obtain, for ε small enough

$$\begin{aligned} A_{\Omega_\varepsilon}(v,q) &\le C_2 \varepsilon^{\frac{1}{2}(p-k)(1-\tau)} A_{\Omega_\varepsilon}(v,r)^\tau \left(\int_{\Omega_\varepsilon} \frac{|v|^p}{d(x)^p}\, dx \right)^{1-\tau} \\ &\le C_2 \varepsilon^{\frac{1}{2}(p-k)(1-\tau)} \left(\tau A(v,r) + C_3(1-\tau)\|v\|^p_{W_0^{1,p}(\Omega)} \right). \end{aligned} \tag{I.22}$$

From the above arguments and since

$$\begin{aligned} B(v,q) &= \int_{\Omega\setminus\Omega_\varepsilon} \mathbb{1}_{\{v\ge M\overline{u}_\lambda\}} K(x) \left(M\overline{u}_\lambda\right)^q v dx + \int_{\Omega_\varepsilon} \mathbb{1}_{\{v\ge M\overline{u}_\lambda\}} K(x) \left(M\overline{u}_\lambda\right)^q v dx \\ &\stackrel{\text{def}}{=} B_{\Omega\setminus\Omega_\varepsilon}(v,q) + B_{\Omega_\varepsilon}(v,q), \end{aligned}$$

we also get

$$B_{\Omega\setminus\Omega_\varepsilon}(v,q) \le C_4 \|v\|_{W_0^{1,p}(\Omega)} \tag{I.23}$$

and

$$B_{\Omega_\varepsilon}(v,q) \le C_5 \varepsilon^{\frac{1}{2}(p-k)(1-\tau)} \left(\tau B(v,r) + C_6(1-\tau)\|v\|^p_{W_0^{1,p}(\Omega)} \right). \tag{I.24}$$

Using inequalities (I.21) to (I.24),

$$\begin{aligned} E_\lambda(v) \ge \frac{1}{2p}\|v\|^p_{W_0^{1,p}(\Omega)} - \lambda \frac{C_1}{q+1}\|v\|^{q+1}_{W_0^{1,p}(\Omega)} - \lambda C_4 \|v\|_{W_0^{1,p}(\Omega)} + \frac{1}{2} B(v,r) \\ + \frac{1}{2(r+1)} A(v,r) - \frac{r}{r+1} C(r) + \lambda \frac{q}{q+1} C(q), \end{aligned} \tag{I.25}$$

for $\varepsilon > 0$ sufficiently small. Together with (I.15), (I.25) implies that E_λ is coercive and bounded from below on $W_0^{1,p}(\Omega)$. So let us define

$$c_\lambda \stackrel{\text{def}}{=} \inf_{v\in W_0^{1,p}(\Omega)} E_\lambda(v)$$

and let $(v_n)_{n\in\mathbb{N}} \subset W_0^{1,p}(\Omega)$ be a minimizing sequence of E_λ, that is to say $E_\lambda(v_n) \xrightarrow[n\to+\infty]{} c_\lambda$. $\left(E_\lambda(v_n)\right)_{n\in\mathbb{N}}$ is bounded and then, $(v_n)_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p}(\Omega)$. Thus, there exist $u_\lambda \in W_0^{1,p}(\Omega)$ and a subsequence $(v_m)_{m\in\mathbb{N}}$ such that $v_m \xrightarrow[m\to+\infty]{} u_\lambda$ weakly in $W_0^{1,p}(\Omega)$, strongly in $L^{q+1}(\Omega)$ and in $L^1(\Omega)$ and a.e. in Ω. Then we get

$$\|u_\lambda\|_{\mathrm{W}_0^{1,p}(\Omega)} \leq \liminf_{m\to+\infty} \|v_m\|_{\mathrm{W}_0^{1,p}(\Omega)}. \tag{I.26}$$

Using Fatou's Lemma and inequality (I.15), it follows that

$$\frac{1}{r}A(u_\lambda, r) + B(u_\lambda, r) \leq \liminf_{m\to+\infty}\left(\frac{1}{r}A(v_m, r) + B(v_m, r)\right) < +\infty. \tag{I.27}$$

Again from Fatou's lemma and inequalities (I.22),(I.24) and (I.26),

$$\begin{aligned} \frac{\lambda}{q+1}A_{\Omega_\varepsilon}(u_\lambda, q) + \lambda B_{\Omega_\varepsilon}(u_\lambda, q) &\leq \liminf_{m\to+\infty}\left(\frac{\lambda}{q+1}A_{\Omega_\varepsilon}(v_m, q) + \lambda B_{\Omega_\varepsilon}(v_m, q)\right) \\ &\leq C_7 \varepsilon^{\frac{1}{2}(p-k)(1-\tau)}. \end{aligned} \tag{I.28}$$

Since $v_m \xrightarrow[m\to+\infty]{} u_\lambda$ in $\mathrm{L}^{q+1}(\Omega)$ and in $\mathrm{L}^1(\Omega)$,

$$A_{\Omega\setminus\Omega_\varepsilon}(v_m, q) \xrightarrow[m\to+\infty]{} A_{\Omega\setminus\Omega_\varepsilon}(u_\lambda, q) \quad \text{and} \quad B_{\Omega\setminus\Omega_\varepsilon}(v_m, q) \xrightarrow[m\to+\infty]{} B_{\Omega\setminus\Omega_\varepsilon}(u_\lambda, q). \tag{I.29}$$

Gathering the estimates (I.26) to (I.29) and using (I.20), we obtain :

$$c_\lambda = \liminf_{m\to+\infty} E(v_m) \geq E_\lambda(u_\lambda) - C_7\varepsilon^{\frac{1}{2}(p-k)(1-\tau)} \geq c_\lambda - C_7\varepsilon^{\frac{1}{2}(p-k)(1-\tau)}.$$

Passing to the limit as $\varepsilon \to 0$, we finally get $E_\lambda(u_\lambda) = c_\lambda$. By definition of c_λ, u_λ satisfies

$$E_\lambda(u_\lambda) = \min_{v\in \mathrm{W}_0^{1,p}(\Omega)} E_\lambda(v)$$

and since E_λ is Gâteaux differentiable, u_λ satisfies the Euler-Lagrange equation associated to E_λ :

$$\forall v \in \mathrm{W}_0^{1,p}(\Omega), \quad \int_\Omega |\nabla u_\lambda|^{p-2}\nabla u_\lambda.\nabla v\, dx = \int_\Omega f_\lambda(x, u_\lambda) v\, dx.$$

In particular, setting $v = (u_\lambda)^- \in \mathrm{W}_0^{1,p}(\Omega)$, by weak maximum principle it follows that $u_\lambda \geq 0$ a.e. in Ω. Moreover, since $M\overline{u}_\lambda$ is a super-solution of (P_λ), for all non-negative $v \in \mathrm{W}_0^{1,p}(\Omega)$,

$$\int_\Omega |\nabla(M\overline{u}_\lambda)|^{p-2}\nabla(M\overline{u}_\lambda).\nabla v\, dx \geq \int_\Omega K(x)\left[\lambda(M\overline{u}_\lambda)^q - (M\overline{u}_\lambda)^r\right] v\, dx.$$

Setting $v = (u_\lambda - M\overline{u}_\lambda)^+ \in \mathrm{W}_0^{1,p}(\Omega)$, we obtain

$$\begin{aligned} 0 &= \int_\Omega \Big(f_\lambda(x, u_\lambda) - K(x)\left[\lambda(M\overline{u}_\lambda)^q - (M\overline{u}_\lambda)^r\right]\Big)(u_\lambda - M\overline{u}_\lambda)^+\ dx \\ &\geq \int_\Omega \Big(|\nabla u_\lambda|^{p-2}\nabla u_\lambda - |\nabla(M\overline{u}_\lambda)|^{p-2}\nabla(M\overline{u}_\lambda)\Big).\nabla\Big((u_\lambda - M\overline{u}_\lambda)^+\Big)\, dx. \end{aligned}$$

Using Lemma 5.1, $\nabla\Big((u_\lambda - M\overline{u}_\lambda)^+\Big) = 0$ a.e. in Ω and by Poincaré's inequality $u_\lambda \le M\overline{u}_\lambda$ a.e. in Ω. Finally

$$I_\lambda(u_\lambda) = E_\lambda(u_\lambda) = \min_{v \in \mathrm{W}_0^{1,p}(\Omega)} E_\lambda(v) \le E_\lambda(\varphi_0) = I_\lambda(\varphi_0) < 0,$$

therefore u_λ is a non-trivial weak solution of (P_λ). □

5.2 Compact support of the solution

In this section we define

$$g_\lambda(t) \stackrel{\text{def}}{=} t^r - \lambda t^q, \quad t \in [0, +\infty) \quad \text{and} \quad a^* \stackrel{\text{def}}{=} \left(\frac{r}{\lambda q}\right)^{\frac{1}{q-r}} \tag{I.30}$$

in such a way that g_λ is positive and increasing on the interval $(0, a^*)$. We first state a result which guarantees the existence of an appropriate super-solution of (P_λ) near the boundary.

Lemma 5.2 *Let $u_\lambda \in \mathrm{W}_0^{1,p}(\Omega) \cap \mathrm{L}^\infty(\Omega)$ be a weak solution of (P_λ). Then $u_\lambda \in \mathscr{C}(\overline{\Omega})$ and there exist $\delta_* > 0$, $M > 0$ and $\alpha \in (1, p')$ such that*

$$u_\lambda(x) \le M\varphi_1(x)^\alpha \quad \textit{in } \Omega_{\delta_*}.$$

In the proof of this lemma, we will use the following weak comparison principle :

Proposition 5.2 *Let us consider the Dirichlet problems :*

$$\begin{cases} -\Delta_p u - b(x,u) = f & \textit{in } \Omega, \\ u = f' & \textit{on } \partial\Omega \end{cases} \tag{I.31}$$

and

$$\begin{cases} -\Delta_p v - b(x,v) = g & \textit{in } \Omega, \\ v = g' & \textit{on } \partial\Omega. \end{cases} \tag{I.32}$$

Assume that $f \le g$ in $\mathrm{L}^{p'}(\Omega)$, $f' \le g'$ in $\mathrm{W}^{\frac{1}{p'},p}(\partial\Omega)$, $u, v \in \mathrm{W}^{1,p}(\Omega)$ are any weak solutions of the Dirichlet problems (I.31) *and* (I.32)*, respectively and $b(x,\cdot) : \mathbb{R} \to \mathbb{R}$ is non-increasing for a.e. $x \in \Omega$. Then, $u \le v$ in Ω.*

Proof. See proposition 2.3 in Cuesta-Takáč [19]. □

Proof. (of Lemma 5.2). According to the previous notations, the set $\omega^* \stackrel{\text{def}}{=} \{x \in \Omega \mid u_\lambda(x) \le a^*\}$ contains a neighbourhood of $\partial\Omega$ and there exists $\delta_0 > 0$ such that $\Omega_{\delta_0} \subset \omega^*$. Since u_λ is bounded, there exists $C^* > 0$ large enough such that $u_\lambda \le C^*\varphi_1$ on $\partial\Omega_{\delta_0}$. Hence, u_λ and $C^*\varphi_1$ satisfy

$$\begin{cases} -\Delta_p u_\lambda \le -\Delta_p\left(C^*\varphi_1\right) & \text{in } \Omega_{\delta_0}, \\ u_\lambda \le C^*\varphi_1 & \text{on } \partial\Omega_{\delta_0}. \end{cases} \tag{I.33}$$

Therefore, by the weak comparison principle $u_\lambda \leq C^*\varphi_1$ in Ω_{δ_0}. From this estimate and the interior regularity result of SERRIN [78], $u_\lambda \in \mathscr{C}(\overline{\Omega})$.

Let $M > 0$ and $\alpha \in (1, p')$, we want to construct a super-solution v to (P_λ) near the boundary such that $v \stackrel{\text{def}}{=} M{\varphi_1}^\alpha$. From to the proof of Lemma 3.2, there exists a $\delta_1 > 0$ only depending on Ω, p, M and α such that :

$$\Delta_p v \sim (M\alpha)^{p-1}(\alpha-1)(p-1)d(x)^{(\alpha-1)(p-1)-1} \quad \text{in } \Omega_{\delta_1} \tag{I.34}$$

and

$$K(x)(v^r - \lambda v^q) \sim M^r L(d(x))d(x)^{\alpha r-k} \quad \text{in } \Omega_{\delta_1}. \tag{I.35}$$

Precisely,

$$\delta_1 \stackrel{\text{def}}{=} \min\left\{\delta^*, \frac{\varepsilon^*}{K_2}\left(\frac{(\alpha-1)(p-1)}{2\lambda_1}\right)^{\frac{1}{p}}, \frac{1}{K_2}\left(\frac{1}{2\lambda}\right)^{\frac{1}{\alpha(q-r)}}\left(\frac{1}{M}\right)^{\frac{1}{\alpha}}\right\},$$

where ε^* and δ^* are defined in (I.8). By definition of δ_1, choosing $\alpha > 1$ small enough, $\delta_1 = \frac{\varepsilon^*}{K_2}\left(\frac{(\alpha-1)(p-1)}{2\lambda_1}\right)^{\frac{1}{p}}$ and we can impose

$$M \leq \left[\frac{\inf\limits_{\delta\leq\delta_1} L(\delta)\delta^{-(\alpha(p-r-1)-(p-k))}}{\alpha^{p-1}(\alpha-1)}\right]^{\frac{1}{p-(1+r)}}. \tag{I.36}$$

Then, by estimates (I.34) and (I.35), v is a super-solution of (P_λ) in Ω_{δ_1}. Moreover, if we set

$$\delta_2 \stackrel{\text{def}}{=} \min\left\{\delta_0, \frac{a^*}{C^*K_2}, \left(\frac{a^*\alpha^{p-1}(\alpha-1)}{C_1 \inf\limits_{\delta<\delta_1} L(\delta)}\right)^{\frac{p-(1+r)}{p-k}}\right\},$$

$u_\lambda \leq a^*$ and $v \leq a^*$ in Ω_{δ_2}. Finally, setting $\delta_* \stackrel{\text{def}}{=} \min\{\delta_1, \delta_2\}$ and choosing α close enough to 1, u_λ and v satisfy :

$$\begin{cases} -\Delta_p v - K(x)g_\lambda(v) \geq 0 & \text{in } \Omega_{\delta_*}, \\ -\Delta_p u_\lambda - K(x)g_\lambda(u_\lambda) = 0 & \text{in } \Omega_{\delta_*}, \\ v \geq u_\lambda & \text{on } \partial\Omega_{\delta_*}. \end{cases}$$

Note that the third assertion is a consequence of (I.33) and (I.36). Since $v \mapsto -K(x)g_\lambda(v)$ is non-increasing in $(0, a^*)$ for all $x \in \Omega_{\delta_*}$, applying the weak comparison principle of Proposition 5.2, it follows that $u_\lambda \leq v$ in Ω_{δ_*}. □

Proposition 5.3 *Let $k \in [1+r, p)$ and let $u_\lambda \in \mathrm{W}_0^{1,p}(\Omega) \cap \mathrm{L}^\infty(\Omega)$ be a weak solution of* (P_λ)*, therefore u_λ has a compact support in Ω.*

Proof. For $s \in [0, a^*]$, we define

$$G_\lambda(s) \stackrel{\text{def}}{=} \int_0^s g_\lambda(t)\,dt.$$

Since $r < p-1$, we have

$$\int_0^{a^*} G_\lambda(s)^{-\frac{1}{p}}\,ds < +\infty. \tag{I.37}$$

Note that this above equation is close to condition (2) in VÁZQUEZ [83] and implies that u_λ may be not positive everywhere in Ω. The compact support principle which is detailed below also appears in the book of PUCCI-SERRIN [74, Chapter 5], where several related results of [71–73, 75] are presented. Precisely, let us fix $\varepsilon < a^*$ (small) and $\delta_\varepsilon \stackrel{\text{def}}{=} (\frac{\varepsilon}{M})^{\frac{1}{\alpha}}$ with M and α defined in Lemma 5.2 in such a way that $u_\lambda < \varepsilon$ in $\Omega_{\delta_\varepsilon}$. Let us define for $t \in [0, a^*]$,

$$h(t) \stackrel{\text{def}}{=} \int_t^\varepsilon G_\lambda(s)^{-\frac{1}{p}}\,ds.$$

h is a $\mathscr{C}^2$ bijection from $[0, a^*]$ to $[h(a^*), h(0)]$ and

$$h'(t) = -G_\lambda(t)^{-\frac{1}{p}} < 0, \quad \text{for } t \in (0, a^*).$$

Then h^{-1} is also twice differentiable on $(h(a^*), h(0))$ and we get,

$$(h^{-1})'(y) = -G_\lambda(h^{-1}(y))^{\frac{1}{p}}$$

and

$$(h^{-1})''(y) = \frac{1}{p}\, g_\lambda\left(h^{-1}(y)\right) G_\lambda\left(h^{-1}(y)\right)^{\frac{2-p}{p}} \quad \text{for } y \in (h(a^*), h(0)).$$

Now let us define

$$j(x) \stackrel{\text{def}}{=} \int_{\varphi_1(x)}^{\inf_{\partial\Omega_{\delta_\varepsilon}} \varphi_1} \left(g_\lambda\left(\frac{s}{2}\right)\frac{s}{2}\right)^{-\frac{1}{p}} ds, \quad \text{for } x \in \Omega_{\delta_\varepsilon}$$

and

$$J(x) \stackrel{\text{def}}{=} \min\{j(x), h(0)\}, \quad \text{for } x \in \Omega_{\delta_\varepsilon}.$$

Remark that $j(x) > h(a^*)$ for $x \in \Omega_{\delta_\varepsilon}$ provided ε is sufficiently small. Indeed,

$$h(a^*) \le C_1\left[\varepsilon^{\frac{p-(r+1)}{p}} - (a^*)^{\frac{p-(r+1)}{p}}\right] < -C_2\varepsilon^{\frac{p-(r+1)}{\alpha p}} \le j(x), \quad \text{for } x \in \Omega_{\delta_\varepsilon},$$

with C_1 and C_2 two positive constants independent of ε.

With all this notations, we finally define the function w in $\Omega_{\delta_\varepsilon}$ by

$$w(x) \stackrel{\text{def}}{=} h^{-1}\left(J(x)\right), \quad \text{for } x \in \Omega_{\delta_\varepsilon}.$$

In other words,

$$\int_{w(x)}^{\varepsilon} G_\lambda(s)^{-\frac{1}{p}}\, ds = J(x), \quad \text{for } x \in \Omega_{\delta_\varepsilon}.$$

Using the last relation, w is non-negative in $\Omega_{\delta_\varepsilon}$ and $w \leq a^*$ in $\Omega_{\delta_\varepsilon}$. Moreover, w vanishes when $d(x)$ is small. Indeed, for $s \in (0, a^*)$,

$$g_\lambda(s)s > G_\lambda(s) > \int_{\frac{s}{2}}^{s} g_\lambda(t)\, dt > \frac{s}{2} g_\lambda\left(\frac{s}{2}\right).$$

Then for $\varepsilon \in (0, a^*)$,

$$\int_0^{\varepsilon} G_\lambda(s)^{-\frac{1}{p}}\, ds < \int_0^{\varepsilon} \left(\frac{s}{2} g_\lambda\left(\frac{s}{2}\right)\right)^{-\frac{1}{p}} ds.$$

So,

$$j(x) \xrightarrow[d(x)\to 0]{} \int_0^{\inf_{\partial\Omega_{\delta_\varepsilon}} \varphi_1} \left(g_\lambda\left(\frac{s}{2}\right)\frac{s}{2}\right)^{-\frac{1}{p}} ds \geq \int_0^{K_1\delta_\varepsilon} \left(g_\lambda\left(\frac{s}{2}\right)\frac{s}{2}\right)^{-\frac{1}{p}} ds > \int_0^{\varepsilon} G_\lambda(s)^{-\frac{1}{p}} ds,$$

for $\varepsilon > 0$ small. Then, from the definitions of J and w, w has a compact support in Ω.

To complete the proof, it is enough to show that $u_\lambda \leq w$ in $\Omega_{\delta_\varepsilon}$. Since w has a compact support, $J \in \mathrm{W}^{1,p}(\Omega_{\delta_\varepsilon})$ and $w = h^{-1} \circ J \in \mathrm{W}^{1,p}(\Omega_{\delta_\varepsilon})$ and satisfies

$$\nabla w = -G_\lambda\,(w)^{\frac{1}{p}}\, \nabla j \qquad \text{in } \mathcal{D}'\left(\Omega_{\delta_\varepsilon}\right).$$

Then,

$$\Delta_p w + G_\lambda(w)^{\frac{1}{p'}} \Delta_p j = \frac{1}{p'}\, g_\lambda(w)|\nabla j|^p \qquad \text{in } \mathcal{D}'\left(\Omega_{\delta_\varepsilon}\right).$$

Provided ε is sufficiently small, we have

$$\frac{1}{p'}|\nabla j|^p = \frac{1}{p'}|\nabla \varphi_1|^p \left[\frac{\varphi_1}{2} g_\lambda\left(\frac{\varphi_1}{2}\right)\right]^{-1} \leqslant K(x) \quad \text{in } \Omega_{\delta_\varepsilon}$$

and

$$\Delta_p j = \frac{1}{p'}|\nabla\varphi_1|^p \left[\frac{\varphi_1}{2} g_\lambda\left(\frac{\varphi_1}{2}\right)\right]^{-\frac{1}{p'}} \left[\frac{1}{2} g_\lambda\left(\frac{\varphi_1}{2}\right) + \frac{\varphi_1}{4}(g_\lambda)'\left(\frac{\varphi_1}{2}\right)\right] + \lambda_1 {\varphi_1}^{p-1}\left[\frac{\varphi_1}{2} g_\lambda\left(\frac{\varphi_1}{2}\right)\right]^{\frac{1}{p'}} \geq 0 \quad \text{in } \Omega_{\delta_\varepsilon}.$$

Hence, $\Delta_p w \leq K(x) g_\lambda(w)$ in $\Omega_{\delta_\varepsilon}$. Moreover, since $g_\lambda \geq 0$ on $\partial\Omega_{\delta_\varepsilon}$, we have $u_\lambda(x) \leq \varepsilon \leq w(x)$ on $\partial\Omega_{\delta_\varepsilon}$. Therefore, from Proposition 5.2, $u_\lambda(x) \leq w(x)$ in $\Omega_{\delta_\varepsilon}$. ☐

Proof. (OF THEOREM 2.2). Since u_λ is compactly supported in Ω, inequality (I.10) is also satisfied when $k \geq 1 + r$, which implies the existence of a critical parameter $\lambda_{**} > 0$ such that (P_λ) has no non-trivial solution for $\lambda < \lambda_{**}$. Thanks to Propositions 5.1 and 5.3 and Remark 5.1, from the regularity result of LIEBERMAN [64] and the same arguments as in Subsection 4.2, Theorem 2.2 follows. ☐

6 Some additional results for problem (P_λ) when $p = 2$

6.1 Introduction and recall

In this last section, we give some additional results on problem (P_λ) concerning the case $p = 2$. In this first chapter we have studied the following quasilinear and singular problem :

$$(\mathrm{P}_\lambda) \begin{cases} -\Delta_p u = \mathbb{1}_{\{u>0\}} K(x) \left(\lambda u^q - u^r\right) & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \quad u \geq 0 & \text{in } \Omega, \end{cases}$$

where Ω is a $\mathscr{C}^2$ bounded domain of $\mathbb{R}^N$, $1 < p < \infty$, $\lambda > 0$ is a positive parameter. In the right-hand side of the equation, the exponents q and r satisfy $-1 < r < q < p - 1$ and $K \in \mathscr{C}(\Omega)$ is a positive function having a singular behaviour near the boundary $\partial\Omega$. Precisely, $K(x) = d(x)^{-k} L(d(x))$ in Ω, with $0 < k < p$ and L a Karamata function, which is a lower positive perturbation satisfying $L \in \mathscr{C}^2((0, D])$ a positive function, with $D \stackrel{\text{def}}{=} \operatorname{diam}(\Omega)$, defined as follows :

$$L(t) = \exp\left(\int_t^D \frac{z(s)}{s} ds\right), \tag{I.38}$$

with $z \in \mathscr{C}([0, D]) \cap \mathscr{C}^1((0, D])$ and $z(0) = 0$. Let us just recall that (I.38) implies that

$$\forall \varepsilon > 0, \quad \lim_{t \to 0^+} t^\varepsilon L(t) = 0 \quad \text{and} \quad \lim_{t \to 0^+} t^{-\varepsilon} L(t) = +\infty. \tag{I.39}$$

The first theorem we have proved in Chapter I is the following.

Theorem 6.1 *When $k < 1 + r$, there exists a constant $\Lambda_1 > 0$ such that :*

1. *For $\lambda > \Lambda_1$, (P_λ) admits a positive weak solution.*
2. *Any weak solution of (P_λ) is $\mathscr{C}^{1,\beta}(\overline{\Omega})$, for some $0 < \beta < 1$.*
3. *For $\lambda < \Lambda_1$, (P_λ) has no positive solution.*

The critical parameter $\Lambda_1 > 0$ is defined as follows :

$$\Lambda_1 \stackrel{\text{def}}{=} \inf\left\{\lambda > 0, \quad u_\lambda > 0 \quad \text{a.e. in } \Omega.\right\}, \tag{I.40}$$

where $u_\lambda \in \mathrm{W}_0^{1,p}(\Omega) \cap \mathrm{L}^\infty(\Omega)$ is a maximal solution to (P_λ) obtained by a sub- and super-solution method. We have $u_\lambda \leq \overline{u}_\lambda$ a.e. in Ω, where $\overline{u}_\lambda \in \mathrm{W}_0^{1,p}(\Omega)$ is a super-solution to (P_λ). Precisely, $\overline{u}_\lambda \stackrel{\text{def}}{=} \overline{M} v$ in Ω, where $\overline{M}$ is a positive constant sufficiently large and v is the unique solution of problem

$$(\mathrm{Q}) \begin{cases} -\Delta_p v = K(x) v^q & \text{in } \Omega, \\ v = 0 \quad \text{on } \partial\Omega, \quad v \geq 0 & \text{in } \Omega. \end{cases}$$

Moreover, from Moser iterations technique, we can prove that $v \in \mathrm{L}^\infty(\Omega)$ and from Lieberman [64], $v \in \mathscr{C}^{1,\alpha}(\overline{\Omega})$, for some $0 < \alpha < 1$. Then, v behaves like the distance to the boundary function in Ω.

We also proved in this paper the existence of a parameter $\lambda_* > 0$ such that

$$\forall \lambda > 0, \quad \lambda < \lambda_* \Longrightarrow u_\lambda \equiv 0 \quad \text{ in } \Omega. \tag{I.41}$$

Accordingly, the natural issues for problem (P) are the existence or non-existence of a non-trivial solution for the critical problem (P_{Λ_1}), the precise behaviour (with respect to the distance to the boundary) of the positive solution u_λ with $\lambda > \Lambda_1$ and the stability of the solutions to (P_λ).

We discuss the above questions in the case where $p = 2$ and

$$-1 < r < 0 \quad \text{ and } \quad 0 < q < 1. \tag{I.42}$$

Let us observe that $t \mapsto \lambda t^q - t^r$ is a concave function in this case.

6.2 Behaviour of the solution $\mathbf{u}_\lambda$

In this context we first get a precise behaviour in Ω of our maximal solution u_λ for $\lambda > \Lambda_1$.

Proposition 6.1 *Assume $\lambda > \Lambda_1$. Then, there exist two constants $C_1, C_2 > 0$ (depending on λ) such that*

$$C_1 d(x) \leq u_\lambda(x) \leq C_2 d(x), \quad \textit{for all } x \in \Omega. \tag{I.43}$$

Proof. Let us choose $\lambda' \in (\Lambda_1, \lambda)$ and consider $\varphi \in \mathscr{C}^2(\Omega) \cap \mathscr{C}^{1,\alpha}(\overline{\Omega})$, for some $0 < \alpha < 1$, solution to

$$\begin{cases} -\Delta \varphi = K(x) u_{\lambda'}^q \quad \text{ in } \Omega, \\ \quad \varphi = 0 \quad \text{ on } \partial\Omega. \end{cases} \tag{I.44}$$

By the Hopf's lemma (See for instance EVANS [37, Lemma p.330]), φ behaves in Ω like the distance function. Moreover, for $\varepsilon > 0$ sufficiently small, $w \overset{\text{def}}{=} u_{\lambda'} + \varepsilon\varphi$ is a sub-solution of (P_λ) in Ω. Indeed, if (I.42) is satisfied and $\lambda' + \varepsilon \leq \lambda$, we have in Ω

$$\begin{aligned} -\Delta w &= K(x) \left\{ (\lambda' + \varepsilon) u_{\lambda'}^q - u_{\lambda'}^r \right\} \\ &\leq K(x)(\lambda w^q - w^r). \end{aligned} \tag{I.45}$$

Then, choosing M sufficiently large in the definition of $\overline{u}_\lambda$ and using the same lower- and upper-solution method as Section 4 of Chapter I, we get

$$w \leq u_\lambda \leq \overline{u}_\lambda \quad \text{ in } \Omega.$$

Since both w and $\overline{u}_\lambda$ behave like the distance function, the proof of Proposition 6.1 is now complete. □

6.3 About the critical problem (P_{Λ_1})

In Theorem 2.1, we prove the existence of a critical value $\Lambda_1 > 0$ for existence of a positive solution to (P_λ). However, it is not clear if there exists a positive solution u_{Λ_1} to (P_{Λ_1}). The present section deals with the positiveness of u_{Λ_1}.

First, let us prove the existence of a non-trivial solution of (P_{Λ_1}). For that, we use the precise behaviour of the solutions of (P_λ), for $\lambda > \Lambda_1$, given in Proposition 6.1. Let $v \in \mathscr{C}_0(\Omega) \cap \mathscr{C}^{1,\alpha}(\overline{\Omega})$, for some $0 < \alpha < 1$, be the unique solution to (Q). Then for $\lambda > \Lambda_1$, we define

$$U_\lambda \stackrel{\text{def}}{=} \lambda^{\frac{1}{1-q}} v \quad \text{in } \Omega. \tag{I.46}$$

This function U_λ is the unique solution of the problem

$$\left(\overline{\mathrm{P}}_\lambda\right) \begin{cases} -\Delta w = \lambda K(x) w^q & \text{in } \Omega, \\ \quad w = 0 \quad \text{on } \partial\Omega, \quad w > 0 & \text{in } \Omega, \end{cases}$$

and behaves in Ω like the distance function. Furthermore, u_λ is also a supersolution to problem (P_λ).

Remark 6.1 *From the lower- and upper-solutions method, we have that $\lambda \mapsto U_\lambda$ is increasing on $(\Lambda_1, +\infty)$.*

We first prove the following lemma :

Lemma 6.1 *Let $\lambda > \Lambda_1$ and let $u_\lambda \in \mathscr{C}^{1,\beta}(\overline{\Omega})$, for some $0 < \beta < 1$ be the positive maximal solution of (P_λ) we proved in Theorem 2.1. Then,*

$$u_\lambda \leq U_\lambda \quad \textit{in } \Omega.$$

Proof. In the proof, we use the uniqueness of the solution to problem (Q). Precisely, let us notice that $\underline{v} \stackrel{\text{def}}{=} u_\lambda$ is a subsolution to (Q). Then, let us define $\overline{v} \stackrel{\text{def}}{=} MV$ in Ω, where $M > 0$ is taken large enough and V is the unique solution of problem

$$\begin{cases} -\Delta V = K(x) & \text{in } \Omega, \\ \quad V = 0 & \text{on } \partial\Omega. \end{cases} \tag{I.47}$$

Using a regularity result due to Gui-Lin [53], $V \in \mathscr{C}_0(\overline{\Omega}) \cap \mathscr{C}^{1,\alpha}(\overline{\Omega})$, for some $0 < \alpha < 1$ and thanks to the Hopf's lemma, V behaves like the distance function in Ω. Then, for $M > 0$ large enough, using the sub-homogeneity of problem (Q), $\overline{v}$ is a supersolution to (Q). Moreover using the behaviour of u_λ given by Proposition 6.1, for M large enough, $\underline{v} \leq \overline{v}$ in Ω. Then, we consider the following monotone iterative scheme : for $n \in \mathbb{N}^*$,

$$(\mathrm{Q}_n) \begin{cases} -\Delta v_n = \lambda K(x) v_{n-1}^q & \text{in } \Omega, \\ \quad v_n = 0 & \text{on } \partial\Omega, \end{cases}$$

with $v_0 \stackrel{\text{def}}{=} \underline{v}$ in Ω. By induction on n, (Q_n) admits a unique solution $v_n \in \mathscr{C}_0(\Omega) \cap \mathscr{C}^{1,\alpha}(\overline{\Omega})$, for some $0 < \alpha < 1$. Moreover, using the weak maximum principle, for all $n \in \mathbb{N}^*$,

$$u_\lambda = \underline{v} \leq v_n \leq v_{n+1} \leq \overline{v} \quad \text{in } \Omega. \tag{I.48}$$

So, for all $x \in \overline{\Omega}$, let us define

$$\tilde{v}(x) \stackrel{\text{def}}{=} \lim_{n\to+\infty} v_n(x).$$

Moreover $(v_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{H}_0^1(\Omega)$, then passing to the limit in (Q_n), $\tilde{v}$ is a weak solution to $\left(\overline{\mathrm{P}}_\lambda\right)$. Passing to the limit in (I.48), $\tilde{v}(x) \sim d(x)$ in Ω. Therefore, from the uniqueness of the solution to $\left(\overline{\mathrm{P}}_\lambda\right)$, $\tilde{v} = U_\lambda$ in Ω. The proof is now complete. □

The next result shows the existence and the positivity of an extremal solution u_{Λ_1} for the problem (P_λ) (u_{Λ_1} may vanish on a Lebesgue's measure-zero set).

Proposition 6.2 *Problem* (P_{Λ_1}) *admits a non-trivial weak solution* $u_{\Lambda_1} \in \mathscr{C}^{1,\beta}(\overline{\Omega})$, *for some* $0<\beta<1$. *Moreover,*

$$\int_\Omega K(x) u_{\Lambda_1}^r \varphi_1 dx < +\infty. \tag{I.49}$$

As a consequence, $u_{\Lambda_1} > 0$ a.e. *in* Ω.

Proof. Let $(\lambda_n)_{n\in\mathbb{N}}$ be a decreasing sequence converging to Λ_1. For all $n \in \mathbb{N}$, let us consider u_{λ_n} the maximal solution to (P_{λ_n}) given in Theorem 2.1. So for all $n \in \mathbb{N}$, $u_{\lambda_{n+1}}$ is a subsolution of $\left(\overline{\mathrm{P}}_{\lambda_n}\right)$ and using Lemma 6.1,

$$u_{\lambda_{n+1}} \le U_{\lambda_{n+1}} \le U_{\lambda_n} \quad \text{ in } \Omega.$$

Then, by the lower- and upper-solution method as it is used in the proof of Theorem 2.1, we construct $\tilde{u}_{\lambda_n}$ solution to (P_{λ_n}) between $u_{\lambda_{n+1}}$ and U_{λ_n}. Hence, by maximality of u_{λ_n}, it follows that

$$0 < u_{\lambda_{n+1}} \le u_{\lambda_n} \le U_{\lambda_0} \quad \text{ in } \Omega. \tag{I.50}$$

So let us define for all $x \in \overline{\Omega}$,

$$u_{\Lambda_1}(x) \stackrel{\text{def}}{=} \lim_{n\to+\infty} u_{\lambda_n}(x) \in [0, U_{\lambda_0}(x)].$$

To prove (I.49), let us choose $\gamma \in (0,1)$, $\varepsilon > 0$ (small enough) and consider the function $\psi \stackrel{\text{def}}{=} (\varphi_1+\varepsilon)^\gamma - \varepsilon^\gamma \in \mathrm{H}_0^1(\Omega)$ as a test function. Then, a direct computation gives

$$-\Delta\psi = -\gamma(\gamma-1)|\nabla\varphi_1|^2(\varphi_1+\varepsilon)^{\gamma-2} + \lambda_1\varphi_1\gamma(\varphi_1+\varepsilon)^{\gamma-1} \ge 0 \quad \text{ in } \Omega. \tag{I.51}$$

For all $n \in \mathbb{N}$,

$$\langle -\Delta u_{\lambda_n}, \psi\rangle_{\mathrm{H}^{-1}(\Omega)\times\mathrm{H}_0^1(\Omega)} = \int_\Omega K(x)(\lambda_n u_{\lambda_n}^q - u_{\lambda_n}^r)\psi\, dx \ge 0. \tag{I.52}$$

Thus, we get

$$\int_\Omega K(x) u_{\lambda_n}^r \psi\, dx \le \lambda_n \int_\Omega K(x) u_{\lambda_n}^q \psi\, dx \tag{I.53}$$

and passing to the limit as $\varepsilon \to 0$ and as $\gamma \to 1$, the Lebegue's dominated convergence theorem yields

$$\int_\Omega K(x) u_{\lambda_n}^r \varphi_1\, dx \le \lambda_n \int_\Omega K(x) u_{\lambda_n}^q \varphi_1\, dx.$$

Finally, since $\forall n \in \mathbb{N}$, $u_{\lambda_n} \le U_{\lambda_0}$ in Ω, we have

$$\int_\Omega K(x) u_{\lambda_n}^r \varphi_1 \, dx \le \Lambda_1 \int_\Omega K(x) U_{\lambda_0}^q \varphi_1 \, dx < +\infty. \tag{I.54}$$

Passing to the limit in (I.54), the monotone convergence theorem provides estimate (I.49)

To complete the proof, we still have to show that u_{Λ_1} is a non-trivial weak solution of the extremal problem (P_{Λ_1}). First, notice that $(u_{\lambda_n})_{n\in\mathbb{N}}$ is bounded in $H_0^1(\Omega)$. Indeed, we have for all $n \in \mathbb{N}$,

$$\int_\Omega |\nabla u_{\lambda_n}|^2 \, dx \le \int_\Omega \lambda_n K(x) u_{\lambda_n}^{q+1} \, dx \le \int_\Omega \lambda_0 K(x) U_{\lambda_0}^{q+1} \, dx < +\infty.$$

So, identifying the limits in $\mathcal{D}'(\Omega)$, up to a subsequence denoted in the same way, $u_{\lambda_n} \underset{n\to+\infty}{\longrightarrow} u_{\Lambda_1}$ in $\mathrm{H}_0^1(\Omega)$ and a.e. in Ω. Let $\varphi \in \mathcal{D}(\Omega)$, then we get

$$\forall n \in \mathbb{N}, \quad \int_\Omega \nabla u_{\lambda_n} . \nabla \varphi \, dx = \int_\Omega K(x)(\lambda_n u_{\lambda_n}^q - u_{\lambda_n}^r)\varphi \, dx. \tag{I.55}$$

In (I.55), it is easy to get the convergence of both the left hand side and the positive part of the right hand side. Concerning the negative part, since $u_{\Lambda_1} > 0$ a.e. in Ω,

$$K(x) u_{\lambda_n}^r \varphi \underset{n\to+\infty}{\longrightarrow} K(x) u_{\Lambda_1}^r \varphi \qquad \text{a.e in } \Omega.$$

Moreover for almost every $x \in \Omega$,

$$|K(x) u_{\lambda_n}^r \varphi| \le K(x) u_{\Lambda_1}^r |\varphi| \in \mathrm{L}^1(\Omega). \tag{I.56}$$

Indeed, observing that

$$K(x) u_{\Lambda_1}^r |\varphi| = \Big(K(x) u_{\Lambda_1}^r \varphi_1\Big) \frac{|\varphi|}{\varphi_1},$$

we get on one hand from estimate (I.49), $K(x) u_{\Lambda_1}^r \varphi_1 \in \mathrm{L}^1(\Omega)$ and on the other hand, $\frac{|\varphi|}{\varphi_1} \in \mathrm{L}^\infty(\Omega)$. So, the Hölder inequality ensures that (I.56) holds. Hence, by the Lebesgue's dominated convergence theorem we pass to the limit when $n \to +\infty$ in (I.55) and it follows that u_{Λ_1} is a non-trivial weak solution to (P_{Λ_1}). Finally, the $\mathscr{C}^{1,\beta}(\overline{\Omega})$ regularity of u_{Λ_1} follows from Theorem 2.1. □

Now, we show the uniqueness of the extremal positive solution u_{Λ_1} to (P_{Λ_1}).

Proposition 6.3 *Let $v \in \mathrm{H}_0^1(\Omega) \cap \mathrm{L}^\infty(\Omega)$ be a positive solution to (P_{Λ_1}). Then, $v = u_{\Lambda_1}$ a.e. in Ω.*

Proof. Let $v \in \mathrm{H}_0^1(\Omega) \cap \mathrm{L}^\infty(\Omega)$ be a positive solution to (P_{Λ_1}) such that $v \not\equiv u_{\Lambda_1}$ in Ω. Since the mapping $t \mapsto \Lambda_1 t^q - t^r$ is (strictly) concave on $(0, +\infty)$, the convex combination

$$w \overset{\text{def}}{=} t u_{\Lambda_1} + (1-t)v, \qquad 0 < t < 1,$$

is a (strict) sub-solution of (P_{Λ_1}) in Ω. We now prove that it implies the existence of a positive solution to a problem $(\mathrm{P}_{\lambda'})$ with $\lambda' < \Lambda_1$ close enough to Λ_1, from which we get a contradiction.

Let $\varphi \in \mathscr{C}^{1,\alpha}(\overline{\Omega})$, for a fixed $0<\alpha<1$, the unique solution to

$$\begin{cases} -\Delta\varphi = K(x)(\Lambda_1 w^q - w^r) \quad \text{in } \Omega, \\ \varphi = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{I.57}$$

By the weak maximum principle, $\varphi \geq w$ in Ω and by the strong maximum principle of Brézis-Nirenberg [14], there exists $\varepsilon > 0$ small enough such that

$$\varphi(x) \geq (w+\varepsilon V)(x) \quad \text{and} \quad (\varphi - \varepsilon V)(x) \geq \varepsilon d(x), \quad \text{for } x \in \Omega.$$

Furthermore,

$$-\Delta\left(\varphi - \varepsilon V\right) \leq K(x)\left[\Lambda_1(\varphi-\varepsilon V)^q - (\varphi - \varepsilon V)^r - \varepsilon\right] \quad \text{in } \Omega,$$

where V is defined in (I.47). Thus, using lower- and upper-solutions method as in Section 4 of this chapter, we prove the existence of $w_1 \in \mathscr{C}^{1,\alpha}(\overline{\Omega})$, for some $0<\alpha<1$, solution of

$$\begin{cases} -\Delta w_1 = K(x)(\Lambda_1 w_1^q - w_1^r - \varepsilon) \quad \text{in } \Omega, \\ w_1 = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{I.58}$$

It follows from the weak maximum principle that, $w_1(x) \geq (\varphi - \varepsilon V)(x) \geq \varepsilon d(x)$ in Ω. Then, let $\lambda' \in (0,\Lambda_1)$ and $\varepsilon' \in \left(0, \frac{\lambda'}{\Lambda_1}\varepsilon\right)$ be such that

$$\left(\frac{\lambda'}{\Lambda_1}\right)^{\frac{1}{r}} w_1 \leq \varepsilon' V + \frac{\lambda'}{\Lambda_1} w_1 \quad \text{in } \Omega.$$

Setting $w_2 \stackrel{\text{def}}{=} \varepsilon' V + \frac{\lambda'}{\Lambda_1} w_1$ in Ω, we get

$$\begin{aligned} -\Delta w_2 &\leq K(x)\left(\lambda' w_1^q - \tfrac{\lambda'}{\Lambda_1} w_1^r - \tfrac{\lambda'}{\Lambda_1}\varepsilon + \varepsilon'\right) \\ &\leq K(x)\left(\lambda' w_2^q - w_2^r\right) \quad \text{in } \Omega. \end{aligned}$$

By choosing λ' close enough to Λ_1, $w_2 \geq w_1$ in Ω. Finally, by a sub- and super-solution method, we conclude on the existence of a positive solution of the problem $(\mathrm{P}_{\lambda'})$, which proves the uniqueness of u_{Λ_1} among the almost everywhere positive solution to (P_{Λ_1}). □

Remark 6.2 *This kind of argument has been introduced by* Brezis-Cazenave-Martel-Ramiandrisoa *[13], for convex non-linearities.*

6.4 About the stability of the solution $\mathbf{u}_\lambda$

Now for $\lambda > \Lambda_1$, let us focus on the stability of the maximal solutions u_λ of Theorem 2.1. For that, let us define the energy functional $\mathscr{E}_\lambda$ by

$$\mathscr{E}_\lambda(v) \stackrel{\text{def}}{=} \int_\Omega |\nabla v|^2\,dx + r\int_\Omega K(x)u_\lambda^{r-1}v^2\,dx - \lambda q\int_\Omega K(x)u_\lambda^{q-1}v^2\,dx,$$

for all $\lambda > \Lambda_1$ and all $v \in \mathrm{H}_0^1(\Omega)$; and set

$$\Lambda(\lambda) \stackrel{\text{def}}{=} \inf_{\substack{v\in \mathrm{H}_0^1(\Omega)\\ \|v\|_{\mathrm{L}^2(\Omega)}=1}} \mathscr{E}_\lambda(v), \tag{I.59}$$

the first eigenvalue of the linearised operator associated to (P_λ).

Definition 6.1 (Stability of the solution u_λ) *The maximal solution u_λ of problem (P_λ) is said to be **stable** (rep. **semi-stable**) if and only if $\Lambda(\lambda) > 0$ (resp. $\Lambda(\lambda) \geq 0$).*

For more details concerning stability of solutions, we refer to the book of L. Dupaigne [36].

First, we observe that $\Lambda(\lambda)$ is well defined thanks to Proposition 6.1 and Hardy's inequality. Indeed, for all $v \in \mathrm{H}_0^1(\Omega)$ and $\varepsilon > 0$ small enough,

$$\begin{aligned} \mathscr{E}_\lambda(v) &\geq \|v\|_{\mathrm{H}_0^1(\Omega)} - \lambda q \int_{\Omega\setminus\Omega_\varepsilon} K(x) u_\lambda^{q-1} v^2\,dx - \lambda q \int_{\Omega_\varepsilon} K(x) u_\lambda^{q-1} v^2\,dx \\ &\geq \|v\|_{\mathrm{H}_0^1(\Omega)} - \lambda q C_\varepsilon \|v\|_{\mathrm{L}^2(\Omega)} - \lambda q \varepsilon^{q+1-k} C \|v\|_{\mathrm{H}_0^1(\Omega)} \\ &\geq \frac{1}{2}\|v\|^2_{\mathrm{H}_0^1(\Omega)} - \lambda q C_\varepsilon \|v\|_{\mathrm{L}^2(\Omega)} \\ &\geq C_0 > -\infty. \end{aligned} \tag{I.60}$$

We now prove that $\Lambda(\lambda) > 0$, for every $\lambda > \Lambda_1$. For that, we use that u_λ is the maximal solution to (P_λ).

