STUDENTS' UNDERSTANDING OF THE CORE CONCEPT OF FUNCTION

by

Hatice Akkoç

A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Education (EdD) in Mathematics Education

University of Warwick

Mathematics Education Research Centre

Institute of Education

May, 2003

List of Tables

List of Figures

List of Equations

Acknowledgments

First of all, I would like to thank to my supervisor, Prof. David Tall, who supported me throughout my degree and contributed so much to my intellectual growth. I also would like to thank to his lovely wife, Susan Tall, both of them, for offering me their house to stay in my last year in the UK. And of course thanks to my father and my mother, who always had a faith in me. Thanks also to people in Mathematics Education Research Centre, in University of Warwick, and especially to PhD students in the SUMINER group, for the intellectual motivation in the discussions and the joy in the conferences. I also would like to thank to Jo Crozier in the Institute of Education for her help in my first year. Thanks also to Professor Sertaç Özenli who encouraged me to study abroad who helped me decide to do a doctoral degree.

I also wish to express sincere appreciation to The Ministry of National Education in Turkey (Milli Eğitim Bakanlığı) who financially supported me throughout my degree and The Council of Higher Education in Turkey (Yükseköğretim Kurulu) who made the arrangements.

The data collection would be impossible without the help of the staff in the schools I visited (Adana Lisesi and Borsa Lisesi), my father who helped with the administration of some of the questionnaires and the students who accepted to be involved in this study.

Declaration

I, the author, declare that this thesis is my own work and has not been peviously presented for any degree at any university. The preliminary results were presented as an EdD Project,

Akkoç, H. (2001) 'The function concept: A Preliminary Study', *Unpublieshed EdD Project 2*, University of Warwick.

as part of this thesis for the degree of EdD (Doctorate of Education).

I also declare that some ideas of the theoretical framework were published in the following paper:

Akkoç, H. & Tall, D.O. (2002) 'The Simplicity, Complexity and Complications of the Function Concept', in *Proceedings of the 26th International Conference on the Psychology of Mathematics Education*, Norwich, UK, Vol. 2, pp. 25-32.

Abstract

This thesis is concerned with students' understanding of the core concept of function which cannot be represented by what is commonly called the multiple representations of functions. The function topic is taught to be the central idea of the whole of mathematics. In that sense, it is a model *of mathematical simplicity*. At the same time it has a richness and has *mathematical complexity*. Because of this nature, for students it is so difficult to grasp. The complexity of the function concept reveals itself as *cognitive complications* for weak students. This thesis investigates why the function concept is so difficult for students.

In the Turkish context, students in high school are introduced to a colloquial definition and are presented with four different aspects of functions, set-correspondence diagrams, sets of ordered pairs, graphs and expressions. The coherency in recognizing these different aspects of functions by focusing on the definitional properties is considered as an indication of an understanding of the core concept of function. Focusing on a sample of a hundred and fourteen students, their responses in the questionnaires are considered to select nine students for individual interviews. The responses from these nine students in the interviews are categorized as they deal with different aspects of functions. The data indicates that there is a spectrum of performance of students. In this spectrum, responses range from the responses which handle the flexibility of the mathematical simplicity and complexity to the responses which are cognitively complicated. Successful students could focus on the definitional properties by using the colloquial definition for all different aspects of functions. Less successful students could use the colloquial definition for only set-correspondence diagrams and sets of ordered pairs and gave complicated responses for the graphs and expressions. Weaker students could not focus on the definitional properties for any aspect of functions.

CHAPTER 1 – INTRODUCTION

This research investigates students' understanding of the function concept in the Turkish context where mathematics is taught in a more formal way. The aim of this thesis is to reveal Turkish students' understanding of the function concept, one of the most fundamental concepts in mathematics. Their understanding is investigated by focusing on Thompson's (1994) notion of the *core concept of function* which cannot be represented by what is commonly called the multiple representations of function. The coherency of recognizing various aspects of functions by focusing on the definitional properties is considered as an indication of the core concept of function.

This thesis suggested a spectrum of performance of students when dealing with different aspects of functions. In this spectrum, very few students strongly focused on the core concept of function. The majority of the students could not focus on the definitional properties.

1.1 Functions in Mathematics

The concept of function is one of the most fundamental concepts in mathematics, which appears from primary school through to university. At a primary level it is given as 'guess my rule' activity. Before university, real-valued functions with one variable are studied. Function as a special kind such as 'continuous' and 'differentiable' is the central underlying concept in calculus (Vinner, 1992). Beyond calculus, in advanced mathematical thinking, functions are used to compare abstract mathematical structures e.g. to show that two sets have the same cardinality, that topologies are homeomorphic, that one group is the homomorphic image of another. The function $y = e^x$ tells us that the additive structure of the real numbers is isomorphic to the multiplicative structure of the positive reals (Selden & Selden, 1992).

In its historical development, the function concept was formed long after mathematicians dealt with the concept in a variety of contexts. Definitions of the function concept such as those proposed by Dirichlet and Bourbaki have taken various forms, from algebraic and analytic relations to any arbitrary correspondence (Cajori; 1980; Sfard, 1991). The Dirichlet definition was first introduced in 1837:

> 'If a variable y is so related to a variable x that whenever a numerical value is assigned to x there is a rule according to which a unique value of y is determined, then y is said to be a function of the independent variable x' (Boyer, 1968, p. 600).

The formal ordered pair definition, the so called the Bourbaki definition (A function *f* is a set of ordered pairs with the property that if $(x, y) \in f$ and $(x, z) \in f$ then $y = z$), was reached in 1939.

There have been different teaching approaches of the concept of function. In textbooks from the middle of 19th century until the middle of the 20th century, the function concept was introduced as a relationship, a correspondence between two variables (numbers only) by an influence of the Dirichlet definition (Bruckheimer *et al.*, 1986). Selden & Selden (1992) claim that the Dirichlet definition facilitates the notions of domain, range (co-domain) and one-to-one-ness. They suggest that although they are technically similar, the Dirichlet definition is more easily grasped than the ordered pair definition. This Bourbaki definition was first introduced in the curriculum within the New Maths movement in 1960's. This set theoretic definition is considered too abstract for a wide range of students as an introduction (Malik, 1980; Bruckheimer *et al.*, 1986; Bakar & Tall, 1992). After the

New Maths movement, the topic of functions was emphasized for the teaching of algebra (Kieran, 1994; Brenner *et al.*, O'Callaghan, 1998; DeMarois & Tall, 1999).

1.2 Background of the study: Turkish curriculum and function concept

1.2.1 Basic facts about the Turkish Education System

In the Turkish education system, compulsory education – so-called "Basic/Primary School" – lasts for eight years. There are two phases of compulsory education; Lower Level Primary Education/Elementary School (a total of five years) and Upper Level Primary Education/Middle School (a total of three years). This is followed by a three year of schooling which is called "high school" or "upper-secondary school". Public schools with access to the public without any exam and tuition fees is three years. Some high schools, such as private and Anatolian High Schools, last for four years; the first year is for foreign language (which is normally English) preparation if desired. Year groups can be named as grade 1 until grade 8 of basic education and grade 1, 2, 3 of High School. The table below summarizes the year groups:

School	Grade	Age
Nursery		5-6
Basic Education Basic/Primary School (Compulsory)	Grade 1	7
	Grade 2	8
	Grade 3	9
	Grade 4	10
	Grade 5	11
	Grade 6	12
	Grade 7	13
	Grade 8	14
Foreign language preparation if desired		
High School	Grade 1	15
	Grade 2	16
	Grade 3	17

Table 1-1. Grades and year groups across Turkish schools

In Turkey, education is centralized by a National Curriculum which is determined by The Ministry of National Education (http://www.meb.gov.tr). The Ministry of National Education specifies a set of textbooks according to the national curriculum. Among those, each school chooses their own textbooks to follow.

1.2.2 Function topic in Turkish context

The topic of “functions”, as a topic on its own, is introduced in the first year of high school. The development of the topic in the national curriculum can be summarized as follows:

	Grade 1	Grade 2	Grade 3
October	Relations. Equivalence and ordered relations. Introduction to the function topic with formal definition. Binary operations.	Trigonometric functions	
November	Compositions of functions, inverse of a function		
February			Finding the domains of the domain of a function Function graphs, 1-1 and onto functions Inverse functions and their graphs
March			Odd and even functions/increasing and decreasing functions/split-domain functions/absolute value functions Integer functions Split-domain absolute value functions
April		Logarithmic functions	
May	Drawing the two degree polynomial functions Drawing a parabola when the two x-intercepts are known Drawing a parabola when the maximum or minimum points and one point are known		

June	The regions on the plane divided by the line in the form of $ax + bx + c = 0$ and the graphical solutions of second order inequalities.		

Table 1-2 The development of the topic of "functions" in the Turkish curriculum.

In the textbook, the definition of function is given as follows:

> Definition: Let A and B be two non-empty sets. A relation from f from A to B is called a *function* if it assigns every element in A to a unique element in B (Demiralp *et al.*, 2000, my translation).

This definition in the textbook is the formal definition translated into words which has colloquial meaning. This definition is followed by a further explanation on the definition which will be called the *colloquial definition*:

A function *f* defined from A to B assigns:

1. All elements in A to elements in B.

2. Every element in A to a unique element in B.

This colloquial definition is followed by a visual explanation as follows:

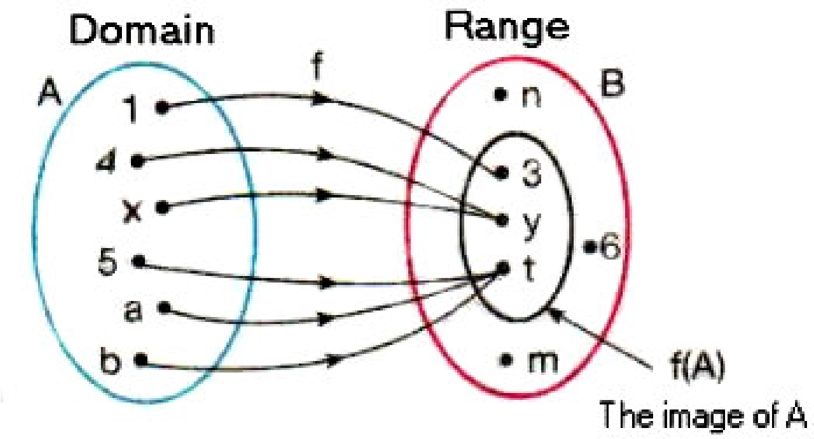


Figure 1-1 - A visual explanation of the function definition

An explanation on the notation is given as follows:

If $x \in A$ and $y \in B$ and if a function *f* from A to B assigns x to y then it is denoted by

$f : A \rightarrow B$, $x \rightarrow y = f(x)$.

'$y = f(x)$' is read as 'y is equal to *f* of x'.

1.3 Organization of the thesis

This thesis is based on two previous EdD Projects (Akkoç, 2000 & Akkoç, 2001). The first project is a general literature review on functions. The second project is a preliminary study to this thesis. This thesis consists of eight chapters, a bibliography and appendices.

Chapter 2 gives a literature review. It has two parts. The first part focuses on a literature review on functions. Previous research which focuses on functions from different theoretical frameworks is discussed. The second part gives a brief account on research on categorization. The second part of the literature review can be also read after reading chapter 3 since the findings from the preliminary study in chapter 3 required a review of the literature on categorization.

Chapter 3 presents brief findings from the preliminary study which helped research questions to be refined (Akkoç, H. 2001, Unpublished EdD Project 2).

Chapter 4 describes the theoretical perspective of this research. Drawing on the literature on functions and categorization, it takes Thompson's (1994) notion of *the core concept of function* as the departure point. The theoretical framework distinguishes between *the simplicity and complexity of the core concept of function* and the *cognitive complications* that students might have. By considering a *prototype-exemplar distinction*, it defines *a* *focus on the core concept of function* as the coherency in focusing on the definitional properties for different aspects of functions.

Chapter 5 defines the methodology. The methodology of this study is a combination of qualitative and quantitative approaches, with a qualitative approach having priority. Research problems which were refined after the preliminary study are stated in this chapter. Description of subjects, methods of data collection are also presented.

Chapter 6 presents the results from the questionnaires. It gives a broader picture for the whole sample of students. Responses from students are categorized and the distribution of these categories across the sample is presented.

Chapter 7 presents the results from the interviews with nine students. The results reveal a spectrum of performances which lead to the categorization of students in the next chapter.

Chapter 8 presents the categorization of students' responses in the spectrum of responses ranging from responses which focus on the simplicity of complexity of the function concept to the responses which are cognitively complicated.

Chapter 9 gives a discussion of the data in relation to the theoretical perspective and the related literature. It is discussed that graphs and expressions, as exemplars of functions, caused much more cognitive complication because of the incidental properties they have while set-correspondence diagrams and sets of ordered pairs caused less complications since they were treated as prototypes of functions.

Chapter 10, concludes the thesis by giving the implications and further research possibilities.

CHAPTER 2 – LITERATURE REVIEW

In formal mathematics, concepts are clearly specified by their definitions. However, students might not always focus on the properties of the definitions when dealing with mathematical concepts. For instance, they might consider some examples of functions as better examples than others. In other words, students might categorize functions in different ways. Considering these, this chapter of literature review is divided into two parts. The first part focuses on the literature review on function concept. The second part gives a brief account of the research on categorization.

2.1 Literature review on function

The topic of functions has been a focus of attention for a few decades. Various research investigates the topic from various theoretical frameworks. Below an account of basic theoretical frameworks is given, including concept definition and concept image, operational and structural conceptions of the function concept, multiple representations of functions, vertical and horizontal growth of the function concept.

2.1.1 Concept definition and concept image

One of the theoretical frameworks to investigate students' understanding of the function concept introduces the notions of *concept definition* and *concept image* and makes a distinction between the two. Tall & Vinner (1981) define *concept definition* as the 'form of words used to specify that concept' (p. 152). A formal concept definition is one accepted by the mathematical community at large. In Vinner (1983), the concept definition is given as the 'verbal definition that accurately explains the concept in a non-circular way' (p. 293). As Tall & Vinner (1981) assert, we can use mathematical concepts without knowing the formal definitions. To explain how this occurs, they define *concept image* as 'the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes' (p. 152). They assert that it is built up over the years by experience, and that different stimuli at different times can activate different parts of the concept image developing them in a way which need not be a coherent whole. Vinner (1992) asserts that specific individuals create idiosyncratic images and also the same individual might react differently to a concept encountered within different situations. In that sense, Tall & Vinner (1981) define the portion of the concept image which is activated at a particular time as the *evoked concept image*.

In his study, Vinner (1983) identifies students' concept images for the function concept, which may conflict with the most general form of the definition:

- A function should be given by one rule. If there are two rules, then there are two functions.
- Ignorance of functions given by several rules when their disjoint domains are not half lines (e.g. $\{x \in R \mid x \leq 2)$ or intervals (e.g. $\{x \in R \mid 1 \leq x \leq 2)$.
- A function should have a 'reasonable' graph.
- Confusing the definition with one-to-oneness.
- Every function is one-to-one.
- Ignorance of the fulfilment of the conditions for functions such as sign or integral part function.
- Ignorance of functions (which are not algebraic) if they are not officially recognized by mathematicians (by giving them a name or denoting them by specific symbols).

In the same study, Vinner (1983) gives the main categories for the students' responses to the question of what is the definition of a function:

- The textbook definition mixed with elements from the concept image cell. Definition by students' own words.

- The function as a rule of correspondence.

- Function as an algebraic expression, a formula, an arithmetical manipulation. Some responses influenced by the textbook definition.

- Some elements of mental pictures e.g. a curved line, correspondence between Venn diagrams, etc. as a definition for the concept.

Students may use their own concept definitions giving idiosyncratic meanings. Tall & Vinner (1981) define *the personal concept definition* as 'the form of words that the student uses for his own explanation of his (evoked) concept image' (p. 152). It is the personal reconstruction by the student of a definition as they suggest. They also call it the "concept definition image".

Students' concept images may or may not focus on the definitional properties. Bakar & Tall (1992) assert that students' understanding of the function concept is reliant on the properties of the families of the prototypical examples rather than the properties of the definition. They claim that everyday concepts such as 'bird' are developed by initially encountering examples, by focusing on the salient features of the concept and by testing other examples against some criteria. They state that 'when the function concept is introduced initially, the examples and non-examples which become prototypes for the function concept are naturally limited in various ways, producing conflicts with the formal definition' (p. 50). In their study with A-level students, they found that positive resonances with prototypes (e.g. recognizing a circle as a function graph since it is familiar) and negative resonances with prototypes (e.g. rejecting a strange looking graph as a graph of a function) produced erroneous responses. They also found that students considered the same function as a function when it is in algebraic form but not in the graphical form or vice versa. For instance, when students are given the algebraic expression and graph of a constant function, they found that 28% of the students considered the constant function as a function in its algebraic and graphical form. 29% of the students say the graph corresponds to a function but the algebraic expression does not, with only 3% the other way round.

To avoid that, Bakar & Tall (1992) suggest that prototypes should be chosen to be as appropriate as possible and a broad spectrum of different representations of function should be provided to prevent the identification of any of these representations with the function concept (see also Sierpinska, 1992).

2.1.2 Operational and structural conception of the function concept

Sfard (1991) discusses structural conception (concepts perceived as objects) and operational conception (concepts perceived as processes) of the function concept. She suggests that objects have static structures, existing somewhere in space and time, and can be manipulated as a whole. Processes, on the other hand, have the potential rather than actual entity. In other words, processes are detailed, dynamic or can be considered as a sequence of actions.

Although Sfard (1991) makes such a distinction, she also suggests that operational and structural conceptions are in fact complementary. This dual nature provides a deep understanding of mathematics. In this duality, the operational conception precedes the structural conception, historically and psychologically in relation to particular concepts such as 'number' and 'function'. In another paper (Sfard, 1992), she asserts that alternating between operational and structural approaches to abstract concepts (just as treating subatomic entities both as particles and waves) is an important skill in mathematics. As an example of this dual nature, Sfard (1991) gives the notion of function. She discusses this duality by considering both the historical development of the function concept and its acquisition by a student (psychologically). She illustrates this dual nature of the function concept by giving three representations of a function: graphical, symbolic and as a computer program. She asserts that a computer program which computes the value of a function for each input of x corresponds to operational conception since it presents computational processes not a unified entity. A graph, on the other hand, corresponds to the structural conception and it can be grasped as an integrated whole, as an object. On the other hand, the symbolic notation, $y = 3x^4$ can be considered both operationally and structurally. Gray & Tall (1994) introduced the term "procept" to explain this duality between process - concept ambiguity. They believe that success in mathematics depends on the ability to think in a flexible way using the ambiguity of the notation. For instance, for the case of function, this flexibility is present when a student could think of two different procedures representing the same function (DeMarois & Tall, 1996, 1999).

Sfard (1992) asserts that structural approach introducing the function concept with the words "A function is a set of ordered pairs such that…" or "A function is a correspondence between two sets of elements which…" may be a source of difficulty for students. She explains that this is because the function concept is not only abstract for students to create meaning for themselves but also the more basic notions of set and element may be too fuzzy to be confidently used and operated upon.

Sfard (1992) found that, for the case of function, the students' conceptions seem closer to operational than to structural even when it is not deliberately promoted. She also found that many students developed pseudostructural (neither operational nor structural) conceptions.

For instance, students associated functions with algebraic expressions or ignored split-domain functions. These findings are in parallel with the concept images of students found by Vinner (1983), which were mentioned above.

