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Contributions au calcul des variations et au Principe du Maximum de Pontryagin en calculs time scale et fractionnaire

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Introduction

En mathématiques, le **calcul des variations** est un ensemble de méthodes permettant la détermination de solutions à des problèmes d'optimisation de quantités traduites en termes de fonctionnelles. De nombreuses applications existent, en particulier dans la recherche de courbes ou surfaces minimales comme celles associées aux théorèmes isopérimétriques, de courbes brachistochrones et de géodésiques. Ces techniques s'appliquent également en **théorie du contrôle optimal** où l'objectif est de déterminer une trajectoire, associée à un système dynamique sur lequel nous pouvons agir au moyen d'un paramètre appelé *contrôle*, minimisant ou maximisant un critère donné. Les systèmes dynamiques considérés sont de natures diverses (équations différentielles, intégrales ou encore stochastiques) et modélisent des problèmes d'origines multiples : aérospatial, automobile, biologie, économie, médecine, etc.

Dans ce manuscrit, nous développons certains aspects de ces deux domaines mathématiques dans les cadres discret, plus généralement **time scale**, et **fractionnaire**. En effet, ces deux outils ont récemment connu un développement considérable dû pour l'un à son application en informatique et pour l'autre à son essor dans des problèmes physiques de diffusion anormale. Que ce soit dans le cadre time scale ou dans le cadre fractionnaire, nos objectifs sont de :

a) développer un calcul des variations et étendre certains résultats classiques (équation d'Euler-Lagrange, théorème de Noether, existence de minimiseur, condition de Helmholtz);

b) établir un Principe du Maximum de Pontryagin (PMP en abrégé) pour des problèmes de contrôle optimal.

Dans ce but, nous adaptons ou généralisons quelques méthodes variationnelles usuelles, allant du simple calcul des variations au principe variationnel d'Ekeland couplé avec la technique des variations-aiguilles, en passant par l'étude d'invariances variationnelles par des groupes de transformations. Les démonstrations des PMPs nous amènent également à employer des théorèmes de point fixe, à démontrer des théorèmes de type Cauchy-Lipschitz et à prendre en considération la technique des multiplicateurs de Lagrange ou encore une méthode basée sur un théorème d'inversion locale conique.

Les mathématiques discrètes sont utilisées pour l'étude de systèmes purement numériques mais aussi pour l'analyse d'approximations discrètes de systèmes continus. Le calcul time scale, introduit en 1988 par S. Hilger, est un outil mathématique qui unifie les résultats de l'analyse continue et de l'analyse discrète. Il permet aussi de couvrir des systèmes posés sur des ensembles combinant structure continue et structure discrète. Il englobe par ailleurs la théorie des équations aux q-différences et permet, de façon plus exotique, l'étude de systèmes posés sur des ensembles complexes tel que l'ensemble de Cantor. Les contributions de ce mémoire à des problèmes variationnels posés sur time scale sont proposées en Partie A.

Le calcul fractionnaire étend les notions classiques de primitive et de dérivée d'ordre entier non nul à tout ordre réel strictement positif. Cette théorie a été initiée dans une lettre écrite par Leibniz en 1695 mais a longtemps été considérée comme une branche purement mathématique. Cependant, depuis quelques décennies, les applications du calcul fractionnaire connaissent un essor dans des domaines scientifiques nombreux et variés. L'introduction d'opérateurs fractionnaires dans des problèmes variationnels est due à F. Riewe en 1996-97. Son étude visait à fournir une structure variationnelle *fractionnaire* à des systèmes dissipatifs n'en admettant pas de classique. Depuis, une littérature conséquente a été consacrée à des problèmes variationnels fractionnaires. Les apports de ce manuscrit dans ce domaine sont donnés en Partie B.

Ce manuscrit est donc composé de deux parties : la Partie A traite de problèmes variationnels posés sur time scale et la Partie B est consacrée à leurs pendants fractionnaires. Dans chacune de ces deux parties, nous suivons (à des différences mineures près) l'organisation suivante :

1. rappels sur les outils mathématiques utilisés (calcul time scale en Partie A et calcul fractionnaire en Partie B) ;
2. détermination de l'équation d'Euler-Lagrange caractérisant les points critiques d'une fonctionnelle Lagrangienne ;
3. énoncé d'un théorème de type Noether assurant l'existence d'une constante de mouvement pour les équations d'Euler-Lagrange admettant une symétrie ;
4. énoncé d'un théorème de type Tonelli assurant l'existence d'un minimiseur pour une fonctionnelle Lagrangienne (et donc, par la même occasion, d'une solution pour l'équation d'Euler-Lagrange associée) sous des hypothèses adéquates de régularité, convexité et coercivité (uniquement en Partie B) ;
5. énoncé d'un PMP (version forte en Partie A, version faible en Partie B) donnant une condition nécessaire pour les trajectoires qui sont solutions de problèmes de contrôle optimal généraux non-linéaires ;
6. détermination d'une condition de Helmholtz caractérisant les équations provenant d'un calcul des variations (uniquement en Partie A et uniquement dans les cas purement continu et purement discret).

Des théorèmes de type Cauchy-Lipschitz et quelques résultats techniques nécessaires à la formulation et à l'étude des problèmes de contrôle optimal sont obtenus dans les Annexes.

Dans la suite de cette introduction, nous traitons séparément les Parties A et B.

Partie A

Dans cette section, nous retraçons brièvement les étapes historiques qui ont conduit à l'étude de problèmes variationnels posés sur time scale. Nous formulons à cette occasion quelques questions qui n'ont pas encore été traitées dans ce domaine et qui ont motivé les études de la Partie A de ce manuscrit. Nous présentons alors succinctement les résultats obtenus dans ce mémoire qui répondent (entièrement ou partiellement) à ces problèmes. Nous proposons également quelques détails sur les difficultés rencontrées et sur les méthodes utilisées.

• **Calcul des variations classique continu.** En analyse fonctionnelle, le calcul des variations est utilisé pour la détermination des points critiques de fonctionnelles. Le célèbre résultat d'Euler et de Lagrange, formulé autour des années 1750, affirme que les points critiques de la fonctionnelle Lagrangienne

$$\begin{array}{rccl} \mathcal{L}: & \mathrm{C}^1([a,b],\mathbb{R}^n) & \longrightarrow & \mathbb{R} \\ & q & \longmapsto & \displaystyle\int_a^b L(q(\tau),\dot{q}(\tau),\tau)\, d\tau, \end{array}$$

où $\dot{q}$ désigne la dérivée de q, sont caractérisés par l'équation différentielle, appelée *équation d'Euler-Lagrange*, suivante (voir [20, p.12]) :

$$\frac{d}{dt}\left[\frac{\partial L}{\partial v}(q,\dot{q},\cdot)\right] = \frac{\partial L}{\partial x}(q,\dot{q},\cdot).$$

Depuis, le calcul des variations a été largement développé, par exemple au cas multidimensionnel avec des dérivées d'ordre supérieur, et est devenu une théorie à part entière avec de très nombreux résultats et de multiples applications. Ce sujet est bien traité dans la littérature, voir par exemple [20, 74, 107].

Nous rappelons en particulier un théorème classique de cette théorie démontré par E. Noether en 1918 (voir [144, 173]) : si le Lagrangien L est invariant par l'action d'un groupe à un paramètre $\Phi : (\theta, x) \in [-\eta, \eta] \times \mathbb{R}^n \longmapsto \Phi(\theta, x) \in \mathbb{R}^n$ de transformations infinitésimales de $\mathbb{R}^n$, c'est-à-dire si pour toute solution q de l'équation d'Euler-Lagrange ci-dessus et si pour tout $t \in [a, b]$, la fonction

$$\theta \longmapsto L\left(\Phi(\theta, q(t)), \frac{d}{dt}[\Phi(\theta, q)](t), t\right)$$

admet une dérivée nulle en $\theta = 0$, alors l'équation d'Euler-Lagrange admet une constante de mouvement donnée par

$$\frac{\partial L}{\partial v}(q, \dot{q}, \cdot) \cdot \frac{\partial \Phi}{\partial \theta}(0, q) = c,$$

pour toute solution q. Dans ce cas, on dit que l'équation d'Euler-Lagrange admet une symétrie. Un cas classique de symétrie est donné par un Lagrangien quadratique associé à une rotation. Ce théorème de Noether présente un fort intérêt puisqu'il propose une loi de conservation explicite (traduisant souvent une quantité physique comme l'énergie totale ou le moment angulaire en mécanique classique) qui permet de réduire ou d'intégrer l'équation différentielle associée par quadrature.

• **Intégrateurs variationnels.** Les équations d'Euler-Lagrange représentent une classe importante de systèmes mécaniques et bénéficient, notamment par le principe de moindre action, d'une interprétation physique naturelle. Les propriétés physiques des systèmes Lagrangiens sont ainsi décrites par leur structure variationnelle qui impose alors de fortes contraintes sur le comportement qualitatif des solutions. D'un point de vue numérique, il semble donc important de préserver cette structure au niveau discret afin de traduire au mieux les caractéristiques Lagrangiennes de ces équations.

J. Cadzow propose en 1970 la construction d'*intégrateurs variationnels*, c'est-à-dire de schémas numériques pour les équations d'Euler-Lagrange, obtenus à partir d'un calcul des variations discret, voir [60]. Plus précisément, pour une équation d'Euler-Lagrange donnée, la construction d'un intégrateur variationnel se décompose en deux étapes :

1. définir une version discrète de la fonctionnelle Lagrangienne associée ;
2. procéder à un calcul des variations sur la fonctionnelle Lagrangienne discrète définie en étape 1.

Cette approche conduit alors à une *équation d'Euler-Lagrange discrète* dont les solutions sont les points critiques de la fonctionnelle Lagrangienne discrète. Cette équation est aussi un *intégrateur variationnel*, c'est-à-dire un schéma numérique pour l'équation d'Euler-Lagrange continue initiale préservant sa structure variationnelle au niveau discret.

Rappelons que cette conservation permet alors de préserver certaines propriétés relatives à la structure variationnelle. Par exemple, les lois de conservation associées aux systèmes Lagrangiens discrets sont étudiées par S. Maeda dans [153, 154]. Plus récemment, d'importants développements des intégrateurs variationnels sont apparus à l'issue des travaux de J. Marsden *et al.* [132, 158, 159] auxquels nous renvoyons pour une étude approfondie.

Afin de rendre nos propos plus concrets, nous allons construire un intégrateur variationnel simple pour l'équation d'Euler-Lagrange historique rappelée dans le paragraphe précédent. Plus précisément, nous allons suivre le cadre proposé par C. Lubich *et al.* dans [101]. Considérons un ensemble discret $\{a = t_0 < \ldots < t_N = b\}$. Nous définissons alors $\mathcal{L}^{\mathrm{d}}$ la version discrète de $\mathcal{L}$ suivante :

$$\mathcal{L}^d(q) = \sum_{k=0}^{N-1} (t_{k+1} - t_k) L(q_k, (\Delta q)_k, t_k),$$

où $(\Delta q)_k = \frac{q_{k+1} - q_k}{t_{k+1} - t_k}$. Par un calcul des variations discret, les points critiques de $\mathcal{L}^{\mathrm{d}}$ sont caractérisés par l'équation d'Euler-Lagrange discrète suivante :

$$\nabla \left[\frac{\partial L}{\partial v}(q, \Delta q, \cdot)\right]_k = \frac{t_{k+1} - t_k}{t_k - t_{k-1}} \frac{\partial L}{\partial x}(q_k, (\Delta q)_k, t_k),$$

où $(\nabla q)_k = \frac{q_k - q_{k-1}}{t_k - t_{k-1}}$. Ce schéma numérique est donc un intégrateur variationnel qui préserve la structure variationnelle de l'équation d'Euler-Lagrange historique dans le sens que ses solutions sont les points critiques de la version discrète $\mathcal{L}^d$ de $\mathcal{L}$. Il est d'ailleurs démontré dans [101] que cet intégrateur variationnel permet de préserver au niveau discret le théorème de Noether dans le sens suivant : si le Lagrangien L est invariant (au sens discret) par l'action d'un groupe à un paramètre Φ de transformations infinitésimales de $\mathbb{R}^n$, alors l'équation d'Euler-Lagrange discrète ci-dessus admet une constante de mouvement donnée par

$$\frac{\partial L}{\partial v}(q_k, (\Delta q)_k, t_k) \cdot \frac{\partial \Phi}{\partial \theta}(0, q_{k+1}) = c,$$

pour toute solution q. Notons en particulier le *shift* qui apparaît dans le théorème de Noether discret, *i.e.* l'émergence du terme q_{k+1}.

• **Calcul time scale.** Le calcul time scale a été introduit dans la thèse [111] de S. Hilger en 1988. Il vise à faire le lien entre analyse continue et analyse discrète. L'idée générale est de démontrer un résultat pour un système dynamique posé sur un sous-ensemble non vide et fermé arbitraire $\mathbb{T}$ de $\mathbb{R}$, $\mathbb{T}$ étant appelé *time scale*. Ainsi, en choisissant par exemple $\mathbb{T} = [a, b]$, le résultat général correspond à un résultat de l'analyse continue tandis que le choix $\mathbb{T} = \{a = t_0 < \ldots < t_N = b\}$ conduit à son pendant discret. Notons cependant qu'il existe bien d'autres time scales comme $\mathbb{N}$, $\mathbb{Z}$ ou $\mathbb{R}$ lui-même. On peut également considérer des time scales combinant structures continues et discrètes comme $[0, 1] \cup \mathbb{N}$ ou même des ensembles plus complexes tel que l'ensemble de Cantor.

L'étude de systèmes dynamiques posés sur time scale a été possible grâce à l'introduction d'opérateurs d'intégration et de dérivation qui coïncident avec les opérateurs standards du calcul continu et du calcul discret. Plus précisément, les notions de Δ- et de ∇-dérivée (définies dans [111]) correspondent

- à la dérivée classique d/dt dans le cas où le time scale est continu, *i.e.* $q^\Delta = q^\nabla = \frac{dq}{dt} = \dot{q}$;
- aux opérateurs de dérivation discrète (définis dans le paragraphe précédent) dans le cas où le time scale est discret, *i.e.* $q^\Delta(t_k) = \Delta q(t_k) = \frac{q(t_{k+1}) - q(t_k)}{t_{k+1} - t_k}$ et $q^\nabla(t_k) = \nabla q(t_k) = \frac{q(t_k) - q(t_{k-1})}{t_k - t_{k-1}}$.

Les notions d'intégrales correspondantes ont également été introduites dans [111] où $\int \Delta\tau$ et $\int \nabla\tau$ coïncident avec l'intégrale usuelle $\int d\tau$ dans le cas où le time scale est continu et valent $\int q(\tau)\Delta\tau = \sum_{k=0}^{N-1}(t_{k+1} - t_k)q(t_k)$ et $\int q(\tau)\nabla\tau = \sum_{k=1}^{N}(t_k - t_{k-1})q(t_k)$ dans le cas où le time scale est discret.

À partir de ces éléments de base, beaucoup de propriétés et de résultats classiques de l'analyse continue et de l'analyse discrète ont été étendus au cas time scale. Nous pouvons citer par exemple des règles de calcul de base comme la formule de Leibniz mais aussi des théories plus fines comme la mesure de Lebesgue et la notion d'intégration qui en découle. Nous renvoyons aux ouvrages [2, 3, 38, 39] de M. Bohner *et al.* pour une étude approfondie du calcul time scale.

Notons que les résultats démontrés sur time scale permettent en particulier une meilleure traduction des discordances qui existent entre un résultat continu et son homologue discret. Notamment, nous verrons que le développement du calcul des variations sur time scale permet de mieux comprendre l'émergence du shift q_{k+1} dans le théorème de Noether discret rappelé au paragraphe précédent.

• **Calcul des variations sur time scale.** L'article pionnier traitant du calcul des variations sur time scale est dû à M. Bohner en 2004 (voir [36]). Il a en particulier caractérisé les points critiques d'une fonctionnelle Lagrangienne de la forme

$$\mathcal{L}(q) = \int_a^b L(q^\sigma(\tau), q^\Delta(\tau), \tau)\ \Delta\tau$$

par l'équation d'Euler-Lagrange suivante :

$$\left[\frac{\partial L}{\partial v}(q^\sigma, q^\Delta, \cdot)\right]^\Delta = \frac{\partial L}{\partial x}(q^\sigma, q^\Delta, \cdot),$$

où $q^\sigma = q \circ \sigma$, σ étant l'opérateur de *saut en avant* associé au time scale $\mathbb{T}$. En particulier, si $\mathbb{T} = [a, b]$, alors $\sigma(t) = t$ et on retrouve le résultat classique d'Euler et de Lagrange. Dans le cas discret $\mathbb{T} = \{a = t_0 < \ldots <$

$t_N = b\}$, on a $\sigma(t_k) = t_{k+1}$ et on retrouve la caractérisation (démontrée dans [11]) des points critiques d'une fonctionnelle Lagrangienne discrète de la forme

$$\mathcal{L}(q) = \sum_{k=0}^{N-1} (t_{k+1} - t_k) L(q(t_{k+1}), \Delta q(t_k), t_k)$$

par l'équation d'Euler-Lagrange discrète suivante :

$$\Delta \left[\frac{\partial L}{\partial v}(q^\sigma, \Delta q, \cdot) \right] (t_k) = \frac{\partial L}{\partial x}(q(t_{k+1}), \Delta q(t_k), t_k).$$

Depuis la publication de [36], de nombreux travaux ont été consacrés au calcul des variations sur time scale, voir par exemple [37, 87, 115, 156, 160]. En particulier, Z. Bartosiewicz et D. Torres [31] ont étendu en 2008 le théorème de Noether au cas time scale. Ils démontrent que si le Lagrangien L est invariant (au sens time scale) par un groupe à un paramètre Φ de transformations infinitésimales de $\mathbb{R}^n$, alors l'équation d'Euler-Lagrange ci-dessus admet une constante de mouvement donnée par

$$\frac{\partial L}{\partial v}(q^\sigma, \Delta q, \cdot) \cdot \frac{\partial \Phi}{\partial \theta}(0, q) = c,$$

pour toute solution q.

Cependant, le cadre proposé par M. Bohner dans [36], que l'on appellera *shifté* en raison de la présence du terme q^σ au lieu de q dans la définition de $\mathcal{L}$, ne couvre pas l'équation d'Euler-Lagrange discrète rappelée dans le paragraphe "Intégrateurs variationnels". Pour couvrir cette dernière, il nous faut considérer la fonctionnelle Lagrangienne *non shiftée* donnée par

$$\mathcal{L}(q) = \int_a^b L(q(\tau), q^\Delta(\tau), \tau) \ \Delta\tau.$$

Ce type de fonctionnelle a été étudiée par R. Hilscher *et al* dans [114, 117] en 2009 et par D. Torres *et al* dans [72, 86] en 2011-12. L'équation d'Euler-Lagrange associée s'écrit alors sous la forme intégrale suivante (voir [114, Théorème 4]) :

$$\frac{\partial L}{\partial v}(q(t), q^\Delta(t), t) = \int_a^{\sigma(t)} \frac{\partial L}{\partial x}(q(\tau), q^\Delta(\tau), \tau) \ \Delta\tau + c.$$

Dans le cas continu, en appliquant l'opérateur d/dt, nous retrouvons le résultat classique d'Euler et de Lagrange. Dans le cas discret, en appliquant l'opérateur ∇, nous retrouvons l'équation d'Euler-Lagrange discrète rappelée dans le paragraphe "Intégrateurs variationnels". La question suivante se pose alors :

Question 1 : *Peut-on ∇-dériver l'équation d'Euler-Lagrange intégrale ci-dessus directement dans le cadre time scale ? Si non, puisque cela est possible dans les cas continu et discret, sur quel type de time scale une telle opération est-elle possible ?*

Nous répondons à ce problème dans les **Chapitres I** et **II** de ce manuscrit. Plus précisément, nous démontrons en premier lieu la formule générale $(q^\sigma)^\nabla(t) = \sigma^\nabla(t) q^\Delta(t)$, valable sous les conditions que l'opérateur σ soit ∇-dérivable et que la fonction q soit Δ-dérivable. Nous en déduisons que si σ est ∇-dérivable, alors nous pouvons ∇-dériver l'équation d'Euler-Lagrange intégrale ci-dessus et obtenir la version *différentielle* suivante :

$$\left[\frac{\partial L}{\partial v}(q, q^\Delta, \cdot) \right]^\nabla (t) = \sigma^\nabla(t) \frac{\partial L}{\partial x}(q(t), q^\Delta(t), t).$$

Dans le cas continu $\mathbb{T} = [a, b]$ (resp. discret $\mathbb{T} = \{a = t_0 < \ldots < t_N = b\}$), l'opérateur σ est ∇-dérivable avec $\sigma^\nabla = 1$ (resp. $\sigma^\nabla(t_k) = \frac{t_{k+1} - t_k}{t_k - t_{k-1}}$) et on retrouve l'équation d'Euler-Lagrange classique du cas continu (resp. discret). Notre résultat répond donc au problème dans le cas d'un time scale vérifiant la ∇-dérivabilité de l'opérateur σ associé. De plus, à l'aide du simple contre-exemple unidimensionnel $L(x, v, t) = x + v^2/2$, il est facile de voir que cette condition est aussi nécessaire. Finalement, nous avons entièrement répondu à la

Question 1. Notons qu'une étude sur la ∇-dérivabilité de l'opérateur σ et ses conséquences sur la structure du time scale, accompagnée d'exemples et de contre-exemples, est proposée en Chapitre I.

Difficultés rencontrées : Comme l'opération de ∇-dérivation de l'équation d'Euler-Lagrange intégrale est directement valable dans les cas continu et discret, la première étape de ce travail a été de comprendre les structures de time scale qui pouvaient faire défaut à cette procédure. La principale difficulté a en fait été d'identifier en termes simples l'ensemble des conditions nécessaires que devait satisfaire le time scale. Finalement, nous avons réduit cet ensemble à la seule condition de ∇-dérivabilité de l'opérateur σ.

L'obtention d'un théorème de type Noether nécessite une équation d'Euler-Lagrange sous forme différentielle. Ainsi, la forme intégrale (donnée par exemple dans [114, Théorème 4]) ne permet pas l'énoncé d'un tel résultat dans le cadre time scale non shifté. La question suivante découle alors de la résolution de la Question 1 :

Question 2 : *Dans le cas où l'opérateur σ est ∇-dérivable, l'équation d'Euler-Lagrange différentielle ci-dessus permet-elle la formulation d'un théorème de type Noether ?*

Une section du **Chapitre II** répond entièrement à cette question. En effet, nous nous plaçons dans le cas où l'opérateur σ est ∇-dérivable et nous démontrons dans un premier temps la formule de Leibniz suivante :

$$(q^\sigma \cdot q')^\nabla = q \cdot q'^\nabla + \sigma^\nabla q^\Delta \cdot q'.$$

Nous en déduisons le théorème de type Noether qui suit : si L est invariant (au sens time scale) par l'action d'un groupe à un paramètre Φ de transformations infinitésimales de $\mathbb{R}^n$, alors l'équation d'Euler-Lagrange *différentielle* obtenue précédemment admet une constante de mouvement donnée par

$$\frac{\partial L}{\partial v}(q, q^\Delta, \cdot) \cdot \frac{\partial \Phi}{\partial \theta}(0, q^\sigma) = c,$$

pour toute solution q. Ce résultat englobe en particulier les théorèmes classiques de Noether obtenus dans les cas continu et discret non shifté. Notons l'émergence du terme q^σ dans la constante de mouvement ci-dessus qui est en adéquation avec l'apparition du terme q_{k+1} dans le cas discret.

Pour compléter ce paragraphe, nous précisons qu'une équation d'Euler-Lagrange sous forme *différentielle* avait déjà été obtenue dans le cadre time scale non shifté, voir [114, Remarque 4]. En effet, à partir de l'équation d'Euler-Lagrange intégrale, l'auteur avait établi la forme différentielle suivante :

$$\left(\frac{\partial L}{\partial v}(q, q^\Delta, \cdot) - \mu \frac{\partial L}{\partial x}(q, q^\Delta, \cdot) \right)^\Delta = \frac{\partial L}{\partial x}(q, q^\Delta, \cdot).$$

Cette formulation présente notamment l'avantage d'être valable sur tout time scale. Cependant, contrairement à la formulation différentielle obtenue plus haut, il semble que celle-ci ne permette pas d'établir un théorème de type Noether. Cette observation donne une justification supplémentaire à la ∇-dérivation de l'équation d'Euler-Lagrange intégrale.

• **Théorie du contrôle optimal sur time scale.** La théorie du contrôle optimal est consacrée à l'étude de systèmes dynamiques contrôlés où l'objectif est de conduire de tels systèmes d'une configuration donnée à une configuration souhaitée tout en minimisant un certain critère. L'obtention d'un principe du maximum par L. Pontryagin *et al.* [40] dans les années 1950 fut une étape très importante pour cette théorie. Ce résultat a encore aujourd'hui un champ large d'applications dans de nombreux domaines. Nous renvoyons aux ouvrages [5, 40, 43, 44, 54, 56, 57, 107, 130, 147, 191, 192, 197] pour une étude approfondie de la théorie du contrôle optimal et de ses applications, essentiellement dans le cas continu.

Le Principe du Maximum de Pontryagin (PMP en abrégé) généralise considérablement le résultat classique d'Euler et de Lagrange formulé deux siècles plus tôt. Plus précisément, considérons le système controlé suivant

$$\dot{q} = f(q, u, t),$$

où $f : \mathbb{R}^n \times \mathbb{R}^m \times [a,b] \longrightarrow \mathbb{R}^n$ et u désigne le contrôle sur le système. Le PMP vise à donner une condition nécessaire sur une trajectoire $q^* : [a,b] \longrightarrow \mathbb{R}^n$, solution du système contrôlé associée à un contrôle $u^* : [a,b] \longrightarrow \mathbb{R}^m$, pour qu'elle minimise le coût

$$\int_a^b f^0(q(\tau), u(\tau), \tau) \, d\tau,$$

où $f^0 : \mathbb{R}^n \times \mathbb{R}^m \times [a,b] \longrightarrow \mathbb{R}^n$, parmi toutes les trajectoires $q : [a,b] \longrightarrow \mathbb{R}^n$ solutions du système contrôlé, chacune associée à un contrôle $u : [a,b] \longrightarrow \mathbb{R}^m$. En introduisant le Hamiltonien $H(x,v,w,w^0,t) = w \cdot f(x,v,t) + w^0 f^0(x,v,t)$, la condition nécessaire donnée par le PMP est l'existence d'un réel $p^0 < 0$ et d'une fonction $p : [a,b] \longrightarrow \mathbb{R}^n$ (appelée vecteur adjoint) satisfaisant $p(a) = p(b) = 0$ tels que q^* et p soient solutions du système Hamiltonien

$$\dot{q}^* = \frac{\partial H}{\partial w}(q^*, u^*, p, p^0, t), \qquad \dot{p} = -\frac{\partial H}{\partial x}(q^*, u^*, p, p^0, t)$$

et tels que la condition de maximisation qui suit soit satisfaite :

$$H(q^*(t), u^*(t), p(t), p^0, t) = \max_{v \in \mathbb{R}^n} H(q^*(t), v, p(t), p^0, t).$$

Rappelons que la condition précédente implique l'égalité

$$\frac{\partial H}{\partial v}(q^*(t), u^*(t), p(t), p^0, t) = 0.$$

Cette dernière condition est décrite en termes de point critique du Hamiltonien (et non plus en termes de maximisation du Hamiltonien) : elle est dite *plus faible*. Dans la littérature, l'obtention de cette seconde condition (plus faible mais également nécessaire) est communément appelée *version faible du PMP*.

Comme pour le calcul des variations, la théorie discrète correspondante a déjà été étudiée et développée. Nous renvoyons aux ouvrages [41, 61, 102, 124, 125, 171, 192] pour une étude approfondie et des applications. Cependant, nous rappelons que les premières études sur le sujet (comme [85]) se sont avérées mathématiquement incorrectes et que l'analogue exact du PMP au niveau discret est faux. Plus précisément, dans un cadre général discret, la condition de maximisation du Hamiltonien n'est plus vérifiée. De nombreux contre-exemples sont fournis dans [41, 171]. Cependant, le PMP faible est quant à lui toujours satisfait, voir [41, Théorème 42.1 p.330]. Notons que dans certains cadres, la condition de maximisation du Hamiltonien peut être récupérée, par exemple pour un Hamiltonien concave en la variable v. En fait, une littérature assez conséquente s'intéresse à la détermination d'une condition suffisante (la moins restrictive possible) permettant de récupérer la condition de maximisation du Hamiltonien dans le cas discret. La notion la plus courante est celle de *convexité directionnelle* imposée sur la dynamique (f, f^0) dans [102, 124, 125].

Le PMP, dans un cadre général, est donc un résultat où la distinction entre le cas continu et le cas discret est relativement forte. Bien que plusieurs études s'intéressent à la théorie du contrôle sur time scale (voir [29, 30, 119, 182, 207]), rares sont les tentatives d'obtention d'un PMP dans ce cadre. R. Hilsher et V. Zeidan sont les premiers à avoir abordé cette question : ils démontrent en 2009 une version faible du PMP sur time scale dans l'article [117]. En 2012, Z. Zhan *et al.* proposent de démontrer une version forte du PMP sous l'hypothèse de convexité directionnelle sur la dynamique (f, f^0) afin d'obtenir une condition de maximisation du Hamiltonien sur tout le time scale, voir [205]. Cependant, la démonstration de leur résultat contient plusieurs erreurs qui sont, à notre avis, rédhibitoires, voir [113] ou [53, Remarque 13] pour plus de détails. Ainsi, la question suivante reste largement ouverte :

Question 3 : *Dans un cadre général (sans hypothèse supplémentaire sur la dynamique), peut-on obtenir un PMP sur time scale recouvrant parfaitement les cas continu et discret ? Autrement dit, peut-on démontrer un tel résultat qui donne la condition de maximisation du Hamiltonien dans le cas continu et la condition plus faible en termes de point critique du Hamiltonien dans le cas discret ?*

Nous répondons à cette question dans le **Chapitre III**. Plus précisément, nous démontrons une version forte du PMP pour un problème de contrôle optimal posé sur time scale. Nous considérons le système dynamique contrôlé

$$q^\Delta = f(q, u, t)$$

où $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{T} \longrightarrow \mathbb{R}^n$, $t \in \mathbb{T}$ où $\mathbb{T}$ est un time scale borné avec $a = \min \mathbb{T}$ et $b = \max \mathbb{T}$ et u désigne le contrôle sur le système. Notre objectif est alors de déterminer une condition nécessaire sur une trajectoire $q^* : \mathbb{T} \longrightarrow \mathbb{R}^n$, solution du système contrôlé associée à un contrôle $u^* : \mathbb{T} \longrightarrow \mathbb{R}^m$, pour qu'elle minimise le coût

$$\int_a^b f^0(q(\tau), u(\tau), \tau)\, \Delta\tau,$$

où $f^0 : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{T} \longrightarrow \mathbb{R}^n$, parmi toutes les trajectoires $q : \mathbb{T} \longrightarrow \mathbb{R}^n$ solutions du système contrôlé, chacune associée à un contrôle $u : \mathbb{T} \longrightarrow \mathbb{R}^m$. Aucune hypothèse n'est imposée sur la dynamique (f, f^0). Afin d'établir un PMP capable d'englober les cas classiques continu et discret, dont les énoncés diffèrent sensiblement, nous séparons les points du time scale en deux catégories : les points isolés à droite (appelés *right-scattered*) et les points non isolés à droite (appelés *right-dense*). Dans le cas continu (resp. discret), tout les points du time scale sont right-dense (resp. right-scattered). En introduisant le Hamiltonien $H(x, v, w, w^0, t) = w \cdot f(x, v, t) + w^0 f^0(x, v, t)$, le résultat principal du Chapitre III est un PMP dont la condition nécessaire est l'existence d'un réel $p^0 < 0$ et d'une fonction $p : \mathbb{T} \longrightarrow \mathbb{R}^n$ (appelée vecteur adjoint) satisfaisant $p(a) = p(b) = 0$ tels que q^* et p soient solutions du système Hamiltonien

$$q^{*\Delta} = \frac{\partial H}{\partial w}(q^*, u^*, p^\sigma, p^0, t), \qquad p^\Delta = -\frac{\partial H}{\partial x}(q^*, u^*, p^\sigma, p^0, t),$$

tels que la condition de maximisation suivante soit satisfaite en (presque) tout point right-dense t :

$$H(q^*(t), u^*(t), p(t), p^0, t) = \max_{v \in \mathbb{R}^n} H(q^*(t), v, p(t), p^0, t)$$

et tels que la condition (plus faible) qui suit soit satisfaite en tout point right-scattered t :

$$\frac{\partial H}{\partial v}(q^*(t), u^*(t), p^\sigma(t), p^0, t) = 0.$$

Ce résultat, valable sur tout time scale, englobe en particulier les énoncés classiques des PMPs obtenus dans les cas continu et discret.

Difficultés rencontrées et méthodes utilisées : Démontrer un PMP requiert l'étude de variations faites sur le contrôle optimal u^* qui est considéré dans L^∞ pour prendre en compte d'éventuelles discontinuités. Cependant, pour faire de telles variations sur le contrôle $u^* \in \mathrm{L}^\infty$ et préserver son admissibilité (*i.e.* que la trajectoire associée soit toujours définie au temps final $t = b$), une théorie générale de Cauchy-Lipschitz complète est nécessaire. Elle doit assurer l'existence et l'unicité d'une solution absolument continue maximale et doit être accompagnée d'un théorème de *sortie de tout compact* en cas de non globalité de cette solution maximale. Bien que des théorèmes de type Cauchy-Lipschitz sur time scale aient déjà été traités dans la littérature (voir *e.g.* [38, 64, 112, 134, 145, 146]), les résultats proposés ne couvrent pas les besoins susmentionnés. La première étape de notre travail a donc été d'adapter la théorie générale de Cauchy-Lipschitz au cas time scale (voir Annexe A). Pour cela, les méthodes classiques ont été revisitées, de l'utilisation d'un théorème de point fixe pour montrer l'existence et l'unicité d'une solution locale jusqu'à l'application du lemme de Zorn pour étendre cette solution locale en une solution maximale. Des conditions adéquates sur la dynamique en termes de continuité localement Lipschitzienne et de régressivité sont requises. Pour finir, on rappelle que l'introduction du vecteur adjoint p dans le PMP conduit à la considération d'un problème de Cauchy *backward* et *shifté*. La théorie de Cauchy-Lipschitz proposée en Annexe A étudie donc des problèmes de Cauchy sur time scale qui sont *forward* et/ou *backward* et *shiftés* ou non.

Comme dans la démonstration historique de L. Pontryagin *et al.* dans [40], nous introduisons dans le Chapitre III des variations de type *aiguille* sur le contrôle optimal u^*. En un point s right-dense, la méthode peut être adaptée sans difficulté majeure : il s'agit de remplacer sur l'intervalle $[s, s + \beta[\cap \mathbb{T}$ la valeur de u^* par une valeur fixe $v \in \mathbb{R}^m$ quelconque et de faire tendre β vers 0. Cette procédure est une variation de type L^1 et elle est rendue possible grâce au fait que s soit un point right-dense et donc non isolé à droite. En revanche, ce type de variation n'est plus possible en un point right-scattered r (car isolé à droite) pour lequel nous envisageons une variation de type L^∞ en remplaçant la valeur de $u(r)$ par $u(r) + \alpha(v - u(r))$, où $v \in \mathbb{R}^m$ quelconque, et en faisant tendre α vers 0. Le fait qu'une variation de type L^∞ soit plus grossière qu'une variation de type L^1 explique pourquoi le résultat final est plus fin en un point right-dense qu'en un point right-scattered.

L'ensemble de ces variations-aiguilles conduit à la construction du cône dit de Pontryagin. Dans la démonstration historique du PMP, l'optimalité de la trajectoire q^* permet, à travers l'application d'un théorème de point fixe de Brouwer (ou similairement d'un théorème d'inversion locale conique), de conclure que ce cône n'est pas égal à $\mathbb{R}^{n+1}$ tout entier et permet donc la construction d'un vecteur orthogonal dont $p^0 < 0$ est la dernière coordonnée. Notre première tentative a été d'adapter cette méthode au cas time scale. Cependant, de sérieuses obstructions liées à la structure générale du time scale (qui peut être complexe comme l'ensemble de Cantor) émergent au niveau de l'application du théorème de point fixe qui requiert la convexité de l'ensemble de définition des paramètres α et β. En effet, un point s right-dense n'admet pas nécessairement d'intervalle à sa droite et donc le paramètre β ne vit pas forcément dans un ensemble convexe. De même, en imposant des contraintes sur la valeur des contrôles u, il n'est pas assuré de pouvoir faire vivre le paramètre α dans un tel convexe. Il semble que ces difficultés ne peuvent être surmontées et cette observation nous a finalement conduits à considérer une toute autre méthode basée sur le principe variationnel d'Ekeland. Cette méthode, introduite par I. Ekeland en 1974 (voir [82]), permet la démonstration d'un PMP approché, noté PMP_ε, sans utiliser de théorème de point fixe. Le PMP exact est ensuite obtenu par passage à la limite. Cette méthode avait d'ores et déjà été appliquée avec succès pour généraliser le PMP au cas de la dimension infinie (ou plus généralement quelconque, voir *e.g.* [149]). Dans le Chapitre III, elle l'est également pour la généralisation du PMP au cas time scale.

Remarque : Pour des raisons de clarté de cette introduction, nous avons volontairement simplifié le problème de contrôle optimal considéré. En réalité, le cadre proposé au Chapitre III suggère :

- des contraintes sur les valeurs du contrôle u. Les conditions obtenues sur le Hamiltonien ci-dessus s'en trouvent légèrement modifiées ;
- des contraintes sur les valeurs initiale et finale des trajectoires q. Des conditions dites de transversalité sont alors associées ;
- le temps final peut être considéré libre. La condition de transversalité correspondante est également établie.

Enfin, une version paramétrée du PMP sur time scale est également proposée.

• **Problèmes inverses de Helmholtz continu et discret.** Un problème inverse bien connu de l'analyse continue est celui formulé par H. Helmholtz [106] en 1887 : trouver une condition nécessaire et suffisante pour laquelle une EDO (ou EDP) est une équation d'Euler-Lagrange et, dans ce cas, trouver toutes les formulations Lagrangiennes possibles. Ce problème a été étudié par de nombreux auteurs et a été complétement résolu par A. Mayer [164] et A. Hirsch [121, 122] une décennie plus tard. La formulation qui suit est due à V. Volterra [200]. Une équation différentielle $O(q) = 0$ est une équation d'Euler-Lagrange si et seulement si les différentielles de O sont toutes auto-adjointes. Nous renvoyons à [180] pour une démonstration moderne et complète de ce résultat.

Ce problème a également été traité au niveau discret par plusieurs auteurs. Nous renvoyons par exemple aux études de D. Opris *et al.* dans [13, 68] et au travail de P. Hydon et de E. Mansfield dans [126]. Le problème traité est alors le suivant : trouver une condition nécessaire et suffisante pour laquelle une équation discrète est une équation d'Euler-Lagrange discrète et, dans ce cas, trouver toutes les formulations Lagrangiennes discrètes possibles. La formulation de V. Volterra a ainsi été adaptée au cas discret.

Cependant, le cadre discret traité dans ces études est très général et le problème n'est pas formulé avec un point de vue "intégrateur variationnel". En effet, un intégrateur variationnel est, par construction, une équation d'Euler-Lagrange discrète (puisqu'il provient d'un calcul des variations discret) mais l'inverse n'est pas vrai : une équation d'Euler-Lagrange discrète n'admet pas forcément une formulation Lagrangienne qui correspond à la discrétisation d'une fonctionnelle Lagrangienne continue. Une discussion sur ce sujet peut être trouvée dans [126, Section 5.3 p.213]. Le problème suivant est donc à notre connaissance encore ouvert :

QUESTION 4 : *Peut-on déterminer une condition nécessaire et suffisante pour laquelle une équation discrète est un intégrateur variationnel, c'est-à-dire un schéma numérique associé à une équation d'Euler-Lagrange continue construit à partir d'un calcul des variations discret ? Et, dans ce cas, peut-on donner toutes les formulations Lagrangiennes discrètes possibles ?*

Dans le **Chapitre IV**, nous répondons *partiellement* à ce problème. Plus précisément, notre résultat principal affirme qu'une équation aux différences finies de type

$$\overline{P}(q(t_p), \Delta q(t_p), \nabla q(t_p), \nabla \circ \Delta q(t_p), t_p, h) = 0$$

est une équation d'Euler-Lagrange discrète de la forme

$$\frac{\partial L_1}{\partial x}(q(t_p), \Delta q(t_p), t_p, h) + \frac{\partial L_2}{\partial x}(q(t_p), \nabla q(t_p), t_p, h) - \nabla \left(\frac{\partial L_1}{\partial v}(q, \Delta q, \cdot, h) \right)(t_p) - \Delta \left(\frac{\partial L_2}{\partial v}(q, \nabla q, \cdot, h) \right)(t_p) = 0,$$

qui caractérise les points critiques de la fonctionnelle Lagrangienne discrète donnée par

$$\mathcal{L}^{\mathrm{d}}(q) = h \sum_{p=0}^{N-1} L_1(q(t_p), \Delta q(t_p), t_p, h) + h \sum_{p=1}^{N} L_2(q(t_p), \nabla q(t_p), t_p, h)$$

si et seulement si $\overline{P}$ satisfait la condition de Helmholtz discrète suivante :

$$\Delta \left(\frac{\partial \overline{P}}{\partial u}(q, \Delta q, \nabla q, \nabla \circ \Delta q, \cdot, h) \right)(t_p) = \frac{\partial \overline{P}}{\partial v_1}(q(t_p), \Delta q(t_p), \nabla q(t_p), \nabla \circ \Delta q(t_p), t_p, h) + \frac{\partial \overline{P}}{\partial v_2}(q(t_{p+1}), \Delta q(t_{p+1}), \nabla q(t_{p+1}), \nabla \circ \Delta q(t_{p+1}), t_{p+1}, h).$$

Dans un tel cas, la démonstration de cet énoncé présente une méthode de construction explicite d'un couple de Lagrangien (L_1, L_2) associé.

Cependant, du fait de la dépendance en h des Lagrangiens L_1 et L_2, nous n'avons pas la garantie que la fonctionnelle Lagrangienne $\mathcal{L}^{\mathrm{d}}$ corresponde à la discrétisation d'une fonctionnelle Lagrangienne continue et donc que l'équation d'Euler-Lagrange discrète associée soit un intégrateur variationnel. En conséquence, nous proposons dans ce chapitre une description de la classe des formulations Lagrangiennes discrètes nulles, c'est-à-dire des couples de Lagrangiens dont l'équation d'Euler-Lagrange discrète associée est nulle. Par linéarité, ce résultat permet, pour un couple de Lagrangien (L_1, L_2) donné, de choisir, parmi une classe de couples (L'_1, L'_2) menant à la même équation d'Euler-Lagrange discrète, un couple particulier (s'il existe) tel que la fonctionnelle Lagrangienne discrète associée corresponde à la discrétisation d'une fonctionnelle Lagrangienne continue et donc que l'équation d'Euler-Lagrange discrète associée soit un intégrateur variationnel.

Ainsi, le problème soulevé en Question 4 reste encore ouvert puisque la condition de Helmholtz discrète donnée dans le Chapitre IV ne caractérise pas directement les équations aux différences finies qui sont des intégrateurs variationnels. En effet, comme mentionné ci-dessus, la recherche (dont le résultat n'est pas garanti) d'une formulation Lagrangienne discrète correspondant à la discrétisation d'une formulation Lagrangienne continue peut s'avérer nécessaire.

Difficultés rencontrées et méthodes utilisées : Notre stratégie a été d'adapter au cas discret une démonstration classique du théorème de Helmholtz. La difficulté principale a finalement été de déterminer un cadre et une démonstration du cas continu qui pouvaient être sujets à une telle adaptation. C'est pourquoi nous consacrons une section entière au cas continu dans le Chapitre IV. La démonstration de notre résultat principal est au final, comme dans le cas continu, une adaptation de la démonstration du théorème classique de Poincaré au cas du calcul des variations discret.

La principale obstruction à la résolution du problème ouvert en Question 4 est que les intégrateurs variationnels dépendent naturellement du pas de discrétisation h. En conséquence, les équations aux différences finies considérées doivent également prendre en compte cette dépendance. Finalement, en adaptant la démonstration de Poincaré au cas du calcul des variations discret, nous sommes amenés à construire des Lagrangiens qui, à leur tour, dépendent de h. Comme expliqué ci-dessus, il n'est alors pas assuré que les formulations Lagrangiennes discrètes obtenues correspondent à des discrétisations de formulations Lagrangiennes continues.

Partie B

Dans cette section, nous donnons quelques éléments bibliographiques menant à l'étude de problèmes variationnels fractionnaires. Nous formulons à cette occasion quelques questions qui n'ont pas encore été traitées dans ce domaine et qui ont motivé les études de la Partie B de ce manuscrit. Nous présentons alors succinctement les réponses obtenues dans ce mémoire et nous proposons quelques détails sur les difficultés rencontrées et sur les méthodes utilisées.

• **Calcul fractionnaire.** Le calcul fractionnaire traite de l'extension des notions classiques d'intégrale et de dérivée d'ordre entier non nul à tout ordre réel strictement positif. La première trace historique traitant de ce sujet nous provient d'une lettre écrite par Leibniz et destinée à L'Hospital en 1695 où l'auteur s'interroge sur l'extension de sa célèbre formule de dérivation d'un produit à des ordres quelconques. Depuis, de nombreux mathématiciens (comme Euler, Fourier, Hadamard, Liouville, Riemann pour ne citer que les plus renommés) ont investi ce champ de recherche. Plusieurs définitions d'intégrales et de dérivées fractionnaires ont ainsi été introduites. Dans ce manuscrit, nous n'utilisons que les notions les plus répandues dans la littérature, à savoir :

– les intégrales fractionnaires (à gauche et à droite) de Riemann-Liouville notées I^{α}_{a+} et I^{α}_{b-} ;
– les dérivées fractionnaires (à gauche et à droite) de Riemann-Liouville notées D^{α}_{a+} et D^{α}_{b-} ;
– les dérivées fractionnaires (à gauche et à droite) de Caputo notées ${}_{c}\mathrm{D}^{\alpha}_{a+}$ et ${}_{c}\mathrm{D}^{\alpha}_{b-}$;

et nous considérons essentiellement des ordres fractionnaires α satisfaisant $0 < \alpha < 1$.

Pendant longtemps, le calcul fractionnaire a été seulement considéré comme une branche purement mathématique. Cependant, depuis quelques décennies, ses applications ont connu un essor dans des domaines scientifiques nombreux et variés : économie [67], biologie [96, 155], acoustique [105], thermodynamique [109], probabilité [148], etc. Son utilisation est particulièrement fructueuse dans des problèmes physiques de diffusion anormale, voir [167, 179, 201, 208, 209]. Par ailleurs, le caractère non local des opérateurs fractionnaires les rend également utiles dans la prise en compte d'effets de mémoire, voir par exemple [24, 25, 81, 183] pour la modélisation de phénomènes viscoélastiques. Nous renvoyons aux monographies [141, 190] pour une étude approfondie du calcul fractionnaire et de ses applications.

• **Calcul des variations fractionnaire.** D'après la condition de Helmholtz, l'équation de l'oscillateur harmonique *avec friction* ne possède pas de structure Lagrangienne, c'est-à-dire qu'il n'est pas possible de l'écrire comme une équation d'Euler-Lagrange. D'un point de vue plus général, F. Riewe soulève la problématique suivante en 1996-97 dans [187, 188] :

« *It is a strange paradox that the most advanced methods of classical mechanics deal only with conservative systems, while almost all classical processes observed in the physical world are nonconservative.* »

L'auteur suggère alors d'incorporer les opérateurs fractionnaires dans le calcul des variations afin de fournir des structures variationnelles *fractionnaires* à des systèmes dissipatifs qui n'en admettent pas de classique. Il se concentre en particulier sur l'obtention d'une structure Lagrangienne fractionnaire pour une équation de friction linéaire du second ordre. Bien que son idée soit pertinente, ses résultats ne sont pas totalement satisfaisants puisque l'objectif initial n'est pas atteint. Néanmoins, dans le même esprit, J. Cresson et P. Inizan proposent en 2012 un cadre variationnel fractionnaire similaire mais plus concluant, voir [71]. En effet, ce cadre, appelé *assymétrique* car basé sur le dédoublement de la variable de la fonctionnelle Lagrangienne, permet par exemple d'obtenir une formulation variationnelle fractionnaire pour l'équation de convection-diffusion, voir [70].

Depuis l'étude de F. Riewe, une littérature conséquente a été consacrée à l'obtention de conditions nécessaires d'optimalité de type Euler-Lagrange pour des fonctionnelles Lagrangiennes de différentes formes et impliquant divers opérateurs fractionnaires, voir [6, 14, 16–19, 27, 69, 174–177]. Nous renvoyons au livre récent [157] de A. Malinowska et D. Torres pour un état de l'art du calcul des variations fractionnaire. Notons que plusieurs articles traitent aussi l'extension du théorème de Noether au cas fractionnaire, voir [21, 69, 89, 92, 172]. Cependant, dans chacune de ses études, une relation (fonctionnelle ou intégrale) est obtenue pour les équations d'Euler-Lagrange fractionnaires admettant une symétrie mais celle-ci ne permet pas d'assurer l'existence d'une constante de mouvement explicite, ce qui est à l'origine l'apport essentiel du théorème classique de Noether.

Pour rendre nos propos plus concrets, rappelons par exemple que les points critiques de la fonctionnelle Lagrangienne fractionnaire

$$\mathcal{L}(q) = \int_a^b L(q(\tau), {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q](\tau), \tau)\, d\tau$$

sont caractérisés par l'équation d'Euler-Lagrange fractionnaire qui suit, voir *e.g.* [16] :

$$\frac{\partial L}{\partial x}(q, {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q], \cdot) + \mathrm{D}^{\alpha}_{b-}\left[\frac{\partial L}{\partial v}(q, {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q], \cdot)\right] = 0.$$

Dans ce contexte, le théorème de type Noether qui suit a été simultanément obtenu en 2007 dans [69, 89] : si le Lagrangien L est invariant (au sens fractionnaire) par l'action d'un groupe à un paramètre Φ de transformations infinitésimales de $\mathbb{R}^n$, alors la relation fonctionnelle suivante est satisfaite :

$$\frac{\partial L}{\partial v}(q, {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q], \cdot) \cdot {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}\left[\frac{\partial \Phi}{\partial \theta}(0, q)\right] - \mathrm{D}^{\alpha}_{b-}\left[\frac{\partial L}{\partial v}(q, {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q], \cdot)\right] \cdot \frac{\partial \Phi}{\partial \theta}(0, q) = 0,$$

pour toute solution q de l'équation d'Euler-Lagrange fractionnaire ci-dessus. Dans le cas limite $\alpha = 1$, la simple formule de Leibniz permet de réécrire cette relation comme une dérivée classique valant 0 et une constante de mouvement est ainsi explicitement obtenue. La question suivante se pose alors naturellement :

Question 5 : *Dans le cas strictement fractionnaire* $0 < \alpha < 1$*, peut-on obtenir une constante de mouvement explicite à partir de cette relation fonctionnelle ?*

Une brève discussion autour d'un résultat que nous avons obtenu dans [48] et qui traite ce problème est proposée en **Chapitre V**. Plus précisément, nous avons proposé dans [48] une *formule de transfert* qui permet la réécriture de la relation fonctionnelle ci-dessus comme une dérivée classique valant 0. Dans ce cas, cette formule de transfert prend la forme suivante :

$$f \cdot {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[g] - \mathrm{D}^{\alpha}_{a+}[f] \cdot g = \frac{d}{dt}\left[\sum_{k \geq 0} (-1)^k \mathrm{I}^{k+1-\alpha}_{a+}[g - g(a)] \cdot f^{(k)} + g^{(k)} \cdot \mathrm{I}^{k+1-\alpha}_{b-}[f]\right].$$

En conséquence, sous certaines hypothèses de régularité des fonctions $f = \frac{\partial L}{\partial v}(q, {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q], \cdot)$ et $g = \frac{\partial \Phi}{\partial \theta}(0, q)$, nous obtenons la constante de mouvement explicite suivante :

$$\sum_{k \geq 0} (-1)^k \mathrm{I}^{k+1-\alpha}_{a+}[g - g(a)](t) \cdot f^{(k)}(t) + g^{(k)}(t) \cdot \mathrm{I}^{k+1-\alpha}_{b-}[f](t) = c.$$

Cependant, dû à son expression en série infinie d'intégrales fractionnaires, cette loi de conservation est très difficilement calculable même dans les cas les plus simples. Il est donc raisonnable de considérer que le problème soulevé en Question 5 est encore ouvert.

Les théorèmes d'existence d'optimiseurs sont des ingrédients essentiels à la résolution de problèmes variationnels. Cependant, les théorèmes d'existence de solutions pour des problèmes variationnels fractionnaires sont rares et ne sont réduits, à notre connaissance, qu'à l'étude de quelques cas particuliers, voir par exemple [129, 143]. La question suivante émerge alors :

Question 6 : *Peut-on formuler des conditions générales sur une fonctionnelle Lagrangienne fractionnaire assurant l'existence d'un minimiseur (et donc, par la même occasion, d'une solution pour l'équation d'Euler-Lagrange fractionnaire associée) ?*

Dans le **Chapitre VI**, nous démontrons une version fractionnaire du théorème de Tonelli qui affirme que, sous des conditions adéquates de régularité, coercivité et convexité, une fonctionnelle Lagrangienne admet un minimiseur. Plus précisément, nous considèrons la fonctionnelle Lagrangienne fractionnaire

$$\begin{array}{rccl} \mathcal{L}: & \mathrm{E} \subset \mathrm{W}^{1,p}([a,b],\mathbb{R}^n) & \longrightarrow & \mathbb{R} \\ & q & \longmapsto & \displaystyle\int_a^b L(q(\tau), {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q](\tau), \dot{q}(\tau), \tau)\, d\tau \end{array}$$

où $0 < \alpha < 1$, $1 < p < \infty$ et E est un sous-ensemble non vide et faiblement fermé de $\mathrm{W}^{1,p}([a,b],\mathbb{R}^n)$. Le Lagrangien L est dit (α,p)-régulier sur E si les conditions suivantes sont satisfaites pour tout $q \in \mathrm{E}$:

$$L(q, {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q], \dot{q}, \cdot) \in \mathrm{L}^1,\ \frac{\partial L}{\partial x}(q, {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q], \dot{q}, \cdot) \in \mathrm{L}^1,\ \frac{\partial L}{\partial v_\alpha}(q, {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q], \dot{q}, \cdot) \in \mathrm{L}^s,\ \frac{\partial L}{\partial v}(q, {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q], \dot{q}, \cdot) \in \mathrm{L}^{p'}$$

avec s satisfaisant $s = \frac{p}{(2-\alpha)p-1}$ si $(1-\alpha)p < 1$, $s > 1$ si $(1-\alpha)p = 1$ et $s = 1$ si $(1-\alpha)p > 1$. Notons que plus p est grand et/ou plus α est proche de 1, moins la notion de (α,p)-régularité est restrictive. Enfin, la fonctionnelle Lagrangienne $\mathcal{L}$ est dite coercive sur E si

$$\lim_{\substack{\|q\|_{\mathrm{W}^{1,p}} \to +\infty \\ q \in \mathrm{E}}} \mathcal{L}(q) = +\infty.$$

Finalement, notre résultat principal est le suivant : si L est (α,p)-régulier, si $\mathcal{L}$ est coercive et si $L(\cdot,\cdot,\cdot,t)$ est convexe pour tout $t \in [a,b]$, alors $\mathcal{L}$ admet un minimiseur.

Les hypothèses de (α,p)-régularité et de coercivité sont des hypothèses très générales. C'est pourquoi nous proposons une étude (plus concrète) sur les Lagrangiens L dont la croissance est bornée par un comportement quasi-polynomial. Plusieurs exemples concrets de Lagrangiens L satisfaisant les conditions de notre résultat principal sont alors obtenus. Cependant, la dernière hypothèse (convexité) est très restrictive. Par conséquent, nous proposons en fin de chapitre des versions de notre résultat principal où l'hypothèse de convexité sur les deux premières variables du Lagrangien est relâchée. En contrepartie, ces versions ne sont valables que sous des hypothèses renforcées au regard de la continuité du Lagrangien L (en termes d'équicontinuité uniforme). Des exemples plus généraux de Lagrangiens L garantissant l'existence d'un minimiseur pour $\mathcal{L}$ sont alors fournis.

Méthodes utilisées : Les techniques employées dans ce chapitre correspondent à une adaptation au cas fractionnaire des méthodes développées dans [63, 73] pour l'existence de minimiseurs pour des fonctionnelles Lagrangiennes classiques (*i.e.* sans dérivée fractionnaire). Le premier travail a été de déterminer les conditions d'intégrabilité qui définissent la notion de (α,p)-régularité la moins restrictive possible. Pour cela, nous utilisons des résultats de régularité sur l'intégrale fractionnaire I^{α}_{a+}, ce qui explique la dépendance de la condition faite sur l'élément s par rapport aux valeurs de α et de p. Enfin, le second travail a été de réduire l'hypothèse de convexité (trop contraignante) à l'aide du théorème d'Ascoli et d'hypothèses plus fortes sur la continuité de L (en termes d'équicontinuité uniforme).

• **Théorie du contrôle optimal fractionnaire.** Une conséquence directe du développement du calcul des variations fractionnaire est l'émergence de travaux traitant de conditions nécessaires d'optimalité pour la théorie du contrôle fractionnaire. L'article pionnier [7] est dû à O. Agrawal en 2004. L'auteur, avec l'aide de la technique des multiplicateurs de Lagrange, formule (en ses termes) « des équations d'Euler-Lagrange pour des problèmes de contrôle optimal fractionnaires ». Ce résultat correspond en fait à la formulation d'un Principe du Maximum de Pontryagin Faible (WPMP en abrégé anglais).

Dans son article, O. Agrawal traite d'équations différentielles contrôlées dépendantes de la dérivée fractionnaire de Riemann-Liouville. Depuis, des études similaires ont été développées : avec dérivée fractionnaire de Caputo [8], avec plusieurs dérivées fractionnaires de Riemann-Liouville de différents ordres [83] ou encore avec la plus haute dérivée d'ordre entier [128]. Notons que le sujet est relativement récent et qu'il n'a pas encore été traité de manière intensive. Pour conclure sur la littérature, notons que G. Frederico et D. Torres ont travaillé sur l'extension du théorème de Noether à ce cadre, voir [88, 90, 91].

Une question naturelle se pose cependant :

Question 7 : *Est-il possible d'obtenir une version forte du Principe du Maximum de Pontryagin pour des problèmes de contrôle optimal fractionnaires ?*

L'objectif initial de notre étude dans le **Chapitre VII** était de répondre à ce problème. Malheureusement, plusieurs obstacles nous empêchent d'adapter au cas fractionnaire la méthode employée au Chapitre III. En effet, le caractère non local des opérateurs fractionnaires entrave l'obtention d'un vecteur de variation associé à une variation-aiguille de type L^1. Hors, ce type de variation est nécessaire à l'obtention d'une condition de maximisation du Hamiltonien. De plus, la définition du vecteur adjoint dans l'énoncé d'un PMP est grandement dépendante de l'existence d'une formule simple de Leibniz. Hors, une telle formule n'existe pas en calcul fractionnaire. Néanmoins, notre étude n'a pas été infructueuse comme l'explique la suite de ce paragraphe.

La méthode des multiplicateurs de Lagrange, utilisée par O. Agrawal dans [7], a été reprise dans les articles ci-dessus dans le but d'obtenir des WPMPs pour divers problèmes de contrôle optimal fractionnaires. Cependant, son utilisation n'a pas encore été totalement justifiée dans la littérature. Par exemple, cette méthode nécessite que la différentielle partielle de la fonction de contraintes soit un isomorphisme entre espaces de Banach mais, à notre connaissance, cette condition n'a pas encore été vérifiée. De plus, comme le montre un contre-exemple fourni dans le Chapitre IX, il s'avère que cette méthode peut même échouer dans certains cadres (à savoir, pour ce contre-exemple, un cadre discret fractionnaire). La question suivante se pose alors :

QUESTION 8 : *Peut-on démontrer un Principe du Maximum de Pontryagin Faible (WPMP) pour un problème de contrôle optimal fractionnaire sans utiliser la méthode des multiplicateurs de Lagrange ?*

Le **Chapitre VII** de ce manuscrit est consacré à une telle démonstration basée uniquement sur un simple calcul des variations fractionnaire. En effet, les vecteurs de variations associés à des variations générales (*i.e.* non localisées) de type L^∞ sont obtenus et nous démontrons alors un WPMP pour un problème de contrôle optimal dont l'équation différentielle non linéaire contrôlée dépend de la dérivée fractionnaire de Caputo. En d'autres termes, nous donnons une démonstration nouvelle et complète du résultat formulé dans [8, 91]. Plus précisément, nous considérons le système dynamique contrôlé

$$ {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q] = f(q,u,t) $$

où $f : \mathbb{R}^n \times \mathbb{R}^m \times [a,b] \longrightarrow \mathbb{R}^n$ et u désigne le contrôle sur le système. Notre objectif est alors de déterminer une condition nécessaire sur une trajectoire $q^* : [a,b] \longrightarrow \mathbb{R}^n$, solution du système contrôlé associée à un contrôle $u^* : [a,b] \longrightarrow \mathbb{R}^m$, pour qu'elle minimise le coût

$$ \int_a^b f^0(q(\tau),u(\tau),\tau)\, d\tau, $$

où $f^0 : \mathbb{R}^n \times \mathbb{R}^m \times [a,b] \longrightarrow \mathbb{R}^n$, parmi toutes les trajectoires $q : [a,b] \longrightarrow \mathbb{R}^n$ solutions du système contrôlé, chacune associée à un contrôle $u : [a,b] \longrightarrow \mathbb{R}^m$. En introduisant le Hamiltonien $H(x,v,w,t) = w \cdot f(x,v,t) + f^0(x,v,t)$, le résultat principal du Chapitre VII est un WPMP dont la condition nécessaire est l'existence d'une fonction $p : [a,b] \longrightarrow \mathbb{R}^n$ (appelée vecteur adjoint) satisfaisant $\mathrm{I}^{1-\alpha}_{b-}p(a) = p(b) = 0$ tels que q^* et p soient solutions du système Hamiltonien fractionnaire

$$ {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q] = \frac{\partial H}{\partial w}(q^*,u^*,p,t), \qquad {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{b-}[p] = \frac{\partial H}{\partial x}(q^*,u^*,p,t) $$

et tels que la condition qui suit soit satisfaite presque partout :

$$ \frac{\partial H}{\partial v}(q^*,u^*,p,t) = 0. $$

Difficultés rencontrées et méthodes utilisées : Comme dans le Chapitre III, les contrôles sont considérés dans L^∞ pour prendre en compte d'éventuelles discontinuités. Avec un tel cadre fractionnaire, la première remarque est que les solutions du système contrôlé doivent être considérées au sens *faible* (*i.e.* solutions de l'équation intégrale). Puis, comme dans le Chapitre III, pour faire des variations sur le contrôle $u^* \in \mathrm{L}^\infty$ et préserver son admissibilité (*i.e.* que la trajectoire associée soit toujours définie au temps final $t = b$), une théorie générale de Cauchy-Lipschitz complète est nécessaire. Bien que des versions fractionnaires des théorèmes de type Cauchy-Lipschitz aient déjà été traitées dans la littérature (voir *e.g.* [12, 77, 78, 104, 136, 137]), les résultats proposés ne couvrent pas les besoins de cette étude. La première étape de ce travail a donc été d'adapter la théorie générale de Cauchy-Lipschitz au cas fractionnaire (voir Annexe C). Les méthodes déjà employées en Annexe A (dans le cadre time scale) ont été revisitées. Des difficultés, dues notamment à la non localité des opérateurs fractionnaires, apparaissent et des modifications nécessaires ont alors été apportées.

Finalement, la preuve de notre résultat principal repose sur un simple calcul des variations (où les variations sont uniquement portées sur le contrôle optimal), sur l'obtention des vecteurs de variations associés et sur la définition d'un vecteur adjoint approprié.

• **Calcul discret fractionnaire.** Bien que plusieurs méthodes aient été développées pour la résolution explicite d'équations différentielles fractionnaires (comme les transformées de Laplace ou de Fourier, voir [141, 168]), ces techniques ne peuvent être étendues à la plupart des équations non linéaires. Par conséquent, des outils d'approximations numériques se sont développés dans la littérature, voir [10, 80, 128]. Ces méthodes sont essentiellement des extensions des méthodes classiques comme le schéma de Adams-Bashforth-Moulton ou l'extrapolation de Richardson dans [79]. Dans ce manuscrit, nous nous concentrons uniquement sur l'utilisation des opérateurs discrets fractionnaires de Grünwald-Letnikov.

Rappelons que A. Grünwald et A. Letnikov ont introduit en 1867-68 une notion de dérivée fractionnaire qui, de manière analogue à la dérivée classique, est définie comme une limite de différences finies. Malgré une approche différente, il s'avère que cette notion coïncide avec celle de Riemann-Liouville et propose donc une méthode naturelle d'approcher numériquement les dérivées fractionnaires de Riemann-Liouville. La notion de Caputo étant liée à celle de Riemann-Liouville par une relation explicite, une méthode d'approximation des dérivées fractionnaires de Caputo en découle. Dans ce manuscrit, nous utilisons finalement les opérateurs discrets fractionnaires suivants :

– les dérivées discrètes fractionnaires Δ^α_{a+} et Δ^α_{b-} (à gauche et à droite) de Grünwald-Letnikov;
– les dérivées discrètes fractionnaires ${}_\mathrm{c}\Delta^\alpha_{a+}$ et ${}_\mathrm{c}\Delta^\alpha_{b-}$ (à gauche et à droite) de Caputo-Grünwald-Letnikov.

Dans le but de définir un schéma numérique pour une équation différentielle impliquant les dérivées fractionnaires de Riemann-Liouville et/ou de Caputo, I. Podlubny [185, 186] suggère de remplacer l'inconnue continue par une inconnue discrète et les opérateurs fractionnaires par les opérateurs discrets fractionnaires cités ci-dessus. Cette méthode, que nous appelons *discrétisation directe*, a été développée pour plusieurs types d'équations, voir par exemple [165, 166] pour une équation de dispersion fractionnaire et [196] pour une équation de diffusion fractionnaire.

• **Calcul des variations discret fractionnaire.** Dû à l'émergence d'une composition entre dérivée fractionnaire à gauche et dérivée fractionnaire à droite dans les équations d'Euler-Lagrange fractionnaires, celles-ci sont particulièrement difficiles à résoudre explicitement. Par conséquent, plusieurs articles développent des simulations numériques pour ces équations. En particulier, la méthode de discrétisation directe (similaire à celle employée par I. Podlubny) est utilisée pour l'oscillateur fractionnaire de Pais-Uhlenbeck dans [28] et pour des équations d'Euler-Lagrange associées à des problèmes de contrôle optimal fractionnaires dans [9, 26, 75, 181].

Comme dans le cas classique, une équation d'Euler-Lagrange fractionnaire possède une structure variationnelle qui lui est intrinsèque et qui impose de fortes contraintes sur le comportement qualitatif de ses solutions. D'un point de vue numérique, il semble donc important de préserver cette structure au niveau discret. Cependant, une discrétisation directe est une procédure algébrique qui ne garantit pas une telle conservation. Nous avons rappelé dans le paragraphe "Intégrateurs variationnels" une méthode usuelle de construction de schémas numériques (appelés *intégrateurs variationnels*) préservant la structure Lagrangienne des équations d'Euler-Lagrange classiques au niveau discret. La question suivante se pose alors :

Question 9 : *Peut-on étendre cette méthode au cas fractionnaire afin de préserver la structure variationnelle des équations d'Euler-Lagrange fractionnaires au niveau discret ?*

Pour répondre à ce problème, le **Chapitre VIII** se propose de construire un intégrateur variationnel pour l'équation d'Euler-Lagrange fractionnaire suivante :

$$\frac{\partial L}{\partial x}(q(t), {}_\mathrm{c}\mathrm{D}^\alpha_{a+}[q](t), t) + \mathrm{D}^\alpha_{b-}\left[\frac{\partial L}{\partial v}(q, {}_\mathrm{c}\mathrm{D}^\alpha_{a+}[q], \cdot)\right](t) = 0$$

qui, on le rappelle, caractérise les points critiques de la fonctionnelle Lagrangienne fractionnaire donnée par

$$\mathcal{L}(q) = \int_a^b L(q(\tau), {}_{\rm c}{\rm D}^{\alpha}_{a+}[q](\tau), \tau)\, d\tau.$$

Comme dans le cas classique, notre stratégie se décompose en deux étapes. En premier lieu, nous définissons une version discrète $\mathcal{L}_h$ de $\mathcal{L}$:

$$\mathcal{L}_h = h \sum_{k=1}^{N} L(q(t_k), {}_{\rm c}\Delta^{\alpha}_{a+}[q](t_k), t_k)$$

et dans un second temps, nous développons un calcul des variations discret fractionnaire afin de caractériser les points critiques de $\mathcal{L}_h$ par l'équation d'Euler-Lagrange discrète fractionnaire suivante :

$$\frac{\partial L}{\partial x}(q(t_k), {}_{\rm c}\Delta^{\alpha}_{a+}[q](t_k), t_k) + \Delta^{\alpha}_{b-}\left[\frac{\partial L}{\partial v}(q, {}_{\rm c}\Delta^{\alpha}_{a+}[q], \cdot)\right](t_k) = 0.$$

Cette équation est donc un intégrateur variationnel pour l'équation d'Euler-Lagrange fractionnaire initiale. En particulier, notons que nous avons en fait démontré que le schéma numérique obtenu naturellement par discrétisation directe coïncide avec un intégrateur variationnel.

Dans le Chapitre VIII, nous déterminons expérimentalement un ordre de convergence de ce schéma numérique dans le cas du Lagrangien de Dirichlet $L(x, v, t) = \frac{1}{2}v^2$. Rappelons que cet exemple est l'un des rares exemples où la solution exacte d'une équation d'Euler-Lagrange fractionnaire est connue quasi-explicitement.

Par ailleurs, nous démontrons dans ce chapitre que le théorème de type Noether fractionnaire obtenu dans [69, 89] (rappelé dans le paragraphe "Calcul des variations fractionnaire") est préservé au niveau discret par cet intégrateur variationnel. Plus précisément, nous démontrons une version discrète fractionnaire de ce résultat : si le Lagrangien L est invariant (au sens discret fractionnaire) par l'action d'un groupe à un paramètre Φ de transformations infinitésimales de $\mathbb{R}^n$, alors la relation fonctionnelle ci-dessous est satisfaite :

$$\frac{\partial L}{\partial v}(q(t_k), {}_{\rm c}\Delta^{\alpha}_{a+}[q](t_k), t_k) \cdot {}_{\rm c}\Delta^{\alpha}_{a+}\left[\frac{\partial \Phi}{\partial \theta}(0, q)\right](t_k) - \Delta^{\alpha}_{b-}\left[\frac{\partial L}{\partial v}(q, {}_{\rm c}\Delta^{\alpha}_{a+}[q], \cdot)\right](t_k) \cdot \frac{\partial \Phi}{\partial \theta}(0, q(t_k)) = 0,$$

pour toute solution q de l'équation d'Euler-Lagrange discrète fractionnaire. Cependant, de manière analogue au cas continu, cette relation fonctionnelle ne fournit pas de constante de mouvement explicite. Nous proposons alors dans [48] une *formule de transfert discrète* permettant de réécrire la relation fonctionnelle ci-dessus comme une dérivée discrète classique valant 0. Une constante de mouvement est ainsi explicitement obtenue et, contrairement au cas continu, elle s'exprime en une somme finie. Par conséquent, cette loi de conservation discrète peut être numériquement calculée. Nous renvoyons à [48] pour de plus amples détails.

• **Théorie du contrôle optimal discret fractionnaire.** Plusieurs articles traitent de simulations numériques pour des problèmes de contrôle optimal fractionnaires. Les méthodes employées sont diverses : polynômes de Legendre [7], formules d'expansion [128], polynômes de Chebychev [135], approximations rationnelles [198], etc. La méthode de discrétisation directe, utilisant les opérateurs discrets fractionnaires de Grünwald-Letnikov sur les équations

$$\begin{aligned} {}_{\rm c}{\rm D}^{\alpha}_{a+}[q^*](t) &= \frac{\partial H}{\partial w}(q^*(t), u^*(t), p(t), t), \\ {}_{\rm c}{\rm D}^{\alpha}_{b-}[p](t) &= \frac{\partial H}{\partial x}(q^*(t), u^*(t), p(t), t) \\ \frac{\partial H}{\partial v}&(q^*(t), u^*(t), p(t), t) = 0 \end{aligned}$$

données par le WPMP, est étudiée dans [26, 75]. Cependant, comme expliqué dans le paragraphe précédent, cette discrétisation algébrique ne préserve pas la structure variationnelle de ces équations. La question suivante se pose alors :

Question 10 : Peut-on construire un schéma numérique pour les équations ci-dessus préservant leur structure variationnelle ?

Nous proposons en **Chapitre IX** de construire un tel schéma en adaptant la méthode de construction d'intégrateurs variationnels au cas d'un problème de contrôle optimal fractionnaire. Concrètement, nous définissons une version discrète du problème de contrôle optimal fractionnaire considéré dans le paragraphe "Théorie du contrôle optimal fractionnaire" et nous démontrons la version discrète du WPMP correspondante. Plus précisément, nous considérons le système dynamique discret contrôlé

$$ {}_{\mathrm{c}}\Delta^{\alpha}_{a+}[q](t_k) = f(q(t_k), u(t_k), t_k) $$

où $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{T} \longrightarrow \mathbb{R}^n$, où $\mathbb{T} = \{a = t_0 < \ldots < t_N = b\}$ et u désigne le contrôle sur le système. Notre objectif est alors de déterminer une condition nécessaire sur une trajectoire $q^* : \mathbb{T} \longrightarrow \mathbb{R}^n$, solution du système contrôlé associée à un contrôle $u^* : \mathbb{T} \longrightarrow \mathbb{R}^m$, pour qu'elle minimise le coût

$$ h \sum_{k=1}^{N} f^0(q(t_k), u(t_k), t_k) $$

où $f^0 : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{T} \longrightarrow \mathbb{R}^n$, parmi toutes les trajectoires $q : \mathbb{T} \longrightarrow \mathbb{R}^n$ solutions du système contrôlé, chacune associée à un contrôle $u : \mathbb{T} \longrightarrow \mathbb{R}^m$. Dans ce cadre discret, le caractère implicite du système contrôlé nous impose de supposer une condition relativement forte sur la fonction f en termes de continuité globalement Lipschitzienne afin d'assurer, pour un contrôle quelconque u, l'existence de la trajectoire q associée. Nous introduisons alors le Hamiltonien $H(x, v, w, t) = w \cdot f(x, v, t) + f^0(x, v, t)$ et nous démontrons un WPMP dont la condition nécessaire est l'existence d'une fonction $p : \mathbb{T} \longrightarrow \mathbb{R}^n$ (appelée vecteur adjoint) satisfaisant $\Delta^{\alpha-1}_{b-} p(a) = p(b) = 0$ tels que q^* et p soient solutions du système Hamiltonien discret fractionnaire

$$ {}_{\mathrm{c}}\Delta^{\alpha}_{a+}[q](t_k) = \frac{\partial H}{\partial w}(q^*(t_k), u^*(t_k), p(t_{k-1}), t_k), \qquad {}_{\mathrm{c}}\Delta^{\alpha}_{b-}[p](t_k) = \frac{\partial H}{\partial x}(q^*(t_k), u^*(t_k), p(t_{k-1}), t_k) $$

et tels que la condition qui suit soit satisfaite :

$$ \frac{\partial H}{\partial v}(q^*(t_k), u^*(t_k), p(t_{k-1}), t_k) = 0. $$

Comme dans le Chapitre VII, la démonstration se base sur un calcul des variations (portant uniquement sur le contrôle), sur l'obtention des vecteurs de variations associés et sur la définition d'un vecteur adjoint adéquat. Ce résultat donne en particulier un schéma numérique pour le problème de contrôle optimal considéré dans le paragraphe "Théorie du contrôle optimal fractionnaire" tout en préservant la structure variationnelle des équations obtenues dans le WPMP. Notons qu'un shift émerge sur le vecteur adjoint p alors que celui-ci n'apparaît pas dans les schémas numériques obtenus par discrétisation directe.

La condition de continuité globalement Lipschitzienne imposée sur f est très restrictive. Cependant, nous démontrons dans le Chapitre IX, à l'aide d'un contre-exemple, qu'une hypothèse de ce type est en fait nécessaire. Nous signalons que nous n'avons pas cherché à optimiser cette condition qui peut probablement être affaiblie.

Pour conclure, une conséquence importante du même contre-exemple est que la méthode des multiplicateurs de Lagrange ne peut pas être appliquée dans certains cadres. Cet argument justifie le besoin d'une justification complète de l'utilisation de cette technique ou de l'obtention du WPMP par une nouvelle preuve comme cela est proposé au Chapitre VII.

Les études menées dans ce manuscrit ont donné lieu à neuf articles dont cinq publiés (ou acceptés pour publication) et quatre soumis :

- *Existence of a weak solution for fractional Euler-Lagrange equations*, paru dans Journal of Mathematical Analysis and Applications, 399(1):239-251, 2013 ;
- *Existence of minimizers for fractional variational problems containing Caputo derivatives*, coécrit avec T. Odzijewicz [1] et D. Torres [1], paru dans Advances in Dynamical Systems and Applications, 8(1):3-12, 2013 ;

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- *A continuous/discrete fractional Noether's theorem*, coécrit avec J. Cresson [2] et I. Greff [2], paru dans Communication in Nonlinear Sciences and Numerical Simulations, 18(4):878-887, 2013 ;
- *Helmholtz's inverse problem of the discrete calculus of variations*, coécrit avec J. Cresson [2], à paraître dans Journal of Difference Equations and Applications ;
- *Variational integrators for fractional Euler-Lagrange equations*, coécrit avec J. Cresson [2], I. Greff [2] et P. Inizan [3], à paraître dans Applied Numerical Analysis and Computational Mathematics ;
- *Non shifted calculus of variations on time scales with* ∇*-differentiable* σ, soumis, preprint arXiv:1302.3623 ;
- *Cauchy-Lipschitz theory for shifted and non shifted* Δ*-Cauchy problems on time scales*, coécrit avec E. Trélat [4], soumis, preprint arXiv:1212.5042v1 ;
- *Pontryagin maximum principle for finite dimensional nonlinear optimal control problems on time scales*, coécrit avec E. Trélat [4], soumis, preprint arXiv:1302.3513 ;
- *Existence of minimizers for generalized Lagrangian functionals and a necessary optimality condition - Application to fractional variational problems*, coécrit avec T. Odzijewicz [1] et D. Torres [1], soumis.

Un glossaire de notations est donné juste après cette introduction. Nous proposons en fin de manuscrit quelques projets de recherche qui font suite aux travaux développés dans ce mémoire.

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Notations

Notations générales

$\mathbb{R}$	Ensemble des nombres réels
$\mathbb{R}^+$ (resp. $\mathbb{R}^-$)	Ensemble des réels positifs (resp. négatifs) ou nuls
$\mathbb{R}^*$	Ensemble des nombres réels non nuls
$\mathbb{N}$	Ensemble des nombres entiers naturels
$\mathbb{N}^*$	Ensemble des nombres entiers naturels non nuls
$\mathbb{Z}$	Ensemble des nombres entiers relatifs
I	Intervalle de $\mathbb{R}$
$\mathrm{card}(A)$	Cardinal d'un ensemble $A \subset \mathbb{R}$
a, b	Nombres réels satisfaisant $a \leq b$ (ou $a < b$)
n	Entier naturel non nul (dimension de l'espace de travail)
$\\| \cdot \\|_{\mathbb{R}^n}$ (ou simplement $\\| \cdot \\|$)	Norme euclidienne de $\mathbb{R}^n$
$\overline{B}(x,R)$	Boule fermée de $\mathbb{R}^n$ centrée en $x \in \mathbb{R}^n$ et de rayon $R \geq 0$
$\mathrm{Int}(A)$	Intérieur d'un ensemble $A \subset \mathbb{R}^n$
$x_1 \cdot x_2$	Produit scalaire dans $\mathbb{R}^n$
M^{T}	Transposée d'une matrice M
$\mathrm{M} \times x$	Produit matrice-vecteur
Φ	Famille à un paramètre de transformations infinitésimales de $\mathbb{R}^n$ (voir Définition II.1)
q	Trajectoires des systèmes dynamiques étudiés (à valeurs dans $\mathbb{R}^n$)
$\mathrm{C}(A,\mathbb{R}^n)$	Ensemble des fonctions continues sur $A \subset \mathbb{R}$ à valeurs dans $\mathbb{R}^n$
$\mathrm{AC}(A,\mathbb{R}^n)$	Ensemble des fonctions absolument continues sur $A \subset \mathbb{R}$ à valeurs dans $\mathbb{R}^n$
$\mathrm{H}^\lambda(A,\mathbb{R}^n)$	Ensemble des fonctions Hölderiennes sur $A \subset \mathbb{R}$ à valeurs dans $\mathbb{R}^n$ d'exposant $0 < \lambda \leq 1$
$\\| \cdot \\|_\infty$	Norme uniforme de $\mathrm{C}(A,\mathbb{R}^n)$ (pour $A \subset \mathbb{R}$ compact)
$\mathrm{C}_a([a,b],\mathbb{R}^n)$ (resp. $\mathrm{C}_b([a,b],\mathbb{R}^n)$)	Sous-ensemble de $\mathrm{C}([a,b],\mathbb{R}^n)$ des fonctions s'annulant en a (resp. en b)
$\mathrm{AC}_a([a,b],\mathbb{R}^n)$ (resp. $\mathrm{AC}_b([a,b],\mathbb{R}^n)$)	Sous-ensemble de $\mathrm{AC}([a,b],\mathbb{R}^n)$ des fonctions s'annulant en a (resp. en b)
$\mathrm{H}^\lambda_a([a,b],\mathbb{R}^n)$ (resp. $\mathrm{H}^\lambda_b([a,b],\mathbb{R}^n)$)	Sous-ensemble de $\mathrm{H}^\lambda([a,b],\mathbb{R}^n)$ des fonctions s'annulant en a (resp. en b)
$\mathscr{C}^k$	Classe usuelle de régularité de fonctions (pour tout $k \in \mathbb{N} \cup \{\infty\}$)
$\mathrm{C}^k([a,b],\mathbb{R}^n)$	Ensemble des fonctions de classe $\mathscr{C}^k$ sur $[a,b]$ à valeurs dans $\mathbb{R}^n$ (pour tout $k \in \mathbb{N} \cup \{\infty\}$)

$\mathrm{C}^\infty_\mathrm{c}([a,b],\mathbb{R}^n)$	Sous-ensemble de $\mathrm{C}^\infty([a,b],\mathbb{R}^n)$ des fonctions à support compact dans $]a,b[$
μ_L	Mesure de Lebesgue usuelle
a.e.	presque partout (almost everywhere)
$\int_a^b d\tau$	Intégrale de Lebesgue usuelle sur l'intervalle $[a,b]$
p (ou r)	Exposant de Lebesgue satisfaisant $1 \leq p \leq \infty$
p' (ou r')	Exposant conjugué de p satisfaisant $p' = \frac{p}{p-1}$
$\mathrm{L}^p([a,b],\mathbb{R}^n)$	Espace de Lebesgue usuel muni de sa norme $\|\cdot\|_{\mathrm{L}^p([a,b],\mathbb{R}^n)}$
$\mathrm{L}^\infty_\mathrm{loc}(A,\mathbb{R}^n)$	$\mathrm{L}^\infty_\mathrm{loc}(A,\mathbb{R}^n) = \cap_{a,b\in A} \mathrm{L}^\infty([a,b],\mathbb{R}^n)$
$\mathrm{W}^{1,p}([a,b],\mathbb{R}^n)$	Espace de Sobolev usuel muni de sa norme $\|\cdot\|_{\mathrm{W}^{1,p}([a,b],\mathbb{R}^n)}$
$\hookrightarrow$	Injection continue
$\hookrightarrow\!\!\!\rightarrow$	Injection compacte

Notations relatives au calcul time scale

$\mathbb{T}$	Time scale *i.e.* sous-ensemble fermé de $\mathbb{R}$
$\mathbb{T}^\kappa$	$\mathbb{T}^\kappa = \mathbb{T}\backslash\{\max \mathbb{T}\}$ si $\mathbb{T}$ admet un maximum isolé et $\mathbb{T}^\kappa = \mathbb{T}$ sinon
$\mathbb{T}_\kappa$	$\mathbb{T}_\kappa = \mathbb{T}\backslash\{\min \mathbb{T}\}$ si $\mathbb{T}$ admet un minimum isolé et $\mathbb{T}_\kappa = \mathbb{T}$ sinon
$\mathbb{T}^\kappa_\kappa$	$\mathbb{T}^\kappa_\kappa = \mathbb{T}^\kappa \cap \mathbb{T}_\kappa$
$A_\mathbb{T}$	$A_\mathbb{T} = A \cap \mathbb{T}$ pour tout ensemble $A \subset \mathbb{R}$
σ	Forward jump operator défini par $\sigma(t) = \inf\{s > t,\ s \in \mathbb{T}\}$
ρ	Backward jump operator défini par $\rho(t) = \sup\{s < t,\ s \in \mathbb{T}\}$
μ	Graininess function définie par $\mu(t) = \sigma(t) - t$
ν	Backward graininess function définie par $\nu(t) = t - \rho(t)$
LS (resp. LD, RS et RD)	Ensemble des points left-scattered (resp. left-dense, right-scattered et right-dense) d'un time scale $\mathbb{T}$ (voir Section I.2.1)
q^σ	$q^\sigma = q \circ \sigma$
q^ρ	$q^\rho = q \circ \rho$
q^Δ (resp. q^∇)	Δ-dérivée (resp. ∇-dérivée) d'une fonction q définie sur $\mathbb{T}$ (voir Section I.2.1)
$\mathrm{C}^{1,\Delta}(\mathbb{T},\mathbb{R}^n)$	Ensemble des fonctions définies sur $\mathbb{T}$ à valeurs dans $\mathbb{R}^n$ admettant une Δ-dérivée continue sur $\mathbb{T}^\kappa$
$\mathrm{C}_\mathrm{rd}(\mathbb{T},\mathbb{R}^n)$	Ensemble des fonctions rd-continues sur $\mathbb{T}$ à valeurs dans $\mathbb{R}^n$ (voir Section I.2.2)
$\mathrm{C}^{1,\Delta}_\mathrm{rd}(\mathbb{T},\mathbb{R}^n)$	Ensemble des fonctions définies sur $\mathbb{T}$ à valeurs dans $\mathbb{R}^n$ admettant une Δ-dérivée rd-continue sur $\mathbb{T}^\kappa$
$\int_a^b \Delta\tau$	Δ-intégrale de Cauchy sur l'intervalle $[a,b[_\mathbb{T}$ (voir Section I.2.2)
μ_Δ	Δ-mesure de Lebesgue (voir Section I.3.1)
Δ-a.e.	Δ-presque partout (Δ-almost everywhere)
$\int_A \Delta\tau$	Δ-intégrale de Lebesgue sur l'ensemble $A \subset \mathbb{T}$
$\mathrm{L}^1_\mathbb{T}(A,\mathbb{R}^n)$	Espace de Banach des fonctions Δ-intégrables sur $A \subset \mathbb{T}$ à valeurs dans $\mathbb{R}^n$ muni de la norme $\|\cdot\|_{\mathrm{L}^1_\mathbb{T}(A,\mathbb{R}^n)}$ (voir Section I.3.1)
$\mathrm{L}^\infty_\mathbb{T}(A,\mathbb{R}^n)$	Espace de Banach des fonctions Δ-essentiellement bornées sur $A \subset \mathbb{T}$ à valeurs dans $\mathbb{R}^n$ muni de la norme $\|\cdot\|_{\mathrm{L}^1_\mathbb{T}(A,\mathbb{R}^n)}$ (voir Section I.3.1)
$\mathrm{L}^\infty_{\mathrm{loc},\mathbb{T}}(\mathbb{T}\backslash\{\sup \mathbb{T}\},\mathbb{R}^n)$	$\mathrm{L}^\infty_{\mathrm{loc},\mathbb{T}}(\mathbb{T}\backslash\{\sup \mathbb{T}\},\mathbb{R}^n) = \cap_{a,b\in\mathbb{T}^2} \mathrm{L}^\infty_\mathbb{T}([a,b[_\mathbb{T},\mathbb{R}^n)$
$\mathscr{L}_{[a,b[_\mathbb{T}}(q)$	Ensemble des Δ-points de Lebesgue dans $[a,b[_\mathbb{T}$ pour une fonction $q \in \mathrm{L}^1_\mathbb{T}([a,b[_\mathbb{T},\mathbb{R}^n)$ (voir Section I.3.2)

Dans les cas particuliers où le time scale est continu ou discret :

$q^\Delta(t) = \dot{q}(t) = \frac{dq}{dt}(t)$ $q^\nabla(t) = \dot{q}(t) = \frac{dq}{dt}(t)$	*dans le cas particulier où $\mathbb{T}$ est un time scale continu*
$q^\Delta(t_k) = \Delta q(t_k) = \frac{q(t_{k+1})-q(t_k)}{t_{k+1}-t_k}$ $q^\nabla(t_k) = \nabla q(t_k) = \frac{q(t_k)-q(t_{k-1})}{t_k-t_{k-1}}$	*dans le cas particulier où $\mathbb{T}$ est un time scale discret*

Notations relatives au calcul fractionnaire

$\alpha > 0$	Exposant fractionnaire (essentiellement $0 < \alpha < 1$ dans ce manuscrit)
$[\alpha]$	Partie entière de $\alpha > 0$ *i.e.* $[\alpha] \in \mathbb{N}$ et $[\alpha] \leq \alpha < [\alpha] + 1$
Γ	Fonction Gamma de Euler, voir Équation (V.15)
$\mathbb{I}_{a+}$	$\mathbb{I}_{a+} = \{I \subset [a, +\infty[$ intervalle tel que $\inf I = a$ et $I \backslash \{a\} \neq \emptyset\}$
$\mathbb{I}_{b-}$	$\mathbb{I}_{a+} = \{I \subset [a, +\infty[$ intervalle tel que $\inf I = a$ et $I \backslash \{a\} \neq \emptyset\}$
I^{α}_{a+} (resp. D^{α}_{a+})	Intégrale (resp. dérivée) fractionnaire de Riemann-Liouville d'ordre α avec limite inférieure a, voir Définition V.1
I^{α}_{b-} (resp. D^{α}_{b-})	Intégrale (resp. dérivée) fractionnaire de Riemann-Liouville d'ordre α avec limite supérieure b, voir Définition V.2
${}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}$ (resp. ${}_{\mathrm{c}}\mathrm{D}^{\alpha}_{b-}$)	Dérivée fractionnaire de Caputo d'ordre α avec limite inférieure a (resp. supérieure b), voir Définitions V.3 et V.4
${}_{\mathrm{GL}}\mathrm{D}^{\alpha}_{a+}$ (resp. ${}_{\mathrm{GL}}\mathrm{D}^{\alpha}_{b-}$)	Dérivée fractionnaire de Grünwald-Letnikov d'ordre α avec limite inférieure a (resp. supérieure b), voir Définition VIII.1
Δ^{α}_{a+} (resp. Δ^{α}_{b-})	Dérivée fractionnaire discrète de Grünwald-Letnikov d'ordre α avec limite inférieure a (resp. supérieure b), voir Définition VIII.2
${}_{\mathrm{c}}\Delta^{\alpha}_{a+}$ (resp. ${}_{\mathrm{c}}\Delta^{\alpha}_{b-}$)	Dérivée fractionnaire discrète de Caputo-Grünwald-Letnikov d'ordre α avec limite inférieure a (resp. supérieure b), voir Définition VIII.3
α_r	$\alpha_0 = 1$ et $\alpha_r = (-\alpha)(1-\alpha)\dots(r-1-\alpha)/r!$ pour tout $r \in \mathbb{N}^*$
α^*_r	$\alpha^*_0 = 1$ et $\alpha^*_r = \sum_{k=0}^{r} \alpha_k = (1-\alpha)\dots(r-\alpha)/r!$ pour tout $r \in \mathbb{N}^*$

Notations relatives au calcul des variations et à la théorie du contrôle optimal

Calcul des variations :

L	Lagrangien (continu, et de classe $\mathscr{C}^1$ en toutes ses variables sauf la dernière)
$\mathcal{L}$	Fonctionnelle Lagrangienne
E	Ensemble (fonctionnel) de définition de $\mathcal{L}$
$D\mathcal{L}(q)(w)$	Différentielle de $\mathcal{L}$ au sens de Gâteaux en q et dans la direction w
$\mathrm{C}^{1,\Delta}_{\mathrm{rd},0}(\mathbb{T}, \mathbb{R}^n)$	Ensemble de variations de $\mathcal{L}$ dans le Chapitre II défini comme le sous-ensemble de $\mathrm{C}^{1,\Delta}_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^n)$ des fonctions s'annulant en a et en b
$\mathrm{C}^1_0([a,b], \mathbb{R}^n)$	Ensemble de variations de $\mathcal{L}$ dans le Chapitre VI défini comme le sous-ensemble de $\mathrm{C}^1([a,b], \mathbb{R}^n)$ des fonctions s'annulant en a et en b

Théorie du contrôle optimal :

PMP	Principe du Maximum de Pontryagin (Pontryagin Maximum Principle)
WPMP	Principe du Maximum de Pontryagin Faible (Weak Pontryagin Maximum Principle)
m	Entier naturel non nul (dimension de l'espace d'arrivée des contrôles)
u	Contrôles du système dynamique contrôlé
$f(x,v,t)$	Fonction de contrainte à valeurs dans $\mathbb{R}^n$ (continue, et de classe $\mathscr{C}^1$ en toutes ses variables sauf la dernière)
$f^0(x,v,t)$	Fonction de coût à valeurs dans $\mathbb{R}$ (continue, et de classe $\mathscr{C}^1$ en toutes ses variables sauf la dernière)
$q(\cdot, u, q_a)$	Variable d'état associé au contrôle u et à la condition initiale q_a
$I(u, q_a)$ (ou $I_{\mathbb{T}}(u, q_a)$)	Intervalle de définition de $q(\cdot, u, q_a)$
$H(x,v,w,w^0,t)$	Hamiltonien associé au problème de contrôle optimal défini par $H(x,v,w,w^0,t) = w \cdot f(x,v,t) + w^0 f^0(x,v,t)$ ($w_0 = 1$ excepté dans le Chapitre III)

Notations spécifiques au Chapitre III

$\mathcal{V}_s^b$	$\mathcal{V}_s^b = \{\beta \geq 0,\ s+\beta \in [s,b]_{\mathbb{T}}\}$ pour tout $s \in \mathrm{RD}$, $b \in \mathbb{T}$ et $s < b$
j	Entier naturel non nul
$\langle \cdot, \cdot \rangle_{\mathbb{R}^n}$	Produit scalaire de $\mathbb{R}^n$ (valable aussi pour $\mathbb{R}^m$ et $\mathbb{R}^j$)
S	Sous-ensemble non vide convexe et fermé de $\mathbb{R}^j$
d_{S}	Fonction distance à S définie par $d_{\mathrm{S}}(x) = \inf_{x' \in \mathrm{S}} \|x - x'\|_{\mathbb{R}^j}$
$\mathrm{P_S}$	Fonction projection sur S (à valeurs dans S) définie par $\|x - \mathrm{P_S}(x)\|_{\mathbb{R}^j} = d_{\mathrm{S}}(x)$
$\mathcal{O}_{\mathrm{S}}(x)$	Orthogonal de S au point $x \in \mathrm{S}$ défini par $\mathcal{O}_{\mathrm{S}}(x) = \{x' \in \mathbb{R}^j,\ \forall x'' \in \mathrm{S},\ \langle x', x'' - x \rangle_{\mathbb{R}^j} \leq 0\}$
$g(x_1, x_2)$	Fonction de contraintes sur les conditions initiale et finale (de la variable d'état) à valeurs dans $\mathbb{R}^j$ (de classe $\mathscr{C}^1$)
Ω	Sous-ensemble non vide et fermé de $\mathbb{R}^m$ (représente l'ensemble des contraintes sur les valeurs du contrôle)
$\mathrm{Aff}(\Omega)$	Sous-espace affine de $\mathbb{R}^m$ engendré par Ω
$\mathcal{D}^\Omega(v, v')$	$\mathcal{D}^\Omega(v, v') = \{0 \leq \alpha \leq 1 \mid v + \alpha(v' - v) \in \Omega\}$ pour tout $v \in \Omega$ et $v' \in \mathbb{R}^m$
$\mathcal{D}^\Omega(v)$	Ensemble des directions Ω-denses à partir du point $v \in \Omega$ défini par $\mathcal{D}^\Omega(v) = \{v' \in \mathbb{R}^m,\ 0$ n'est pas isolé dans $\mathcal{D}^\Omega(v, v')\}$
$\mathcal{D}^\Omega_{\mathrm{stab}}(v)$	Ensemble des directions Ω-denses stables à partir du point $v \in \Omega$ défini par $\mathcal{D}^\Omega_{\mathrm{stab}}(v) = \{v' \in \mathcal{D}^\Omega(v),\ \exists \varepsilon > 0,\ \forall v'' \in \overline{B}(v, \varepsilon) \cap \Omega,\ v' \in \mathcal{D}^\Omega(v'')\}$
$\overline{\mathrm{Co}}(\mathcal{D}^\Omega_{\mathrm{stab}}(v))$	Cône convexe fermé de $\mathbb{R}^m$ engendré par $\mathcal{D}^\Omega_{\mathrm{stab}}(v)$
$\mathrm{Opp}(v)$	Symétrique de $\overline{\mathrm{Co}}(\mathcal{D}^\Omega_{\mathrm{stab}}(v))$ par la symétrie de centre $v \in \Omega$ *i.e.* $\mathrm{Opp}(v) = \{2v - v' \mid v' \in \overline{\mathrm{Co}}(\mathcal{D}^\Omega_{\mathrm{stab}}(v))\}$
$\mathcal{U}$	$\mathcal{U} = \mathrm{L}^\infty_{\mathrm{loc},\mathbb{T}}(\mathbb{T} \backslash \{\sup \mathbb{T}\}, \mathbb{R}^m)$
$\mathcal{UQ}^b_{\mathrm{ad}}$	Ensemble des couples $(u, q_a) \in \mathcal{U} \times \mathbb{R}^n$ tel que u est un contrôle admissible sur $[a,b]_{\mathbb{T}}$ associé à la condition initiale q_a
$\mathrm{E}(u, q_a, R)$	$\mathrm{E}(u, q_a, R) = \{(u', q'_a) \in \mathcal{U} \times \mathbb{R}^n \mid \|u' - u\|_{\mathrm{L}^1_{\mathbb{T}}([a,b[_{\mathbb{T}}, \mathbb{R}^m)} \leq \nu_R,\ \|u'\|_{\mathrm{L}^\infty_{\mathbb{T}}([a,b[_{\mathbb{T}}, \mathbb{R}^m)} \leq R,\ \|q'_a - q_a\|_{\mathbb{R}^n} \leq \eta_R\}$ pour $(u, q_a) \in \mathcal{UQ}^b_{\mathrm{ad}}$ et $R > \|u\|_{\mathrm{L}^\infty_{\mathbb{T}}([a,b[_{\mathbb{T}}, \mathbb{R}^m)}$
$F_{(u,q_a,R)}$	$F_{(u,q_a,R)} : (u', q'_a) \in \mathrm{E}(u, q_a, R) \longrightarrow q(\cdot, u', q'_a) \in \mathrm{C}([a,b]_{\mathbb{T}}, \mathbb{R}^n)$
$\Pi = (r, y)$	Paramètre définissant une variation aiguille du contrôle u en un point $r \in \mathrm{RS}$ et dans la direction $y \in \mathcal{D}^\Omega(u(r))$
$u_\Pi(\cdot, \alpha)$	Variation aiguille du contrôle u associée au paramètre Π, voir Section III.3.2.2
$F_{(u,q_a,\Pi)}$	$F_{(u,q_a,\Pi)} : \alpha \in \mathcal{D}^\Omega(u(r), y) \longrightarrow q(\cdot, u_\Pi(\cdot, \alpha), q_a) \in \mathrm{C}([a,b]_{\mathbb{T}}, \mathbb{R}^n)$
$\omega_\Pi(\cdot, u, q_a)$	Vecteur de variation associé au couple $(u, q_a) \in \mathcal{UQ}^b_{\mathrm{ad}}$ et à la variation aiguille paramétrée par Π *i.e.* valeur de la différentielle de $F_{(u,q_a,\Pi)}$ en $\alpha = 0$, voir Section III.3.2.2
$\mathrm{II} = (s, z)$	Paramètre définissant une variation aiguille du contrôle u en un point $s \in \mathrm{RD}$ avec la valeur $z \in \Omega$
$u_{\mathrm{II}}(\cdot, \beta)$	Variation aiguille du contrôle u suivant le paramètre II, voir Section III.3.2.3
$F_{(u,q_a,\mathrm{II})}$	$F_{(u,q_a,\mathrm{II})} : \beta \in \mathcal{V}^b_s \longrightarrow q(\cdot, u_{\mathrm{II}}(\cdot, \beta), q_a) \in \mathrm{C}([a,b]_{\mathbb{T}}, \mathbb{R}^n)$
$\omega_{\mathrm{II}}(\cdot, u, q_a)$	Vecteur de variation associé au couple $(u, q_a) \in \mathcal{UQ}^b_{\mathrm{ad}}$ et à la variation aiguille paramétrée par II *i.e.* valeur de la différentielle de $F_{(u,q_a,\mathrm{II})}$ en $\beta = 0$, voir Section III.3.2.3
q'_a	Vecteur de $\mathbb{R}^n$ utile pour former une variation sur une condition initiale q_a
$F_{(u,q_a,q'_a)}$	$F_{(u,q_a,q'_a)} : \gamma \geq 0 \longrightarrow q(\cdot, u, q_a + \gamma q'_a) \in \mathrm{C}([a,b]_{\mathbb{T}}, \mathbb{R}^n)$
$\omega_{q'_a}(\cdot, u, q_a)$	Vecteur de variation associé au couple $(u, q_a) \in \mathcal{UQ}^b_{\mathrm{ad}}$ et à la variation de la condition initiale dans la direction q'_a *i.e.* valeur de la différentielle de $F_{(u,q_a,q'_a)}$ en $\gamma = 0$, voir Section III.3.2.4
$\bar{f}$, $\bar{g}$, $\bar{q}$, $\bar{x}$	Relatifs à l'augmentation du système en Section III.3.3

E^R_Ω	$\mathrm{E}^R_\Omega = \{(u,\bar{q}_a) \in \mathcal{U} \times \mathbb{R}^{n+1} \mid \bar{q}_a = (q_a,0),\ (u,q_a) \in \mathrm{E}(u^*,\bar{q}^*_a,R),\ u \in \mathrm{L}^\infty_\mathbb{T}([a,b[_\mathbb{T};\Omega)\}$
$J^R_\varepsilon(u,\bar{q}_a)$	Fonctionnelle de pénalisation définie par $J^R_\varepsilon : (u,\bar{q}_a) \in \mathrm{E}^R_\Omega \longrightarrow \big(\max(q^0(b,u,\bar{q}_a) - q^{0*}(b) + \varepsilon, 0)^2 + d^2_\mathrm{S}(\bar{g}(\bar{q}_a, \bar{q}(b,u,\bar{q}_a)))\big)^{1/2} \in \mathbb{R}^+$
$\mathcal{L}^R_{[a,b[_\mathbb{T}}$	$\mathcal{L}^R_{[a,b[_\mathbb{T}} = \cap_{k\in\mathbb{N}} \mathcal{L}_{[a,b[_\mathbb{T}}(f(q(\cdot,u^R_{\varepsilon_k},q^R_{a,\varepsilon_k}),u^R_{\varepsilon_k},t)) \cap \mathcal{L}_{[a,b[_\mathbb{T}}(f(q(\cdot,u^*,q^*_a),u^*,t)) \cap \{t \in [a,b[_\mathbb{T},\ u^R_{\varepsilon_k}(t)$ converge vers $u^*(t)\}$
$\mathcal{L}_{[a,b[_\mathbb{T}}$	$\mathcal{L}_{[a,b[_\mathbb{T}} = \cap_{\ell\in\mathbb{N}} \mathcal{L}^{R_\ell}_{[a,b[_\mathbb{T}}$

Notations spécifiques au Chapitre IV

$O^{a,b}(q)$	$O^{a,b}(q)(t) = \overline{O}(q(t),\dot{q}(t),\ddot{q}(t),t)$ pour tout $t \in [a,b]$ avec $\overline{O} : (x,v,u,t) \in \mathbb{R}^4 \longrightarrow \overline{O}(x,v,u,t) \in \mathbb{R}$
$O^{a,b}(q) = 0$	Équation différentielle d'ordre 2 sur l'intervalle $[a,b]$
$DO^{a,b}(q)(w)$	Différentielle de $O^{a,b}$ au sens de Gâteaux en q et dans la direction w
$\mathscr{T}^\mathrm{f}_\mathrm{uni}$	$\mathscr{T}^\mathrm{f}_\mathrm{uni} = \{\mathbb{T} \subset \mathbb{R},\ 5 \leq \mathrm{card}(\mathbb{T}) < \infty$ et $\exists h > 0,\ \forall t \in \mathbb{T}^\kappa,\ \mu(t) = h\}$
$\mathbb{T} \in \mathscr{T}^\mathrm{f}_\mathrm{uni}$	$\mathbb{T} = \{t_p\}_{p=0,\dots,N}$ intervalle uniformément fini avec $N = \mathrm{card}(\mathbb{T}) - 1$ et $h = (t_N - t_0)/N$
$P^\mathbb{T}(q)$	$P^\mathbb{T}(q)(t_p) = \overline{P}(q(t_p),\Delta q(t_p),\nabla q(t_p),\nabla\circ\Delta q(t_p),t_p,h)$ pour tout $p = 1,\dots,N-1$ avec $\overline{P} : (x,v_1,v_2,u,t,\xi) \in \mathbb{R}^5 \times \mathbb{R}^+_* \longrightarrow \overline{P}(x,v_1,v_2,u,t,\xi)$
$P^\mathbb{T}(q) = 0$	Équation aux différences finies d'ordre 2 sur l'intervalle uniformément fini $\mathbb{T}$
$DP^\mathbb{T}(q)(w)$	Différentielle de $P^\mathbb{T}$ au sens de Gâteaux en q et dans la direction w
$\mathrm{C}_0(\mathbb{T},\mathbb{R})$	$\mathrm{C}_0(\mathbb{T},\mathbb{R}) = \{w \in \mathrm{C}(\mathbb{T},\mathbb{R}),\ w(t_0) = w(t_N) = 0\}$
$\mathrm{C}_{00}(\mathbb{T},\mathbb{R})$	$\mathrm{C}_{00}(\mathbb{T},\mathbb{R}) = \{w \in \mathrm{C}(\mathbb{T},\mathbb{R}),\ w(t_0) = w(t_1) = w(t_{N-1}) = w(t_N) = 0\}$
$\star_p$	$\star_p = (q(t_p),\Delta q(t_p),\nabla q(t_p),\nabla\circ\Delta q(t_p),t_p,h)$

Notations spécifiques au Chapitre VII

I_f	Intervalle inclus dans $[a,+\infty[$ satisfaisant $\min I_f = a$ et $I_f \backslash \{a\} \neq \emptyset$
$\mathcal{U}$	$\mathcal{U} = \mathrm{L}^\infty_\mathrm{loc}(I_f,\mathbb{R}^m)$
$\mathcal{UQ}^b_\mathrm{ad}$	Ensemble des couples $(u,q_a) \in \mathcal{U} \times \mathbb{R}^n$ tel que u est un contrôle admissible sur $[a,b]$ associé à la condition initiale q_a
$\mathrm{E}(u,q_a)$	$\mathrm{E}(u,q_a) = \{(u',q'_a) \in \mathcal{U}\times\mathbb{R}^n \mid \Vert u'-u\Vert_{\mathrm{L}^\infty([a,b],\mathbb{R}^m)} \leq \nu,\ \Vert q'_a - q_a\Vert_{\mathbb{R}^n} \leq \eta\}$ pour $(u,q_a) \in \mathcal{UQ}^b_\mathrm{ad}$
$F_{(u,q_a)}$	$F_{(u,q_a)} : (u',q'_a) \in \mathrm{E}(u,q_a) \longrightarrow q(\cdot,u',q'_a) \in \mathrm{C}([a,b],\mathbb{R}^n)$
$F_{(u,q_a,\bar{u})}$	$F_{(u,q_a,\bar{u})} : \varepsilon \in [-\varepsilon_0,\varepsilon_0] \longrightarrow q(\cdot,u+\varepsilon\bar{u},q_a) \in \mathrm{C}([a,b],\mathbb{R}^n)$ où $\bar{u} \in \mathrm{C}^\infty_\mathrm{c}([a,b],\mathbb{R}^m)$
$\omega_{\bar{u}}(\cdot,u,q_a)$	Vecteur de variation associé au couple $(u,q_a) \in \mathcal{UQ}^b_\mathrm{ad}$ et à la variation générale associée à $\bar{u} \in \mathrm{C}^\infty_\mathrm{c}([a,b],\mathbb{R}^m)$ *i.e.* valeur de la différentielle de $F_{(u,q_a,\bar{u})}$ en $\varepsilon = 0$, voir Section VII.3.1
$F_{(u,q_a,\bar{q}_a)}$	$F_{(u,q_a,\bar{q}_a)} : \gamma \in [-\gamma_0,\gamma_0] \longrightarrow q(\cdot,u,q_a+\gamma\bar{q}_a) \in \mathrm{C}([a,b],\mathbb{R}^n)$ où $\bar{q}_a \in \mathbb{R}^n$
$\omega_{\bar{q}_a}(\cdot,u,q_a)$	Vecteur de variation associé au couple $(u,q_a) \in \mathcal{UQ}^b_\mathrm{ad}$ et à la variation de la condition initiale dans la direction $\bar{q}_a \in \mathbb{R}^n$ *i.e.* valeur de la différentielle de $F_{(u,q_a,\bar{q}_a)}$ en $\gamma = 0$, voir Section VII.3.1

Notations spécifiques au Chapitre IX

$\mathbb{T}$	$\mathbb{T} = \{t_k\}_{k=0,\dots,N} = \{a+kh\}_{k=0,\dots,N}$ intervalle uniformément fini où $N \geq 2$ et $h = (b-a)/N$
$F^k_{\bar{u}}$	$F^k_{\bar{u}} : \varepsilon \in [-\varepsilon_0,\varepsilon_0] \longrightarrow q(t_k,u+\varepsilon\bar{u},q_a) \in \mathbb{R}^n$ où $\bar{u} \in \mathrm{C}(\mathbb{T},\mathbb{R}^m)$

$\omega_{\bar{u}}(\cdot, u, q_a)$	Vecteur de variation associé au couple $(u, q_a) \in \mathrm{C}(\mathbb{T}, \mathbb{R}^n) \times \mathbb{R}^n$ et à la variation générale associée à $\bar{u} \in \mathrm{C}(\mathbb{T}, \mathbb{R}^m)$ *i.e.* valeur de la différentielle de $F^k_{\bar{u}}$ en $\varepsilon = 0$, voir Section IX.3.1
$F^k_{\bar{q}_a}$	$F^k_{\bar{q}_a} : \gamma \in [-\gamma_0, \gamma_0] \longrightarrow q(t_k, u, q_a + \gamma \bar{q}_a) \in \mathbb{R}^n$ où $\bar{q}_a \in \mathbb{R}^n$
$\omega_{\bar{q}_a}(\cdot, u, q_a)$	Vecteur de variation associé au couple $(u, q_a) \in \mathrm{C}(\mathbb{T}, \mathbb{R}^n) \times \mathbb{R}^n$ et à la variation de la condition initiale dans la direction $\bar{q}_a \in \mathbb{R}^n$ *i.e.* valeur de la différentielle de $F^k_{\bar{q}_a}$ en $\gamma = 0$, voir Section IX.3.1

Première partie

Contributions au calcul des variations et au Principe du Maximum de Pontryagin en calcul time scale

Chapitre I

Rappels et compléments sur le calcul time scale

La majorité des rappels qui suivent est extraite des ouvrages [2, 4, 38, 39] de M. Bohner et al. auxquels nous renvoyons pour une étude approfondie de la théorie du calcul time scale.