6.4.1 Study of a regularised problem

Let $\varepsilon_0 > 0$. So, for $0 < \varepsilon < \varepsilon_0$, we consider the following perturbed problem :

$$(\mathrm{P}_{\lambda,\varepsilon}) \begin{cases} -\Delta u_\varepsilon = \lambda K(x)(u_\varepsilon + \varepsilon)^q - \dfrac{K(x) u_\varepsilon}{(u_\varepsilon + \varepsilon)^{1-r}} & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \quad u \geq 0 \quad \text{in } \Omega, \end{cases}$$

Let us prove that $(\mathrm{P}_{\lambda,\varepsilon})$ admits a maximal solution. Observe that u_λ the maximal solution to (P_λ) constructed in Theorem 2.1 is a subsolution of $(\mathrm{P}_{\lambda,\varepsilon})$. To get a suitable supersolution of problem $(\mathrm{P}_{\lambda,\varepsilon})$, we consider the following problem :

$$\left(\overline{\mathrm{P}}_{\lambda,\varepsilon}\right) \begin{cases} -\Delta v = \lambda K(x)(v + \varepsilon)^q & \text{in } \Omega, \\ v = 0 \quad \text{on } \partial\Omega, \quad v \geq 0 \quad \text{in } \Omega, \end{cases}$$

Proposition 6.4 *Problem $\left(\overline{\mathrm{P}}_{\lambda,\varepsilon}\right)$ has a maximal solution $\overline{u}_{\lambda,\varepsilon} \in \mathscr{C}^{1,\alpha}(\overline{\Omega})$, for some $0 < \alpha < 1$ satisfying*

$$U_\lambda \leq \overline{u}_{\lambda,\varepsilon} \leq MV \quad \text{in } \Omega, \tag{I.61}$$

where U_λ *and* V *are respectively defined in* (I.46) *and* (I.47) *and* $M > 0$ *is chosen large enough. Moreover, for* $0 < \varepsilon' \le \varepsilon < \varepsilon_0$,

$$\overline{u}_{\lambda,\varepsilon'} \le \overline{u}_{\lambda,\varepsilon} \quad \textit{in } \Omega. \tag{I.62}$$

Proof. Proposition 6.4 follows from the lower- and upper-solution method. Indeed, U_λ is a subsolution of $\left(\overline{\mathrm{P}}_{\lambda,\varepsilon}\right)$ independent of ε. Moreover, since V is bounded in Ω there exists $C > 0$ independent of M and ε such that

$$\lambda\left(V + \frac{\varepsilon}{M}\right)^q \le C \quad \text{in } \Omega.$$

Then,

$$-\Delta(MV) - K(x)(MV+\varepsilon)^q \ge M^q K(x)\left[M^{1-q} - C\right] \ge 0 \quad \text{in } \Omega,$$

for $M > 0$ large enough. Thus, MV is a supersolution to $\left(\overline{P}_{\lambda,\varepsilon}\right)$ and the existence of the maximal solution $\overline{u}_{\lambda,\varepsilon}$ follows. For $\varepsilon' \in (0,\varepsilon)$, $\overline{u}_{\lambda,\varepsilon'}$ is a subsolution of $\left(\overline{\mathrm{P}}_{\lambda,\varepsilon}\right)$ such that $\overline{u}_{\lambda,\varepsilon'} \le MV$ in Ω. Therefore, from the maximality of the solution $\overline{u}_{\lambda,\varepsilon}$, (I.62) follows. □

Now, we state the following theorem :

Theorem 6.2 *Problem* $(\mathrm{P}_{\lambda,\varepsilon})$ *has a maximal solution* $u_{\lambda,\varepsilon} \in \mathscr{C}^{1,\alpha}(\overline{\Omega})$, *for some* $0 < \alpha < 1$, *such that*

$$u_\lambda \le u_{\lambda,\varepsilon} \le \overline{u}_{\lambda,\varepsilon} \quad \textit{in } \Omega. \tag{I.63}$$

Moreover for $0 < \varepsilon' \le \varepsilon < \varepsilon_0$, *we have*

$$u_{\lambda,\varepsilon'} \le u_{\lambda,\varepsilon} \quad \textit{in } \Omega. \tag{I.64}$$

Proof. We consider the following iterative scheme :

$$\left(\mathrm{P}^n_{\lambda,\varepsilon}\right)\begin{cases} -\Delta u^n_\varepsilon + \dfrac{K(x)u^n_\varepsilon}{(u^{n-1}_\varepsilon+\varepsilon)^{1-r}} = \lambda K(x)(u^{n-1}_\varepsilon+\varepsilon)^q & \text{in } \Omega,\\ u^n_\varepsilon = 0 \quad \text{on } \partial\Omega, \quad u^n_\varepsilon \ge 0 \quad \text{in } \Omega, \end{cases}$$

with $u^0_\varepsilon = \overline{u}_{\lambda,\varepsilon}$. By induction on n, $(\mathrm{P}^n_{\lambda,\varepsilon})$ admits a unique solution $u^n_\varepsilon \in \mathscr{C}^2(\Omega)\cap\mathscr{C}(\overline{\Omega})$. Indeed, for $n = 1$ we get a solution u^1_ε of $(\mathrm{P}^1_{\lambda,\varepsilon})$ as a minimizer of the functional E_1 defined for all $v \in \mathrm{H}^1_0(\Omega)$ by

$$E_1(v) \overset{\text{def}}{=} \int_\Omega |\nabla v|^2\,dx + \frac{1}{2}\int_\Omega \frac{K(x)v^2}{(\overline{u}_{\lambda,\varepsilon}+\varepsilon)^{1-r}}\,dx - \lambda\int_\Omega K(x)(\overline{u}_{\lambda,\varepsilon}+\varepsilon)^q v\,dx. \tag{I.65}$$

Moreover,

$$-\Delta\left(u^1_\varepsilon - \overline{u}_{\lambda,\varepsilon}\right) + K(x)\left[\frac{u^1_\varepsilon - \overline{u}_{\lambda,\varepsilon}}{(\overline{u}_{\lambda,\varepsilon}+\varepsilon)^{1-r}}\right] \le 0$$

in $\mathrm{H}^{-1}(\Omega)$. Then by the weak maximum principle, $u^1_\varepsilon \le \overline{u}_{\lambda,\varepsilon}$ in Ω. And similarly, $u_\lambda \le u^1_\varepsilon$ in Ω. Now, let $n \in \mathbb{N}^*$. By the same method we prove the existence of u^n_ε solution of $(\mathrm{P}^n_{\lambda,\varepsilon})$ such that $u_\lambda \le u^n_\varepsilon \le \overline{u}_{\lambda,\varepsilon}$

in Ω. Moreover, we have

$$-\Delta\left(u_\varepsilon^{n+1}-u_\varepsilon^n\right)+K(x)\left[\frac{u_\varepsilon^{n+1}}{(u_\varepsilon^n+\varepsilon)^{1-r}}-\frac{u_\varepsilon^n}{(u_\varepsilon^{n-1}+\varepsilon)^{1-r}}\right]=K(x)\left[(u_\varepsilon^n)^q-(u_\varepsilon^{n-1})^q\right] \tag{I.66}$$

in $\mathrm{H}^{-1}(\Omega)$. So choosing $(u_\varepsilon^{n+1}-u_\varepsilon^n)^+\in \mathrm{H}_0^1(\Omega)$, we get

$$\int_\Omega K(x)\left[(u_\varepsilon^n)^q-(u_\varepsilon^{n-1})^q\right](u_\varepsilon^{n+1}-u_\varepsilon^n)^+\,dx\le 0$$

and

$$\int_\Omega K(x)\left[\frac{u_\varepsilon^{n+1}}{(u_\varepsilon^n+\varepsilon)^{1-r}}-\frac{u_\varepsilon^n}{(u_\varepsilon^{n-1}+\varepsilon)^{1-r}}\right](u_\varepsilon^{n+1}-u_\varepsilon^n)^+\,dx$$
$$\ge\int_\Omega K(x)\left[\frac{u_\varepsilon^{n+1}-u_\varepsilon^n}{(u_\varepsilon^n+\varepsilon)^{1-r}}\right](u_\varepsilon^{n+1}-u_\varepsilon^n)^+\,dx\ge 0$$

Hence finally,

$$\forall n\in\mathbb{N}^*,\qquad \underline{u}_\lambda\le u_\varepsilon^{n+1}\le u_\varepsilon^n\le\overline{u}_{\lambda,\varepsilon}\quad \text{ in }\Omega. \tag{I.67}$$

For all $x\in\Omega$, we define $u_{\lambda,\varepsilon}(x)=\lim_{n\to\infty}u_\varepsilon^n(x)$. We also have for all $n\in\mathbb{N}^*$,

$$\begin{aligned}\int_\Omega|\nabla u_{\lambda,\varepsilon}^n|^2\,dx &\le \int_\Omega K(x)(u_\varepsilon^{n-1}+\varepsilon)^q u_\varepsilon^n\,dx\\ &\le \int_\Omega K(x)(\overline{u}_{\lambda,\varepsilon}+\varepsilon)^q\overline{u}_{\lambda,\varepsilon}\,dx<+\infty.\end{aligned}$$

Hence, $(u_\varepsilon^n)_{n\in\mathbb{N}^*}$ is bounded in $\mathrm{H}_0^1(\Omega)$. Therefore, $u_{\lambda,\varepsilon}\in\mathrm{H}_0^1(\Omega)$ and up to a subsequence denoted in the same way, $u_\varepsilon^n \underset{n\to+\infty}{\rightharpoonup} u_{\lambda,\varepsilon}$ in $\mathrm{H}_0^1(\Omega)$ and a.e. in Ω. So passing to the limit in $(\mathrm{P}_{\lambda,\varepsilon}^n)$, $u_{\lambda,\varepsilon}$ is a weak solution of $(\mathrm{P}_{\lambda,\varepsilon})$ satisfying (I.63). Finally, the $\mathscr{C}^{1,\alpha}(\overline{\Omega})$ regularity of $u_{\lambda,\varepsilon}$ follows from Theorem 1.1 in Gui-Lin [53]. Now, let $0<\varepsilon'\le\varepsilon<\varepsilon$. Then, for $n=1$ we have

$$-\Delta\left(u_{\varepsilon'}^1-u_\varepsilon^1\right)+K(x)\left[\frac{u_{\varepsilon'}^1-u_\varepsilon^1}{(\overline{u}_{\lambda,\varepsilon}+\varepsilon)^{1-r}}\right]\le\lambda K(x)\left[(\overline{u}_{\lambda,\varepsilon}+\varepsilon')^q-(\overline{u}_{\lambda,\varepsilon}+\varepsilon)^q\right]\le 0.$$

Then by the weak maximum principle, $u_{\varepsilon'}^1\le u_\varepsilon^1$ in Ω. For $n\in\mathbb{N}^*$, by induction we have

$$\begin{aligned}&-\Delta\left(u_{\varepsilon'}^n-u_\varepsilon^n\right)+K(x)\left[\frac{u_{\varepsilon'}^n-u_\varepsilon^n}{(u_\varepsilon^{n-1}+\varepsilon)^{1-r}}\right]\\ &\qquad\le -\Delta\left(u_{\varepsilon'}^n-u_\varepsilon^n\right)+K(x)\left[\frac{u_{\varepsilon'}^n}{(u_{\varepsilon'}^{n-1}+\varepsilon')^{1-r}}-\frac{u_\varepsilon^n}{(u_\varepsilon^{n-1}+\varepsilon)^{1-r}}\right]\\ &\qquad= \lambda K(x)\left[\left(u_{\varepsilon'}^{n-1}+\varepsilon'\right)^q-\left(u_\varepsilon^{n-1}+\varepsilon\right)^q\right]\le 0.\end{aligned}$$

Hence, $u_{\varepsilon'}^n\le u_\varepsilon^n$ in Ω and passing to the limit as $n\to+\infty$, we finally get (I.64). □

6.4.2 Semi-stability of the maximal solution $\mathbf{u}_{\lambda,\varepsilon}$

Let $u_{\lambda,\varepsilon}$ be the maximal solution of $(\mathrm{P}_{\lambda,\varepsilon})$ obtained above and let us define the first eigenvalue of the linearised operator associated to $(\mathrm{P}_{\lambda,\varepsilon})$:

$$\Lambda_\varepsilon(\lambda) \stackrel{\text{def}}{=} \inf_{\substack{v \in \mathrm{H}_0^1(\Omega) \\ \|v\|_{\mathrm{L}^2(\Omega)}=1}} \mathscr{E}_{\lambda,\varepsilon}(v), \tag{I.68}$$

where $\mathscr{E}_{\lambda,\varepsilon}(v)$ is defined for all $v \in \mathrm{H}_0^1(\Omega)$ by

$$\begin{aligned}\mathscr{E}_{\lambda,\varepsilon}(v) \stackrel{\text{def}}{=} \int_\Omega |\nabla v|^2\,dx - \lambda q \int_\Omega \frac{K(x)v^2}{(u_{\lambda,\varepsilon}+\varepsilon)^{1-q}}\,dx \\ + \int_\Omega \frac{K(x)v^2}{(u_{\lambda,\varepsilon}+\varepsilon)^{1-r}}\,dx + (r-1)\int_\Omega \frac{K(x)u_{\lambda,\varepsilon}v^2}{(u_{\lambda,\varepsilon}+\varepsilon)^{2-r}}\,dx.\end{aligned} \tag{I.69}$$

Proposition 6.5 *There exists* $\Phi_\varepsilon \in \mathscr{H} \stackrel{\text{def}}{=} \{v \in \mathrm{H}_0^1(\Omega) \mid \|v\|_{\mathrm{L}^2} = 1\}$*, non-negative a.e. in* Ω *such that*

$$\mathscr{E}_{\lambda,\varepsilon}(\Phi_\varepsilon) = \min_{v \in \mathscr{H}} \mathscr{E}_{\lambda,\varepsilon}(v). \tag{I.70}$$

Hence, $\Phi_\varepsilon \in \mathscr{C}^{1,\alpha}(\overline{\Omega})$*, for some* $0 < \alpha < 1$ *and satisfies*

$$\begin{cases} -\Delta\Phi_\varepsilon = \Lambda_\varepsilon(\lambda)\Phi_\varepsilon + f'_{\lambda,\varepsilon}(u_{\lambda,\varepsilon})\Phi_\varepsilon \quad \text{in } \Omega, \\ \Phi_\varepsilon = 0 \quad \text{on } \partial\Omega, \quad \Phi_\varepsilon \geq 0 \quad \text{in } \Omega, \end{cases} \tag{I.71}$$

where

$$\forall v \in \mathrm{H}_0^1(\Omega), \quad f_{\lambda,\varepsilon}(v) \stackrel{\text{def}}{=} \lambda K(x)(v+\varepsilon)^q - \frac{K(x)v}{(v+\varepsilon)^{1-r}}. \tag{I.72}$$

Proof. By sake of clarity, we denote in (I.69)

$$\mathscr{E}_{\lambda,\varepsilon}(v) \stackrel{\text{def}}{=} \|v\|^2_{\mathrm{H}_0^1(\Omega)} - \lambda q I_1^\varepsilon(v) + I_2^\varepsilon(v) + (r-1)I_3^\varepsilon(v). \tag{I.73}$$

Using the Hardy's inequality, we get a similar estimate to (I.60) for $\mathscr{E}_{\lambda,\varepsilon}$ More precisely, for all $v \in \mathscr{H}$,

$$\int_\Omega |\nabla v|^2\,dx - \lambda q \int_\Omega K(x)(u_{\lambda,\varepsilon}+\varepsilon)^{q-1}v^2\,dx + \int_\Omega \frac{K(x)v^2}{(u_{\lambda\varepsilon}+\varepsilon)^{1-r}}\,dx + (r-1)\int_\Omega \frac{K(x)u_{\lambda,\varepsilon}v^2}{(u_{\lambda\varepsilon}+\varepsilon)^{2-r}}\,dx$$

and $\Lambda_\varepsilon(\lambda) \in \mathbb{R}$. So, let $(v_n)_{n\in\mathbb{N}} \subset \mathscr{H}$ be an associated minimizing sequence. We have, $(v_n)_{n\in\mathbb{N}}$ is bounded in $\mathscr{H}$ (see (I.60)). Therefore, there exist $\Phi_\varepsilon \in \mathscr{H}$ and a subsequence still denoted $(v_n)_{n\in\mathbb{N}}$ such that $v_n \underset{n\to+\infty}{\longrightarrow} \Phi_\varepsilon$ weakly in $\mathrm{H}_0^1(\Omega)$ and strongly in $\mathrm{L}^2(\Omega)$. and a.e. in Ω. Then we get

$$\|\Phi_\varepsilon\|_{\mathrm{H}_0^1(\Omega)} \leq \liminf_{n\to+\infty} \|v_n\|_{\mathrm{H}_0^1(\Omega)}.$$

Moreover, by Hardy's inequality $\left(\frac{v_n}{d(x)}\right)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^2(\Omega)$; therefore up to a subsequence $\frac{v_n}{d(x)} \underset{n\to+\infty}{\longrightarrow} \frac{\Phi_\varepsilon}{d(x)}$ in $\mathrm{L}^2(\Omega)$. Then, writing

$$I_1^\varepsilon(v_n) = \int_\Omega K(x)d(x)(u_{\lambda,\varepsilon}+\varepsilon)^{q-1}\left(\frac{v_n}{d(x)}\right)v_n\,dx$$

we get that $I_1^\varepsilon(v_n) \underset{n\to+\infty}{\longrightarrow} I_1^\varepsilon(\Phi_\varepsilon)$. And similarly,

$$I_2^\varepsilon(v_n) \underset{n\to+\infty}{\longrightarrow} I_2^\varepsilon(\Phi_\varepsilon) \quad \text{and} \quad I_3^\varepsilon(v) \underset{n\to+\infty}{\longrightarrow} I_3^\varepsilon(\Phi_\varepsilon).$$

Hence, $\mathscr{E}_{\lambda,\varepsilon}(\Phi_\varepsilon) \le \liminf_{n\to+\infty} \mathscr{E}_{\lambda,\varepsilon}(v_n)$ and

$$\mathscr{E}_{\lambda,\varepsilon}(\Phi_\varepsilon) = \min_{v\in\mathscr{H}} \mathscr{E}_{\lambda,\varepsilon}(v) = \Lambda_\varepsilon(\lambda).$$

Since for all $v\in\mathscr{H}$, $\mathscr{E}_{\lambda,\varepsilon}(v) = \mathscr{E}_{\lambda,\varepsilon}(|v|)$, we can assume $\Phi_\varepsilon \ge 0$ a.e in Ω. From variational arguments, Φ_ε is a weak solution to (I.71). Finally the $\mathscr{C}^{1,\alpha}(\overline{\Omega})$ Hölder regularity of Φ_ε follows from Gui-lin [53, Theorem 1.1]. □

Proposition 6.6 *Let $\lambda > 0$ and $0 < \varepsilon < \varepsilon_0$. Then, the solution $u_{\lambda,\varepsilon}$ of $(\mathrm{P}_{\lambda,\varepsilon})$ is semi-stable.*

Proof. Let us argue by contradiction suppose that $\Lambda_\varepsilon(\lambda) < 0$. Let $\varepsilon' > 0$ and consider $\underline{u}_{\lambda,\varepsilon} \stackrel{\text{def}}{=} u_{\lambda,\varepsilon} + \varepsilon'\Phi_\varepsilon$. Then, we have

$$-\Delta \underline{u}_{\lambda,\varepsilon} = f_{\lambda,\varepsilon}(u_{\lambda,\varepsilon}) + \varepsilon' f'_{\lambda,\varepsilon}(u_{\lambda,\varepsilon})\Phi_\varepsilon + \varepsilon'\Lambda_\varepsilon(\lambda)\Phi_\varepsilon \quad \text{in } \Omega,$$

with $f_{\lambda,\varepsilon}$ defined in (I.71). And by a Taylor-Lagrange expansion

$$f_{\lambda,\varepsilon}(\underline{u}_{\lambda,\varepsilon}) = f_{\lambda,\varepsilon}(u_{\lambda,\varepsilon}) + \varepsilon' f'_{\lambda,\varepsilon}(u_{\lambda,\varepsilon})\Phi_\varepsilon + \frac{1}{2}(\varepsilon'\Phi_\varepsilon)^2 f''_{\lambda,\varepsilon}(u_{\lambda,\varepsilon} + \theta\varepsilon'\Phi_\varepsilon) \quad \text{in } \Omega,$$

with $\theta \in (0,1)$ and

$$\forall v \in H_0^1(\Omega), \quad f''_{\lambda,\varepsilon}(v) = \lambda q(q-1)\frac{K(x)}{(v+\varepsilon)^{2-q}} + r(1-r)\frac{K(x)}{(v+\varepsilon)^{2-r}} - \varepsilon(1-r)(2-r)\frac{K(x)}{(v+\varepsilon)^{3-r}}.$$

By Theorem 2.1 in Gui-Lin [53], $\Phi_\varepsilon(x) \sim d(x)$ in Ω, therefore there exists a positive constant C independent of Θ and ε' such that

$$\left|\Phi_\varepsilon^2 f''_{\lambda,\varepsilon}(u_{\lambda,\varepsilon} + \theta\varepsilon'\Phi_\varepsilon)\right| \le C \quad \text{in } \Omega. \tag{I.74}$$

So, choosing ε' small enough,

$$\varepsilon'\Lambda_\varepsilon(\lambda)\Phi_\varepsilon < \frac{1}{2}(\varepsilon'\Phi_\varepsilon)^2 f''_{\lambda,\varepsilon}(u_{\lambda,\varepsilon} + \theta\varepsilon'\Phi_\varepsilon) \quad \text{in } \Omega$$

and $\underline{u}_{\lambda,\varepsilon}$ is a subsolution of $(\mathrm{P}_{\lambda,\varepsilon})$. Moreover, using the Brézis-Nirenberg [14] strong maximum principle, $u_{\lambda,\varepsilon} < \overline{u}_{\lambda,\varepsilon_0}$ in Ω and since $\Phi_\varepsilon \in \mathscr{C}^{1,\alpha}(\overline{\Omega})$, for $\varepsilon' > 0$ sufficiently small, we have

$$\underline{u}_{\lambda,\varepsilon} \leq \overline{u}_{\lambda,\varepsilon_0} \quad \text{in } \Omega.$$

Hence, using the same sub and supersolution technique, we get the existence of $\tilde{u}_{\lambda,\varepsilon}$ weak solution of $(\mathrm{P}_{\lambda,\varepsilon})$ such that

$$\underline{u}_{\lambda,\varepsilon} \leq \tilde{u}_{\lambda,\varepsilon} \leq \overline{u}_{\lambda,\varepsilon_0} \quad \text{in } \Omega,$$

which contradicts the maximality of $u_{\lambda,\varepsilon}$ in Theorem 6.2. □

6.4.3 Semi-stability of the solution $\mathbf{u}_\lambda$ for $\lambda \geq \mathbf{\Lambda_1}$

To prove the semi-stability of the maximal solution u_λ of (P_λ), we pass to the limit when $\varepsilon \to 0^+$. Indeed, from (I.61), (I.62), (I.63) and (I.64), let us define for all $x \in \Omega$,

$$\tilde{U}_\lambda(x) \stackrel{\text{def}}{=} \lim_{\varepsilon \to 0+} \overline{u}_{\lambda,\varepsilon}(x) \quad \text{and} \quad \tilde{u}_\lambda(x) \stackrel{\text{def}}{=} \lim_{\varepsilon \to 0+} u_{\lambda,\varepsilon}(x).$$

Then, passing to the limit in (I.63), we get

$$u_\lambda \leq \tilde{u}_\lambda \leq \tilde{U}_\lambda \quad \text{in } \Omega. \tag{I.75}$$

We also have for all $\varepsilon \in (0, \varepsilon_0)$ and all $v \in \mathrm{H}_0^1(\Omega)$,

$$\int_\Omega \nabla \overline{u}_{\lambda,\varepsilon}.\nabla v\, dx = \lambda \int_\Omega K(x)(\overline{u}_{\lambda,\varepsilon} + \varepsilon)^q v\, dx. \tag{I.76}$$

So choosing $\overline{u}_{\lambda,\varepsilon}$ as test function, we get

$$\begin{aligned} \int_\Omega |\nabla \overline{u}_{\lambda,\varepsilon}|^2\, dx &= \lambda \int_\Omega K(x)(\overline{u}_{\lambda,\varepsilon} + \varepsilon)^q \overline{u}_{\lambda,\varepsilon}\, dx. \\ &\leq \lambda M \int_\Omega K(x)(MV + \varepsilon_0)^q V\, dx < +\infty. \end{aligned}$$

Then, $(\overline{u}_{\lambda,\varepsilon})_{\varepsilon>0}$ is bounded in $\mathrm{H}_0^1(\Omega)$. So, up to a subsequence, passing to the limit when $\varepsilon \to 0^+$ in (I.61) and in (I.76), $\tilde{U}_\lambda$ is a weak solution of $\left(\overline{\mathrm{P}}_\lambda\right)$ satisfying

$$U_\lambda \leq \tilde{U}_\lambda \leq MV \quad \text{in } \Omega.$$

Hence, by uniqueness of a such solution of $\left(\overline{\mathrm{P}}_\lambda\right)$, $U_\lambda = \tilde{U}_\lambda$ in Ω. Similarly, $(u_{\lambda,\varepsilon})_{\varepsilon>0}$ is bounded in $\mathrm{H}_0^1(\Omega)$ and then $\tilde{u}_\lambda$ is a weak solution to (P_λ) such that

$$u_\lambda \leq \tilde{u}_\lambda \leq U_\lambda \quad \text{in } \Omega.$$

Hence, since u_λ is a maximal solution, it follows that $\tilde{u}_\lambda \equiv u_\lambda$ in Ω.

So finally, since $\Lambda_\varepsilon(\lambda) \geq 0$, $\mathscr{E}_{\lambda,\varepsilon}(v) \geq 0$ for all $v \in \mathscr{H}$. With the notations used in (I.73),

$$I_1^\varepsilon(v) \xrightarrow[\varepsilon\to 0^+]{} \int_\Omega K(x) u_\lambda^{r-1} v^2\,dx \quad \text{and} \quad I_2^\varepsilon(v) + (r-1) I_3^\varepsilon(v) \xrightarrow[\varepsilon\to 0^+]{} r\int_\Omega K(x) u_\lambda^{q-1} v^2\,dx.$$

Hence, $\Lambda(\lambda) = \lim\limits_{\varepsilon\to 0^+} \Lambda_\varepsilon(\lambda) \geq 0$, which proves the semi-stability of u_λ. Moreover, by inequality (I.50) and Dini's theorem, $u_\lambda \xrightarrow[\lambda\to\Lambda_1^+]{} u_{\Lambda_1}$ in $\mathrm{L}^\infty(\Omega)$. So, we also have $\Lambda(\Lambda_1) \geq 0$.

6.4.4 Stability of $\mathbf{u}_\lambda$ for lambda $> \mathbf{\Lambda_1}$

Finally, let us prove that $\Lambda(\lambda) > 0$ for $\lambda > \Lambda_1$. For that we introduce the following new perturbed problem :

$$\left(\mathrm{P}_\lambda^\theta\right) \begin{cases} -\Delta u = K(x)\left(\lambda u^q - u^r + \theta\right) & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \quad u \geq 0 & \text{in } \Omega, \end{cases}$$

with $\theta \in \mathbb{R}$. As above, we first show the existence of a branch of maximal solutions denoted $u_\lambda^\theta \in \mathscr{C}^{1,\alpha}(\overline{\Omega})$ to problem $\left(\mathrm{P}_\lambda^\theta\right)$ for $\lambda > \Lambda_{1,\theta}$, where

$$\Lambda_{1,\theta} \stackrel{\text{def}}{=} \inf\left\{\lambda > 0 \;\middle|\; \left(\mathrm{P}_\lambda^\theta\right) \text{ has a positive solution a.e. in } \Omega\right\}. \tag{I.77}$$

As above, we have

$$\Lambda^\theta(\lambda) \stackrel{\text{def}}{=} \inf_{v\in\mathscr{H}} \mathscr{E}^\theta(v) \geq 0,$$

with $v \in \mathrm{H}_0^1(\Omega)$ and

$$\mathscr{E}^\theta(v) \stackrel{\text{def}}{=} \int_\Omega |\nabla u|^2\,dx + r\int_\Omega K(x)\left(u_\lambda^\theta\right)^{r-1} u^2\,dx - \lambda q\int_\Omega K(x)\left(u_\lambda^\theta\right)^{q-1} u^2\,dx.$$

We now prove the following result :

Lemma 6.2 *Assume $\lambda > \Lambda_1$. Then,*

1. *there exists $\theta_0 < 0$ such that $u_\lambda^{\theta_0} > 0$ a.e. in Ω ;*
2. *the mapping $\theta \longmapsto \Lambda^\theta(\lambda)$ is increasing on $(\theta_0, +\infty)$.*

Proof.

1. By Proposition 6.1, for $\lambda > \Lambda_1$ there exist two constants $C_1, C_2 > 0$ depending on λ such that

$$C_1 d(x) \leq u_\lambda(x) \leq C_2 d(x), \quad \text{for all } x \in \Omega.$$

Then, let us choose $\lambda' \in (\Lambda_1, \lambda)$ and ε small enough to satisfy

$$\frac{\lambda}{\lambda'} u_{\lambda'} \geq u_{\lambda'} + \varepsilon V \quad \text{a.e. in } \Omega \tag{I.78}$$

and

$$\left[\frac{\lambda}{\lambda'} - \left(\frac{\lambda}{\lambda'}\right)^{\frac{1}{r}}\right] u_{\lambda'} \geq \varepsilon V \quad \text{a.e. in } \Omega, \tag{I.79}$$

with V solution to (I.47). Defining $\underline{w} \stackrel{\text{def}}{=} \frac{\lambda}{\lambda'} u_{\lambda'} - \varepsilon V$, we get

$$-\Delta \underline{w} \leq K(x)\left(\lambda \underline{w}^q - \underline{w}^r - \varepsilon\right) \quad \text{in } \Omega$$

and as in the proof of Proposition 6.3, we prove the existence of $w \in \mathscr{C}^{1,\alpha}(\overline{\Omega})$, for some $0 < \alpha < 1$, solution to

$$\begin{cases} -\Delta w = K(x)\left(\lambda w^q - w^r - \varepsilon\right) & \text{in } \Omega, \\ w = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{I.80}$$

It suffices to choose $\theta_0 \in (-\varepsilon, 0)$ to get assertion *1.* of the Lemma 6.2.

2. The second assertion of Lemma 6.2 follows from the strong maximum principle from which we get that, for $\theta < \theta'$ the positive maximal solutions to $\left(\mathrm{P}_\lambda^\theta\right)$ and $\left(\mathrm{P}_\lambda^{\theta'}\right)$ satisfy

$$u_\lambda^\theta < u_\lambda^{\theta'} \quad \text{in } \Omega. \tag{I.81}$$

Then, noting that as previously, or every $\theta \in (\theta_0, +\infty)$, the infimum

$$\Lambda^\theta(\lambda) \stackrel{\text{def}}{=} \inf_{u \in \mathscr{H}} \left\{ \int_\Omega |\nabla u|^2 \, dx + r \int_\Omega K(x) \left(u_\lambda^\theta\right)^{r-1} u^2 \, dx - \lambda q \int_\Omega K(x) \left(u_\lambda^\theta\right)^{q-1} u^2 \, dx \right\}$$

is achieved for an element $\Phi^\theta \in H_0^1(\Omega)$, we finally get $\Lambda^{\theta'}(\lambda) > \Lambda^\theta(\lambda)$. □

Thanks to this lemma, $\Lambda(\lambda) = \Lambda^0(\lambda) > \Lambda^{-\varepsilon}(\lambda) \geq 0$, which completes the proof.

Remark 6.3 *When $\lambda > \Lambda_1$, u_λ is the unique positive and semi-stable solution of* (P_λ)*. Indeed, let us suppose there exists an other positive and semi stable solution $v_\lambda \in \mathrm{H}_0^1(\Omega)$, therefore by the strong maximum principle $v_\lambda < u_\lambda$ in Ω. By hypothesis, for every $u \in \mathrm{H}_0^1(\Omega)$,*

$$\int_\Omega K(x)\left(\lambda q v_\lambda^{q-1} - r v_\lambda^{r-1}\right) u^2 \, dx \leq \int_\Omega |\nabla u|^2 \, dx. \tag{I.82}$$

Choosing $u = u_\lambda - v_\lambda \in \mathrm{H}_0^1(\Omega)$ this estimate becomes

$$\int_\Omega K(x)\left(\lambda q v_\lambda^{q-1} - r v_\lambda^{r-1}\right)(u_\lambda - v_\lambda)^2 \, dx \leq \int_\Omega |\nabla(u_\lambda - v_\lambda)|^2 \, dx. \tag{I.83}$$

Since, u_λ and v_λ both are solution to (P_λ)*, then we also have*

$$\int_\omega K(x)\left[\lambda\left(u_\lambda^q - v_\lambda^q\right) - \left(u_\lambda^r - v_\lambda^r\right)\right](u_\lambda - v_\lambda) \, dx = \int_\Omega |\nabla(u_\lambda - v_\lambda)|^2 \, dx. \tag{I.84}$$

Combining (I.83) *and* (I.84)*, we get*

$$\int_\Omega K(x)(u_\lambda - v_\lambda)\left\{\left[\left(\lambda u_\lambda^q - u_\lambda^r\right) - \left(\lambda u_\lambda^q - u_\lambda^r\right)\right] - \left(\lambda q v_\lambda^{q-1} - r v_\lambda^{r-1}\right)(u_\lambda - v_\lambda)\right\} dx \geq 0, \tag{I.85}$$

which contradicts $(u_\lambda - v_\lambda) > 0$ in Ω because by concavity of $t \mapsto \lambda t^q - t^r$,

$$\left[\left(\lambda u_\lambda^q - u_\lambda^r\right) - \left(\lambda u_\lambda^q - u_\lambda^r\right)\right] - \left(\lambda q v_\lambda^{q-1} - r v_\lambda^{r-1}\right)(u_\lambda - v_\lambda) \leq 0 \quad \text{in } \Omega.$$

Therefore u_λ *is the unique solution among the positive and semi-stable solutions of* (P_λ).

Chapitre II

Study of elliptic quasilinear and singular systems

Nous présentons ici les résultats issus de Giacomoni-Hernández-Sauvy [45], travail réalisé en collaboration avec Jacques Giacomoni et Jesús Hernández, Professeur à l'Unversité Autonome de Madrid.

1 Introduction

In this chapter we are interested in the following quasilinear elliptic and singular system,

$$
\text{(P)}\begin{cases} -\Delta_p u = f_1(x,u,v) \quad \text{dans } \Omega\,; \quad u|_{\partial\Omega} = 0, \quad u > 0 \quad \text{in } \Omega, \\ \\ -\Delta_q v = f_2(x,u,v) \quad \text{dans } \Omega\,; \quad v|_{\partial\Omega} = 0, \quad v > 0 \quad \text{in } \Omega. \end{cases}
$$

Here, Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 2$ with $\mathscr{C}^2$ boundary $\partial\Omega$, $\Delta_r u \overset{\text{def}}{=} \text{div}(|\nabla u|^{r-2}\nabla u)$ denotes the r-Laplace operator and $1 < p, q < \infty$. In the right-hand sides, f_1 and f_2 are two Carathéodory functions in $\Omega \times (\mathbb{R}_+^* \times \mathbb{R}_+^*)$ possibly singular. More precisely, for every $(t_1,t_2) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ and for almost every $x \in \Omega$, we assume that

(H_1) $f_1(\cdot,t_1,t_2)$ and $f_2(\cdot,t_1,t_2)$ are Lebesgue measurable in Ω,

(H_2) $f_1(x,\cdot,\cdot)$ and $f_2(x,\cdot,\cdot)$ are in $\mathscr{C}^1(\mathbb{R}_+^* \times \mathbb{R}_+^*)$.

We aim to establish the existence of a positive weak solutions pair to problem (P) using the Schauder Fixed Point Theorem. Namely, if we can compose two order-reversing mappings,

$$
(u,v) \mapsto T_1(u,v) \overset{\text{def}}{=} \tilde{u} \quad \text{and} \quad (u,v) \mapsto T_2(u,v) \overset{\text{def}}{=} \tilde{v}, \tag{II.1}
$$

where $\tilde{u} \in \mathrm{W}_0^{1,p}(\Omega)$ and $\tilde{v} \in \mathrm{W}_0^{1,q}(\Omega)$ are defined to be the (unique) positive weak solution to the Dirichlet problems

$$
-\Delta_p \tilde{u} + h_1(x,\tilde{u}) = f_1(x,u,v) + h_1(x,u) \quad \text{in } \Omega; \quad \tilde{u}|_{\partial\Omega=0}, \quad \tilde{u} > 0 \quad \text{in } \Omega, \tag{II.2}
$$

$$
-\Delta_q \tilde{v} + h_2(x,\tilde{v}) = f_2(x,u,v) + h_2(x,v) \quad \text{in } \Omega; \quad \tilde{v}|_{\partial\Omega=0}, \quad \tilde{v} > 0 \quad \text{in } \Omega, \tag{II.3}
$$

respectively, in suitable conical shells of positive cones in $\mathrm{W}_0^{1,p}(\Omega)$ and $\mathrm{W}_0^{1,q}(\Omega)$, with appropriate functions h_1 and h_2 ; then any fixed point of the mapping

$$(u,v) \mapsto T(u,v) \stackrel{\text{def}}{=} (T_1(u,v), T_2(u,v)) \tag{II.4}$$

is a positive weak solution pair to (P) and *vice-versa*. To prove that T is well defined and invariant in some conical shell, we use monotonicity methods together with the existence of sub- and super-solutions which prescribe the behaviour of the right-hand side singular non-linearities, namely f_1 and f_2, near the boundary $\partial\Omega$. The continuity and the compactness in $\mathscr{C}^{0,\alpha}(\overline{\Omega}) \times \mathscr{C}^{0,\alpha}(\overline{\Omega})$ for some suitable $0 < \alpha < 1$ follow from the regularity result Theorem 1.1 in Giacomoni-Schindler-Takáč [48] we recall in Appendix A (see Theorem 0.1). We derive further uniqueness results in case where the system (P) is competitive or cooperative (see Theorem 3.3). To establish the uniqueness of a positive pair of solutions to (P), it is essential that the mapping T is *sub-homogeneous*. In the cooperative and "strong" singular case, we also prove the existence of very weak solutions in $\mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega) \times \mathrm{W}^{1,q}_{\mathrm{loc}}(\Omega)$ (see Theorem 2.2).

Quasilinear elliptic systems have been quite intensively investigated in the literature with various methods. In De Thélin-Vélin [25], the authors take advantage of the variational structure of the problem to apply variational methods. In Clément-Manásevich-Mitidieri [17], a blow up argument combined with a Liouville theorem yields universal *a priori* bounds. Then, the existence of solutions is obtained by a topological degree argument (see also the review article De Figueiredo [22]). In Cuesta-Takáč [19], the key ingredients to prove existence of solutions are the Strong Comparison Principle and Kreĭn-Rutman theorem for homogeneous non-linear mappings. While dealing with sub-homogeneous systems, one usually appeals the method of sub- and super-solutions.

Related problems for ***singular*** quasilinear systems have been also studied in Lee-Shivaji-Ye [62] and Giacomoni-Hernández-Mouassaoui [44]. Accordingly, we study in this chapter a more general situation that handles more singular cases. We point out additionally that in the present work non-linearities f_1 and f_2 are not necessary non-negative.

The case of singular semi-linear systems ($p = q = 2$) has been studied even more frequently in Choi-McKenna [15], [16],Ghergu [42], Hernández-Mancebo [55], and Ni [68]. We refer to Hernández-Mancebo-Vega [56] for additional references on the subject.

Let us recall the main notations and definitions we will use throughout this chapter :

1. To $r > 1$ we associate $r' \stackrel{\text{def}}{=} \frac{r}{r-1} > 1$ and we denote by $\mathrm{W}^{-1,r'}(\Omega)$ the dual space of $\mathrm{W}^{1,r}_0(\Omega)$ with respect to the standard inner product in $\mathrm{L}^2(\Omega)$.

2. We denote by $d(x) \stackrel{\text{def}}{=} \inf_{y\in\partial\Omega} d(x,y)$, the distance from $x \in \Omega$ to $\partial\Omega$.

3. We denote by $D \stackrel{\text{def}}{=} \sup_{x,y\in\Omega} d(x,y)$, the diameter of the domain Ω.

4. Let $f, g : \Omega \longrightarrow [0,+\infty]$ be two non-negative functions of $\mathrm{L}^1_{\mathrm{loc}}(\Omega)$. Then, we write

$$f(x) \sim g(x) \qquad \text{in } \Omega$$

if there exist two positive constants C_1 and C_2 such that for almost every $x \in \Omega$,

$$C_1 g(x) \le f(x) \le C_2 g(x).$$

5. The function $\varphi_{1,r} \in \mathrm{W}_0^{1,r}(\Omega)$ denotes the positive and L^r-renormalized eigenfunction corresponding to the first eigenvalue of $-\Delta_r$,

$$\lambda_{1,r} \stackrel{\text{def}}{=} \inf \left\{ \int_\Omega |\nabla v|^r dx \in \mathbb{R}_+ \quad \middle| \quad v \in \mathrm{W}_0^{1,r}(\Omega) \quad \text{and} \quad \int_\Omega |v|^r dx = 1 \right\}.$$

It is a weak solution of the following eigenvalue problem :

$$-\Delta_r w = \lambda_{1,r} w^{r-1} \text{ in } \Omega; \quad w|_{\partial\Omega} = 0, \quad w > 0 \quad \text{in } \Omega.$$

Using Moser iterations, $\varphi_{1,r} \in \mathrm{L}^\infty(\Omega)$ and using the Hölder regularity result in Lieberman [64], $\varphi_{1,r} \in \mathscr{C}^{1,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$. Moreover the strong maximum and boundary principles from Vázquez [83], guarantee that $\varphi_{1,r}$ satisfies

$$\varphi_{1,r}(x) \sim d(x) \quad \text{in } \Omega. \tag{II.5}$$

6. We say that a Lebesgue measurable function $f : \Omega \to \mathbb{R}$ is ***locally uniformly positive*** if $\operatorname*{ess\,inf}_{x \in K} f(x) > 0$ holds over every compact set $K \subset \Omega$.

7. In this chapter, we primarily look for ***positive weak solution pairs*** (***positive solutions***, for short) of problem (P), that is, pairs of functions $(u, v) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ with both u and v locally uniformly positive and each satisfying the respective equation in problem (P) in the weak sense. More precisely, given $1 < r < \infty$ and $f \in \mathrm{W}^{-1,r'}(\Omega)$, we say that a function $u \in \mathrm{W}_0^{1,r}(\Omega)$ satisfies the equation

$$-\Delta_r u = f \quad \text{in } \Omega \tag{II.6}$$

in the weak sense if u is locally uniformly positive and satisfies

$$\forall w \in \mathrm{W}_0^{1,r}(\Omega), \quad \int_\Omega |\nabla u|^{r-2} \nabla u . \nabla w \, dx = \langle f, w \rangle_{\mathrm{W}^{-1,r'}(\Omega) \times \mathrm{W}_0^{1,r}(\Omega)}.$$

In the case where the existence of positive solutions of (P) cannot be established, we discuss the existence of weaker solutions. Then, we say that $(u, v) \in \mathrm{W}_{\mathrm{loc}}^{1,p}(\Omega) \times \mathrm{W}_{\mathrm{loc}}^{1,q}(\Omega)$ is a ***positive very weak solution pair*** of (P) if both u and v are locally uniformly positive and satisfy the respective equation in problem (P) in the distributions sense.

In the three last points, for $1 < r < +\infty$, $\mathscr{A}_r(\Omega)$ represents the space $\mathrm{W}_0^{1,r}(\Omega)$ or the space $\mathrm{W}_{\mathrm{loc}}^{1,r}(\Omega)$.

8. Let $\underline{w}, \overline{w} \in \mathscr{A}_r(\Omega)$, two locally uniformly positive functions such that $\underline{w} \le \overline{w}$ a.e. in Ω. We define

the convex and closed set, for the $\mathscr{C}(\overline{\Omega})$ topology

$$[\underline{w},\overline{w}] \stackrel{\text{def}}{=} \left\{ w \in \mathscr{C}(\overline{\Omega}) \quad | \quad \underline{w} \le w \le \overline{w} \quad \text{a.e. in } \Omega \right\}.$$

9. Let $\underline{u},\overline{u} \in \mathscr{A}_p(\Omega)$ and $\underline{v},\overline{v} \in \mathscr{A}_q(\Omega)$ four locally uniformly positive functions such that $\underline{u} \le \overline{u}$ a.e. in Ω and $\underline{v} \le \overline{v}$ a.e. in Ω. The couples $(\underline{u},\underline{v})$ and $(\overline{u},\overline{v})$ are said to be ***sub- and super-solutions*** pairs to (P) if the following inequalities are satisfied in the sense of distributions

$$-\Delta_p \underline{u} \le f_1(x,\underline{u},v) \quad \text{in } \Omega, \quad \text{for any } v \in [\underline{v},\overline{v}], \tag{II.7}$$

$$-\Delta_q \underline{v} \le f_2(x,u,\underline{v}) \quad \text{in } \Omega, \quad \text{for any } u \in [\underline{u},\overline{u}], \tag{II.8}$$

$$-\Delta_p \overline{u} \ge f_1(x,\overline{u},v) \quad \text{in } \Omega, \quad \text{for any } v \in [\underline{v},\overline{v}], \tag{II.9}$$

$$-\Delta_q \overline{v} \ge f_2(x,u,\overline{v}) \quad \text{in } \Omega, \quad \text{for any } u \in [\underline{u},\overline{u}]. \tag{II.10}$$

10. Let $(\underline{u},\underline{v})$, $(\overline{u},\overline{v}) \in \mathscr{A}_p(\Omega) \times \mathscr{A}_q(\Omega)$ be respectively sub- and super-solutions pairs to (P). Then, the conical shell $[\underline{u},\overline{u}] \times [\underline{v},\overline{v}]$ is denoted by $\mathcal{C}$.

The chapter is organised as follows. The next section contains the statements and the proofs of our main results (Theorem 2.1 and Theorem 2.2). Different applications of Theorems 2.1 and 2.2 arising in population dynamics models are given in Section 3. Appendix A contains the regularity result (Theorem 0.1) used to prove Hölder continuity of solutions. Theorem 0.1 is proved in Giacomoni-Schindler-Takáč [48].

2 General results

Theorem 2.1 *Let $(\underline{u},\underline{v}),(\overline{u},\overline{v}) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ be sub- and super-solutions pairs to* (P) *and assume in addition that the following conditions hold :*

1. *there exist constants $k_1,k_2 > 0$ and $\delta_1,\delta_2 \in \mathbb{R}$ such that*

$$|f_1(x,u,v)| \le k_1 d(x)^{\delta_1} \quad \text{and} \quad |f_2(x,u,v)| \le k_2 d(x)^{\delta_2} \quad \text{in } \Omega \times \mathcal{C}, \tag{II.11}$$

2. *there exist constants $C_1,C_2 > 0$ and $b_1,b_2 > 0$ such that*

$$\overline{u} \le C_1 d(x)^{b_1} \quad \text{and} \quad \overline{v} \le C_2 d(x)^{b_2} \quad \text{in } \Omega, \tag{II.12}$$

3. *and there exist $\kappa_1,\kappa_2 > 0$ and $\alpha_1,\alpha_2 > 0$ such that*

$$\left| \frac{\partial f_1}{\partial u}(x,u,v) \right| \le \kappa_1 d(x)^{\delta_1-\alpha_1} \quad \text{in } \Omega \times \mathcal{C}, \tag{II.13}$$

$$\left|\frac{\partial f_2}{\partial v}(x,u,v)\right| \leq \kappa_2 d(x)^{\delta_2-\alpha_2} \quad in\ \Omega\times\mathcal{C}, \tag{II.14}$$

with the following conditions on the coefficients

$$\delta_1 > -2+\frac{1}{p}+(\alpha_1-b_1)^+, \qquad \delta_2 > -2+\frac{1}{q}+(\alpha_2-b_2)^+. \tag{II.15}$$

Then, there exists a positive weak solutions pair $(u,v)\in \mathrm{W}_0^{1,p}(\Omega)\times \mathrm{W}_0^{1,q}(\Omega)$ *such that* $(u,v)\in\mathcal{C}$.

Remark 2.1 *Instead of conditions* (II.13) *and* (II.14), *as in* Giacomoni-Hernández-Mouassaoui *[44], we can rather suppose that there exist* $\kappa_1,\kappa_2>0$ *and* $\alpha_1,\alpha_2>0$ *such that for all* $(u,v)\in\mathcal{C}$,

$$w\mapsto f_1(x,w,v)+\kappa_1 d(x)^{\delta_1-\alpha_1}w^{p-1} \ \textit{is non decreasing on } [\underline{u},\overline{u}],$$

$$w\mapsto f_2(x,u,w)+\kappa_2 d(x)^{\delta_2-\alpha_2}w^{q-1} \ \textit{is non decreasing on } [\underline{v},\overline{v}].$$

Replacing condition (II.15) *by*

$$\delta_1 > -2-\frac{1}{p}+(\alpha_1-(p-1)b_1)^+, \qquad \delta_2 > -2+\frac{1}{q}+(\alpha_2-(q-1)\,b_2)^+,$$

we get the same result and the condition is sharper if $p,q>2$. *For that, it suffices to replace the first equation of the problem* (Q), *given below, by*

$$-\Delta_p w+\tilde{g}_1(x,w) = f_1(x,u,v)+\kappa_1 d(x)^{\delta_1-\alpha_1}u^{p-1} \ \ in\ \ \Omega,$$

with $\tilde{g}_1:\Omega\times\mathbb{R}\to\mathbb{R}_+^*$ *the cut-off function defined as follows :*

$$\tilde{g}_1(x,z) \stackrel{\text{def}}{=} \begin{cases} \kappa_1 d(x)^{\delta_1-\alpha_1}\overline{u}^{p-1} & \textit{if } z\geq \overline{u}(x),\\ \kappa_1 d(x)^{\delta_1-\alpha_1}z^{p-1} & \textit{if } z\in[0,\overline{u}(x)]\,,\\ 0 & \textit{if } z\leq 0\end{cases} \tag{II.16}$$

and proceed similarly for the second equation of (P).

Proof. Let $(u,v)\in\mathcal{C}$. We first prove the existence of $T_1(u,v)\in \mathrm{W}_0^{1,p}(\Omega)$, where $T_1(u,v)$ is defined in (II.2) with $h_1(x,u)\stackrel{\text{def}}{=}\kappa_1 d(x)^{\delta_1-\alpha_1}u$ in $\Omega\times[\underline{u},\overline{u}]$. For that, let us introduce the following problem :

$$(\mathrm{Q})\begin{cases} -\Delta_p w+g_1(x,w) = f_1(x,u,v)+\kappa_1 d(x)^{\delta_1-\alpha_1}u \ \text{ in } \ \Omega,\\ \\ w|_{\partial\Omega}=0, \quad w>0 \quad \text{in } \Omega,\end{cases}$$

with $g_1:\Omega\times\mathbb{R}\to\mathbb{R}_+^*$ the cut-off function defined as follows :

$$g_1(x,z) \stackrel{\text{def}}{=} \begin{cases} \kappa_1 d(x)^{\delta_1-\alpha_1}\overline{u} & \text{if } z\geq \overline{u}(x),\\ \kappa_1 d(x)^{\delta_1-\alpha_1}z & \text{if } z\in[0,\overline{u}(x)]\,,\\ 0 & \text{if } z\leq 0.\end{cases} \tag{II.17}$$

Then, g_1 is a Carathéodory function on $\Omega \times \mathbb{R}$. Thus, for $(x,s) \in \Omega \times \mathbb{R}$, setting $G_1(x,s) \stackrel{\text{def}}{=} \int_0^s g_1(x,z)dz$, we consider the following functional : $\forall w \in \mathrm{W}_0^{1,p}(\Omega)$,

$$E(w) \stackrel{\text{def}}{=} \frac{1}{p}\int_\Omega |\nabla w|^p\,dx + \int_\Omega G_1(x,w)\,dx - \int_\Omega \Big(f_1(x,u,v) + \kappa_1 d(x)^{\delta_1-\alpha_1}u\Big)\,w\,dx.$$

By assumption (II.15) and Hardy's inequality, E is well defined in $\mathrm{W}_0^{1,p}(\Omega)$ and for all $w \in \mathrm{W}_0^{1,p}(\Omega)$,

$$E(w) \geq \frac{1}{p}\|w\|^p_{\mathrm{W}_0^{1,p}(\Omega)} - C\left\|\Big(f_1(x,u,v) + \kappa_1 d(x)^{\delta_1-\alpha_1}u\Big)\,d(x)\right\|_{\mathrm{L}^{p'}(\Omega)}\|w\|_{\mathrm{W}_0^{1,p}(\Omega)}. \tag{II.18}$$

So, let us define

$$I \stackrel{\text{def}}{=} \inf_{w\in \mathrm{W}_0^{1,p}(\Omega)} E(w) \tag{II.19}$$

and let $(w_n)_{n\in\mathbb{N}} \subset \mathrm{W}_0^{1,p}(\Omega)$ be a minimizing sequence of E, *i.e.* $\lim_{n\to\infty} E(w_n) = I$. Using (II.18), $(w_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{W}_0^{1,p}(\Omega)$, therefore there exists a subsequence $(w_{n_k})_{k\in\mathbb{N}}$ and $\tilde{u} \in \mathrm{W}_0^{1,p}(\Omega)$ such that $w_{n_k} \underset{k\to\infty}{\rightharpoonup} \tilde{u}$ in $\mathrm{W}_0^{1,p}(\Omega)$ and a.e. in Ω. Therefore,

$$\liminf_{k\to\infty} \|w_{n_k}\|_{\mathrm{W}_0^{1,p}(\Omega)} \geq \|\tilde{u}\|_{\mathrm{W}_0^{1,p}(\Omega)}$$

and using Fatou's lemma,

$$\liminf_{k\to\infty}\int_\Omega G_1(x,w_{n_k})dx \geq \int_\Omega \liminf_{k\to\infty} G_1(x,w_{n_k})\,dx = \int_\Omega G_1(x,\tilde{u})dx.$$

Hence, $E(\tilde{u}) = I$ and $\tilde{u}$ is a solution to the Euler-Lagrange equation associated to E_0, that is :

$$\int_\Omega |\nabla\tilde{u}|^{p-2}\nabla\tilde{u}.\nabla w\,dx + \int_\Omega g_1(x,\tilde{u})w\,dx = \int_\Omega \Big(f_1(x,u,v) + \kappa_1 d(x)^{\delta_1-\alpha_1}u\Big)\,w\,dx, \tag{II.20}$$

for any $w \in \mathrm{W}_0^{1,p}(\Omega)$. Now let us prove that $\tilde{u} \in [\underline{u},\overline{u}]$. Combining (II.7) and (II.20), we get for all $w \in \mathrm{W}_0^{1,p}(\Omega)^+ \stackrel{\text{def}}{=} \{w \in \mathrm{W}_0^{1,p}(\Omega) \mid w \geq 0 \text{ a.e in } \Omega\}$,

$$\begin{aligned}\int_\Omega \Big(|\nabla\tilde{u}|^{p-2}\nabla\tilde{u} - |\nabla\underline{u}|^{p-2}\nabla\underline{u}\Big).\nabla w\,dx + \int_\Omega (g_1(x,\tilde{u}) - g_1(x,\underline{u}))\,w\,dx \\ \geq \int_\Omega \Big[\Big(f_1(x,u,v) + \kappa_1 d(x)^{\delta_1-\alpha_1}u\Big) - \Big(f_1(x,\underline{u},v) + \kappa_1 d(x)^{\delta_1-\alpha_1}\underline{u}\Big)\Big]\,w\,dx.\end{aligned} \tag{II.21}$$

By assumption (II.13), applying this inequality with $w = (\tilde{u}-\underline{u})^- \in \mathrm{W}_0^{1,p}(\Omega)^+$, we get $\tilde{u} \geq \underline{u}$ a.e. in Ω. Similarly, combining (II.9) and (II.20) we also get $\tilde{u} \leq \overline{u}$ a.e. in Ω. Then, $\tilde{u}$ satisfies the equation

$$-\Delta_p\tilde{u} + \kappa_1 d(x)^{\delta_1-\alpha_1}\tilde{u} = f_1(x,u,v) + \kappa_1 d(x)^{\delta_1-\alpha_1}u \quad \text{ in } \Omega, \tag{II.22}$$

in the weak sense. Moreover, using a classical local regularity result in Serrin [78], $\tilde{u} \in \mathscr{C}^{1,\gamma}(K)$ for some $\gamma > 0$ in any compact subset K of Ω. So using inequality (II.12), $\tilde{u} \in \mathscr{C}(\overline{\Omega})$, which gives us that $\tilde{u} \in [\underline{u},\overline{u}]$. Finally, by the weak comparison principle, $\tilde{u}$ is the unique function in the conical shell $[\underline{u},\overline{u}]$

satisfying (II.22). Then, the mapping $T_1 : (u,v) \mapsto \tilde{u}$ is well-defined from $\mathcal{C}$ to $[\underline{u}, \overline{u}]$. In the same spirit, we get the existence of the mapping $T_2 : (u,v) \mapsto \tilde{v}$ defined from $\mathcal{C}$ to $[\underline{v}, \overline{v}]$, where $\tilde{v}$ is the unique weak solution in $[\underline{v}, \overline{v}]$ of

$$-\Delta_p \tilde{v} + \kappa_2 d(x)^{\delta_2 - \alpha_2} \tilde{v} = f_2(x,u,v) + \kappa_2 d(x)^{\delta_2-\alpha_2} v \quad \text{ in } \Omega. \tag{II.23}$$

This proves that the operator T defined in (II.4) is well-defined, with values in $\mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ and makes invariant the conical shell $\mathcal{C}$.