2.1.3 Multiple representations of functions

Multiple representations of functions (e.g. graphical, algebraic, tabular representations) have been a focus of attention especially with the availability of computers and graphical calculators (Confrey, 1994; Kaput, 1992; Keller & Hirsch, 1998; Leinhardt *et al.*, 1990; Yerushalmy, 1991, and so on). Research on multiple representations of functions assumed that if students could link various representations they would have a better understanding of the function concept. The assumption was that part of the meaning is best conveyed by each well-chosen representation and links between various representations will aid understanding the whole message. Kaput (1988) asserts that making links between various representations will reduce the isolation of each topic to be learnt and provide a more coherent and unified view (cited in Goldenberg, 1988). Considering this fact, the literature focuses on the importance of moving among different representations of functions. Yerushalmy (1991) suggests that the ability to operate between several linked representation systems is crucial for students to understand a new concept. Keller & Hirsch (1998) claim that the connection between a concept and its representation is constructed from a student's preferred representations. They discuss calculus students' preferences for multiple representations of functions by comparing contextualised and non-contextualised settings using "Representation Preference Tests". In their study they found that students preferred to use the equation in purely mathematical situations while they preferred to use the table or the graph (graph being preferred in high level questions) in contextualised tasks, since they try to make sense of the situation by reasoning from the contextual information. As a consequence, this influences their preference for a representation. However, when working on a purely mathematical situation they tend to use taught recipes. They claim that students shifted from 'using table of values' to ' using graph' when dealing with a task requiring more interpretation. They also found that students preferred to use the graph or the table of values on tasks with less formal, more intuitive language, while they preferred to use the equation when formal language is used. This shallow symbol manipulation is also found in Monk's (1992) study. Monk (1992) suggests that students rarely assign much meaning to what they are doing. For instance, he uses a physical model to represent a functional situation and found that students 'focus on the shape of a graph as having primary significance over particular meaning of the axes' (p. 193). Keller & Hirsch (1998) claim that by the availability of multiple representations, especially in technology-rich situations, students' preferences for different types of representations become less tied to whether the task is contextualised or not. With multiple representations, it may also be possible for students to tie their higher order thinking skills in contextualised settings to the purely mathematical situations. However, various studies indicate that it is difficult to move flexibly between different representations (e.g. Even 1998, Yerushalmy 1991, Leinhardt *et al.* 1990; Hitt, 1998).

2.1.4 Action-process conception of functions

Another theoretical framework to investigate students' learning of the function concept is the action-process conception. (Beineke *et al.*, 1992; Breidenbach *et al.*, 1992; Dubinsky & Harel, 1992). This theoretical framework is based on APOS theory (acronym for **A**ction, **P**rocess, **O**bject and **S**chema) which was first mentioned in Dubinsky (1991) and formulated later in Cottrill *et al.* (1996). (Readers can refer to Czarnocha *et al.* (1999) for a further discussion of the theory or Tall (1999) for a critique of the theory).

Breidenbach *et al.* (1992) assert that understanding the function concept must include a process conception. They distinguish between 'action' and 'process'. Action requires an explicit recipe or formula, a step-by-step manner. When there is no necessity, for an individual, to run all the specific steps in an action, then they suggest that the action has been interiorized to become a process. Thus, to have a process conception, one does not need an explicit recipe or absolute certainty of the transformation. They suggest that, an action is relatively external to our thinking, while a process is more internal.

Action and process conceptions can be 'best regarded as opposite ends of a continuum, rather than two fully differentiated conceptions' (Selden & Selden, 1992, p. 3). Students who have an action conception of functions are most likely to handle only algebraic operations. Students in the middle of the continuum would admit logical branching and non-algebraic procedures. Finally, students who have a full process conception might not require an explicit algorithm at all (Selden & Selden, 1992).

Students need to treat functions as objects or entities and as elements of sets, need to act upon and transform functions e.g. when finding derivatives and anti-derivatives. However, students do not treat functions as objects but rather they carry out operations automatically (Selden & Selden, 1992).

Breidenbach *et al.* (1992) discusses the following three indications of a better process conception. Firstly, students in their study could work not only with functions having numbers as the domain and the range but also with, for example, Boolean valued functions. The second indication is the students' ability to imagine certain operations such as adding, composing two functions, or reversing a function. The third indication of the process conception of function is the ability to de-encapsulate the objects and represent their processes, when dealing with functions as objects (e.g. when functions themselves are the elements of the domain of a function).

2.1.5 A criticism of multiple representations of functions

The notion of multiple representations of the function concept is criticised by Thompson (1994). The following quote is very crucial:

> …the idea of multiple representations, as currently construed, has not been carefully thought out, and the primary construct needing explication is the very idea of a representation… the core concept of "function" is not represented by any of what are commonly called the multiple representations of function, but instead our making connections among representational activities produces a subjective sense of invariance…it may be wrongheaded to focus on graphs, expressions, or tables as representations of function. We should instead focus on them as representations of something that, from students' perspective, is representable, such as aspects of a specific situation. (Thompson, 1994, p. 39)

Thompson (1994) suggests that we should focus on different representations as representation of something that is representable (from students' perspective) rather than treating graphs, expressions, or tables as representations of function. He claims that if students do not realise that something remains the same as they move among different representations then they see each representation as a "topic" to be learned in isolation. As Sierpinska (1992) suggests, focusing on different aspects of functions as representing the same concept is fundamental for the understanding function. However, this is not easy to achieve since students encounter different representations in a variety of contexts in their schooling as Eisenberg (1991) mentions.

As a parallel to Thompson's (1994) critique, it is worth mentioning DeMarois, McGowen & Tall's (2000a) notion of "function plus". They suggest that students assign some extra properties to the function concept in different contexts. DeMarois, McGowen & Tall (2000a) assert that it is not the function concept itself which is studied, but rather it is a special kind of function such as linear, quadratic, trigonometric, given by a formula, differentiable etc. Instead of the term 'function', they use the term 'function plus', where 'plus' refers to the additional properties which change the nature of the function concept. A linear function, for instance, is uniquely determined by two pairs of input-output. In other words, the whole set of ordered pairs can be determined by the two ordered pairs. They mention that the "plus" is extremely subtle if the graph of a function in R is considered. In that case it is assumed that the elements of the domain and range, the real numbers, are ordered. This is an extra property that a function may carry. In other words, the concept imagery is gained from the examples of "function plus" and this is likely to lead to conflicting concept images with the core concept of function.

2.1.6 Vertical and horizontal growth

Theoretical frameworks in two different directions, operational and structural conception of functions, action-process conceptions on the one hand and multiple representations of functions on the other, have been combined together by the work of Beineke *et al.* (1992), Arcavi & Schoenfeld (1992), DeMarois & Tall (1999). Beineke *et al.* (1992) considers horizontal (the breadth of students' concept image) and vertical growth (the depth of students' formal understanding) of function concept. DeMarois & Tall (1999) explain the breadth of students' concept image by illustrating two dimensions of function concept: 'the links between various representational facets of the function concept and the layers or levels of compression in process-object encapsulation' (p. 257).

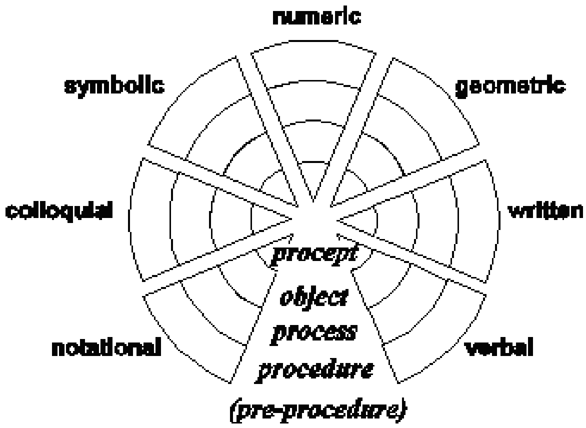


Figure 2-1. Facets and layers of the function concept (DeMarois and Tall, 1999, p. 258).

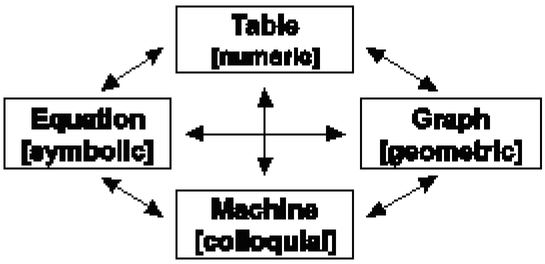


Figure 2-2. Possible links between function facets (DeMarois and Tall, 1999, p. 258).

DeMarois & Tall (1999) found that students enrolled in a developmental algebra course improved their flexibility in layers and facets as a result of a function machine approach. Interestingly, students are more successful with the symbolic facet compared to graph facet both in the pre and post tests.

2.2 Literature review on categorization

Mathematically, the category of functions is determined by the definitional properties possessed by all the category members. As mentioned in the previous section on the literature review on functions, students' focus of attention is not always on the definitional properties. They might consider some aspects as better examples of functions e.g. functions which have reasonable graphs (Vinner, 1983). Therefore, students might categorize functions in different ways. In that sense, Bakar & Tall (1992) assert that students' understanding of the function concept is reliant on the properties of the families of the prototypical examples rather than the properties of the definition.

The next sections of the literature review, therefore, focuses on the issue of categorization. It gives a brief account of different views such as prototype and exemplar views of categorizations.

2.2.1 Importance of categorization

Categorization has great importance for cognition. Lakoff (1987a) emphasizes that

> …there is nothing more basic than categorization to our thought, perception, action, and speech. Every time we see something as a kind of thing…we are employing categories. Whenever we intentionally perform any kind of action…we are using categories. The particular action we perform on that occasion is a kind of motor activity…it is in a particular category of motor actions…any time we either produce or understand any utterance of any reasonable length, we are employing dozens if not hundreds of categories: categories of speech sounds, of words, of phrases and clauses, as well as conceptual categories (Lakoff, 1987a, pp. 5-6).

'Categories allow us to access and use relevant knowledge, even for items we have never encountered before' (Ross & Makin, 1999, p. 205). Rosch (1977) emphasizes that categorization is important since it allows organisms to deal with the diversity of stimuli and therefore allows them to treat non-identical stimuli equivalently. She states that

> This important function would, thus, seem to be a prime target for theoretical accounts – by what principles do humans divide up the world in the way they do? Why do we, for example, have "red" and "orange" which are considered two different colours and "cats" and "dogs" which are considered two different animals while other cultures may cut up these domains in different ways? (Rosch, 1977, pp. 1-2)

Categorization is also important for its implications for reasoning. As Lakoff (1987a) suggests categorization is important since our understanding of categorization is closely related to our understanding of reasoning that is 'every view of reason must have an associated account of categorization' (p. 8).

2.2.2 *Different views of categorization*

There are different views of categorization, that is to say that there are different ways of explaining how category knowledge is represented. Early understanding of categorization was Aristotelian in nature: 'Categories are logical, clearly bounded entities, whose membership is defined by an item's possession of a simple set of critical features, in which all instances possessing the critical attributes have a full and equal degree of membership' (Rosch, 1975, p. 193). This view is called the *classical view of categorization*. The classical view of categorization explains well-defined categories such as "square": 'Any closed figure with four equal sides and four equal angles is a square, and all squares have these properties' (Ross & Makin, 1999, p. 208). This view of categorization has implications for reasoning. As Lakoff (1987a) suggests the classical view implies a view of reason as disembodied symbol-manipulation. The classical view disregards the role of imaginative processes such as metaphors, metonymy, and mental imagery.

Lakoff (1987a) gives a critical review of ideas on categorization which challenged the classical view, from the work of Wittgenstein in the 1950's to the work of Rosch in the 1970's. As he points out, the classical view of categorization has been taken for granted until the studies of Rosch and her associates. The studies of Eleanor Rosch provided a general perspective on categorization. Considering the example from a perceptual domain, the concept of colour, Rosch (1975) suggested that categories might have fuzzy boundaries. Like the category of colour, some categories are not represented as a set of critical features with clear-cut boundaries but rather in terms of a *prototype*. Rosch explains a prototype as the 'clearest cases' or 'best examples' (Rosch, 1975) or people's judgements of goodness of membership in the category (Rosch, 1978). There are different views on the sources of prototype effects. Lakoff (1987b) gives an account of two common interpretations of prototype effects:

> The effects=Structure Interpretation: Goodness-of-example ratings are a direct reflection of degree of category membership.
>
> The prototype=Representation Interpretation: Categories are represented in the mind in terms of prototypes (that is, best examples). Degrees of category membership for other entities are determined by their degree of similarity to the prototype (p. 64).

In the literature, there are two interpretations of "the prototype=representation interpretation". These are the *prototype* and *exemplar views of categorization* as discussed by Ross & Makin (1999). Briefly speaking, the distinction between two views is 'whether the knowledge underlying cognitive performance is a general abstraction built up from earlier experiences (prototype view) or is a function of more specific instances (exemplar view)' (p. 206). After these views, the *rational model*, a combination of prototype and exemplar views is put forward. Below, a summary account of these three models is given.

2.2.2.1 Prototype model

The prototype model assumes 'a summary representation of the category, called a prototype, which consists of some central tendency of the features of the category members' (p. 208). Classification is determined by the similarity to the prototype. The following formula from Hampton (1995) formally represent how the similarity to the prototype is computed:

$$S(A,t) = \sum_{i=1}^{n}(w_i \times v_{it})$$

$S(A,t)$: the similarity of t to category A, which for a prototype view means the similarity of t to the prototype of A.
w_i: the weight of the *i*th feature in the prototype.
v_{it}: the extent to which item *t* possesses the feature *i*. ($0 \le w_i$, $v_{it} \le 1$)

Equation 2-1. Prototype model

Ross & Makin (1999) emphasize two important characteristics of the prototype view. First, similarity has an additive function across the features. Second, each instance of a category is classified with the same single summary representation, the prototype. On the other hand, these two characteristics cause some problems. First, instances that are similar to studied instances are classified more accurately. Thus, more information than just similarity to the prototype is used. Second, not only the central tendency of features, but also the range of values of each feature or the correlations of features with each other are used in classification.

2.2.2.2 Exemplar model

The exemplar model 'assumes that the categories consist of a set of exemplars and that the classification of new instances is by their similarity to these stored exemplars' (Ross & Makin, 1999, p. 212). As Ross & Makin (1999) mention the most prominent exemplar model is the "context model" of Medin and Schaffer (1978). In this model, there is not a single summary representation as in the prototype model, but a collection of instance representations. Ross & Makin (1999) gives the formula, from Medin and Schaffer (1978), for the similarity in exemplar model as shown below:

$$S(A,t) = \sum_{a \in A} S(a,t) \qquad S(a,t) = \prod_{i=1}^{n} s_i$$

$S(A,t)$: the similarity of *t* to category A. $S(a,t) =$ similarity of *t* to the exemplar a.
s_i : similarity of *i*th feature.

Equation 2-2. Exemplar model

As understood from the formula above, an important characteristic of the exemplar model is that the similarity to each exemplar has a multiplicative function.

Ross & Makin (1999) give various problems of the exemplar model. First of all, exemplar model takes away the categoriness of the category. Second, it is questionable that abstractions are never used for classifying an unfamiliar instance. Third, 'it is not clear how to apply the exemplar view to some of the issues in category research, such as hierarchical effects or basic levels' (p. 215).

Ross & Makin (1999) also discuss that the two characteristics of the exemplar model which contrast to the prototype model provide useful information for categorization. First, the exemplar model allows for *selective* use of knowledge. That is, the most relevant information arises through focusing on the most similar exemplars. Second, the exemplar model takes into account *relational* information such as features which are possessed by more than one exemplar.

2.2.2.3 Rational model

Ross & Makin (1999) discuss the rational model, a combination of prototype and exemplar view, of Anderson (1991). According to this model, there are *miniprototypes* of exemplar clusters. When a new item is encountered, the model determines whether to add it to an existing cluster or start a new cluster. Therefore, the determination is made by calculating the similarity of the new item to each of the various existing clusters. Then the new item is assigned to the most similar cluster.

2.2.3 Going beyond prototype effects

Different views of categorizations discussed so far disregard the problem of which features to attend and also the existence of interactional properties. In other words, how particular features are chosen in the first place?

Lakoff (1987b) emphasizes that prototype effects exist but they do not imply anything direct about the nature of categorization. He suggests that the existence of prototype effects in ungraded and classical categories is due to the idealized character of cognitive models.

Although prototype effects indicate nothing direct about the nature of categorization, studies of Rosch and her co-workers have been interpreted as if prototype effects reflect the nature of categorization (Lakoff, 1987b). The fact that prototype effects indicate nothing direct about human reasoning is explained by Osherson and Smith (1981) and Armstrong, Gleitman, and Gleitman (1983) by turning back to the classical view of categorization. Lakoff (1987b) criticizes their discussion since their explanations do not consider metonymic sources of prototype effects:

> 'A major source of such effects is metonymy – a situation in which some subcategory or member or submodel is used (often for some limited and immediate purpose) to comprehend the category as a whole. In other words, these are cases where a part (subcategory or member or submodel) stands for the whole category – in reasoning, recognition, and so on. Within the theory of cognitive models, such cases are represented by metonymic models' (p. 71).

2.2.4 Final remarks

Two terms, *prototype* and *exemplar*, are used to establish the theoretical framework as will be discussed in chapter 4. It is useful to mention that the exact mathematical formulas for prototype and exemplar models are given just to emphasize the similarities and differences between the two models. The main implication from these two numerical models is that the prototype model is additive and exemplar model is exclusive. Because the prototype model is additive, the more the properties of an instance match the prototype, the more likely it is to be considered as a member of the category. However, the exemplar model is exclusive since the similarity of an instance has a multiplicative function. A zero entry will cause the final result to be zero. Hence if an instance fails to have an important property shared by all the exemplars, then it is immediately excluded from the category.

The theoretical framework does not lead into further discussion on these two models based on these formulas. Rather, the aim of focusing on these two views is to make a distinction between prototypes representing general ideas and exemplars referring to more specific cases.

CHAPTER 3 – FINDINGS FROM THE PRELIMINARY STUDY

In this chapter, results from a previous project which was part of the EdD degree are reported briefly. Although it is a separate project, it was seen as a preliminary study for this thesis. Here, mainly the results from the qualitative data from the interviews will be reported since they gave shape to the research questions for this main study.

3.1 Results from the preliminary study

The theoretical framework for the preliminary study was based on the notion of an 'informally operable definition' derived from the work of Bills & Tall (1998). They define an 'operable definition' if students make appropriate logical deductions by focusing on the definitional properties. The notion of 'informally operable definition' is defined if a student could successfully decide whether or not the given item is a function by focusing on the definitional properties. The term 'informal' refers to the fact that the formal definition is not the focus of attention. Rather it is the colloquial definition as given in section 1.2.2 on page 5. The term 'operable' is used when a student could decide whether or not the given item is a function by focusing on the definitional properties.

Subjects in the preliminary study came from three different grades (grade 1, 2, 3 of high school) and three different high schools in Turkey. There were different types of schools; one public school, one private school and one selective school. A hundred questionnaires were analyzed and based on that analysis eight students were chosen for individual interviews. These eight students represented a spectrum of performance in terms of the number of correct answers in the questionnaires.

3.1.1 Results from the interviews:

The interviews have two sections. In the first section, students were given various items and asked to decide whether or not they are functions and explain how they make their decisions. They were given the following:

- A set of ordered pairs
- Correspondence between two set diagrams
- Graphs with coloured domains
- Equations
- A verbal statement

They were also given, in succession, three forms of a constant function:

- $y = 4$
- $y = 4$ (for all values of x)
- $y = 4$ for $x \geq 2$

In the second section of the interview, students were asked to explain reasons behind their answers to the certain questions in the questionnaire which asked them to decide about various forms of functions such as equations, graphs, set diagrams, and set of ordered pairs.

The analysis of the interview data focused on how students make decisions for a variety of items. Firstly, different forms of functions are compared by focusing on responses from eight students for each form. Secondly, the eight students are compared by focusing on each student's overall responses in the interviews.

The first conclusion from the analysis of the interview data is that the students are more likely to 'informally operate' with the colloquial definition for set-correspondence diagrams and sets of ordered pairs than graphs and expressions. Here is a summary for how students responded for each form:

3.1.1.1 Set-correspondence diagrams:

The eight students in the interviews were given the following set-correspondence diagrams:

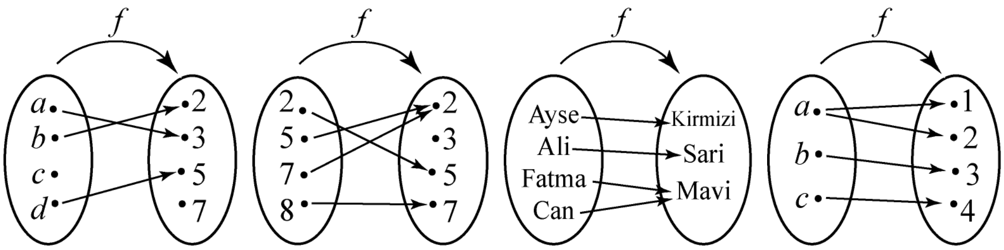


Figure 3-1. Set-correspondence diagrams in the preliminary questionnaire

Seven students referred to the definitional properties in the context of set-correspondence diagrams. Three of them successfully operated with the colloquial definition. In other words, these three students have 'informally operable definition' in that context.