La Section I.4 traite du cas particulier des time scales avec opérateur σ ∇-dérivable et présente de nouveaux résultats extraits de Bourdin L., Non shifted calculus of variations on time scales with Nabla-differentiable Sigma, preprint arXiv:1302.3623. *En particulier, sous certaines hypothèses de régularité, nous démontrons l'égalité suivante :*

$$(q^\sigma)^\nabla = \sigma^\nabla q^\Delta,$$

et nous déduisons la formule de Leibniz qui suit :

$$(q^\sigma \cdot q')^\nabla = q \cdot q'^\nabla + \sigma^\nabla q^\Delta \cdot q'.$$

Ces deux formules seront utilisées dans le Chapitre II dans le domaine du calcul des variations sur time scale.

I.1 Introduction

The *time scale* theory was introduced in 1988 by S. Hilger in his PhD thesis [111] (under the supervision of B. Auldbach) in order to unify the discrete and continuous analyses. The general idea is to prove a general result for a dynamical equation on an arbitrary non-empty *closed* subset $\mathbb{T}$ of $\mathbb{R}$. Such a subset $\mathbb{T}$ is the so-called *time scale*. Choosing $\mathbb{T} = \mathbb{R}$ yields to a continuous-time analysis while choosing $\mathbb{T} = \mathbb{Z}$ leads to its discrete-time counterpart.

There exist many other time scales like $\mathbb{N}$, $[0,1] \cup \{2,3,4\}$, the Cantor set, etc. As a consequence, the time scale theory allows to treat more general models of processes, *e.g.* involving both continuous- and discrete-time elements. We refer to [93, 163] for an application to a dynamical population whose generations do not overlap. Another example of application is to consider $\mathbb{T} = \{0\} \cup \lambda^{\mathbb{N}}$ with $0 < \lambda < 1$ allowing to cover the quantum calculus [131]. More exotically, the time scale calculus also treats dynamical systems where the time evolves along a set of a complex nature which may even be a Cantor set. Hence, by proving a result on a general time scale, one has a much more general result than just continuous- and discrete-time versions, so *unification* and *extension* can be given as the two main features of the time scale theory.

S. Hilger defined in [111] the notions of Δ- and ∇-derivatives for functions q defined on a time scale $\mathbb{T}$ and derived fundamental results (*e.g.* product and quotient rules, Leibniz formula, etc.). In particular, note that:

- in the continuous setting $\mathbb{T} = \mathbb{R}$, the definitions of the Δ- and ∇-operators coincide with the classical notion of derivative d/dt, *i.e.* $q^\Delta = q^\nabla = \dot{q} = dq/dt$;
- in the discrete case $\mathbb{T} = \mathbb{Z}$, the Δ-derivative coincides with the forward difference operator, *i.e.* $q^\Delta(t) = \Delta q(t) = q(t+1) - q(t)$. The ∇-derivative is the backward one, *i.e.* $q^\nabla(t) = \nabla q(t) = q(t) - q(t-1)$.

S. Hilger also introduced the associated notions of Δ- and ∇-integrals that coincide with the usual integral in the case $\mathbb{T} = \mathbb{R}$ and that are summations in the case $\mathbb{T} = \mathbb{Z}$.

Many results concerning differential equations carry over quite easily to corresponding results for difference equations, while other results seem to be completely different in nature from their discrete counterparts. The study of dynamical equations on time scale reveals such discrepancies. Let us give a concrete and simple example showing how the time scale theory allows to close the gap between continuous and discrete analyses. It is well known that nonzero solutions of the second order differential equation $d/dt(a\dot{q})(t) + b(t)q(t) = 0$ on $\mathbb{T} = \mathbb{R}$ are associated, via the Riccati substitution $z = a\dot{q}/q$, to solutions of the differential Riccati equation

$$\dot{z}(t) + b(t) + \frac{z^2(t)}{a(t)} = 0. \tag{I.1.1}$$

In the discrete case $\mathbb{T} = \mathbb{Z}$, the well-known second order difference equation $\Delta(a\Delta q)(t) + b(t)q(t+1) = 0$ has nonzero solutions that are associated, via the substitution $z = a\Delta q/q$, to the solutions of the discrete Riccati equation

$$\Delta z(t) + b(t) + \frac{z^2(t)}{z(t) + a(t)} = 0. \tag{I.1.2}$$

The techniques of the time scale calculus unify these two previous results and also explain why the discrete Riccati equation has a different form from its continuous counterpart. Indeed, in the general time scale setting, it turns out that nonzero solutions for the dynamical equation $(aq^{\Delta})^{\Delta}(t) + b(t)q^{\sigma}(t) = 0$ are associated, via the substitution $z = aq^{\Delta}/q$, to the solutions of the time scale Riccati equation

$$z^{\Delta}(t) + b(t) + \frac{z^2(t)}{\mu(t)z(t) + a(t)} = 0. \tag{I.1.3}$$

Hence, since $\mu = 0$ in the case $\mathbb{T} = \mathbb{R}$ and $\mu = 1$ in the case $\mathbb{T} = \mathbb{Z}$, the time scale Riccati equation explains more precisely why the Riccati equation looks different depending on whether we are in the discrete or continuous framework. Furthermore, note that the result is also extended to any time scale.

Many theories of standard discrete and continuous calculus have been extended to the time scale setting. We can cite the Mean Value Theorem, Taylor developments, Gronwall's inequality, Hölder's inequality, etc. We refer to the surveys [2, 3, 38, 39] of M. Bohner *et al* for more examples. However, due to the recency of the field, the basic nonlinear theory on time scale is yet to be developed and refined. Part A of the present manuscript contributes to the calculus of variations and to the optimal control theory posed on time scale. The purpose of this chapter is to recall basic definitions, notations and properties on time scale calculus.

Organization of the chapter. Section I.2 is concerned with basic notions and properties on the structure of time scales, Δ-differentiability and Cauchy Δ-integration. In Section I.3, we give recalls on Lebesgue Δ-measure, Lebesgue Δ-integration and absolute continuity on time scale. Finally, Section I.4 deals with time scales with ∇-differentiable σ and presents some results extracted from [45] useful for Chapter II.

I.2 Basic notions and properties

In the whole Part A, $\mathbb{T}$ denotes a *time scale*, *i.e.* an arbitrary non-empty closed subset of $\mathbb{R}$. For any $A \subset \mathbb{R}$, we denote by $A_{\mathbb{T}} = A \cap \mathbb{T}$. In particular, an interval of $\mathbb{T}$ is defined by $I_{\mathbb{T}}$ where I is an interval of $\mathbb{R}$.

Section I.2.1 is devoted to basic recalls on time scales and Δ-differentiability and Section I.2.2 to notions and properties of the Cauchy Δ-integral. The following sections are both extracted from the survey [38].

I.2.1 Structure of time scales, Δ- and ∇-differentiabilities

The backward and forward jump operators ρ, $\sigma : \mathbb{T} \longrightarrow \mathbb{T}$ are respectively defined by

$$\begin{aligned} \rho(t) &= \sup\{s \in \mathbb{T},\ s < t\}, \\ \sigma(t) &= \inf\{s \in \mathbb{T},\ s > t\}, \end{aligned} \tag{I.2.4}$$

for every $t \in \mathbb{T}$, where we put $\rho(\min \mathbb{T}) = \min \mathbb{T}$ (resp. $\sigma(\max \mathbb{T}) = \max \mathbb{T}$) whenever $\mathbb{T}$ admits a minimum (resp. a maximum). A point $t \in \mathbb{T}$ is said to be left-scattered (resp. right-scattered) if $\rho(t) < t$ (resp. $\sigma(t) > t$). A point $t \in \mathbb{T}$ is said to be left-dense (resp. right-dense) if $\rho(t) = t$ and $t > \inf \mathbb{T}$ (resp. $\sigma(t) = t$ and $t < \sup \mathbb{T}$). Let LD (resp. LS, RD and RS) denote the set of all left-dense (resp. left-scattered, right-dense and right-scattered) points of $\mathbb{T}$. The graininess (resp. backward graininess) function $\mu : \mathbb{T} \longrightarrow \mathbb{R}^+$ (resp. $\nu : \mathbb{T} \longrightarrow \mathbb{R}^+$) is defined by $\mu(t) = \sigma(t) - t$ (resp. $\nu(t) = t - \rho(t)$) for any $t \in \mathbb{T}$.

We set $\mathbb{T}_\kappa = \mathbb{T}\backslash\{\min \mathbb{T}\}$ whenever $\mathbb{T}$ admits a right-scattered minimum, and $\mathbb{T}_\kappa = \mathbb{T}$ otherwise. Similarly, we set $\mathbb{T}^\kappa = \mathbb{T}\backslash\{\max \mathbb{T}\}$ whenever $\mathbb{T}$ admits a left-scattered maximum, and $\mathbb{T}^\kappa = \mathbb{T}$ otherwise. Finally we denote by $\mathbb{T}^\kappa_\kappa = \mathbb{T}^\kappa \cap \mathbb{T}_\kappa$.

A function $q : \mathbb{T} \longrightarrow \mathbb{R}^n$, where $n \in \mathbb{N}^*$, is said to be Δ-differentiable at $t \in \mathbb{T}^\kappa$ (resp. ∇-differentiable at $t \in \mathbb{T}_\kappa$) if the following limit exists in $\mathbb{R}^n$:

$$\lim_{\substack{s\to t \\ s\neq\sigma(t)}} \frac{q(\sigma(t)) - q(s)}{\sigma(t) - s} \quad \left(\text{resp. } \lim_{\substack{s\to t \\ s\neq\rho(t)}} \frac{q(s) - q(\rho(t))}{s - \rho(t)} \right). \tag{I.2.5}$$

In such a case, this limit is denoted by $q^\Delta(t)$ (resp. $q^\nabla(t)$). The following results on Δ-differentiability are respectively proved in [38, Theorem 1.16 p.5], [38, Corollary 1.20 p.8] and [38, Corollary 1.68 p.25]. Analogous results for ∇-differentiability are also valid.

Proposition I.1. *Let $q : \mathbb{T} \longrightarrow \mathbb{R}^n$ and $t \in \mathbb{T}^\kappa$. The following properties hold:*

1. *if q is Δ-differentiable at t, then q is continuous at t.*
2. *if $t \in \mathrm{RS}$ and if q is continuous at t, then q is Δ-differentiable at t with*

$$q^\Delta(t) = \frac{q(\sigma(t)) - q(t)}{\mu(t)}. \tag{I.2.6}$$

3. *if $t \in \mathrm{RD}$, then q is Δ-differentiable at t if and only if the following limit exists in $\mathbb{R}^n$:*

$$\lim_{\substack{s\to t \\ s\neq t}} \frac{q(t) - q(s)}{t - s}. \tag{I.2.7}$$

In such a case, this limit is equal to $q^\Delta(t)$.

Proposition I.2 (Leibniz formula)**.** *Let q, $q' : \mathbb{T} \longrightarrow \mathbb{R}^n$. If q and q' are Δ-differentiable at $t \in \mathbb{T}^\kappa$, then the scalar product $q \cdot q'$ is Δ-differentiable at t and the following Leibniz formula holds:*

$$(q \cdot q')^\Delta(t) = q^\Delta(t) \cdot q'(\sigma(t)) + q(t) \cdot q'^\Delta(t) = q^\Delta(t) \cdot q'(t) + q(\sigma(t)) \cdot q'^\Delta(t). \tag{I.2.8}$$

Proposition I.3. *Let $q : \mathbb{T} \longrightarrow \mathbb{R}^n$. Then, q is Δ-differentiable on $\mathbb{T}^\kappa$ with $q^\Delta = 0$ if and only if there exists $c \in \mathbb{R}^n$ such that $q(t) = c$ for every $t \in \mathbb{T}$.*

For every $t \in \mathrm{RS}$, it follows from Proposition I.1 that a function q is Δ-differentiable at t if and only if q is continuous at t.

From Proposition I.1, every Δ-differentiable function on $\mathbb{T}^\kappa$ is continuous on $\mathbb{T}$. In what follows, we denote by $\mathrm{C}(\mathbb{T}, \mathbb{R}^n)$ the functional space of continuous functions on $\mathbb{T}$ with values in $\mathbb{R}^n$ and by $\mathrm{C}^{1,\Delta}(\mathbb{T}, \mathbb{R}^n)$ the functional subspace of all Δ-differentiable functions on $\mathbb{T}^\kappa$ with continuous Δ-derivative.

I.2.2 Rd-continuous functions and Cauchy Δ-integral

A function is said to be rd-continuous on $\mathbb{T}$ if it is continuous at every $t \in \mathrm{RD}$ and if it admits a left-sided limit at every $t \in \mathrm{LD}$, see [38, Definition 1.58 p.22]. $\mathrm{C_{rd}}(\mathbb{T}, \mathbb{R}^n)$ denotes the functional space of rd-continuous functions on $\mathbb{T}$ with values in $\mathbb{R}^n$ and $\mathrm{C}^{1,\Delta}_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^n)$ denotes the functional subspace of Δ-differentiable functions on $\mathbb{T}^\kappa$ with rd-continuous Δ-derivative. The following results are proved in [38, Theorem 1.60 p.22]:

- σ is rd-continuous.
- if $q \in \mathrm{C_{rd}}(\mathbb{T}, \mathbb{R}^n)$, then the composition $q^\sigma = q \circ \sigma$ is rd-continuous.
- if $q \in \mathrm{C_{rd}}(\mathbb{T}, \mathbb{R}^n)$, then the composition $\varphi \circ q$ with any continuous function φ is rd-continuous.
- if q, $q' \in \mathrm{C_{rd}}(\mathbb{T}, \mathbb{R}^n)$, then the scalar product $q \cdot q' \in \mathrm{C_{rd}}(\mathbb{T}, \mathbb{R})$.

Every function $q \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T}^\kappa, \mathbb{R}^n)$ admits a Δ-antiderivative $Q \in \mathrm{C}^{1,\Delta}_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^n)$ in the sense that $Q^\Delta = q$ on $\mathbb{T}^\kappa$, see [38, Theorem 1.74 p.27]. Then, for every a, $b \in \mathbb{T}$, the Cauchy Δ-integral of q is defined by

$$\int_a^b q(\tau)\ \Delta\tau = Q(b) - Q(a). \tag{I.2.9}$$

In particular, for every $q \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T}^\kappa, \mathbb{R}^n)$ and every $a \in \mathbb{T}$, the function Q, defined by $Q(t) = \int_a^t q(\tau)\Delta\tau$ for every $t \in \mathbb{T}$, is the unique Δ-antiderivative of q vanishing at $t = a$.

From the Leibniz formula given in Proposition I.2, one can easily derive the following integration by parts formula. For every q, $q' \in \mathrm{C}^{1,\Delta}_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^n)$ and every a, $b \in \mathbb{T}$, it holds

$$\int_a^b q^\Delta(\tau) \cdot q'(\tau)\ \Delta\tau = q(b) \cdot q'(b) - q(a) \cdot q'(a) - \int_a^b q^\sigma(\tau) \cdot q'^\Delta(\tau)\ \Delta\tau. \tag{I.2.10}$$

I.3 Recalls on Lebesgue Δ-integrability

The concept of Cauchy Δ-integration, introduced by means of Δ-antiderivatives of rd-continuous functions, was recalled in Section I.2.2. As in the continuous case, this notion can be extended to a larger class of functions by considering measure theory and Lebesgue integration, see [66, 103] for a general theory. Such an extension for time scales was briefly considered in [22] and also independently in [23].

A Lebesgue Δ-measure on time scale is introduced by G. Guseinov in [99, Section 5] in terms of Carathéodory extension and a Lebesgue Δ-integration is derived. We give basic recalls on this concept in Section I.3.1. A. Cabada and D.R. Vivero give in [59] an expression of this Lebesgue Δ-integral as an usual Lebesgue integral and deduce criterions for absolute continuity on time scale in [58]. Section I.3.2 is devoted to reminders on absolutely continuous functions on time scale.

I.3.1 Lebesgue Δ-measure and Lebesgue Δ-integral

Recall that the set of right-scattered points RS is at most countable, see [59, Lemma 3.1].

Let μ_Δ denote the Lebesgue Δ-measure on $\mathbb{T}$ defined in terms of Carathéodory extension, see [99, Section 5] or [39, Chapter 5 p.157]. In particular, for all elements a, b of $\mathbb{T}$ such that $a \leq b$, one has $\mu_\Delta([a, b[_\mathbb{T}) = b - a$. Recall that $A \subset \mathbb{T}$ is a μ_Δ-measurable set of $\mathbb{T}$ if and only if A is an usual μ_L-measurable set of $\mathbb{R}$, where μ_L denotes the usual Lebesgue measure, see [59, Proposition 3.1]. Moreover, if $A \subset \mathbb{T} \backslash \{\sup \mathbb{T}\}$, then

$$\mu_\Delta(A) = \mu_L(A) + \sum_{r \in A \cap \mathrm{RS}} \mu(r). \tag{I.3.11}$$

Let $A \subset \mathbb{T}$. A property is said to hold Δ-almost everywhere (shortly Δ-a.e.) on A if it holds for every $t \in A \backslash A_0$, where $A_0 \subset A$ is some μ_Δ-measurable subset of $\mathbb{T}$ satisfying $\mu_\Delta(A_0) = 0$. In particular, since $\mu_\Delta(\{r\}) = \mu(r) > 0$ for every $r \in \mathrm{RS}$, we conclude that if a property holds Δ-a.e. on A, then it holds for every $r \in A \cap \mathrm{RS}$.

The Lebesgue Δ-integral, also denoted by $\int \Delta\tau$, is defined as the Lebesgue integral associated with the Lebesgue Δ-measure μ_Δ. In particular, all theorems of the general Lebesgue integration theory hold, including the Lebesgue dominated convergence theorem. Finally, let us mention that the Lebesgue Δ-integral coincides with the Cauchy one on rd-continuous functions.

Let $A \subset \mathbb{T} \backslash \{\sup \mathbb{T}\}$ be a μ_Δ-measurable set of $\mathbb{T}$ and let $n \in \mathbb{N}^*$. Consider a function q defined Δ-a.e. on A with values in $\mathbb{R}^n$. Let $\tilde{A} = A \cup]r, \sigma(r)[_{r \in A \cap \mathrm{RS}}$, and let $\tilde{q}$ be the extension of q defined μ_L-a.e. on $\tilde{A}$ by

$$\tilde{q}(t) = \begin{cases} q(t) & \text{if} \quad t \in A \\ q(r) & \text{if} \quad t \in]r, \sigma(r)[\text{ for every } r \in A \cap \mathrm{RS}. \end{cases} \tag{I.3.12}$$

From [59, Proposition 4.1], q is μ_Δ-measurable on A if and only if $\tilde{q}$ is μ_L-measurable on $\tilde{A}$.

For every μ_Δ-measurable sets $A \subset \mathbb{T} \backslash \{\sup \mathbb{T}\}$ and every $n \in \mathbb{N}^*$, recall the definitions of the following functional spaces:

– The functional space $\mathrm{L}^1_{\mathbb{T}}(A,\mathbb{R}^n)$ is the set of all functions q defined Δ-a.e. on A, with values in $\mathbb{R}^n$, that are μ_Δ-measurable on A and such that

$$\int_A \|q(\tau)\| \; \Delta\tau < +\infty, \tag{I.3.13}$$

where $\|\cdot\|$ denotes the usual Euclidean norm of $\mathbb{R}^n$. Endowed with the norm $\|q\|_{\mathrm{L}^1_{\mathbb{T}}(A,\mathbb{R}^n)} = \int_A \|q(\tau)\|\Delta\tau$, it is a Banach space, see [4, Theorem 2.5]. If $q \in \mathrm{L}^1_{\mathbb{T}}(A,\mathbb{R}^n)$ then

$$\int_A q(\tau) \; \Delta\tau = \int_{\tilde{A}} \tilde{q}(\tau) \; d\tau = \int_A q(\tau) \; d\tau + \sum_{r\in A\cap \mathrm{RS}} \mu(r)q(r), \tag{I.3.14}$$

see [59, Theorems 5.1 and 5.2].

– The functional space $\mathrm{L}^\infty_{\mathbb{T}}(A,\mathbb{R}^n)$ is the set of all functions q defined Δ-a.e. on A, with values in $\mathbb{R}^n$, that are μ_Δ-measurable on A and such that

$$\sup_{\tau\in A} \operatorname{ess} \|q(\tau)\| < +\infty. \tag{I.3.15}$$

Endowed with the norm $\|q\|_{\mathrm{L}^\infty_{\mathbb{T}}(A,\mathbb{R}^n)} = \sup_{\tau\in A} \operatorname{ess} \|q(\tau)\|$, it is a Banach space, see [4, Theorem 2.5]. Note that if A is bounded then $\mathrm{L}^\infty_{\mathbb{T}}(A,\mathbb{R}^n) \subset \mathrm{L}^1_{\mathbb{T}}(A,\mathbb{R}^n)$.

Finally, we introduce the functional space $\mathrm{L}^\infty_{\mathrm{loc},\mathbb{T}}(\mathbb{T}\backslash\{\sup\mathbb{T}\},\mathbb{R}^n)$ of all functions q defined Δ-a.e. on $\mathbb{T}\backslash\{\sup\mathbb{T}\}$, with values in $\mathbb{R}^n$, that are μ_Δ-measurable on $\mathbb{T}\backslash\{\sup\mathbb{T}\}$ and such that $q \in \mathrm{L}^\infty_{\mathbb{T}}([a,b[_{\mathbb{T}},\mathbb{R}^m)$ for all elements a, $b \in \mathbb{T}$ such that $a < b$.

I.3.2 Absolutely continuous functions

Let a, $b \in \mathbb{T}$ such that $a < b$. In the sequel, $\mathrm{AC}([a,b]_{\mathbb{T}},\mathbb{R}^n)$ denotes the functional space of all absolutely continuous functions on $[a,b]_{\mathbb{T}}$ with values in $\mathbb{R}^n$. The two following results hold:

Proposition I.4. *Let $t_0 \in [a,b]_{\mathbb{T}}$ and $q : [a,b]_{\mathbb{T}} \longrightarrow \mathbb{R}^n$. Then, $q \in \mathrm{AC}([a,b]_{\mathbb{T}},\mathbb{R}^n)$ if and only if the following conditions are both satisfied:*

1. *q is Δ-differentiable Δ-a.e. on $[a,b[_{\mathbb{T}}$ and $q^\Delta \in \mathrm{L}^1_{\mathbb{T}}([a,b[_{\mathbb{T}},\mathbb{R}^n)$;*
2. *for every $t \in [a,b]_{\mathbb{T}}$, it holds*

$$q(t) = q(t_0) + \int_{[t_0,t[_{\mathbb{T}}} q^\Delta(\tau) \; \Delta\tau \tag{I.3.16}$$

whenever $t \geq t_0$, and

$$q(t) = q(t_0) - \int_{[t,t_0[_{\mathbb{T}}} q^\Delta(\tau) \; \Delta\tau \tag{I.3.17}$$

whenever $t \leq t_0$.

This result can easily be derived from [58, Theorem 4.1]. By combining Proposition I.4 and the usual Lebesgue point theory in $\mathbb{R}$, we infer the following result, see also [206] for a similar one.

Proposition I.5. *Let $t_0 \in [a,b]_{\mathbb{T}}$ and $q \in \mathrm{L}^1_{\mathbb{T}}([a,b[_{\mathbb{T}},\mathbb{R}^n)$. Let Q be the function defined on $[a,b]_{\mathbb{T}}$ by*

$$Q(t) = \int_{[t_0,t[_{\mathbb{T}}} q(\tau) \; \Delta\tau \tag{I.3.18}$$

if $t \geq t_0$, and by

$$Q(t) = -\int_{[t,t_0[_{\mathbb{T}}} q(\tau) \; \Delta\tau \tag{I.3.19}$$

if $t \leq t_0$. Then $Q \in \mathrm{AC}([a,b]_{\mathbb{T}},\mathbb{R}^n)$ and $Q^\Delta = q$ Δ-a.e. on $[a,b[_{\mathbb{T}}$.

Note the following properties:

– if $q \in \mathrm{AC}([a,b]_{\mathbb{T}},\mathbb{R}^n)$ with $q^\Delta = 0$ Δ-a.e. on $[a,b[_{\mathbb{T}}$, then q is constant on $[a,b]_{\mathbb{T}}$;
– if q, $q' \in \mathrm{AC}([a,b]_{\mathbb{T}},\mathbb{R}^n)$, then the scalar product $q \cdot q' \in \mathrm{AC}([a,b]_{\mathbb{T}},\mathbb{R})$.

In what follows, for every $q \in \mathrm{L}^1_{\mathbb{T}}([a,b[_{\mathbb{T}},\mathbb{R}^n)$, we denote by

$$\mathscr{L}_{[a,b[_{\mathbb{T}}}(q) = \{t \in [a,b[_{\mathbb{T}},\ t \text{ is a } \Delta\text{-Lebesgue point of } q\}. \tag{I.3.20}$$

From Proposition I.5, the equality $\mu_\Delta(\mathscr{L}_{[a,b[_{\mathbb{T}}}(q)) = b-a$ holds.

I.4 Complements on time scales with ∇-differentiable σ

In this section, some contributions to the study of time scales with ∇-differentiable σ are provided. Precisely, we give characterizations of time scales with continuous σ (Section I.4.1) and with ∇-differentiable σ (Section I.4.2). Some concrete examples and remarks are given. Finally, in Section I.4.3, a formula for ∇-derivatives of type $(q^\sigma)^\nabla$ is stated (see Theorem I.1) and a Leibniz formula is deduced (see Proposition I.8).

These results are introduced in order to be applied in Chapter II.

I.4.1 Continuity of σ

We first prove the following characterizations of the continuity of σ at a point $t \in \mathbb{T}_\kappa$.

Proposition I.6. *Let $t \in \mathbb{T}_\kappa$. The following properties are equivalent:*

1. *σ is continuous at t;*
2. *$\sigma \circ \rho(t) = t$;*
3. *$t \notin \mathrm{RS} \cap \mathrm{LD}$.*

Proof. Let us prove that 1. implies 2.. By contradiction, we assume that $\sigma \circ \rho(t) \neq t$. Necessarily, we have $t \in \mathrm{RS}\cap\mathrm{LD}$. As a consequence, $t \neq \min\mathbb{T}$ (if it exists) since $t \in \mathbb{T}_\kappa$. Then, let $(s_k) \subset \mathbb{T}$ be a sequence such that $s_k < t$ for any $k \in \mathbb{N}$ and $s_k \to t$. Thus, we have $\sigma(s_k) < t < \sigma(t)$ for any $k \in \mathbb{N}$ and consequently, $(\sigma(s_k))$ does not tend to $\sigma(t)$. This is a contradiction with the continuity of σ at t.

Let us prove that 2. implies 3.. If $t \in \mathrm{RS}\cap\mathrm{LD}$, then $\sigma\circ\rho(t) = \sigma(t) \neq t$.

Let us prove that 3. implies 1.. By contradiction, we assume that σ is not continuous at t. As a consequence, there exists $\varepsilon > 0$ and a monotone sequence (s_k) such that $s_k \to t$ and $|\sigma(t)-\sigma(s_k)| \geq \varepsilon$ for every $k \in \mathbb{N}$. Firstly, let us assume that (s_k) is decreasing. Then, we have $t < s_k < s_{k-1}$ and then $t \leq \sigma(t) \leq \sigma(s_k) \leq s_{k-1}$ for any $k \in \mathbb{N}^*$. It is a contradiction since $s_{k-1} \to t$. Secondly, let us assume that (s_k) is increasing. As a consequence, $t \in \mathrm{LD}$ and then $t \in \mathrm{RD}$ (see 3.). Finally, we have $s_{k-1} < s_k < t$ and then, $s_{k-1} < \sigma(s_k) \leq t = \sigma(t)$ for any $k \in \mathbb{N}^*$. It is a contradiction since $s_{k-1} \to t$. In the two cases, we have obtained a contradiction. ☐

If $\mathbb{T}$ admits a minimum, note that σ is continuous at $\min\mathbb{T}$. Indeed, if $\min\mathbb{T} \in \mathrm{RS}$, then $\min\mathbb{T}$ is isolated and thus, σ is continuous at $\min\mathbb{T}$. If $\min\mathbb{T} \in \mathrm{RD}$, then $\min\mathbb{T} \in \mathbb{T}_\kappa$, $\min\mathbb{T} \notin \mathrm{RS}\cap\mathrm{LD}$ and Proposition I.6 concludes.

Consequently, σ is continuous at $t \in \mathbb{T}$ if and only if $t = \min\mathbb{T}$ (if it exists) or (non-exclusive) $t \notin \mathrm{RS}\cap\mathrm{LD}$. Finally, σ is continuous on $\mathbb{T}$ if and only if $\mathrm{RS}\cap\mathrm{LD}\backslash\{\inf\mathbb{T}\} = \emptyset$. It means that σ is continuous on $\mathbb{T}$ if and only if no point of $\mathbb{T}\backslash\{\inf\mathbb{T}\}$ is right-scattered and left-dense. A similar remark is already done in [38, Example 1.55].

Let us give some examples and counterexamples.

Example I.1.
1. If $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = [0,1]$, σ is continuous on $\mathbb{T}$.
2. If $\mathbb{T} = \mathbb{Z}$ or $\mathrm{card}(\mathbb{T}) < \infty$, σ is continuous on $\mathbb{T}$.
3. If $\mathbb{T} = \mathbb{Z}^- \cup [1,+\infty[$, σ is continuous on $\mathbb{T}$.
4. If $\mathbb{T} = \mathbb{R}^- \cup \{1/k,\ k \in \mathbb{N}^*\} \cup [2,+\infty[$, σ is continuous on $\mathbb{T}$.
5. If $\mathbb{T} = [0,1]\cup[2,3]$, σ is not continuous at $1 \in \mathrm{RS}\cap\mathrm{LD}\backslash\{0\}$.
6. If $\mathbb{T}$ is the usual Cantor set (see [38, Example 1.47 p.18]), σ is not continuous at $1/3 \in \mathrm{RS}\cap\mathrm{LD}\backslash\{0\}$.

Remark I.1. If σ is continuous on $\mathbb{T}$, then every $t \in \mathrm{RS}$ is isolated. In such a case, every function defined on $\mathbb{T}$ is directly continuous and Δ-differentiable at every $t \in \mathrm{RS}$, see Proposition I.1. Consequently, $\mathrm{C_{rd}}(\mathbb{T},\mathbb{R}^n)$ coincides with $\mathrm{C}(\mathbb{T},\mathbb{R}^n)$ and similarly, $\mathrm{C}^{1,\Delta}_{\mathrm{rd}}(\mathbb{T},\mathbb{R}^n)$ coincides with $\mathrm{C}^{1,\Delta}(\mathbb{T},\mathbb{R}^n)$.

I.4.2 ∇-differentiability of σ

From Proposition I.6, we derive the following result.

Proposition I.7. *The following properties are satisfied:*

1. *if σ is continuous at $t \in \mathrm{LS}$, then σ is directly ∇-differentiable at t with $\sigma^\nabla(t) = \mu(t)/\nu(t)$.*
2. *if σ is continuous on $\mathbb{T}$, then σ is ∇-differentiable on $\mathbb{T}_\kappa$ if and only if for every $t \in \mathrm{LD} \cap \mathbb{T}_\kappa$, the following limit exist in $\mathbb{R}$:*
$$\lim_{\substack{s\to t\\ s\neq t}} \frac{\sigma(s)-t}{s-t}. \tag{I.4.21}$$
In such a case, this limit is equal to $\sigma^\nabla(t)$.

Proof. Let us prove the first point. From Proposition I.1 and since σ is continuous at $t \in \mathrm{LS} \subset \mathbb{T}_\kappa$, σ is directly ∇-differentiable at t with
$$\sigma^\nabla(t) = \frac{\sigma(t)-\sigma(\rho(t))}{\nu(t)} = \frac{\sigma(t)-t}{\nu(t)} = \frac{\mu(t)}{\nu(t)}, \tag{I.4.22}$$
since $\sigma \circ \rho(t) = t$ from Proposition I.6.

Let us prove the second point. Since σ is continuous on $\mathbb{T}$, σ is directly ∇-differentiable at every $t \in \mathrm{LS}$ from the first point. Consequently, σ is ∇-differentiable on $\mathbb{T}_\kappa$ if and only if σ is ∇-differentiable at every $t \in \mathrm{LD} \cap \mathbb{T}_\kappa$ *i.e.* if and only if for every $t \in \mathrm{LD} \cap \mathbb{T}_\kappa$, the following limit exists in $\mathbb{R}$:
$$\lim_{\substack{s\to t\\ s\neq t}} \frac{\sigma(s)-\sigma(t)}{s-t}. \tag{I.4.23}$$
To conclude, it is sufficient to note that the continuity of σ implies $\mathrm{LD} \cap \mathbb{T}_\kappa \subset \mathrm{RD}$, see Proposition I.6. The proof is complete. ☐

Let us give some examples of time scale with ∇-differentiable σ.

Example I.2.
1. If $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = [0,1]$, σ is ∇-differentiable on $\mathbb{T}_\kappa$ with $\sigma^\nabla = 1$.
2. If $\mathbb{T} = \mathbb{Z}$ or $\mathrm{card}(\mathbb{T}) < \infty$, σ is ∇-differentiable on $\mathbb{T}_\kappa$ with $\sigma^\nabla = \mu/\nu$.
3. If $\mathbb{T} = \{0\} \cup \{z_k,\ k \in \mathbb{N}\}$ where (z_k) is a decreasing positive sequence tending to 0 and if $\lim_{k\to\infty} z_{k-1}/z_k$ exists (denoted by ℓ), then σ is ∇-differentiable on $\mathbb{T}_\kappa$. In particular, we have $\sigma^\nabla(0) = \ell$. Indeed, let $(s_k) \subset \mathbb{T}$ be a positive sequence tending to 0. Then, for every $k \in \mathbb{N}$, there exists $p_k \in \mathbb{N}$ such that $s_k = z_{p_k}$. Since $s_k \to 0$, we have $p_k \to +\infty$. Finally, we obtain
$$\lim_{k\to\infty} \frac{\sigma(s_k)-0}{s_k-0} = \lim_{k\to\infty} \frac{z_{p_k-1}}{z_{p_k}} = \ell. \tag{I.4.24}$$
4. Application: if $\mathbb{T} = \{0\} \cup \{1/r^k,\ k \in \mathbb{N}\}$ with $r > 1$, then σ is ∇-differentiable on $\mathbb{T}_\kappa$. In particular, we have $\sigma^\nabla(0) = r$.
5. Similarly to 3., we can prove that if $\mathbb{T} = \{0\} \cup \{z_k,\ k \in \mathbb{N}\}$ where (z_k) is an increasing negative sequence tending to 0 and if $\lim_{k\to\infty} z_{k+1}/z_k$ exists (denoted by ℓ), then σ is ∇-differentiable on $\mathbb{T}_\kappa$. In particular, we have $\sigma^\nabla(0) = \ell$.
6. Application: if $\mathbb{T} = \{0\} \cup \{-1/k,\ k \in \mathbb{N}^*\}$, then σ is ∇-differentiable on $\mathbb{T}_\kappa$. In particular, we have $\sigma^\nabla(0) = 1$.
7. Similarly to 3., we can prove that if $\mathbb{T} = [-1,0] \cup \{z_k,\ k \in \mathbb{N}\}$ where (z_k) is a decreasing positive sequence tending to 0 and if $\lim_{k\to\infty} z_{k-1}/z_k = 1$, then σ is ∇-differentiable on $\mathbb{T}_\kappa$. In particular, we have $\sigma^\nabla(0) = 1$.
8. Application: if $\mathbb{T} = [-1,0] \cup \{1/k^2,\ k \in \mathbb{N}^*\}$, then σ is ∇-differentiable on $\mathbb{T}_\kappa$. In particular, we have $\sigma^\nabla(0) = 1$.

9. Similarly to 3., we can prove that if $\mathbb{T} = \{0\} \cup \{z_k^-,\ k \in \mathbb{N}\} \cup \{z_k^+,\ k \in \mathbb{N}\}$ where (z_k^-) (resp. (z_k^+)) is an increasing negative (resp. decreasing positive) sequence tending to 0 and if $\lim_{k\to\infty} z_{k+1}^-/z_k^- = \lim_{k\to\infty} z_{k-1}^+/z_k^+ = \ell$, then σ is ∇-differentiable on $\mathbb{T}_\kappa$. In particular, we have $\sigma^\nabla(0) = \ell$. Note that, in such a case, we can only have $\ell = 1$ since $z_{k+1}^-/z_k^- < 1 < z_{k-1}^+/z_k^+$ for every $k \in \mathbb{N}$.
10. Application: if $\mathbb{T} = \{0\} \cup \{z_k^-,\ k \in \mathbb{N}\} \cup \{z_k^+,\ k \in \mathbb{N}\}$ with $z_k^- = -1/k$ and $z_k^+ = 1/k^2$ for every $k \in \mathbb{N}^*$, then σ is ∇-differentiable on $\mathbb{T}_\kappa$. In particular, we have $\sigma^\nabla(0) = 1$.

Let us give some examples of time scale with continuous but non ∇-differentiable σ.

Example I.3. 1. If $\mathbb{T} = \{0\} \cup \{1/k!,\ k \in \mathbb{N}\}$, then σ is not ∇-differentiable in 0 since $k!/(k-1)! = k$ tends to $+\infty$.
2. If $\mathbb{T} = [-1, 0] \cup \{1/2^k,\ k \in \mathbb{N}\}$, then σ is not ∇-differentiable in 0 since $2^k/2^{k-1} = 2$ does not tend to 1.
3. If $\mathbb{T} = \{0\} \cup \{\pm 1/2^k,\ k \in \mathbb{N}\}$, then σ is not ∇-differentiable in 0 since $2^k/2^{k-1} = 2$, $2^k/2^{k+1} = 1/2$ and $2 \neq 1/2$.

Examples I.1, I.2 and I.3 allow us to get a better understanding of the restrictions imposed on a time scale by the ∇-differentiability of σ. Indeed, we conclude that such a time scale has to satisfy the following properties:
– Due to the continuity of σ, no point (except $\min \mathbb{T}$ if it exists) can be right-scattered and left-dense.
– Due to the ∇-differentiability of σ, the density in a dense point is not "too weak" in contrary to 1. in Example I.3. Moreover, in a left and right-dense point, the left and the right densities have to be "homogeneous" with limit equal to 1, as in 8., 10. of Example I.2 and in contrary to 2., 3. of Example I.3.

I.4.3 Main result and corollaries

The most important result of Section I.4 is the following.

Theorem I.1 (Main result)**.** *Let $q : \mathbb{T} \longrightarrow \mathbb{R}^n$ and let $t \in \mathbb{T}^\kappa_\kappa$. If the two following properties are satisfied:*
– *σ is ∇-differentiable at t;*
– *q is Δ-differentiable at t;*
then, q^σ is ∇-differentiable at t with $(q^\sigma)^\nabla(t) = \sigma^\nabla(t) q^\Delta(t)$.

Proof. Since σ is continuous at t, $\sigma \circ \rho(t) = t$ from Proposition I.6. We distinguish two cases: $t \in \mathrm{LS}$ and $t \in \mathrm{LD}$.
– Firstly, we consider that $t \in \mathrm{LS}$. Since σ is continuous at t, we have $\sigma^\nabla(t) = \mu(t)/\nu(t)$, see Proposition I.7. If moreover $t \in \mathrm{RS}$, then $q^\Delta(t) = (q(\sigma(t)) - q(t))/\mu(t)$ and since t is isolated, q^σ is ∇-differentiable at t with
$$(q^\sigma)^\nabla(t) = \frac{q^\sigma(t) - q^\sigma(\rho(t))}{\nu(t)} = \frac{q(\sigma(t)) - q(t)}{\nu(t)} = \frac{\mu(t)}{\nu(t)} q^\Delta(t) = \sigma^\nabla(t) q^\Delta(t). \tag{I.4.25}$$
Else $t \in \mathrm{RD}$, since σ is continuous at t and since q is continuous at $t = \sigma(t)$, we deduce that q^σ is continuous at $t \in \mathrm{LS}$. Then, from Proposition I.1, q^σ is ∇-differentiable at t with
$$(q^\sigma)^\nabla(t) = \frac{q^\sigma(t) - q^\sigma(\rho(t))}{\nu(t)} = \frac{q(\sigma(t)) - q(t)}{\nu(t)} = 0, \tag{I.4.26}$$
since $\sigma(t) = t$. However, in this case, we have $\sigma^\nabla(t) = \mu(t)/\nu(t) = 0$. Consequently, we also retrieve $(q^\sigma)^\nabla(t) = \sigma^\nabla(t) q^\Delta(t)$ in this case.
– Secondly, we consider that $t \in \mathrm{LD}$. Since σ is continuous at t and since $t \in \mathbb{T}_\kappa$, then $t \in \mathrm{RD}$ from Proposition I.6. Finally, since q is Δ-differentiable at t and since σ is ∇-differentiable at t, we get
$$\lim_{\substack{s \to t \\ s \neq t}} \frac{q^\sigma(s) - q^\sigma(t)}{s - t} = \lim_{\substack{s \to t \\ s \neq t}} \frac{\sigma(s) - t}{s - t} \ \frac{q(\sigma(s)) - q(t)}{\sigma(s) - t} = \sigma^\nabla(t) q^\Delta(t). \tag{I.4.27}$$
In the previous limit, since σ is continuous at $t \in \mathrm{LD} \cap \mathrm{RD}$, we have used that $s \to t$, $s \neq t$ implies that $\sigma(s) \to \sigma(t) = t$, $\sigma(s) \neq t$.

The proof is complete. □

The following corollary is directly derived from Theorem I.1.

Corollary I.1. *Let $q : \mathbb{T} \longrightarrow \mathbb{R}^n$. If the following properties are satisfied:*
– *σ is ∇-differentiable on $\mathbb{T}_\kappa$;*
– *q is Δ-differentiable on $\mathbb{T}^\kappa$;*
then, q^σ is ∇-differentiable at every $t \in \mathbb{T}^\kappa_\kappa$ with $(q^\sigma)^\nabla(t) = \sigma^\nabla(t) q^\Delta(t)$.

We conclude this chapter with the following Leibniz formula.

Proposition I.8 (Leibniz formula)**.** *Let q, $q' : \mathbb{T} \longrightarrow \mathbb{R}^n$ and $t \in \mathbb{T}^\kappa_\kappa$. If the following properties are satisfied:*
– *σ is ∇-differentiable at t;*
– *q is Δ-differentiable at t;*
– *q' is ∇-differentiable at t;*
then, the scalar product $q^\sigma \cdot q'$ is ∇-differentiable at t and the following Leibniz formula holds:

$$(q^\sigma \cdot q')^\nabla(t) = q(t) \cdot q'^\nabla(t) + \sigma^\nabla(t) q^\Delta(t) \cdot q'(t). \tag{I.4.28}$$

Proof. Since σ is continuous at $t \in \mathbb{T}_\kappa$, recall that $\sigma \circ \rho(t) = t$ from Proposition I.6. From Theorem I.1, q^σ is ∇-differentiable at t with $(q^\sigma)^\nabla(t) = \sigma^\nabla(t) q^\Delta(t)$. Finally, from the usual Leibniz formula on time scales (see Proposition I.2), we have $q^\sigma \cdot q'$ is ∇-differentiable at t with

$$(q^\sigma \cdot q')^\nabla(t) = q^\sigma(\rho(t)) \cdot q'^\nabla(t) + (q^\sigma)^\nabla(t) \cdot q'(t) = q(t) \cdot q'^\nabla(t) + \sigma^\nabla(t) q^\Delta(t) \cdot q'(t). \tag{I.4.29}$$

The proof is complete. □

Chapitre II

Contributions au calcul des variations non shifté sur time scale avec opérateur σ ∇-dérivable

Dans le domaine du calcul des variations sur time scale, une équation d'Euler-Lagrange sous forme intégrale est habituellement obtenue pour caractériser les points critiques d'une fonctionnelle Lagrangienne non shiftée, voir [114, Théorème 4].

Dans ce chapitre, nous démontrons que la ∇-dérivabilité de l'opérateur σ est une condition nécessaire et suffisante sur le time scale pour pouvoir ∇-dériver cette équation d'Euler-Lagrange intégrale et ainsi obtenir une version différentielle. Précisément, sous cette hypothèse et grâce aux résultats du Chapitre I, nous démontrons que les points critiques d'une fonctionnelle Lagrangienne non shiftée du type

$$\mathcal{L}(q) = \int_a^b L(q(\tau), q^\Delta(\tau), \tau) \, \Delta\tau$$

sont caractérisés par l'équation d'Euler-Lagrange différentielle suivante (voir Théorème II.1) :

$$\left[\frac{\partial L}{\partial v}(q, q^\Delta, \cdot) \right]^\nabla (t) = \sigma^\nabla(t) \frac{\partial L}{\partial x}(q(t), q^\Delta(t), t).$$

De plus, la formule de Leibniz introduite au Chapitre I appliquée à cette forme différentielle nous permet d'établir un théorème de type Noether assurant l'existence d'une constante de mouvement explicite pour les équations d'Euler-Lagrange différentielles admettant une symétrie (voir Théorème II.2).

Ces résultats englobent les cas classiques continu et discret non shifté. Ce chapitre est issu de Bourdin L., Non shifted calculus of variations on time scales with Nabla-differentiable Sigma, preprint arXiv:1302.3623.

II.1 Introduction

In this chapter, $\mathbb{T}$ denotes a bounded time scale with $a = \min \mathbb{T}$ and $b = \max \mathbb{T}$. We assume that $\text{card}(\mathbb{T}) \geq 3$ ensuring that $\mathbb{T}^\kappa_\kappa \neq \emptyset$.

Context in shifted calculus of variations. The pioneering work on calculus of variations on time scale is due to M. Bohner in 2004, see [36]. In particular, he obtains a necessary condition for local optimizers of Lagrangian functionals of type

$$\mathcal{L}(q) = \int_a^b L(q^\sigma(\tau), q^\Delta(\tau), \tau) \, \Delta\tau. \tag{II.1.1}$$

Precisely, he characterizes the critical points of $\mathcal{L}$ as the solutions of the following differential Euler-Lagrange equation, see [36, Theorem 4.2]:

$$\left[\frac{\partial L}{\partial v}(q^\sigma, q^\Delta, \cdot) \right]^\Delta (t) = \frac{\partial L}{\partial x}(q^\sigma(t), q^\Delta(t), t). \tag{II.1.2}$$

As mentioned in [36], this work recovers the usual continuous case $\mathbb{T} = [a, b]$ where the critical points of Lagrangian functionals of type

$$\mathcal{L}(q) = \int_a^b L(q(\tau), \dot{q}(\tau), \tau)\, d\tau \tag{II.1.3}$$

are characterized by the solutions of the celebrated continuous Euler-Lagrange equation (see *e.g.* [20, p.12]) given by

$$\frac{d}{dt}\left[\frac{\partial L}{\partial v}(q, \dot{q}, \cdot)\right](t) = \frac{\partial L}{\partial x}(q(t), \dot{q}(t), t). \tag{II.1.4}$$

Moreover, the work of M. Bohner in [36] also encompasses the following discrete case $\mathbb{T} = \{a = t_0 < t_1 < \ldots < t_N = b\}$ where the critical points of discrete Lagrangian functionals of type

$$\mathcal{L}(q) = \sum_{k=0}^{N-1} (t_{k+1} - t_k) L(q(t_{k+1}), \Delta q(t_k), t_k), \tag{II.1.5}$$

where Δ is given by $\Delta q(t_k) = (q(t_{k+1}) - q(t_k))/(t_{k+1} - t_k)$, are characterized by the solutions of the well known discrete Euler-Lagrange equation (see *e.g.* [11]) given by

$$\Delta\left[\frac{\partial L}{\partial v}(q^\sigma, \Delta q, \cdot)\right](t_k) = \frac{\partial L}{\partial x}(q(t_{k+1}), \Delta q(t_k), t_k). \tag{II.1.6}$$

In what follows, $\mathcal{L}$ (defined in (II.1.1)) is called *shifted* Lagrangian functional in reference to the presence of $q^\sigma(\tau)$ (instead of $q(\tau)$) in its definition. This characteristic has no consequence on the continuous case. Nevertheless, in the discrete case, let us notice the presence of $q(t_{k+1})$ instead of the more natural $q(t_k)$. We will see that this difference is of importance at the discrete level and *a fortiori* at the time scale one too.

Since the publication of [36], the *shifted* calculus of variations on time scale is widely investigated in several directions: with double integral [37], with higher-order Δ-derivatives [87], with non fixed boundary conditions and transversality conditions [115], with double integral mixing Δ- and ∇-derivatives [156], with higher-order ∇-derivatives [160], etc. Actually, the *shifted* variational problems are particularly suitable because of the emergence of a shift in the integration by parts formula on time scale (see Equality (I.2.10)).

Context in non shifted calculus of variations. The *shifted* calculus of variations on time scale developed in [36] does not cover the following discrete calculus of variations: the critical points of discrete (non shifted) Lagrangian functionals of type

$$\mathcal{L}(q) = \sum_{k=0}^{N-1} (t_{k+1} - t_k) L(q(t_k), \Delta q(t_k), t_k) \tag{II.1.7}$$

are characterized by the solutions of the well known discrete Euler-Lagrange equation (see *e.g.* [101]) given by

$$\nabla\left[\frac{\partial L}{\partial v}(q, \Delta q, \cdot)\right](t_k) = \frac{t_{k+1} - t_k}{t_k - t_{k-1}} \frac{\partial L}{\partial x}(q(t_k), \Delta q(t_k), t_k), \tag{II.1.8}$$

where ∇ is given by $\nabla q(t_k) = (q(t_k) - q(t_{k-1}))/(t_k - t_{k-1})$. The above discrete Euler-Lagrange equation (II.1.8) corresponds to the variational integrator constructed and studied in [101, 159]. It is an efficient numerical scheme for the continuous Euler-Lagrange equation (II.1.4) preserving its variational structure and relative properties at the discrete level. Moreover, as it is well known in numerical analysis, the emergence of the composition between the operators ∇ and Δ provides a higher order of convergence.

Up to our knowledge, only few references treat on *non shifted* calculus of variations on time scale, see [114] or [72, 86]. In these papers, the critical points of (non shifted) Lagrangian functionals of type

$$\mathcal{L}(q) = \int_a^b L(q(\tau), q^\Delta(\tau), \tau)\, \Delta\tau \tag{II.1.9}$$

are characterized by the solutions of the following integral Euler-Lagrange equation, see [114, Theorem 4]:

$$\frac{\partial L}{\partial v}(q(t), q^{\Delta}(t), t) = \int_a^{\sigma(t)} \frac{\partial L}{\partial x}(q(\tau), q^{\Delta}(\tau), \tau)\, \Delta\tau + c. \tag{EL$_{\text{int}}$}$$

The objective of this chapter is to ∇-differentiate (EL_{int}) in order to get a *differential* Euler-Lagrange equation of type

$$\left[\frac{\partial L}{\partial v}(q, q^{\Delta}, \cdot)\right]^{\nabla}(t) = \omega(t)\frac{\partial L}{\partial x}(q(t), q^{\Delta}(t), t) \tag{EL$_{\text{diff}}$}$$

that encompasses the classical continuous and non shifted discrete cases given by (II.1.4) and (II.1.8). In Theorem II.1, we prove that the ∇-differentiability of σ is a sufficient condition on the time scale in order to ∇-differentiate (EL_{int}). Moreover, Example II.1 proves that this assumption is also necessary. We observe that the ∇-differentiability of σ directly emerges in (EL_{diff}) since $\omega = \sigma^{\nabla}$. Let us mention that this hypothesis is not a loss of generality since it is satisfied in the continuous case $\mathbb{T} = [a, b]$ (with $\sigma^{\nabla} = 1$) and in the discrete case $\text{card}(\mathbb{T}) < \infty$ (with $\sigma^{\nabla}(t_k) = (t_{k+1} - t_k)/(t_k - t_{k-1})$), see Section I.4. As a consequence, our main result encompasses both the usual continuous Euler-Lagrange equation (II.1.4) and its non shifted discrete counterpart (II.1.8).

Derivations of Noether-type results. In the *shifted* calculus of variations, the existence of constant of motion for differential Euler-Lagrange equations (II.1.2) is studied in [31], see also [161]. The common strategy is to extend the celebrated Noether's theorem [144, 173] to the time scale setting. Precisely, under some invariance assumption on the Lagrangian L, the authors prove that a conservation law can be obtained.

In the *non shifted* calculus of variations, the non differential form of (EL_{int}) was an obstruction to develop the same strategy. A direct application of the differential form (EL_{diff}) is to provide a Noether-type theorem, see Theorem II.2. This result is based on the Leibniz formula introduced in Proposition I.8.

Remark. For sake of completeness of this introduction, we mention that an Euler-Lagrange equation of *differential* form in the non shifted time scale case is obtained in [114, Remark 4]. Precisely, the author characterizes the critical points of $\mathcal{L}$ (defined in (II.1.9)) as the solutions of the following differential Euler-Lagrange equation:

$$\left(\frac{\partial L}{\partial v}(q, q^{\Delta}, \cdot) - \mu\frac{\partial L}{\partial x}(q, q^{\Delta}, \cdot)\right)^{\Delta}(t) = \frac{\partial L}{\partial x}(q(t), q^{\Delta}(t), t). \tag{II.1.10}$$

The advantage of this result is to be valid on every time scale. Nevertheless, the obtaining of a Noether-type result from this differential shape remains an open problem. This observation gives a particular interest for the ∇-differentiation of (EL_{int}) and the differential formulation (EL_{diff}).

Organization of the chapter. Section II.2 is devoted to recalls on non shifted calculus of variations on time scale. The main results of this chapter are stated in Section II.3. Precisely, in Section II.3.1, in the case where σ is ∇-differentiable, the differential Euler-Lagrange equation of type (EL_{diff}) is given as a characterization for critical points of the (non shifted) Lagrangian functional $\mathcal{L}$ defined in (II.1.9), see Theorem II.1. In Section II.3.2, from this differential form of Euler-Lagrange equation and from the Leibniz formula introduced in Proposition I.8, we prove in Theorem II.2 a Noether-type theorem.

II.2 Recalls on non shifted calculus of variations on general time scales

This section is devoted to reminders on the non shifted calculus of variations on time scale developed in [114], see also [86] and [72, Section 9]. For the reader's convenience, some sketches of proofs are recalled.

Let $\text{C}^{1,\Delta}_{\text{rd},0}(\mathbb{T}, \mathbb{R}^n) = \{w \in \text{C}^{1,\Delta}_{\text{rd}}(\mathbb{T}, \mathbb{R}^n),\ w(a) = w(b) = 0\}$ and let E be a non empty subset of $\text{C}^{1,\Delta}_{\text{rd}}(\mathbb{T}, \mathbb{R}^n)$ open in the $\text{C}^{1,\Delta}_{\text{rd},0}(\mathbb{T}, \mathbb{R}^n)$-direction *i.e.*

$$\forall q \in \text{E},\ \forall w \in \text{C}^{1,\Delta}_{\text{rd},0}(\mathbb{T}, \mathbb{R}^n),\ \exists \eta > 0,\ \forall \varepsilon \in [-\eta, \eta],\ q + \varepsilon w \in \text{E}. \tag{II.2.11}$$

Let L be a Lagrangian *i.e.* a continuous map of class $\mathscr{C}^1$ in its two first variables

$$L: \begin{array}{ccc} \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{T}^\kappa & \longrightarrow & \mathbb{R} \\ (x,v,t) & \longmapsto & L(x,v,t) \end{array} \tag{II.2.12}$$

and let $\mathcal{L}$ be the following (non shifted) Lagrangian functional:

$$\mathcal{L}: \begin{array}{ccc} \mathrm{E} \subset \mathrm{C}^{1,\Delta}_{\mathrm{rd}}(\mathbb{T},\mathbb{R}^n) & \longrightarrow & \mathbb{R} \\ q & \longmapsto & \displaystyle\int_a^b L(q(\tau), q^\Delta(\tau), \tau)\, \Delta\tau. \end{array} \tag{II.2.13}$$

The aim of this section is to give a necessary condition for local optimizers of $\mathcal{L}$. Thus, let us introduce the following notions and notations:

– $\mathrm{C}^{1,\Delta}_{\mathrm{rd},0}(\mathbb{T},\mathbb{R}^n)$ is the set of *variations* of $\mathcal{L}$.
– $q \in \mathrm{E}$ is said to be a *critical point* of $\mathcal{L}$ if $D\mathcal{L}(q)(w) = 0$ for every variation $w \in \mathrm{C}^{1,\Delta}_{\mathrm{rd},0}(\mathbb{T},\mathbb{R}^n)$, where $D\mathcal{L}(q)(w)$ denotes the Gâteaux-differential of $\mathcal{L}$ at q in direction w.

In particular, if q is a local optimizer of $\mathcal{L}$, then q is a critical point of $\mathcal{L}$.

Remark II.1. Note that the consideration of E allows to assume (or not) general boundary conditions on the variational problem. For example, one can consider:

– $\mathrm{E} = \mathrm{C}^{1,\Delta}_{\mathrm{rd}}(\mathbb{T},\mathbb{R}^n)$ (no boundary conditions);
– $\mathrm{E} = \{q \in \mathrm{C}^{1,\Delta}_{\mathrm{rd}}(\mathbb{T},\mathbb{R}^n),\ q(a) = q_a\}$ where $q_a \in \mathbb{R}^n$ (initial condition);
– $\mathrm{E} = \{q \in \mathrm{C}^{1,\Delta}_{\mathrm{rd}}(\mathbb{T},\mathbb{R}^n),\ g(q(a),q(b)) \in \mathrm{S}\}$ where $g : \mathbb{R}^n \times \mathbb{R}^n \longmapsto \mathbb{R}^j$, $j \in \mathbb{N}^*$ and S is a non empty subset of $\mathbb{R}^j$ (mixing of initial and final conditions).

In order to follow the usual strategy of calculus of variations, we first need to recall the following time scale version of the celebrated du Bois-Reymond lemma originally proved in [36, Lemma 4.1].

Lemma II.1 (du Bois-Reymond)**.** *Let $q \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T}^\kappa,\mathbb{R}^n)$. Then, the equality*

$$\int_a^b q(\tau) \cdot w^\Delta(\tau)\, \Delta\tau = 0 \tag{II.2.14}$$

holds for every $w \in \mathrm{C}^{1,\Delta}_{\mathrm{rd},0}(\mathbb{T},\mathbb{R}^n)$ if and only if there exists $c \in \mathbb{R}^n$ such that $q(t) = c$ for every $t \in \mathbb{T}^\kappa$.

Proof. The sufficient condition is obvious. Let us prove the necessary one. Let $Q \in \mathrm{C}^{1,\Delta}_{\mathrm{rd}}(\mathbb{T},\mathbb{R}^n)$ be the unique Δ-antiderivative of q vanishing at $t = a$. Then, let $c = Q(b)/(b-a) \in \mathbb{R}^n$ and $w \in \mathrm{C}^{1,\Delta}_{\mathrm{rd},0}(\mathbb{T},\mathbb{R}^n)$ defined by $w(t) = Q(t) - (t-a)c$ for every $t \in \mathbb{T}$. In particular, $w^\Delta = q - c$. As a consequence, it holds

$$\int_a^b q(\tau) \cdot (q(\tau) - c)\, \Delta\tau = \int_a^b \|q(\tau) - c\|^2\, \Delta\tau + c \cdot \int_a^b w^\Delta(\tau)\, \Delta\tau = \int_a^b \|q(\tau) - c\|^2\, \Delta\tau = 0. \tag{II.2.15}$$

The proof is complete. □

From regularity hypotheses on L, the following result can be easily derived from arguments of uniform continuity, uniform convergence and using [39, Theorem 5.32 p.135] or [99, Theorem 3.11].

Lemma II.2. *$\mathcal{L}$ is Gâteaux-differentiable at every $q \in \mathrm{E}$ in every direction $w \in \mathrm{C}^{1,\Delta}_{\mathrm{rd},0}(\mathbb{T})$ with*

$$D\mathcal{L}(q)(w) = \int_a^b \frac{\partial L}{\partial x}(q(\tau), q^\Delta(\tau), \tau) \cdot w(\tau) + \frac{\partial L}{\partial v}(q(\tau), q^\Delta(\tau), \tau) \cdot w^\Delta(\tau)\, \Delta\tau. \tag{II.2.16}$$

Finally, we recall the following characterization of the critical points of $\mathcal{L}$, see [114, Theorem 4].

Proposition II.1. *Let $q \in \mathrm{E}$. Then, q is a critical point of $\mathcal{L}$ if and only if there exists $c \in \mathbb{R}^n$ such that*

$$\frac{\partial L}{\partial v}(q(t), q^\Delta(t), t) = \int_a^{\sigma(t)} \frac{\partial L}{\partial x}(q(\tau), q^\Delta(\tau), \tau)\, \Delta\tau + c, \tag{EL$_{\mathrm{int}}$}$$

for every $t \in \mathbb{T}^\kappa$.

Proof. For every $q \in \mathrm{E}$, let $F_q \in \mathrm{C}^{1,\Delta}_{\mathrm{rd}}(\mathbb{T},\mathbb{R}^n)$ denote the unique Δ-antiderivative of $\partial L/\partial x(q,q^\Delta,\cdot) \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T}^\kappa,\mathbb{R}^n)$ vanishing at $t=a$. In particular, $F^\sigma_q \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T}^\kappa,\mathbb{R}^n)$. From Lemma II.2, q is a critical point of $\mathcal{L}$ if and only if

$$\forall w \in \mathrm{C}^{1,\Delta}_{\mathrm{rd},0}(\mathbb{T},\mathbb{R}^n),\ \int_a^b F^\Delta_q(\tau)\cdot w(\tau) + \frac{\partial L}{\partial v}(q(\tau),q^\Delta(\tau),\tau)\cdot w^\Delta(\tau)\ \Delta\tau = 0 \tag{II.2.17}$$

if and only if (using the integration by parts formula (I.2.10))

$$\forall w \in \mathrm{C}^{1,\Delta}_{\mathrm{rd},0}(\mathbb{T},\mathbb{R}^n),\ \int_a^b \left(\frac{\partial L}{\partial v}(q(\tau),q^\Delta(\tau),\tau) - F^\sigma_q(\tau)\right)\cdot w^\Delta(\tau)\ \Delta\tau = 0 \tag{II.2.18}$$

if and only if (from Lemma II.1) there exists $c \in \mathbb{R}^n$ such that:

$$\frac{\partial L}{\partial v}(q(t),q^\Delta(t),t) - F^\sigma_q(t) = c, \tag{II.2.19}$$

for every $t \in \mathbb{T}^\kappa$. The proof is complete. □

Hence, Proposition II.1 provides a necessary condition for local optimizers of $\mathcal{L}$. Precisely, if q is a local optimizer of $\mathcal{L}$, then there exists $c \in \mathbb{R}^n$ such that q satisfies the integral Euler-Lagrange equation ($\mathrm{EL_{int}}$). We refer to Example II.1 for an application of Proposition II.1.

II.3 Main results on time scales with ∇-differentiable σ

In this section, from the results obtained in Chapter I (see Section I.4), we obtain an Euler-Lagrange equation of differential form in Theorem II.1 and we provide in Theorem II.2 a Noether-type theorem giving rise to an explicit conservation law for every differential Euler-Lagrange equation admitting a symmetry.

II.3.1 A differential Euler-Lagrange equation

In this section, our aim is to prove that the ∇-differentiability of σ is a sharp assumption on the time scale in order to ∇-differentiate ($\mathrm{EL_{int}}$) and to rewrite Proposition II.1 with a differential Euler-Lagrange equation of type ($\mathrm{EL_{diff}}$). Applying Proposition II.1 with Corollary I.1 and Proposition I.3 leads to:

Theorem II.1. *We assume that σ is ∇-differentiable on $\mathbb{T}_\kappa$ and let $q \in \mathrm{E}$. Then, q is a critical point of $\mathcal{L}$ if and only if q is a solution of the following differential Euler-Lagrange equation:*

$$\left[\frac{\partial L}{\partial v}(q,q^\Delta,\cdot)\right]^\nabla(t) = \sigma^\nabla(t)\frac{\partial L}{\partial x}(q(t),q^\Delta(t),t), \tag{$\mathrm{EL_{diff}}$}$$

for every $t \in \mathbb{T}^\kappa_\kappa$.

This theorem recovers both the usual continuous and discrete Euler-Lagrange equations given by (II.1.4) and (II.1.8) in Introduction of this chapter. Indeed, as it is mentioned in Example I.2 in Section I.4.2, the following properties are satisfied:

- if $\mathbb{T}=[a,b]$, then σ is ∇-differentiable with $\sigma^\nabla = 1$;
- if $\mathrm{card}(\mathbb{T}) < \infty$, then σ is ∇-differentiable with $\sigma^\nabla = \mu/\nu$.

The following example shows that the ∇-differentiability of σ is moreover necessary for the validity of Theorem II.1.

Example II.1. Let us consider $n=1$, $\mathrm{E} = \mathrm{C}^{1,\Delta}_{\mathrm{rd}}(\mathbb{T},\mathbb{R}^n)$, $L(x,v,t) = x + v^2/2$ and $q \in \mathrm{E}$ defined by $q(t) = \int_a^t \sigma(\tau)\Delta\tau$ for every $t \in \mathbb{T}$. Since q satisfies ($\mathrm{EL_{int}}$) with $c=a$, we obtain that q is a critical point of $\mathcal{L}$, see Proposition II.1. However, $\partial L/\partial v(q,q^\Delta,\cdot) = q^\Delta = \sigma$ and consequently, Theorem II.1 is not valid if σ is not ∇-differentiable.

We refer to Example I.1 of Section I.4.1 for examples of time scale with continuous and non continuous σ. We refer to Examples I.2 and I.3 of Section I.4.2 for examples of time scale with ∇-differentiable and non ∇-differentiable σ.

II.3.2 A Noether-type theorem

Conservation laws of dynamical systems are generally associated with some physical quantities like total energy or angular momentum in mechanical systems and sometimes, they also can be used to reduce or integrate by quadrature the equation.

In this section, under the ∇-differentiability of σ and from the Leibniz formula introduced in Proposition I.8, we prove a Noether-type theorem providing an explicit constant of motion for differential Euler-Lagrange equations ($\mathrm{EL_{diff}}$) admitting a symmetry. We first review the definition of a one-parameter family of infinitesimal transformations of $\mathbb{R}^n$.

Definition II.1. Let $\eta > 0$. A map Φ is said to be a one-parameter family of infinitesimal transformations of $\mathbb{R}^n$ if Φ is a map of class $\mathscr{C}^2$

$$\begin{array}{rccc} \Phi: & [-\eta,\eta]\times\mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & (\theta,x) & \longmapsto & \Phi(\theta,x), \end{array} \tag{II.3.20}$$

such that $\Phi(0,\cdot) = \mathrm{Id}_{\mathbb{R}^n}$.

The action of a one-parameter family of infinitesimal transformations of $\mathbb{R}^n$ on a Lagrangian allows us to introduce the notion of symmetry for a differential Euler-Lagrange equation ($\mathrm{EL_{diff}}$).

Definition II.2. Let Φ be a one-parameter family of infinitesimal transformations of $\mathbb{R}^n$. A Lagrangian L is said to be invariant under the action of Φ if for every solution $q \in \mathrm{C}^{1,\Delta}_{\mathrm{rd}}(\mathbb{T},\mathbb{R}^n)$ of ($\mathrm{EL_{diff}}$) and every $t \in \mathbb{T}^\kappa_\kappa$, the map

$$\theta \longmapsto L(\Phi(\theta,q(t)),\Phi(\theta,q)^\Delta(t),t) \tag{II.3.21}$$

has a null derivative in 0. In such a case, Φ is said to be a symmetry of the differential Euler-Lagrange equation ($\mathrm{EL_{diff}}$).

The most classical examples of invariance of a Lagrangian under the action of a one-parameter family of infinitesimal transformations of $\mathbb{R}^n$ are given by quadratic Lagrangian and rotations.

Example II.2. Let us consider $n = 2$, $L(x,v,t) = \|x\|^2 + \|y\|^2$, $\eta = \pi > 0$ and Φ defined by:

$$\begin{array}{rccc} \Phi: & [-\pi,\pi]\times\mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ & (\theta,x_1,x_2) & \longmapsto & \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \end{array} \tag{II.3.22}$$

Then, for every $q \in \mathrm{C}^{1,\Delta}_{\mathrm{rd}}(\mathbb{T},\mathbb{R}^n)$, every $(\theta,t) \in [-\pi,\pi]\times\mathbb{T}^\kappa_\kappa$, we have $\Phi(\theta,q)^\Delta(t) = \Phi(\theta,q^\Delta(t))$. Consequently, for every $q \in \mathrm{C}^{1,\Delta}_{\mathrm{rd}}(\mathbb{T},\mathbb{R}^n)$ and every $\mathbb{T}^\kappa_\kappa$, one can easily prove that the map

$$\theta \longmapsto L(\Phi(\theta,q(t)),\Phi(\theta,q)^\Delta(t),t) \tag{II.3.23}$$

is independent of θ and then has a null derivative in 0.

Finally, we prove the following Noether-type theorem.

Theorem II.2 (Noether)**.** *Let us assume that σ is ∇-differentiable on $\mathbb{T}_\kappa$ and let Φ be a one-parameter family of infinitesimal transformations of $\mathbb{R}^n$. If L is invariant under the action of Φ, then for every solution $q \in \mathrm{C}^{1,\Delta}_{\mathrm{rd}}(\mathbb{T},\mathbb{R}^n)$ of* ($\mathrm{EL_{diff}}$)*, there exists $c \in \mathbb{R}$ such that*

$$\frac{\partial L}{\partial v}(q(t),q^\Delta(t),t)\cdot\frac{\partial\Phi}{\partial\theta}(0,q^\sigma(t)) = c, \tag{II.3.24}$$

for every $t \in \mathbb{T}^\kappa$.

Proof. Let $q \in \mathrm{C}^{1,\Delta}_{\mathrm{rd}}(\mathbb{T},\mathbb{R}^n)$ be a solution of ($\mathrm{EL_{diff}}$). Let us differentiate at $\theta = 0$ the map given by (II.3.21) and invert the operators Δ and $\partial/\partial\theta$ from $\Phi(0,\cdot) = \mathrm{Id}_{\mathbb{R}^n}$ and from the $\mathscr{C}^2$-regularity of Φ. It can be noted that this last operation is not obvious and needs some technical calculations. We obtain for every $t \in \mathbb{T}^\kappa_\kappa$

$$\frac{\partial L}{\partial x}(q(t),q^\Delta(t),t)\cdot\frac{\partial\Phi}{\partial\theta}(0,q(t)) + \frac{\partial L}{\partial v}(q(t),q^\Delta(t),t)\cdot\frac{\partial\Phi}{\partial\theta}(0,q)^\Delta(t) = 0. \tag{II.3.25}$$

Multiplying (II.3.25) by $\sigma^\nabla(t)$ and using that q is solution of (EL_{diff}) on $\mathbb{T}^\kappa_\kappa$ leads to

$$\left[\frac{\partial L}{\partial v}(q, q^\Delta, \cdot)\right]^\nabla(t) \cdot \frac{\partial \Phi}{\partial \theta}(0, q(t)) + \sigma^\nabla(t)\frac{\partial L}{\partial v}(q(t), q^\Delta(t), t) \cdot \frac{\partial \Phi}{\partial \theta}(0, q)^\Delta(t) = 0, \tag{II.3.26}$$

for every $t \in \mathbb{T}^\kappa_\kappa$. Finally, from the Leibniz formula introduced in Proposition I.8, it holds

$$\left[\frac{\partial L}{\partial v}(q, q^\Delta, \cdot) \cdot \frac{\partial \Phi}{\partial \theta}(0, q)^\sigma\right]^\nabla(t) = 0, \tag{II.3.27}$$

for every $t \in \mathbb{T}^\kappa_\kappa$. Proposition I.3 concludes the proof. □

This theorem both encompasses the usual Noether's theorems given in the continuous case (see *e.g.* [20, p.88]) and in the (non shifted) discrete case (see *e.g.* [101, Theorem 6.4]). For an example of application of Theorem II.2, one can consider the framework of Example II.2.

Chapitre III

Principe du Maximum de Pontryagin pour des problèmes de contrôle optimal non linéaires posés sur time scale

Nous démontrons une version forte du Principe du Maximum de Pontryagin (PMP en abrégé) pour des problèmes de contrôle optimal non linéaires posés sur time scale. Le temps final pourra être fixé ou non. Nous établissons également les conditions de transversalité relatives aux conditions de bord considérées. Notre résultat, étant valable sur tout time scale, recouvre en particulier les versions classiques du PMP en temps continu et en temps discret. La preuve s'appuie sur le principe variationnel d'Ekeland.

Ce chapitre est issu de Bourdin L. et Trélat E., Pontryagin Maximum Principle for finite dimensional nonlinear optimal control problems on time scales, preprint arXiv:1302.3513. *Par ailleurs, des résultats de type Cauchy-Lipschitz sur time scale introduits dans* Bourdin L. et Trélat E., Cauchy-Lipschitz theory for shifted and non shifted Δ-Cauchy problems on time scales, preprint arXiv:1212.5042 *sont rappelés en Annexe A et sont appliqués à plusieurs reprises dans cette étude.*

III.1 Introduction

Optimal control theory is concerned with the analysis of controlled dynamical systems, where one aims at steering such a system from a given configuration to some desired target one by minimizing or maximizing some criterion. The Pontryagin Maximum Principle (denoted in short PMP), established at the end of the fifties for finite dimensional general nonlinear continuous-time dynamics (see [40], and see [94] for the history of this discovery), is the milestone of the classical optimal control theory. It provides a first-order necessary condition for optimality, by asserting that any optimal trajectory must be the projection of an extremal. The PMP then reduces the search of optimal trajectories to a boundary value problem posed on extremals. Optimal control theory, and in particular the PMP, have an immense field of applications in various domains, and it is not our aim here to list them. We refer the reader to textbooks on optimal control such as [5, 40, 43, 44, 54, 56, 57, 107, 130, 147, 191, 192, 197] for many examples of theoretical or practical applications, essentially in a continuous-time setting.

Right after this discovery, the corresponding theory has been developed for discrete-time dynamics, under appropriate convexity assumptions (see *e.g.* [102, 124, 125]), leading to a version of the PMP for discrete-time optimal control problems. The considerable development of the discrete-time control theory was motivated by many potential applications *e.g.* to digital systems or in view of discrete approximations in numerical simulations of differential controlled systems. We refer the reader to the textbooks [41, 61, 171, 192] for details on this theory and many examples of applications. It can be noted that some early works devoted to the discrete-time PMP (like [85]) are mathematically incorrect. Indeed, many counterexamples were provided in [41] (see also [171]), showing that, as is now well known, the exact analogous of the continuous-time PMP does not hold at the discrete level. More precisely, the maximization condition of the PMP cannot be expected to hold in a general discrete-time case. Nevertheless a weaker condition can be derived (see [41, Theorem 42.1 p. 330]). This is the context of what is usually referred to as the *weak PMP*. Note that a wide literature is devoted to the introduction of convexity assumptions on the dynamics allowing one to recover the maximization condition in the discrete case (such as the concept of *directional convexity* used in [61, 124, 125]).

Few attempts have been made to derive a PMP on time scale. In [117, 118], the authors establish a *weak* PMP for shifted and non-shifted controlled systems on time scale. We mention that a closely related topic is the Hamilton-Jacobi theory for optimal control problems on time scale from [119]. A strong version of the PMP is claimed in [205] but several arguments thereof are erroneous (see [113] or [53, Remark 13] for details).

The objective of the present chapter is to state and prove a strong version of the PMP on time scale, available for general nonlinear dynamics, and without assuming any unnecessary Lipschitz or convexity conditions. Our statement is as general as possible, and encompasses the classical continuous-time PMP that can be found *e.g.* in [40, 147] as well as versions of discrete-time PMP's mentioned above. In accordance with all known results, the maximization condition is obtained at right-dense points of the time scale and a weaker one (similar to [41, Theorem 42.1 p. 330]) is derived at right-scattered points. Moreover, we consider general constraints on the initial and final values of the state variable and we derive the resulting transversality conditions. We provide as well a version of the PMP for optimal control problems with parameters.

Organization of the chapter. In Section III.2, we first define some appropriate notions such as the notion of stable Ω-dense direction in Subsection III.2.1. In Subsection III.2.2 we settle the notion of admissible control and define general optimal control problems on time scales. Our main result (Pontryagin Maximum Principle, Theorem III.1) is stated in Subsection III.2.3, and we analyze and comment the results in a series of remarks. Section III.3 is devoted to the proof of Theorem III.1. First, in Subsection III.3.1 we make some preliminary comments explaining which obstructions may appear when dealing with general time scales, and why we were led to a proof based on Ekeland's Variational Principle. In Subsection III.3.2, after having shown that the set of admissible controls is open, we define needle-like variations at right-dense and right-scattered points and derive some properties. In Subsection III.3.3, we apply Ekeland's Variational Principle to a well chosen functional in an appropriate complete metric space and then prove the PMP.

III.2 Main result

Throughout this chapter, $\mathbb{T}$ denotes a lower bounded time scale such that $\mathrm{card}(\mathbb{T}) \geq 2$. We denote by $a = \min \mathbb{T}$.

Notations: In the sequel, for every $n \in \mathbb{N}^*$, the notations $\langle \cdot, \cdot \rangle_{\mathbb{R}^n}$ and $\| \cdot \|_{\mathbb{R}^n}$ respectively stand for the usual scalar product and Euclidean norm of $\mathbb{R}^n$. For every $x \in \mathbb{R}^n$ and every $R \geq 0$, the notation $\overline{B}(x,R)$ stands for the closed ball of $\mathbb{R}^n$ centered at x and with radius R.

For all elements $s \in \mathrm{RD}$, $b \in \mathbb{T}$ such that $s < b$, we introduce the following set:

$$\mathcal{V}_s^b = \{\beta \geq 0,\ s + \beta \in [s,b]_{\mathbb{T}}\}. \tag{III.2.1}$$

Note that 0 is not isolated in $\mathcal{V}_s^b$. In particular, for all $(c,d) \in \mathbb{T}^2$ such that $c < d$, it holds

$$\lim_{\substack{\beta \to 0^+ \\ \beta \in \mathcal{V}_s^d}} \frac{1}{\beta} \int_{[s,s+\beta[_{\mathbb{T}}} q(\tau)\ \Delta\tau = q(s), \tag{III.2.2}$$

for every $q \in \mathrm{L}^1_{\mathbb{T}}([c,d[_{\mathbb{T}}, \mathbb{R}^n)$ and every $s \in \mathscr{L}_{[c,d[_{\mathbb{T}}}(q) \cap \mathrm{RD}$, see Proposition I.5.

Remark III.1. Note that the analogous result for $s \in \mathscr{L}_{[c,d[_{\mathbb{T}}}(q) \cap \mathrm{LD}$ is not true in general. Indeed, let $q \in \mathrm{L}^1_{\mathbb{T}}([c,d[_{\mathbb{T}}, \mathbb{R}^n)$ and assume that there exists a point $s \in [c,d[_{\mathbb{T}} \cap \mathrm{LD} \cap \mathrm{RS}$. Since $\mu_\Delta(\{s\}) = \mu(s) > 0$, one has $s \in \mathscr{L}_{[c,d[_{\mathbb{T}}}(q)$. Nevertheless the limit $\frac{1}{\beta}\int_{[s-\beta,s[_{\mathbb{T}}} q(\tau)\,\Delta\tau$ as $\beta \to 0^+$, with $s - \beta \in \mathbb{T}$, is not necessarily equal to $q(s)$. For instance, consider $\mathbb{T} = [0,1] \cup \{2\}$, $s = 1$ and q defined on $\mathbb{T}$ by $q(t) = 0$ for every $t \neq 1$ and $q(1) = 1$.

III.2.1 Topological preliminaries

Let $m \in \mathbb{N}^*$ and let Ω be a non empty closed subset of $\mathbb{R}^m$. In this section, we introduce the notion of stable Ω-dense direction. In our main result, the set Ω stands for the set of pointwise constraints on the controls.

Definition III.1. Let $v \in \Omega$ and $v' \in \mathbb{R}^m$.

1. We set $\mathcal{D}^\Omega(v, v') = \{0 \leq \alpha \leq 1 \mid v + \alpha(v' - v) \in \Omega\}$. Note that $0 \in \mathcal{D}^\Omega(v, v')$.
2. We say that v' is a Ω-dense direction from v if 0 is not isolated in $\mathcal{D}^\Omega(v, v')$. The set of all Ω-dense directions from v is denoted by $\mathcal{D}^\Omega(v)$.
3. We say that v' is a stable Ω-dense direction from v if there exists $\varepsilon > 0$ such that $v' \in \mathcal{D}^\Omega(v'')$ for every $v'' \in \overline{B}(v, \varepsilon) \cap \Omega$. The set of all stable Ω-dense directions from v is denoted by $\mathcal{D}^\Omega_{\text{stab}}(v)$.

In other words, $v' \in \mathcal{D}^\Omega_{\text{stab}}(v)$ means that v' is a Ω-dense direction from v'' for every $v'' \in \Omega$ in a neighbourhood of v. In what follows, we denote by Int the interior of a subset. We have the following obvious properties.

1. If $v \in \text{Int}(\Omega)$, then $\mathcal{D}^\Omega_{\text{stab}}(v) = \mathbb{R}^m$.
2. If $\Omega = \{v\}$ then $\mathcal{D}^\Omega_{\text{stab}}(v) = \{v\}$;
3. If Ω is convex then $\Omega \subset \mathcal{D}^\Omega_{\text{stab}}(v)$ for every $v \in \Omega$.

For every $v \in \Omega$, we denote by $\overline{\text{Co}}(\mathcal{D}^\Omega_{\text{stab}}(v))$ the closed convex cone of vertex v spanned by $\mathcal{D}^\Omega_{\text{stab}}(v)$, with the agreement that $\overline{\text{Co}}(\mathcal{D}^\Omega_{\text{stab}}(v)) = \{v\}$ whenever $\mathcal{D}^\Omega_{\text{stab}}(v) = \emptyset$. In particular, it holds $v \in \overline{\text{Co}}(\mathcal{D}^\Omega_{\text{stab}}(v))$ for every $v \in \Omega$.

Although elementary, since these notions are new (up to our knowledge), before proceeding with our main result (stated in Section III.2.3) we provide some simple examples illustrating these notions. Since $\mathcal{D}^\Omega_{\text{stab}}(v) = \overline{\text{Co}}(\mathcal{D}^\Omega_{\text{stab}}(v)) = \mathbb{R}^m$ for every $v \in \text{Int}(\Omega)$, we focus on elements $v \in \partial\Omega$ in the examples below.

Example III.1. Assume that $m = 1$. The closed convex subsets Ω of $\mathbb{R}$ having a nonempty interior and such that $\partial\Omega \neq \emptyset$ are closed intervals upper or lower bounded and not reduced to a singleton. If Ω is lower bounded then $\mathcal{D}^\Omega_{\text{stab}}(\min\Omega) = \overline{\text{Co}}(\mathcal{D}^\Omega_{\text{stab}}(\min\Omega)) = [\min\Omega, +\infty[$, and if Ω is upper bounded then $\mathcal{D}^\Omega_{\text{stab}}(\max\Omega) = \overline{\text{Co}}(\mathcal{D}^\Omega_{\text{stab}}(\max\Omega)) =]-\infty, \max\Omega]$.

Example III.2. Assume that $m = 2$ and let Ω be the convex set of $v = (v_1, v_2) \in \mathbb{R}^2$ such that $v_1 \geq 0$, $v_2 \geq 0$ and $v_1^2 + v_2^2 \leq 1$ (see Figure III.1). The stable Ω-dense directions for elements $v \in \partial\Omega$ are given by:

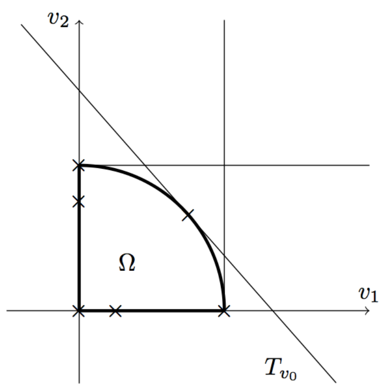

Figure III.1

- if $v = (0, 0)$, then $\mathcal{D}^\Omega_{\text{stab}}(v) = \overline{\text{Co}}(\mathcal{D}^\Omega_{\text{stab}}(v)) = (\mathbb{R}^+)^2$;
- if $v = (0, v_0)$ with $0 < v_0 < 1$, then $\mathcal{D}^\Omega_{\text{stab}}(v) = \overline{\text{Co}}(\mathcal{D}^\Omega_{\text{stab}}(v)) = \mathbb{R}^+ \times \mathbb{R}$;
- if $v = (v_0, 0)$ with $0 < v_0 < 1$, then $\mathcal{D}^\Omega_{\text{stab}}(v) = \overline{\text{Co}}(\mathcal{D}^\Omega_{\text{stab}}(v)) = \mathbb{R} \times \mathbb{R}^+$;
- if $v = (0, 1)$, then $\mathcal{D}^\Omega_{\text{stab}}(v) = \{(v_1, v_2) \in \mathbb{R}^2 \mid v_1 \geq 0, v_2 < 1\} \cup \{v\}$ and $\overline{\text{Co}}(\mathcal{D}^\Omega_{\text{stab}}(v)) = \{(v_1, v_2) \in \mathbb{R}^2 \mid v_1 \geq 0, v_2 \leq 1\}$;

- if $v = (1, 0)$, then $\mathcal{D}^{\Omega}_{\text{stab}}(v) = \{(v_1, v_2) \in \mathbb{R}^2 \mid v_1 < 1, v_2 \geq 0\} \cup \{v\}$ and $\overline{\text{Co}}(\mathcal{D}^{\Omega}_{\text{stab}}(v)) = \{(v_1, v_2) \in \mathbb{R}^2 \mid v_1 \leq 1, v_2 \geq 0\}$;
- if $v = (v_0, \sqrt{1 - v_0^2})$ with $0 < v_0 < 1$, then $\mathcal{D}^{\Omega}_{\text{stab}}(v)$ is the union of $\{v\}$ and of the strict hypograph of T_{v_0}, and $\overline{\text{Co}}(\mathcal{D}^{\Omega}_{\text{stab}}(v))$ is the hypograph of T_{v_0}.

Remark III.2. Let Ω be a non empty closed convex subset of $\mathbb{R}^m$ and let $\text{Aff}(\Omega)$ denote the smallest affine subspace of $\mathbb{R}^m$ containing Ω. For every $v \in \partial\Omega$ that is not a corner point, $\overline{\text{Co}}(\mathcal{D}^{\Omega}_{\text{stab}}(v))$ is the half-space of $\text{Aff}(\Omega)$ delimited by the tangent hyperplane (in $\text{Aff}(\Omega)$) of Ω at v, and containing Ω.

Example III.3. Assume that $m = 2$ and let Ω be the set of $v = (v_1, v_2) \in \mathbb{R}^2$ such that $v_2 \leq |v_1|$ (see Figure III.2). The stable Ω-dense directions for elements $v \in \partial\Omega$ are given by:

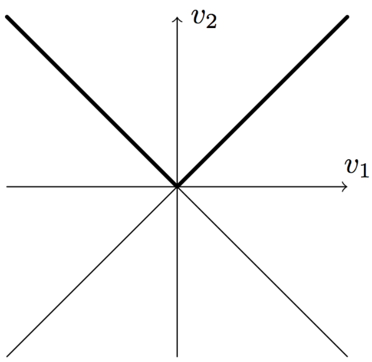

Figure III.2

- if $v = (v_0, |v_0|)$ with $v_0 < 0$, then $\mathcal{D}^{\Omega}(v) = \mathcal{D}^{\Omega}_{\text{stab}}(v) = \{(v_1, v_2) \in \mathbb{R}^2 \mid v_2 \leq -v_1\}$;
- if $v = (v_0, |v_0|)$ with $v_0 > 0$, then $\mathcal{D}^{\Omega}(v) = \mathcal{D}^{\Omega}_{\text{stab}}(v) = \{(v_1, v_2) \in \mathbb{R}^2 \mid v_2 \leq v_1\}$;
- if $v = (0, 0)$, then $\mathcal{D}^{\Omega}(v) = \Omega$, $\mathcal{D}^{\Omega}_{\text{stab}}(v) = \{(v_1, v_2) \in \mathbb{R}^2 \mid v_2 \leq -|v_1|\}$;

Note that, in all cases, $\mathcal{D}^{\Omega}_{\text{stab}}(v)$ is a closed convex cone of vertex v and therefore $\overline{\text{Co}}(\mathcal{D}^{\Omega}_{\text{stab}}(v)) = \mathcal{D}^{\Omega}_{\text{stab}}(v)$.

Example III.4. Assume that $m = 2$ and let Ω be the set of $v = (v_1, v_2) \in \mathbb{R}^2$ such that $v_2 \leq v_1^2$ (see Figure III.3). Let $v_0 \in \mathbb{R}$ and let $T_{v_0}(v_1) = v_0(2v_1 - v_0)$ denote the graph of the tangent to Ω at the point $v = (v_0, v_0^2)$. It is easy to see that $\mathcal{D}^{\Omega}(v)$ is the hypograph of T_{v_0}, that $\mathcal{D}^{\Omega}_{\text{stab}}(v)$ is the strict hypograph of T_{v_0} (note that

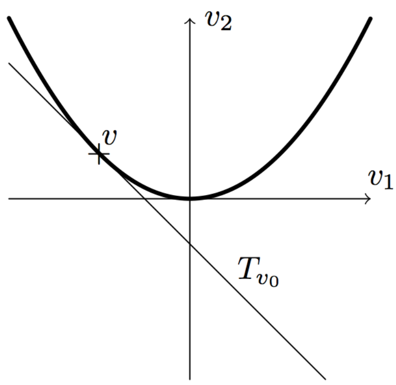

Figure III.3

$v \notin \mathcal{D}^{\Omega}_{\text{stab}}(v)$), and that $\overline{\text{Co}}(\mathcal{D}^{\Omega}_{\text{stab}}(v))$ is the hypograph of T_{v_0}.

Remark III.3. The above example shows that it may happen that $v \notin \mathcal{D}^{\Omega}_{\text{stab}}(v)$. Actually, it may happen that $\mathcal{D}^{\Omega}_{\text{stab}}(v) = \emptyset$. For example, if Ω is the unit sphere of $\mathbb{R}^2$, then $\mathcal{D}^{\Omega}_{\text{stab}}(v) = \emptyset$ for every $v \in \Omega$, and hence $\overline{\text{Co}}(\mathcal{D}^{\Omega}_{\text{stab}}(v)) = \{v\}$.