Now, the continuity and the compactness of T follow from a regularity result of Giacomoni-Schindler-Takáč [48] we recall in Appendix A. Indeed, let $(u_n, v_n)_{n\in\mathbb{N}} \subset \mathcal{C}$ and $(u,v) \in \mathcal{C}$ such that the sequence $(u_n, v_n) \to (u,v)$ in $\mathscr{C}(\overline{\Omega}) \times \mathscr{C}(\overline{\Omega})$ as $n \to +\infty$. Then, from Theorem 0.1 and assumptions (II.11), $(T_1(u_n,v_n) = \tilde{u}_n)_{n\in\mathbb{N}}$ is bounded in $\mathscr{C}^{0,\alpha}(\overline{\Omega})$, for some $0 < \alpha < 1$. By Ascoli-Arzelà theorem, there exists a sub-sequence $(\tilde{u}_{n_k})_{k\in\mathbb{N}}$ and $\tilde{u} \in [\underline{u}, \overline{u}]$ such that $\tilde{u}_{n_k} \to \tilde{u}$ uniformly in $\overline{\Omega}$ as $k \to \infty$. Moreover, using the local regularity result in Serrin [78], $(\tilde{u}_{n_k})_{k\in\mathbb{N}}$ is bounded in $\mathscr{C}^{1,\gamma}(K)$ for some $\gamma > 0$ and for any compact subset K of Ω which entails that up to a subsequence denoted again $(\tilde{u}_{n_k})_{k\in\mathbb{N}}$ such that $\nabla \tilde{u}_{n_k} \to \nabla \tilde{u}$ uniformly in K as $k \to +\infty$. Then, $\tilde{u}$ satisfies

$$-\Delta_p \tilde{u} + \kappa_1 d(x)^{\delta_1 - \alpha_1} \tilde{u} = f_1(x,u,v) + \kappa_1 d(x)^{\delta_1-\alpha_1} u \quad \text{ in } \Omega \tag{II.24}$$

in the distributions sense. Moreover, since $\tilde{u} \le \overline{u}$ a.e in Ω, $f_1(x,u,v) + \kappa_1 d(x)^{\delta_1-\alpha_1}(u - \tilde{u}) \in \mathrm{W}^{-1,p'}(\Omega)$, which implies that $\tilde{u} \in \mathrm{W}_0^{1,p}(\Omega)$. Hence $\tilde{u} \in [\underline{u}, \overline{u}]$ and is a weak solution of (II.24). By uniqueness of a such solution in $[\underline{u}, \overline{u}]$, it follows that $\tilde{u} = T_1(u,v)$ and all the sequence $(\tilde{u}_n)_{n\in\mathbb{N}}$ converges to $\tilde{u}$ in $\mathscr{C}(\overline{\Omega})$. The same arguments hold to prove that $T_2(u_n,v_n) \to T_2(u,v)$ uniformly in $\overline{\Omega}$ as $n \to +\infty$. Then, $T : \mathcal{C} \to \mathcal{C}$ is continuous. Finally, it easy from the compact embedding of $\mathscr{C}^{0,\alpha}(\overline{\Omega})$ in $\mathscr{C}(\overline{\Omega})$ to get the compactness of T. Applying the Schauder Fixed Point Theorem to T in $\mathcal{C}$, the proof of Theorem 2.1 is now complete. □

We now give a more general result which guarantees the existence of a "very weak" positive solutions pair, in the cooperative case, when the inequalities (II.15) may not be satisfied.

Theorem 2.2 *Assume that* (P) *is a **cooperative system**, i.e.*

$$\frac{\partial f_1}{\partial v}(x,u,v) > 0 \quad \text{and} \quad \frac{\partial f_2}{\partial u}(x,u,v) > 0 \quad \text{ in } \Omega \times \mathbb{R}_+^* \times \mathbb{R}_+^*. \tag{II.25}$$

Let $(\underline{u}, \underline{v}), (\overline{u}, \overline{v}) \in \left[\mathscr{C}(\overline{\Omega}) \cap \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega)\right] \times \left[\mathscr{C}(\overline{\Omega}) \cap \mathrm{W}^{1,q}_{\mathrm{loc}}(\Omega)\right]$ *be sub- and super-solutions pairs to* (P). *Assume in addition that the following conditions hold :*

1. *there exist constants* $C_1, C_2 > 0$ *and* $b_1, b_2 > 0$ *such that*

$$\overline{u} \le C_1 d(x)^{b_1} \quad \text{and} \quad \overline{v} \le C_2 d(x)^{b_2} \quad \text{in } \Omega, \tag{II.26}$$

2. there exist $\kappa_1, \kappa_2 > 0$ and $\delta_1, \delta_2 \in \mathbb{R}$ such that

$$\left|\frac{\partial f_1}{\partial u}(x,u,v)\right| \leq \kappa_1 d(x)^{\delta_1} \quad \textit{and} \quad \left|\frac{\partial f_2}{\partial v}(x,u,v)\right| \leq \kappa_2 d(x)^{\delta_2} \quad \textit{in } \Omega \times \mathcal{C}. \tag{II.27}$$

Then, there exists a positive very weak solution pair $(u,v) \in \left[\mathrm{L}^\infty(\Omega) \cap \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega)\right] \times \left[\mathrm{L}^\infty(\Omega) \cap \mathrm{W}^{1,q}_{\mathrm{loc}}(\Omega)\right]$ to (P) *such that $(u,v) \in \mathcal{C}$.*

Remark 2.2 *Since f_1 and f_2 are continuous with respect to the two last variables in $\mathbb{R}^*_+ \times \mathbb{R}^*_+$, assumptions* (II.26) *and* (II.27) *imply that for any $K \subset\subset \Omega$, there exist $C_K, C'_K > 0$ such that*

$$|f_1(x,u,v)| \leq C_K \quad \textit{and} \quad |f_2(x,u,v)| \leq C'_K \quad \textit{in } K \times \mathcal{C}. \tag{II.28}$$

Proof. Since Ω is a smooth domain, we can introduce $(\Omega_n)_{n\in\mathbb{N}^*} \subset \Omega$ an increasing sequence of smooth sub-domains of Ω such that $\Omega_n \underset{n\to\infty}{\longrightarrow} \Omega$ in the Hausdorff topology with

$$\forall n \in \mathbb{N}^*, \quad \frac{1}{n+1} < \mathrm{dist}(\partial\Omega, \partial\Omega_n) < \frac{1}{n}.$$

Then, for all $n \in \mathbb{N}^*$ we consider the following iterative scheme :

$$(\mathrm{P}_n)\begin{cases} -\Delta_p u_n + \kappa_1 d(x)^{\delta_1} u_n = f_1(x, \tilde{u}_{n-1}, \tilde{v}_{n-1}) + \kappa_1 d(x)^{\delta_1}\tilde{u}_{n-1}; \quad u_n > 0 \quad \text{in } \Omega_n, \\ -\Delta_q v_n + \kappa_2 d(x)^{\delta_2} v_n = f_2(x, \tilde{u}_{n-1}, \tilde{v}_{n-1}) + \kappa_2 d(x)^{\delta_2}\tilde{v}_{n-1}; \quad v_n > 0 \quad \text{in } \Omega_n, \\ u_n|_{\partial\Omega_n} = \underline{u} \quad \text{and} \quad v_n|_{\partial\Omega_n} = \underline{v}, \end{cases}$$

with initial data $u_0 = \underline{u}$ and $v_0 = \underline{v}$ in Ω_0 and for all $n \in \mathbb{N}$,

$$\tilde{u}_n \stackrel{\text{def}}{=} \mathbb{1}_{\Omega_n}.u_n + \mathbb{1}_{\Omega\setminus\Omega_n}.\underline{u} \quad \text{and} \quad \tilde{v}_n \stackrel{\text{def}}{=} \mathbb{1}_{\Omega_n}.v_n + \mathbb{1}_{\Omega\setminus\Omega_n}.\underline{v} \quad \text{in} \quad \Omega.$$

By induction on $n \in \mathbb{N}^*$, (P_n) has a solution $(u_n, v_n) \in \mathrm{W}^{1,p}(\Omega_n) \times \mathrm{W}^{1,q}(\Omega_n)$ satisfying for all $n \in \mathbb{N}^*$,

$$\underline{u} \leq \tilde{u}_n \leq \tilde{u}_{n+1} \leq \overline{u} \quad \text{and} \quad \underline{v} \leq \tilde{v}_n \leq \tilde{v}_{n+1} \leq \overline{v} \quad \text{a.e. in } \Omega. \tag{II.29}$$

Indeed, using estimates (II.26) and (II.28),

$$f_1(x, \underline{u}, \underline{v}) + \kappa_1 d(x)^{\delta_1}\underline{u} \in \mathrm{L}^\infty(\Omega_1) \hookrightarrow \mathrm{W}^{-1,p'}(\Omega_1)$$

and since $\underline{u} \in \mathrm{W}^{1,p}(\Omega_1) \hookrightarrow \mathrm{W}^{1/p',p}(\partial\Omega_1)$ in the sense of the traces, we get $u_1 \in \mathrm{W}^{1,p}(\Omega_1)$ as a minimum of the functional E_1 defined for $w \in \mathrm{W}^{1,p}(\Omega_1)$ by

$$E_1(w) \stackrel{\text{def}}{=} \frac{1}{p}\int_{\Omega_1} |\nabla(w+\underline{u})|^p\,dx + \frac{\kappa_1}{2}\int_{\Omega_1} d(x)^{\delta_1}(w+\underline{u})^2\,dx - \int_{\Omega_1}(f_1(x,\underline{u},\underline{v}) + \kappa_1 d(x)^{\delta_1}\underline{u})w\,dx. \tag{II.30}$$

Since the operator $u \mapsto -\Delta_p u + \kappa_1 d(x)^{\delta_1} u$ is monotone in $\mathrm{W}^{1,p}(\Omega_1)$, applying the weak comparison principle we get

$$\underline{u} \leq u_1 \leq \overline{u} \quad \text{a.e. in } \Omega_1.$$

Using the same arguments as above, we prove the existence of $v_1 \in \mathrm{W}^{1,q}(\Omega_1)$ satisfying $\underline{v} \leq v_1 \leq \overline{v}$ a.e. in Ω_1. Now, let us fix $n \in \mathbb{N}^*$ and suppose that for all $k \leq n$, (P_k) has a solution $(u_k, v_k) \in \mathrm{W}^{1,p}(\Omega_k) \times \mathrm{W}^{1,q}(\Omega_k)$ satisfying (II.29). The existence of positive solutions of (P_{n+1}), $(u_{n+1}, v_{n+1}) \in \mathrm{W}^{1,p}(\Omega_{n+1}) \times \mathrm{W}^{1,q}(\Omega_{n+1})$ satisfying

$$\underline{u} \leq u_{n+1} \leq \overline{u} \quad \text{and} \quad \underline{v} \leq v_{n+1} \leq \overline{v} \quad \text{a.e. in } \Omega_{n+1},$$

can be established using similar techniques as above. To prove the monotonicity of the sequences $(\tilde{u}_m)_{m\in\mathbb{N}^*}$ and $(\tilde{v}_m)_{m\in\mathbb{N}^*}$, we remark that $\tilde{u}_n \in \mathrm{W}^{1,p}(\Omega_{n+1})$ and satisfies

$$-\Delta_p \tilde{u}_n + \kappa_1 d(x)^{\delta_1} \tilde{u}_n \leq f_1(x, \tilde{u}_{n-1}, \tilde{v}_{n-1}) + \kappa_1 d(x)^{\delta_1} \tilde{u}_{n-1} \quad \text{in } \Omega_{n+1}, \tag{II.31}$$

in the weak sense. Then, using (II.31) together with (II.28), we deduce from the previous inequality that,

$$-\Delta_p \tilde{u}_n + \kappa_1 d(x)^{\delta_1} \tilde{u}_n \leq f_1(x, \tilde{u}_{n-1}, \tilde{v}_n) + \kappa_1 d(x)^{\delta_1} \tilde{u}_{n-1} \quad \text{in } \Omega_{n+1},$$

in the weak sense. Hence, by estimate (II.27) and from the weak comparison principle applied in $\mathrm{W}^{1,p}(\Omega_{n+1})$, we obtain

$$\tilde{u}_n \leq u_{n+1} \quad \text{a.e. in } \Omega_{n+1}.$$

Similarly, we get the existence and the behaviour of v_{n+1}. Then, for almost every $x \in \Omega$, we define

$$u(x) = \lim_{n\to\infty} \tilde{u}_n(x) \quad \text{and} \quad v(x) = \lim_{n\to\infty} \tilde{v}_n(x).$$

Moreover, using a classical local regularity result of Serrin [78], $\tilde{u}_n, \tilde{v}_n \in \mathscr{C}^{1,\gamma}_{\mathrm{loc}}(\Omega_n)$ for some $0 < \gamma < 1$ and $\nabla \tilde{u}_n \underset{n\to\infty}{\longrightarrow} \nabla u$ and $\nabla \tilde{v}_n \underset{n\to\infty}{\longrightarrow} \nabla v$, uniformly in any compact set K of Ω. Thus, $(u, v) \in [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$ and passing to the limit in (P_n), (u, v) is a solution of (P) in the sense of distributions. □

3 Applications

3.1 Example 1

In this section we focus on the following quasilinear elliptic and singular system,

$$(\mathrm{P}) \begin{cases} -\Delta_p u = K_1(x) u^{a_1} v^{b_1} \quad \text{in } \Omega\,; \quad u|_{\partial\Omega} = 0, \quad u > 0 \quad \text{in } \Omega, \\ \\ -\Delta_q v = K_2(x) v^{a_2} u^{b_2} \quad \text{in } \Omega\,; \quad v|_{\partial\Omega} = 0, \quad v > 0 \quad \text{in } \Omega. \end{cases}$$

In this problem,

1. The exponents $a_1 < p-1$, $a_2 < q-1$ and $b_1, b_2 \neq 0$ satisfy the sub-homogeneous condition

$$(p-1-a_1)(q-1-a_2) - |b_1 b_2| > 0, \tag{II.32}$$

which is equivalent to the existence of a positive constant $\sigma > 0$ such that

$$(p-1-a_1) - \sigma|b_1| > 0 \quad \text{and} \quad \sigma(q-1-a_2) - |b_2| > 0. \tag{II.33}$$

2. K_1, K_2 are two positive functions in Ω satisfying

$$K_1(x) = d(x)^{-k_1} L_1(d(x)) \quad \text{and} \quad K_2(x) = d(x)^{-k_2} L_2(d(x)) \quad \text{in } \Omega, \tag{II.34}$$

with $0 \leq k_1 < p$, $0 \leq k_2 < q$ and for $i = 1, 2$, L_i a lower perturbation in $\mathscr{C}^2((0, D])$ (D the diameter of the domain Ω), of the form :

$$\forall t \in (0, D],\ L_i(t) = \exp\left(\int_t^D \frac{z_i(s)}{s} ds\right), \tag{II.35}$$

with $z_i \in \mathscr{C}([0, D]) \cap \mathscr{C}^1((0, D])$ and $z_i(0) = 0$.

Remark 3.1 *Recall here the main properties of functions defined by* (II.35) *:*

(a) Let us notice that (II.35) *implies that*

$$\forall \varepsilon > 0, \qquad \lim_{t \to 0^+} t^{-\varepsilon} L_i(t) = +\infty \quad \text{and} \quad \lim_{t \to 0^+} t^{\varepsilon} L_i(t) = 0. \tag{II.36}$$

(b) Definition (II.35) *also implies that*

$$\lim_{t \to 0^+} \frac{t L_i'(t)}{L_i(t)} = 0 \quad \text{and} \quad \lim_{t \to 0^+} \frac{t L_i''(t)}{L_i'(t)} = -1.$$

(c) If L_1, L_2 are two functions satisfying (II.35)*, then for any $\alpha, \beta \in \mathbb{R}$, the function ${L_1}^{\alpha}.{L_2}^{\beta}$ also satisfies* (II.35)*.*

(d) Such functions L_1, L_2 defined as above belong to the Karamata Class (see Karamata *[58]).*

Example 3.1 *Let $m \in \mathbb{N}^*$ and $A >> D$ large enough. Let us define*

$$\forall t \in (0, D], \qquad L_i(t) = \prod_{n=1}^{m} \left(\log_n\left(\frac{A}{t}\right)\right)^{\mu_n},$$

where, $\log_n \overset{\text{def}}{=} \log \circ \cdots \circ \log$ *(n times) and $\mu_n > 0$. Then L_i satisfies* (II.35).

In our study, $b_1 \neq 0$ and $b_2 \neq 0$. In the case where $b_1 > 0$ and $b_2 > 0$, the expression of the right-hand sides of the two coupled equations in system (P) defines a ***cooperative*** interaction between the two components (species) u and v :

$$\frac{\partial}{\partial v}\left(K_1(x) u^{a_1} v^{b_1}\right) = b_1 K_1(x) u^{a_1} v^{b_1 - 1} > 0, \tag{II.37}$$

$$\frac{\partial}{\partial u}\Big(K_2(x)v^{a_2}u^{b_2}\Big) = b_2K_2(x)v^{a_2}u^{b_2-1} > 0. \tag{II.38}$$

In the case where $b_1 < 0$ and $b_2 < 0$, both partial derivative in (II.37) and (II.38) are negative and the expression of the right-hand sides of the two coupled equations of (P) define a ***competitive*** interaction between u and v.

First, we discuss the existence of positive weak solutions pairs to problem (P). For that, regarding Theorem 2.1, we take

$$f_1(x,u,v) = K_1(x)u^{a_1}v^{b_1}, \quad f_2(x,u,v) = K_2(x)v^{a_2}u^{b_2}$$

and construct suitable sub- and super-solutions pairs of (P) in $\mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$.

Then, in the cases where (P) is either competitive or cooperative, we investigate the uniqueness of such positive weak solutions pairs. For that, it is essential that the mappings $T_1 \circ T_2$ and $T_2 \circ T_1$ (where T_1 and T_2 are defined in (II.1)) are ***sub-homogeneous***, which is equivalent to condition (II.32).

3.1.1 Preliminary results

Let $1 < r < \infty$, $\delta < r-1$ and $K : x \longmapsto d(x)^{-k}L(d(x))$, with $0 \le k < r$ and L a perturbation function satisfying (II.35). In view of constructing suitable pairs of sub- and super-solutions to (P), we first introduce the following problem :

$$-\Delta_r w = K(x)w^\delta \quad \text{in } \Omega; \quad w|_{\partial\Omega} = 0, \quad w > 0 \quad \text{in } \Omega. \tag{II.39}$$

Proposition 3.1 *Under the above hypothesis, we have :*

1. *If $k-1 < \delta < r-1$, problem* (II.39) *has a unique positive weak solution $\psi \in \mathrm{W}_0^{1,r}(\Omega)$ that satisfies the following estimate :*

$$\psi(x) \sim d(x) \qquad \text{in } \Omega. \tag{II.40}$$

In addition, we have $\psi \in \mathscr{C}^{1,\alpha}(\overline{\Omega})$, for some $0 < \alpha < 1$.

2. *If $\delta = k-1$, problem* (II.39) *has a unique positive weak solution $\psi \in \mathrm{W}_0^{1,r}(\Omega)$ that satisfies the following estimate :*

$$\psi(x) \sim d(x)\left(\int_{d(x)}^{D} L(t)t^{-1}\,dt\right)^{\frac{1}{r-k}} \qquad \text{in } \Omega. \tag{II.41}$$

In addition, we have $\psi \in \mathscr{C}^{0,\alpha}(\overline{\Omega})$, for some $0 < \alpha < 1$.

3. *If $k-2+\frac{k-1}{r-1} < \delta < k-1$, problem* (II.39) *has a unique positive weak solution $\psi \in \mathrm{W}_0^{1,r}(\Omega)$ that*

satisfies the following estimate :

$$\psi(x) \sim d(x)^{\frac{r-k}{r-1-\delta}} L(d(x))^{\frac{1}{r-1-\delta}} \qquad in\ \Omega. \tag{II.42}$$

In addition, we have $\psi \in \mathscr{C}^{0,\alpha}(\overline{\Omega})$*, for some* $0 < \alpha < 1$*.*

4. *If* $\delta \leq k - 2 + \frac{k-1}{r-1}$*, problem* (II.39) *has at least one positive weak solution* $\psi \in \mathrm{W}^{1,r}_{\mathrm{loc}}(\Omega) \cap \mathscr{C}_0(\overline{\Omega})$ *that satisfies the following estimate :*

$$\psi(x) \sim d(x)^{\frac{r-k}{r-1-\delta}} L(d(x))^{\frac{1}{r-1-\delta}} \qquad in\ \Omega. \tag{II.43}$$

Proof. See Lemma 3.3 in Giacomoni-Mâagli-Sauvy [46] or Section 1 of Appendix B. □

Remark 3.2 *In 4. above, it can be proved that* $\forall\, \gamma > \frac{(r-1)(r-1-\delta)}{r(r-k)}$, $\psi^\gamma \in \mathrm{W}^{1,r}_0(\Omega)$*.*

We give now a weak comparison principle used to establish the uniqueness of a positive weak solutions pair of (P).

Theorem 3.1 *Let* $K : \Omega \to \mathbb{R}_+$ *be a* $\mathrm{L}^1_{\mathrm{loc}}(\Omega)$ *function and* $\delta < r - 1$*. Assume* $u, v \in \mathrm{W}^{1,r}_0(\Omega) \cap \mathrm{L}^\infty(\Omega)$ *are two locally uniformly positive functions satisfying the sub- and super-solution inequalities :*

$$-\Delta_r u \leq K(x) u^\delta \quad and \quad -\Delta_r v \geq K(x) v^\delta \qquad in\ \Omega, \tag{II.44}$$

in the sense of distributions (i.e. Radon measures) in $\mathrm{W}^{-1,r'}(\Omega)$*. Then*

1. *If* $\delta < 0$*, inequality* $u \leq v$ *holds a.e. in* Ω*.*
2. *If* $\delta > 0$ *and if we suppose in addition that there exist* $C_1, C_2 > 0$ *and a locally uniformly positive function* $w_0 \in \mathrm{L}^\infty(\Omega)$ *such that* $C_1 w_0 \leq u, v \leq C_2 w_0$ *a.e. in* Ω *and*

$$\int_\Omega K(x) {w_0}^{\delta+1}\, dx < +\infty, \tag{II.45}$$

inequality $u \leq v$ *holds a.e. in* Ω*.*

To prove this theorem, we use the well-known inequality in Lemma 3.1 and the Díaz-Saa inequality (see Díaz-Saa [33]).

Lemma 3.1 *There exists a constant* $C_r > 0$ *such that, for all* $x, y \in \mathbb{R}^N$*,*

$$(|x|^r - |y|^r - r|x|^{r-2}x).(y-x) \geq \begin{cases} C_r|x-y|^r & if \quad r \geq 2, \\ C_r \dfrac{|x-y|^2}{(|x|+|y|)^{2-r}} & if \quad 1 < r < 2. \end{cases}$$

Proof. See Lemma 4.2 in Lindqvist [65]. □

Proof. (OF THEOREM 3.1)

1. If $\delta < 0$, we wish to prove that the function $w = (u - v)^+$ satisfies $w = 0$ a.e. in Ω. First notice that $0 \le w \in \mathrm{W}_0^{1,r}(\Omega)$. Applying the duality between $\mathrm{W}_0^{1,r}(\Omega)$ and $\mathrm{W}^{-1,r'}(\Omega)$, respectively, to w and the the difference
$$-\Delta_r u + \Delta_r v \le K(x)\left(u^\delta - v^\delta\right)$$
which is ≤ 0 on the set $\Omega_+ \stackrel{\text{def}}{=} \{x \in \Omega, \quad w(x) > 0\}$, we obtain
$$\int_{\Omega_+} \left(|\nabla u|^{r-2}\nabla u - |\nabla v|^{r-2}\nabla v\right).(\nabla u - \nabla v)\,dx = \int_{\Omega} \left(|\nabla u|^{r-2}\nabla u - |\nabla v|^{r-2}\nabla v\right).\nabla w\,dx \le 0.$$
This forces $\nabla w = 0$ a.e. in Ω_+ and, consequently, also in Ω. Since $w \in \mathrm{W}_0^{1,r}(\Omega)$, we conclude that $w = 0$ a.e. in Ω as claimed, that is, $u \le v$ a.e. in Ω.

2. If $0 < \delta < r - 1$, following some ideas in LINDQUIST [65] (see also DRÁBEK-HERNÁNDEZ [35]), we use the Díaz-Saa inequality.
More precisely, for $\varepsilon > 0$, we set $u_\varepsilon \stackrel{\text{def}}{=} u + \varepsilon$ and $v_\varepsilon \stackrel{\text{def}}{=} v + \varepsilon$ in Ω and we define
$$\phi \stackrel{\text{def}}{=} \frac{{u_\varepsilon}^r - {v_\varepsilon}^r}{{u_\varepsilon}^{r-1}} \quad \text{and} \quad \psi \stackrel{\text{def}}{=} \frac{{v_\varepsilon}^r - {u_\varepsilon}^r}{{v_\varepsilon}^{r-1}} \quad \text{in } \Omega.$$
Then, $\frac{u_\varepsilon}{v_\varepsilon}, \frac{v_\varepsilon}{u_\varepsilon} \in \mathrm{L}^\infty(\Omega)$ and $\phi, \psi \in \mathrm{W}_0^{1,r}(\Omega)$ with
$$\nabla\phi = \left[1 + (r-1)\left(\frac{v_\varepsilon}{u_\varepsilon}\right)^r\right]\nabla u - r\left(\frac{v_\varepsilon}{u_\varepsilon}\right)^{r-1}\nabla v \quad \text{in } \Omega, \tag{II.46}$$
$$\nabla\psi = \left[1 + (r-1)\left(\frac{u_\varepsilon}{v_\varepsilon}\right)^r\right]\nabla v - r\left(\frac{u_\varepsilon}{v_\varepsilon}\right)^{r-1}\nabla u \quad \text{in } \Omega. \tag{II.47}$$
Setting $\Omega_+ \stackrel{\text{def}}{=} \{x \in \Omega, \quad u(x) > v(x)\}$, we have that $\phi > 0$ and $\psi < 0$ in Ω_+ and
$$\int_{\Omega_+} |\nabla u|^{r-2}\nabla u.\nabla\phi\,dx \le \int_{\Omega_+} K(x)u^\delta\phi\,dx < +\infty,$$
$$\int_{\Omega_+} |\nabla v|^{r-2}\nabla v.\nabla\psi\,dx \le \int_{\Omega_+} K(x)v^\delta\psi\,dx < +\infty.$$
Using equalities (II.46) and (II.47) and the fact that
$$|\nabla \ln u_\varepsilon| = \frac{|\nabla u|}{u_\varepsilon} \quad \text{and} \quad |\nabla \ln v_\varepsilon| = \frac{|\nabla v|}{v_\varepsilon} \quad \text{in } \Omega, \tag{II.48}$$

we get

$$\begin{aligned}\int_{\Omega_+}|\nabla u|^{r-2}\nabla u.\nabla\phi\,dx+\int_{\Omega_+}|\nabla v|^{r-2}\nabla v.\nabla\psi\,dx&\\ &=\int_{\Omega_+}({u_\varepsilon}^r-{v_\varepsilon}^r)(|\nabla\ln u_\varepsilon|^r-|\nabla\ln v_\varepsilon|^r)\,dx\\ &-\int_{\Omega_+}r{v_\varepsilon}^r|\nabla\ln u_\varepsilon|^{r-2}(\nabla\ln u_\varepsilon).\left(\nabla\ln v_\varepsilon-\nabla\ln u_\varepsilon\right)\,dx\\ &-\int_{\Omega_+}r{u_\varepsilon}^r|\nabla\ln v_\varepsilon|^{r-2}(\nabla\ln v_\varepsilon).\left(\nabla\ln u_\varepsilon-\nabla\ln v_\varepsilon\right)\,dx.\end{aligned}$$

(a) If $r\geq 2$, from Lemma 3.1, it follows that

$$\begin{aligned}\int_{\Omega_+}|\nabla u|^{r-2}\nabla u.\nabla\phi\,dx+\int_{\Omega_+}|\nabla v|^{r-2}\nabla v.\nabla\psi\,dx&\\ &\geq\int_{\Omega_+}({u_\varepsilon}^r-{v_\varepsilon}^r)(|\nabla\ln u_\varepsilon|^r-|\nabla\ln v_\varepsilon|^r)\,dx\\ &+\int_{\Omega_+}{v_\varepsilon}^r\left(|\nabla\ln u_\varepsilon|^r-|\nabla\ln v_\varepsilon|^r+C_r|\nabla\ln u_\varepsilon-\nabla\ln v_\varepsilon|^r\right)\,dx\\ &+\int_{\Omega_+}{u_\varepsilon}^r\left(|\nabla\ln v_\varepsilon|^r-|\nabla\ln u_\varepsilon|^r+C_r|\nabla\ln u_\varepsilon-\nabla\ln v_\varepsilon|^r\right)\,dx\\ &=C_r\int_{\Omega_+}|u_\varepsilon\nabla v_\varepsilon-v_\varepsilon\nabla u_\varepsilon|^r\left(\frac{1}{{u_\varepsilon}^r}+\frac{1}{v_\varepsilon^r}\right)\,dx.\end{aligned}$$

(b) If $1<r<2$, Lemma 3.1 entails

$$\begin{aligned}\int_{\Omega_+}|\nabla u|^{r-2}\nabla u.\nabla\phi\,dx+\int_{\Omega_+}|\nabla v|^{r-2}\nabla v.\nabla\psi\,dx&\\ &\geq C_r\int_{\Omega_+}\frac{|u_\varepsilon\nabla v_\varepsilon-v_\varepsilon\nabla u_\varepsilon|^2}{(u_\varepsilon|\nabla v_\varepsilon|+v_\varepsilon|\nabla u_\varepsilon|)^{2-r}}\left(\frac{1}{{u_\varepsilon}^r}+\frac{1}{{v_\varepsilon}^r}\right)\,dx.\end{aligned}$$

In the right-hand side, we get

$$\int_{\Omega_+}K(x)\left(u^\delta\phi+v^\delta\psi\right)\,dx=\int_{\Omega_+}K(x)\left[\frac{u^\delta}{u^{r-1}}\left(\frac{u}{u_\varepsilon}\right)^{r-1}-\frac{v^\delta}{v^{r-1}}\left(\frac{v}{v_\varepsilon}\right)^{r-1}\right]({u_\varepsilon}^r-{v_\varepsilon}^r)\,dx.$$

Then, since $\frac{u}{u_\varepsilon}\to 1$, $\frac{v}{v_\varepsilon}\to 1$ as $\varepsilon\to 0^+$ a.e. in Ω, we get from (II.45) and Lebesgue's Theorem that

$$\lim_{\varepsilon\to 0^+}\int_{\Omega_+}K(x)\left(u^\delta\phi+v^\delta\psi\right)\,dx\leq 0.$$

By Fatou's Lemma and using the above estimates, we obtain in the both cases that $|u\nabla v-v\nabla u|=0$ a.e. in Ω_+, from which we get that on each connected component set ω of Ω_+, there exists $k>0$ such that $u=kv$ a.e. in ω. From sub- an super-solution inequalities (II.44) we have,

$$\begin{aligned}k^r\int_\omega K(x)v^{\delta+1}\,dx&\leq k^r\int_\omega|\nabla v|^r\,dx=\int_\omega|\nabla u|^r\,dx\\ &\leq\int_\omega K(x)u^{\delta+1}\,dx=k^{\delta+1}\int_\omega K(x)v^{\delta+1}\,dx.\end{aligned}\tag{II.49}$$

Hence, $k \leq 1$ which implies that $u \leq v$ a.e. in Ω_+ and then, from the definition of Ω_+, $u \leq v$ a.e. in Ω. □

3.1.2 Main results

Theorem 3.2 *Assume that the exponents $a_1 < p-1$, $a_2 < q-1$ and $b_1, b_2 \neq 0$ in problem* (P) *satisfy the hypothesis* (II.32)*.*

1. *Set*

$$\alpha_1 = \frac{q-1-a_2}{(p-1-a_1)(q-1-a_2)-b_1b_2}, \qquad \alpha_2 = \frac{p-1-a_1}{(p-1-a_1)(q-1-a_2)-b_1b_2},$$

$$\beta_1 = \frac{b_1}{(p-1-a_1)(q-1-a_2)-b_1b_2}, \qquad \beta_2 = \frac{b_2}{(p-1-a_1)(q-1-a_2)-b_1b_2},$$

$$\gamma_1 = \frac{(p-k_1)(q-1-a_2)+(q-k_2)b_1}{(p-1-a_1)(q-1-a_2)-b_1b_2}, \qquad \gamma_2 = \frac{(q-k_2)(p-1-a_1)+(p-k_1)b_2}{(p-1-a_1)(q-1-a_2)-b_1b_2}$$

and assume that

$$1-\frac{1}{p} < \gamma_1 < 1 \quad \textit{and} \quad 1-\frac{1}{q} < \gamma_2 < 1. \tag{II.50}$$

Then, problem (P) *possesses positive solutions $(u,v) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ that satisfy the following estimates :*

$$u(x) \sim d(x)^{\gamma_1} L_1(d(x))^{\alpha_1} L_2(d(x))^{\beta_1} \quad \textit{in } \Omega, \tag{II.51}$$

$$v(x) \sim d(x)^{\gamma_2} L_2(d(x))^{\alpha_2} L_1(d(x))^{\beta_2} \quad \textit{in } \Omega. \tag{II.52}$$

In addition, we have $(u,v) \in \mathscr{C}^{0,\alpha}(\overline{\Omega}) \times \mathscr{C}^{0,\alpha}(\overline{\Omega})$, for some $0 < \alpha < 1$.

2. *Now assume that*

$$k_1 - 1 < a_1 + b_1 < p-1 \quad \textit{and} \quad k_2 - 1 < a_2 + b_2 < q-1. \tag{II.53}$$

Then, problem (P) *possesses positive solutions $(u,v) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ that satisfy the following estimates :*

$$u(x) \sim d(x) \quad \textit{and} \quad v(x) \sim d(x) \quad \textit{in } \Omega. \tag{II.54}$$

In addition, we have $(u,v) \in \mathscr{C}^{1,\alpha}(\overline{\Omega}) \times \mathscr{C}^{1,\alpha}(\overline{\Omega})$, for some $0 < \alpha < 1$.

3. *Set*

$$\gamma = \frac{p-k_1+b_1}{p-1-a_1}$$

and assume that

$$1-\frac{1}{p} < \gamma < 1 \quad \textit{and} \quad k_2 - 1 < a_2 + b_2\gamma < q-1. \tag{II.55}$$

Then, problem (P) *possesses positive solutions* $(u, v) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ *that satisfy the following estimates :*

$$u(x) \sim d(x)^{\gamma} L_1(d(x))^{\frac{1}{p-1-a_1}} \quad \text{and} \quad v(x) \sim d(x) \quad \text{in } \Omega. \tag{II.56}$$

In addition, we have $(u, v) \in \mathscr{C}^{0,\alpha}(\overline{\Omega}) \times \mathscr{C}^{1,\alpha}(\overline{\Omega})$*, for some* $0 < \alpha < 1$*.*

4. *Symmetrically to part 3. above, set*

$$\gamma = \frac{q - k_2 + b_2}{q - 1 - a_2}$$

and assume that

$$k_1 - 1 < a_1 + b_1\gamma < p - 1 \quad \text{and} \quad 1 - \frac{1}{q} < \gamma < 1. \tag{II.57}$$

Then, problem (P) *possesses positive solutions* $(u, v) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ *that satisfy the following estimates :*

$$u(x) \sim d(x) \quad \text{and} \quad v(x) \sim d(x)^{\gamma} L_2(d(x))^{\frac{1}{q-1-a_2}} \quad \text{in } \Omega. \tag{II.58}$$

In addition, we have $(u, v) \in \mathscr{C}^{1,\alpha}(\overline{\Omega}) \times \mathscr{C}^{0,\alpha}(\overline{\Omega})$*, for some* $0 < \alpha < 1$*.*

Theorem 3.3 *Let* $a_1 < p - 1$*,* $a_2 < q - 1$ *and* $b_1, b_2 \neq 0$ *satisfying the sub-homogeneity hypothesis* (II.32)*. Assume that* (P) *is either a competitive or a cooperative system, i.e.* $\boldsymbol{b_1 b_2 > 0}$*. Then, each solution provided by Theorem 3.2 is unique.*

The cooperative case is further analysed in the following result :

Theorem 3.4 *Let us suppose that the exponents* $a_1 < p - 1$*,* $a_2 < q - 1$ *and* $b_1, b_2 \neq 0$ *satisfy the sub-homogeneity hypothesis* (II.32)*. Moreover, assume that* (P) *is a cooperative system, i.e.,* $\boldsymbol{b_1 > 0}$ *and* $\boldsymbol{b_2 > 0}$*.*

1. *Set*

$$\gamma_1 = \frac{(p - k_1)(q - 1 - a_2) + (q - k_2)b_1}{(p - 1 - a_1)(q - 1 - a_2) - b_1 b_2}, \qquad \gamma_2 = \frac{(q - k_2)(p - 1 - a_1) + (p - k_1)b_2}{(p - 1 - a_1)(q - 1 - a_2) - b_1 b_2} \tag{II.59}$$

and assume that one of the three following conditions are satisfied :

$$0 < \gamma_1 \leq 1 - \frac{1}{p} \quad \text{and} \quad 0 < \gamma_2 \leq 1 - \frac{1}{q}, \tag{II.60}$$

$$1 - \frac{1}{p} < \gamma_1 < 1 \quad \text{and} \quad 0 < \gamma_2 \leq 1 - \frac{1}{q}, \tag{II.61}$$

$$0 < \gamma_1 \leq 1 - \frac{1}{p} \quad \text{and} \quad 1 - \frac{1}{q} < \gamma_2 < 1. \tag{II.62}$$

Then, problem (P) *nevertheless admits positive solutions* $(u,v) \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega) \times \mathrm{W}^{1,q}_{\mathrm{loc}}(\Omega)$ *in the distributions sense satisfying the estimates* (II.51) *and* (II.52).

2. *Set*

$$\gamma = \frac{p - k_1 + b_1}{p - 1 - a_1} \tag{II.63}$$

and assume that

$$0 < \gamma \leq 1 - \frac{1}{p} \quad \text{and} \quad k_2 - 1 < a_2 + b_2\gamma < q - 1. \tag{II.64}$$

Then, problem (P) *admits positive solutions* $(u,v) \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega) \times \mathrm{W}^{1,q}_{0}(\Omega)$ *in the distributions sense satisfying the estimates* (II.51) *and* (II.52).

3. *Symmetrically to part 2. above, set*

$$\gamma = \frac{q - k_2 + b_2}{q - 1 - a_2} \tag{II.65}$$

and assume that

$$k_1 - 1 < a_1 + b_1\gamma < p - 1 \quad \text{and} \quad 0 < \gamma \leq 1 - \frac{1}{q}. \tag{II.66}$$

Then, problem (P) *possesses positive solutions* $(u,v) \in \mathrm{W}^{1,p}_{0}(\Omega) \times \mathrm{W}^{1,q}_{\mathrm{loc}}(\Omega)$ *in the distributions sense that satisfy the estimates given in* (II.58).

The next result deals with some limiting cases :

Theorem 3.5 *Assume that the exponents* $a_1 < p-1, a_2 < q-1$ *and* $b_1, b_2 \neq 0$ *satisfy the sub-homogeneity hypothesis* (II.32).

1. *Assume that*

$$a_1 + b_1 = k_1 - 1 \quad \text{and} \quad k_2 - 1 \leq a_2 + b_2 < q - 1. \tag{II.67}$$

Then, for all $\varepsilon > 0$ *small enough, there exist* $C_1, C_2 > 0$ *and* $C'_1, C'_2 > 0$ *such that problem* (P) *possesses positive solutions* $(u,v) \in \mathrm{W}^{1,p}_{0}(\Omega) \times \mathrm{W}^{1,q}_{0}(\Omega)$ *that satisfy the following estimates :*

$$C_1 d(x) \leq u \leq C_2 d(x)^{1-\varepsilon} \quad \text{and} \quad C'_1 d(x) \leq v \leq C'_2 d(x)^{1-\varepsilon\sigma} \quad \text{in } \Omega, \tag{II.68}$$

where $\sigma > 0$ *is given in* (II.33).*In addition, we have* $(u,v) \in \mathscr{C}^{0,\alpha}(\overline{\Omega}) \times \mathscr{C}^{0,\alpha}(\overline{\Omega})$, *for some* $0 < \alpha < 1$.

2. *Symmetrically, assume that*

$$a_2 + b_2 = k_2 - 1 \quad \text{and} \quad k_1 - 1 \leq a_1 + b_1 < q - 1. \tag{II.69}$$

Then, for all $\varepsilon > 0$ *small enough, there exist* $C_1, C_2 > 0$ *and* $C'_1, C'_2 > 0$ *such that problem* (P) *possesses positive solutions* $(u,v) \in \mathrm{W}^{1,p}_{0}(\Omega) \times \mathrm{W}^{1,q}_{0}(\Omega)$ *that satisfy the following estimates :*

$$C_1 d(x) \leq u \leq C_2 d(x)^{1-\varepsilon} \quad \text{and} \quad C'_1 d(x) \leq v \leq C'_2 d(x)^{1-\varepsilon\sigma} \quad \text{in } \Omega. \tag{II.70}$$

In addition, we have $(u,v) \in \mathscr{C}^{0,\alpha}(\overline{\Omega}) \times \mathscr{C}^{0,\alpha}(\overline{\Omega})$, *for some* $0 < \alpha < 1$.

3. *Let us abbreviate*

$$\gamma = \frac{p - k_1 + b_1}{p - 1 - a_1}$$

and assume that

$$1 - \frac{1}{p} < \gamma < 1 \quad \text{and} \quad a_2 + b_2\gamma = k_2 - 1. \tag{II.71}$$

Then, for all $\varepsilon > 0$ *small enough, there exist* $C_1, C_2 > 0$ *and* $C_1', C_2' > 0$ *such that problem* (P) *possesses positive solutions* $(u,v) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ *that satisfy the following estimates :*

$$C_1 d(x)^{\gamma+\varepsilon} \leq u \leq C_2 d(x)^{\gamma-\varepsilon} \ \text{ and } \ C_1' d(x) \leq v \leq C_2' d(x)^{1-\varepsilon\sigma} \quad \text{in } \Omega. \tag{II.72}$$

In addition, we have $(u,v) \in \mathscr{C}^{0,\alpha}(\overline{\Omega}) \times \mathscr{C}^{0,\alpha}(\overline{\Omega})$, *for some* $0 < \alpha < 1$.

4. *Symmetrically, let us abbreviate*

$$\gamma = \frac{q - k_2 + b_2}{q - 1 - a_2}$$

and assume that

$$a_1 + b_1\gamma = k_2 - 1 \quad \text{and} \quad 1 - \frac{1}{q} < \gamma < 1. \tag{II.73}$$

Then, for all $\varepsilon > 0$ *small enough, there exist* $C_1, C_2 > 0$ *and* $C_1', C_2' > 0$ *such that problem* (P) *possesses positive solutions* $(u,v) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ *that satisfy the following estimates :*

$$C_1 d(x) \leq u \leq C_2 d(x)^{1-\varepsilon} \ \text{ and } \ C_1' d(x)^{\gamma+\varepsilon\sigma} \leq v \leq C_2' d(x)^{\gamma-\varepsilon\sigma} \quad \text{in } \Omega. \tag{II.74}$$

In addition, we have $(u,v) \in \mathscr{C}^{0,\alpha}(\overline{\Omega}) \times \mathscr{C}^{0,\alpha}(\overline{\Omega})$, *for some* $0 < \alpha < 1$.

3.1.3 Proof of Theorem 3.2

Thanks to Proposition 3.1, we apply Theorem 2.1 with a suitable choice of sub- and super-solutions pairs $(\underline{u},\underline{v}), (\overline{u},\overline{v}) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ in the following form :

$$\underline{u} = m\psi_1 \quad \text{and} \quad \overline{u} = m^{-1}\psi_1 \quad \text{in } \Omega,$$

$$\underline{v} = m^{\sigma}\psi_2 \quad \text{and} \quad \overline{v} = m^{-\sigma}\psi_2 \quad \text{in } \Omega,$$

where $\sigma > 0$ is given in (II.33), $0 < m < 1$ is an appropriate constant small enough and $\psi_1 \in \mathrm{W}_0^{1,p}(\Omega)$, $\psi_2 \in \mathrm{W}_0^{1,q}(\Omega)$ are given by Proposition 3.1 as the respective unique solutions of problems

$$-\Delta_p w = d(x)^{-k_1}\mathcal{L}_1(d(x))w^{\delta_1} \quad \text{in } \Omega; \quad w|_{\partial\Omega} = 0, \quad w > 0 \quad \text{in } \Omega, \tag{II.75}$$

$$-\Delta_q w = d(x)^{-k_2}\mathcal{L}_2(d(x))w^{\delta_2} \quad \text{in } \Omega; \quad w|_{\partial\Omega} = 0, \quad w > 0 \quad \text{in } \Omega, \tag{II.76}$$

satisfying some cone conditions we specify below. In the following alternatives, we choose suitable perturbations $\mathcal{L}_1, \mathcal{L}_2$ as in (II.35) and suitable values of exponents $k_1 - 2 + \frac{k_1-1}{p-1} < \delta_1 < p-1$ and $k_2 - 2 + \frac{k_2-1}{q-1} < \delta_2 < q-1$ in order to satisfy

$$-\Delta_p \psi_1 \sim K_1(x){\psi_1}^{a_1}{\psi_2}^{b_1} \quad \text{and} \quad -\Delta_q \psi_2 \sim K_2(x){\psi_2}^{a_2}{\psi_1}^{b_2} \quad \text{in } \Omega, \tag{II.77}$$

which provide us the inequalities (II.7) to (II.10) in order to apply Theorem 2.1.