3.1.1.2 Sets of ordered pairs

The eight students in the interviews were given the following set of ordered pairs:

$$f:\{1,2,3,7,9\} \to R, \ f = \{(1,3),(2,5),(3,2),(7,-1),(9,1)\}$$

They were asked to explain why they consider or reject it as a function. When deciding about this, two students informally operated with the colloquial definition. Three other students could focus on the assignment between the domain and the range.

3.1.1.3 Graphs

Students were asked to explain their answers to various graphs in the questionnaires and graphs with coloured domain in the interviews such as the following:

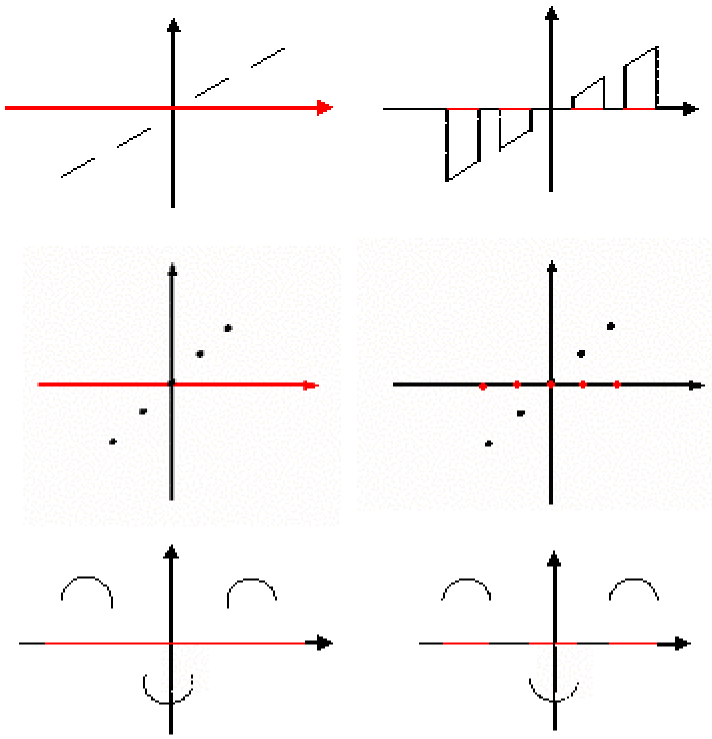


Figure 3-2. Coloured-domain graphs in preliminary interviews.

In the context of graphs, students are more reluctant to refer to the definitional properties. Two students could see the role of the domain (which was coloured as red). One student used the vertical line test but could only focus on the graph. She could not focus on the elements in the domain which were not assigned to any elements in the range. When compared to set diagrams and set of ordered pairs, in the context of graphs they easily develop prototypical examples. If the presented items do not fit their existing prototypes (e.g. graphs in one piece or graphs with smooth shapes) they do not consider them as functions.

3.1.1.4 Expressions

The most problematic context was 'function as an expression'. Students responded to the expressions such as:

- $y = 4$
- $y = 4$ for all values of x
- $y = 4$ for x ≥ 2
- $f : R \rightarrow R$, $f(x) = \sqrt{x}$
- $f : R \rightarrow R$, $f(x) = \frac{1}{x}$
- $y = 0$ (if x is a rational number)
- $y = 0$ (if x is a rational number)
  $y = 1$ (if x is an irrational number)

None of the students could successfully use the definitional properties when dealing with function as an expression. Students had great difficulty especially for the case of constant function. None of the students considered all three forms of the constant function as a function and gave idiosyncratic meanings to the three forms.

The second conclusion was based on the comparison of the students for their overall responses in the interview. When each student's response to all questions in the interview is considered, two students' responses were distinguished to be more coherent in focusing on the definitional properties for most of the questions. These two students did not rely on properties of prototypical examples as the other students often did.

3.1.2 Results from the questionnaire

The main result that concerns our main study is that the frequencies for the correct responses for the items of set diagram pictures are remarkably higher than the frequencies for the items of graphs, expressions and sets of ordered pairs (Akkoç, 2001). The other remarkable result was that the frequency percent of the correct answers for the split-domain function (which is problematic for students as previous research indicated e.g. Vinner, 1983) is higher (72%) than the other expressions. When asked for a definition, 50% of students referred to the colloquial definition which was given them after they were introduced to the formal definition. 40% of them had all the definitional properties of a function.

3.2 Refining the research problem

The findings from the preliminary study gave shape to the research problem as will be discussed in chapter 5 which describes the methodology. The fundamental finding from the preliminary study was that more students operated informally with their personal concept definitions for the set diagram pictures and sets of ordered pairs compared with graphs and expressions. On the other hand, for graphs and expressions, most of the students relied on the properties of prototypical examples instead of the definitional properties.

For these reasons, in the main study a more refined analysis was carried out by focusing on the distinction between a "prototype" and an "exemplar". In the Turkish curriculum, graphs and expressions appear in various clusters. For instance, trigonometric functions are studied as a cluster of related examples at a particular stage of the curriculum (see table 1.2). Therefore, this has a tendency to emphasise the exemplar view with several distinct examples in each cluster rather than a more general prototypical case.

CHAPTER 4 – THEORETICAL FRAMEWORK

4.1 Overview

Drawing on the literature on functions and categorization and the results from the preliminary study, the theoretical framework aims to investigate students' understanding of the function concept. A theoretical framework is established to investigate the sources of difficulties when students deal with different aspects of functions. Thompson's (1994) notion of *core concept of function* is taken of as a starting point for the theoretical framework. Starting with the notion of the core concept of function, the theoretical framework takes into account some implications of the research on categorization. It makes a distinction between a *prototype* and an *exemplar*. Forming a framework by making such a distinction between prototype and exemplar of a function, the aim of this research is not to validate that these two views of categorization are correct nor to claim that human beings categorize things one way or the other. Rather the purpose is to explore what happens when students cannot focus on the core concept of function with the help of making such a distinction.

4.2 Departure point: core concept of function

The departure point for the theoretical framework is Thompson's (1994) notion of *core concept of function*. As mentioned in section 2.1.5 in the literature review on functions, Thompson (1994) critizes the notion of the multiple-representations of the function concept and emphasizes a distinction between multiple-representations of functions and the core concept of function. He claims that if students do not realise that something remains the same as they move among different representations then they see each representation as a 'topic' to be learned in isolation. Even when students have various links between different representations of functions, overall these links may not imply the core concept of function. Therefore, investigating students' understanding of the function concept by focusing on the notion of core concept of function becomes more crucial.

There are various aspects of a function as given below:

- Formal set of ordered pair definition*
- A colloquial definition (in everyday language)*
- A function box (input-output box)
- A set of ordered pairs (considered set-theoretically)*
- A set diagram (two sets and arrows between them)*
- A table of values
- Graphs (drawn by hand or computer)*
- Expressions*

Starred aspects are the focus of attention in the Turkish context. Therefore, these will be focused as the data in this study.

4.3 Simplicity and complexity of the core concept of function

Mathematically, the core concept of function has both its *simplicity* and *complexity*. The words "simplicity" or "simple" will be used in a particular way that may be different from their use in everyday language. It is simple in the sense that the properties of it are minimal. That is, given two sets, we assign each element in the first set with a unique element in the second set. It is complex in the sense that it has a richness and it gives access to a variety of ideas (Akkoç & Tall, 2002). In other words, it acts as a unifying concept for different mathematical ideas. Some students focus on the essential of the concept definition which is central to the wider complexity. However, for many other students, the function concept may continue to be *cognitively complicated* in the sense that poorly connected ideas continue to persist without being coherently linked. For instance, some students focus on the details in different contexts, therefore could not overcome the complications.

4.4 Different aspects of functions: Prototypes versus exemplars of functions

Mathematically, the function concept belongs to a clear-cut category, the category of function. Something is either a function or not. However, for students some aspects are better examples of functions and these better examples are different for each student. Therefore, the category of function might be fuzzy for a student. As discussed in the literature review chapter, a category can be represented by prototypes or by exemplars, prototypes representing general ideas and exemplars as more specific cases. To explore the complications of the function concept, the theoretical framework makes a distinction between prototypes and exemplars.

The fundamental finding from the preliminary study was that the core concept of function is not the focus of attention for most of the students when dealing with different aspects of functions. Students had much more difficulty with graphs and expressions compared to set-correspondence diagrams and sets of ordered pairs. The personal concept definition of a student is more likely to be 'informally operable' for set-correspondence diagrams and sets of ordered pairs than graphs and expressions. This finding was explained by the fact that students deal with set diagrams and sets of ordered pairs differently from graphs and expressions. Instead of using the definitional properties, students rely on prototypical examples of graphs and expressions. However, when dealing with set diagrams and sets of ordered pairs, they do not develop prototypical examples, but are more focused on the core concept of function by using the definitional properties. Therefore, in the preliminary study, a distinction between using the definitional properties and relying on prototypical examples was made. Considering these results, the theoretical framework of this study focuses on this distinction from a different point of view. Instead of using the term “prototypical example”, the term “exemplar” is used to emphasize that graphs and expressions are more specific cases (Akkoç & Tall, 2002).

Two variations of Prototype=Representation interpretation of prototype effects for categorization are chosen to be a starting point to distinguish between set-correspondence diagrams and sets of ordered pairs on the one hand, and graphs and expressions on the other. As discussed in the literature review in section 2.2.2, one variation of Prototype=Representation interpretation suggests that prototype is an abstraction, say a schema or a feature bundle. A second variation suggests that the prototype is an exemplar, that is, a particular example (Lakoff, 1987b). Instead of using the term ‘prototypical examples’ for graphs and expressions as in the preliminary study, function graphs and expressions are treated as exemplars, as more specific cases. The aim of making a distinction between prototypes and exemplars is not to claim that human beings categorize by developing prototypes or exemplars. The reason for starting with a theoretical framework which makes such a distinction is that some aspects of functions are taught in a prototypical way while some aspects (such as graphs and expressions) are taught in clusters. Function graphs and expressions are given to students in different clusters such as linear, polynomial, logarithmic, exponential functions. These are taught as different topics at different stages in the curriculum.

4.5 Core concept of function and prototype – exemplar distinction

So what is the relationship between the core concept of function and prototype and exemplar distinction? As discussed in the preliminary study, students who could not use their personal concept definitions, heavily relied on the prototypical examples. As discussed in Akkoç & Tall (2002), students' responses revealed a spectrum of performance ranging from students who have strong focuses on the core concept of function to students who can hardly refer to definitional properties for different aspects of functions. Furthermore, students' responses to prototypical aspects of functions differed from their responses to exemplars. Exemplars of functions caused more complications.

The theoretical framework of this study considers coherency in recognizing different aspects of functions (both prototypes and exemplars) correctly with a strong focus on the definitional properties as an indication of the ability of focusing on the core concept of function as a cognitive unit. Therefore in the analysis this coherency is considered to categorize the performance of students.

CHAPTER 5 – METHODOLOGY

5.1 Overview

This study is followed by a previous Ed.D. project which was considered as a preliminary phase to this main study. The aim of the preliminary study was to refine the research problem, which was defined roughly in the beginning of the research. Findings from the preliminary study, as presented in chapter 3, indicated that students are more likely to operate informally with the colloquial definition for particular aspects of functions (set-correspondence diagrams and sets of ordered pairs). The suggested explanation for this phenomenon was that students develop prototypical examples for the other aspects such as graphs and expressions instead of referring to the colloquial definition. To investigate the phenomenon, the research problem was refined considering the theoretical framework and research questions were established.

The data came from two sources; the questionnaires which were administered to a sample of a hundred and fourteen students (in grade 3 in two high schools in Turkey) and the interviews with nine students which were selected from the whole sample. These nine students to be interviewed were selected to represent deviant cases as well as the typical cases.

5.2 Statement of the research problem

This research focuses on students' understanding of the core concept of function as they recognize graphs, expressions, set-correspondence diagrams, and sets of ordered pairs as functions. The notion of 'core concept of function' has great importance for the research problem. Many research studies on functions do not make a distinction between the concept which is focused in a particular context (e.g. linear graphs) and the core concept of function. Thompson (1994) emphasizes that 'the core concept of "function" can not be represented by any of what are commonly called the multiple representations of functions' (p. 39). However, when focusing on students' understanding of the core concept of function as a research problem, it is unavoidable to focus on each aspect of function in isolation. To be able to investigate students' understanding of the core concept of function, coherency in the way a student reasons about different aspects of functions will be considered as an indication of an understanding of the core concept.

Considering the previous research on functions and the results from the preliminary study, a set of research questions and subquestions are established. The main four research questions are as follows:

1. Do students use the core concept of function to recognize a function?

1a – Do students use the formal definition or colloquial definition or any other method to respond to different aspects of functions?

2. Whatever the response is, what do they do to recognize a function?

2a – Which parts of the concept image are evoked for each aspect of a function?

2b – Does a student use vertical line test for graphs? If so, is the use of vertical line test for graphs procedural or conceptual?

3. How do the various aspects of a function play their part?

3a – Do students develop clusters of exemplars for graphs and expressions?

3b – Do students see set-correspondence diagrams and ordered pairs as prototypes to abstract definitional properties? If so, how do they use it in a prototypical way?

3c – How does a student use the definitional properties for a given aspect of a function?

3d – How is a student's overall response to different aspects of functions effected by the subtle differences among various aspects?

4.What do these three research questions imply for students' understanding of the core concept of function?

4a – How coherent is a student's response as s/he move from one aspect to the other?

4b – How do students who give coherent responses to different aspects of functions cope with this?

5.3 Defining the methodology

This research is mainly qualitative. However, quantitative methods were also combined in the research. There are reasons for giving priority to qualitative approach. First of all, the nature of the research problem, focusing on the understanding of core concept of function, requires a qualitative inquiry. Research questions focus on students' understanding of the core concept of function for various aspects. When doing this, students' evoked concept images for each aspect of function are investigated. Quantitative methods are not seen as suitable for that purpose since they focus on the outcome or product rather than the process (Denzin & Lincoln, 1994). Furthermore, the process that is focused aims to reveal a student's thinking during reasoning. Therefore, quantitative methods would be insufficient for this aim. Also, quantitative methods require predetermined variables and look for relations between them. This could not reveal what is in a student's mind. Sub-questions of research question 3, which begin with 'how' can not be answered with quantitative methods.

Although the approach to research is qualitative, quantitative methods are also combined to the research. There are various reasons for that.

The first reason is to select subjects for the interview. Subjects are selected by a questionnaire since it allows the researcher to make a selection from a bigger sample. To reduce the threats to the validity of quantitative data from the questionnaire, open-ended questions are included in the questionnaire. In the questionnaire students are asked to give reasons for their answers so that the selection can be based on their reasoning as well as the number of correct answers.

Brannen (1992) discusses two other purposes of combining methods: complementary and integrative. For complementary purposes, qualitative and quantitative approaches are used in relation to a different research problem or different aspect of a research problem. For integrative purposes, qualitative and quantitative approaches focus on the same research problem and enhance claims concerning the validity of the conclusions that could be reached about the data. In this study, both purposes are relevant.

For complementary purposes, some of the research questions require quantitative methods since they are hypothetical. To explain whether students develop exemplars of function graphs, one exemplar cluster, the sine function, is chosen. Various graphs are chosen to be either in that cluster or a combination of two exemplars from two different clusters. It was hypothesized that if a student has that particular exemplar cluster then s/he would consider other graphs as a function which are in that exemplar cluster. For integrative purpose, quantitative and qualitative data will be used for triangulation to validate the conclusions to be reached.

5.4 Method of data collection

5.4.1 Questionnaire

Research involves the distribution of a hundred and fourteen questionnaires. The questionnaire has two purposes. The first purpose is to select students for the interview. The questionnaire was seen to be appropriate for selection since it allows the researcher to choose students from a wide population. To be able to make a more precise selection, students are asked the reasons for their answers.

The second purpose of the questionnaire is to obtain a secondary source of data to triangulate with the qualitative data. Therefore, some aspects of the research problem could be looked at over a bigger sample.

5.4.1.1 Subjects

Unlike the preliminary questionnaire (which was administered to grade 1, 2, 3 students in high school) the scope of the main study was restricted to only grade 3 of high school (17 year-old students) since the purpose is not to compare different grades where students have different backgrounds on functions. Rather, the focus is on grade 3 students. Ideally the aim is to produce a representative sample of students in grade 3 in Turkey. However, only two schools were chosen due to the time and sources available. In grade 3, there are three different subject groups. The table below shows the distribution of the subjects across different subjects and schools:

	Maths and Science	Social Subjects	Turkish and Mathematics	Total
Özel Adana Lisesi	22	–	19	41
Borsa Lisesi	18	42	13	73
Total	40	42	32	114

Table 5-1. Distribution of students in the sample across different subjects and schools.

The first school, Özel Adana Lisesi, is a private school where parents need to pay tuition fees. Private schools are better schools in the sense that they have better resources with better teachers. The rate of students going to the university is higher than from public schools.

The second school, Borsa Lisesi, is a public school. Students do not need to pay for tuition fees. Compared to the private ones, the population in the classroom is very high and the success for entering university is lower.

In both types of school, teachers follow a national curriculum. Textbooks to follow are announced by a board in The Ministry of National Education. Both schools in this study follow the same textbook (Demiralp *et al.*, 2000).

5.4.1.2 Sampling

Among the probability sampling types, multi-stage sampling is chosen. As Denscombe (1998) mentions, multi-stage sampling involves identifying an initial sample (possibly a cluster, possibly not) and then choosing a sample from among those in the initial level sample. This study focuses on grade 3 students in Turkey who have been studying functions for three years. Therefore all subjects were chosen from grade 3 students. Students studying three different subjects are chosen to be three clusters and nearly equal numbers of students from each cluster are chosen. Two schools mentioned above were chosen for practical reasons such as availability in terms of access.

5.4.1.3 Procedure of administration of the questionnaire

Students were allowed a time of a whole session (around 40 minutes) to complete the questionnaire. Since I was a full-time student in the UK and since my time available in Turkey was limited, a certain number of questionnaires were sent to Turkey by post prior to my arrival to save some time. While administering the rest of the questionnaires by myself, I have coded the questionnaires, which were already administered and ready to code on my arrival.

5.4.1.4 Content of the questionnaire

The questionnaire can be found in Appendix A1. It includes nine question and focuses on four different aspects of functions:

- Graphs
- Expressions
- Correspondence between two set diagrams
- A set of ordered pairs

Various items of these aspects are included and students are asked whether they are functions or not. The reasons behind the answers are also asked after each item. There are also questions, which ask students to give a couple of examples of function graphs and expressions; a question which asks students to think of a graph and draw if they can see it in their minds, a question which asks them to write down an equation that comes to their minds. Finally, the last question asks them to write the definition of a function.

5.4.1.5 Rationale for the questions included in the questionnaire

Below a rationale is given for explaining why each question is chosen for the questionnaire:

Question1: Give a couple of examples of functions.

Question 1 is included to investigate which examples come to students' minds when they think about a function and to see whether their evoked concept images are particular aspects of a function.

Question2: Think of a graph of a function in your mind.

Can you see it?

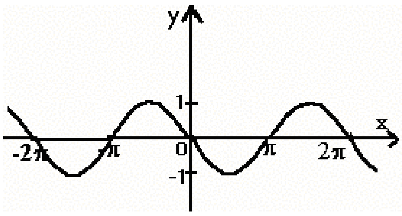
î Yes
î No

Now draw a sketch of the function here:

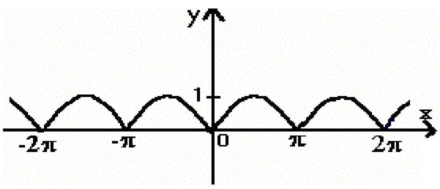
This question is included to reveal each individual student's typical exemplar of a function graph.