Example III.5. Assume that $m=2$. We set $\Omega = \cup_{k\in\mathbb{N}}\overline{\Omega}_k \cup \overline{\Omega}_\infty$, where $\Omega_k = \{(v_1,(1-v_1)/2^k) \mid 0<v_1<1\}$ for every $k\in\mathbb{N}$, and $\Omega_\infty = \{(v_1,0) \mid 0<v_1<1\}$ (see Figure III.4). Note that Ω has an empty interior. Denote by $\overline{v}=(1,0)$. We have the following properties:

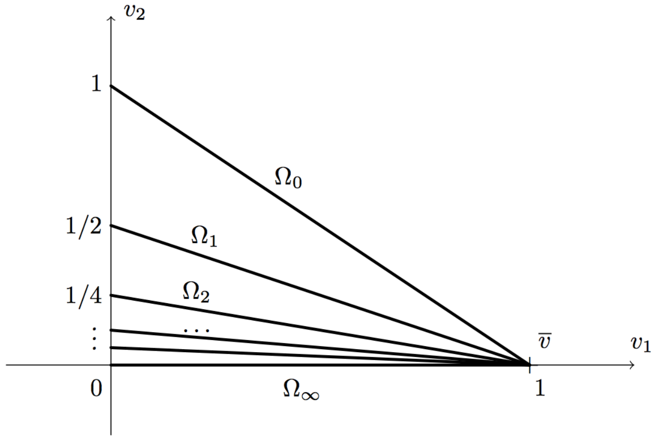

Figure III.4

- if $v\in\Omega_k$ with $k\in\mathbb{N}$, then $\overline{\mathrm{Co}}(\mathcal{D}^\Omega_{\mathrm{stab}}(v)) = \mathcal{D}^\Omega_{\mathrm{stab}}(v) = \mathcal{D}^\Omega(v) = \{(v_1,(1-v_1)/2^k) \mid v_1\in\mathbb{R}\}$;
- if $v=(0,1/2^k)$ with $k\in\mathbb{N}$, then $\overline{\mathrm{Co}}(\mathcal{D}^\Omega_{\mathrm{stab}}(v)) = \mathcal{D}^\Omega_{\mathrm{stab}}(v) = \mathcal{D}^\Omega(v) = \{(v_1,(1-v_1)/2^k) \mid v_1\geq 0\}$;
- if $v=(v_1,0)$ with $0<v_1<1$, then $\mathcal{D}^\Omega(v)=\mathbb{R}\times\mathbb{R}^+$ and $\mathcal{D}^\Omega_{\mathrm{stab}}(v)=\{\overline{v}\}$, and thus $\overline{\mathrm{Co}}(\mathcal{D}^\Omega_{\mathrm{stab}}(v)) = [v_1,+\infty[\times\{0\}$;
- if $v=(0,0)$, then $\mathcal{D}^\Omega(v)=(\mathbb{R}^+)^2$ and $\mathcal{D}^\Omega_{\mathrm{stab}}(v)=\{\overline{v}\}$, and thus $\overline{\mathrm{Co}}(\mathcal{D}^\Omega_{\mathrm{stab}}(v))=\mathbb{R}^+\times\{0\}$;
- if $v=\overline{v}$, then $\mathcal{D}^\Omega(\overline{v}) = \cup_{k\in\mathbb{N}}\{(v_1,(1-v_1)/2^k) \mid v_1\leq 1\}\cup\{(v_1,0)\mid v_1\leq 1\}$ and $\overline{\mathrm{Co}}(\mathcal{D}^\Omega_{\mathrm{stab}}(v)) = \mathcal{D}^\Omega_{\mathrm{stab}}(\overline{v}) = \{\overline{v}\}$.

III.2.2 General nonlinear optimal control problem on time scales

Let n and m be nonzero integers, and let Ω be a non empty closed subset of $\mathbb{R}^m$. Throughout the chapter, we consider the general nonlinear controlled system posed on the time scale $\mathbb{T}$

$$q^\Delta(t) = f(q(t),u(t),t), \tag{III.2.3}$$

where $f:\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{T}\longrightarrow\mathbb{R}^n$, $(x,v,t)\longmapsto f(x,v,t)$ is a continuous function of class $\mathscr{C}^1$ with respect to its two first variables, and where the control functions u belong to $\mathrm{L}^\infty_{\mathrm{loc},\mathbb{T}}(\mathbb{T}\backslash\{\sup\mathbb{T}\};\Omega)$.

Before defining an optimal control problem associated with the controlled system (III.2.3), the first question that has to be addressed is the question of the existence and uniqueness of a solution of (III.2.3), for a given control function u and a given initial condition $q(a)=q_a\in\mathbb{R}^n$. Since there did not exist up to now in the existing literature any Cauchy-Lipschitz like theorem, sufficiently general to cover such a situation, in the companion Appendix A (or in [52]), we derived a Cauchy-Lipschitz theorem for general nonlinear systems posed on time scale, providing existence and uniqueness of the maximal solution of a given Δ-Cauchy problem under suitable assumptions like regressivity and local Lipschitz continuity. Moreover, we also discussed some related issues like the behavior of maximal solutions at terminal points.

Setting $\mathcal{U}=\mathrm{L}^\infty_{\mathrm{loc},\mathbb{T}}(\mathbb{T}\backslash\{\sup\mathbb{T}\};\mathbb{R}^m)$, according to Theorem A.1 (see also [52, Theorem 1]), for every $(u,q_a)\in\mathcal{U}\times\mathbb{R}^n$, there exists a unique maximal solution of (III.2.3) with the initial condition $q(a)=q_a$. This maximal solution is denoted by $q(\cdot,u,q_a)$ and is defined on the maximal interval $I_\mathbb{T}(u,q_a)$. Recall that $q(t,u,q_a) = q_a+\int_{[a,t[_\mathbb{T}} f(q(\tau,u,q_a),u(\tau),\tau)\,\Delta\tau$, for every $t\in I_\mathbb{T}(u,q_a)$, see Lemma A.1 or [52, Lemma 1]. Moreover, one has:
- either $I_\mathbb{T}(u,q_a)=\mathbb{T}$, that is, $q(\cdot,u,q_a)$ is a *global* solution of (III.2.3);
- or $I_\mathbb{T}(u,q_a)=[a,b[_\mathbb{T}$ where $b\in\mathbb{T}\backslash\{a\}$ is a left-dense point of $\mathbb{T}$, and in this case, $q(\cdot,u,q_a)$ is not bounded on $I_\mathbb{T}(u,q_a)$.

We refer to Theorem A.2 or [52, Theorem 2] for more details. The above results are instrumental to define the concept of admissible control.

Definition III.2. For every $q_a \in \mathbb{R}^n$, the control $u \in \mathcal{U}$ is said to be *admissible* on $[a,b[_{\mathbb{T}}$ for some given $b \in \mathbb{T}\backslash\{a\}$ whenever $q(\cdot,u,q_a)$ is well defined on $[a,b]_{\mathbb{T}}$, that is, $b \in I_{\mathbb{T}}(u,q_a)$.

We are now in a position to define rigorously a general optimal control problem on the time scale $\mathbb{T}$ (denoted in short $(\mathbf{OCP})_{\mathbb{T}}$). Let $j \in \mathbb{N}^*$ and S be a non empty closed convex subset of $\mathbb{R}^j$. Let $f^0 : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{T} \longrightarrow \mathbb{R}, (x,v,t) \longmapsto f^0(x,v,t)$ be a continuous function of class $\mathscr{C}^1$ with respect to its two first variables, and $g : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^j, (x_1,x_2) \longmapsto g(x_1,x_2)$ be a function of class $\mathscr{C}^1$. In what follows the subset S and the function g account for constraints on the initial and final conditions of the control problem.

Definition of $(\mathbf{OCP})_{\mathbb{T}}$: determine a trajectory q^* defined on $[a,b^*]_{\mathbb{T}}$, solution of

$$q^{\Delta}(t) = f(q(t),u(t),t), \tag{III.2.3}$$

and associated with a control $u^* \in \mathrm{L}^\infty_{\mathbb{T}}([a,b^*[_{\mathbb{T}};\Omega)$, minimizing the cost

$$\int_{[a,b[_{\mathbb{T}}} f^0(q(\tau),u(\tau),\tau)\,\Delta\tau \tag{III.2.4}$$

over all possible trajectories q defined on $[a,b]_{\mathbb{T}}$, solutions of (III.2.3) and associated with an admissible control $u \in \mathrm{L}^\infty_{\mathbb{T}}([a,b[_{\mathbb{T}};\Omega)$, with $b \in \mathbb{T}\backslash\{a\}$, and satisfying $g(q(a),q(b)) \in \mathrm{S}$. The final time can be fixed or not. If it is fixed then $b^* = b$ in $(\mathbf{OCP})_{\mathbb{T}}$.

III.2.3 Pontryagin Maximum Principle

In the statement below, the *orthogonal* of S at a point $x \in \mathrm{S}$ is defined by

$$\mathcal{O}_{\mathrm{S}}(x) = \{x' \in \mathbb{R}^j \mid \forall x'' \in \mathrm{S},\ \langle x', x''-x\rangle_{\mathbb{R}^j} \leq 0\}. \tag{III.2.5}$$

It is a closed convex cone containing 0. The *Hamiltonian* of the optimal control problem $(\mathbf{OCP})_{\mathbb{T}}$ is the function $H : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{T} \longrightarrow \mathbb{R}$ defined by $H(x,v,w,w^0,t) = \langle w, f(x,v,t)\rangle_{\mathbb{R}^n} + w^0 f^0(x,v,t)$.

Theorem III.1 (Pontryagin Maximum Principle)**.** *Let $b^* \in \mathbb{T}\backslash\{a\}$. If the trajectory q^*, defined on $[a,b^*]_{\mathbb{T}}$ and associated with a control $u^* \in \mathrm{L}^\infty_{\mathbb{T}}([a,b^*[_{\mathbb{T}};\Omega)$, is a solution of $(\mathbf{OCP})_{\mathbb{T}}$, then there exist $p^0 \leq 0$ and $\psi \in \mathbb{R}^j$, with $(p^0,\psi) \neq (0,0)$, and there exists a mapping $p \in \mathrm{AC}([a,b^*]_{\mathbb{T}},\mathbb{R}^n)$ (called adjoint vector), such that it holds*

$$q^{*\Delta}(t) = \frac{\partial H}{\partial w}(q^*(t),u^*(t),p^\sigma(t),p^0,t), \qquad p^{\Delta}(t) = -\frac{\partial H}{\partial x}(q^*(t),u^*(t),p^\sigma(t),p^0,t), \tag{III.2.6}$$

for Δ-a.e. $t \in [a,b^*[_{\mathbb{T}}$. Moreover, it holds

$$\Big\langle \frac{\partial H}{\partial v}(q^*(r),u^*(r),p^\sigma(r),p^0,r), v-u^*(r)\Big\rangle_{\mathbb{R}^m} \leq 0, \tag{III.2.7}$$

for every $r \in [a,b^*[_{\mathbb{T}} \cap \mathrm{RS}$ and every $v \in \overline{\mathrm{Co}}(\mathcal{D}^\Omega_{\mathrm{stab}}(u^*(r)))$, and

$$H(q^*(s),u^*(s),p^\sigma(s),p^0,s) = \max_{v\in\Omega} H(q^*(s),v,p^\sigma(s),p^0,s), \tag{III.2.8}$$

for Δ-a.e. $s \in [a,b^*[_{\mathbb{T}} \cap \mathrm{RD}$.

Besides, one has the transversality conditions on the initial and final adjoint vector

$$p(a) = -\left(\frac{\partial g}{\partial x_1}(q^*(a),q^*(b^*))\right)^{\mathrm{T}} \times \psi, \qquad p(b^*) = \left(\frac{\partial g}{\partial x_2}(q^*(a),q^*(b^*))\right)^{\mathrm{T}} \times \psi, \tag{III.2.9}$$

and $-\psi \in \mathcal{O}_{\mathrm{S}}(g(q^*(a),q^*(b^*)))$.

Furthermore, if the final time b^* is not fixed in $(\mathbf{OCP})_{\mathbb{T}}$, and if additionally b^* belongs to the interior of $\mathbb{T}$ for the topology of $\mathbb{R}$, then

$$\max_{v\in\Omega} H(q^*(b^*),v,p^\sigma(b^*),p^0,b^*) = 0. \tag{III.2.10}$$

Proof. See Section III.3. □

Before proceeding with a series of remarks and comments, we provide:
– a version of the PMP for optimal control problems with parameters (see Remark III.4);
– a version of the PMP with an additional necessary optimality condition in the case where the final time is free and the Hamiltonian is autonomous (see Remark III.5).

Remark III.4 (PMP for optimal control problems with parameters)**.** Let Λ be a Banach space. We consider the general nonlinear controlled system with parameters posed on the time scale $\mathbb{T}$

$$q^{\Delta}(t) = f(\lambda, q(t), u(t), t), \tag{III.2.11}$$

where $f : \Lambda \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{T} \longrightarrow \mathbb{R}^n, (\lambda, x, v, t) \longmapsto f(\lambda, x, v, t)$ is a continuous function of class $\mathscr{C}^1$ with respect to its three first variables, and where $u \in \mathcal{U}$ as before. The notion of admissibility is defined as before. Let $f^0 : \Lambda \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{T} \longrightarrow \mathbb{R}, (\lambda, x, v, t) \longmapsto f^0(\lambda, x, v, t)$ be a continuous function of class $\mathscr{C}^1$ with respect to its three first variables, and $g : \Lambda \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^j, (\lambda, x_1, x_2) \longmapsto g(\lambda, x_1, x_2)$ be a function of class $\mathscr{C}^1$.

We consider the optimal control problem on $\mathbb{T}$, denoted in short $(\mathbf{OCP})^{\lambda}_{\mathbb{T}}$, of determining a trajectory q^* defined on $[a, b^*]_{\mathbb{T}}$, solution of (III.2.11) and associated with a control $u^* \in \mathrm{L}^{\infty}_{\mathbb{T}}([a, b^*[_{\mathbb{T}}; \Omega)$ and with a parameter $\lambda^* \in \Lambda$, minimizing the cost

$$\int_{[a,b[_{\mathbb{T}}} f^0(\lambda, q(\tau), u(\tau), \tau)\, \Delta\tau$$

over all possible trajectories q defined on $[a, b]_{\mathbb{T}}$, solutions of (III.2.11) and associated with $\lambda \in \Lambda$ and with an admissible control $u \in \mathrm{L}^{\infty}_{\mathbb{T}}([a, b[_{\mathbb{T}}; \Omega)$, with $b \in \mathbb{T}\backslash\{a\}$, and satisfying $g(\lambda, q(a), q(b)) \in \mathrm{S}$. The final time can be fixed or not.

The *Hamiltonian* of $(\mathbf{OCP})^{\lambda}_{\mathbb{T}}$ is the function $H : \Lambda \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{T} \longrightarrow \mathbb{R}$ defined by

$$H(\lambda, x, v, w, w^0, t) = \langle w, f(\lambda, x, v, t)\rangle_{\mathbb{R}^n} + w^0 f^0(\lambda, x, v, t).$$

If the trajectory q^*, defined on $[a, b^*]_{\mathbb{T}}$ and associated with a control $u^* \in \mathrm{L}^{\infty}_{\mathbb{T}}([a, b^*[_{\mathbb{T}}; \Omega)$ and with a parameter $\lambda^* \in \Lambda$, is a solution of $(\mathbf{OCP})^{\lambda}_{\mathbb{T}}$, then all conclusions of Theorem III.1 hold, and moreover

$$\int_{[a,b^*[_{\mathbb{T}}} \frac{\partial H}{\partial \lambda}(\lambda^*, q^*(t), u^*(t), p^{\sigma}(t), p^0, t)\, \Delta t + \Big\langle \frac{\partial g}{\partial \lambda}(\lambda^*, q^*(a), q^*(b^*)), \psi \Big\rangle_{\mathbb{R}^j} = 0. \tag{III.2.12}$$

This additional statement is proved in Section III.3.3.2 and allows to prove the PMP for optimal control problems with free final time and autonomous Hamiltonian given in Remark III.5.

Remark III.5 (PMP for optimal control problems with free final time and autonomous Hamiltonian)**.** Let us assume that the final time b^* is not fixed in $(\mathbf{OCP})_{\mathbb{T}}$, and that additionally b^* belongs to the interior of $\mathbb{T}$ for the topology of $\mathbb{R}$. Moreover, we assume that H is autonomous (that is, does not depend on t). Then all conclusions of Theorem III.1 hold, and moreover

$$\int_{[a,b^*[_{\mathbb{T}}} H(q^*(t), u^*(t), p^{\sigma}(t), p^0)\, \Delta t = 0. \tag{III.2.13}$$

This additional statement is proved as well in Section III.3.3.2.

Remark III.6. As is well known, the Lagrange multiplier (p^0, ψ) (and thus the triple (p, p^0, ψ)) is defined up to a multiplicative scalar. Defining an *extremal* as a quadruple (q, u, p, p^0) solution of the above equations, an extremal is said to be *normal* whenever $p^0 \neq 0$ and *abnormal* whenever $p^0 = 0$. The component p^0 corresponds to the Lagrange multiplier associated with the cost function. In the normal case $p^0 \neq 0$, it is usual to normalize the Lagrange multiplier so that $p^0 = -1$. Finally, note that the convention $p^0 \leq 0$ in the PMP leads to a maximization condition of the Hamiltonian (the convention $p^0 \geq 0$ would lead to a minimization condition).

Remark III.7. Without loss of generality, we consider in this chapter optimal control problems defined with the notions of Δ-derivative and Δ-integral. These notions are naturally associated with right-dense and right-scattered points in the basic properties of calculus (see Section I.2.1). Therefore, when using a Δ-derivative in the definition of $(\mathbf{OCP})_{\mathbb{T}}$, one cannot hope to derive in general, for instance, a maximization condition at left-dense points (see Remark III.1).

Note that all results of this chapter can be as well stated in terms of ∇-derivative, ∇-integral, left-dense and left-scattered points. We refer to the duality principle [62], which describes how to obtain a result in the nabla time scale setting from the delta one and vice versa.

Remark III.8. In the classical continuous case, it is well known that the maximized Hamiltonian along an optimal extremal, that is, the function $t \longmapsto \max_{v\in\Omega} H(q^*(t), v, p^\sigma(t), p^0, t)$, is Lipschitzian on $[a, b^*]$, and if the dynamics are autonomous (that is, if H does not depend on t) then this function is constant. Moreover, if the final time is free then the maximized Hamiltonian vanishes at the final time.

In the discrete-time setting and a fortiori in the general time scale setting, none of these properties do hold any more in general (see Examples III.6 and III.8 below). The non constant feature is due in particular to the fact that the usual formula of derivative of a composition does not hold in general time scale calculus.

Remark III.9. The PMP is derived here in a general framework. We do not make any particular assumption on the time scale $\mathbb{T}$, and do not assume that the set of control constraints Ω is convex or compact. In Section III.3.1, we discuss the strategy of proof of Theorem III.1 and we explain how the generality of the framework led us to choose a method based on a variational principle rather than one based on a fixed-point theorem.

We do not make any convexity assumption on the dynamics (f, f^0). As a consequence, and as is well known in the discrete case (see *e.g.* [41, p. 50–63]), at right-scattered points the maximization condition (III.2.8) does not hold true in general and must be weakened into (III.2.7) (see Remark III.11).

Remark III.10. In a right-scattered point r, note that if $u^*(r) \in \mathrm{Int}(\Omega)$ then Inequality (III.2.7) leads to

$$\frac{\partial H}{\partial v}(q^*(r), u^*(r), p^\sigma(r), p^0, r) = 0.$$

Consequently, in the case $\Omega = \mathbb{R}^m$, this equality holds true at every right-scattered point and also at right-dense points from the maximization condition. This is the context of what is usually referred to as the *weak PMP*, see [117, Theorem 9.4] or [118].

In particular, the statement of this *weak PMP* allows to recover the integral Euler-Lagrange equation obtained in [114, Theorem 4] and recalled in Chapter II in the case of a normal extremal and with $(f(x, v, t), f^0(x, v, t)) = (v, L(x, v, t))$.

Remark III.11. In the classical continuous-time case, all points are right-dense and consequently, Theorem III.1 generalizes the usual continuous-time PMP where the maximization condition (III.2.8) is valid μ_L-almost everywhere (see [40, Theorem 6 p. 67]).

In the discrete-time setting, the possible failure of the maximization condition is a well known fact (see *e.g.* [41, p. 50–63]), and a fortiori in the time scale setting, the maximization condition cannot be expected to hold in general at right-scattered points (see counterexamples below).

Many works have been devoted to derive a PMP in the discrete-time setting (see *e.g.* [35, 41, 61, 102, 124, 125, 171]). Since the maximization condition cannot be expected to hold true in general for discrete-time optimal control problems, it must be replaced with a weaker condition, of the kind (III.2.7), involving the derivative of H with respect to v. Such a kind of inequality is provided in [41, Theorem 42.1 p. 330] for finite horizon problems and in [35] for infinite horizon problems. In [124, 125, 192], the authors assume *directional convexity*, that is, for all $(v_1, v_2) \in \Omega^2$ and every $\theta \in [0, 1]$, there exists $v_\theta \in \Omega$ such that

$$f(x, v_\theta, t) = \theta f(x, v_1, t) + (1-\theta) f(x, v_2, t), \quad f^0(x, v_\theta, t) \leq \theta f^0(x, v_1, t) + (1-\theta) f^0(x, v_2, t),$$

for every $x \in \mathbb{R}^n$ and every $t \in \mathbb{T}$; and under this assumption they derive the maximization condition in the discrete-time case (see also [61] and [192, p. 235]). Note that this assumption is satisfied whenever Ω is convex, the dynamics f is affine with respect to v, and f^0 is convex in v (which implies that H is concave in v).

However, note that, under additional assumptions, (III.2.7) implies the maximization condition. More precisely, let $r \in [a,b[_{\mathbb{T}} \cap \mathrm{RS}$ and let (q^*,u^*,p,p^0) be the optimal extremal of Theorem III.1. Let $r \in [a,b[_{\mathbb{T}} \cap \mathrm{RS}$. If the function $v \longmapsto H(q^*(r),v,p^\sigma(r),p^0,r)$ is concave on $\mathbb{R}^m$, then the inequality (III.2.7) implies that

$$H(q^*(r),u^*(r),p^\sigma(r),p^0,r) = \max_{v \in \overline{\mathrm{Co}}(\mathcal{D}^\Omega_{\mathrm{stab}}(u^*(r)))} H(q^*(r),v,p^\sigma(r),p^0,r).$$

If moreover $\Omega \subset \overline{\mathrm{Co}}(\mathcal{D}^\Omega_{\mathrm{stab}}(u^*(r)))$ (this is the case if Ω is convex), since $u^*(r) \in \Omega$, it follows that

$$H(q^*(r),u^*(r),p^\sigma(r),p^0,r) = \max_{v \in \Omega} H(q^*(r),v,p^\sigma(r),p^0,r).$$

Therefore, in particular, if H is concave in v and Ω is convex then the maximization condition holds as well at every right-scattered point.

Remark III.12. It is interesting to note that, if H is convex in v then a certain minimization condition can be derived at every right-scattered point, as follows.

For every $v \in \Omega$, let $\mathrm{Opp}(v) = \{2v - v' \mid v' \in \overline{\mathrm{Co}}(\mathcal{D}^\Omega_{\mathrm{stab}}(v))\}$ denote the symmetric of $\overline{\mathrm{Co}}(\mathcal{D}^\Omega_{\mathrm{stab}}(v))$ with respect to the center v. It obviously follows from (III.2.7) that

$$\left\langle \frac{\partial H}{\partial v}(q^*(r),u^*(r),p^\sigma(r),p^0,r), v - u^*(r) \right\rangle_{\mathbb{R}^m} \geq 0, \tag{III.2.14}$$

for every $r \in [a,b[_{\mathbb{T}} \cap \mathrm{RS}$ and every $v \in \mathrm{Opp}(u^*(r))$. If H is convex in v on $\mathbb{R}^m$, then the inequality (III.2.14) implies that

$$H(q^*(r),u^*(r),p^\sigma(r),p^0,r) = \min_{v \in \mathrm{Opp}(u^*(r))} H(q^*(r),v,p^\sigma(r),p^0,r) \tag{III.2.15}$$

We next provide several very simple examples illustrating the previous remarks.

Example III.6. Here we give a counterexample showing that, although the final time is not fixed, the maximized Hamiltonian may not vanish.

Set $\mathbb{T} = \mathbb{N}$, $n = m = 1$, $f(x,v,t) = v$, $f^0(x,v,t) = 1$, $\Omega = [0,1]$, $j = 2$, $g(x_1,x_2) = (x_1,x_2)$ and $\mathrm{S} = \{0\} \times \{3/2\}$. The corresponding optimal control problem is the problem of steering the discrete-time control one-dimensional system $q(k+1) = q(k) + u(k)$ from $q(0) = 0$ to $q(b) = 3/2$ in minimal time, with control constraints $0 \leq u(k) \leq 1$. It is clear that the minimal time is $b^* = 2$, and that any control u such that $0 \leq u(0) \leq 1$, $0 \leq u(1) \leq 1$, and $u(0) + u(1) = 3/2$, is optimal.

Among these optimal controls, consider u^* defined by $u^*(0) = 1/2$ and $u^*(1) = 1$. Consider ψ, $p^0 \leq 0$ and p the adjoint vector whose existence is asserted by the PMP. Since $u^*(0) \in \mathrm{Int}(\Omega)$, it follows from (III.2.7) that $p(1) = 0$. The Hamiltonian is $H(x,v,w,w^0,t) = wv + w^0$, and since it is independent of x, it follows that p is constant and thus equal to 0. In particular, $p(0) = p(2) = 0$ and hence $\psi = 0$. From the nontriviality condition $(p^0,\psi) \neq (0,0)$ we infer that $p^0 \neq 0$. Therefore the maximized Hamiltonian at the final time is here equal to p^0 and thus is not equal to 0.

Example III.7. Here we give a counterexample (in the spirit of [41, Examples 10.1-10.4 p. 59–62]) showing the failure of the maximization condition at right-scattered points.

Set $\mathbb{T} = \{0,1,2\}$, $n = m = 1$, $f(x,v,t) = v - x$, $f^0(x,v,t) = 2x^2 - v^2$, $\Omega = [0,1]$, $j = 1$, $g(x_1,x_2) = x_1$ and $\mathrm{S} = \{0\}$. Any solution of the resulting control system is such that $q(0) = 0$, $q(1) = u(0)$, $q(2) = u(1)$, and its cost is equal to $u(0)^2 - u(1)^2$. It follows that the optimal control u^* is unique and is such that $u^*(0) = 0$ and $u^*(1) = 1$. The Hamiltonian is $H(q,u,p,p^0,t) = x(v-x) + p^0(2x^2 - v^2)$. Consider ψ, $p^0 \leq 0$ and p the adjoint vector whose existence is asserted by the PMP. Since g does not depend on x_2, it follows that $p(2) = 0$, and from the extremal equations we infer that $p(1) = 0$ and $p(0) = 0$. Therefore $\psi = 0$ and hence $p^0 \neq 0$ (nontriviality condition) and we can assume that $p^0 = -1$. It follows that the maximized Hamiltonian is equal to $-p^0 = 1$ at $r = 0,1,2$, whereas $H(q^*(0),u^*(0),p(1),p^0,0) = 0$. In particular, the maximization condition (III.2.8) is not satisfied at $r = 0 \in \mathrm{RS}$ (note that it is however satisfied at $r = 1$).

Note that, in accordance with the fact that H is convex in v and $\mathrm{Opp}(u^*(0)) =]-\infty,0]$ and $\mathrm{Opp}(u^*(1)) = [1,+\infty[$, the minimization condition (III.2.15) is indeed satisfied (see Remark III.12).

Example III.8. Here we give a counterexample in which, although the Hamiltonian is autonomous (independent of t), the maximized Hamiltonian is not constant over $\mathbb{T}$.

Set $\mathbb{T} = \{0,1,2\}$, $n = m = 1$, $f(q,u,t) = v - x$, $f^0(q,u,t) = (v^2 - x^2)/2$, $j = 1$, $g(x_1,x_2) = x_1$, $\mathrm{S} = \{1\}$, $\Omega = [0,1]$ and $b = 2$. Any solution of the resulting control system is such that $q(0) = 1$, $q(1) = u(0)$, $q(2) = u(1)$, and its cost is equal to $(u(1)^2 - 1)/2$. It follows that any control u such that $u(1) = 0$ is optimal (the value of $u(0)$ is arbitrary). Consider the optimal control u^* defined by $u^*(0) = u^*(1) = 0$, and let q^* be the corresponding trajectory. Then $q^*(0) = 1$ and $q^*(1) = q^*(2) = 0$. The Hamiltonian is $H(x,v,w,w^0,t) = w(v-x)+w^0(v^2-x^2)/2$. Consider ψ, $p^0 \leq 0$ and p the adjoint vector whose existence is asserted by the PMP. Since g does not depend on x_2, it follows that $p(2) = 0$, and from the extremal equations we infer that $p(1) = 0$ and $p(0) = -p^0$. In particular, from the nontriviality condition one has $p^0 \neq 0$ and we can assume that $p^0 = -1$. Therefore $H(q^*(0),v,p(1),p^0,0) = 1/2 - v^2$ and $H(q^*(1),v,p(2),p^0,1) = -v^2/2$, and it easily follows that that the maximization condition holds at $r = 0$ and $r = 1$. This is in accordance with the fact that H is concave in v and Ω is convex. Moreover, the maximized Hamiltonian is equal to $1/2$ at $r = 0$, and to 0 at $r = 1$ and $r = 2$.

III.3 Proof of the main result

This section is devoted to the proof of our main result (Theorem III.1). Firstly, Section III.3.1 contains preliminary comments on the strategy used in our reasoning. In particular, we explain the reasons that led us to use a proof based on a variational principle rather than a fixed point theorem. In Section III.3.2, we introduce needle-like variations on admissible controls and relative results that are instrumental (with the use of the Ekeland's Variational Principle) in order to prove Theorem III.1 in Section III.3.3.

Numerous lemmas are only technical or very similar each other. Consequently, some proofs are only detailed in Appendix B.

III.3.1 Preliminary comments

There exist several proofs of the continuous-time PMP in the literature. Mainly they can be classified as variants of two different approaches: the first of which consists of using a fixed point argument, and the second consists of using Ekeland's Variational Principle.

More precisely, the classical (and historical) proof of [40] relies on the use of the so-called needle-like variations combined with a fixed point Brouwer argument (see also [107, 147]). There exist variants, relying on the use of a conic version of the Implicit Function Theorem (see [5] or [100, 193]), the proof of which being however based on a fixed point argument. The proof of [54] uses a separation theorem (Hahn-Banach arguments) for cones combined with the Brouwer fixed point theorem. We could cite many other variants, all of them relying, at some step, on a fixed point argument.

The proof of [82] is of a different nature and follows from the combination of needle-like variations with Ekeland's Variational Principle. It does not rely on a fixed point argument. By the way note that this proof leads as well to an approximate PMP (see [82]), and withstands generalizations to the infinite dimensional setting (see *e.g.* [149]).

Note that, in all cases, needle-like variations are used to generate the so-called Pontryagin cone, serving as a first-order convex approximation of the reachable set. The adjoint vector is then constructed by propagating backward in time a Lagrange multiplier which is normal to this cone. Roughly, needle-like variations are kinds of perturbations of the reference control in L^1 topology (perturbations with arbitrary values, over small intervals of time) which generate perturbations of the trajectories in $\mathscr{C}^0$ topology.

Due to obvious topological obstructions, it is evident that the classical strategy of needle-like variations combined with a fixed point argument cannot hold in general in the time scale setting. At least one should distinguish between dense points and scattered points of $\mathbb{T}$. But even this distinction is not sufficient. Indeed, when applying the Brouwer fixed point Theorem to the mapping built on needle-like variations (see [40, 147]), it appears to be crucial that the domain of this mapping be convex. Roughly speaking, this domain consists of the product of the intervals of the spikes (intervals of perturbation). This requirement obviously excludes the scattered points of a time scale (which have anyway to be treated in another way), but even at some right-dense point $s \in \mathrm{RD}$, there does not necessarily exist $\varepsilon > 0$ such that $[s, s+\varepsilon] \subset \mathbb{T}$. At such a point we can only ensure that 0 is not isolated in the set $\{\beta \geq 0 \mid s+\beta \in \mathbb{T}\}$. In our opinion this basic obstruction makes impossible the use of a fixed point argument in order to derive the PMP on a general time scale. Of course to overcome this difficulty one can assume that the μ_Δ-measure of right-dense points not admitting a right interval included in $\mathbb{T}$ is zero. This assumption is however not very natural and would rule out time scales such as a generalized Cantor set having a positive μ_L-measure. Another serious difficulty that we are faced with on a general time scale is the technical fact that the formula (III.2.2), accounting for Lebesgue points, is valid only for β such that $s+\beta \in \mathbb{T}$. Actually if $s+\beta \notin \mathbb{T}$ then (III.2.2) is not true any more in general (it is very easy to construct a time scale $\mathbb{T}$ for which (III.2.2) fails whenever $s+\beta \notin \mathbb{T}$, even with $q=1$). Note that the concept of Lebesgue point is instrumental in the classical proof of the PMP in order to ensure that the needle-like variations can be built at different times [1] (see [40, 147]). On a general time scale this technical point would raise a serious issue [2].

The proof of the PMP that we provide in this chapter is based on Ekeland's Variational Principle, which permits to avoid the above obstructions and happens to be well adapted for the proof of a general PMP on time scale. It requires however the treatment of other kinds of technicalities, one of them being the concept of stable Ω-dense direction that we were led to introduce. Another point is that Ekeland's Variational Principle requires a complete metric space, which has led us to assume that Ω is closed (see Footnote 3).

III.3.2 Needle-like variations of admissible controls

Let $b \in \mathbb{T}\backslash\{a\}$. Following the definition of an admissible control (see Definition III.2), we denote by $\mathcal{UQ}^b_{\mathrm{ad}}$ the set of all $(u, q_a) \in \mathcal{U} \times \mathbb{R}^n$ such that u is an admissible control on $[a,b]_\mathbb{T}$ associated with the initial condition q_a. It is endowed with the distance

$$d_{\mathcal{UQ}^b_{\mathrm{ad}}}((u,q_a),(u',q'_a)) = \|u-u'\|_{\mathrm{L}^1_\mathbb{T}([a,b[_\mathbb{T},\mathbb{R}^m)} + \|q_a - q'_a\|_{\mathbb{R}^n}. \tag{III.3.16}$$

Throughout the section, we consider $(u, q_a) \in \mathcal{UQ}^b_{\mathrm{ad}}$ with $u \in \mathrm{L}^\infty_\mathbb{T}([a,b[_\mathbb{T};\Omega)$ and the corresponding solution $q(\cdot, u, q_a)$ of (III.2.3) with $q(a) = q_a$. This Section III.3.2 is devoted to define appropriate variations of (u, q_a), instrumental in order to prove the PMP. We present some preliminary topological results in Section III.3.2.1. Then we define needle-like variations of u in Sections III.3.2.2 and III.3.2.3, respectively at a right-scattered point and at a right-dense point and derive some useful properties. Finally in Section III.3.2.4 we make some variations of the initial condition q_a.

III.3.2.1 Preliminaries

In the first lemma below, we state that $\mathcal{UQ}^b_{\mathrm{ad}}$ is open. Actually we prove a stronger result, by showing that $\mathcal{UQ}^b_{\mathrm{ad}}$ contains a neighborhood of any of its point in $\mathrm{L}^1_\mathbb{T}$ topology, which is useful in order to define needle-like variations.

The proofs of the two following lemmas are detailed in Section B.1.1 of Appendix B.

Lemma III.1. *Let $R > \|u\|_{\mathrm{L}^\infty_\mathbb{T}([a,b[_\mathbb{T},\mathbb{R}^m)}$. There exist $\nu_R > 0$ and $\eta_R > 0$ such that the set*

$$\mathrm{E}(u,q_a,R) = \{(u',q'_a) \in \mathcal{U}\times\mathbb{R}^n \mid \|u'-u\|_{\mathrm{L}^1_\mathbb{T}([a,b[_\mathbb{T},\mathbb{R}^m)} \leq \nu_R,\ \|u'\|_{\mathrm{L}^\infty_\mathbb{T}([a,b[_\mathbb{T},\mathbb{R}^m)} \leq R,\ \|q'_a - q_a\|_{\mathbb{R}^n} \leq \eta_R\}$$

is contained in $\mathcal{UQ}^b_{\mathrm{ad}}$.

Lemma III.2. *With the notations of Lemma III.1, the mapping*

$$\begin{array}{rccc} F_{(u,q_a,R)}: & (\mathrm{E}(u,q_a,R), d_{\mathcal{UQ}^b_{\mathrm{ad}}}) & \longrightarrow & (\mathrm{C}([a,b]_\mathbb{T},\mathbb{R}^n), \|\cdot\|_\infty) \\ & (u',q'_a) & \longmapsto & q(\cdot,u',q'_a) \end{array}$$

is Lipschitzian. In particular, for every $(u',q'_a) \in \mathrm{E}(u,q_a,R)$, $q(\cdot,u',q'_a)$ converges uniformly to $q(\cdot,u,q_a)$ on $[a,b]_\mathbb{T}$ when u' tends to u in $\mathrm{L}^1_\mathbb{T}([a,b[_\mathbb{T},\mathbb{R}^m)$ and q'_a tends to q_a in $\mathbb{R}^n$.

1. More precisely, what is used in the approximate continuity property (see *e.g.* [84]).
2. We are actually able to overcome this difficulty by considering multiple variations at right-scattered points, however this requires to assume that the set Ω is locally convex. The proof that we present further does not require such an assumption.

III.3.2.2 Needle-like variation of u at a right-scattered point

Let $r \in [a,b[_{\mathbb{T}} \cap \mathrm{RS}$ and let $y \in \mathcal{D}^{\Omega}(u(r))$. We define the needle-like variation $\Pi = (r,y)$ of u at the right-scattered point r by

$$u_{\Pi}(t,\alpha) = \begin{cases} u(r) + \alpha(y - u(r)) & \text{if} \quad t = r, \\ u(t) & \text{if} \quad t \neq r. \end{cases}$$

for every $\alpha \in \mathcal{D}^{\Omega}(u(r),y)$. It follows from Section III.2.1 that $u_{\Pi}(\cdot,\alpha) \in \mathrm{L}^{\infty}_{\mathbb{T}}([a,b[_{\mathbb{T}};\Omega)$. Let us give the following series of technical results proved in Section B.1.2 of Appendix B.

Lemma III.3. *There exists $\alpha_0 > 0$ such that $(u_{\Pi}(\cdot,\alpha), q_a) \in \mathcal{UQ}^b_{\mathrm{ad}}$, for every $\alpha \in \mathcal{D}^{\Omega}(u(r),y) \cap [0,\alpha_0]$.*

Lemma III.4. *The mapping*

$$\begin{array}{rccc} F_{(u,q_a,\Pi)} : & (\mathcal{D}^{\Omega}(u(r),y) \cap [0,\alpha_0], |\cdot|) & \longrightarrow & (\mathrm{C}([a,b]_{\mathbb{T}},\mathbb{R}^n), \|\cdot\|_{\infty}) \\ & \alpha & \longmapsto & q(\cdot, u_{\Pi}(\cdot,\alpha), q_a) \end{array}$$

is Lipschitzian. In particular, for every $\alpha \in \mathcal{D}^{\Omega}(u(r),y) \cap [0,\alpha_0]$, $q(\cdot,u_{\Pi}(\cdot,\alpha),q_a)$ converges uniformly to $q(\cdot,u,q_a)$ on $[a,b]_{\mathbb{T}}$ as α tends to 0.

We define the so-called *variation vector* $\omega_{\Pi}(\cdot,u,q_a)$ associated with the needle-like variation $\Pi = (r,y)$ as the unique solution on $[\sigma(r),b]_{\mathbb{T}}$ of the linear Δ-Cauchy problem

$$\omega^{\Delta}(t) = \frac{\partial f}{\partial x}(q(t,u,q_a),u(t),t) \times \omega(t), \quad \omega(\sigma(r)) = \mu(r)\frac{\partial f}{\partial v}(q(r,u,q_a),u(r),r) \times (y - u(r)). \tag{III.3.17}$$

The existence and uniqueness of $\omega_{\Pi}(\cdot,u,q_a)$ are ensured by Theorem A.3 (see also [52, Theorem 3]).

Proposition III.1. *The mapping*

$$\begin{array}{rccc} F_{(u,q_a,\Pi)} : & (\mathcal{D}^{\Omega}(u(r),y) \cap [0,\alpha_0], |\cdot|) & \longrightarrow & (\mathrm{C}([\sigma(r),b]_{\mathbb{T}},\mathbb{R}^n), \|\cdot\|_{\infty}) \\ & \alpha & \longmapsto & q(\cdot, u_{\Pi}(\cdot,\alpha), q_a) \end{array} \tag{III.3.18}$$

is differentiable at 0, and it holds $DF_{(u,q_a,\Pi)}(0) = \omega_{\Pi}(\cdot,u,q_a)$.

Lemma III.5. *Let $R > \|u\|_{\mathrm{L}^{\infty}_{\mathbb{T}}([a,b[_{\mathbb{T}},\mathbb{R}^m)}$ and let $(u_k,q_{a,k})_{k\in\mathbb{N}}$ be a sequence of $\mathrm{E}(u,q_a,R)$. If u_k converges to u Δ-a.e. on $[a,b[_{\mathbb{T}}$ and $q_{a,k}$ converges to q_a in $\mathbb{R}^n$ as k tends to $+\infty$, then $\omega_{\Pi}(\cdot,u_k,q_{a,k})$ converges uniformly to $\omega_{\Pi}(\cdot,u,q_a)$ on $[\sigma(r),b]_{\mathbb{T}}$ as k tends to $+\infty$.*

Remark III.13. It is interesting to note that, since $u_k(r)$ converges to $u(r)$ as k tends to $+\infty$, if we assume that $y \in \mathcal{D}^{\Omega}_{\mathrm{stab}}(u(r))$, then $y \in \mathcal{D}^{\Omega}(u_k(r))$ for k sufficiently large.

III.3.2.3 Needle-like variation of u at a right-dense point

The definition of a needle-like variation at a Lebesgue right-dense point is very similar to the classical continuous-time case. Let $s \in \mathscr{L}_{[a,b[_{\mathbb{T}}}(f(q(\cdot,u,q_a),u(\cdot),\cdot)) \cap \mathrm{RD}$ and $z \in \Omega$. We define the needle-like variation $\amalg = (s,z)$ of u at s by

$$u_{\amalg}(t,\beta) = \begin{cases} z & \text{if} \quad t \in [s,s+\beta[_{\mathbb{T}}, \\ u(t) & \text{if} \quad t \notin [s,s+\beta[_{\mathbb{T}}. \end{cases}$$

for every $\beta \in \mathcal{V}^b_s$ (here, we use the notations introduced in the beginning of Section III.2). Note that $u_{\amalg}(\cdot,\beta) \in \mathrm{L}^{\infty}_{\mathbb{T}}([a,b[_{\mathbb{T}};\Omega)$. Let us give the following series of technical results proved in Section B.1.3 of Appendix B.

Lemma III.6. *There exists $\beta_0 > 0$ such that $(u_{\amalg}(\cdot,\beta), q_a) \in \mathcal{UQ}^b_{\mathrm{ad}}$ for every $\beta \in \mathcal{V}^b_s \cap [0,\beta_0]$.*

Lemma III.7. *The mapping*

$$\begin{array}{rccc} F_{(u,q_a,\amalg)} : & (\mathcal{V}^b_s \cap [0,\beta_0], |\cdot|) & \longrightarrow & (\mathrm{C}([a,b]_{\mathbb{T}},\mathbb{R}^n), \|\cdot\|_{\infty}) \\ & \beta & \longmapsto & q(\cdot, u_{\amalg}(\cdot,\beta), q_a) \end{array}$$

is Lipschitzian. In particular, for every $\beta \in \mathcal{V}^b_s \cap \overline{B}(0,\beta_0)$, $q(\cdot,u_{\amalg}(\cdot,\beta),q_a)$ converges uniformly to $q(\cdot,u,q_a)$ on $[a,b]_{\mathbb{T}}$ as β tends to 0.

According to Theorem A.3 (see also [52, Theorem 3]), we define the *variation vector* $\omega_{\Pi}(\cdot,u,q_a)$ associated with the needle-like variation $\Pi=(s,z)$ as the unique solution on $[s,b]_{\mathbb{T}}$ of the linear Δ-Cauchy problem

$$\omega^{\Delta}(t)=\frac{\partial f}{\partial x}(q(t,u,q_a),u(t),t)\times\omega(t),\quad \omega(s)=f(q(s,u,q_a),z,s)-f(q(s,u,q_a),u(s),s). \tag{III.3.19}$$

Proposition III.2. *For every $\delta\in\mathcal{V}^b_s\backslash\{0\}$, the mapping*

$$\begin{array}{rccl} F^{\delta}_{(u,q_a,\Pi)}: & (\mathcal{V}^b_s\cap[0,\beta_0],|\cdot|) & \longrightarrow & (\mathrm{C}([s+\delta,b]_{\mathbb{T}},\mathbb{R}^n),\|\cdot\|_\infty) \\ & \beta & \longmapsto & q(\cdot,u_{\Pi}(\cdot,\beta),q_a) \end{array} \tag{III.3.20}$$

is differentiable at 0*, and one has* $DF^{\delta}_{(u,q_a,\Pi)}(0)=\omega_{\Pi}(\cdot,u,q_a)$.

Lemma III.8. *Let $R>\|u\|_{\mathrm{L}^\infty_{\mathbb{T}}([a,b[_{\mathbb{T}},\mathbb{R}^m)}$ and let $(u_k,q_{a,k})_{k\in\mathbb{N}}$ be a sequence of* $\mathrm{E}(u,q_a,R)$*. If u_k converges to u Δ-a.e. on $[a,b[_{\mathbb{T}}$ and $q_{a,k}$ converges to q_a as k tends to $+\infty$, then $\omega_{\Pi}(\cdot,u_k,q_{a,k})$ converges uniformly to $\omega_{\Pi}(\cdot,u,q_a)$ on $[s,b]_{\mathbb{T}}$ as k tends to $+\infty$.*

III.3.2.4 Variation of the initial condition q_a

Let $q'_a\in\mathbb{R}^n$. Let us give the following series of technical results proved in Section B.1.4 of Appendix B.

Lemma III.9. *There exists $\gamma_0>0$ such that $(u,q_a+\gamma q'_a)\in\mathcal{UQ}^b_{\mathrm{ad}}$ for every $\gamma\in[0,\gamma_0]$.*

Lemma III.10. *The mapping*

$$\begin{array}{rccl} F_{(u,q_a,q'_a)}: & ([0,\gamma_0],|\cdot|) & \longrightarrow & (\mathrm{C}([a,b]_{\mathbb{T}},\mathbb{R}^n),\|\cdot\|_\infty) \\ & \gamma & \longmapsto & q(\cdot,u,q_a+\gamma q'_a) \end{array}$$

is Lipschitzian. In particular, for every $\gamma\in[0,\gamma_0]$, $q(\cdot,u,q_a+\gamma q'_a)$ converges uniformly to $q(\cdot,u,q_a)$ on $[a,b]_{\mathbb{T}}$ as γ tends to 0.

According to Theorem A.3 (see also [52, Theorem 3]), we define the *variation vector* $\omega_{q'_a}(\cdot,u,q_a)$ associated with the perturbation q'_a as the unique solution on $[a,b]_{\mathbb{T}}$ of the linear Δ-Cauchy problem

$$\omega^{\Delta}(t)=\frac{\partial f}{\partial x}(q(t,u,q_a),u(t),t)\times\omega(t),\quad \omega(a)=q'_a. \tag{III.3.21}$$

Proposition III.3. *The mapping*

$$\begin{array}{rccl} F_{(u,q_a,q'_a)}: & ([0,\gamma_0],|\cdot|) & \longrightarrow & (\mathrm{C}([a,b]_{\mathbb{T}},\mathbb{R}^n),\|\cdot\|_\infty) \\ & \gamma & \longmapsto & q(\cdot,u,q_a+\gamma q'_a) \end{array} \tag{III.3.22}$$

is differentiable at 0*, and one has* $DF_{(u,q_a,q'_a)}(0)=\omega_{q'_a}(\cdot,u,q_a)$.

Lemma III.11. *Let $R>\|u\|_{\mathrm{L}^\infty_{\mathbb{T}}([a,b[_{\mathbb{T}},\mathbb{R}^m)}$ and let $(u_k,q_{a,k})_{k\in\mathbb{N}}$ be a sequence of elements of* $\mathrm{E}(u,q_a,R)$*. If u_k converges to u Δ-a.e. on $[a,b[_{\mathbb{T}}$ and $q_{a,k}$ converges to q_a in $\mathbb{R}^n$ as k tends to $+\infty$, then $\omega_{q'_a}(\cdot,u_k,q_{a,k})$ converges uniformly to $\omega_{q'_a}(\cdot,u,q_a)$ on $[a,b]_{\mathbb{T}}$ as k tends to $+\infty$.*

III.3.3 Proof of PMP

Throughout this section we consider $(\mathbf{OCP})_{\mathbb{T}}$ with a fixed final time $b\in\mathbb{T}\backslash\{a\}$. We proceed as is very usual (see *e.g.* [40, 147]) by considering the *augmented controlled system* in $\mathbb{R}^{n+1}$

$$\bar{q}^{\Delta}(t)=\bar{f}(\bar{q}(t),u(t),t), \tag{III.3.23}$$

with $\bar{q}=(q,q^0)^{\mathrm{T}}\in\mathbb{R}^n\times\mathbb{R}$, the augmented state, and $\bar{f}:\mathbb{R}^{n+1}\times\mathbb{R}^m\times\mathbb{T}\longrightarrow\mathbb{R}^{n+1}$, the augmented dynamics, defined by $\bar{f}(\bar{x},v,t)=(f(x,v,t),f^0(x,v,t))^{\mathrm{T}}$, where $\bar{x}=(x,x^0)$. The additional coordinate q^0 stands for the cost, and we always impose as an initial condition $q^0(a)=0$, so that $q^0(b)=\int_{[a,b[_{\mathbb{T}}}f^0(q(\tau),u(\tau),\tau)\,\Delta\tau$. The function $\bar{g}:\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}\longrightarrow\mathbb{R}^j$ is defined by $\bar{g}(\bar{x}_1,\bar{x}_2)=g(x_1,x_2)$, where $\bar{x}_i=(x_i,x^0_i)$ for $i=1,2$. Note that

$\bar{f}$ does not depend on x^0 and that $\bar{g}$ does not depend on x_1^0 nor on x_2^0. Note as well that the Hamiltonian of $(\mathbf{OCP})_{\mathbb{T}}$ is written as $H(x,v,w,w^0,t) = \langle \bar{w}, \bar{f}(\bar{x},v,t)\rangle_{\mathbb{R}^{n+1}}$.

With these notations, $(\mathbf{OCP})_{\mathbb{T}}$ consists of determining a trajectory $\bar{q}^* = (q^*, q^{0*})$ defined on $[a,b]_{\mathbb{T}}$, solution of (III.3.23) and associated with a control $u^* \in \mathrm{L}^\infty_{\mathbb{T}}([a,b[_{\mathbb{T}};\Omega)$, minimizing $q^0(b)$ over all possible trajectories $\bar{q} = (q,q^0)$ defined on $[a,b]_{\mathbb{T}}$, solutions of (III.3.23) and associated with an admissible control $u \in \mathrm{L}^\infty_{\mathbb{T}}([a,b[_{\mathbb{T}};\Omega)$ and satisfying $\bar{g}(\bar{q}(a),\bar{q}(b)) \in \mathrm{S}$.

In what follows, let $\bar{q}^*$ be such an optimal trajectory. Set $q_a^* = q^*(a)$. We are going to apply first Ekeland's Variational Principle to a well chosen functional in an appropriate complete metric space, and then, using needle-like variations as defined previously (applied to the augmented system, that is, with the dynamics $\bar{f}$), we are going to derive some inequalities, finally resulting into the desired statement of the PMP.

III.3.3.1 Application of Ekeland's Variational Principle

For the completeness, we recall Ekeland's Variational Principle.

Theorem III.2 ([82]). *Let $(\mathrm{E}, d_{\mathrm{E}})$ be a complete metric space and $J : \mathrm{E} \longrightarrow \mathbb{R}\cup\{+\infty\}$ be a lower semi-continuous function which is bounded below. Let $\varepsilon > 0$ and $u^* \in \mathrm{E}$ such that $J(u^*) \leq \inf_{u\in\mathrm{E}} J(u) + \varepsilon$. Then there exists $u_\varepsilon \in \mathrm{E}$ such that $d_{\mathrm{E}}(u_\varepsilon, u^*) \leq \sqrt{\varepsilon}$ and $J(u_\varepsilon) \leq J(u) + \sqrt{\varepsilon} d_{\mathrm{E}}(u,u_\varepsilon)$ for every $u \in \mathrm{E}$.*

Recall from Lemma III.1 that, for $R > \|u^*\|_{\mathrm{L}^\infty_{\mathbb{T}}([a,b[_{\mathbb{T}},\mathbb{R}^m)}$, the set $\mathrm{E}(u^*, \bar{q}_a^*, R)$ defined in this lemma is contained in $\mathcal{UQ}^b_{\mathrm{ad}}$. To take into account the set Ω of constraints on the controls, we define

$$\mathrm{E}^R_\Omega = \{(u,\bar{q}_a) \in \mathcal{U}\times\mathbb{R}^{n+1} \mid \bar{q}_a = (q_a,0),\ (u,q_a) \in \mathrm{E}(u^*,\bar{q}_a^*,R),\ u \in \mathrm{L}^\infty_{\mathbb{T}}([a,b[_{\mathbb{T}};\Omega)\}.$$

Using the fact that Ω is closed [3], it clearly follows from the (partial) converse of Lebesgue's Dominated Convergence Theorem that $(\mathrm{E}^R_\Omega, d_{\mathcal{UQ}^b_{\mathrm{ad}}})$ is a complete metric space.

Before applying Ekeland's Variational Principle in this space, let us introduce several notations and recall basic facts in order to handle the convex set S. We denote by d_{S} the distance function to S defined by $d_{\mathrm{S}}(x) = \inf_{x'\in\mathrm{S}} \|x-x'\|_{\mathbb{R}^j}$, for every $x \in \mathbb{R}^j$. Recall that, for every $x\in\mathbb{R}^j$, there exists a unique element $\mathrm{P}_{\mathrm{S}}(x) \in \mathrm{S}$ (projection of x onto S) such that $d_{\mathrm{S}}(x) = \|x - \mathrm{P}_{\mathrm{S}}(x)\|_{\mathbb{R}^j}$. It is characterized by the property $\langle x - \mathrm{P}_{\mathrm{S}}(x), x' - \mathrm{P}_{\mathrm{S}}(x)\rangle_{\mathbb{R}^j} \leq 0$ for every $x' \in \mathrm{S}$. Moreover, the projection mapping P_{S} is 1-Lipschitz continuous. Furthermore, it holds $x - \mathrm{P}_{\mathrm{S}}(x) \in \mathcal{O}_{\mathrm{S}}(\mathrm{P}_{\mathrm{S}}(x))$ for every $x\in\mathbb{R}^j$ (where $\mathcal{O}_{\mathrm{S}}(x)$ is defined by (III.2.5)). We recall the following obvious lemmas.

Lemma III.12. *Let $(x_k)_{k\in\mathbb{N}}$ be a sequence of points of $\mathbb{R}^j$ and $(\zeta_k)_{k\in\mathbb{N}}$ be a sequence of nonnegative real numbers such that $x_k \to x \in \mathrm{S}$ and $\zeta_k(x_k - \mathrm{P}_{\mathrm{S}}(x_k)) \to x' \in \mathbb{R}^j$ as $k\to+\infty$. Then $x' \in \mathcal{O}_{\mathrm{S}}(x)$.*

Lemma III.13. *The function $x \longmapsto d^2_{\mathrm{S}}(x)$ is differentiable on $\mathbb{R}^j$ with differential given by $(x)(x') \longmapsto 2\langle x - \mathrm{P}_{\mathrm{S}}(x), x'\rangle_{\mathbb{R}^j}$.*

We are now in a position to apply Ekeland's Variational Principle. For every $\varepsilon > 0$ such that $\sqrt{\varepsilon} < \min(\nu_R,\eta_R)$, we consider the functional $J^R_\varepsilon : (\mathrm{E}^R_\Omega, d_{\mathcal{UQ}^b_{\mathrm{ad}}}) \longrightarrow \mathbb{R}^+$ defined by

$$J^R_\varepsilon(u,\bar{q}_a) = \left(\max(q^0(b,u,\bar{q}_a) - q^{0*}(b) + \varepsilon, 0)^2 + d^2_{\mathrm{S}}(\bar{g}(\bar{q}_a, \bar{q}(b,u,\bar{q}_a)))\right)^{1/2}.$$

Since $F_{(u^*,\bar{q}_a^*,R)}$ (by Lemma III.2), $\bar{g}$ and d_{S} are continuous, it follows that J^R_ε is continuous on $(\mathrm{E}^R_\Omega, d_{\mathcal{UQ}^b_{\mathrm{ad}}})$. Moreover, one has $J^R_\varepsilon(u^*,\bar{q}^*_a) = \varepsilon$ and $J^R_\varepsilon(u,\bar{q}_a) > 0$ for every $(u,\bar{q}_a) \in \mathrm{E}^R_\Omega$. It follows from Ekeland's Variational Principle that, for every $\varepsilon > 0$ such that $\sqrt{\varepsilon} < \min(\nu_R,\eta_R)$, there exists $(u^R_\varepsilon, \bar{q}^R_{a,\varepsilon}) \in \mathrm{E}^R_\Omega$ such that $d_{\mathcal{UQ}^b_{\mathrm{ad}}}((u^R_\varepsilon,\bar{q}^R_{a,\varepsilon}),(u^*,\bar{q}^*_a)) \leq \sqrt{\varepsilon}$ and

$$-\sqrt{\varepsilon} d_{\mathcal{UQ}^b_{\mathrm{ad}}}((u,\bar{q}_a),(u^R_\varepsilon,\bar{q}^R_{a,\varepsilon})) \leq J^R_\varepsilon(u,\bar{q}_a) - J^R_\varepsilon(u^R_\varepsilon,\bar{q}^R_{a,\varepsilon}), \tag{III.3.24}$$

3. Note that the assumption Ω closed is used (only) here in a crucial way. In the proof of the classical continuous-time PMP this assumption is not required because the Ekeland distance which is then used is defined by $\rho(u,v) = \mu_L(\{t\in[a,b] \mid u(t)\neq v(t)\})$, and obviously the set of measurable functions $u : [a,b] \longrightarrow \Omega$ endowed with this distance is complete, under the sole assumption that Ω is measurable. In the discrete-time setting and a fortiori in the general time scale setting, this distance cannot be used any more. Here we use the distance $d_{\mathcal{UQ}^b_{\mathrm{ad}}}$ defined by (III.3.16) but then to ensure completeness it is required to assume that Ω is closed.

for every $(u,\bar{q}_a)\in \mathrm{E}^R_\Omega$. In particular, u^R_ε converges to u^* in $\mathrm{L}^1_{\mathbb{T}}([a,b[_{\mathbb{T}},\mathbb{R}^m)$ and $\bar{q}^R_{a,\varepsilon}$ converges to $\bar{q}^*_a$ as ε tends to 0. Besides, setting

$$\psi^{0R}_\varepsilon = \frac{-1}{J^R_\varepsilon(u^R_\varepsilon,\bar{q}^R_{a,\varepsilon})}\max(q^0(b,u^R_\varepsilon,\bar{q}^R_{a,\varepsilon})-q^{0*}(b)+\varepsilon,0)\leq 0 \tag{III.3.25}$$

and

$$\psi^R_\varepsilon = \frac{-1}{J^R_\varepsilon(u^R_\varepsilon,\bar{q}^R_{a,\varepsilon})}\big(\bar{g}(\bar{q}^R_{a,\varepsilon},\bar{q}(b,u^R_\varepsilon,\bar{q}^R_{a,\varepsilon}))-\mathrm{P_S}(\bar{g}(\bar{q}^R_{a,\varepsilon},\bar{q}(b,u^R_\varepsilon,\bar{q}^R_{a,\varepsilon})))\big)\in\mathbb{R}^j, \tag{III.3.26}$$

note that $|\psi^{0R}_\varepsilon|^2+\|\psi^R_\varepsilon\|^2_{\mathbb{R}^j}=1$ and $-\psi^R_\varepsilon\in\mathcal{O}_\mathrm{S}(\mathrm{P_S}(\bar{g}(\bar{q}^R_{a,\varepsilon},\bar{q}(b,u^R_\varepsilon,\bar{q}^R_{a,\varepsilon}))))$.

Using a compactness argument, the continuity of $F_{(u^*,\bar{q}^*_a,R)}$ and the $\mathscr{C}^1$-regularity of $\bar{g}$, and the (partial) converse of the Dominated Convergence Theorem, we infer that there exists a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ of positive real numbers converging to 0 such that $u^R_{\varepsilon_k}$ converges to u^* Δ-a.e. on $[a,b[_{\mathbb{T}}$, $\bar{q}^R_{a,\varepsilon_k}$ converges to $\bar{q}^*_a$, $\bar{g}(\bar{q}^R_{a,\varepsilon_k},\bar{q}(b,u^R_{\varepsilon_k},\bar{q}^R_{a,\varepsilon_k}))$ converges to $\bar{g}(\bar{q}^*_a,\bar{q}^*(b))\in\mathrm{S}$, $\partial\bar{g}/\partial\bar{x}_1(\bar{q}^R_{a,\varepsilon_k},\bar{q}(b,u^R_{\varepsilon_k},\bar{q}^R_{a,\varepsilon_k}))$ converges to $\partial\bar{g}/\partial\bar{x}_1(\bar{q}^*_a,\bar{q}^*(b))$, $\partial\bar{g}/\partial\bar{x}_2(\bar{q}^R_{a,\varepsilon_k},\bar{q}(b,u^R_{\varepsilon_k},\bar{q}^R_{a,\varepsilon_k}))$ converges to $\partial\bar{g}/\partial\bar{x}_2(\bar{q}^*_a,\bar{q}^*(b))$, $\psi^{0R}_{\varepsilon_k}$ converges to some $\psi^{0R}\leq 0$, and $\psi^R_{\varepsilon_k}$ converges to some $\psi^R\in\mathbb{R}^j$ as k tends to $+\infty$, with $|\psi^{0R}|^2+\|\psi^R\|^2_{\mathbb{R}^j}=1$ and $-\psi^R\in\mathcal{O}_\mathrm{S}(\bar{g}(\bar{q}^*_a,\bar{q}^*(b)))$ (see Lemma III.12).

In the three next lemmas, we use the inequality (III.3.24) respectively with needle-like variations of $u^R_{\varepsilon_k}$ at right-scattered points and then at right-dense points, and variations of $\bar{q}^R_{a,\varepsilon_k}$, and infer some crucial inequalities by taking the limit in k. Note that these variations were defined in Section III.3.2 for any dynamics f, and that we apply them here to the augmented system (III.3.23), associated with the augmented dynamics $\bar{f}$.

Lemma III.14. *For every $r\in[a,b[_{\mathbb{T}}\cap\mathrm{RS}$ and every $y\in\mathcal{D}^\Omega_{\mathrm{stab}}(u^*(r))$, considering the needle-like variation $\Pi=(r,y)$ at the right-scattered point r as defined in Section III.3.2.2, it holds*

$$\psi^{0R}\omega^0_\Pi(b,u^*,\bar{q}^*_a)+\Big\langle\Big(\frac{\partial\bar{g}}{\partial x_2}(\bar{q}^*_a,\bar{q}^*(b))\Big)^{\mathrm{T}}\times\psi^R,\omega_\Pi(b,u^*,\bar{q}^*_a)\Big\rangle_{\mathbb{R}^n}\leq 0, \tag{III.3.27}$$

where the variation vector $\bar{\omega}_\Pi=(\omega_\Pi,\omega^0_\Pi)$ is defined by (III.3.17) *(replacing f with $\bar{f}$).*

Proof. Since $u^R_{\varepsilon_k}$ converges to u^* Δ-a.e. on $[a,b[_{\mathbb{T}}$, it follows that $u^R_{\varepsilon_k}(r)$ converges to $u^*(r)$ as k tends to $+\infty$. Hence $y\in\mathcal{D}^\Omega(u^R_{\varepsilon_k}(r))$ and $\|u^R_{\varepsilon_k}(r)\|_{\mathbb{R}^m}<R$ for k sufficiently large. Fixing such a large integer k, we recall that $u^R_{\varepsilon_k,\Pi}(\cdot,\alpha)\in\mathrm{L}^\infty_{\mathbb{T}}([a,b[_{\mathbb{T}};\Omega)$ for every $\alpha\in\mathcal{D}^\Omega(u^R_{\varepsilon_k}(r),y)$, and

$$\begin{aligned}\|u^R_{\varepsilon_k,\Pi}(\cdot,\alpha)\|_{\mathrm{L}^\infty_{\mathbb{T}}([a,b[_{\mathbb{T}},\mathbb{R}^m)} &\leq \max(\|u^R_{\varepsilon_k}\|_{\mathrm{L}^\infty_{\mathbb{T}}([a,b[_{\mathbb{T}},\mathbb{R}^m)},\|u^R_{\varepsilon_k,\Pi}(r,\alpha)\|_{\mathbb{R}^m})\\ &\leq \max(R,\|u^R_{\varepsilon_k}(r)\|_{\mathbb{R}^m}+\alpha\|y-u^R_{\varepsilon_k}(r)\|_{\mathbb{R}^m}),\end{aligned}$$

and

$$\begin{aligned}\|u^R_{\varepsilon_k,\Pi}(\cdot,\alpha)-u^*\|_{\mathrm{L}^1_{\mathbb{T}}([a,b[_{\mathbb{T}},\mathbb{R}^m)} &\leq \|u^R_{\varepsilon_k,\Pi}(\cdot,\alpha)-u^R_{\varepsilon_k}\|_{\mathrm{L}^1_{\mathbb{T}}([a,b[_{\mathbb{T}},\mathbb{R}^m)}+\|u^R_{\varepsilon_k}-u^*\|_{\mathrm{L}^1_{\mathbb{T}}([a,b[_{\mathbb{T}},\mathbb{R}^m)}\\ &\leq \alpha\mu(r)\|y-u^R_{\varepsilon_k}(r)\|_{\mathbb{R}^m}+\sqrt{\varepsilon_k}.\end{aligned}$$

Therefore $(u^R_{\varepsilon_k,\Pi}(\cdot,\alpha),\bar{q}^R_{a,\varepsilon_k})\in\mathrm{E}^R_\Omega$ for every $\alpha\in\mathcal{D}^\Omega(u^R_{\varepsilon_k}(r),y)$ sufficiently small. It then follows from (III.3.24) that

$$-\sqrt{\varepsilon_k}\|u^R_{\varepsilon_k,\Pi}(\cdot,\alpha)-u^R_{\varepsilon_k}\|_{\mathrm{L}^1_{\mathbb{T}}([a,b[_{\mathbb{T}},\mathbb{R}^m)}\leq J^R_k(u^R_{\varepsilon_k,\Pi}(\cdot,\alpha),\bar{q}^R_{a,\varepsilon_k})-J^R_k(u^R_{\varepsilon_k},\bar{q}^R_{a,\varepsilon_k}),$$

and thus

$$-\sqrt{\varepsilon_k}\mu(r)\|y-u^R_{\varepsilon_k}(r)\|_{\mathbb{R}^m}\leq\frac{J^R_k(u^R_{\varepsilon_k,\Pi}(\cdot,\alpha),\bar{q}^R_{a,\varepsilon_k})^2-J^R_k(u^R_{\varepsilon_k},\bar{q}^R_{a,\varepsilon_k})^2}{\alpha(J^R_k(u^R_{\varepsilon_k,\Pi}(\cdot,\alpha),\bar{q}^R_{a,\varepsilon_k})+J^R_k(u^R_{\varepsilon_k},\bar{q}^R_{a,\varepsilon_k}))}.$$

Using Proposition III.1, since $\bar{g}$ does not depend on x^0_2, we infer that

$$\begin{aligned}\lim_{\alpha\to 0}&\frac{J^R_k(u^R_{\varepsilon_k,\Pi}(\cdot,\alpha),\bar{q}^R_{a,\varepsilon_k})^2-J^R_k(u^R_{\varepsilon_k},\bar{q}^R_{a,\varepsilon_k})^2}{\alpha}\\ &=2\max(q^0(b,u^R_{\varepsilon_k},\bar{q}^R_{a,\varepsilon_k})-q^{0*}(b)+\varepsilon_k,0)\,\omega^0_\Pi(b,u^R_{\varepsilon_k},\bar{q}^R_{a,\varepsilon_k})\\ &\quad+2\Big\langle\bar{g}(\bar{q}^R_{a,\varepsilon_k},\bar{q}(b,u^R_{\varepsilon_k},\bar{q}^R_{a,\varepsilon_k}))-\mathrm{P_S}(\bar{g}(\bar{q}^R_{a,\varepsilon_k},\bar{q}(b,u^R_{\varepsilon_k},\bar{q}^R_{a,\varepsilon_k}))),\\ &\qquad\frac{\partial\bar{g}}{\partial x_2}(\bar{q}^R_{a,\varepsilon_k},\bar{q}(b,u^R_{\varepsilon_k},\bar{q}^R_{a,\varepsilon_k}))\times\omega_\Pi(b,u^R_{\varepsilon_k},\bar{q}^R_{a,\varepsilon_k})\Big\rangle_{\mathbb{R}^j}.\end{aligned}$$

Since $J_k^R(u^R_{\varepsilon_k,\Pi}(\cdot,\alpha),\bar{q}^R_{a,\varepsilon_k})$ converges to $J_k^R(u^R_{\varepsilon_k},\bar{q}^R_{a,\varepsilon_k})$ as α tends to 0, using (III.3.25) and (III.3.26) it follows that

$$-\sqrt{\varepsilon_k}\mu(r)\|y-u^R_{\varepsilon_k}(r)\|_{\mathbb{R}^m} \leq -\psi^{0R}_{\varepsilon_k}\omega^0_\Pi(b,u^R_{\varepsilon_k},\bar{q}^R_{a,\varepsilon_k}) - \Big\langle \Big(\frac{\partial \bar{g}}{\partial x_2}(\bar{q}^R_{a,\varepsilon_k},\bar{q}(b,u^R_{\varepsilon_k},\bar{q}^R_{a,\varepsilon_k}))\Big)^{\mathrm{T}} \times \psi^R_{\varepsilon_k}, \omega_\Pi(b,u^R_{\varepsilon_k},\bar{q}^R_{a,\varepsilon_k})\Big\rangle_{\mathbb{R}^n}.$$

By letting k tend to $+\infty$, and using Lemma III.5, the lemma follows. □

Denote by $\mathscr{L}^R_{[a,b[_{\mathbb{T}}}$ the set of times $t\in[a,b[_{\mathbb{T}}$ such that $t\in\mathscr{L}_{[a,b[_{\mathbb{T}}}(f(q(\cdot,u^*,q^*_a),u^*,t))$, such that $t\in\mathscr{L}_{[a,b[_{\mathbb{T}}}(f(q(\cdot,u^R_{\varepsilon_k},q^R_{a,k}),u^R_{\varepsilon_k},t))$ for every $k\in\mathbb{N}$, and such that $u^R_{\varepsilon_k}(t)$ converges to $u^*(t)$ as k tends to $+\infty$. It holds $\mu_\Delta(\mathscr{L}^R_{[a,b[_{\mathbb{T}}})=\mu_\Delta([a,b[_{\mathbb{T}})=b-a$.

Lemma III.15. *For every $s\in\mathscr{L}^R_{[a,b[_{\mathbb{T}}}\cap\mathrm{RD}$ and any $z\in\Omega\cap\overline{B}_{\mathbb{R}^m}(0,R)$, considering the needle-like variation $\Pi=(s,z)$ as defined in Section III.3.2.3, it holds*

$$\psi^{0R}\omega^0_\Pi(b,u^*,\bar{q}^*_a)+\Big\langle\Big(\frac{\partial\bar{g}}{\partial x_2}(\bar{q}^*_a,\bar{q}^*(b))\Big)^{\mathrm{T}}\times\psi^R,\omega_\Pi(b,u^*,\bar{q}^*_a)\Big\rangle_{\mathbb{R}^n}\leq 0, \tag{III.3.28}$$

where the variation vector $\bar{\omega}_\Pi=(\omega_\Pi,\omega^0_\Pi)$ is defined by (III.3.19) *(replacing f with $\bar{f}$).*

Proof. Similar to the proof of Lemma III.14. We refer to Section B.2.1 of Appendix B. □

Lemma III.16. *For every $\bar{q}_a\in\mathbb{R}^n\times\{0\}$, considering the variation of initial point as defined in Section III.3.2.4, it holds*

$$\psi^{0R}\omega^0_{\bar{q}_a}(b,u^*,\bar{q}^*_a)+\Big\langle\Big(\frac{\partial\bar{g}}{\partial x_2}(\bar{q}^*_a,\bar{q}^*(b))\Big)^{\mathrm{T}}\times\psi^R,\omega_{\bar{q}_a}(b,u^*,\bar{q}^*_a)\Big\rangle_{\mathbb{R}^n} \leq -\Big\langle\Big(\frac{\partial\bar{g}}{\partial x_1}(\bar{q}^*_a,\bar{q}^*(b))\Big)^{\mathrm{T}}\times\psi^R,q_a\Big\rangle_{\mathbb{R}^n}, \tag{III.3.29}$$

where the variation vector $\bar{\omega}_{\bar{q}_a}=(\omega_{\bar{q}_a},\omega^0_{\bar{q}_a})$ is defined by (III.3.21) *(replacing f with $\bar{f}$).*

Proof. Similar to the proof of Lemma III.14. We refer to Section B.2.2 of Appendix B. □

At this step, we have obtained in the three previous lemmas the three fundamental inequalities (III.3.27), (III.3.28) and (III.3.29), valid for any $R>\|u^*\|_{\mathrm{L}^\infty_{\mathbb{T}}([a,b[_{\mathbb{T}},\mathbb{R}^m)}$. Recall that $|\psi^{0R}|^2+\|\psi^R\|^2_{\mathbb{R}^j}=1$ and $-\psi^R\in\mathcal{O}_{\mathrm{S}}(\bar{g}(\bar{q}^*_a,\bar{q}^*(b)))$. Then, considering a sequence of real numbers R_ℓ converging to $+\infty$ as ℓ tends to $+\infty$, we infer that there exist $\psi^0\leq 0$ and $\psi\in\mathbb{R}^j$ such that ψ^{0R_ℓ} converges to ψ^0 and ψ^{R_ℓ} converges to ψ as ℓ tends to $+\infty$, and moreover $|\psi^0|^2+\|\psi\|^2_{\mathbb{R}^j}=1$ and $-\psi\in\mathcal{O}_{\mathrm{S}}(\bar{g}(\bar{q}^*_a,\bar{q}^*(b)))$ (since $\mathcal{O}_{\mathrm{S}}(\bar{g}(\bar{q}^*_a,\bar{q}^*(b)))$ is a closed subset of $\mathbb{R}^j$).

We set $\mathscr{L}_{[a,b[_{\mathbb{T}}}=\bigcap_{\ell\in\mathbb{N}}\mathscr{L}^{R_\ell}_{[a,b[_{\mathbb{T}}}$. Note that $\mu_\Delta(\mathscr{L}_{[a,b[_{\mathbb{T}}})=\mu_\Delta([a,b[_{\mathbb{T}})=b-a$. Taking the limit in ℓ in (III.3.27), (III.3.28) and (III.3.29), we get the following lemma.

Lemma III.17. *We have the following variational inequalities.*

For every $r\in[a,b[_{\mathbb{T}}\cap\mathrm{RS}$, and every $y\in\mathcal{D}^\Omega_{\mathrm{stab}}(u^*(r))$, it holds

$$\psi^0\omega^0_\Pi(b,u^*,\bar{q}^*_a)+\Big\langle\Big(\frac{\partial\bar{g}}{\partial x_2}(\bar{q}^*_a,\bar{q}^*(b))\Big)^{\mathrm{T}}\times\psi,\omega_\Pi(b,u^*,\bar{q}^*_a)\Big\rangle_{\mathbb{R}^n}\leq 0, \tag{III.3.30}$$

where the variation vector $\bar{\omega}_\Pi=(\omega_\Pi,\omega^0_\Pi)$ associated with the needle-like variation $\Pi=(r,y)$ is defined by (III.3.17) *(replacing f with $\bar{f}$);*

For every $s\in\mathscr{L}_{[a,b[_{\mathbb{T}}}\cap\mathrm{RD}$, and every $z\in\Omega$, it holds

$$\psi^0\omega^0_\Pi(b,u^*,\bar{q}^*_a)+\Big\langle\Big(\frac{\partial\bar{g}}{\partial x_2}(\bar{q}^*_a,\bar{q}^*(b))\Big)^{\mathrm{T}}\times\psi,\omega_\Pi(b,u^*,\bar{q}^*_a)\rangle_{\mathbb{R}^n}\leq 0, \tag{III.3.31}$$

where the variation vector $\bar{\omega}_\Pi=(\omega_\Pi,\omega^0_\Pi)$ associated with the needle-like variation $\Pi=(s,z)$ is defined by (III.3.19) *(replacing f with $\bar{f}$);*

For every $\bar{q}_a \in \mathbb{R}^n \times \{0\}$, it holds

$$\psi^0 \omega^0_{\bar{q}_a}(b,u^*,\bar{q}^*_a) + \Big\langle \Big(\frac{\partial \bar{g}}{\partial x_2}(\bar{q}^*_a,\bar{q}^*(b)) \Big)^{\mathrm{T}} \times \psi, \omega_{\bar{q}_a}(b,u^*,\bar{q}^*_a) \Big\rangle_{\mathbb{R}^n} \leq -\Big\langle \Big(\frac{\partial \bar{g}}{\partial x_1}(\bar{q}^*_a,\bar{q}^*(b)) \Big)^{\mathrm{T}} \times \psi, q_a \Big\rangle_{\mathbb{R}^n}, \quad \text{(III.3.32)}$$

where the variation vector $\bar{\omega}_{\bar{q}_a} = (\omega_{\bar{q}_a}, \omega^0_{\bar{q}_a})$ associated with the variation $\bar{q}_a$ of the initial point q^*_a is defined by (III.3.21) *(replacing f with $\bar{f}$).*

This result concludes the application of Ekeland's Variational Principle. The last step of the proof consists of deriving the PMP from these inequalities.

III.3.3.2 End of the proof

We define $\bar{p} = (p, p^0)$ as the unique solution on $[a,b]_{\mathbb{T}}$ of the backward shifted linear Δ-Cauchy problem

$$\bar{p}^{\Delta}(t) = -\Big(\frac{\partial \bar{f}}{\partial \bar{x}}(\bar{q}^*(t),u^*(t),t) \Big)^{\mathrm{T}} \times \bar{p}^{\sigma}(t), \quad \bar{p}(b) = \Big(\Big(\frac{\partial \bar{g}}{\partial x_2}(\bar{q}^*_a,\bar{q}^*(b)) \Big)^{\mathrm{T}} \times \psi, \psi^0 \Big)^{\mathrm{T}}.$$

The existence and uniqueness of $\bar{p}$ are ensured by Theorem A.6 (see also [52, Theorem 6]). Since $\bar{f}$ does not depend on x^0, it is clear that p^0 is constant, still denoted by p^0 (with $p^0 = \psi^0$).

Right-scattered points. Let $r \in [a,b[_{\mathbb{T}} \cap \mathrm{RS}$ and $y \in \mathcal{D}^{\Omega}_{\mathrm{stab}}(u^*(r))$. Since $t \longmapsto \langle \bar{\omega}_{\Pi}(t,u^*,\bar{q}^*_a), \bar{p}(t) \rangle_{\mathbb{R}^{n+1}}$ is absolutely continuous, it holds $\langle \bar{p}, \bar{\omega}_{\Pi}(\cdot,u^*,\bar{q}^*_a) \rangle^{\Delta}_{\mathbb{R}^{n+1}} = 0$ Δ-almost everywhere on $[\sigma(r),b[_{\mathbb{T}}$ (see Leibniz formula in Proposition I.2) and hence this function is constant on $[\sigma(r),b]_{\mathbb{T}}$. It thus follows from (III.3.30) that

$$\begin{aligned} \langle \bar{p}(\sigma(r)), \bar{\omega}_{\Pi}(\sigma(r),u^*,\bar{q}^*_a) \rangle_{\mathbb{R}^{n+1}} &= \langle \bar{\omega}(b), \bar{\omega}_{\Pi}(b,u^*,\bar{q}^*_a) \rangle_{\mathbb{R}^{n+1}} \\ &= \psi^0 \omega^0_{\Pi}(b,u^*,\bar{q}^*_a) + \Big\langle \Big(\frac{\partial \bar{g}}{\partial x_2}(\bar{q}^*_a,\bar{q}^*(b)) \Big)^{\mathrm{T}} \times \psi, \omega_{\Pi}(b,u^*,\bar{q}^*_a) \Big\rangle_{\mathbb{R}^n} \leq 0, \quad \text{(III.3.33)} \end{aligned}$$

and since $\bar{\omega}_{\Pi}(\sigma(r),u^*,\bar{q}^*_a) = \mu(r) \frac{\partial \bar{f}}{\partial v}(\bar{q}^*(r),u^*(r),r) \times (y - u^*(r))$, we finally get

$$\Big\langle \frac{\partial H}{\partial v}(\bar{q}^*(r),u^*(r),\bar{p}^{\sigma}(r),r), y - u^*(r) \Big\rangle_{\mathbb{R}^m} \leq 0.$$

Since this inequality holds for every $y \in \mathcal{D}^{\Omega}_{\mathrm{stab}}(u^*(r))$, we easily prove that it holds as well for every $v \in \overline{\mathrm{Co}}(\mathcal{D}^{\Omega}_{\mathrm{stab}}(u^*(r)))$. This proves (III.2.7).

Right-dense points. Let $s \in \mathscr{L}_{[a,b[_{\mathbb{T}}} \cap \mathrm{RD}$ and $z \in \Omega$. Since $t \longmapsto \langle \bar{\omega}_{\mathrm{II}}(t,u^*,\bar{q}^*_a), \bar{p}(t) \rangle_{\mathbb{R}^{n+1}}$ is an absolutely continuous function, the Leibniz formula given in Proposition I.2 leads to $\langle \bar{p}, \bar{\omega}_{\mathrm{II}}(\cdot,u^*,\bar{q}^*_a) \rangle^{\Delta}_{\mathbb{R}^{n+1}} = 0$ Δ-almost everywhere on $[s,b[_{\mathbb{T}}$, and hence this function is constant on $[s,b]_{\mathbb{T}}$. It thus follows from (III.3.31) that

$$\begin{aligned} \langle \bar{p}(s), \bar{\omega}_{\mathrm{II}}(s,u^*,\bar{q}^*_a) \rangle_{\mathbb{R}^{n+1}} &= \langle \bar{p}(b), \bar{\omega}_{\mathrm{II}}(b,u^*,\bar{q}^*_a) \rangle_{\mathbb{R}^{n+1}} \\ &= \psi^0 \omega^0_{\mathrm{II}}(b,u^*,\bar{q}^*_a) + \Big\langle \Big(\frac{\partial \bar{g}}{\partial x_2}(\bar{q}^*_a,\bar{q}^*(b)) \Big)^{\mathrm{T}} \times \psi, \omega_{\mathrm{II}}(b,u^*,\bar{q}^*_a) \Big\rangle_{\mathbb{R}^n} \leq 0, \quad \text{(III.3.34)} \end{aligned}$$

and since $\bar{\omega}_{\mathrm{II}}(s,u^*,\bar{q}^*_a) = \bar{f}(\bar{q}^*(s),z,s) - \bar{f}(\bar{q}(s),u^*(s),s)$, we finally get

$$\langle \bar{p}(s), \bar{f}(\bar{q}^*(s),z,s) \rangle_{\mathbb{R}^{n+1}} \leq \langle \bar{p}(s), \bar{f}(\bar{q}(s),u^*(s),s) \rangle_{\mathbb{R}^{n+1}}.$$

Since this inequality holds for every $z \in \Omega$, the maximization condition (III.2.8) follows.

Transversality conditions. The transversality condition on the adjoint vector p at the final time b has been obtained by definition (note that $-\psi \in \mathcal{O}_{\mathrm{S}}(\bar{g}(\bar{q}^*_a,\bar{q}^*(b)))$ as mentioned previously). Let us now establish the transversality condition at the initial time a (left-hand equality of (III.2.9)). Let $\bar{q}_a \in \mathbb{R}^n \times \{0\}$. With the same arguments as before, we prove that the function $t \longmapsto \langle \bar{\omega}_{\bar{q}_a}(t, u^*, \bar{q}_a^*), \bar{p}(t) \rangle_{\mathbb{R}^{n+1}}$ is constant on $[a, b]_{\mathbb{T}}$. It thus follows from (III.3.32) that

$$\begin{aligned} \langle \bar{p}(a), \bar{\omega}_{\bar{q}_a}(a, u^*, \bar{q}_a^*) \rangle_{\mathbb{R}^{n+1}} &= \langle \bar{p}(b), \bar{\omega}_{\bar{q}_a}(b, u^*, \bar{q}_a^*) \rangle_{\mathbb{R}^{n+1}} \\ &= \psi^0 \omega^0_{\bar{q}_a}(b, u^*, \bar{q}_a^*) + \Big\langle \Big(\frac{\partial \bar{g}}{\partial x_2}(\bar{q}_a^*, \bar{q}^*(b)) \Big)^{\mathrm{T}} \times \psi, \omega_{\bar{q}_a}(b, u^*, \bar{q}_a^*) \Big\rangle_{\mathbb{R}^n} \\ &\leq -\Big\langle \Big(\frac{\partial \bar{g}}{\partial x_1}(\bar{q}_a^*, \bar{q}^*(b)) \Big)^{\mathrm{T}} \times \psi, q_a \Big\rangle_{\mathbb{R}^n}, \end{aligned}$$

and since $\bar{\omega}_{\bar{q}_a}(a, u^*, \bar{q}_a^*) = \bar{q}_a = (q_a, 0)$, we finally get

$$\Big\langle p(a, u^*, \bar{q}_a^*) + \Big(\frac{\partial \bar{g}}{\partial x_1}(\bar{q}_a^*, \bar{q}^*(b)) \Big)^{\mathrm{T}} \times \psi, q_a \Big\rangle_{\mathbb{R}^n} \leq 0.$$

Since this inequality holds for every $\bar{q}_a \in \mathbb{R}^n \times \{0\}$, the left-hand equality of (III.2.9) follows.

Free final time. Assume that the final time is not fixed in $(\mathbf{OCP})_{\mathbb{T}}$, and let b^* be the final time associated with the optimal trajectory q^*. We assume moreover that b^* belongs to the interior of $\mathbb{T}$ for the topology of $\mathbb{R}$. The proof of (III.2.10) then goes exactly as in the classical continuous-time case, and thus we do not provide any details. It suffices to consider variations of the final time b in a neighbourhood of b^*, and to modify accordingly the functional of Section III.3.3.1 to which Ekeland's Variational Principle is applied.

PMP with parameters (Remark III.4). To obtain the statement it suffices to apply the PMP to the control system associated with the dynamics $\tilde{f}(\lambda, x, v, t) = (f(\lambda, x, v, t), 0)^{\mathrm{T}}$, with the extended state $\tilde{q} = (\lambda, q)$. In other words, we add to the control system the equation $\lambda^{\Delta}(t) = 0$ (this is a standard method to derive a parametrized version of the PMP). Applying the PMP then yields an adjoint vector $\tilde{p} = (p_\lambda, p)$, where p clearly satisfies all conclusions of Theorem III.1 (except (III.2.13)), and $p_\lambda^{\Delta}(t) = -\frac{\partial H}{\partial \lambda}(\lambda^*, q^*(t), u^*(t), p^\sigma(t), p^0, t)$ Δ-almost everywhere. From this last equation it follows that $p_\lambda(b) - p_\lambda(a) = -\int_{[a,b^*[} \frac{\partial H}{\partial \lambda}(\lambda^*, q^*(t), u^*(t), p^\sigma(t), p^0, t)\, \Delta t$, and then (III.2.12) follows from the already established transversality conditions.

Free final time and autonomous Hamiltonian (Remark III.5). To derive (III.2.13), we consider the change of variable $\tilde{t} = (t - a)/(b - a)$. The crucial remark is that, since it is an *affine* change of variable, Δ-derivatives of compositions work in the time scale setting as in the time-continuous case. Then it suffices to consider the resulting optimal control problem as a parametrized one with parameter b lying in a neighbourhood of b^*. Then (III.2.13) follows from the additional condition (III.2.12) of the PMP with parameters (see Remark III.4).

Chapitre IV

Problème inverse de Helmholtz en calcul des variations discret

Dans ce chapitre, nous formulons et nous résolvons un problème inverse de type Helmholtz discret. Plus précisément, considérant des time scales uniformément finis $\mathbb{T} = \{t_0 < \ldots < t_N\}$ *(i.e. avec* $t_{p+1} - t_p = h$ *pour tout* p*), notre résultat principal affirme qu'une équation aux différences finies de type*

$$\overline{P}(q(t_p), \Delta q(t_p), \nabla q(t_p), \nabla \circ \Delta q(t_p), t_p, h) = 0$$

est une équation d'Euler-Lagrange discrète de la forme

$$\frac{\partial L_1}{\partial x}(q(t_p), \Delta q(t_p), t_p, h) + \frac{\partial L_2}{\partial x}(q(t_p), \nabla q(t_p), t_p, h) - \nabla \left(\frac{\partial L_1}{\partial v}(q, \Delta q, \cdot, h) \right)(t_p) - \Delta \left(\frac{\partial L_2}{\partial v}(q, \nabla q, \cdot, h) \right)(t_p) = 0,$$

qui caractérise les points critiques de la fonctionnelle Lagrangienne discrète donnée par

$$\mathcal{L}^{\mathrm{d}}(q) = h \sum_{p=0}^{N-1} L_1(q(t_p), \Delta q(t_p), t_p, h) + h \sum_{p=1}^{N} L_2(q(t_p), \nabla q(t_p), t_p, h),$$

si et seulement si $\overline{P}$ *satisfait la condition de Helmholtz discrète suivante :*

$$\Delta \left(\frac{\partial \overline{P}}{\partial u}(q, \Delta q, \nabla q, \nabla \circ \Delta q, \cdot, h) \right)(t_p) = \frac{\partial \overline{P}}{\partial v_1}(q(t_p), \Delta q(t_p), \nabla q(t_p), \nabla \circ \Delta q(t_p), t_p, h) + \frac{\partial \overline{P}}{\partial v_2}(q(t_{p+1}), \Delta q(t_{p+1}), \nabla q(t_{p+1}), \nabla \circ \Delta q(t_{p+1}), t_{p+1}, h).$$

Dans un tel cas, la démonstration de cet énoncé présente une méthode explicite de construction d'un couple de Lagrangien (L_1, L_2) *associé.*

À cause de la dépendance en h *des Lagrangiens* L_1 *et* L_2*, nous n'avons pas la garantie que la fonctionnelle Lagrangienne* $\mathcal{L}^{\mathrm{d}}$ *corresponde à la discrétisation d'une fonctionnelle Lagrangienne continue et donc que l'équation d'Euler-Lagrange discrète associée soit un intégrateur variationnel. En conséquence, nous proposons dans ce chapitre une description de la classe des formulations Lagrangiennes discrètes nulles, c'est-à-dire des couples de Lagrangiens dont l'équation d'Euler-Lagrange discrète associée est nulle. Par linéarité, ce résultat permet, pour un couple de Lagrangien* (L_1, L_2) *donné, de choisir, parmi une classe de couples* (L'_1, L'_2) *menant à la même équation d'Euler-Lagrange discrète, un couple particulier (s'il existe) tel que la fonctionnelle Lagrangienne discrète associée corresponde à la discrétisation d'une fonctionnelle Lagrangienne continue et donc que l'équation d'Euler-Lagrange discrète associée soit un intégrateur variationnel.*

La Section IV.2 de ce chapitre est consacrée à des rappels sur le problème inverse continu de Helmholtz qui est aujourd'hui totalement résolu. L'étude présentée en Section IV.3 (cas discret) est extraite de Bourdin L. et Cresson J., Helmholtz's inverse problem of the discrete calculus of variations, accepté pour publication dans Journal of Difference Equations and Applications.

IV.1 Introduction

A classical problem in Analysis is the well known *Helmholtz inverse problem of the calculus of variations* formulated in 1887, see [106]: find a necessary and sufficient condition for a (partial) differential equation to be an Euler-Lagrange equation and, in this case, find all the possible Lagrangian formulations. This problem has been studied by numerous authors and has been completely solved by A. Mayer [164] and A. Hirsch [121, 122] in 1896/97. The formulation that we use is due to V. Volterra [200]. Precisely, let O be a differential operator. Then, the (partial) differential equation $O(q) = 0$ is an Euler-Lagrange equation if and only if all differentials of O are self-adjoint. This condition is usually called *Helmholtz condition*. We refer to [180] for a modern presentation and a complete proof of this theorem.

Recently, an increasing activity has been devoted to discrete versions of the calculus of variations in the context of the geometric numerical integration, see *e.g.* [101, 159]. In this context, a suitable method is suggested to provide a numerical scheme for a given continuous Euler-Lagrange equation. It consists in two steps:

1. define a discrete version of the Lagrangian functional associated;
2. form a discrete calculus of variations on the discrete Lagrangian functional defined in step 1.

This procedure leads to a discrete Euler-Lagrange equation characterizing the discrete critical points of the discrete Lagrangian functional. This discrete equation is called *variational integrator* and is a numerical scheme for the initial continuous Euler-Lagrange equation preserving its intrinsic variational structure at the discrete level. In particular, numerous properties relative to this variational structure are preserved at the discrete level, including the Noether's theorem for example. We refer to the book [101] and to the review paper [159] for more details.

Several discrete Helmholtz inverse problems have been studied and solved. We refer in particular to the works of Albu-Opris [13], Cracium-Opris [68] and Hydon-Mansfield [126]. However, in each of these papers, the discrete Helmholtz problem considered is not formulated in terms of variational integrators. Precisely, the discrete Lagrangian functionals considered are very general and are not intended to correspond to discretizations of continuous Lagrangian functionals. As a consequence, these papers cannot be used to provide an answer to a discrete Helmholtz inverse problem posed in the context of geometric numerical integration. A discussion on this issue can be found in [126, Section 5.3 p.213-214].

In this chapter, we are interested in second order finite differences equations of type

$$\overline{P}(q(t_p), \Delta q(t_p), \nabla q(t_p), \nabla \circ \Delta q(t_p), t_p, h) = 0 \tag{IV.1.1}$$

defined on a finite uniform time scale $\mathbb{T} = \{t_0 < t_1 < \ldots < t_N\}$ (*i.e.* with $t_p - t_{p-1} = h$ for every $p = 1, \ldots, N$) and where Δ and ∇ are given by $\Delta q(t_p) = (q(t_{p+1}) - q(t_p))/h$ and $\nabla q(t_p) = (q(t_p) - q(t_{p-1}))/h$. Our aim is to solve the following *discrete Helmholtz inverse problem*: find a necessary and sufficient condition for a second order finite differences equation to be a discrete Euler-Lagrange equation of type

$$\frac{\partial L_1}{\partial x}(q(t_p), \Delta q(t_p), t_p, h) + \frac{\partial L_2}{\partial x}(q(t_p), \nabla q(t_p), t_p, h) \\ - \nabla\left(\frac{\partial L_1}{\partial v}(q, \Delta q, \cdot, h)\right)(t_p) - \Delta\left(\frac{\partial L_2}{\partial v}(q, \nabla q, \cdot, h)\right)(t_p) = 0 \tag{IV.1.2}$$

characterizing the critical points of the discrete Lagrangian functionals of type

$$\mathcal{L}^{\mathrm{d}}(q) = h \sum_{p=0}^{N-1} L_1(q(t_p), \Delta q(t_p), t_p, h) + h \sum_{p=1}^{N} L_2(q(t_p), \nabla q(t_p), t_p, h), \tag{IV.1.3}$$

where (L_1, L_2) is a couple of Lagrangian.

Towards this goal, we introduce a *discrete Helmholtz condition* (Definition IV.10) and Theorem IV.4 proves that it characterizes the second order finite differences equations of type (IV.1.1) that can be written as discrete Euler-Lagrange equations of type (IV.1.2). Moreover, in such a case, the proof of Theorem IV.4 gives an explicit method in order to construct an associated couple of Lagrangian (L_1, L_2).

The fourth variable of the Lagrangian L_1 and L_2 makes them dependent on the step size of discretization h. Then, $\mathcal{L}^{\mathrm{d}}$ is not necessarily a discrete version of a continuous Lagrangian functional of type $\mathcal{L}(q) = \int_{t_0}^{t_N} L(q(\tau),\dot{q}(\tau),\tau)d\tau$, where $\dot{q}$ denotes the derivative of q, and the discrete Euler-Lagrange equation associated is not necessarily a variational integrator. Nevertheless, this problem can be overcome. Indeed, Theorem IV.5 gives a characterization of null couples of Lagrangian (*i.e.* couples of Lagrangian leading to the null discrete Euler-Lagrange equation). From linearity, this result allows, for a given couple (L_1, L_2), to choose, among a class of couples (L'_1, L'_2) leading to the same discrete Euler-Lagrange equation, a discrete Lagrangian formulation corresponding to a discretization of a continuous one (if it exists). In such a case, we conclude that the discrete Euler-Lagrange equation associated is a variational integrator.

Remark IV.1. In this chapter, we are concerned with structures of continuous and discrete dynamical systems and our aim is not to deal with regularity issues. As a consequence, we assume that all maps considered are sufficiently smooth in order to make valid all calculations.

For sake of simplicity of the notations, we only treat the unidimensional case $n = 1$. However, all results can be similarly derived to the case $n \geq 2$.

Organization of the chapter. Section IV.2 is devoted to reminders on the classical continuous Helmholtz problem. The purpose of this preliminary section is to develop a framework that we adapt to the discrete case in Section IV.3. Section IV.4 concludes the chapter with some comments and perspectives.

IV.2 Reminders on the continuous case

In Section IV.2.1, we formulate rigorously the classical continuous Helmholtz inverse problem for second order differential equations. In Section IV.2.2, we recall the usual continuous Helmholtz condition characterizing the second order differential equations that can be written as Euler-Lagrange equations. Finally, Section IV.2.3 is devoted to the characterization of null Lagrangian describing the class of possible Lagrangian formulations for a given Euler-Lagrange equation.

IV.2.1 Formulation of the classical Helmholtz inverse problem

For every a, $b \in \mathbb{R}$ with $a < b$, $\mathrm{C}^\infty([a,b],\mathbb{R})$ denotes the usual set of all functions infinitely differentiable on $[a,b]$ with values in $\mathbb{R}$.

Definition IV.1. A second order differential equation on an interval $[a,b]$ with $a < b$ is defined by $O^{a,b}(q) = 0$ where O is a second order differential operator *i.e.*

$$O : a < b \longmapsto \begin{array}{rcl} O^{a,b} : \quad \mathrm{C}^\infty([a,b],\mathbb{R}) & \longrightarrow & \mathrm{C}^\infty([a,b],\mathbb{R}) \\ q & \longmapsto & \begin{array}{rcl} O^{a,b}(q) : \quad [a,b] & \longrightarrow & \mathbb{R} \\ t & \longmapsto & \overline{O}(q(t),\dot{q}(t),\ddot{q}(t),t) \end{array} \end{array} \tag{IV.2.4}$$

where $\dot{q}$ (resp. $\ddot{q}$) denotes the first (resp. second) derivative of q and where $\overline{O}$ is a map

$$\begin{array}{rrcl} \overline{O} : & \mathbb{R}^4 & \longrightarrow & \mathbb{R} \\ & (x,v,u,t) & \longmapsto & \overline{O}(x,v,u,t). \end{array} \tag{IV.2.5}$$

Let L be a smooth Lagrangian

$$\begin{array}{rrcl} L : & \mathbb{R}^3 & \longrightarrow & \mathbb{R} \\ & (x,v,t) & \longmapsto & L(x,v,t). \end{array} \tag{IV.2.6}$$

For every a, $b \in \mathbb{R}$ such that $a < b$, we define the following Lagrangian functional:

$$\begin{array}{rrcl} \mathcal{L}^{a,b} : & \mathrm{C}^\infty([a,b],\mathbb{R}) & \longrightarrow & \mathbb{R} \\ & q & \longmapsto & \displaystyle\int_a^b L(q(\tau),\dot{q}(\tau),\tau)\, d\tau. \end{array} \tag{IV.2.7}$$

Let $\mathrm{C}^\infty_\mathrm{c}([a,b],\mathbb{R}) = \{w \in \mathrm{C}^\infty([a,b],\mathbb{R})$ with compact support in $]a,b[\}$ be the set of variations of $\mathcal{L}^{a,b}$. A curve $q \in \mathrm{C}^\infty([a,b],\mathbb{R})$ is said to be a critical point of $\mathcal{L}^{a,b}$ if $D\mathcal{L}^{a,b}(q)(w) = 0$ for every variation $w \in \mathrm{C}^\infty_\mathrm{c}([a,b],\mathbb{R})$. Recall that $D\mathcal{L}^{a,b}(q)(w)$ denotes the Gâteaux-differential of $\mathcal{L}^{a,b}$ at q in direction w. An usual calculus of variations characterizes the critical points of $\mathcal{L}^{a,b}$ as the solutions of the well known Euler-Lagrange equation given by

$$\frac{\partial L}{\partial x}(q,\dot{q},\cdot) - \frac{d}{dt}\left(\frac{\partial L}{\partial v}(q,\dot{q},\cdot)\right) = 0 \tag{IV.2.8}$$

that can also be written under the following form:

$$\frac{\partial L}{\partial x}(q,\dot{q},\cdot) - \frac{\partial^2 L}{\partial x \partial v}(q,\dot{q},\cdot)\dot{q} - \frac{\partial^2 L}{\partial v^2}(q,\dot{q},\cdot)\ddot{q} - \frac{\partial^2 L}{\partial t \partial v}(q,\dot{q},\cdot) = 0. \tag{EL}$$

As a consequence, the Euler-Lagrange equation (EL) is a second order differential equation $O^{a,b}(q) = 0$ associated with the map $\overline{O}$ given by

$$\overline{O}(x,v,u,t) = \frac{\partial L}{\partial x}(x,v,t) - \frac{\partial^2 L}{\partial x \partial v}(x,v,t)v - \frac{\partial^2 L}{\partial v^2}(x,v,t)u - \frac{\partial^2 L}{\partial t \partial v}(x,v,t). \tag{IV.2.9}$$

The classical Helmholtz inverse problem [106] treats on the following inverse problem: *if we consider a second order differential equation, how to know if it corresponds to an Euler-Lagrange equation of type* (EL)? For sake of rigorousness of the formulation of the problem, we introduce the following definition.