Alternative 1 : We look for positive solutions $(u, v) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ to (P) by making the "Ansatz" that

$$u(x) \sim d(x)^{\gamma_1} L_1(d(x))^{\alpha_1} L_2(d(x))^{\beta_1} \quad \text{in } \Omega,$$

$$v(x) \sim d(x)^{\gamma_2} L_2(d(x))^{\alpha_2} L_1(d(x))^{\beta_2} \quad \text{in } \Omega,$$

for some $\gamma_1 \in (1-\frac{1}{p}, 1)$, $\gamma_2 \in (1-\frac{1}{q}, 1)$ and $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$. For that, we take in (II.75) and (II.76)

$$k_1 - 2 + \frac{k_1 - 1}{p-1} < \delta_1 < k_1 - 1 \quad \text{and} \quad \mathrm{k}_2 - 2 + \frac{k_2-1}{q-1} < \delta_2 < k_2 - 1, \tag{II.78}$$

$$\mathcal{L}_1 = {L_1}^{\lambda_1}.{L_2}^{\mu_1} \quad \text{and} \quad \mathcal{L}_2 = {L_2}^{\lambda_2}.{L_1}^{\mu_2} \quad \text{in } \Omega,$$

where λ_1, λ_2, μ_1, $\mu_2 \in \mathbb{R}$ are suitable exponents we fix later. By Proposition 3.1, $\psi_1 \in \mathrm{W}_0^{1,p}(\Omega)$, $\psi_2 \in \mathrm{W}_0^{1,q}(\Omega)$ and satisfy

$$\psi_1(x) \sim d(x)^{\frac{p-k_1}{p-1-\delta_1}} L_1(d(x))^{\frac{\lambda_1}{p-1-\delta_1}} L_2(d(x))^{\frac{\mu_1}{p-1-\delta_1}} \quad \text{in } \Omega, \tag{II.79}$$

$$\psi_2(x) \sim d(x)^{\frac{q-k_2}{q-1-\delta_2}} L_2(d(x))^{\frac{\lambda_2}{q-1-\delta_2}} L_1(d(x))^{\frac{\mu_2}{q-1-\delta_2}} \quad \text{in } \Omega. \tag{II.80}$$

In view of satisfying estimates given in (II.77), the comparison of the term $-\Delta_p\psi_1$ with $K_1(x){\psi_1}^{a_1}{\psi_2}^{b_1}$ on one side, and the term $-\Delta_q\psi_2$ with $K_2(x){\psi_2}^{a_2}{\psi_1}^{b_2}$ on the other side, imposes the exponents $\lambda_1, \lambda_2, \mu_1, \mu_2$ and δ_1, δ_2 to satisfy the following system :

$$\begin{cases} \delta_1 \frac{p-k_1}{p-1-\delta_1} = a_1 \frac{p-k_1}{p-1-\delta_1} + b_1 \frac{q-k_2}{q-1-\delta_2}, & \delta_2 \frac{q-k_2}{q-1-\delta_2} = a_2 \frac{q-k_2}{q-1-\delta_2} + b_2 \frac{p-k_1}{p-1-\delta_1}, \\ \lambda_1 \frac{p-1}{p-1-\delta_1} = 1 + a_1 \frac{\lambda_1}{p-1-\delta_1} + b_1 \frac{\mu_2}{q-1-\delta_2}, & \lambda_2 \frac{q-1}{q-1-\delta_2} = 1 + b_2 \frac{\mu_1}{p-1-\delta_1} + a_2 \frac{\lambda_2}{q-1-\delta_2}, \\ \mu_1 \frac{p-1}{p-1-\delta_1} = a_1 \frac{\mu_1}{p-1-\delta_1} + b_1 \frac{\lambda_2}{q-1-\delta_2}, & \mu_2 \frac{q-1}{q-1-\delta_2} = b_2 \frac{\lambda_1}{p-1-\delta_1} + a_2 \frac{\mu_2}{q-1-\delta_2}. \end{cases}$$

Then, we get

$$\gamma_1 = \frac{p-k_1}{p-1-\delta_1} = \frac{(p-k_1)(q-1-a_2) + (q-k_2)b_1}{(p-1-a_1)(q-1-a_2) - b_1 b_2}, \tag{II.81}$$

$$\gamma_2 = \frac{q-k_2}{q-1-\delta_2} = \frac{(q-k_2)(p-1-a_1)+(p-k_1)b_2}{(p-1-a_1)(q-1-a_2)-b_1b_2}, \tag{II.82}$$

$$\alpha_1 = \frac{\lambda_1}{p-1-\delta_1} = \frac{q-1-a_2}{(p-1-a_1)(q-1-a_2)-b_1b_2}, \tag{II.83}$$

$$\alpha_2 = \frac{\lambda_2}{q-1-\delta_2} = \frac{p-1-a_1}{(p-1-a_1)(q-1-a_2)-b_1b_2}, \tag{II.84}$$

$$\beta_1 = \frac{\mu_1}{p-1-\delta_1} = \frac{b_1}{(p-1-a_1)(q-1-a_2)-b_1b_2}, \tag{II.85}$$

$$\beta_2 = \frac{\mu_2}{q-1-\delta_2} = \frac{b_1}{(p-1-a_1)(q-1-a_2)-b_1b_2}, \tag{II.86}$$

which imply estimate (II.77). Moreover, inequalities (II.50) are then equivalent to inequalities (II.78). Let $(u,v) \in [\underline{u},\overline{u}] \times [\underline{v},\overline{v}]$. On one hand, we have

$$-\Delta_p \underline{u} \leq m^{p-1} C_1 L_1(d(x))^{\lambda_1+\delta_1\gamma_1} L_2(d(x))^{\mu_1+\delta_1\beta_1} d(x)^{\delta_1\gamma_1-k_1} \text{ in } \Omega,$$

$$-\Delta_q \underline{v} \leq m^{\sigma(q-1)} C'_1 L_2(d(x))^{\lambda_2+\delta_2\alpha_2} L_1(d(x))^{\mu_2+\delta_2\beta_2} d(x)^{\delta_2\gamma_2-k_2} \text{ in } \Omega.$$

On the other hand,

$$K_1(x)\underline{u}^{a_1} v^{b_1} \geq C_2 m^{a_1+\sigma|b_1|} \Lambda_1(d(x)) d(x)^{a_1\gamma_1+b_1\gamma_2-k_1} \text{ in } \Omega$$

with $\Lambda_1 = {L_1}^{1+a_1\alpha_1+b_1\beta_2} . {L_2}^{a_1\beta_1+b_1\alpha_2}$. Similarly,

$$K_2(x)\underline{v}^{a_2} u^{b_2} \geq C'_2 m^{\sigma a_2+|b_2|} \Lambda_2(d(x)) d(x)^{a_2\gamma_2+b_2\gamma_1-k_2} \text{ in } \Omega,$$

with $\Lambda_2 = {L_2}^{1+a_2\alpha_2+b_2\beta_1} . {L_1}^{a_2\beta_2+b_2\alpha_1}$. Then, under condition (II.33) and using (II.81) to (II.86), $(\underline{u},\underline{v})$ is a sub-solutions pair of problem (P), for m small enough. Next,

$$-\Delta_p \overline{u} \geq m^{1-p} C_3 L_1(d(x))^{\lambda_1+\delta_1\gamma_1} L_2(d(x))^{\mu_1+\delta_1\beta_1} d(x)^{\delta_1\gamma_1-k_1} \text{ in } \Omega,$$

$$-\Delta_q \overline{v} \geq m^{\sigma(1-q)} C'_3 L_2(d(x))^{\lambda_2+\delta_2\alpha_2} L_1(d(x))^{\mu_2+\delta_2\beta_2} d(x)^{\delta_2\gamma_2-k_2} \text{ in } \Omega.$$

Furthermore,

$$K_1(x)\overline{u}^{a_1} v^{b_1} \leq C_4 m^{-a_1-\sigma|b_1|} \Lambda_1(d(x)) d(x)^{a_1\gamma_1+b_1\gamma_2-k_1} \text{ in } \Omega.$$

Similarly,

$$K_2(x)\overline{v}^{a_2} u^{b_2} \leq C'_4 m^{-\sigma a_2-|b_2|} \Lambda_2(d(x)) d(x)^{a_2\gamma_2+b_2\gamma_1-k_2} \text{ in } \Omega.$$

Then under (II.33) and thanks to (II.81) to (II.86), $(\overline{u}, \overline{v})$ is a super-solutions pair of problem (P), for m small enough. Therefore estimates (II.7) to (II.10) hold. Let us check that conditions (II.11) to (II.15) of Theorem 2.1 are satisfied. By estimates (II.79) and (II.80) and using the properties of the perturbations L_1 and L_2 given in point (a) and (c) of Remark 3.1, for all $\varepsilon > 0$ there exist positive constants C_1, C_2 and C_1', C_2' such that

$$C_1 d(x)^{\gamma_1} \leq \underline{u}, \overline{u} \leq C_2 d(x)^{\gamma_1 - \varepsilon} \quad \text{and} \quad C_1' d(x)^{\gamma_2} \leq \underline{v}, \overline{v} \leq C_2' d(x)^{\gamma_2 - \varepsilon} \quad \text{in } \Omega.$$

In addition, thanks to (II.81) to (II.86), there exist positive constants κ_1, κ_2 such that

$$|f_1(x, u, v)| = K_1(x) u^{a_1} v^{b_1} \leq \kappa_1 d(x)^{\delta_1 \gamma_1 - k_1 - \varepsilon} \quad \text{in } \Omega \times \mathcal{C},$$

$$|f_2(x, u, v)| = K_2(x) v^{a_2} u^{b_2} \leq \kappa_2 d(x)^{\delta_2 \gamma_2 - k_2 - \varepsilon} \quad \text{in } \Omega \times \mathcal{C}$$

and

$$\left|\frac{\partial f_1}{\partial u}(x, u, v)\right| = |a_1| K_1(x) u^{a_1 - 1} v^{b_1} \leq \kappa_1 d(x)^{(\delta_1 \gamma_1 - k_1 - \varepsilon) - \gamma_1} \quad \text{in } \Omega \times \mathcal{C},$$

$$\left|\frac{\partial f_2}{\partial v}(x, u, v)\right| = |a_2| K_2(x) v^{a_2 - 1} u^{b_2} \leq \kappa_2 d(x)^{(\delta_2 \gamma_2 - k_2 - \varepsilon) - \gamma_2} \quad \text{in } \Omega \times \mathcal{C}.$$

Since $\gamma_1 \in (1 - \frac{1}{p}, 1)$ and $\gamma_2 \in (1 - \frac{1}{q}, 1)$, inequalities (II.15) hold for ε small enough. Then, applying Theorem 2.1 we conclude about the existence of positive solutions to (P) in $\mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ satisfying the estimates (II.51) and (II.52).

Finally, using Theorem 0.1, we get that any positive weak solutions pair to (P) in the conical shell $\mathcal{C}$ belongs to $\mathscr{C}^{0,\alpha}(\overline{\Omega}) \times \mathscr{C}^{0,\alpha}(\overline{\Omega})$, for some $0 < \alpha < 1$. This proves *1.* of Theorem 3.2.

Alternative 2 : In this part, we look for positive solutions $(u, v) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ by making the "Ansatz" that both function u and v behave like the distance function $d(x)$ for $x \in \Omega$ near the boundary $\partial\Omega$. For that, similarly as in *Alternative 1*, we take in (II.75) and (II.76)

$$k_1 - 1 < \delta_1 < p - 1 \quad \text{and} \quad \mathrm{k}_2 - 1 < \delta_2 < q - 1, \tag{II.87}$$

$$\mathcal{L}_1 = L_1 \quad \text{and} \quad \mathcal{L}_2 = L_2 \quad \text{in } \Omega.$$

By Proposition 3.1, $\psi_1 \in \mathrm{W}_0^{1,p}(\Omega)$, $\psi_2 \in \mathrm{W}_0^{1,q}(\Omega)$ and satisfy

$$\psi_1(x) \sim d(x) \quad \text{and} \quad \psi_2(x) \sim d(x) \quad \text{in } \Omega.$$

In view of satisfying estimates given in (II.77), we fix δ_1 and δ_2 as follows :

$$\delta_1 = a_1 + b_1 \quad \text{and} \quad \delta_2 = a_2 + b_2. \tag{II.88}$$

Then, (II.77) holds and inequalities given in (II.87) entail (II.53). The rest of the proof is as in *Alternative 1.* This proves *2.* of Theorem 3.2.

Alternative 3 : Now we combine our methods from *Alternative 1* and *Alternative 2.* We search for positive solutions $(u,v) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ to problem (P) by again making the "Ansatz" that

$$u(x) \sim d(x)^{\gamma} L_1(d(x))^{\alpha} L_2(d(x))^{\beta} \text{ in } \Omega,$$

for some $\gamma \in (1-\frac{1}{p},1)$ and $\alpha, \beta \in \mathbb{R}$, and v behave like the distance function in Ω. For that, we take in (II.75) and (II.76)

$$k_1 - 2 + \frac{k_1-1}{p-1} < \delta_1 < k_1 - 1 \quad \text{and} \quad \mathrm{k}_2 - 1 < \delta_2 < q-1, \tag{II.89}$$

$$\mathcal{L}_1 = {L_1}^{\lambda_1}.{L_2}^{\mu_1} \quad \text{and} \quad \mathcal{L}_2 = {L_2}^{\lambda_2}.{L_1}^{\mu_2} \quad \text{in } \Omega,$$

where λ_1, λ_2, μ_1, μ_2 $\in \mathbb{R}$ are suitable exponents to be fixed. By Proposition 3.1, $\psi_1 \in \mathrm{W}_0^{1,p}(\Omega)$, $\psi_2 \in \mathrm{W}_0^{1,q}(\Omega)$ and satisfy

$$\psi_1(x) \sim d(x)^{\frac{p-k_1}{p-1-\delta_1}} L_1(d(x))^{\frac{\lambda_1}{p-1-\delta_1}} L_2(d(x))^{\frac{\mu_1}{p-1-\delta_1}} \quad \text{and} \quad \psi_2(x) \sim d(x) \quad \text{in } \Omega.$$

In view of (II.77), the exponents have to satisfy

$$\begin{cases} \delta_1 \frac{p-k_1}{p-1-\delta_1} = a_1 \frac{p-k_1}{p-1-\delta_1} + b_1, & \delta_2 = b_2 \frac{p-k_1}{p-1-\delta_1} + a_2, \\ \lambda_1 \frac{p-1}{p-1-\delta_1} = a_1 \frac{\lambda_1}{p-1-\delta_1} + 1, & \lambda_2 = b_2 \frac{\mu_1}{p-1-\delta_1} + 1, \\ \mu_1 \frac{p-1}{p-1-\delta_1} = a_1 \frac{\mu_1}{p-1-\delta_1}, & \mu_2 = b_2 \frac{\lambda_1}{p-1-\delta_1}. \end{cases}$$

Hence, we obtain particularly

$$\gamma = \tfrac{p-k_1}{p-1-\delta_1} = \tfrac{p-k_1+b_1}{p-1-a_1} \quad \text{and} \quad \delta_2 = a_2 + b_2 \tfrac{p-k_1+b_1}{p-1-a_1},$$

$$\alpha = \tfrac{\lambda_1}{p-1-\delta_1} = \tfrac{1}{p-1-\delta_1} \quad \text{and} \quad \beta = \tfrac{\mu_1}{p-1-\delta_1} = 0.$$

The rest of the proof is as in *Alternative 1.* This proves *3.* of Theorem 3.2 and *4.* is the corresponding symmetric case of *3.*. □

3.1.4 Proof of Theorem 3.3

To prove uniqueness of solutions, we apply a classical argument of Krasnoselskii [60]. Let $(u,v), (\tilde{u},\tilde{v}) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$, be two distinct positive weak solutions pairs to problem (P) in the conical shell $\mathcal{C} = [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$, where $(\underline{u}, \underline{v})$, $(\overline{u}, \overline{v})$ are given in the proof of Theorem 3.2. This means that $T(u,v)=(u,v)$ and $T(\tilde{u},\tilde{v})=(\tilde{u},\tilde{v})$, which implies that, $T_1\circ T_2(u)=u$, $T_2\circ T_1(v)=v$ and $T_1\circ T_2(\tilde{u})=\tilde{u}$, $T_1\circ T_2(\tilde{v})=\tilde{v}$, respectively. Let us define

$$C_{\max}\overset{\text{def}}{=}\sup\{C\in\mathbb{R}_+,\quad C\tilde{u}\le u\quad\text{and}\quad C\tilde{v}\le v\quad\text{a.e. in }\Omega\}. \tag{II.90}$$

$$T_1\circ T_2(C_{\max}\tilde{u})=(C_{\max})^{\frac{b_1}{p-1-a_1}\cdot\frac{b_2}{q-1-a_2}}T_1\circ T_2(\tilde{u})=(C_{\max})^{\frac{b_1}{p-1-a_1}\cdot\frac{b_2}{q-1-a_2}}\tilde{u},$$

$$T_2\circ T_1(C_{\max}\tilde{v})=(C_{\max})^{\frac{b_2}{q-1-a_2}\cdot\frac{b_1}{p-1-a_1}}T_2\circ T_1(\tilde{v})=(C_{\max})^{\frac{b_2}{q-1-a_2}\cdot\frac{b_1}{p-1-a_2}}\tilde{v}.$$

Therefore, by Theorem 3.1, both mappings $T_1\circ T_2$ and $T_2\circ T_1$ being (pointwise) order-preserving, we arrive at

$$u=T_1\circ T_2(u)\ge T_1\circ T_2(C_{\max}\tilde{u})=(C_{\max})^{\frac{b_1}{p-1-a_1}\cdot\frac{b_2}{q-1-a_2}}\tilde{u}, \tag{II.91}$$

$$v=T_2\circ T_1(v)\ge T_2\circ T_1(C_{\max}\tilde{v})=(C_{\max})^{\frac{b_2}{q-1-a_2}\cdot\frac{b_1}{p-1-a_1}}\tilde{v}. \tag{II.92}$$

From $0<C_{\max}<1$ combined with the sub-homogeneity condition (II.32) we deduce that

$$C'_{\max}\overset{\text{def}}{=}(C_{\max})^{\frac{b_1}{p-1-a_1}\cdot\frac{b_2}{q-1-a_2}}>C_{\max},$$

which contradicts the maximality of the constant $C_{\max}$ in (II.90), by inequalities (II.91) and (II.92). Then, $C_{\max}\ge 1$ which entails $\tilde{u}\le u$ and $\tilde{v}\le v$ a.e. in Ω. Interchanging the roles of (u,v) and $(\tilde{u},\tilde{v})$, we finally get $(u,v)=(\tilde{u},\tilde{v})$ a.e. in Ω. ☐

Proof. (of Theorem 3.4) The proof is similar to the proof of Theorem 3.2, so we omit it. ☐

3.1.5 Proof of Theorem 3.5

Alternative 1 : Assume that $a_1+b_1=k_1-1$ and $k_2-1\le a_2+b_2<q-1$. We look for positive sub- and super-solutions pairs $(\underline{u},\underline{v})$, $(\overline{u},\overline{v})$ in the form :

$$\underline{u}=m\psi_1\quad\text{and}\quad\overline{u}=m^{-1}(\varphi_{1,p})^{1-\varepsilon}\quad\text{in }\Omega,$$

$$\underline{v}=m^{\sigma}\psi_2\quad\text{and}\quad\overline{v}=m^{-\sigma}(\varphi_{1,q})^{1-\sigma\varepsilon}\quad\text{in }\Omega,$$

where $\sigma>0$ is given by (II.33), $\varepsilon<1$ and $m<1$ are appropriate positive constants small enough and $\psi_1\in \mathrm{W}_0^{1,p}(\Omega)$ and $\psi_2\in\mathrm{W}_0^{1,q}(\Omega)$ are the respective solutions to

$$-\Delta_p w=K_1(x)w^{\delta_1}\quad\text{in }\Omega;\quad w|_{\partial\Omega}=0,\quad w>0\quad\text{in }\Omega,$$

$$-\Delta_q w=K_2(x)w^{\delta_2}\quad\text{in }\Omega;\quad w|_{\partial\Omega}=0,\quad w>0\quad\text{in }\Omega,$$

with $k_1 - 1 < \delta_1 < p - 1$ and $a_2 + b_2 < \delta_2 < q - 1$. By Proposition 3.1, both ψ_1 and ψ_2 behave like the distance function in Ω. Let us remark that by estimate (II.5), $\underline{u} \leq \overline{u}$ and $\underline{v} \leq \overline{v}$ in Ω, for m small enough. Now, let $1 < r < \infty$ and $\gamma \in (0, 1)$, then we have

$$\begin{aligned} -\Delta_r \left[(\varphi_{1,r})^\gamma\right] &= \gamma^{r-1} \left[\lambda_{1,r}(\varphi_{1,r})^{\gamma(r-1)} - (\gamma - 1)(r - 1)(\varphi_{1,r})^{(\gamma-1)(r-1)-1}|\nabla\varphi_{1,r}|^r\right] \\ &= \gamma^{r-1}(\varphi_{1,r})^{-(1-\gamma)(r-1)-1} \left[\lambda_{1,r}(\varphi_{1,r})^r + (1 - \gamma)(r - 1)|\nabla\varphi_{1,r}|^r\right] \end{aligned}$$

in Ω. By estimate (II.5), we conclude that

$$-\Delta_r \left[(\varphi_{1,r}(x))^\gamma\right] \sim d(x)^{-(1-\gamma)(r-1)-1} \quad \text{in } \Omega. \tag{II.93}$$

So, let $(u, v) \in [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$. On one hand, we have

$$-\Delta_p\underline{u} \leq m^{p-1}C_1K_1(x)d(x)^{\delta_1} \quad \text{and} \quad -\Delta_q\underline{v} \leq m^{q-1}C_1'K_2(x)d(x)^{\delta_2} \quad \text{in}\,\Omega.$$

On the other hand, we also have

$$\begin{aligned} K_1(x)\underline{u}^{a_1}v^{b_1} &\geq \begin{cases} m^{a_1+\sigma b_1}K_1(x){\psi_1}^{a_1}{\psi_2}^{b_1} & \text{if } b_1 > 0, \\ m^{a_1-\sigma b_1}K_1(x){\psi_1}^{a_1}(\varphi_{1,q})^{b_1(1-\varepsilon\sigma)} & \text{if } b_1 < 0, \end{cases} \\ &\geq m^{a_1+\sigma|b_1|}C_2K_1(x)d(x)^{k_1-1+\varepsilon\sigma b_1^-} \quad \text{in } \Omega, \end{aligned}$$

in Ω. Similarly, we get

$$K_2(x)\underline{v}^{a_2}u^{b_2} \geq m^{\sigma a_2+|b_2|}C_2'K_2(x)d(x)^{a_2+b_2+\varepsilon b_2^-} \quad \text{in } \Omega.$$

Then, for m and ε small enough, $(\underline{u}, \underline{v})$ is a sub-solutions pair of problem (P). Similarly, using estimate (II.93), we obtain

$$-\Delta_p\overline{u} \geq m^{1-p}C_3d(x)^{-1-\varepsilon(p-1)} \quad \text{and} \quad -\Delta_q\overline{v} \geq m^{\sigma(1-q)}C_3'd(x)^{-1-\varepsilon\sigma(q-1)} \quad \text{in } \Omega.$$

Furthermore, by (II.36), for any $\varepsilon' > 0$, there exists $C_4 = C_4(\varepsilon') > 0$ such that

$$\begin{aligned} K_1(x)\overline{u}^{a_1}v^{b_1} &\leq \begin{cases} m^{-(a_1+\sigma b_1)}K_1(x)(\varphi_{1,p})^{a_1(1-\varepsilon)}(\varphi_{1,q})^{b_1(1-\varepsilon\sigma)} & \text{if } b_1 > 0, \\ m^{-(a_1-\sigma b_1)}K_1(x)(\varphi_{1,p})^{a_1(1-\varepsilon)}{\psi_2}^{b_1} & \text{if } b_1 < 0, \end{cases} \\ &\leq m^{-(a_1+\sigma|b_1|)}C_4d(x)^{-1-\varepsilon(a_1+\sigma b_1^+)-\varepsilon'} \quad \text{in } \Omega, \end{aligned}$$

Similarly, we have

$$K_2(x)\underline{v}^{a_2}u^{b_2} \leq m^{-(\sigma a_2+|b_2|)}C_4'd(x)^{-k_1+a_2+b_2-\varepsilon(\sigma a_2+b_2^+)-\varepsilon'} \quad \text{in } \Omega,$$

with $C'_4 = C'_4(\varepsilon')$. Then, for m, ε and ε' small enough, $(\overline{u}, \overline{v})$ is a super-solutions pair of problem (P). Applying Theorem 2.1, we get the existence of positive solutions $(u, v) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ of (P) satisfying (II.99). This proves *1.* of Theorem 3.5.

Alternative 2 : When $k_1 - 1 \leq a_1 + b_1 < q - 1$ and $a_2 + b_2 = k_2 - 1$, interchanging the role of u and v, the proof of *2.* is the same as above.

Alternative 3 : Assume that (II.66) is satisfied. To prove *3.*, we follow the proof in *Alternative 1.* We construct positive sub- and super-solutions pairs $(\underline{u}, \underline{v})$, $(\overline{u}, \overline{v}) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ in the form

$$\underline{u} = m(\varphi_{1,p})^{\gamma+\varepsilon},\ \overline{u} = m^{-1}(\varphi_{1,p})^{\gamma-\varepsilon} \text{ and } \underline{v} = m^{\sigma}\psi,\ \overline{v} = m^{-\sigma}(\varphi_{1,q})^{1-\sigma\varepsilon} \text{ in } \Omega,$$

where $\sigma > 0$ is given by (II.33), and ε, m are appropriate positive constants small enough and $\psi \in \mathrm{W}_0^{1,q}(\Omega)$ is the solution (see Proposition 3.1) of

$$-\Delta_q w = K_2(x) w^{\delta} \quad \text{in } \Omega; \quad w|_{\partial\Omega} = 0, \quad w > 0 \quad \text{in } \Omega,$$

with $a_2 + \gamma b_2 < \delta < q - 1$. *4.* is the symmetric case of *3.* by interchanging the role of u and v. Finally, from Theorem 0.1, we get the Hölder regularity of (u, v). □

3.2 Example 2

We consider now the following singular system

$$(\mathrm{P}) \begin{cases} -\Delta_p u = u^{a_1} v^{b_1} - u^{\alpha_1} v^{\beta_1} & \text{in } \Omega\,; \quad u|_{\partial\Omega} = 0, \quad u > 0 \quad \text{in } \Omega, \\ \\ -\Delta_q v = v^{a_2} u^{b_2} - v^{\alpha_2} u^{\beta_2} & \text{in } \Omega\,; \quad v|_{\partial\Omega} = 0, \quad v > 0 \quad \text{in } \Omega, \end{cases}$$

where the above exponents satisfy

$$(p - 1 - a_1) - \sigma|b_1| > 0 \quad \text{and} \quad (\alpha_1 - a_1) - \sigma(|\beta_1| - |b_1|) > 0, \tag{II.94}$$

$$\sigma(q - 1 - a_2) - |b_2| > 0 \quad \text{and} \quad \sigma(\alpha_2 - a_2) - (|\beta_2| - |b_2|) > 0, \tag{II.95}$$

for some constant $\sigma > 0$. Then, we have the following result :

Theorem 3.6

1. Let

$$\gamma_1 = \frac{p(q - 1 - a_2) + q b_1}{(p - 1 - a_1)(q - 1 - a_2) - b_1 b_2},\ \ \gamma_2 = \frac{q(p - 1 - a_1) + p b_2}{(p - 1 - a_1)(q - 1 - a_2) - b_1 b_2} \tag{II.96}$$

and assume that

$$1-\frac{1}{p}<\gamma_1<1 \quad \text{and} \quad (\alpha_1-a_1)\gamma_1+(\beta_1-b_1)\gamma_2>0, \tag{II.97}$$

$$1-\frac{1}{q}<\gamma_2<1 \quad \text{and} \quad (\alpha_2-a_2)\gamma_2+(\beta_2-b_2)\gamma_1>0. \tag{II.98}$$

Then, problem (P) *has a positive solution* $(u,v)\in \mathrm{W}_0^{1,p}(\Omega)\times \mathrm{W}_0^{1,q}(\Omega)$ *satisfying*

$$u(x)\sim d(x)^{\gamma_1} \quad \text{and} \quad v(x)\sim d(x)^{\gamma_2} \quad \text{in } \Omega. \tag{II.99}$$

In addition, we have $(u,v)\in \mathscr{C}^{0,\alpha}(\overline{\Omega})\times \mathscr{C}^{0,\alpha}(\overline{\Omega})$, *for some* $0<\alpha<1$.

2. *Assume that*

$$-1<a_1+b_1<p-1 \quad \text{and} \quad (\alpha_1-a_1)+(\beta_1-b_1)>0, \tag{II.100}$$

$$-1<a_2+b_2<q-1 \quad \text{and} \quad (\alpha_2-a_2)+(\beta_2-b_2)>0. \tag{II.101}$$

Then, (P) *has a positive solution* $(u,v)\in \mathrm{W}_0^{1,p}(\Omega)\times \mathrm{W}_0^{1,q}(\Omega)$ *satisfying*

$$u(x)\sim d(x) \quad \text{and} \quad v(x)\sim d(x) \quad \text{in } \Omega. \tag{II.102}$$

In addition, we have $(u,v)\in \mathscr{C}^{1,\alpha}(\overline{\Omega})\times \mathscr{C}^{1,\alpha}(\overline{\Omega})$, *for some* $0<\alpha<1$.

3. *Let*

$$\gamma=\frac{p+b_1}{p-1-a_1} \tag{II.103}$$

and assume that

$$1-\frac{1}{p}<\gamma<1 \quad \text{and} \quad (\alpha_1-a_1)\gamma+(\beta_1-b_1)>0, \tag{II.104}$$

$$-1<a_2+b_2\gamma<p-1 \quad \text{and} \quad (\alpha_2-a_2)+(\beta_2-b_2)\gamma>0. \tag{II.105}$$

Then, (P) *has a positive solution* $(u,v)\in \mathrm{W}_0^{1,p}(\Omega)\times \mathrm{W}_0^{1,q}(\Omega)$ *satisfying*

$$u(x)\sim d(x)^{\gamma} \quad \text{and} \quad v(x)\sim d(x) \quad \text{in } \Omega. \tag{II.106}$$

In addition, we have $(u,v)\in \mathscr{C}^{0,\alpha}(\overline{\Omega})\times \mathscr{C}^{1,\alpha}(\overline{\Omega})$, *for some* $0<\alpha<1$.

4. *Symmetrically, set*

$$\gamma=\frac{q+b_2}{q-1-a_2} \tag{II.107}$$

and assume that

$$-1<a_1+b_1\gamma<p-1 \quad \text{and} \quad (\alpha_1-a_1)+(\beta_1-b_1)\gamma>0, \tag{II.108}$$

$$1 - \frac{1}{q} < \gamma < 1 \quad \textit{and} \quad (\alpha_2 - a_2)\gamma + (\beta_2 - b_2) > 0. \tag{II.109}$$

Then, (P) *has a positive solution* $(u, v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ *satisfying*

$$u(x) \sim d(x) \quad \textit{and} \quad v(x) \sim d(x)^{\gamma} \quad \textit{in } \Omega. \tag{II.110}$$

In addition, we have $(u, v) \in \mathscr{C}^{1,\alpha}(\overline{\Omega}) \times \mathscr{C}^{0,\alpha}(\overline{\Omega})$, *for some* $0 < \alpha < 1$.

Proof. We apply Theorem 2.1 with

$$\underline{u} = m\psi_1, \quad \overline{u} = m^{-1}\psi_1 \quad \text{and} \quad \underline{v} = m^{\sigma}\psi_2, \quad \overline{v} = m^{-\sigma}\psi_2 \quad \text{in } \Omega,$$

where $\sigma > 0$ is the constant given in (II.94) and (II.95), $m < 1$ is a positive constant small enough and $\psi_1 \in W_0^{1,p}(\Omega)$, $\psi_2 \in W_0^{1,q}(\Omega)$ are given by Proposition 3.1 as the respective unique solutions of problems

$$-\Delta_p w = w^{\delta_1} \quad \text{in } \Omega; \quad w|_{\partial\Omega} = 0, \quad w > 0 \quad \text{in } \Omega,$$

$$-\Delta_q w = w^{\delta_2} \quad \text{in } \Omega; \quad w|_{\partial\Omega} = 0, \quad w > 0 \quad \text{in } \Omega,$$

satisfying some cone conditions we precise below. In the following *Alternatives*, we choose $-2 - \frac{1}{p-1} < \delta_1 < p - 1$ and $-2 - \frac{1}{q-1} < \delta_2 < q - 1$ such that

$$-\Delta_p\psi_1 \sim {\psi_1}^{a_1}{\psi_2}^{b_1} \quad \text{and} \quad -\Delta_q\psi_2 \sim {\psi_2}^{a_2}{\psi_1}^{b_2} \quad \text{in } \Omega. \tag{II.111}$$

Alternative 1 : Assume that conditions (II.97) and (II.98) hold. Then, arguing as in *Alternative 1* in the proof of Theorem 3.2, we choose $-2 - \frac{1}{p-1} < \delta_1 < -1$ and $-2 - \frac{1}{q-1} < \delta_2 < -1$ unique solutions pair of the following system :

$$\frac{\delta_1 p}{p - 1 - \delta_1} = \frac{a_1 p}{p - 1 - \delta_1} + \frac{b_1 q}{q - 1 - \delta_2} \quad \text{and} \quad \frac{\delta_2 q}{q - 1 - \delta_2} = \frac{a_2 q}{q - 1 - \delta_2} + \frac{b_2 p}{p - 1 - \delta_2}.$$

Since

$$\psi_1(x) \sim d(x)^{\gamma_1} \quad \text{and} \quad \psi_2(x) \sim d(x)^{\gamma_2} \quad \text{in } \Omega,$$

where $\gamma_1 = \frac{p}{p-1-\delta_1}$ and $\gamma_2 = \frac{q}{q-1-\delta_2}$ are given by (II.96), estimates (II.111) follows. Let $(u, v) \in [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$. First, we have

$$-\Delta_p\underline{u} \leq m^{p-1}C_1 d(x)^{\delta_1\gamma_1} \quad \text{and} \quad -\Delta_q\underline{v} \leq m^{\sigma(q-1)}C_1' d(x)^{\delta_2\gamma_2} \quad \text{in } \Omega. \tag{II.112}$$

In the other hand, by (II.94) and (II.97),

$$\begin{aligned} \underline{u}^{a_1}v^{b_1} - \underline{u}^{\alpha_1}v^{\beta_1} &\geq m^{a_1+\sigma|b_1|}{\psi_1}^{a_1}{\psi_2}^{b_1}\left[1 - m^{\alpha_1-a_1-\sigma(|\beta_1|-|b_1|)}{\psi_1}^{\alpha_1-a_1}{\psi_2}^{\beta_1-b_1}\right] \\ &\geq m^{a_1+\sigma|b_1|}C_2 d(x)^{a_1\gamma_1+b_1\gamma_2} \quad \text{in } \Omega. \end{aligned} \tag{II.113}$$

for m small enough. By (II.95) and (II.98), we also have

$$\underline{v}^{a_2}u^{b_2} - \underline{v}^{\alpha_2}u^{\beta_2} \geq m^{\sigma a_2+|b_2|}C_2' d(x)^{a_2\gamma_2+b_2\gamma_1} \quad \text{in } \Omega, \tag{II.114}$$

for m small enough. Then, under conditions (II.94), (II.95), (II.97) and (II.98) and for m small enough, $(\underline{u}, \underline{v})$ is a sub-solutions pair of problem (P).

Similarly, we have

$$-\Delta_p \overline{u} \geq m^{1-p}C_3 d(x)^{\delta_1\gamma_1} \quad \text{and} \quad -\Delta_q \overline{v} \geq m^{\sigma(1-q)}C_3' d(x)^{\delta_2\gamma_2} \quad \text{in } \Omega. \tag{II.115}$$

In addition,

$$\overline{u}^{a_1}v^{b_1} - \overline{u}^{\alpha_1}v^{\beta_1} \leq m^{-a_1-\sigma|b_1|}{\psi_1}^{a_1}{\psi_2}^{b_1} \leq m^{-a_1-\sigma|b_1|}C_4 d(x)^{a_1\gamma_1+b_1\gamma_2} \tag{II.116}$$

in Ω. We obtain further

$$\overline{v}^{a_2}u^{b_2} - \overline{v}^{\alpha_2}u^{\beta_2} \leq m^{-\sigma a_2-|b_2|}C_4' d(x)^{a_2\gamma_2+b_2\gamma_1} \quad \text{in } \Omega. \tag{II.117}$$

Then, under conditions (II.94), (II.95) and for m small enough, $(\underline{u}, \underline{v})$ is a super-solutions pair of problem (P).

Applying Theorem 2.1, we get the existence of positive solutions $(u, v) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ of (P) satisfying (II.99). Again from Theorem 0.1, (u, v) are Hölder continuous. This proves the assertion *1.*.

Alternative 2 : Now, assume that conditions (II.100) and (II.101) are satisfied. Then, we choose $\delta_1 = a_1 + b_1$ and $\delta_2 = a_2 + b_2$. By Proposition 3.1, since

$$\psi_1(x) \sim d(x) \quad \text{and} \quad \psi_2(x) \sim d(x) \quad \text{in } \Omega,$$

estimates (II.111) hold. Instead of inequalities (II.112), we have in this case

$$-\Delta_p \underline{u} \leq m^{p-1}C_1 d(x)^{a_1+b_1} \text{ and } -\Delta_q \underline{v} \leq m^{\sigma(q-1)}C_1' d(x)^{a_2+b_2} \quad \text{in } \Omega.$$

From (II.94), (II.95), (II.100) and (II.101), we get for any $(u, v) \in [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$:

$$\underline{u}^{a_1}v^{b_1} - \underline{u}^{\alpha_1}v^{\beta_1} \geq m^{a_1+\sigma|b_1|}C_2 d(x)^{a_1+b_1} \quad \text{in } \Omega,$$

$$\underline{v}^{a_2}u^{b_2} - \underline{v}^{\alpha_2}u^{\beta_2} \geq m^{\sigma a_2+|b_2|}C_2' d(x)^{a_2+b_2} \quad \text{in } \Omega,$$

for m small enough. Then, under conditions (II.94), (II.95), (II.100), (II.101) and for m small enough, $(\underline{u}, \underline{v})$ is a sub-solution pair of problem (P). Instead of inequalities (II.115), we have in this case in Ω,

$$-\Delta_p \overline{u} \geq m^{1-p} C_3 d(x)^{a_1+b_1} \quad \text{and} \quad -\Delta_q \overline{v} \geq m^{\sigma(1-q)} C_3' d(x)^{a_2+b_2}.$$

In addition, instead of inequalities (II.116) and (II.117), we get

$$\overline{u}^{a_1} v^{b_1} - \overline{u}^{\alpha_1} v^{\beta_1} \leq m^{-a_1-\sigma|b_1|} C_4 d(x)^{a_1+b_1},$$

$$\overline{v}^{a_2} u^{b_2} - \overline{v}^{\alpha_2} u^{\beta_2} \leq m^{-\sigma a_2-|b_2|} C_4' d(x)^{a_2+b_2},$$

in Ω. Then, under conditions (II.94), (II.95) and for m small enough, $(\overline{u}, \overline{v})$ is a super-solution pair of problem (P). Then, we conclude as in the *Alternative 1* and *2.* is proved.

Alternative 3 : Now, assume conditions (II.104) and (II.105) hold. Then, arguing as in the proof of Theorem 3.2, we choose $-2-\frac{1}{p} < \delta_1 < -1$ and $-1 < \delta_2 < q-1$ unique solutions pair of the following system :

$$\frac{\delta_1 p}{p-1-\delta_1} = \frac{a_1 p}{p-1-\delta_1} + b_1 \quad \text{and} \quad \delta_2 = a_2 + \frac{b_2 p}{p-1-\delta_2}.$$

Estimates in (II.111) hold since

$$\psi_1(x) \sim d(x)^{\gamma} \quad \text{and} \quad \psi_2(x) \sim d(x) \quad \text{in } \Omega,$$

with γ given by (II.103). Instead of inequalities (II.112), we have in this case

$$-\Delta_p \underline{u} \leq m^{p-1} C_1 d(x)^{\delta_1 \gamma} \quad \text{and} \quad -\Delta_q \underline{v} \leq m^{\sigma(q-1)} C_1' d(x)^{\delta_2} \quad \text{in } \Omega.$$

From (II.94), (II.95), (II.104) and (II.105), we obtain now

$$\underline{u}^{a_1} v^{b_1} - \underline{u}^{\alpha_1} v^{\beta_1} \geq m^{a_1+\sigma|b_2|} C_2 d(x)^{a_1\gamma+b_1} \quad \text{in } \Omega,$$

$$\underline{v}^{a_2} u^{b_2} - \underline{v}^{\alpha_2} u^{\beta_2} \geq m^{\sigma a_2+|b_2|} C_2' d(x)^{a_2+b_2\gamma} \quad \text{in } \Omega,$$

for m small enough. Then, under conditions (II.94), (II.95), (II.104), (II.105) and for m small enough, $(\underline{u}, \underline{v})$ is a sub-solution pair of problem (P). Instead of (II.115), we have

$$-\Delta_p \overline{u} \geq m^{1-p} C_3 d(x)^{\delta_1 \gamma} \quad \text{and} \quad -\Delta_p \overline{v} \geq m^{\sigma(1-q)} C_3' d(x)^{\delta_2} \quad \text{in}\, \Omega.$$

And inequalities (II.116) are replaced by

$$\overline{u}^{a_1} v^{b_1} - \overline{u}^{\alpha_1} v^{\beta_1} \leq m^{-a_1-\sigma|b_1|} C_4 d(x)^{a_1\gamma+b_1} \quad \text{in } \Omega,$$

$$\overline{v}^{a_2}u^{b_2} - \overline{v}^{\alpha_2}u^{\beta_2} \leq m^{-\sigma a_2 - |b_2|}C_4' d(x)^{a_2+b_2\gamma} \quad \text{in } \Omega.$$

Then, under conditions (II.94), (II.95) and for m small enough, $(\overline{u}, \overline{v})$ is a super-solution pair of problem (P). We conclude as in the *Alternative 1*. Thus, *3.* is proved. Note that *4.* is the symmetric case of *3.* by interchanging u and v. □

We can further prove similarly (we omit the proof) :

Theorem 3.7 *Assume that conditions* (II.94) *and* (II.95) *are satisfied.*

1. *Assume that*

$$a_1 + b_1 = -1 \quad \text{and} \quad (\alpha_1 - a_1) + (\beta_1 - b_1) > 0, \tag{II.118}$$

$$-1 \leq a_2 + b_2 < q - 1 \quad \text{and} \quad (\alpha_2 - a_2) + (\beta_2 - b_2) > 0. \tag{II.119}$$

Then, for all $\varepsilon > 0$ small enough, there exist $C_1, C_2 > 0$ and $C_1', C_2' > 0$ such that (P) *admits positive solutions $(u, v) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ satisfying :*

$$C_1 d(x) \leq u \leq C_2 d(x)^{1-\varepsilon} \quad \text{and} \quad C_1' d(x) \leq v \leq C_2' d(x)^{1-\varepsilon\sigma} \quad \text{in } \Omega, \tag{II.120}$$

with $\sigma > 0$ is given in (II.33)*. In addition, we have $(u, v) \in \mathscr{C}^{0,\alpha}(\overline{\Omega}) \times \mathscr{C}^{0,\alpha}(\overline{\Omega})$, for some $0 < \alpha < 1$.*

2. *Symmetrically, assume that*

$$-1 \leq a_1 + b_1 < q - 1 \quad \text{and} \quad (\alpha_1 - a_1) + (\beta_1 - b_1) > 0, \tag{II.121}$$

$$a_2 + b_2 = -1 \quad \text{and} \quad (\alpha_2 - a_2) + (\beta_2 - b_2) > 0. \tag{II.122}$$

Then, for all $\varepsilon > 0$ small enough, there exist $C_1, C_2 > 0$ and $C_1', C_2' > 0$ such that (P) *admits positive solutions $(u, v) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ satisfying :*

$$C_1 d(x) \leq u \leq C_2 d(x)^{1-\varepsilon} \quad \text{and} \quad C_1' d(x) \leq v \leq C_2' d(x)^{1-\varepsilon\sigma} \quad \text{in } \Omega. \tag{II.123}$$

In addition, we have $(u, v) \in \mathscr{C}^{0,\alpha}(\overline{\Omega}) \times \mathscr{C}^{0,\alpha}(\overline{\Omega})$, for some $0 < \alpha < 1$.

3. *Let*

$$\gamma = \frac{p + b_1}{p - 1 - a_1}$$

and assume that

$$1 - \frac{1}{p} < \gamma < 1 \quad \text{and} \quad (\alpha_1 - a_1)\gamma + (\beta_1 - b_1) > 0, \tag{II.124}$$

$$a_2 + b_2\gamma = -1 \quad \text{and} \quad (\alpha_2 - a_2) + (\beta_2 - b_2)\gamma > 0. \tag{II.125}$$

Then, for all $\varepsilon > 0$ small enough, there exist $C_1, C_2 > 0$ and $C_1', C_2' > 0$ such that (P) *admits positive solutions $(u, v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ satisfying :*

$$C_1 d(x)^{\gamma+\varepsilon} \leq u \leq C_2 d(x)^{\gamma-\varepsilon} \quad \textit{and} \quad C_1' d(x) \leq v \leq C_2' d(x)^{1-\varepsilon\sigma} \quad \textit{in } \Omega. \tag{II.126}$$

In addition, we have $(u, v) \in \mathscr{C}^{0,\alpha}(\overline{\Omega}) \times \mathscr{C}^{0,\alpha}(\overline{\Omega})$, for some $0 < \alpha < 1$.

4. *Symmetrically, let*

$$\gamma = \frac{q + b_2}{q - 1 - a_2}$$

and assume that

$$a_1 + b_1\gamma = -1 \quad \textit{and} \quad (\alpha_1 - a_1) + (\beta_1 - b_1)\gamma > 0, \tag{II.127}$$

$$1 - \frac{1}{q} < \gamma < 1 \quad \textit{and} \quad (\alpha_2 - a_2)\gamma + (\beta_2 - b_2) > 0. \tag{II.128}$$

Then, for all $\varepsilon > 0$ small enough, there exist $C_1, C_2 > 0$ and $C_1', C_2' > 0$ such that (P) *admits positive solutions $(u, v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ satisfying :*

$$C_1 d(x) \leq u \leq C_2 d(x)^{1-\varepsilon} \quad \textit{and} \quad C_1' d(x)^{\gamma+\varepsilon\sigma} \leq v \leq C_2' d(x)^{\gamma-\varepsilon\sigma} \; \textit{in } \Omega. \tag{II.129}$$

In addition, we have $(u, v) \in \mathscr{C}^{0,\alpha}(\overline{\Omega}) \times \mathscr{C}^{0,\alpha}(\overline{\Omega})$, for some $0 < \alpha < 1$.

3.3 Example 3

In this section, we consider the following singular competition system

$$(\mathrm{P}) \begin{cases} -\Delta_p u = \lambda_1 u^{\alpha_1} - u^{\beta_1} - \mu_1 u^{a_1} v^{b_1} \quad \text{in } \Omega\,; \quad u|_{\partial\Omega} = 0, \quad u > 0 \quad \text{in } \Omega, \\ \\ -\Delta_q v = \lambda_2 v^{\alpha_2} - v^{\beta_2} - \mu_2 v^{a_2} u^{b_2} \quad \text{in } \Omega\,; \quad v|_{\partial\Omega} = 0, \quad v > 0 \quad \text{in } \Omega, \end{cases}$$

where λ_1, λ_2 and μ_1, μ_2 are positive and $\alpha_1, \alpha_2, \beta_1, \beta_2, a_1, a_2, b_1, b_2$ satisfy

$$-2 - \frac{1}{p-1} < \alpha_1 < p - 1, \quad \alpha_1 < \beta_1 \quad \text{and} \quad a_1 - \alpha_1 - \sigma|b_1| > 0, \tag{II.130}$$

$$-2 - \frac{1}{q-1} < \alpha_2 < q - 1, \quad \alpha_2 < \beta_2 \quad \text{and} \quad \sigma(a_2 - \alpha_2) - |b_2| > 0, \tag{II.131}$$

for some constant $\sigma > 0$. Then, we have

Theorem 3.8 1. *Assume that*

$$-2 - \frac{1}{p-1} < \alpha_1 < -1 \quad \textit{and} \quad \frac{(a_1 - \alpha_1)p}{p - 1 - \alpha_1} + \frac{b_1 q}{q - 1 - \alpha_2} > 0, \tag{II.132}$$

$$-2 - \frac{1}{q-1} < \alpha_2 < -1 \quad \textit{and} \quad \frac{(a_2 - \alpha_2)q}{q - 1 - \alpha_2} + \frac{b_2 p}{p - 1 - \alpha_1} > 0. \tag{II.133}$$

Then, (P) *admits positive solutions* $(u,v) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ *satisfying :*

$$u(x) \sim d(x)^{\frac{p}{p-1-\alpha_1}} \quad \textit{and} \quad v(x) \sim d(x)^{\frac{q}{q-1-\alpha_2}} \quad \textit{in } \Omega. \tag{II.134}$$

In addition, we have $(u,v) \in \mathscr{C}^{0,\alpha}(\overline{\Omega}) \times \mathscr{C}^{0,\alpha}(\overline{\Omega})$*, for some* $0 < \alpha < 1$.

2. Assume that

$$-1 < \alpha_1 < p-1 \quad \textit{and} \quad a_1 - \alpha_1 + b_1 > 0, \tag{II.135}$$

$$-1 < \alpha_2 < q-1 \quad \textit{and} \quad a_2 - \alpha_2 + b_2 > 0. \tag{II.136}$$

Then, (P) *admits positive solutions* $(u,v) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ *satisfying :*

$$u(x) \sim d(x) \quad \textit{and} \quad v(x) \sim d(x) \quad \textit{in } \Omega. \tag{II.137}$$

In addition, we have $(u,v) \in \mathscr{C}^{1,\alpha}(\overline{\Omega}) \times \mathscr{C}^{1,\alpha}(\overline{\Omega})$*, for some* $0 < \alpha < 1$.

3. Assume that

$$-2 - \frac{1}{p-1} < \alpha_1 < -1 \quad \textit{and} \quad (a_1 - \alpha_1 + b_1)p - b_1(\alpha_1 + 1) > 0, \tag{II.138}$$

$$-1 < \alpha_2 < q-1 \quad \textit{and} \quad (a_2 - \alpha_2 + b_2)p - (a_2 - \alpha_2)(\alpha_1 + 1) > 0. \tag{II.139}$$

Then, (P) *admits positive solutions* $(u,v) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ *satisfying :*

$$u(x) \sim d(x)^{\frac{p}{p-1-\alpha_1}} \quad \textit{and} \quad v(x) \sim d(x) \quad \textit{in } \Omega. \tag{II.140}$$

In addition, we have $(u,v) \in \mathscr{C}^{0,\alpha}(\overline{\Omega}) \times \mathscr{C}^{1,\alpha}(\overline{\Omega})$*, for some* $0 < \alpha < 1$.

4. Symmetrically, assume that

$$-1 < \alpha_1 < p-1 \quad \textit{and} \quad (a_1 - \alpha_1 + b_1)q - (a_1 - \alpha_1)(\alpha_2 + 1) > 0, \tag{II.141}$$

$$-2 - \frac{1}{q-1} < \alpha_2 < -1 \quad \textit{and} \quad (a_2 - \alpha_2 + b_2)q - b_2(\alpha_2 + 1) > 0. \tag{II.142}$$

Then, (P) *admits positive solutions* $(u,v) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ *satisfying :*

$$u(x) \sim d(x) \quad \textit{and} \quad v(x) \sim d(x)^{\frac{q}{q-1-\alpha_2}} \quad \textit{in } \Omega. \tag{II.143}$$

In addition, we have $(u,v) \in \mathscr{C}^{1,\alpha}(\overline{\Omega}) \times \mathscr{C}^{0,\alpha}(\overline{\Omega})$*, for some* $0 < \alpha < 1$.

Proof. We apply Theorem 2.1 with

$$\underline{u} = m\psi_1, \quad \overline{u} = m^{-1}\psi_1 \quad \text{and} \quad \underline{v} = m^{\sigma}\psi_2, \quad \overline{v} = m^{-\sigma}\psi_2 \quad \text{in } \Omega, \tag{II.144}$$

where $\sigma > 0$ is the constant given in (II.130) and (II.131), $m < 1$ is a suitable small positive constant and $\psi_1 \in \mathrm{W}_0^{1,p}(\Omega)$, $\psi_2 \in \mathrm{W}_0^{1,q}(\Omega)$ are (given by Proposition 3.1) the respective unique solutions of problems

$$-\Delta_p w = w^{\alpha_1} \quad \text{in } \Omega; \quad w|_{\partial\Omega} = 0, \quad w > 0 \quad \text{in } \Omega,$$

$$-\Delta_q w = w^{\alpha_2} \quad \text{in } \Omega; \quad w|_{\partial\Omega} = 0, \quad w > 0 \quad \text{in } \Omega.$$

Alternative 1 : Assume conditions (II.132) and (II.133) are satisfied. Then, from Proposition 3.1, we get

$$\psi_1(x) \sim d(x)^{\frac{p}{p-1-\alpha_1}} \quad \text{and} \quad \psi_2(x) \sim d(x)^{\frac{q}{q-1-\alpha_2}} \quad \text{in } \Omega.$$

Let us prove that, for m small enough, $(\underline{u}, \underline{v})$ and $(\overline{u}, \overline{v})$ are respectively sub- and super-solutions pairs of (P). Let $(u, v) \in [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$. We have

$$-\Delta_p \underline{u} \leq m^{p-1} C_1 d(x)^{\frac{\alpha_1 p}{p-1-\alpha_1}} \text{ and } -\Delta_q \underline{v} \leq m^{\sigma(q-1)} C_1' d(x)^{\frac{\alpha_2 q}{q-1-\alpha_2}} \quad \text{in } \Omega. \tag{II.145}$$

From (II.130) and (II.132), we obtain :

$$\begin{aligned} &\lambda_1 \underline{u}^{\alpha_1} - \underline{u}^{\beta_1} - \mu_1 \underline{u}^{a_1} v^{b_1} \\ &\geq \lambda_1 (m\psi_1)^{\alpha_1} \left[1 - \tfrac{1}{\lambda_1} (m\psi_1)^{\beta_1 - \alpha_1} - \tfrac{\mu_1}{\lambda_1} (m\psi_1)^{a_1 - \alpha_1} \left(m^{-\sigma \operatorname{sign}(b_1)} \psi_2 \right)^{b_1} \right] \\ &\geq \tfrac{\lambda_1}{2} m^{\alpha_1} C_2 d(x)^{\frac{\alpha_1 p}{p-1-\alpha_1}}, \end{aligned} \tag{II.146}$$

for m small enough. In addition, from (II.131) and (II.133), we get :

$$\lambda_2 \underline{v}^{\alpha_2} - \underline{v}^{\beta_2} - \mu_2 \underline{v}^{a_2} u^{b_2} \geq \frac{\lambda_1}{2} m^{\sigma \alpha_2} C_2' d(x)^{\frac{\alpha_2 q}{q-1-\alpha_2}} \quad \text{in } \Omega, \tag{II.147}$$

for m small enough. Then, under conditions (II.132), (II.133) and for m small enough, $(\underline{u}, \underline{v})$ is a sub-solutions pair of problem (P). We also get

$$-\Delta_p \overline{u} \geq m^{1-p} C_3 d(x)^{\frac{\alpha_1 p}{p-1-\alpha_1}} \text{ and } -\Delta_q \overline{v} \geq m^{\sigma(1-q)} C_3' d(x)^{\frac{\alpha_2 q}{q-1-\alpha_2}} \quad \text{in } \Omega. \tag{II.148}$$

Similarly, one has

$$\lambda_1 \overline{u}^{\alpha_1} - \overline{u}^{\beta_1} - \mu_1 \overline{u}^{a_1} v^{b_1} \leq \lambda_1 m^{-\alpha_1} C_4 d(x)^{\frac{\alpha_1 p}{p-1-\alpha_1}} \quad \text{in } \Omega, \tag{II.149}$$

$$\lambda_2 \overline{v}^{\alpha_2} - \overline{v}^{\beta_2} - \mu_2 \overline{v}^{a_2} u^{b_2} \leq \lambda_2 m^{-\sigma \alpha_2} C_4' d(x)^{\frac{\alpha_2 q}{q-1-\alpha_2}} \quad \text{in } \Omega. \tag{II.150}$$

Then, for m small enough, $(\overline{u}, \overline{v})$ is a super-solutions pair of problem (P).