Question 3: Below various graphs are given. Which of the following graphs are graphs of a function of x from R to R? Tick as appropriate. Give reasons for your answers.

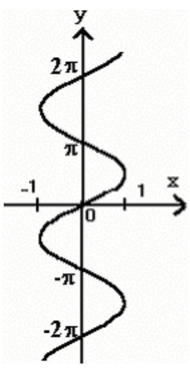
a)



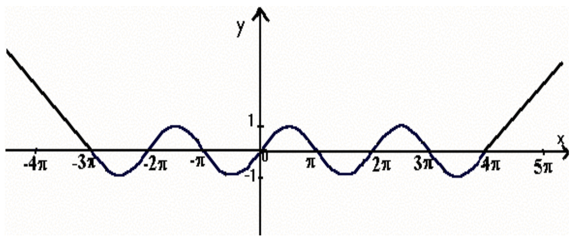
b)



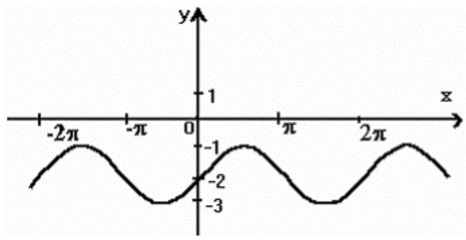
c)



d)

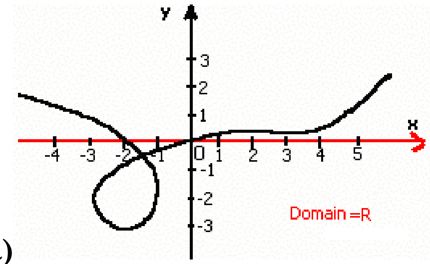


e)

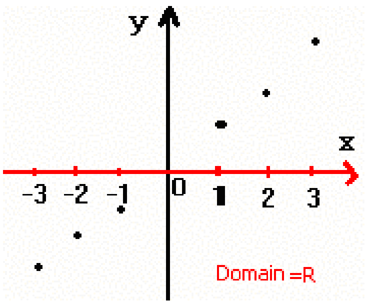


This question focuses on the issue of exemplars of graphs. Four graphs (3a, 3b, 3c, 3e) are chosen to be symbolically in the same exemplar cluster (cluster of $f(x) = \sin x$): $f(x) = -\sin x$, $f(x) = |\sin x|$, $f(y) = -\sin y$, $f(x) = \sin x - 2$. Item 3d is chosen to be a graph which is a combination of two exemplars, namely the graph of *sine* function and a linear graph.

Question4: Below various graphs are given. The domain is coloured as red. Which of the following graphs are graphs of a function of *x*? Tick as appropriate. Give the reason for your answer.

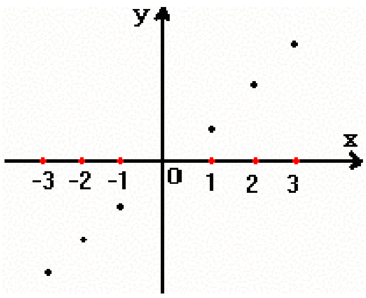


a)

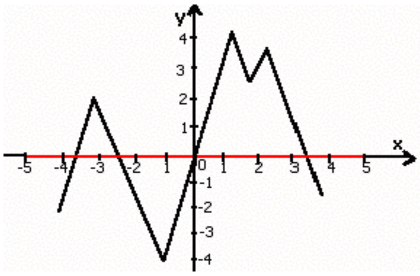


b)

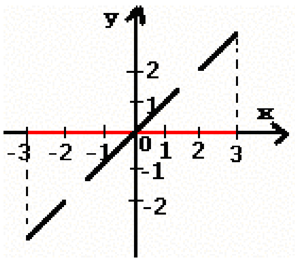
c)



d)



e)



Question 4 focuses on the graphs with coloured domains. The aim of this question is to see whether or not students consider the role of the domain when recognizing graphs of functions.

Question 5: Write down a function equation which comes into your mind immediately.

This question is included to see what kind of exemplars of function expressions that students have.

Question 6: Below various equations are given. Which of the following equations represent a function of *x*? Tick as appropriate. Give the reasons for your answers.

a) $f: \mathrm{R} \to \mathrm{R}$, $f(\mathrm{x})=\sqrt{x^2-16}$

b) $f: \mathrm{R} \to \mathrm{R}$, $\mathrm{x}^2+\mathrm{y}^2=1$

c) $y=5$

d) $y=5$ (for $\mathrm{x} \geq 2$)

e) $y=5$ (for all values of x)

f) $f: \mathrm{R}^+ \to \mathrm{R}$, $f(\mathrm{x})=|\mathrm{x}^2-4|$

g) $f: \mathrm{R} \to \mathrm{R}$,

$$f(\mathrm{x})=\begin{cases} 1, & \text{if } \mathrm{x}^2-2\mathrm{x}+1>0 \\ 0, & \text{if } \mathrm{x}^2-2\mathrm{x}+1=0 \\ -1, & \text{if } \mathrm{x}^2-2\mathrm{x}+1<0 \end{cases}$$

This question includes various expressions which are exemplars and non exemplars of a function. Items 6a and 6b are chosen to be expressions which are not functions but similar to examples of functions as an expression. Item 6g, which is actually $f(x)=sign(x^2-2x+1)$, is included to see whether it is considered as an exemplar. It is hypothesized that students might not consider it as a function without the symbolic clue of *sign*.

Three cases of constant function (6c, 6d, 6e) are also included. The domain of each function is not specified. This is done on purpose. By doing that it is aimed to explore the students' side of the didactical contract about the notion of domain. It was assumed that students might or might not need to ask about the domain.

Question 7: A={1,2,3,4} B={1,2,3}are given. Which of the set of ordered pairs are functions from A to B? Tick as appropriate. Give reasons for your answers?

a) f : A → B f ={(1,1), (2,1), (3,2), (4,2)}

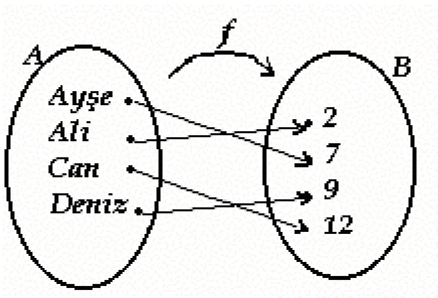
b) g : A → B g ={(1,1), (1,2), (2,2), (3,3), (4,3)}

c) h : A → B h = {(1,1),(2,2)}

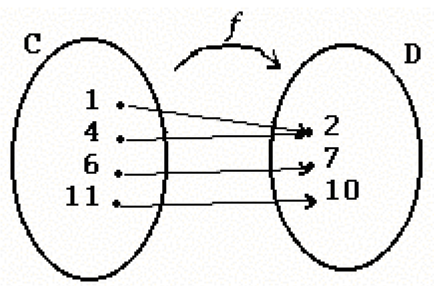
This question is asked to investigate how students deal with sets of ordered pairs and whether they can focus on the definitional properties.

Question 8: Which of the following are functions? Tick as appropriate. Give reasons to your answers.

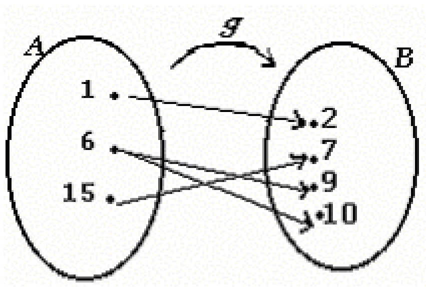
a)



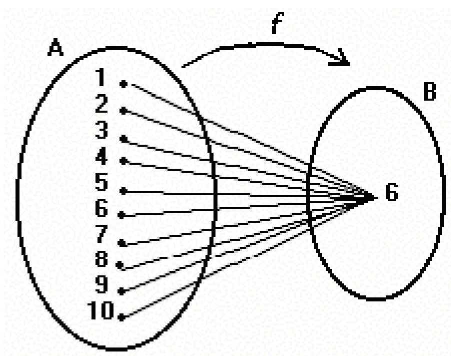
b)



c)



c)



This question is included to investigate how students deal with set correspondence diagrams and whether they can focus on the definitional properties.

Question9: Give the definition of a function.

This question is included to investigate students' personal concept definitions.

5.4.2 Interview

The second phase of the data collection involved interviews with nine students. Interviews are semi-structured. A set of questions was asked to all students. This standardized structure of the interview has the purpose of making comparisons between students for their overall successes and making comparisons between different aspects of functions (See Appendix B1).

5.4.2.1 Rationale for interview questions

The interview schedule can be found in Appendix B1. One set-correspondence diagram and a set of ordered pairs, both of which are not functions, were included. It was thought that counter examples could assess their understanding better. For graphs and expressions, it was thought that a larger number of items should be included. By doing this, it is possible to eliminate the exemplar effect from a single item in students' responses. Therefore, it would be more secure to say that students' responses reveal a coherency if they could do so. As in the questionnaire, two types of graphs were included. The first type is the graphs defined from R to R. The second type of graph is the coloured-domain graph. Part of the x-axis is coloured with red to refer the domain of the function. Some of the first type of graph were chosen to be familiar e.g. linear and sine graphs. Some graphs were chosen to be non-exemplar graphs. These non-exemplar graphs are not functions. On purpose, they were drawn in a way that it is difficult to distinguish if there are two corresponding values for a value in the domain. therefore, it was aimed to distinguish students who have a strong focus. Coloured-domain graphs were included to see whether students focus on the domain.

Four of the expressions are the same as in the questionnaire. One expression, "$f: R \to R$, $f(x) = \sin x - 2$", is included to make a comparison to the results for its graph.

Finally, transformation from "$f: R \to R$, $f(x) = 5$" to other aspects of functions are asked to see the links between different aspects of a function. "$f(x) = 5$" is chosen for a purpose. The constant function is a singular case since the term "variable" is often associated with a function and the constant function does not vary. It is thought that the transformations from the constant function to other aspects might reveal students' understanding of the constant function as well as links between function as an expression and the other aspects.

5.4.2.2 Selecting students for the interview

Since I had to administer the questionnaire and carry out the interviews in a limited amount of time available while I was in Turkey, I had to choose the sample for the interview before the deeper analysis of the questionnaire. Selection of students for the interview is based on theoretical sampling. As Mason (1996) asserts theoretical sampling means selecting a sample on the basis of their relevance to the research questions and theoretical positions to be able to build in certain characteristics or criteria which help to develop and test the theory and explanation. Therefore, the criteria for selection are based on the research problem under investigation. As discussed in chapter 3, findings from the preliminary study indicated that students' personal concept definitions are more likely to be operable for set-correspondence diagrams and sets of ordered pairs than graphs and expressions. Students' responses to the questions in the questionnaire were considered in that sense. First, considering the total number of correct answers to questions 3, 4, 6, 7, 8 (a total of 24 items), various numbers of correct answers (high, average, and low) were selected. Considering these students, reasons behind answers were considered to see whether they could use the definition for various aspects of functions. Nine students were selected revealing a spectrum of performance. When selecting students, a few deviant cases were considered as well as the typical cases supporting the results from the preliminary study. As Silverman (2000) mentions deviant cases are the negative instances as defined by the theory and they can offer a crucial test of a theory. A typical case could be either where a student is successful or unsuccessful for all questions or where a student is successful with only set-correspondence diagrams and/or sets of ordered pairs. A deviant case is where a student is successful for graphs and/or expressions but not for set-correspondence diagrams and/or sets of ordered pairs. The table 5.2 below summarises the characteristics of the sample for the interview with deviant cases shaded:

	Number of correct answers (among 24 items)	Set diagrams (Four items)	Ordered pairs (Three items)	Graphs (Ten items)	Expressions (Seven items)
Ali	18	CD for non-function items	CD for two items	CD for coloured domain items/Finding formulas	No
Aysel	17	No explanation	No explanation	CD for some items	No explanation
Ahmet	14	CD	CD	CD/VLT	CD (wrong)
Arif	13	CD	SD/CD (wrong for some)	CD for some	CD for some
Cem	12	No explanation	No explanation	VH	No explanation
Belma	9	CD	SD	No explanation	No explanation
Belgin	8	CD	CD for some/CD wrong	CD for some	No explanation
Deniz	5	No explanation	No explanation	VH	No explanation
Demet	0	No explanation	No explanation	No explanation	No explanation

Table 5-2. Number of correct answers and a spectrum of different responses to reasons behind answers for students selected for the interview.

Abbreviations: CD: Colloquial definition, SD: Set diagram (drawing a set diagram), VLT: Vertical Line Test, VH: Visual hints. (Shaded rows stand for the deviant cases).

Aysel was considered as a deviant case since she used the colloquial definition only for graphs but not set diagrams and ordered pairs. Arif was also considered as a deviant case since he used the colloquial definition wrongly for some of the set of ordered pair items but used correctly for some of the graph and expression items. Similarly, Belgin used the colloquial definition wrongly for the set of ordered pairs and used the colloquial definition for some of the graph items.

A detailed account of how each student responded to each item is given in Appendix B2. The reader may note that the names begin with letters A, B, C, D. This relates to the choice of pseudonyms at a later stage of analysis which will be discussed in chapter 8.

5.4.2.3 Background of the students

The table below shows some background information of students involved in the interviews.

Name	Grade	Subject	School
Ali	3	Maths and Science	Adana Lisesi
Aysel	3	Maths and Science	Adana Lisesi
Ahmet	3	Maths and Science	Adana Lisesi
Arif	3	Turkish and Maths	Borsa Lisesi
Belma	3	Turkish and Maths	Borsa Lisesi
Belgin	3	Turkish and Maths	Borsa Lisesi
Cem	3	Social Subjects	Borsa Lisesi
Demet	3	Social Subjects	Borsa Lisesi
Deniz	3	Social Subjects	Borsa Lisesi

Table 5-3. Background of students in the interview

Each student's responses to the questions in the questionnaire can be found in Appendix B2.

5.4.2.4 Interviewing technique

Interviews were semi-structured (See Appendix B.1). All questions are asked to all nine students. After each question, follow-up questions are asked to reveal the reasons behind students' answers. Some features of clinical interviewing method are considered. First of all, clinical interviewing aims to understand the underlying thinking of a child, to enter the child's mind rather than evaluating in the same way that a test evaluates. Therefore, in clinical interviews questions like "How did you do this?" and "Why?" are asked. Questions are asked from a student-centred point of view e.g. "What is your way of adding the numbers?" (Ginsburg, 2000). Therefore, in this follow-up questioning part of the interview, a "thinking aloud" approach is used. The following phrases are emphasized in the beginning of the interview:

> "I want you to think aloud, and tell me what is in your mind. It is not important to answer right or wrong. Try to tell me what is going on in your head".

These phrases are also emphasized before each question. If students could not say anything, they are allowed a sufficient time to think.

A second feature of a clinical interviewing technique, *strength of conviction*, is also taken into account. As Ginsburg (2000) discusses, Piaget pointed out that children tend to say what they believe the adult wants to hear. Piaget's methods of "repetition" and "countersuggestion" aim to obtain a strength of conviction (Ginsburg, 2000). Therefore, the phrases like "It is not important to answer right or wrong. Try to tell me what is going on in your head" are repeated throughout the interview. If a student explains successfully why a given item is a function or not, s/he is asked a non-function item as a countersuggestion. Also, when a student gives an explanation to answer successfully, s/he is asked the same question from a different angle with a counter-suggestion to his/her response to obtain persistency in the responses. If a student seems to be reluctant, they are encouraged to tell what comes into his/her mind, right or wrong. If s/he still does not respond, then the next question is asked and that question is asked later on again.

5.4.2.5 Procedure of interviewing

In the first school, Özel Adana Lisesi, three students were interviewed. In the second school, Borsa Lisesi, six students were interviewed. Students to be interviewed were informed on the day of the interviewing. They were interviewed during the Physical Education and Religious Education sessions. Permission was obtained from their teachers.

All interviews were audio taped. Permissions from the students were obtained. All of the students accepted being recorded. However, it was observed that the students in the second school, Borsa Lisesi, were a bit anxious about the recording while the other three students in Özel Adana Lisesi were very relaxed.

5.5 A framework for analysis

The analysis is carried out in two parts. Firstly, how students as the whole sample deal with different aspects of functions is investigated. To do that, descriptive statistics are used to give the percentages of correct answers to the questions and the percentages of different kinds of explanations they give for their answers. These results from the questionnaires are presented in chapter 6. Secondly, the main data analysis will focus on nine students who were chosen for the individual interviews. The aim is to categorize their overall responses. This will be done by preparing a grid which summarizes their responses to different questions in the interview. Responses of all students to different aspects of functions is summarized in chapter 7. These results will be considered to prepare the grid so that a categorization is made by focusing on each student's overall responses.

5.6 Validity and reliability

In this section, the validity and reliability of the questionnaire and the interview are discussed. Bell (1999) defines reliability as 'the extent to which a test or procedure produces similar results under constant conditions on all occasions' (p. 103). As Anderson & Arsenault (1998) emphasize, the data in educational research must be reliable if the analysis is to have any meaning. Validity is the extent to which an item measures or describes what it is supposed to measure or describe (Bell, 1999). Although unreliability implies lack of validity, reliability does not imply validity. Anderson & Arsenault (1998) discuss two kinds of validity: internal and external validity. Internal validity is related to the truthfulness of the results. External validity is concerned with the generalizability of the obtained results.

If we consider the reliability of the questionnaire, as mentioned by Anderson & Arsenault (1998), most straightforward multiple-choice questions are answered consistently therefore would have higher reliability. In questions 3, 4, 6, 7, 8, there are different kinds of multiple-choice items which ask students to choose from three options: "function", "not a function", "I don't know". Although data from multiple-choice items are considered to be reliable, they may lack validity (Anderson & Arsenault, 1998). To increase the validity of the questionnaire, open-ended sections are included. Students are asked to give reasons for their answers. However, including a lot of open-ended sections may result in a high no-response rate.

One other aspect of validity of this study is concerned with the curriculum. In the curriculum, students may not have met with the type of questions given in the questionnaires and interviews. These questions ask them to reason about an item to decide whether or not it is a function. In the curriculum the emphasis is on the mechanics of the procedures rather than the meaning of the function concept. However, for the purpose of this research, students were asked these questions to investigate whether they can focus on the definitional properties. At the same time, the unfamiliarity of the questions they needed to answer is taken into account when considering the validity of the results. To do that, students are asked why they responded the way they did both in the questionnaires and interviews.

One concern about the validity of the questionnaire arose through a weakness in design. A set correspondence diagram in question 1 and a set of ordered pairs in question 2 are presented to students. However, both of them are non-function items with the same reason (an element in the domain is assigned to more than one element in the range). Therefore, it is difficult to investigate whether a student checks all parts of the colloquial definition. This could be eliminated in a future study by including items which are functions and items which are not functions with a different reason (e.g. where not all of the elements in the domain are assigned to an element in the range).

The coding in the analysis in this research was checked in collaboration with my supervisor. The reliability of the results can be tested by analyzing and coding students' responses by multiple readers. It should have been further tested by analysis and coding by multiple readers, but this was not possible in the given time-frame.

CHAPTER 6 – RESULTS FROM THE QUESTIONNAIRES

Although the main purpose of the questionnaire was to choose students for individual interviews, the questionnaire is also used to investigate how the whole sample of students deal with functions. This chapter gives a brief account of the procedure of coding and the results from the questionnaire. The results have two parts. In the first part, there are results from the closed-ended questions. In the second part, there are results for the open-ended questions. Various categories emerged from the responses to the open-ended questions. Later on, they are considered together with the categories from the responses in the interview. Both sets of categories are considered together to form a grid which reflects students' overall responses to various aspects of functions.

6.1 Coding the questionnaire

There are two kinds of questions in the questionnaire; closed and open-ended. Closed questions are pre-determined. For open-ended questions the responses are put into various categories.