Definition IV.2. Let O be a second order differential operator. Then, O is said to be a second order Euler-Lagrange operator if there exists a Lagrangian L such that for every $a < b$ and every $q \in \mathrm{C}^\infty([a,b],\mathbb{R})$, it holds

$$\forall t \in [a,b],\ O^{a,b}(q)(t) = \frac{\partial L}{\partial x}(q(t),\dot{q}(t),t) - \frac{d}{dt}\left(\frac{\partial L}{\partial v}(q,\dot{q},\cdot)\right)(t). \tag{IV.2.10}$$

From this definition, the classical Helmholtz inverse problem can be formulated as follows.

Classical Helmholtz inverse problem: Give a necessary and sufficient condition on a second order differential operator in order to be a second order Euler-Lagrange operator.

IV.2.2 Usual and explicit Helmholtz conditions

This section is based on results of [180] in order to solve the classical Helmholtz inverse problem defined in the previous section.

Definition IV.3 (Self-adjointness)**.** Let O be a second order differential operator, $a < b$ and $q \in \mathrm{C}^\infty([a,b],\mathbb{R})$. The Gâteaux-differential of $O^{a,b}$ at q denoted by $DO^{a,b}(q)$ is said to be self-adjoint if for every $(w,z) \in \mathrm{C}^\infty_\mathrm{c}([a,b],\mathbb{R}) \times \mathrm{C}^\infty([a,b],\mathbb{R})$, it holds

$$\int_a^b DO^{a,b}(q)(w)(\tau)z(\tau)\ d\tau = \int_a^b w(\tau)DO^{a,b}(q)(z)(\tau)\ d\tau. \tag{IV.2.11}$$

Definition IV.4 (Usual Helmholtz condition)**.** Let O be a second order differential operator. Then, O is said to be satisfying the Helmholtz condition if for every $a < b$ and every $q \in \mathrm{C}^\infty([a,b],\mathbb{R})$, $DO^{a,b}(q)$ is self-adjoint.

Theorem IV.1. *Let O be a second order differential operator. Then, O is a second order Euler-Lagrange operator if and only if O satisfies the Helmholtz condition.*

This theorem can be formulated in a more general framework. Indeed, it is also valid for higher-order differential equations and even for partial differential equations. We refer to [180] for a complete study. However, the usual Helmholtz condition is relatively abstract. Since we focus on second order differential equations, we recall that the following **explicit** Helmholtz condition is equivalent to the **usual** Helmholtz condition.

Definition IV.5 (Explicit Helmholtz condition)**.** Let O be a second order differential operator. Then, O is said to be satisfying the Helmholtz condition if for every $a < b$ and every $q \in \mathrm{C}^\infty([a,b],\mathbb{R})$, it holds

$$\forall t \in [a,b], \ \frac{d}{dt}\left(\frac{\partial \overline{O}}{\partial u}(q,\dot{q},\ddot{q},\cdot)\right)(t) = \frac{\partial \overline{O}}{\partial v}(q(t),\dot{q}(t),\ddot{q}(t),t). \tag{IV.2.12}$$

This formulation has the advantage to present an explicit equality easily and quickly verifiable. We refer to Example IV.1 for an application. Finally, the following theorem holds and since we adapt the proof to the discrete case in Section IV.3, we recall in details the arguments for the reader's convenience.

Theorem IV.2. *Let O be a second order differential operator. Then, O is a second order Euler-Lagrange operator if and only if O satisfies the explicit Helmholtz condition.*

Proof. Let us prove that the condition is necessary. Let O be a second order Euler-Lagrange operator and let L be the Lagrangian associated. Thus, it holds

$$\overline{O}(x,v,u,t) = \frac{\partial L}{\partial x}(x,v,t) - \frac{\partial^2 L}{\partial x \partial v}(x,v,t)v - \frac{\partial^2 L}{\partial v^2}(x,v,t)u - \frac{\partial^2 L}{\partial t \partial v}(x,v,t). \tag{IV.2.13}$$

One can easily check that the Helmholtz condition (Definition IV.5) is satisfied.

Now, let us prove that the condition is sufficient. Let O be a second order differential operator satisfying the Helmholtz condition. We define the following *augmented* Lagrangian:

$$\begin{array}{rccl} L_0 : & \mathbb{R}^4 & \longrightarrow & \mathbb{R} \\ & (x,v,u,t) & \longmapsto & x \displaystyle\int_0^1 \overline{O}(\lambda x, \lambda v, \lambda u, t)\, d\lambda \end{array} \tag{IV.2.14}$$

and for every $a < b$, we define the associated *augmented* Lagrangian functional:

$$\begin{array}{rccl} \mathcal{L}_0^{a,b} : & \mathrm{C}^\infty([a,b],\mathbb{R}) & \longrightarrow & \mathbb{R} \\ & q & \longmapsto & \displaystyle\int_a^b L_0(q(\tau),\dot{q}(\tau),\ddot{q}(\tau),\tau)\, d\tau. \end{array} \tag{IV.2.15}$$

On the one hand, an usual calculus of variations leads to

$$\begin{aligned} D\mathcal{L}_0^{a,b}(q)(w) = \int_a^b \Bigg[& \frac{\partial L_0}{\partial x}(q(\tau),\dot{q}(\tau),\ddot{q}(\tau),\tau) \\ & - \frac{d}{dt}\left(\frac{\partial L_0}{\partial v}(q,\dot{q},\ddot{q},\cdot)\right)(\tau) + \frac{d^2}{dt^2}\left(\frac{\partial L_0}{\partial u}(q,\dot{q},\ddot{q},\cdot)\right)(\tau)\Bigg] w(\tau)\, d\tau, \end{aligned} \tag{IV.2.16}$$

for every $a < b$ and every $(q,w) \in \mathrm{C}^\infty([a,b],\mathbb{R}) \times \mathrm{C}^\infty_\mathrm{c}([a,b],\mathbb{R})$. On the other hand, from the definition of L_0, we have for every $a < b$ and every $(q,w) \in \mathrm{C}^\infty([a,b],\mathbb{R}) \times \mathrm{C}^\infty_\mathrm{c}([a,b],\mathbb{R})$

$$\mathcal{L}_0^{a,b}(q) = \int_a^b q(\tau) \int_0^1 O^{a,b}(\lambda q)(\tau)\, d\lambda\, d\tau \tag{IV.2.17}$$

and consequently,

$$D\mathcal{L}_0^{a,b}(q)(w) = \int_a^b w(\tau) \int_0^1 O^{a,b}(\lambda q)(\tau)\, d\lambda\, d\tau + \int_a^b q(\tau) \int_0^1 DO^{a,b}(\lambda q)(\lambda w)(\tau)\, d\lambda\, d\tau. \tag{IV.2.18}$$

Using the Helmholtz condition (Definition IV.4), we obtain

$$\begin{aligned} D\mathcal{L}_0^{a,b}(q)(w) &= \int_a^b w(\tau) \int_0^1 O^{a,b}(\lambda q)(\tau)\, d\lambda\, d\tau + \int_0^1 \int_a^b q(\tau) DO^{a,b}(\lambda q)(\lambda w)(\tau)\, d\tau\, d\lambda \\ &= \int_a^b w(\tau) \int_0^1 O^{a,b}(\lambda q)(\tau)\, d\lambda\, d\tau + \int_0^1 \int_a^b \lambda w(\tau) DO^{a,b}(\lambda q)(q)(\tau)\, d\tau\, d\lambda \\ &= \int_a^b w(\tau) \int_0^1 O^{a,b}(\lambda q)(\tau)\, d\lambda\, d\tau + \int_a^b w(\tau) \int_0^1 \lambda DO^{a,b}(\lambda q)(q)(\tau)\, d\lambda\, d\tau. \end{aligned}$$

Finally, an integration by parts with respect to λ (in the second double-integral) leads to

$$D\mathcal{L}_0^{a,b}(q)(w) = \int_a^b O^{a,b}(q)(\tau)w(\tau)\, d\tau. \tag{IV.2.19}$$

From Equalities (IV.2.16) and (IV.2.19), for every $a < b$ and every $q \in \mathrm{C}^\infty([a,b],\mathbb{R})$, we conclude that the following equality holds on $[a,b]$:

$$O^{a,b}(q) = \frac{\partial L_0}{\partial x}(q,\dot{q},\ddot{q},\cdot) - \frac{d}{dt}\left(\frac{\partial L_0}{\partial v}(q,\dot{q},\ddot{q},\cdot)\right) + \frac{d^2}{dt^2}\left(\frac{\partial L_0}{\partial u}(q,\dot{q},\ddot{q},\cdot)\right). \tag{IV.2.20}$$

However, the proof is not complete yet since the Euler-Lagrange operator obtained is *augmented*. O is a second order differential operator. Then, the development of the derivatives in the right-hand side of (IV.2.20) leads to $\partial^2 L_0/\partial u^2 = 0$. Then, L_0 rewrites as $L_0(x,v,u,t) = \varphi(x,v,t)u + \psi(x,v,t)$. Let $\Phi(x,\cdot,t)$ be an antiderivative of $\varphi(x,\cdot,t)$ for every $(x,t) \in \mathbb{R}^2$. The following equality holds on $[a,b]$ for every $a < b$ and every $q \in \mathrm{C}^\infty([a,b],\mathbb{R})$:

$$\begin{aligned} L_0(q,\dot{q},\ddot{q},\cdot) &= \varphi(q,\dot{q},\cdot)\ddot{q} + \psi(q,\dot{q},\cdot) && \text{(IV.2.21)} \\ &= \psi(q,\dot{q},\cdot) - \frac{\partial\Phi}{\partial x}(q,\dot{q},\cdot)\dot{q} - \frac{\partial\Phi}{\partial t}(q,\dot{q},\cdot) + \frac{d}{dt}(\Phi(q,\dot{q},\cdot)). && \text{(IV.2.22)} \end{aligned}$$

Finally, we introduce the following Lagrangian

$$\begin{array}{rccl} L: & \mathbb{R}^3 & \longrightarrow & \mathbb{R} \\ & (x,v,t) & \longmapsto & \psi(x,v,t) - \dfrac{\partial\Phi}{\partial x}(x,v,t)v - \dfrac{\partial\Phi}{\partial t}(x,v,t). \end{array} \tag{IV.2.23}$$

Hence, we obtain that for every $a < b$ and every $q \in \mathrm{C}^\infty([a,b],\mathbb{R})$

$$\begin{aligned} \mathcal{L}_0^{a,b}(q) &= \int_a^b L(q(\tau),\dot{q}(\tau),\tau) + \frac{d}{dt}(\Phi(q,\dot{q},\cdot))(\tau)\, d\tau \\ &= \int_a^b L(q(\tau),\dot{q}(\tau),\tau)\, d\tau + \Phi(q(b),\dot{q}(b),b) - \Phi(q(a),\dot{q}(a),a). \end{aligned} \tag{IV.2.24}$$

Thus, from a calculus of variations, we obtain that for every $a < b$ and for every $(q,w) \in \mathrm{C}^\infty([a,b],\mathbb{R}) \times \mathrm{C}_c^\infty([a,b],\mathbb{R})$

$$D\mathcal{L}_0^{a,b}(q)(w) = \int_a^b \left[\frac{\partial L}{\partial x}(q(\tau),\dot{q}(\tau),\tau) - \frac{d}{dt}\left(\frac{\partial L}{\partial v}(q,\dot{q},\cdot)\right)(\tau)\right] w(\tau)\, d\tau. \tag{IV.2.25}$$

Finally, from Equalities (IV.2.19) and (IV.2.25) available for every $a < b$ and for every $(q,w) \in \mathrm{C}^\infty([a,b],\mathbb{R}) \times \mathrm{C}_c^\infty([a,b],\mathbb{R})$, the proof is complete. □

Hence, Theorem IV.2 characterizes second order differential equations that can be written as second order Euler-Lagrange equations of type (EL). Moreover, the proof of Theorem IV.2 gives an explicit method in order to provide a corresponding Lagrangian L. Let us detail the historical example given by the oscillator.

Example IV.1 (Oscillator equation)**.** Let α, β and $\gamma \in \mathbb{R}$. The second order differential equation given by $\alpha\ddot{q}+\beta\dot{q}+\gamma q = 0$ is usually called oscillator equation and is associated with the map $\overline{O}(x,v,u,t) = \alpha u+\beta v+\gamma x$. A simple calculation proves that the oscillator equation satisfies the Helmholtz condition (Definition IV.5) if and only if $\beta = 0$. As a consequence, from Theorem IV.2, the oscillator equation can be written as an Euler-Lagrange equation of type (EL) if and only if $\beta = 0$. Recall that, in the case where $\beta = 0$, the oscillator equation is said to be *without friction*. This is in accordance with the well known fact that *dissipative* systems are generally not covered by Euler-Lagrange equations.

Now, let us assume that $\beta = 0$. From the proof of Theorem IV.2, we can construct an explicit Lagrangian L such that the Euler-Lagrange equation associated is the oscillator equation without friction. Indeed, we first define $L_0(x,v,u,t) = x\int_0^1 \overline{O}(\lambda x,\lambda v,\lambda u,t)d\lambda = (\alpha x/2)u + (\gamma x^2/2)$. Then, we define $\Phi(x,v,t) = \alpha xv/2$ (antiderivative of $(\alpha x/2)$ with respect to the variable v). Finally, we define $L(x,v,t) = (\gamma x^2 - \alpha v^2)/2$ and one can easily check that the Euler-Lagrange equation associated with L is given by $\alpha\ddot{q} + \gamma q = 0$.

IV.2.3 Characterization of null Lagrangian

Note that a given Euler-Lagrange equation (EL) is not associated with a unique Lagrangian. For example, a Lagrangian L and the Lagrangian given by $L+c$, where $c \in \mathbb{R}$, give the same Euler-Lagrange equation. In this section, our aim is to describe the class of possible Lagrangian formulations for a given second order Euler-Lagrange equation.

Definition IV.6. Two Lagrangian L, L' are said to be equivalent (denoted by $L \sim L'$) if they lead to the same Euler-Lagrange equation (EL). From the linearity of the Euler-Lagrange equation with respect to the associated Lagrangian, we obtain that $\sim$ defines an equivalence relation. Finally, a Lagrangian L is said to be null if $L \sim 0$.

The aim of this section is to give a complete description of the equivalent class of a given Lagrangian L. Then, it is sufficient to give a complete description of the equivalent class of 0. Note that a Lagrangian L is null if and only if for every $a < b$, every curve $q \in \mathrm{C}^\infty([a,b],\mathbb{R})$ is solution of the Euler-Lagrange equation (EL) associated with L.

Theorem IV.3. *Let L be a Lagrangian. Then, L is a null Lagrangian if and only if there exists a map $f : (x,t) \in \mathbb{R}^2 \longrightarrow f(x,t) \in \mathbb{R}$ such that*

$$\forall a < b,\ \forall q \in \mathrm{C}^\infty([a,b],\mathbb{R}),\ \forall t \in [a,b],\ L(q(t),\dot{q}(t),t) = \frac{d}{dt}(f(q,\cdot))(t). \tag{IV.2.26}$$

This condition can also be written as $L(x,v,t) = (\partial f/\partial x)(x,t)v + (\partial f/\partial t)(x,t)$.

Proof. Let us prove that the condition is sufficient. Let us assume that there exists a map $f : (x,t) \in \mathbb{R}^2 \longrightarrow f(x,t) \in \mathbb{R}$ such that $L(x,v,t) = (\partial f/\partial x)(x,t)v + (\partial f/\partial t)(x,t)$. Then, for every $a < b$ and every $q \in \mathrm{C}^\infty([a,b],\mathbb{R})$, the following equality holds on $[a,b]$:

$$\frac{\partial L}{\partial x}(q,\dot{q},\cdot) - \frac{d}{dt}\left(\frac{\partial L}{\partial v}(q,\dot{q},\cdot)\right) = 0. \tag{IV.2.27}$$

As a consequence, the Lagrangian L is null.

Let us prove that the condition is necessary. Let us assume that L is null. Then, for every $a < b$ and every $q \in \mathrm{C}^\infty([a,b],\mathbb{R})$, we have

$$\frac{\partial L}{\partial x}(q,\dot{q},\cdot) - \frac{\partial^2 L}{\partial x \partial v}(q,\dot{q},\cdot)\dot{q} - \frac{\partial^2 L}{\partial v^2}(q,\dot{q},\cdot)\ddot{q} - \frac{\partial^2 L}{\partial t \partial v}(q,\dot{q},\cdot) = 0. \tag{IV.2.28}$$

Recall that for every $(x,v,u,t) \in \mathbb{R}^3 \times [a,b]$, there exists a curve $q \in \mathrm{C}^\infty([a,b],\mathbb{R})$ such that $q(t) = x$, $\dot{q}(t) = v$ and $\ddot{q}(t) = u$. Thus, for every $a < b$ and every $(x,v,u,t) \in \mathbb{R}^3 \times [a,b]$, it holds:

$$\frac{\partial L}{\partial x}(x,v,t) - \frac{\partial^2 L}{\partial x \partial v}(x,v,t)v - \frac{\partial^2 L}{\partial v^2}(x,v,t)u - \frac{\partial^2 L}{\partial t \partial v}(x,v,t) = 0. \tag{IV.2.29}$$

We conclude that necessarily $\partial^2 L/\partial v^2 = 0$. As a consequence, L can be rewritten as $L(x,v,t) = \varphi(x,t)v + \psi(x,t)$. From Equality (IV.2.29), we obtain $\partial\varphi/\partial t = \partial\psi/\partial x$. From the Poincaré's theorem, we conclude that there exists a map $f : (x,t) \in \mathbb{R}^2 \longrightarrow f(x,t) \in \mathbb{R}$ such that $\partial f/\partial x = \varphi$ and $\partial f/\partial t = \psi$. It completes the proof. □

Let us give an example.

Example IV.2. Let us consider the framework of Example IV.1 with $\beta = 0$. From Theorem IV.3, for every map $f : (x,t) \in \mathbb{R}^2 \longrightarrow f(x,t) \in \mathbb{R}$, the Euler-Lagrange equation associated with the Lagrangian L' defined by $L'(x,v,t) = L(x,v,t) + (\partial f/\partial x)(x,t)v + (\partial f/\partial t)(x,t)$ is the oscillator equation without friction. For example, let us take $f(x,t) = \sin(tx)/2$ and define $L'(x,v,t) = (\gamma x^2 + t\cos(tx)v + x\cos(tx) - \alpha v^2)/2$. One can easily check that the Euler-Lagrange equation associated with L' is given by $\alpha\ddot{q} + \gamma q = 0$.

IV.3 A discrete Helmholtz inverse problem

The aim of this section is to solve the discrete Helmholtz inverse problem presented in Introduction of this chapter. Our strategy is to adapt the framework and the results of the previous section to the discrete case. The whole section is extracted from [47].

Section IV.3.1 is devoted to basic recalls and notations on finite uniform time scales and on discrete derivatives. In Section IV.3.2, we introduce the notion of second order finite differences equations and we give a rigorous formulation to the discrete Helmholtz inverse problem. In Section IV.3.3, we introduce a discrete Helmholtz condition (Definition IV.10) and Theorem IV.4 proves that it characterizes the second order finite differences equations that can be written as second order discrete Euler-Lagrange equations. In Section IV.3.4, we give a characterization of null couples of Lagrangian (*i.e.* couples of Lagrangian leading to the null discrete Euler-Lagrange equation), see Theorem IV.5. From linearity, this result allows, for a given couple (L_1, L_2), to choose, among a class of couples (L'_1, L'_2) leading to the same discrete Euler-Lagrange equation, a discrete Lagrangian formulation corresponding to a discretization of a continuous one (if it exists). In such a case, we conclude that the discrete Euler-Lagrange equation associated is a variational integrator.

IV.3.1 Reminders about finite uniform time scales

In the whole chapter, $\mathscr{T}^{\mathrm{f}}_{\mathrm{uni}}$ denotes the set of all finite uniform time scales containing at least five elements *i.e.*

$$\mathscr{T}^{\mathrm{f}}_{\mathrm{uni}} := \{\mathbb{T} \subset \mathbb{R},\ 5 \leq \mathrm{card}(\mathbb{T}) < \infty \text{ and } \exists h > 0,\ \forall t \in \mathbb{T}^{\kappa},\ \mu(t) = h\}. \tag{IV.3.30}$$

For every $\mathbb{T} \in \mathscr{T}^{\mathrm{f}}_{\mathrm{uni}}$, we denote by $\mathbb{T} = \{t_p\}_{p=0,\dots,N}$ where $N = \mathrm{card}(\mathbb{T}) - 1$ is an integer such that $N \geq 4$. Note that $h = t_{p+1} - t_p$ for every $p = 0, \dots, N-1$ and that $\mathbb{T}^{\kappa} = \{t_p\}_{p=0,\dots,N-1}$, $\mathbb{T}_{\kappa} = \{t_p\}_{p=1,\dots,N}$ and $\mathbb{T}^{\kappa}_{\kappa} = \{t_p\}_{p=1,\dots,N-1}$. Recall that every function defined on $\mathbb{T}$ is directly continuous. Consequently, the set of all functions defined on $\mathbb{T}$ with values in $\mathbb{R}$ is denoted by $\mathrm{C}(\mathbb{T}, \mathbb{R})$. Moreover, recall that for every $q \in \mathrm{C}(\mathbb{T}, \mathbb{R})$, the following properties are satisfied:

- q is Δ-differentiable on $\mathbb{T}^{\kappa}$ with $\Delta q(t_p) = (q(t_{p+1}) - q(t_p))/h$ for every $p = 0, \dots, N-1$.
- q is ∇-differentiable on $\mathbb{T}_{\kappa}$ with $\nabla q(t_p) = (q(t_p) - q(t_{p-1}))/h$ for every $p = 1, \dots, N$.
- Δq (resp. ∇q) is ∇-differentiable (resp. Δ-differentiable) on $\mathbb{T}^{\kappa}_{\kappa}$ with $\nabla \circ \Delta q(t_p) = \Delta \circ \nabla q(t_p) = (q(t_{p+1}) - 2q(t_p) + q(t_{p-1}))/h^2$ for every $p = 1, \dots, N-1$.

Note that the operators ∇ and Δ commute and the composition corresponds to the usual discrete centered approximation of d^2/dt^2.

For every $\mathbb{T} \in \mathscr{T}^{\mathrm{f}}_{\mathrm{uni}}$, let $\mathrm{C}_0(\mathbb{T}, \mathbb{R})$ and $\mathrm{C}_{00}(\mathbb{T}, \mathbb{R})$ be the following sets:

- $\mathrm{C}_0(\mathbb{T}, \mathbb{R}) = \{w \in \mathrm{C}(\mathbb{T}, \mathbb{R}),\ w(t_0) = w(t_N) = 0\}$;
- $\mathrm{C}_{00}(\mathbb{T}, \mathbb{R}) = \{w \in \mathrm{C}(\mathbb{T}, \mathbb{R}),\ w(t_0) = w(t_1) = w(t_{N-1}) = w(t_N) = 0\}$.

Let us recall the classical discrete versions of the Leibniz formula and the integration by parts formula, see respectively [169, Paragraph 2.51 p.34-35] and [169, Paragraph 2.64].

Lemma IV.1 (Discrete Leibniz formulas). *Let $\mathbb{T} \in \mathscr{T}^{\mathrm{f}}_{\mathrm{uni}}$. For every q, $w \in \mathrm{C}(\mathbb{T}, \mathbb{R})$, it holds*

$$\forall p = 0, ..., N-1,\quad \Delta(qw)(t_p) = \Delta q(t_p) w(t_p) + q(t_{p+1}) \Delta w(t_p) \tag{IV.3.31}$$

and

$$\forall p = 1, ..., N,\quad \nabla(qw)(t_p) = \nabla q(t_p) w(t_p) + q(t_{p-1}) \nabla w(t_p). \tag{IV.3.32}$$

Finally, for every $p = 1, ..., N-1$, it holds

$$\nabla \circ \Delta(qw)(t_p) = \nabla \circ \Delta q(t_p) w(t_p) + q(t_p) \nabla \circ \Delta w(t_p) + \Delta q(t_p) \Delta w(t_p) + \nabla q(t_p) \nabla w(t_p). \tag{IV.3.33}$$

Lemma IV.2 (Discrete integration by parts formulas). *Let $\mathbb{T} \in \mathscr{T}^{\mathrm{f}}_{\mathrm{uni}}$ and let $q \in \mathrm{C}(\mathbb{T}, \mathbb{R})$. Then, for every $w \in \mathrm{C}_0(\mathbb{T}, \mathbb{R})$, it holds*

$$h \sum_{p=0}^{N-1} q(t_p) \Delta w(t_p) = -h \sum_{p=1}^{N-1} \nabla q(t_p) w(t_p) \tag{IV.3.34}$$

and

$$h\sum_{p=1}^{N} q(t_p)\nabla w(t_p) = -h\sum_{p=1}^{N-1} \Delta q(t_p) w(t_p). \tag{IV.3.35}$$

Moreover, for every $w \in \mathrm{C}_{00}(\mathbb{T},\mathbb{R})$, *it holds*

$$h\sum_{p=1}^{N-1} q(t_p)\nabla\circ\Delta w(t_p) = h\sum_{p=2}^{N-2} \nabla\circ\Delta q(t_p) w(t_p). \tag{IV.3.36}$$

IV.3.2 Formulation of the discrete Helmholtz inverse problem

In this section, we formulate rigorously the discrete Helmholtz inverse problem. Let us first define the second order finite differences equations.

Definition IV.7. A second order finite differences equation on $\mathbb{T} \in \mathscr{T}^{\mathrm{f}}_{\mathrm{uni}}$ is defined by $P^{\mathbb{T}}(q) = 0$ where P is a second order finite differences operator *i.e.*

$$P : \mathbb{T} \in \mathscr{T}^{\mathrm{f}}_{\mathrm{uni}} \longmapsto P^{\mathbb{T}} : \begin{array}{ccc} \mathrm{C}(\mathbb{T},\mathbb{R}) & \longrightarrow & \mathrm{C}(\mathbb{T}^{\kappa}_{\kappa},\mathbb{R}) \\ q & \longmapsto & P^{\mathbb{T}}(q) \end{array} \tag{IV.3.37}$$

where

$$\forall p = 1,...,N-1,\ P^{\mathbb{T}}(q)(t_p) = \overline{P}(q(t_p),\Delta q(t_p),\nabla q(t_p),\nabla\circ\Delta q(t_p),t_p,h) \tag{IV.3.38}$$

and

$$\overline{P} : \begin{array}{ccc} \mathbb{R}^5 \times \mathbb{R}^*_+ & \longrightarrow & \mathbb{R} \\ (x,v_1,v_2,u,t,\xi) & \longmapsto & \overline{P}(x,v_1,v_2,u,t,\xi). \end{array} \tag{IV.3.39}$$

Let (L_1,L_2) be a couple of smooth Lagrangian *i.e.*

$$L_i : \begin{array}{ccc} \mathbb{R}^3 \times \mathbb{R}^*_+ & \longrightarrow & \mathbb{R} \\ (x,v,t,\xi) & \longmapsto & L_i(x,v,t,\xi), \end{array} \tag{IV.3.40}$$

for $i = 1,2$. For every $\mathbb{T} \in \mathscr{T}^{\mathrm{f}}_{\mathrm{uni}}$, we define the following discrete Lagrangian functional:

$$\mathcal{L}^{\mathbb{T}} : \begin{array}{ccl} \mathrm{C}(\mathbb{T},\mathbb{R}) & \longrightarrow & \mathbb{R} \\ q & \longmapsto & h\displaystyle\sum_{p=0}^{N-1} L_1(q(t_p),\Delta q(t_p),t_p,h) + h\sum_{p=1}^{N} L_2(q(t_p),\nabla q(t_p),t_p,h). \end{array} \tag{IV.3.41}$$

Let $\mathrm{C}_0(\mathbb{T},\mathbb{R})$ be the set of variations of $\mathcal{L}^{\mathbb{T}}$. Then, a discrete curve $q \in \mathrm{C}(\mathbb{T},\mathbb{R})$ is said to be a critical point of $\mathcal{L}^{\mathbb{T}}$ if $D\mathcal{L}^{\mathbb{T}}(q)(w) = 0$ for every variation $w \in \mathrm{C}_0(\mathbb{T},\mathbb{R})$. Recall that $D\mathcal{L}^{\mathbb{T}}(q)(w)$ denotes the Gâteaux-differential of $\mathcal{L}^{\mathbb{T}}$ at q in direction w. An usual discrete calculus of variations leads to the characterization of the critical points of $\mathcal{L}^{\mathbb{T}}$ as the solutions of the following discrete Euler-Lagrange equation:

$$\frac{\partial L_1}{\partial x}(q(t_p),\Delta q(t_p),t_p,h) + \frac{\partial L_2}{\partial x}(q(t_p),\nabla q(t_p),t_p,h) - \nabla\left(\frac{\partial L_1}{\partial v}(q,\Delta q,\cdot,h)\right)(t_p) - \Delta\left(\frac{\partial L_2}{\partial v}(q,\nabla q,\cdot,h)\right)(t_p) = 0. \tag{IV.3.42}$$

that can be written as follows:

$$\begin{aligned} &\frac{\partial L_1}{\partial x}(q(t_p),\Delta q(t_p),t_p,h) + \frac{\partial L_2}{\partial x}(q(t_p),\nabla q(t_p),t_p,h) \\ &\quad - \sum_{i+j+k\geq 1} \frac{(-h)^{i+j+k-1}}{i!j!k!}\frac{\partial^{i+j+k+1}L_1}{\partial x^i\partial v^{j+1}\partial t^k}(q(t_p),\Delta q(t_p),t_p,h)(\nabla q(t_p))^i(\nabla\circ\Delta q)^j \\ &\quad - \sum_{i+j+k\geq 1} \frac{h^{i+j+k-1}}{i!j!k!}\frac{\partial^{i+j+k+1}L_2}{\partial x^i\partial v^{j+1}\partial t^k}(q(t_p),\nabla q(t_p),t_p,h)(\Delta q(t_p))^i(\nabla\circ\Delta q)^j = 0. \end{aligned} \tag{EL$^{\mathrm{d}}$}$$

As a consequence, the discrete Euler-Lagrange equation (EL^{d}) is a second order finite differences equation $P^{\mathbb{T}}(q) = 0$ associated with the map $\overline{P}$ given by

$$\overline{P}(x, v_1, v_2, u, t, \xi) = \frac{\partial L_1}{\partial x}(x, v_1, t, \xi) + \frac{\partial L_2}{\partial x}(x, v_2, t, \xi) \\ - \sum_{i+j+k \geq 1} \frac{(-\xi)^{i+j+k-1}}{i!j!k!} \frac{\partial^{i+j+k+1} L_1}{\partial x^i \partial v^{j+1} \partial t^k}(x, v_1, t, \xi) v_2^i u^j \\ - \sum_{i+j+k \geq 1} \frac{\xi^{i+j+k-1}}{i!j!k!} \frac{\partial^{i+j+k+1} L_2}{\partial x^i \partial v^{j+1} \partial t^k}(x, v_2, t, \xi) v_1^i u^j. \quad \text{(IV.3.43)}$$

The discrete Helmholtz inverse problem treats on the following issue: *if we consider a second order finite differences equation, how to know if it is a second order discrete Euler-Lagrange of type* (EL^{d})? For sake of rigorousness of the formulation of the problem, we introduce the following definition.

Definition IV.8. Let P be a second order finite differences operator. Then, P is said to be a second order discrete Euler-Lagrange operator if there exists a couple (L_1, L_2) of Lagrangian such that for every $\mathbb{T} \in \mathscr{T}^{\mathrm{f}}_{\mathrm{uni}}$ and every $q \in \mathrm{C}(\mathbb{T}, \mathbb{R})$, it holds

$$P^{\mathbb{T}}(q)(t_p) = \frac{\partial L_1}{\partial x}(q(t_p), \Delta q(t_p), t_p, h) + \frac{\partial L_2}{\partial x}(q(t_p), \nabla q(t_p), t_p, h) \\ - \nabla \left(\frac{\partial L_1}{\partial v}(q, \Delta q, \cdot, h) \right)(t_p) - \Delta \left(\frac{\partial L_2}{\partial v}(q, \nabla q, \cdot, h) \right)(t_p) = 0, \quad \text{(IV.3.44)}$$

for every $p = 1, \dots, N-1$.

With the help of this definition, the discrete Helmholtz inverse problem can be formulated as follows.

Discrete Helmholtz inverse problem: Find a necessary and sufficient condition on a second order finite differences operator in order to be a second order discrete Euler-Lagrange operator.

IV.3.3 A discrete Helmholtz condition

In this section, we adapt the strategy of Section IV.2 to the discrete case.

Definition IV.9 (Self-adjointness)**.** Let P be a second order finite differences operator, $\mathbb{T} \in \mathscr{T}^{\mathrm{f}}_{\mathrm{uni}}$ and $q \in \mathrm{C}(\mathbb{T}, \mathbb{R})$. The Gâteaux-differential of $P^{\mathbb{T}}$ at q denoted by $DP^{\mathbb{T}}(q)$ is said to be self-adjoint if for every $(w, z) \in \mathrm{C}_{00}(\mathbb{T}, \mathbb{R}) \times \mathrm{C}(\mathbb{T}, \mathbb{R})$, it holds

$$h \sum_{p=1}^{N-1} DP^{\mathbb{T}}(q)(w)(t_p) z(t_p) = h \sum_{p=2}^{N-2} w(t_p) DP^{\mathbb{T}}(q)(z)(t_p). \quad \text{(IV.3.45)}$$

Definition IV.10 (Discrete Helmholtz condition)**.** Let P be a second order finite differences operator. Then, P is said to be satisfying the discrete Helmholtz condition if for every $\mathbb{T} \in \mathscr{T}^{\mathrm{f}}_{\mathrm{uni}}$ and every $q \in \mathrm{C}(\mathbb{T}, \mathbb{R})$, it holds

$$\Delta \left(\frac{\partial \overline{P}}{\partial u}(q, \Delta q, \nabla q, \nabla \circ \Delta q, \cdot, h) \right)(t_p) = \frac{\partial \overline{P}}{\partial v_1}(q(t_p), \Delta q(t_p), \nabla q(t_p), \nabla \circ \Delta q(t_p), t_p, h) \\ + \frac{\partial \overline{P}}{\partial v_2}(q(t_{p+1}), \Delta q(t_{p+1}), \nabla q(t_{p+1}), \nabla \circ \Delta q(t_{p+1}), t_{p+1}, h), \quad \text{(IV.3.46)}$$

for every $p = 1, \dots, N-2$.

The proof of the main result (Theorem IV.4) needs the following series of Lemmas. The proofs of Lemmas IV.3, IV.4 and IV.5 are detailed in Section IV.3.5.

Lemma IV.3. *Let P be a second order finite differences operator satisfying the discrete Helmholtz condition. Then, for every $\mathbb{T} \in \mathscr{T}^{\mathrm{f}}_{\mathrm{uni}}$ and every $q \in \mathrm{C}(\mathbb{T}, \mathbb{R})$, $DP^{\mathbb{T}}(q)$ is self-adjoint.*

Lemma IV.4. *Let P be a second order finite differences operator satisfying the discrete Helmholtz condition. Then, there exist φ, $\psi : (x,v,t,\xi) \in \mathbb{R}^3 \times \mathbb{R}^*_+ \longmapsto \varphi(x,v,t,\xi), \psi(x,v,t,\xi) \in \mathbb{R}$ such that*

$$\forall (x,v_1,v_2,t,\xi) \in \mathbb{R}^4 \times \mathbb{R}^*_+,\ \overline{P}\left(x,v_1,v_2,\frac{v_1-v_2}{\xi},t,\xi\right) = \varphi(x,v_1,t,\xi) + \psi(x,v_2,t,\xi). \tag{IV.3.47}$$

Lemma IV.5. *Let P be a second order finite differences operator. If there exists a couple of Lagrangian (L_1,L_2) such that for every $\mathbb{T} \in \mathscr{T}^{\mathrm{f}}_{\mathrm{uni}}$ and for every $q \in \mathrm{C}(\mathbb{T},\mathbb{R})$, it holds*

$$\begin{aligned} P^{\mathbb{T}}(q)(t_p) = \frac{\partial L_1}{\partial x}(q(t_p),\Delta q(t_p),t_p,h) + \frac{\partial L_2}{\partial x}(q(t_p),\nabla q(t_p),t_p,h) \\ - \nabla\left(\frac{\partial L_1}{\partial v}(q,\Delta q,\cdot,h)\right)(t_p) - \Delta\left(\frac{\partial L_2}{\partial v}(q,\nabla q,\cdot,h)\right)(t_p) = 0, \end{aligned} \tag{IV.3.48}$$

for every $p = 2,\dots,N-2$, then P is a second order discrete Euler-Lagrange operator associated with (L_1,L_2).

Finally, let us give the following characterization of second order finite differences equations that can be written as second order discrete Euler-Lagrange equations of type (EL^{d}).

Theorem IV.4. *Let P be a second order finite differences operator. Then, P is a second order discrete Euler-Lagrange operator if and only if P satisfies the discrete Helmholtz condition.*

Proof. Let us introduce the following notation:

$$\star_p = (q(t_p),\Delta q(t_p),\nabla q(t_p),\nabla\circ\Delta q(t_p),t_p,h) \tag{IV.3.49}$$

for every $\mathbb{T} \in \mathscr{T}^{\mathrm{f}}_{\mathrm{uni}}$, every $q \in \mathrm{C}(\mathbb{T},\mathbb{R})$ and every $p = 1,\dots,N-1$.

Let us prove that the condition is necessary. Let P be a second order discrete Euler-Lagrange operator and let (L_1,L_2) be the couple of Lagrangian associated. Then, we have

$$\begin{aligned} \overline{P}(x,v_1,v_2,u,t,\xi) = \frac{\partial L_1}{\partial x}(x,v_1,t,\xi) + \frac{\partial L_2}{\partial x}(x,v_2,t,\xi) \\ - \sum_{i+j+k\geq 1} \frac{(-\xi)^{i+j+k-1}}{i!j!k!}\frac{\partial^{i+j+k+1}L_1}{\partial x^i \partial v^{j+1}\partial t^k}(x,v_1,t,\xi)v_2^i u^j \\ - \sum_{i+j+k\geq 1} \frac{\xi^{i+j+k-1}}{i!j!k!}\frac{\partial^{i+j+k+1}L_2}{\partial x^i \partial v^{j+1}\partial t^k}(x,v_2,t,\xi)v_1^i u^j. \end{aligned} \tag{IV.3.50}$$

As a consequence, the three following equalities hold for every $\mathbb{T} \in \mathscr{T}^{\mathrm{f}}_{\mathrm{uni}}$ and every $q \in \mathrm{C}(\mathbb{T},\mathbb{R})$:

$$\frac{\partial \overline{P}}{\partial u}(\star_p) = -\frac{\partial^2 L_1}{\partial v^2}(q(t_{p-1}),\Delta q(t_{p-1}),t_{p-1},h) - \frac{\partial^2 L_2}{\partial v^2}(q(t_{p+1}),\nabla q(t_{p+1}),t_{p+1},h), \tag{IV.3.51}$$

$$\frac{\partial \overline{P}}{\partial v_1}(\star_p) = \frac{\partial^2 L_1}{\partial v\partial x}(q(t_p),\Delta q(t_p),t_p,h) - \nabla\left(\frac{\partial^2 L_1}{\partial v^2}(q,\Delta q,\cdot,h)\right)(t_p) - \frac{\partial^2 L_2}{\partial x\partial v}(q(t_{p+1}),\nabla q(t_{p+1}),t_{p+1},h), \tag{IV.3.52}$$

$$\frac{\partial \overline{P}}{\partial v_2}(\star_p) = \frac{\partial^2 L_2}{\partial v\partial x}(q(t_p),\nabla q(t_p),t_p,h) - \frac{\partial^2 L_1}{\partial x\partial v}(q(t_{p-1}),\Delta q(t_{p-1}),t_{p-1},h) - \Delta\left(\frac{\partial^2 L_2}{\partial v^2}(q,\nabla q,\cdot,h)\right)(t_p), \tag{IV.3.53}$$

for every $p = 1,\dots,N-1$. It is then easy to check that P satisfies the discrete Helmholtz condition (Definition IV.10).

Now, let us prove that the condition is sufficient. Let P be a second order finite differences operator satisfying the discrete Helmholtz condition and let φ, ψ be the two maps given by Lemma IV.4. Let L_1, L_2 be the Lagrangian defined by

$$\begin{array}{rccl} L_1 : & \mathbb{R}^3 \times \mathbb{R}^*_+ & \longrightarrow & \mathbb{R} \\ & (x,v,t,\xi) & \longmapsto & x\displaystyle\int_0^1 \varphi(\lambda x,\lambda v,t,\xi)\,d\lambda \end{array} \tag{IV.3.54}$$

and

$$\begin{array}{rccl} L_2: & \mathbb{R}^3 \times \mathbb{R}^*_+ & \longrightarrow & \mathbb{R} \\ & (x,v,t,\xi) & \longmapsto & x \displaystyle\int_0^1 \psi(\lambda x, \lambda v, t, \xi)\, d\lambda, \end{array} \tag{IV.3.55}$$

and for every $\mathbb{T} \in \mathscr{T}^{\mathrm{f}}_{\mathrm{uni}}$, let $\mathcal{L}^{\mathbb{T}}$ be the following discrete Lagrangian functional:

$$\begin{array}{rccl} \mathcal{L}^{\mathbb{T}}: & \mathrm{C}(\mathbb{T},\mathbb{R}) & \longrightarrow & \mathbb{R} \\ & q & \longmapsto & h \displaystyle\sum_{p=0}^{N-1} L_1(q(t_p), \Delta q(t_p), t_p, h) + h \sum_{p=1}^{N} L_2(q(t_p), \nabla q(t_p), t_p, h). \end{array} \tag{IV.3.56}$$

On the one hand, an usual discrete calculus of variations gives

$$D\mathcal{L}^{\mathbb{T}}(q)(w) = h \sum_{p=2}^{N-2} \Bigg[\frac{\partial L_1}{\partial x}(q(t_p), \Delta q(t_p), t_p, h) + \frac{\partial L_2}{\partial x}(q(t_p), \nabla q(t_p), t_p, h) \\ -\nabla \left(\frac{\partial L_1}{\partial v}(q, \Delta q, \cdot, h) \right)(t_p) - \Delta \left(\frac{\partial L_2}{\partial v}(q, \nabla q, \cdot, h) \right)(t_p) \Bigg] w(t_p), \tag{IV.3.57}$$

for every $\mathbb{T} \in \mathscr{T}^{\mathrm{f}}_{\mathrm{uni}}$ and every $(q,w) \in \mathrm{C}(\mathbb{T},\mathbb{R}) \times \mathrm{C}_{00}(\mathbb{T},\mathbb{R})$. On the other hand, from the definitions of L_1 and L_2, we have for every $\mathbb{T} \in \mathscr{T}^{\mathrm{f}}_{\mathrm{uni}}$ and every $(q,w) \in \mathrm{C}(\mathbb{T},\mathbb{R}) \times \mathrm{C}_{00}(\mathbb{T},\mathbb{R})$

$$\mathcal{L}^{\mathbb{T}}(q) = h[L_1(q(t_0), \Delta q(t_0), t_0, h) + L_2(q(t_N), \nabla q(t_N), t_N, h)] + h \sum_{p=1}^{N-1} q(t_p) \int_0^1 P^{\mathbb{T}}(\lambda q)(t_p)\, d\lambda. \tag{IV.3.58}$$

and consequently

$$D\mathcal{L}^{\mathbb{T}}(q)(w) = h \sum_{k=1}^{N-1} w(t_p) \int_0^1 P^{\mathbb{T}}(\lambda q)(t_p)\, d\lambda + h \sum_{k=1}^{N-1} q(t_p) \int_0^1 DP^{\mathbb{T}}(\lambda q)(\lambda w)(t_p)\, d\lambda. \tag{IV.3.59}$$

Using the discrete Helmholtz condition (Definition IV.10) and Lemma IV.3 in the second sum, we obtain

$$D\mathcal{L}^{\mathbb{T}}(q)(w) = h \sum_{k=2}^{N-2} w(t_p) \int_0^1 P^{\mathbb{T}}(\lambda q)(t_p)\, d\lambda + h \sum_{k=2}^{N-2} w(t_p) \int_0^1 \lambda DP^{\mathbb{T}}(\lambda q)(q)(t_p)\, d\lambda. \tag{IV.3.60}$$

Finally, an integration by parts with respect to λ (in the second integral) leads to

$$D\mathcal{L}^{\mathbb{T}}_0(q)(w) = h \sum_{k=2}^{N-2} P^{\mathbb{T}}(q)(t_p) w(t_p). \tag{IV.3.61}$$

As a consequence, Equalities (IV.3.57) and (IV.3.61) leads to

$$P^{\mathbb{T}}(q)(t_p) = \frac{\partial L_1}{\partial x}(q(t_p), \Delta q(t_p), t_p, h) + \frac{\partial L_2}{\partial x}(q(t_p), \nabla q(t_p), t_p, h) \\ - \nabla \left(\frac{\partial L_1}{\partial v}(q, \Delta q, \cdot, h) \right)(t_p) - \Delta \left(\frac{\partial L_2}{\partial v}(q, \nabla q, \cdot, h) \right)(t_p), \tag{IV.3.62}$$

for every $p = 2, \ldots, N-2$, every $\mathbb{T} \in \mathscr{T}^{\mathrm{f}}_{\mathrm{uni}}$ and every $q \in \mathrm{C}(\mathbb{T},\mathbb{R})$. Lemma IV.5 concludes the proof. □

The proof of Theorem IV.4 gives a method to provide an explicit couple of Lagrangian (L_1, L_2). Let us study the example of the discrete oscillator.

Example IV.3 (Discrete oscillator equation)**.** Let α, β_1, β_2 and $\gamma \in \mathbb{R}$. The second order finite differences equation given by

$$\alpha \frac{q(t_{p+1}) - 2q(t_p) + q(t_{p-1})}{h^2} + \beta_1 \frac{q(t_{p+1}) - q(t_p)}{h} + \beta_2 \frac{q(t_p) - q(t_{p-1})}{h} + \gamma q(t_p) = 0, \tag{IV.3.63}$$

for every $p = 1, \dots, N-1$, is called discrete oscillator equation and is associated with the map $\overline{P}(x, v_1, v_2, u, t, \xi) = \alpha u + \beta_1 v_1 + \beta_2 v_2 + \gamma x$. The discrete oscillator equation satisfies the discrete Helmholtz condition (Definition IV.10) if and only if $\beta_1 + \beta_2 = 0$. As a consequence, from Theorem IV.4, the discrete oscillator equation can be written as a discrete Euler-Lagrange equation of type (EL^{d}) if and only if $\beta_1 + \beta_2 = 0$.

Let us assume that $\beta_1 + \beta_2 = 0$ and let us denote by $\beta = \beta_1 = -\beta_2$. From the proof of Theorem IV.4, we can construct a couple of Lagrangian (L_1, L_2) such that the discrete Euler-Lagrange equation associated is the discrete oscillator equation. Indeed, note that $\overline{P}(x, v_1, v_2, \frac{v_1 - v_2}{\xi}, t, \xi) = \varphi(x, v_1, t, \xi) + \psi(x, v_2, t, \xi)$ where $\varphi(x, v, t, \xi) = \alpha \frac{v}{\xi} + \beta v + \gamma x$ and $\psi(x, v, t, \xi) = -\alpha \frac{v}{\xi} - \beta v$. Then, we define $L_1(x, v, t, \xi) = \frac{x}{2}(\alpha \frac{v}{\xi} + \beta v + \gamma x)$ and $L_2(x, v, t, \xi) = -\frac{x}{2}(\alpha \frac{v}{\xi} + \beta v)$. One can easily check that the discrete Euler-Lagrange equation associated with (L_1, L_2) is the discrete oscillator equation.

Remark IV.2. Example IV.3 provides a couple of Lagrangian (L_1, L_2) such that the discrete Euler-Lagrange equation associated is the discrete oscillator equation with $\beta_1 + \beta_2 = 0$. Nevertheless, the discrete Lagrangian formulation associated to (L_1, L_2) does not correspond to a discretization of a continuous one of type $\mathcal{L}(q) = \int_{t_0}^{t_N} L(q(\tau), \dot{q}(\tau), \tau) d\tau$, where $\dot{q}$ is the derivative of q. Indeed, (L_1, L_2) does not admit a limit when ξ tends to 0^+ and then a continuum Lagrangian given by $L(x, v, t) = \lim_{\xi \to 0^+} L'_1(x, v, t, \xi) + \lim_{\xi \to 0^+} L'_2(x, v, t, \xi)$ cannot be defined. Then, we cannot affirm that the discrete oscillator equation with $\beta_1 + \beta_2 = 0$ corresponds to a variational integrator associated with a continuous Euler-Lagrange equation.

However, the following section allow us to choose, among a family of couples of Lagrangian leading to the discrete oscillator equation with $\beta_1 + \beta_2 = 0$, a couple of Lagrangian admitting a continuum limit. Consequently, the discrete oscillator equation with $\beta_1 + \beta_2 = 0$ corresponds to a variational integrator associated with a continuous Euler-Lagrange equation. We refer to Example IV.4 for more details.

IV.3.4 Characterization of null couples of Lagrangian

As in the continuous case, a discrete Euler-Lagrange equation (EL^{d}) is not associated with a unique couple of Lagrangian (L_1, L_2). The aim of this section is to describe the class of possible Lagrangian formulations for a given second order discrete Euler-Lagrange equation.

Definition IV.11. Two couples of Lagrangian (L_1, L_2), (L'_1, L'_2) are said to be equivalent (denoted by $(L_1, L_2) \sim (L'_1, L'_2)$) if they lead to the same discrete Euler-Lagrange equation (EL^{d}). From the linearity of the discrete Euler-Lagrange equation with respect to the associated couple of Lagrangian, we obtain that $\sim$ defines an equivalence relation. Finally, a couple of Lagrangian (L_1, L_2) is said to be null if $(L_1, L_2) \sim (0, 0)$.

In order to give a complete description of the equivalent class of a given couple of Lagrangian (L_1, L_2), it is sufficient to have a complete description of the equivalent class of $(0, 0)$. Note that a couple of Lagrangian (L_1, L_2) is null if and only if for every $\mathbb{T} \in \mathscr{T}^{\mathrm{f}}_{\mathrm{uni}}$, every discrete curve $q \in \mathrm{C}(\mathbb{T}, \mathbb{R})$ is solution of the discrete Euler-Lagrange equation (EL^{d}) associated with (L_1, L_2).

Theorem IV.5. *Let (L_1, L_2) be a couple of Lagrangian. Then, (L_1, L_2) is a null couple of Lagrangian if and only if there exist two maps $f : (x, t, \xi) \in \mathbb{R}^2 \times \mathbb{R}^*_+ \longmapsto f(x, t, \xi) \in \mathbb{R}$ and $g : (t, \xi) \in \mathbb{R} \times \mathbb{R}^*_+ \longmapsto g(t, \xi) \in \mathbb{R}$ such that for every $\mathbb{T} \in \mathscr{T}^{\mathrm{f}}_{\mathrm{uni}}$ and every $q \in \mathrm{C}(\mathbb{T}, \mathbb{R})$, it holds*

$$L_1(q(t_p), \Delta q(t_p), t_p, h) + L_2(q(t_{p+1}), \nabla q(t_{p+1}), t_{p+1}, h) = \Delta(f(q, \cdot, h))(t_p) + g(t_p, h), \tag{IV.3.64}$$

for every $p = 0, \dots, N-1$.

Proof. Let us prove that the condition is sufficient. In such a case, for every $\mathbb{T} \in \mathscr{T}^{\mathrm{f}}_{\mathrm{uni}}$ and every $q \in \mathrm{C}(\mathbb{T}, \mathbb{R})$, the discrete Lagrangian functional $\mathcal{L}^{\mathbb{T}}$ associated with (L_1, L_2) is given by

$$\begin{aligned} \mathcal{L}^{\mathbb{T}}(q) &= h \sum_{p=0}^{N-1} L_1(q(t_p), \Delta q(t_p), t_p, h) + h \sum_{p=1}^{N} L_2(q(t_p), \nabla q(t_p), t_p, h) && \text{(IV.3.65)} \\ &= h \sum_{p=0}^{N-1} \Big[L_1(q(t_p), \Delta q(t_p), t_p, h) + L_2(q(t_{p+1}), \nabla q(t_{p+1}), t_{p+1}, h) \Big] && \text{(IV.3.66)} \\ &= h \sum_{p=0}^{N-1} \Big[\Delta(f(q, \cdot, h))(t_p) + g(t_p, h) \Big] && \text{(IV.3.67)} \\ &= f(q(t_N), t_N, h) - f(q(t_0), t_0, h) + h \sum_{p=0}^{N-1} g(t_p, h). && \text{(IV.3.68)} \end{aligned}$$

Finally, we see that every $q \in \mathrm{C}(\mathbb{T}, \mathbb{R})$ is a critical point of $\mathcal{L}^{\mathbb{T}}$. This concludes that the couple (L_1, L_2) is null.

Now, let us prove that the condition is necessary. Let us assume that (L_1, L_2) is a null couple of Lagrangian. Then, for every $\mathbb{T} \in \mathscr{T}^{\mathrm{f}}_{\mathrm{uni}}$ and every $q \in \mathrm{C}(\mathbb{T}, \mathbb{R})$, it holds

$$\begin{aligned} &\frac{\partial L_1}{\partial x}(q(t_p), \Delta q(t_p), t_p, h) + \frac{\partial L_2}{\partial x}(q(t_p), \nabla q(t_p), t_p, h) \\ &\qquad - \nabla \left(\frac{\partial L_1}{\partial v}(q, \Delta q, \cdot, h) \right)(t_p) - \Delta \left(\frac{\partial L_2}{\partial v}(q, \nabla q, \cdot, h) \right)(t_p) = 0, \quad \text{(IV.3.69)} \end{aligned}$$

for every $p = 1, \ldots, N-1$. As a consequence, for every $(y_{-1}, y_0, y_1, t, \xi) \in \mathbb{R}^4 \times \mathbb{R}^*_+$, it holds

$$\begin{aligned} &\frac{\partial L_1}{\partial x}\left(y_0, \frac{y_1 - y_0}{\xi}, t, \xi \right) + \frac{\partial L_2}{\partial x}\left(y_0, \frac{y_0 - y_{-1}}{\xi}, t, \xi \right) \\ &\qquad + \frac{1}{\xi} \left[\frac{\partial L_1}{\partial v}\left(y_{-1}, \frac{y_0 - y_{-1}}{\xi}, t - \xi, \xi \right) - \frac{\partial L_1}{\partial v}\left(y_0, \frac{y_1 - y_0}{\xi}, t, \xi \right) \right] \\ &\qquad + \frac{1}{\xi} \left[\frac{\partial L_2}{\partial v}\left(y_0, \frac{y_0 - y_{-1}}{\xi}, t, \xi \right) - \frac{\partial L_2}{\partial v}\left(y_1, \frac{y_1 - y_0}{\xi}, t + \xi, \xi \right) \right] = 0. \quad \text{(IV.3.70)} \end{aligned}$$

Differentiating the last equality with respect to y_1 (or y_{-1}), we obtain for every $(y_0, y_1, t, \xi) \in \mathbb{R}^3 \times \mathbb{R}^*_+$

$$\begin{aligned} &\frac{1}{\xi} \frac{\partial^2 L_1}{\partial v \partial x}\left(y_0, \frac{y_1 - y_0}{\xi}, t, \xi \right) - \frac{1}{\xi^2} \frac{\partial^2 L_1}{\partial v^2}\left(y_0, \frac{y_1 - y_0}{\xi}, t, \xi \right) \\ &\qquad - \frac{1}{\xi} \frac{\partial^2 L_2}{\partial x \partial v}\left(y_1, \frac{y_1 - y_0}{\xi}, t + \xi, \xi \right) - \frac{1}{\xi^2} \frac{\partial^2 L_2}{\partial v^2}\left(y_1, \frac{y_1 - y_0}{\xi}, t + \xi, \xi \right) = 0. \quad \text{(IV.3.71)} \end{aligned}$$

Let us define the following map:

$$\begin{aligned} \ell : \quad \mathbb{R}^3 \times \mathbb{R}^*_+ &\longrightarrow \mathbb{R} \\ (y_0, y_1, t, \xi) &\longmapsto L_1\left(y_0, \frac{y_1 - y_0}{\xi}, t, \xi \right) + L_2\left(y_1, \frac{y_1 - y_0}{\xi}, t + \xi, \xi \right). \end{aligned} \quad \text{(IV.3.72)}$$

Equality (IV.3.71) leads to $\partial^2 \ell / \partial y_1 \partial y_0 = 0$ and consequently, there exist two maps ℓ_1, $\ell_2 : \mathbb{R}^2 \times \mathbb{R}^*_+ \longrightarrow \mathbb{R}$ such that $\ell(y_0, y_1, t, \xi) = \ell_1(y_0, t, \xi) + \ell_2(y_1, t, \xi)$. As a consequence, for every $\mathbb{T} \in \mathscr{T}^{\mathrm{f}}_{\mathrm{uni}}$ and every $q \in \mathrm{C}(\mathbb{T}, \mathbb{R})$, the discrete Lagrangian functional $\mathcal{L}^{\mathbb{T}}$ associated with (L_1, L_2) is given by

$$\mathcal{L}^{\mathbb{T}}(q) = h \sum_{p=0}^{N-1} \Big[\ell_1(q(t_p), t_p, h) + \ell_2(q(t_{p+1}), t_p, h) \Big]. \quad \text{(IV.3.73)}$$

For every $\mathbb{T} \in \mathscr{T}^{\mathrm{f}}_{\mathrm{uni}}$ and every $(q, w) \in \mathrm{C}(\mathbb{T}, \mathbb{R}) \times \mathrm{C}_0(\mathbb{T}, \mathbb{R})$, it holds

$$D\mathcal{L}^{\mathbb{T}}(q)(w) = h \sum_{p=1}^{N-1} \left[\frac{\partial \ell_1}{\partial y_0}(q(t_p), t_p, h) + \frac{\partial \ell_2}{\partial y_0}(q(t_p), t_{p-1}, h) \right] w(t_p). \quad \text{(IV.3.74)}$$

Since (L_1,L_2) is a null couple of Lagrangian, for every $\mathbb{T}\in\mathscr{T}^{\mathrm{f}}_{\mathrm{uni}}$, every discrete curve $q\in \mathrm{C}(\mathbb{T},\mathbb{R})$ is a critical point of $\mathcal{L}^{\mathbb{T}}$ *i.e.*

$$\frac{\partial \ell_1}{\partial y_0}(q(t_p),t_p,h)+\frac{\partial \ell_2}{\partial y_0}(q(t_p),t_{p-1},h)=0, \tag{IV.3.75}$$

for every $p=1,\dots,N-1$. Consequently, for every $(y_0,t,\xi)\in\mathbb{R}^2\times\mathbb{R}^*_+$, it holds $\partial\ell_1/\partial y_0(y_0,t,\xi)+\partial\ell_2/\partial y_0(y_0,t-\xi,\xi)=0$. We conclude that there exists $g:(t,\xi)\in\mathbb{R}\times\mathbb{R}^*_+\longmapsto g(t,\xi)\in\mathbb{R}$ such that for every $(y_0,t,\xi)\in\mathbb{R}^2\times\mathbb{R}^*_+$, it holds $\ell_1(y_0,t,\xi)+\ell_2(y_0,t-\xi,\xi)=g(t,\xi)$. Finally, for every $\mathbb{T}\in\mathscr{T}^{\mathrm{f}}_{\mathrm{uni}}$ and every $q\in\mathrm{C}(\mathbb{T},\mathbb{R})$, we obtain

$$\begin{aligned}L_1(q(t_p),\Delta q(t_p),t_p,h)&+L_2(q(t_{p+1}),\nabla q(t_{p+1}),t_{p+1},h)=\ell(q(t_p),q(t_{p+1}),t_p,h)\\&=\ell_1(q(t_p),t_p,h)+\ell_2(q(t_{p+1}),t_p,h)=\ell_2(q(t_{p+1}),t_p,h)-\ell_2(q(t_p),t_p-h,h)+g(t_p,h),\end{aligned} \tag{IV.3.76}$$

for every $p=0,\dots,N-1$. To complete the proof, we define $f:(x,t,\xi)\in\mathbb{R}^2\times\mathbb{R}^*_+\longmapsto \xi\ell_2(x,t-\xi,\xi)\in\mathbb{R}$. □

Example IV.4. Let us consider the framework of Example IV.3 with $\beta_1+\beta_2=0$ and let $\beta=\beta_1=-\beta_2$. Example IV.3 shows that the discrete Euler-Lagrange equation associated with $L_1(x,v,t,\xi)=\frac{x}{2}(\alpha\frac{v}{\xi}+\beta v+\gamma x)$ and $L_2(x,v,t,\xi)=-\frac{x}{2}(\alpha\frac{v}{\xi}+\beta v)$ is the discrete oscillator equation. Nevertheless, as mentioned in Remark IV.2, the discrete Lagrangian formulation associated to (L_1,L_2) does not correspond to a discretization of a continuous one. Then, we cannot affirm that the discrete oscillator equation with $\beta_1+\beta_2=0$ corresponds to a variational integrator of a continuous Euler-Lagrange equation.

It is then interesting to look for a couple of Lagrangian (L'_1,L'_2) in the equivalent class of (L_1,L_2) admitting a continuum limit. From Theorem IV.5, the equivalent class of (L_1,L_2) is reduced to the family of couples (L'_1,L'_2) satisfying

$$\begin{aligned}L'_1(q(t_p),\Delta q(t_p),t_p,h)&+L'_2(q(t_{p+1}),\nabla q(t_{p+1}),t_{p+1},h)\\&=L_1(q(t_p),\Delta q(t_p),t_p,h)+L_2(q(t_{p+1}),\nabla q(t_{p+1}),t_{p+1},h)+\Delta(f(q,\cdot,h))(t_p)+g(t_p,h),\end{aligned} \tag{IV.3.77}$$

for every $\mathbb{T}\in\mathscr{T}^{\mathrm{f}}_{\mathrm{uni}}$, every $q\in\mathrm{C}(\mathbb{T},\mathbb{R})$, every $p=0,\dots,N-1$ and for some maps f and g.

Note that for every $\mathbb{T}\in\mathscr{T}^{\mathrm{f}}_{\mathrm{uni}}$, every $q\in\mathrm{C}(\mathbb{T},\mathbb{R})$ and every $p=0,\dots,N-1$, it holds

$$\begin{aligned}L_1(q(t_p),\Delta q(t_p),t_p,h)&+L_2(q(t_{p+1}),\nabla q(t_{p+1}),t_{p+1},h)\\&=\frac{1}{2}\left[\alpha q(t_p)\frac{\Delta q(t_p)}{h}+\beta q(t_p)\Delta q(t_p)+\gamma q(t_p)^2-\alpha q(t_{p+1})\frac{\Delta q(t_p)}{h}+\beta q(t_{p+1})\Delta q(t_p)\right]\\&=\frac{1}{2}\Big[-\alpha(\Delta q(t_p))^2-\beta h\Delta q(t_p)+\gamma q(t_p)^2\Big].\end{aligned} \tag{IV.3.78}$$

Then, let us choose $L'_1(x,v,t,\xi)=(-\alpha v^2-\beta\xi v+\gamma x^2)/2$ and $L'_2=f=g=0$. Then, (L'_1,L'_2) belongs to the equivalent class of (L_1,L_2) and the discrete Euler-Lagrange equation associated with (L'_1,L'_2) is the discrete oscillator equation. Moreover, (L'_1,L'_2) admits a continuum limit when $\xi\to 0$ and the continuum Lagrangian associated is given by $L(x,v,t)=\lim_{\xi\to 0^+}L'_1(x,v,t,\xi)+\lim_{\xi\to 0^+}L'_2(x,v,t,\xi)=(\gamma x^2-\alpha v^2)/2$. As a consequence, we conclude that the discrete oscillator equation (with $\beta_1+\beta_2=0$) is a variational integrator for the continuous oscillator equation without friction given in Example IV.1.

IV.3.5 Proofs of Lemmas IV.3, IV.4 and IV.5

This section is devoted to the detailed proofs of Lemmas IV.3, IV.4 and IV.5.

Lemma IV.6 (Lemma IV.3)**.** *Let P be a second order finite differences operator satisfying the discrete Helmholtz condition. Then, for every $\mathbb{T}\in\mathscr{T}^{\mathrm{f}}_{\mathrm{uni}}$ and every $q\in\mathrm{C}(\mathbb{T},\mathbb{R})$, $DP^{\mathbb{T}}(q)$ is self-adjoint.*

Proof. Let $\mathbb{T}\in\mathscr{T}^{\mathrm{f}}_{\mathrm{uni}}$ and let $q\in\mathrm{C}(\mathbb{T},\mathbb{R})$. Then, for every $w\in\mathrm{C}(\mathbb{T},\mathbb{R})$, it holds

$$DP^{\mathbb{T}}(q)(w)(t_p)=\frac{\partial\overline{P}}{\partial x}(\star_p)w(t_p)+\frac{\partial\overline{P}}{\partial v_1}(\star_p)\Delta w(t_p)+\frac{\partial\overline{P}}{\partial v_2}(\star_p)\nabla w(t_p)+\frac{\partial\overline{P}}{\partial u}(\star_p)\nabla\circ\Delta w(t_p), \tag{IV.3.79}$$

for every $p = 1, \ldots, N-1$ and where $\star$ is defined in the proof of Theorem IV.4. Finally, from discrete integration by parts formulas given in Lemma IV.2, we obtain for every $(w, z) \in \mathrm{C}_{00}(\mathbb{T}, \mathbb{R}) \times \mathrm{C}(\mathbb{T}, \mathbb{R})$

$$h \sum_{p=1}^{N-1} DP^{\mathbb{T}}(q)(w)(t_p) z(t_p) = h \sum_{p=2}^{N-2} \left[\frac{\partial \overline{P}}{\partial x}(\star_p) z(t_p) - \nabla \left(\frac{\partial \overline{P}}{\partial v_1}(\star) z \right)(t_p) \right. \\ \left. - \Delta \left(\frac{\partial \overline{P}}{\partial v_2}(\star) z \right)(t_p) + \nabla \circ \Delta \left(\frac{\partial \overline{P}}{\partial u}(\star) z \right)(t_p) \right] w(t_p). \quad \text{(IV.3.80)}$$

The discrete Leibniz formulas given in Lemma IV.1 give that $h \sum_{p=1}^{N-1} DP^{\mathbb{T}}(q)(w)(t_p) z(t_p)$ is equal to

$$h \sum_{p=2}^{N-2} \left[\left(\frac{\partial \overline{P}}{\partial x}(\star_p) - \nabla \frac{\partial \overline{P}}{\partial v_1}(\star)(t_p) - \Delta \frac{\partial \overline{P}}{\partial v_2}(\star)(t_p) + \nabla \circ \Delta \frac{\partial \overline{P}}{\partial u}(\star)(t_p) \right) z(t_p) \right. \\ + \left(\Delta \frac{\partial \overline{P}}{\partial u}(\star)(t_p) - \frac{\partial \overline{P}}{\partial v_2}(\star_{p+1}) \right) \Delta z(t_p) + \left(\nabla \frac{\partial \overline{P}}{\partial u}(\star)(t_p) - \frac{\partial \overline{P}}{\partial v_1}(\star_{p-1}) \right) \nabla z(t_p) \\ \left. + \frac{\partial \overline{P}}{\partial u}(\star_p) \nabla \circ \Delta z(t_p) \right] w(t_p). \quad \text{(IV.3.81)}$$

Finally, from the discrete Helmholtz condition given in Definition IV.10, we obtain that

$$h \sum_{p=1}^{N-1} DP^{\mathbb{T}}(q)(w)(t_p) z(t_p) \\ = h \sum_{p=2}^{N-2} \left[\frac{\partial \overline{P}}{\partial x}(\star_p) z(t_p) + \frac{\partial \overline{P}}{\partial v_1}(\star_p) \Delta z(t_p) + \frac{\partial \overline{P}}{\partial v_2}(\star_p) \nabla z(t_p) + \frac{\partial \overline{P}}{\partial u}(\star_p) \nabla \circ \Delta z(t_p) \right] w(t_p) \\ = h \sum_{p=2}^{N-2} DP^{\mathbb{T}}(q)(z)(t_p) w(t_p). \quad \text{(IV.3.82)}$$

The proof is complete. ☐

Lemma IV.7 (Lemma IV.4)**.** *Let P be a second order finite differences operator satisfying the discrete Helmholtz condition. Then, there exist φ, $\psi : (x, v, t, \xi) \in \mathbb{R}^3 \times \mathbb{R}^*_+ \longmapsto \varphi(x, v, t, \xi), \psi(x, v, t, \xi) \in \mathbb{R}$ such that*

$$\forall (x, v_1, v_2, t, \xi) \in \mathbb{R}^4 \times \mathbb{R}^*_+, \ \overline{P}\left(x, v_1, v_2, \frac{v_1 - v_2}{\xi}, t, \xi \right) = \varphi(x, v_1, t, \xi) + \psi(x, v_2, t, \xi). \quad \text{(IV.3.83)}$$

Proof. From the discrete Helmholtz condition, for every $(y_{-1}, y_0, y_1, y_2, t, \xi) \in \mathbb{R}^5 \times \mathbb{R}^*_+$, it holds

$$\frac{1}{\xi} \left[\frac{\partial \overline{P}}{\partial u} \left(y_1, \frac{y_2 - y_1}{\xi}, \frac{y_1 - y_0}{\xi}, \frac{y_2 - 2y_1 + y_0}{\xi^2}, t + \xi, \xi \right) \right. \\ \left. - \frac{\partial \overline{P}}{\partial u} \left(y_0, \frac{y_1 - y_0}{\xi}, \frac{y_0 - y_{-1}}{\xi}, \frac{y_1 - 2y_0 + y_{-1}}{\xi^2}, t, \xi \right) \right] \\ - \frac{\partial \overline{P}}{\partial v_1} \left(y_0, \frac{y_1 - y_0}{\xi}, \frac{y_0 - y_{-1}}{\xi}, \frac{y_1 - 2y_0 + y_{-1}}{\xi^2}, t, \xi \right) \\ - \frac{\partial \overline{P}}{\partial v_2} \left(y_1, \frac{y_2 - y_1}{\xi}, \frac{y_1 - y_0}{\xi}, \frac{y_2 - 2y_1 + y_0}{\xi^2}, t + \xi, \xi \right) = 0. \quad \text{(IV.3.84)}$$

Differentiating the last equality with respect to y_{-1} (or y_2), we obtain

$$\frac{1}{\xi^2}\frac{\partial^2 \overline{P}}{\partial v_2 \partial u}\left(y_0, \frac{y_1-y_0}{\xi}, \frac{y_0-y_{-1}}{\xi}, \frac{y_1-2y_0+y_{-1}}{\xi^2}, t, \xi\right) - \frac{1}{\xi^3}\frac{\partial^2 \overline{P}}{\partial u^2}\left(y_0, \frac{y_1-y_0}{\xi}, \frac{y_0-y_{-1}}{\xi}, \frac{y_1-2y_0+y_{-1}}{\xi^2}, t, \xi\right) + \frac{1}{\xi}\frac{\partial^2 \overline{P}}{\partial v_2 \partial v_1}\left(y_0, \frac{y_1-y_0}{\xi}, \frac{y_0-y_{-1}}{\xi}, \frac{y_1-2y_0+y_{-1}}{\xi^2}, t, \xi\right) - \frac{1}{\xi^2}\frac{\partial^2 \overline{P}}{\partial u \partial v_1}\left(y_0, \frac{y_1-y_0}{\xi}, \frac{y_0-y_{-1}}{\xi}, \frac{y_1-2y_0+y_{-1}}{\xi^2}, t, \xi\right) = 0. \quad \text{(IV.3.85)}$$

Then, we conclude that for every $(x, v_1, v_2, t, \xi) \in \mathbb{R}^4 \times \mathbb{R}_+^*$, it holds

$$\frac{1}{\xi}\frac{\partial^2 \overline{P}}{\partial v_2 \partial u}\left(x, v_1, v_2, \frac{v_1-v_2}{\xi}, t, \xi\right) - \frac{1}{\xi^2}\frac{\partial^2 \overline{P}}{\partial u^2}\left(x, v_1, v_2, \frac{v_1-v_2}{\xi}, t, \xi\right) + \frac{\partial^2 \overline{P}}{\partial v_2 \partial v_1}\left(x, v_1, v_2, \frac{v_1-v_2}{\xi}, t, \xi\right) - \frac{1}{\xi}\frac{\partial^2 \overline{P}}{\partial u \partial v_1}\left(x, v_1, v_2, \frac{v_1-v_2}{\xi}, t, \xi\right) = 0. \quad \text{(IV.3.86)}$$

Finally, let us introduce the following map:

$$\begin{array}{rccl} \ell : & \mathbb{R}^4 \times \mathbb{R}_+^* & \longrightarrow & \mathbb{R} \\ & (x, v_1, v_2, t, \xi) & \longmapsto & \overline{P}\left(x, v_1, v_2, \dfrac{v_1-v_2}{\xi}, t, \xi\right). \end{array} \quad \text{(IV.3.87)}$$

From Equality (IV.3.86), we obtain that $\partial^2 \ell / \partial v_1 \partial v_2 = 0$. As a consequence, there exist φ, ψ : $(x, v, t, \xi) \in \mathbb{R}^3 \times \mathbb{R}_+^* \longmapsto \varphi(x, v, t, \xi), \psi(x, v, t, \xi) \in \mathbb{R}$ such that $\ell(x, v_1, v_2, t, \xi) = \varphi(x, v_1, t, \xi) + \psi(x, v_2, t, \xi)$. The proof is complete. □

Lemma IV.8 (Lemma IV.5)**.** *Let P be a second order finite differences operator. If there exists a couple of Lagrangian (L_1, L_2) such that for every $\mathbb{T} \in \mathscr{T}^{\mathrm{f}}_{\mathrm{uni}}$ and for every $q \in \mathrm{C}(\mathbb{T}, \mathbb{R})$, it holds*

$$P^{\mathbb{T}}(q)(t_p) = \frac{\partial L_1}{\partial x}(q(t_p), \Delta q(t_p), t_p, h) + \frac{\partial L_2}{\partial x}(q(t_p), \nabla q(t_p), t_p, h) - \nabla\left(\frac{\partial L_1}{\partial v}(q, \Delta q, \cdot, h)\right)(t_p) - \Delta\left(\frac{\partial L_2}{\partial v}(q, \nabla q, \cdot, h)\right)(t_p) = 0, \quad \text{(IV.3.88)}$$

for every $p = 2, \dots, N-2$, then P is a second order discrete Euler-Lagrange operator associated with (L_1, L_2).

Démonstration. Let $\mathbb{T} \in \mathscr{T}^{\mathrm{f}}_{\mathrm{uni}}$ and $q \in \mathrm{C}(\mathbb{T}, \mathbb{R})$. It is sufficient to prove that Equality (IV.3.88) is also valid for $p = 1$ (the validity for $p = N-1$ can be derived in a similar way). Let us denote by $\mathbb{T}' \in \mathscr{T}^{\mathrm{f}}_{\mathrm{uni}}$ the finite uniform time scale defined by $\mathbb{T}' = \{t'_p\}_{p=0,\dots,N}$ where $t'_p = t_p - h$ for every $p = 0, \dots, N$. Similarly, we denote by $q' \in \mathrm{C}(\mathbb{T}', \mathbb{R})$ defined by $q'(t'_p) = q(t'_p)$ for every $p = 1, \dots, N$ and $q'(t'_0) = 0$. Then, we have:

$$P^{\mathbb{T}}(q)(t_1) = \overline{P}(q(t_1), \Delta q(t_1), \nabla q(t_1), \nabla \circ \Delta q(t_1), t_1, h) = \overline{P}(q'(t'_2), \Delta q'(t'_2), \nabla q'(t'_2), \nabla \circ \Delta q'(t'_2), t'_2, h) = P^{\mathbb{T}'}(q')(t'_2). \quad \text{(IV.3.89)}$$

We can apply Equality (IV.3.88) for $\mathbb{T}'$, q' and $p = 2$. Then, it holds:

$$P^{\mathbb{T}}(q)(t_1) = \frac{\partial L_1}{\partial x}(q'(t'_2), \Delta q'(t'_2), t'_2, h) + \frac{\partial L_2}{\partial x}(q'(t'_2), \nabla q'(t'_2), t'_2, h) - \nabla\left(\frac{\partial L_1}{\partial v}(q', \Delta q', \cdot, h)\right)(t'_2) - \Delta\left(\frac{\partial L_2}{\partial v}(q', \nabla q', \cdot, h)\right)(t'_2) = 0. \quad \text{(IV.3.90)}$$

Finally, replacing the value of q' by q and (t'_1, t'_2, t'_3) by (t_0, t_1, t_2), we obtain:

$$P^{\mathbb{T}}(q)(t_1) = \frac{\partial L_1}{\partial x}(q(t_1), \Delta q(t_1), t_1, h) + \frac{\partial L_2}{\partial x}(q(t_1), \nabla q(t_1), t_1, h) \\ - \nabla \left(\frac{\partial L_1}{\partial v}(q, \Delta q, \cdot, h) \right)(t_1) - \Delta \left(\frac{\partial L_2}{\partial v}(q, \nabla q, \cdot, h) \right)(t_1) = 0. \quad \text{(IV.3.91)}$$

The proof is complete. □

IV.4 Comments and perspectives

We conclude this chapter with some remarks and outlooks.

1. Section IV.3.2 shows that every second order discrete Euler-Lagrange equation (EL^{d}) is a second order finite differences equation. Actually, the form of (EL^{d}) and the strategy pursued in the continuous case motivate such a definition (Definition IV.7) of second order finite differences equation. Nevertheless, it can be noted that Theorem IV.4 can be applied on general discrete equations. Indeed, in order to apply Theorem IV.4 on a general discrete equation of type $\overline{Q}(q(t_{p-1}), q(t_p), q(t_{p+1}), t_{p-1}, t_p, t_{p-1}, h) = 0$, one has just to first rewrite this discrete equation as the second order finite differences equation associated with the map $\overline{P}(x, v_1, v_2, u, t, \xi) = \overline{Q}(x - \xi v_2, x, x + \xi v_1, t - \xi, t, t + \xi, \xi)$. Finally, note that a general discrete equation depending on other elements than $q(t_{p+1})$, $q(t_p)$, $q(t_{p-1})$, t_{p+1}, t_p, t_{p-1} and h (like t_{p-2} or $q(t_{p+2})$) cannot be a second order discrete Euler-Lagrange equation of type (EL^{d}) anyway.
2. A short-term perspective is to derive all the analogous results of this chapter in the higher-dimensional case $n \geq 2$. Nevertheless, we recall that the explicit continuous Helmholtz condition (see Definition IV.5 in the case $n = 1$) is a little more complicated in the higher-dimensional case $n \geq 2$. Namely, it is made up of several equalities in contrary to only one in the case $n = 1$. Let us mention that the same observation can be made at the discrete level. However, in order to overcome all these equalities, one can still formulate the discrete Helmholtz condition in terms of self-adjointness of differentials of the discrete operator.
3. Despite that the results of this chapter provide some answers to a discrete Helmholtz inverse problem regarding variational integrators, they are not totally satisfactory. Indeed, in the case where a discrete equation satisfies the discrete Helmholtz condition, despite that we have provided a method to construct all discrete Lagrangian formulations leading to this discrete Euler-Lagrange equation, there is no guarantee that one of them admits a continuum limit or, more generally, corresponds to a discretization of a continuous Lagrangian formulation. Consequently, we cannot directly validate (or invalidate) that this discrete Euler-Lagrange equation corresponds to a variational integrator.

   This issue shows that the following problem is still open: find a necessary and sufficient condition for a second order finite differences equation to be a variational integrator.

   We conclude this remark with the following inverse problem even more ambitious: find a necessary and sufficient condition for a second order finite differences equation to be a second order discrete Euler-Lagrange equation associated with a couple of Lagrangian independent of ξ. In particular, such a discrete Lagrangian formulation corresponds to the discretization of a continuous one.
4. In a more general point of view, it would be interesting to formulate and solve the Helmholtz inverse problem posed on a general time scale. Nevertheless, as it is well known in time scale calculus, many difficulties emerge concerning Δ- and ∇-derivatives of compositions, which are instrumental in the usual derivation of Helmholtz conditions. The formulation of such a Helmholtz inverse problem requires a differential form of the Euler-Lagrange equation. If one considers the differential Euler-Lagrange equation obtained in Chapter II, one has to assume the ∇-differentiability of the forward jump operator σ and due to the symmetry of the problem, one has to also assume the Δ-differentiability of the backward jump operator ρ. We mention that the differential Euler-Lagrange equation obtained in [114, Remark 4] can be considered without assuming these two previous hypotheses. To conclude, despite of some direct obstructions (as above cited), the resolution of a Helmholtz inverse problem posed on time scale seems quite possible.

Deuxième partie

Contributions au calcul des variations et au Principe du Maximum de Pontryagin Faible en calcul fractionnaire

Chapitre V

Rappels sur le calcul fractionnaire et son application en calcul des variations

La majorité des rappels qui suivent en Sections V.1 et V.2 est extraite des ouvrages [141, 190] de A. Kilbas et al. auxquels nous renvoyons pour une étude approfondie de la théorie du calcul fractionnaire.

La Section V.3 est consacrée à quelques rappels sur l'application du calcul fractionnaire en calcul des variations. Nous réferrons le lecteur intéressé par ce sujet au livre récent [157] de A. Malinowska et D. Torres. En particulier, une équation d'Euler-Lagrange fractionnaire (Section V.3.1) et un théorème de type Noether (Section V.3.2) sont rappelés. Une brève discussion est proposée en Section V.3.2 autour d'un résultat issu de Bourdin L., Cresson J. et Greff I., A continuous/discrete fractional Noether's theorem, Communications in Nonlinear Science and Numerical Simulation, 18(4):878-887, 2013 *portant sur l'obtention d'une constante de mouvement explicite à partir de ce théorème de type Noether.*

V.1 Introduction

The *fractional calculus* is the mathematical area dealing with the generalization of the classical notions of integral and derivative to any real order. In this introduction, we give a brief panorama of the historical steps of the elaboration of this concept and we present some past and current applications in various scientific branches, in particular in calculus of variations. Let us first give some reference works on fractional calculus that have inspired this introduction.

The first reference book [179] on fractional calculus, developing some mathematical aspects and applications, was written by K. Oldham and J. Spanier in 1974. In 1993, K. Miller and B. Ross have treated fractional differential equations in [168]. The monographs [190] of A. Kilbas, O. Marichev and S. Samko in 1987 and [141] of A. Kilbas, H. Srivastava and J. Trujillo in 2006 are essential books on fractional calculus, treating on mathematical aspects with rigorous proofs, in particular concerning regularity issues, on fractional differential equations and containing some applications. We also refer to [81, 127, 185] and some chapters of [97, 110, 162] containing handy introductions to fractional calculus. Finally, we also mention [152] for the recent history of the fractional calculus.

Brief historical overview of the elaboration of the fractional calculus. The fractional calculus seems to be originally introduced in 1695 in a letter written by Leibniz to L'Hospital where he suggested to generalize his celebrated formula of the k^{th}-derivative of a product (with $k \in \mathbb{N}$) to any positive real k. In another letter to Bernoulli, Leibniz mentioned derivatives of *general order*.

In 1730, Euler was the second renowned mathematician to work on this topic. Indeed, after introducing his well known Gamma function (denoted by Γ) that generalizes the factorial function to any positive real, Euler defined the derivative of order α of the function $t \longmapsto t^\beta$ in the case $0 < \alpha < \beta + 1$. Precisely, his idea was to generalize the formula

$$\left(\frac{d}{dt}\right)^k [t^{k_1}] = \frac{k_1!}{(k_1-k)!} t^{k_1-k}, \tag{V.1}$$

valid for every couple of integers (k, k_1) such that $k \leq k_1$, by introducing the following definition:

$$\left(\frac{d}{dt}\right)^{\alpha}[t^{\beta}] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}, \tag{V.2}$$

for every $0 < \alpha < \beta+1$. Then, he naturally extended this fractional derivative to every function written in power series. We will see that this approach is still very close to the most recurrent notion of fractional derivative in the recent literature, namely the notion of Riemann-Liouville (see Example V.1). The name *fractional* seems to come from the work of Euler, since the author considered rational numbers α and β. Nowadays, the modern fractional calculus considers real orders (even complex orders) but the appellation *fractional* has been kept.

In 1822, Fourier defined a fractional derivative for functions written in Fourier series. Precisely, Fourier suggested to generalize the formula

$$\left(\frac{d}{dt}\right)^{k}[\cos(t)] = \cos\left(t + \frac{k}{2}\right), \tag{V.3}$$

valid for every integer k, to any positive real k. Applying the similar idea for the sine function, Fourier extended this fractional derivative to every function written in Fourier series.

Liouville, author of eight articles on fractional calculus between 1832 and 1837, is the first mathematician who worked in detail on this topic. In particular, Liouville extended the formula

$$\left(\frac{d}{dt}\right)^{k}[e^{at}] = a^k e^{at}, \tag{V.4}$$

valid for every integer k, to any positive real k. Then, he naturally extended this fractional derivative to every function that can be written in exponential series.

In 1847, instead of generalizing a formula of k^{th}-derivative, Riemann suggested another strategy based on a generalization of a formula of k^{th}-integral. Hence, a fractional integral is obtained and then, the corresponding fractional derivative is defined as the composition between the classical derivative and the fractional integral. We refer to Sections V.2.1 and V.2.2 for more details. Note that Riemann did not completely introduce this approach. Actually, the definition of the most recurrent notion of fractional operator (namely the notion of Riemann-Liouville) was originally introduced in the work of Sonin in 1869.

As we will see, the fractional derivative of Riemann-Liouville presents a singularity at the initial point. As a consequence, Caputo suggested in 1967 a slight modification. He defined a fractional derivative as a fractional integral of Riemann-Liouville composed with a classical derivative (in this exact order). Hence, he obtained a similar fractional derivative with more interesting regularity properties. The notion of Caputo is also a very recurrent tool in the modern literature on fractional calculus. In Part B, we mainly study variational problems involving fractional derivatives of Caputo type.

Many notions of fractional operator have been introduced during the XIX^{th} and the XX^{th} centuries. The work of Grünwald and Letnikov in 1867-68 introduces another approach based on a limit of finite differences, analogously to the definition of the classical derivative. This notion is particularly suitable in order to define discrete fractional operators and to study discrete counterparts of fractional problems. We refer to Chapters VIII and IX for more details and examples.

All these notions are not disconnected. Actually, in most cases, it can be proved that two different notions coincide or are correlated by an explicit formula. For example, fractional operators of Riemann-Liouville and Grünwald-Letnikov coincide. On the other hand, Caputo fractional derivatives and Riemann-Liouville's ones are linked by a simple formula, see Remark V.6. Finally, let us mention that the first works of Euler, Fourier and Liouville are encompassed (depending on some parameters) by the modern fractional calculus.

Some applications of fractional calculus in various scientific branches. For a long time, the fractional calculus was only considered as a pure mathematical branch. In 1974, a first conference on fractional calculus has been organized by B. Ross at the University of New Haven (Connecticut, USA). Since then, the fractional calculus and its applications experience a boom in several scientific fields. The uses are so varied that it seems difficult to give a complete overview of the current researches involving fractional operators. We refer to [108, 189] for a large panorama of applications of fractional calculus.

We first mention the work [1] of Abel in 1823 solving the generalized version of the tautochrone problem (also called Abel's mechanical problem). He considered a particle moving without friction on a curve in uniform gravity. Abel proved that the final time (where the particle arrives at the lowest point of the curve), written dependently on the altitude, is equal to the fractional derivative (in the Caputo sense) of order $1/2$ of the curvilinear abscissa.

The fractional calculus is widely applied in the physical context of anomalous diffusion. For instance, K. Oldham and J. Spanier proved in 1970 that the diffusion flux is proportional to the fractional derivative (in the Caputo sense) of order $1/2$ of the physical parameter (like temperature), see [178]. The fractional derivatives substitute the classical ones in the diffusion equations in order to model the motion of particles in some anomalous diffusion phenomena. Precisely, in the case where the diffusion front grows linearly with respect to time (*i.e.* $t \longmapsto ct$), the Brownian motion of particles can be described by the usual diffusion equation. Nevertheless, in more complex systems, the diffusion front grows non linearly with respect to time but grows in terms of $t \longmapsto ct^\alpha$ with $\alpha > 0$. The motion of particles is not Brownian any more but it can be described by a stochastic process called Continuous Time Random Walk (CTRW) introduced by E. Montroll and G. Weiss in [170]. Then, the usual diffusion equation is not adapted any more but many generalizations have been considered. In particular, the classical derivatives are often replaced by fractional derivatives in time and/or in space. W. Wyss is the first one to study fractional diffusion equation in [201]. We refer to the complete review [167] of R. Metzler and J. Klafter for details on anomalous diffusion, CTRW and fractional diffusion equations. Anomalous diffusion phenomena appear in a lot of varied fields (and consequently, so does the fractional calculus). For example, we refer to [95, 208, 209] for applications of fractional operators in fluid mechanics in heterogeneous porous media. Otherwise, numerous chaotic systems present anomalous diffusion phenomena. G. Zaslavsky studied them in detail and contributed to the elaboration of fractional diffusion equations modelling these phenomena, see [203, 204].

The fractional operators, due to their non local characteristic, are also used in order to take into account of memory effects. For instance, some materials, like polymers (gum, rubber), present an intermediate behaviour between viscosity and elasticity. Note that the so-called viscoelasticity is modelled by a fractional differential equation of order $\alpha = 1/2$ in [24, 25]. We also refer to [81, 183] for more details concerning viscoelasticity and fractional derivatives.

Hence, the fractional calculus has countless applications. We refer to the following studies in wave mechanic [15], economy [67], biology [96, 155], acoustic [105], thermodynamic [109], probability [148], etc. In a more general point of view, fractional differential equations are even considered as an alternative model to non-linear differential equations, see [42].

Despite their ubiquity in many scientific areas, there are not so many examples where the use of fractional derivatives can be fully justified. This comes from several difficulties connected with the signification of the fractional calculus itself and in particular from the fact that the dynamical meaning of fractional derivatives (independently of the definition which is used) is not so clear. However, A. Stanislavsky [194] proved that the introduction of a stochastic internal time (a "slow" time) transforms the usual derivative operator into a Caputo fractional one. Hence, the transition classical/fractional can be the effect of a change of time. Nevertheless, the main drawback is still to connect such a new time with the underlying problem.

Fractional calculus of variations. Thanks to the classical Helmholtz condition, we have seen in Chapter IV that it is impossible to write the oscillator equation with friction as a classical Euler-Lagrange equation. In a more general point of view, F. Riewe raised in 1996-97 the following problematic in [187, 188]:

It is a strange paradox that the most advanced methods of classical mechanics deal only with conservative systems, while almost all classical processes observed in the physical world are nonconservative.

Then, he suggested to incorporate fractional operators in the calculus of variations in order to derive fractional variational structures for dissipative systems. Despite his idea was very relevant, his results are not totally satisfactory, we refer to Section V.3.3 for a discussion on this issue. In the same spirit, J. Cresson and P. Inizan have recently introduced in [71] a similar but more conclusive fractional variational framework based on the splitting in two of the variable of the Lagrangian functional. This framework is so-called *asymmetric*. As an application, the authors of [70] obtain an asymmetric fractional variational structure for the convection-diffusion equation. We give more details in Section V.3.3.

Since the study of F. Riewe, a comprehensive literature has been devoted to necessary optimality conditions of type Euler-Lagrange in several directions, see [6, 14, 16–19, 27, 69, 174–177]. Concerning the state of the art on fractional calculus of variations and associated fractional Euler-Lagrange equations, we refer to the recent book [157] of D. Torres and A. Malinowska. Several works on Noether-type theorems for fractional Euler-Lagrange equations can be found in [21, 69, 89, 92, 172]. But, in each of these papers, authors can only provide a relation (functional or integral) that does not ensure the existence of an explicit constant of motion, which is the main concern of the classical Noether's theorem. The difficulty comes from the non existence of a simple fractional Leibniz rule. A result in this direction is introduced in [48] and is briefly addressed in Section V.3.2.

Organization of the chapter. In Section V.2, we follow the strategy of Riemann leading to the standard definitions of fractional operators of Riemann-Liouville and Caputo. Then, basic properties of these operators are recalled, in particular on regularity issues. Section V.3 is devoted to the fractional calculus of variations where a fractional Euler-Lagrange equation characterizing the critical points of a functional involving Caputo fractional derivative is derived, see Section V.3.1. Then, corresponding Noether-type theorems are recalled in Section V.3.2. Finally, a discussion on the works of F. Riewe, J. Cresson and P. Inizan concludes this chapter in Section V.3.3.

V.2 Recalls on fractional operators of Riemann-Liouville and Caputo

Section V.2.1 is devoted to give formulations of k^{th}-integrals and k^{th}-derivatives (with $k \in \mathbb{N}^*$) of a function. These formulations are generalized to any positive real k in Section V.2.2 and then, the definitions of fractional operators of Riemann-Liouville and Caputo are obtained. Finally, Section V.2.3 recalls basic regularity properties of these operators. All definitions and results are extracted from [141, 190].

In what follows, $n \in \mathbb{N}^*$ denotes the dimension and for every interval $I \subset \mathbb{R}$ and every $k \in \mathbb{N}^*$, we denote by $\mathrm{C}^k(I,\mathbb{R}^n)$ the set of all functions $q : I \longrightarrow \mathbb{R}^n$ of class $\mathscr{C}^k$ on I. Finally, for every $a \in \mathbb{R}$ and every $b \in \mathbb{R}$, we introduce the two following sets:

$$\mathbb{I}_{a+} = \{I \subset [a,+\infty[\text{ interval such that } \inf I = a \text{ and } I\backslash\{a\} \neq \emptyset\} \tag{V.5}$$

and

$$\mathbb{I}_{b-} = \{I \subset]-\infty,b] \text{ interval such that } \sup I = b \text{ and } I\backslash\{b\} \neq \emptyset\}. \tag{V.6}$$

For example, let $(a,b) \in \mathbb{R}^2$ such that $a < b$. Then, $]a,b] \in \mathbb{I}_{a+} \cap \mathbb{I}_{b-}$ and $]-\infty,b] \in \mathbb{I}_{b-}$.

V.2.1 Preliminaries on k^{th}-integrals and k^{th}-derivatives

In this section, the calculations are made formally, without any consideration on regularity issues. Let $a \in \mathbb{R}$, $I \in \mathbb{I}_{a+}$ and let $q : I \longrightarrow \mathbb{R}^n$ be a sufficiently smooth function. Let us denote by $\mathrm{I}^1_{a+}[q]$ the antiderivative of q vanishing at $t = a$ *i.e.*

$$\forall t \in I,\ \mathrm{I}^1_{a+}[q](t) = \int_a^t q(\tau)\ d\tau. \tag{V.7}$$

Then, denoting by $\mathrm{I}^k_{a+} = \mathrm{I}^1_{a+} \circ \ldots \circ \mathrm{I}^1_{a+}$ (k-times) for every $k \in \mathbb{N}^*$, one can easily prove by induction and from the Fubini's theorem that

$$\forall t \in I,\ \mathrm{I}^k_{a+}[q](t) = \frac{1}{(k-1)!} \int_a^t (t-\tau)^{k-1} q(\tau)\ d\tau. \tag{V.8}$$

Remark V.1. Note that, in contrary to the classical derivative d/dt, I^k_{a+} is not a local operator since $\mathrm{I}^k_{a+}[q](t)$ depends on all values $q(\tau)$ with $\tau \leq t$.

Then, for every $k \in \mathbb{N}^*$, $\mathrm{I}^k_{a+}[q]$ is usually called the *left* integral with inferior limit a of order k of q. Finally, for every $k \in \mathbb{N}^*$, recall that the k^{th}-derivative of q satisfies both

$$\forall t \in I,\ \left(\frac{d}{dt}\right)^k [q](t) = \left(\frac{d}{dt}\right)^{k+1} \left[\mathrm{I}^1_{a+}[q]\right](t) \tag{V.9}$$

and, if $a \in I$, we also have

$$\forall t \in I,\ \left(\frac{d}{dt}\right)^k [q](t) - \left(\frac{d}{dt}\right)^k [q](a) = \mathrm{I}^1_{a+} \left[\left(\frac{d}{dt}\right)^{k+1} [q]\right](t). \tag{V.10}$$

We can derive *right* counterparts of the previous results. Namely, let $b \in \mathbb{R}$, $I \in \mathbb{I}_{b-}$ and let $q : I \longrightarrow \mathbb{R}^n$ be a sufficiently smooth function. We denote by $\mathrm{I}^1_{b-}[q]$ the minus antiderivative of q vanishing at $t = b$ *i.e.*

$$\forall t \in I,\ \mathrm{I}^1_{b-}[q](t) = \int_t^b q(\tau)\ d\tau. \tag{V.11}$$

Then, denoting by $\mathrm{I}^k_{b-} = \mathrm{I}^1_{b-} \circ \ldots \circ \mathrm{I}^1_{b-}$ (k-times) for every $k \in \mathbb{N}^*$, one can easily prove that

$$\forall t \in I,\ \mathrm{I}^k_{b-}[q](t) = \frac{1}{(k-1)!} \int_t^b (\tau - t)^{k-1} q(\tau)\ d\tau. \tag{V.12}$$

For every $k \in \mathbb{N}^*$, $\mathrm{I}^k_{b-}[q]$ is usually called the *right* integral with superior limit b of order k of q. Moreover, the k^{th}-derivative of q satisfies both

$$\forall t \in I,\ \left(-\frac{d}{dt}\right)^k [q](t) = \left(-\frac{d}{dt}\right)^{k+1} \left[\mathrm{I}^1_{b-}[q]\right](t) \tag{V.13}$$

and, if $b \in I$, we also have

$$\forall t \in I,\ \left(-\frac{d}{dt}\right)^k [q](t) - \left(-\frac{d}{dt}\right)^k [q](b) = \mathrm{I}^1_{b-} \left[\left(-\frac{d}{dt}\right)^{k+1} [q]\right](t). \tag{V.14}$$

In Section V.2.2, we generalize Equations (V.12), (V.13) and (V.14) to any positive real k in order to define *right* fractional operators of Riemann-Liouville and Caputo.

V.2.2 Definitions of fractional operators of Riemann-Liouville and Caputo

Recall that the Gamma function of Euler, denoted by Γ and given by

$$\forall \alpha > 0,\ \Gamma(\alpha) = \int_0^{+\infty} \tau^{\alpha-1} e^{-\tau}\ d\tau, \tag{V.15}$$

generalizes the factorial function to any positive real in the sense that Γ is defined on whole $\mathbb{R}^*_+$ and satisfies $\Gamma(k) = (k-1)!$ for every $k \in \mathbb{N}^*$.

In what follows, let $[\alpha]$ be the floor of α for any $\alpha > 0$.

Fractional integrals and derivatives of Riemann-Liouville. The left and right fractional integrals and derivatives of Riemann-Liouville are generalizations of the formulas given by Equations (V.8), (V.9), (V.12) and (V.13). They are given with the help of the Gamma function of Euler in the following standard definitions.

Definition V.1. Let $a \in \mathbb{R}$, $I \in \mathbb{I}_{a+}$ and $q : I \longrightarrow \mathbb{R}^n$. The left fractional integral of Riemann-Liouville with inferior limit a of order $\alpha > 0$ of q is given by

$$\forall t \in I,\ \mathrm{I}^{\alpha}_{a+}[q](t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} q(\tau)\, d\tau, \tag{V.16}$$

provided that the right-hand side term is well defined. The left fractional derivative of Riemann-Liouville with inferior limit a of order $\alpha > 0$ of q is given by

$$\forall t \in I,\ \mathrm{D}^{\alpha}_{a+}[q](t) = \left(\frac{d}{dt}\right)^{[\alpha]+1} \left[\mathrm{I}^{[\alpha]+1-\alpha}_{a+}[q]\right](t), \tag{V.17}$$

provided that the right-hand side term is well defined.

Definition V.2. Let $b \in \mathbb{R}$, $I \in \mathbb{I}_{b-}$ and $q : I \longrightarrow \mathbb{R}^n$. The right fractional integral of Riemann-Liouville with superior limit b of order $\alpha > 0$ of q is given by

$$\forall t \in I,\ \mathrm{I}^{\alpha}_{b-}[q](t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau-t)^{\alpha-1} q(\tau)\, d\tau, \tag{V.18}$$

provided that the right-hand side term is well defined. The right fractional derivative of Riemann-Liouville with superior limit b of order $\alpha > 0$ of q is given by

$$\forall t \in I,\ \mathrm{D}^{\alpha}_{b-}[q](t) = \left(-\frac{d}{dt}\right)^{[\alpha]+1} \left[\mathrm{I}^{[\alpha]+1-\alpha}_{b-}[q]\right](t), \tag{V.19}$$

provided that the right-hand side term is well defined.

The above definitions recover Equalities (V.8), (V.9), (V.12) and (V.13) whenever $\alpha = k \in \mathbb{N}^*$. Hence, the operators I^{α}_{a+}, D^{α}_{a+}, I^{α}_{b-} and D^{α}_{b-} are generalizations to any positive real order of the classical notions of integral and derivative.

Example V.1. The left fractional integral and derivative of Riemann-Liouville of order $\alpha > 0$ for $q(t) = (t-a)^{\beta}$ with $\beta > -1$ are given by:
- $\mathrm{I}^{\alpha}_{a+}[q](t) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}(t-a)^{\beta+\alpha}$ for every $t \in [a, +\infty[$ if $\beta + \alpha \geq 0$ and for every $t \in]a, +\infty[$ if $\beta + \alpha < 0$;
- $\mathrm{D}^{\alpha}_{a+}[q](t) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha}$ for every $t \in [a, +\infty[$ if $\beta - \alpha \geq 0$ and for every $t \in]a, +\infty[$ if $-1 < \beta - \alpha < 0$.