Applying Theorem 2.1, we get the existence of positive solutions $(u, v) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ of (P) satisfying (II.134). From Theorem 0.1, we get the Hölder regularity of u and v. This proves *1.*.

Alternative 2 : Now, let conditions (II.135) and (II.136) be satisfied. Then,

$$\psi_1(x) \sim d(x) \quad \text{and} \quad \psi_2(x) \sim d(x) \quad \text{in } \Omega.$$

Let $(u, v) \in [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$. Instead of (II.145), we now get

$$-\Delta_p \underline{u} \leq m^{p-1} C_1 d(x)^{\alpha_1} \quad \text{and} \quad -\Delta_q \underline{v} \leq m^{\sigma(q-1)} C_1' d(x)^{\alpha_2} \quad \text{in } \Omega.$$

From (II.130), (II.131), (II.135) and (II.136), instead of (II.146) and (II.147), we have

$$\lambda_1 \underline{u}^{\alpha_1} - \underline{u}^{\beta_1} - \mu_1 \underline{u}^{a_1} v^{b_1} \geq \frac{\lambda_1}{2} m^{\alpha_1} C_2 d(x)^{\alpha_1} \quad \text{in } \Omega,$$

$$\lambda_2 \underline{v}^{\alpha_2} - \underline{v}^{\beta_2} - \mu_2 \underline{v}^{a_2} u^{b_2} \geq \frac{\lambda_2}{2} m^{\sigma\alpha_2} C_2' d(x)^{\alpha_2} \quad \text{in } \Omega,$$

for m small enough. Then, under conditions (II.135), (II.136) and for m small enough, $(\underline{u}, \underline{v})$ is a sub-solutions pair of problem (P). Instead of (II.148), we have

$$-\Delta_p \overline{u} \geq m^{1-p} C_3 d(x)^{\alpha_1} \quad \text{and} \quad -\Delta_q \overline{v} \geq m^{\sigma(1-q)} C_3' d(x)^{\alpha_2} \quad \text{in } \Omega.$$

Furthermore, the following inequalities

$$\lambda_1 \overline{u}^{\alpha_1} - \overline{u}^{\beta_1} - \mu_1 \overline{u}^{a_1} v^{b_1} \leq \lambda_1 m^{-\alpha_1} C_4 d(x)^{\alpha_1} \ \text{ in } \Omega,$$

$$\lambda_2 \overline{v}^{\alpha_2} - \overline{v}^{\beta_2} - \mu_2 \overline{v}^{a_2} u^{b_2} \leq \lambda_1 m^{-\sigma\alpha_2} C_4' d(x)^{\alpha_2} \ \text{ in } \Omega$$

replace (II.149) and (II.150). Then, for m small enough, $(\overline{u}, \overline{v})$ is a super-solutions pair of problem (P). We conclude as in the Alternative 1 and *2.* is proved.

Alternative 3 : Now, assume that conditions (II.138) and (II.139) are satisfied. Then,

$$\psi_1(x) \sim d(x)^{\frac{p}{p-1-\alpha_1}} \quad \text{and} \quad \psi_2(x) \sim d(x) \quad \text{in } \Omega.$$

Let $(u, v) \in [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$. Instead of (II.145), we have

$$-\Delta_p \underline{u} \leq m^{p-1} C_1 d(x)^{\frac{\alpha_1 p}{p-1-\alpha_1}} \quad \text{and} \quad -\Delta_q \underline{v} \leq m^{\sigma(q-1)} C_1' d(x)^{\alpha_2} \quad \text{in } \Omega.$$

From (II.130), (II.131), (II.138) and (II.139), instead of (II.146) and (II.147), we get

$$\lambda_1 \underline{u}^{\alpha_1} - \underline{u}^{\beta_1} - \mu_1 \underline{u}^{a_1} v^{b_1} \geq \frac{\lambda_1}{2} m^{\alpha_1} C_2 d(x)^{\frac{\alpha_1 p}{p-1-\alpha_1}} \quad \text{in } \Omega,$$

$$\lambda_2 \underline{v}^{\alpha_2} - \underline{v}^{\beta_2} - \mu_2 \underline{v}^{a_2} u^{b_2} \geq \frac{\lambda_2}{2} m^{\sigma\alpha_2} C_2' d(x)^{\alpha_2} \quad \text{in } \Omega,$$

for m small enough. Then, under conditions (II.138), (II.139) and for m small enough, $(\underline{u}, \underline{v})$ is a sub-solutions pair of problem (P). Finally, Instead of (II.148), we have

$$-\Delta_p \overline{u} \geq m^{1-p} C_3 d(x)^{\frac{\alpha_1 p}{p-1-\alpha_1}} \quad \text{and} \quad -\Delta_q \overline{v} \geq m^{\sigma(1-q)} C_3' d(x)^{\alpha_2} \quad \text{in } \Omega.$$

Instead of (II.149) and (II.150), we obtain

$$\lambda_1 \overline{u}^{\alpha_1} - \overline{u}^{\beta_1} - \mu_1 \overline{u}^{a_1} v^{b_1} \leq \lambda_1 m^{-\alpha_1} C_4 d(x)^{\frac{\alpha_1 p}{p-1-\alpha_1}} \quad \text{in } \Omega,$$

$$\lambda_2 \overline{v}^{\alpha_2} - \overline{v}^{\beta_2} - \mu_2 \overline{v}^{a_2} u^{b_2} \leq \lambda_1 m^{-\sigma\alpha_2} C_4' d(x)^{\alpha_2} \quad \text{in } \Omega.$$

Then, for m small enough, $(\overline{u}, \overline{v})$ is a super-solutions pair of problem (P). Then, we conclude as in the *Alternative 1.* Thus, *3.* and by symmetry *4.* are proved. ☐

Concerning the above theorem, we analyse further some limiting cases. The proof of the next result follows the proof of Theorem 3.5. So we omit it.

Theorem 3.9

1. *Let*

$$\alpha_1 = -1 \quad \textit{and} \quad (a_1 - \alpha_1 + b_1)q - (a_1 - \alpha_1)(\alpha_2 + 1) > 0, \tag{II.151}$$

$$-2 - \frac{1}{q-1} < \alpha_2 < -1 \quad \textit{and} \quad (a_2 - \alpha_2 - b_2)q - b_2(\alpha_2 + 1) > 0. \tag{II.152}$$

Then, (P) *admits positive solutions* $(u, v) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ *satisfying :*

$$u(x) \sim d(x) \left| \ln(d(x)) \right|^{\frac{1}{p}} \quad \textit{and} \quad v(x) \sim d(x)^{\frac{q}{q-1-\alpha_2}} \quad \textit{in } \Omega. \tag{II.153}$$

In addition, we have $(u, v) \in \mathscr{C}^{0,\alpha}(\overline{\Omega}) \times \mathscr{C}^{0,\alpha}(\overline{\Omega})$, *for some* $0 < \alpha < 1$.

2. *Let*

$$\alpha_1 = -1 \quad \textit{and} \quad a_1 - \alpha_1 + b_1 > 0, \tag{II.154}$$

$$\alpha_2 = -1 \quad \textit{and} \quad a_2 - \alpha_2 + b_2 > 0. \tag{II.155}$$

Then, (P) *admits positive solutions* $(u, v) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ *satisfying :*

$$u(x) \sim d(x) \left| \ln(d(x)) \right|^{\frac{1}{p}} \quad \textit{and} \quad v(x) \sim d(x) \left| \ln(d(x)) \right|^{\frac{1}{q}} \quad \textit{in } \Omega. \tag{II.156}$$

In addition, we have $(u, v) \in \mathscr{C}^{0,\alpha}(\overline{\Omega}) \times \mathscr{C}^{0,\alpha}(\overline{\Omega})$, *for some* $0 < \alpha < 1$.

3. *Let*

$$\alpha_1 = -1 \quad \textit{and} \quad a_1 - \alpha_1 + b_1 > 0, \tag{II.157}$$

$$-1 < \alpha_2 < q - 1 \quad \textit{and} \quad a_2 - \alpha_2 + b_2 > 0. \tag{II.158}$$

Then, (P) *admits positive solutions* $(u,v) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ *satisfying :*

$$u(x) \sim d(x)|\ln(d(x))|^{\frac{1}{p}} \quad \text{and} \quad v(x) \sim d(x) \quad \text{in } \Omega. \tag{II.159}$$

In addition, we have $(u,v) \in \mathscr{C}^{0,\alpha}(\overline{\Omega}) \times \mathscr{C}^{1,\alpha}(\overline{\Omega})$, *for some* $0<\alpha<1$.

3.4 Example 4

We consider next the following singular symbiosis system

$$(\mathrm{P}) \begin{cases} -\Delta_p u = \lambda_1 u^{\alpha_1} - u^{\beta_1} + \mu_1 u^{a_1} v^{b_1} \quad \text{in } \Omega\,; \quad u|_{\partial\Omega} = 0, \quad u>0 \quad \text{in } \Omega, \\ \\ -\Delta_q v = \lambda_2 v^{\alpha_2} - v^{\beta_2} + \mu_2 v^{a_2} u^{b_2} \quad \text{in } \Omega\,; \quad v|_{\partial\Omega} = 0, \quad v>0 \quad \text{in } \Omega, \end{cases}$$

where $\lambda_1, \lambda_2 > 0$ and $\mu_1, \mu_2 > 0$ and $\alpha_1, \alpha_2, \beta_1, \beta_2, a_1, a_2, b_1, b_2$ satisfy

$$-2 - \frac{1}{p-1} < \alpha_1 < p-1, \quad \alpha_1 < \beta_1 \quad \text{and} \quad a_1 - \alpha_1 + \sigma|b_1| > 0, \tag{II.160}$$

$$-2 - \frac{1}{q-1} < \alpha_2 < q-1, \quad \alpha_2 < \beta_2 \quad \text{and} \quad \sigma(a_2 - \alpha_2) + |b_2| > 0, \tag{II.161}$$

for some constant $\sigma > 0$. Then, we get the same results about existence of positive solutions as those in the previous section.

Proof. We use the same strategy as Theorem 3.8's proof. We will only justify here the choice of sub- and super-solutions in assertion *1.*. Let conditions (II.160) and (II.161) be satisfied. Let us consider sub- and super-solution pairs $(\underline{u}, \underline{v})$ and $(\overline{u}, \overline{v})$ defined by (II.144) and let $(u,v) \in [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$. Then, From (II.160) and (II.132), we have

$$\begin{aligned} \lambda_1 \underline{u}^{\alpha_1} - \underline{u}^{\beta_1} + \mu_1 \underline{u}^{a_1} v^{b_1} \quad &\geq \quad \lambda_1 (m\psi_1)^{\alpha_1} \left[1 - \tfrac{1}{\lambda_1}(m\psi_1)^{\beta_1 - \alpha_1}\right] \\ &\geq \quad \tfrac{\lambda_1}{2} m^{\alpha_1} C_2 d(x)^{\frac{\alpha_1 p}{p-1-\alpha_1}} \quad \text{in } \Omega, \end{aligned} \tag{II.162}$$

for m small enough. Similarly, from (II.161) and (II.133), we get

$$\lambda_1 \underline{v}^{\alpha_2} - \underline{v}^{\beta_2} + \mu_2 \underline{v}^{a_2} u^{b_2} \geq \frac{\lambda_2}{2} m^{\sigma\alpha_2} C_2' d(x)^{\frac{\alpha_2 p}{q-1-\alpha_2}} \quad \text{in } \Omega, \tag{II.163}$$

for m small enough. Then, for m small enough, $(\underline{u}, \underline{v})$ is a sub-solution pair of (P). From (II.160) and (II.132), we have

$$\begin{aligned} \lambda_1 \overline{u}^{\alpha_1} - \overline{u}^{\beta_1} + \mu_1 \overline{u}^{a_1} v^{b_1} \quad &\leq \quad \lambda_1 (m^{-1}\psi_1)^{\alpha_1} + \mu_1 (m^{-1}\psi_1)^{\beta_1} \\ &\leq \quad \lambda_1 \left(m^{-1}\psi_1\right)^{\alpha_1} + \mu_1 \left(m^{-1}\psi_1\right)^{a_1} \left(m^{-\sigma \operatorname{sign}(b_1)}\psi_2\right)^{b_1} \\ &\leq \quad 2\lambda_1 m^{-\alpha_1} C_4 d(x)^{\frac{\alpha_1 p}{p-1-\alpha_1}} \quad \text{in } \Omega, \end{aligned} \tag{II.164}$$

for m small enough. Similarly, from (II.161) and (II.133), we have

$$\lambda_1 \overline{v}^{\alpha_2} - \overline{v}^{\beta_2} + \mu_2 \overline{v}^{a_2} u^{b_2} \leq 2\lambda_2 m^{-\sigma\alpha_2} C_4' d(x)^{\frac{\alpha_2 p}{q-1-\alpha_2}} \quad \text{in } \Omega, \tag{II.165}$$

for m small enough. Then, for m small enough, $(\overline{u}, \overline{v})$ is a super-solutions pair of (P). The modifications are similar in the three others alternatives. □

3.5 Example 5

We consider next the following singular predator-prey system

$$(\mathrm{P}) \begin{cases} -\Delta_p u = \lambda_1 u^{\alpha_1} - u^{\beta_1} - \mu_1 u^{a_1} v^{b_1} \quad \text{in } \Omega\,; \quad u|_{\partial\Omega} = 0, \quad u > 0 \quad \text{in } \Omega, \\ \\ -\Delta_q v = \lambda_2 v^{\alpha_2} - v^{\beta_2} + \mu_2 v^{a_2} u^{b_2} \quad \text{in } \Omega\,; \quad v|_{\partial\Omega} = 0, \quad v > 0 \quad \text{in } \Omega, \end{cases}$$

where $\lambda_1, \lambda_2 > 0$ and $\mu_1, \mu_2 > 0$ and $\alpha_1, \alpha_2, \beta_1, \beta_2, a_1, a_2, b_1, b_2$ satisfy

$$-2 - \frac{1}{p-1} < \alpha_1 < p - 1, \quad \alpha_1 < \beta_1 \quad \text{and} \quad a_1 - \alpha_1 - \sigma|b_1| > 0, \tag{II.166}$$

$$-2 - \frac{1}{q-1} < \alpha_2 < q - 1, \quad \alpha_2 < \beta_2 \quad \text{and} \quad \sigma(a_2 - \alpha_2) + |b_2| > 0, \tag{II.167}$$

for some constant $\sigma > 0$. Then, we derive the same results about existence of positive solutions as those in Section 3.3. We omit the proofs here.

Chapitre III

Study of an absorption phenomenon for a singular and parabolic problem.

Nous présentons ici les résultats issus de Giacomoni-Sauvy-Shmarev [47], travail réalisé en collaboration avec Jacques Giacomoni et Sergey Shmarev, Professeur à l'Université d'Oviedo.

1 Introduction

1.1 Statement of the problem

Let Ω be a bounded domain of $\mathbb{R}^d$, $d \geq 2$, with a smooth boundary and let $T > 0$. Setting $Q \stackrel{\text{def}}{=} (0,T) \times \Omega$ and $\Gamma \stackrel{\text{def}}{=} (0,T) \times \partial\Omega$, we consider the following quasilinear and singular parabolic problem :

$$(\mathrm{P}) \begin{cases} \partial_t u - \Delta_p u + \mathbb{1}_{\{u>0\}} u^{-\beta} = f(x,u) & \text{in } Q, \\ u = 0 & \text{on } \Gamma, \\ u(0,\cdot) = u_0 & \text{in } \Omega. \end{cases}$$

In this problem, $\Delta_p u \stackrel{\text{def}}{=} \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $\beta \in (0,1)$ and the initial datum satisfies

$$u_0 \in \mathrm{W}_0^{1,p}(\Omega) \cap \mathrm{L}^\infty(\Omega) \quad \text{and} \quad u_0 \geq 0 \quad \text{a.e. in } \Omega. \tag{III.1}$$

We denote by $\mathbb{1}_{\{u>0\}}$ the characteristic function of the set $\{(t,x) \in Q \mid u(t,x) > 0\}$ and we tacitly assume that $\mathbb{1}_{\{u>0\}} u^{-\beta} = 0$ whenever $u = 0$. In the right hand side of the first equation, f satisfies the following conditions :

1. $f : \Omega \times [0,+\infty) \to \mathbb{R}$ is a Carathéodory function, locally Lipschitz with respect to the second variable uniformly in x and such that

$$\forall \text{ a.e. } x \in \Omega, \qquad f(x,0) = 0. \tag{III.2}$$

2. There exists a non-decreasing and locally Lipschitz function $g : [0,+\infty) \to [0,+\infty)$ such that

$$\forall \text{ a.e. } x \in \Omega,\ \forall w \in [0,+\infty), \qquad |f(x,w)| \leq g(w). \tag{III.3}$$

Problem (P) appears in the limiting situation of models describing enzymatic kinetics (see Banks [8]) and in the Langmuir-Hinshelwood model of heterogeneous chemical catalyst (see Aris-Cho-Carr [6] and also Díaz [26]). It has already been studied for the heat equation, *i.e.* with $p = 2$, by Deng-Levine [24], Fila-Hushof-Quittner [38], Fila-Kawohl [39], Fila-Levine-Vázquez [40] and Levine [63], with Dirichlet boundary conditions equal to 1 on Γ; and by Phillips in the whole space $\mathbb{R}^d$ (see [70]). Winkler has also studied in [85, 86] other related parabolic problems in a non divergence form and with a singular absorption term. The most striking phenomenon that can occur due to the singular absorption term is that, even staring with a positive initial condition u_0, a solution may vanish in finite time. This behaviour is called ***"quenching"*** and was first observed in the pioneering paper of Kawarada [59]. The contributions quoted above have studied the occurrence of such quenching phenomena in various situations. Qualitative properties of solutions (asymptotic behaviour, uniqueness, stability, etc...) and the profile of solutions near the quenching points are also studied in detail in these papers. In Davila-Montenegro [21], the authors have considered the equation in (P) with $p = 2$, homogeneous boundary conditions on Γ and $u_0 \in \mathrm{L}^\infty(\Omega) \cap \mathscr{C}(\Omega)$ and $u_0 \geq 0$ a.e. in Ω. The authors first prove the existence of a weak solution of (P) (see Theorem 1.1 in [21]). For that, they study an approximated problem obtained by a suitable regularization of the singular nonlinearity and thanks to uniform a priori estimates they are able to pass to the limit to obtain a weak solution to the initial problem. Then, they further investigate the global behaviour of weak solutions. Precisely under a sub-linear asymptotic behaviour of the nonlinearity f, they prove a quenching behaviour of the weak solution u, in the sense that the measure of the vanishing set $\{(t, x) \in [0, +\infty) \times \Omega \mid u(t, x) = 0\}$ is positive. As it is shown in [21], the quenching phenomenon is strongly related to the non-existence of positive solutions to the stationary problem associated to (P) (see Theorems 1.6 and 1.7). The properties of the stationary solutions are described in Davila-Montenegro [20] where in particular the existence of compact support solutions is proved under some additional conditions on f.

In the present paper we investigate (P) in the quasilinear situation (i.e. $1 < p < \infty$) and for a more general class of nonlinearities f. We first prove the existence of a weak solution to (P). For that, we follow a similar approach as in [21] based on the study of approximated problems but in contrast with [21] to get convergence of solutions to approximated problems, we employ monotonicity arguments instead of estimates from Hölder regularity theory for semilinear parabolic equations. In order to get the monotonicity property, we are forced to take a specific choice of the approximations (see Sections 1.2 and 2). We point out that since the nonlinearity is discontinuous, we need a careful analysis of the behaviour of the approximated solutions to get the convergence to a solution to (P) (see Subsections 2.1 and 2.2).

Next, under the sub-linear asymptotic behaviour (III.6), we prove the extinction of the solution u in Ω, in finite time. The quenching behaviour is established by proving a differential inequality on the energy $t \mapsto \|u(t)\|_{\mathrm{L}^2(\Omega)}$ derived from inequalities of Gagliardo-Nirenberg-type. We highlight that for a quite large class of reaction f we get a complete quenching phenomenon in respect to Theorem 1.6 in [21].

This paper is organised as follows : in the next section, we give the main results concerning the existence and the asymptotic behaviour of the weak solutions of (P) (see Theorems 1.1 and 1.2) and the existence and the uniqueness of a weak solution of the perturbed problem (P_ε) (see Theorem 1.3). In Section 2, we introduce the perturbed problem ($\mathrm{P}_{\varepsilon,\eta}$). Using a semi-discretization in time method, we prove the existence and the uniqueness of the weak solutions to problem (P_ε) for which we only require u_0 to satisfy (III.10) (see Theorem 2.2). In Section 3, using monotone arguments, we give the proof of Theorems 1.3 and 1.1, passing to the limit as $\eta \to 0$ and then as $\varepsilon \to 0$ in the perturbed problems ($\mathrm{P}_{\varepsilon,\eta}$) and ($\mathrm{P}_\varepsilon$).

Finally, in Section 4, assuming that f satisfies (III.6), we give the proof of the quenching behaviour of the weak solutions of (P) (see Theorem 1.2).

1.2 Definitions and main results

Definition 1.1 *Let us define the function space*

$$\mathscr{U} \stackrel{\text{def}}{=} \left\{ v \in \mathrm{L}^\infty\left(0,T;\mathrm{W}_0^{1,p}(\Omega)\right) \cap \mathrm{L}^\infty(Q) \quad \middle| \quad \partial_t v \in \mathrm{L}^2(Q) \right\}.$$

Definition 1.2 *A function $u \in \mathscr{U}$ is called* ***weak solution of problem* (P)** *if :*

1. $u \geq 0$ a.e. *in Q.*
2. *For every test function $\varphi \in \mathscr{D}(Q)$,*

$$\int_Q \partial_t u \varphi \, dz + \int_Q |\nabla u|^{p-2}\nabla u.\nabla\varphi \, dz + \int_Q \mathbb{1}_{\{u>0\}} u^{-\beta}\varphi \, dz = \int_Q f(x,u)\varphi \, dz, \qquad \text{(III.4)}$$

   where $dz \stackrel{\text{def}}{=} dx\,ds$.

3. $u(0,\cdot) = u_0$ a.e. *in Ω.*

Remark 1.1

1. *Point 3. of the above definition is meaningful. Indeed, we have the compact embedding of* $\mathrm{W}_0^{1,p}(\Omega)$ *in* $\mathrm{L}^2(\Omega)$ *if* $p > \frac{2d}{d+2}$ *or in* $\mathrm{L}^q(\Omega)$*, for any* $1 \leq q < \frac{dp}{d-p}$*, if* $p \leq \frac{2d}{d+2}$*. Then, from Aubin-Simon compactness result (see* Aubin *[7] ot* Simon *[79]),* $\mathscr{U} \hookrightarrow \mathscr{C}([0,T],\mathrm{L}^r(\Omega))$*, with* $r \stackrel{\text{def}}{=} \min\{2,q\}$.
2. *In fact,* $\mathscr{U} \hookrightarrow \mathscr{C}([0,T],\mathrm{L}^s(\Omega))$*, for any* $s \in [1,+\infty)$*. Indeed, since* $\mathscr{U} \hookrightarrow \mathrm{L}^\infty(Q)$*, it suffices to use the following interpolation inequality (see* Brézis *[12]) when* $s > r$ *:*

$$\|v\|_{\mathrm{L}^s(\Omega)} \leq \|v\|^\alpha_{\mathrm{L}^q(\Omega)} \times \|v\|^{1-\alpha}_{\mathrm{L}^\infty(\Omega)},$$

   with $\alpha = \frac{r}{s} < 1$.

Our two main results are the following :

Theorem 1.1 *Assume that u_0 satisfies condition* (III.1)*. Then, there exists $T^* > 0$ such that for any $T < T^*$, problem* (P) *has at least one weak solution $u \in \mathscr{U}$. Moreover, u satisfies the following energy* *identity : for any* $t \in [0, T]$,

$$\frac{1}{2}\|u(t)\|^2_{\mathrm{L}^2(\Omega)} - \frac{1}{2}\|u_0\|^2_{\mathrm{L}^2(\Omega)} + \int_0^t \int_\Omega |\nabla u|^p \, dz + \int_0^t \int_\Omega u^{1-\beta} \, dz = \int_0^t \int_\Omega f(x,u)u \, dz, \tag{III.5}$$

with the natural notation $u(t) \stackrel{\text{def}}{=} u(t,\cdot)$ a.e. *in* Ω.

Theorem 1.2 *Assume that* u_0 *satisfies condition* (III.1). *Furthermore, assume that* $p \geq \frac{2d}{d+2}$ *and instead of condition* (III.3), *the asymptotic behaviour of* f *is more precisely given by*

$$\forall w \in [0, +\infty), \qquad |f(x,w)| \leq \alpha w^{p-1} + C_\alpha, \tag{III.6}$$

where $\alpha + C_\alpha < \lambda_1$ *and* $C_\alpha < 1$, *with* λ_1, *the first eigenvalue of the p-Laplace operator defined by*

$$\lambda_1 \stackrel{\text{def}}{=} \inf\left\{ \int_\Omega |\nabla v|^p \, dx \in \mathbb{R}^+ \quad \middle| \quad v \in \mathrm{W}^{1,p}_0(\Omega), \quad \int_\Omega |v|^p \, dx = 1 \right\}.$$

Then, for any $T > 0$, (P) *has a weak solution*

$$u \in \tilde{\mathscr{U}} \stackrel{\text{def}}{=} \left\{ v \in \mathrm{L}^\infty\left(0, T; \mathrm{W}^{1,p}_0(\Omega)\right) \cap \mathrm{L}^\infty\left(0, T; \mathrm{L}^2(\Omega)\right) \quad \middle| \quad \partial_t v \in \mathrm{L}^2(Q) \right\},$$

in the sense of Definition 1.2 and which satisfies (III.5). *Moreover, there exists a constant* $T_* > 0$ *only depending on* p, d, Ω, $\|u\|_{\mathrm{L}^\infty(Q)}$ *and* $\|u_0\|_{\mathrm{L}^2(\Omega)}$ *such that*

$$\forall t > T_*, \qquad \|u(t)\|_{\mathrm{L}^2(\Omega)} = 0.$$

Therefore, u *vanishes in finite time.*

The proof of Theorem 1.1 is based on an approximation method. Let $\varepsilon > 0$ and consider the following regularised problem :

$$(\mathrm{P}_\varepsilon) \begin{cases} \partial_t u_\varepsilon - \Delta_p u_\varepsilon &= h_\varepsilon(x, u_\varepsilon) \quad \text{in } Q, \\ u_\varepsilon &= 0 \quad \text{on } \Gamma, \\ u_\varepsilon(0,\cdot) &= u_0 \quad \text{in } \Omega, \end{cases}$$

where $h_\varepsilon : \Omega \times [0, +\infty) \to \mathbb{R}$ is defined by $h_\varepsilon(x,w) \stackrel{\text{def}}{=} f(x,w) - g_\varepsilon(w)$, with g_ε an approximation of the singular term, defined as follows :

$$g_\varepsilon(w) = \begin{cases} 0 & \text{if } w = 0, \\ \varepsilon^{-\beta} & \text{if } w \in (0, \varepsilon), \\ w^{-\beta} & \text{if } w \geq \varepsilon. \end{cases} \tag{III.7}$$

Concerning existence and uniqueness of the weak solution to (P_ε), we get the following result :

Theorem 1.3 *Assume that* u_0 *satisfies condition* (III.1). *Then, there exists* $T^* > 0$ *such that for any* $T < T^*$, *problem* (P_ε) *has a unique weak solution* $u_\varepsilon \in \mathscr{U}$. *Moreover,* u_ε *satisfies the following energy* *identity : for any* $t \in [0,T]$,

$$\frac{1}{2}\|u_\varepsilon(t)\|^2_{\mathrm{L}^2(\Omega)} - \frac{1}{2}\|u_0\|^2_{\mathrm{L}^2(\Omega)} + \int_0^t \int_\Omega |\nabla u_\varepsilon|^p \, dz + \int_0^t \int_\Omega g_\varepsilon(u_\varepsilon) u_\varepsilon \, dz = \int_0^t \int_\Omega f(x, u_\varepsilon) u_\varepsilon \, dz, \qquad \text{(III.8)}$$

Remark 1.2 *Since* $g_\varepsilon(u_\varepsilon) \in \mathrm{L}^\infty(Q) \hookrightarrow \mathrm{L}^{p'}\left(0,T;\mathrm{W}^{-1,p'}(\Omega)\right)$, *the weak solution* u_ε *of* (P_ε) *is also a weak solution in the sense of Definition 2.2 we will introduce in the next section.*

Since the nonlinearity in the equation of Problem (P_ε) is discontinuous, we need to introduce a new auxilary problem studied in the next subsection.

2 Study of a regularised problem

Let $\varepsilon > \eta > 0$. We consider the following regularised problem :

$$(\mathrm{P}_{\varepsilon,\eta}) \begin{cases} \partial_t u_{\varepsilon,\eta} - \Delta_p u_{\varepsilon,\eta} = h_{\varepsilon,\eta}(x, u_{\varepsilon,\eta}) & \text{in } Q, \\ u_{\varepsilon,\eta} = 0 & \text{on } \Gamma, \\ u_{\varepsilon,\eta}(0,\cdot) = u_0 & \text{in } \Omega, \end{cases}$$

where $h_{\varepsilon,\eta} : \Omega \times [0,+\infty) \to \mathbb{R}$ is defined by $h_{\varepsilon,\eta}(x,w) \stackrel{\text{def}}{=} f(x,w) - g_{\varepsilon,\eta}(w)$, with $g_{\varepsilon,\eta}$ a Lipschitz approximation of the singular term, defined as follows :

$$g_{\varepsilon,\eta}(w) = \begin{cases} \varepsilon^{-\beta}\eta^{-1}w & \text{if } w \in [0,\eta)\,, \\ \varepsilon^{-\beta} & \text{if } w \in [\eta,\varepsilon)\,, \\ w^{-\beta} & \text{if } w \geq \varepsilon. \end{cases} \qquad \text{(III.9)}$$

In this section, we only assume that

$$u_0 \in \mathrm{L}^\infty(\Omega) \quad \text{and} \quad u_0 \geq 0 \quad \text{a.e. in } \Omega \qquad \text{(III.10)}$$

and we look for solutions of problem $(\mathrm{P}_{\varepsilon,\eta})$ in the following sense :

Definition 2.1 *Let us define the function space*

$$\mathscr{V} \stackrel{\text{def}}{=} \left\{ v \in \mathrm{L}^p\left(0,T;\mathrm{W}^{1,p}_0(\Omega)\right) \cap \mathrm{L}^\infty(Q) \;\middle|\; \partial_t v \in \mathrm{L}^{p'}\left(0,T;\mathrm{W}^{-1,p'}(\Omega)\right) \right\}.$$

Definition 2.2 *A function* $u_{\varepsilon,\eta} \in \mathscr{V}$ *is called* ***weak solution of problem*** $(\mathbf{P}_{\varepsilon,\eta})$ *if :*

1. $u_{\varepsilon,\eta} \geq 0$ a.e. *in* Q.
2. *For every test function* $\varphi \in \mathrm{L}^p\left(0,T;\mathrm{W}^{1,p}_0(\Omega)\right)$,

$$\int_0^T \langle \partial_t u_{\varepsilon,\eta}(s), \varphi(s)\rangle \, ds + \int_Q |\nabla u_{\varepsilon,\eta}|^{p-2}\nabla u_{\varepsilon,\eta}.\nabla\varphi \, dz = \int_Q h_{\varepsilon,\eta}(x, u_{\varepsilon,\eta})\varphi \, dz, \qquad \text{(III.11)}$$

where $\langle \cdot\,,\cdot\rangle$ *denotes the dual product between* $\mathrm{W}^{-1,p'}(\Omega)$ *and* $\mathrm{W}^{1,p}_0(\Omega)$.

3. $u_{\varepsilon,\eta}(0,.) = u_0$ *a.e. in* Ω.

Remark 2.1

1. *Point 3. of the above definition is meaningful since* $\mathscr{V} \hookrightarrow \mathscr{C}\left([0,T], \mathrm{L}^2(\Omega)\right)$ *(see* Barbu *[9, Lemma 4.1, Theorem 4.2 p. 167-168]). Moreover, for every* $v, w \in \mathscr{V}$ *and every* $t_1,\, t_2 \in [0,T]$*, we have the following identification :*

$$\int_\Omega v(t_2)w(t_2)\,dx - \int_\Omega v(t_1)w(t_1)\,dx = \int_{t_1}^{t_2} \langle \partial_t v(s), w(s)\rangle\,ds + \int_{t_1}^{t_2} \langle \partial_t w(s), v(s)\rangle\,ds.$$

In particular for $v = w$,

$$\frac{1}{2}\|v(t_2)\|^2_{\mathrm{L}^2(\Omega)} - \frac{1}{2}\|v(t_1)\|^2_{\mathrm{L}^2(\Omega)} = \int_{t_1}^{t_2} \langle \partial_t v(s), v(s)\rangle\,ds. \qquad \text{(III.12)}$$

2. *Assume that* $u_{\varepsilon,\eta} \in \mathscr{V}$ *satisfies* (III.11)*. Then, it also satisfies for any* $t \in [0,T)$ *and any test function* $\varphi \in \mathrm{L}^p\left(0,T;\mathrm{W}^{1,p}_0(\Omega)\right)$,

$$\int_0^t \langle \partial_t u_{\varepsilon,\eta}(s), \varphi(s)\rangle\,ds + \int_0^t\int_\Omega |\nabla u_{\varepsilon,\eta}|^{p-2}\nabla u_{\varepsilon,\eta}.\nabla\varphi\,dz = \int_0^t\int_\Omega h_{\varepsilon,\eta}(x,u_{\varepsilon,\eta})\varphi\,dz. \qquad \text{(III.13)}$$

Indeed, let us define $\varphi_k : s \mapsto h_k(s)\varphi(s) \in \mathrm{W}^{1,p}_0(\Omega)$*, with* $k > 0$ *small enough and* h_k *defined on* $[0,T]$ *as follows :*

$$h_k(s) = \begin{cases} 1 & \text{if } s \in [0,t), \\ 1 - \frac{1}{k}(s-t) & \text{if } s \in [t, t+k), \\ 0 & \text{if } s \in [t+k, T]. \end{cases}$$

Noticing that $\varphi_k \underset{k\to 0}{\longrightarrow} \mathbb{1}_{\{s\in[0,t]\}}\varphi$ *in* $\mathrm{L}^p\left(0,T;\mathrm{W}^{1,p}_0(\Omega)\right)$*, we get* (III.13) *taking* φ_k *as test function in* (III.11) *and passing to the limit as* $k \to 0$.

Then, we have the following existence result :

Theorem 2.1 *Assume that* u_0 *satisfies* (III.10)*. Then, there exists* $T^* > 0$ *such that for any* $T < T^*$ *and any* $\varepsilon > \eta > 0$*, problem* $(\mathrm{P}_{\varepsilon,\eta})$ *has a unique weak solution* $u_{\varepsilon,\eta} \in \mathscr{V}$*. Moreover,* $u_{\varepsilon,\eta} \in \mathscr{C}\left((0,T], \mathrm{W}^{1,p}_0(\Omega)\right)$, $\sqrt{t}\partial_t u_{\varepsilon,\eta} \in \mathrm{L}^2(Q)$ *and satisfy the following energy identity :* $\forall t \in (0,T]$,

$$\int_0^t\int_\Omega s\left(\partial_t u_{\varepsilon,\eta}\right)^2\,dz + \frac{t}{p}\int_\Omega |\nabla u_{\varepsilon,\eta}(t)|^p\,dx - \frac{1}{p}\int_0^t\int_\Omega |\nabla u_{\varepsilon,\eta}|^p\,dz$$
$$= t\int_\Omega H_{\varepsilon,\eta}(x,u_{\varepsilon,\eta}(t))\,dx - \int_0^t\int_\Omega H_{\varepsilon,\eta}(x,u_{\varepsilon,\eta})\,dz, \qquad \text{(III.14)}$$

with for almost every $x \in \Omega$ *and every* $w \in [0,+\infty)$,

$$H_{\varepsilon,\eta}(x,w) \overset{\mathrm{def}}{=} \int_0^w h_{\varepsilon,\eta}(x,v)\,dv.$$

If u_0 satisfies condition (III.1), we have the following result :

Theorem 2.2 *Assume that u_0 satisfies* (III.1). *Then, there exists $T^* > 0$ such that for any $T < T^*$ and any $\varepsilon > \eta > 0$, problem* $(\mathrm{P}_{\varepsilon,\eta})$ *has a unique weak solution $u_{\varepsilon,\eta} \in \mathscr{V}$. Moreover, $u_{\varepsilon,\eta} \in \mathscr{C}\left([0,T], \mathrm{W}_0^{1,p}(\Omega)\right)$, $\partial_t u_{\varepsilon,\eta} \in \mathrm{L}^2(Q)$ and satisfy the following energy identity : $\forall t \in [0,T]$,*

$$\int_0^t \int_\Omega (\partial_t u_{\varepsilon,\eta})^2 \, dz + \frac{1}{p}\int_\Omega |\nabla u_{\varepsilon,\eta}(t)|^p \, dx - \frac{1}{p}\int_\Omega |\nabla u_0|^p \, dx = \int_\Omega H_{\varepsilon,\eta}\left(x, u_{\varepsilon,\eta}(t)\right) dx - \int_\Omega H_{\varepsilon,\eta}(x,u_0)\, dx. \quad \text{(III.15)}$$

Remark 2.2 *In Theorem 2.2, we prove that $u_{\varepsilon,\eta} \in \mathrm{L}^\infty\left(0,T; \mathrm{W}_0^{1,p}(\Omega)\right)$ and $\partial_t u_{\varepsilon,\eta} \in \mathrm{L}^2(Q)$. Then, $u_{\varepsilon,\eta} \in \mathscr{U}$ and is also a weak solution of* $(\mathrm{P}_{\varepsilon,\eta})$ *in the sense of Definition 1.2.*

To prove those two theorems, we apply a semi-discretization in time method as it is explained in the following subsection.

2.1 Semi-discretization in time

Let $N >> 1$ be a large enough integer. Let us define $\Delta_t \overset{\text{def}}{=} \frac{T}{N}$ and consider the uniform subdivision of the interval $[0,T]$:

$$0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T, \quad \text{(III.16)}$$

where for $n \in \{0, \cdots, N\}$, $t_n \overset{\text{def}}{=} n\Delta_t$. Then for $n \in \{1, \cdots, N\}$, we consider the following iterative scheme :

$$\left(\mathrm{P}^n_{\varepsilon,\eta}\right) \begin{cases} \dfrac{u^n_{\varepsilon,\eta} - u^{n-1}_{\varepsilon,\eta}}{\Delta_t} - \Delta_p u^n_{\varepsilon,\eta} + K_{\varepsilon,\eta} u^n_{\varepsilon,\eta} = h_{\varepsilon,\eta}\left(x, u^{n-1}_{\varepsilon,\eta}\right) + K_{\varepsilon,\eta} u^{n-1}_{\varepsilon,\eta} & \text{in } \Omega, \\ u^n_{\varepsilon,\eta} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $u^0_{\varepsilon,\eta} = u_0$ a.e. in Ω and $K_{\varepsilon,\eta} > 0$ is a suitable constant we fix later. By induction on n, we prove the following result :

Proposition 2.1 *Assume that $N \geq T\varepsilon^{-\beta}\eta^{-1}$. Then, for every $n \in \{1, \cdots, N\}$, problem $\left(\mathrm{P}^n_{\varepsilon,\eta}\right)$ has a unique non-negative weak solution $u^n_{\varepsilon,\eta} \in \mathscr{C}^{1,\alpha}(\overline{\Omega})$, for some $0 < \alpha < 1$.*

Proof. For $n = 1$, we define for all $v \in \mathrm{W}_0^{1,p}(\Omega) \cap \mathrm{L}^2(\Omega)$,

$$\mathrm{E}_1(v) = \frac{1}{2\Delta_t}\int_\Omega v^2 \, dx - \frac{1}{\Delta_t}\int_\Omega u^0_{\varepsilon,\eta} v \, dx + \frac{1}{p}\int_\Omega |\nabla v|^p \, dx + \frac{K_{\varepsilon,\eta}}{2}\int_\Omega v^2 \, dx - \int_\Omega h_{\varepsilon,\eta}\left(x, u^0_{\varepsilon,\eta}\right) v \, dx - K_{\varepsilon,\eta}\int_\Omega u^0_{\varepsilon,\eta} v \, dx.$$

E_1 is well defined on the whole space $\mathrm{W}_0^{1,p}(\Omega) \cap \mathrm{L}^2(\Omega)$, convex, coercive (with respect to the natural norm $\|\cdot\|_{\mathrm{W}_0^{1,p}(\Omega)} + \|\cdot\|_{\mathrm{L}^2(\Omega)}$ of $\mathrm{W}_0^{1,p}(\Omega) \cap \mathrm{L}^2(\Omega)$) and lower semi-continuous. Then, there exists a unique global minimizer $u^1_{\varepsilon,\eta} \in \mathrm{W}_0^{1,p}(\Omega) \cap \mathrm{L}^2(\Omega)$ of E_1 which is a weak solution of $\left(\mathrm{P}^1_{\varepsilon,\eta}\right)$. Furthermore, since

$u^0_{\varepsilon,\eta} \in \mathrm{L}^\infty(\Omega)$,

$$h_{\varepsilon,\eta}\left(x, u^0_{\varepsilon,\eta}\right) + K_{\varepsilon,\eta} u^0_{\varepsilon,\eta} + \frac{u^0_{\varepsilon,\eta}}{\Delta_t} \in \mathrm{L}^\infty(\Omega).$$

So, by a well-known regularity result from LIEBERMAN [64], $u^1_{\varepsilon,\eta}$ belongs to the Hölder's space $\mathscr{C}^{1,\alpha}(\overline{\Omega})$, for some $0 < \alpha < 1$. Finally, for $N \geq T\varepsilon^{-\beta}\eta^{-1}$ the function $w \mapsto \frac{w}{\Delta_t} - g_{\varepsilon,\eta}(w)$ is non negative on $[0, +\infty)$. Therefore, it follows from the weak comparison principle that $u^1_{\varepsilon,\eta} \geq 0$ in Ω.

Now arguing by induction, let $n \in \{2, \cdots, N\}$ and let us suppose that there exists $u^{n-1}_{\varepsilon,\eta} \in \mathscr{C}^{1,\alpha}(\overline{\Omega})$ a non-negative weak solution to $\left(\mathrm{P}^{n-1}_{\varepsilon,\eta}\right)$. We consider the functional E_n defined for all $v \in \mathrm{W}^{1,p}_0(\Omega) \cap \mathrm{L}^2(\Omega)$ by

$$\mathrm{E}_n(v) = \frac{1}{2\Delta_t}\int_\Omega v^2\,dx - \frac{1}{\Delta_t}\int_\Omega u^{n-1}_{\varepsilon,\eta} v\,dx + \frac{1}{p}\int_\Omega |\nabla v|^p\,dx + \frac{K_{\varepsilon,\eta}}{2}\int_\Omega v^2\,dx - \int_\Omega h_{\varepsilon,\eta}\left(x, u^{n-1}_{\varepsilon,\eta}\right) v\,dx - K_{\varepsilon,\eta}\int_\Omega u^{n-1}_{\varepsilon,\eta} v\,dx. \quad \text{(III.17)}$$

Similarly to the case $n = 1$, E_n has a unique non-negative local minimizer $u^n_{\varepsilon,\eta} \in \mathscr{C}^{1,\alpha}(\overline{\Omega})$, which a weak solution to problem $\left(\mathrm{P}^n_{\varepsilon,\eta}\right)$. □

From now, we can define two approximate solutions of problem $(\mathrm{P}_{\varepsilon,\eta})$.

Definition 2.3 *Let* $u_{\Delta_t} \in \mathrm{L}^\infty\left(0, T; \mathrm{W}^{1,p}_0(\Omega)\right) \cap \mathrm{L}^\infty(Q)$ *and* $\tilde{u}_{\Delta_t} \in \mathrm{W}^{1,\infty}\left(0, T; \mathrm{L}^\infty(\Omega)\right)$ *respectively be the piecewise constant and affine approximate solutions to* $(\mathrm{P}_{\varepsilon,\eta})$ *define as follows :* $\forall n \in \{1, \cdots, N\}$, $\forall t \in (t_{n-1}, t_n]$,

$$u_{\Delta_t}(t, \cdot) = u^n_{\varepsilon,\eta} \quad \textit{and} \quad \tilde{u}_{\Delta_t}(t, \cdot) = u^{n-1}_{\varepsilon,\eta} + \frac{t - t_{n-1}}{\Delta_t}\left(u^n_{\varepsilon,\eta} - u^{n-1}_{\varepsilon,\eta}\right) \quad \text{a.e. } \textit{in } \Omega.$$

And $u_{\Delta_t}(0, \cdot) = \tilde{u}_{\Delta_t}(0, \cdot) = u^0_{\varepsilon,\eta}$ a.e. *in* Ω.

We start by giving a $\mathrm{L}^\infty(Q)$-bound for u_{Δ_t} and $\tilde{u}_{\Delta_t}$.

2.2 $\mathrm{L}^\infty(Q)$-bound for u_{Δ_t} and $\tilde{u}_{\Delta_t}$.

In this subsection, we construct an upper barrier function to problems (P), (P_ε) and $(\mathrm{P}_{\varepsilon,\eta})$, which provides a L^∞-bound for the approximations u_{Δ_t} and $\tilde{u}_{\Delta_t}$. For that, we consider the following ordinary differential equation :

$$\text{(Q)}\begin{cases} y'(t) &= g(y(t)), \qquad t \in [0, +\infty), \\ y(0) &= \|u_0\|_{\mathrm{L}^\infty(\Omega)}, \end{cases}$$

with $g : [0, +\infty) \to [0, +\infty)$ given in (III.3). By the Cauchy-Lipschitz theorem, there exists $T^* > 0$ such that (Q) has a unique solution $y \in \mathscr{C}^1([0, T^*), \mathbb{R})$. Similarly to the previous semi-discretization, let $N >> 1$ be a large enough integer and $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$ be the regular subdivision of the interval $[0, T]$, defined in (III.16). Then, let us consider the following Euler's explicit scheme : $\forall n \in \{1, \cdots, N\}$,

$$\frac{y^n - y^{n-1}}{\Delta_t} = g\left(y^{n-1}\right)$$

and $y^0 = y(0)$. So, let us define the following approximate solution of (Q) :

Definition 2.4 *Let* $\tilde{y}_{\Delta_t} \in \mathrm{W}^{1,\infty}((0,T))$ *the piecewise affine approximate solution of* (Q) *on* $[0,T]$ *defined as follows :* $\forall n \in \{1, \cdots, N\}$, $\forall t \in (t_{n-1}, t_n]$,

$$\tilde{y}_{\Delta_t}(t) = y^{n-1} + \frac{t - t_{n-1}}{\Delta_t}(y^n - y^{n-1}),$$

with $\tilde{y}_{\Delta_t}(0) = y^0$.

Lemma 2.1 *For all* $t \in [0,T]$, *we have that* $\tilde{y}_{\Delta_t}(t) \leq y(t)$.

Proof. First, $\tilde{y}_{\Delta_t}(0) = y(0)$ and for $t \in (0, t_1]$,

$$(\tilde{y}_{\Delta_t})'(t) = \frac{y^1 - y^0}{\Delta_t} \leq g(y(0)) = y'(0) \leq y'(t),$$

since $(y^n)_{n\in\{0,\cdots,N\}}$ is an increasing sequence and y' is an increasing function in $[0, +\infty)$, by the monotonicity of g. Hence, $\tilde{y}_{\Delta_t} \leq y$ in $[0, t_1]$. Now, let $n \in \{2, \cdots, N\}$ and assume that $\tilde{y}_{\Delta_t} \leq y$ in $[0, t_{n-1}]$. Therefore, for $t \in (t_{n-1}, t_n]$,

$$(\tilde{y}_{\Delta_t})'(t) = \frac{y^n - y^{n-1}}{\Delta_t} \leq g\left(\tilde{y}_{\Delta_t}(t_{n-1})\right) \leq g(y(t_{n-1})) = y'(t_{n-1}) \leq y'(t).$$

Hence, $\tilde{y}_{\Delta_t} \leq y$ in $[0, t_n]$, which proves this lemma arguing by induction on n. □

Proposition 2.2 *Assume that* $T < T^*$. *Then, the approximate solutions to* $(\mathrm{P}_{\varepsilon,\eta})$, $u_{\Delta_t}, \tilde{u}_{\Delta_t} \in \mathrm{L}^\infty(Q)$ *and are bounded independently of* Δ_t, η *and* ε. *Precisely,*

$$\|u_{\Delta_t}\|_{\mathrm{L}^\infty(Q)} \leq y(T) \quad \text{and} \quad \|\tilde{u}_{\Delta_t}\|_{\mathrm{L}^\infty(Q)} \leq y(T).$$

Proof. First, for almost every $x \in \Omega$,

$$u_{\Delta_t}(0, x) = u_0(x) \leq \|u_0\|_{\mathrm{L}^\infty(\Omega)} = y(0) \leq y(T).$$

Then for $n = 1$, by definition of u_ε^1 and y^1 and assumption (III.3),

$$\frac{u^1_{\varepsilon,\eta} - u^0_{\varepsilon,\eta}}{\Delta_t} - \frac{y^1 - y^0}{\Delta_t} - \Delta_p\left(u^1_{\varepsilon,\eta} - y^1\right) + K_{\varepsilon,\eta}\left(u^1_{\varepsilon,\eta} - y^1\right) \leq \left[g\left(u^0_{\varepsilon,\eta}\right) - g\left(y^0\right)\right] + K_{\varepsilon,\eta}\left(u^0_{\varepsilon,\eta} - y^0\right) \quad \text{in } \Omega.$$

So, since g is non-decreasing, it follows by the weak comparison principle that $u^1_{\varepsilon,\eta} \leq y^1$ a.e. in Ω. Now, let $n \in \{2, \cdots, N\}$ and assume that $u^{n-1}_{\varepsilon,\eta} \leq y^{n-1}$ a.e. in Ω. Then,

$$\frac{u^n_{\varepsilon,\eta} - u^{n-1}_{\varepsilon,\eta}}{\Delta_t} - \frac{y^n - y^{n-1}}{\Delta_t} - \Delta_p\left(u^n_{\varepsilon,\eta} - y^n\right) + K_{\varepsilon,\eta}\left(u^n_{\varepsilon,\eta} - y^n\right) \leq \left[g\left(u^{n-1}_{\varepsilon,\eta}\right) - g\left(y^{n-1}\right)\right] + K_{\varepsilon,\eta}\left(u^{n-1}_{\varepsilon,\eta} - y^{n-1}\right) \quad \text{in } \Omega.$$

Hence, using again the weak comparison principle, $u^n_{\varepsilon,\eta} \le y^n$ a.e. in Ω, which proves by induction on n that

$$\forall n \in \{1, \cdots, N\}, \quad u^n_{\varepsilon,\eta} \le y^n \quad \text{a.e. in } \Omega. \tag{III.18}$$

Therefore using Lemma 2.1 and (III.18), for $n \in \{1, \cdots, N\}$ and $t \in (t_{n-1}, t_n]$, it follows that

$$u_{\Delta_t}(t) \le \tilde{y}_{\Delta_t}(t_n) \le \tilde{y}_{\Delta_t}(T) \le y(T) \quad \text{a.e. in } \Omega.$$

Finally, since for $n \in \{1, \cdots, N\}$ and $t \in (t_{n-1}, t_n]$,

$$\tilde{u}_{\Delta_t}(t) \le \max\{u_{\Delta_t}(t_{n-1}), u_{\Delta_t}(t_n)\} \quad \text{a.e. in } \Omega,$$

we also have $\tilde{u}_{\Delta_t}(t) \le y(T)$ a.e. in Ω. □

We now establish ***energy estimates*** which provide suitable bounds for u_{Δ_t} and $\tilde{u}_{\Delta_t}$ independent of Δ_t.