6.1.1 Pre-coded closed questions:

The questionnaire can be found in Appendix A1. It involves twenty-four closed questions (3a, 3b, 3c, 3d, 3e, 4a, 4b, 4c, 4d, 4e, 6a, 6b, 6c, 6d, 6e, 6f, 6g, 7a, 7b, 7c, 8a, 8b, 8c, 8d). For each of these items students are asked to tick from the following:

- Function
- Not a function
- I don't know

The responses for these items are coded to SPSS as nominal variables:

Function/Not a function/I don't know/No response

6.2 Results from the questionnaire

6.2.1 Question 1

In question 1, students were asked to give a couple of examples of functions. Mostly, they referred to functions as formulas, graphs and set-correspondence diagrams. Table 6.1 below summarizes their responses:

Responses to question 1

	Frequency	Percent
Writing formulas	25	21.9
Drawing graphs	21	18.4
Giving set correspondence diagram	18	15.7
Writing composition of functions	12	10.5
Giving examples of one-to-one and onto functions	9	7.8
Writing notations such as f(x), gof(x)	5	4.3
Other	3	2.6
No responses	21	18.4
Total	114	100.0

Table 6-1. Frequency counts and percentages of categories of examples of functions given by students in question 1 in the questionnaire.

6.2.2 Function as a graph

6.2.2.1 Question 2

In question 2, students are asked to think of a graph and to draw it if they can see it in their minds. In the Table 6–2 below, frequencies for the categories of students' responses are given:

Responses to question 2

	Frequency	Percent
Parabola	33	28.9
Straight line	10	8.8
Polynomial	8	7.0
Other	16	14.0
No graph	47	41.2
Total	114	100.0

Table 6-2. Frequencies and percentages of categories of examples of graphs given by students in question 2 in the questionnaire.

Assuming the fact that people are more likely to list more representative examples when asked to draw examples of category members (Lakoff, 1987a), it can be said that parabola, straight line and polynomial graphs are more representative examples of graphs for students.

6.2.2.2 Question 3

In Question 3 students are presented with five graphs. Frequencies of the answers for each item are presented in Table 6–3 below. In each cell the first row represents the percentages (with frequencies in parenthesis) of answers. The second row represents the percentages (with frequencies in parenthesis) of students (in the whole population) who give an explanation with their answers. Correct answers are marked in bold:

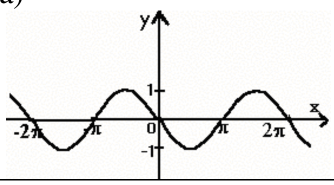
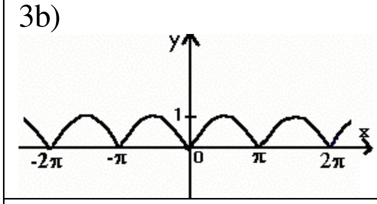
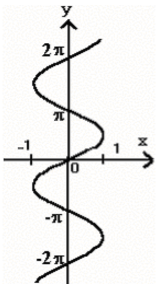
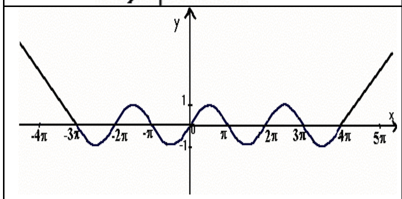
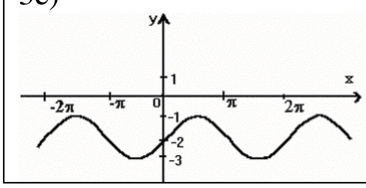
Frequency % (count) Explanation[% (count)] N=114 Question 3	Function	Not a function	I don't know	No response
3a) 	**47.4% (54)** **[30.7% (35)]**	14% (16) [11.4% (13)]	37.7% (43)	0.9% (1)
3b) 	**22.8% (26)** **[11.4% (13)]**	25.4% (29) [15.8% (18)]	50% (57)	1.8% (2)
3c) 	21.9% (25) [8.8% (10)]	**27.2% (31)** **[18.4% (21)]**	50% (57)	0.9% (1)
 3d)	**29.8% (34)** **[13.2% (15)]**	17.5% (20) [11.4% (13)]	49.1% (56)	3.5% (4)
3e) 	**13.2% (15)** **[4.4% (5)]**	39.5% (45) [25.4% (29)]	43.9%(50)	3.5% (4)

Table 6-3. Percentages and frequencies of answers to graphs in question 3.

The results from the questionnaire indicate that the percentages of correct answers to 3a (graph of $f(x) = -\sin x$, which was chosen as an exemplar), is the highest (47.4% - 54) among other graphs. The percentages decline to 22.8% (26), 27.2% (31), 29.8% (34) for the other graphs in 3b, 3c, 3d. The lowest percentage is for 3e, 13.2% (15). These results are consistent with the hypothesis that the graph of $f(x) = -\sin x$ is a more central exemplar than the graph of $f(x) = \sin x - 2$. A more detailed analysis of the responses given for these two graphs is made in the next chapter where the results from the interviews are presented.

6.2.2.3 Reasons for responses to question 3

The following categories emerged from students' explanations for their responses to items 3a, 3b, 3c, 3d, 3e in question 3. Examples of students' verbal explanations are given for each category:

- *Colloquial definition*: Use of the colloquial definition. Making statements to check the definitional properties:

"No element is assigned to more than one element", "for $x \in R$, y takes a value between 1 and -1" (3a).

- *Colloquial definition wrongly used*: Either recalling the colloquial definition wrongly or using it in a wrong way:

"Same values takes different values. For instance, it should be, $f(5\pi)=f(2\pi)$" (3d).

- *First impression/general appearance of the graph*:

"this shape doesn't look like a function", "a graph can not be like this" (3a).

"that's a wrong graph", "I haven't seen such a graph like this before, like mountains in a row, like Taurus Mountains" (3d).

- *Specific visual hints*:

"it intersects x axis at various places", "function can not be negative on y axis" (3a).

"A function can't go only upwards" (3b).

"it's (the graph) on the same surface (probably referring to *y* axis)" (3c).

"it's not a function since it has nothing related to *x* axis", "it doesn't touch to *x* axis", "it only passes through *y* axis" (3e).

- *Other*

- *No explanation*: Responses like 'I don't like maths' and 'no responses' are considered in this category.

Students' explanations are presented in detail in Appendix A2.1. The distribution of students' explanations for their answers across the categories above are presented in the table below:

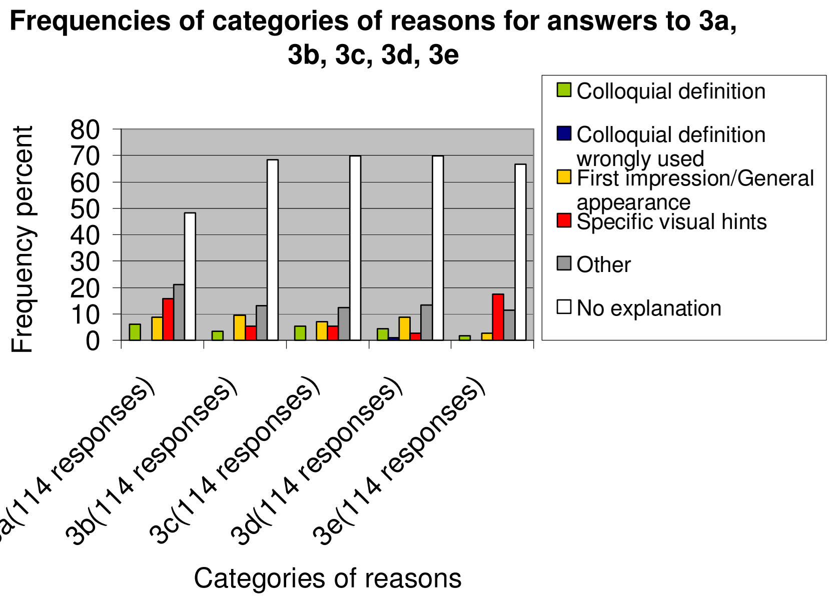


Table 6-4. Frequencies of categories of reasons for answers to 3a, 3b, 3c, 3d, 3e.

As seen in the table above, very few students refered to the colloquial definition. The table below summarizes students explanations for the correct answers:

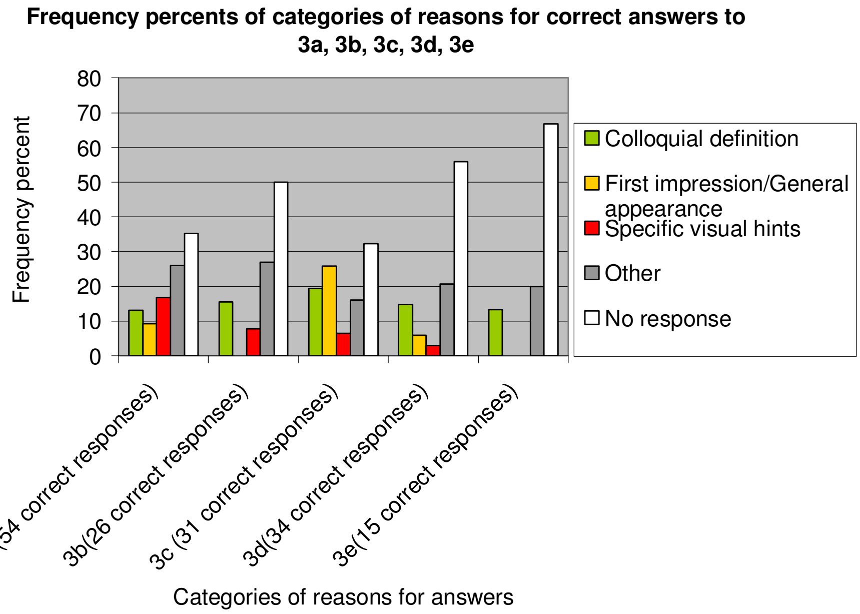


Table 6-5. Frequencies of categories of reasons for correct answers to 3a, 3b, 3c, 3d, 3e.

As seen in the table above, not all students who correctly gave the correct answers used the colloquial definition. Some of them focused on these graphs as exemplars by relying on their general appearances or specific visual hints. None of the students used the colloquial definition to recognize the graph in 3e (graph of $f(x) = \sin x - 2$) as a function. Most of the students correctly rejected it as a function did so because of the specific visual hints e.g. the graph being under the x-axis.

Those students who answered incorrectly mostly relied on the general appearances of the graphs and especially the specific visual hints as seen in the table below:

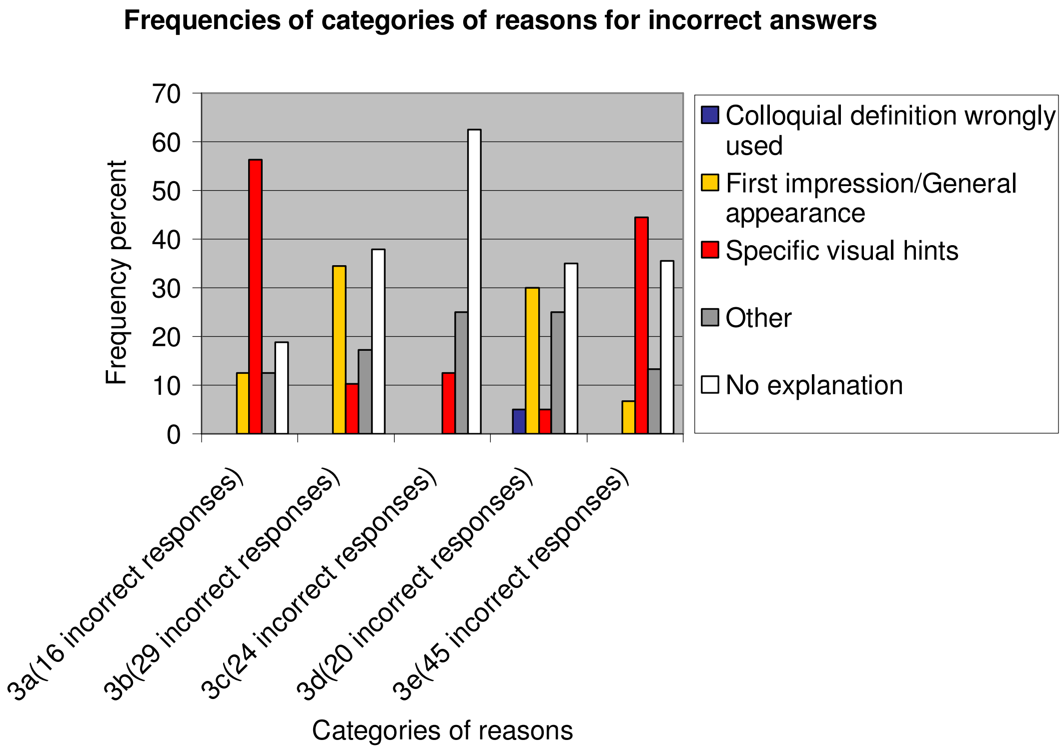


Table 6-6. Frequencies of categories of reasons for incorrect answers to 3a, 3b, 3c, 3d, 3e.

6.2.2.4 Question 4 – Coloured-domain graphs

In question 4, there are graphs with domains which are coloured as red. Table 6–7 below shows the frequencies (of responses) for each item. In each cell the first row represents the percentages (with frequencies in parenthesis) of answers. The second row represents the percentages (with frequencies in parenthesis) of students (in the whole population) who give an explanation with their answers. The correct answers are marked in bold:

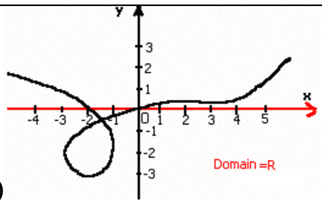
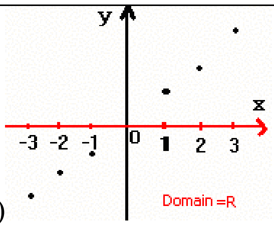
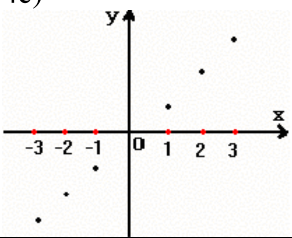
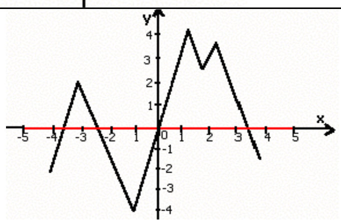
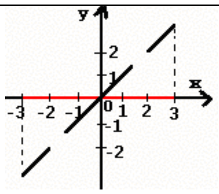
Frequency % (count) Explanation[% (count)] / Question 3 N=114	Function	Not a function	I don't know	No response
4a) 	7.9% (9) [0% (0)]	**50.9% (58) [30.7% (35)]**	39.5% (45)	1.8% (2)
4b) 	42.1% (48) [28.1% (32)]	**20.2% (23) [12.3% (14)]**	35.1% (40)	2.6% (3)
4c) 	**39.5% (45) [22.8% (26)]**	12.3% (14) [7% (8)]	39.5% (45)	8.8% (10)
4d) 	15% (17) [5.3% (6)]	**32.7% (37) [16.8% (19)]**	49.6% (56)	2.7% (3)
4e) 	33.3% (38) [14.9% (17)]	**22.8% (26) [16.7% (19)]**	43.0% (49)	0.9% (1)

Table 6-7. Percentages and frequencies of answers to graphs in question 4.

6.2.2.5 Reasons for responses to question 4

The following categories emerged from students' explanations for their responses to items 4a, 4b, 4c, 4d, 4e in question 4. Examples of students' verbal explanations are given for each category:

- Colloquial definition:

"an element in the domain can not be assigned to more than one element in the range" (4a).

"there can't be elements left in the domain" (4b).

"some elements of the domain do not have corresponding values" (4e).

- Colloquial definition wrongly used:

"It's a function since all elements are assigned to each other" (4b).

"it's not a function since elements in the domain are assigned to more than one element" (4c).

"all of the elements in the domain are assigned to an element" (4d).

- First impression/general appearance of the graph:

"it looks like familiar" (4c).

"there can't be a function like this, like a graph of a beating heart" (4d).

- Specific visual hints:

"the graph doesn't pass through from integers on the x axis such as 3 or 4" (4d).

- Other
- No explanation

Examples of students' verbal explanations are presented in detail in Appendix A2.2. The distribution of students' explanations for their answers across the categories above are presented in the table below:

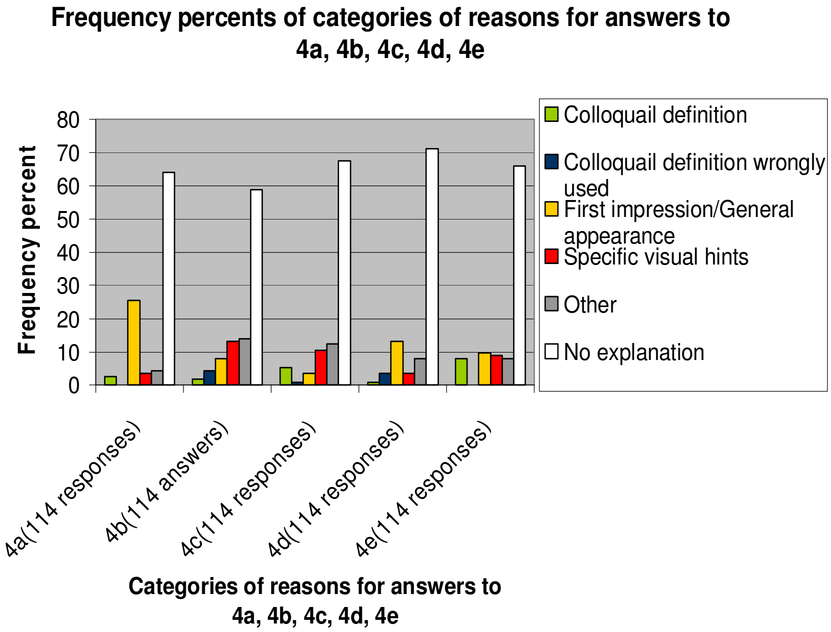


Table 6-8. Frequencies of categories of reasons for answers to 4a, 4b, 4c, 4d, 4e.

The results indicate that students tend to reason about graphs by looking at their general appearances or specific visual hints but not using the colloquial definition even when the domain is mentioned. As seen in the table below, even the students who gave correct answers relied mostly on the general appearances of the graph or specific visual hints:

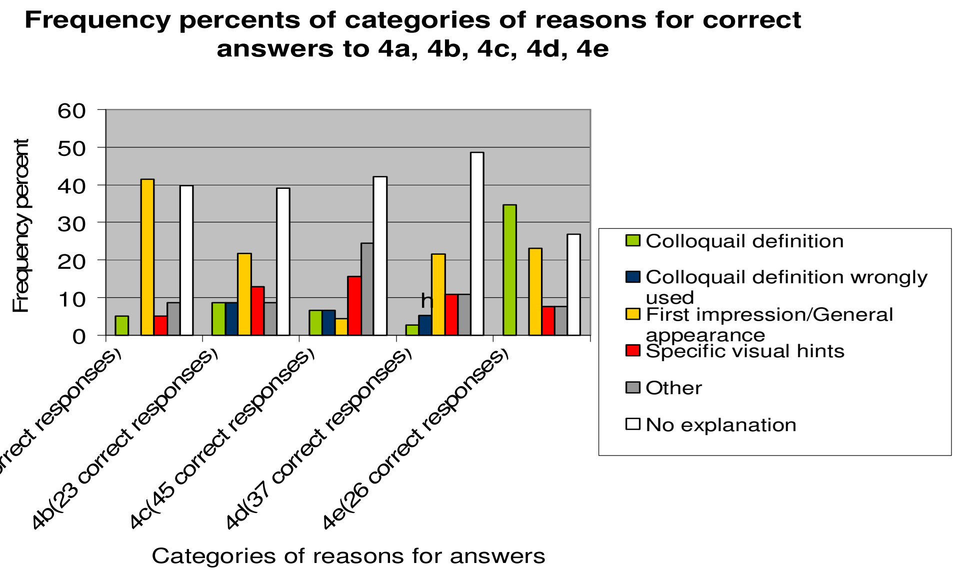


Table 6-9. Frequencies of categories of reasons for correct answers to 4a, 4b, 4c, 4d, 4e.