Remark V.2. Similar results can be derived for right fractional integral and derivative of Riemann-Liouville of order $\alpha > 0$ with $q(t) = (b-t)^{\beta}$ and $\beta > -1$.

From Example V.1, some remarks can be done on fractional derivatives of Riemann-Liouville:

1. The fractional derivative of a constant in the Riemann-Liouville sense is not zero! For instance, from Example V.1, it holds $\mathrm{D}^{1/2}_{a+}[1](t) = (\Gamma(1/2)\sqrt{t-a})^{-1} \neq 0$ for every $t > a$.
2. From the previous point, we also remark that the left fractional derivative of Riemann-Liouville presents a singularity in $t = a$, even for a constant function. Actually, it is well known in fractional calculus that this singularity emerges for each regular function q such that $q(a) \neq 0$. Indeed, let q be a regular map written in power series *i.e.* $q(t) = \sum_{k \geq 0} q^{(k)}(a)(t-a)^k/k!$ for every $t \geq a$. From Example V.1 and for every $0 < \alpha < 1$, we obtain that $\mathrm{D}^{\alpha}_{a+}[q]$ is well defined in $t = a$ if and only if $q(a) = 0$.

The previous remarks both motivate the introduction of Caputo fractional derivatives.

Fractional derivatives of Caputo. The fractional derivatives in the sense of Caputo, also based on fractional integrals of Riemann-Liouville, are given as follows.

Definition V.3. Let $a \in \mathbb{R}$, $I \in \mathbb{I}_{a+}$ and $q : I \longrightarrow \mathbb{R}^n$. The left fractional derivative of Caputo with inferior limit a of order $\alpha > 0$ of q is given by

$$\forall t \in I,\ {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q](t) = \mathrm{I}^{[\alpha]+1-\alpha}_{a+}\left[\left(\frac{d}{dt}\right)^{[\alpha]+1}[q]\right](t), \tag{V.20}$$

provided that the right-hand side term is well defined.

Definition V.4. Let $b \in \mathbb{R}$, $I \in \mathbb{I}_{b-}$ and $q : I \longrightarrow \mathbb{R}^n$. The right fractional derivative of Caputo with superior limit b of order $\alpha > 0$ of q is given by

$$\forall t \in I,\ {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{b-}[q](t) = \mathrm{I}^{[\alpha]+1-\alpha}_{b-}\left[\left(-\frac{d}{dt}\right)^{[\alpha]+1}[q]\right](t), \tag{V.21}$$

provided that the right-hand side term is well defined.

Note that the above definitions recover Equalities (V.10) and (V.14) whenever $\alpha = k \in \mathbb{N}^*$. As a consequence, the fractional derivatives of Caputo are not exactly generalizations of the classical derivative. Actually, ${}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}$ is a generalization of the operator $q \longmapsto (d/dt)^k[q] - (d/dt)^k[q](a)$ and ${}_{\mathrm{c}}\mathrm{D}^{\alpha}_{b-}$ of the operator $q \longmapsto (-d/dt)^k[q] - (-d/dt)^k[q](b)$. However, let us mention that the Caputo fractional derivative is an approximation of the classical derivative d/dt when $0 < \alpha < 1$ and $\alpha \to 1$, see [190, p.51].

Let us make the following comments:

1. In contrary to the Riemann-Liouville's case, a Caputo fractional derivative of a constant is zero. The notion of Caputo preserves this important characteristic of the classical derivative.
2. In contrary to the Riemann-Liouville's case, the Caputo fractional derivative does not present singularities. Indeed, we recall that, for every $0 < \alpha < 1$ and under other appropriate hypotheses, the equality ${}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q] = \mathrm{D}^{\alpha}_{a+}[q - q(a)]$ holds, see Section V.2.3. Consequently, the singularity of the Riemann-Liouville fractional derivative is cancelled by "forcing" the value $q(a) = 0$.

Finally, let us mention that the fractional integrals are not local operators and then, the fractional derivatives are not local neither. Note that the class of functions considered is strongly dependent on this characteristic. Indeed, from Example V.1, we see that $\mathrm{I}^{\alpha}_{a+}[(\cdot - a)^{\beta}]$ is well defined on $]a, +\infty[$ if and only if $\beta > -1$ in order to ensure the integrability of the integrand. As a consequence, for $\alpha > 0$ and $\alpha \notin \mathbb{N}$, $\mathrm{D}^{\alpha}_{a+}[(\cdot - a)^{\beta}]$ can be considered only if $\beta > -1$ while $(\cdot - a)^{\beta}$ is infinitely differentiable for every $\beta \in \mathbb{R}$.

V.2.3 Properties of fractional operators of Riemann-Liouville and Caputo for $0 < \alpha < 1$

The whole Part B deals with fractional variational problems with $0 < \alpha < 1$. Many difficulties already emerge at this step. Moreover, many results in the case $0 < \alpha < 1$ can be extended to the case $1 < \alpha$ and $\alpha \notin \mathbb{N}$ by assuming more regularities on considered curves and by rewriting α as $[\alpha] + (\alpha - [\alpha])$ where $0 < \alpha - [\alpha] < 1$.

Consequently, in what follows, we only focus on the case $0 < \alpha < 1$. In this special case, the fractional derivatives of Riemann-Liouville and Caputo simply rewrite:

$$\mathrm{D}^{\alpha}_{a+}[q] = \frac{d}{dt}\Big[\mathrm{I}^{1-\alpha}_{a+}[q]\Big],\ \mathrm{D}^{\alpha}_{b-}[q] = -\frac{d}{dt}\Big[\mathrm{I}^{1-\alpha}_{b-}[q]\Big],\ {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q] = \mathrm{I}^{1-\alpha}_{a+}[\dot{q}] \text{ and } {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{b-}[q] = -\mathrm{I}^{1-\alpha}_{b-}[\dot{q}], \tag{V.22}$$

where $\dot{q}$ denotes the derivative of q *i.e.* $d/dt[q]$.

Moreover, in this section, we only consider functions $q : I \longrightarrow \mathbb{R}^n$ with $I = [a, b]$ where $a < b$. We denote by $\mathrm{H}^{\lambda}([a, b], \mathbb{R}^n)$ the set of all Hölder continuous functions on $[a, b]$ with exponent $0 < \lambda \leq 1$. $\mathrm{C}_a([a, b], \mathbb{R}^n)$ denotes the set of all functions $q \in \mathrm{C}([a, b], \mathbb{R}^n)$ such that $q(a) = 0$. Similarly, we denote by $\mathrm{C}_b([a, b], \mathbb{R}^n)$ the set of all functions $q \in \mathrm{C}([a, b], \mathbb{R}^n)$ such that $q(b) = 0$. We define in a similar way the sets $\mathrm{AC}_a([a, b], \mathbb{R}^n)$, $\mathrm{AC}_b([a, b], \mathbb{R}^n)$, $\mathrm{H}^{\lambda}_a([a, b], \mathbb{R}^n)$ and $\mathrm{H}^{\lambda}_b([a, b], \mathbb{R}^n)$. Finally, for every $1 \leq p \leq \infty$, $\mathrm{L}^p([a, b], \mathbb{R}^n)$ denotes the usual Lebesgue space of p-integrable functions endowed with its usual norm $\|\cdot\|_{\mathrm{L}^p([a,b],\mathbb{R}^n)}$.

Regularity of fractional integrals and derivatives. This paragraph is concerned with regularity issues of left fractional integral and derivatives. The following properties are also valid for the *right* fractional operators. All the following results are extracted from [141, p.69–79] and from [190, p.32–103].

We first address properties for integrable functions q. The following statements are instrumental in order to derive a Tonelli-type theorem in Chapter VI.

Proposition V.1. *For every $0<\alpha<1$ and every $1 \leq p \leq +\infty$, the operator I^{α}_{a+} is linear and continuous from $\mathrm{L}^p([a,b],\mathbb{R}^n)$ to $\mathrm{L}^p([a,b],\mathbb{R}^n)$. Precisely, for every $q \in \mathrm{L}^p([a,b],\mathbb{R}^n)$, the following inequality holds:*

$$\|\mathrm{I}^{\alpha}_{a+}[q]\|_{\mathrm{L}^p([a,b],\mathbb{R}^n)} \leq \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} \|q\|_{\mathrm{L}^p([a,b],\mathbb{R}^n)}. \tag{V.23}$$

Proposition V.2. *Let $0<\alpha<1$, $1 \leq p \leq +\infty$ and $q \in \mathrm{L}^p([a,b],\mathbb{R}^n)$. The following statement are satisfied:*

- *if $0<\alpha<1=p$, then $\mathrm{I}^{\alpha}_{a+}[q] \in \mathrm{L}^r([a,b],\mathbb{R}^n)$ for every $1 \leq r < \frac{1}{1-\alpha}$;*
- *if $0<\alpha<(1/p)<1$, then $\mathrm{I}^{\alpha}_{a+}[q] \in \mathrm{L}^r([a,b],\mathbb{R}^n)$ for every $1 \leq r \leq \frac{p}{1-\alpha p}$;*
- *if $0<\alpha=(1/p)<1$, then $\mathrm{I}^{\alpha}_{a+}[q] \in \mathrm{L}^r([a,b],\mathbb{R}^n)$ for every $1 \leq r < +\infty$;*
- *if $0 \leq (1/p) < \alpha < 1$, then $\mathrm{I}^{\alpha}_{a+}[q] \in \mathrm{H}^{\alpha-(1/p)}_a([a,b],\mathbb{R}^n) \subset \mathrm{L}^{\infty}([a,b],\mathbb{R}^n)$.*

Remark V.3. The larger p is and/or the closer α is to 1, then the more integrable $\mathrm{I}^{\alpha}_{a+}[q]$ is.

Remark V.4. Actually, for every $0<\alpha<1$ and every $1<p\leq+\infty$, it is proved that:

- if $0<\alpha<(1/p)<1$, then the operator I^{α}_{a+} is linear and continuous from $\mathrm{L}^p([a,b],\mathbb{R}^n)$ to $\mathrm{L}^r([a,b],\mathbb{R}^n)$ for every $1 \leq r \leq \frac{p}{1-\alpha p}$;
- if $0<\alpha=(1/p)<1$, then the operator I^{α}_{a+} is linear and continuous from $\mathrm{L}^p([a,b],\mathbb{R}^n)$ to $\mathrm{L}^r([a,b],\mathbb{R}^n)$ for every $1 \leq r < +\infty$;
- if $0 \leq (1/p) < \alpha < 1$, then the operator I^{α}_{a+} is linear and continuous from $\mathrm{L}^p([a,b],\mathbb{R}^n)$ to $\mathrm{L}^r([a,b],\mathbb{R}^n)$ for every $1 \leq r \leq +\infty$.

We are now interested in regularity properties of fractional integrals of continuous functions.

Proposition V.3. *Let $0<\alpha<1$ and let $q \in \mathrm{C}([a,b],\mathbb{R}^n)$. Then, $\mathrm{I}^{\alpha}_{a+}[q] \in \mathrm{C}_a([a,b],\mathbb{R}^n)$.*

Proposition V.4. *Let $0<\alpha<1$ and let $q \in \mathrm{AC}([a,b],\mathbb{R}^n)$. Then, $\mathrm{I}^{\alpha}_{a+}[q] \in \mathrm{AC}_a([a,b],\mathbb{R}^n)$ and the following equality is satisfied almost everywhere on $[a,b]$:*

$$\frac{d}{dt}\Big[\mathrm{I}^{\alpha}_{a+}[q]\Big](t) = \mathrm{I}^{\alpha}_{a+}[\dot{q}](t) + \frac{q(a)}{(t-a)^{1-\alpha}\Gamma(\alpha)}. \tag{V.24}$$

The two most important consequences of the previous propositions are the following.

Remark V.5. For every $0<\alpha<1$ and every $q \in \mathrm{C}^1([a,b],\mathbb{R}^n)$, ${}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q] \in \mathrm{C}_a([a,b],\mathbb{R}^n)$.

Remark V.6. For every $0<\alpha<1$ and every $q \in \mathrm{AC}([a,b],\mathbb{R}^n)$, $\mathrm{D}^{\alpha}_{a+}[q]$ and ${}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q]$ are defined almost everywhere on $[a,b]$ and the following equality holds:

$$\mathrm{D}^{\alpha}_{a+}[q](t) = {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q](t) + \frac{q(a)}{(t-a)^{\alpha}\Gamma(1-\alpha)}. \tag{V.25}$$

Actually, the previous equality can be summarized as follows:

$${}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q] = \mathrm{D}^{\alpha}_{a+}[q-q(a)]. \tag{V.26}$$

In particular, if $q \in \mathrm{AC}_a([a,b],\mathbb{R}^n)$, then ${}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q] = \mathrm{D}^{\alpha}_{a+}[q]$.

Relations between fractional operators. The composition between fractional operators is of interest in calculus of variations involving fractional derivatives. In particular, a composition rule of type $\mathrm{D}^{1/2} \circ \mathrm{D}^{1/2} = d/dt$ has attractive applications to formulate variational structures for some dissipative systems, see Section V.3.3.

Let us give some composition rules between left fractional operators. The counterparts for right ones are also valid. The most important property concerns the composition between fractional integrals. Indeed, from the following statement proved in [190, p.34], corollaries on compositions between fractional derivatives and fractional integrals are easily derivable.

Proposition V.5. *For every $\alpha > 0$, $\beta > 0$ and every $q \in \mathrm{L}^1([a,b],\mathbb{R}^n)$, it holds $\mathrm{I}^{\alpha}_{a+} \circ \mathrm{I}^{\beta}_{a+}[q] = \mathrm{I}^{\alpha+\beta}_{a+}[q]$ almost everywhere.*

Corollary V.1. *For every $0 < \alpha < 1$, the following equalities are satisfied almost everywhere:*
- *if $q \in \mathrm{L}^1([a,b],\mathbb{R}^n)$, then $\mathrm{D}^{\alpha}_{a+} \circ \mathrm{I}^{\alpha}_{a+}[q] = q$;*
- *if $q \in \mathrm{AC}([a,b],\mathbb{R}^n)$, then $\mathrm{I}^{\alpha}_{a+} \circ {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q] = q - q(a)$;*
- *if $q \in \mathrm{AC}([a,b],\mathbb{R}^n)$, then $\mathrm{I}^{\alpha}_{a+} \circ \mathrm{D}^{\alpha}_{a+}[q] = {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+} \circ \mathrm{I}^{\alpha}_{a+}[q] = q$.*

Corollary V.2. *For every $0 < \alpha < 1$ and every $q \in \mathrm{AC}([a,b],\mathbb{R}^n)$, it holds $\mathrm{D}^{1-\alpha}_{a+} \circ {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q] = \dot{q}$ almost everywhere. In particular, note that $\mathrm{D}^{1/2}_{a+} \circ {}_{\mathrm{c}}\mathrm{D}^{1/2}_{a+}[q] = \dot{q}$ almost everywhere.*

Proposition V.6 (Fractional integration by parts)**.** *Let $\alpha > 0$, $q_1 \in \mathrm{L}^p([a,b],\mathbb{R}^n)$ and $q_2 \in \mathrm{L}^r([a,b],\mathbb{R}^n)$. If $(1/p) + (1/r) < 1 + \alpha$, then it holds*

$$\int_a^b \mathrm{I}^{\alpha}_{a+}[q_1](\tau) \cdot q_2(\tau)\, d\tau = \int_a^b q_1(\tau) \cdot \mathrm{I}^{\alpha}_{b-}[q_2](\tau)\, d\tau. \tag{V.27}$$

This fractional integration by parts formula is proved in [190, p.34] and is instrumental in order to derive fractional Euler-Lagrange equations characterizing the critical points of Lagrangian functionals involving fractional operators.

Remark V.7. There are no simple generalizations of product and composition rules for fractional derivatives. Namely, in general $\mathrm{D}^{\alpha}[q_1 \cdot q_2] \neq \mathrm{D}^{\alpha}[q_1] \cdot q_2 + q_1 \cdot \mathrm{D}^{\alpha}[q_2]$ and $\mathrm{D}^{\alpha}[q_1 \circ q_2] \neq \mathrm{D}^{\alpha}[q_2] \cdot \mathrm{D}^{\alpha}[q_1] \circ q_2$. Some formulas have been suggested in both cases but their complexities make difficult their uses in practice.

V.3 Some recalls and complements on fractional calculus of variations

This section is devoted to recalls on calculus of variations for Lagrangian functionals involving Caputo fractional derivatives.

V.3.1 A fractional Euler-Lagrange equation

We recall the usual method leading to the fractional Euler-Lagrange equation characterizing the critical points of a Lagrangian functional involving a Caputo fractional derivative of order $0 < \alpha < 1$, see *e.g.* [16].

Let $a < b$ and let $\mathrm{C}^1_0([a,b],\mathbb{R}^n) = \{w \in \mathrm{C}^1([a,b],\mathbb{R}^n),\ w(a) = w(b) = 0\}$. We consider E a non empty subset of $\mathrm{C}^1([a,b],\mathbb{R}^n)$ open in the $\mathrm{C}^1_0([a,b],\mathbb{R}^n)$-direction *i.e.*

$$\forall q \in \mathrm{E},\ \forall w \in \mathrm{C}^1_0([a,b],\mathbb{R}^n),\ \exists \eta > 0,\ \forall \varepsilon \in [-\eta,\eta],\ q + \varepsilon w \in \mathrm{E}. \tag{V.28}$$

Let L be a Lagrangian *i.e.* a continuous map of class $\mathscr{C}^1$ in its two first variables

$$\begin{array}{rccl} L: & \mathbb{R}^n \times \mathbb{R}^n \times [a,b] & \longrightarrow & \mathbb{R} \\ & (x,v,t) & \longmapsto & L(x,v,t) \end{array} \tag{V.29}$$

and let $\mathcal{L}$ be the following fractional Lagrangian functional:

$$\begin{array}{rccl} \mathcal{L}: & \mathrm{E} \subset \mathrm{C}^1([a,b],\mathbb{R}^n) & \longrightarrow & \mathbb{R} \\ & q & \longmapsto & \displaystyle\int_a^b L(q(\tau), {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q](\tau), \tau)\, d\tau. \end{array} \tag{V.30}$$

The aim of this section is to give a necessary condition for local optimizers of $\mathcal{L}$. Let us introduce the following notions:

– $\mathrm{C}^1_0([a,b],\mathbb{R}^n)$ is the set of *variations* of $\mathcal{L}$.
– $q \in \mathrm{E}$ is said to be a *critical point* of $\mathcal{L}$ if $D\mathcal{L}(q)(w) = 0$ for every variation $w \in \mathrm{C}^1_0([a,b],\mathbb{R}^n)$. Recall that $D\mathcal{L}(q)(w)$ denotes the Gâteaux-differential of $\mathcal{L}$ at q in direction w.

In particular, if q is a local optimizer of $\mathcal{L}$, then q is a critical point of $\mathcal{L}$.

Remark V.8. Thanks to the regularity of the Caputo fractional derivative, note that $\mathcal{L}(q)$ is well defined for every $q \in \mathrm{C}^1([a,b],\mathbb{R}^n)$ since ${}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q] \in \mathrm{C}_a([a,b],\mathbb{R}^n)$.

It can be noted that the singularity of the left fractional derivative of Riemann-Liouville at $t = a$ is an obstruction to the study of Lagrangian functionals involving such a fractional derivative. For instance, in the very usual case $n = 1$, $\alpha = 1/2$ and $L(x,v,t) = v^2$ (Dirichlet Lagrangian), one can easily see that

$$\int_a^b L(q(\tau), \mathrm{D}^{1/2}_{a+}[q](\tau), \tau)\, d\tau = \int_a^b \mathrm{D}^{1/2}_{a+}[q](\tau)^2\, d\tau \tag{V.31}$$

is well defined if and only if $q(a) = 0$. Indeed, we recall that ${}_{\mathrm{c}}\mathrm{D}^{1/2}_{a+}[q] \in \mathrm{C}_a([a,b],\mathbb{R})$ and $\mathrm{D}^{1/2}_{a+}[q](t) = {}_{\mathrm{c}}\mathrm{D}^{1/2}_{a+}[q](t) + q(a)(\pi(t-a))^{-1/2}$, see Remark V.6. As a consequence, a calculus of variations is possible only if $q(a) = 0$ leading to the Caputo case.

Remark V.9. The consideration of E allows to assume (or not) general boundary conditions on the variational problem. For example, one can consider:

– $\mathrm{E} = \mathrm{C}^1([a,b],\mathbb{R}^n)$ (no boundary conditions);
– $\mathrm{E} = \{q \in \mathrm{C}^1([a,b],\mathbb{R}^n),\ q(a) = q_a\}$ where $q_a \in \mathbb{R}^n$ (initial condition);
– $\mathrm{E} = \{q \in \mathrm{C}^1([a,b],\mathbb{R}^n),\ g(q(a),q(b)) \in \mathrm{S}\}$ where $g : \mathbb{R}^n \times \mathbb{R}^n \longmapsto \mathbb{R}^j$, $j \in \mathbb{N}^*$ and S is a non empty subset of $\mathbb{R}^j$ (mixing of initial and final conditions).

Some arguments of uniform continuity, uniform convergence and regularity hypotheses on L lead to the differentiability of $\mathcal{L}$.

Lemma V.1. *Let $0 < \alpha < 1$. Then, $\mathcal{L}$ is Gâteaux-differentiable at every $q \in \mathrm{E}$ in every direction $w \in \mathrm{C}^1_0([a,b],\mathbb{R}^n)$ with*

$$D\mathcal{L}(q)(w) = \int_a^b \frac{\partial L}{\partial x}(q(\tau), {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q](\tau), \tau) \cdot w(\tau) + \frac{\partial L}{\partial v}(q(\tau), {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q](\tau), \tau) \cdot {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[w](\tau)\, d\tau. \tag{V.32}$$

Theorem V.1. *Let $0 < \alpha < 1$ and let $q \in \mathrm{E}$. Then, q is a critical point of $\mathcal{L}$ if and only if q is a solution of the following fractional Euler-Lagrange equation:*

$$\frac{\partial L}{\partial x}(q(t), {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q](t), t) + \mathrm{D}^{\alpha}_{b-}\left[\frac{\partial L}{\partial v}(q, {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q], \cdot)\right](t) = 0, \tag{EL$^\alpha$}$$

for every $t \in [a,b]$.

Proof. The proof is recalled for the reader's convenience. Let $q \in \mathrm{E}$. We denote by $F_q \in \mathrm{C}^1_a([a,b],\mathbb{R}^n)$ the antiderivative of $\partial L/\partial x(q, {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q], \cdot) \in \mathrm{C}([a,b],\mathbb{R}^n)$ vanishing at $t = a$. From Lemma V.1, q is a critical point of $\mathcal{L}$ if and only if

$$\forall w \in \mathrm{C}^1_0([a,b],\mathbb{R}^n),\ \int_a^b \dot{F}_q(\tau) \cdot w(\tau) + \frac{\partial L}{\partial v}(q(\tau), {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q](\tau), \tau) \cdot {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[w](\tau)\, \Delta\tau = 0 \tag{V.33}$$

if and only if, using the usual and the fractional integration by parts (see Proposition V.6) formulas,

$$\forall w \in \mathrm{C}^1_0([a,b],\mathbb{R}^n),\ \int_a^b \left(\mathrm{I}^{1-\alpha}_{b-}\left[\frac{\partial L}{\partial v}(q, {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q], \cdot)\right](\tau) - F_q(\tau)\right) \cdot \dot{w}(\tau)\, d\tau = 0 \tag{V.34}$$

if and only if (from the du Bois-Reymond Lemma) in the case $\mathbb{T} = [a,b]$) there exists $c \in \mathbb{R}^n$ such that:

$$\mathrm{I}^{1-\alpha}_{b-}\left[\frac{\partial L}{\partial v}(q, {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q], \cdot)\right](t) - F_q(t) = c, \tag{V.35}$$

for every $t \in [a,b]$. The proof is complete since $F_q \in \mathrm{C}^1([a,b],\mathbb{R}^n)$. □

Hence, Theorem V.1 provides a necessary condition for local optimizers of $\mathcal{L}$. Precisely, if q is a local optimizer of $\mathcal{L}$, then q is a solution of the fractional Euler-Lagrange equation (EL^α).

Example V.2 (Dirichlet)**.** Let $n = 1$, $[a, b] = [0, 1]$ and $L(x, v, t) = v^2/2$. Then, the fractional Euler-Lagrange equation (EL^α) is given by $\mathrm{D}^\alpha_{b-} \circ {}_\mathrm{c}\mathrm{D}^\alpha_{a+}[q] = 0$. Assuming that $1/2 < \alpha < 1$, the unique solution q satisfying $q(0) = 0$ and $q(1) = 1$ is given by

$$\forall t \in [0, 1],\ q(t) = (2\alpha - 1) \int_0^t \frac{d\tau}{[(1-\tau)(t-\tau)]^{1-\alpha}}. \tag{V.36}$$

We refer to [6, Example 1] for more details.

Remark V.10. We can note that Theorem V.1 is not exactly a generalization of Theorem II.1 (considered in the classical continuous case $\alpha = 1$ and $\mathbb{T} = [a, b]$). Indeed, we recall that ${}_\mathrm{c}\mathrm{D}^1_{a+}[q] = \dot{q} - \dot{q}(a)$ and consequently, we do not recover exactly the definition of the (non shifted) Lagrangian functional $\mathcal{L}$ given in Equation (II.2.13). Nevertheless, we recall that the Caputo fractional derivative is an approximation of the classical derivative when $0 < \alpha < 1$ and $\alpha \to 1$, see [190, p.51].

V.3.2 A Noether-type theorem and complements

In this section, a Noether-type theorem originally and simultaneously derived in [69, 89] is recalled. The idea is to extend the strategy of Noether to the fractional case. Namely, we are interested in fractional Euler-Lagrange equations (EL^α) admitting a symmetry.

As in the classical context, the notion of symmetry for fractional Euler-Lagrange equations (EL^α) is introduced via the action of a one-parameter family of infinitesimal transformations of $\mathbb{R}^n$ (see Definition II.1) on Lagrangian.

Definition V.5. Let Φ be a one-parameter family of infinitesimal transformations of $\mathbb{R}^n$ (see Definition II.1) and let $0 < \alpha < 1$. A Lagrangian L is said to be invariant under the action of Φ if for every solution $q \in \mathrm{C}^1([a, b], \mathbb{R}^n)$ of (EL^α) and every $t \in [a, b]$, the map

$$\theta \longmapsto L(\Phi(\theta, q(t)), {}_\mathrm{c}\mathrm{D}^\alpha_{a+}[\Phi(\theta, q)](t), t) \tag{V.37}$$

has a null derivative in 0. In such a case, Φ is said to be a symmetry of the fractional Euler-Lagrange equation (EL^α).

As in the classical case, the most classical examples of invariance of a Lagrangian under the action of a one-parameter family of infinitesimal transformations of $\mathbb{R}^n$ are given by quadratic Lagrangian and rotations, see Example II.2.

Proposition V.7 (Noether)**.** *Let $0 < \alpha < 1$ and let Φ be a one-parameter family of infinitesimal transformations of $\mathbb{R}^n$. If L is invariant under the action of Φ, then for every solution $q \in \mathrm{C}^1([a, b], \mathbb{R}^n)$ of (EL^α), the following equality holds:*

$$\frac{\partial L}{\partial v}(q(t), {}_\mathrm{c}\mathrm{D}^\alpha_{a+}[q](t), t) \cdot {}_\mathrm{c}\mathrm{D}^\alpha_{a+}\left[\frac{\partial \Phi}{\partial \theta}(0, q)\right](t) - \mathrm{D}^\alpha_{b-}\left[\frac{\partial L}{\partial v}(q, {}_\mathrm{c}\mathrm{D}^\alpha_{a+}[q], \cdot)\right](t) \cdot \frac{\partial \Phi}{\partial \theta}(0, q(t)) = 0, \tag{V.38}$$

for every $t \in [a, b]$.

Proof. The proof is recalled for the reader's convenience. Let $q \in \mathrm{C}^1([a, b], \mathbb{R}^n)$ be a solution of (EL^α). Let us differentiate in 0 the map given by (V.37) and let us invert the operators ${}_\mathrm{c}\mathrm{D}^\alpha_{a+}$ and $\partial/\partial\theta$ from $\Phi(0, \cdot) = \mathrm{Id}_{\mathbb{R}^n}$ and from the $\mathscr{C}^2$-regularity of Φ. Note that this last operation is not obvious and needs some technical calculations. We obtain for every $t \in [a, b]$

$$\frac{\partial L}{\partial x}(q(t), {}_\mathrm{c}\mathrm{D}^\alpha_{a+}[q](t), t) \cdot \frac{\partial \Phi}{\partial \theta}(0, q(t)) + \frac{\partial L}{\partial v}(q(t), {}_\mathrm{c}\mathrm{D}^\alpha_{a+}[q](t), t) \cdot {}_\mathrm{c}\mathrm{D}^\alpha_{a+}\left[\frac{\partial \Phi}{\partial \theta}(0, q)\right](t) = 0. \tag{V.39}$$

Using that q is a solution of (EL^α) on $[a, b]$, the proof is complete. □

As mentioned in Remark V.10, Proposition V.7 is not exactly a generalization of Theorem II.2 (considered in the classical continuous case $\alpha = 1$ and $\mathbb{T} = [a, b]$) for several reasons. Firstly, we use a Caputo fractional derivative that does not coincide exactly with the classical derivative in the case $\alpha = 1$, see Remark V.10. But above all, Proposition V.7 does not provide an explicit constant of motion, which is the main concern of the classical Noether's theorem. Indeed, in contrary to the classical case (see proof of Theorem II.2), no simple Leibniz formula in the fractional case $0 < \alpha < 1$ allows to provide an explicit constant of motion from Equality (V.38).

From this last observation, a *transfer formula* is introduced in [48] allowing to write the left-hand term of Equality (V.38) as an explicit derivative. Precisely, let f, g be sufficiently smooth functions (analytic functions for example), see [48] for more details on sufficient assumptions. The following transfer formula is then satisfied:

$$f \cdot {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[g] - \mathrm{D}^{\alpha}_{a+}[f] \cdot g = \frac{d}{dt} \left[\sum_{k \geq 0} (-1)^k \mathrm{I}^{k+1-\alpha}_{a+}[g - g(a)] \cdot f^{(k)} + g^{(k)} \cdot \mathrm{I}^{k+1-\alpha}_{b-}[f] \right]. \tag{V.40}$$

Hence, under the hypotheses of Proposition V.7 and assuming that $f = \partial L/\partial v(q, {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q], \cdot)$ and $g = \partial \Phi/\partial \theta(0, q)$ are sufficiently smooth, we conclude that there exists $c \in \mathbb{R}$ such that

$$\sum_{k \geq 0} (-1)^k \mathrm{I}^{k+1-\alpha}_{a+}[g - g(a)](t) \cdot f^{(k)}(t) + g^{(k)}(t) \cdot \mathrm{I}^{k+1-\alpha}_{b-}[f](t) = c, \tag{V.41}$$

for every $t \in [a, b]$. Hence, an explicit constant of motion is derived from Proposition V.7 for fractional Euler-Lagrange equations (EL^{α}) admitting a symmetry. Nevertheless, analogously to the product and composition rules for fractional derivatives, this transfer formula is written in terms of infinite series and consequently, this does not allow an easy calculation of the conservation law.

A similar strategy is developed in [48] for discrete fractional Euler-Lagrange equations admitting a symmetry (see Chapter VIII). In the discrete case, the constant of motion is written with a *finite* summation and therefore, the numerical calculation of the discrete conservation law is possible. We refer to [48] for more details.

V.3.3 Some comments on the work of F. Riewe

In 1996-97, the original idea of F. Riewe was to provide variational structures for dissipative equations by incorporating fractional operators in the calculus of variations, see [187, 188]. Indeed, in the case $\alpha = 1/2$, such an approach raises a composition of type $\mathrm{D}^{1/2} \circ \mathrm{D}^{1/2}$ in the fractional Euler-Lagrange equations associated. Nevertheless, as we have seen in Theorem V.1, this composition mixes a *right* fractional derivative and a *left* one *i.e.* $\mathrm{D}^{1/2}_{b-} \circ \mathrm{D}^{1/2}_{a+}$. Unfortunately, such a composition is not equal to the classical derivative d/dt. Therefore, despite that the idea of F. Riewe was very relevant, his work is not totally satisfactory since its main objective, *i.e.* to obtain a fractional variational structure for a dissipative system, has not been reached.

In order to illustrate the idea of F. Riewe, let us look for a variational structure for the dissipative differential equation $q - \dot{q} = 0$, on an interval $[a, b]$ (with $a < b$) in the case $n = 1$. The solutions are given by $q(t) = ce^t$ with $c \in \mathbb{R}$. This differential equation does not satisfy the classical Helmholtz condition (see Section IV.2) and consequently, it cannot be written as a classical Euler-Lagrange equation. The general idea of F. Riewe is to take $\alpha = 1/2$ and to consider the following fractional Lagrangian functional:

$$\begin{array}{rccl} \mathcal{L}: & \mathrm{C}^1([a, b], \mathbb{R}) & \longrightarrow & \mathbb{R} \\ & q & \longmapsto & \displaystyle\int_a^b [q(\tau)^2 + {}_{\mathrm{c}}\mathrm{D}^{1/2}_{a+}[q](\tau)^2]/2 \, d\tau, \end{array} \tag{V.42}$$

associated with the Lagrangian $L(x, v, t) = (x^2 + v^2)/2$. From Theorem V.1, the critical points of $\mathcal{L}$ are then characterized by the fractional Euler-Lagrange equation $q + \mathrm{D}^{1/2}_{b-} \circ {}_{\mathrm{c}}\mathrm{D}^{1/2}_{a+}[q] = 0$. Nevertheless, since $\mathrm{D}^{1/2}_{b-} \circ {}_{\mathrm{c}}\mathrm{D}^{1/2}_{a+} \neq -d/dt$, we do not recover $q - \dot{q} = 0$. Actually, many formulations (with different fractional operators or different Lagrangian) can be attempted but it clearly seems that an asymmetric composition always emerges and this is a strong obstruction avoiding a positive conclusion to the framework of F. Riewe.

However, with the same objective, J. Cresson and P. Inizan suggest in [71] to modify the framework of Section V.3.1 by splitting in two the variable of the functional. Let us recall briefly their idea. They consider a Lagrangian L and they introduce the following *asymmetric* fractional Lagrangian functional:

$$\begin{array}{rrcl} \mathcal{L}_{\text{asy}}: & \mathrm{C}^1([a,b],\mathbb{R}^n)\times \mathrm{C}^1_0([a,b],\mathbb{R}^n) & \longrightarrow & \mathbb{R} \\ & (q,w) & \longmapsto & \displaystyle\int_a^b L(q(\tau)+w(\tau), {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q](\tau) - {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{b-}[w](\tau),\tau)\, d\tau. \end{array} \tag{V.43}$$

Then, $(q,0)$ (with $q \in \mathrm{C}^1([a,b],\mathbb{R}^n)$) is said to be a critical point of $\mathcal{L}_{\text{asy}}$ in the $\{0\}\times \mathrm{C}^1_0([a,b],\mathbb{R}^n)$-direction if $D\mathcal{L}_{\text{asy}}(q,0)(0,w)=0$ for every $w\in \mathrm{C}^1_0([a,b],\mathbb{R}^n)$. With this *asymmetric* framework, the authors prove that $(q,0)$ is a critical point of $\mathcal{L}_{\text{asy}}$ in the $\{0\}\times \mathrm{C}^1_0([a,b],\mathbb{R}^n)$-direction if and only if q is a solution of the following *asymmetric* fractional Euler-Lagrange equation:

$$\mathrm{D}^{\alpha}_{a+}\left[\frac{\partial L}{\partial v}(q,{}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q],\cdot)\right](t) = \frac{\partial L}{\partial x}(q(t),{}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q](t),t), \tag{$\mathrm{EL}^{\alpha}_{\text{asy}}$}$$

for every $t\in[a,b]$. The key point is the emergence of the composition $\mathrm{D}^{\alpha}_{a+}\circ {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}$ (instead of $\mathrm{D}^{\alpha}_{b-}\circ {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}$). It allows to obtain dissipative terms in ($\mathrm{EL}^{\alpha}_{\text{asy}}$) in the case $\alpha=1/2$. For example, authors of [70] give an *asymmetric* fractional variational structure for the convection-diffusion equation. Note that this *asymmetric* framework is also relevant for the dissipative equation $q-\dot{q}=0$. Indeed, taking $\alpha=1/2$ and $L(x,v,t)=(x^2+v^2)/2$, the *asymmetric* fractional Euler-Lagrange equation ($\mathrm{EL}^{\alpha}_{\text{asy}}$) is given by $q-\dot{q}=0$ since $\mathrm{D}^{1/2}_{a+}\circ {}_{\mathrm{c}}\mathrm{D}^{1/2}_{a+}[q]=\dot{q}$ (Corollary V.2).

Numerous perspectives can be investigated from the results of J. Cresson and P. Inizan. Indeed, one could attempt to enrich the results on dissipative equations by extending the mathematical theory of the classical calculus of variations to the *asymmetric* fractional one. To conclude this chapter, we note that every linear (partial) differential equation with constant coefficients possesses an *asymmetric* fractional variational structure. Nevertheless, strong obstructions emerge in the case of more complicated equations like Navier-Stokes. An ambitious outlook would be to construct a framework providing a variational structure covering such complex systems.

Chapitre VI

Existence de minimiseurs pour des fonctionnelles Lagrangiennes dépendantes de dérivées classique et fractionnaire de Caputo

Depuis la fin du XX$^{\text{ème}}$ siècle, une littérature conséquente a été développée autour du calcul des variations fractionnaire. Nous renvoyons par exemple aux articles [6, 14–19, 21, 27, 69, 89, 92, 172, 174–177] et aux références qu'ils contiennent. Cependant, les résultats traitant de l'existence de solutions pour des problèmes variationnels fractionnaires sont rares, voir [129, 143].

Dans ce chapitre, nous démontrons une version fractionnaire du théorème de Tonelli. Plus précisément, sous des hypothèses adéquates de régularité, coercivité et convexité, nous établissons l'existence d'un minimiseur pour des fonctionnelles Lagrangiennes dépendantes de dérivées classique et fractionnaire de type Caputo. Nous déterminons également la condition nécessaire d'optimalité de type Euler-Lagrange correspondante.

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VI.1 Introduction

As mentioned in Chapter V, the fractional calculus of variations was originally investigated by F. Riewe in 1996-97, see [187, 188]. Since then, it has attracted the attention of many authors and a significant literature regarding necessary optimality conditions and Noether's theorems has been developed, see *e.g.* [6, 14–19, 21, 27, 69, 89, 92, 172, 174–177] and references therein. We also refer to the recent book [157] for a state of the art.

Nevertheless, the results addressed to the existence of solutions for variational problems involving fractional operators are rare. As far as we know, they are only discussed in particular cases, see *e.g.* [129, 143]. However, existence theorems are essential ingredients of the deductive method for solving variational problems, which starts with the proof of existence, proceeds with application of optimality conditions, and finishes examining the candidates to arrive to a solution. These arguments make the question of existence an emergent topic, which requires serious attention and more interest.

We treated this question of existence in [46] for fractional calculus of variations involving Riemann-Liouville fractional derivatives. Then, we applied the similar method in [51] for variational problems involving a Riemann-Liouville fractional integral, a classical derivative and a Caputo fractional derivative. In each of these papers, the strategy is the same and is widely inspired from [63, 74] where the existence of solutions for *classical* variational problems, *i.e.* without fractional derivatives, is studied. Actually, the main results of these articles are nothing else but fractional analogues of the classical Tonelli theorem. Finally, note that the method has also been successfully applied in [50] for *generalized* variational problems, *i.e.* depending on general linear operators.

In this chapter, we recall the framework and the results provided in [51]. Precisely, we obtain sufficient conditions ensuring the existence of a minimizer for Lagrangian functionals involving classical and Caputo fractional derivatives. The main result (given in Theorem VI.1) is based on suitable assumptions of regularity, coercivity and convexity. Some improvements have been added in this chapter with respect to the paper [51]. They are concerned with

- the weakening of the regularity assumption, by using regularity properties of the fractional operators seen in Section V.2.3;
- the weakening of the convexity assumption, by assuming stronger hypotheses on the continuity of the Lagrangian and by using the Ascoli's theorem, see Section VI.4.

The corresponding fractional Euler-Lagrange equation is derived in Section VI.3.

Organization of the chapter. The framework and the most important results of this study are stated in Section VI.2. Precisely, the Tonelli-type theorem (main result) is given in Section VI.2.1 with general assumptions of regularity, coercivity and convexity. In Sections VI.2.2 and VI.2.3, we provide more concrete sufficient assumptions for regularity and coercivity, regarding quasi-polynomial behaviours of the Lagrangian. Examples are given in Section VI.2.4. Section VI.3 is devoted to the derivation of the corresponding necessary optimality condition of type Euler-Lagrange. Finally, Section VI.4 concludes this chapter with improvements of the convexity assumption in the general Tonelli-type theorem by assuming stronger hypotheses on the continuity of the Lagrangian. Examples are also provided in this last section.

VI.2 A Tonelli-type theorem

Let $(a,b) \in \mathbb{R}^2$ with $a < b$ and $n \in \mathbb{N}^*$. The notation $\|\cdot\|$ stands for the usual Euclidean norm of $\mathbb{R}^n$. The set of all continuous functions on $[a,b]$ with values in $\mathbb{R}^n$ is denoted by $\mathrm{C}([a,b],\mathbb{R}^n)$ and is endowed with its usual uniform norm $\|\cdot\|_\infty$. In what follows, for every $1 \leq r \leq \infty$, we denote by

- $\mathrm{L}^r([a,b],\mathbb{R}^n)$ the usual space of r-Lebesgue integrable functions endowed with its usual norm $\|\cdot\|_{\mathrm{L}^r}$;
- $\mathrm{W}^{1,r}([a,b],\mathbb{R}^n)$ the usual r-Sobolev space endowed with its usual norm $\|\cdot\|_{\mathrm{W}^{1,r}}$.

Along the work, we consider $1 < p < \infty$ and p' its adjoint *i.e.* $p' = p/(p-1)$. We recall that $\mathrm{W}^{1,p}([a,b],\mathbb{R}^n)$ is a reflexive Banach space and that the compact embedding $\mathrm{W}^{1,p}([a,b],\mathbb{R}^n) \hookrightarrow\!\!\!\!\to \mathrm{C}([a,b],\mathbb{R}^n)$ holds (see [55] for more details).

Our main goal is to give sufficient conditions ensuring the existence of a minimizer for the fractional Lagrangian functional

$$\begin{array}{rrcl} \mathcal{L}: & \mathrm{E} \subset \mathrm{W}^{1,p}([a,b],\mathbb{R}^n) & \longrightarrow & \mathbb{R} \\ & q & \longmapsto & \displaystyle\int_a^b L(q(\tau), {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q](\tau), \dot{q}(\tau), \tau)\, d\tau, \end{array} \tag{VI.1}$$

where E is a non empty weakly closed subset of $\mathrm{W}^{1,p}([a,b],\mathbb{R}^n)$, ${}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q]$ is the Caputo fractional derivative of q of order $0 < \alpha < 1$, $\dot{q}$ is the classical derivative of q and L is a Lagrangian *i.e.* a continuous function of class $\mathscr{C}^1$ in its three first variables

$$\begin{array}{rrcl} L: & (\mathbb{R}^n)^3 \times [a,b] & \longrightarrow & \mathbb{R} \\ & (x, v_\alpha, v, t) & \longmapsto & L(x, v_\alpha, v, t). \end{array} \tag{VI.2}$$

Note that the consideration of E allows to study this variational problem with or without general boundary conditions.

VI.2.1 Main result

Using general assumptions of regularity, coercivity and convexity, we prove a fractional analogue of the classical Tonelli theorem, ensuring the existence of a minimizer for $\mathcal{L}$.

Definition VI.1. We say that L is (α,p)-regular if

- $L(q, {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q], \dot{q}, \cdot) \in \mathrm{L}^1([a,b],\mathbb{R}^n)$;
- $\frac{\partial L}{\partial x}(q, {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q], \dot{q}, \cdot) \in \mathrm{L}^1([a,b],\mathbb{R}^n)$;
- $\frac{\partial L}{\partial v_\alpha}(q, {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q], \dot{q}, \cdot) \in \mathrm{L}^s([a,b],\mathbb{R}^n)$ for s satisfying

  - $s = \frac{p}{(2-\alpha)p-1}$ if $(1-\alpha)p < 1$;
  - $s > 1$ if $(1-\alpha)p = 1$;
  - $s = 1$ if $(1-\alpha)p > 1$;
- $\frac{\partial L}{\partial v}(q, {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q], \dot{q}, \cdot) \in \mathrm{L}^{p'}([a,b],\mathbb{R}^n)$,

for every $q \in \mathrm{E}$.

If $(1-\alpha)p < 1$, then $\frac{p}{(2-\alpha)p-1} > 1$. Consequently, the larger p is and/or the closer α is to 0, then the less restrictive the (α,p)-regularity is in terms of integrability of $\partial L/\partial v_\alpha(q, {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q], \dot{q}, \cdot)$ and $\partial L/\partial v(q, {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q], \dot{q}, \cdot)$.

Definition VI.2. $\mathcal{L}$ is said to be coercive on E if

$$\lim_{\substack{\|q\|_{\mathrm{W}^{1,p}}\to\infty \\ q\in\mathrm{E}}} \mathcal{L}(q) = +\infty. \tag{VI.3}$$

The main result of this chapter is the following.

Theorem VI.1 (Tonelli)**.** *If the following assumptions are satisfied:*
- *L is (α,p)-regular;*
- *$\mathcal{L}$ is coercive on* E*;*
- *$L(\cdot,\cdot,\cdot,t)$ is convex on $(\mathbb{R}^n)^3$ for every $t \in [a,b]$,*

then there exists a minimizer for $\mathcal{L}$.

The proof of Theorem VI.1 needs the following lemma.

Lemma VI.1. *Let $(q_k)_{k\in\mathbb{N}}$ be a weakly convergent sequence of* $\mathrm{W}^{1,p}([a,b],\mathbb{R}^n)$ *and let $\bar{q}$ denote its weak limit. Moreover, let s satisfying*
- *$s = \frac{p}{(2-\alpha)p-1}$ if $(1-\alpha)p < 1$;*
- *$s > 1$ if $(1-\alpha)p = 1$;*
- *$s = 1$ if $(1-\alpha)p > 1$.*

Denoting s' the adjoint of s, the following statements are satisfied:
- *$(q_k)_{k\in\mathbb{N}}$ is strongly convergent to $\bar{q}$ in $\mathrm{L}^\infty([a,b],\mathbb{R}^n)$;*
- *$({}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q_k])_{k\in\mathbb{N}}$ is weakly convergent to ${}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[\bar{q}]$ in $\mathrm{L}^{s'}([a,b],\mathbb{R}^n)$;*
- *$(\dot{q}_k)_{k\in\mathbb{N}}$ is weakly convergent to $\dot{\bar{q}}$ in $\mathrm{L}^p([a,b],\mathbb{R}^n)$.*

Proof. The first statement is due to the compact embedding $\mathrm{W}^{1,p}([a,b],\mathbb{R}^n) \hookrightarrow\!\!\!\!\rightarrow \mathrm{C}([a,b],\mathbb{R}^n)$. The last one is obvious. For the second property, we have to distinguish three cases and use the weakly convergence of $(\dot{q}_k)_{k\in\mathbb{N}}$ to $\dot{\bar{q}}$ in $\mathrm{L}^p([a,b],\mathbb{R}^n)$.

Case $(1-\alpha)p < 1$: From Remark V.4, we infer that $({}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q_k])_{k\in\mathbb{N}} = (\mathrm{I}^{1-\alpha}_{a+}[\dot{q}_k])_{k\in\mathbb{N}}$ is weakly convergent to ${}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[\bar{q}] = \mathrm{I}^{1-\alpha}_{a+}[\dot{\bar{q}}]$ in $\mathrm{L}^r([a,b],\mathbb{R}^n)$ for $r = \frac{p}{1-(1-\alpha)p}$. To conclude, one has to see that $s' = r$.

Case $(1-\alpha)p = 1$: From Remark V.4, we infer that $({}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q_k])_{k\in\mathbb{N}} = (\mathrm{I}^{1-\alpha}_{a+}[\dot{q}_k])_{k\in\mathbb{N}}$ is weakly convergent to ${}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[\bar{q}] = \mathrm{I}^{1-\alpha}_{a+}[\dot{\bar{q}}]$ in $\mathrm{L}^r([a,b],\mathbb{R}^n)$ for every $1 \le r < \infty$ and consequently in $\mathrm{L}^{s'}([a,b],\mathbb{R}^n)$.

Case $(1-\alpha)p > 1$: From Remark V.4, we infer that $({}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q_k])_{k\in\mathbb{N}} = (\mathrm{I}^{1-\alpha}_{a+}[\dot{q}_k])_{k\in\mathbb{N}}$ is weakly convergent to ${}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[\bar{q}] = \mathrm{I}^{1-\alpha}_{a+}[\dot{\bar{q}}]$ in $\mathrm{L}^\infty([a,b],\mathbb{R}^n)$. □

Proof of Theorem VI.1. Since L is (α,p)-regular, $L(q, {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q], \dot{q}, \cdot) \in \mathrm{L}^1([a,b],\mathbb{R}^n)$ and $\mathcal{L}(q)$ exists in $\mathbb{R}$ for every $q \in \mathrm{E}$. Let $(q_k)_{k\in\mathbb{N}} \subset \mathrm{E}$ be a minimizing sequence of $\mathcal{L}$ *i.e.*

$$\lim_{k\to\infty} \mathcal{L}(q_k) = \inf_{q\in\mathrm{E}} \mathcal{L}(q) < +\infty. \tag{VI.4}$$

The coercivity of $\mathcal{L}$ implies the boundedness of $(q_k)_{k\in\mathbb{N}}$ in $\mathrm{W}^{1,p}([a,b],\mathbb{R}^n)$ that is reflexive. Consequently, up to a subsequence, $(q_k)_{k\in\mathbb{N}}$ is weakly convergent in $\mathrm{W}^{1,p}([a,b],\mathbb{R}^n)$. By $\bar{q}$ we denote its weak limit. Then $\bar{q} \in \mathrm{E}$

since E is a weakly closed subset of $\mathrm{W}^{1,p}([a,b],\mathbb{R}^n)$. From the convexity of L, it holds for every $k \in \mathbb{N}$

$$\mathcal{L}(q_k) \geq \mathcal{L}(\bar{q}) + \int_a^b \Bigg[\frac{\partial L}{\partial x}(\bar{q}(\tau), {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[\bar{q}](\tau), \dot{q}(\tau), \tau) \cdot (q_k(\tau) - \bar{q}(\tau)) + \frac{\partial L}{\partial v_\alpha}(\bar{q}(\tau), {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[\bar{q}](\tau), \dot{q}(\tau), \tau) \cdot ({}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q_k](\tau) - {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[\bar{q}](\tau)) + \frac{\partial L}{\partial v}(\bar{q}(\tau), {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[\bar{q}](\tau), \dot{q}(\tau), \tau) \cdot (\dot{q}_k(\tau) - \dot{\bar{q}}(\tau)) \Bigg] \, d\tau. \quad \text{(VI.5)}$$

Finally, from the (α,p)-regularity of L and from Lemma VI.1, we deduce the following inequality by passing to the limit on k:

$$\inf_{q \in \mathrm{E}} \mathcal{L}(q) \geq \mathcal{L}(\bar{q}), \quad \text{(VI.6)}$$

which concludes the proof. □

We give more concrete sufficient conditions (in terms of quasi-polynomial behaviours) on the Lagrangian L that imply its regularity (see Section VI.2.2) and the coercivity of $\mathcal{L}$ (see Section VI.2.3). Examples are provided in Section VI.2.4.

The most important constraint of Theorem VI.1 is the convexity hypothesis. In Section VI.4, some improved versions of Theorem VI.1 with weaker convexity assumptions are given. This is allowed by stronger hypotheses on the continuity of L and the Ascoli's theorem.

VI.2.2 Sufficient conditions for a (α,p)-regular Lagrangian L

Recall that for every $q \in \mathrm{W}^{1,p}([a,b],\mathbb{R}^n)$, the following properties are satisfied:
- $q \in \mathrm{L}^{\infty}([a,b],\mathbb{R}^n)$;
- ${}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q] \in \mathrm{L}^{r}([a,b],\mathbb{R}^n)$ for
  - $r = \frac{p}{1-(1-\alpha)p}$ if $(1-\alpha)p < 1$;
  - any $1 \leq r < \infty$ if $(1-\alpha)p = 1$;
  - $r = \infty$ if $(1-\alpha)p > 1$;
- $\dot{q} \in \mathrm{L}^{p}([a,b],\mathbb{R}^n)$.

From these observations, we introduce the following sets $\mathcal{P}^{(\alpha,p)}_M$ of quasi-polynomial functions.

Definition VI.3. Let $M \geq 1$ and let $P : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times [a,b] \longrightarrow \mathbb{R}^+$. Then,
- in the case $(1-\alpha)p < 1$, P belongs to $\mathcal{P}^{(\alpha,p)}_M$ if P writes as

$$P(x,v_\alpha,v,t) = \sum_{k=1}^{N} \varphi_k(x,t) \|v_\alpha\|^{d_{\alpha,k}} \|v\|^{d_{1,k}} \quad \text{(VI.7)}$$

  with $d_{\alpha,k}(1-(1-\alpha)p) + d_{1,k} \leq p/M$;
- in the case $(1-\alpha)p = 1$, P belongs to $\mathcal{P}^{(\alpha,p)}_M$ if P writes as

$$P(x,v_\alpha,v,t) = \sum_{k=1}^{N} \varphi_k(x,t) \|v_\alpha\|^{d_{\alpha,k}} \|v\|^{d_{1,k}} \quad \text{(VI.8)}$$

  with $d_{\alpha,k} = 0$ and $d_{1,k} \leq p/M$, or with $d_{\alpha,k} \neq 0$ and $d_{1,k} < p/M$;
- in the case $(1-\alpha)p > 1$, P belongs to $\mathcal{P}^{(\alpha,p)}_M$ if P writes as

$$P(x,v_\alpha,v,t) = \sum_{k=1}^{N} \varphi_k(x,v_\alpha,t) \|v\|^{d_{1,k}} \quad \text{(VI.9)}$$

  with $d_{1,k} \leq p/M$;

where $N \in \mathbb{N}^*$, φ_k are continuous functions with values in $\mathbb{R}^+$ and $d_{\alpha,k}$, $d_{1,k} \in \mathbb{R}^+$.

Remark VI.1. Let us mention that if $P \in \mathcal{P}_M^{(\alpha,p)}$, then $P \in \mathcal{P}_M^{(\alpha_1,p)}$ for every $0 < \alpha_1 \leq \alpha < 1$.

The following lemma shows the interest of sets $\mathcal{P}_M^{(\alpha,p)}$.

Lemma VI.2. *Let $M \geq 1$ and $P \in \mathcal{P}_M^{(\alpha,p)}$. Then, it holds*

$$P(q, {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q], \dot{q}, \cdot) \in \mathrm{L}^M([a,b], \mathbb{R}^+)$$

for every $q \in \mathrm{W}^{1,p}([a,b], \mathbb{R}^n)$.

Proof. We only prove the first case $(1-\alpha)p < 1$. The other ones are obtained in an analogous way. Let $q \in \mathrm{W}^{1,p}([a,b], \mathbb{R}^n)$ and let $k \in \{1, \ldots, N\}$. Then, $\|{}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q]\|^{d_{\alpha,k}} \in \mathrm{L}^{p/[d_{\alpha,k}(1-(1-\alpha)p)]}([a,b], \mathbb{R}^+)$, $\varphi_k(q, \cdot) \in \mathrm{L}^\infty([a,b], \mathbb{R}^+)$ and $\|\dot{q}\|^{d_k} \in \mathrm{L}^{p/d_k}([a,b], \mathbb{R}^+)$. Consequently, it holds

$$\varphi_k(q, \cdot)\|{}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q]\|^{d_{\alpha,k}}\|\dot{q}\|^{d_k} \in \mathrm{L}^{p/[d_{\alpha,k}(1-(1-\alpha)p)+d_k]}([a,b], \mathbb{R}^+). \tag{VI.10}$$

The proof is complete. □

This lemma leads to the following proposition.

Proposition VI.1 (Regularity of L)**.** *If there exist $P \in \mathcal{P}_1^{(\alpha,p)}$, $P_0 \in \mathcal{P}_1^{(\alpha,p)}$, $P_\alpha \in \mathcal{P}_s^{(\alpha,p)}$ and $P_1 \in \mathcal{P}_{p'}^{(\alpha,p)}$ such that*
- $|L(x, v_\alpha, v, t)| \leq P(x, v_\alpha, v, t)$;
- $\left\| \dfrac{\partial L}{\partial x}(x, v_\alpha, v, t) \right\| \leq P_0(x, v_\alpha, v, t)$;
- $\left\| \dfrac{\partial L}{\partial v_\alpha}(x, v_\alpha, v, t) \right\| \leq P_\alpha(x, v_\alpha, v, t)$;
- $\left\| \dfrac{\partial L}{\partial v}(x, v_\alpha, v, t) \right\| \leq P_1(x, v_\alpha, v, t)$,

where s satisfies
- $s = \frac{p}{(2-\alpha)p-1}$ *in the case* $(1-\alpha)p < 1$;
- $s > 1$ *in the case* $(1-\alpha)p = 1$;
- $s = 1$ *in the case* $(1-\alpha)p > 1$,

then L is (α,p)-regular.

Here also, we see that the bigger p is and/or the closer α is to 0, then the less restrictive the assumptions of Proposition VI.1 are. From Remark VI.1, if one concludes the (α,p)-regularity of L from Proposition VI.1, then L is also (α_1,p)-regular for every $0 < \alpha_1 \leq \alpha < 1$. We refer to Section VI.2.4 for examples.

VI.2.3 Sufficient condition for a coercive functional $\mathcal{L}$

The notion of coercivity of $\mathcal{L}$ is strongly dependent on the considered set E. Therefore, in this section, we consider an example of set E and we give a sufficient condition on L ensuring the coercivity of $\mathcal{L}$ on E.

Namely, in this section, we consider $q_a \in \mathbb{R}^n$ and

$$\mathrm{E} = \mathrm{W}^{1,p}_{q_a}([a,b], \mathbb{R}^n) = \{q \in \mathrm{W}^{1,p}([a,b], \mathbb{R}^n),\ q(a) = q_a \in \mathbb{R}^n\}, \tag{VI.11}$$

corresponding to the variational problem with initial condition. From the compact embedding $\mathrm{W}^{1,p}([a,b], \mathbb{R}^n) \hookrightarrow\!\!\!\!\rightarrow \mathrm{C}([a,b], \mathbb{R}^n)$, we deduce that $\mathrm{W}^{1,p}_{q_a}([a,b], \mathbb{R}^n)$ is a weakly closed subset of $\mathrm{W}^{1,p}([a,b], \mathbb{R}^n)$. The following lemma is instrumental to give a sufficient condition for coercivity of $\mathcal{L}$ on $\mathrm{W}^{1,p}_{q_a}([a,b], \mathbb{R}^n)$.

Lemma VI.3. *There exist γ_0, $\gamma_1 \geq 0$ such that*
- $\|q\|_{\mathrm{L}^p} \leq \gamma_0 \|\dot{q}\|_{\mathrm{L}^p} + \gamma_1$;
- $\|{}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q]\|_{\mathrm{L}^p} \leq \gamma_0 \|\dot{q}\|_{\mathrm{L}^p} + \gamma_1$;

for every $q \in \mathrm{W}^{1,p}_{q_a}([a,b], \mathbb{R}^n)$.

Proof. The second point comes from Proposition V.1. The first point comes from the continuous embeddings $\mathrm{L}^\infty([a,b],\mathbb{R}^n) \hookrightarrow \mathrm{L}^p([a,b],\mathbb{R}^n) \hookrightarrow \mathrm{L}^1([a,b],\mathbb{R}^n)$. Then, it is sufficient to take the maxima of the emerged constants. □

Precisely, this lemma states the *affine domination* of the term $\|\dot{q}\|_{\mathrm{L}^p}$ on the terms $\|q\|_{\mathrm{L}^p}$ and $\|{}_{\mathrm{c}}\mathrm{D}^\alpha_{a+}[q]\|_{\mathrm{L}^p}$ for any $q \in \mathrm{W}^{1,p}_{q_a}([a,b],\mathbb{R}^n)$. This characteristic of $\mathrm{W}^{1,p}_{q_a}([a,b],\mathbb{R}^n)$ leads us to the following proposition.

Proposition VI.2 (Coercivity of $\mathcal{L}$)**.** *Let us assume that L satisfies:*

$$L(x,v_\alpha,v,t) \geq c_0\|v\|^p + \sum_{k=1}^{N} c_k \|x\|^{d_{0,k}} \|v_\alpha\|^{d_{\alpha,k}} \|v\|^{d_{1,k}}, \tag{VI.12}$$

where $c_0 > 0$, $N \in \mathbb{N}^*$, $c_k \in \mathbb{R}$ and $d_{0,k}$, $d_{\alpha,k}$, $d_{1,k} \in \mathbb{R}^+$ satisfying $d_{0,k} + d_{\alpha,k} + d_{1,k} < p$. Then, $\mathcal{L}$ is coercive on $\mathrm{W}^{1,p}_{q_a}([a,b],\mathbb{R}^n)$.

Proof. Let us define $r_k = p/(d_{0,k} + d_{\alpha,k} + d_{1,k}) > 1$ and let r'_k be the adjoint of r_k. The Hölder's inequality leads to

$$\mathcal{L}(q) \geq c_0\|\dot{q}\|^p_{\mathrm{L}^p} - \sum_{k=1}^{N} (b-a)^{1/r'_k} |c_k| \|q\|^{d_{0,k}}_{\mathrm{L}^p} \|{}_{\mathrm{c}}\mathrm{D}^\alpha_{a+}[q]\|^{d_{\alpha,k}}_{\mathrm{L}^p} \|\dot{q}\|^{d_{1,k}}_{\mathrm{L}^p} \tag{VI.13}$$

for every $q \in \mathrm{W}^{1,p}_{q_a}([a,b],\mathbb{R}^n)$. From Lemma VI.3 and from the hypothesis $d_{0,k} + d_{\alpha,k} + d_{1,k} < p$, we conclude that

$$\lim_{\substack{\|\dot{q}\|_{\mathrm{L}^p} \to \infty \\ q \in \mathrm{W}^{1,p}_{q_a}([a,b],\mathbb{R}^n)}} \mathcal{L}(q) = +\infty. \tag{VI.14}$$

Still from Lemma VI.3, we see that $\|\dot{q}\|_{\mathrm{L}^p} \to \infty$ is equivalent to $\|q\|_{\mathrm{W}^{1,p}} \to \infty$ in $\mathrm{W}^{1,p}_{q_a}([a,b],\mathbb{R}^n)$. Then, the proof is complete. □

Remark VI.2. As usual in the classical theory, the shape of the Lagrangian L fixes the coefficient p in order to ensure the coercivity of $\mathcal{L}$. In the fractional context, we can adjust the coefficient α in order to also ensure the (α,p)-regularity of L. We refer to Section VI.2.4 for examples.

Remark VI.3. One can develop similar results for variational problems with final condition or with mixing of initial and final conditions. For more general examples of set E, in order to get coercivity, a structure of E implying the *domination* of one of terms q, ${}_{\mathrm{c}}\mathrm{D}^\alpha_{a+}[q]$ and $\dot{q}$ has to be associated with a Lagrangian controlled from below by a map preserving this *domination*.

Remark VI.4. An additional difficulty emerges if the Lagrangian functional does not involve the classical derivative but only a fractional one. In this context, one has to introduce a functional space that is reflexive and that ensures the *affine domination* of the term $\mathrm{D}^\alpha_{a+}[q]$ on q. We refer to [46] for such an example.

VI.2.4 Examples

The sufficient conditions provided in Sections VI.2.2 and VI.2.3 concern quasi-polynomial behaviours. Consequently, in this section, we consider such a convex Lagrangian.

Example VI.1. Let $q_a \in \mathbb{R}^n$ and $\mathrm{E} = \mathrm{W}^{1,p}_{q_a}([a,b],\mathbb{R}^n)$. We consider the following convex Lagrangian:

$$L(x,v_\alpha,v,t) = \|x\|^{d_0} + \|v_\alpha\|^{d_\alpha} + \|v\|^p, \tag{VI.15}$$

with d_0, $d_\alpha > 1$. Proposition VI.2 gives the coercivity of $\mathcal{L}$ on $\mathrm{W}^{1,p}_{q_a}([a,b],\mathbb{R}^n)$. Proposition VI.1 gives the (α,p)-regularity of L for every $0 < \alpha < 1$ satisfying $\alpha \leq \frac{p+(p-1)d_\alpha}{pd_\alpha}$. Finally, Theorem VI.1 gives the existence of a minimizer for $\mathcal{L}$ for every $0 < \alpha < 1$ satisfying $\alpha \leq \frac{p+(p-1)d_\alpha}{pd_\alpha}$.

In particular, since $\frac{1}{p'} \leq \frac{p+(p-1)d_\alpha}{pd_\alpha}$ for every $d_\alpha > 1$, note that $\mathcal{L}$ admits a minimizer for every $0 < \alpha \leq \frac{1}{p'}$ independently of the value of $d_\alpha > 1$.

This example shows the importance of the regularity results on the operator $\mathrm{I}^{1-\alpha}_{a+}$ given in Proposition V.2 and also of the embedding $\mathrm{W}^{1,p}([a,b],\mathbb{R}^n) \hookrightarrow \mathrm{L}^\infty([a,b],\mathbb{R}^n)$. Indeed, without these results, we could obtain the existence of a minimizer only for $d_0 \leq p$ and $d_\alpha \leq p$. Thanks to these regularity properties, we can deduce existence of a minimizer for a larger set of situations. Let us give the conclusions of the previous example in the special cases $p = 2$ and $p = 4$.

Example VI.2 (p=2)**.** Let $q_a \in \mathbb{R}^n$ and $\mathrm{E} = \mathrm{W}^{1,2}_{q_a}([a,b],\mathbb{R}^n)$. We consider the following convex Lagrangian:

$$L(x, v_\alpha, v, t) = \|x\|^{d_0} + \|v_\alpha\|^{d_\alpha} + \|v\|^2, \tag{VI.16}$$

with d_0, $d_\alpha > 1$. Then, $\mathcal{L}$ admits a minimizer for every $0 < \alpha < 1$ such that $\alpha \leq \frac{2+d_\alpha}{2d_\alpha}$. In particular, $\mathcal{L}$ admits a minimizer for every $0 < \alpha \leq 1/2$ independently of the value of $d_\alpha > 1$.

In the case $d_\alpha = 2$, $\mathcal{L}$ admits a minimizer for every $0 < \alpha < 1$. In the case $d_\alpha = 4$, $\mathcal{L}$ admits a minimizer for every $0 < \alpha \leq 3/4$.

Example VI.3 (p=4)**.** Let $q_a \in \mathbb{R}^n$ and $\mathrm{E} = \mathrm{W}^{1,4}_{q_a}([a,b],\mathbb{R}^n)$. We consider the following convex Lagrangian:

$$L(x, v_\alpha, v, t) = \|x\|^{d_0} + \|v_\alpha\|^{d_\alpha} + \|v\|^4, \tag{VI.17}$$

with d_0, $d_\alpha > 1$. Then, $\mathcal{L}$ admits a minimizer for every $0 < \alpha < 1$ such that $\alpha \leq \frac{4+3d_\alpha}{4d_\alpha}$. In particular, $\mathcal{L}$ admits a minimizer for every $0 < \alpha \leq 3/4$ independently of the value of $d_\alpha > 1$.

In the case $d_\alpha = 4$, $\mathcal{L}$ admits a minimizer for every $0 < \alpha < 1$. In the case $d_\alpha = 8$, $\mathcal{L}$ admits a minimizer for every $0 < \alpha \leq 7/8$.

As mentioned before, the convexity hypothesis of Theorem VI.1 is very restrictive. Consequently, Theorem VI.1 can only cover this kind of examples. However, in Section VI.4, we provide improved versions of Theorem VI.1 with weaker convexity assumptions by assuming stronger hypotheses on the continuity of L. These versions are accompanied by more general examples.

VI.3 Necessary optimality condition of Euler-Lagrange type

The aim of this section is to provide a characterization of the critical points of $\mathcal{L}$ (defined on E) as it is done in Section V.3.1 in the context of C^1-variable. In particular, such a result directly gives a necessary condition for the minimizer given in Theorem VI.1.

Throughout this section, we denote by $\mathrm{C}^\infty_\mathrm{c}([a,b],\mathbb{R}^n)$ the set of all infinitely differentiable functions with compact support in $]a,b[$ and we assume additionally that

- L satisfies the assumptions of Proposition VI.1. In particular, L is (α,p)-regular.
- E is open in the $\mathrm{C}^\infty_\mathrm{c}([a,b],\mathbb{R}^n)$-direction *i.e.*

$$\forall q \in \mathrm{E},\ \forall w \in \mathrm{C}^\infty_\mathrm{c}([a,b],\mathbb{R}^n),\ \exists 0 < \eta \leq 1,\ \forall \varepsilon \in [-\eta,\eta],\ q + \varepsilon w \in \mathrm{E}. \tag{VI.18}$$

In what follows, $\mathrm{C}^\infty_\mathrm{c}([a,b],\mathbb{R}^n)$ is the set of *variations* of $\mathcal{L}$ and $q \in \mathrm{E}$ is said to be a critical point of $\mathcal{L}$ if $D\mathcal{L}(q)(w) = 0$ for every variations $w \in \mathrm{C}^\infty_\mathrm{c}([a,b],\mathbb{R}^n)$. Recall that $D\mathcal{L}(q)(w)$ denotes the Gâteaux-differential of $\mathcal{L}$ at q in direction w.

In this section, we first prove the differentiability of $\mathcal{L}$ at every $q \in \mathrm{E}$ in every direction $w \in \mathrm{C}^\infty_\mathrm{c}([a,b],\mathbb{R}^n)$. Then, we characterize the critical points of $\mathcal{L}$ as the weak solutions of a fractional Euler-Lagrange equation. In particular, a necessary condition for a curve $q \in \mathrm{E}$ to be a minimizer of $\mathcal{L}$ is to be a weak solution of this fractional Euler-Lagrange equation.

Note that *weak solution* means *solution of the equation almost everywhere on* $[a,b]$.

Proposition VI.3. *We assume that L satisfies the assumptions of Proposition VI.1. Then, $\mathcal{L}$ is differentiable at every $q \in \mathrm{E}$ and every $w \in \mathrm{C}^\infty_\mathrm{c}([a,b],\mathbb{R}^n)$ with*

$$D\mathcal{L}(q)(w) = \int_a^b \Bigg[\frac{\partial L}{\partial x}(q(\tau), {}_\mathrm{c}\mathrm{D}^\alpha_{a+}[q](\tau), \dot{q}(\tau), \tau) \cdot w(\tau) + \frac{\partial L}{\partial v_\alpha}(q(\tau), {}_\mathrm{c}\mathrm{D}^\alpha_{a+}[q](\tau), \dot{q}(\tau), \tau) \cdot {}_\mathrm{c}\mathrm{D}^\alpha_{a+}[w](\tau) + \frac{\partial L}{\partial v}(q(\tau), {}_\mathrm{c}\mathrm{D}^\alpha_{a+}[q](\tau), \dot{q}(\tau), \tau) \cdot \dot{w}(\tau) \Bigg]\, d\tau.$$

In order to prove Proposition VI.3, we first need the following lemma.

Lemma VI.4. *Let $M \geq 1$ and $P \in \mathcal{P}_M^{(\alpha,p)}$. Then, for every $q \in \mathrm{E}$ and every $w \in \mathrm{C}_\mathrm{c}^\infty([a,b],\mathbb{R}^n)$, there exist $u \in \mathrm{L}^M([a,b],\mathbb{R}^+)$ such that for every $\varepsilon \in [-\eta,\eta]$, it holds*

$$P(q+\varepsilon w, {}_\mathrm{c}\mathrm{D}^\alpha_{a+}[q] + \varepsilon {}_\mathrm{c}\mathrm{D}^\alpha_{a+}[w], \dot{q}+\varepsilon\dot{w}, \cdot) \leq u. \tag{VI.19}$$

Proof. We only prove this result in the case $(1-\alpha)p < 1$. The other ones are derived in a similar way. Let $q \in \mathrm{E}$ and $w \in \mathrm{C}_\mathrm{c}^\infty([a,b],\mathbb{R}^n)$. For every $k = 1,\ldots,N$, for almost every $t \in [a,b]$ and every $\varepsilon \in [-\eta,\eta]$, it holds

$$\begin{aligned}\varphi_k(q(t)+\varepsilon w(t),t)\|{}_\mathrm{c}\mathrm{D}^\alpha_{a+}[q](t) + \varepsilon {}_\mathrm{c}\mathrm{D}^\alpha_{a+}[w](t)\|^{d_{\alpha,k}}\|\dot{q}(t)+\varepsilon\dot{w}(t)\|^{d_{1,k}} \\ \leq c_k(\|{}_\mathrm{c}\mathrm{D}^\alpha_{a+}[q](t)\|^{d_{\alpha,k}} + \|{}_\mathrm{c}\mathrm{D}^\alpha_{a+}[w](t)\|^{d_{\alpha,k}})(\|\dot{q}(t)\|^{d_{1,k}} + \|\dot{w}(t)\|^{d_{1,k}}),\end{aligned} \tag{VI.20}$$

where $c_k = 2^{d_{\alpha,k}+d_{1,k}} \max_{[a,b]\times[-\eta,\eta]} \varphi_k(q(t)+\varepsilon w(t),t)$. From the assumptions made on P (see Definition VI.3), we conclude that the right-hand side term belongs to $\mathrm{L}^M([a,b],\mathbb{R}^+)$ and is independent of $\varepsilon \in [-\eta,\eta]$. The proof is complete. ☐

Proof of Proposition VI.3. We only prove this result in the case $(1-\alpha)p < 1$. The other ones can be derived in a similar way. Let $q \in \mathrm{E}$ and $w \in \mathrm{C}_\mathrm{c}^\infty([a,b],\mathbb{R}^n)$. We define

$$f_{q,w}(t,\varepsilon) = L(q(t)+\varepsilon w(t), {}_\mathrm{c}\mathrm{D}^\alpha_{a+}[q](t) + \varepsilon {}_\mathrm{c}\mathrm{D}^\alpha_{a+}[w](t), \dot{q}(t)+\varepsilon\dot{w}(t), t) \tag{VI.21}$$

for every $\varepsilon \in [-\eta,\eta]$ and almost every $t \in [a,b]$. Then, we define

$$\begin{array}{rccl} g_{q,w} : & [-\eta,\eta] & \longrightarrow & \mathbb{R} \\ & \varepsilon & \longmapsto & \mathcal{L}(q+\varepsilon w) = \displaystyle\int_a^b f_{q,w}(\tau,\varepsilon)\, d\tau. \end{array} \tag{VI.22}$$

Let us prove that $g_{q,w}$ is differentiable in 0 using the classical theorem of differentiation under the integral sign. Indeed, for almost every $t \in [a,b]$, $f_{q,w}(t,\cdot)$ is differentiable on $[-\eta,\eta]$ with

$$\frac{\partial f_{q,w}}{\partial \varepsilon}(t,\varepsilon) = \frac{\partial L}{\partial x}(\star_\varepsilon)\cdot w(t) + \frac{\partial L}{\partial v_\alpha}(\star_\varepsilon)\cdot {}_\mathrm{c}\mathrm{D}^\alpha_{a+}[w](t) + \frac{\partial L}{\partial v}(\star_\varepsilon)\cdot \dot{w}(t), \tag{VI.23}$$

where $\star_\varepsilon = (q(t)+\varepsilon w(t), {}_\mathrm{c}\mathrm{D}^\alpha_{a+}[q](t) + \varepsilon {}_\mathrm{c}\mathrm{D}^\alpha_{a+}[w](t), \dot{q}(t)+\varepsilon\dot{w}(t), t)$. Then, since L satisfies the assumptions of Proposition VI.1 and from Lemma VI.4, there exist $u_0 \in \mathrm{L}^1([a,b],\mathbb{R}^+)$, $u_\alpha \in \mathrm{L}^s([a,b],\mathbb{R}^+)$ (where $s = \frac{p}{(2-\alpha)p-1}$) and $u_1 \in \mathrm{L}^{p'}([a,b],\mathbb{R}^+)$ such that, for every $\varepsilon \in [-\eta,\eta]$ and for almost every $t \in [a,b]$, it holds

$$\left|\frac{\partial f_{q,w}}{\partial \varepsilon}(t,\varepsilon)\right| \leq u_0(t)\|w(t)\| + u_\alpha(t)\|{}_\mathrm{c}\mathrm{D}^\alpha_{a+}[w](t)\| + u_1(t)\|\dot{w}(t)\|. \tag{VI.24}$$

Since $w \in \mathrm{L}^\infty([a,b],\mathbb{R}^n)$, ${}_\mathrm{c}\mathrm{D}^\alpha_{a+}[w] \in \mathrm{L}^{s'}([a,b],\mathbb{R}^n)$ ($s' = \frac{p}{1-(1-\alpha)p}$) and $\dot{w} \in \mathrm{L}^p([a,b],\mathbb{R}^n)$, we conclude that the right-hand side term belongs to $\mathrm{L}^1([a,b],\mathbb{R}^+)$ and is independent of ε. Consequently, from the theorem of differentiation under the integral sign, we obtain that $g_{q,w}$ is differentiable with

$$\forall \varepsilon \in [-\eta,\eta],\ \frac{dg_{q,w}}{d\varepsilon}(\varepsilon) = \int_a^b \frac{\partial f_{q,w}}{\partial \varepsilon}(\tau,\varepsilon)\, d\tau. \tag{VI.25}$$

The proof is completed taking $\varepsilon = 0$ in the previous equality. ☐

Finally, Proposition VI.3 leads to the following theorem.

Theorem VI.2. *We assume that L satisfies the assumptions of Proposition VI.1. Let $q \in \mathrm{E}$. Then, q is a critical point of $\mathcal{L}$ if and only if q is a weak solution of the fractional Euler-Lagrange equation given by*

$$\frac{d}{dt}\left[\frac{\partial L}{\partial v}(q, {}_\mathrm{c}\mathrm{D}^\alpha_{a+}[q], \dot{q}, \cdot) + \mathrm{I}^{1-\alpha}_{b-}\left[\frac{\partial L}{\partial v_\alpha}(q, {}_\mathrm{c}\mathrm{D}^\alpha_{a+}[q], \dot{q}, \cdot)\right]\right] = \frac{\partial L}{\partial x}(q, {}_\mathrm{c}\mathrm{D}^\alpha_{a+}[q], \dot{q}, \cdot). \tag{EL$^{\alpha,1}$}$$

Proof. Let $q \in \mathrm{E}$. Since L is (α, p)-regular, $\partial L/\partial x(q, {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q], \dot{q}, \cdot) \in \mathrm{L}^1([a,b],\mathbb{R}^n)$. We denote by F_q an absolutely continuous antiderivative of $\partial L/\partial x(q, {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q], \dot{q}, \cdot)$. From Proposition VI.3 and using the fractional integration by parts (Proposition V.6) and an usual one, we obtain that q is a critical point of $\mathcal{L}$ if and only if

$$\forall w \in \mathrm{C}^{\infty}_{\mathrm{c}}([a,b],\mathbb{R}^n), \int_a^b \left[\frac{\partial L}{\partial v}(q(\tau), {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q](\tau), \dot{q}(\tau), \tau) + \mathrm{I}^{1-\alpha}_{b-}\left[\frac{\partial L}{\partial v_\alpha}(q, {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q], \dot{q}, \cdot)\right](\tau) - F_q(\tau) \right] \cdot \dot{w}(\tau)\, d\tau = 0, \quad \text{(VI.26)}$$

if and only if there exists $c \in \mathbb{R}^n$ such that

$$\frac{\partial L}{\partial v}(q(t), {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q](t), \dot{q}(t), t) + \mathrm{I}^{1-\alpha}_{b-}\left[\frac{\partial L}{\partial v_\alpha}(q, {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q], \dot{q}, \cdot)\right](t) = F_q(t) + c, \quad \text{(VI.27)}$$

for almost every $t \in [a,b]$, see [55]. Since F_q is absolutely continuous, the proof is complete. ☐

Finally, combining Theorems VI.1 and VI.2, we obtain the following corollary stating the existence of a minimizer for $\mathcal{L}$ and giving a necessary condition for this minimizer.

Corollary VI.1. *If the following assumptions are satisfied:*
- *L satisfies the assumptions of Proposition VI.1;*
- *$\mathcal{L}$ is coercive on* E*;*
- *$L(\cdot,\cdot,\cdot,t)$ is convex on $(\mathbb{R}^n)^3$ for every $t \in [a,b]$;*

then there exists a minimizer for $\mathcal{L}$ that is a weak solution of the fractional Euler-Lagrange equation ($\mathrm{EL}^{\alpha,1}$).

Example VI.4. Let $q_a \in \mathbb{R}^n$, $p = 2$ and $\mathrm{E} = \mathrm{W}^{1,2}_{q_a}([a,b],\mathbb{R}^n)$. We consider the following Lagrangian L:

$$L(x, v_\alpha, v, t) = \frac{1}{2}(\|x\|^2 + \|v_\alpha\|^2 + \|v\|^2). \quad \text{(VI.28)}$$

Then, $\mathcal{L}$ admits a minimizer $\bar{q}$ for every $0 < \alpha < 1$, from Example VI.2. Moreover, $\bar{q}$ is a weak solution of

$$\frac{d}{dt}\Big[\dot{q} + \mathrm{I}^{1-\alpha}_{b-}\big[{}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q]\big]\Big] = q. \quad \text{(VI.29)}$$

VI.4 Some improvements for Theorem VI.1

The convexity hypothesis in Theorem VI.1 is a strong constraint. In this section, our aim is to prove two versions of Theorem VI.1 releasing the convexity hypothesis in the first and/or second variable. This is allowed by stronger assumptions on the continuity of L in terms of uniform equicontinuity.

Note that the releasing of the convexity assumption in the second variable is valid only in the case $(1-\alpha)p > 1$ and that the Ascoli's theorem is instrumental to arrive to such a result, see Section VI.4.2.

VI.4.1 Releasing of the convexity assumption in the first variable

We first recall the following definition.

Definition VI.4. We say that $(L(\cdot, v_\alpha, v, t))_{(v_\alpha,v,t)\in(\mathbb{R}^n)^2\times[a,b]}$ is uniformly equicontinuous if

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall (x_1,x_2) \in (\mathbb{R}^n)^2,\ \|x_1 - x_2\| \leq \delta \Longrightarrow \forall (v_\alpha, v, t) \in (\mathbb{R}^n)^2 \times [a,b],\ |L(x_1, v_\alpha, v, t) - L(x_2, v_\alpha, v, t)| \leq \varepsilon. \quad \text{(VI.30)}$$

This condition is satisfied by every Lagrangian L with bounded $\partial L/\partial x$ for example. This notion leads to the following theorem.

Theorem VI.3 (Tonelli)**.** *If the following assumptions are satisfied:*
- *$(L(\cdot, v_\alpha, v, t))_{(v_\alpha,v,t)\in(\mathbb{R}^n)^2\times[a,b]}$ is uniformly equicontinuous;*

– *L is (α, p)-regular;*
– *$\mathcal{L}$ is coercive on* E*;*
– *$L(x, \cdot, \cdot, t)$ is convex on $(\mathbb{R}^n)^2$ for every $(x,t) \in \mathbb{R}^n \times [a,b]$;*

then there exists a minimizer for $\mathcal{L}$.

Proof. With a similar proof as the one of Theorem VI.1, we construct a weakly convergent sequence $(q_k)_{k\in\mathbb{N}} \subset \mathrm{E}$ in $\mathrm{W}^{1,p}([a,b],\mathbb{R}^n)$ with $\bar{q} \in \mathrm{E}$ its weak limit, and such that $\lim_{k\to\infty} \mathcal{L}(q_k)$ tends to $\inf_{q\in\mathrm{E}} \mathcal{L}(q)$. Let $\varepsilon > 0$. Let $\delta > 0$ given by Definition VI.4. The compact embedding $\mathrm{W}^{1,p}([a,b],\mathbb{R}^n) \hookrightarrow \mathrm{C}([a,b],\mathbb{R}^n)$ gives the existence of $K \in \mathbb{N}$ such that for every $k \geq K$, $\|q_k - \bar{q}\|_{\mathrm{L}^\infty} \leq \delta$. Consequently, for every $k \geq K$ and for almost every $t \in [a,b]$, it holds

$$|L(q_k(t), {}_{\mathrm{c}}\mathrm{D}^\alpha_{a+}[q_k](t), \dot{q}_k(t), t) - L(\bar{q}(t), {}_{\mathrm{c}}\mathrm{D}^\alpha_{a+}[q_k](t), \dot{q}_k(t), t)| \leq \varepsilon. \tag{VI.31}$$

Then, for every $k \geq K$, we have

$$\mathcal{L}(q_k) \geq \int_a^b L(\bar{q}(\tau), {}_{\mathrm{c}}\mathrm{D}^\alpha_{a+}[q_k](\tau), \dot{q}_k(\tau), \tau)\, d\tau - (b-a)\varepsilon. \tag{VI.32}$$

From the convexity hypothesis and using the same strategy as in the proof of Theorem VI.1, we obtain by passing to the limit on k

$$\inf_{q\in\mathrm{E}} \mathcal{L}(q) \geq \mathcal{L}(\bar{q}) - (b-a)\varepsilon. \tag{VI.33}$$

Since this last equality is true for every $\varepsilon > 0$, the proof is complete. □

Remark VI.5. Actually, the uniform equicontinuity of $(L(\cdot, v_\alpha, v, t))_{(v_\alpha,v,t)\in(\mathbb{R}^n)^2\times[a,b]}$ also allows to weaken the assumption of (α,p)-regularity since the hypothesis $\partial L/\partial x(q, {}_{\mathrm{c}}\mathrm{D}^\alpha_{a+}[q], \dot{q}, \cdot) \in \mathrm{L}^1([a,b],\mathbb{R}^n)$ is not necessary in the proof of Theorem VI.3.

Such a version of the Tonelli theorem allows to consider examples of the following type.

Example VI.5. Let $n = 1$, $p = 4$, $q_a \in \mathbb{R}$ and $\mathrm{E} = \mathrm{W}^{1,4}_{q_a}([a,b],\mathbb{R})$. We consider the following Lagrangian:

$$L(x, v_\alpha, v, t) = \frac{1}{4}(\sin(x) + v_\alpha^4 + v^4), \tag{VI.34}$$

which is not convex in its first variable. Note that $(L(\cdot, v_\alpha, v, t))_{(v_\alpha,v,t)\in\mathbb{R}^2\times[a,b]}$ is uniformly equicontinuous. From Proposition VI.2, $\mathcal{L}$ is coercive on $\mathrm{W}^{1,4}_{q_a}([a,b],\mathbb{R})$ and from Proposition VI.1, L is (α,p)-regular for every $0 < \alpha < 1$. Consequently, from Theorem VI.3, $\mathcal{L}$ admits a minimizer for every $0 < \alpha < 1$ and from Theorem VI.2, this minimizer is solution of the following fractional Euler-Lagrange equation

$$\frac{d}{dt}\Big[\dot{q}^3 + \mathrm{I}^{1-\alpha}_{b-}\big[{}_{\mathrm{c}}\mathrm{D}^\alpha_{a+}[q]\big]\Big] = \cos(q)/4. \tag{VI.35}$$

VI.4.2 Releasing of the convexity assumption in the two first variables in the case $(1-\alpha)p > 1$

Using the Ascoli's theorem, we prove the following result.

Proposition VI.4. *We assume that $(1-\alpha)p > 1$. Let $(q_k)_{k\in\mathbb{N}}$ be a weakly convergent sequence of $\mathrm{W}^{1,p}([a,b],\mathbb{R}^n)$ and let $\bar{q}$ denotes its weak limit. Then, ${}_{\mathrm{c}}\mathrm{D}^\alpha_{a+}[q_k]$ is strongly convergent to ${}_{\mathrm{c}}\mathrm{D}^\alpha_{a+}[\bar{q}]$ in $\mathrm{L}^\infty([a,b],\mathbb{R}^n)$.*

In order to prove this proposition, we first need the following lemma corresponding to the fourth statement of Proposition V.2.

Lemma VI.5. *We assume that $(1-\alpha)p > 1$. Let $q \in \mathrm{L}^p([a,b],\mathbb{R}^n)$. Then, for every t_1, $t_2 \in [a,b]$, it holds*

$$\|\mathrm{I}^{1-\alpha}_{a+}[q](t_2) - \mathrm{I}^{1-\alpha}_{a+}[q](t_1)\| \leq \frac{2\|q\|_{\mathrm{L}^p}}{\Gamma(1-\alpha)(1-\alpha p')^{1/p'}} |t_2 - t_1|^{1-\alpha-(1/p)}. \tag{VI.36}$$

Proof. Let $a \leq t_1 \leq t_2 \leq b$. Then, we have

$$\Gamma(1-\alpha)\|\mathrm{I}^{1-\alpha}_{a+}[q](t_2) - \mathrm{I}^{1-\alpha}_{a+}[q](t_1)\| = \left\| \int_a^{t_2} (t_2-\tau)^{-\alpha} q(\tau)\, d\tau - \int_a^{t_1} (t_1-\tau)^{-\alpha} q(\tau)\, d\tau \right\|$$
$$\leq \left\| \int_{t_1}^{t_2} (t_2-\tau)^{-\alpha} q(\tau)\, d\tau \right\| + \left\| \int_a^{t_1} ((t_2-\tau)^{-\alpha} - (t_1-\tau)^{-\alpha}) q(\tau)\, d\tau \right\| \quad \text{(VI.37)}$$

Using Hölder's inequalities, it holds

$$\Gamma(1-\alpha)\|\mathrm{I}^{1-\alpha}_{a+}[q](t_2) - \mathrm{I}^{1-\alpha}_{a+}[q](t_1)\|$$
$$\leq \|q\|_{\mathrm{L}^p} \left(\int_{t_1}^{t_2} (t_2-\tau)^{-\alpha p'}\, d\tau \right)^{1/p'} + \|q\|_{\mathrm{L}^p} \left(\int_a^{t_1} ((t_1-\tau)^{-\alpha} - (t_2-\tau)^{-\alpha})^{p'}\, d\tau \right)^{1/p'}$$
$$\leq \|q\|_{\mathrm{L}^p} \left[\left(\int_{t_1}^{t_2} (t_2-\tau)^{-\alpha p'}\, d\tau \right)^{1/p'} + \left(\int_a^{t_1} (t_1-\tau)^{-\alpha p'} - (t_2-\tau)^{-\alpha p'}\, d\tau \right)^{1/p'} \right], \quad \text{(VI.38)}$$

which completes the proof by calculations. □

Proof of Proposition VI.4. Since $(q_k)_{k\in\mathbb{N}}$ is weakly convergent to $\bar{q}$ in $\mathrm{W}^{1,p}([a,b],\mathbb{R}^n)$, then $(\dot{q}_k)_{k\in\mathbb{N}}$ is weakly convergent to $\dot{\bar{q}}$ in $\mathrm{L}^p([a,b],\mathbb{R}^n)$ and then is bounded in $\mathrm{L}^p([a,b],\mathbb{R}^n)$. Moreover, since $\mathrm{I}^{1-\alpha}_{a+}$ is a linear continuous operator (Proposition V.1), $({}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q_k])_{k\in\mathbb{N}}$ converges weakly to ${}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[\bar{q}]$ in $\mathrm{L}^p([a,b],\mathbb{R}^n)$. Since $(\dot{q}_k)_{k\in\mathbb{N}}$ is bounded in $\mathrm{L}^p([a,b],\mathbb{R}^n)$, we deduce from Lemma VI.5 and from the classical Ascoli's theorem that $({}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q_k])_{k\in\mathbb{N}}$ is relatively compact in $\mathrm{C}([a,b],\mathbb{R}^n)$. Consequently, there exists a subsequence of $({}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q_k])_{k\in\mathbb{N}}$ converging strongly in $\mathrm{L}^\infty([a,b],\mathbb{R}^n)$ and its limit is ${}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[\bar{q}]$ by uniqueness of the weak limit. Now, let us prove by contradiction that the whole sequence $({}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q_k])_{k\in\mathbb{N}}$ converges strongly to ${}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[\bar{q}]$ in $\mathrm{L}^\infty([a,b],\mathbb{R}^n)$. If not, there exist $\varepsilon > 0$ and a subsequence $({}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q_{k_m}])_{m\in\mathbb{N}}$ such that $\|{}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[\bar{q}] - {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q_{k_m}]\|_{\mathrm{L}^\infty} \geq \varepsilon$ for every $m \in \mathbb{N}$. However, $(q_{k_m})_{m\in\mathbb{N}}$ is a subsequence of $(q_k)_{k\in\mathbb{N}}$. Then, $(q_{k_m})_{m\in\mathbb{N}}$ converges weakly to $\bar{q}$ in $\mathrm{W}^{1,p}([a,b],\mathbb{R}^n)$. With the same reasoning, we obtain that $({}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q_{k_m}])_{m\in\mathbb{N}}$ admits a subsequence converging strongly to ${}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[\bar{q}]$ in $\mathrm{L}^\infty([a,b],\mathbb{R}^n)$. This is a contradiction with $\|{}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[\bar{q}] - {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q_{k_m}]\|_{\mathrm{L}^\infty} \geq \varepsilon$ for every $m \in \mathbb{N}$. The proof is complete. □

Finally, let us recall the following definition.

Definition VI.5. We say that $(L(\cdot,\cdot,v,t))_{(v,t)\in\mathbb{R}^n\times[a,b]}$ is uniformly equicontinuous if

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall (x_1,x_2) \in (\mathbb{R}^n)^2,\ \forall (v_{\alpha,1},v_{\alpha,2}) \in (\mathbb{R}^n)^2,\ \|x_1 - x_2\| \leq \delta,\ \|v_{\alpha,1} - v_{\alpha,2}\| \leq \delta$$
$$\Longrightarrow \forall (v,t) \in \mathbb{R}^n \times [a,b],\ |L(x_1,v_{\alpha,1},v,t) - L(x_2,v_{\alpha,2},v,t)| \leq \varepsilon. \quad \text{(VI.39)}$$

This condition is satisfied by every Lagrangian L with bounded $\partial L/\partial x$ and bounded $\partial L/\partial v_\alpha$ for example. This notion leads to the following theorem.

Theorem VI.4. *If the following assumptions are satisfied:*
- $(1-\alpha)p > 1$*;*
- $(L(\cdot,\cdot,v,t))_{(v,t)\in\mathbb{R}^n\times[a,b]}$ *is uniformly equicontinuous;*
- L *is* (α,p)*-regular;*
- $\mathcal{L}$ *is coercive on* E*;*
- $L(x,v_\alpha,\cdot,t)$ *is convex on* $\mathbb{R}^n$ *for every* $(x,v_\alpha,t) \in (\mathbb{R}^n)^2 \times [a,b]$*;*

then there exists a minimizer for $\mathcal{L}$*.*

Proof. With a similar proof as the one of Theorem VI.1, we construct a weakly convergent sequence $(q_k)_{k\in\mathbb{N}} \subset \mathrm{E}$ in $\mathrm{W}^{1,p}([a,b],\mathbb{R}^n)$ with $\bar{q} \in \mathrm{E}$ its weak limit, and such that $\lim_{k\to\infty} \mathcal{L}(q_k)$ tends to $\inf_{q\in\mathrm{E}} \mathcal{L}(q)$. Let $\varepsilon > 0$. Let $\delta > 0$ given by Definition VI.5. From the compact embedding $\mathrm{W}^{1,p}([a,b],\mathbb{R}^n) \hookrightarrow \mathrm{C}([a,b],\mathbb{R}^n)$ and from Proposition VI.4, there exists $K \in \mathbb{N}$ such that for every $k \geq K$, $\|q_k - \bar{q}\|_{\mathrm{L}^\infty} \leq \delta$ and $\|{}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q_k] - {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[\bar{q}]\|_{\mathrm{L}^\infty} \leq \delta$. Consequently, for every $k \geq K$ and for almost every $t \in [a,b]$, it holds

$$|L(q_k(t), {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q_k](t), \dot{q}_k(t), t) - L(\bar{q}(t), {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[\bar{q}](t), \dot{q}_k(t), t)| \leq \varepsilon. \quad \text{(VI.40)}$$

Then, for every $k \geq K$, we have

$$\mathcal{L}(q_k) \geq \int_a^b L(\bar{q}(\tau), {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[\bar{q}](\tau), \dot{q}_k(\tau), \tau)\, d\tau - (b-a)\varepsilon. \tag{VI.41}$$

From the convexity hypothesis and using the same strategy of the proof of Theorem VI.1, we obtain by passing to the limit on k

$$\inf_{q \in \mathrm{E}} \mathcal{L}(q) \geq \mathcal{L}(\bar{q}) - (b-a)\varepsilon. \tag{VI.42}$$

Since this last equality is true for every $\varepsilon > 0$, the proof is complete. ☐

Remark VI.6. The condition $(1-\alpha)p > 1$ and the uniform equicontinuity of $(L(\cdot,\cdot,v,t))_{(v,t)\in\mathbb{R}^n\times[a,b]}$ also allows to weaken the assumption of (α,p)-regularity since the hypotheses $\partial L/\partial x(q, {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q], \dot{q}, \cdot) \in \mathrm{L}^1([a,b],\mathbb{R}^n)$ and $\partial L/\partial v_\alpha(q, {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q], \dot{q}, \cdot) \in \mathrm{L}^1([a,b],\mathbb{R}^n)$ are not necessary in the proof of Theorem VI.4.

Such a version of the Tonelli theorem allows to consider examples of the following type.

Example VI.6. Let $n = 1$, $p = 2$, $q_a \in \mathbb{R}$ and $\mathrm{E} = \mathrm{W}^{1,2}_{q_a}([a,b],\mathbb{R})$. We consider the following Lagrangian:

$$L(x, v_\alpha, v, t) = \sin(v_\alpha) - \sqrt{1+x^2} + \frac{1}{2}v^2, \tag{VI.43}$$

which is not convex in its two first variables. Note that $(L(\cdot,\cdot,v,t))_{(v,t)\in\mathbb{R}\times[a,b]}$ is equicontinuous. From Proposition VI.2, $\mathcal{L}$ is coercive on $\mathrm{W}^{1,2}_{q_a}([a,b],\mathbb{R})$. From Proposition VI.1, L is (α,p)-regular for every $0 < \alpha < 1$. Finally, from Theorem VI.4, we conclude that $\mathcal{L}$ admits a minimizer for every $0 < \alpha < 1/2$.

The aim of this chapter was to give a general framework in order to study the existence of solution for fractional variational problems. As a consequence, Theorems VI.1, VI.3 and VI.4 treat on general Lagrangian and consequently they are based on relatively strong hypotheses. Therefore, numerous Lagrangian do not satisfy their sufficient assumptions, in particular the convexity assumption in the third variable. We can cite the unidimensional ($n = 1$) Bolza's example associated with the Lagrangian $L(x, v_\alpha, v, t) = (v^2-1)^2 + v_\alpha^4 + x^4$. However, as usual with variational methods, these sufficient conditions can often be weakened accordingly to the specific problem studied. Note that the method developed in this chapter can be similarly applied in a lot of various situations as it is done in [46], [50] and [51].

Chapitre VII

Principe du Maximum de Pontryagin Faible pour des problèmes de contrôle optimal non linéaires avec dérivée fractionnaire de Caputo

Dans ce chapitre, nous donnons une nouvelle preuve du Principe du Maximum de Pontryagin Faible (WPMP en abrégé) pour des problèmes de contrôle optimal non linéaires avec dérivée fractionnaire de Caputo.

En effet, la technique des multiplicateurs de Lagrange habituellement utilisée dans la littérature nécessite des conditions qui ne sont pas vérifiées – notamment en termes d'isomorphisme entre espaces de Banach de la différentielle partielle de la fonction de contrainte. D'ailleurs, un contre-exemple donné dans le Chapitre IX, dans un cadre discret fractionnaire, montre que cette méthode ne peut pas être employée dans certains cadres.

Notre démonstration du WPMP s'appuie sur un calcul des variations (portant uniquement sur le contrôle et sur la condition initiale) et sur l'introduction d'un vecteur adjoint adéquat. La condition initiale peut être fixée ou non, auquel cas la condition de transversalité correspondante est établie.

Des résultats de type Cauchy-Lipschitz pour des équations différentielles avec dérivée fractionnaire de Caputo nécessaires à ce chapitre sont démontrés en Annexe C.

VII.1 Introduction

Optimal control theory, and in particular the Pontryagin Maximum Principle (denoted in short PMP), have an immense field of applications in various areas. We refer to the introduction of Chapter III for a short content and some reference works on this topic. In the same way, we refer to the introduction of Chapter V for a brief historical overview and some recalls on fractional calculus and its applications in numerous diversified scientific domains. As mentioned in Chapter V, the fractional calculus of variations was originally investigated by F. Riewe in 1996-97, see [187, 188]. Since then, it has attracted the attention of many authors and a significant literature regarding necessary optimality conditions and Noether's theorems has been developed, see *e.g.* [6, 14–19, 21, 27, 69, 89, 92, 172, 174–177] and references therein. We also refer to the recent book [157] for a state of the art.