2.3 *A priori* estimates.

It is easy to see that $u^n_{\varepsilon,\eta}$ is a solution to the following Euler-Lagrange equation associated to E_n : $\forall v \in \mathrm{W}^{1,p}_0(\Omega) \cap \mathrm{L}^2(\Omega)$,

$$\int_\Omega \frac{u^n_{\varepsilon,\eta} - u^{n-1}_{\varepsilon,\eta}}{\Delta_t} v\, dx + \int_\Omega |\nabla u^n_{\varepsilon,\eta}|^{p-2} \nabla u^n_{\varepsilon,\eta}.\nabla v\, dx + K_{\varepsilon,\eta} \int_\Omega u^n_{\varepsilon,\eta} v\, dx$$
$$= \int_\Omega h_{\varepsilon,\eta}\left(x, u^{n-1}_{\varepsilon,\eta}\right) v\, dx + K_{\varepsilon,\eta} \int_\Omega u^{n-1}_{\varepsilon,\eta} v\, dx. \tag{III.19}$$

We fix $K_{\varepsilon,\eta}$ in such a way that $w \longmapsto h_{\varepsilon,\eta}(x, w) + K_{\varepsilon,\eta} w$ is increasing on $[0, y(T)]$, with y defined in Subsection 2.2 and $T < T^*$. Precisely,

$$K_{\varepsilon,\eta} \stackrel{\text{def}}{=} \mathrm{Lip}(f) + \varepsilon^{-\beta} \eta^{-1}, \tag{III.20}$$

with $\mathrm{Lip}(f)$, the Lipschitz constant of f in $[0, y(T)]$.

Energy estimate 1 : Below, we establish the following proposition :

Proposition 2.3 *The approximate solutions to* $(\mathrm{P}_{\varepsilon,\eta})$, u_{Δ_t} *and* $\tilde{u}_{\Delta_t}$, *are bounded independently of* Δ_t *in* $\mathrm{L}^p\left(0, T; \mathrm{W}^{1,p}_0(\Omega)\right)$.

Proof. The first energy estimate is obtained by taking $u^n_{\varepsilon,\eta}$ as test function in each equation (III.19) (for all $n \in \{1, \cdots, N\}$), summing all those equations and multiplying by Δ_t. Hence we get,

$$\sum_{n=1}^{N}\int_\Omega\left(u_{\varepsilon,\eta}^n-u_{\varepsilon,\eta}^{n-1}\right)u_{\varepsilon,\eta}^n\,dx+\Delta_t\sum_{n=1}^{N}\int_\Omega|\nabla u_{\varepsilon,\eta}^n|^p\,dx$$
$$+K_{\varepsilon,\eta}\Delta_t\sum_{n=1}^{N}\int_\Omega u_{\varepsilon,\eta}^n\left(u_{\varepsilon,\eta}^n-u_{\varepsilon,\eta}^{n-1}\right)\,dx=\Delta_t\sum_{n=1}^{N}\int_\Omega h_{\varepsilon,\eta}\left(x,u_{\varepsilon,\eta}^{n-1}\right)u_{\varepsilon,\eta}^n\,dx. \quad \text{(III.21)}$$

On one hand,

$$\begin{aligned}\sum_{n=1}^{N}\int_\Omega\left(u_{\varepsilon,\eta}^n-u_{\varepsilon,\eta}^{n-1}\right)u_{\varepsilon,\eta}^n\,dx &= \frac{1}{2}\sum_{n=1}^{N}\int_\Omega\left[\left(u_{\varepsilon,\eta}^n-u_{\varepsilon,\eta}^{n-1}\right)^2+\left(u_{\varepsilon,\eta}^n\right)^2-\left(u_{\varepsilon,\eta}^{n-1}\right)^2\right]dx\\ &= \frac{1}{2}\sum_{n=1}^{N}\int_\Omega\left(u_{\varepsilon,\eta}^n-u_{\varepsilon,\eta}^{n-1}\right)^2dx+\frac{1}{2}\int_\Omega\left(u_{\varepsilon,\eta}^N\right)^2\,dx-\frac{1}{2}\int_\Omega {u_0}^2\,dx.\end{aligned}$$

And on the other hand, using Proposition 2.2 and the asymptotic behaviour of f given in assumption (III.3), we get that there exist $C_1>0$ and $C_2>0$ independent of Δ_t such that

$$\Delta_t\sum_{n=1}^{N}\int_\Omega h_{\varepsilon,\eta}\left(x,u_{\varepsilon,\eta}^{n-1}\right)u_{\varepsilon,\eta}^n\,dx\le C_1 \;\text{ and }\; K_{\varepsilon,\eta}\Delta_t\sum_{n=1}^{N}\int_\Omega\left|u_{\varepsilon,\eta}^n\left(u_{\varepsilon,\eta}^n-u_{\varepsilon,\eta}^{n-1}\right)\right|\,dx\le C_2.$$

So, gathering inequalities above, (III.21) yields

$$\frac{1}{2}\sum_{n=1}^{N}\int_\Omega\left(u_{\varepsilon,\eta}^n-u_{\varepsilon,\eta}^{n-1}\right)^2\,dx+\frac{1}{2}\int_\Omega\left(u_{\varepsilon,\eta}^N\right)^2\,dx+\|u_{\Delta_t}\|^p_{\mathrm{L}^p\left(0,T;\mathrm{W}_0^{1,p}(\Omega)\right)}\le C_1+C_2+\frac{1}{2}\|u_0\|^2_{\mathrm{L}^2(\Omega)}. \quad \text{(III.22)}$$

Moreover, we can easily check that

$$\|\tilde u_{\Delta_t}\|_{\mathrm{L}^p\left(0,T;\mathrm{W}_0^{1,p}(\Omega)\right)}\le 2^p(2^p+1)\|u_{\Delta_t}\|_{\mathrm{L}^p\left(0,T;\mathrm{W}_0^{1,p}(\Omega)\right)}.$$

Hence, $\tilde u_{\Delta_t}$ is also bounded in $\mathrm{L}^p\left(0,T;\mathrm{W}_0^{1,p}(\Omega)\right)$ independently of Δ_t. □

Energy estimate 2 : Below, we establish the following proposition :

Proposition 2.4 *Let $\varepsilon'\in(0,T)$. Then, the approximate solutions to $(\mathrm{P}_{\varepsilon,\eta})$, u_{Δ_t} and $\tilde u_{\Delta_t}$, are bounded independently of Δ_t in $\mathrm{L}^\infty\left(\varepsilon',T;\mathrm{W}_0^{1,p}(\Omega)\right)$. Moreover, $\partial_t\tilde u_{\Delta_t}$ is also bounded independently of Δ_t in $\mathrm{L}^2\left(\varepsilon',T;\mathrm{L}^2(\Omega)\right)$.*

Proof. Let $N'\in\{2,\cdots,N\}$. The second energy estimate is obtained by multiplying each equation (III.19), for $n\in\{2,\cdots,N'\}$, by $\frac{1}{2}\left(t_n+t_{n-1}\right)\left(u_{\varepsilon,\eta}^n-u_{\varepsilon,\eta}^{n-1}\right)$ and summing all those equations. Hence, we get

$$\begin{aligned}
\frac{\Delta_t}{2}\sum_{n=2}^{N'}(t_n+t_{n-1})\int_\Omega\left(\frac{u^n_{\varepsilon,\eta}-u^{n-1}_{\varepsilon,\eta}}{\Delta_t}\right)^2 dx&\\
+\frac{1}{2}\sum_{n=2}^{N'}(t_n+t_{n-1})\int_\Omega|\nabla u^n_{\varepsilon,\eta}|^{p-2}\nabla u^n_{\varepsilon,\eta}.\left(\nabla u^n_{\varepsilon,\eta}-\nabla u^{n-1}_{\varepsilon,\eta}\right)dx&\\
+\frac{K_{\varepsilon,\eta}}{2}\sum_{n=2}^{N'}(t_n+t_{n-1})\int_\Omega u^n_{\varepsilon,\eta}\left(u^n_{\varepsilon,\eta}-u^{n-1}_{\varepsilon,\eta}\right)dx&\\
=\frac{1}{2}\sum_{n=2}^{N'}(t_n+t_{n-1})\int_\Omega\left(h_{\varepsilon,\eta}\left(x,u^{n-1}_{\varepsilon,\eta}\right)+K_{\varepsilon,\eta}u^{n-1}_{\varepsilon,\eta}\right)\left(u^n_{\varepsilon,\eta}-u^{n-1}_{\varepsilon,\eta}\right)dx.&\quad\text{(III.23)}
\end{aligned}$$

In the left hand side of this identity, let us remark that

$$\frac{\Delta_t}{2}\sum_{n=2}^{N'}(t_n+t_{n-1})\int_\Omega\left(u^n_{\varepsilon,\eta}-u^{n-1}_{\varepsilon,\eta}\right)^2 dx=\left\|\sqrt{t}\partial_t\tilde{u}_{\Delta_t}\right\|^2_{\mathrm{L}^2(0,T_{N'};\mathrm{L}^2(\Omega))}-\int_\Omega\left(u^1_{\varepsilon,\eta}-u^0_{\varepsilon,\eta}\right)^2dx.\quad\text{(III.24)}$$

Moreover, since $v\mapsto\int_\Omega|\nabla v|^p\,dx$ and $v\mapsto\frac{1}{2}\int_\Omega v^2\,dx$ are convex in $\mathrm{W}^{1,p}_0(\Omega)\cap\mathrm{L}^2(\Omega)$, we have in (III.23),

$$\begin{aligned}
&\frac{1}{2}\sum_{n=2}^{N'}(t_n+t_{n-1})\int_\Omega|\nabla u^n_{\varepsilon,\eta}|^{p-2}\nabla u^n_{\varepsilon,\eta}.\left(\nabla u^n_{\varepsilon,\eta}-\nabla u^{n-1}_{\varepsilon,\eta}\right)dx\\
&\geq\frac{1}{2p}\sum_{n=2}^{N'}(t_n+t_{n-1})\int_\Omega\left(|\nabla u^n_{\varepsilon,\eta}|^p-\left|\nabla u^{n-1}_{\varepsilon,\eta}\right|^p\right)dx\\
&=\frac{1}{p}\sum_{n=2}^{N'}\int_\Omega\left(t_n\left|\nabla u^n_{\varepsilon,\eta}\right|^p dx-t_{n-1}\left|\nabla u^{n-1}_{\varepsilon,\eta}\right|^p dx\right)-\frac{\Delta_t}{2p}\sum_{n=2}^{N'}\int_\Omega\left(\left|\nabla u^n_{\varepsilon,\eta}\right|^p+\left|\nabla u^{n-1}_{\varepsilon,\eta}\right|^p\right)dx\\
&=\frac{1}{p}\int_\Omega\left(t_{N'}\left|\nabla u^{N'}_{\varepsilon,\eta}\right|^p dx-\Delta_t\left|\nabla u^1_{\varepsilon,\eta}\right|^p dx\right)-\frac{\Delta_t}{2p}\sum_{n=2}^{N'}\int_\Omega\left(\left|\nabla u^n_{\varepsilon,\eta}\right|^p+\left|\nabla u^{n-1}_{\varepsilon,\eta}\right|^p\right)dx\\
&\geq\frac{1}{p}\left(t_{N'}\int_\Omega\left|\nabla u^{N'}_{\varepsilon,\eta}\right|^p dx-\Delta_t\int_\Omega\left|\nabla u^1_{\varepsilon,\eta}\right|^p dx\right)-\frac{1}{p}\|u_{\Delta_t}\|^p_{\mathrm{L}^p\left(0,t_{N'};\mathrm{W}^{1,p}_0(\Omega)\right)}
\end{aligned}\qquad\text{(III.25)}$$

and

$$\begin{aligned}
\frac{K_{\varepsilon,\eta}}{2}\sum_{n=2}^{N'}(t_n+t_{n-1})\int_\Omega u^n_{\varepsilon,\eta}\left(u^n_{\varepsilon,\eta}-u^{n-1}_{\varepsilon,\eta}\right)dx&\\
\geq\frac{K_{\varepsilon,\eta}}{4}\sum_{n=2}^{N'}(t_n+t_{n-1})\int_\Omega\left[\left(u^n_{\varepsilon,\eta}\right)^2-\left(u^{n-1}_{\varepsilon,\eta}\right)^2\right]dx.&\quad\text{(III.26)}
\end{aligned}$$

In the right hand side of (III.23), due to the choice of $K_{\varepsilon,\eta}$ in (III.20), the mapping $w \mapsto H_{\varepsilon,\eta}(x,w) + \frac{K_{\varepsilon,\eta}}{2} w^2$, where $H_{\varepsilon,\eta}$ is defined in Theorem 2.1, is convex. So, similarly to (III.25) we get

$$\begin{aligned}
&\frac{1}{2}\sum_{n=2}^{N'}(t_n+t_{n-1})\int_\Omega \left(h_{\varepsilon,\eta}\left(x,u_{\varepsilon,\eta}^{n-1}\right)+K_{\varepsilon,\eta}u_{\varepsilon,\eta}^{n-1}\right)\left(u_{\varepsilon,\eta}^{n}-u_{\varepsilon,\eta}^{n-1}\right)dx\\
&\quad\le \frac{1}{2}\sum_{n=2}^{N'}(t_n+t_{n-1})\int_\Omega \left[H_{\varepsilon,\eta}\left(x,u_{\varepsilon,\eta}^{n}\right)-H_{\varepsilon,\eta}\left(x,u_{\varepsilon,\eta}^{n-1}\right)\right]dx\\
&\qquad\qquad + \frac{K_{\varepsilon,\eta}}{4}\sum_{n=2}^{N'}(t_n+t_{n-1})\int_\Omega \left[\left(u_{\varepsilon,\eta}^{n}\right)^2-\left(u_{\varepsilon,\eta}^{n-1}\right)^2\right]dx\\
&\quad= t_{N'}\int_\Omega H_{\varepsilon,\eta}\left(x,u_{\varepsilon,\eta}^{N'}\right)dx-\Delta_t\int_\Omega H_{\varepsilon,\eta}\left(x,u_{\varepsilon,\eta}^{1}\right)dx\\
&\qquad -\frac{\Delta_t}{2}\sum_{n=2}^{N'}\int_\Omega\left[H_{\varepsilon,\eta}\left(x,u_{\varepsilon,\eta}^{n}\right)-H_{\varepsilon,\eta}\left(x,u_{\varepsilon,\eta}^{n-1}\right)\right]dx\\
&\qquad\qquad + \frac{K_{\varepsilon,\eta}}{4}\sum_{n=2}^{N'}(t_n+t_{n-1})\int_\Omega \left[\left(u_{\varepsilon,\eta}^{n}\right)^2-\left(u_{\varepsilon,\eta}^{n-1}\right)^2\right]dx. \qquad \text{(III.27)}
\end{aligned}$$

So, gathering estimates (III.24) to (III.27), equality (III.23) gives

$$\begin{aligned}
&\left\|\sqrt{t}\partial_t\tilde{u}_{\Delta_t}\right\|^2_{\mathrm{L}^2(0,T_N;\mathrm{L}^2(\Omega))}+\frac{t_{N'}}{2}\int_\Omega\left|\nabla u_{\varepsilon,\eta}^{N'}\right|^p dx-\frac{1}{p}\|u_{\Delta_t}\|_{\mathrm{L}^p\left(0,t_{N'};\mathrm{W}_0^{1,p}(\Omega)\right)}\\
&\quad\le \frac{1}{2}\int_\Omega\left(u_{\varepsilon,\eta}^{1}-u_{\varepsilon,\eta}^{0}\right)^2dx+\frac{\Delta_t}{p}\int_\Omega\left|\nabla u_{\varepsilon,\eta}^{1}\right|^p dx+t_{N'}\int_\Omega H_{\varepsilon,\eta}\left(x,u_{\varepsilon,\eta}^{N'}\right)dx\\
&\quad -\Delta_t\int_\Omega H_{\varepsilon,\eta}\left(x,u_{\varepsilon,\eta}^{1}\right)dx-\frac{\Delta_t}{2}\sum_{n=2}^{N'}\int_\Omega\left[H_{\varepsilon,\eta}\left(x,u_{\varepsilon,\eta}^{n}\right)-H_{\varepsilon,\eta}\left(x,u_{\varepsilon,\eta}^{n-1}\right)\right]dx. \qquad \text{(III.28)}
\end{aligned}$$

By Propositions 2.2 and 2.3, there exist $C_3 > 0$ and $C_4 > 0$ independent of Δ_t and N' such that $\|u_{\Delta_t}\|_{\mathrm{L}^p\left(0,T_{N'};\mathrm{W}_0^{1,p}(\Omega)\right)} \le C_3$ and

$$\begin{aligned}
&\frac{1}{2}\int_\Omega\left(u_{\varepsilon,\eta}^{1}-u_{\varepsilon,\eta}^{0}\right)^2dx+\frac{\Delta_t}{p}\int_\Omega\left|\nabla u_{\varepsilon,\eta}^{1}\right|^p dx+t_{N'}\int_\Omega H_{\varepsilon,\eta}\left(x,u_{\varepsilon,\eta}^{N'}\right)dx\\
&\quad -\Delta_t\int_\Omega H_{\varepsilon,\eta}\left(x,u_{\varepsilon,\eta}^{1}\right)dx-\frac{\Delta_t}{2}\sum_{n=2}^{N'}\int_\Omega\left[H_{\varepsilon,\eta}\left(x,u_{\varepsilon,\eta}^{n}\right)-H_{\varepsilon,\eta}\left(x,u_{\varepsilon,\eta}^{n-1}\right)\right]dx \le C_4.
\end{aligned}$$

Hence, for $N = N'$, from (III.28) we arrive at

$$\left\|\sqrt{t}\partial_t\tilde{u}_{\Delta_t}\right\|^2_{\mathrm{L}^2(Q)} \le C_3 + C_4 \qquad \text{(III.29)}$$

and then $\sqrt{t}\partial_t\tilde{u}_{\Delta_t}$ is bounded in $\mathrm{L}^2(Q)$ independently of Δ_t. This implies that, for all $\varepsilon' \in (0,T)$, $\partial_t\tilde{u}_{\Delta_t}$ is bounded in $\mathrm{L}^2\left(\varepsilon', T; \mathrm{L}^2(\Omega)\right)$ independently of Δ_t. Now, going back to equality (III.28), for any $N' \in \{1, \cdots, N\}$ we obtain

$$\frac{t_{N'}}{2p}\int_\Omega\left|\nabla u_{\varepsilon,\eta}^{N'}\right|^p dx \le C_3 + C_4 + \left\|\sqrt{t}\partial_t\tilde{u}_{\Delta_t}\right\|^2_{\mathrm{L}^2(Q)}, \qquad \text{(III.30)}$$

with C_3, C_4 independent of Δ_t and N'. In the other hand,

$$\|t|\nabla u_{\Delta_t}|^p\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^1(\Omega))} = \max_{N'\in\{1,\cdots,N\}} \left(t_{N'} \int_\Omega |\nabla u_{\Delta_t}|^p \, dx \right).$$

Therefore, together with the boundedness of $\sqrt{t}\partial_t \tilde{u}_{\Delta_t}$ in $\mathrm{L}^2(Q)$, inequality (III.30) yields $t|\nabla u_{\Delta_t}|^p$ is bounded in $\mathrm{L}^\infty\left(0,T;\mathrm{L}^1(\Omega)\right)$ independently of Δ_t. This implies that, for all $\varepsilon' \in (0,T)$, u_{Δ_t} is bounded in $\mathrm{L}^\infty\left(\varepsilon',T;\mathrm{W}_0^{1,p}(\Omega)\right)$, from which we obtain that $\tilde{u}_{\Delta_t}$ is also bounded in $\mathrm{L}^\infty\left(\varepsilon',T;\mathrm{W}_0^{1,p}(\Omega)\right)$. □

Remark 2.3 *In the case where $u_0 \in \mathrm{W}_0^{1,p}(\Omega) \cap \mathrm{L}^\infty(\Omega)$, to get the second energy estimate, it suffices to multiply each equation* (III.19)*, for $n \in \{,\cdots,N'\}$, by $u_{\varepsilon,\eta}^n - u_{\varepsilon,\eta}^{n-1}$ and summing those equations from $n=1$ to N'. Then, inequality* (III.27) *becomes*

$$\|\partial_t \tilde{u}_{\Delta_t}\|^2_{\mathrm{L}^2(0,t_{N'};\mathrm{L}^2(\Omega))} + \frac{1}{p}\left\|u_{\varepsilon,\eta}^{N'}\right\|^p_{\mathrm{W}_0^{1,p}(\Omega)} - \frac{1}{p}\|u_0\|^p_{\mathrm{W}_0^{1,p}(\Omega)} \leq \int_\Omega H_{\varepsilon,\eta}\left(x, u_{\varepsilon,\eta}^{N'}\right) dx - \int_\Omega H_{\varepsilon,\eta}(x,u_0)\, dx \quad \text{(III.31)}$$

and thanks to the boundedness of u_{Δ_t} in $\mathrm{L}^\infty(Q)$, we get the following proposition :

Proposition 2.5 *Assume that u_0 satisfies condition* (III.1)*. Then, for any $T < T^*$, the approximate solutions to* ($\mathrm{P}_{\varepsilon,\eta}$)*, u_{Δ_t} and $\tilde{u}_{\Delta_t}$, are bounded independently of Δ_t in $\mathrm{L}^\infty\left(0,T;\mathrm{W}_0^{1,p}(\Omega)\right)$. Moreover, $\partial_t\tilde{u}_{\Delta_t}$ is also bounded independently of Δ_t in $\mathrm{L}^2(Q)$.*

2.4 Proof of Theorems 2.1 and 2.2 :

Proposition 2.6 *Assume that u_0 satisfies condition* (III.10)*. Then, for any $T < T^*$ and any $\varepsilon > 0$ there exists a non-negative $u_{\varepsilon,\eta} \in \mathscr{V}$ satisfying* (III.11)*.*

Proof. From Propositions 2.2, 2.3 and 2.4, it follows that for any $\varepsilon' > 0$, there exists

$$u_{\varepsilon,\eta} \in \mathrm{L}^p\left(0,T;\mathrm{W}_0^{1,p}(\Omega)\right) \cap \mathrm{L}^\infty(Q) \cap \mathrm{L}^\infty\left(\varepsilon',T;\mathrm{W}_0^{1,p}(\Omega)\right)$$

and $\tilde{u}_{\varepsilon,\eta} \in \mathrm{L}^p\left(0,T;\mathrm{W}_0^{1,p}(\Omega)\right) \cap \mathrm{L}^\infty(Q)$ with $\partial_t\tilde{u}_{\varepsilon,\eta} \in \mathrm{L}^2\left(\varepsilon',T;\mathrm{L}^2(\Omega)\right)$ such that,up to a subsequence

$$u_{\Delta_t} \xrightarrow[\Delta_t\to 0]{*} u_{\varepsilon,\eta} \quad \text{and} \quad \tilde{u}_{\Delta_t} \xrightarrow[\Delta_t\to 0]{*} \tilde{u}_{\varepsilon,\eta} \quad \text{in } \mathrm{L}^\infty(Q), \quad \text{(III.32)}$$

$$u_{\Delta_t} \xrightarrow[\Delta_t\to 0]{} u_{\varepsilon,\eta} \quad \text{and} \quad \tilde{u}_{\Delta_t} \xrightarrow[\Delta_t\to 0]{} \tilde{u}_{\varepsilon,\eta} \quad \text{in } \mathrm{L}^p\left(0,T;\mathrm{W}_0^{1,p}(\Omega)\right) \quad \text{(III.33)}$$

and

$$u_{\Delta_t} \xrightarrow[\Delta_t\to 0]{*} u_{\varepsilon,\eta} \quad \text{in } \mathrm{L}^\infty\left(\varepsilon',T;\mathrm{W}_0^{1,p}(\Omega)\right), \quad \text{(III.34)}$$

$$\partial_t\tilde{u}_{\Delta_t} \xrightarrow[\Delta_t\to 0]{} \partial_t\tilde{u}_{\varepsilon,\eta} \quad \text{in } \mathrm{L}^2\left(\varepsilon',T;\mathrm{L}^2(\Omega)\right). \quad \text{(III.35)}$$

Furthermore, since $\sqrt{t}\partial_t u_{\varepsilon,\eta}$ is bounded in $\mathrm{L}^2(\Omega)$ (see (III.29) in energy estimate 2), identifying the limits in $\mathscr{D}'(\Omega)$, we get

$$\sqrt{t}\partial_t \tilde{u}_{\Delta_t} \xrightarrow[\Delta_t \to 0]{} \sqrt{t}\partial_t \tilde{u}_{\varepsilon,\eta} \qquad \text{in } \mathrm{L}^2(Q)\,. \tag{III.36}$$

For any $\varepsilon' \in (0,T)$,

$$\tilde{u}_{\Delta_t} \in \mathscr{U}_{\varepsilon'} \stackrel{\text{def}}{=} \left\{ v \in \mathrm{L}^\infty\left(\varepsilon', T; \mathrm{W}_0^{1,p}(\Omega)\right) \cap \mathrm{L}^\infty(Q), \quad \middle| \quad \partial_t v \in \mathrm{L}^2\left(\varepsilon', T; \mathrm{L}^2(\Omega)\right)\right\}.$$

On one hand, we have the compact embedding of $\mathrm{W}_0^{1,p}(\Omega)$ in $\mathrm{L}^2(\Omega)$ if $p > \frac{2d}{d+2}$ or in $\mathrm{L}^q(\Omega)$, for any $1 \leq q < \frac{dp}{d-p}$, if $p \leq \frac{2d}{d+2}$; and on the other hand, $\mathscr{U}_{\varepsilon'} \hookrightarrow \mathrm{L}^\infty(Q)$. Then similarly to Remark 1.1, we get that $\mathscr{U}_{\varepsilon'} \hookrightarrow\!\!\!\!\rightarrow \mathscr{C}\left([\varepsilon', T], \mathrm{L}^2(\Omega)\right)$. Hence, up to a subsequence,

$$\tilde{u}_{\Delta_t} \xrightarrow[\Delta_t \to 0]{} \tilde{u}_{\varepsilon,\eta} \quad \text{in } \mathscr{C}\left([\varepsilon', T], \mathrm{L}^2(\Omega)\right). \tag{III.37}$$

Moreover, for any $\varepsilon' \in (0,T)$, for any $N >> N_0$ (large enough), there exist a unique $N' = N'(N) \in \{1, \cdots, N-1\}$ such that $\varepsilon' \in (t_{N'}, t_{N'+1}]$. Then,

$$\begin{aligned}
\|u_{\Delta_t} - \tilde{u}_{\Delta_t}\|_{\mathrm{L}^\infty(\varepsilon',T;\mathrm{L}^2(\Omega))} &\leq 2 \max_{n \in \{N', \cdots, N\}} \|u^n_{\varepsilon,\eta} - u^{n-1}_{\varepsilon,\eta}\|_{\mathrm{L}^2(\Omega)} \\
&\leq 2\Delta_t \sum_{n=N'}^{N} \left\| \frac{u^n_{\varepsilon,\eta} - u^{n-1}_{\varepsilon,\eta}}{\Delta_t} \right\|_{\mathrm{L}^2(\Omega)} \\
&\leq 2\sqrt{\Delta_t} \|\partial_t \tilde{u}_{\Delta_t}\|_{\mathrm{L}^2(t_{N'}(N_0),T;\mathrm{L}^2(\Omega))} \xrightarrow[\Delta_t \to 0]{} 0
\end{aligned}$$

since $\partial_t \tilde{u}_{\Delta_t}$ is bounded in $\mathrm{L}^2\left(t'_N(N_0), T; \mathrm{L}^2(\Omega)\right)$. Together with (III.37), it follows that,

$$\|u_{\Delta_t} - \tilde{u}_{\varepsilon,\eta}\|_{\mathrm{L}^\infty(\varepsilon',T;\mathrm{L}^2(\Omega))} \leq \|u_{\Delta_t} - \tilde{u}_{\Delta_t}\|_{\mathrm{L}^\infty(\varepsilon',T;\mathrm{L}^2(\Omega))} + \|\tilde{u}_{\Delta_t} - \tilde{u}_{\varepsilon,\eta}\|_{\mathscr{C}([\varepsilon',T];L^2(\Omega))} \xrightarrow[\Delta_t \to 0]{} 0. \tag{III.38}$$

Therefore, identifying the limits we get

$$\forall \varepsilon' \in (0,T), \qquad u_{\varepsilon,\eta} = \tilde{u}_{\varepsilon,\eta} \quad \text{in } \mathrm{L}^\infty\left(\varepsilon', T; \mathrm{L}^2(\Omega)\right),$$

which implies that $u_{\varepsilon,\eta} = \tilde{u}_{\varepsilon,\eta}$ a.e. in Q. Furthermore, (III.37) and (III.38) also imply that, up to a subsequence, $u_{\Delta_t}, \tilde{u}_{\Delta_t} \xrightarrow[\Delta_t \to 0]{} u_{\varepsilon,\eta}$ a.e. in Q, which involves that $u_{\varepsilon,\eta} \geq 0$ a.e. in Q.

So now, let us pass to the limit as $\Delta_t \to 0$ in the discrete scheme. We have in $\mathrm{L}^{p'}\left(0, T; \mathrm{W}^{-1,p'}(\Omega)\right)$,

$$\partial_t \tilde{u}_{\Delta_t} - \Delta_p u_{\Delta_t} + K_{\varepsilon,\eta} u_{\Delta_t} = h_{\varepsilon,\eta}\left(x, u^\tau_{\Delta_t}\right) + K_{\varepsilon,\eta} u^\tau_{\Delta_t}, \tag{III.39}$$

with the notation $u^{\tau}_{\Delta_t}(t) \overset{\text{def}}{=} u_{\Delta_t}(t-\Delta_t)$ a.e. in Ω, for every $t \in (0,T)$. Then for any $\varepsilon' \in (0,T)$,

$$\begin{aligned}\int_{\varepsilon'}^{T}\int_{\Omega}\partial_t\tilde{u}_{\Delta_t}\left(\tilde{u}_{\Delta_t}-u_{\varepsilon,\eta}\right)\,dz+\int_{\varepsilon'}^{T}\int_{\Omega}\left|\nabla u_{\Delta_t}\right|^{p-2}\nabla u_{\Delta_t}.\nabla(u_{\Delta_t}-u_{\varepsilon,\eta})\,dz\\ +K_{\varepsilon,\eta}\int_{\varepsilon'}^{T}\int_{\Omega}\left(u_{\Delta_t}-u^{\tau}_{\Delta_t}\right)\left(u_{\Delta_t}-u_{\varepsilon,\eta}\right)dz\\ =\int_{\varepsilon'}^{T}\int_{\Omega}h_{\varepsilon,\eta}\left(x,u^{\tau}_{\Delta_t}\right)\left(u_{\Delta_t}-u_{\varepsilon,\eta}\right)dz+\int_{\varepsilon'}^{T}\int_{\Omega}\partial_t\tilde{u}_{\Delta_t}\left(\tilde{u}_{\Delta_t}-u_{\Delta_t}\right)\,dz.\end{aligned} \tag{III.40}$$

Using (III.35), (III.37) and Proposition 2.2, we get

$$\int_{\varepsilon'}^{T}\int_{\Omega}\partial_t\tilde{u}_{\Delta_t}\left(\tilde{u}_{\Delta_t}-u_{\Delta_t}\right)\,dz\xrightarrow[\Delta_t\to 0]{}0\quad\text{and}\quad\int_{\varepsilon'}^{T}\int_{\Omega}\left(u_{\Delta_t}-u^{\tau}_{\Delta_t}\right)\left(u_{\Delta_t}-u_{\varepsilon,\eta}\right)dz\xrightarrow[\Delta_t\to 0]{}0. \tag{III.41}$$

On other hand by (III.33) and (III.37), we also have that

$$\int_{\varepsilon'}^{T}\int_{\Omega}\partial_t u_{\varepsilon,\eta}\left(\tilde{u}_{\Delta_t}-u_{\varepsilon,\eta}\right)\,dz\xrightarrow[\Delta_t\to 0]{}0\ \text{ and }\ \int_{\varepsilon'}^{T}\int_{\Omega}|\nabla u_{\varepsilon,\eta}|^{p-2}\nabla u_{\varepsilon,\eta}.\nabla(u_{\Delta_t}-u_{\varepsilon,\eta})\,dz\xrightarrow[\Delta_t\to 0]{}0. \tag{III.42}$$

Therefore gathering (III.40) to (III.42), we can write

$$\begin{aligned}\int_{\varepsilon'}^{T}\int_{\Omega}\left(\partial_t\tilde{u}_{\Delta_t}-\partial_t u_{\varepsilon,\eta}\right)\left(\tilde{u}_{\Delta_t}-u_{\varepsilon,\eta}\right)\,dz\\ +\int_{\varepsilon'}^{T}\int_{\Omega}\left(\left|\nabla u_{\Delta_t}\right|^{p-2}\nabla u_{\Delta_t}-\left|\nabla u_{\varepsilon,\eta}\right|^{p-2}\nabla u_{\varepsilon,\eta}\right).\nabla(u_{\Delta_t}-u_{\varepsilon,\eta})\,dz\\ =\int_{\varepsilon'}^{T}\int_{\Omega}h_{\varepsilon,\eta}\left(x,u^{\tau}_{\Delta_t}\right)\left(u_{\Delta_t}-u_{\varepsilon,\eta}\right)dz+\mathcal{O}_{\Delta_t},\end{aligned} \tag{III.43}$$

with $\mathcal{O}_{\Delta_t}\xrightarrow[\Delta_t\to 0]{}0$. Thanks to (III.37) we then have

$$\begin{aligned}\int_{\varepsilon'}^{T}\int_{\Omega}\left(\partial_t\tilde{u}_{\Delta_t}-\partial_t u_{\varepsilon,\eta}\right)\left(\tilde{u}_{\Delta_t}-u_{\varepsilon,\eta}\right)\,dz\\ =\frac{1}{2}\|\tilde{u}_{\Delta_t}(T)-u_{\varepsilon,\eta}(T)\|^2_{\mathrm{L}^2(\Omega)}-\frac{1}{2}\|\tilde{u}_{\Delta_t}(\varepsilon')-u_{\varepsilon,\eta}(\varepsilon')\|^2_{\mathrm{L}^2(\Omega)}\xrightarrow[\Delta_t\to 0]{}0.\end{aligned} \tag{III.44}$$

From Proposition 2.2 and the asymptotic behaviour of f given in (III.3) and (III.38), we have

$$\int_{\varepsilon'}^{T}\int_{\Omega}h_{\varepsilon,\eta}\left(x,u^{\tau}_{\Delta_t}\right)\left(u_{\Delta_t}-u_{\varepsilon,\eta}\right)dz\le C\int_{\varepsilon'}^{T}\int_{\Omega}\left(u_{\Delta_t}-u_{\varepsilon,\eta}\right)dz\xrightarrow[\Delta_t\to 0]{}0, \tag{III.45}$$

with $C=g(T)+\varepsilon^{-\beta}$. Therefore, gathering (III.44) and (III.45) it follows from (III.43) that

$$\int_{\varepsilon'}^{T}\int_{\Omega}\left(\left|\nabla u_{\Delta_t}\right|^{p-2}\nabla u_{\Delta_t}-\left|\nabla u_{\varepsilon,\eta}\right|^{p-2}\nabla u_{\varepsilon,\eta}\right).(\nabla u_{\Delta_t}-\nabla u_{\varepsilon,\eta})\,dz\xrightarrow[\Delta_t\to 0]{}0. \tag{III.46}$$

Let us recall a well-known inequality for the p-Laplacian operator :

Lemma 2.2 *Let $u,v\in \mathrm{W}^{1,p}_0(\Omega)\setminus\{0\}$. Then,*

1. *if $p \geq 2$, there exists a constant $C_1 > 0$, independent of u and v such that*

$$\int_\Omega \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right).(\nabla u - \nabla v)\,dx \geq C_1\|u-v\|^p_{\mathrm{W}_0^p(\Omega)}.$$

2. *And if $1 < p < 2$, there exists a constant $C_2 > 0$, independent of u and v such that*

$$\int_\Omega \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right).(\nabla u - \nabla v)\,dx \geq C_2\frac{\|u-v\|^2_{\mathrm{W}_0^p(\Omega)}}{\left(\|u\|_{\mathrm{W}_0^p(\Omega)} + \|v\|_{\mathrm{W}_0^p(\Omega)}\right)^{2-p}}.$$

Proof. See Simon [79]. □

Then, using this lemma in (III.46), it finally follows that $u_{\Delta_t} \xrightarrow[\Delta_t\to 0]{} u_{\varepsilon,\eta}$ in $\mathrm{L}^p\left(\varepsilon', T; \mathrm{W}_0^{1,p}(\Omega)\right)$ (for any $\varepsilon' \in (0,T)$), which provides that $-\Delta_p u_{\Delta_t} \xrightarrow[\Delta_t\to 0]{} -\Delta_p u_{\varepsilon,\eta}$ in $\mathscr{D}'(Q)$. Moreover thanks to (III.38), $K_{\varepsilon,\eta}u^\tau_{\Delta_t} \xrightarrow[\Delta_t\to 0]{} K_{\varepsilon,\eta}u_{\varepsilon,\eta}$ in $\mathscr{D}'(Q)$ and $h_{\varepsilon,\eta}\left(x, u^\tau_{\Delta_t}\right) \xrightarrow[\Delta_t\to 0]{} h_{\varepsilon,\eta}(x, u_{\varepsilon,\eta})$ in $\mathscr{D}'(Q)$. Then, together with (III.35), passing to the limit as $\Delta_t \to 0$ in (III.39), $u_{\varepsilon,\eta}$ satisfies

$$\partial_t u_{\varepsilon,\eta} - \Delta_p u_{\varepsilon,\eta} = h_{\varepsilon,\eta}(x, u_{\varepsilon,\eta}) \quad \text{in } \mathscr{D}'(Q). \tag{III.47}$$

Since $\Delta_p u_{\varepsilon,\eta} + h_{\varepsilon,\eta}(x, u_{\varepsilon,\eta}) \in \mathrm{L}^{p'}\left(0, T; \mathrm{W}^{-1,p'}(\Omega)\right)$, $u_{\varepsilon,\eta} \in \mathscr{V}$ and satisfies (III.11). □

Next, we have the following result :

Proposition 2.7 *Let $\varphi \in \mathrm{W}_0^{1,p}(\Omega)$. Let T_{Δ_t} and $T_{\varepsilon,\eta}$ be two functions defined in $[0,T]$ by : $\forall t \in [0,T]$,*

$$T_{\Delta_t}(t) = \int_\Omega \tilde{u}_{\Delta_t}(t,x)\varphi(x)\,dx \quad \text{and} \quad T_{\varepsilon,\eta}(t) = \int_\Omega u_{\varepsilon,\eta}(t,x)\varphi(x)\,dx.$$

Then up to a subsequence, $T_{\Delta_t} \xrightarrow[\Delta_t\to 0]{} T_{\varepsilon,\eta}$, uniformly in $[0,T]$.

Remark 2.4 *Thanks to this proposition, we get in particular passing to the limit for $t = 0$,*

$$\forall\varphi \in \mathrm{W}_0^{1,p}(\Omega), \qquad \int_\Omega (u_{\varepsilon,\eta}(0,x) - u_0(x))\,\varphi(x)\,dx = 0.$$

Therefore, it follows that $u_{\varepsilon,\eta}(0,\cdot) = u_0$ a.e. in Ω.

Proof. To prove this proposition, we apply the Ascoli-Arzelà theorem to $(T_{\Delta_t})_{\Delta_t>0} \subset \mathscr{C}([0,T],\mathbb{R})$. Since $\tilde{u}_{\Delta_t}$ is bounded in $\mathrm{L}^\infty(Q)$ independently of Δ_t,

$$\sup_{\Delta_t>0} \|T_{\Delta_t}\|_{\mathscr{C}([0,T],\mathbb{R})} < +\infty.$$

Let $t_1 < t_2 \in [0,T]$ and let $k \in \mathbb{N}^*$ large enough. For all $t \in [0,T]$ and a.e. $x \in \Omega$, we define

$$\psi_k(t) \stackrel{\text{def}}{=} h_k(t)\varphi \quad \text{a.e. in } \Omega, \qquad \text{with } h_k(t) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } 0 \le t < t_1, \\ k(t-t_1) & \text{if } t_1 \le t < t_1 + \frac{1}{k}, \\ 1 & \text{if } t_1 + \frac{1}{k} \le t < t_2 - \frac{1}{k}, \\ k(t_2 - t) & \text{if } t_2 - \frac{1}{k} \le t < t_2, \\ 0 & \text{if } t_2 \le t \le T. \end{cases}$$

Since $\psi_k \in \mathrm{W}^{1,1}\left(0,T;\mathrm{W}_0^{1,p}(\Omega)\right)$, integrating by parts we get,

$$\left|\int_Q \partial_t \tilde{u}_{\Delta_t}\psi_k\,dz\right| = \left|k\int_{t_2-\frac{1}{k}}^{t_2}\int_\Omega \tilde{u}_{\Delta_t}\varphi\,dz - k\int_{t_1}^{t_1+\frac{1}{k}}\int_\Omega \tilde{u}_{\Delta_t}\varphi\,dz\right| \xrightarrow[k\to+\infty]{} \left|T_{\Delta_t}(t_1) - T_{\Delta_t}(t_2)\right|. \quad \text{(III.48)}$$

On the other side, by the Hölder's inequality,

$$\left|\int_Q |\nabla u_{\Delta_t}|^{p-2}\nabla u_{\Delta_t}.\nabla\psi_k\,dz\right| \le \|u_{\Delta_t}\|^{p-1}_{\mathrm{L}^p\left(0,T;\mathrm{W}_0^{1,p}(\Omega)\right)}\left(\int_Q |h_k(t)\nabla\varphi(x)|^p\,dxdt\right)^{\frac{1}{p}}.$$

Since u_{Δ_t} is bounded in $\mathrm{L}^p\left(0,T;\mathrm{W}^{1,p}(\Omega)\right)$, it follows that

$$\left|\int_Q |\nabla u_{\Delta_t}|^{p-2}\nabla u_{\Delta_t}.\nabla\psi_k\,dz\right| \le C_1|t_1 - t_2|, \quad \text{(III.49)}$$

with $C_1 > 0$ independent of Δ_t, k, t_1 and t_2. Moreover, since $h_{\varepsilon,\eta}\left(x, u^\tau_{\Delta_t}\right)$ is bounded in $\mathrm{L}^\infty(Q)$ independently of Δ_t, there exists two constants $C_2 > 0$ independent of Δ_t, k, t_1 and t_2 such that

$$\left|\int_Q h_{\varepsilon,\eta}\left(x, u^\tau_{\Delta_t}\right)\psi_k\,dz\right| \le C_2|t_1 - t_2|. \quad \text{(III.50)}$$

Gathering (III.48) to (III.50) and passing to the limit as $k \to +\infty$ in (III.39), it follows that

$$|T_{\Delta_1}(t_1) - T_{\Delta_t}(t_2)| \le (C_1 + C_2)|t_1 - t_2|.$$

Hence, $(T_{\Delta_t})_{\Delta_t>0}$ is equicontinuous and applying the Ascoli-Arzelà theorem, the proof is now completed. □

The uniqueness of weak solutions to $(\mathrm{P}_{\varepsilon,\eta})$ follows from the Gronwall's lemma.