Students' responses in the interviews reveal a more deeper understanding of their reasoning about coloured-domain graphs as will be discussed in the next chapter.

6.2.3 Function as an expression

6.2.3.1 Question 5

In question 5, students are asked to write down a function equation which comes into their minds immediately. The following categories emerged from students' responses:

Responses to question 5

	Frequency	Percent
Polynomial expressions	27	23.6
Linear expressions	26	22.8
Other	10	8.7
No response	51	44.7
Total	114	100.0

Table 6-10. Frequencies and percentages of categories of the examples of expressions given by students in question 5 in the questionnaire.

Assuming the fact that people are more likely to list more representative examples when asked to draw examples of category members (Lakoff, 1987a), it can be said that polynomial and linear expressions are more representative examples of expressions.

6.2.3.2 Question 6

In Question 6 students are presented with various expressions. Frequencies of answers for each item is presented in Table 6–11 below. In each cell the first row represents the percentages (with frequencies in parenthesis) of answers. The second row represents the percentages (with frequencies in parenthesis) of students (in the whole population) who give an explanation with their answers. The correct answers are marked in bold:

N=114 Question 6 \ Frequency % (count) Explanation[%(count)]	Function	Not a function	I don't know	No response
6a) $f: R \to R \quad f(x) = \sqrt{x^2 - 16}$	44.2% (50) [19.5% (22)]	**10.6 (12)** **[8% (9)]**	42.5% (48)	2.7% (3)
6b) $f: R \to R$, $x^2 + y^2 = 1$	34.2 (39) [14.9% (17)]	**18.4 (21)** **[7.9% (9)]**	42.1% (48)	5.3% (6)
6c) $y = 5$	**24.8% (28)** **[16.8% (19)]**	35.4% (40) [14.2% (16)]	37.2% (42)	2.7% (3)
6d) $y = 5$ (for x≥ 2)	**21.9% (25)** **[9.6% (11)]**	24.6% (28) [10.5% (12)]	50.9% (58)	2.6% (3)
6e) $y = 5$ (for all values of x)	**21.9% (25)** **[11.4% (13)]**	20.2% (23) [7.9% (9)]	52.6% (60)	5.3% (6)
6f) $f: R^+ \to R$, $f(x) = \lvert x^2 - 4 \rvert$	**46.5% (53)** **[23.7% (27)]**	5.3% (6) [0.9% (1)]	43.0% (49)	5.3 (6)
6g) $f: R \to R$ $f(x)=\begin{cases} 1, & \text{if } x^2-2x+1>0 \\ 0, & \text{if } x^2-2x+1=0 \\ -1, & \text{if } x^2-2x+1<0 \end{cases}$	**47.4% (54)** **[28.1% (32)]**	4.4% (5) [1.8 (2)]	38.6% (44)	9.6 (11)

Table 6-11. Percentages and frequencies of answers to expressions in question 6.

6.2.3.3 Reasons for responses to question 6

Students' explanations for their answers revealed different categories from those for the graph questions. The distribution of students' explanations for their answers across the categories and examples of students' written explanations are presented in Appendix A2.3. The results indicate that students' explanations are very complicated. Very few of them used the colloquial definition for their answers.

Very few students (10.6%, 18.4%) rejected the two non function expressions, "$f : R \to R$, $f(x) = \sqrt{x^2 - 16}$" and "$f : R \to R,\ x^2 + y^2 = 1$", as a function. Very few of them used the colloquial definition as an explanation. For "$f : R \to R,\ f(x) = \sqrt{x^2 - 16}$", only 5 out of the 12 students (41.7% of those who correctly rejected it as a function) used the colloquial definition. Similarly, for "$x^2 + y^2 = 1$", 2 out of the 21 students (9.5% of those who correctly rejected it as a function) used the colloquial definition. Students who used the colloquial definition to reject those two expressions, gave some other explanations based on specific hints such as the existence of a square root in an expression or absence of "*f*" at the front. Reasons for incorrect answers for "$x^2 + y^2 = 1$" are different than "$f : R \to R$, $f(x) = \sqrt{x^2 - 16}$". Among those students who incorrectly considered "$f : R \to R$, $f(x) = \sqrt{x^2 - 16}$" as a function, 56% (28 out of 50) of them gave no explanation and 16% (8 out of 50) of them found specific values for x and/or $f(x)$. 25.6% (10 out of 39) of them considered "$x^2 + y^2 = 1$" as a function since it is an equation and has an unknown and 10.3% (4 out of 39) of them because of specific hints such as the existence of "$f : R \to R$". Although the percentages of correct answers for 6f and 6g are higher than the other expressions, some students focused on them as exemplars. 46.5% (53 out of 114) of the students considered 6f as a function. 7.5% (4 out of 53) of them considered it as a specific example, namely the absoloute value function. 47.4% (54 out of 114) of the students correctly considered 6g as a function. 7.4% of those (4 out of 54) considered it as a split-domain function and 16.7% (9 out of 54) as a signum-function.

6.2.4 Function as a set of ordered pairs – Question 7

In Question 7, students are presented with three sets of ordered pairs. Frequencies of the answers for each item are presented in Table 6–12 below. In each cell the first row represents the percentages (with frequencies in parenthesis) of answers. The second row represents the percentages (with frequencies in parenthesis) of students (in the whole population) who give an explanation with their answers. The correct answers are marked in bold:

Frequency % (count) Explanation[%(count)] / N=114 Question 7 $A=\{1,2,3,4\}$ $B=\{1,2,3\}$	Function	Not a function	I don't know	No response
7a) $f: A \to B$ $f=\{(1,1),(2,1),(3,2),(4,2)\}$	**47.4% (54) [37.7% (43]**	10.5% (12) [3.5% (4)]	37.7% (43)	4.4% (5)
7b) $g: A \to B$ $g=\{(1,1),(1,2),(2,2),(4,3)\}$	31.6% (36) [19.3% (22)]	**27.2% (31) [22.8% (26)]**	38.6% (44)	2.6% (3)
7c) $h: A \to B$, $h=\{(1,1),(2,2)$	43.9% (50) [29.8% (34)]	**14% (16) [7.9% (9)]**	37.7% (43)	4.4% (5)

Table 6-12. Percentages and frequencies of answers in question 7.

6.2.4.1 Reasons for the responses to question 7

The following categories emerged from students' explanations for their responses to items 7a, 7b, 7c in question 7. Examples of students' verbal explanations are given for each category:

- Colloquial definition:

"it's not a function, it's a relation, 1 has two different values" (7b).

- Colloquial definition wrongly used:

"for a value of x in *A*, there is a value in *B*. There aren't two elements for the same value" (7c).

- Specific visual hints:

"it's a function, since it says $A \to B$" (7b), "it's a function, since it says $A \to B$" (7c).

- *One to one:*

"The set of ordered pair is a one to one function" (7c).

- *Drawing a set diagram:* Drawing set diagram pictures for the given sets of ordered pairs.
- Other
- No explanation

Examples of students' verbal explanations are presented in detail in Appendix A2.4. The distribution of students' explanations for their answers across the categories are presented in the table below:

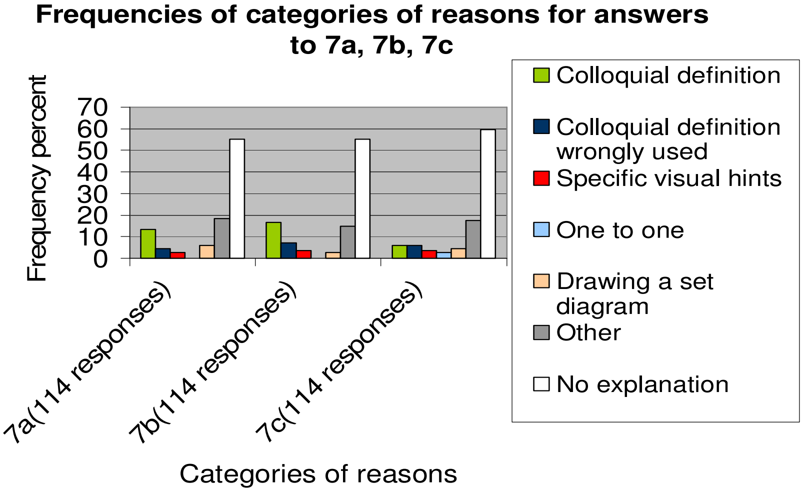


Table 6-13. Frequencies of categories of reasons for answers to 7a, 7b, 7c.

The results indicate that students are more likely to use the colloquial definition for the sets of ordered pairs compared to the function graphs and expressions. The table below illustrates the reasons behind the correct answers:

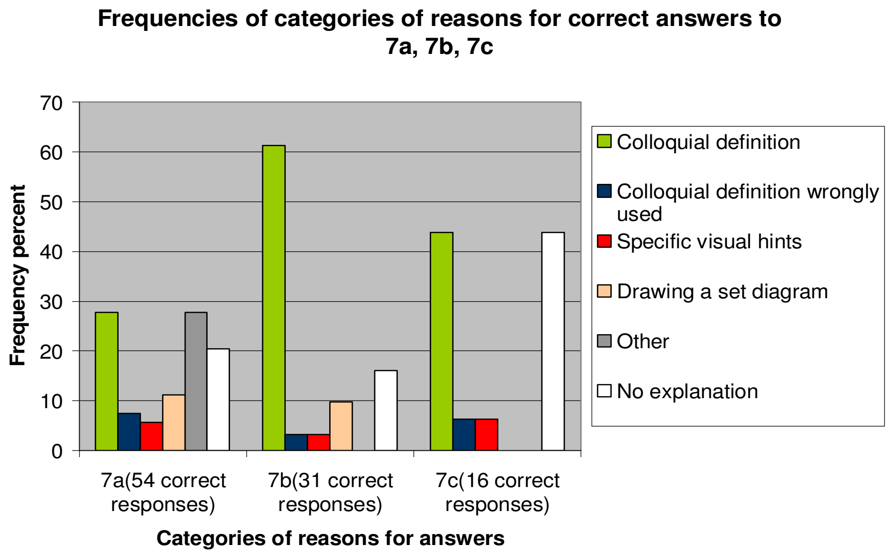


Table 6-14. Frequencies of categories of reasons for correct answers to 7a, 7b, 7c.

The results indicate that percentages of correct answers to the function and non-function sets of ordered pairs are different. It is higher for those ordered pairs which are functions than those which are not. Of those giving explanations, a higher percentage of students give a correct explanation for the non-function items (61.3%, 44%) than for the function item (28%).This might be because some students might have considered the first set of ordered pairs, $f : A \to B$, $f = \{(1,1),(2,1),(3,2),(4,2)\}$, as a function considering that *any* set of ordered pair is a function. On the other hand, rejecting non-function items as functions is more difficult since it requires checking the definitional properties.

The table below illustrates the reasons behind the incorrect answers:

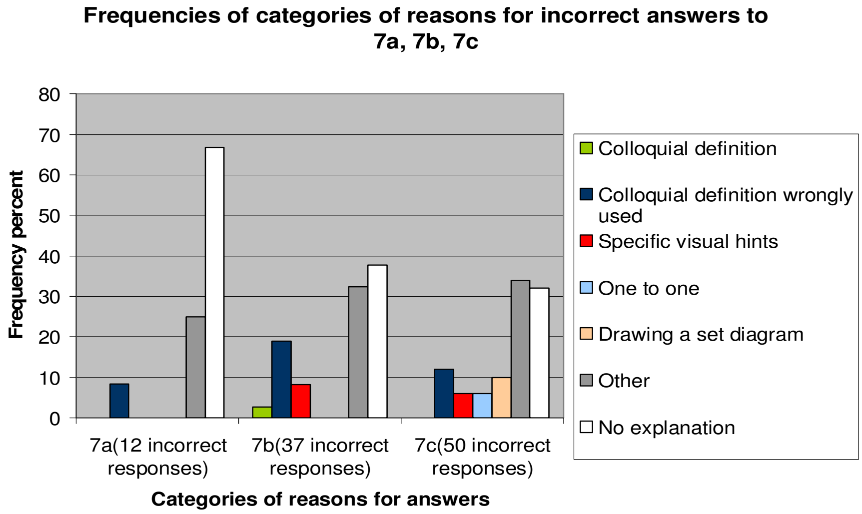


Table 6-15. Frequencies of categories of reasons for incorrect answers to 7a, 7b, 7c.

As seen in the table above, those students, who answered incorrectly, tended to use the colloquial definition wrongly rather than focusing on the specific visual hints.

6.2.5 Function as a set-correspondence diagram – Question 8

In Question 8, students are presented with four items of set-correspondence diagrams. Percentages for each item are presented in Table 6–16 below. In each cell the first row represents the percentages (with frequencies in parenthesis) of answers. The second row represents the percentages (with frequencies in parenthesis) of students (in the whole population) who give an explanation with their answers. The correct answers are marked in bold:

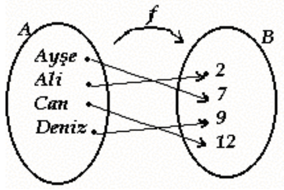
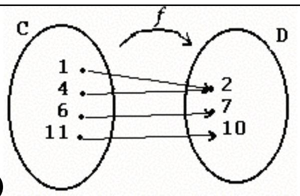
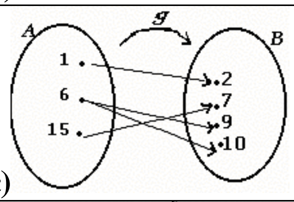
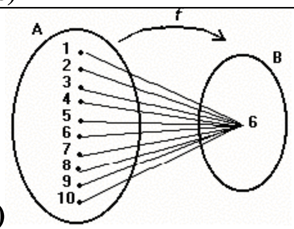
Frequency % (count) Explanation[% (count)] N=114 Question 8	Function	Not a function	I don't know	No response
 **8a)**	**67.5% (77) [50.9%(58)]**	7.9% (9) [4.4% (5)]	21.9% (25)	2.6% (3)
 **8b)**	**64% (73) [39.5%(45)]**	3.5% (4) [1.8% (2)]	26.3% (30)	6.1% (7)
 **8c)**	27.2%(31) [20.2% (23]	**39.5% (45) [28.1% (32)]**	28.1% (32)	5.3% (6)
 **8d)**	**66.7% (76) [47.4%(54)]**	6.1% (7) [3.5% (4)]	23.7% (27)	3.5% (4)

Table 6-16. Percentages and frequencies of answers in question 8.

Results as presented in table above indicate that frequencies of correct answers to set diagrams which are functions are higher than the frequencies of correct answers for the other aspects of functions, 67.5%, 62.3%, 66.7%. However, the percentages of correct answers to the set diagram which is not a function declines to 39.5%. If we look at incorrect answers, it can be seen that the percentage for the non-function set diagram is higher than the function set diagrams.

6.2.5.1 *Reasons for the responses to set diagrams - question 8*

The following categories emerged from students' explanations for their responses to items 8a, 8b, 8c, 8d in question 8. Examples of students' verbal explanations are given for each category:

- Colloquial definition:

"an element in *A* can not be assigned to more than one element in *B*" (8c).

- Colloquial definition wrongly used:

"(it's a function) an element in the domain can be assigned to more than one element in the range" (8c).

- Specific visual hints:

"(it's a function) names are connected to numbers" (8a).

"(it's a function) since two lines can intersect with each other" (8b).

- One to one and onto-ness:

"it's an onto function", "it is a one-to-one and onto function" (8c).

- Constant function: In this category, there are students who considered the set correspondence diagram in 8d as a constant function.
- Other
- No explanation

Examples of students' verbal explanations are presented in detail in Appendix A2.5. The distribution of students' explanations for their answers across the categories are presented in the table below:

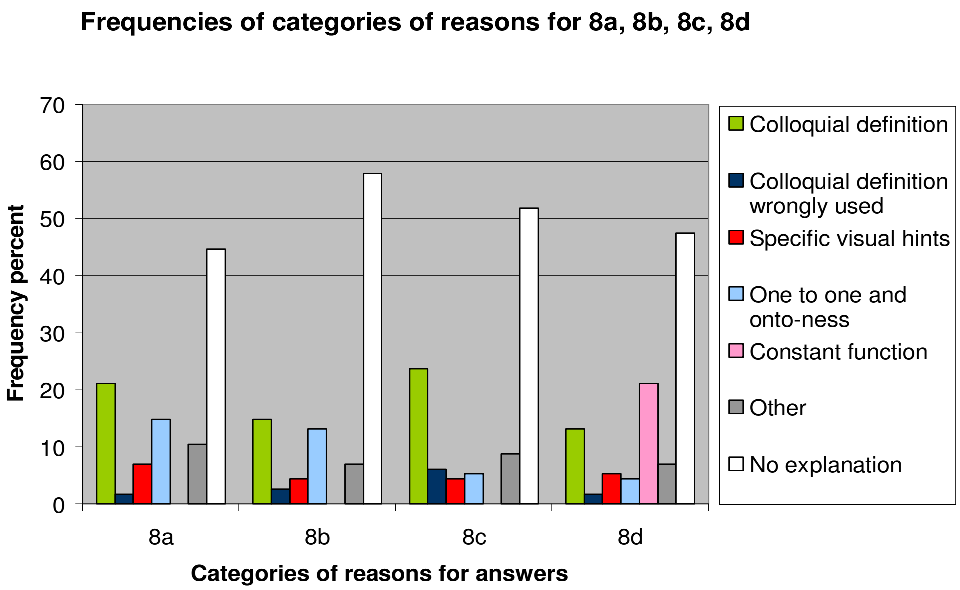


Table 6-17. Frequencies of categories of reasons for answers to 8a, 8b, 8c, 8d.

These results indicate that the percentages of the students who used the colloquial definition for their explanations for the set-correspondence diagrams are the highest among other aspects of functions.

The table below illustrates the reasons behind the corrects answers:

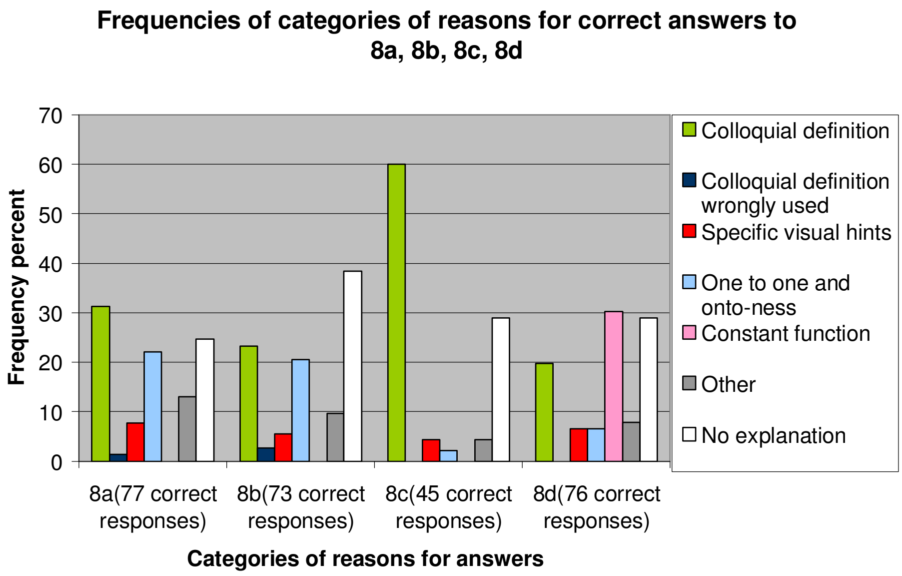


Table 6-18. Frequencies of categories of reasons for correct answers to 8a, 8b, 8c, 8d.

For the non-function set diagram item, 60% of the correct answers are followed by an explanation based on the colloquial definition. On the other hand, this percentage declines to (31.2% for 8a, 23.3% for 8b, 19.7% for 8d) for the set diagrams which are functions.

As seen in the table below, students who gave incorrect answers mostly used the colloquial definition in a wrong way or relied on the specific visual hints.