A direct consequence is the emergence of studies on necessary optimality conditions in fractional control theory. The pioneering work using the calculus of variations on fractional optimal control problems is due to O. Agrawal [7] in 2004. He obtains, with the additional help of the Lagrange multiplier technique, a necessary optimality condition of type Euler-Lagrange. This result corresponds to the formulation of a *Weak* Pontryagin Maximum Principle (denoted in short WPMP). The denomination *Weak* refers to the substitution of the maximization condition of the Hamiltonian (obtained in the classical PMP, see *e.g.* [40]) by a weaker condition in terms of critical point of the Hamiltonian. Note that the denomination *WPMP* is not used in [7]: O. Agrawal speaks of *Euler-Lagrange equations for fractional optimal control problems.*

In [7], O. Agrawal considers general controlled systems defined with a Riemann-Liouville fractional derivative. Since, formulations of WPMPs have been investigated in several directions: Caputo fractional derivative [8], several Riemann-Liouville fractional derivatives of different orders [83], fractional controlled differential equations with integer highest order [128], time-optimal control problems for fractional systems [199], etc. Other studies on this topic can be found in the references of these papers. Nevertheless, due to the recency of this field, it can be noted that the subject is not extensively investigated yet. To conclude on the literature, we mention that G. Frederico and D. Torres also investigate conservation laws adapting the classical strategy of Noether to fractional optimal control problems admitting a symmetry, see [88, 90, 91].

In each of the previous papers, the authors use the Lagrange multiplier technique. Nevertheless, the crucial transition from optimization with constraint to optimization without constraint, via the introduction of a Lagrange multiplier, requires some conditions that are not checked. For example, the partial differential of the constraint function needs to be an isomorphism between Banach spaces. Besides, a counterexample studied in Chapter IX, in a discrete fractional case, shows that the Lagrange multiplier technique cannot be applied in some frameworks.

Hence, the main contribution of this chapter is to give a new and complete proof of the WPMP for optimal control problems with Caputo fractional derivative. Our proof does not use the Lagrange multiplier technique: it is only based on a calculus of variations (on the control and the initial condition) and on the introduction of an appropriate adjoint vector.

Remark VII.1. The initial (but not reached) objective of the present work was to provide a strong version of the PMP for fractional optimal control problems. Unfortunately, due to the non locality of the fractional operators, many difficulties first emerge in the derivation of variation vectors associated with needle-like variations (see needle-like variations in Section III.3.2.3 in the time scale context). Anyway, other obstructions would appear in the definition of the adjoint vector since we cannot ensure that the product of a given variation vector and the adjoint vector is constant. This is due to the lack of a simple fractional Leibniz formula.

Organization of the chapter. In Section VII.2.1, we settle the notion of admissible control and define a general optimal control problem involving a Caputo fractional derivative. Our main result (Weak Pontryagin Maximum Principle, Theorem VII.1) is stated in Section VII.2.2. Section VII.3 is devoted to the proof of Theorem VII.1. In Section VII.3.1, we show that the set of admissible controls is open, we define variations on the control and the initial condition and derive some useful properties. Finally, in Section VII.3.2, we apply a calculus of variations proving the WPMP.

VII.2 Main result

The notation $\|\cdot\|$ stands for the Euclidean norms of $\mathbb{R}^n$ for every $n \in \mathbb{N}^*$. Finally, for every $n \in \mathbb{N}^*$, every $x \in \mathbb{R}^n$ and every $R \geq 0$, the notation $\overline{B}(x,R)$ stands for the closed ball of $\mathbb{R}^n$ centered at x with radius R.

VII.2.1 General nonlinear optimal control problem with Caputo fractional derivative

Throughout this chapter, let $a \in \mathbb{R}$ and n, $m \in \mathbb{N}^*$. We denote by ${}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}$ the left fractional derivative of Caputo of order $0 < \alpha < 1$ given in Definition V.3. We consider the general fractional nonlinear controlled system

$$ {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q](t) = f(q(t), u(t), t), \tag{VII.1} $$

where $f : \mathbb{R}^n \times \mathbb{R}^m \times I_f \longrightarrow \mathbb{R}^n, (x,v,t) \longmapsto f(x,v,t)$ is a continuous function of class $\mathscr{C}^1$ with respect to its two first variables, $I_f \subset [a,+\infty[$ is an interval satisfying $a = \min I_f$ and $I_f \backslash \{a\} \neq \emptyset$ and where the control functions u belong to $\mathcal{U} = \mathrm{L}^{\infty}_{\mathrm{loc}}(I_f, \mathbb{R}^m)$. We recall that $u \in \mathrm{L}^{\infty}_{\mathrm{loc}}(I_f, \mathbb{R}^m)$ means that $u \in \mathrm{L}^{\infty}([c,d], \mathbb{R}^m)$ for all $(c,d) \in I_f$ such that $c < d$.

Before defining an optimal control problem associated with the control system (VII.1), the first question that has to be addressed is the question of the existence and uniqueness of a solution of (VII.1), for a given control function $u \in \mathcal{U}$ and a given initial condition $q(a) = q_a \in \mathbb{R}^n$. Since it did not exist up to now in the existing literature any Cauchy-Lipschitz like theorem, sufficiently general to cover such a situation, in the companion Appendix C, we derive a general Cauchy-Lipschitz theorem for general fractional nonlinear systems, ensuring the existence and the uniqueness of a maximal *weak* solution for a given Cauchy problem with Caputo fractional derivative under suitable assumptions like local Lipschitz continuity. We recall that the term *weak solution* means *solution of the integral formulation of the fractional differential equation*, see Definition C.4. In Appendix C, we also discuss some related issues like the behavior of maximal weak solutions at terminal points. This last point is instrumental to ensure the validity of the calculus of variations leading to our main result.

According to Theorem C.1, for every $(u, q_a) \in \mathcal{U} \times \mathbb{R}^n$, there exists a unique maximal weak solution of (VII.1) with the initial condition $q(a) = q_a$. This maximal weak solution is denoted by $q(\cdot, u, q_a)$ and is defined on the maximal interval $I(u, q_a) \subset I_f$. Recall that $q(t, u, q_a) = q_a + \mathrm{I}^\alpha_{a+}[f(q(\cdot, u, q_a), u, \cdot)](t)$ for every $t \in I(u, q_a)$, see Definition C.4. Moreover, one has:

– either $I(u, q_a) = I_f$, that is, $q(\cdot, u, q_a)$ is a *global* weak solution of (VII.1);
– or $I(u, q_a) = [a, b[$ where $b \in I_f \backslash \{a\}$, and in this case, $q(\cdot, u, q_a)$ is not bounded on $I(u, q_a)$.

We refer to Theorem C.2 for more details. The above results are instrumental to define the concept of admissible control.

Definition VII.1. For every $q_a \in \mathbb{R}^n$, the control $u \in \mathcal{U}$ is said to be *admissible* on $[a, b]$ for some given $b \in I_f \backslash \{a\}$ whenever $q(\cdot, u, q_a)$ is well defined on $[a, b]$, that is, $b \in I(u, q_a)$.

We are now in a position to define rigorously a general optimal control problem with Caputo fractional derivative (denoted in short $(\mathbf{OCP})^\alpha$). Let $f^0 : \mathbb{R}^n \times \mathbb{R}^m \times I_f \longrightarrow \mathbb{R}, (x, v, t) \longmapsto f^0(x, v, t)$ be a continuous function of class $\mathscr{C}^1$ with respect to its two first variables and let $b \in I_f \backslash \{a\}$.

Definition of $(\mathbf{OCP})^\alpha$: determine a trajectory q^* defined on $[a, b]$, weak solution of

$$ {}_{\mathrm{c}}\mathrm{D}^\alpha_{a+}[q](t) = f(q(t), u(t), t), \tag{VII.1} $$

and associated with a control $u^* \in \mathrm{L}^\infty([a, b], \mathbb{R}^m)$ minimizing the cost

$$ \int_a^b f^0(q(\tau), u(\tau), \tau)\, d\tau \tag{VII.2} $$

over all possible trajectories q defined on $[a, b]$, weak solutions of (VII.1) and associated with an admissible control $u \in \mathrm{L}^\infty([a, b], \mathbb{R}^m)$. The initial condition can be fixed or not.

VII.2.2 Weak Pontryagin Maximum Principle

The *Hamiltonian* associated with the fractional optimal control problem $(\mathbf{OCP})^\alpha$ is the function $H : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times [a, b] \longrightarrow \mathbb{R}$ defined by $H(x, v, w, t) = w \cdot f(x, v, t) + f^0(x, v, t)$. The main result of this chapter is the following WPMP proved in Section VII.3.

Theorem VII.1 (Weak Pontryagin Maximum Principle)**.** *If the trajectory q^*, defined on $[a, b]$ and associated with a control $u^* \in \mathrm{L}^\infty([a, b], \mathbb{R}^m)$, is a solution of $(\mathbf{OCP})^\alpha$, then there exists a mapping $p \in \mathrm{H}^\alpha_b([a, b], \mathbb{R}^n)$ (called adjoint vector), such that q^* and p are weak solutions on $[a, b]$ of*

$$ {}_{\mathrm{c}}\mathrm{D}^\alpha_{a+}[q^*](t) = \frac{\partial H}{\partial w}(q^*(t), u^*(t), p(t), t), \qquad {}_{\mathrm{c}}\mathrm{D}^\alpha_{b-}[p](t) = \frac{\partial H}{\partial x}(q^*(t), u^*(t), p(t), t). \tag{VII.3} $$

Moreover, it holds

$$ \frac{\partial H}{\partial v}(q^*(t), u^*(t), p(t), t) = 0_{\mathbb{R}^m}, \tag{VII.4} $$

for almost every $t \in [a, b]$. Furthermore, if the initial condition is free, one has the following transversality condition

$$ \int_a^b \frac{\partial H}{\partial x}(q^*(\tau), u^*(\tau), p(\tau), \tau)\, d\tau = 0_{\mathbb{R}^n}. \tag{VII.5} $$

As mentioned in the introduction of this chapter, a similar result is already derived in [8, 91] where the proofs are based on the Lagrange multiplier technique but some necessary conditions are not checked. We give in Section VII.3 an entire proof of Theorem VII.1 only based on a calculus of variations (on the control and the initial condition) and on the introduction of the appropriate adjoint vector.

Remark VII.2. The transversality condition (VII.5) can also be written as

$$\mathrm{I}_{b-}^{1-\alpha}[p](a) = 0. \tag{VII.6}$$

VII.3 Proof of Theorem VII.1

In Section VII.3.1, we introduce variations on the control and the initial condition and we derive relative results that are instrumental in order to prove Theorem VII.1 in Section VII.3.2 with the help of a calculus of variations.

VII.3.1 Variations on admissible controls

Following the definition of an admissible control (see Definition VII.1), we denote by $\mathcal{UQ}^b_{\mathrm{ad}}$ the set of all $(u, q_a) \in \mathcal{U} \times \mathbb{R}^n$ such that u is an admissible control on $[a, b]$ associated with the initial condition q_a. It is endowed with the distance

$$d_{\mathcal{UQ}^b_{\mathrm{ad}}}((u, q_a), (u', q'_a)) = \|u - u'\|_{\mathrm{L}^\infty([a,b],\mathbb{R}^m)} + \|q_a - q'_a\|. \tag{VII.7}$$

Throughout this section, we consider $(u, q_a) \in \mathcal{UQ}^b_{\mathrm{ad}}$ and the corresponding weak solution $q(\cdot, u, q_a)$ of (VII.1) with $q(a) = q_a$. The proofs of the following results are only technical or very similar each other. Consequently, the details are given in Appendix D.

Preliminaries. In the first lemma below, we state that $\mathcal{UQ}^b_{\mathrm{ad}}$ is open. Precisely, we prove that $\mathcal{UQ}^b_{\mathrm{ad}}$ contains a neighbourhood of any of its point in L^∞ topology, which is useful in order to define variations of (u, q_a). The proofs of the two following lemmas are detailed in Section D.1 of Appendix D.

Lemma VII.1. *There exist $\nu > 0$ and $\eta > 0$ such that the set*

$$\mathrm{E}(u, q_a) = \{(u', q'_a) \in \mathcal{U} \times \mathbb{R}^n \mid \|u' - u\|_{\mathrm{L}^\infty([a,b],\mathbb{R}^m)} \leq \nu,\ \|q'_a - q_a\| \leq \eta\} \tag{VII.8}$$

is contained in $\mathcal{UQ}^b_{\mathrm{ad}}$.

This lemma is crucial since it ensures that a perturbation sufficiently small (in $\mathrm{L}^\infty \times \mathbb{R}^n$-sense) of an admissible couple (u, q_a) preserves its admissibility. This result guaranties the validity of the calculus of variations made in Section VII.3.2.

Lemma VII.2. *With the notations of Lemma VII.1, the mapping*

$$\begin{array}{rccc} F_{(u,q_a)}: & (\mathrm{E}(u, q_a), d_{\mathcal{UQ}^b_{\mathrm{ad}}}) & \longrightarrow & (\mathrm{C}([a, b], \mathbb{R}^n), \|\cdot\|_\infty) \\ & (u', q'_a) & \longmapsto & q(\cdot, u', q'_a) \end{array}$$

is Lipschitzian. In particular, for every $(u', q'_a) \in \mathrm{E}(u, q_a)$, $q(\cdot, u', q'_a)$ converges uniformly to $q(\cdot, u, q_a)$ on $[a, b]$ when u' tends to u in $\mathrm{L}^\infty([a, b], \mathbb{R}^m)$ and q'_a tends to q_a in $\mathbb{R}^n$.

Variation on u. Let $\bar{u} \in \mathrm{C}^\infty_{\mathrm{c}}([a, b], \mathbb{R}^m)$. Let us give the following series of technical results proved in Section D.2 of Appendix D.

Lemma VII.3. *There exists $\varepsilon_0 > 0$ such that $(u + \varepsilon\bar{u}, q_a) \in \mathcal{UQ}^b_{\mathrm{ad}}$, for every $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$.*

Lemma VII.4. *The mapping*

$$\begin{array}{rccc} F_{(u,q_a,\bar{u})}: & ([-\varepsilon_0, \varepsilon_0], |\cdot|) & \longrightarrow & (\mathrm{C}([a, b], \mathbb{R}^n), \|\cdot\|_\infty) \\ & \varepsilon & \longmapsto & q(\cdot, u + \varepsilon\bar{u}, q_a) \end{array}$$

is Lipschitzian. In particular, for every $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$, $q(\cdot, u + \varepsilon\bar{u}, q_a)$ converges uniformly to $q(\cdot, u, q_a)$ on $[a, b]$ as ε tends to 0.

We define the so-called *variation vector* $\omega_{\bar{u}}(\cdot, u, q_a)$, associated with the variation $\bar{u}$, as the unique weak solution on $[a, b]$ of the affine fractional Cauchy problem

$$ {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[\omega](t) = \frac{\partial f}{\partial x}(q(t, u, q_a), u(t), t) \times \omega(t) + \frac{\partial f}{\partial v}(q(t, u, q_a), u(t), t) \times \bar{u}(t), \quad \omega(a) = 0. \tag{VII.9} $$

The existence and uniqueness of $\omega_{\bar{u}}(\cdot, u, q_a)$ are ensured by Theorem C.3.

Proposition VII.1. *The mapping*

$$ \begin{array}{rccl} F_{(u,q_a,\bar{u})} : & ([-\varepsilon_0, \varepsilon_0], |\cdot|) & \longrightarrow & (\mathrm{C}([a,b], \mathbb{R}^n), \|\cdot\|_\infty) \\ & \varepsilon & \longmapsto & q(\cdot, u + \varepsilon \bar{u}, q_a) \end{array} \tag{VII.10} $$

is differentiable at 0*, and it holds* $DF_{(u,q_a,\bar{u})}(0) = \omega_{\bar{u}}(\cdot, u, q_a)$.

Variation on the initial condition q_a. Let $\bar{q}_a \in \mathbb{R}^n$. Let us give the following series of technical results proved in Section D.3 of Appendix D.

Lemma VII.5. *There exists* $\gamma_0 > 0$ *such that* $(u, q_a + \gamma \bar{q}_a) \in \mathcal{UQ}^b_{\mathrm{ad}}$ *for every* $\gamma \in [-\gamma_0, \gamma_0]$.

Lemma VII.6. *The mapping*

$$ \begin{array}{rccl} F_{(u,q_a,\bar{q}_a)} : & ([-\gamma_0, \gamma_0], |\cdot|) & \longrightarrow & (\mathrm{C}([a,b], \mathbb{R}^n), \|\cdot\|_\infty) \\ & \gamma & \longmapsto & q(\cdot, u, q_a + \gamma \bar{q}_a) \end{array} $$

is Lipschitzian. In particular, for every $\gamma \in [-\gamma_0, \gamma_0]$, $q(\cdot, u, q_a + \gamma \bar{q}_a)$ *converges uniformly to* $q(\cdot, u, q_a)$ *on* $[a, b]$ *as* γ *tends to* 0.

According to Theorem C.3, we define the *variation vector* $\omega_{\bar{q}_a}(\cdot, u, q_a)$, associated with the perturbation $\bar{q}_a$, as the unique weak solution on $[a, b]$ of the linear fractional Cauchy problem

$$ {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[\omega](t) = \frac{\partial f}{\partial x}(q(t, u, q_a), u(t), t) \times \omega(t), \quad \omega(a) = \bar{q}_a. \tag{VII.11} $$

Proposition VII.2. *The mapping*

$$ \begin{array}{rccl} F_{(u,q_a,\bar{q}_a)} : & ([-\gamma_0, \gamma_0], |\cdot|) & \longrightarrow & (\mathrm{C}([a,b], \mathbb{R}^n), \|\cdot\|_\infty) \\ & \gamma & \longmapsto & q(\cdot, u, q_a + \gamma \bar{q}_a) \end{array} \tag{VII.12} $$

is differentiable at 0*, and one has* $DF_{(u,q_a,\bar{q}_a)}(0) = \omega_{\bar{q}_a}(\cdot, u, q_a)$.

VII.3.2 Proof of the WPMP

Let us prove Theorem VII.1 and let $q^*_a = q^*(a)$, in particular $q(\cdot, u^*, q^*_a) = q^*$ and $[a, b] \subset I(u^*, q^*_a)$. We define $p \in \mathrm{H}^{\alpha}_{b}([a, b], \mathbb{R}^n)$ as the unique weak solution of the fractional affine Cauchy problem given by

$$ {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{b-}[p](t) = \left(\frac{\partial f}{\partial x}(q^*(t), u^*(t), t) \right)^{\mathrm{T}} \times p(t) + \frac{\partial f_0}{\partial x}(q^*(t), u^*(t), t), \qquad p(b) = 0. $$

The existence and uniqueness of p are ensured by the counterpart of Theorem C.3 for right fractional derivatives.

Firstly, we note that the introduction of p ensures that q^* and p are weak solutions on $[a, b]$ of

$$ {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q^*](t) = \frac{\partial H}{\partial w}(q^*(t), u^*(t), p(t), t), \qquad {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{b-}[p](t) = \frac{\partial H}{\partial x}(q^*(t), u^*(t), p(t), t). \tag{VII.13} $$

Proof of Equality (VII.4)**.** Let $\bar{u} \in \mathrm{C}_\mathrm{c}^\infty([a,b],\mathbb{R}^m)$. From Lemma VII.3, there exists $\varepsilon_0 > 0$ sufficiently small such that the function

$$\Phi_{\bar{u}}(\varepsilon) = \int_a^b f^0(q(\tau, u^* + \varepsilon\bar{u}, q_a^*), u^*(\tau) + \varepsilon\bar{u}(\tau), \tau)\, d\tau.$$

is well defined for every $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$. The differentiability of $\Phi_{\bar{u}}$ at $\varepsilon = 0$ comes from Lemma VII.4, Proposition VII.1 and from some arguments of uniform continuity and uniform convergence. It holds

$$\Phi'_{\bar{u}}(0) = \int_a^b \frac{\partial f^0}{\partial x}(q^*(\tau), u^*(\tau), \tau) \cdot \omega_{\bar{u}}(\tau, u^*, q_a^*) + \frac{\partial f^0}{\partial v}(q^*(\tau), u^*(\tau), \tau) \cdot \bar{u}(\tau)\, d\tau. \tag{VII.14}$$

Then, it holds

$$\begin{aligned}\Phi'_{\bar{u}}(0) = \int_a^b \Bigg(\left(\left(\frac{\partial f}{\partial x}(q^*(\tau), u^*(\tau), \tau)\right)^{\mathrm{T}} \times p(\tau) + \frac{\partial f^0}{\partial x}(q^*(\tau), u^*(\tau), \tau) \right) \cdot \omega_{\bar{u}}(\tau, u^*, q_a^*) \\ + \frac{\partial f^0}{\partial v}(q^*(\tau), u^*(\tau), \tau) \cdot \bar{u}(\tau) - \left(\frac{\partial f}{\partial x}(q^*(\tau), u^*(\tau), \tau) \times \omega_{\bar{u}}(\tau, u^*, q_a^*) \right) \cdot p(\tau)\, d\tau.\end{aligned} \tag{VII.15}$$

We recall that $\omega_{\bar{u}}(\cdot, u^*, q_a^*)$ and p satisfy

$$\omega_{\bar{u}}(t, u^*, q_a^*) = \mathrm{I}_{a+}^\alpha \left[\frac{\partial f}{\partial x}(q^*, u^*, \cdot) \times \omega_{\bar{u}}(\cdot, u^*, q_a^*) + \frac{\partial f}{\partial v}(q^*, u^*, \cdot) \times \bar{u} \right](t) \tag{VII.16}$$

and

$$p(t) = \mathrm{I}_{b-}^\alpha \left[\left(\frac{\partial f}{\partial x}(q^*, u^*, \cdot) \right)^{\mathrm{T}} \times p + \frac{\partial f_0}{\partial x}(q^*, u^*, \cdot) \right](t) \tag{VII.17}$$

for every $t \in [a,b]$. As a consequence, using the fractional integration by parts given in Proposition V.6, we obtain

$$\Phi'_{\bar{u}}(0) = \int_a^b \left(\left(\frac{\partial f}{\partial v}(q^*(\tau), u^*(\tau), \tau) \right)^{\mathrm{T}} \times p(\tau) + \frac{\partial f^0}{\partial v}(q^*(\tau), u^*(\tau), \tau) \right) \cdot \bar{u}(\tau)\, d\tau \tag{VII.18}$$

that is

$$\Phi'_{\bar{u}}(0) = \int_a^b \frac{\partial H}{\partial v}(q^*(\tau), u^*(\tau), p(\tau), \tau) \cdot \bar{u}(\tau)\, d\tau. \tag{VII.19}$$

Finally, since q^* is a solution of the fractional optimal control problem $(\mathbf{OCP})^\alpha$, $\Phi_{\bar{u}}$ admits a minimum at $\varepsilon = 0$ and then $\Phi'_{\bar{u}}(0) = 0$. Consequently, we deduce that

$$\int_a^b \frac{\partial H}{\partial v}(q^*(\tau), u^*(\tau), p(\tau), \tau) \cdot \bar{u}(\tau)\, d\tau = 0 \tag{VII.20}$$

for every $\bar{u} \in \mathrm{C}_\mathrm{c}^\infty([a,b],\mathbb{R}^m)$ which concludes the proof of Equality (VII.4).

Proof of Equality (VII.5)**.** Let $\bar{q}_a \in \mathbb{R}^n$. From Lemma VII.5, there exists $\gamma_0 > 0$ sufficiently small such that the function

$$\Phi_{\bar{q}_a}(\gamma) = \int_a^b f^0(q(\tau, u^*, q_a^* + \gamma\bar{q}_a), u^*(\tau), \tau)\, d\tau.$$

is well defined for every $\gamma \in [-\gamma_0, \gamma_0]$. Lemma VII.6, Proposition VII.2 and some arguments of uniform continuity and uniform convergence leads to the differentiability of $\Phi_{\bar{q}_a}$ at $\gamma = 0$ with

$$\Phi'_{\bar{q}_a}(0) = \int_a^b \frac{\partial f^0}{\partial x}(q^*(\tau), u^*(\tau), \tau) \cdot \omega_{\bar{q}_a}(\tau, u^*, q_a^*)\, d\tau. \tag{VII.21}$$

Then, it holds

$$\begin{aligned}\Phi'_{\bar{q}_a}(0) = \int_a^b \Bigg(\left(\left(\frac{\partial f}{\partial x}(q^*(\tau), u^*(\tau), \tau)\right)^{\mathrm{T}} \times p(\tau) + \frac{\partial f^0}{\partial x}(q^*(\tau), u^*(\tau), \tau) \right) \cdot \omega_{\bar{q}_a}(\tau, u^*, q_a^*) \\ - \left(\frac{\partial f}{\partial x}(q^*(\tau), u^*(\tau), \tau) \times \omega_{\bar{q}_a}(\tau, u^*, q_a^*) \right) \cdot p(\tau)\, d\tau.\end{aligned} \tag{VII.22}$$

We recall that $\omega_{\bar{q}_a}(\cdot, u^*, q_a^*)$ and p satisfy

$$\omega_{\bar{q}_a}(t, u^*, q_a^*) = \bar{q}_a + \mathrm{I}^{\alpha}_{a+} \left[\frac{\partial f}{\partial x}(q^*, u^*, \cdot) \times \omega_{\bar{q}_a}(\cdot, u^*, q_a^*) \right](t) \tag{VII.23}$$

and

$$p(t) = \mathrm{I}^{\alpha}_{b-} \left[\left(\frac{\partial f}{\partial x}(q^*, u^*, \cdot) \right)^{\mathrm{T}} \times p + \frac{\partial f_0}{\partial x}(q^*, u^*, \cdot) \right](t) \tag{VII.24}$$

for every $t \in [a, b]$. As a consequence, using the fractional integration by parts of Proposition V.6, we obtain

$$\Phi'_{\bar{q}_a}(0) = \int_a^b \left(\left(\frac{\partial f}{\partial x}(q^*(\tau), u^*(\tau), \tau) \right)^{\mathrm{T}} \times p(\tau) + \frac{\partial f^0}{\partial x}(q^*(\tau), u^*(\tau), \tau) \right) \cdot \bar{q}_a \, d\tau \tag{VII.25}$$

that is

$$\Phi'_{\bar{q}_a}(0) = \int_a^b \frac{\partial H}{\partial x}(q^*(\tau), u^*(\tau), p(\tau), \tau) \cdot \bar{q}_a \, d\tau. \tag{VII.26}$$

Now, let us assume that $(\mathbf{OCP})^{\alpha}$ is considered with free initial condition. Since q^* is a solution of $(\mathbf{OCP})^{\alpha}$, $\Phi_{\bar{q}_a}$ admits a minimum at $\gamma = 0$ and then $\Phi'_{\bar{q}_a}(0) = 0$. Consequently, we deduce that

$$\int_a^b \frac{\partial H}{\partial x}(q^*(\tau), u^*(\tau), p(\tau), \tau) \cdot \bar{q}_a \, d\tau = 0 \tag{VII.27}$$

for every $\bar{q}_a \in \mathbb{R}^n$ which concludes the proof of Equality (VII.5).

Chapitre VIII

Intégrateur variationnel pour des équations d'Euler-Lagrange fractionnaires

Dans cette étude inspirée de Bourdin L., Cresson J., Greff I. et Inizan P., Variational integrators for fractional Euler-Lagrange equations, accepté pour publication dans Applied Numerical Analysis and Computational Mathematics*, nous construisons un intégrateur variationnel pour l'équation d'Euler-Lagrange fractionnaire* (EL^α) *obtenue dans le Chapitre V. Plus précisémment, à l'aide des dérivées fractionnaires discrètes de Grünwald-Letnikov, nous construisons un schéma numérique pour* (EL^α) *préservant au niveau discret la structure variationelle intrinsèque de l'équation.*

Comme dans le cas classique, nous démontrons que cet intégrateur variationnel permet de préserver au niveau discret le théorème de type Noether énoncé dans la Proposition V.7.

VIII.1 Introduction

Several methods have been proposed to find the exact solutions of fractional (partial) differential equations, as Laplace, Mellin or Fourier transforms, see *e.g.* [141, 168]. However, these methods cannot be extended to most of nonlinear fractional (partial) differential equations. As a consequence, there has been a growing interest to develop numerical schemes for such equations, see *e.g.* [10, 80, 128]. These numerical methods are essentially extensions of classical ones, see *e.g.* the extension of the Adams-Bashforth-Moulton scheme or of the Richardson extrapolation in [79].

The notion of Grünwald-Letnikov fractional derivatives is defined as a limit of finite differences and coincides with the Riemann-Liouville's one on a wide class of functions. As a consequence, this notion is particularly suitable to define discrete fractional operators approximating the Riemann-Liouville fractional derivatives. In order to define a numerical scheme for a given (partial) differential equation involving Riemann-Liouville fractional derivatives, I. Podlubny in [185, 186] substitutes the continuous unknowns by discrete ones and replaces the fractional derivatives by the discrete Grünwald-Letnikov operators. This method is widely used in different fields. For example, we refer to [165, 166] for applications on fractional dispersion equations and to [196] for a fractional diffusion equation.

Due to the emergence of an asymmetric composition between left and right fractional operators, the fractional Euler-Lagrange equations are particularly difficult to solve explicitly. Consequently, it is of interest to develop efficient numerical schemes for such dynamical systems. In [28], authors apply the same method than I. Podlubny and provide numerical simulations for the fractional Pais-Uhlenbeck oscillator. The articles [9, 26, 75, 181] also apply this method on Euler-Lagrange equations for fractional optimal control problems. Another direct discretization of a particular fractional Euler-Lagrange equation is studied in [34].

Nevertheless, the fractional Euler-Lagrange equations admit a variational structure in the sense that they derive from a calculus of variations on a functional. This structure is intrinsic and induces strong constraints on the qualitative behaviour of the solutions. It is then important to preserve this structure at the discrete level. However, the numerical schemes previously mentioned are obtained via a direct discretization, that is an *algebraic* procedure only based on the differential writing of the equation. Consequently, there is no guarantee that the intrinsic variational structure of the equation is preserved.

We have seen in Chapter IV that the construction of a variational integrator is a suitable method in order to provide a numerical scheme for a given classical Euler-Lagrange equation preserving its variational structure at the discrete level. This method is well studied in [101, 159] for example. The aim of this chapter is to extend this method to the fractional case. Namely, we construct a variational integrator for the fractional Euler-Lagrange equation (EL^α) obtained in Chapter V. In the classical case, a variational integrator presents many interesting properties, as the conservation at the discrete level of constants of motion given by the classical Noether's theorem, see [101, 159]. In Section VIII.4, we prove that the variational integrator extended to the fractional case also allows to preserve the fractional Noether-type result stated in Proposition V.7.

Remark VIII.1. The construction of the variational integrator for (EL^α) leads us to develop a discrete calculus of variations involving the discrete fractional operators of Grünwald-Letnikov. A similar procedure is also developed in [32, 33] with other operators (called left and right fractional differences) without the point of view of variational integrators.

Organization of the chapter. Section VIII.2 is devoted to recalls on (discrete) Grünwald-Letnikov fractional derivatives. In Section VIII.3, we construct a variational integrator for (EL^α) and some numerical experiments are provided for the fractional Dirichlet example. In Section VIII.4, we prove that the variational integrator constructed allows to preserve at the discrete level the fractional Noether-type result stated in Proposition V.7.

VIII.2 Reminders about Grünwald-Letnikov fractional derivatives

This section is devoted to recalls on fractional operators of Grünwald-Letnikov (1867). All definitions and all results are extracted from [141, 185, 190].

VIII.2.1 Definitions of Grünwald-Letnikov fractional derivatives

In what follows, $n \in \mathbb{N}^*$ denotes the dimension and $(a,b) \in \mathbb{R}^2$ with $a < b$. Let $q : [a,b] \longrightarrow \mathbb{R}^n$ be a sufficiently smooth function. Recall that the classical derivative of q is defined by

$$\forall t \in [a,b],\ \frac{d}{dt}[q](t) = \lim_{h\to 0} \frac{q(t)-q(t-h)}{h}. \tag{VIII.1}$$

By induction on $k \in \mathbb{N}^*$, one can prove that the k^{th}-derivative of q is given by

$$\forall t \in [a,b],\ \left(\frac{d}{dt}\right)^k [q](t) = \lim_{h\to 0} \frac{1}{h^k} \sum_{r=0}^{k} \frac{(-k)(1-k)\dots(r-1-k)}{r!} q(t-rh). \tag{VIII.2}$$

The notions of Grünwald-Letnikov fractional derivatives are based on the generalization of (VIII.2). For every $\alpha > 0$, we define $\alpha_0 = 1$ and $\alpha_r = (-\alpha)(1-\alpha)\dots(r-1-\alpha)/r!$ for every $r \in \mathbb{N}^*$.

Definition VIII.1. Let $q : [a,b] \longrightarrow \mathbb{R}^n$. The left fractional derivative of Grünwald-Letnikov with inferior limit a of order $\alpha > 0$ of q is defined by

$$\forall t \in]a,b],\ {}_{\mathrm{GL}}\mathrm{D}^\alpha_{a+}[q](t) = \lim_{\substack{h\to 0^+\\ ph=t-a}} \frac{1}{h^\alpha} \sum_{r=0}^{p} \alpha_r q(t-rh), \tag{VIII.3}$$

provided that the right-hand side is well defined. The right fractional derivative of Grünwald-Letnikov with superior limit b of order $\alpha > 0$ of q is defined by

$$\forall t \in [a,b[,\ {}_{\mathrm{GL}}\mathrm{D}^\alpha_{b-}[q](t) = \lim_{\substack{h\to 0^+\\ ph=b-t}} \frac{1}{h^\alpha} \sum_{r=0}^{p} \alpha_r q(t+rh), \tag{VIII.4}$$

provided that the right-hand side is well defined.

Despite that the generalizations of Riemann-Liouville and Grünwald-Letnikov are constructed in a different way, they actually coincide for every $q \in \mathrm{C}^{[\alpha]+1}([a,b],\mathbb{R}^n)$ where $[\alpha]$ denotes the floor of $\alpha > 0$, see [185] for details. Hence, the definitions of the Grünwald-Letnikov fractional derivatives suggest an handy way to define discrete approximations of Riemann-Liouville fractional derivatives.

VIII.2.2 Discrete fractional derivatives of Grünwald-Letnikov

Let us introduce some discrete elements and notations available in the whole chapter. Let $N \geq 2$, let $h = (b-a)/N$ be the step size of the discretization and let $\mathbb{T} = \{t_k\}_{k=0,\dots,N} = \{a+kh\}_{k=0,\dots,N}$ be the usual regular partition of the interval $[a,b]$. Similarly to Section IV.3, we take anew the notations of the time scale setting. Since all functions defined on $\mathbb{T}$ with values in $\mathbb{R}^n$ are automatically continuous, we denote by $\mathrm{C}(\mathbb{T},\mathbb{R}^n)$ the set of all functions defined on $\mathbb{T}$ with values in $\mathbb{R}^n$.

Definition VIII.2. Let $q \in \mathrm{C}(\mathbb{T},\mathbb{R}^n)$. The left discrete fractional derivative of Grünwald-Letnikov of inferior limit a of order $\alpha > 0$ of q is defined by

$$\forall k = 1,\dots,N,\ \Delta^\alpha_{a+}[q](t_k) = \frac{1}{h^\alpha}\sum_{r=0}^{k} \alpha_r q(t_{k-r}). \tag{VIII.5}$$

The right discrete fractional derivative of Grünwald-Letnikov of superior limit b of order $\alpha > 0$ of q is defined by

$$\forall k = 0,\dots,N-1,\ \Delta^\alpha_{b-}[q](t_k) = \frac{1}{h^\alpha}\sum_{r=0}^{N-k} \alpha_r q(t_{k+r}). \tag{VIII.6}$$

Δ^α_{a+} (resp. Δ^α_{b-}) is an approximation of order 1 of the left (resp. right) Riemann-Liouville fractional derivative D^α_{a+} (resp. D^α_{b-}). We refer to [77, 185] for more details.

Remark VIII.2. With respect to the notations of the time scale setting, note that Δ^α_{a+} (resp. Δ^α_{b-}) is a mapping from $\mathrm{C}(\mathbb{T},\mathbb{R}^n)$ to $\mathrm{C}(\mathbb{T}_\kappa,\mathbb{R}^n)$ (resp. from $\mathrm{C}(\mathbb{T},\mathbb{R}^n)$ to $\mathrm{C}(\mathbb{T}^\kappa,\mathbb{R}^n)$).

Remark VIII.3. Note that Δ^α_{a+} and $-\Delta^\alpha_{b-}$ recover the backward and forward Euler approximations of the classical derivative d/dt in the special case $\alpha = 1$.

In this chapter, we are interested in the discretization of a fractional Euler-Lagrange equation (EL^α) involving a Caputo fractional derivative ${}_{\mathrm{c}}\mathrm{D}^\alpha_{a+}$ of order $0 < \alpha < 1$. We recall that for every $q \in \mathrm{C}^1([a,b],\mathbb{R}^n)$, it holds ${}_{\mathrm{c}}\mathrm{D}^\alpha_{a+}[q] = \mathrm{D}^\alpha_{a+}[q - q(a)]$, see Remark V.6. As a consequence, we introduce the following definitions.

Definition VIII.3. Let $q \in \mathrm{C}(\mathbb{T},\mathbb{R}^n)$. The left discrete fractional derivative of Caputo-Grünwald-Letnikov of inferior limit a of order $0 < \alpha < 1$ of q is defined by

$$\forall k = 1,\dots,N,\ {}_{\mathrm{c}}\Delta^\alpha_{a+}[q](t_k) = \Delta^\alpha_{a+}[q - q(a)](t_k). \tag{VIII.7}$$

The right discrete fractional derivative of Caputo-Grünwald-Letnikov of superior limit b of order $0 < \alpha < 1$ of q is defined by

$$\forall k = 0,\dots,N-1,\ {}_{\mathrm{c}}\Delta^\alpha_{b-}[q](t_k) = \Delta^\alpha_{b-}[q - q(b)](t_k). \tag{VIII.8}$$

Hence, the discrete operator ${}_{\mathrm{c}}\Delta^\alpha_{a+}$ (resp. ${}_{\mathrm{c}}\Delta^\alpha_{b-}$) is an approximation of the left (resp. right) Caputo fractional derivative ${}_{\mathrm{c}}\mathrm{D}^\alpha_{a+}$ (resp. ${}_{\mathrm{c}}\mathrm{D}^\alpha_{b-}$) for $0 < \alpha < 1$.

VIII.3 Variational integrator for fractional Euler-Lagrange equations

In what follows, we assume that $0 < \alpha < 1$. Let us recall the framework and the results obtained in Section V.3.1. Let L be a Lagrangian *i.e.* a continuous map of class $\mathscr{C}^1$ in its two first variables

$$\begin{array}{rccc} L: & \mathbb{R}^n \times \mathbb{R}^n \times [a,b] & \longrightarrow & \mathbb{R} \\ & (x,v,t) & \longmapsto & L(x,v,t) \end{array} \tag{VIII.9}$$

and let $\mathcal{L}$ be the following fractional Lagrangian functional:

$$\begin{array}{rccc} \mathcal{L}: & \mathrm{C}^1([a,b],\mathbb{R}^n) & \longrightarrow & \mathbb{R} \\ & q & \longmapsto & \displaystyle\int_a^b L(q(\tau), {}_{\mathrm{c}}\mathrm{D}^\alpha_{a+}[q](\tau), \tau)\, d\tau, \end{array} \tag{VIII.10}$$

where ${}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q]$ is the Caputo fractional derivative of q of order α that is equal to $\mathrm{D}^{\alpha}_{a+}[q-q(a)]$ for every $q \in \mathrm{C}^1([a,b],\mathbb{R}^n)$. Let $\mathrm{C}^1_0([a,b],\mathbb{R}^n) = \{w \in \mathrm{C}^1([a,b],\mathbb{R}^n),\ w(a)=w(b)=0\}$ be the set of variations of $\mathcal{L}$. A curve $q \in \mathrm{C}^1([a,b],\mathbb{R}^n)$ is said to be a *critical point* of $\mathcal{L}$ if $D\mathcal{L}(q)(w)=0$ for every variation $w \in \mathrm{C}^1_0([a,b],\mathbb{R}^n)$, where $D\mathcal{L}(q)(w)$ is the Gâteaux-differential of $\mathcal{L}$ at q in direction w. In particular, if q is a local optimizer of $\mathcal{L}$, then q is a critical point of $\mathcal{L}$. Finally, in Section V.3.1, we have obtained the following characterization of the critical points of $\mathcal{L}$

Theorem VIII.1. *Let $0<\alpha<1$ and let $q \in \mathrm{C}^1([a,b],\mathbb{R}^n)$. Then, q is a critical point of $\mathcal{L}$ if and only if q is a solution of the following fractional Euler-Lagrange equation:*

$$\frac{\partial L}{\partial x}(q(t), {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q](t), t) + \mathrm{D}^{\alpha}_{b-}\left[\frac{\partial L}{\partial v}(q, {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q], \cdot)\right](t) = 0, \tag{EL$^\alpha$}$$

for every $t \in [a,b]$.

Remark VIII.4. An usual and algebraic way to obtain a discrete version of a fractional differential equation involving Riemann-Liouville fractional derivatives of order $0<\alpha<1$ is to replace the curves $q \in \mathrm{C}^1([a,b],\mathbb{R}^n)$ by discrete curves $q \in \mathrm{C}(\mathbb{T},\mathbb{R}^n)$ and to substitute the fractional derivatives D^{α}_{a+} and D^{α}_{b-} by the discrete counterparts Δ^{α}_{a+} and Δ^{α}_{b-}. This method is widely used, see *e.g.* [9, 28, 75, 165, 166, 181, 185, 186]. Such a direct discretization of (EL$^\alpha$) leads to the following numerical scheme:

$$\frac{\partial L}{\partial x}(q(t_k), {}_{\mathrm{c}}\Delta^{\alpha}_{a+}[q](t_k), t_k) + \Delta^{\alpha}_{b-}\left[\frac{\partial L}{\partial v}(q, {}_{\mathrm{c}}\Delta^{\alpha}_{a+}[q], \cdot)\right](t_k) = 0, \tag{VIII.11}$$

for every $k=1,\ldots,N-1$. As mentioned in the introduction of this chapter, the fractional Euler-Lagrange equation (EL$^\alpha$) admits a variational structure that is intrinsic and induces strong constraints on the qualitative behaviour of the solutions. It seems then important to preserve this structure at the discrete level. However, the numerical scheme (VIII.11) is obtained via an *algebraic* procedure only based on the differential shape of (EL$^\alpha$). Consequently, there is no guarantee that the intrinsic variational structure of (EL$^\alpha$) is preserved at the discrete level.

The aim of the next section is to construct a variational integrator for (EL$^\alpha$). Such a variational procedure allows to preserve the variational structure of the equation at the discrete level.

VIII.3.1 Construction of a variational integrator

The construction of a variational integrator for (EL$^\alpha$) is made up of two steps:

Step 1: define a discrete version of the functional $\mathcal{L}$;

Step 2: develop a discrete calculus of variations on the discrete functional defined in Step 1.

Hence, a discrete equation is obtained and it is called *variational integrator*. It is a numerical scheme for (EL$^\alpha$) preserving the variational structure at the discrete level in the sense that *the discrete solutions correspond to the discrete critical points of the discrete version of the initial functional* $\mathcal{L}$. In this section, we follow the two steps above described.

First step. Considering the usual Gaussian quadrature formula, we define a discrete version of $\mathcal{L}$ by

$$\begin{array}{rccl} \mathcal{L}_h : & \mathrm{C}(\mathbb{T},\mathbb{R}^n) & \longrightarrow & \mathbb{R} \\ & q & \longmapsto & h\displaystyle\sum_{k=1}^{N} L(q(t_k), {}_{\mathrm{c}}\Delta^{\alpha}_{a+}[q](t_k), t_k). \end{array} \tag{VIII.12}$$

Second step. Let $\mathrm{C}_0(\mathbb{T},\mathbb{R}^n) = \{w \in \mathrm{C}(\mathbb{T},\mathbb{R}^n),\ w(a)=w(b)=0\}$ denote the set of *discrete variations* of $\mathcal{L}_h$. Then, $q \in \mathrm{C}(\mathbb{T},\mathbb{R}^n)$ is said to be a *discrete critical point* of $\mathcal{L}_h$ if it satisfies $D\mathcal{L}_h(q)(w)=0$ for every discrete variations $w \in \mathrm{C}_0(\mathbb{T},\mathbb{R}^n)$. Recall that $D\mathcal{L}_h(q)(w)$ denotes the Gâteaux-differential of $\mathcal{L}_h$ at q in the direction w.

As in the continuous case, in order to obtain a characterization of the discrete critical points of $\mathcal{L}_h$, we need an integration by parts formula. The following lemma is originally introduced in [49].

Lemma VIII.1 (Discrete fractional integration by parts)**.** *For every $q \in \mathrm{C}(\mathbb{T},\mathbb{R}^n)$ and every $w \in \mathrm{C}_0(\mathbb{T},\mathbb{R}^n)$, it holds*

$$h\sum_{k=1}^{N} q(t_k)\cdot \Delta^{\alpha}_{a+}[w](t_k) = h\sum_{k=1}^{N-1} \Delta^{\alpha}_{b-}[q](t_k)\cdot w(t_k). \tag{DFIBP}$$

Proof. Since $w(a)=w(b)=0$, the following equalities hold:

$$\begin{aligned} h^{\alpha}\sum_{k=1}^{N} q(t_k)\cdot \Delta^{\alpha}_{a+}[w](t_k) &= \sum_{k=1}^{N}\sum_{r=0}^{k} \alpha_r q(t_k)\cdot w(t_{k-r}) = \sum_{k=0}^{N}\sum_{r=0}^{k} \alpha_r q(t_k)\cdot w(t_{k-r}) \\ = \sum_{r=0}^{N}\sum_{k=r}^{N} \alpha_r q(t_k)\cdot w(t_{k-r}) &= \sum_{r=0}^{N}\sum_{k=0}^{N-r} \alpha_r q(t_{k+r})\cdot w(t_k) = \sum_{k=0}^{N}\sum_{r=0}^{N-k} \alpha_r q(t_{k+r})\cdot w(t_k) \\ = \sum_{k=1}^{N-1}\sum_{r=0}^{N-k} \alpha_r q(t_{k+r})\cdot w(t_k) &= h^{\alpha}\sum_{k=1}^{N-1} \Delta^{\alpha}_{b-}[q](t_k)\cdot w(t_k). \end{aligned} \tag{VIII.13}$$

Multiplying by $h^{1-\alpha}$, the proof is complete. ☐

Finally, using Lemma VIII.1 and developing a discrete calculus of variations leads to the following result originally proved in [49].

Theorem VIII.2. *Let $q \in \mathrm{C}(\mathbb{T},\mathbb{R}^n)$. Then, q is a discrete critical point of $\mathcal{L}_h$ if and only if q is a solution of the following discrete fractional Euler-Lagrange equation:*

$$\frac{\partial L}{\partial x}(q(t_k), {}_{\mathrm{c}}\Delta^{\alpha}_{a+}[q](t_k), t_k) + \Delta^{\alpha}_{b-}\left[\frac{\partial L}{\partial v}(q, {}_{\mathrm{c}}\Delta^{\alpha}_{a+}[q], \cdot)\right](t_k) = 0, \tag{EL$^{\alpha}_h$}$$

for every $k = 1,\dots,N-1$.

Proof. Let $q \in \mathrm{C}(\mathbb{T},\mathbb{R}^n)$. For every $w \in \mathrm{C}_0(\mathbb{T},\mathbb{R}^n)$, we define the following function:

$$\begin{array}{rccl} \varphi_w : & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & \varepsilon & \longmapsto & \mathcal{L}_h(q+\varepsilon w) = h\displaystyle\sum_{k=1}^{N} L(q(t_k)+\varepsilon w(t_k), {}_{\mathrm{c}}\Delta^{\alpha}_{a+}[q](t_k) + \varepsilon\,{}_{\mathrm{c}}\Delta^{\alpha}_{a+}[w](t_k), t_k). \end{array} \tag{VIII.14}$$

Since $D\mathcal{L}_h(q)(w) = \dot{\varphi}_w(0)$ and since ${}_{\mathrm{c}}\Delta^{\alpha}_{a+}[w](t_k) = \Delta^{\alpha}_{a+}[w]$, it holds

$$\begin{aligned} D\mathcal{L}_h(q)(w) = h\sum_{k=1}^{N}\Bigg[&\frac{\partial L}{\partial x}(q(t_k), \Delta^{\alpha}_{a+}[q-q(a)](t_k), t_k)\cdot w(t_k) \\ &+ \frac{\partial L}{\partial v}(q(t_k), \Delta^{\alpha}_{a+}[q-q(a)](t_k), t_k)\cdot \Delta^{\alpha}_{a+}[w](t_k)\Bigg]. \end{aligned} \tag{VIII.15}$$

Since $w \in \mathrm{C}_0(\mathbb{T},\mathbb{R}^n)$ and using the discrete fractional integration by parts given in Lemma VIII.1, we obtain

$$D\mathcal{L}_h(q)(w) = h\sum_{k=1}^{N-1}\left[\frac{\partial L}{\partial x}(q(t_k), {}_{\mathrm{c}}\Delta^{\alpha}_{a+}[q](t_k), t_k) + \Delta^{\alpha}_{b-}\left[\frac{\partial L}{\partial v}(q, {}_{\mathrm{c}}\Delta^{\alpha}_{a+}[q], \cdot)\right](t_k)\right]\cdot w(t_k), \tag{VIII.16}$$

which completes the proof. ☐

Theorem VIII.2 completes the Step 2 of the construction of a variational integrator. Precisely, the discrete fractional Euler-Lagrange equation (EL$^{\alpha}_h$) is the variational integrator constructed for the fractional Euler-Lagrange equation (EL$^{\alpha}$). It is a numerical scheme preserving the variational structure of (EL$^{\alpha}$) in the sense that the discrete solutions of (EL$^{\alpha}_h$) coincide with the discrete critical points of the discrete version $\mathcal{L}_h$ of $\mathcal{L}$.

Remark VIII.5. We note that the discrete Euler-Lagrange equation (EL$^{\alpha}_h$) coincides with the numerical scheme (VIII.11) obtained with a direct discretization. Note that such a phenomena is not obvious. For example, in the classical case $\alpha = 1$, it is not clear to know how to replace d/dt at the discrete level. Indeed, one can choose Δ^1_{a+} or $-\Delta^1_{b-}$ or a mixing of the two of them. In the fractional case, the non locality of the fractional operators does not permit such a choice.

VIII.3.2 Some numerical considerations on the fractional Dirichlet example

As mentioned before, the fractional Euler-Lagrange equations are particularly difficult to solve explicitly. Only few examples provide exact solutions and furthermore not in an explicit way. Due to the additional complexity of fractional operators, the purpose of this section is only to obtain some experimental results on a concrete example and not to provide numerical analyses.

Namely, we consider the fractional Dirichlet example studied in [6] and recalled in Example V.2. In this example, a *quasi-explicit* solution is known. Indeed, let us consider the interval $[a,b] = [0,1]$, $n = 1$ and the following Lagrangian:

$$\begin{array}{rccc} L: & \mathbb{R} \times \mathbb{R} \times [0,1] & \longrightarrow & \mathbb{R} \\ & (x,v,t) & \longmapsto & \dfrac{1}{2}v^2. \end{array} \tag{VIII.17}$$

The associated fractional Euler-Lagrange equation (EL^α) is given by $\mathrm{D}^\alpha_{b-}\left[{}_\mathrm{c}\mathrm{D}^\alpha_{a+}[q]\right] = 0$. Under the boundary assumptions $q(0) = 0$ and $q(1) = 1$ and for every $1/2 < \alpha < 1$, this equation admits a unique solution denoted by $\tilde{q}$ and given by

$$\tilde{q}(t) = (2\alpha - 1) \int_0^t \frac{1}{[(1-x)(t-x)]^{1-\alpha}} \, dx, \tag{VIII.18}$$

for every $t \in [0,1]$. Note that the exact solution $\tilde{q}$ is not explicitly given since it is only written with an improper integral. As a consequence, the following numerical results consider an approximation of $\tilde{q}$ using high-order global adaptive quadrature.

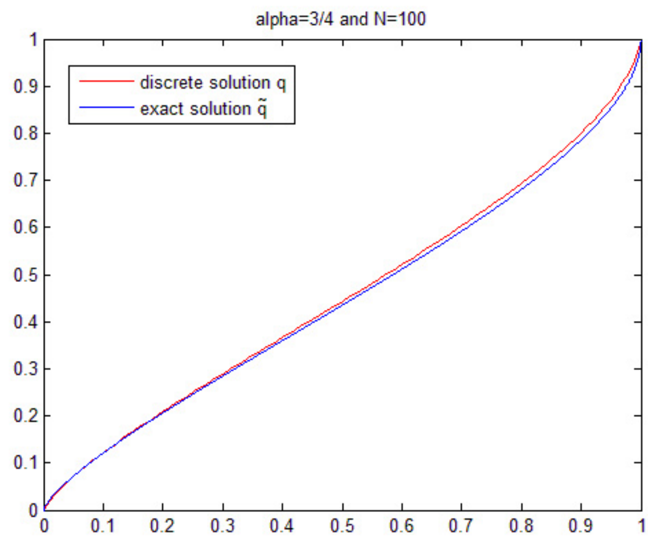
Let us solve the associated discrete fractional Euler-Lagrange equation (EL^α_h) given by

$$\forall k = 1, \ldots, N-1, \; \Delta^\alpha_{b-}\left[{}_\mathrm{c}\Delta^\alpha_{a+}[q]\right](t_k) = 0, \tag{VIII.19}$$

with the boundary values $q(0) = 0$ and $q(1) = 1$. For any $1/2 < \alpha < 1$ and every $N \geq 2$, the ℓ^∞-error and the ℓ^2-error between the exact solution $\tilde{q}$ and the discrete one $q \in \mathrm{C}(\mathbb{T}, \mathbb{R})$ are defined by

$$\mathrm{Err}^\infty_{\alpha,N} = \max_{k=0,\ldots,N} |q(t_k) - \tilde{q}(t_k)| \text{ and } \mathrm{Err}^2_{\alpha,N} = \sqrt{h \sum_{k=0}^{N} |q(t_k) - \tilde{q}(t_k)|^2}, \tag{VIII.20}$$

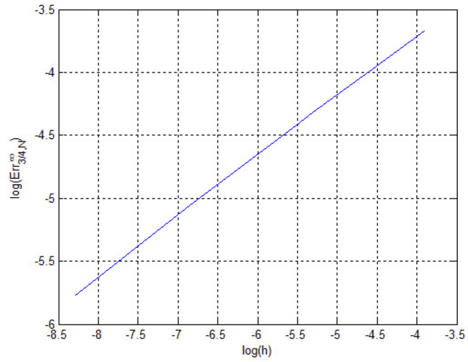
where $h = 1/N$. We confront q and $\tilde{q}$ for $\alpha = 3/4$ and $N = 100$. The solutions are displayed on the following picture:

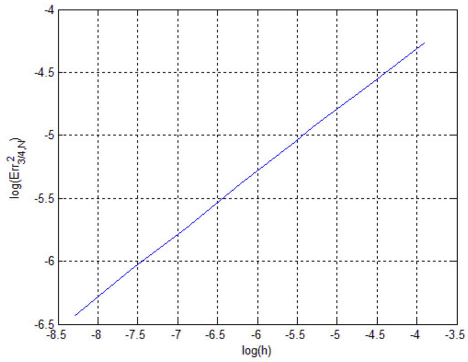


The errors $\mathrm{Err}^\infty_{\alpha,N}$ and $\mathrm{Err}^2_{\alpha,N}$ for $\alpha = 3/4$ and for varying N are given in the following table:

N	50	100	200	250	500	1000	2000	4000
$\text{Err}^{\infty}_{3/4,N}$	0.0255	0.0185	0.0134	0.0120	0.0086	0.0062	0.0044	0.0031
$\text{Err}^{2}_{3/4,N}$	0.0140	0.0100	0.0072	0.0064	0.0046	0.0032	0.0023	0.0016

Finally, the graphic representations of $\log(\text{Err}^{\infty}_{3/4,N})$ and $\log(\text{Err}^{2}_{3/4,N})$ with respect to $\log(h)$ are respectively given on the following pictures:





We clearly obtain two affine functions with the common slope $\lambda_{3/4} \simeq 0.51$. In this case, we conclude that we obtain an experimental convergence of order $\lambda_{3/4} \simeq 0.51$.

Now, let us study the evolution of the errors $\text{Err}^{\infty}_{\alpha,N}$ and $\text{Err}^{2}_{\alpha,N}$ and the slope λ_α with respect to α. For $1/2 < \alpha < 1$, the errors $\text{Err}^{\infty}_{\alpha,N}$ and $\text{Err}^{2}_{\alpha,N}$ are given in the following table:

$\alpha \backslash N$	50	100	200	250	500	1000	2000	4000
0.55	0.3811	0.3679	0.3527	0.3475	0.3310	0.3141	0.2974	0.2810
	0.1746	0.1560	0.1393	0.1344	0.1203	0.1079	0.0972	0.0878
0.6	0.1988	0.1788	0.1596	0.1537	0.1363	0.1204	0.1061	0.0933
	0.0969	0.0825	0.0701	0.0665	0.0566	0.0483	0.0413	0.0354
0.65	0.1023	0.0856	0.0711	0.0669	0.0552	0.0453	0.0372	0.0304
	0.0525	0.0423	0.0340	0.0317	0.0255	0.0205	0.0165	0.0133
0.7	0.0517	0.0403	0.0311	0.0286	0.0219	0.0168	0.0129	0.0098
	0.0276	0.0210	0.0159	0.0146	0.0110	0.0083	0.0063	0.0048
0.75	0.0255	0.0185	0.0134	0.0120	0.0086	0.0062	0.0044	0.0031
	0.0140	0.0100	0.0072	0.0064	0.0046	0.0032	0.0023	0.0016
0.8	0.0122	0.0083	0.0056	0.0049	0.0033	0.0022	0.0015	0.00097
	0.0067	0.0045	0.0030	0.0027	0.0018	0.0012	0.0008	0.00052
0.85	0.0055	0.0035	0.0022	0.0019	0.0012	0.00074	0.00046	0.00028
	0.0030	0.0019	0.0012	0.0010	0.0006	0.0004	0.00025	0.00015
0.9	0.0022	0.0013	0.00079	0.00066	0.00039	0.00022	0.00013	0.000075
	0.0012	0.0007	0.00041	0.00035	0.0002	0.00012	0.00007	0.00004
0.95	0.00068	0.00038	0.00021	0.00017	0.000095	0.000052	0.000028	0.000015
	0.00037	0.0002	0.00011	0.00009	0.00005	0.000026	0.000014	0.000007

We clearly see that the closer α is to 1, then the better the approximation is. Now, let us study the evolution of slope λ_α with respect to α:

α	0.55	0.6	0.65	0.7	0.75	0.8	0.85	0.9	0.95
λ_α	0.11	0.2	0.3	0.39	0.51	0.62	0.72	0.8	0.91

For this example, we conclude that we obtain an experimental convergence of order $\lambda_\alpha = 2\alpha - 1$.

VIII.4 Conservation of a fractional Noether-type result

As in the classical continuous case, a symmetry for a discrete fractional Euler-Lagrange equation (EL^{α}_{h}) is based on the action of a one-parameter family of infinitesimal transformations of $\mathbb{R}^n$ (see Definition II.1) on the Lagrangian L associated.

Definition VIII.4. Let Φ be a one-parameter family of infinitesimal transformations of $\mathbb{R}^n$ (see Definition II.1) and let $0 < \alpha < 1$. A Lagrangian L is said to be invariant under the action of Φ if for every solution $q \in \mathrm{C}(\mathbb{T}, \mathbb{R}^n)$ of (EL_h^α) and every $k = 1, \ldots, N-1$, the map

$$\theta \longmapsto L(\Phi(\theta, q(t_k)), {}_{\mathrm{c}}\Delta^\alpha_{a+}[\Phi(\theta, q)](t_k), t_k) \qquad \text{(VIII.21)}$$

has a null derivative in 0. In such a case, Φ is said to be a symmetry of the discrete fractional Euler-Lagrange equation (EL_h^α).

As in the classical continuous case, the most classical examples of invariance of a Lagrangian under the action of a one-parameter family of infinitesimal transformations of $\mathbb{R}^n$ are given by quadratic Lagrangian and rotations, see Example II.2.

With this notion of symmetry, we state the following discrete counterpart of Proposition V.7 originally derived in [48].

Proposition VIII.1 (Noether-type)**.** *Let $0 < \alpha < 1$ and let Φ be a one-parameter family of infinitesimal transformations of $\mathbb{R}^n$. If L is invariant under the action of Φ, then for every solution $q \in \mathrm{C}(\mathbb{T}, \mathbb{R}^n)$ of (EL_h^α), the following equality holds:*

$$\frac{\partial L}{\partial v}(q(t_k), {}_{\mathrm{c}}\Delta^\alpha_{a+}[q](t_k), t_k) \cdot {}_{\mathrm{c}}\Delta^\alpha_{a+}\left[\frac{\partial \Phi}{\partial \theta}(0, q)\right](t_k) - \Delta^\alpha_{b-}\left[\frac{\partial L}{\partial v}(q, {}_{\mathrm{c}}\Delta^\alpha_{a+}[q], \cdot)\right](t_k) \cdot \frac{\partial \Phi}{\partial \theta}(0, q(t_k)) = 0, \quad \text{(VIII.22)}$$

for every $k = 1, \ldots, N-1$.

Proof. Let $q \in \mathrm{C}(\mathbb{T}, \mathbb{R}^n)$ be a solution of (EL_h^α). Let us differentiate in 0 the map given by (VIII.21) and let us invert the operators ${}_{\mathrm{c}}\Delta^\alpha_{a+}$ and $\partial/\partial\theta$. It can be noted that this last operation is easier in the discrete case than in the continuous one. We obtain for every $k = 1, \ldots, N-1$

$$\frac{\partial L}{\partial x}(q(t_k), {}_{\mathrm{c}}\Delta^\alpha_{a+}[q](t_k), t_k) \cdot \frac{\partial \Phi}{\partial \theta}(0, q(t_k)) + \frac{\partial L}{\partial v}(q(t_k), {}_{\mathrm{c}}\Delta^\alpha_{a+}[q](t_k), t_k) \cdot {}_{\mathrm{c}}\Delta^\alpha_{a+}\left[\frac{\partial \Phi}{\partial \theta}(0, q)\right](t_k) = 0. \quad \text{(VIII.23)}$$

Using that q is a solution of (EL_h^α), the proof is complete. □

From this last result, we conclude that the discrete fractional Euler-Lagrange equation (EL_h^α), that is a variational integrator for (EL^α), allows to preserve at the discrete level the fractional Noether-type result stated in Proposition V.7.

Remark VIII.6. Nevertheless, as in the fractional continuous setting, Proposition VIII.1 does not provide an explicit constant of motion, which is the main concern of the classical Noether's theorem. Indeed, in contrary to the classical case (see the proof of Theorem II.2), no simple Leibniz formula in the discrete fractional case $0 < \alpha < 1$ allows to provide an explicit constant of motion from Equality (VIII.22). From this last observation, we introduce in [48] a *discrete transfer formula* allowing to write Equality (VIII.22) as an explicit discrete derivative. As a consequence, we can derive from Proposition VIII.1 an explicit constant of motion for discrete fractional Euler-Lagrange equations (EL_h^α) admitting a symmetry. However, in contrary to the continuous fractional case (see Section V.3.2), the discrete transfer formula is written with a *finite* sum and therefore, the numerical computation of the discrete conservation law is possible. We refer to [48] for more details.

Chapitre IX

Principe du Maximum de Pontryagin Faible pour des problèmes de contrôle optimal non linéaires discrets avec dérivée fractionnaire de Caputo-Grünwald-Letnikov

L'objectif de ce chapitre est de construire un schéma numérique pour le problème de contrôle optimal fractionnaire $(\mathbf{OCP})^\alpha$ *(défini en Chapitre VII) préservant sa structure variationnelle au niveau discret. À l'instar du Chapitre VIII pour les équations d'Euler-Lagrange fractionnaires, notre construction nous amène à définir un problème de contrôle optimal discret fractionnaire et à démontrer le Principe du Maximum de Pontryagin Faible (WPMP en abrégé anglais) correspondant.*

Dans un premier temps, la discrétisation naturelle de l'équation contrôlée donne lieu à une équation discrète implicite. En conséquence, le WPMP est démontré sous une condition relativement restrictive en termes de continuité globalement Lipschitzienne de la fonction de contrainte. Cependant, un contre-exemple montre qu'une hypothèse de ce type est indispensable. Par ailleurs, ce même contre-exemple montre que la méthode des multiplicateurs de Lagrange ne peut pas être appliquée dans certains contextes. Comme dans le cas continu (voir Chapitre VII), notre démonstration ne repose donc pas sur cette méthode mais sur un calcul des variations (portant sur le contrôle et la condition initiale) et sur l'introduction d'un vecteur adjoint adéquat.

Dans le but de fournir un WPMP sans aucune condition sur la fonction de contrainte, nous concluons ce chapitre par l'étude d'un problème de contrôle optimal discret fractionnaire shifté. En effet, dans ce cadre, l'équation discrète contrôlée est explicite et le WPMP correspondant est alors vérifié pour des fonctions de contraintes plus générales.

IX.1 Introduction

As mentioned in Chapter VIII, solving fractional (partial) differential equations is a difficult task. As a consequence, there has been a growing interest to develop numerical methods for these equations, see *e.g.* [10, 80, 128]. The fractional equations deriving from some variational problems often involve both left and right fractional operators and are consequently even more difficult to solve. Consequently, several papers are concerned with numerical simulations for fractional optimal control problems. The methods use different type of approximations: Legendre polynomials [7], expansion formulas [128], Chebychev polynomials [135], rational approximations [198], etc.

The method of direct discretization using discrete fractional derivatives of Grünwald-Letnikov is also applied, see *e.g.* [26, 75]. Precisely, adapted to the fractional optimal control problem $(\mathbf{OCP})^\alpha$ studied in Chapter VII, these investigations are concerned with the discretization of the three following equations provided by the Weak

Pontryagin Maximum Principle (denoted in short WPMP and stated in Theorem VII.1):

$$\begin{aligned} {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{a+}[q^*](t) &= \frac{\partial H}{\partial w}(q^*(t), u^*(t), p(t), t), && \text{(IX.1)} \\ {}_{\mathrm{c}}\mathrm{D}^{\alpha}_{b-}[p](t) &= \frac{\partial H}{\partial x}(q^*(t), u^*(t), p(t), t), && \text{(IX.2)} \\ \frac{\partial H}{\partial v}(q^*(t), u^*(t), p(t), t) &= 0. && \text{(IX.3)} \end{aligned}$$

Namely, the authors directly replace the fractional derivatives by discrete Grünwald-Letnikov counterparts. As explained in Chapter VIII, this algebraic discretization is only based on the differential writing of the equations and consequently, there is no guarantee that the variational structure of $(\mathbf{OCP})^{\alpha}$ is preserved at the discrete level.

The objective of this chapter is to construct a numerical scheme for the three previous equations preserving the variational structure of $(\mathbf{OCP})^{\alpha}$. Similarly to Chapter VIII for fractional Euler-Lagrange equations, our strategy consists in two steps:

1. define a discrete version of the fractional optimal control problem $(\mathbf{OCP})^{\alpha}$;
2. state the corresponding discrete WPMP.

Hence, the discrete WPMP leads to a numerical scheme for Equations (IX.1), (IX.2) and (IX.3) preserving the variational structure of $(\mathbf{OCP})^{\alpha}$ in the sense that the discrete solutions are candidate to be moreover solutions of the discrete version of $(\mathbf{OCP})^{\alpha}$. We refer to Section IX.2 for more details.

The proof of the discrete WPMP (Theorem IX.1) is the discrete analogue of the proof of Theorem VII.1. Nevertheless, an additional difficulty emerges in the discrete case due to the *implicit* shape of the discrete fractional controlled system, see Equation (IX.4). A relatively restrictive assumption on the constraint function (in terms of globally Lipschitz continuity) is assumed in order to overcome this difficulty, see Assumption (IX.5). However, a counterexample (Section IX.2.3) shows that an assumption of this type is necessary. Another remark derives from the same counterexample: the Lagrange multiplier technique can fail in some frameworks. As a consequence, our proof is not based on this method but only on a calculus of variations (on the control and the initial condition) and on the introduction of an appropriate adjoint vector, see Section IX.3.

In order to study a framework without restrictive assumption on the constraint function, we consider a *shifted* discrete controlled system and we derive the corresponding WPMP in Section IX.4. In this *shifted* setting, the discrete fractional controlled system is *explicit* and consequently, this context allows to release the assumption of globally Lipschitz continuity and to consider general constraint functions.

Organization of the chapter. In Subsection IX.2.1, a general discrete optimal control problem with a Caputo-Grünwald-Letnikov fractional derivative (*i.e.* the discrete version of $(\mathbf{OCP})^{\alpha}$) is defined under an assumption on the constraint function. The corresponding discrete WPMP (Theorem IX.1) is stated in Subsection IX.2.2. The counterexample above mentioned is detailed in Subsection IX.2.3. Section IX.3 is devoted to the proof of Theorem IX.1. Finally, a general *shifted* discrete optimal control problem with a Caputo-Grünwald-Letnikov fractional derivative is considered in Subsection IX.4 and the corresponding WPMP valid for general constraint functions is stated.

IX.2 Main result

Throughout this chapter, let $(a,b) \in \mathbb{R}^2$ with $a < b$ and m, $n \in \mathbb{N}^*$. The notation $\|\cdot\|$ stands for the Euclidean norms of $\mathbb{R}^m$ and $\mathbb{R}^n$.

Let $N \geq 2$, let $h = (b-a)/N$ be the step size of the discretization and let $\mathbb{T} = \{t_k\}_{k=0,\dots,N} = \{a+kh\}_{k=0,\dots,N}$ be the usual regular partition of the interval $[a,b]$. Note that we take anew the notations of the time scale setting. Since all functions defined on $\mathbb{T}$ with values in $\mathbb{R}^n$ are automatically continuous, we denote by $\mathrm{C}(\mathbb{T},\mathbb{R}^n)$ the set of all functions defined on $\mathbb{T}$ with values in $\mathbb{R}^n$.

IX.2.1 Discrete optimal control problem with a Caputo-Grünwald-Letnikov fractional derivative

Let ${}_{\mathrm{c}}\Delta^{\alpha}_{a+}$ denote the left discrete fractional derivative of Caputo-Grünwald-Letnikov of order $0 < \alpha < 1$ given by Definition VIII.3. In this chapter, we consider the general discrete fractional nonlinear controlled system

$$ {}_{\mathrm{c}}\Delta^{\alpha}_{a+}[q](t_k) = f(q(t_k), u(t_k), t_k), \tag{IX.4} $$

for every $k = 1, \dots, N$, where $f : \mathbb{R}^n \times \mathbb{R}^m \times [a,b] \longrightarrow \mathbb{R}^n, (x,v,t) \longmapsto f(x,v,t)$ is a continuous function of class $\mathscr{C}^1$ with respect to its two first variables and where the control functions u belong to $\mathrm{C}(\mathbb{T}, \mathbb{R}^m)$.

Before defining an optimal control problem associated with the controlled system (IX.4), the first question that has to be addressed is the question of the existence and uniqueness of a solution of (IX.4), for a given control function u and a given initial condition $q(a) = q_a \in \mathbb{R}^n$. This question is here more crucial than in the continuous case since Equation (IX.4) is an *implicit* equation and consequently, the existence of a solution, even local, is not ensured. Moreover, for an initial condition $q_a \in \mathbb{R}^n$ fixed, there exist examples where a couple $(q,u) \in \mathrm{C}(\mathbb{T}, \mathbb{R}^n) \times \mathrm{C}(\mathbb{T}, \mathbb{R}^m)$ solution of (IX.4) exists and is unique. Then, the couple (q,u) is optimal independently of the optimal control problem considered. Nevertheless, no variation can be done on the control u and consequently, no WPMP can be derived. Actually, a counterexample is provided in Section IX.2.3 and the two following comments emerge:

- In order to ensure the validity of a variation on the control or on the initial condition, an hypothesis on the constraint function f have to be assumed. In this section, we consider that f is globally Lipschitz continuous in its first variable *i.e.*

$$ \exists 0 \leq L < h^{-\alpha},\ \forall (x_1,x_2) \in (\mathbb{R}^n)^2,\ \forall (v,t) \in \mathbb{R}^m \times [a,b],\ \|f(x_1,v,t) - f(x_2,v,t)\| \leq L \|x_1 - x_2\|. \tag{IX.5} $$

- The Lagrange multiplier technique cannot be used in some frameworks. As a consequence, the proof of the WPMP is not based on the Lagrange multiplier technique but only on a calculus of variations (on the control and the initial condition) and on the introduction of an appropriate adjoint vector.

Assumption (IX.5) implies the following statement.

Lemma IX.1. *We assume that f is globally Lipschitz continuous in the sense of* (IX.5). *For every $(u, q_a) \in \mathrm{C}(\mathbb{T}, \mathbb{R}^m) \times \mathbb{R}^n$, there exists a unique solution $q \in \mathrm{C}(\mathbb{T}, \mathbb{R}^m)$ of* (IX.4) *satisfying the initial condition $q(a) = q_a$. This solution is denoted by $q(\cdot, u, q_a)$.*

Proof. Let $(u, q_a) \in \mathrm{C}(\mathbb{T}, \mathbb{R}^m) \times \mathbb{R}^n$. We construct $q(\cdot, u, q_a)$ by induction. Indeed, we first choose $q(a, u, q_a) = q_a$ and then, for every $k = 1, \dots, N$, we construct $q(t_k, u, q_a)$ as the unique fixed point of the following mapping:

$$ x \in \mathbb{R}^n \longmapsto h^{\alpha} f(x, u(t_k), t_k) + \alpha^*_k q_a - \sum_{r=1}^{k} \alpha_r q(t_{k-r}, u, q_a) \in \mathbb{R}^n, \tag{IX.6} $$

where α^*_k is defined by $\alpha^*_k = \sum_{r=0}^{k} \alpha_r = (1-\alpha)(2-\alpha)\dots(k-\alpha)/k! = (\alpha - 1)_k$ for every $k = 1, \dots, N$ and $\alpha^*_0 = \alpha_0 = 1$. From Assumption (IX.5), the previous mapping is contractive and the Banach fixed point theorem concludes the proof. □

We are now in a position to define rigorously a general discrete optimal control problem with a Caputo-Grünwald-Letnikov fractional derivative (denoted in short $(\mathbf{OCP})^{\alpha}_{h}$). Let $f^0 : \mathbb{R}^n \times \mathbb{R}^m \times [a,b] \longrightarrow \mathbb{R}, (x,v,t) \longmapsto f^0(x,v,t)$ be a continuous function of class $\mathscr{C}^1$ with respect to its two first variables.

Definition of $(\mathbf{OCP})^{\alpha}_{h}$: determine a trajectory $q^* \in \mathrm{C}(\mathbb{T}, \mathbb{R}^n)$ solution of

$$ {}_{\mathrm{c}}\Delta^{\alpha}_{a+}[q](t_k) = f(q(t_k), u(t_k), t_k), \tag{IX.4} $$

and associated with a control $u^* \in \mathrm{C}(\mathbb{T}, \mathbb{R}^m)$ minimizing the cost

$$ h \sum_{k=1}^{N} f^0(q(t_k), u(t_k), t_k) \tag{IX.7} $$

over all possible trajectories $q \in \mathrm{C}(\mathbb{T}, \mathbb{R}^n)$ solutions of (IX.4) and associated with a control $u \in \mathrm{C}(\mathbb{T}, \mathbb{R}^m)$. The initial condition can be fixed or not.

Remark IX.1. Except Assumption (IX.5) on the constraint function f, note that $(\mathbf{OCP})^\alpha_h$ is a discrete version of $(\mathbf{OCP})^\alpha$. In the following section, we state the corresponding discrete WPMP. Hence, a numerical scheme for Equations (IX.1), (IX.2) and (IX.3) preserving the variational structure of $(\mathbf{OCP})^\alpha$ is obtained.

IX.2.2 Weak Pontryagin Maximum Principle

The *Hamiltonian* of the discrete fractional optimal control problem $(\mathbf{OCP})^\alpha_h$ is the function $H : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times [a,b] \longrightarrow \mathbb{R}$ defined by $H(x,v,w,t) = w \cdot f(x,v,t) + f^0(x,v,t)$. The main result of this chapter is Theorem IX.1 and is proved in Section IX.3.

Theorem IX.1 (Weak Pontryagin Maximum Principle)**.** *We assume that f is globally Lipschitz continuous in the sense of* (IX.5)*. If the trajectory $q^* \in \mathrm{C}(\mathbb{T}, \mathbb{R}^n)$, associated with a control $u^* \in \mathrm{C}(\mathbb{T}, \mathbb{R}^m)$, is a solution of $(\boldsymbol{OCP})^\alpha_h$, then there exists a mapping $p \in \mathrm{C}(\mathbb{T}, \mathbb{R}^n)$ (called adjoint vector), such that $p(b) = 0$ and such that q^* and p are solutions of*

$$ {}_{\mathrm{c}}\Delta^\alpha_{a+}[q^*](t_k) = \frac{\partial H}{\partial w}(q^*(t_k), u^*(t_k), p(t_{k-1}), t_k), \quad {}_{\mathrm{c}}\Delta^\alpha_{b-}[p](t_{k-1}) = \frac{\partial H}{\partial x}(q^*(t_k), u^*(t_k), p(t_{k-1}), t_k), \tag{IX.8} $$

for every $k = 1, \ldots, N$. Moreover, it holds

$$ \frac{\partial H}{\partial v}(q^*(t_k), u^*(t_k), p(t_{k-1}), t_k) = 0_{\mathbb{R}^m}, \tag{IX.9} $$

for every $k = 1, \ldots, N$. Furthermore, if the initial condition is free, one has the following transversality condition

$$ h \sum_{k=1}^{N} \frac{\partial H}{\partial x}(q^*(t_k), u^*(t_k), p(t_{k-1}), t_k) = 0_{\mathbb{R}^n}. \tag{IX.10} $$

Theorem IX.1 provides in particular a numerical scheme, given by Equations (IX.8) and (IX.9), for the continuous fractional optimal control problem $(\mathbf{OCP})^\alpha$ studied in Chapter VII. Analogously to the variational integrator constructed in Chapter VIII for fractional Euler-Lagrange equations, this numerical scheme allows to preserve at the discrete level the variational structure of $(\mathbf{OCP})^\alpha$ in the sense that the discrete solutions are candidate to be moreover solutions of the discrete version $(\mathbf{OCP})^\alpha_h$ of $(\mathbf{OCP})^\alpha$.

A natural phenomena, similar to the time scale setting (see Theorem III.1), is the emergence of a *shift* on the adjoint vector. The numerical schemes constructed *e.g.* in [26, 75] derive from an algebraic direct discretization and consequently, the shift on the adjoint vector is not considered. As a consequence, the numerical schemes suggested in these papers do not ensure the conservation of the variational structure of $(\mathbf{OCP})^\alpha$ at the discrete level.

Remark IX.2. Assumption (IX.5) is necessary at several occasions to prove Theorem IX.1 and is not only used for ensuring the existence and uniqueness of the solution of (IX.4) for every couple $(u, q_a) \in \mathrm{C}(\mathbb{T}, \mathbb{R}^m) \times \mathbb{R}^n$.

Remark IX.3. Assumption (IX.5) is relatively restrictive and maybe can be weakened but this is not the aim of this chapter. However, recall that a WPMP is derived in Section IX.4 for *shifted* discrete optimal control problems with a Caputo-Grünwald-Letnikov fractional derivative. In this shifted case, the controlled system is *explicit* and consequently, one can consider general constraint functions f.

IX.2.3 A counterexample and comments

Let us consider $n = m = 1$, $a = 0$, $b \in \mathbb{N} \backslash \{0, 1\}$ and $N = b \geq 2$. Then, we have $\mathbb{T} = \{0, 1, \ldots, N\}$ and $h = 1$. Let f and f^0 be the functions given by $f(x,v,t) = x^2 + v^2 + x$ and $f^0(x,v,t) = v$. Let us note that f does not satisfy Assumption (IX.5). We consider the discrete fractional optimal control problem $(\mathbf{OCP})^\alpha_h$ with fixed initial condition $q_a = 0$. Since the value $u(a)$ does not intervene in $(\mathbf{OCP})^\alpha_h$, we omit it and we consider controls belonging to $\mathrm{C}(\mathbb{T}_\kappa, \mathbb{R}^m)$.

The only possible control $u \in \mathrm{C}(\mathbb{T}_\kappa, \mathbb{R}^m)$ ensuring the existence of a solution $q \in \mathrm{C}(\mathbb{T}, \mathbb{R}^n)$ of (IX.4) satisfying the initial condition $q(a) = 0$ is the null control *i.e.* $u(t_k) = 0$ for every $k = 1, \dots, N$. Then, the corresponding solution q is the null function *i.e.* $q(t_k) = 0$ for every $k = 0, \dots, N$.

As a consequence, the null couple (q, u) is automatically a solution of $(\mathbf{OCP})^\alpha_h$ but for every $p \in \mathrm{C}(\mathbb{T}, \mathbb{R}^n)$, we have

$$\frac{\partial H}{\partial v}(q(t_k), u(t_k), p(t_{k-1}), t_k) = 1 \neq 0, \tag{IX.11}$$

for every $k = 1, \dots, N$. Hence, the WPMP stated in Theorem IX.1 is not valid in this framework. This is due to the lack of Assumption (IX.5). In fact, it is not possible to make a variation on the optimal control u since it is the unique one ensuring the existence of a solution q of (IX.4).

Remark IX.4. This counterexample shows the failure of the Lagrange multiplier technique. Indeed, Equations (IX.8) and (IX.9) still characterize the critical points of the following augmented functional

$$\begin{aligned}(q, u, p) \in \mathrm{C}(\mathbb{T}, \mathbb{R}^n) \times \mathrm{C}(\mathbb{T}, \mathbb{R}^m) \times \mathrm{C}(\mathbb{T}, \mathbb{R}^n) \longmapsto \sum_{k=1}^{N} \Big[& f^0(q(t_k), u(t_k), t_k) \\ & + p(t_{k-1})({}_{\mathrm{c}}\Delta^\alpha_{a+}[q](t_k) - f(q(t_k), u(t_k), t_k))\Big].\end{aligned} \tag{IX.12}$$

Nevertheless, the solution of the optimal control problem $(\mathbf{OCP})^\alpha_h$ is not a critical point of (IX.12).

IX.3 Proof of Theorem IX.1

In this section, we assume that f is globally Lipschitz continuous in the sense of (IX.5). In Section IX.3.1, we introduce variations on the control and the initial condition and we derive relative properties in order to prove Theorem IX.1 in Section IX.3.2.

IX.3.1 Variations on admissible controls

Throughout the section, we consider $(u, q_a) \in \mathrm{C}(\mathbb{T}, \mathbb{R}^m) \times \mathbb{R}^n$ and the corresponding solution $q(\cdot, u, q_a) \in \mathrm{C}(\mathbb{T}, \mathbb{R}^n)$ of (IX.4) satisfying $q(a) = q_a$. We first define variations on u and derive some useful properties. Next, a paragraph is devoted to the variations on the initial condition q_a.

Variation on u. Let $\bar{u} \in \mathrm{C}(\mathbb{T}, \mathbb{R}^m)$. Let us give the following series of technical results with their proofs.

Lemma IX.2. *We assume that f is globally Lipschitz continuous in the sense of* (IX.5)*. For every $k = 0, \dots, N$, the mapping*

$$\begin{array}{rccl} F^k_{\bar{u}}: & [-1,1] & \longrightarrow & \mathbb{R}^n \\ & \varepsilon & \longmapsto & q(t_k, u + \varepsilon\bar{u}, q_a) \end{array} \tag{IX.13}$$

is continuous at 0 *i.e.* $q(t_k, u + \varepsilon\bar{u}, q_a)$ *tends to* $q(t_k, u, q_a)$ *when* ε *tends to* 0.

Proof. We prove by induction on k that $q(t_k, u + \varepsilon\bar{u}, q_a)$ tends to $q(t_k, u, q_a)$ when ε tends to 0. For $k = 0$, this property is obviously satisfied since $q(a, u + \varepsilon\bar{u}, q_a) = q_a$ for every $\varepsilon \in [-1, 1]$. Now, let $k \in \{1, \dots, N\}$ and let us assume that this property is satisfied for every $r = 0, \dots, k-1$. For every $\varepsilon \in [-1, 1]$, it holds

$$\begin{aligned} q(t_k, u + \varepsilon\bar{u}, q_a) - q(t_k, u, q_a) = h^\alpha [& f(q(t_k, u + \varepsilon\bar{u}, q_a), u(t_k) + \varepsilon\bar{u}(t_k), t_k) - f(q(t_k, u, q_a), u(t_k), t_k)] \\ & - \sum_{r=1}^{k} \alpha_r (q(t_{k-r}, u + \varepsilon\bar{u}, q_a) - q(t_{k-r}, u, q_a)). \end{aligned} \tag{IX.14}$$

Hence, we have

$$\begin{aligned}\|q(t_k,u+\varepsilon\bar{u},q_a)-q(t_k,u,q_a)\| \\ \leq h^\alpha\|f(q(t_k,u+\varepsilon\bar{u},q_a),u(t_k)+\varepsilon\bar{u}(t_k),t_k)-f(q(t_k,u,q_a),u(t_k)+\varepsilon\bar{u}(t_k),t_k)\| \\ +h^\alpha\|f(q(t_k,u,q_a),u(t_k)+\varepsilon\bar{u}(t_k),t_k)-f(q(t_k,u,q_a),u(t_k),t_k)\| \\ +\sum_{r=1}^{k}|\alpha_r|\|q(t_{k-r},u+\varepsilon\bar{u},q_a)-q(t_{k-r},u,q_a)\|. \quad \text{(IX.15)}\end{aligned}$$

Then, from Assumption (IX.5), we obtain

$$\begin{aligned}(1-h^\alpha L)\|q(t_k,u+\varepsilon\bar{u},q_a)-q(t_k,u,q_a)\| \\ \leq h^\alpha\|f(q(t_k,u,q_a),u(t_k)+\varepsilon\bar{u}(t_k),t_k)-f(q(t_k,u,q_a),u(t_k),t_k)\| \\ +\sum_{r=1}^{k}|\alpha_r|\|q(t_{k-r},u+\varepsilon\bar{u},q_a)-q(t_{k-r},u,q_a)\|, \quad \text{(IX.16)}\end{aligned}$$

which concludes the proof by induction. □

We define the so-called *variation vector* $\omega_{\bar{u}}(\cdot,u,q_a)$, associated with the variation $\bar{u}$, as the unique solution of the affine discrete fractional Cauchy problem

$${}_{\mathrm{c}}\Delta^{\alpha}_{a+}[\omega](t_k)=\frac{\partial f}{\partial x}(q(t_k,u,q_a),u(t_k),t_k)\times\omega(t_k)+\frac{\partial f}{\partial v}(q(t_k,u,q_a),u(t_k),t_k)\times\bar{u}(t_k), \quad \text{(IX.17)}$$

for every $k=1,\dots,N$, with the initial condition $\omega(a)=0$. The existence and uniqueness of $\omega_{\bar{u}}(\cdot,u,q_a)$ are ensured by Assumption (IX.5) and can be derived in a similar way than in Lemma IX.1.

Proposition IX.1. *We assume that f is globally Lipschitz continuous in the sense of* (IX.5). *For every $k=0,\dots,N$, the mapping*

$$\begin{array}{rccc} F^k_{\bar{u}}: & [-1,1] & \longrightarrow & \mathbb{R}^n \\ & \varepsilon & \longmapsto & q(t_k,u+\varepsilon\bar{u},q_a)\end{array} \quad \text{(IX.18)}$$

is differentiable at 0*, and it holds* $DF^k_{\bar{u}}(0)=\omega_{\bar{u}}(t_k,u,q_a)$.

Proof. We prove by induction on k that

$$\frac{q(t_k,u+\varepsilon\bar{u},q_a)-q(t_k,u,q_a)}{\varepsilon} \quad \text{(IX.19)}$$

tends to $\omega_{\bar{u}}(t_k,u,q_a)$ when ε tends to 0 for every $k=0,\dots,N$. For $k=0$, this property is obviously satisfied. Now, let $k\in\{1,\dots,N\}$ and let us assume that this property is satisfied for every $r=0,\dots,k-1$. For every $\varepsilon\in[-1,1]$, $\varepsilon\neq 0$, it holds

$$\begin{aligned}\frac{q(t_k,u+\varepsilon\bar{u},q_a)-q(t_k,u,q_a)}{\varepsilon}-\omega_{\bar{u}}(t_k,u,q_a) \\ =h^\alpha\Bigg[\frac{f(q(t_k,u+\varepsilon\bar{u},q_a),u(t_k)+\varepsilon\bar{u}(t_k),t_k)-f(q(t_k,u,q_a),u(t_k),t_k)}{\varepsilon} \\ -\frac{\partial f}{\partial x}(q(t_k,u,q_a),u(t_k),t_k)\times\omega_{\bar{u}}(t_k,u,q_a)-\frac{\partial f}{\partial v}(q(t_k,u,q_a),u(t_k),t_k)\times\bar{u}(t_k)\Bigg] \\ -\sum_{r=1}^{k}\alpha_r\left(\frac{q(t_{k-r},u+\varepsilon\bar{u},q_a)-q(t_{k-r},u,q_a)}{\varepsilon}-\omega_{\bar{u}}(t_{k-r},u,q_a)\right). \quad \text{(IX.20)}\end{aligned}$$

From the Mean Value Theorem, for every $\varepsilon \in [-1,1]$, $\varepsilon \neq 0$, there exists $(\theta_1^k(\varepsilon), \theta_2^k(\varepsilon)) \in \mathbb{R}^n \times \mathbb{R}^m$ that is a convex combination of $(q(t_k, u+\varepsilon\bar{u}, q_a), u(t_k)+\varepsilon\bar{u}(t_k))$ and $(q(t_k,u,q_a), u(t_k))$ such that

$$\begin{aligned}
&\frac{q(t_k, u+\varepsilon\bar{u}, q_a) - q(t_k,u,q_a)}{\varepsilon} - \omega_{\bar{u}}(t_k,u,q_a) \\
&\quad = h^\alpha \frac{\partial f}{\partial x}(\theta_1^k(\varepsilon), \theta_2^k(\varepsilon), t_k) \times \left[\frac{q(t_k, u+\varepsilon\bar{u}, q_a) - q(t_k,u,q_a)}{\varepsilon} - \omega_{\bar{u}}(t_k,u,q_a) \right] \\
&\quad + h^\alpha \left[\frac{\partial f}{\partial x}(\theta_1^k(\varepsilon), \theta_2^k(\varepsilon), t_k) - \frac{\partial f}{\partial x}(q(t_k,u,q_a), u(t_k), t_k) \right] \times \omega_{\bar{u}}(t_k,u,q_a) \\
&\quad + h^\alpha \left[\frac{\partial f}{\partial v}(\theta_1^k(\varepsilon), \theta_2^k(\varepsilon), t_k) - \frac{\partial f}{\partial v}(q(t_k,u,q_a), u(t_k), t_k) \right] \times \bar{u}(t_k) \\
&\quad - \sum_{r=1}^{k} \alpha_r \left(\frac{q(t_{k-r}, u+\varepsilon\bar{u}, q_a) - q(t_{k-r},u,q_a)}{\varepsilon} - \omega_{\bar{u}}(t_{k-r},u,q_a) \right). \qquad \text{(IX.21)}
\end{aligned}$$

Note that $(\theta_1^k(\varepsilon), \theta_2^k(\varepsilon))$ tends to $(q(t_k,u,q_a), u(t_k))$ when ε tends to 0 from Lemma IX.2. Then, Assumption (IX.5) leads to

$$\begin{aligned}
&(1-h^\alpha L) \left\| \frac{q(t_k, u+\varepsilon\bar{u}, q_a) - q(t_k,u,q_a)}{\varepsilon} - \omega_{\bar{u}}(t_k,u,q_a) \right\| \\
&\quad \leq h^\alpha \left\| \frac{\partial f}{\partial x}(\theta_1^k(\varepsilon), \theta_2^k(\varepsilon), t_k) - \frac{\partial f}{\partial x}(q(t_k,u,q_a), u(t_k), t_k) \right\| \|\omega_{\bar{u}}(t_k,u,q_a)\| \\
&\quad + h^\alpha \left\| \frac{\partial f}{\partial v}(\theta_1^k(\varepsilon), \theta_2^k(\varepsilon), t_k) - \frac{\partial f}{\partial v}(q(t_k,u,q_a), u(t_k), t_k) \right\| \|\bar{u}(t_k)\| \\
&\quad + \sum_{r=1}^{k} |\alpha_r| \left\| \frac{q(t_{k-r}, u+\varepsilon\bar{u}, q_a) - q(t_{k-r},u,q_a)}{\varepsilon} - \omega_{\bar{u}}(t_{k-r},u,q_a) \right\|, \qquad \text{(IX.22)}
\end{aligned}$$

which concludes the proof by induction. □

Variation on the initial condition q_a. Let $\bar{q}_a \in \mathbb{R}^n$. Let us give the following series of technical results with their proofs.

Lemma IX.3. *We assume that f is globally Lipschitz continuous in the sense of* (IX.5). *For every $k = 0, \dots, N$, the mapping*

$$\begin{array}{rccl} F_{\bar{q}_a}^k : & [-1,1] & \longrightarrow & \mathbb{R}^n \\ & \gamma & \longmapsto & q(t_k, u, q_a+\gamma\bar{q}_a) \end{array} \qquad \text{(IX.23)}$$

is continuous at 0 *i.e. $q(t_k,u,q_a+\gamma\bar{q}_a)$ tends to $q(t_k,u,q_a)$ when γ tends to* 0.

Proof. Our aim is to prove by induction on k that $q(t_k,u,q_a+\gamma\bar{q}_a)$ tends to $q(t_k,u,q_a)$ when γ tends to 0. For $k=0$, it is obvious since $q(a,u,q_a+\gamma\bar{q}_a) = q_a+\gamma\bar{q}_a$ for every $\gamma \in [-1,1]$. Now, let $k \in \{1,\dots,N\}$ and let us assume that this property is satisfied for every $r = 0,\dots,k-1$. For every $\gamma \in [-1,1]$, it holds

$$\begin{aligned}
q(t_k,u,q_a+\gamma\bar{q}_a) - q(t_k,u,q_a) &= h^\alpha [f(q(t_k,u,q_a+\gamma\bar{q}_a), u(t_k), t_k) - f(q(t_k,u,q_a), u(t_k), t_k)] \\
&\quad + \gamma\alpha_k^*\bar{q}_a - \sum_{r=1}^{k} \alpha_r (q(t_{k-r},u,q_a+\gamma\bar{q}_a) - q(t_{k-r},u,q_a)). \qquad \text{(IX.24)}
\end{aligned}$$

Then, from Assumption (IX.5), we obtain

$$\begin{aligned}
&(1-h^\alpha L)\|q(t_k,u,q_a+\gamma\bar{q}_a) - q(t_k,u,q_a)\| \\
&\quad \leq |\gamma||\alpha_k^*|\|\bar{q}_a\| + \sum_{r=1}^{k} |\alpha_r| \|q(t_{k-r},u,q_a+\gamma\bar{q}_a) - q(t_{k-r},u,q_a)\|, \qquad \text{(IX.25)}
\end{aligned}$$

which concludes the proof by induction. □

We define the so-called *variation vector* $\omega_{\bar{q}_a}(\cdot,u,q_a)$, associated with the variation $\bar{q}_a$, as the unique solution of the linear discrete fractional Cauchy problem

$$ {}_{\mathrm{c}}\Delta^{\alpha}_{a+}[\omega](t_k) = \frac{\partial f}{\partial x}(q(t_k,u,q_a),u(t_k),t_k) \times \omega(t_k), \tag{IX.26}$$

for every $k=1,\dots,N$, with the initial condition $\omega(a)=\bar{q}_a$. The existence and uniqueness of $\omega_{\bar{q}_a}(\cdot,u,q_a)$ are ensured by Assumption (IX.5) and can be derived in a similar way than in Lemma IX.1.

Proposition IX.2. *We assume that f is globally Lipschitz continuous in the sense of* (IX.5). *For every $k=0,\dots,N$, the mapping*

$$\begin{array}{rccl} F^k_{\bar{q}_a}: & [-1,1] & \longrightarrow & \mathbb{R}^n \\ & \gamma & \longmapsto & q(t_k,u,q_a+\gamma\bar{q}_a) \end{array} \tag{IX.27}$$

is differentiable at 0, and it holds $DF^k_{\bar{q}_a}(0)=\omega_{\bar{q}_a}(t_k,u,q_a)$.

Proof. Our aim is to prove by induction on k that

$$\frac{q(t_k,u,q_a+\gamma\bar{q}_a)-q(t_k,u,q_a)}{\gamma} \tag{IX.28}$$

tends to $\omega_{\bar{q}_a}(t_k,u,q_a)$ when γ tends to 0 for every $k=0,\dots,N$. For $k=0$, this property is obviously satisfied. Now, let $k\in\{1,\dots,N\}$ and let us assume that this property is satisfied for every $r=0,\dots,k-1$. For every $\gamma\in[-1,1]$, $\gamma\neq 0$, it holds

$$\begin{aligned} &\frac{q(t_k,u,q_a+\gamma\bar{q}_a)-q(t_k,u,q_a)}{\gamma} - \omega_{\bar{q}_a}(t_k,u,q_a) \\ &\qquad = h^{\alpha}\Bigg[\frac{f(q(t_k,u,q_a+\gamma\bar{q}_a),u(t_k),t_k)-f(q(t_k,u,q_a),u(t_k),t_k)}{\gamma} \\ &\qquad\qquad -\frac{\partial f}{\partial x}(q(t_k,u,q_a),u(t_k),t_k)\times\omega_{\bar{q}_a}(t_k,u,q_a)\Bigg] \\ &\qquad\qquad -\sum_{r=1}^{k}\alpha_r\left(\frac{q(t_{k-r},u,q_a+\gamma\bar{q}_a)-q(t_{k-r},u,q_a)}{\gamma}-\omega_{\bar{q}_a}(t_{k-r},u,q_a)\right). \end{aligned} \tag{IX.29}$$

From the Mean Value Theorem, for every $\gamma\in[-1,1]$, $\gamma\neq 0$, there exists $\theta^k(\gamma)\in\mathbb{R}^n$ that is a convex combination of $q(t_k,u,q_a+\gamma\bar{q}_a)$ and $q(t_k,u,q_a)$ such that

$$\begin{aligned} &\frac{q(t_k,u,q_a+\gamma\bar{q}_a)-q(t_k,u,q_a)}{\gamma} - \omega_{\bar{q}_a}(t_k,u,q_a) \\ &\qquad = h^{\alpha}\frac{\partial f}{\partial x}(\theta^k(\gamma),u(t_k),t_k)\times\left[\frac{q(t_k,u,q_a+\gamma\bar{q}_a)-q(t_k,u,q_a)}{\gamma}-\omega_{\bar{q}_a}(t_k,u,q_a)\right] \\ &\qquad\quad + h^{\alpha}\left[\frac{\partial f}{\partial x}(\theta^k(\gamma),u(t_k),t_k)-\frac{\partial f}{\partial x}(q(t_k,u,q_a),u(t_k),t_k)\right]\times\omega_{\bar{q}_a}(t_k,u,q_a) \\ &\qquad\quad -\sum_{r=1}^{k}\alpha_r\left(\frac{q(t_{k-r},u,q_a+\gamma\bar{q}_a)-q(t_{k-r},u,q_a)}{\varepsilon}-\omega_{\bar{q}_a}(t_{k-r},u,q_a)\right). \end{aligned} \tag{IX.30}$$

Note that $\theta^k(\gamma)$ tends to $q(t_k,u,q_a)$ when γ tends to 0, see Lemma IX.3. Then, from Assumption (IX.5), we obtain

$$\begin{aligned} &(1-h^{\alpha}L)\left\|\frac{q(t_k,u,q_a+\gamma\bar{q}_a)-q(t_k,u,q_a)}{\gamma}-\omega_{\bar{q}_a}(t_k,u,q_a)\right\| \\ &\qquad \leq h^{\alpha}\left\|\frac{\partial f}{\partial x}(\theta^k(\gamma),u(t_k),t_k)-\frac{\partial f}{\partial x}(q(t_k,u,q_a),u(t_k),t_k)\right\|\|\omega_{\bar{q}_a}(t_k,u,q_a)\| \\ &\qquad\quad +\sum_{r=1}^{k}|\alpha_r|\left\|\frac{q(t_{k-r},u,q_a+\gamma\bar{q}_a)-q(t_{k-r},u,q_a)}{\gamma}-\omega_{\bar{q}_a}(t_{k-r},u,q_a)\right\|, \end{aligned} \tag{IX.31}$$

which concludes the proof by induction. □

IX.3.2 Proof of the WPMP

We assume that the hypotheses of Theorem IX.1 are satisfied. We denote by $q_a^* = q^*(a)$. In particular, one has $q(\cdot, u^*, q_a^*) = q^*$. We define $p \in \mathrm{C}(\mathbb{T}, \mathbb{R}^n)$ as the unique solution of the discrete fractional affine Cauchy problem given by

$$ {}_{\mathrm{c}}\Delta^{\alpha}_{b-}[p](t_k) = \left(\frac{\partial f}{\partial x}(q^*(t_{k+1}), u^*(t_{k+1}), t_{k+1}) \right)^{\mathrm{T}} \times p(t_k) + \frac{\partial f_0}{\partial x}(q^*(t_{k+1}), u^*(t_{k+1}), t_{k+1}), $$

for every $k = 0, \ldots, N-1$ with the final condition $p(b) = 0$. The existence and uniqueness of p are ensured by Assumption (IX.5) and can be derived in a similar way than in Lemma IX.1.

Firstly, we note that the introduction of p ensures that q^* and p are solutions of

$$ {}_{\mathrm{c}}\Delta^{\alpha}_{a+}[q^*](t_k) = \frac{\partial H}{\partial w}(q^*(t_k), u^*(t_k), p(t_{k-1}), t_k), \quad {}_{\mathrm{c}}\Delta^{\alpha}_{b-}[p](t_{k-1}) = \frac{\partial H}{\partial x}(q^*(t_k), u^*(t_k), p(t_{k-1}), t_k), \tag{IX.32} $$

for every $k = 1, \ldots, N$. Before continuing, we first need to introduce a new discrete fractional integration by parts. We do not detail the proof since it is very similar to the proof of Lemma VIII.1.