Proposition 2.8 *The unique weak solution $u_{\varepsilon,\eta}$ of problem* $(\mathrm{P}_{\varepsilon,\eta})$ *belongs to* $\mathscr{C}\left((0,T],\mathrm{W}_0^{1,p}(\Omega)\right)$ *and satisfies identity* (III.14) *of Theorem 2.1.*

Proof. First, since $u_{\varepsilon,\eta} \in \mathscr{C}\left([0,T],\mathrm{L}^2(\Omega)\right) \cap \mathrm{L}^\infty\left(\varepsilon',T;\mathrm{W}_0^{1,p}(\Omega)\right)$ for any $\varepsilon' > 0$, it follows that $u_{\varepsilon,\eta} : (0,T] \to \mathrm{W}_0^{1,p}(\Omega)$ is weakly continuous. Then, for any $t_0 \in (0,T]$,

$$\|u_{\varepsilon,\eta}(t_0)\|_{\mathrm{W}_0^{1,p}(\Omega)} \le \liminf_{t\to t_0}\|u_{\varepsilon,\eta}(t)\|_{\mathrm{W}_0^{1,p}(\Omega)}. \quad \text{(III.51)}$$

Let $t \in (0,T]$, then for any $N >> N_0$ (large enough), there exists a unique $N' = N'(N) \in \{2, \cdots, N\}$ such that $t \in (t_{N'-1}, t_{N'}]$. In view of passing to the limit in inequality (III.28) as $\Delta_t \to 0$, let us remark that the sequence $(t_{N'})_{N>>N_0}$ is decreasing and

$$t_{N'} \xrightarrow[\Delta_t \to 0]{} t. \tag{III.52}$$

In the left hand side of inequality (III.28), thanks to (III.36) and (III.52),

$$\int_0^t \int_\Omega s\left(\partial_t u_{\varepsilon,\eta}\right)^2 \, dz \leq \liminf_{\Delta_t \to 0} \left\| \sqrt{t}\partial_t \tilde{u}_{\Delta_t} \right\|^2_{\mathrm{L}^2(0,t_{N'};\mathrm{L}^2(\Omega))}. \tag{III.53}$$

Thanks to (III.34) and (III.52),

$$\frac{t}{p}\int_\Omega |\nabla u_{\varepsilon,\eta}(t)|^p \, dx \leq \liminf_{\Delta_t \to 0} \left(\frac{t_{N'}}{p} \int_\Omega \left| \nabla u^{N'}_{\varepsilon,\eta} \right|^p \, dx \right). \tag{III.54}$$

And thanks to (III.33) and (III.52), by Lebesgue's dominated convergence theorem,

$$\|u_{\Delta_t}\|_{\mathrm{L}^p\left(0,t_{N'};\mathrm{W}^{1,p}_0(\Omega)\right)} \xrightarrow[\Delta_t \to 0]{} \int_0^t \int_\Omega |\nabla u_{\varepsilon,\eta}|^p \, dz. \tag{III.55}$$

In the right hand side of inequality (III.28), thanks to Proposition 2.2 and (III.38)

$$\Delta_t \int_\Omega H_{\varepsilon,\eta}\left(x, u^1_{\varepsilon,\eta}\right) \, dx \xrightarrow[\Delta_t \to 0]{} 0 \quad \text{and} \quad \frac{1}{2}\int_\Omega \left(u^1_{\varepsilon,\eta} - u^0_{\varepsilon,\eta}\right)^2 \, dx \xrightarrow[\Delta_t \to 0]{} 0. \tag{III.56}$$

Thanks to Proposition 2.2, energy estimate 1 and (III.56),

$$\Delta_t \int_\Omega \left|\nabla u^1_{\varepsilon,\eta}\right|^p \, dx = \Delta_t \int_\Omega h_{\varepsilon,\eta}\left(x, u^0_{\varepsilon,\eta}\right) u^1_{\varepsilon,\eta} \, dx - (1 - K_{\varepsilon,\eta}\Delta_t) \int_\Omega \left(u^1_{\varepsilon,\eta} - u^0_{\varepsilon,\eta}\right) u^1_{\varepsilon,\eta} \, dx \xrightarrow[\Delta_t \to 0]{} 0. \tag{III.57}$$

And by Lebesgue's dominated convergence theorem,

$$t_{N'} \int_\Omega H_{\varepsilon,\eta}\left(x, u^{N'}_{\varepsilon,\eta}\right) \, dx \xrightarrow[\Delta_t \to 0]{} t \int_\Omega H_{\varepsilon,\eta}\left(x, u_{\varepsilon,\eta}(t)\right) \, dx \tag{III.58}$$

and

$$\frac{\Delta_t}{2} \sum_{n=2}^{N'} \int_\Omega \left| H_{\varepsilon,\eta}\left(x, u^n_{\varepsilon,\eta}\right) + H_{\varepsilon,\eta}\left(x, u^{n-1}_{\varepsilon,\eta}\right) \right| \, dx \xrightarrow[\Delta_t \to 0]{} \int_0^t \int_\Omega H_{\varepsilon,\eta}(x, u_{\varepsilon,\eta}) \, dz. \tag{III.59}$$

So, gathering estimates (III.53) to (III.59) and passing to the limit inferior in (III.28), we get

$$\begin{aligned} \int_0^t \int_\Omega s\left(\partial_t u_{\varepsilon,\eta}\right)^2 \, dz + \frac{t}{p}\int_\Omega |\nabla u_{\varepsilon,\eta}(t)|^p \, dx - \frac{1}{p}\int_0^t \int_\Omega |\nabla u_{\varepsilon,\eta}|^p \, dz \\ \leq t \int_\Omega H_{\varepsilon,\eta}(x, u_{\varepsilon,\eta}(t)) \, dx - \int_0^t \int_\Omega H_{\varepsilon,\eta}(x, u_{\varepsilon,\eta}) \, dz. \end{aligned} \tag{III.60}$$

Now, let $t_0 \in (0,T)$. Similarly to the above analysis for problem $(\mathrm{P}_{\varepsilon,\eta})$ and making a semi-dicretization in time on $(t_0, T) \times \Omega$ changing the initial datum in the iterative scheme $(\mathrm{P}^n_{\varepsilon,\eta})$ by $v^0_{\varepsilon,\eta} = u_{\varepsilon,\eta}(t_0)$ a.e. in Ω, and the initial datum in (Q) by $y(t_0) = \|u_{\varepsilon,\eta}(t_0)\|_{\mathrm{L}^\infty}$, we can easily adapt the proofs of Propositions 2.2 to 2.8 to show the existence and the uniqueness of $v_{\varepsilon,\eta} \in \mathscr{V}_{t_0}$, with

$$\mathscr{V}_{t_0} \overset{\text{def}}{=} \left\{ v \in \mathrm{L}^p\left(t_0, T; \mathrm{W}_0^{1p}(\Omega)\right) \cap \mathrm{L}^\infty((t_0,T)\times\Omega) \quad \middle| \quad \partial_t v \in \mathrm{L}^{p'}\left(t_0, T; \mathrm{W}^{-1,p'}(\Omega)\right)\right\},$$

which is the unique weak solution (in the sense of Definition 2.2 replacing 0 by t_0 and $\mathscr{V}$ by $\mathscr{V}_{t_0}$) to problem

$$\begin{cases} \partial_t v_{\varepsilon,\eta} - \Delta_p v_{\varepsilon,\eta} &= h_{\varepsilon,\eta}(x, v_{\varepsilon,\eta}) \quad \text{in } (t_0,T)\times\Omega, \\ v_{\varepsilon,\eta} &= 0 \quad \text{on } (t_0,T)\times\partial\Omega, \\ v_{\varepsilon,\eta}(t_0,\cdot) &= u_{\varepsilon,\eta}(t_0) \quad \text{in } \Omega, \end{cases}$$

satisfying for any $t \in (t_0, T]$,

$$\int_{t_0}^t \int_\Omega s\left(\partial_t v_{\varepsilon,\eta}\right)^2 dz + \frac{t}{p}\int_\Omega |\nabla v_{\varepsilon,\eta}(t)|^p\,dx - \frac{t_0}{p}\int_\Omega |\nabla v_{\varepsilon,\eta}(t)|^p\,dx - \frac{1}{p}\int_{t_0}^t\int_\Omega |\nabla v_{\varepsilon,\eta}|^p\,dz$$
$$\leq t\int_\Omega H_{\varepsilon,\eta}(x, v_{\varepsilon,\eta}(t))\,dx - t_0\int_\Omega H_{\varepsilon,\eta}(x, v_{\varepsilon,\eta}(t_0))\,dx - \int_{t_0}^t\int_\Omega H_{\varepsilon,\eta}(x, v_{\varepsilon,\eta})\,dz. \quad \text{(III.61)}$$

From Gronwall's lemma again and (III.13), $v_{\varepsilon,\eta} = u_{\varepsilon,\eta}$ a.e. in $(t_0,T)\times\Omega$. Then, $u_{\varepsilon,\eta}$ also satisfies inequality (III.61) for any $t \in (t_0, T]$. Finally, from (III.61) and Lebesgue's dominated convergence theorem, it follows that,

$$\limsup_{t\to t_0^+} \|u_{\varepsilon,\eta}(t)\|_{\mathrm{W}_0^{1,p}(\Omega)} \leq \|u_{\varepsilon,\eta}(t_0)\|_{\mathrm{W}_0^{1,p}(\Omega)} \quad \text{(III.62)}$$

and then, coupling (III.62) with (III.51), $u_{\varepsilon,\eta} : (0,T] \to \mathrm{W}_0^{1,p}(\Omega)$ is right continuous. Let us now prove the left continuity. For that, let $k \in (0, t-t_0]$. Since $\partial_t u_{\varepsilon,\eta} \in \mathrm{L}^2\left(t_0, T; \mathrm{L}^2(\Omega)\right)$, taking

$$\tau_k(u_{\varepsilon,\eta}) : s \longmapsto \frac{s}{k}\left(u_{\varepsilon,\eta}(s+k) - u_{\varepsilon,\eta}(s)\right) \in \mathrm{L}^p\left(0, T; \mathrm{W}_0^{1,p}(\Omega)\right) \cap \mathrm{L}^\infty(Q)$$

as test function in (III.13) and using convexity arguments, we get that

$$\int_{t_0}^t\int_\Omega \partial_t u_{\varepsilon,\eta}\tau_k\left(u_{\varepsilon,\eta}\right) dz + \frac{1}{kp}\int_{t_0}^t\int_\Omega s\left(|\nabla u_{\varepsilon,\eta}(s+k)|^p - |\nabla u_{\varepsilon,\eta}(s)|^p\right) dxds$$
$$\geq \frac{1}{k}\int_{t_0}^t\int_\Omega h_{\varepsilon,\eta}(x, u_{\varepsilon,\eta})\tau_k(u_{\varepsilon,\eta})\,dz. \quad \text{(III.63)}$$

From (III.63), it follows that

$$\frac{1}{kp}\int_{t_0}^t\int_\Omega s\left(|\nabla u_{\varepsilon,\eta}(s+k)|^p - |\nabla u_{\varepsilon,\eta}(s)|^p\right) dxds = \frac{1}{kp}\int_t^{t+k}\int_\Omega s|\nabla u_{\varepsilon,\eta}(s)|^p\,dxds$$
$$-\frac{1}{kp}\int_{t_0}^{t_0+k}\int_\Omega s|\nabla u_{\varepsilon,\eta}(s)|^p\,dxds - \frac{1}{p}\int_{t_0+k}^{t+k}\int_\Omega |\nabla u_{\varepsilon,\eta}(s)|^p\,dxds \quad \text{(III.64)}$$

Moreover, from the right-continuity of $u_{\varepsilon,\eta}$ from $(0,T]$ to $\mathrm{W}_0^{1,p}(\Omega)$ and the Lebesgue's dominated convergence theorem, we have

$$\frac{1}{kp}\int_{t_0}^{t_0+k}\int_\Omega s|\nabla u_{\varepsilon,\eta}(s)|^p\,dxds \xrightarrow[\Delta_t\to 0]{} \frac{t_0}{p}\int_\Omega |\nabla u_{\varepsilon,\eta}(t_0)|^p\,dx, \tag{III.65}$$

$$\frac{1}{kp}\int_{t}^{t+k}\int_\Omega s|\nabla u_{\varepsilon,\eta}(s)|^p\,dxds \xrightarrow[\Delta_t\to 0]{} \frac{t}{p}\int_\Omega |\nabla u_{\varepsilon,\eta}(t)|^p\,dx, \tag{III.66}$$

$$\frac{1}{p}\int_{t_0+k}^{t+k}\int_\Omega |\nabla u_{\varepsilon,\eta}(s)|^p\,dxds \xrightarrow[\Delta_t\to 0]{} \frac{1}{p}\int_{t_0}^{t}\int_\Omega |\nabla u_{\varepsilon,\eta}|^p\,dz, \tag{III.67}$$

$$\int_{t_0}^{t}\int_\Omega \partial_t u_{\varepsilon,\eta}\tau_k(u_{\varepsilon,\eta})\,dz \xrightarrow[\Delta_t\to 0]{} \int_{t_0}^{t}\int_\Omega s\left(\partial_t u_{\varepsilon,\eta}\right)^2\,dz \tag{III.68}$$

and

$$\frac{1}{k}\int_{t_0}^{t}\int_\Omega h_{\varepsilon,\eta}(x,u_{\varepsilon,\eta})\tau_k(u_{\varepsilon,\eta})\,dz \xrightarrow[\Delta_t\to 0]{} \int_{t_0}^{t}\int_\Omega s h_{\varepsilon,\eta}(x,u_{\varepsilon,\eta})\partial_t u_{\varepsilon,\eta}\,dz. \tag{III.69}$$

Let us notice that integrating by parts,

$$\int_{t_0}^{t}\int_\Omega s h_{\varepsilon,\eta}(x,u_{\varepsilon,\eta})\partial_t u_{\varepsilon,\eta}\,dz = \int_\Omega H_{\varepsilon,\eta}(x,u_{\varepsilon,\eta}(t))\,dx \\ - \int_\Omega H_{\varepsilon,\eta}(x,u_{\varepsilon,\eta}(t_0))\,dx - \int_{t_0}^{t}\int_\Omega H_{\varepsilon,\eta}(x,u_{\varepsilon,\eta})\,dz. \tag{III.70}$$

Hence, gathering (III.64) to (III.70) and passing to the limit as $\Delta_t \to 0$ in (III.63), we get

$$\int_0^t\int_\Omega s\left(\partial_t u_{\varepsilon,\eta}\right)^2\,dz + \frac{t}{p}\int_\Omega|\nabla u_{\varepsilon,\eta}(t)|^p\,dx - \frac{t_0}{p}\int_\Omega|\nabla u_{\varepsilon,\eta}(t)|^p\,dx - \frac{1}{p}\int_{t_0}^{t}\int_\Omega|\nabla u_{\varepsilon,\eta}|^p\,dz \\ \geq t\int_\Omega H_{\varepsilon,\eta}(x,u_{\varepsilon,\eta}(t))\,dx - t_0\int_\Omega H_{\varepsilon,\eta}(x,u_{\varepsilon,\eta}(t))\,dx - \int_{t_0}^{t}\int_\Omega H_{\varepsilon,\eta}(x,u_{\varepsilon,\eta})\,dz. \tag{III.71}$$

Hence, together with (III.61), inequality (III.14) is in fact an equality. Furthermore, we get that $u_{\varepsilon,\eta} \in \mathscr{C}\left((0,T],\mathrm{W}_0^{1,p}(\Omega)\right)$. Finally, to prove that estimate (III.14) holds, let us remark that for any $t \in (t_0,T]$,

$$\int_0^t\int_\Omega s\left(\partial_t u_{\varepsilon,\eta}\right)^2\,dz + \frac{t}{p}\int_\Omega|\nabla u_{\varepsilon,\eta}(t)|^p\,dx - \frac{1}{p}\int_{t_0}^{t}\int_\Omega|\nabla u_{\varepsilon,\eta}|^p\,dz \\ \geq t\int_\Omega H_{\varepsilon,\eta}(x,u_{\varepsilon,\eta}(t))\,dx - t_0\int_\Omega H_{\varepsilon,\eta}(x,u_{\varepsilon,\eta}(t))\,dx - \int_{t_0}^{t}\int_\Omega H_{\varepsilon,\eta}(x,u_{\varepsilon,\eta})\,dz. \tag{III.72}$$

Then, passing to the limit as $t_0 \to 0$, we get by the Lebesgue's dominated convergence theorem that

$$\int_0^t\int_\Omega s\left(\partial_t u_{\varepsilon,\eta}\right)^2\,dz + \frac{t}{p}\int_\Omega|\nabla u_{\varepsilon,\eta}(t)|^p\,dx - \frac{1}{p}\int_{0}^{t}\int_\Omega|\nabla u_{\varepsilon,\eta}|^p\,dz \\ \geq t\int_\Omega H_{\varepsilon,\eta}(x,u_{\varepsilon,\eta}(t))\,dx - \int_{0}^{t}\int_\Omega H_{\varepsilon,\eta}(x,u_{\varepsilon,\eta})\,dz. \tag{III.73}$$

Together with (III.60), inequality (III.73) implies (III.14). □

Proof. (OF THEOREM 2.1) Finally, Theorem 2.1 follows from Propositions 2.6 to 2.8. □

Proof. (OF THEOREM 2.2) The proof of Theorem 2.2 is similar. Indeed, from Proposition 2.5, (III.34) and (III.35) imply

$$u_{\Delta_t} \xrightarrow[\Delta_t \to 0]{*} u_{\varepsilon,\eta} \qquad \text{in } \mathrm{L}^\infty\left(0,T;\mathrm{W}_0^{1,p}(\Omega)\right)$$

and

$$\partial_t \tilde{u}_{\Delta_t} \xrightarrow[\Delta_t \to 0]{} \partial_t u_{\varepsilon,\eta} \qquad \text{in } \mathrm{L}^2\left(Q\right).$$

So, passing to the limit as $\Delta_t \to 0$ in (III.31) and following the same strategy as in the proof of Proposition 2.8, we get (III.15). Since, $u_{\varepsilon,\eta} \in \mathscr{C}\left([0,T],\mathrm{L}^2(\Omega)\right) \cap \mathrm{L}^\infty\left(0,T;\mathrm{W}_0^{1,p}(\Omega)\right)$, we get that $u_{\varepsilon,\eta} \in \mathscr{C}\left([0,T],\mathrm{W}_0^{1,p}(\Omega)\right)$. □

3 Existence of weak solutions of problems (P_ε) and (P)

We recall that in this section, u_0 satisfies condition (III.1).

3.1 Proof of Theorem 1.3 :

We first derive uniform bounds for solutions to problem ($\mathrm{P}_{\varepsilon,\eta}$).

Proposition 3.1 *$u_{\varepsilon,\eta}$ and $\partial_t u_{\varepsilon,\eta}$ are respectively bounded in $\mathrm{L}^\infty\left(0,T;\mathrm{W}_0^{1,p}(\Omega)\right) \cap \mathrm{L}^\infty(Q)$ and in $\mathrm{L}^2(Q)$, independently of η.*

Proof. Thanks to Proposition 2.2, passing to the limit as $\Delta_t \to 0$, the weak solutions of problems ($\mathrm{P}_{\varepsilon,\eta}$) are bounded in $\mathrm{L}^\infty(Q)$, independently of η. Therefore, by (III.15) we have that for any $t \in [0,T]$,

$$\int_0^t \int_\Omega \left(\partial_t u_{\varepsilon,\eta}\right)^2 \, dz + \frac{1}{p}\int_\Omega |\nabla u_{\varepsilon,\eta}(t)|^p \, dx \le \int_\Omega F(x,u_{\varepsilon,\eta}(t))\, dx + \frac{1}{p}\int_\Omega |\nabla u_0|^p \, dx \le C,$$

with $F(x,w) \stackrel{\text{def}}{=} \displaystyle\int_0^w f(x,v)\, dv$ and $C > 0$ independent of η and t. Hence, $u_{\varepsilon,\eta}$ is also bounded in $\mathrm{L}^\infty\left(0,T;\mathrm{W}_0^{1,p}(\Omega)\right)$ and $\partial_t u_{\varepsilon,\eta}$ is bounded in $\mathrm{L}^2(Q)$, independently of η. □

From Proposition 3.1, there exists $u_\varepsilon \in \mathscr{U}$ such that, up to a subsequence,

$$u_{\varepsilon,\eta} \xrightarrow[\eta \to 0]{*} u_\varepsilon \qquad \text{in } \mathrm{L}^\infty\left(0,T;\mathrm{W}_0^{1,p}(\Omega)\right) \cap \mathrm{L}^\infty(Q) \tag{III.74}$$

and

$$\partial_t u_{\varepsilon,\eta} \xrightarrow[\eta \to 0]{} \partial_t u_\varepsilon \qquad \text{in } \mathrm{L}^2\left(Q\right). \tag{III.75}$$

So, using Aubin-Simon compactness result we have that up to a subsequence,

$$u_{\varepsilon,\eta} \xrightarrow[\eta \to 0]{} u_\varepsilon \qquad \text{in } \mathscr{C}\left([0,T];\mathrm{L}^2(\Omega)\right) \tag{III.76}$$

and up to an other subsequence,

$$u_{\varepsilon,\eta} \xrightarrow[\eta\to 0]{} u_\varepsilon \qquad \text{a.e. in } Q, \tag{III.77}$$

which implies that $u_\varepsilon \geq 0$ a.e. in Q and $u_\varepsilon(0,\cdot) = u_0$ a.e. in Ω. Moreover, let us remark that since $(u_{\varepsilon,\eta})_{\eta>0}$ is bounded in $\mathrm{L}^p\left(0,T;\mathrm{W}_0^{1,p}(\Omega)\right)$, we get that the sequence $\left(|\nabla u_{\varepsilon,\eta}|^{p-2}\nabla u_{\varepsilon,\eta}\right)_{\eta>0}$ is bounded in $\mathrm{L}^{p'}(Q)^d$. Therefore, there exists $V_\varepsilon \in \mathrm{L}^{p'}(Q)^d$ such that, up to a subsequence,

$$|\nabla u_{\varepsilon,\eta}|^{p-2}\nabla u_{\varepsilon,\eta} \xrightarrow[\eta\to 0]{} V_\varepsilon \qquad \text{in } \mathrm{L}^{p'}(Q)^d. \tag{III.78}$$

In other words,

$$-\Delta_p u_{\varepsilon,\eta} \xrightarrow[\eta\to 0]{} \psi_\varepsilon \qquad \text{in } \mathscr{D}'(Q), \tag{III.79}$$

where $\psi_\varepsilon \in \mathrm{L}^{p'}\left(0,T;\mathrm{W}^{-1,p'}(\Omega)\right)$ is defined for all $v \in \mathrm{L}^p\left(0,T;\mathrm{W}_0^{1,p}(\Omega)\right)$ by

$$\langle\langle \psi_\varepsilon, v\rangle\rangle \stackrel{\text{def}}{=} \int_Q V_\varepsilon.\nabla v\, dz,$$

with $\langle\langle\cdot,\cdot\rangle\rangle$, the dual product between $\mathrm{L}^{p'}\left(0,T;\mathrm{W}^{-1,p'}(\Omega)\right)$ and $\mathrm{L}^p\left(0,T;\mathrm{W}_0^{1,p}(\Omega)\right)$. Then, let us identify the limit vector valued function V_ε.

Proposition 3.2 *We have that* $V_\varepsilon = |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon$ a.e. in Q.

Proof. By definition of $g_{\varepsilon,\eta}$ in (III.9), the sequence $\left(g_{\varepsilon,\eta}(u_{\varepsilon,\eta})\right)_{\eta>0}$ is bounded in $\mathrm{L}^\infty(Q) \hookrightarrow \mathrm{L}^2(Q)$, independently of η. Then, there exists $\phi_\varepsilon \in \mathrm{L}^\infty(Q)$ such that, up to a subsequence

$$g_{\varepsilon,\eta}(u_{\varepsilon,\eta}) \xrightarrow[\eta\to 0]{} \phi_\varepsilon \qquad \text{in } \mathrm{L}^2(Q). \tag{III.80}$$

Since f is a Carathéodory function satisfying (III.3), by (III.77) we have that

$$f(x,u_{\varepsilon,\eta}) \xrightarrow[\eta\to 0]{} f(x,u_\varepsilon) \quad \text{in } \mathscr{D}'(Q). \tag{III.81}$$

Then, gathering (III.75), (III.79), (III.113), (III.80) and (III.81) and passing to the limit as $\eta \to 0$ in (III.47), we have now :

$$\partial_t u_\varepsilon + \psi_\varepsilon + \phi_\varepsilon = f(x,u_\varepsilon) \quad \text{in } \mathscr{D}'(Q). \tag{III.82}$$

In order to identify the limit vector-valued function V_ε defined in (III.78), we apply the Minty trick (see Lions [66, p. 160-161]) thanks to the monotonicity of the p-Laplace operator. Indeed, we have for every $v \in \mathrm{L}^p\left(0,T;\mathrm{W}_0^{1,p}(\Omega)\right)$,

$$X_{\varepsilon,\eta} \stackrel{\text{def}}{=} \int_0^T\int_\Omega \left(|\nabla u_{\varepsilon,\eta}|^{p-2}\nabla u_{\varepsilon,\eta} - |\nabla v|^{p-2}\nabla v\right).\nabla(u_{\varepsilon,\eta} - v)\, dz \geq 0.$$

On one hand, taking $\varphi = u_{\varepsilon,\eta} \in \mathrm{L}^p\left(0,T;\mathrm{W}_0^{1,p}(\Omega)\right)$ as a test function in (III.11), we have

$$X_{\varepsilon,\eta} = \int_Q f(x,u_{\varepsilon,\eta})u_{\varepsilon,\eta}\,dz - \int_Q g_\varepsilon(u_{\varepsilon,\eta})u_{\varepsilon,\eta}\,dz + \frac{1}{2}\|u_0\|^2_{\mathrm{L}^2(\Omega)} - \frac{1}{2}\|u_{\varepsilon,\eta}(T)\|^2_{\mathrm{L}^2(\Omega)} - \int_Q |\nabla u_{\varepsilon,\eta}|^{p-2}\nabla u_{\varepsilon,\eta}.\nabla v\,dz - \int_Q |\nabla v|^{p-2}\nabla v.\nabla(u_{\varepsilon,\eta}-v)\,dz.$$

Let us remark that, from (III.76) and (III.80),

$$\int_Q g_{\varepsilon,\eta}(u_{\varepsilon,\eta})u_{\varepsilon,\eta}\,dz \underset{\eta\to 0}{\longrightarrow} \int_Q \phi_\varepsilon u_\varepsilon\,dz.$$

Then, using (III.76) and passing to the limit as $\eta \to 0$, we get

$$0 \leq \lim_{\eta\to 0} X_{\varepsilon,\eta} = \int_Q f(x,u_\varepsilon)u_\varepsilon\,dz - \int_Q \phi_\varepsilon u_\varepsilon\,dz + \frac{1}{2}\|u_0\|^2_{\mathrm{L}^2(\Omega)} - \frac{1}{2}\|u_\varepsilon(T)\|^2_{\mathrm{L}^2(\Omega)} - \int_Q V_\varepsilon.\nabla v\,dz - \int_Q |\nabla v|^{p-2}\nabla v.\nabla(u_\varepsilon - v)\,dz. \quad \text{(III.83)}$$

On the other hand, taking $\varphi = u_\varepsilon \in \mathrm{L}^p\left(0,T;\mathrm{W}_0^{1,p}(\Omega)\right)$ as a test function in (III.11), we have

$$\int_Q \partial_t u_{\varepsilon,\eta}u_\varepsilon\,dz + \int_Q |\nabla u_{\varepsilon,\eta}|^{p-2}\nabla u_{\varepsilon,\eta}.\nabla u_\varepsilon\,dz + \int_Q g_{\varepsilon,\eta}(u_{\varepsilon,\eta})u_\varepsilon\,dz = \int_Q f(x,u_{\varepsilon,\eta})u_\varepsilon\,dz. \quad \text{(III.84)}$$

Let us also remark that from (III.80), one has

$$\int_Q g_{\varepsilon,\eta}(u_{\varepsilon,\eta})u_\varepsilon\,dz \underset{\eta\to 0}{\longrightarrow} \int_Q \phi_\varepsilon u_\varepsilon\,dz.$$

So, together with (III.75), (III.76) and (III.81), passing to the limit as $\eta \to 0$ in (III.84) we get

$$\frac{1}{2}\|u_\varepsilon(T)\|^2_{\mathrm{L}^2(\Omega)} - \frac{1}{2}\|u_0\|^2_{\mathrm{L}^2(\Omega)} + \int_Q V_\varepsilon.\nabla u_\varepsilon\,dz + \int_Q \phi_\varepsilon u_\varepsilon\,dz = \int_Q f(x,u_\varepsilon)u_\varepsilon\,dz. \quad \text{(III.85)}$$

Gathering (III.83) and (III.85), we conclude that for every $v \in \mathrm{L}^p\left(0,T;\mathrm{W}_0^{1,p}(\Omega)\right)$,

$$\int_Q \left(V_\varepsilon - |\nabla v|^{p-2}\nabla v\right).\nabla(u_\varepsilon - v)\,dz \geq 0.$$

Choosing now $v = u_\varepsilon + \lambda\, w$ with $\lambda > 0$, $w \in \mathrm{L}^p\left(0,T;\mathrm{W}_0^{1,p}(\Omega)\right)$, we arrive at

$$\int_Q \left(V_\varepsilon - |\nabla(u_\varepsilon + \lambda w)|^{p-2}\nabla(u_\varepsilon + \lambda w)\right).\nabla w\,dz \leq 0.$$

Passing to the limit as $\lambda \to 0^+$ in this inequality, we get from Lebesgue's dominated convergence theorem that

$$\int_Q \left(V_\varepsilon - |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\right).\nabla w\,dz \leq 0.$$

Since this inequality holds for any $w \in \mathrm{L}^p\left(0,T;\mathrm{W}_0^{1,p}(\Omega)\right)$, we conclude that $V_\varepsilon = |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon$ a.e. in Q. □

Now, we complete the proof of Theorem 1.3.

Proof. (OF THEOREM 1.3) From (III.82) and Proposition 3.2, we have that $u_\varepsilon \in \mathscr{U}$ and satisfies

$$\partial_t u_\varepsilon - \Delta_p u_\varepsilon + \phi_\varepsilon = f(x,u_\varepsilon) \quad \text{in } \mathscr{D}'(Q). \tag{III.86}$$

Let us notice that $f(x,u_\varepsilon) + \Delta_p u_\varepsilon - \phi_\varepsilon \in \mathrm{L}^{p'}\left(0,T,\mathrm{W}^{-1,p'}(\Omega)\right)$. Thus, (III.86) is also satisfied in $\mathrm{L}^{p'}\left(0,T,\mathrm{W}^{-1,p'}(\Omega)\right)$. Now, to prove the existence of a weak solution of problem (P_ε), it remains to identify ϕ_ε as $g_\varepsilon(u_\varepsilon)$ a.e. in Q, where g_ε is defined in (III.7). From (III.77), we can easily prove that

$$g_{\varepsilon,\eta}(u_{\varepsilon,\eta}) \underset{\eta\to 0}{\longrightarrow} g_\varepsilon(u_\varepsilon) \quad \text{a.e. in } \{u_\varepsilon > 0\}. \tag{III.87}$$

So, let us denote ϕ_ε by

$$\phi_\varepsilon = g_\varepsilon(u_\varepsilon) + \mathbb{1}_{\{u_\varepsilon = 0\}}\chi_\varepsilon \quad \text{a.e. in } Q. \tag{III.88}$$

Since $g_{\varepsilon,\eta}$ is non-negative, (III.80) forces ϕ_ε to be non-negative a.e. in Q. And then, $\mathbb{1}_{\{u_\varepsilon=0\}}\chi_\varepsilon \geq 0$ a.e. in Q. On one hand, taking $g_{\varepsilon,\eta}(u_\varepsilon) \in \mathrm{L}^p\left(0,T,\mathrm{W}_0^{1,p}(\Omega)\right)$ as a test function in (III.86), we get

$$\int_Q \partial_t u_\varepsilon g_{\varepsilon,\eta}(u_\varepsilon)\,dz + \int_0^T |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon.\nabla\left(g_{\varepsilon,\eta}(u_\varepsilon)\right)\,dz + \int_Q \phi_\varepsilon g_{\varepsilon,\eta}(u_\varepsilon)\,dz = \int_Q f(x,u_\varepsilon)g_{\varepsilon,\eta}(u_\varepsilon)\,dz. \tag{III.89}$$

Setting $G_{\varepsilon,\eta}(w) \overset{\text{def}}{=} \int_0^w g_{\varepsilon,\eta}(v)\,dv$ and $G_\varepsilon(w) \overset{\text{def}}{=} \int_0^w g_\varepsilon(v)dv$, we obtain from above :

$$\int_\Omega G_{\varepsilon,\eta}(u_\varepsilon(T))\,dx - \int_\Omega G_{\varepsilon,\eta}(u_0)\,dx + \int_Q |\nabla u_\varepsilon|^p g'_{\varepsilon,\eta}(u_\varepsilon)\,dz + \int_Q \phi_\varepsilon g_{\varepsilon,\eta}(u_\varepsilon)\,dz = \int_Q f(x,u_\varepsilon)g_{\varepsilon,\eta}(u_\varepsilon)\,dz. \tag{III.90}$$

Let us observe that $g_{\varepsilon,\eta}(u_\varepsilon) \underset{\eta\to 0}{\longrightarrow} g_\varepsilon(u_\varepsilon)$ a.e. in Q. Then, by the Lebesgue's dominated convergence theorem we get

$$\int_\Omega G_{\varepsilon,\eta}(u_\varepsilon(T))\,dx \underset{\eta\to 0}{\longrightarrow} \int_\Omega G_\varepsilon(u_\varepsilon(T))\,dx \;\text{ and }\; \int_\Omega G_{\varepsilon,\eta}(u_0)\,dx \to \int_\Omega G_\varepsilon(u_0)\,dx, \tag{III.91}$$

$$\int_Q \phi_\varepsilon g_{\varepsilon,\eta}(u_\varepsilon)\,dz \underset{\eta\to 0}{\longrightarrow} \int_Q (g_\varepsilon(u_\varepsilon))^2\,dz \tag{III.92}$$

and

$$\int_Q f(x,u_\varepsilon)g_{\varepsilon,\eta}(u_\varepsilon)\,dz \underset{\eta\to 0}{\longrightarrow} \int_Q f(x,u_\varepsilon)g_\varepsilon(u_\varepsilon)\,dz. \tag{III.93}$$

In addition, we have

$$\int_Q |\nabla u_\varepsilon|^p g'_{\varepsilon,\eta}(u_\varepsilon)\,dz \underset{\eta\to 0}{\longrightarrow} \int_Q \mathbb{1}_{\{u_\varepsilon>0\}}|\nabla u_\varepsilon|^p g'_\varepsilon(u_\varepsilon)\,dz \tag{III.94}$$

Indeed, by definition of $g_{\varepsilon,\eta}$, we have

$$g'_{\varepsilon,\eta}(u_\varepsilon) = \begin{cases} \varepsilon^{-\beta}\eta^{-1} & \text{if } u_\varepsilon \in [0,\eta), \\ 0 & \text{if } u_\varepsilon \in [\eta,\varepsilon), \\ -\beta u_\varepsilon^{-(\beta+1)} & \text{if } u_\varepsilon \geq \varepsilon, \end{cases}$$

which means that to prove (III.94) with a fixed ε amounts to proving that

$$I_{\varepsilon,\eta} \stackrel{\text{def}}{=} \varepsilon^{-\beta}\eta^{-1} \int_{Q\cap\{0\leq u_\varepsilon\leq\eta\}} |\nabla u_\varepsilon|^p \, dz \xrightarrow[\eta\to 0]{} 0.$$

Taking $\min\{u_\varepsilon,\eta\} \in \mathrm{L}^p\left(0,T,\mathrm{W}_0^{1,p}(\Omega)\right)$ as test function in (III.86), we get

$$\frac{1}{2}\int_\Omega \left(\min\{u_\varepsilon(T),\eta\}\right)^2 \, dx + \eta\varepsilon^\beta I_{\varepsilon,\eta} + \int_Q \phi_\varepsilon \min\{u_\varepsilon,\eta\} \, dz$$
$$= \frac{1}{2}\int_\Omega \left(\min\{u_0,\eta\}\right)^2 \, dx + \int_Q f(x,u_\varepsilon)\min\{u_\varepsilon,\eta\} \, dz$$

whence, dropping the terms that do not change the sign, we obtain :

$$\eta\varepsilon^\beta I_{\varepsilon,\eta} \leq \frac{1}{2}\int_\Omega \left(\min\{u_0,\eta\}\right)^2 \, dx + \eta \int_{Q\cap\{0\leq u_\varepsilon\leq\eta\}} |f(x,u_\varepsilon)| \, dz. \tag{III.95}$$

Since f is locally Lipschitz with respect to the second variable and $(u_\varepsilon)_{\varepsilon>0}$ is bounded in $\mathrm{L}^\infty(Q)$, it follows that

$$\eta\varepsilon^\beta I_{\varepsilon,\eta} \leq \eta^2\mathcal{L}^d(\Omega)\left(\frac{1}{2} + T\mathrm{Lip(f)}\right).$$

Therefore, $I_{\varepsilon,\eta} \xrightarrow[\eta\to 0]{} 0$ and (III.94) holds. Then, gathering estimates (III.91) to (III.94) and passing to the limit as $\eta \to 0$ in (III.90), we get

$$\int_\Omega G_\varepsilon(u_\varepsilon(T)) \, dx - \int_\Omega G_\varepsilon(u_0) \, dx + \int_Q \mathbb{1}_{\{u_\varepsilon>0\}}|\nabla u_\varepsilon|^p g'_\varepsilon(u_\varepsilon) \, dz$$
$$+ \int_Q \left(g_\varepsilon(u_\varepsilon)\right)^2 \, dz = \int_Q f(x,u_\varepsilon)g_\varepsilon(u_\varepsilon) \, dz. \tag{III.96}$$

In the other hand, taking $g_{\varepsilon,\eta}(u_{\varepsilon,\eta}) \in \mathrm{L}^p\left(0,T;\mathrm{W}_0^{1,p}(\Omega)\right)$ in (III.11), we get that

$$\int_\Omega G_{\varepsilon,\eta}(u_{\varepsilon,\eta}(T)) \, dx - \int_\Omega G_{\varepsilon,\eta}(u_0) \, dx + \int_Q |\nabla u_{\varepsilon,\eta}|^p g'_{\varepsilon,\eta}(u_{\varepsilon,\eta}) \, dz$$
$$+ \int_Q (g_{\varepsilon,\eta}(u_{\varepsilon,\eta}))^2 \, dz = \int_Q f(x,u_{\varepsilon,\eta})g_{\varepsilon,\eta}(u_{\varepsilon,\eta}) \, dz. \tag{III.97}$$

Using assumption (III.2) and (III.77), it follows from the Lebesgue's dominated convergence theorem that

$$\int_\Omega G_{\varepsilon,\eta}(u_{\varepsilon,\eta}(T)) \, dx \xrightarrow[\eta\to 0]{} \int_\Omega G_\varepsilon(u_\varepsilon(T)) \, dx, \tag{III.98}$$

$$\int_\Omega G_{\varepsilon,\eta}(u_0)\,dx \xrightarrow[\eta\to 0]{} \int_\Omega G_\varepsilon(u_0)\,dx \tag{III.99}$$

and

$$\int_Q f(x,u_{\varepsilon,\eta})g_{\varepsilon,\eta}(u_{\varepsilon,\eta})\,dz \xrightarrow[\eta\to 0]{} \int_Q f(x,u_\varepsilon)g_\varepsilon(u_\varepsilon)\,dz. \tag{III.100}$$

From (III.80), we also have

$$\int_Q \phi_\varepsilon^2\,dz = \int_Q \left(\mathbb{1}_{\{u_\varepsilon>0\}}\chi_\varepsilon^2 + g_\varepsilon(u_\varepsilon)^2\right)\,dz \le \liminf_{\eta\to 0}\int_Q (g_{\varepsilon,\eta}(u_{\varepsilon,\eta}))^2\,dz. \tag{III.101}$$

Furthermore, repeating the same arguments giving (III.94), we have that

$$\begin{aligned}\liminf_{\eta\to 0}\int_Q |\nabla u_{\varepsilon,\eta}|^p g'_{\varepsilon,\eta}(u_{\varepsilon,\eta})\,dz &\ge \lim_{\eta\to 0}\int_Q \mathbb{1}_{\{u_\varepsilon>0\}}|\nabla u_{\varepsilon,\eta}|^p g'_{\varepsilon,\eta}(u_{\varepsilon,\eta})\,dz\\ &= \int_Q \mathbb{1}_{\{u_\varepsilon>0\}}|\nabla u_\varepsilon|^p g'_\varepsilon(u_\varepsilon)\,dz. \end{aligned}\tag{III.102}$$

Then, from (III.98) to (III.102) and equalities (III.96) and (III.97), it follows that

$$\limsup_{\eta\to 0}\int_Q (g_{\varepsilon,\eta}(u_{\varepsilon,\eta}))^2\,dz \le \int_Q (g_\varepsilon(u_\varepsilon))^2\,dz. \tag{III.103}$$

Finally, combining inequalities (III.101) and (III.103), we obtain $\mathbb{1}_{\{u_\varepsilon>0\}}\chi_\varepsilon = 0$ a.e. in Q and $\phi_\varepsilon = g_\varepsilon(u_\varepsilon)$ a.e. in Q.

To complete the proof of Theorem 1.3, it remains to prove the uniqueness of the weak solution. For that, let $v_\varepsilon \in \mathscr{U}$ be an another solution to (P_ε). On one hand, let us notice that for any $\varepsilon > \eta > 0$ we have

$$\forall w \in [0,+\infty), \qquad g_{\varepsilon,\eta}(w) \le g_\varepsilon(w).$$

From (III.13) and Remark 1.2, this means that for every $t \in [0,T]$ and every non negative test function $\varphi \in \mathrm{L}^p\left(0,T;\mathrm{W}_0^{1,p}(\Omega)\right)$,

$$\begin{aligned}&\int_0^t\int_\Omega \partial_t v_\varepsilon \varphi\,dz + \int_0^t\int_\Omega |\nabla v_\varepsilon|^{p-2}\nabla v_\varepsilon.\nabla\varphi\,dz\\ &\qquad\le \int_0^t\int_\Omega f(x,v_\varepsilon)\varphi\,dz - \int_0^t\int_\Omega g_{\varepsilon,\eta}(v_\varepsilon)\varphi\,dz = \int_0^t\int_\Omega h_{\varepsilon,\eta}(x,v_\varepsilon)\varphi\,dz.\end{aligned}$$

And then,

$$\begin{aligned}&\int_0^t\int_\Omega \partial_t\left(v_\varepsilon - u_{\varepsilon,\eta}\right)\varphi\,dz + \int_0^t\int_\Omega \left(|\nabla v_\varepsilon|^{p-2}\nabla v_\varepsilon - |\nabla u_{\varepsilon,\eta}|^{p-2}\nabla u_{\varepsilon,\eta}\right).\nabla\varphi\,dz\\ &\qquad\le \int_0^t\int_\Omega \left(h_{\varepsilon,\eta}(x,v_\varepsilon) - h_{\varepsilon,\eta}(x,u_{\varepsilon,\eta})\right)\varphi\,dz.\end{aligned}\tag{III.104}$$

Let us recall that $u_{\varepsilon,\eta}$ is bounded independently of η and ε in $\mathrm{L}^\infty(Q)$. Therefore, taking $(v_\varepsilon - u_{\varepsilon,\eta})^+ \in \mathrm{L}^p\left(0,T;\mathrm{W}_0^{1,p}(\Omega)\right)$ as a test function in (III.112), we get

$$\forall t \in [0,T], \qquad \Phi'(t) \le C_\varepsilon \Phi(t),$$

where $\Phi : t \mapsto \int_0^t \int_\Omega \mathbb{1}_{\{v_\varepsilon > u_{\varepsilon,\eta}\}} (v_\varepsilon - u_{\varepsilon,\eta})^2 \, dz$. Moreover, since $v_\varepsilon(0) = u_{\varepsilon,\eta}(0) = u_0$ a.e. in Ω, $\Phi(0) = 0$. Hence, it follows from Gronwall's lemma that $\Phi \equiv 0$ in $[0,T]$, which implies that for every $t \in [0,T]$, $u_{\varepsilon,\eta}(t) \ge v_\varepsilon(t)$ a.e. in Ω. And then, using (III.77) and passing to the limit as $\eta \to 0$ in this inequality we get that $u_\varepsilon \ge v_\varepsilon$ a.e. in Q.

On the other hand, we have for any non-negative $\varphi \in \mathrm{L}^p\left(0,T;\mathrm{W}_0^{1,p}(\Omega)\right)$,

$$\int_0^t \int_\Omega \partial_t u_\varepsilon \varphi \, dz + \int_0^t \int_\Omega |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon . \nabla \varphi \, dz \le \int_0^t \int_\Omega \left[f(x,u_\varepsilon) - \mathbb{1}_{\{v_\varepsilon > 0\}} g_\varepsilon(u_\varepsilon) \right] \varphi \, dz. \tag{III.105}$$

And then,

$$\begin{aligned} \int_0^t \int_\Omega \partial_t (u_\varepsilon - v_\varepsilon) \varphi \, dz &+ \int_0^t \int_\Omega \left(|\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon - |\nabla v_\varepsilon|^{p-2} \nabla v_\varepsilon \right) . \nabla \varphi \, dz \\ &\le \int_0^t \int_\Omega \left[(f(x,u_\varepsilon) - f(x,v_\varepsilon)) - \mathbb{1}_{\{v_\varepsilon > 0\}} (g_\varepsilon(u_\varepsilon) - g_\varepsilon(v_\varepsilon)) \right] \varphi \, dz. \end{aligned} \tag{III.106}$$

Then, taking $(u_\varepsilon - v_\varepsilon)^+ \in \mathrm{L}^p\left(0,T;\mathrm{W}_0^{1,p}(\Omega)\right)$ as a test function, (III.104) becomes :

$$\begin{aligned} \int_0^t \int_\Omega \partial_t (u_\varepsilon - v_\varepsilon) \varphi \, dz &+ \int_0^t \int_\Omega \left(|\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon - |\nabla v_\varepsilon|^{p-2} \nabla v_\varepsilon \right) . \nabla \varphi \, dz \\ &\le \int_0^t \int_\Omega \left[(f(x,u_\varepsilon) - f(x,v_\varepsilon)) + \varepsilon^{-(\beta+1)} (u_\varepsilon - v_\varepsilon) \right] \varphi \, dz. \end{aligned} \tag{III.107}$$

However $(x,w) \longmapsto f(x,w) + \varepsilon^{-(\beta+1)} w$ is locally Lipschitz with respect to the second variable on $\Omega \times [0,+\infty)$. Then arguing as above, we get that $u_\varepsilon \le v_\varepsilon$ a.e. in Q and finally $u_\varepsilon = v_\varepsilon$ a.e. in Q. □

Remark 3.1 *$u_\varepsilon \in \mathscr{V}$ and is also a weak solution to* (P_ε) *in the sense of Definition 2.2.*

3.2 Proof of Theorem 1.1 :

First, using the arguments as in the above subsection, we prove the the following proposition :

Proposition 3.3 *u_ε and $\partial_t u_\varepsilon$ are respectively bounded in* $\mathrm{L}^\infty\left(0,T;\mathrm{W}_0^{1,p}(\Omega)\right) \cap \mathrm{L}^\infty(Q)$ *and in* $\mathrm{L}^2(Q)$*, independently of ε.*

From this result, there exists of a non negative $u \in \mathscr{U}$ satisfying $u(0,\cdot) = u_0$ a.e in Ω and such that up to a subsequence,

$$\partial_t u_\varepsilon \underset{\varepsilon\to 0}{\rightharpoonup} \partial_t u \qquad \text{in } \mathrm{L}^2(Q), \tag{III.108}$$

$$u_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} u \qquad \text{a.e. in } Q \tag{III.109}$$

and

$$|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \underset{\varepsilon\to 0}{\rightharpoonup} V \qquad \text{in } \mathrm{L}^{p'}(Q)^d; \tag{III.110}$$

In other words,

$$-\Delta_p u_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \psi \qquad \text{in } \mathscr{D}'(Q), \tag{III.111}$$

where $\psi \in \mathrm{L}^{p'}\left(0,T;\mathrm{W}^{-1,p'}(\Omega)\right)$ is defined for all $v \in \mathrm{L}^p\left(0,T;\mathrm{W}_0^{1,p}(\Omega)\right)$ by

$$\langle\langle \psi, v\rangle\rangle \overset{\text{def}}{=} \int_Q V.\nabla v\, dz.$$

First, we establish the following result on the monotonicity of the sequence $(u_\varepsilon)_{\varepsilon>0}$.

Proposition 3.4 *Let $\varepsilon > \varepsilon' > 0$. Then, for almost every $t \in [0,T]$, $u_\varepsilon(t) \geq u_{\varepsilon'}(t)$* a.e. *in Ω.*

Proof. For arbitrary $\varepsilon > \varepsilon' > \eta > 0$, let us notice that

$$\forall w \in [0,+\infty), \qquad g_{\varepsilon,\eta}(w) \leq g_{\varepsilon',\eta}(w).$$

From (III.13), this means that for every $t \in [0,T]$ and every non negative $\varphi \in \mathrm{L}^p\left(0,T;\mathrm{W}_0^{1,p}(\Omega)\right)$,

$$\begin{aligned}\int_0^t\int_\Omega \partial_t u_{\varepsilon',\eta}\varphi\, dz + \int_0^t\int_\Omega |\nabla u_{\varepsilon',\eta}|^{p-2}\nabla u_{\varepsilon',\eta}.\nabla\varphi\, dz \\ \leq \int_0^t\int_\Omega f(x,u_{\varepsilon',\eta})\varphi\, dz - \int_0^t\int_\Omega g_{\varepsilon,\eta}(u_{\varepsilon',\eta})\varphi\, dz = \int_0^t\int_\Omega h_{\varepsilon,\eta}(x,u_{\varepsilon',\eta})\varphi\, dz.\end{aligned}$$

And then,

$$\begin{aligned}\int_0^t\int_\Omega \partial_t\left(u_{\varepsilon',\eta} - u_{\varepsilon,\eta}\right)\varphi\, dz + \int_0^t\int_\Omega \left(|\nabla u_{\varepsilon',\eta}|^{p-2}\nabla u_{\varepsilon',\eta} - |\nabla u_{\varepsilon,\eta}|^{p-2}\nabla u_{\varepsilon,\eta}\right).\nabla\varphi\, dz \\ \leq \int_0^t\int_\Omega \left(h_{\varepsilon,\eta}(x,u_{\varepsilon',\eta}) - h_{\varepsilon,\eta}(x,u_{\varepsilon,\eta})\right)\varphi\, dz.\end{aligned} \tag{III.112}$$

Let us recall that $u_{\varepsilon,\eta}$ is bounded independently of η and ε in $\mathrm{L}^\infty(Q)$. Therefore, taking $(u_{\varepsilon',\eta}-u_{\varepsilon,\eta})^+ \in \mathrm{L}^p\left(0,T;\mathrm{W}_0^{1,p}(\Omega)\right)$ as a test function in (III.112), we get

$$\forall t \in [0,T], \qquad \Phi'(t) \leq C_\varepsilon \Phi(t),$$

where $\Phi : t \mapsto \int_0^t \int_\Omega \mathbb{1}_{\{u_{\varepsilon',\eta} > u_{\varepsilon,\eta}\}} \left(u_{\varepsilon',\eta} - u_{\varepsilon,\eta}\right)^2 \, dz$. Moreover, since $u_{\varepsilon,\eta}(0) = u_{\varepsilon',\eta}(0) = u_0$ a.e. in Ω, $\Phi(0) = 0$. Hence, it follows from Gronwall's lemma that $\Phi \equiv 0$ in $[0, T]$, which implies that for every $t \in [0, T]$, $u_{\varepsilon,\eta}(t) \geq u_{\varepsilon',\eta}(t)$ a.e. in Ω. To complete the proof, we use (III.77) and we pass to the limit as $\eta \to 0$ in this inequality. □

Now, we can prove the following result :

Proposition 3.5 *Let $\varphi \in \mathscr{D}(Q)$. Then, $\mathbb{1}_{\{u>0\}} u^{-\beta} \varphi \in \mathrm{L}^1(Q)$ and*

$$\int_Q g_\varepsilon(u_\varepsilon) \varphi \, dz \xrightarrow[\varepsilon \to 0]{} \int_Q \mathbb{1}_{\{u>0\}} u^{-\beta} \varphi \, dz. \tag{III.113}$$

Proof. Let $\varphi \in \mathscr{D}(\Omega)$ and (t_0, x_0) in Q. Let us differentiate two cases :

1. If (t_0, x_0) is such that there exists $\varepsilon_0 > 0$ such that $u_{\varepsilon_0}(t_0, x_0) = 0$. Using Proposition 3.4, for $\varepsilon < \varepsilon_0$ we get that $u_\varepsilon(t_0, x_0) = 0$. Then, $u(t_0, x_0) = 0$ and

$$g_\varepsilon\left(u_\varepsilon(t_0, x_0)\right) = 0 \xrightarrow[\varepsilon \to 0]{} 0 = \mathbb{1}_{\{u>0\}} u(t_0, x_0)^{-\beta}.$$

2. If (t_0, x_0) is such that for any $\varepsilon > 0$, $u_\varepsilon(t_0, x_0) > 0$. Then remarking that g_ε is non-increasing in $(0, +\infty)$, it follows from Proposition 3.4 that the sequence $\left(g_\varepsilon(u_\varepsilon(t_0, x_0))\right)_{\varepsilon>0}$ is non-decreasing. Then, we define the following measurable function $g : Q \to [0, +\infty]$ by : $\forall (t, x) \in Q$ a.e.,

$$g(t, x) \stackrel{\text{def}}{=} \lim_{\varepsilon \to 0} g_\varepsilon(u_\varepsilon(t, x)) \in [0, +\infty].$$

Then, for any $\phi \in \mathscr{D}(\Omega)$, using Remark 3.1 and taking $|\phi| \in \mathrm{L}^p\left(0, T; \mathrm{W}_0^{1,p}(\Omega)\right)$ as test function in (III.11),

$$\int_Q g_\varepsilon(u_\varepsilon) |\phi| \, dz = \int_Q f(x, u_\varepsilon) |\phi| \, dz - \int_Q |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon . \nabla |\phi| \, dz - \int_Q \partial_t u_\varepsilon |\phi| \, dz.$$

However by Proposition 3.3, $(u_\varepsilon)_{\varepsilon>0}$ is bounded in $\mathrm{L}^p\left(0, T; \mathrm{W}_0^{1,p}(\Omega)\right) \cap \mathrm{L}^\infty(Q)$ on one hand, and $(\partial_t u_\varepsilon)_{\varepsilon>0}$ is bounded in $\mathrm{L}^2(Q)$ on the other hand. Therefore, it follows that the right hand side of the above equation is bounded independently of ε. Then, from the monotone convergence theorem, we get

$$\int_Q g |\phi| \, dz = \lim_{\varepsilon \to 0} \int_Q g_\varepsilon(u_\varepsilon) |\phi| \, dz < +\infty. \tag{III.114}$$

Therefore, $g\phi \in \mathrm{L}^1(Q)$ for any $\phi \in \mathscr{D}(\Omega)$. This implies that,

$$\mathscr{L}^d\left(\{(t, x) \in Q \mid g(t, x) = +\infty\}\right) = 0.$$

So, we can assume that $g(t_0, x_0) \in (0, +\infty)$ and in this case we necessarily have that $g(t_0, x_0) = u(t_0, x_0)^{-\beta}$.

Therefore, $g_\varepsilon(u_\varepsilon)\varphi \underset{\varepsilon\to 0}{\longrightarrow} \mathbb{1}_{\{u>0\}}u^{-\beta}\varphi$ a.e. in Q. Finally, by Proposition 3.4 and (III.114) with $\phi = \varphi$, we have a.e. in Q

$$|g_\varepsilon(u_\varepsilon)\varphi| \le \mathbb{1}_{\{u>0\}}u^{-\beta}|\varphi| \in \mathrm{L}^1(\Omega).$$

Then, applying the Lebesgue's dominated convergence theorem, (III.113) holds. □

Proof. (OF THEOREM 1.1) Since f is a Carathéodory function satisfying (III.3), by (III.109)

$$f(x,u_\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} f(x,u) \quad \text{in } \mathscr{D}'(Q). \tag{III.115}$$

Gathering (III.108), (III.111), (III.113) and (III.115) and passing to the limit as $\varepsilon \to 0$ in (III.86), we have now :

$$\partial_t u + \psi + \mathbb{1}_{\{u>0\}}u^{-\beta} = f(x,u) \quad \text{in } \mathscr{D}'(Q).$$

Finally, to get (III.4) it remains to identify the limit vector-valued function V defined in (III.110). For that, we use Remark 3.1 and similarly to the proof of Proposition 3.2, we apply the Minty's argument. From this, we also obtain (III.5), using similar arguments as in the proof of Theorem 1.3 ; which completes the proof. □

4 Quenching in a finite time

In this section, we assume that f satisifies (III.6). We first prove the existence of a global solution of problem (P) :

Proposition 4.1 *Assume that u_0 satisfies condition* (III.1) *and that f satisfies condition* (III.6). *Then, for any $T > 0$,* (P) *has a weak solution in*

$$\tilde{\mathscr{U}} \overset{\text{def}}{=} \left\{ v \in \mathrm{L}^\infty\left(0,T;\mathrm{W}_0^{1,p}(\Omega)\right) \quad \middle| \quad \partial_t v \in \mathrm{L}^2(Q) \right\},$$

in the sense of Definition 1.2.