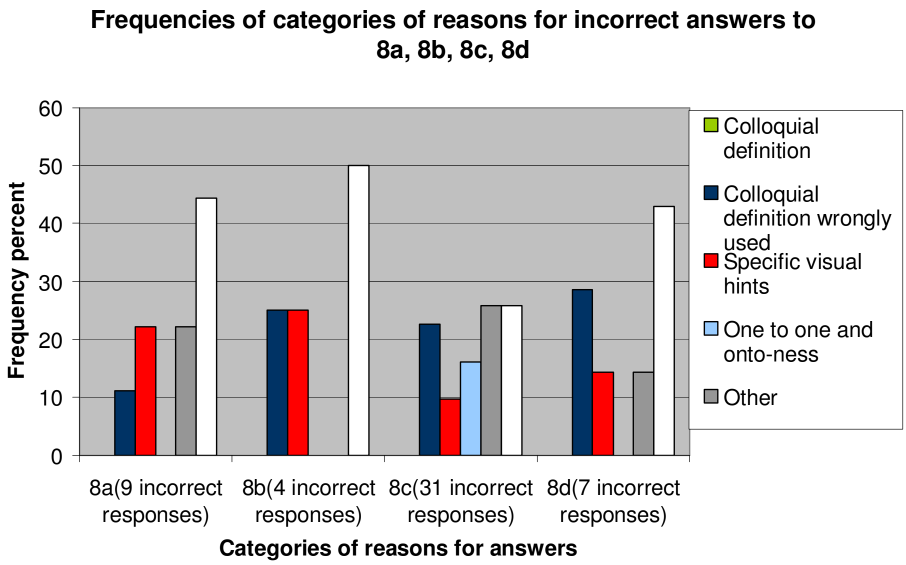


Table 6-19. Frequencies of categories of reasons for incorrect answers to 8a, 8b, 8c, 8d.

6.3 Comparing different aspects of functions

To compare the responses to different aspects of functions, correct responses are calculated with a new variable which represents the number of correct answers to all items in each question. Frequency tables are presented in Appendix A2.7. These results indicated that students are more successful with the set-correspondence diagrams and the sets of ordered pairs compared to the graphs and the expressions. 1.8% of the students answered graphs items in question 3 correctly. None of the students answered all coloured-domain graphs in question 4 correctly. Similarly none of the students answered all expression questions correctly. The percentages increase for questions of sets of ordered pairs and set diagrams.

18.4% of the students responded correctly to all set of ordered pairs item in question 7. Similarly, 19.3% of the students correctly answer all set diagram items in question 8.

6.4 Definition of function – Question 9

Students' responses to question 9 in the questionnaire are considered as their personal concept definitions as described by Tall & Vinner (1981). None of the students gave the formal definition of a function. Although a formal mathematical language is used for some parts of some responses such as "$A \neq \emptyset \wedge B \neq \emptyset,\ \beta \subset (A \times B)$", these are followed by a description. Responses revealed the following categories:

Colloquial definition: Responses in the form of a colloquial definition which includes all of the definitional properties are considered in this category. E.g. "(A function) is to write a relation from one set to the other with the condition that there can not be elements left in the domain and one element can not have more one value".

Incomplete colloquial definition: Although some responses are in the form of a colloquial definition, they do not involve all of the definitional properties or did not involve correct properties. E.g. "$A \neq \emptyset \wedge B \neq \emptyset,\ \beta \subset (A \times B)$". If the relation β does not leave any elements left and elements are assigned to each other then the relation β is a function in $A \times B$. It is denoted by *f*, *g*, or *h*" or "It is a relation which does not have no gap in the domain".

Other: e.g. "It is defined from $f : R \to R$".

The percentages for each category are presented in table 6.20 below:

Response to question 9 - Definition of function

	Frequency	Percent
Colloquial definition	12	10.5
Incomplete colloquial definition	15	13.2
Other	14	12.3
No response	73	64.0
Total	114	100.0

Table 6-20. Percentages and frequency counts of the responses given for the definition of a function.

Most students used the colloquial definition, a definition in an everyday language. However, among those, only 10.5% focused on all properties of the definition correctly. 13.2% of the students either could not focus on all properties of the definition or could not remember the properties correctly. In contrast to the preliminary study, the no response rate is very high at 64%.

6.5 A note on the no responses

Although the "no response" rates are very low for the closed-ended questions, the percentages for "I don't know" option reveal that there are a lot of students who could not decide about the given items (See tables 6.3, 6.7, 6.11, 6.12, 6.16). This might be because students either could not decide about the items or they simply did not respond. No response rates to open-ended questions are very high as shown in table 6–21 below:

	All items in question3	All items in question4	All items in question6	All items in question7	All items in question8	All items
Percentage % (Frequency)	36.8% (42)	36% (41)	43% (49)	45.6% (52)	34.2% (39)	11.4% (13)

Table 6-21. No responses for reasons for the responses in the questionnaires.

This might be because they gave answers without any reasons or just preferred not to respond since it might be problematic for them to give explanations for each answer.

6.6 A summary of chapter 6

The quantitative data obtained from the questionnaire revealed that a hundred and fourteen students in this study responded differently for different aspects of functions. They were more successful with the set-correspondence diagrams and sets of ordered pairs compared to the graphs and expressions. Very few students used the colloquial definition. Furthermore, higher percentages of students used the colloquial definition for the set-correspondence diagrams and the sets of ordered pairs. One limitation of these results is that the no response rate for the explanations is high as discussed in section 6.2.8. However, inverview results, demonstrating responses from nine students as discussed in the next chapter, draw a more complete picture of how students reason about different aspects of functions. The results from the interviews with nine students are presented in chapter 7 which leads a categorization of the students' responses in chapter 8.

CHAPTER 7 –RESULTS FROM THE INTERVIEWS

Although there is a high rate of no responses for the explanations in the questionnaires, interview results drew a different picture. The results reveal a range of different explanations similar to the results from the preliminary study. A categorization of responses is presented in the next chapter. This chapter gives an account of the analysis of the data obtained from the interviews with nine students.

The first step for the analysis of the interviews is description. A descriptive summary is made to manage the interview data (Patton, 1990). As presented in the following section, the results for each question emerged from the descriptive summary.

A remark about the drawings of the functions in the interview questions should be made. The diagrams have implicit meanings that are not true absolutely. As Hardy (1967) states, the quality of the drawings does not matter if the reader has the same sophistication:

> 'Let us suppose that I am giving a lecture on some system of geometry, such as ordinary Euclidean geometry, that I draw figures on the blackboard to stimulate the imagination of my audience, rough drawings of straight lines or circles or ellipses. It is plain, first, that the truth of the theorems which I prove is in no way affected by the quality of my drawings. Their function is merely to bring home my meaning to my hearers, and, if I can do that, there would be no gain in having them redrawn by the most skilful draughtsman. They are pedagogical illustrations, not part of the real subject-matter of the lecture' (Hardy, 1967, p. 125).

Therefore, the responses from the students in the interviews should be evaluated considering the implicit meanings of the drawings presented to them.

7.1 The results from the interviews

In the following sections, the results for each question in the interview are presented. Students' responses revealed a spectrum of performance. A detailed discussion on the characteristics of this spectrum is given in the next chapter. However, at this point it should be mentioned that there are four different categories in the spectrum. As well as for the ethical reasons, students' names were altered so that the initial letters, A, B, C, D refer to the categories from the top to the bottom. In the first category, there are four students Ali, Ahmet, Aysel and Arif who could focus on the definitional properties by mostly using the colloquial definition. In the second category, there are two students, Belma and Belgin, who could use the colloquial definition only for the set-correspondence diagrams and the sets of ordered pairs. These two students gave complicated responses when dealing with the graphs and expressions. They focused on the properties of the graphs and expressions which are irrelevant to the core concept of function. In the third category, there is one student, Cem, who used the colloquial definition wrongly for the set-correspondence diagrams and the sets of ordered pairs. For the graphs and expressions, he gave complicated responses which did not focus on the definitional properties. In the fourth category, there are two students, Deniz and Demet, who could not focus on the definitional properties for any aspects of the function concept.

The following sections present the results in summary tables followed by students' explanations.

7.1.1 Set-correspondence diagram

In the interview, all students were shown a set-correspondence diagram as shown in Figure 7–1 below:

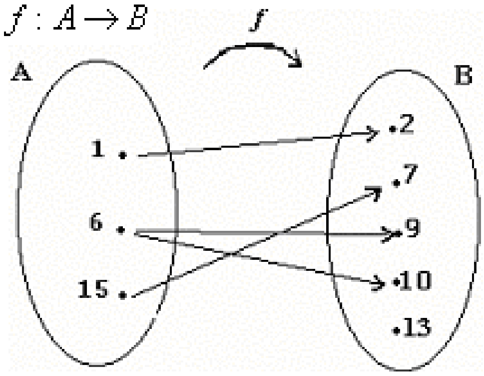


Figure 7-1. The set-correspondence diagram in the interview.

Students were asked whether it was a function or not. They were then asked to explain the reasons for their answers. Table 7–1 below summarizes all students' responses:

	Function or not	Explanation
Ali	Not a function	Colloquial definition
Aysel	Not a function	Colloquial definition
Ahmet	Not a function	Colloquial definition
Belma	Not a function	Colloquial definition
Belgin	Not a function	Colloquial definition
Arif	Not a function	Colloquial definition
Cem	Function	Colloquial definition wrongly used
Deniz	Not a function	Visual hints
Demet	Not a function	Visual hints

Table 7-1. A summary of students' responses to the set-correspondence diagram.

Six out of nine students explained why they did not consider the given set diagram as a function in terms of the colloquial definition:

> 'It is not a function. 6 has two values' (Ali).
>
> 'Not a function…6 is an element in *A* and it goes to both 9 and 10' (Aysel).
>
> 'Not a function…6 has two values. 6 has two different values in the range' (Ahmet).
>
> 'It is not a function…for the same reason. This number 6' (Belma).
>
> 'There can be elements left (13 in the range), but elements in the domain can not be assigned to more than one element in the range' (Belgin).

‘6 has two corresponding values…13 can be left unassigned’ (Arif).

One student (Cem) answered in terms of the colloquial definition, which he remembered wrongly. He said that it is a function since 6 (in the domain) could go to 9 and 10. He was then asked to give a counter example, which is not a function.

He drew the set diagram in the Figure 7–2 below and said that it was not a function ‘because it (6) goes to more than two elements:

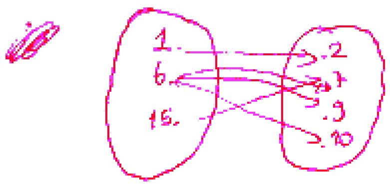


Figure 7-2. Cem’s written explanation for the set-correspondence diagram.

The other two students (Demet and Deniz) focused on the visual properties of the diagrams which are irrelevant to the core concept of function. Demet did not consider the set diagram picture as a function since the arrows intersected one another. To make clear what she meant, she was asked to give a counter example, which could be a function. She then drew the set diagrams as seen in Figure 7–3 below where the arrows do not intersect one another:

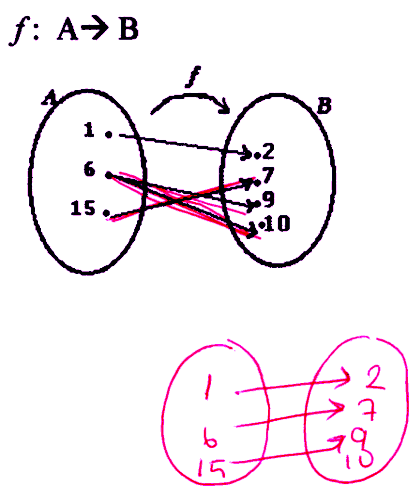


Figure 7-3. Demet’s written explanation for the set-correspondence diagram.

Deniz did not consider the given set diagram as a function for a similar reason:

> ‘For me, it is not a function, because arrows are in a mess…6, 9, 10 they are in a mess, it is not a function’ (Deniz).

To make clear what he meant, he was asked to give a counter example, which could be a function. Drawing the set diagrams as shown in Figure 7–4 below, he said that the first diagram he drew could be a function since the directions of arrows are from *A* to *B*, but not the second one:

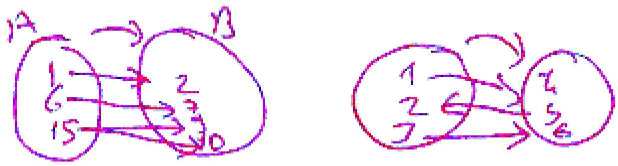


Figure 7-4. Deniz’s written explanation for the set-correspondence diagram.

7.1.2 Sets of ordered pairs

In the interview, all students were shown a set of ordered pairs as shown below:

$A = \{1,2,3,4\}$

$f : A \to R$, $f = \{(1,1), (1,2), (2,2), (3,3), (4,3)\}$

Students were asked whether it was a function or not. They were then asked to explain the reasons for their answers. Table 7–2 below summarizes all students’ responses and their explanations:

	Function or not	Explanation
Belma	Not a function	Colloquial definition
Aysel	Not a function	Colloquial definition with an explanation with vertical line test
Ahmet	Not a function	Colloquial definition using a set-correspondence diagram
Arif	Not a function	Colloquial definition using a set-correspondence diagram
Ali	First considered as a function then changed his mind	Colloquial definition wrongly used. When reminded of 1 having two different values, he correctly used the colloquial definition.
Cem	Not a function	Colloquial definition wrongly used

Deniz	Not a function	Numbers of elements of ordered pairs is not equal to numbers of elements of the domain.
Belgin	Not a function	No explanation
Demet	Function	Ploting a point and joining it to the origin.

Table 7-2. A summary of students' responses to the set of ordered pairs in the interview.

Eight out of nine students did not consider this set of ordered pairs as a function. It was correctly rejected as a function by five students using the colloquial definition. Unlike the use of the colloquial definition for the set diagrams, for the set of ordered pairs the colloquial definition was used with an explanation referring to other aspects of functions. Aysel used the colloquial definition followed by an explanation with the vertical line test:

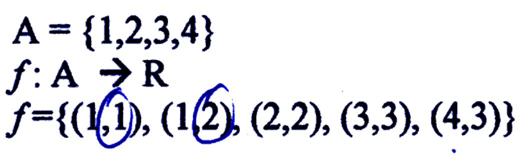


Figure 7-5. Aysel's written explanation for the set of ordered pair.

> 'Every value is given, there are no elements left in the domain, but 1 goes to two values, and this is like the line intersecting the function twice, it can not go to both 1 and 2, it is not a function' (Aysel).

Ahmet and Arif used the set-correspondence diagram in a prototypical way. In other words, they used the set-correspondence diagram to represent general ideas. Ahmet referred to the set diagram when using the colloquial definition as shown in Figure 7–6 below:

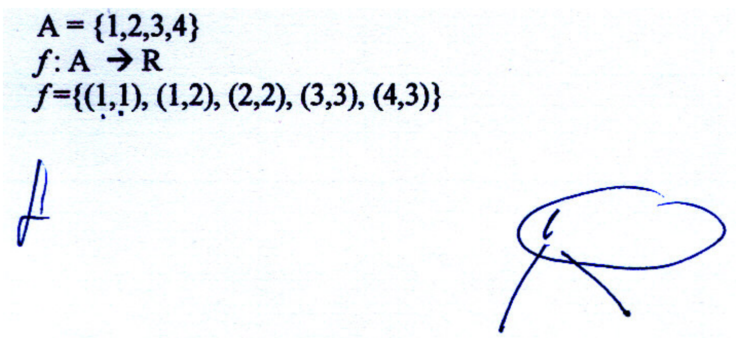


Figure 7-6. Ahmet's written explanation for the set of ordered pairs.

> 'For the value of 1 it is 1, it is in R, 1 can not have a second value, it is not a function' (Ahmet).

Arif also used the colloquial definition by drawing a set diagram as shown in Figure 7–7 below:

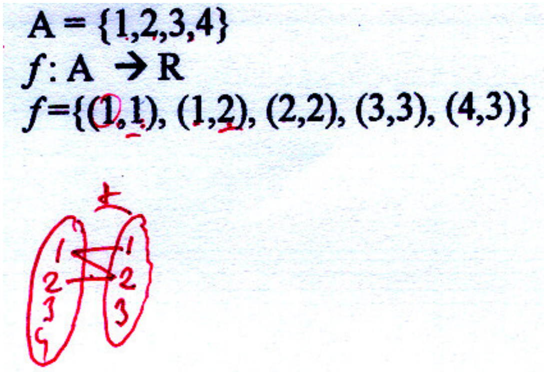


Figure 7-7. Arif's written explanation for the set of ordered pair.

Referring to the set diagram question where 6 has two different values, Arif said that 1 has two corresponding values since there are (1,1) and (1,2) in the set. Therefore he did not consider it as a function.

One student (Ali) used the colloquial definition wrongly to consider it as a function:

> 'From *A* to *R*, 1 to 1, 1 to 2, 2 to 2, 3 to 3, 4 to 3…it is a function…because every element in the domain has a corresponding value, 4, 3, 2, 1 all have' (Ali).

When reminded that 1 has two values he changed his mind and explained as follows:

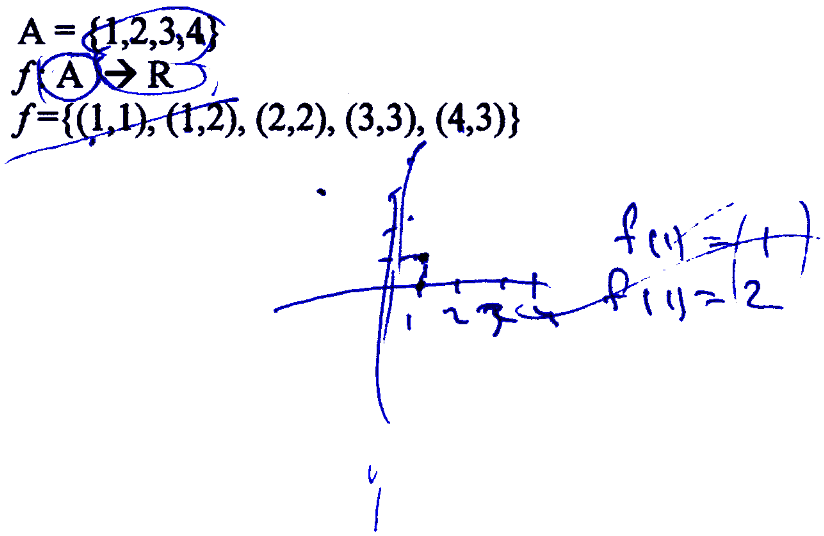


Figure 7-8. Ali's written explanation for the set of ordered pair.

‘Every element in the domain has a corresponding value, but 1 has two corresponding values, in other words when $f(1)=1$, also $f(1)=2$, therefore it can not be a function…but there can be elements left in the range’ (Ali).

Belma directly used the colloquial definition as follows:

‘From A to f, one set is given, every set, a set in R is assigned to its elements. 1 with 2, 2 with 3, 3 with 4, and 4 corresponding to. It is not a function…because an element in A is assigned to more than one element, it can not be a function’ (Belma).

One student (Cem) used the colloquial definition wrongly to consider it not as a function:

‘From 1 to 1, from 1 to 2, from 3 to 3, from 4 from 3…it is not a function…because in brackets only 3 is given, it is (3,3), it shouldn’t be (4,3)’ (Cem).

He did not consider it as a function since two different values are assigned to 3 in the range. To understand how he decided, he was asked a counter example which can be a function. He explained as follows as shown in Figure 7–9:

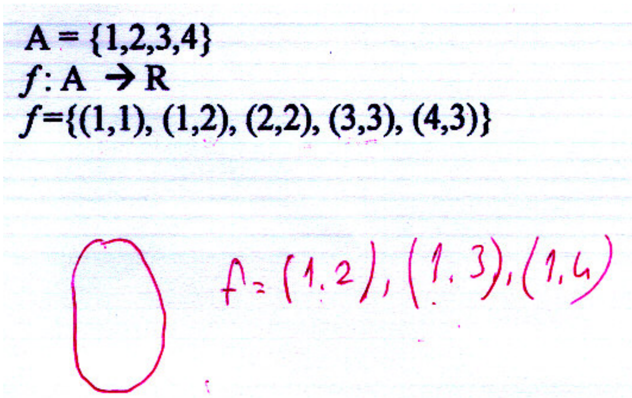


Figure 7-9. Cem’s written explanation for the set of ordered pair.