Lemma IX.4 (Discrete fractional integration by part)**.** *Let q_1, $q_2 \in \mathrm{C}(\mathbb{T}, \mathbb{R}^n)$ such that $q_1(a) = q_2(b) = 0$. Then, it holds*

$$ h \sum_{k=1}^{N} {}_{\mathrm{c}}\Delta^{\alpha}_{a+}[q_1](t_k) \cdot q_2(t_{k-1}) = h \sum_{k=1}^{N} q_1(t_k) \cdot {}_{\mathrm{c}}\Delta^{\alpha}_{b-}[q_2](t_{k-1}). \tag{IX.33} $$

Proof of Equality (IX.9)**.** Let $\bar{u} \in \mathrm{C}(\mathbb{T}, \mathbb{R}^m)$. From Lemma IX.2 and Proposition IX.1, one has the differentiability of the function

$$ \Phi_{\bar{u}}(\varepsilon) = h \sum_{k=1}^{N} f^0(q(t_k, u^* + \varepsilon\bar{u}, q_a^*), u^*(t_k) + \varepsilon\bar{u}(t_k), t_k) $$

at $\varepsilon = 0$ with

$$ \Phi'_{\bar{u}}(0) = h \sum_{k=1}^{N} \frac{\partial f^0}{\partial x}(q^*(t_k), u^*(t_k), t_k) \cdot \omega_{\bar{u}}(t_k, u^*, q_a^*) + \frac{\partial f^0}{\partial v}(q^*(t_k), u^*(t_k), t_k) \cdot \bar{u}(t_k). \tag{IX.34} $$

Then, it holds

$$ \begin{aligned} \Phi'_{\bar{u}}(0) = h \sum_{k=1}^{N} \Bigg(\left(\left(\frac{\partial f}{\partial x}(q^*(t_k), u^*(t_k), t_k) \right)^{\mathrm{T}} \times p(t_{k-1}) + \frac{\partial f^0}{\partial x}(q^*(t_k), u^*(t_k), t_k) \right) \cdot \omega_{\bar{u}}(t_k, u^*, q_a^*) \\ + \frac{\partial f^0}{\partial v}(q^*(t_k), u^*(t_k), t_k) \cdot \bar{u}(t_k) - \left(\frac{\partial f}{\partial x}(q^*(t_k), u^*(t_k), t_k) \times \omega_{\bar{u}}(t_k, u^*, q_a^*) \right) \cdot p(t_{k-1}) \Bigg). \end{aligned} \tag{IX.35} $$

We recall that $\omega_{\bar{u}}(\cdot, u^*, q_a^*)$ and p satisfy

$$ {}_{\mathrm{c}}\Delta^{\alpha}_{a+}[\omega_{\bar{u}}(\cdot, u^*, q_a^*)](t_k) = \frac{\partial f}{\partial x}(q^*(t_k), u^*(t_k), t_k) \times \omega_{\bar{u}}(t_k, u^*, q_a^*) + \frac{\partial f}{\partial v}(q^*(t_k), u^*(t_k), t_k) \times \bar{u}(t_k) \tag{IX.36} $$

and

$$ {}_{\mathrm{c}}\Delta^{\alpha}_{b-}[p](t_{k-1}) = \left(\frac{\partial f}{\partial x}(q^*(t_k), u^*(t_k), t_k) \right)^{\mathrm{T}} \times p(t_{k-1}) + \frac{\partial f_0}{\partial x}(q^*(t_k), u^*(t_k), t_k), $$

for every $k = 1, \ldots, N$. As a consequence, using the discrete fractional integration by parts given in Lemma IX.4, we obtain

$$ \Phi'_{\bar{u}}(0) = h \sum_{k=1}^{N} \left(\left(\frac{\partial f}{\partial v}(q^*(t_k), u^*(t_k), t_k) \right)^{\mathrm{T}} \times p(t_{k-1}) + \frac{\partial f^0}{\partial v}(q^*(t_k), u^*(t_k), t_k) \right) \cdot \bar{u}(t_k) \tag{IX.37} $$

that is

$$\Phi'_{\bar{u}}(0) = h \sum_{k=1}^{N} \frac{\partial H}{\partial v}(q^*(t_k), u^*(t_k), p(t_{k-1}), t_k) \cdot \bar{u}(t_k). \tag{IX.38}$$

Finally, since q^* is a solution of the discrete fractional optimal control problem $(\mathbf{OCP})_h^\alpha$, $\Phi_{\bar{u}}$ admits a minimum at $\varepsilon = 0$ and then $\Phi'_{\bar{u}}(0) = 0$. Consequently, we deduce that

$$h \sum_{k=1}^{N} \frac{\partial H}{\partial v}(q^*(t_k), u^*(t_k), p(t_{k-1}), t_k) \cdot \bar{u}(t_k) = 0 \tag{IX.39}$$

for every $\bar{u} \in \mathrm{C}(\mathbb{T}, \mathbb{R}^m)$ which concludes the proof of Equality (IX.9).

Proof of Equality (IX.10)**.** Let $\bar{q}_a \in \mathbb{R}^n$. From Lemma IX.3 and Proposition IX.2, one has the differentiability of the function

$$\Phi_{\bar{q}_a}(\gamma) = h \sum_{k=1}^{N} f^0(q(t_k, u^*, q_a^* + \gamma \bar{q}_a), u^*(t_k), t_k)$$

at $\gamma = 0$ with

$$\Phi'_{\bar{q}_a}(0) = h \sum_{k=1}^{N} \frac{\partial f^0}{\partial x}(q^*(t_k), u^*(t_k), t_k) \cdot \omega_{\bar{q}_a}(t_k, u^*, q_a^*). \tag{IX.40}$$

Then, it holds

$$\Phi'_{\bar{q}_a}(0) = h \sum_{k=1}^{N} \left(\left(\frac{\partial f}{\partial x}(q^*(t_k), u^*(t_k), t_k) \right)^{\mathrm{T}} \times p(t_{k-1}) + \frac{\partial f^0}{\partial x}(q^*(t_k), u^*(t_k), t_k) \right) \cdot \omega_{\bar{q}_a}(t_k, u^*, q_a^*) \\ - \left(\frac{\partial f}{\partial x}(q^*(t_k), u^*(t_k), t_k) \times \omega_{\bar{q}_a}(t_k, u^*, q_a^*) \right) \cdot p(t_{k-1}). \tag{IX.41}$$

We recall that $\omega_{\bar{q}_a}(\cdot, u^*, q_a^*)$ and p satisfy

$$_{\mathrm{c}}\Delta^\alpha_{a+}[\omega_{\bar{q}_a}(\cdot, u^*, q_a^*)](t_k) = \frac{\partial f}{\partial x}(q^*(t_k), u^*(t_k), t_k) \times \omega_{\bar{q}_a}(t_k, u^*, q_a^*) \tag{IX.42}$$

and

$$_{\mathrm{c}}\Delta^\alpha_{b-}[p](t_{k-1}) = \left(\frac{\partial f}{\partial x}(q^*(t_k), u^*(t_k), t_k) \right)^{\mathrm{T}} \times p(t_{k-1}) + \frac{\partial f_0}{\partial x}(q^*(t_k), u^*(t_k), t_k),$$

for every $k = 1, \ldots, N$. As a consequence, using the discrete fractional integration by parts given in Lemma IX.4, we obtain

$$\Phi'_{\bar{q}_a}(0) = h \sum_{k=1}^{N} \bar{q}_a \cdot {_{\mathrm{c}}\Delta^\alpha_{b-}}[p](t_{k-1}) = h \bar{q}_a \cdot \sum_{k=1}^{N} \frac{\partial H}{\partial x}(q^*(t_k), u^*(t_k), p(t_{k-1}), t_k). \tag{IX.43}$$

Now, let us assume that $(\mathbf{OCP})_h^\alpha$ is considered with free initial condition. Since q^* is a solution of $(\mathbf{OCP})_h^\alpha$, $\Phi_{\bar{q}_a}$ admits a minimum at $\gamma = 0$ and then $\Phi'_{\bar{q}_a}(0) = 0$. Consequently, we deduce that

$$\bar{q}_a \cdot \left(h \sum_{k=1}^{N} \frac{\partial H}{\partial x}(q^*(t_k), u^*(t_k), p(t_{k-1}), t_k) \right) = 0 \tag{IX.44}$$

for every $\bar{q}_a \in \mathbb{R}^n$ which concludes the proof of Equality (IX.10).

IX.4 A WPMP for shifted discrete fractional optimal control problems

In order to study a framework without restrictive assumption on the constraint function, we consider in this section a *shifted* discrete controlled system and we derive the corresponding WPMP. Precisely, we consider the general *shifted* discrete fractional nonlinear controlled system

$$ {}_{\mathrm{c}}\Delta^{\alpha}_{a+}[q](t_{k+1}) = f(q(t_k), u(t_k), t_k), \tag{IX.45} $$

for every $k = 0, \dots, N-1$, where $f : \mathbb{R}^n \times \mathbb{R}^m \times [a,b] \longrightarrow \mathbb{R}^n, (x,v,t) \longmapsto f(x,v,t)$ is a continuous function of class $\mathscr{C}^1$ with respect to its two first variables and where the control functions u belong to $\mathrm{C}(\mathbb{T}, \mathbb{R}^m)$.

In contrary to the framework considered in Section IX.2, Equation (IX.45) is an *explicit* equation. As a consequence, the existence and uniqueness of a solution of (IX.45) with the initial condition $q(a) = q_a$ is guaranteed for every $(u, q_a) \in \mathrm{C}(\mathbb{T}, \mathbb{R}^m) \times \mathbb{R}^n$. Moreover, this solution can be given in an explicit way and consequently, all calculations of variations on the control or on the initial condition are easier. In this section, we do not consider any additional assumption on the constraint function f.

In this shifted framework, we define a general shifted discrete optimal control problem with a Caputo-Grünwald-Letnikov fractional derivative (denoted in short $(\mathbf{OCP}\mathrm{shift})^{\alpha}_{\mathbf{h}}$) as follows. Let $f^0 : \mathbb{R}^n \times \mathbb{R}^m \times [a,b] \longrightarrow \mathbb{R}, (x,v,t) \longmapsto f^0(x,v,t)$ be a continuous function of class $\mathscr{C}^1$ with respect to its two first variables.

Definition of $(\mathbf{OCP}\mathrm{shift})^{\alpha}_{\mathbf{h}}$: determine a trajectory $q^* \in \mathrm{C}(\mathbb{T}, \mathbb{R}^n)$ solution of

$$ {}_{\mathrm{c}}\Delta^{\alpha}_{a+}[q](t_{k+1}) = f(q(t_k), u(t_k), t_k), \tag{IX.45} $$

and associated with a control $u^* \in \mathrm{C}(\mathbb{T}, \mathbb{R}^m)$ minimizing the cost

$$ h \sum_{k=0}^{N-1} f^0(q(t_k), u(t_k), t_k) \tag{IX.46} $$

over all possible trajectories $q \in \mathrm{C}(\mathbb{T}, \mathbb{R}^n)$ solutions of (IX.45) and associated with a control $u \in \mathrm{C}(\mathbb{T}, \mathbb{R}^m)$. The initial condition can be fixed or not.

The *Hamiltonian* of the discrete shifted fractional optimal control problem $(\mathbf{OCP}\mathrm{shift})^{\alpha}_{\mathbf{h}}$ is the function $H : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times [a,b] \longrightarrow \mathbb{R}$ defined by $H(x,v,w,t) = w \cdot f(x,v,t) + f^0(x,v,t)$.

Theorem IX.2 (Weak Pontryagin Maximum Principle)**.** *If the trajectory $q^* \in \mathrm{C}(\mathbb{T}, \mathbb{R}^n)$, associated with a control $u^* \in \mathrm{C}(\mathbb{T}, \mathbb{R}^m)$, is a solution of $(\mathbf{OCP}\mathrm{shift})^{\alpha}_{\mathbf{h}}$, then there exists a mapping $p \in \mathrm{C}(\mathbb{T}, \mathbb{R}^n)$ (called adjoint vector), such that $p(b) = 0$ and such that q^* and p are solutions of*

$$ {}_{\mathrm{c}}\Delta^{\alpha}_{a+}[q^*](t_{k+1}) = \frac{\partial H}{\partial w}(q^*(t_k), u^*(t_k), p(t_{k+1}), t_k), \quad {}_{\mathrm{c}}\Delta^{\alpha}_{b-}[p](t_k) = \frac{\partial H}{\partial x}(q^*(t_k), u^*(t_k), p(t_{k+1}), t_k), \tag{IX.47} $$

for every $k = 0, \dots, N-1$. Moreover, it holds

$$ \frac{\partial H}{\partial v}(q^*(t_k), u^*(t_k), p(t_{k+1}), t_k) = 0_{\mathbb{R}^m}, \tag{IX.48} $$

for every $k = 0, \dots, N-1$. Furthermore, if the initial condition is free, one has the following transversality condition

$$ h \sum_{k=0}^{N-1} \frac{\partial H}{\partial x}(q^*(t_k), u^*(t_k), p(t_{k+1}), t_k) = 0_{\mathbb{R}^n}. \tag{IX.49} $$

Proof. The proof is very similar to the proof of Theorem IX.1 developed in the previous section. Actually, the calculations are even easier since the solutions of (IX.45) can be written in an explicit way. Consequently, we do not detail the proof of this theorem. □

Conclusion et perspectives

Après avoir rappelé les principaux résultats obtenus dans les Parties A et B, nous soulevons quelques problèmes ouverts et perspectives.

1. Problèmes ouverts en calcul time scale

Dans la Partie A, nous avons obtenu les résultats suivants :
- équation d'Euler-Lagrange sous forme différentielle (Théorème II.1) et théorème de Noether (Théorème II.2) en calcul des variations non shifté sur time scale ;
- version forte du Principe du Maximum de Pontryagin sur time scale (Théorème III.1) ;
- condition de Helmholtz discrète (voir Section IV.3).

Les démonstrations font apparaître deux obstacles majeurs à l'extension de résultats classiques au cadre time scale : la structure différentielle et le caractère arbitraire du time scale, qui en font aussi sa force. De ces observations découlent les problèmes suivants.

1.1. Autour du calcul des variations. Il n'existe pas, à notre connaissance, de résultat d'existence de solutions pour des problèmes variationnels sur time scale.

Problème 1 : *Énoncer un théorème de type Tonelli pour des fonctionnelles Lagrangiennes (shiftées ou non) sur time scale.*

Il s'agit d'adapter au cadre time scale l'étude faite au Chapitre VI. Nous pourrons alors utiliser les résultats de R. Agarwal *et al.* dans [4] autour des espaces de Sobolev sur time scale où la réfléxivité de ces espaces est établie.

Suite à la formulation différentielle de l'équation d'Euler-Lagrange dans le Chapitre II, nous posons les problèmes suivants :

Problème 2 : *Formuler une condition de type Helmholtz dans le cadre time scale.*

Pour cela, nous pourrons considérer l'équation d'Euler-Lagrange différentielle obtenue dans le Chapitre II ou encore celle obtenue dans [114, Remarque 4] pour un cadre plus général (puisque la ∇-dérivabilité de l'opérateur σ n'est pas requise pour ce résultat).

Problème 3 : *Formuler un théorème de type Noether pour l'équation d'Euler-Lagrange différentielle obtenue dans [114, Remarque 4].*

1.2. Autour du Principe du Maximum de Pontryagin (PMP en abrégé). Certaines techniques usuelles de démonstrations ne s'étendent pas au cadre time scale quand celui-ci est supposé général. C'est le cas du théorème de point fixe de Brouwer pour l'extension du PMP au cadre time scale (voir Section III.3.1). Dans ce cas, il est possible de contourner la difficulté par le principe variationnel d'Ekeland (voir Section III.3.3).

Problème 4 : *Généraliser le PMP obtenu dans le Chapitre III au cas de trajectoires à valeurs dans un espace de Banach quelconque.*

On renvoie à [149] pour l'utilisation du principe variationnel d'Ekeland dans la démonstration du PMP classique au cas de la dimension infinie.

Problème 5 : *Formuler un PMP où la condition de maximisation du Hamiltonien est obtenue sur (presque) tout le time scale grâce à l'hypothèse de convexité directionnelle*[1] *sur la dynamique* (f, f^0).

Une tentative a été proposée dans [205] mais plusieurs erreurs, notamment sur l'utilisation de la convexité directionnelle, ont été commises. Nous renvoyons à [113] et [53, Remarque 13] pour plus de détails.

1.3. Time scale et analyse numérique. Le calcul time scale a été initié par S. Hilger dans le but d'unifier l'analyse continue et l'analyse discrète. L'analyse numérique est une branche très dévelopée du calcul discret où les estimations entre solutions continues (dites *exactes*) et solutions discrètes (dites *approchées*) sont particulièrement recherchées. Une piste de recherche naturelle est alors la suivante :

Problème 6 : *Formuler des estimations entre la solution d'un système posé sur un time scale* $\mathbb{T}_1$ *et la solution du même problème considéré sur un second time scale* $\mathbb{T}_2 \subset \mathbb{T}_1$.

Ces estimations pourraient par exemple être données en fonction de la norme uniforme de $\mu^{\mathbb{T}_2} - \mu^{\mathbb{T}_1}$. Au delà de l'extension des résultats classiques de l'analyse numérique au cas time scale, ce projet propose l'idée de *choix* sur le time scale :

Problème 7 : *Développer des résultats d'optimisation du time scale en fonction du problème considéré et en fonction de certaines contraintes.*

Par exemple, dans le cas d'un schéma numérique pour une équation différentielle, l'optimisation du time scale porte sur l'approximation de la solution continue et les contraintes sont que le time scale est fini et que les valeurs de μ sont minorées.

2. Problèmes ouverts en calcul fractionnaire

Dans la Partie B, nous avons obtenu les résultats suivants :
- existence de points critiques en calcul des variations fractionnaire (Théorème VI.1) ;
- version faible du Principe du Maximum de Pontryagin fractionnaire (Théorème VII.1) ;
- construction d'intégrateurs variationnels pour des problèmes variationnels fractionnaires (Chapitres VIII et IX).

Les démonstrations font apparaître deux obstacles majeurs à l'extension de résultats classiques au cadre fractionnaire : l'absence de formule de Leibniz simple et le caractère non local des opérateurs fractionnaires. En contrepartie, le choix du paramètre α donne plus de flexibilité à certains cadres variationnels classiques. De ces observations découlent les problèmes suivants.

2.1. Autour d'une formule de Leibniz. Le théorème de type Noether fractionnaire (Proposition V.7) fait apparaître le problème suivant :

Problème 8 : *Développer les résultats sur les couples de fonctions* (f, g) *satisfaisant une relation du type* $\mathrm{D}^{\alpha}_{a+} f \cdot g - f \cdot \mathrm{D}^{\alpha}_{b-} g = 0$.

1. Voir Remarque III.11 pour la définition de convexité directionnelle. Cette hypothèse permet de récupérer la condition de maximisation du Hamiltonien dans le PMP discret.

Ce problème est dû au fait que la formule d'intégration par parties fractionnaire n'est pas une conséquence d'une formule de Leibniz mais du théorème de Fubini.

La complexité de la formule de Leibniz fractionnaire (voir [185, Section 2.7.2]) induit le problème suivant :

Problème 9 : *Définir des vecteurs adjoints pour des problèmes de Cauchy linéaires fractionnaires.*

La résolution de ce problème constitue une étape importante pour l'obtention d'une version forte du PMP fractionnaire.

2.2. Perturbation localisée des équations différentielles fractionnaires. La non localité des opérateurs fractionnaires est attractive pour la modélisation de systèmes avec effets de mémoire. Cependant, elle amène au problème suivant :

Problème 10 : *Obtenir des vecteurs de variations associés à des variations localisées sur des équations différentielles fractionnaires.*

La résolution de ce problème constitue une autre étape importante pour l'obtention d'une version forte du PMP fractionnaire.

2.3. Calcul fractionnaire et analyse numérique. Dû à leur caractère non local, le calcul explicite des opérateurs fractionnaires est délicat. Il est donc important d'obtenir des approximations numériques précises de ces quantités.

Problème 11 : *Construire des opérateurs discrets fractionnaires d'ordre plus élevé que les notions de Grünwald-Letnikov.*

On renvoie à [81, 150] pour des exemples de réponse à ce problème. Plus généralement, on peut penser à étendre la démarche de S. Hilger, reliant calcul différentiel et calcul aux différences, au cadre fractionnaire :

Problème 12 : *Définir une notion de dérivée fractionnaire sur time scale.*

La résolution de ces deux problèmes permettrait la construction d'intégrateurs variationnels plus élaborés pour les problèmes variationnels fractionnaires en suivant l'approche des Chapitres VIII et IX. Le Problème 12 est traité dans [32, 33] mais les résultats peuvent encore être développés.

2.4. Autour de problèmes variationnels fractionnaires. Les résultats d'existence de solutions pour des équations différentielles fractionnaires admettant une structure variationnelle sont rares et ne sont réduits qu'à quelques cas particuliers, voir *e.g.* [129, 143]. Dans le Chapitre VI, nous avons donc proposé la formulation de conditions générales suffisantes sur les fonctionnelles Lagrangiennes pour qu'elles admettent un minimiseur et, par la même occasion, que les équations d'Euler-Lagrange fractionnaires associées admettent une solution.

Le calcul fractionnaire permet une flexibilité sur le paramètre α qui n'existe pas dans les cadres classiques. Nous avons constaté que, là où la situation est relativement figée dans les cadres classiques quant à la régularité d'un Lagrangien donné, le calcul fractionnaire permet un "ajustement" du paramètre α pour assurer cette régularité tout en préservant le caractère coercif de la fonctionnelle Lagrangienne. Cette observation, couplée à l'introduction récente d'espaces de Hilbert fractionnaires, conduit à la perspective suivante :

Problème 13 : *Développer des cadres Hilbertiens permettant de démontrer des résultats d'existence et d'unicité de solutions pour des équations différentielles fractionnaires.*

Un travail en cours, en collaboration avec D. Idczak (Faculty of Mathematics and Computer Science, University of Lodz Banacha, Lodz, Poland), démontre un tel résultat à partir des théorèmes classiques de Lax-Milgram et de Stampacchia.

L'origine du calcul des variations fractionnaire se trouve dans la volonté de fournir une structure variationnelle fractionnaire à des systèmes dissipatifs qui n'en admettent pas de classique. Dans [70, 71], il est montré que l'équation de convection-diffusion (à coefficients constants) admet une structure variationnelle fractionnaire dite *assymétrique* (car la variable de la fonctionnelle est dédoublée). De manière générale, nous pouvons démontrer que toute équation différentielle linéaire (à coefficients constants) admet une telle structure.

Problème 14 : *Déterminer une structure variationnelle fractionnaire assymétrique pour des équations plus complexes comme l'équation de Navier-Stokes.*

La recherche de nouvelles propriétés (variationnelles) des équations dissipatives dans ce cadre donne lieu aux pistes de recherche suivantes :

Problème 15 : *Développer les résultats autour des fonctionnelles Lagrangiennes fractionnaires assymétriques.*

En particulier, nous pourrions développer des résultats d'existence de points critiques (au sens de [70]) de ces fonctionnelles.

Annexe A : Théorie de Cauchy-Lipschitz sur time scale pour des Δ-problèmes de Cauchy shiftés et non shiftés

Cette annexe complète certains aspects de la théorie de Cauchy-Lipschitz pour des systèmes généraux non linéaires posés sur time scale nécessaires à la démonstration au Chapitre III d'un Principe du Maximum de Pontryagin en théorie du contrôle optimal sur time scale.

Nous introduisons la notion de solution absolument continue d'un Δ-problème de Cauchy (shifté ou non) puis la notion de solution maximale. Par la suite, nous démontrons un théorème de type Cauchy-Lipschitz assurant l'existence et l'unicité de la solution maximale pour un Δ-problème de Cauchy donné. Comme dans les cas continu et discret, ce théorème n'est valable qu'à partir de conditions adéquates comme la régressivité et/ou la continuité localement Lipschitzienne de la dynamique. Pour finir, nous complétons notre étude par des résultats relatifs au comportement des solutions maximales au voisinage de leurs points terminaux.

Cette annexe est extraite de Bourdin L. et Trélat E., Cauchy-Lipschitz theory for shifted and non shifted Δ-Cauchy problems on time scales, preprint arXiv:1212.5042v1.

A.1 Introduction

Some Cauchy-Lipschitz (Picard-Lindelöf) type results on a general time scale $\mathbb{T}$ are provided in [38, 64, 112, 134, 145, 146] where the authors prove the existence and uniqueness of solutions for Δ-Cauchy problems of the form

$$q^{\Delta} = f(q,t), \quad q(t_0) = q_0, \tag{A.1}$$

where $t_0 \in \mathbb{T}$. Note that papers are devoted to Δ-Cauchy problems with parameter in [120] and with time delays in [133]. Many authors are also interested in shifted Δ-Cauchy problems of the type

$$q^{\Delta} = f(q^{\sigma},t), \quad q(t_0) = q_0. \tag{A.2}$$

Such shifted problems are often used as models in the existing literature (see *e.g.* [31, 117, 156], [116, Remark 3.9] and [120, Remark 3.6]), because they appear in adjoint equations accordingly to the emergence of a shift in the Leibniz formula (see Proposition I.2 or [38, Corollary 1.20 p.8]). Nevertheless, to the best of our knowledge, there does not exist a general Cauchy-Lipschitz theory on time scale that is fully complete in order to be applied to problems arising for example in control theory [1].

Let us recall briefly the bibliographical context on the Cauchy-Lipschitz theory on time scale. The first result on Δ-Cauchy problems is due to S. Hilger in [112, Paragraph 5], who derived the existence and uniqueness of $\mathrm{C}^{1,\Delta}_{\mathrm{rd}}$-solutions for continuous dynamics. This framework is not suitable for general control problems where controls are measurable functions that have discontinuities in general. Note that similar frameworks and results

1. Actually, the present study is motivated by the needs of completing the existing results on Cauchy-Lipschitz theory on time scale, in order to investigate general non linear control systems on time scale, and more precisely to derive a general version of the Pontryagin Maximum Principle in optimal control in Chapter III.

are provided in [38, Paragraph 8.2], in [134, 146, 195] and references therein. In [64, 145], the authors respectively treat weak continuous and Carathéodory dynamics living in a general Banach space. Note that they only treat the non shifted case where q_0 is an initial condition, that is, solutions are only defined for $t \geq t_0$. In view of considering adjoint equations, it is of interest to study backward Δ-Cauchy problems where q_0 is a final condition, for which solutions are defined for $t \leq t_0$. As is very well known in time scale calculus, the solvability of such backward non shifted Δ-Cauchy problems requires a *regressivity* assumption on the dynamics (see *e.g.* [38, 112] and [116, Remark 3.8]). This important issue is not addressed in these two articles. Another issue which is not addressed is the fact that the usual Cauchy-Lipschitz theory treats Cauchy problems constraining the solutions to take values in an open subset Ω of $\mathbb{R}^n$ (see *e.g.* [65, 123]). Finally, up to our knowledge, the notion of extension of a solution on time scale, and the behavior of the maximal solution at terminal points, have not been studied. Similarly, we are not aware of articles treating both shifted and non shifted general nonlinear Δ-Cauchy problems.

This appendix is thus devoted to fill an existing gap of the literature, and to provide a general Cauchy-Lipschitz theory on time scale generalizing the basic notions and results of the classical continuous theory surveyed *e.g.* in [65, 123]. Precisely, we first introduce the notion of an *absolutely continuous solution.* Then we define the concept of *extension* of a solution, and of *maximal* and *global* solutions in the time scale context. We establish a general version of the Cauchy-Lipschitz theorem (existence and uniqueness of the maximal solution, also referred to as Picard-Lindelöf theorem) for dynamics posed on a time scale, under regressivity and local Lipschitz continuity assumptions, for shifted and non shifted general nonlinear Δ-Cauchy problems in the following framework:

- f is a general Δ-Carathéodory function, where Δ-measure μ_Δ on a time scale $\mathbb{T}$ is defined in terms of Carathéodory extension in [39, Chapter 5], see also some details in Section I.3;
- q_0 is not necessarily an initial or a final condition;
- the solutions take their values in an open subset Ω of $\mathbb{R}^n$.

We also investigate the globality feature of the maximal solution. Our results are established first for general non shifted Δ-Cauchy problems (A.1) and then for shifted ones (A.2).

Our study uses the work of A. Cabada and D. Vivero in [58], who proved a criterion for absolutely continuous functions written as the Δ-integral of their Δ-derivatives, see details in Section I.3.2. Their result allows us to give a Δ-integral characterization of the solutions of Δ-Cauchy problems which is instrumental in our proofs.

Notice that analogous results on ∇-Cauchy problems (ρ-shifted or not) can be derived in a similar way.

Organization of this appendix. In Section A.2, we define the notions of a solution, of an extension of a solution, of a maximal and a global solution for general non shifted Δ-Cauchy problems. Under suitable assumptions on the dynamics, we establish a Cauchy-Lipschitz theorem and then investigate the behavior of the maximal solution at its terminal points. Section A.3 is devoted to establish similar results for *shifted* Δ-Cauchy problems.

A.2 General non shifted Δ-Cauchy problem

Notations: In the whole study, $\mathbb{T}$ denotes a time scale such that $\text{card}(\mathbb{T}) \geq 2$. In the sequel, the notation $\|\cdot\|$ stands for the Euclidean norm of $\mathbb{R}^n$ where $n \in \mathbb{N}^*$. Finally, for every $x \in \mathbb{R}^n$ and every $R \geq 0$, the notation $\overline{B}(x,R)$ stands for the closed ball of $\mathbb{R}^n$ centered at x and with radius R.

Throughout this section we consider the general non shifted Δ-Cauchy problem

$$(\Delta\text{-CP}) \qquad \begin{aligned} q^\Delta(t) &= f(q(t),t), \\ q(t_0) &= q_0, \end{aligned}$$

where $t_0 \in \mathbb{T}$, $q_0 \in \Omega$, where Ω is a non empty open subset of $\mathbb{R}^n$, and $f : \Omega \times \mathbb{T}\backslash\{\sup \mathbb{T}\} \longrightarrow \mathbb{R}^n$, $(x,t) \longmapsto f(x,t)$ is a Δ-Carathéodory function. The notation $\mathcal{K}$ stands for the set of compact subsets of Ω.

A.2.1 Preliminaries

In what follows it will be important to distinguish between three cases:

1. $t_0 = \min \mathbb{T}$;
2. $t_0 = \max \mathbb{T}$;
3. $t_0 \neq \inf \mathbb{T}$ and $t_0 \neq \sup \mathbb{T}$.

Indeed, the interval of definition of a solution of (Δ-CP) depends on the specific case under consideration. If $t_0 = \min \mathbb{T}$, then a solution can only *go forward* since $]-\infty, t_0[_{\mathbb{T}} = \emptyset$. If $t_0 = \max \mathbb{T}$, then a solution can only *go backward*. If $t_0 \neq \inf \mathbb{T}$ and $t_0 \neq \sup \mathbb{T}$, then a solution can go *backward* and *forward*.

Definition A.1. For all $(a, b) \in \mathbb{T}^2$, we say that $a \trianglelefteq t_0 \trianglelefteq b$ if

- $a = t_0 < b$ in the case $t_0 = \min \mathbb{T}$;
- $a < t_0 = b$ in the case $t_0 = \max \mathbb{T}$;
- $a < t_0 < b$ in the case $t_0 \neq \inf \mathbb{T}$ and $t_0 \neq \sup \mathbb{T}$.

If $a \trianglelefteq t_0 \trianglelefteq b$ then $[a, b]_{\mathbb{T}}$ is a potential interval of definition for a solution of (Δ-CP). Due to this difference of intervals, it is required to make different assumptions on f accordingly, whence the following series of definitions.

Definition A.2. The function f is said to be *locally bounded on* $\Omega \times \mathbb{T} \backslash \{\sup \mathbb{T}\}$ if, for every $K \in \mathcal{K}$, for all $(a, b) \in \mathbb{T}^2$ such that $a < b$, there exists $M \geq 0$ such that

$$\|f(x, t)\| \leq M, \tag{H_∞}$$

for every $x \in K$ and for Δ-a.e. $t \in [a, b[_{\mathbb{T}}$. In what follows this property will be referred to as (H_∞).

Definition A.3. The function f is said to be *locally Lipschitz continuous with respect to the first variable at right-dense points* if, for every $\overline{x} \in \Omega$ and every right-dense point $\overline{t} \in \mathbb{T} \backslash \{\sup \mathbb{T}\}$, there exist $R > 0$, $\delta > 0$ and $L \geq 0$ such that $\overline{B}(\overline{x}, R) \subset \Omega$ and $\overline{t} + \delta \in \mathbb{T}$, and such that

$$\|f(x_1, t) - f(x_2, t)\| \leq L\|x_1 - x_2\|, \tag{$\mathrm{H}^{\mathrm{rd}}_{\mathrm{loc-Lip}}$}$$

for all $x_1, x_2 \in \overline{B}(\overline{x}, R)$ and for Δ-a.e. $t \in [\overline{t}, \overline{t} + \delta[_{\mathbb{T}}$. In what follows this property will be referred to as $(\mathrm{H}^{\mathrm{rd}}_{\mathrm{loc-Lip}})$.

Definition A.4. The function f is said to be *forward Ω-stable at right-scattered points* if the mapping

$$\begin{array}{rccc} G^+(t): & \Omega & \longrightarrow & \mathbb{R}^n \\ & x & \longmapsto & x + \mu(t) f(x, t) \end{array} \tag{$\mathrm{H}^{\mathrm{forw}}_{\mathrm{stab}}$}$$

takes its values in Ω, for every $t \in \mathrm{RS}$. In what follows this property will be referred to as $(\mathrm{H}^{\mathrm{forw}}_{\mathrm{stab}})$.

Definition A.5. The function f is said to be *locally Lipschitz continuous with respect to the first variable at left-dense points* if, for every $\overline{x} \in \Omega$ and every left-dense point $\overline{t} \in \mathbb{T} \backslash \{\inf \mathbb{T}\}$, there exist $R > 0$, $\delta > 0$ and $L \geq 0$ such that $\overline{B}(\overline{x}, R) \subset \Omega$ and $\overline{t} - \delta \in \mathbb{T}$ and such that

$$\|f(x_1, t) - f(x_2, t)\| \leq L\|x_1 - x_2\|, \tag{$\mathrm{H}^{\mathrm{ld}}_{\mathrm{loc-Lip}}$}$$

for all $x_1, x_2 \in \overline{B}(\overline{x}, R)$ and for Δ-a.e. $t \in [\overline{t} - \delta, \overline{t}[_{\mathbb{T}}$. In what follows this property will be referred to as $(\mathrm{H}^{\mathrm{ld}}_{\mathrm{loc-Lip}})$.

Definition A.6. The function f is said to be *backward regressive at right-scattered points* if

$$G^+(t) \text{ is invertible}, \tag{$\mathrm{H}^{\mathrm{back}}_{\mathrm{regr}}$}$$

for every $t \in \mathrm{RS}$. In what follows this property will be referred to as $(\mathrm{H}^{\mathrm{back}}_{\mathrm{regr}})$.

Assumption (H_∞) is instrumental to provide a Δ-integral characterization of the solutions of (Δ-CP) (see Lemma A.1). The other assumptions play a role in order to *go forward* or *backward* for a solution of a non shifted Δ-Cauchy problem. More precisely, $(\mathrm{H}^{\mathrm{rd}}_{\mathrm{loc-Lip}})$ and $(\mathrm{H}^{\mathrm{forw}}_{\mathrm{stab}})$ allow to go forward, and $(\mathrm{H}^{\mathrm{ld}}_{\mathrm{loc-Lip}})$ and $(\mathrm{H}^{\mathrm{back}}_{\mathrm{regr}})$ allow to go backward (see the proofs of Propositions A.1 and A.2 for more details).

In view of investigating global solutions, the following definition is also useful.

Definition A.7. The function f is said to be *globally Lipschitz continuous in its first variable* if for all $(a,b) \in \mathbb{T}^2$ such that $a < b$, there exists $L \geq 0$ such that

$$\|f(x_1,t) - f(x_2,t)\| \leq L\|x_1 - x_2\|. \tag{$\mathrm{H}^{\mathrm{glob}}_{\mathrm{Lip}}$}$$

for all $x_1, x_2 \in \Omega$ and for Δ-a.e. $t \in [a,b[_{\mathbb{T}}$. In what follows this property will be referred to as $(\mathrm{H}^{\mathrm{glob}}_{\mathrm{Lip}})$.

A.2.2 Definition of a maximal solution

We first define the notion of a solution of (Δ-CP) on an interval $[a,b]_{\mathbb{T}}$ with $a \trianglelefteq t_0 \trianglelefteq b$.

Definition A.8. Let $(a,b) \in \mathbb{T}^2$ be such that $a \trianglelefteq t_0 \trianglelefteq b$ and let $q : [a,b]_{\mathbb{T}} \longrightarrow \Omega$. The couple $(q,[a,b]_{\mathbb{T}})$ is said to be a solution of (Δ-CP) if $q \in \mathrm{AC}([a,b]_{\mathbb{T}})$, if $q(t_0) = q_0$, and if $q^\Delta(t) = f(q(t),t)$ for Δ-a.e. $t \in [a,b[_{\mathbb{T}}$.

Note that, if $(q,[a,b]_{\mathbb{T}})$ is a solution of (Δ-CP), then $(q,[a',b']_{\mathbb{T}})$ is as well a solution of (Δ-CP) for all $a',b' \in [a,b]_{\mathbb{T}}$ satisfying $a' \trianglelefteq t_0 \trianglelefteq b'$.

In view of defining the notion of a solution of (Δ-CP) on more general intervals, we set

$$\mathbb{I} = \{ I_{\mathbb{T}} \mid \exists a,b \in I_{\mathbb{T}},\ a \trianglelefteq t_0 \trianglelefteq b \}. \tag{A.3}$$

The set $\mathbb{I}$ is the set of potential intervals of $\mathbb{T}$ for a solution of (Δ-CP).

Definition A.9. Let $I_{\mathbb{T}} \in \mathbb{I}$ and let $q : I_{\mathbb{T}} \longrightarrow \Omega$. The couple $(q, I_{\mathbb{T}})$ is said to be a solution of (Δ-CP) if $(q,[a,b]_{\mathbb{T}})$ is a solution of (Δ-CP) for all $a,b \in I_{\mathbb{T}}$ satisfying $a \trianglelefteq t_0 \trianglelefteq b$.

Finally, we define the concept of a maximal solution.

Definition A.10. Let $(q, I_{\mathbb{T}})$ and $(q_1, I^1_{\mathbb{T}})$ be two solutions of (Δ-CP). The solution $(q_1, I^1_{\mathbb{T}})$ is said to be an extension of the solution $(q, I_{\mathbb{T}})$ if $I_{\mathbb{T}} \subset I^1_{\mathbb{T}}$ and $q_1 = q$ on $I_{\mathbb{T}}$. A solution $(q, I_{\mathbb{T}})$ of (Δ-CP) is said to be maximal if, for every extension $(q_1, I^1_{\mathbb{T}})$ of $(q, I_{\mathbb{T}})$, there holds $I^1_{\mathbb{T}} = I_{\mathbb{T}}$. A solution $(q, I_{\mathbb{T}})$ of (Δ-CP) is said to be global if $I_{\mathbb{T}} = \mathbb{T}$.

Note that, if $(q, I_{\mathbb{T}})$ is a global solution of (Δ-CP), then $(q, I_{\mathbb{T}})$ is a maximal solution of (Δ-CP).

A.2.3 Main results

Recall that we consider the general non shifted Δ-Cauchy problem

$$(\Delta\text{-CP}) \qquad \begin{aligned} q^\Delta(t) &= f(q(t),t), \\ q(t_0) &= q_0, \end{aligned}$$

where $t_0 \in \mathbb{T}$, $q_0 \in \Omega$, where Ω is a non empty open subset of $\mathbb{R}^n$, and $f : \Omega \times \mathbb{T} \backslash \{\sup \mathbb{T}\} \longrightarrow \mathbb{R}^n$, $(x,t) \longmapsto f(x,t)$ is a Δ-Carathéodory function. We have the following general Cauchy-Lipschitz result.

Theorem A.1. *We make the following assumptions on the dynamics f, depending on t_0.*

1. *If $t_0 = \min \mathbb{T}$, then we assume that*
   - *f satisfies (H_∞), that is, f is locally bounded on $\Omega \times \mathbb{T}\backslash\{\sup \mathbb{T}\}$;*
   - *f satisfies $(\mathrm{H}^{\mathrm{rd}}_{\mathrm{loc-Lip}})$, that is, f is locally Lipschitz continuous with respect to the first variable at right-dense points;*
   - *f satisfies $(\mathrm{H}^{\mathrm{forw}}_{\mathrm{stab}})$, that is, f is forward Ω-stable at right-scattered points.*
2. *If $t_0 = \max \mathbb{T}$, then we assume that*
   - *f satisfies (H_∞), that is, f is locally bounded on $\Omega \times \mathbb{T}\backslash\{\sup \mathbb{T}\}$;*
   - *f satisfies $(\mathrm{H}^{\mathrm{ld}}_{\mathrm{loc-Lip}})$, that is, f is locally Lipschitz continuous with respect to the first variable at left-dense points;*
   - *f satisfies $(\mathrm{H}^{\mathrm{back}}_{\mathrm{regr}})$, that is, f is backward regressive in right-scattered points.*
3. *If $t_0 \neq \inf \mathbb{T}$ and $t_0 \neq \sup \mathbb{T}$, then we assume that*

- *f satisfies* (H_∞)*, that is, f is locally bounded on* $\Omega \times \mathbb{T}\backslash\{\sup \mathbb{T}\}$*;*
- *f satisfies* $(\mathrm{H}^{\mathrm{rd}}_{\mathrm{loc-Lip}})$*, that is, f is locally Lipschitz continuous with respect to the first variable at right-dense points;*
- *f satisfies* $(\mathrm{H}^{\mathrm{forw}}_{\mathrm{stab}})$*, that is, f is forward Ω-stable at right-scattered points;*
- *f satisfies* $(\mathrm{H}^{\mathrm{ld}}_{\mathrm{loc-Lip}})$*, that is, f is locally Lipschitz continuous with respect to the first variable at left-dense points;*
- *f satisfies* $(\mathrm{H}^{\mathrm{back}}_{\mathrm{regr}})$*, that is, f is and backward regressive in right-scattered points.*

Then, the non shifted Δ-Cauchy problem (Δ-CP) *has a unique maximal solution $(q, I_\mathbb{T})$. Moreover, $(q, I_\mathbb{T})$ is the maximal extension of any other solution of* (Δ-CP)*.*

This theorem is proved in Section A.4.1. The following result gives information on the behavior of a maximal solution at its terminal points.

Theorem A.2. *Under the assumptions of Theorem A.1, let $(q, I_\mathbb{T})$ be the maximal solution of the non shifted Δ-Cauchy problem* (Δ-CP)*. Then either $I_\mathbb{T} = \mathbb{T}$, that is, the maximal solution $(q, I_\mathbb{T})$ is global, or the maximal solution is not global and then*

1. *if $t_0 = \min \mathbb{T}$ then $I_\mathbb{T} = [t_0, b[_\mathbb{T}$ where $b \in]t_0, +\infty[_\mathbb{T}$ is a left-dense point of $\mathbb{T}$;*
2. *if $t_0 = \max \mathbb{T}$ then $I_\mathbb{T} =]a, t_0]_\mathbb{T}$ where $a \in]-\infty, t_0[_\mathbb{T}$ is a right-dense point of $\mathbb{T}$;*
3. *if $t_0 \neq \inf \mathbb{T}$ and $t_0 \neq \sup \mathbb{T}$ then $I_\mathbb{T} =]a, +\infty[_\mathbb{T}$ or $I_\mathbb{T} =]-\infty, b[_\mathbb{T}$ or $I_\mathbb{T} =]a, b[_\mathbb{T}$, where $a \in]-\infty, t_0[_\mathbb{T}$ is a right-dense point of $\mathbb{T}$ and $b \in]t_0, +\infty[_\mathbb{T}$ is a left-dense point of $\mathbb{T}$;*

and moreover, for every $K \in \mathcal{K}$ there exists $t \in I_\mathbb{T}$ (close to a or b depending on the cases listed above) such that $q(t) \in \Omega \setminus K$.

This theorem is proved in Section A.4.2. It states that the maximal solution must go out of any compact of Ω near its terminal points whenever it is not global.

The following last result states that, under global Lipschitz assumption, the maximal solution is global.

Theorem A.3. *If $t_0 = \min \mathbb{T}$, $\Omega = \mathbb{R}^n$, if f satisfies* (H_∞)*, that is, f is locally bounded on $\mathbb{R}^n \times \mathbb{T}\backslash\{\sup \mathbb{T}\}$, and if f satisfies* $(\mathrm{H}^{\mathrm{glob}}_{\mathrm{Lip}})$*, that is, f is globally Lipschitz continuous in its first variable, then the non shifted Δ-Cauchy problem* (Δ-CP) *has a unique maximal solution $(q, I_\mathbb{T})$, which is moreover global.*

The proof is done in Section A.4.3.

Remark A.1. As an application of Theorem A.3, we recover the well known fact that, in the linear case

$$q^\Delta(t) = h(t) \times q(t),$$

where $h : \mathbb{T}\backslash\{\sup \mathbb{T}\} \longrightarrow \mathbb{R}^{n\times n}$ such that $h \in \mathrm{L}^\infty_\mathbb{T}([a, b[_\mathbb{T}, \mathbb{R}^{n\times n})$ for all $(a, b) \in \mathbb{T}^2$ with $a < b$, solutions are global.

A.2.4 Further comments

In this section, we provide simple examples (in the one-dimensional case $n = 1$) showing the sharpness of the assumptions made in Theorem A.1. Indeed, if one of these assumptions is not satisfied, then the existence or the uniqueness of the maximal solution is no more guaranteed.

Example A.1 (Lack of Assumption $(\mathrm{H}^{\mathrm{rd}}_{\mathrm{loc-Lip}})$ in the first case)**.** Let $\mathbb{T} = [0, +\infty[$, $\Omega = \mathbb{R}$, $t_0 = 0$, $q_0 = 0$ and $f : \mathbb{R} \times \mathbb{T} \longrightarrow \mathbb{R}$ be defined by $f(x, t) = 2\sqrt{|x|}$. The function f obviously satisfies $(\mathrm{H}^{\mathrm{forw}}_{\mathrm{stab}})$ since $\mathrm{RS} = \emptyset$, however it does not satisfy $(\mathrm{H}^{\mathrm{rd}}_{\mathrm{loc-Lip}})$. The corresponding Δ-Cauchy problem (Δ-CP) has two global solutions q_1 and q_2 given by $q_1(t) = 0$ and $q_2(t) = t^2$, for every $t \in \mathbb{T}$.

This example shows that, in the absence of Assumption $(\mathrm{H}^{\mathrm{rd}}_{\mathrm{loc-Lip}})$, the uniqueness of the maximal solution is not guaranteed.

Example A.2 (Lack of Assumption ($\mathrm{H}^{\mathrm{forw}}_{\mathrm{stab}}$) in the first case)**.** Let $\mathbb{T} = \{0,1\}$, $\Omega =]-1,1[$, $t_0 = 0$, $q_0 = 0$ and $f : \Omega \times \{0\} \longrightarrow \mathbb{R}$ be defined by $f(x,t) = 1$. The function f obviously satisfies ($\mathrm{H}^{\mathrm{rd}}_{\mathrm{loc-Lip}}$) since $\mathbb{T}\backslash\{\sup \mathbb{T}\} = \{0\}$ does not admit any right-dense point of $\mathbb{T}$, however it does not satisfy ($\mathrm{H}^{\mathrm{forw}}_{\mathrm{stab}}$) since $x+1 \notin \Omega$ for $x \in [0,1[$. Since $q(0) = 0$ and $q(1) = q(0) + \mu(0)f(q(0),0)$ imply $q(1) = 1 \notin \Omega$, we conclude that (Δ-CP) does not admit any solution.

Therefore, in the absence of Assumption ($\mathrm{H}^{\mathrm{forw}}_{\mathrm{stab}}$), (Δ-CP) may fail to have a solution.

Example A.3 (Lack of Assumption ($\mathrm{H}^{\mathrm{ld}}_{\mathrm{loc-Lip}}$) in the second case)**.** Let $\mathbb{T} =]-\infty,0]$, $\Omega = \mathbb{R}$, $t_0 = 0$, $q_0 = 0$ and $f : \mathbb{R} \times \mathbb{T} \longrightarrow \mathbb{R}$ be defined by $f(x,t) = -2\sqrt{|x|}$. The function f obviously satisfies ($\mathrm{H}^{\mathrm{back}}_{\mathrm{regr}}$) since $\mathrm{RS} = \emptyset$, however it does not satisfy ($\mathrm{H}^{\mathrm{ld}}_{\mathrm{loc-Lip}}$). The corresponding Δ-Cauchy problem (Δ-CP) ha two global solutions q_1 and q_2 given by $q_1(t) = 0$ and $q_2(t) = t^2$ for every $t \in \mathbb{T}$.

This example shows that, in the absence of Assumption ($\mathrm{H}^{\mathrm{ld}}_{\mathrm{loc-Lip}}$), the uniqueness of the maximal solution is not guaranteed.

Example A.4 (Lack of Assumption ($\mathrm{H}^{\mathrm{back}}_{\mathrm{regr}}$) in the second case)**.** Let $\mathbb{T} = \{0,1\}$, $\Omega = \mathbb{R}$, $t_0 = 1$, $q_0 \in \mathbb{R}$ and $f : \mathbb{R}\times\{0\} \longrightarrow \mathbb{R}$ be defined by $f(x,t) = -x$. The function f obviously satisfies ($\mathrm{H}^{\mathrm{ld}}_{\mathrm{loc-Lip}}$) since $\mathbb{T}\backslash\{\inf \mathbb{T}\} = \{1\}$ does not admit any left-dense point of $\mathbb{T}$, however it does not satisfy ($\mathrm{H}^{\mathrm{back}}_{\mathrm{regr}}$) since $G^+(0) = 0$. As a consequence, if $q_0 \neq 0$, (Δ-CP) does not admit any solution. Indeed, $q(1) = q_0$ and $q(1) = q(0)+\mu(0)f(q(0),0)$ imply $q(1) = 0$, which is a contradiction. If $q_0 = 0$, we obtain an infinite number of global solutions. Indeed, any function q defined on $\mathbb{T}$ with $q(1) = 0$ is then a global solution of (Δ-CP).

A.3 General shifted Δ-Cauchy problem

Throughout this section we consider the general *shifted* Δ-Cauchy problem

$$(\Delta\text{-CP}^\sigma) \qquad \begin{aligned} q^\Delta(t) &= f(q^\sigma(t),t), \\ q(t_0) &= q_0, \end{aligned}$$

where $t_0 \in \mathbb{T}$, $q_0 \in \Omega$, where Ω is a non empty open subset of $\mathbb{R}^n$ and $f : \Omega \times \mathbb{T}\backslash\{\sup \mathbb{T}\} \longrightarrow \mathbb{R}^n, (x,t) \longmapsto f(x,t)$ is a Δ-Carathéodory function.

The results of the section follow the same lines as in the previous section. Therefore we do not give any proof nor counterexamples as above. Some comments are however done in Section A.4.4.

A.3.1 Preliminaries and definition of a maximal solution

As in Section A.2, it will be important to distinguish between the three following cases: $t_0 = \min \mathbb{T}$; $t_0 = \max \mathbb{T}$; $t_0 \neq \inf \mathbb{T}$ and $t_0 \neq \sup \mathbb{T}$. With respect to Section A.2.1, we introduce two additional concepts.

Definition A.11. The function f is said to be *backward Ω-stable at right-scattered points* if the mapping

$$\begin{array}{rccl} G^-(t): & \Omega & \longrightarrow & \mathbb{R}^n \\ & x & \longmapsto & x - \mu(t)f(x,t) \end{array} \qquad (\mathrm{H}^{\mathrm{back}}_{\mathrm{stab}})$$

takes its values in Ω, for every $t \in \mathrm{RS}$. In what follows this property will be referred to as ($\mathrm{H}^{\mathrm{back}}_{\mathrm{stab}}$).

Definition A.12. The function f is said to be *forward regressive at right-scattered points* if

$$G^-(t) : \Omega \longrightarrow \mathbb{R}^n \text{ is invertible,} \qquad (\mathrm{H}^{\mathrm{forw}}_{\mathrm{regr}})$$

for every $t \in \mathrm{RS}$. In what follows this property will be referred to as ($\mathrm{H}^{\mathrm{forw}}_{\mathrm{regr}}$).

These above assumptions play a role in order to *go forward* or *backward* for a solution of a shifted Δ-Cauchy problem. Precisely, ($\mathrm{H}^{\mathrm{rd}}_{\mathrm{loc-Lip}}$) and ($\mathrm{H}^{\mathrm{forw}}_{\mathrm{regr}}$) allow to *go forward.* Similarly, ($\mathrm{H}^{\mathrm{ld}}_{\mathrm{loc-Lip}}$) and ($\mathrm{H}^{\mathrm{back}}_{\mathrm{stab}}$) allow to *go backward.*

Definition A.13. Let $(a,b) \in \mathbb{T}^2$ satisfying $a \trianglelefteq t_0 \trianglelefteq b$ and let $q : [a,b]_\mathbb{T} \longrightarrow \Omega$. The couple $(q,[a,b]_\mathbb{T})$ is said to be a solution of (Δ-CP$^\sigma$) if $q \in \mathrm{AC}([a,b]_\mathbb{T})$, $q(t_0) = q_0$, and $q^\Delta(t) = f(q^\sigma(t),t)$ for Δ-a.e. $t \in [a,b[_\mathbb{T}$.

Definition A.14. Let $I_\mathbb{T} \in \mathbb{I}$ and let $q : I_\mathbb{T} \longrightarrow \Omega$. The couple $(q, I_\mathbb{T})$ is said to be a solution of (Δ-CP$^\sigma$) if $(q,[a,b]_\mathbb{T})$ is a solution of (Δ-CP$^\sigma$) for all $a,b \in I_\mathbb{T}$ satisfying $a \trianglelefteq t_0 \trianglelefteq b$.

Definition A.15. Let $(q, I_\mathbb{T})$ and $(q_1, I^1_\mathbb{T})$ be two solutions of (Δ-CP$^\sigma$). The solution $(q_1, I^1_\mathbb{T})$ is said to be an *extension* of the solution $(q, I_\mathbb{T})$ if $I_\mathbb{T} \subset I^1_\mathbb{T}$ and $q_1 = q$ on $I_\mathbb{T}$. A solution $(q, I_\mathbb{T})$ of (Δ-CP$^\sigma$) is said to be *maximal* if, for every extension $(q_1, I^1_\mathbb{T})$ of $(q, I_\mathbb{T})$, there holds $I^1_\mathbb{T} = I_\mathbb{T}$. A solution $(q, I_\mathbb{T})$ of (Δ-CP$^\sigma$) is said to be *global* if $I_\mathbb{T} = \mathbb{T}$.

A.3.2 Main results

Recall that we consider the general shifted Δ-Cauchy problem

$$(\Delta\text{-CP}^\sigma) \qquad \begin{aligned} q^\Delta(t) &= f(q^\sigma(t),t), \\ q(t_0) &= q_0, \end{aligned}$$

where $t_0 \in \mathbb{T}$, $q_0 \in \Omega$ where Ω is a non empty open subset of $\mathbb{R}^n$ and $f : \Omega \times \mathbb{T}\backslash\{\sup \mathbb{T}\} \longrightarrow \mathbb{R}^n, (x,t) \longmapsto f(x,t)$ is a Δ-Carathéodory function.

Theorem A.4. *We make the following assumptions on the dynamics f, depending on t_0.*

1. *If $t_0 = \min \mathbb{T}$, then we assume that*
   - *f satisfies (H_∞), that is, f is locally bounded on $\Omega \times \mathbb{T}\backslash\{\sup \mathbb{T}\}$;*
   - *f satisfies $(\mathrm{H}^{\mathrm{rd}}_{\mathrm{loc-Lip}})$, that is, f is locally Lipschitz continuous with respect to the first variable at right-dense points;*
   - *f satisfies $(\mathrm{H}^{\mathrm{forw}}_{\mathrm{regr}})$, that is, f is forward regressive in right-scattered points.*
2. *If $t_0 = \max \mathbb{T}$, then we assume that*
   - *f satisfies (H_∞), that is, f is locally bounded on $\Omega \times \mathbb{T}\backslash\{\sup \mathbb{T}\}$;*
   - *f satisfies $(\mathrm{H}^{\mathrm{ld}}_{\mathrm{loc-Lip}})$, that is, f is locally Lipschitz continuous with respect to the first variable at left-dense points;*
   - *f satisfies $(\mathrm{H}^{\mathrm{back}}_{\mathrm{stab}})$, that is, f is backward Ω-stable in right-scattered points.*
3. *If $t_0 \neq \inf \mathbb{T}$ and $t_0 \neq \sup \mathbb{T}$, then we assume that*
   - *f satisfies (H_∞), that is, f is locally bounded on $\Omega \times \mathbb{T}\backslash\{\sup \mathbb{T}\}$;*
   - *f satisfies $(\mathrm{H}^{\mathrm{rd}}_{\mathrm{loc-Lip}})$, that is, f is locally Lipschitz continuous with respect to the first variable at right-dense points;*
   - *f satisfies $(\mathrm{H}^{\mathrm{forw}}_{\mathrm{regr}})$, that is, f is forward regressive at right-scattered points;*
   - *f satisfies $(\mathrm{H}^{\mathrm{ld}}_{\mathrm{loc-Lip}})$, that is, f is locally Lipschitz continuous with respect to the first variable at left-dense points;*
   - *f satisfies $(\mathrm{H}^{\mathrm{back}}_{\mathrm{stab}})$, that is, f is backward Ω-stable at right-scattered points.*

Then the shifted Δ-Cauchy problem (Δ-CP$^\sigma$) has a unique maximal solution $(q, I_\mathbb{T})$. Moreover, $(q, I_\mathbb{T})$ is the maximal extension of any other solution of (Δ-CP$^\sigma$)

Theorem A.5. *Under the assumptions of Theorem A.4, let $(q, I_\mathbb{T})$ be the maximal solution of the shifted Δ-Cauchy problem (Δ-CP$^\sigma$). Then either $I_\mathbb{T} = \mathbb{T}$, that is, the maximal solution $(q, I_\mathbb{T})$ is global, or the maximal solution is not global and then*

1. *if $t_0 = \min \mathbb{T}$ then $I_\mathbb{T} = [t_0, b[_\mathbb{T}$ where $b \in]t_0, +\infty[_\mathbb{T}$ is a left-dense point of $\mathbb{T}$;*
2. *if $t_0 = \max \mathbb{T}$ then $I_\mathbb{T} =]a, t_0]_\mathbb{T}$ where $a \in]-\infty, t_0[_\mathbb{T}$ is a right-dense point of $\mathbb{T}$;*
3. *if $t_0 \neq \inf \mathbb{T}$ and $t_0 \neq \sup \mathbb{T}$ then $I_\mathbb{T} =]a, +\infty[_\mathbb{T}$ or $I_\mathbb{T} =]-\infty, b[_\mathbb{T}$ or $I_\mathbb{T} =]a, b[_\mathbb{T}$ where $a \in]-\infty, t_0[_\mathbb{T}$ is a right-dense point of $\mathbb{T}$ and $b \in]t_0, +\infty[_\mathbb{T}$ is a left-dense point of $\mathbb{T}$;*

and moreover, for every $K \in \mathcal{K}$ there exists $t \in I_\mathbb{T}$ (close to a or b depending on the cases listed above) such that $q(t) \in \Omega \setminus K$.

Theorem A.6. *If* $t_0 = \max \mathbb{T}$, $\Omega = \mathbb{R}^n$, *if* f *satisfies* (H_∞), *that is,* f *is locally bounded on* $\mathbb{R}^n \times \mathbb{T}\backslash\{\sup \mathbb{T}\}$, *and if* f *satisfies* $(\mathrm{H}^{\mathrm{glob}}_{\mathrm{Lip}})$, *that is,* f *is globally Lipschitz continuous in its first variable, then, the shifted* Δ*-Cauchy problem* $(\Delta\text{-CP}^\sigma)$ *has a unique maximal solution* $(q, I_\mathbb{T})$, *which is moreover global.*

Remark A.2. As in Remark A.1, in the linear case the maximal solution of any shifted Δ-Cauchy problem is automatically global.

A.4 Proofs of the results

In this section, we detail the proof of Theorem A.1, A.2 and A.3 and we give some additional comments on the shifted case.

A.4.1 Proof of Theorem A.1

If f satisfies (H_∞), then for all $(a, b) \in \mathbb{T}^2$ such that $a < b$, there holds

$$f(q,t) \in \mathrm{L}^\infty_\mathbb{T}([a,b[_\mathbb{T}, \mathbb{R}^n) \subset \mathrm{L}^1_\mathbb{T}([a,b[_\mathbb{T}, \mathbb{R}^n), \tag{A.4}$$

for every $q \in \mathrm{C}([a,b]_\mathbb{T}, \mathbb{R}^n)$. Then, from Section I.3.2, we have the following Δ-integral characterization of the solutions of (Δ-CP).

Lemma A.1. *Let* $I_\mathbb{T} \in \mathbb{I}$ *and let* $q : I_\mathbb{T} \longrightarrow \Omega$. *If* f *satisfies* (H_∞), *then the couple* $(q, I_\mathbb{T})$ *is a solution of* (Δ-CP) *if and only if for all* $a, b \in I_\mathbb{T}$ *satisfying* $a \trianglelefteq t_0 \trianglelefteq b$, *one has* $q \in \mathrm{C}([a,b]_\mathbb{T}, \mathbb{R}^n)$ *and*

$$q(t) = \begin{cases} q_0 + \int_{[t_0,t[_\mathbb{T}} f(q(\tau),\tau)\,\Delta\tau & \text{if} \quad t \geq t_0, \\ q_0 - \int_{[t,t_0[_\mathbb{T}} f(q(\tau),\tau)\,\Delta\tau & \text{if} \quad t \leq t_0, \end{cases}$$

for every $t \in [a,b]_\mathbb{T}$.

This characterization allows one to prove the following result.

Lemma A.2. *If* f *satisfies* (H_∞), *then every solution of* (Δ-CP) *can be extended to a maximal solution.*

Proof. Let $(q, I_\mathbb{T})$ be a solution of (Δ-CP). Let us define the non empty set $\mathscr{F}$ of extensions of $(q, I_\mathbb{T})$. The set $\mathscr{F}$ is ordered by

$$(q_1, I^1_\mathbb{T}) \leq (q_2, I^2_\mathbb{T}) \text{ if and only if } (q_2, I^2_\mathbb{T}) \text{ is an extension of } (q_1, I^1_\mathbb{T}).$$

Let us prove that $\mathscr{F}$ is inductive. Let $\mathscr{G} = \{(q_p, I^p_\mathbb{T})\}_{p\in\mathscr{P}}$ be a non empty totally ordered subset of $\mathscr{F}$. Let us prove that $\mathscr{G}$ admits an upper bound.

Let us define $\overline{I} = \cup_{p\in\mathscr{P}} I^p$. This is an interval of $\mathbb{R}$, since $t_0 \in \cap_{p\in\mathscr{P}} I^p$. Then $\overline{I}_\mathbb{T} = \cup_{p\in\mathscr{P}} I^p_\mathbb{T} \in \mathbb{I}$. For every $t \in \overline{I}_\mathbb{T}$, there exists $p \in \mathscr{P}$ such that $t \in I^p_\mathbb{T}$ and, since $\mathscr{G}$ is totally ordered, if $t \in I^{p_1}_\mathbb{T} \cap I^{p_2}_\mathbb{T}$ then $q_{p_1}(t) = q_{p_2}(t)$. Consequently, we can define $\overline{q}$ by

$$\forall t \in \overline{I}_\mathbb{T},\ \overline{q}(t) = q_p(t) \in \Omega \text{ where } t \in I^p_\mathbb{T}. \tag{A.5}$$

Our aim is to prove that $(\overline{q}, \overline{I}_\mathbb{T})$ is a solution of (Δ-CP). Let $a, b \in \overline{I}_\mathbb{T}$ satisfying $a \trianglelefteq t_0 \trianglelefteq b$. Since $\mathscr{G}$ is totally ordered, there exists $p \in \mathscr{P}$ such that $[a,b]_\mathbb{T} \subset I^p_\mathbb{T}$ and $\overline{q} = q_p$ on $[a,b]_\mathbb{T}$. Since $(q_p, I^p_\mathbb{T})$ is a solution of (Δ-CP), we obtain that q_p satisfies the necessary and sufficient condition of Lemma A.1 on $[a,b]_\mathbb{T}$. Consequently, this holds true as well for $\overline{q}$ on $[a,b]_\mathbb{T}$. Finally, since this last sentence is true for all $a, b \in \overline{I}_\mathbb{T}$ satisfying $a \trianglelefteq t_0 \trianglelefteq b$, we infer from Lemma A.1 that $(\overline{q}, \overline{I}_\mathbb{T})$ is a solution of (Δ-CP). Since $(\overline{q}, \overline{I}_\mathbb{T})$ is obviously an extension of any element of $\mathscr{G}$, we obtain that $\mathscr{G}$ admits an upper bound and then, $\mathscr{F}$ is inductive.

Finally, $\mathscr{F}$ is a non empty ordered inductive set and consequently, from the classical Zorn lemma, admits a maximal element. The proof is complete. □

Proposition A.1 (Existence of a local solution)**.** *There exist* $a, b \in \mathbb{T}$ *satisfying* $a \trianglelefteq t_0 \trianglelefteq b$ *and* $q : [a,b]_\mathbb{T} \longrightarrow \Omega$ *such that* $(q, [a,b]_\mathbb{T})$ *is a solution of* (Δ-CP).

Proof. We only prove this proposition in the third case of Theorem A.1 (the two first cases are derived similarly) for which $t_0 \neq \inf \mathbb{T}$ and $t_0 \neq \sup \mathbb{T}$. We distinguish between four situations.

First case: t_0 is a left- and a right-scattered point of $\mathbb{T}$. In this case, it is sufficient to consider $a = \rho(t_0) \in]-\infty, t_0[_{\mathbb{T}}$, $b = \sigma(t_0) \in]t_0, +\infty[_{\mathbb{T}}$ and the function q defined on $[a,b]_{\mathbb{T}} = \{a, t_0, b\}$ with values in Ω by $q(a) = G^+(a)^{-1}(q_0)$, $q(t_0) = q_0$ and $q(b) = G^+(t_0)(q_0)$. We note that $q(a)$ is well-defined in Ω from $(\mathrm{H}^{\mathrm{back}}_{\mathrm{regr}})$ and $q(b) \in \Omega$ from $(\mathrm{H}^{\mathrm{forw}}_{\mathrm{stab}})$.

Second case: t_0 is a left- and a right-dense point of $\mathbb{T}$. Let R', δ' and L' associated with q_0 and t_0 in $(\mathrm{H}^{\mathrm{ld}}_{\mathrm{loc-Lip}})$ and let R'', δ'' and L'' associated with q_0 and t_0 in $(\mathrm{H}^{\mathrm{rd}}_{\mathrm{loc-Lip}})$. We define $R = \min(R', R'') > 0$ and $L = \max(L', L'') \geq 0$. Let M associated with $\overline{B}(q_0, R) \in \mathcal{K}$ and $[t_0 - \delta', t_0 + \delta''[_{\mathbb{T}}$ in (H_∞). Consider $0 < \delta_1 \leq \delta'$ and $0 < \delta_2 \leq \delta''$ such that $a = t_0 - \delta_1 \in]-\infty, t_0[_{\mathbb{T}}$, $b = t_0 + \delta_2 \in]t_0, +\infty[_{\mathbb{T}}$ and δ_1 and δ_2 are sufficiently small in order to have $\max(\delta_1, \delta_2) M \leq R$ and $\max(\delta_1, \delta_2) L < 1$. Then, we can construct the $\max(\delta_1, \delta_2)L$-contraction map with respect to the norm $\| \cdot \|_\infty$

$$\begin{array}{rccc} F: & \mathrm{C}([a,b]_{\mathbb{T}}, \overline{B}(q_0,R)) & \longrightarrow & \mathrm{C}([a,b]_{\mathbb{T}}, \overline{B}(q_0,R)) \\ & q & \longmapsto & F(q), \end{array}$$

with

$$\begin{array}{rccc} F(q): & [a,b]_{\mathbb{T}} & \longrightarrow & \overline{B}(q_0,R) \\ & t & \longmapsto & \begin{cases} q_0 + \int_{[t_0,t[_{\mathbb{T}}} f(q(\tau),\tau)\, \Delta\tau & \text{if} \quad t \geq t_0 \\ q_0 - \int_{[t,t_0[_{\mathbb{T}}} f(q(\tau),\tau)\, \Delta\tau & \text{if} \quad t \leq t_0. \end{cases} \end{array}$$

It follows from the Banach fixed point theorem that F has a unique fixed point denoted by q, and then $(q, [a,b]_{\mathbb{T}})$ is a solution of (Δ-CP).

Third case: t_0 is a left-scattered and a right-dense point of $\mathbb{T}$. Let R, δ and L associated with q_0 and t_0 in $(\mathrm{H}^{\mathrm{rd}}_{\mathrm{loc-Lip}})$. Let M associated with $\overline{B}(q_0, R) \in \mathcal{K}$ and $[t_0, t_0 + \delta[_{\mathbb{T}}$ in (H_∞). Consider $0 < \delta_1 \leq \delta$ such that $b = t_0 + \delta_1 \in]t_0, +\infty[_{\mathbb{T}}$ and δ_1 is sufficiently small in order to have $\delta_1 M \leq R$ and $\delta_1 L < 1$. Then, we can construct the $\delta_1 L$-contraction map with respect to the norm $\| \cdot \|_\infty$

$$\begin{array}{rccc} F: & \mathrm{C}([t_0,b]_{\mathbb{T}}, \overline{B}(q_0,R)) & \longrightarrow & \mathrm{C}([t_0,b]_{\mathbb{T}}, \overline{B}(q_0,R)) \\ & q & \longmapsto & F(q) \end{array}$$

with

$$\begin{array}{rccc} F(q): & [t_0,b]_{\mathbb{T}} & \longrightarrow & \overline{B}(q_0,R) \\ & t & \longmapsto & q_0 + \displaystyle\int_{[t_0,t[_{\mathbb{T}}} f(q(\tau),\tau)\, \Delta\tau. \end{array}$$

It follows from the Banach fixed point theorem that F has a unique fixed point denoted by q defined on $[t_0, b]_{\mathbb{T}}$. Finally, since t_0 is a left-scattered point of $\mathbb{T}$ and from $(\mathrm{H}^{\mathrm{back}}_{\mathrm{regr}})$, we define $a = \rho(t_0) \in]-\infty, t_0[_{\mathbb{T}}$ and $q(a) = G^+(a)^{-1}(q_0) \in \Omega$. We have thus obtained a solution $(q, [a,b]_{\mathbb{T}})$ of (Δ-CP).

Fourth case: t_0 is a left-dense and a right-scattered point of $\mathbb{T}$. Let R, δ and L associated with q_0 and t_0 in $(\mathrm{H}^{\mathrm{ld}}_{\mathrm{loc-Lip}})$. Let M associated with $\overline{B}(q_0, R) \in \mathcal{K}$ and $[t_0 - \delta, t_0[_{\mathbb{T}}$ in (H_∞). Consider $0 < \delta_1 \leq \delta$ such that $a = t_0 - \delta_1 \in]-\infty, t_0[_{\mathbb{T}}$ and δ_1 is sufficiently small in order to have $\delta_1 M \leq R$ and $\delta_1 L < 1$. Then, we can construct the $\delta_1 L$-contraction map with respect to the norm $\| \cdot \|_\infty$

$$\begin{array}{rccc} F: & \mathrm{C}([a,t_0]_{\mathbb{T}}, \overline{B}(q_0,R)) & \longrightarrow & \mathrm{C}([a,t_0]_{\mathbb{T}}, \overline{B}(q_0,R)) \\ & q & \longmapsto & F(q) \end{array}$$

with

$$\begin{array}{rccc} F(q): & [a,t_0]_{\mathbb{T}} & \longrightarrow & \overline{B}(q_0,R) \\ & t & \longmapsto & q_0 - \displaystyle\int_{[t,t_0[_{\mathbb{T}}} f(q(\tau),\tau)\, \Delta\tau. \end{array}$$

It follows from the Banach fixed point theorem that F admits a unique fixed point denoted by q defined on $[a, t_0]_{\mathbb{T}}$. Since t_0 is a right-scattered point of $\mathbb{T}$, and from $(\mathrm{H}^{\mathrm{forw}}_{\mathrm{stab}})$, we define $b = \sigma(t_0) \in]t_0, +\infty[_{\mathbb{T}}$ and $q(b) = G^+(t_0)(q_0) \in \Omega$. We have thus obtained a solution $(q, [a,b]_{\mathbb{T}})$ of (Δ-CP). $\square$

From Lemma A.2, we can extend the solution given in Proposition A.1 and we obtain the existence of a maximal solution. The following result proves that it is unique.

Proposition A.2 (Local uniqueness of a solution). *Let* $(q_1, I^1_{\mathbb{T}})$ *and* $(q_2, I^2_{\mathbb{T}})$ *be two solutions of* (Δ-CP). *Then,* $q_1 = q_2$ *on* $I^1_{\mathbb{T}} \cap I^2_{\mathbb{T}}$.

Proof. As before, we only prove this proposition in the third case of Theorem A.1. We denote by $I = I^1 \cap I^2$ (interval of $\mathbb{R}$). One can easily prove that $I_{\mathbb{T}} = I^1_{\mathbb{T}} \cap I^2_{\mathbb{T}} \in \mathbb{I}$. It is sufficient to prove $q_1 = q_2$ on $[a,b]_{\mathbb{T}}$ for all $a, b \in I_{\mathbb{T}}$ satisfying $a \trianglelefteq t_0 \trianglelefteq b$. Let a, $b \in I_{\mathbb{T}}$ satisfying $a \trianglelefteq t_0 \trianglelefteq b$. Set

$$A = \{t \in [a,t_0]_{\mathbb{T}},\ q_1(t) \neq q_2(t)\},$$

and

$$B = \{t \in [t_0,b]_{\mathbb{T}},\ q_1(t) \neq q_2(t)\}.$$

Let us prove by contradiction that $A \cup B = \emptyset$. Assume that $A \neq \emptyset$ and let $\bar{t} = \sup A$. Note that $\bar{t} \in [a,t_0]_{\mathbb{T}}$ (since $\mathbb{T}$ is closed) and that $q_1 = q_2$ on $]\bar{t}, t_0]_{\mathbb{T}}$. In order to raise a contradiction, we first derive the four following facts.

1. *Fact 1:* $\bar{t} < t_0$. If t_0 is a left-scattered point of $\mathbb{T}$, this claim is obvious since $q_1(t_0) = q_2(t_0) = q_0$ and $q_1(\rho(t_0)) = q_2(\rho(t_0)) = G^+(\rho(t_0))^{-1}(q_0)$ from $(\mathrm{H}^{\mathrm{back}}_{\mathrm{regr}})$. If t_0 is a left-dense point of $\mathbb{T}$, let R, δ and L associated with q_0 and t_0 in $(\mathrm{H}^{\mathrm{ld}}_{\mathrm{loc-Lip}})$. Let M associated with $\overline{B}(q_0,R) \in \mathcal{K}$ and $[t_0-\delta, t_0[_{\mathbb{T}}$ in (H_∞). Consider $0 < \delta_1 \leq \delta$ such that $c = t_0 - \delta_1 \in [a, t_0[_{\mathbb{T}}$ and δ_1 is sufficiently small in order to have $\delta_1 M \leq R$, $\delta_1 L < 1$ and q_1, $q_2 \in \mathrm{C}([c,t_0]_{\mathbb{T}}, \overline{B}(q_0,R))$. Since q_1 and q_2 are solutions of (Δ-CP) on $[a,b]_{\mathbb{T}}$, they are in particular fixed points of the $\delta_1 L$-contraction map

$$\begin{array}{rccl} F: & \mathrm{C}([c,t_0]_{\mathbb{T}}, \overline{B}(q_0,R)) & \longrightarrow & \mathrm{C}([c,t_0]_{\mathbb{T}}, \overline{B}(q_0,R)) \\ & q & \longmapsto & F(q) \end{array}$$

   with

$$\begin{array}{rccl} F(q): & [c,t_0]_{\mathbb{T}} & \longrightarrow & \overline{B}(q_0,R) \\ & t & \longmapsto & q_0 - \displaystyle\int_{[t,t_0[_{\mathbb{T}}} f(q(\tau),\tau)\,\Delta\tau. \end{array}$$

   Since F has a unique fixed point from the Banach fixed point theorem, we conclude that $q_1 = q_2$ on $[c,t_0]_{\mathbb{T}}$. Hence $\bar{t} < t_0$.
2. *Fact 2:* $q_1(\bar{t}) = q_2(\bar{t})$. If $\bar{t}$ is a right-scattered point of $\mathbb{T}$, then $\sigma(\bar{t})$ is a left-scattered point of $\mathbb{T}$ and $q_1(\sigma(\bar{t})) = q_2(\sigma(\bar{t}))$. As a consequence, $q_1(\bar{t}) = q_2(\bar{t}) = G^+(\bar{t})^{-1}(q_1(\sigma(\bar{t})))$. If $\bar{t}$ is a right-dense point of $\mathbb{T}$, then $q_1(\bar{t}) = q_2(\bar{t})$ from the continuity of q_1 and q_2 and since $q_1 = q_2$ on $]\bar{t}, t_0]_{\mathbb{T}}$.
3. *Fact 3:* $\bar{t} > a$. Indeed, if $\bar{t} = a$ then $A = \emptyset$ since $q_1(\bar{t}) = q_2(\bar{t})$;
4. *Fact 4:* $\bar{t}$ *is a left-dense point of* $\mathbb{T}$. Indeed, if $\bar{t}$ were to be a left-scattered point of $\mathbb{T}$, since $q_1(\bar{t}) = q_2(\bar{t})$, then $q_1(\rho(\bar{t})) = q_2(\rho(\bar{t})) = G^+(\rho(\bar{t}))^{-1}(q_1(\bar{t}))$ and then it would raise a contradiction with the definition of $\bar{t}$.

Let us denote by $\overline{x} = q_1(\bar{t}) = q_2(\bar{t})$. Let R, δ and L associated with $\bar{t}$ and $\overline{x}$ in $(\mathrm{H}^{\mathrm{ld}}_{\mathrm{loc-Lip}})$. Let M associated with $\overline{B}(\overline{x},R) \in \mathcal{K}$ and $[\bar{t}-\delta, \bar{t}[_{\mathbb{T}}$ in (H_∞). Consider $0 < \delta_1 \leq \delta$ such that $c_0 = \bar{t} - \delta_1 \in [a, \bar{t}[_{\mathbb{T}}$ and δ_1 is sufficiently small in order to have $\delta_1 M \leq R$, $\delta_1 L < 1$ and q_1, $q_2 \in \mathrm{C}([c_0,\bar{t}]_{\mathbb{T}}, \overline{B}(\overline{x},R))$. Since q_1 and q_2 are solutions of (Δ-CP) on $[a,b]_{\mathbb{T}}$, they are in particular fixed points of the $\delta_1 L$-contraction map

$$\begin{array}{rccl} F_0: & \mathrm{C}([c_0,\bar{t}]_{\mathbb{T}}, \overline{B}(\overline{x},R)) & \longrightarrow & \mathrm{C}([c_0,\bar{t}]_{\mathbb{T}}, \overline{B}(\overline{x},R)) \\ & q & \longmapsto & F_0(q) \end{array}$$

with

$$\begin{array}{rccl} F_0(q): & [c_0,\bar{t}]_{\mathbb{T}} & \longrightarrow & \overline{B}(\overline{x},R) \\ & t & \longmapsto & \overline{x} - \displaystyle\int_{[t,\bar{t}[_{\mathbb{T}}} f(q(\tau),\tau)\,\Delta\tau. \end{array}$$

Since F_0 has a unique fixed point from the Banach fixed point theorem, we conclude that $q_1 = q_2$ on $[c_0,\bar{t}]_{\mathbb{T}}$, and this is a contradiction. Consequently $A = \emptyset$.

In the same way, we prove that $B = \emptyset$ and the proof is complete. □

Theorem A.1 follows from Lemma A.2, Propositions A.1 and A.2.

A.4.2 Proof of Theorem A.2

Proposition A.3. *Under the assumptions of Theorem A.1, let* $(q, I_{\mathbb{T}})$ *be the maximal solution of* (Δ-CP)*. Then either* $I_{\mathbb{T}} = \mathbb{T}$*, that is, the solution* $(q, I_{\mathbb{T}})$ *is global, or*

1. *if* $t_0 = \min \mathbb{T}$ *then* $I_{\mathbb{T}} = [t_0, b[_{\mathbb{T}}$ *where* $b \in]t_0, +\infty[_{\mathbb{T}}$ *is a left-dense point of* $\mathbb{T}$*;*
2. *if* $t_0 = \max \mathbb{T}$ *then* $I_{\mathbb{T}} =]a, t_0]_{\mathbb{T}}$ *where* $a \in]-\infty, t_0[_{\mathbb{T}}$ *is a right-dense point of* $\mathbb{T}$*;*
3. *if* $t_0 \neq \inf \mathbb{T}$ *and* $t_0 \neq \sup \mathbb{T}$ *then* $I_{\mathbb{T}} =]a, +\infty[_{\mathbb{T}}$ *or* $I_{\mathbb{T}} =]-\infty, b[_{\mathbb{T}}$ *or* $I_{\mathbb{T}} =]a, b[_{\mathbb{T}}$*, where* $a \in]-\infty, t_0[_{\mathbb{T}}$ *is a right-dense point of* $\mathbb{T}$ *and* $b \in]t_0, +\infty[_{\mathbb{T}}$ *is a left-dense point of* $\mathbb{T}$*.*

Proof. We only prove this proposition in the first case of Theorem A.1 (the other ones are derived similarly).

Let us first prove that if $I_{\mathbb{T}} = [t_0, b]_{\mathbb{T}}$ then $b = \max \mathbb{T}$ (and thus $I_{\mathbb{T}} = \mathbb{T}$). By contradiction, assume that $I_{\mathbb{T}} = [t_0, b]_{\mathbb{T}}$ with $b < \sup \mathbb{T}$. Consider the Δ-Cauchy problem

$$z^{\Delta}(t) = f(z(t), t), \quad z(b) = q(b).$$

As in Proposition A.1, we can prove that it has a solution $(z, [b, b_1]_{\mathbb{T}})$ with $b_1 \in]b, +\infty[_{\mathbb{T}}$. Then, we define q_1 by

$$q_1(t) = \begin{cases} q(t) & \text{if} \quad t \in [t_0, b]_{\mathbb{T}}, \\ z(t) & \text{if} \quad t \in [b, b_1]_{\mathbb{T}}, \end{cases} \tag{A.6}$$

for every $t \in [t_0, b_1]_{\mathbb{T}}$. Then $q_1 \in \mathrm{C}([t_0, b_1]_{\mathbb{T}}, \Omega)$ and one can easily prove that

$$q_1(t) = q_0 + \int_{[t_0, t[_{\mathbb{T}}} f(q_1(\tau), \tau) \, \Delta\tau.$$

for every $t \in [t_0, b_1]_{\mathbb{T}}$. It follows from Lemma A.1 that $(q_1, [t_0, b_1]_{\mathbb{T}})$ is a solution of (Δ-CP) and is a strict extension of $(q, [t_0, b]_{\mathbb{T}})$. It is a contradiction with the maximality of $(q, [t_0, b]_{\mathbb{T}})$.

If $I_{\mathbb{T}} = [t_0, b[_{\mathbb{T}}$ with b a left-scattered point of $\mathbb{T}$, then $I_{\mathbb{T}} = [a, \rho(b)]_{\mathbb{T}}$ with $\rho(b) < \sup \mathbb{T}$ and we recover to the previous contradiction. ☐

Lemma A.3. *Under the assumptions of Theorem A.1, let* $(q, I_{\mathbb{T}})$ *be the maximal solution of* (Δ-CP)*. If* $(q, I_{\mathbb{T}})$ *is not global, then* q *cannot be continuously extended with a value in* Ω *at* $t = a$ *or at* $t = b$ *(see Proposition A.3 for* a *and* b*).*

Proof. We only prove this lemma in the first case of Theorem A.1. By contradiction, let us assume that q can be continuously extended with a value in Ω at $t = b$, that is, $\lim_{t \to b, \, t \in [t_0, b[_{\mathbb{T}}} q(t) = q_b \in \Omega$. Then, we define q_1 by

$$q_1(t) = \begin{cases} q(t) & \text{if} \quad t \in [t_0, b[_{\mathbb{T}} \\ q_b & \text{if} \quad t = b, \end{cases}$$

for every $t \in [t_0, b]_{\mathbb{T}}$. In particular $q_1 \in \mathrm{C}([t_0, b]_{\mathbb{T}}, \Omega)$. Our aim is to prove that $(q_1, [t_0, b]_{\mathbb{T}})$ is a solution of (Δ-CP).

Since $(q, [t_0, b[_{\mathbb{T}})$ is a solution of (Δ-CP), it follows from Lemma A.1 that

$$q_1(t) = q(t) = q_0 + \int_{[t_0, t[_{\mathbb{T}}} f(q(\tau), \tau) \, \Delta\tau = q_0 + \int_{[t_0, t[_{\mathbb{T}}} f(q_1(\tau), \tau) \, \Delta\tau, \tag{A.7}$$

for every $b' \in]t_0, b[_{\mathbb{T}}$ and every $t \in [t_0, b']_{\mathbb{T}}$. Since $f(q_1, t) \in \mathrm{L}^1_{\mathbb{T}}([t_0, b[_{\mathbb{T}}, \mathbb{R}^n)$ (see (A.4)), we infer from Lebesgue's dominated convergence theorem that

$$q_1(b) = q_b = q_0 + \int_{[t_0, b[_{\mathbb{T}}} f(q_1(\tau), \tau) \, \Delta\tau.$$

Therefore (A.7) also holds for $b' = b$. It follows from Lemma A.1 that $(q_1, [t_0, b]_{\mathbb{T}})$ is a solution of (Δ-CP) and is a strict extension of $(q, [t_0, b[_{\mathbb{T}})$. It is a contradiction with the maximality of $(q, [t_0, b[_{\mathbb{T}})$. ☐

Lemma A.4. *Under the assumptions of Theorem A.1, let* $(q, I_{\mathbb{T}})$ *be the maximal solution of* (Δ-CP). *If* $(q, I_{\mathbb{T}})$ *is not global, then for every* $K \in \mathcal{K}$ *there exists* $t \in I_{\mathbb{T}}$ *(close to* a *or* b *depending on the cases listed in the theorem) such that* $q(t) \in \Omega \setminus K$.

Proof. We only prove this lemma in the first case of Theorem A.1. By contradiction, assume that there exists $K \in \mathcal{K}$ such that q takes its values in K on $I_{\mathbb{T}} = [t_0, b[_{\mathbb{T}}$ with b a left dense point of $\mathbb{T}$. Consider $M \geq 0$ associated with $K \in \mathcal{K}$ and $[t_0, b[_{\mathbb{T}}$ in (H_∞). For all $t_1 \leq t_2$ elements of $[t_0, b[_{\mathbb{T}}$, one has

$$\|q(t_2) - q(t_1)\| \leq \int_{[t_1,t_2[_{\mathbb{T}}} \|f(q(\tau), \tau)\| \, \Delta\tau \leq M(t_2 - t_1).$$

Therefore q is Lipschitz continuous and thus uniformly continuous on $[t_0, b[_{\mathbb{T}}$ with b a left-dense point of $\mathbb{T}$. Hence q can be continuously extended at $t = b$ with a value $q_b \in \mathbb{R}^n$. Moreover, since q takes its values in the compact $K \subset \Omega$, it follows that $q_b \in \Omega$. Using Lemma A.3, this raises a contradiction. □

The proof of Theorem A.2 follows from Proposition A.3 and Lemma A.4.

A.4.3 Proof of Theorem A.3

Note that since $\Omega = \mathbb{R}^n$ and since f satisfies ($\mathrm{H}^{\mathrm{glob}}_{\mathrm{Lip}}$), f automatically satisfies ($\mathrm{H}^{\mathrm{forw}}_{\mathrm{stab}}$) and ($\mathrm{H}^{\mathrm{rd}}_{\mathrm{loc-Lip}}$). Since $t_0 = \min \mathbb{T}$, (Δ-CP) admits a unique maximal solution from Theorem A.1. Proving that this maximal solution is global requires the following result.

Lemma A.5. *If* $t_0 = \min \mathbb{T}$ *then*

$$\int_{[t_0,t[_{\mathbb{T}}} (\tau - t_0)^k \, \Delta\tau \leq \frac{(t - t_0)^{k+1}}{k+1},$$

for every $k \in \mathbb{N}$ *and every* $t \in \mathbb{T}$.

Proof. One has

$$\int_{[t_0,t[_{\mathbb{T}}} (\tau - t_0)^k \, \Delta\tau = \int_{[t_0,t[_{\mathbb{T}}} (\tau - t_0)^k \, d\tau + \sum_{r \in [t_0,t[_{\mathbb{T}} \cap \mathrm{RS}} \mu(r)(r - t_0)^k,$$

for every $k \in \mathbb{N}$ and every $t \in \mathbb{T}$. Since

$$\sum_{r \in [t_0,t[_{\mathbb{T}} \cap \mathrm{RS}} \mu(r)(r - t_0)^k = \sum_{r \in [t_0,t[_{\mathbb{T}} \cap \mathrm{RS}} \int_{]r,\sigma(r)[} (r - t_0)^k \, d\tau \leq \sum_{r \in [t_0,t[_{\mathbb{T}} \cap \mathrm{RS}} \int_{]r,\sigma(r)[} (\tau - t_0)^k \, d\tau,$$

it follows that

$$\int_{[t_0,t[_{\mathbb{T}}} (\tau - t_0)^k \, \Delta\tau \leq \int_{[t_0,t[} (\tau - t_0)^k \, d\tau = \frac{(t - t_0)^{k+1}}{k+1},$$

and the proof is complete. □

For every $b \in \mathbb{T} \backslash \{t_0\}$, we define the mapping

$$\begin{array}{rccl} F_b : & \mathrm{C}([t_0, b]_{\mathbb{T}}, \mathbb{R}^n) & \longrightarrow & \mathrm{C}([t_0, b]_{\mathbb{T}}, \mathbb{R}^n) \\ & q & \longmapsto & F(q) \end{array}$$

with

$$\begin{array}{rccl} F_b(q) : & [t_0, b]_{\mathbb{T}} & \longrightarrow & \mathbb{R}^n \\ & t & \longmapsto & q_0 + \displaystyle\int_{[t_0,t[_{\mathbb{T}}} f(q(\tau), \tau) \, \Delta\tau. \end{array}$$

From Lemma A.5 and Assumption ($\mathrm{H}^{\mathrm{glob}}_{\mathrm{Lip}}$), one can easily prove by induction that

$$\|F_b^k(q_1)(t) - F_b^k(q_2)(t)\| \leq \frac{L^k}{k!} \|q_1 - q_2\|_\infty (t - t_0)^k,$$

for every $k \in \mathbb{N}^*$, all $q_1, q_2 \in \mathrm{C}([t_0,b]_\mathbb{T},\mathbb{R}^n)$, and every $t \in [t_0,b]_\mathbb{T}$. Then,

$$\|F_b^k(q_1) - F_b^k(q_2)\|_\infty \leq \frac{(L(b-t_0))^k}{k!} \|q_1 - q_2\|_\infty,$$

for every $k \in \mathbb{N}^*$, all $q_1, q_2 \in \mathrm{C}([t_0,b]_\mathbb{T},\mathbb{R}^n)$. Therefore F_b admits a contraction iterate and thus has a unique fixed point that is a solution on $[t_0,b]_\mathbb{T}$ of (Δ-CP). In the case of a bounded time scale $\mathbb{T}$, it suffices to take $b = \max \mathbb{T}$. In the case where $\mathbb{T}$ is not bounded, it suffices to make b tend to $+\infty$. This last comment concludes the proof of Theorem A.3.

A.4.4 Further comments for the shifted case

An important remark in the *shifted* case is the following. Let $(a,b) \in \mathbb{T}^2$ satisfying $a \trianglelefteq t_0 \trianglelefteq b$ and let $q : [a,b]_\mathbb{T} \longrightarrow \Omega$. Since $\sigma(t) \in [a,b]_\mathbb{T}$ for every $t \in [a,b[_\mathbb{T}$, q^σ is well defined on $[a,b[_\mathbb{T}$. This remark permits to derive all results of Section A.2 in a similar way since Δ-integrals are considered on intervals of the form $[a,b[_\mathbb{T}$.

For example, if f satisfies (H_∞), then for all $(a,b) \in \mathbb{T}^2$ such that $a < b$,

$$f(q^\sigma, t) \in \mathrm{L}^\infty_\mathbb{T}([a,b[_\mathbb{T},\mathbb{R}^n) \subset \mathrm{L}^1_\mathbb{T}([a,b[_\mathbb{T},\mathbb{R}^n),$$

for every $q \in \mathrm{C}([a,b]_\mathbb{T},\mathbb{R}^n)$. This remark permits to prove (from Section I.3.2) the following Δ-integral characterization of the solutions of (Δ-CP$^\sigma$).

Lemma A.6. *Let $I_\mathbb{T} \in \mathbb{I}$ and $q : I_\mathbb{T} \longrightarrow \Omega$. If f satisfies* (H_∞)*, then the couple $(q, I_\mathbb{T})$ is a solution of* (Δ-CP$^\sigma$) *if and only if for all $a,b \in I_\mathbb{T}$ satisfying $a \trianglelefteq t_0 \trianglelefteq b$, one has $q \in \mathrm{C}([a,b]_\mathbb{T},\mathbb{R}^n)$ and*

$$q(t) = \begin{cases} q_0 + \int_{[t_0,t[_\mathbb{T}} f(q^\sigma(\tau),\tau)\ \Delta\tau & \text{if} \quad t \geq t_0, \\ q_0 - \int_{[t,t_0[_\mathbb{T}} f(q^\sigma(\tau),\tau)\ \Delta\tau & \text{if} \quad t \leq t_0. \end{cases}$$

for every $t \in [a,b]_\mathbb{T}$.

All results permitting to prove Theorems A.4 and A.5 can be derived as in Section A.2. Nevertheless, in order to derive Theorem A.6, the following result is required.

Lemma A.7. *If $t_0 = \max \mathbb{T}$ then*

$$\int_{[t,t_0[_\mathbb{T}} (t_0 - \sigma(\tau))^k\ \Delta\tau \leq \frac{(t_0-t)^{k+1}}{k+1},$$

for every $k \in \mathbb{N}$ and every $t \in \mathbb{T}$.

Proof. One has

$$\int_{[t,t_0[_\mathbb{T}} (t_0 - \sigma(\tau))^k\ \Delta\tau = \int_{[t,t_0[_\mathbb{T}} (t_0 - \tau)^k\ d\tau + \sum_{r \in [t,t_0[_\mathbb{T} \cap \mathrm{RS}} \mu(r)(t_0 - \sigma(r))^k,$$

for every $k \in \mathbb{N}$ and every $t \in \mathbb{T}$. Since

$$\sum_{r \in [t,t_0[_\mathbb{T} \cap \mathrm{RS}} \mu(r)(t_0 - \sigma(r))^k = \sum_{r \in [t,t_0[_\mathbb{T} \cap \mathrm{RS}} \int_{]r,\sigma(r)[} (t_0 - \sigma(r))^k\ d\tau \leq \sum_{r \in [t,t_0[_\mathbb{T} \cap \mathrm{RS}} \int_{]r,\sigma(r)[} (t_0 - \tau)^k\ d\tau,$$

we infer that

$$\int_{[t,t_0[_\mathbb{T}} (t_0 - \sigma(\tau))^k\ \Delta\tau \leq \int_{[t,t_0[} (t_0 - \tau)^k\ d\tau = \frac{(t_0-t)^{k+1}}{k+1},$$

and the statement follows. □

Annexe B : Preuves techniques du Chapitre III

Cette annexe est consacrée aux démonstrations des résultats techniques du Chapitre III. Par conséquent, les notations utilisées correspondent aux notations du Chapitre III. Les preuves qui suivent sont toutes extraites de Bourdin L. et Trélat E., Pontryagin Maximum Principle for finite dimensional nonlinear optimal control problems on time scales, preprint arXiv:1302.3513.

We first recall that a generalized exponential function on $\mathbb{T}$ is defined by

$$\forall L \geq 0, \ \forall s, t \in \mathbb{T}, \ e_L(t, s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(L) \ \Delta\tau\right), \tag{B.1}$$

where

$$\xi_{\mu(\tau)}(L) = \begin{cases} \dfrac{1}{\mu(\tau)} \log(1 + L\mu(\tau)) \text{ if } \mu(\tau) > 0 \\ \\ L \text{ if } \mu(\tau) = 0. \end{cases} \tag{B.2}$$

We refer to [38, Chapter 2.2] for a detailed content on generalized exponential functions. Note that $\xi_{\mu(\cdot)}(L)$ is rd-continuous on $\mathbb{T}$. For any $L \geq 0$ and any $s \in \mathbb{T}$, $e_L(\cdot, s)$ is a continuous, positive and increasing function on $\mathbb{T}$. Moreover, it holds:

$$\forall L \geq 0, \ \forall r, s, t \in \mathbb{T}, \ e_L(t, s) e_L(s, r) = e_L(t, r). \tag{B.3}$$

Finally, recall that $e_L(\cdot, s) \in \mathrm{C}^{1,\Delta}(\mathbb{T}, \mathbb{R})$ with $e_L^{\Delta}(\cdot, s) = L e_L(\cdot, s)$, see [38, Theorem 2.33 p.59].

We refer to [38, Chapter 6.1] for different Gronwall inequalities on time scales. Let us recall the following one, see [38, Theorem 6.4 p.256].

Lemma B.1 (Gronwall). *Let a, $b \in \mathbb{T}$ such that $a < b$ and $q \in \mathrm{C_{rd}}([a, b]_{\mathbb{T}}, \mathbb{R})$ such that:*

$$\exists L_1, L_2 \geq 0, \ \forall t \in [a, b]_{\mathbb{T}}, \ 0 \leq q(t) \leq L_1 + L_2 \int_a^t q(\tau) \ \Delta\tau. \tag{B.4}$$

Then, $0 \leq q(t) \leq L_1 e_{L_2}(t, a)$ for every $t \in [a, b]_{\mathbb{T}}$.

B.1 Proofs of Section III.3.2

B.1.1 Proofs of Section III.3.2.1

Proof of Lemma III.1 Let $R > \|u\|_{\mathrm{L}^\infty_{\mathbb{T}}([a,b[_{\mathbb{T}}, \mathbb{R}^m)}$. By continuity of $q(\cdot, u, q_a)$ on $[a, b]_{\mathbb{T}}$, the set

$$K = \{(x, v, t) \in \mathbb{R}^n \times \overline{B}_{\mathbb{R}^m}(0, R) \times [a, b]_{\mathbb{T}} \mid \|x - q(t, u, q_a)\|_{\mathbb{R}^n} \leq 1\}$$

is a compact subset of $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{T}$. Therefore $\|\partial f / \partial x\|$ and $\|\partial f / \partial v\|$ are bounded by some $L \geq 0$ on K and moreover L is chosen such that

$$\|f(x_1, v_1, t) - f(x_2, v_2, t)\|_{\mathbb{R}^n} \leq L(\|x_1 - x_2\|_{\mathbb{R}^n} + \|v_1 - v_2\|_{\mathbb{R}^m}), \tag{B.5}$$

for all (x_1, v_1, t) and (x_2, v_2, t) in K. Let $\nu_R > 0$ and $0 < \eta_R < 1$ such that $(\eta_R + \nu_R L)e_L(b,a) < 1$. Note that K, L, ν_R and η_R depend on (u, q_a, R).

Let $(u', q'_a) \in \mathrm{E}(u, q_a, R)$. We denote by $I'_{\mathbb{T}}$ the interval of definition of $q(\cdot, u', q'_a)$ satisfying $a \in I'_{\mathbb{T}}$ and $I'_{\mathbb{T}} \backslash \{a\} \neq \emptyset$. It suffices to prove that $b \in I'_{\mathbb{T}}$. By contradiction, assume that the set $A = \{t \in I'_{\mathbb{T}} \cap [a,b]_{\mathbb{T}} \mid \|q(t,u',q'_a) - q(t,u,q_a)\|_{\mathbb{R}^n} > 1\}$ is not empty and set $t_1 = \inf A$. Since $\mathbb{T}$ is closed, $t_1 \in I'_{\mathbb{T}} \cap [a,b]_{\mathbb{T}}$ and $[a,t_1]_{\mathbb{T}} \subset I'_{\mathbb{T}} \cap [a,b]_{\mathbb{T}}$. If t_1 is a minimum then $\|q(t_1,u',q'_a) - q(t_1,u,q_a)\|_{\mathbb{R}^n} > 1$. If t_1 is not a minimum then $t_1 \in \mathrm{RD}$ and by continuity we have $\|q(t_1,u',q'_a) - q(t_1,u,q_a)\|_{\mathbb{R}^n} \geq 1$. Moreover there holds $t_1 > a$ since $\|q(a,u',q'_a) - q(a,u,q_a)\|_{\mathbb{R}^n} = \|q'_a - q_a\|_{\mathbb{R}^n} \leq \eta_R < 1$. Hence $\|q(\tau,u',q'_a) - q(\tau,u,q_a)\|_{\mathbb{R}^n} \leq 1$ for every $\tau \in [a,t_1[_{\mathbb{T}}$. Therefore $(q(\tau,u',q'_a), u'(\tau), \tau)$ and $(q(\tau,u,q_a), u(\tau), \tau)$ are elements of K for Δ-a.e. $\tau \in [a,t_1[_{\mathbb{T}}$. Since there holds

$$q(t,u',q'_a) - q(t,u,q_a) = q'_a - q_a + \int_{[a,t[_{\mathbb{T}}} \left(f(q(\tau,u',q'_a), u'(\tau), \tau) - f(q(\tau,u,q_a), u(\tau), \tau) \right) \Delta\tau,$$

for every $t \in I'_{\mathbb{T}} \cap [a,b]_{\mathbb{T}}$, it follows from (B.5) and from Lemma B.1 that, for every $t \in [a,t_1]_{\mathbb{T}}$,

$$\begin{aligned}
\|q(t,u',q'_a) - q(t,u,q_a)\|_{\mathbb{R}^n} &\leq \|q'_a - q_a\|_{\mathbb{R}^n} + L \int_{[a,t[_{\mathbb{T}}} \|u'(\tau) - u(\tau)\|_{\mathbb{R}^m} \Delta\tau \\
&\quad + L \int_{[a,t[_{\mathbb{T}}} \|q(\tau,u',q'_a) - q(\tau,u,q_a)\|_{\mathbb{R}^n} \Delta\tau \\
&\leq (\|q'_a - q_a\|_{\mathbb{R}^n} + L\|u' - u\|_{\mathrm{L}^1_{\mathbb{T}}([a,b[_{\mathbb{T}},\mathbb{R}^m)}) e_L(b,a) \\
&\leq (\eta_R + \nu_R L) e_L(b,a) < 1.
\end{aligned}$$

This raises a contradiction at $t = t_1$. Therefore A is empty and thus $q(\cdot,u',q'_a)$ is bounded on $I'_{\mathbb{T}} \cap [a,b]_{\mathbb{T}}$. It follows from Theorem A.2 (see also [52, Theorem 2]) that $b \in I'_{\mathbb{T}}$, that is, $(u',q'_a) \in \mathcal{UQ}^b_{\mathrm{ad}}$.