Proof. Going back to energy estimate 1 in Subsection 2.3, we have that for any $T > 0$ and any $N' \in \{1, \cdots, N\}$,

$$\frac{1}{2}\int_\Omega \left(u_{\varepsilon,\eta}^{N'}\right)^2 dx + \Delta_t \sum_{n=1}^{N'} \int_\Omega |\nabla u_{\varepsilon,\eta}^n|^p\, dx \le \frac{1}{2}\|u_0\|^2_{\mathrm{L}^2(\Omega)} + \\ K_{\varepsilon,\eta}\Delta_t \sum_{n=1}^{N'} \int_\Omega u_{\varepsilon,\eta}^n \left(u_{\varepsilon,\eta}^{n-1} - u_{\varepsilon,\eta}^n\right) + \Delta_t \sum_{n=1}^{N'} \int_\Omega \left[\alpha\left(u_{\varepsilon,\eta}^{n-1}\right)^{p-1} + C_\alpha\right] u_{\varepsilon,\eta}^n\, dx. \tag{III.116}$$

Let $\eta > 0$. Then for $N' = N$, from Young's and Hölder's inequalities, there exist $C_{1,\eta},\ C_{2,\eta} > 0$ independent of Δ_t such that

$$K_{\varepsilon,\eta}\Delta_t \sum_{n=1}^{N'} \int_\Omega u_{\varepsilon,\eta}^n \left|u_{\varepsilon,\eta}^{n-1} - u_{\varepsilon,\eta}^n\right| \le \eta\|u_{\Delta_t}\|^p_{\mathrm{L}^p(0,T;\mathrm{L}^p(\Omega))} + C_{1,\eta}$$

and

$$\Delta_t \sum_{n=1}^{N'} \int_\Omega \left[\alpha \left(u_{\varepsilon,\eta}^{n-1}\right)^{p-1} + C_\alpha\right] u_{\varepsilon,\eta}^n \, dx \le (\alpha+\eta)\|u_{\Delta_t}\|^p_{\mathrm{L}^p(0,T;\mathrm{L}^p(\Omega))} + C_{2,\eta}.$$

Furthermore, from Poincaré's inequality we have

$$\forall v \in \mathrm{W}_0^{1,p}(\Omega), \qquad \lambda_1 \|v\|^p_{\mathrm{L}^p(\Omega)} \le \|\nabla v\|^p_{\mathrm{L}^p(\Omega)}. \tag{III.117}$$

Then in (III.116),

$$\frac{\lambda_1 - (\alpha+2\eta)}{\lambda_1}\|u_{\Delta_t}\|^p_{\mathrm{L}^p(0,T;\mathrm{L}^p(\Omega))} \le \frac{1}{2}\|u_0\|^2_{\mathrm{L}^2(\Omega)} + C_{1,\eta} + C_{2,\eta}$$

and since $\alpha < \lambda_1$, choosing $\eta > 0$ small enough, u_{Δ_t} is bounded in $\mathrm{L}^p\left(0,T;\mathrm{W}_0^{1,p}(\Omega)\right)$. Now taking into account (III.116), it follows

$$\frac{1}{2}\int_\Omega \left(u_{\varepsilon,\eta}^{N'}\right)^2 dx \le \frac{1}{2}\|u_0\|^2_{\mathrm{L}^2(\Omega)} + \frac{\alpha+2\eta}{\lambda_1}\|u_{\Delta_t}\|^p_{\mathrm{L}^p(0,T;\mathrm{L}^p(\Omega))} + C_{1,\eta} + C_{2,\eta}.$$

Hence, u_{Δ_t} is bounded in $\mathrm{L}^\infty\left(0,T;\mathrm{L}^2(\Omega)\right)$ independently of Δ_t. From these estimates, similarly to Sections 2 and 3, we can prove the existence of a weak solution $u \in \tilde{\mathscr{U}} \hookrightarrow \mathscr{C}\left([0,T],\mathrm{L}^2(\Omega)\right)$ satisfying Definition 1.2 and estimate (III.5). □

We now focus on the asymptotic behaviour of this weak solution.

4.1 The energy equality

Note that identity (III.5) and condition (III.6) imply that : for every t_1, $t_2 \in [0,T]$,

$$\frac{1}{2}\|u(t_2)\|^2_{\mathrm{L}^2(\Omega)} - \frac{1}{2}\|u(t_1)\|^2_{\mathrm{L}^2(\Omega)} + \int_{t_1}^{t_2}\int_\Omega \left(|\nabla u|^p + u^{1-\beta}\right) dz \le \int_{t_1}^{t_2}\int_\Omega \alpha u^p + C_\alpha u \, dz. \tag{III.118}$$

Let us take $t_1 = t$, $t_2 = t+h$ with $h > 0$ and write (III.118) in the form

$$\frac{1}{2h}\left(\|u(t+h)\|^2_{\mathrm{L}^2(\Omega)} - \|u(t)\|^2_{\mathrm{L}^2(\Omega)}\right) + \frac{1}{h}\int_t^{t+h}\int_\Omega \left(|\nabla u|^p + u^{1-\beta}\right) dz \le \frac{1}{h}\int_t^{t+h}\int_\Omega \left(\alpha u^p + C_\alpha u\right) dz.$$

Since $u \in \tilde{\mathscr{U}}$ and satisfies (III.5), we have the inclusions

$$\int_\Omega \left(|\nabla u|^p + u^{1-\beta}\right) dx \in \mathrm{L}^1(0,T) \qquad \text{and} \qquad \int_\Omega \left(\alpha u^p + C_\alpha u\right) dx \in \mathrm{L}^1(0,T).$$

By the Lebesgue differentiation theorem for a.e. $t \in (0,T)$ there exist

$$\lim_{h\to 0}\frac{1}{h}\int_t^{t+h}\int_\Omega \left(|\nabla u|^p + u^{1-\beta}\right) dxds = \int_\Omega \left(|\nabla u(t)|^p + u(t)^{1-\beta}\right) dx$$

and

$$\lim_{h\to 0}\frac{1}{h}\int_t^{t+h}\int_\Omega (\alpha u^p + C_\alpha u)\; dxds = \int_\Omega (\alpha u(t)^p + C_\alpha u(t))\; dx,$$

and (III.118) leads to the following relation : $\forall$ a.e. $t\in(0,T)$,

$$\frac{d}{dt}\left(\frac{1}{2}\|u(t)\|^2_{\mathrm{L}^2(\Omega)}\right) + \int_\Omega |\nabla u(t)|^p\, dx + \int_\Omega u(t)^{1-\beta}\, dx \le \alpha\int_\Omega u(t)^p\, dx + C_\alpha \int_\Omega u(t)\, dx.$$

4.2 Ordinary differential inequality for the energy function

Let us introduce the function z defined for all $t\in[0,T]$ by $z(t) \stackrel{\text{def}}{=} \|u(t)\|^2_{\mathrm{L}^2(\Omega)}$ and write the previous equality in the form : $\forall$ a.e. $t\in(0,T)$,

$$\frac{1}{2}\, z'(t) + \int_\Omega |\nabla u(t)|^p\, dx + \int_\Omega u(t)^{1-\beta}\, dx \le \alpha\int_\Omega u(t)^p\, dx + C_\alpha \int_\Omega u(t)\, dx. \tag{III.119}$$

Since $\beta\in(0,1)$, let us remark in (III.119) that

$$\int_\Omega u(t)\, dx \le \int_\Omega u(t)^p\, dx + \int_\Omega u(t)^{1-\beta}\, dx.$$

Then, we get

$$\frac{1}{2}\, z'(t) + \int_\Omega |\nabla u(t)|^p\, dx + (1-C_\alpha)\int_\Omega u(t)^{1-\beta}\, dx - (C_\alpha+\alpha)\int_\Omega u(t)^p\, dx \le 0. \tag{III.120}$$

Then, from (III.117) and (III.120) we get : $\forall$ a.e. $t\in(0,T)$,

$$\frac{1}{2}\, z'(t) + C_1\int_\Omega |\nabla u(t)|^p\, dx + C_2\int_\Omega u(t)^{1-\beta}\, dx \le 0, \tag{III.121}$$

where $C_1 = 1-\frac{\alpha+C_\alpha}{\lambda_1} > 0$ and $C_2 = 1-C_\alpha > 0$. We appeal the well-known interpolation inequality :

Lemma 4.1 *(Gagliardo-Nirenberg)*
Let $1<p<+\infty$. If $p\ge d$, let $r\in[1,+\infty)$ and if $d>p$, let $r\in\left[1,\frac{dp}{d-p}\right]$. Then, there exists a constant $C>0$ only depending on p, r, d and $\mathcal{L}^d(\Omega)$ such that for every $v\in \mathrm{W}^{1,p}_0(\Omega)$,

$$\|v\|_{\mathrm{L}^r(\Omega)} \le C\,\|\nabla v\|^\theta_{\mathrm{L}^p(\Omega)}\|v\|^{1-\theta}_{\mathrm{L}^1(\Omega)}, \tag{III.122}$$

with $\theta = \dfrac{1-\frac{1}{r}}{\frac{1}{d}-\frac{1}{p}+1}\in(0,1)$.

Proof. See Friedman [41] or Nirenberg [69]. □

Proposition 4.2 *Let $u\in\tilde{\mathscr{U}}$ be a weak solution of problem* (P) *satisfying* (III.5)*. Then, if $p\ge\frac{2d}{d+2}$,* *the function* $z : t \mapsto \|u(t)\|^2_{\mathrm{L}^2(\Omega)}$ *satisfies the differential inequality*

$$\begin{cases} z'(t) + K\, z^{\gamma}(t) \le 0 & t \in (0,T) \text{ a.e.}, \\ z(0) = z_0 \ge 0, \end{cases} \tag{III.123}$$

with the exponents

$$\gamma = \frac{1}{2\left(\frac{\theta}{p} + 1 - \theta\right)} \in (0,1), \qquad \theta = \frac{\frac{1}{2}}{\frac{1}{d} - \frac{1}{p} + 1} \in (0,1)$$

and

$$K = \left[C^{\gamma} C_1^{1-\theta} \left(C_2 \|u\|^{-\beta}_{\mathrm{L}^\infty(Q)}\right)^{\frac{\theta}{p}}\right]^{-1},$$

where the constants $C_1, C_2 > 0$ *are given in* (III.121) *and* $C = C\left(d, p, \mathcal{L}^d(\Omega)\right)$ *in* $(III.122)$.

Proof. By assumption $p \ge \frac{2d}{d+2}$; which yields $\frac{dp}{d-p} \ge 2$. In this case inequality (III.122) holds with $r = 2$. Denoting $M = \|u\|_{\mathrm{L}^\infty(Q)}$, we may estimate : for almost every $t \in (0,T)$,

$$\begin{aligned} C_1^{\frac{\theta}{p}} \left(C_2 M^{-\beta}\right)^{1-\theta} \|u(t)\|_{\mathrm{L}^2(\Omega)} &\le C_1^{\frac{\theta}{p}} \left(C_2 M^{-\beta}\right)^{1-\theta} \left[C \|\nabla u(t)\|^{\theta}_{\mathrm{L}^p(\Omega)} \|u(t)\|^{1-\theta}_{\mathrm{L}^1(\Omega)}\right] \\ &= C \left(C_1 \int_\Omega |\nabla u(t)|^p \, dx\right)^{\frac{\theta}{p}} \left(C_2 \int_\Omega M^{-\beta}\, u(t)\, dx\right)^{1-\theta} \\ &\le C \left(C_1 \int_\Omega |\nabla u(t)|^p \, dx + C_2 \int_\Omega M^{-\beta}\, u(t)\, dx\right)^{\frac{\theta}{p}+1-\theta}, \end{aligned}$$

with $\theta = \dfrac{\frac{1}{2}}{\frac{1}{n} - \frac{1}{p} + 1} \in (0,1)$, constants C_1 and C_2 from (III.121) and constant C from (III.122). Noting that

$$\int_\Omega u(t)^{1-\beta}\, dx \ge M^{-\beta} \int_\Omega u(t)\, dx,$$

we now have

$$C_1^{\frac{\theta}{p}} \left(C_2 M^{-\beta}\right)^{1-\theta} \|u(t)\|_{\mathrm{L}^2(\Omega)} \le C \left(C_1 \int_\Omega |\nabla u(t)|^p \, dx + C_2 \int_\Omega u(t)^{1-\beta}\, dx\right)^{\frac{\theta}{p}+1-\theta}.$$

Writing this inequality in the form

$$\left[C^{\gamma} C_1^{1-\theta} \left(C_2 M^{-\beta}\right)^{\frac{\theta}{p}}\right]^{-1} z^{\gamma}(t) \le C_1 \int_\Omega |\nabla u(t)|^p \, dx + C_2 \int_\Omega u(t)^{1-\beta}\, dx$$

and plugging to (III.121), we obtain (III.123). □

Proposition 4.3 *Let* $z : (0,T) \to \mathbb{R}$ *be a non-negative function satisfying inequality* $(III.123)$ *with* $\gamma \in (0,1)$. *Then,*

$$\forall t \ge T_*, \qquad z(t) = 0, \tag{III.124}$$

where $T_* = z_0^{1-\gamma} \left[K(1-\gamma)\right]^{-1}$, *with* K *defined in Proposition 4.2.*

Proof. First, note that if $z_0 = 0$, (III.124) directly holds. Now, if $z_0 > 0$, there exists an interval $(0, \tau)$ such that $z(t) > 0$ for all $t \in [0, \tau)$. Arguing by contradiction, let us assume that

$$\xi \stackrel{\text{def}}{=} \sup\{\tau > 0 \quad | \quad \forall t \in [0, \tau), \quad z(t) > 0\} > T_*.$$

Dividing the both terms of inequality (III.123) by $z^\gamma(t)$, we get

$$\frac{1}{1-\gamma}\left(z^{1-\gamma}(t)\right)' \leq -K$$

and then integrating over the interval $(0, t)$ with $t \in (T_*, \xi)$:

$$z^{1-\gamma}(t) \leq z_0^{1-\gamma} - K(1-\gamma)t.$$

Notice that by virtue of inequality (III.123), $z'(t) \leq 0$ for almost every t and the function z is non-increasing. On the other hand, since $z(t)$ is nonnegative and $t \mapsto z_0^{1-\gamma} - K(1-\gamma)t$ is monotone decreasing, we have

$$\forall t \geq T_*, \qquad 0 \leq z(t) \leq z_0^{1-\gamma} - K(1-\gamma)t < 0,$$

which is impossible unless $T_* \geq \xi$. Thus, $z(T_*) = 0$ and Proposition 4.3 follows from the monotonicity of z. □

Proof. (of Theorem 1.2) Theorem 1.2 immediately follows from Propositions 4.1 and 4.3. □

Annexe A

A useful Hölder regularity result

Let Ω be a $\mathscr{C}^2$ bounded domain of $\mathbb{R}^N$, $N \geq 2$. We consider the following quasilinear elliptic boundary value problem,

$$(\mathrm{P}) \begin{cases} -\mathrm{div}(\varphi(x, \nabla u)) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \Omega. \end{cases}$$

In the left hand side of the first equation of (P), for $x \in \Omega$ and $u \in \mathrm{W}_0^{1,p}(\Omega)$,

$$\mathrm{div}(\varphi(x, \nabla u)) \stackrel{\mathrm{def}}{=} \sum_{i=1}^{N} \frac{\partial \varphi_i}{\partial x_i}(x, \nabla u(x)), \tag{A.1}$$

with values in $\mathrm{W}^{-1,p'}(\Omega)$. Moreover, the components φ_i of the vector field $\varphi : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$, $\varphi = (\varphi_1, \cdots, \varphi_N)$, are functions of x and $\eta \in \mathbb{R}^N$, such that for $i, j \in \{1, \cdots, N\}$, $\varphi_i \in \mathscr{C}(\Omega \times \mathbb{R}^N)$ and $\frac{\partial \varphi_i}{\partial \eta_j} \in \mathscr{C}(\Omega \times (\mathbb{R}^N \setminus \{0\}))$. We assume that φ satisfies the following ellipticity and growth conditions :

There exist some constants $\gamma > 0$, $\kappa \in [0, 1]$, $\Gamma > 0$ and $\beta \in (0, 1)$, such that for all $x, y \in \Omega$, all $\eta \in \mathbb{R}^N \setminus \{0\}$ and $\xi \in \mathbb{R}^N$,

$$\varphi_i(x, 0) = 0, \quad \text{for } i = 1, \cdots, N, \tag{A.2}$$

$$\sum_{i,j=1}^{N} \frac{\partial \varphi_i}{\partial \eta_j}(x, \eta)\xi_i\xi_j \geq \gamma(\kappa + |\eta|)^{p-2}|\xi|^2, \tag{A.3}$$

$$\sum_{i,j=1}^{N} \left| \frac{\partial \varphi_i}{\partial \eta_j}(x, \eta) \right| \leq \Gamma(\kappa + |\eta|)^{p-2}, \tag{A.4}$$

$$\sum_{i=1}^{N} |\varphi_i(x, \eta) - \varphi_i(y, \eta)| \leq \Gamma(1 + |\eta|)^p |x - y|^\beta. \tag{A.5}$$

We remark that condition (A.2) through (A.5) are motivated by the elliptic boundary value problem,

$$(\mathrm{P}) \begin{cases} -\Delta_p u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Omega. \end{cases}$$

In the right hand side of the first equation of (P), f is a $\mathrm{L}^{\infty}_{\mathrm{loc}}(\Omega)$ function such that there exist two constants $C > 0$ and $\delta > 0$ satisfying

$$|f(x)| \leq Cd(x)^{-\delta}, \qquad \text{a.e. in } \Omega. \tag{A.6}$$

Then, we have the following Hölder regularity result on the solutions of (P).

Theorem 0.1 *Assume that φ satisfies the structural hypotheses* (A.2) *through* (A.5) *and f satisfies the growth hypothesis* (A.6)*. Let $u \in \mathrm{W}^{1,p}_0(\Omega)$ be a positive weak solution of* (P)*. Let $\overline{u} \in \mathrm{W}^{1,p}_0(\Omega)$ be a super-solutions of* (P) *such that*

$$-\mathrm{div}\big(\varphi\,(x, \overline{u}(x))\,\big) \geq |f(x)| \qquad \text{in } \Omega, \tag{A.7}$$

in the sense of distributions in $\mathrm{W}^{-1,r'}(\Omega)$.

1. *Assume that $0 < \delta < 1$ and that there exists $C' > 0$ such that*

$$0 \leq u \leq \overline{u} \leq C'd(x) \qquad \text{a.e } in\ \Omega. \tag{A.8}$$

Then, there exists constants $0 < \alpha < \beta$ and $M > 0$, depending solely on Ω, p and N, on the constant γ, Γ and β in (A.3) *through* (A.5)*, on the constants C and δ in* (A.6) *and on the constant C' in* (A.8)*, such that u satisfies $u \in \mathscr{C}^{1,\alpha}(\overline{\Omega})$ and*

$$\|u\|_{\mathscr{C}^{1,\alpha}(\overline{\Omega})} \leq M.$$

2. *Assume that there exists $C' > 0$ such that*

$$0 \leq u \leq \overline{u} \leq C'd(x)^{\delta'} \qquad \text{a.e } in\ \Omega, \tag{A.9}$$

with $0 < \delta' < \delta$. Finally, let $0 < \alpha < \beta$ be an arbitrary number such that

$$0 < \alpha < \frac{p}{p - 1 + \delta/\delta'} < 1.$$

Then, there exists a constant $M > 0$, depending solely on Ω, p and N, on the constant γ, Γ and β in (A.3) *through* (A.5)*, on the constants C and δ in* (A.6) *and on the constants C' and δ' in* (A.9) *such that u satisfies $u \in \mathscr{C}^{0,\alpha}(\overline{\Omega})$*

$$\|u\|_{\mathscr{C}^{0,\alpha}(\overline{\Omega})} \leq M.$$

Proof. For point *1.*, the proof is quite similar to the Theorem B.1's in Giacomoni-Schindler-Takáč [49]. Indeed, to overcome the non-positivity of f, we add conditions (A.7) and (A.8). Then, introducing the same boundary value problem as (B.14) in [49], instead of inequality (B.16), we get here

$$|u(x) - v(x)| \leq \overline{u}(x) \leq Cx_N \quad \text{for all } x = (x', x_N) \in B_R^+(0). \tag{A.10}$$

Then, estimate (B.17) still holds and the end of the proof is similar. For point *2.*, the proof is quite similar to the Theorem 1.1's in Giacomoni-Schindler-Takáč [48]. Precisely, to overcome the non-positivity of f, we add conditions (A.7) and (A.9). Then, introducing the same boundary value problem as (12) in [48], instead of inequality (14), we get here

$$|u(x) - v(x)| \leq \overline{u}(x) \leq Cx_N^{\delta'} \quad \text{for all } x = (x', x_N) \in B_R^+(0). \tag{A.11}$$

Then, estimate (15) still holds and the end of the proof is similar. □

Annexe B

Proofs of some results of Chapter I

1 Proof of Proposition 3.1

1.1 When $\mathbf{0 \leq k < 1 + q}$

By (I.5), $\underline{v} \overset{\text{def}}{=} m\varphi_1$ is a sub-solution of (Q) in Ω for $m > 0$ small enough. Now let us define

$$f(x) \overset{\text{def}}{=} Md(x)^{-(k-q)}L\left(d(x)\right) \quad \text{in } \Omega,$$

with $M > 0$. Let $(k-q)^+ < \delta < 1$, therefore $0 < f(x) \leq C_1 d(x)^{-\delta}$ in Ω. Thus if we consider the problem

$$\left(\overline{\mathrm{Q}}\right)\begin{cases} -\Delta_p v = f & \text{in } \Omega, \\ \quad v = 0 & \text{on } \partial\Omega, \quad v > 0 \quad \text{in } \Omega, \end{cases}$$

by a result of Giacomoni, Schindler and Takáč [49] $\left(\overline{\mathrm{Q}}\right)$ has a unique solution $\overline{v} \in \mathscr{C}^{1,\alpha}(\overline{\Omega})$, with $0 < \alpha < 1$ and $\overline{v} \sim d(x)$ in Ω. Therefore, $-\Delta_p \overline{v} \geq K(x)\overline{v}^q$ in Ω for $M > 0$ sufficiently large. Hence we get that both sub- and super-solution of the problem $\left(\overline{\mathrm{Q}}\right)$, namely $\underline{v}$ and $\overline{v}$, behave like the distance function $d(x)$ near $\partial\Omega$. Using the sub- and super-solution method as in Section 4 of Chapter I, we get a solution $v \in \mathscr{C}^{1,\alpha}(\overline{\Omega})$ satisfying

$$v(x) \sim d(x) \quad \text{in } \Omega.$$

Now let $w \in \mathrm{W}_0^{1,p}(\Omega)$ be a solution to $\left(\overline{\mathrm{Q}}\right)$ satisfying $w(x) \sim d(x)$ in Ω. Then, using Theorem 0.1 of Appendix A, $w \in \mathscr{C}^{1,\alpha}(\Omega)$ and we can define

$$C_{\max} \overset{\text{def}}{=} \sup\left\{C > 0 \mid Cw \leq v \ \text{ in } \Omega\right\} \in \mathbb{R}.$$

It is easy to see that $C_{\max} > 0$ and $C_{\max}w \leq v$ in Ω, so for all $x \in \Omega$ we get

$$-\Delta_p\left((C_{\max})^{\frac{q}{p-1}}w(x)\right) = K(x)\left(C_{\max}w(x)\right)^q \leq K(x)v(x)^q = -\Delta_p(v(x)).$$

If we suppose $C_{\max} < 1$, the weak maximum principle implies $(C_{\max})^{\frac{q}{p-1}}w \leq v$ in Ω. However, since $C_{\max} < 1$ and $q < p-1$, we have that $(C_{\max})^{\frac{q}{p-1}} > C_{\max}$. Therefore

$$C_{\max}w < (C_{\max})^{\frac{q}{p-1}}w \leqslant v \quad \text{in } \Omega,$$

which contradicts the definition of $C_{\max}$. So $C_{\max} \geq 1$ and $w \leq C_{\max} w \leq v$ in Ω. Interchanging the role of w and v, we finally get that $w = v$ and this proves the uniqueness of the solution of $\left(\overline{\mathrm{Q}}\right)$ in the convex set

$$\Lambda_1 \stackrel{\text{def}}{=} \left\{ w \in \mathrm{W}_0^{1,p}(\Omega) \;\middle|\; w(x) \sim d(x) \quad \text{in } \Omega \right\}.$$

1.2 When $\mathbf{1+q \leq k \leq p}$

For $t \in (0, D]$ we define

$$\Theta(t) \stackrel{\text{def}}{=} \exp\left(\int_t^D \frac{y(s)}{s} ds\right),$$

with $y \in \mathscr{C}([0,D]) \cap \mathscr{C}^1((0,D])$ such that $y(0) = 0$ and $\lim_{t \to 0^+} \frac{ty'(t)}{y(t)} = 0$. Then,

$$\lim_{t \to 0^+} \frac{t\Theta'(t)}{\Theta(t)} = 0 \quad \text{and} \quad \lim_{t \to 0^+} \frac{t\Theta''(t)}{\Theta'(t)} = -1. \tag{B.1}$$

Let $\beta \in [0,1]$, for $x \in \Omega$ we also define $\psi(x) \stackrel{\text{def}}{=} \varphi_1(x)^\beta \Theta(\varphi_1(x))$ in Ω. Then,

$$\begin{aligned} -\Delta_p \psi = &\Big(\Theta(\varphi_1)\Big)^{p-1} {\varphi_1}^{(\beta-1)(p-1)-1} \left(\beta + \frac{\varphi_1 \Theta'(\varphi_1)}{\Theta(\varphi_1)}\right)^{p-2} \times \\ &\left[\left(\beta + \frac{\varphi_1 \Theta'(\varphi_1)}{\Theta(\varphi_1)}\right) \lambda_1 {\varphi_1}^p + (p-1)|\nabla \varphi_1|^p \left(\beta(1-\beta) - 2\beta \frac{\varphi_1 \Theta'(\varphi_1)}{\Theta(\varphi_1)} - \frac{\varphi_1^2 \Theta''(\varphi_1)}{\Theta(\varphi_1)}\right)\right]. \end{aligned}$$

We now distinguish the following cases :

First case : $\mathbf{0 < \beta < 1}$ ***:*** There exists $\varepsilon > 0$ sufficiently small such that for $x \in \Omega_\varepsilon$,

$$\frac{\beta}{2} \leq \beta + \frac{\varphi_1(x)\Theta'(\varphi_1(x))}{\Theta(\varphi_1(x))} \leq \frac{3\beta}{2}$$

and

$$\frac{\beta(1-\beta)}{2} \leq \beta(1-\beta) - 2\beta \frac{\varphi_1(x)\Theta'(\varphi_1(x))}{\Theta(\varphi_1(x))} - \frac{\varphi_1(x)^2 \Theta''(\varphi_1(x))}{\Theta(\varphi_1(x))} \leq \frac{3}{2}\beta(1-\beta).$$

Therefore we get

$$-\Delta_p \psi(x) \sim \Theta(\varphi_1(x))^{p-1} \varphi_1(x)^{(\beta-1)(p-1)-1} \quad \text{in } \Omega,$$

which implies

$$\Big(-\Delta_p \psi(x)\Big) \psi(x)^{-q} \sim \Theta(\varphi_1(x))^{p-(1+q)} \varphi_1(x)^{(\beta-1)(p-1)-1-q\beta} \quad \text{in } \Omega.$$

When $1+q < k < p$, if we choose $\beta = \frac{p-k}{p-(1+q)} \in (0,1)$ and $y(t) = \frac{z(t)}{p-(1+q)}$ for $t \in [0,D]$, w satisfies

$$\begin{cases} \Big(-\Delta_p \psi(x)\Big) \psi(x)^{-q} \sim K(x) & \text{in } \Omega, \\ w = 0 \quad \text{on } \partial\Omega, & w > 0 \text{ in } \Omega. \end{cases}$$

Therefore there exist $C_1, C_2 > 0$ such that $C_1\psi$ and $C_2\psi$ are respectively sub- and super-solutions of the problem (Q). Thus, (Q) has a solution $v \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega) \cap \mathscr{C}_0(\overline{\Omega})$ satisfying

$$v(x) \sim d(x)^{\frac{p-k}{p-(1+q)}} L\left(d(x)\right)^{\frac{1}{p-(1+q)}} \qquad \text{in } \Omega. \tag{B.2}$$

Using the same arguments as section 1.1, we get the uniqueness of the solution in the set

$$\Lambda_2 \overset{\text{def}}{=} \left\{ w \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega) \cap \mathscr{C}_0(\overline{\Omega}) \quad \middle| \quad w(x) \sim d(x)^{\frac{p-k}{p-(1+q)}} L\left(d(x)\right)^{\frac{1}{p-(1+q)}} \qquad \text{in } \Omega \right\}.$$

Moreover, $\overline{u}_\lambda \in \mathrm{W}^{1,p}_0(\Omega)$ if and only if the right hand term in the equation of problem (Q) is $\mathrm{W}^{-1,p'}(\Omega)$, *i.e.* if and only if there exists a constant $C > 0$ such that

$$\forall w \in \mathrm{W}^{1,p}_0(\Omega), \qquad \left| \int_\Omega K(x)\overline{u}_\lambda(x)^q w(x) dx \right| \leq C\|w\|_{\mathrm{W}^{1,p}_0(\Omega)}.$$

Using estimate (B.2), Hardy's and Hölder's inequalities and property (I.4), this condition is satisfied if $k < 1 + q + \frac{p-(1+q)}{p}$. Then in this case, from regularity Theorem 0.1 of Appendix A, we get that $v \in \mathrm{W}^{1,p}_0(\Omega) \cap \mathscr{C}^{1,\alpha}(\overline{\Omega})$, for some $0 < \alpha < 1$. Moreover, by (B.2), we have

$$\int_\Omega K(x) \left(\overline{u}_\lambda(x)\right)^{q+1} dx < +\infty$$

only if $k \leq 1 + q + \frac{p-(1+q)}{p}$. Then, for $k > 1 + q + \frac{p-(1+q)}{p}$, $\overline{u}_\lambda \notin \mathrm{W}^{1,p}_0(\Omega)$.

Second case : $\beta = 1$: The computation of $-\Delta_p w$ gives

$$-\Delta_p\psi = \Theta'(\varphi_1)\left(\Theta(\varphi_1)\right)^{p-2}\left(1 + \frac{\varphi_1\Theta'(\varphi_1)}{\Theta(\varphi_1)}\right)^{p-2} \times \\ \left[\left(1 + \frac{\Theta(\varphi_1)}{\varphi_1\Theta'(\varphi_1)}\right)\lambda_1{\varphi_1}^p + (p-1)|\nabla\varphi_1|^p\left(-2 - \frac{\varphi_1\Theta''(\varphi_1)}{\Theta'(\varphi_1)}\right)\right].$$

We choose Θ such that

$$C_1{\varphi_1}^p \leq -\frac{\Theta(\varphi_1){\varphi_1}^{p-1}}{\Theta'(\varphi_1)} \leq C_2{\varphi_1}^{p-1}$$

near the boundary, that is equivalent to require

$$C_1 t \leq -\frac{\Theta(t)}{\Theta'(t)} \leq C_2, \text{ for } t > 0 \text{ small enough.} \tag{B.3}$$

Hence,

$$\Big(-\Delta_p\psi(x)\Big)\psi(x)^{-q} \sim -\Theta'\left(\varphi_1(x)\right)\Theta\left(\varphi_1(x)\right)^{p-2-q}\varphi_1(x)^{-q} \qquad \text{in } \Omega.$$

To get

$$\Big(-\Delta_p\psi(x)\Big)\psi(x)^{-q} \sim \varphi_1(x)^{-k}L\left(\varphi_1(x)\right) \qquad \text{in } \Omega,$$

we require

$$t^{-(1+q)}y(t)\Theta(t)^{p-(q+1)} \sim t^{-k}\left(\int_t^D \frac{z(s)}{s}ds\right) \quad \text{in } (0,D].$$

This condition can be satisfied only if $k = 1+q$. Then taking

$$\Theta(t) = \left(\int_t^D s^{-1}L(s)ds\right)^{\frac{1}{p-(1+q)}}, \; 0 < t \leq D,$$

Θ satisfies conditions (B.1) and (B.3). Thus, if $k = 1+q$ and

$$\psi(x) = \varphi_1(x)\left(\int_{\varphi_1(x)}^D L(t)t^{-1}dt\right)^{\frac{1}{p-(1+q)}} \quad \text{in } \Omega,$$

there exist $C_1, C_2 > 0$ such that $C_1\psi$ and $C_2\psi$ are respectively sub- and super-solutions of (Q). Then, from the same sub- and super solution method as above and using Theorem 0.1 of Appendix A, there exists a solution $v \in \mathrm{W}_0^{1,p}(\Omega) \cap \mathscr{C}^{0,\alpha}(\overline{\Omega})$ of (Q) satisfying

$$v(x) \sim d(x)\left(\int_{d(x)}^D L(t)t^{-1}dt\right)^{\frac{1}{p-(1+q)}} \quad \text{in } \Omega.$$

Using the same argument as in section 1.1 we get the uniqueness of the solution in the set

$$\Lambda_3 \stackrel{\text{def}}{=} \left\{ w \in \mathrm{W}_0^{1,p}(\Omega) \cap \mathscr{C}(\overline{\Omega}) \;\middle|\; w(x) \sim d(x)\left(\int_{d(x)}^D L(t)t^{-1}dt\right)^{\frac{1}{p-(1+q)}} \quad \text{in } \Omega \right\}.$$

Third case : $\beta = 0$: In this case, we get

$$-\Delta_p\psi = \varphi(x)^{-1}\left(\Theta'(\varphi_1)\right)^{p-1}\left[\lambda_1{\varphi_1}^p - (p-1)|\nabla\varphi_1|^p\frac{\varphi_1\Theta''(\varphi_1)}{\Theta(\varphi_1)}\right].$$

Hence,

$$\Big(-\Delta_p\psi(x)\Big)\psi(x)^{-q} \sim \varphi_1(x)^{-1}\Theta'(\varphi_1(x))^{p-1}\Theta(\varphi_1(x))^{-q} \quad \text{in } \Omega.$$

Similarly as the previous case, to get

$$\varphi_1(x)^{-1}\Theta'(\varphi_1(x))^{p-1}\Theta(\varphi_1(x))^{-q} \sim \varphi_1(x)^{-k}L(\varphi(x)) \quad \text{in } \Omega,$$

we require

$$t^{-p}\Theta(t)^{p-(1+q)}\Big(-y(t)\Big)^{p-1} \sim t^{-k}\exp\left(\int_t^D \frac{z(s)}{s}dt\right) \quad \text{in } (0,t].$$

This condition can be satisfied only if $k = p$. Then if condition (I.11) holds and if we choose

$$\Theta(t) = \exp\left(\int_t^D \frac{y(s)}{s}ds\right) = C\left(\int_0^t s^{-1}L(s)^{\frac{1}{p-1}}ds\right)^{\frac{p-1}{p-(1+q)}}, \; 0 < t \leq D,$$

we get that Θ satisfies conditions (B.1) and (B.3). Thus if $k = p$ and

$$\psi(x) = C\left(\int_0^{\varphi_1(x)} t^{-1}L(t)^{\frac{1}{p-1}}dt\right)^{\frac{p-1}{p-(1+q)}},$$

there exists $C_1, C_2 > 0$ such that $C_1\psi$ and $C_2\psi$ are respectively sub- and super-solutions of (Q) and there exists a solution $v \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega) \cap \mathscr{C}_0(\overline{\Omega})$ of (Q) satisfying

$$v(x) \sim \left(\int_0^{d(x)} s^{-1}L(s)^{\frac{1}{p-1}}ds\right)^{\frac{p-1}{p-(1+q)}} \quad \text{in } \Omega.$$

Using the same argument as section 1.1, we get the uniqueness of the solution in the set

$$\Lambda_4 \overset{\text{def}}{=} \left\{ w \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega) \cap \mathscr{C}_0(\overline{\Omega}) \;\middle|\; w(x) \sim \left(\int_0^{d(x)} s^{-1}L(s)^{\frac{1}{p-1}}ds\right)^{\frac{p-1}{p-(1+q)}} \quad \text{in } \Omega \right\}.$$

2 Proof of Proposition 3.2

To prove this proposition, we need the two following lemmas :

Lemma 2.1 (Picone's Identity)
Let $u, v \in \mathscr{C}^1(\overline{\Omega})$ two positive functions satisfying the Hopf's lemma. Then,

$$L(u,v) \overset{\text{def}}{=} |\nabla u|^p + (p-1)\left(\frac{u}{v}\right)^p |\nabla v|^p - p\left(\frac{u}{v}\right)^{p-1} |\nabla v|^{p-2}\nabla v.\nabla u$$

satisfies $L(u,v) \geq 0$ in Ω and $L(u,v) = R(u,v)$ where

$$R(u,v) \overset{\text{def}}{=} |\nabla u|^p - |\nabla v|^{p-2}\nabla v.\nabla\left(\frac{u^p}{v^{p-1}}\right).$$

Moreover, $L(u,v) = 0$ in Ω if and only if there exists $C > 0$ such that $u = Cv$ in Ω.

Proof. See Theorem 1.1 in ALLEGRETTO-HUANG [1]. □

Lemma 2.2 (Díaz-Saa inequality)
For $i = 1,2$ let $w_i \in \mathrm{L}^\infty(\Omega)$ such that $w_i \geq 0$ a.e. in Ω, ${w_i}^{\frac{1}{p}} \in \mathrm{W}^{1,p}(\Omega)$, $\Delta_p\left({w_i}^{\frac{1}{p}}\right) \in \mathrm{L}^\infty(\Omega)$ and $w_1 = w_2$ on $\partial\Omega$. Moreover if $\frac{w_1}{w_2}, \frac{w_2}{w_1} \in \mathrm{L}^\infty(\Omega)$, we have the inequality

$$\int_\Omega \left(\frac{-\Delta_p\left({w_1}^{\frac{1}{p}}\right)}{{w_1}^{\frac{p-1}{p}}} + \frac{\Delta_p\left({w_2}^{\frac{1}{p}}\right)}{{w_2}^{\frac{p-1}{p}}}\right)(w_1 - w_2)\,dx \geq 0. \tag{B.4}$$

Futhermore, (B.4) *becomes an equality if and only if there exists $C > 0$ such that $w_1 = Cw_2$ a.e. in Ω.*

Proof. See Lemme 2 in DÍAZ-SAA [33]. □

Proof. (OF PROPOSITION 3.2). We argue by contradiction. If Proposition 3.2 does not hold, there exist $\overline{v} \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega) \cap \mathscr{C}_0(\overline{\Omega})$ weak solution of (Q), $\eta > 0$ and $\varepsilon > 0$ satisfying $\overline{v} \geq \eta \underline{u}$ a.e. in Ω and (I.12) holds.

First step, when* $\mathbf{q \geq 0}$ *: We consider the following perturbed problem :

$$(\mathrm{Q}_n) \begin{cases} -\Delta_p v = K_n(x) v^q, \quad v > 0 \quad \text{in } \Omega, \\ \quad v = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $(K_n)_{n\in\mathbb{N}} \subset \mathrm{L}^\infty(\Omega)$ is increasing sequence satisfying $K_n \xrightarrow[n\to+\infty]{} K$ a.e. in Ω. We will prove there exists a unique solution of (Q_n) in $\mathrm{W}^{1,p}_0(\Omega)$ and show that this solution is $\mathscr{C}^{1,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$. Let us consider the functional,

$$I_n(u) \stackrel{\text{def}}{=} \int_\Omega |\nabla u|^p dx, \qquad u \in V \stackrel{\text{def}}{=} \left\{ w \in \mathrm{W}^{1,p}_0(\Omega) \quad \Big| \quad \int_\Omega K_n(x) w^{q+1} dx = 1 \right\}.$$

From the compactness embedding $\mathrm{W}^{1,p}_0(\Omega) \hookrightarrow \mathrm{L}^{q+1}(\Omega)$, there exists a non-negative and non-trivial $\tilde{v}_n \in \mathrm{W}^{1,p}_0(\Omega)$ satisfying

$$I_n(\tilde{v}_n) = \min_{u\in V} I_n(u).$$

Therefore, from the Lagrange multiplier rule, there exists a Lagrange multiplier $\lambda_n > 0$ such that

$$\begin{cases} -\Delta_p \tilde{v}_n = \lambda_n K_n(x) \left(\tilde{v}_n\right)^q \quad \text{in } \Omega, \\ \quad \tilde{v}_n = 0 \quad \text{on } \partial\Omega. \end{cases}$$

By homogeneity of the p-Laplacian operator, if we define

$$v_n \stackrel{\text{def}}{=} (\lambda_n)^{\frac{1}{p-(1+q)}} \tilde{v}_n \in \mathrm{W}^{1,p}_0(\Omega),$$

v_n satisfies

$$\begin{cases} -\Delta_p v_n = K_n(x) {v_n}^q, \quad \text{in } \Omega, \\ \quad v_n = 0 \quad \text{on } \partial\Omega, \end{cases}$$

Since $q < p - 1$ and $K_n \in \mathrm{L}^\infty(\Omega)$, using Moser iterations we prove that $v_n \in \mathrm{L}^\infty(\Omega)$ and from the regularity result in LIEBERMAN [64], $v_n \in \mathscr{C}^{1,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$; $v_n > 0$ in Ω from the strong maximum principle in VÁZQUEZ [83] and v_n is a solution of (Q_n).

Now, let us prove the uniqueness of a such solution. Therefore, for that, we use for that the Díaz-Saa inequality (B.4). So let $u_n \in \mathscr{C}^{1,\alpha}(\overline{\Omega})$ be an other solution of (Q_n), then

$$\int_\Omega \left(\frac{-\Delta_p u_n}{{u_n}^{p-1}} + \frac{\Delta_p v_n}{{v_n}^{p-1}} \right) \left({u_n}^p - {v_n}^p\right) dx \geq 0, \tag{B.5}$$

which implies

$$\int_\Omega K_n(x) \left(\frac{1}{{u_n}^{p-(1+q)}} - \frac{1}{{v_n}^{p-(1+q)}} \right) \left({u_n}^p - {v_n}^p\right) dx = 0.$$

Then inequality (B.5) becomes an equality, therefore by Lemma B.4 there exits $C > 0$ such that $u_n = Cv_n$ in Ω. Furthermore, by homogeneity arguments, $-\Delta_p(Cv_n) \neq K_n(x)(Cv_n)^q$ in Ω if $C \neq 1$, so $u_n = v_n$ in Ω and we get the uniqueness.

Now, we will prove that for all $n \in \mathbb{N}$ $v_n \leq \overline{v}$. For that, we apply the sub- and super-solution method in a compact subset of Ω. So let us fix $n \in \mathbb{N}$ and define $(\Omega_m)_{m\in\mathbb{N}^*}$ an increasing sequence of smooth sub-domains of Ω such that $\Omega_m \xrightarrow[m\to+\infty]{} \Omega$ in the Hausdorff topology with

$$\forall m \in \mathbb{N}^*, \quad \frac{1}{m+1} < \mathrm{dist}(\partial\Omega, \partial\Omega_m) < \frac{1}{m}.$$

Then we consider the following sequence of problems :

$$(\mathrm{Q}_{n,m}) \begin{cases} -\Delta_p u = K_n(x)v^q \quad \text{in } \Omega_m, \\ \quad v = \eta\underline{v} \quad \text{on } \partial\Omega_m, \quad v > 0 \quad \text{in } \Omega_m, \end{cases}$$

with $\underline{v} \in \mathrm{W}_0^{1,p}(\Omega) \cap \mathscr{C}(\overline{\Omega})$ the sub-solution of (P). Since $\overline{v} \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega) \cap \mathrm{L}^\infty(\Omega)$ and $\overline{v} \geq \eta\underline{v}$ in Ω and using the same arguments as in the proof of Proposition 4.1, for all $m \in \mathbb{N}$ there exists $v_{n,m} \in \mathrm{W}^{1,p}(\Omega_m) \cap \mathscr{C}(\overline{\Omega_m})$ unique solution of $(\mathrm{Q}_{n,m})$. Moreover, $v_{n,m}$ satisfies

$$\eta\underline{v} \leq v_{n,m} \leq \overline{v} \quad \text{in } \Omega.$$

Now, setting $\tilde{v}_{n,m}$ the extension of $v_{n,m}$ by 0 in $\Omega \setminus \Omega_m$, the sequence $(\tilde{v}_{n,m})_{m\in\mathbb{N}^*}$ is an increasing sequence which converges pointwise to an element $u_n \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega) \cap \mathscr{C}_0(\overline{\Omega})$ solution of (Q_n), by similar arguments as in the proof of Proposition 4.1. Then, the uniqueness argument implies $u_n = v_n$ in Ω and then

$$\forall n \in \mathbb{N}, \quad v_n \leq \overline{v} \quad \text{in } \Omega.$$

First step, when* $\mathbf{q < 0}$ *: Let us define the following problem :

$$(\mathrm{Q}'_n) \begin{cases} -\Delta_p v = K_n(x)\left(v + \frac{1}{n}\right)^q, \quad v > 0 \quad \text{in } \Omega, \\ \quad v = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Using a similar method as step 1, we get the existence and the uniqueness of a sequence of weak solutions of (Q'_n) in $\mathrm{W}_0^{1,p}(\Omega)$.

Second step : Applying Picone's Identity with $u = {\varphi_1}^\beta \in \mathscr{C}^1(\overline{\Omega})$, where $\beta = \frac{p-1+\varepsilon}{p}$ and $v = v_n \in \mathscr{C}^1(\overline{\Omega})$, we get

$$0 \leq \int_\Omega |\nabla u|^p - |\nabla v|^{p-2}\nabla v.\nabla\left(\frac{u^p}{v^{p-1}}\right) dx. \tag{B.6}$$

1. Then for $q \geq 0$, we get

$$\beta^p \int_\Omega |\nabla \varphi_1|^p {\varphi_1}^{(\beta-1)p} dx = \int_\Omega \left| \nabla {\varphi_1}^\beta \right|^p dx$$
$$\geq \int_\Omega |\nabla v_n|^{p-2} \nabla v_n . \nabla \left(\frac{{\varphi_1}^{\beta p}}{{v_n}^{p-1}} \right) dx = \int_\Omega K_n(x) \frac{{\varphi_1}^{\beta p}}{{v_n}^{p-(q+1)}} dx.$$

Therefore, passing to the limit as $n \to +\infty$, there exists $C > 0$ such that

$$\inf_{y \in \omega} \frac{1}{\overline{v}(y)^{p-(q+1)}} \int_\omega K(x) {\varphi_1}^{p-1+\varepsilon} dx \leq C, \quad \forall \omega \subset\subset \Omega.$$

This inequality does not hold for ω close enough to Ω, *i.e* when $\text{dist}(\Omega, \omega)$ is sufficiently small, because

$$\int_\Omega K(x) {\varphi_1}^{p-1+\varepsilon} dx = +\infty,$$

by assumption.

2. When $q < 0$ arguing similarly as in the first case, we get

$$\beta^p \int_\Omega |\nabla \varphi_1|^p {\varphi_1}^{(\beta-1)p} dx \geq \int_\Omega K_n(x) \frac{{\varphi_1}^{\beta p}}{\left(v_n + \frac{1}{n}\right)^{p-(q+1)}} dx.$$

Therefore passing to the limit as $n \to +\infty$,

$$\inf_{y \in \omega} \frac{1}{(\overline{v}(y) + 1)^{p-(q+1)}} \int_\omega K(x) {\varphi_1}^{p-1+\varepsilon} dx \leq C, \quad \forall \omega \subset\subset \Omega.$$

And we conclude as above. □

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Résumé : Cette thèse s'inscrit dans le domaine mathématique de **l'analyse des équations aux dérivées partielles non-linéaires**. Plus précisément, nous avons fait ici l'étude de **problèmes quasi-linéaires singuliers**. Le terme "singulier" fait référence à l'intervention d'une non-linéarité qui explose au bord du domaine où l'équation est posée. La présence d'une telle singularité entraine un manque de régularité et donc de compacité des solutions qui ne nous permet pas d'appliquer directement les méthodes classiques de l'analyse non-linéaire pour démontrer l'**existence de solutions** et discuter des propriétés de **régularité** et de **comportement asymptotique** de ces solutions. Pour contourner cette difficulté, nous sommes amenés à établir des estimations *a priori* très fines au voisinage du bord du domaine en combinant diverses méthodes : méthodes de monotonie (reliée au principe du maximum), méthodes variationnelles, argument de convexité, méthodes de point fixe et semi-discrétisation en temps. A travers, l'étude de trois problèmes-modèle faisant intervenir l'**opérateur *p*-Laplacien**, nous avons montré comment ces différentes méthodes pouvaient être mises en œuvre. Les résultats que nous avons obtenus sont décrits dans les trois chapitres de cette thèse :

- Dans le Chapitre I, nous avons étudié un **problème d'absorption elliptique singulier**. En utilisant des **méthodes de sur- et sous solutions** et des **méthodes variationnelles**, nous établissons des résultats d'existence de solutions. Par des **méthodes de comparaison locale**, nous démontrons également la propriété de **support compact** de ces solutions, pour de fortes singularités.
- Dans le Chapitre II, nous étudions le cas d'un **système d'équations quasi-linéaires singulières**. Par des arguments de **point fixe** et de **monotonie**, nous démontrons deux résultats généraux d'existence de solutions. Dans un deuxièmes temps, nous faisons une analyse plus détaillée de systèmes du type Gierer-Meinhardt modélisant des phénomènes biologiques. Des résultats d'unicité ainsi que des estimations précises sur le comportement des solutions sont alors obtenus.
- Dans le Chapitre III, nous faisons l'étude d'un **problème d'absorption, parabolique singulier**. Nous établissons par une méthode de **semi-discrétisation en temps** des résultats d'existence de solutions. Grâce à des inégalités d'énergie, nous démontrons également l'**extinction en temps fini** de ces solutions.

Mots clés : opérateur p-Laplacien, problèmes singuliers, problèmes/systèmes elliptiques, problèmes paraboliques, solution à support compact, stabilité des solutions, extinction en temps fini.

Abstract : This thesis deals with the mathematical field of **nonlinear partial differential equations analysis**. More precisely, we focus on **quasilinear and singular problems**. By singularity, we mean that the problems that we have considered involve a nonlinearity in the equation which blows-up near the boundary. This singular pattern gives rise to a lack of regularity and compactness that prevent the straightforward applications of classical methods in nonlinear analysis used for proving **existence of solutions** and for establishing the **regularity properties** and the **asymptotic behaviour** of the solutions. To overcome this difficulty, we establish estimations on the precise behaviour of the solutions near the boundary combining several techniques : monotonicity method (related to the maximum principle), variational method, convexity arguments, fixed point methods and semi-discretization in time. Throughout the study of three problems involving the ***p*-Laplacian operator**, we show how to apply this different methods. The three chapters of this dissertation the describes results we get :

- In Chapter I, we study a **singular elliptic absorption problem**. By using sub- and super-solutions and variational methods, we prove the existence of the solutions. In the case of a strong singularity, by using **local comparison techniques**, we also prove that the **compact support** of the solution.
- In Chapter II, we study a **singular elliptic system**. By using **fixed point** and **monotonicity** arguments, we establish two general theorems on the existence of solution. In a second time, we more precisely analyse the Gierer-Meinhardt systems which model some biological phenomena. We prove some results about the uniqueness and the precise behaviour of the solutions.
- In Chapter III, we study a **singular parabolic absorption problem**. By using a **semi-discretization in time method**, we establish the existence of a solution. Moreover, by using **differential energy inequalities**, we prove that the solution **vanishes in finite time**. This phenomenon is called **"quenching"**.

Keywords : p-Laplacian operator, singular problems, elliptic problems/systems, parabolic problems, compact support solutions, stability of the solutions, quenching problems.