Remark B.1. Let $(u', q'_a) \in \mathrm{E}(u, q_a, R)$. With the notations of the above proof, since $I'_{\mathbb{T}} \cap [a,b]_{\mathbb{T}} = [a,b]_{\mathbb{T}}$ and A is empty, we infer that $\|q(t,u',q'_a) - q(t,u,q_a)\| \leq 1$, for every $t \in [a,b]_{\mathbb{T}}$. Therefore $(q(t,u',q'_a), u'(t), t) \in K$ for every $(u',q'_a) \in \mathrm{E}(u,q_a,R)$ and for Δ-a.e. $t \in [a,b[_{\mathbb{T}}$.

Proof of Lemma III.2 Let (u', q'_a) and (u'', q''_a) be elements of $\mathrm{E}(u, q_a, R) \subset \mathcal{UQ}^b_{\mathrm{ad}}$. It follows from Remark B.1 that $(q(\tau,u'',q''_a), u''(\tau), \tau)$ and $(q(\tau,u',q'_a), u'(\tau), \tau)$ are elements of K for Δ-a.e. $\tau \in [a,b[_{\mathbb{T}}$. Following the same arguments as in the previous proof, it follows from (B.5) and from Lemma B.1 that, for every $t \in [a,b]_{\mathbb{T}}$,

$$\|q(t,u'',q''_a) - q(t,u',q'_a)\|_{\mathbb{R}^n} \leq (\|q''_a - q'_a\|_{\mathbb{R}^n} + L\|u'' - u'\|_{\mathrm{L}^1_{\mathbb{T}}([a,b[_{\mathbb{T}},\mathbb{R}^m)}) e_L(b,a).$$

The lemma follows.

B.1.2 Proofs of Section III.3.2.2

Proof of Lemma III.3 Let $R = \max(\|u\|_{\mathrm{L}^\infty_{\mathbb{T}}([a,b[_{\mathbb{T}},\mathbb{R}^m)}, \|u(r)\|_{\mathbb{R}^m} + \|y\|_{\mathbb{R}^m}) + 1 > \|u\|_{\mathrm{L}^\infty_{\mathbb{T}}([a,b[_{\mathbb{T}},\mathbb{R}^m)}$. We use the notations K, L, ν_R and η_R, associated with (u, q_a, R), defined in Lemma III.1 and in its proof.

One has $\|u_\Pi(\cdot,\alpha)\|_{\mathrm{L}^\infty_{\mathbb{T}}([a,b[_{\mathbb{T}},\mathbb{R}^m)} \leq R$ for every $\alpha \in \mathcal{D}^\Omega(u(r),y)$, and

$$\|u_\Pi(\cdot,\alpha) - u\|_{\mathrm{L}^1_{\mathbb{T}}([a,b[_{\mathbb{T}},\mathbb{R}^m)} = \mu(r)\|u_\Pi(r,\alpha) - u(r)\|_{\mathbb{R}^m} = \alpha\mu(r)\|y - u(r)\|_{\mathbb{R}^m}.$$

Hence, there exists $\alpha_0 > 0$ such that $\|u_\Pi(\cdot,\alpha) - u\|_{\mathrm{L}^1_{\mathbb{T}}([a,b[_{\mathbb{T}},\mathbb{R}^m)} \leq \nu_R$ for every $\alpha \in \mathcal{D}^\Omega(u(r),y) \cap [0,\alpha_0]$, and hence $(u_\Pi(\cdot,\alpha), q_a) \in \mathrm{E}(u,q_a,R)$. The claim follows then from Lemma III.1.

Proof of Lemma III.4 We use the notations of proof of Lemma III.3. It follows from Lemma III.2 that there exists $C \geq 0$ (the Lipschitz constant of $F_{(u,q_a,R)}$) such that

$$\begin{aligned}
\|q(\cdot,u_\Pi(\cdot,\alpha^2),q_a) - q(\cdot,u_\Pi(\cdot,\alpha^1),q_a)\|_\infty &\leq C d_{\mathcal{UQ}^b_{\mathrm{ad}}}((u_\Pi(\cdot,\alpha^2),q_a),(u_\Pi(\cdot,\alpha^1),q_a)) \\
&= C|\alpha^2 - \alpha^1|\mu(r)\|y - u(r)\|_{\mathbb{R}^m},
\end{aligned}$$

for all α^1 and α^2 in $\mathcal{D}^\Omega(u(r),y) \cap [0,\alpha_0]$. The lemma follows.

Proof of Proposition III.1 We use the notations of proof of Lemma III.3. In Remark B.1, we have seen that $(q(t,u_\Pi(\cdot,\alpha),q_a),u_\Pi(t,\alpha),t)\in K$ for every $\alpha\in\mathcal{D}^\Omega(u(r),y)\cap[0,\alpha_0]$ and for Δ-a.e. $t\in[a,b[_\mathbb{T}$. For every $\alpha\in\mathcal{D}^\Omega(u(r),y)\cap]0,\alpha_0]$ and every $t\in[\sigma(r),b]_\mathbb{T}$, we define

$$\varepsilon_\Pi(t,\alpha)=\frac{q(t,u_\Pi(\cdot,\alpha),q_a)-q(t,u,q_a)}{\alpha}-\omega_\Pi(t,u,q_a).$$

It suffices to prove that $\varepsilon_\Pi(\cdot,\alpha)$ converges uniformly to 0 on $[\sigma(r),b]_\mathbb{T}$ as α tends to 0. The function $\varepsilon_\Pi(\cdot,\alpha)$ is absolutely continuous on $[\sigma(r),b]_\mathbb{T}$ for every $\alpha\in\mathcal{D}^\Omega(u(r),y)\cap]0,\alpha_0]$, and for every $t\in[\sigma(r),b]_\mathbb{T}$ we have $\varepsilon_\Pi(t,\alpha)=\varepsilon_\Pi(\sigma(r),\alpha)+\int_{[\sigma(r),t[_\mathbb{T}}\varepsilon^\Delta_\Pi(\tau,\alpha)\,\Delta\tau$, where

$$\varepsilon^\Delta_\Pi(t,\alpha)=\frac{f(q(t,u_\Pi(\cdot,\alpha),q_a),u(t),t)-f(q(t,u,q_a),u(t),t)}{\alpha}-\frac{\partial f}{\partial x}(q(t,u,q_a),u(t),t)\times\omega_\Pi(t,u,q_a),\tag{B.6}$$

for Δ-a.e. $t\in[\sigma(r),b[_\mathbb{T}$. It follows from the Mean Value Theorem applied for Δ-a.e. $t\in[\sigma(r),b[_\mathbb{T}$ to the function defined by $\varphi_t(\theta)=f((1-\theta)q(t,u,q_a)+\theta q(t,u_\Pi(\cdot,\alpha),q_a),u(t),t)$ for every $\theta\in[0,1]$, that there exists $\theta_\Pi(t,\alpha)\in\mathbb{R}^n$, belonging to the segment of extremities $q(t,u,q_a)$ and $q(t,u_\Pi(\cdot,\alpha),q_a)$, such that

$$\varepsilon^\Delta_\Pi(t,\alpha)=\frac{\partial f}{\partial x}(\theta_\Pi(t,\alpha),u(t),t)\times\varepsilon_\Pi(t,\alpha)+\left(\frac{\partial f}{\partial x}(\theta_\Pi(t,\alpha),u(t),t)-\frac{\partial f}{\partial x}(q(t,u,q_a),u(t),t)\right)\times\omega_\Pi(t,u,q_a).\tag{B.7}$$

Since $(\theta_\Pi(t,\alpha),u(t),t)\in K$ for Δ-a.e. $t\in[\sigma(r),b[_\mathbb{T}$, it follows that $\|\varepsilon^\Delta_\Pi(t,\alpha)\|\le\chi_\Pi(t,\alpha)+L\|\varepsilon_\Pi(t,\alpha)\|$, where $\chi_\Pi(t,\alpha)=\big\|\big(\frac{\partial f}{\partial x}(\theta_\Pi(t,\alpha),u(t),t)-\frac{\partial f}{\partial x}(q(t,u,q_a),u(t),t)\big)\times\omega_\Pi(t,u,q_a)\big\|$. Therefore, one has

$$\|\varepsilon_\Pi(t,\alpha)\|_{\mathbb{R}^n}\le\|\varepsilon_\Pi(\sigma(r),\alpha)\|_{\mathbb{R}^n}+\int_{[\sigma(r),b[_\mathbb{T}}\chi_\Pi(\tau,\alpha)\,\Delta\tau+L\int_{[\sigma(r),t[_\mathbb{T}}\|\varepsilon_\Pi(\tau,\alpha)\|_{\mathbb{R}^n}\,\Delta\tau,$$

for every $t\in[\sigma(r),b]_\mathbb{T}$. It follows from Lemma B.1 that $\|\varepsilon_\Pi(t,\alpha)\|_{\mathbb{R}^n}\le\Upsilon_\Pi(\alpha)e_L(b,\sigma(r))$, for every $t\in[\sigma(r),b]_\mathbb{T}$, where $\Upsilon_\Pi(\alpha)=\|\varepsilon_\Pi(\sigma(r),\alpha)\|_{\mathbb{R}^n}+\int_{[\sigma(r),b[_\mathbb{T}}\chi_\Pi(\tau,\alpha)\,\Delta\tau$.

To conclude, it remains to prove that $\Upsilon_\Pi(\alpha)$ converges to 0 as α tends to 0. First, since $\theta_\Pi(\cdot,\alpha)$ converges uniformly to $q(\cdot,u,q_a)$ on $[\sigma(r),b]_\mathbb{T}$ as α tends to 0, and since $\partial f/\partial x$ is uniformly continuous on K, we infer that $\int_{[\sigma(r),b[_\mathbb{T}}\chi_\Pi(\tau,\alpha)\,\Delta\tau$ converges to 0 as α tends to 0. Second, it is easy to see that $\|\varepsilon_\Pi(\sigma(r),\alpha)\|_{\mathbb{R}^n}$ converges to 0 as α tends to 0. The conclusion follows.

Proof of Lemma III.5 We use the notations K, L, ν_R and η_R, associated with (u,q_a,R), defined in Lemma III.1 and in its proof.

Consider the absolutely continuous function defined by $\Phi_k(t)=\omega_\Pi(t,u_k,q_{a,k})-\omega_\Pi(t,u,q_a)$ for every $k\in\mathbb{N}$ and every $t\in[\sigma(r),b]_\mathbb{T}$. Let us prove that Φ_k converges uniformly to 0 on $[\sigma(r),b]_\mathbb{T}$ as k tends to $+\infty$. One has

$$\Phi_k(t)=\Phi_k(\sigma(r))+\int_{[\sigma(r),t[_\mathbb{T}}\frac{\partial f}{\partial x}(q(\tau,u_k,q_{a,k}),u_k(\tau),\tau)\times\Phi_k(\tau)\,\Delta\tau+\int_{[\sigma(r),t[_\mathbb{T}}\left(\frac{\partial f}{\partial x}(q(\tau,u_k,q_{a,k}),u_k(\tau),\tau)-\frac{\partial f}{\partial x}(q(\tau,u,q_a),u(\tau),\tau)\right)\times\omega_\Pi(\tau,u,q_a)\,\Delta\tau,\tag{B.8}$$

for every $t\in[\sigma(r),b]_\mathbb{T}$ and every $k\in\mathbb{N}$. Since $(u_k,q_{a,k})\in\mathrm{E}(u,q_a,R)$ for every $k\in\mathbb{N}$, it follows from Remark B.1 that $(q(t,u_k,q_{a,k}),u_k(t),t)\in K$ and $(q(t,u,q_a),u(t),t)\in K$ for Δ-a.e. $t\in[a,b[_\mathbb{T}$. Hence it follows from Lemma B.1 that

$$\|\Phi_k(t)\|_{\mathbb{R}^n}\le(\|\Phi_k(\sigma(r))\|_{\mathbb{R}^n}+\vartheta_k)e_L(b,\sigma(r)),$$

for every $t\in[\sigma(r),b]_\mathbb{T}$, where

$$\vartheta_k=\int_{[\sigma(r),b[_\mathbb{T}}\left\|\frac{\partial f}{\partial x}(q(\tau,u_k,q_{a,k}),u_k(\tau),\tau)-\frac{\partial f}{\partial x}(q(\tau,u,q_a),u(\tau),\tau)\right\|_{\mathbb{R}^{n,n}}\|\omega_\Pi(\tau,u,q_a)\|_{\mathbb{R}^n}\,\Delta\tau.$$

Since $\mu_\Delta(\{r\})=\mu(r)>0$, $u_k(r)$ converges to $u(r)$ as k tends to $+\infty$. Moreover, $(u_k,q_{a,k})$ converges to (u,q_a) in $(\mathrm{E}(u,q_a,R),d_{\mathcal{UQ}^b_{\mathrm{ad}}})$ and, from Lemma III.2, $q(\cdot,u_k,q_{a,k})$ converges uniformly to $q(\cdot,u,q_a)$ on $[a,b]_\mathbb{T}$ as k tends to $+\infty$. We infer that $\Phi_k(\sigma(r))$ converges to 0 as k tends to $+\infty$, and from the Lebesgue dominated convergence theorem we conclude that ϑ_k converges to 0 as k tends to $+\infty$. The lemma follows.

B.1.3 Proofs of Section III.3.2.3

Proof of Lemma III.6 Let $R = \max(\|u\|_{\mathrm{L}^\infty_{\mathbb{T}}([a,b[_{\mathbb{T}},\mathbb{R}^m)}, \|z\|_{\mathbb{R}^m}) + 1 > \|u\|_{\mathrm{L}^\infty_{\mathbb{T}}([a,b[_{\mathbb{T}},\mathbb{R}^m)}$. We use the notations K, L, ν_R and η_R, associated with (u, q_a, R), defined in Lemma III.1 and in its proof.

For every $\beta \in \mathcal{V}^b_s$ one has $\|u_{\mathrm{II}}(\cdot,\beta)\|_{\mathrm{L}^\infty_{\mathbb{T}}([a,b[_{\mathbb{T}},\mathbb{R}^m)} \leq R$ and

$$\|u_{\mathrm{II}}(\cdot,\beta) - u\|_{\mathrm{L}^1_{\mathbb{T}}([a,b[_{\mathbb{T}},\mathbb{R}^m)} = \int_{[s,s+\beta[_{\mathbb{T}}} \|z - u(\tau)\|_{\mathbb{R}^m}\, \Delta\tau \leq 2R\beta.$$

Hence, there exists $\beta_0 > 0$ such that for every $\beta \in \mathcal{V}^b_s \cap [0,\beta_0]$, $\|u_{\mathrm{II}}(\cdot,\beta) - u\|_{\mathrm{L}^1_{\mathbb{T}}([a,b[_{\mathbb{T}},\mathbb{R}^m)} \leq \nu_R$ and thus $(u_{\mathrm{II}}(\cdot,\beta), q_a) \in \mathrm{E}(u, q_a, R)$. The conclusion then follows from Lemma III.1.

Proof of Lemma III.7 We use the notations of proof of Lemma III.6. From Lemma III.2, there exists $C \geq 0$ (Lipschitz constant of $F_{(u,q_a,R)}$) such that

$$\begin{aligned}\|q(\cdot, u_{\mathrm{II}}(\cdot,\beta^2), q_a) - q(\cdot, u_{\mathrm{II}}(\cdot,\beta^1), q_a)\|_\infty &\leq C d_{\mathrm{U}\mathcal{Q}^b_{\mathrm{ad}}}((u_{\mathrm{II}}(\cdot,\beta^2), q_a), (u_{\mathrm{II}}(\cdot,\beta^1), q_a)) \\ &\leq 2CR|\beta^2 - \beta^1|,\end{aligned}$$

for all β^1 and β^2 in $\mathcal{V}^b_s \cap [0,\beta_0]$. The lemma follows.

Proof of Proposition III.2 We use the notations of proof of Lemma III.6. In Remark B.1, we have seen that $(q(t, u_{\mathrm{II}}(\cdot,\beta), q_a), u_{\mathrm{II}}(t,\beta), t)$ and $(q(t, u_{\mathrm{II}}(\cdot,\beta), q_a), z, t)$ belong to K for every $\beta \in \mathcal{V}^b_s \cap [0,\beta_0]$ and for Δ-a.e. $t \in [a,b[_{\mathbb{T}}$. For every $\beta \in \mathcal{V}^b_s \cap]0,\beta_0]$ and every $t \in [s+\beta, b]_{\mathbb{T}}$, we define

$$\varepsilon_{\mathrm{II}}(t,\beta) = \frac{q(t, u_{\mathrm{II}}(\cdot,\beta), q_a) - q(t, u, q_a)}{\beta} - \omega_{\mathrm{II}}(t, u, q_a).$$

It suffices to prove that $\varepsilon_{\mathrm{II}}(\cdot,\beta)$ converges uniformly to 0 on $[s+\beta, b]_{\mathbb{T}}$ as β tends to 0 (note that, for every $\delta \in \mathcal{V}^b_s \backslash \{0\}$, it suffices to consider $\beta \leq \delta$). For every $\beta \in \mathcal{V}^b_s \cap]0,\beta_0]$, the function $\varepsilon_{\mathrm{II}}(\cdot,\beta)$ is absolutely continuous on $[s+\beta, b]_{\mathbb{T}}$ and $\varepsilon_{\mathrm{II}}(t,\beta) = \varepsilon_{\mathrm{II}}(s+\beta,\beta) + \int_{[s+\beta,t[_{\mathbb{T}}} \varepsilon^\Delta_{\mathrm{II}}(\tau,\beta)\, \Delta\tau$, for every $t \in [s+\beta, b]_{\mathbb{T}}$, where

$$\varepsilon^\Delta_{\mathrm{II}}(t,\beta) = \frac{f(q(t, u_{\mathrm{II}}(\cdot,\beta), q_a), u(t), t) - f(q(t,u,q_a), u(t), t)}{\beta} - \frac{\partial f}{\partial x}(q(t,u,q_a), u(t), t) \times \omega_{\mathrm{II}}(t, u, q_a). \tag{B.9}$$

for Δ-a.e. $t \in [s+\beta, b[_{\mathbb{T}}$. As in the proof of Proposition III.1, it follows from the Mean Value Theorem that, for Δ-a.e. $t \in [s+\beta, b[_{\mathbb{T}}$, there exists $\theta_{\mathrm{II}}(t,\beta) \in \mathbb{R}^n$, belonging to the segment of extremities $q(t,u,q_a)$ and $q(t, u_{\mathrm{II}}(\cdot,\beta), q_a)$, such that

$$\begin{aligned}\varepsilon^\Delta_{\mathrm{II}}(t,\beta) = {} & \frac{\partial f}{\partial x}(\theta_{\mathrm{II}}(t,\beta), u(t), t) \times \varepsilon_{\mathrm{II}}(t,\beta) \\ & + \left(\frac{\partial f}{\partial x}(\theta_{\mathrm{II}}(t,\beta), u(t), t) - \frac{\partial f}{\partial x}(q(t,u,q_a), u(t), t) \right) \times \omega_{\mathrm{II}}(t,u,q_a).\end{aligned} \tag{B.10}$$

Since $(\theta_{\mathrm{II}}(t,\beta), u(t), t) \in K$ for Δ-a.e. $t \in [s+\beta, b[_{\mathbb{T}}$, it follows that $\|\varepsilon^\Delta_{\mathrm{II}}(t,\beta)\| \leq \chi_{\mathrm{II}}(t,\beta) + L\|\varepsilon_{\mathrm{II}}(t,\beta)\|$, where $\chi_{\mathrm{II}}(t,\beta) = \big\|\big(\frac{\partial f}{\partial x}(\theta_{\mathrm{II}}(t,\beta), u(t), t) - \frac{\partial f}{\partial x}(q(t,u,q_a), u(t), t)\big) \times \omega_{\mathrm{II}}(t,u,q_a)\big\|$. Therefore, one has

$$\|\varepsilon_{\mathrm{II}}(t,\beta)\|_{\mathbb{R}^n} \leq \|\varepsilon_{\mathrm{II}}(s+\beta,\beta)\|_{\mathbb{R}^n} + \int_{[s+\beta,b[_{\mathbb{T}}} \chi_{\mathrm{II}}(\tau,\beta)\, \Delta\tau + L \int_{[s+\beta,t[_{\mathbb{T}}} \|\varepsilon_{\mathrm{II}}(\tau,\beta)\|_{\mathbb{R}^n}\, \Delta\tau,$$

for every $t \in [s+\beta, b]_{\mathbb{T}}$, and it follows from Lemma B.1 that $\|\varepsilon_{\mathrm{II}}(t,\beta)\| \leq \Upsilon_{\mathrm{II}}(\beta) e_L(b,s)$, for every $t \in [s+\beta, b]_{\mathbb{T}}$, where $\Upsilon_{\mathrm{II}}(\beta) = \|\varepsilon_{\mathrm{II}}(s+\beta,\beta)\|_{\mathbb{R}^n} + \int_{[s+\beta,b[_{\mathbb{T}}} \chi_{\mathrm{II}}(\tau,\beta)\, \Delta\tau$.

To conclude, it remains to prove that $\Upsilon_{\mathrm{II}}(\beta)$ converges to 0 as β tends to 0. First, since $\theta_{\mathrm{II}}(\cdot,\beta)$ converges uniformly to $q(\cdot,u,q_a)$ on $[s+\beta, b]_{\mathbb{T}}$ as β tends to 0 and since $\partial f/\partial x$ is uniformly continuous on K, we infer that $\int_{[s+\beta,b[_{\mathbb{T}}} \chi_{\mathrm{II}}(\tau,\beta)\, \Delta\tau$ converges to 0 as β tends to 0. Second, let us prove that $\|\varepsilon_{\mathrm{II}}(s+\beta,\beta)\|_{\mathbb{R}^n}$ converges to 0 as β tends to 0. By continuity, $\omega_{\mathrm{II}}(s+\beta, u, q_a)$ converges to $\omega_{\mathrm{II}}(s, u, q_a)$ as β to 0. Moreover, since $q(\cdot, u_{\mathrm{II}}(\cdot,\beta), q_a)$

converges uniformly to $q(\cdot,u,q_a)$ on $[a,b]_{\mathbb{T}}$ as β tends to 0 and since f is uniformly continuous on K, it follows that $f(q(\cdot,u_{\text{II}}(\cdot,\beta),q_a),z,t)$ converges uniformly to $f(q(\cdot,u,q_a),z,t)$ on $[a,b]_{\mathbb{T}}$ as β tends to 0. Therefore, it suffices to note that

$$\frac{1}{\beta}\int_{[s,s+\beta[_{\mathbb{T}}} f(q(\tau,u,q_a),z,\tau)-f(q(\tau,u,q_a),u(\tau),\tau)\,\Delta\tau$$

converges to $\omega_{\text{II}}(s,u,q_a)=f(q(s,u,q_a),z,s)-f(q(s,u,q_a),u(s),s)$ as β tends to 0 since s is a Δ-Lebesgue point of $f(q(\cdot,u,q_a),z,t)$ and of $f(q(\cdot,u,q_a),u,t)$. Then $\|\varepsilon_{\text{II}}(s+\beta,\beta)\|$ converges to 0 as β tends to 0, and hence $\Upsilon_{\text{II}}(\beta)$ converges to 0 as well.

Proof of Lemma III.8 The proof is similar to the one of Lemma III.7, replacing $\sigma(r)$ with s.

B.1.4 Proofs of Section III.3.2.4

Proof of Lemma III.9 Let $R=\|u\|_{\mathrm{L}^\infty_{\mathbb{T}}([a,b[_{\mathbb{T}},\mathbb{R}^m)}+1>\|u\|_{\mathrm{L}^\infty_{\mathbb{T}}([a,b[_{\mathbb{T}},\mathbb{R}^m)}$. We use the notations K, L, ν_R and η_R, associated with (u,q_a,R), defined in Lemma III.1 and in its proof.

There exists $\gamma_0>0$ such that $\|q_a+\gamma q'_a-q_a\|_{\mathbb{R}^n}=\gamma\|q'_a\|_{\mathbb{R}^n}\leq\eta_R$ for every $\gamma\in[0,\gamma_0]$, and hence $(u,q_a+\gamma q'_a)\in\mathrm{E}(u,q_a,R)$. Then the claim follows from Lemma III.1.

Proof of Lemma III.10 We use the notations of proof of Lemma III.9. From Lemma III.2, there exists $C\geq 0$ (Lipschitz constant of $F_{(u,q_a,R)}$) such that

$$\begin{aligned}\|q(\cdot,u,q_a+\gamma^2 q'_a)-q(\cdot,u,q_a+\gamma^1 q'_a)\|_\infty &\leq C d_{\mathcal{UQ}^b_{\text{ad}}}((u,q_a+\gamma^2 q'_a),(u,q_a+\gamma^1 q'_a))\\ &= C|\gamma^2-\gamma^1|\|q'_a\|_{\mathbb{R}^n}.\end{aligned}$$

for all γ^1 and γ^2 in $[0,\gamma_0]$.

Proof of Proposition III.3 We use the notations of proof of Lemma III.9. Note that, from Remark B.1, $(q(t,u,q_a+\gamma q'_a),u(t),t)\in K$ for every $\gamma\in[0,\gamma_0]$ and for Δ-a.e. $t\in[a,b[_{\mathbb{T}}$. For every $\gamma\in]0,\gamma_0]$ and every $t\in[a,b]_{\mathbb{T}}$, we define

$$\varepsilon_{q'_a}(t,\gamma)=\frac{q(t,u,q_a+\gamma q'_a)-q(t,u,q_a)}{\gamma}-\omega_{q'_a}(t,u,q_a).$$

It suffices to prove that $\varepsilon_{q'_a}(\cdot,\gamma)$ converges uniformly to 0 on $[a,b]_{\mathbb{T}}$ as γ tends to 0. For every $\gamma\in]0,\gamma_0]$, the function $\varepsilon_{q'_a}(\cdot,\gamma)$ is absolutely continuous on $[a,b]_{\mathbb{T}}$, and $\varepsilon_{q'_a}(t,\gamma)=\varepsilon_{q'_a}(a,\gamma)+\int_{[a,t[_{\mathbb{T}}}\varepsilon^{\Delta}_{q'_a}(\tau,\gamma)\,\Delta\tau$, for every $t\in[a,b]_{\mathbb{T}}$, where

$$\varepsilon^{\Delta}_{q'_a}(t,\gamma)=\frac{f(q(t,u,q_a+\gamma q'_a),u(t),t)-f(q(t,u,q_a),u(t),t)}{\gamma}-\frac{\partial f}{\partial x}(q(t,u,q_a),u(t),t)\times\omega_{q'_a}(t,u,q_a), \tag{B.11}$$

for Δ-a.e. $t\in[a,b[_{\mathbb{T}}$. As in the proof of Proposition III.1, it follows from the Mean Value Theorem that, for Δ-a.e. $t\in[a,b[_{\mathbb{T}}$, there exists $\theta_{q'_a}(t,\gamma)\in\mathbb{R}^n$, belonging to the segment of extremities $q(t,u,q_a)$ and $q(t,u,q_a+\gamma q'_a)$, such that

$$\begin{aligned}\varepsilon^{\Delta}_{q'_a}(t,\gamma)=\frac{\partial f}{\partial x}(\theta_{q'_a}(t,\gamma),u(t),t)\times\varepsilon_{q'_a}(t,\gamma)&\\ +\left(\frac{\partial f}{\partial x}(\theta_{q'_a}(t,\gamma),u(t),t)-\frac{\partial f}{\partial x}(q(t,u,q_a),u(t),t)\right)\times\omega_{q'_a}(t,u,q_a).&\end{aligned} \tag{B.12}$$

Since $(\theta_{q'_a}(t,\gamma),u(t),t)\in K$ for Δ-a.e. $t\in[a,b[_{\mathbb{T}}$, it follows that

$$\|\varepsilon^{\Delta}_{q'_a}(t,\gamma)\|_{\mathbb{R}^n}\leq\chi_{q'_a}(t,\gamma)+L\|\varepsilon_{q'_a}(t,\gamma)\|_{\mathbb{R}^n},$$

where $\chi_{q'_a}(t,\gamma)=\left\|\left(\frac{\partial f}{\partial x}(\theta_{q'_a}(t,\gamma),u(t),t)-\frac{\partial f}{\partial x}(q(t,u,q_a),u(t),t)\right)\times\omega_{q'_a}(t,u,q_a)\right\|_{\mathbb{R}^n}$. Hence

$$\|\varepsilon_{q'_a}(t,\gamma)\|_{\mathbb{R}^n}\leq\|\varepsilon_{q'_a}(a,\gamma)\|_{\mathbb{R}^n}+\int_{[a,b[_{\mathbb{T}}}\chi_{q'_a}(\tau,\gamma)\,\Delta\tau+L\int_{[a,t[_{\mathbb{T}}}\|\varepsilon_{q'_a}(\tau,\gamma)\|_{\mathbb{R}^n}\,\Delta\tau,$$

for every $t \in [a,b]_{\mathbb{T}}$, and it follows from Lemma B.1 that $\|\varepsilon_{q'_a}(t,\gamma)\| \leq \Upsilon_{q'_a}(\gamma) e_L(b,a)$, for every $t \in [a,b]_{\mathbb{T}}$, where $\Upsilon_{q'_a}(\gamma) = \|\varepsilon_{q'_a}(a,\gamma)\|_{\mathbb{R}^n} + \int_{[a,b[_{\mathbb{T}}} \chi_{q'_a}(\tau,\gamma)\, \Delta\tau$.

To conclude, it remains to prove that $\Upsilon_{q'_a}(\gamma)$ converges to 0 as γ tends to 0. First, since $\theta_{q'_a}(\cdot,\gamma)$ converges uniformly to $q(\cdot,u,q_a)$ on $[a,b]_{\mathbb{T}}$ as γ tends to 0 and since $\partial f/\partial x$ is uniformly continuous on K, we infer that $\int_{[a,b[_{\mathbb{T}}} \chi_{q'_a}(\tau,\gamma)\, \Delta\tau$ tends to 0 when $\gamma \to 0$. Second, it is easy to see that $\varepsilon_{q'_a}(a,\gamma) = 0$ for every $\gamma \in]0,\gamma_0]$. The conclusion follows.

Proof of Lemma III.11 The proof is similar to the one of Lemma III.7, replacing $\sigma(r)$ with a.

B.2 Proofs of Section III.3.3.1

B.2.1 Proof of Lemma III.15

For every $k \in \mathbb{N}$ and any $\beta \in \mathcal{V}^b_s$, we recall that $u^R_{\varepsilon_k,\Pi}(\cdot,\beta) \in \mathrm{L}^\infty_{\mathbb{T}}([a,b[_{\mathbb{T}},\Omega)$ and

$$\|u^R_{\varepsilon_k,\Pi}(\cdot,\beta)\|_{\mathrm{L}^\infty_{\mathbb{T}}([a,b[_{\mathbb{T}},\mathbb{R}^m)} \leq \max(\|u^R_{\varepsilon_k}\|_{\mathrm{L}^\infty_{\mathbb{T}}([a,b[_{\mathbb{T}},\mathbb{R}^m)}, \|z\|_{\mathbb{R}^m}) \leq R,$$

and

$$\begin{aligned} \|u^R_{\varepsilon_k,\Pi}(\cdot,\beta) - u^*\|_{\mathrm{L}^1_{\mathbb{T}}([a,b[_{\mathbb{T}},\mathbb{R}^m)} &\leq \|u^R_{\varepsilon_k,\Pi}(\cdot,\beta) - u^R_{\varepsilon_k}\|_{\mathrm{L}^1_{\mathbb{T}}([a,b[_{\mathbb{T}},\mathbb{R}^m)} + \|u^R_{\varepsilon_k} - u^*\|_{\mathrm{L}^1_{\mathbb{T}}([a,b[_{\mathbb{T}},\mathbb{R}^m)} \\ &\leq 2R\beta + \sqrt{\varepsilon_k}. \end{aligned}$$

Therefore $(u^R_{\varepsilon_k,\Pi}(\cdot,\beta), \bar{q}^R_{a,\varepsilon_k}) \in \mathrm{E}^R_\Omega$ for $\beta \in \mathcal{V}^b_s$ sufficiently small. It then follows from (III.3.24) that

$$-\sqrt{\varepsilon_k}\|u^R_{\varepsilon_k,\Pi}(\cdot,\beta) - u^R_{\varepsilon_k}\|_{\mathrm{L}^1_{\mathbb{T}}([a,b[_{\mathbb{T}},\mathbb{R}^m)} \leq J^R_k(u^R_{\varepsilon_k,\Pi}(\cdot,\beta), \bar{q}^R_{a,\varepsilon_k}) - J^R_k(u^R_{\varepsilon_k}, \bar{q}^R_{a,\varepsilon_k}),$$

and thus

$$-2R\sqrt{\varepsilon_k} \leq \frac{J^R_k(u^R_{\varepsilon_k,\Pi}(\cdot,\beta), \bar{q}^R_{a,\varepsilon_k})^2 - J^R_k(u^R_{\varepsilon_k}, \bar{q}^R_{a,\varepsilon_k})^2}{\beta(J^R_k(u^R_{\varepsilon_k,\Pi}(\cdot,\beta), \bar{q}^R_{a,\varepsilon_k}) + J^R_k(u^R_{\varepsilon_k}, \bar{q}^R_{a,\varepsilon_k}))}.$$

Using Proposition (III.2), since $\bar{g}$ does not depend on x^0_2, we infer that

$$\begin{aligned} \lim_{\beta\to 0} & \frac{J^R_k(u^R_{\varepsilon_k,\Pi}(\cdot,\beta), \bar{q}^R_{a,\varepsilon_k})^2 - J^R_k(u^R_{\varepsilon_k}, \bar{q}^R_{a,\varepsilon_k})^2}{\beta} \\ &= 2\max(q^0(b,u^R_{\varepsilon_k},\bar{q}^R_{a,\varepsilon_k}) - q^{0*}(b) + \varepsilon_k, 0)\, w^0_\Pi(b,u^R_{\varepsilon_k},\bar{q}^R_{a,\varepsilon_k}) \\ &\quad + 2\Big\langle \bar{g}(\bar{q}^R_{a,\varepsilon_k}, \bar{q}(b,u^R_{\varepsilon_k},\bar{q}^R_{a,\varepsilon_k})) - \mathrm{P_S}(\bar{g}(\bar{q}^R_{a,\varepsilon_k}, \bar{q}(b,u^R_{\varepsilon_k},\bar{q}^R_{a,\varepsilon_k}))), \\ &\qquad \frac{\partial \bar{g}}{\partial x_2}(\bar{q}^R_{a,\varepsilon_k}, \bar{q}(b,u^R_{\varepsilon_k},\bar{q}^R_{a,\varepsilon_k})) \times \omega_\Pi(b,u^R_{\varepsilon_k},\bar{q}^R_{a,\varepsilon_k}) \Big\rangle_{\mathbb{R}^j}. \end{aligned}$$

Since $J^R_k(u^R_{\varepsilon_k,\Pi}(\cdot,\beta), \bar{q}^R_{a,\varepsilon_k})$ converges to $J^R_k(u^R_{\varepsilon_k}, \bar{q}^R_{a,\varepsilon_k})$ as α tends to 0, using (III.3.25) and (III.3.26) it follows that

$$-2R\sqrt{\varepsilon_k} \leq -\psi^{0R}_{\varepsilon_k}\omega^0_\Pi(b,u^R_{\varepsilon_k},\bar{q}^R_{a,\varepsilon_k}) - \Big\langle \Big(\frac{\partial \bar{g}}{\partial x_2}(\bar{q}^R_{a,\varepsilon_k}, \bar{q}(b,u^R_{\varepsilon_k},\bar{q}^R_{a,\varepsilon_k}))\Big)^{\mathrm{T}} \times \psi^R_{\varepsilon_k}, \omega_\Pi(b,u^R_{\varepsilon_k},\bar{q}^R_{a,\varepsilon_k}) \Big\rangle_{\mathbb{R}^n}.$$

By letting k tend to $+\infty$, and using Lemma III.8, the lemma follows.

B.2.2 Proof of Lemma III.16

For every $k \in \mathbb{N}$ and every $\gamma \geq 0$, one has

$$\|\bar{q}^R_{a,\varepsilon_k} + \gamma\bar{q}_a - \bar{q}^*_a\|_{\mathbb{R}^n} \leq \gamma\|\bar{q}_a\|_{\mathbb{R}^n} + \|\bar{q}^R_{a,\varepsilon_k} - \bar{q}^*_a\|_{\mathbb{R}^n} \leq \gamma\|\bar{q}_a\|_{\mathbb{R}^n} + \sqrt{\varepsilon_k}.$$

Therefore $(u^R_{\varepsilon_k}, \bar{q}^R_{a,\varepsilon_k} + \gamma\bar{q}_a) \in \mathrm{E}^R_\Omega$ for $\gamma \geq 0$ sufficiently small. It then follows from (III.3.24) that

$$-\sqrt{\varepsilon_k}\|\bar{q}^R_{a,\varepsilon_k} + \gamma\bar{q}_a - \bar{q}^R_{a,\varepsilon_k}\|_{\mathbb{R}^n} \leq J^R_k(u^R_{\varepsilon_k}, \bar{q}^R_{a,\varepsilon_k} + \gamma\bar{q}_a) - J^R_k(u^R_{\varepsilon_k}, \bar{q}^R_{a,\varepsilon_k}),$$

and thus

$$-\sqrt{\varepsilon_k}\|\bar{q}_a\|_{\mathbb{R}^n} \leq \frac{J_k^R(u_{\varepsilon_k}^R, \bar{q}_{a,\varepsilon_k}^R + \gamma\bar{q}_a)^2 - J_k^R(u_{\varepsilon_k}^R, \bar{q}_{a,\varepsilon_k}^R)^2}{\gamma(J_k^R(u_{\varepsilon_k}^R, \bar{q}_{a,\varepsilon_k}^R + \gamma\bar{q}_a) + J_k^R(u_{\varepsilon_k}^R, \bar{q}_{a,\varepsilon_k}^R))}.$$

Using Proposition (III.3), since $\bar{g}$ does not depend on x_1^0 and x_2^0, we infer that

$$\begin{aligned}\lim_{\gamma\to 0} \frac{J_k^R(u_{\varepsilon_k}^R, \bar{q}_{a,\varepsilon_k}^R + \gamma\bar{q}_a)^2 - J_k^R(u_{\varepsilon_k}^R, \bar{q}_{a,\varepsilon_k}^R)^2}{\gamma}&\\ = 2\max(q^0(b, u_{\varepsilon_k}^R, \bar{q}_{a,\varepsilon_k}^R) - q^{0*}(b) + \varepsilon_k, 0)\, \omega_{\bar{q}_a}^0(b, u_{\varepsilon_k}^R, \bar{q}_{a,\varepsilon_k}^R)&\\ + 2\Big\langle \bar{g}(\bar{q}_{a,\varepsilon_k}^R, \bar{q}(b, u_{\varepsilon_k}^R, \bar{q}_{a,\varepsilon_k}^R)) - \mathrm{P_S}(\bar{g}(\bar{q}_{a,\varepsilon_k}^R, \bar{q}(b, u_{\varepsilon_k}^R, \bar{q}_{a,\varepsilon_k}^R))), \frac{\partial \bar{g}}{\partial x_1}(\bar{q}_{a,\varepsilon_k}^R, \bar{q}(b, u_{\varepsilon_k}^R, \bar{q}_{a,\varepsilon_k}^R)) \times q_a \Big\rangle_{\mathbb{R}^j}&\\ + 2\Big\langle \bar{g}(\bar{q}_{a,\varepsilon_k}^R, \bar{q}(b, u_{\varepsilon_k}^R, \bar{q}_{a,\varepsilon_k}^R)) - \mathrm{P_S}(\bar{g}(\bar{q}_{a,\varepsilon_k}^R, \bar{q}(b, u_{\varepsilon_k}^R, \bar{q}_{a,\varepsilon_k}^R))),&\\ \frac{\partial \bar{g}}{\partial x_2}(\bar{q}_{a,\varepsilon_k}^R, \bar{q}(b, u_{\varepsilon_k}^R, \bar{q}_{a,\varepsilon_k}^R)) \times \omega_{\bar{q}_a}(b, u_{\varepsilon_k}^R, \bar{q}_{a,\varepsilon_k}^R) \Big\rangle_{\mathbb{R}^j}&.\end{aligned}$$

Since $J_k^R(u_{\varepsilon_k}^R, \bar{q}_{a,\varepsilon_k}^R + \gamma\bar{q}_a)$ converges to $J_k^R(u_{\varepsilon_k}^R, \bar{q}_{a,\varepsilon_k}^R)$ as γ tends to 0, using (III.3.25) and (III.3.26) it follows that

$$\begin{aligned}-\sqrt{\varepsilon_k}\|\bar{q}_a\| \leq -\psi_{\varepsilon_k}^{0R}\omega_{\bar{q}_a}^0(b, u_{\varepsilon_k}^R, \bar{q}_{a,\varepsilon_k}^R) - \Big\langle \Big(\frac{\partial \bar{g}}{\partial x_1}(\bar{q}_{a,\varepsilon_k}^R, \bar{q}(b, u_{\varepsilon_k}^R, \bar{q}_{a,\varepsilon_k}^R))\Big)^{\mathrm{T}} \times \psi_{\varepsilon_k}^R, q_a \Big\rangle_{\mathbb{R}^n}&\\ - \Big\langle \Big(\frac{\partial \bar{g}}{\partial x_2}(\bar{q}_{a,\varepsilon_k}^R, \bar{q}(b, u_{\varepsilon_k}^R, \bar{q}_{a,\varepsilon_k}^R))\Big)^{\mathrm{T}} \times \psi_{\varepsilon_k}^R, \omega_{\bar{q}_a}(b, u_{\varepsilon_k}^R, \bar{q}_{a,\varepsilon_k}^R) \Big\rangle_{\mathbb{R}^n}&.\end{aligned}$$

By letting k tend to $+\infty$, and using Lemma III.11, the lemma follows.

Annexe C : Théorie de Cauchy-Lipschitz pour des problèmes de Cauchy avec dérivée fractionnaire de Caputo

Cette annexe complète certains aspects de la théorie de Cauchy-Lipschitz pour des systèmes généraux non linéaires avec dérivée fractionnaire de Caputo nécessaires à la démonstration au Chapitre VII d'un Principe du Maximum de Pontryagin Faible en théorie du contrôle optimal fractionnaire.

Nous introduisons la notion de solution faible d'un problème de Cauchy avec dérivée fractionnaire de Caputo puis la notion de solution faible maximale. Par la suite, nous démontrons un théorème de type Cauchy-Lipschitz assurant l'existence et l'unicité de la solution faible maximale pour un problème de Cauchy avec dérivée fractionnaire de Caputo donné. Comme dans le cas classique, ce théorème ne sera valable qu'à partir de conditions adéquates comme la continuité localement Lipschitzienne de la dynamique. Pour finir, nous complétons cette étude par des résultats relatifs au comportement des solutions faibles maximales au voisinage de leurs points terminaux.

C.1 Introduction

In this introduction, we first give a brief overview of the existing results on existence and uniqueness of solutions for fractional differential equations widely inspired by the survey [142] and by [141, Chapter 3]. Next, we devote a paragraph to detail the framework and the contributions of the present appendix.

Brief overview on the existing literature. Most of the investigations of this field involve fractional differential equations with the Riemann-Liouville fractional derivative D^{α}_{a+} considered on a finite interval $[a,b]$ with $b>a$, with general fractional order $\alpha>0$. Nevertheless, for sake of simplicity of this appendix, we only consider the case $0<\alpha<1$. Hence, in the literature, the "model" nonlinear differential equation of fractional order $0<\alpha<1$ on the finite interval $[a,b]$ has the form

$$\begin{aligned} \mathrm{D}^{\alpha}_{a+}[q](t) &= f(q(t),t), \\ \lim_{t\to a^+} \mathrm{I}^{1-\alpha}_{a+}[q](t) &= q_a. \end{aligned} \tag{C.1}$$

Essentially (and as in the classical theory), the investigations of the above fractional Cauchy problem are based on the integral formulation

$$q(t) = \frac{q_a}{(t-a)^{1-\alpha}\Gamma(\alpha)} + \mathrm{I}^{\alpha}_{a+}[f(q,\cdot)](t). \tag{C.2}$$

The first paper treating on this topic is due to E. Pitcher and W. Sewell [184] in 1938. They investigate the case where $q_a=0$ and f is a continuous function satisfying a boundedness and a global Lipschitz continuity assumption. Despite that E. Pitcher and W. Sewell present the original idea of reducing the differential problem to an integral one, their main result, providing the existence of a global continuous solution of the integral equation (C.2), is based on an erroneous proof. However, under the same kind of assumptions on f (but without

$q_a = 0$), M. Al-Bassam [12] uses the method of successive approximations in 1965 in order to well establish the existence of a global continuous solution of the integral equation (C.2). Nevertheless, the hypotheses on f (in particular the boundedness) are very strong and avoid to apply this result in the simple example $f(x,t) = x$. M. Al-Bassam seems to be the first author to indicate that the method of contractive mapping could be applied to prove the uniqueness of the solution. In 1996, D. Delbosco and L. Rodino [76] consider an initial condition of type $q(a) = q_a$ instead of $\lim_{t\to a^+} \mathrm{I}^{1-\alpha}_{a+}[q](t) = q_a$. Under some continuity assumption on f and using a fixed point theorem, they prove that the fractional Cauchy problem admits at least a local continuous solution. This result corresponds to a fractional version of the classical Peano theorem. Under a global Lipschitz continuity assumption, they moreover prove that the solution is unique and global. Note that N. Hayek *et al* [104] apply the same argument and obtain the same last result for the more usual initial condition $\lim_{t\to a^+} \mathrm{I}^{1-\alpha}_{a+}[q](t) = q_a$. Recall that A. Kilbas *et al* establish existence and uniqueness results in spaces of integrable functions [137] and in weighted spaces of continuous functions [136]. Actually, the subject is widely treated in several directions. We can cite [77, 168] for other examples of studies.

As mentioned in [141, Chapter 3], the differential equations involving Caputo fractional derivative have not been studied extensively. In a first period, only particular cases have been investigated in the view of giving explicitly the exact solutions, see *e.g.* the works of R. Gorenflo *et al* in [97, 98, 151]. In 2002, K. Diethelm and N. Ford [78] study the "model" nonlinear differential equation with Caputo fractional derivative of order $0 < \alpha < 1$ on the finite interval $[a, b]$ given by

$$\begin{aligned} {}_{c}\mathrm{D}^{\alpha}_{a+}[q](t) &= f(q(t), t), \\ q(a) &= q_a. \end{aligned} \tag{C.3}$$

They prove the existence and uniqueness of a local continuous solution under the assumption of continuity and local Lipschitz continuity of f. They also investigate the dependence of the solution with respect to the initial condition and to the function f. A. Kilbas and S. Marzan [138, 139] also study the above fractional Cauchy problem via its integral formulation

$$q(t) = q_a + \mathrm{I}^{\alpha}_{a+}[f(q, \cdot)](t) \tag{C.4}$$

and prove existence and uniqueness of a global continuous solution in the case of continuous and global Lipschitz continuous function f. Furthermore, Kilbas *et al* also investigate the issue of boundary condition in any $t \in [a, b]$ (*i.e.* non necessary in $t = a$), see [141].

Numerous studies have also been devoted to existence and uniqueness results for differential equations involving other notions of fractional operator. For example, we can cite the study [140] with Hadamard fractional derivatives.

Contributions of the present appendix in the Caputo case. To the best of our knowledge, there does not exist a general Cauchy-Lipschitz theory with Caputo fractional derivative that is fully complete in order to be applied to problems arising for example in fractional control theory [1] (*e.g.* with non continuous controls). This appendix is thus devoted to fill an existing gap of the literature, and to provide a general Cauchy-Lipschitz theory with Caputo fractional derivative generalizing the basic notions and results of the classical continuous theory surveyed *e.g.* in [65, 123]. Namely, we study the fractional Cauchy problem (C.3) in the following framework:

- f is a general Carathéodory function (non necessary continuous in its second variable);
- the fractional Cauchy problem is posed on a general interval with lower bound a (*i.e.* the interval is not necessarily bounded);
- the solutions take their values in a non empty open subset Ω of $\mathbb{R}^n$.

Hence, we first introduce the notion of a *weak solution* that corresponds to continuous solution of the integral formulation (C.4). Due to the non regularity of f, we cannot ensure the equivalence between (C.3) and (C.4) since we only deal with Hölderian (and non necessary absolutely continuous) solutions. We refer to the remark made after Definition C.4 for more details. Then, we define the concept of *extension* of a weak solution, and of *maximal* and *global* weak solution. We establish a general version of the Cauchy-Lipschitz theorem (existence

1. Actually, the present study is motivated by the needs of completing the existing results on fractional Cauchy-Lipschitz theory, in order to investigate general non linear control systems with Caputo fractional derivative, and more precisely to derive a fractional version of the Weak Pontryagin Maximum Principle in optimal control in Chapter VII.

and uniqueness of the maximal weak solution, also referred to as Picard-Lindelöf theorem) under local Lipschitz continuity assumption. We also investigate the globality feature of the maximal weak solution. Finally, we discuss some related issues like the behavior of maximal weak solutions at terminal points. Precisely, we state that the maximal weak solution must go out of any compact of Ω near its terminal point whenever it is not global.

Organization of this appendix. In Section C.2, we define the notions of a weak solution, of an extension of a weak solution, of a maximal and a global weak solution for general Cauchy problems with Caputo fractional derivative. Under suitable assumptions on the dynamics, we establish a Cauchy-Lipschitz theorem and then investigate the behavior of the maximal weak solution at its terminal points. Finally, Section C.3 is devoted to the detailed proofs of the previous results.

C.2 General study and main results

Notations: In the whole study, $a \in \mathbb{R}$, $n \in \mathbb{N}^*$ and $0 < \alpha < 1$. In the sequel, the notation $\| \cdot \|$ stands for the Euclidean norm of $\mathbb{R}^n$. For every $x \in \mathbb{R}^n$ and every $R \geq 0$, the notation $\overline{B}(x, R)$ stands for the closed ball of $\mathbb{R}^n$ centered at x and with radius R.

Throughout this section, we consider the general fractional Cauchy problem

$$(\mathrm{CP}^\alpha) \qquad \begin{aligned} {}_{\mathrm{c}}\mathrm{D}^\alpha_{a+}[q](t) &= f(q(t), t), \\ q(a) &= q_a, \end{aligned}$$

with $q_a \in \Omega$, where Ω is a non empty open subset of $\mathbb{R}^n$, $f : \Omega \times I_f \longrightarrow \mathbb{R}^n, (x, t) \longmapsto f(x, t)$ is a Carathéodory function and $I_f \subset [a, +\infty[$ is an interval such that $a = \min I_f$ and $I_f \backslash \{a\} \neq \emptyset$. The notation $\mathcal{K}$ stands for the set of compact subsets of Ω.

C.2.1 Preliminaries and properties of function f

As in the classical Cauchy-Lipschitz theory, the existence and uniqueness of a solution of (CP^α) requires some assumptions on f, whence the following series of definitions.

Definition C.1. The function f is said to be *locally bounded on* $\Omega \times I_f$ if, for every $K \in \mathcal{K}$, for all $(c, d) \in I_f^2$ such that $c < d$, there exists $M \geq 0$ such that

$$\| f(x, t) \| \leq M, \tag{H_∞}$$

for every $x \in K$ and for almost every $t \in [c, d]$. In what follows this property will be referred to as (H_∞).

Definition C.2. The function f is said to be *locally Lipschitz continuous with respect to its first variable* if, for every $\overline{x} \in \Omega$ and every $\overline{t} \in I_f$, there exist $R > 0$, $\delta > 0$ and $L \geq 0$ such that $\overline{B}(\overline{x}, R) \subset \Omega$ and such that

$$\| f(x_1, t) - f(x_2, t) \| \leq L \| x_1 - x_2 \|, \tag{$\mathrm{H}^{\mathrm{loc}}_{\mathrm{Lip}}$}$$

for all $x_1, x_2 \in \overline{B}(\overline{x}, R)$ and for almost every $t \in [\overline{t} - \delta, \overline{t} + \delta] \cap I_f$. In what follows this property will be referred to as $(\mathrm{H}^{\mathrm{loc}}_{\mathrm{Lip}})$.

Assumption (H_∞) is instrumental to define the notion of *weak solution* of (CP^α) via an integral formulation, see Definition C.4. In view of investigating global weak solutions, the following definition is also useful.

Definition C.3. The function f is said to be *globally Lipschitz continuous in its first variable* if for all $(c, d) \in I_f^2$ such that $c < d$, there exists $L \geq 0$ such that

$$\| f(x_1, t) - f(x_2, t) \| \leq L \| x_1 - x_2 \|, \tag{$\mathrm{H}^{\mathrm{glob}}_{\mathrm{Lip}}$}$$

for all $x_1, x_2 \in \Omega$ and for almost every $t \in [c, d]$.

C.2.2 Definition of a maximal weak solution

We first define the notion of a weak solution of (CP^α) on an interval $[a,b]$ with $b \in I_f \backslash \{a\}$.

Definition C.4. Let $b \in I_f \backslash \{a\}$ and let $q : [a,b] \longrightarrow \Omega$. The couple $(q,[a,b])$ is said to be a *weak solution* of (CP^α) if $q \in \mathrm{C}([a,b],\mathbb{R}^n)$ and if

$$\forall t \in [a,b],\ q(t) = q_a + \mathrm{I}^\alpha_{a+}[f(q,\cdot)](t). \tag{C.5}$$

From Assumption (H_∞), note that for every $b \in I_f \backslash \{a\}$ and every $q \in \mathrm{C}([a,b],\mathbb{R}^n)$, we have $f(q,\cdot) \in \mathrm{L}^\infty([a,b],\mathbb{R}^n)$. As a consequence, from Proposition V.2, a weak solution $(q,[a,b])$ of (CP^α) automatically belongs to $\mathrm{H}^\alpha([a,b],\mathbb{R}^n)$ and satisfies $q(a) = q_a$. The word *weak* refers to the integral formulation (C.5) of (CP^α). However, note that the differential equation ${}_{\mathrm{c}}\mathrm{D}^\alpha_{a+}[q] = f(q,\cdot)$ is satisfied almost everywhere on $[a,b]$ by every absolutely continuous weak solution $(q,[a,b])$. To prove this, we refer to Remark V.6 and Corollary V.1.

Note that, if $(q,[a,b])$ is a weak solution of (CP^α), then $(q,[a,b'])$ is as well a weak solution of (CP^α) for all $b' \in]a,b]$. Then, in view of defining the notion of a weak solution of (CP^α) on more general intervals, we set:

$$\mathbb{I}^f_{a+} = \{I \subset I_f \text{ interval such that } a = \min I,\ I\backslash\{a\} \neq \emptyset\}. \tag{C.6}$$

The set $\mathbb{I}^f_{a+}$ is the set of potential intervals for a weak solution of (CP^α).

Definition C.5. Let $I \in \mathbb{I}^f_{a+}$ and let $q : I \longrightarrow \Omega$. The couple (q,I) is said to be a weak solution of (CP^α) if $(q,[a,b])$ is a weak solution of (CP^α) for all $b \in I\backslash\{a\}$.

Finally, we define the concept of a maximal weak solution.

Definition C.6. Let (q,I) and (q_1,I^1) be two weak solutions of (CP^α). The solution (q_1,I^1) is said to be an extension of the solution (q,I) if $I \subset I^1$ and $q_1 = q$ on I. A solution (q,I) of (CP^α) is said to be maximal if, for every extension (q_1,I^1) of (q,I), there holds $I^1 = I$. Finally, a solution (q,I) of (CP^α) is said to be global if $I = I_f$.

Note that a global weak solution of (CP^α) is automatically maximal.

C.2.3 Main results

Recall that we consider the general fractional Cauchy problem

$$(\mathrm{CP}^\alpha) \qquad \begin{aligned} {}_{\mathrm{c}}\mathrm{D}^\alpha_{a+}[q](t) &= f(q(t),t), \\ q(a) &= q_a, \end{aligned}$$

with $q_a \in \Omega$, where Ω is a non empty open subset of $\mathbb{R}^n$, $f : \Omega \times I_f \longrightarrow \mathbb{R}^n, (x,t) \longmapsto f(x,t)$ is a Carathéodory function and $I_f \subset [a,+\infty[$ is an interval such that $a = \min I_f$ and $I_f\backslash\{a\} \neq \emptyset$. We have the following general Cauchy-Lipschitz result.

Theorem C.1. *We assume that f satisfies (H_∞), that is, f is locally bounded on $\Omega \times I_f$, and that f satisfies $(\mathrm{H}^{\mathrm{loc}}_{\mathrm{Lip}})$, that is, f is locally Lipschitz continuous with respect to its first variable. Then, the Cauchy problem (CP^α) has a unique maximal weak solution (q,I). Moreover, (q,I) is the maximal extension of any other weak solution of (CP^α).*

This theorem is proved in Section C.3.1. The following result gives information on the behavior of a maximal solution at its terminal points.

Theorem C.2. *Under the assumptions of Theorem C.1, let (q,I) be the maximal weak solution of the Cauchy problem (CP^α). Then either $I = I_f$, that is, the maximal weak solution (q,I) is global, or the maximal weak solution is not global and then $I = [a,b[$ with $b \in I_f$ and moreover, for every $K \in \mathcal{K}$, there exists $t \in I$ (close to b) such that $q(t) \in \Omega\backslash K$.*

This theorem is proved in Section C.3.2. It states that the maximal weak solution must go out of any compact of Ω near its terminal point whenever it is not global.

The following last result states that, under global Lipschitz continuity assumption, the maximal weak solution is global.

Theorem C.3. *We assume that $\Omega = \mathbb{R}^n$, f satisfies (H_∞), that is, f is locally bounded on $\mathbb{R}^n \times I_f$ and that f satisfies $(\mathrm{H}^{\mathrm{glob}}_{\mathrm{Lip}})$, that is, f is globally Lipschitz continuous in its first variable. Then, the Cauchy problem (CP^α) has a unique maximal solution (q, I), which is moreover global i.e. $I = I_f$.*

The proof is done in Section C.3.3.

Remark C.1. As an application of Theorem C.3, we recover the well known fact that, in the linear case

$$ {}_{\mathrm{c}}\mathrm{D}^\alpha_{a+}[q](t) = h(t) \times q(t), $$

where $h : I_f \longrightarrow \mathbb{R}^{n\times n}$ such that $h \in \mathrm{L}^\infty([c,d], \mathbb{R}^{n\times n})$ for all $(c,d) \in I_f^2$ with $c < d$, solutions are global.

We conclude this section with a simple example showing the sharpness of the assumption $(\mathrm{H}^{\mathrm{loc}}_{\mathrm{Lip}})$ made in Theorem C.1.

Example C.1. Let $a = 0$, $n = 1$, $\Omega = \mathbb{R}$, $q_0 = 0$, $I_f = [0, +\infty[$ and $f : \mathbb{R} \times I_f \longrightarrow \mathbb{R}$ be defined by

$$ f(x,t) = \frac{\alpha}{1-\beta} \, \frac{\Gamma(\frac{\alpha}{1-\beta})}{\Gamma(\frac{\alpha}{1-\beta} + 1 - \alpha)} |x|^\beta, \tag{C.7} $$

where β satisfies $0 < 1-\beta \leq \alpha < 1$. The function f does not satisfy $(\mathrm{H}^{\mathrm{loc}}_{\mathrm{Lip}})$. The corresponding Cauchy problem (CP^α) has two global weak solutions q_1 and q_2 given by $q_1(t) = 0$ and $q_2(t) = t^{\alpha/(1-\beta)}$, for every $t \in I_f$.

This example shows that, in the absence of Assumption $(\mathrm{H}^{\mathrm{loc}}_{\mathrm{Lip}})$, the uniqueness of the maximal weak solution is not guaranteed.

C.3 Proofs of the results

In this section, we detail the proofs of Theorems C.1, C.2 and C.3.

C.3.1 Proof of Theorem C.1

In this section, we assume that the hypotheses of Theorem C.1 are satisfied. We first prove the following lemma using the classical Zorn lemma.

Lemma C.1. *Every weak solution of (CP^α) can be extended to a maximal weak solution.*

Proof. Let (q, I) be a weak solution of (CP^α). Let us define the non empty set $\mathscr{F}$ of extensions of (q, I). The set $\mathscr{F}$ is ordered by

$$ (q_1, I^1) \leq (q_2, I^2) \text{ if and only if } (q_2, I^2) \text{ is an extension of } (q_1, I^1). $$

Let us prove that $\mathscr{F}$ is inductive. Let $\mathscr{G} = \{(q_p, I^p)\}_{p\in\mathscr{P}}$ be a non empty totally ordered subset of $\mathscr{F}$. Let us prove that $\mathscr{G}$ admits an upper bound.

Let us define $\overline{I} = \cup_{p\in\mathscr{P}} I^p \in \mathbb{I}^f_{a+}$. For every $t \in \overline{I}$, there exists $p \in \mathscr{P}$ such that $t \in I^p$ and, since $\mathscr{G}$ is totally ordered, if $t \in I^{p_1} \cap I^{p_2}$ then $q_{p_1}(t) = q_{p_2}(t)$. Consequently, we can define $\overline{q}$ by

$$ \forall t \in \overline{I}, \ \overline{q}(t) = q_p(t) \in \Omega \text{ where } t \in I^p. \tag{C.8} $$

Our aim is to prove that $(\overline{q}, \overline{I})$ is a weak solution of (CP^α). Let $b \in \overline{I}\backslash\{a\}$, there exists $p \in \mathscr{P}$ such that $[a,b] \subset I^p$ and $\overline{q} = q_p$ on $[a,b]$. Since (q_p, I^p) is a weak solution of (CP^α), this holds true as well for $\overline{q}$ on $[a,b]$. Finally, since this last sentence is true for all $b \in \overline{I}\backslash\{a\}$, we infer that $(\overline{q}, \overline{I})$ is a weak solution of (CP^α). Since

$(\overline{q}, \overline{I})$ is obviously an extension of any element of $\mathscr{G}$, we obtain that $\mathscr{G}$ admits an upper bound and then, $\mathscr{F}$ is inductive.

Finally, $\mathscr{F}$ is a non empty ordered inductive set and consequently, from the classical Zorn lemma, admits a maximal element. The proof is complete. □

Proposition C.1 (Existence of a local solution)**.** *There exist* $b \in I_f \backslash \{a\}$ *and* $q : [a, b] \longrightarrow \Omega$ *such that* $(q, [a, b])$ *is a weak solution of* (CP^α)*.*

Proof. Let R, δ and L associated with q_a and a in $(\mathrm{H}^{\mathrm{loc}}_{\mathrm{Lip}})$. We assume that δ is sufficiently small in order to have $[a, a+\delta] \subset I_f$. Let M associated with $\overline{B}(q_a, R) \in \mathcal{K}$ and $[a, a+\delta]$ in (H_∞). Consider $0 < \delta_1 \leq \delta$ and $b = a + \delta_1$ such that δ_1 is sufficiently small in order to have $\delta_1^\alpha M/\Gamma(1+\alpha) \leq R$ and $\delta_1^\alpha L/\Gamma(1+\alpha) < 1$. Then, we construct the $\delta_1^\alpha L/\Gamma(1+\alpha)$-contraction map with respect to the norm $\| \cdot \|_\infty$

$$F : \begin{array}{ccc} \mathrm{C}([a,b], \overline{B}(q_a, R)) & \longrightarrow & \mathrm{C}([a,b], \overline{B}(q_a, R)) \\ q & \longmapsto & F(q) \end{array}$$

with

$$F(q) : \begin{array}{ccc} [a,b] & \longrightarrow & \overline{B}(q_a, R) \\ t & \longmapsto & q_a + \mathrm{I}^\alpha_{a+}[f(q, \cdot)](t). \end{array}$$

It follows from the Banach fixed point theorem that F has a unique fixed point denoted by q defined on $[a, b]$. We have thus obtained a weak solution $(q, [a, b])$ of (CP^α). □

From Lemma C.1, we can extend the solution given in Proposition C.1 and we obtain the existence of a maximal solution. The following result proves that it is unique. Note that the non locality of the fractional operator I^α_{a+} is responsible for modifications in the following proof with respect to the classical case seen in Proposition A.2.

Proposition C.2 (Local uniqueness of a solution)**.** *Let* (q_1, I^1) *and* (q_2, I^2) *be two weak solutions of* (CP^α)*. Then,* $q_1 = q_2$ *on* $I^1 \cap I^2$*.*

Proof. In this proof, we assume that $I^1 \subset I^2$ and we prove that $q_2 = q_1$ on I^1. It is sufficient to prove that $q_2 = q_1$ on $[a, b]$ for all $b \in I^1 \backslash \{a\}$. Then, let $b \in I^1 \backslash \{a\}$. We set

$$A = \{t \in [a, b],\ q_1(t) \neq q_2(t)\}.$$

Let us prove by contradiction that $A = \emptyset$. Assume that $A \neq \emptyset$ and let $\overline{t} = \inf A$. Note that $\overline{t} \in [a, b]$ and that $q_2 = q_1$ on $[a, \overline{t}[$. In order to raise a contradiction, we first derive the three following facts.

1. *Fact 1:* $\overline{t} > a$*.* As we have seen in the previous proof, (CP^α) admits a unique weak solution defined in a right neighbourhood of a. As a consequence, $q_2 = q_1$ in this right neighbourhood and thus, $\overline{t} > a$.
2. *Fact 2:* $q_2(\overline{t}) = q_1(\overline{t})$*.* This fact derives from the continuity of q_1 and q_2 and from the equality $q_2 = q_1$ on $[a, \overline{t}[\neq \emptyset$.
3. *Fact 3:* $\overline{t} < b$*.* This fact derives from $q_2 = q_1$ on $[a, \overline{t}]$ and from $A \neq \emptyset$.

Let K_1 be the image of q_1 on $[a, \overline{t}]$. In particular, $K_1 \in \mathcal{K}$. Let M_1 associated with K_1 and $[a, \overline{t}]$ in (H_∞). Let us denote by $\overline{x} = q_1(\overline{t}) = q_2(\overline{t}) \in \Omega$. Let R, δ and L associated with $\overline{t}$ and $\overline{x}$ in $(\mathrm{H}^{\mathrm{loc}}_{\mathrm{Lip}})$. We assume that δ is sufficiently small in order to have $\overline{t} + \delta \in]\overline{t}, b]$. Let M associated with $\overline{B}(\overline{x}, R) \in \mathcal{K}$ and $[\overline{t}, \overline{t} + \delta]$ in (H_∞). Consider $0 < \delta_1 \leq \delta$ and $d = \overline{t} + \delta_1 \in]\overline{t}, b]$ such that δ_1 is sufficiently small in order to have $\delta_1^\alpha (M_1 + M)/\Gamma(1+\alpha) \leq R$, $\delta_1^\alpha L/\Gamma(1+\alpha) < 1$ and q_1, $q_2 \in \mathrm{C}([\overline{t}, d], \overline{B}(\overline{x}, R))$. Finally, we introduce the following complete set with respect to $\| \cdot \|_\infty$:

$$\mathrm{C}^\star = \{q \in \mathrm{C}([a, d], \mathbb{R}^n),\ q = q_1 \text{ on } [a, \overline{t}],\ q \in \mathrm{C}([\overline{t}, d], \overline{B}(\overline{x}, R))\}. \tag{C.9}$$

In particular, we have q_1, $q_2 \in \mathrm{C}^\star$. Moreover, since q_1 and q_2 are weak solutions of (CP^α) on $[a, d]$, they are in particular fixed points of the $\delta_1^\alpha L/\Gamma(1+\alpha)$-contraction map

$$F : \begin{array}{ccc} \mathrm{C}^\star & \longrightarrow & \mathrm{C}^\star \\ q & \longmapsto & F(q) \end{array}$$

with

$$F(q): \begin{array}{ccc} [a,d] & \longrightarrow & \mathbb{R}^n \\ t & \longmapsto & q_a + \mathrm{I}^{\alpha}_{a+}[f(q,\cdot)](t). \end{array}$$

Since F has a unique fixed point from the Banach fixed point theorem, we conclude that $q_1 = q_2$ on $[\bar{t}, d]$, and this is a contradiction since $\bar{t} = \inf A$. Consequently $A = \emptyset$ and the proof is complete. □

Theorem C.1 follows from Lemma C.1, Propositions C.1 and C.2.

C.3.2 Proof of Theorem C.2

In this section, we assume that the hypotheses of Theorem C.1 are satisfied. Note that, due to the non locality of the fractional operator I^{α}_{a+}, the following proof is modified with respect to its classical counterpart seen in Proposition A.3.

Proposition C.3. *Let (q,I) be the maximal weak solution of* (CP$^\alpha$)*. Then either $I = I_f$, that is, the weak solution (q,I) is global, or $I = [a,b[$ with $b \in I_f$.*

Proof. We prove that if $I = [a,b]$, then $b = \max I_f$ (and thus $I = I_f$). By contradiction, assume that $I = [a,b]$ with $b < \sup I_f$. Let K_1 be the image of q on $[a,b]$. In particular, $K_1 \in \mathcal{K}$. Let M_1 associated with K_1 and $[a,b]$ in (H_∞). Let us denote by $\overline{x} = q(b) \in \Omega$. Let R, δ and L associated with b and $\overline{x}$ in $(\mathrm{H}^{\mathrm{loc}}_{\mathrm{Lip}})$. We assume that δ is sufficiently small in order to have $b+\delta \in I_f$. Let M associated with $\overline{B}(\overline{x},R) \in \mathcal{K}$ and $[b,b+\delta]$ in (H_∞). Consider $0 < \delta_1 \leq \delta$ and $d = b + \delta_1 \in I_f$ such that δ_1 is sufficiently small in order to have $\delta_1^\alpha (M_1 + M)/\Gamma(1+\alpha) \leq R$, $\delta_1^\alpha L/\Gamma(1+\alpha) < 1$. Finally, we introduce the following complete set with respect to $\|\cdot\|_\infty$:

$$\mathrm{C}^{\star\star} = \{q_1 \in \mathrm{C}([a,d],\mathbb{R}^n),\ q_1 = q \text{ on } [a,b],\ q_1 \in \mathrm{C}([b,d],\overline{B}(\overline{x},R))\}. \tag{C.10}$$

Then, we consider the $\delta_1^\alpha L/\Gamma(1+\alpha)$-contraction map

$$F: \begin{array}{ccc} \mathrm{C}^{\star\star} & \longrightarrow & \mathrm{C}^{\star\star} \\ q_1 & \longmapsto & F(q_1) \end{array}$$

with

$$F(q_1): \begin{array}{ccc} [a,d] & \longrightarrow & \mathbb{R}^n \\ t & \longmapsto & q_a + \mathrm{I}^{\alpha}_{a+}[f(q_1,\cdot)](t). \end{array}$$

From the Banach fixed point theorem, F has a unique fixed point that is a weak solution of (CP$^\alpha$) on $[a,d]$ and is a strict extension of $(q,[a,b])$. Thus, we have obtained a contradiction with the maximality of $(q,[a,b])$ and the proof is complete. □

Lemma C.2. *Let (q,I) be the maximal weak solution of* (CP$^\alpha$)*. If (q,I) is not global, then q cannot be continuously extended with a value in Ω at $t = b$ (see Proposition C.3 for b).*

Proof. By contradiction, let us assume that q can be continuously extended with a value in Ω at $t = b$, that is, $\lim_{t\to b,\ t\in[a,b[} q(t) = q_b \in \Omega$. Then, we define q_1 by

$$q_1(t) = \begin{cases} q(t) & \text{if} \quad t \in [a,b[\\ q_b & \text{if} \quad t = b, \end{cases}$$

for every $t \in [a,b]$. In particular, $q_1 \in \mathrm{C}([a,b],\Omega)$. Our aim is to prove that $(q_1,[a,b])$ is a weak solution of (CP$^\alpha$). Since $(q,[a,b[)$ is a weak solution of (CP$^\alpha$), it holds

$$q_1(t) = q(t) = q_a + \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(q(\tau),\tau)\, d\tau = q_a + \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(q_1(\tau),\tau)\, d\tau, \tag{C.11}$$

for every $t \in [a,b[$. Since $f(q_1,\cdot) \in \mathrm{L}^\infty([a,b],\mathbb{R}^n)$, we infer from the Lebesgue's dominated convergence theorem that

$$q_1(b) = q_b = q_a + \frac{1}{\Gamma(\alpha)} \int_a^b (b-\tau)^{\alpha-1} f(q_1(\tau),\tau)\, d\tau.$$

Therefore (C.11) also holds for $t = b$. It follows that $(q_1,[a,b])$ is a weak solution of (CP$^\alpha$) and is a strict extension of $(q,[a,b[)$. It is a contradiction with the maximality of $(q,[a,b[)$. The proof is complete. □

Lemma C.3. *Let (q, I) be the maximal weak solution of* (CP^α)*. If (q, I) is not global, then for every $K \in \mathcal{K}$ there exists $t \in I$ (close to b, see Proposition C.3 for b) such that $q(t) \in \Omega \backslash K$.*

Proof. By contradiction, we assume that there exists $K \in \mathcal{K}$ such that q takes its values in K on $I = [a, b[$. Consider $M \geq 0$ associated with $K \in \mathcal{K}$ and $[a, b]$ in (H_∞). For all $t_1 \leq t_2$ elements of $[a, b[$, one has

$$\|q(t_2) - q(t_1)\| \leq \frac{2M}{\Gamma(1+\alpha)}(t_2 - t_1)^\alpha.$$

Therefore, q is uniformly continuous on $[a, b[$. Hence, q can be continuously extended at $t = b$ with a value $q_b \in \mathbb{R}^n$. Moreover, since q takes its values in the compact $K \subset \Omega$, it follows that $q_b \in \Omega$. Using Lemma C.2, this raises a contradiction. □

The proof of Theorem C.2 follows from Proposition C.3 and Lemma C.3.

C.3.3 Proof of Theorem C.3

In this section, we assume that the hypotheses of Theorem C.3 are satisfied. Note that since $\Omega = \mathbb{R}^n$ and since f satisfies $(\mathrm{H}^{\mathrm{glob}}_{\mathrm{Lip}})$, f automatically satisfies $(\mathrm{H}^{\mathrm{loc}}_{\mathrm{Lip}})$. Then, (CP^α) admits a unique maximal weak solution from Theorem C.1. Now, let us prove that this solution is global. For every $b \in I_f \backslash \{a\}$, we define the mapping

$$\begin{array}{rccc} F_b : & \mathrm{C}([a,b], \mathbb{R}^n) & \longrightarrow & \mathrm{C}([a,b], \mathbb{R}^n) \\ & q & \longmapsto & F(q) \end{array}$$

with

$$\begin{array}{rccc} F_b(q) : & [a,b] & \longrightarrow & \mathbb{R}^n \\ & t & \longmapsto & q_a + \mathrm{I}^\alpha_{a+}[f(q,\cdot)](t). \end{array}$$

From Assumption $(\mathrm{H}^{\mathrm{glob}}_{\mathrm{Lip}})$, one can easily prove by induction that

$$\|F^k_b(q_1)(t) - F^k_b(q_2)(t)\| \leq \frac{L^k}{\Gamma(1+k\alpha)}\|q_1 - q_2\|_\infty (t-a)^{k\alpha},$$

for every $k \in \mathbb{N}^*$, all $q_1, q_2 \in \mathrm{C}([a,b], \mathbb{R}^n)$, and every $t \in [a, b]$. Then,

$$\|F^k_b(q_1) - F^k_b(q_2)\|_\infty \leq \frac{(L(b-a)^\alpha)^k}{\Gamma(1+k\alpha)}\|q_1 - q_2\|_\infty,$$

for every $k \in \mathbb{N}^*$, all q_1, $q_2 \in \mathrm{C}([a,b], \mathbb{R}^n)$. Therefore, F_b admits a contraction iterate and thus has a unique fixed point that is a weak solution on $[a, b]$ of (CP^α). In the case where I_f is closed and bounded, it suffices to take $b = \max I_f$. In the contrary case, it suffices to make b tend to $\sup I_f$. This last comment concludes the proof of Theorem C.3 and this appendix.

Annexe D : Preuves techniques de la Section VII.3.1 du Chapitre VII

Cette annexe est consacrée aux démonstrations des résultats techniques de la Section VII.3.1 du Chapitre VII. Par conséquent, les notations utilisées correspondent aux notations du Chapitre VII.

We first recall the definition of the Mittag-Leffler function E_α with parameter $0 < \alpha < 1$:

$$\forall t \in \mathbb{R},\ E_\alpha(t) = \sum_{k\in\mathbb{N}} \frac{t^k}{\Gamma(\alpha k + 1)}. \tag{D.1}$$

Recall that the Mittag-Leffler function corresponds to a fractional version of the exponential function. We refer to [141, 190] for more details. In particular, it is instrumental to prove the following fractional version of the Gronwall's lemma.

Lemma D.1 (Gronwall)**.** *Let $q \in \mathrm{C}([a,b],\mathbb{R})$ satisfying*

$$\exists L_1, L_2 \geq 0,\ \forall t \in [a,b],\ 0 \leq q(t) \leq L_1 + L_2 \mathrm{I}^\alpha_{a+}[q](t). \tag{D.2}$$

Then, q satisfies

$$\forall t \in [a,b],\ 0 \leq q(t) \leq L_1 E_\alpha(L_2(t-a)^\alpha). \tag{D.3}$$

We refer to [77, 202] for a detailed proof of Lemma D.1.

D.1 Proofs of Lemmas VII.1 and VII.2

Proof of Lemma VII.1 Let $R = \|u\|_{\mathrm{L}^\infty([a,b],\mathbb{R}^m)} + 1$. By continuity of $q(\cdot, u, q_a)$ on $[a,b]$, the set

$$K = \{(x,v,t) \in \mathbb{R}^n \times \overline{B}(0,R) \times [a,b] \mid \|x - q(t,u,q_a)\| \leq 1\}$$

is a compact subset of $\mathbb{R}^n \times \mathbb{R}^m \times I_f$. Therefore, $\|\partial f/\partial x\|$ and $\|\partial f/\partial v\|$ are bounded by some $L \geq 0$ on K and moreover L is chosen such that

$$\|f(x_1,v_1,t) - f(x_2,v_2,t)\| \leq L(\|x_1 - x_2\| + \|v_1 - v_2\|), \tag{D.4}$$

for all (x_1,v_1,t) and (x_2,v_2,t) in K. Let $0 < \nu < 1$ and $0 < \eta < 1$ such that

$$\left(\eta + \frac{L(b-a)^\alpha}{\Gamma(1+\alpha)}\nu\right) E_\alpha(L(b-a)^\alpha) < 1. \tag{D.5}$$

Let $(u', q'_a) \in \mathrm{E}(u, q_a)$. In particular, note that the values of u' are contained in $\overline{B}(0,R)$. We denote by I' the interval of definition of $q(\cdot, u', q'_a)$ satisfying $a \in I'$ and $I' \backslash \{a\} \neq \emptyset$. It suffices to prove that $b \in I'$. By contradiction, assume that the set $A = \{t \in I' \cap [a,b] \mid \|q(t,u',q'_a) - q(t,u,q_a)\| > 1\}$ is not empty and set $t_1 = \inf A$. In particular, we have $[a, t_1] \subset I' \cap [a, b]$ and $\|q(t_1, u', q'_a) - q(t_1, u, q_a)\| \geq 1$ by continuity. Since $\|q(a, u', q'_a) - q(a, u, q_a)\| = \|q'_a - q_a\| \leq \eta < 1$, we conclude that $t_1 > a$. Finally, we have obtained $\|q(\tau, u', q'_a) - q(\tau, u, q_a)\| \leq 1$ for every $\tau \in [a, t_1]$. Therefore $(q(\tau, u', q'_a), u'(\tau), \tau)$ and $(q(\tau, u, q_a), u(\tau), \tau)$ are elements of K for almost every $\tau \in [a, t_1]$. Since there holds

$$q(t, u', q'_a) - q(t, u, q_a) = q'_a - q_a + \mathrm{I}^{\alpha}_{a+}[f(q(\cdot, u', q'_a), u', \cdot) - f(q(\cdot, u, q_a), u, \cdot)](t),$$

for every $t \in I' \cap [a, b]$, it follows from (D.4) and from Lemma D.1 that, for every $t \in [a, t_1]$,

$$\begin{aligned} \|q(t, u', q'_a) - q(t, u, q_a)\| &\leq \|q'_a - q_a\| + L\mathrm{I}^{\alpha}_{a+}[\|u' - u\|](t) + L\mathrm{I}^{\alpha}_{a+}[\|q(\cdot, u', q'_a) - q(\cdot, u, q_a)\|](t) \\ &\leq \left(\|q'_a - q_a\| + \frac{L(b-a)^\alpha}{\Gamma(1+\alpha)} \|u' - u\|_{\mathrm{L}^\infty([a,b],\mathbb{R}^m)} \right) E_\alpha(L(b-a)^\alpha) \\ &\leq \left(\eta + \frac{L(b-a)^\alpha}{\Gamma(1+\alpha)} \nu \right) E_\alpha(L(b-a)^\alpha) < 1. \end{aligned}$$

This raises a contradiction at $t = t_1$. Therefore A is empty and thus $q(\cdot, u', q'_a)$ is bounded on $I' \cap [a, b]$. It follows from Theorem C.2 that $b \in I'$, that is, $(u', q'_a) \in \mathcal{UQ}^b_{\mathrm{ad}}$.

Remark D.1. Let $(u', q'_a) \in \mathrm{E}(u, q_a)$. With the notations of the above proof, since $I' \cap [a, b] = [a, b]$ and A is empty, we infer that $\|q(t, u', q'_a) - q(t, u, q_a)\| \leq 1$, for every $t \in [a, b]$. Therefore $(q(t, u', q'_a), u'(t), t) \in K$ for every $(u', q'_a) \in \mathrm{E}(u, q_a)$ and for almost every $t \in [a, b]$.

Proof of Lemma VII.2 Let (u', q'_a) and (u'', q''_a) be elements of $\mathrm{E}(u, q_a) \subset \mathcal{UQ}^b_{\mathrm{ad}}$. It follows from Remark D.1 that $(q(\tau, u'', q''_a), u''(\tau), \tau)$ and $(q(\tau, u', q'_a), u'(\tau), \tau)$ are elements of K for almost every $t \in [a, b]$. Following the same arguments as in the previous proof, it follows from (D.4) and from Lemma D.1 that, for every $t \in [a, b]$,

$$\|q(t, u'', q''_a) - q(t, u', q'_a)\| \leq \left(\|q''_a - q'_a\| + \frac{L(b-a)^\alpha}{\Gamma(1+\alpha)} \|u'' - u'\|_{\mathrm{L}^\infty([a,b],\mathbb{R}^m)} \right) E_\alpha(L(b-a)^\alpha).$$

The lemma follows.

D.2 Proofs of Lemmas VII.3, VII.4 and Proposition VII.1

Proof of Lemma VII.3 We use the notations K, L, ν and η, associated with (u, q_a), defined in Lemma VII.1 and in its proof.

For $\varepsilon_0 > 0$ sufficiently small, we have $\|u + \varepsilon\bar{u} - u\|_{\mathrm{L}^\infty([a,b],\mathbb{R}^m)} \leq \nu$ for all $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$. Hence, $(u + \varepsilon\bar{u}, q_a) \in E(u, q_a)$ for all $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$. The claim follows then from Lemma VII.1.

Proof of Lemma VII.4 We use the notations of proof of Lemma VII.3. It follows from Lemma VII.2 that there exists $C \geq 0$ (the Lipschitz constant of $F_{(u,q_a)}$) such that

$$\begin{aligned} \|q(\cdot, u + \varepsilon_2\bar{u}, q_a) - q(\cdot, u + \varepsilon_1\bar{u}, q_a)\|_\infty &\leq C d_{\mathcal{UQ}^b_{\mathrm{ad}}}((u + \varepsilon_2\bar{u}, q_a), (u + \varepsilon_1\bar{u}, q_a)) \\ &= C|\varepsilon^2 - \varepsilon^1| \|\bar{u}\|_{\mathrm{L}^\infty([a,b],\mathbb{R}^m)}, \end{aligned}$$

for all ε^1 and ε^2 in $[-\varepsilon_0, \varepsilon_0]$. The lemma follows.

Proof of Proposition VII.1 We use the notations of proof of Lemma VII.3. Recall that $(q(t, u+\varepsilon\bar{u}, q_a), u(t)+\varepsilon\bar{u}(t), t) \in K$ for every $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ and for almost every $t \in [a, b]$, see Remark D.1. For every $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$, $\varepsilon \neq 0$ and every $t \in [a, b]$, we define

$$\varphi(t, \varepsilon) = \frac{q(t, u + \varepsilon\bar{u}, q_a) - q(t, u, q_a)}{\varepsilon} - \omega_{\bar{u}}(t, u, q_a).$$

It suffices to prove that $\varphi(\cdot,\varepsilon)$ converges uniformly to 0 on $[a,b]$ as ε tends to 0. For every $\varepsilon \in [-\varepsilon_0,\varepsilon_0]$, $\varepsilon \neq 0$ and every $t \in [a,b]$, it holds $\varphi(t,\varepsilon) = \mathrm{I}^{\alpha}_{a+}\left[\star(\cdot,\varepsilon)\right](t)$ where for almost every $\tau \in [a,b]$ and every $\varepsilon \in [-\varepsilon_0,\varepsilon_0]$, $\varepsilon \neq 0$, $\star(\tau,\varepsilon)$ is defined by

$$\begin{aligned}\star(\tau,\varepsilon) = {} & \frac{f(q(\tau,u+\varepsilon\bar{u},q_a),u(\tau)+\varepsilon\bar{u}(\tau),\tau) - f(q(\tau,u,q_a),u(\tau),\tau)}{\varepsilon} \\ & - \frac{\partial f}{\partial x}(q(\tau,u,q_a),u(\tau),\tau) \times \omega_{\bar{u}}(\tau,u,q_a) - \frac{\partial f}{\partial v}(q(\tau,u,q_a),u(\tau),\tau) \times \bar{u}(\tau). \end{aligned} \tag{D.6}$$

From the Mean Value Theorem, for almost every $\tau \in [a,b]$ and every $\varepsilon \in [-\varepsilon_0,\varepsilon_0]$, $\varepsilon \neq 0$, there exists $(\theta_1(\tau,\varepsilon),\theta_2(\tau,\varepsilon)) \in \mathbb{R}^n \times \mathbb{R}^m$ a convex combination of $(q(\tau,u,q_a),u(\tau))$ and $(q(\tau,u+\varepsilon\bar{u},q_a),u(\tau)+\varepsilon\bar{u}(\tau))$, such that

$$\begin{aligned}\star(\tau,\varepsilon) = {} & \frac{\partial f}{\partial x}(\theta_1(\tau,\varepsilon),\theta_2(\tau,\varepsilon),\tau) \times \varphi(\tau,\varepsilon) \\ & + \left[\frac{\partial f}{\partial x}(\theta_1(\tau,\varepsilon),\theta_2(\tau,\varepsilon),\tau) - \frac{\partial f}{\partial x}(q(\tau,u,q_a),u(\tau),\tau)\right] \times \omega_{\bar{u}}(\tau,u,q_a) \\ & + \left[\frac{\partial f}{\partial v}(\theta_1(\tau,\varepsilon),\theta_2(\tau,\varepsilon),\tau) - \frac{\partial f}{\partial v}(q(\tau,u,q_a),u(\tau),\tau)\right] \times \bar{u}(\tau). \end{aligned} \tag{D.7}$$

Since $(\theta_1(\tau,\varepsilon),\theta_2(\tau,\varepsilon),\tau) \in K$ for almost every $\tau \in [a,b]$, it follows that $\|\star(\tau,\varepsilon)\| \leq L\|\varphi(\tau,\varepsilon)\| + \chi_1(\tau,\varepsilon) + \chi_2(\tau,\varepsilon)$, where

$$\chi_1(\tau,\alpha) = \left\| \left[\frac{\partial f}{\partial x}(\theta_1(\tau,\varepsilon),\theta_2(\tau,\varepsilon),\tau) - \frac{\partial f}{\partial x}(q(\tau,u,q_a),u(\tau),\tau)\right] \times \omega_{\bar{u}}(\tau,u,q_a) \right\|$$

and

$$\chi_2(\tau,\alpha) = \left\| \left[\frac{\partial f}{\partial v}(\theta_1(\tau,\varepsilon),\theta_2(\tau,\varepsilon),\tau) - \frac{\partial f}{\partial v}(q(\tau,u,q_a),u(\tau),\tau)\right] \times \bar{u}(\tau) \right\|.$$

Therefore, one has

$$\|\varphi(t,\varepsilon)\| \leq \mathrm{I}^{\alpha}_{a+}[\chi_1(\cdot,\varepsilon)](b) + \mathrm{I}^{\alpha}_{a+}[\chi_2(\cdot,\varepsilon)](b) + L\mathrm{I}^{\alpha}_{a+}[\|\varphi(\cdot,\varepsilon)\|](t) \tag{D.8}$$

for every $t \in [a,b]$. It follows from Lemma D.1 that $\|\varphi(t,\varepsilon)\| \leq \Upsilon(\varepsilon)E_\alpha(L(b-a)^\alpha)$, for every $t \in [a,b]$, where $\Upsilon(\varepsilon) = \mathrm{I}^{\alpha}_{a+}[\chi_1(\cdot,\varepsilon)](b) + \mathrm{I}^{\alpha}_{a+}[\chi_2(\cdot,\varepsilon)](b)$.

To conclude, it remains to prove that $\Upsilon(\varepsilon)$ converges to 0 as ε tends to 0. First, since $(\theta_1(\cdot,\varepsilon),\theta_2(\cdot,\varepsilon))$ converges to $(q(\cdot,u,q_a),u)$ in $\mathrm{L}^\infty([a,b],\mathbb{R}^n) \times \mathrm{L}^\infty([a,b],\mathbb{R}^m)$ as ε tends to 0, and since $\partial f/\partial x$ is uniformly continuous on K, we infer that $\mathrm{I}^{\alpha}_{a+}[\chi_1(\cdot,\varepsilon)](b)$ converges to 0 as ε tends to 0. Similarly, we prove that $\mathrm{I}^{\alpha}_{a+}[\chi_2(\cdot,\varepsilon)](b)$ converges to 0 as ε tends to 0. The conclusion follows.

D.3 Proofs of Lemmas VII.5, VII.6 and Proposition VII.2

Proof of Lemma VII.5 We use the notations K, L, ν and η, associated with (u,q_a), defined in Lemma VII.1 and in its proof. There exists $\gamma_0 > 0$ such that $\|q_a + \gamma\bar{q}_a - q_a\| = \gamma\|\bar{q}_a\| \leq \eta$ for every $\gamma \in [-\gamma_0,\gamma_0]$, and hence $(u, q_a+\gamma\bar{q}_a) \in \mathrm{E}(u,q_a)$. Then the claim follows from Lemma VII.1.

Proof of Lemma VII.6 We use the notations of proof of Lemma VII.5. From Lemma VII.2, there exists $C \geq 0$ (Lipschitz constant of $F_{(u,q_a)}$) such that

$$\begin{aligned}\|q(\cdot,u,q_a+\gamma^2\bar{q}_a) - q(\cdot,u,q_a+\gamma^1\bar{q}_a)\|_\infty & \leq C d_{\mathcal{U}\mathcal{Q}^b_{\mathrm{ad}}}((u,q_a+\gamma^2\bar{q}_a),(u,q_a+\gamma^1\bar{q}_a)) \\ & = C|\gamma^2-\gamma^1|\|\bar{q}_a\|. \end{aligned}$$

for all γ^1 and γ^2 in $[-\gamma_0,\gamma_0]$.

Proof of Proposition VII.2 We use the notations of proof of Lemma VII.5. Recall that $(q(t,u,q_a+\gamma\bar{q}_a),u(t),t)\in K$ for every $\gamma\in[-\gamma_0,\gamma_0]$ and for almost every $t\in[a,b]$, see Remark D.1. For every $\gamma\in[-\gamma_0,\gamma_0]$, $\gamma\neq 0$ and every $t\in[a,b]$, we define

$$\varphi(t,\gamma)=\frac{q(t,u,q_a+\gamma\bar{q}_a)-q(t,u,q_a)}{\gamma}-\omega_{\bar{q}_a}(t,u,q_a).$$

It suffices to prove that $\varphi(\cdot,\gamma)$ converges uniformly to 0 on $[a,b]$ as γ tends to 0. For every $\gamma\in[-\gamma_0,\gamma_0]$, $\gamma\neq 0$ and every $t\in[a,b]$, it holds $\varphi(t,\gamma)=\mathrm{I}^{\alpha}_{a+}\left[\star(\cdot,\gamma)\right](t)$ where for almost every $\tau\in[a,b]$ and every $\gamma\in[-\gamma_0,\gamma_0]$, $\gamma\neq 0$, $\star(\tau,\gamma)$ is defined by

$$\star(\tau,\gamma)=\frac{f(q(t,u,q_a+\gamma\bar{q}_a),u(\tau,\tau)-f(q(\tau,u,q_a),u(\tau),\tau)}{\gamma}-\frac{\partial f}{\partial x}(q(\tau,u,q_a),u(\tau),\tau)\times\omega_{\bar{q}_a}(\tau,u,q_a). \tag{D.9}$$

From the Mean Value Theorem, for almost every $\tau\in[a,b]$ and every $\gamma\in[-\gamma_0,\gamma_0]$, $\gamma\neq 0$, there exists $\theta(\tau,\gamma)\in\mathbb{R}^n$ that is a convex combination of $q(\tau,u,q_a)$ and $q(t,u,q_a+\gamma\bar{q}_a)$, such that

$$\begin{aligned}\star\,(\tau,\gamma)=\;&\frac{\partial f}{\partial x}(\theta_1(\tau,\gamma),u(\tau),\tau)\times\varphi(\tau,\gamma)\\&+\left[\frac{\partial f}{\partial x}(\theta_1(\tau,\gamma),u(\tau),\tau)-\frac{\partial f}{\partial x}(q(\tau,u,q_a),u(\tau),\tau)\right]\times\omega_{\bar{u}}(\tau,u,q_a).\end{aligned} \tag{D.10}$$

Since $\theta(\tau,\gamma)\in K$ for almost every $\tau\in[a,b]$, it follows that $\|\star(\tau,\gamma)\|\leq L\|\varphi(\tau,\gamma)\|+\chi(\tau,\gamma)$, where

$$\chi(\tau,\alpha)=\left\|\left[\frac{\partial f}{\partial x}(\theta(\tau,\gamma),u(\tau),\tau)-\frac{\partial f}{\partial x}(q(\tau,u,q_a),u(\tau),\tau)\right]\times\omega_{\bar{u}}(\tau,u,q_a)\right\|.$$

Therefore, one has

$$\|\varphi(t,\gamma)\|\leq\mathrm{I}^{\alpha}_{a+}[\chi(\cdot,\gamma)](b)+L\mathrm{I}^{\alpha}_{a+}[\|\varphi(\cdot,\gamma)\|](t) \tag{D.11}$$

for every $t\in[a,b]$. It follows from Lemma D.1 that $\|\varphi(t,\gamma)\|\leq\Upsilon(\gamma)E_\alpha(L(b-a)^\alpha)$, for every $t\in[a,b]$, where $\Upsilon(\gamma)=\mathrm{I}^{\alpha}_{a+}[\chi(\cdot,\gamma)](b)$.

To conclude, it remains to prove that $\Upsilon(\gamma)$ converges to 0 as γ tends to 0. First, since $\theta(\cdot,\gamma)$ converges uniformly to $q(\cdot,u,q_a)$ on $[a,b]$ as γ tends to 0, and since $\partial f/\partial x$ is uniformly continuous on K, we infer that $\mathrm{I}^{\alpha}_{a+}[\chi(\cdot,\gamma)](b)$ converges to 0 as γ tends to 0. The conclusion follows.

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Résumé

Cette thèse est une contribution au **calcul des variations** et à la **théorie du contrôle optimal** dans les cadres discret, plus généralement **time scale**, et **fractionnaire**. Ces deux domaines ont récemment connu un développement considérable dû pour l'un à son application en informatique et pour l'autre à son essor dans des problèmes physiques de diffusion anormale. Que ce soit dans le cadre time scale ou dans le cadre fractionnaire, nos objectifs sont de :

a) développer un calcul des variations et étendre quelques résultats classiques (voir plus bas) ;

b) établir un **principe du maximum de Pontryagin** (PMP en abrégé) pour des problèmes de contrôle optimal.

Dans ce but, nous généralisons plusieurs méthodes variationnelles usuelles, allant du simple calcul des variations au *principe variationnel d'Ekeland* (couplé avec la technique des *variations-aiguilles*), en passant par l'étude d'*invariances variationnelles* par des groupes de transformations. Les démonstrations des PMPs nous amènent également à employer des *théorèmes de point fixe* et à prendre en considération la technique des *multiplicateurs de Lagrange* ou encore une méthode basée sur un *théorème d'inversion locale conique*.

Ce manuscrit est donc composé de deux parties : la Partie A traite de problèmes variationnels posés sur time scale et la Partie B est consacrée à leurs pendants fractionnaires. Dans chacune de ces deux parties, nous suivons l'organisation suivante :

1. détermination de l'**équation d'Euler-Lagrange** caractérisant les points critiques d'une fonctionnelle Lagrangienne ;
2. énoncé d'un **théorème de type Noether** assurant l'existence d'une constante de mouvement pour les équations d'Euler-Lagrange admettant une symétrie ;
3. énoncé d'un **théorème de type Tonelli** assurant l'existence d'un minimiseur pour une fonctionnelle Lagrangienne et donc, par la même occasion, d'une solution pour l'équation d'Euler-Lagrange associée (uniquement en Partie B) ;
4. énoncé d'un **PMP** (version forte en Partie A, version faible en Partie B) donnant une condition nécessaire pour les trajectoires qui sont solutions de problèmes de contrôle optimal généraux non-linéaires ;
5. détermination d'une **condition de type Helmholtz** caractérisant les équations provenant d'un calcul des variations (uniquement en Partie A et uniquement dans les cas purement continu et purement discret).

Des **théorèmes de type Cauchy-Lipschitz** nécessaires à l'étude de problèmes de contrôle optimal sont démontrés en Annexe.

Mots clés : calcul des variations ; contrôle optimal ; calcul time scale ; calcul fractionnaire ; équation d'Euler-Lagrange ; théorème de Noether ; condition de Helmholtz ; résultat d'existence ; principe du maximum de Pontryagin ; intégrateurs variationnels ; principe variationnel d'Ekeland ; variations-aiguilles ; conditions de transversalité ; théorème de type Cauchy-Lipschitz.

Abstract

This dissertation deals with the mathematical fields called **calculus of variations** and **optimal control theory**. More precisely, we develop some aspects of these two domains in discrete, more generally **time scale**, and **fractional** frameworks. Indeed, these two settings have recently experience a significant development due to its applications in computing for the first one and to its emergence in physical contexts of anomalous diffusion for the second one. In both frameworks, our goals are:

a) to develop a calculus of variations and extend some classical results (see below);

b) to state a **Pontryagin maximum principle** (denoted in short PMP) for optimal control problems.

Towards these purposes, we generalize several classical variational methods, including the *Ekeland's variational principle* (combined with *needle-like variations*) as well as *variational invariances* via the action of groups of transformations. Furthermore, the investigations for PMPs lead us to use *fixed point theorems* and to consider the *Lagrange multiplier technique* and a method based on a *conic implicit function theorem*.

This manuscript is made up of two parts : Part A deals with variational problems on time scale and Part B is devoted to their fractional analogues. In each of these parts, we follow (with minor differences) the following organization:

1. obtaining of an **Euler-Lagrange equation** characterizing the critical points of a Lagrangian functional;
2. statement of a **Noether-type theorem** ensuring the existence of a constant of motion for Euler-Lagrange equations admitting a symmetry;
3. statement of a **Tonelli-type theorem** ensuring the existence of a minimizer for a Lagrangian functional and, consequently, of a solution for the corresponding Euler-Lagrange equation (only in Part B);
4. statement of a **PMP** (strong version in Part A and weak version in Part B) giving a necessary condition for the solutions of general nonlinear optimal control problems;
5. obtaining of a **Helmholtz condition** characterizing the equations deriving from a calculus of variations (only in Part A and only in the purely continuous and purely discrete cases).

Some **Picard-Lindelöf type theorems** necessary for the analysis of optimal control problems are obtained in Appendices.

Keywords: calculus of variations; optimal control theory; time scale calculus; fractional calculus; Euler-Lagrange equation; Noether's theorem; Helmholtz condition; existence result; Pontryagin maximum principle; variational integrators; Ekeland's variational principle; needle-like variations; transversality conditions; Picard-Lindelöf theorem.