‘It is a function because they are in order’ (Cem).

Deniz did not consider the set of ordered pair as a function since the number of elements of ordered pairs is not equal to the number of elements of the domain.

‘It is not a function…elements of *A* are known, it is from *A* to *R*, *R* is not known. (1,1),(1,2), so elements of this (*R*) are not known. If *R* was {1,2,3,4} then it would be a function…here first is 1, second should be 2 but it is 1 here, third should be 3 but it is 2...the number of elements (of *A*) is 4 but here (in set of ordered pairs) is 5…there are two 1’s, therefore it is not a function’ (Deniz).

When he was asked to give a counter example which can be a function, he wrote 'f = (1,1), (2,2), (3,3), (4,4)' as shown in Figure 7–10 below:

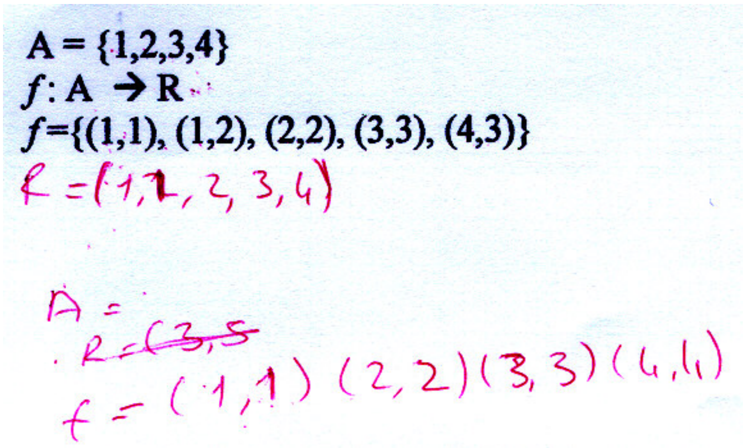


Figure 7-10. Deniz's written explanation for the set of ordered pair.

One student (Belgin) gave no explanation for why she did not consider the set of ordered pairs as a function. Only one student (Demet) considered it as a function. She plotted (1,2) and joined it to the origin as shown in Figure 7–11 below:

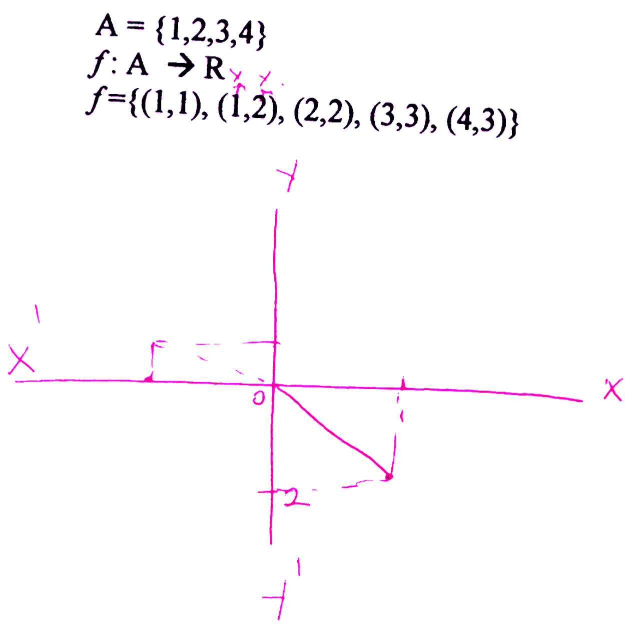


Figure 7-11. Demet's written explanation for the set of ordered pair.

'Function…the function is where these are joined together'.

7.1.3 Straight line graph

In the interview, all students were shown a straight line graph as shown in Figure 7–12 below:

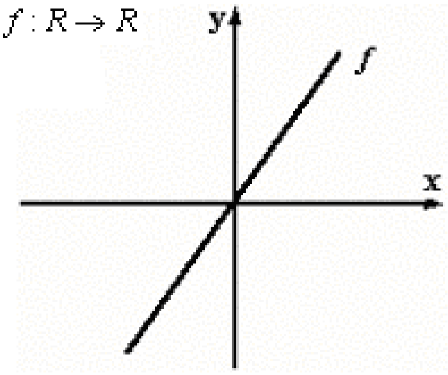


Figure 7-12. Straight line graph in the interview.

Students were asked whether it was a function or not. They were then asked to explain the reasons for their answers. Table 7–3 summarizes all students' responses and their explanations:

	Function or not	Explanation
Ali	Function	Exemplar based focus followed by colloquial definition/ use of set diagram
Aysel	Function	Exemplar based focus followed by colloquial definition
Ahmet	Function	Vertical line test with reference to the colloquial definition
Belma	Function	Action on the graph (assigning numbers on x to the numbers on y)
Belgin	Function	Action on the graph (assigning numbers on x to the numbers on y)
Arif	Function	Action on the graph (confused with the domain and range / assigning numbers on x and y with each other)
Cem	Function	Visual hints
Demet	Function	No explanation
Deniz	Could not decide	No explanation

Table 7-3. A summary of students' responses to the straight line graph in the interview

Eight out of nine students considered the straight line graph as a function. Two of them (Ali and Aysel) used the colloquial definition to explain their answers. Both of them referred to the exemplars of straight lines.

For instance, Ali said that it was $f(x) = x$. He was then asked to think of it as if he did not know that it was the graph of $f(x) = x$. He then responded in terms of the colloquial definition by drawing a set diagram picture as shown in Figure 7–13 below:

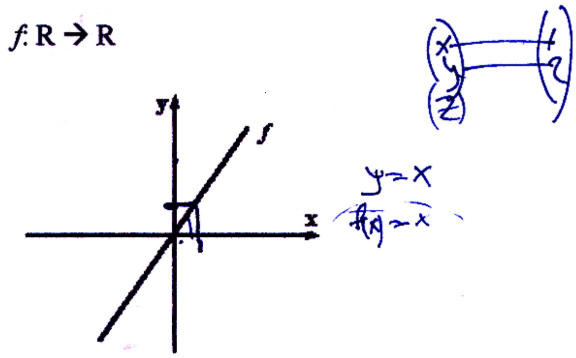


Figure 7-13. Ali's written explanations for the straight line graph.

> 'There shouldn't be elements left in the domain…I said $f = x$ but it's not like this. I have to know the slope…every x value has an image in y, the definition'. (Ali)

Similarly, Aysel first referred to a cluster of exemplars, $y = ax$. She then continued to respond in terms of the colloquial definition:

> '…function definition, it's a special relation. Every element in the domain goes to only one element, there are not elements left in the domain. Everything in x, since it goes to infinity, all elements in x find their places in the function. Furthermore, one value in x does not go to more than one y. Therefore, it's a function'. (Aysel)

Ahmet used the colloquial definition by applying the vertical line test as shown in Figure 7–14 below and said:

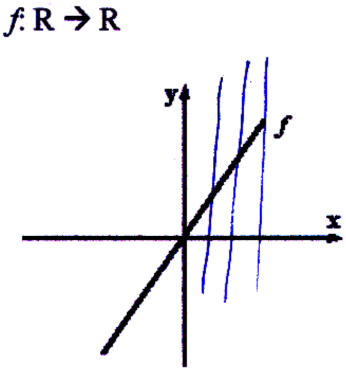


Figure 7-14. Ahmet's written explanations for the straight line graph.

> 'It's a function because for every x value, there is a y value. I can't see two y values for x here. Do we draw lines parallel to x, or to y? One of them. If we draw verticals to y and if it intersects at one point…if it intersected at two points then it wouldn't be a function…if it intersected twice, then there would be two y values for an x'. (Ahmet)

Three students (Belma, Belgin, Arif) assigned a few numbers on x axis to the numbers on y axis. Belma considered it as a function and explained on the first graph in Figure 7–15:

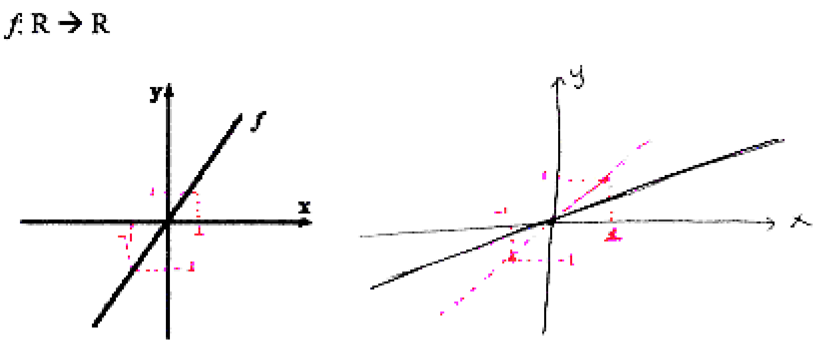


Figure 7-15. Belma's written explanations for the straight line graph.

> 'It's a function…because…from R to R, from the elements of the set of real numbers to other elements. In other words every element is met with its element…when we give 1 for $f(x)$, x is 1'.

She was then shown another straight line with a different slope (the second graph in Figure 7–15 above). She again assigned 1 to 1 and drew $y = x$ rather than focusing on the given straight line.

Belgin also considered the straight line as a function with a similar reason:

> 'because every value…values in the domain are assigned to the values in the range…for instance, if we give 1 for x, then y is 1…there are no elements left in the domain. This is a function' (Belgin).

However, when she was given a straight line passing through the origin with a different slope as shown in Figure 7–16 below, she did not consider it as a function. She plotted the point (1,1) and (referring to 1 on the x axis) said that there is an element left in the domain.

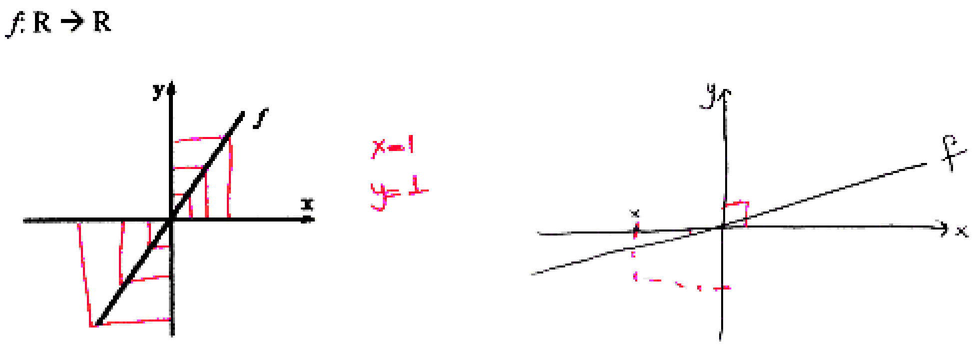


Figure 7-16. Belgin's written explanations for the straight line graph.

Arif assigned 3 on the x axis to 1 in y axis. However he seemed to be confused with the aspects of domain and range:

> "This is a function. Because…if we draw parallel lines here, for every y, an image of y in x, and if we draw on x, an image of x in y. Suppose y is 1, and x is 3…therefore it's a function" (Arif).

Considering the straight line graph as a function, one student (Cem) focused on the visual properties of the graph.

He realized that there weren't numbers on the axes. Therefore, I have put numbers on the axes. He then drew lines from negative numbers on the negative y axis to the graph as shown in the Figure 7–.17 below. However, he did not assign these values to the numbers on the x axis:

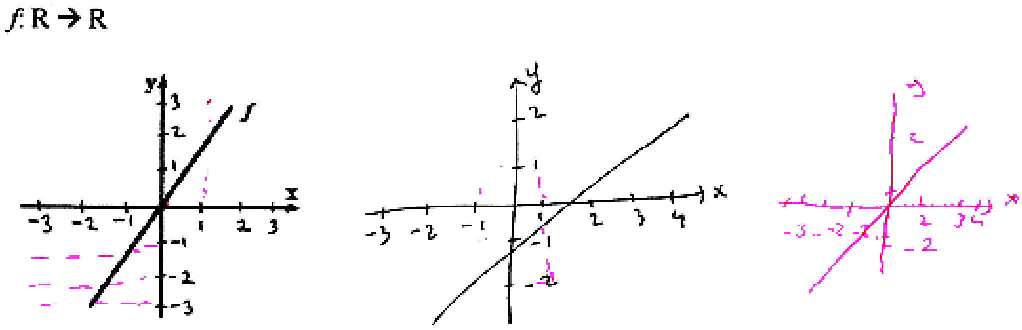


Figure 7-17. Cem's written explanations for the straight line graph.

He was then given a straight line, which did not pass through the origin as shown in Figure 17 above. Although he considered this as a function, he could not explain the reason correctly:

'The lines coming vertically from here also come here…to –2'. (Cem)

7.1.4 Straight lines in three pieces

In the interview, all students were shown a straight line graph in three pieces, as shown in Figure 7–18 below:

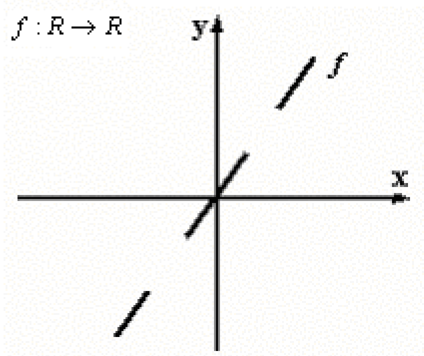


Figure 7-18. Straight line in three pieces.

Students were asked whether it was a function or not. They were then asked to explain the reasons for their answers. Table 7–4 summarizes all students' responses and their explanations:

	Function or not	Explanation
Ali	Not a function	Colloquial definition
Aysel	Not a function	Colloquial definition
Arif	Not a function	Colloquial definition
Ahmet	Function (on a restricted domain)	Colloquial definition by explaining it with a set-correspondence diagram
Belma	Not sure	Visual hints
Demet	Not a function	Visual hints
Deniz	Not a function	Visual hints
Cem	Not a function	Visual hints
Belgin	Function	No clear explanation

Table 7-4. A summary of students' responses to the straight line graph in three pieces in the interview

Four out of nine students responded (Ali, Aysel, Arif) in terms of the colloquial definition:

> 'There are elements left. The points here do not have images…here there are gaps' (Ali) (See Figure 7– 19).

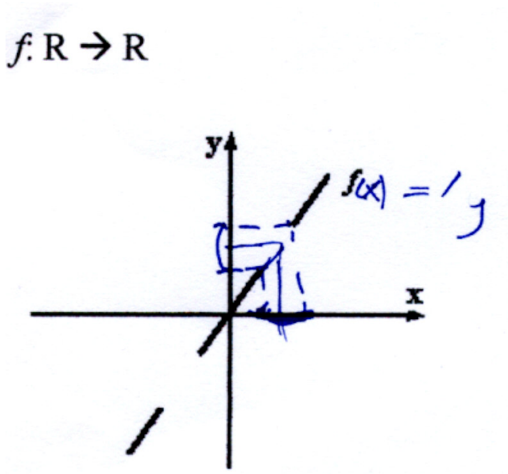


Figure 7-19. Ali's written explanation for the straight line graph in three pieces.

> 'It's not a function, because there are elements left in the domain. They don't find their places on the function, that's why for instance let's say 2 here. 2 is left, there's nothing for $f(2)$. Since it's not defined it's not a function' (Aysel). (See Figure 7–20).

f: R → R

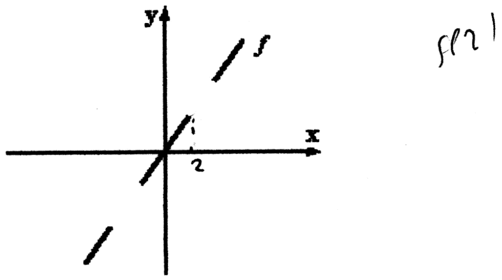


Figure 7-20. Aysel's written explanation for the straight line graph in three pieces.

> 'It's not a function…the ones in this area of gap…that x gap. For instance there is 3 there, it doesn't have an image' (Arif) (See Figure 7–21).

f: R → R

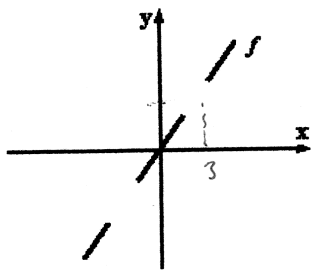


Figure 7-21. Arif's written explanation for the straight line graph in three pieces.

One student, Ahmet, considered it as a function in a certain domain. He explained his response by drawing a set diagram picture as shown in Figure 7–22 below:

f: R → R

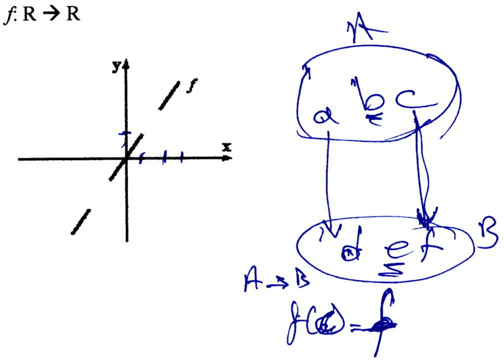


Figure 7-22. Ahmet's written explanation for the straight line graph in three pieces.

> 'from c to f, it's a function, also from a to d. These are left (b and e), but these are functions' (Ahmet).

His explanation was interpreted as he considered the set-correspondence diagram he drew as a function by restricting the domain (excluding b from the domain) so that it becomes a function. His explanation was considered as the use of the colloquial definition. He used the set-correspondence diagram to explain how he used the colloquial definition.

Four out of nine students (Belma, Demet, Deniz, Cem) focused on the visual hints. Belma, could not decide whether it is a function or not and focused on the gaps on the graph without any reference to the definitional properties. Demet, Deniz and Cem did not consider it as a function since the graph was in separate pieces.

One student, Belgin, could not explain her answer. She considered it as a function:

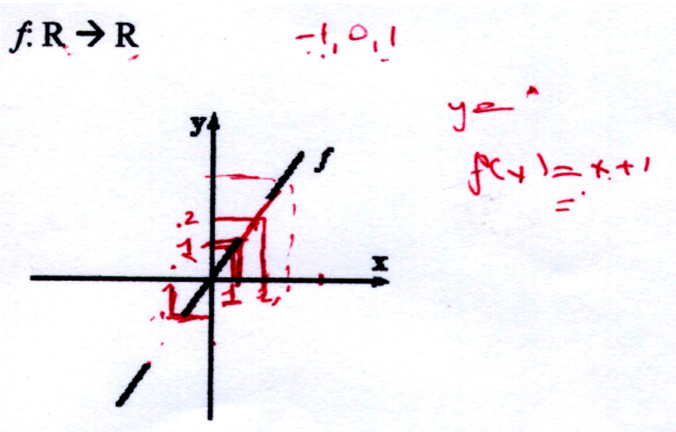


Figure 7-23. Belgin's written explanation for the straight line graph in three pieces.

'This is a function. For instance, 1 is included…there are elements left' (Belgin).

However, when she was asked which elements were left in the domain she could not explain:

'I have no idea about this. I only know that this is a function' (Belgin).

7.1.5 Points on $y = x$ with the domain of projected points

In the interview, all students were shown a graph as shown in Figure 7–24 below:

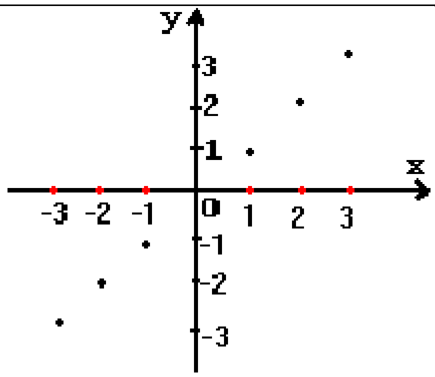


Figure 7-24. Points on $y = x$ with the domain of projected points.

Students were told that the points coloured as red was the domain They were then asked whether it was a function or not. They were then asked to explain the reasons for their answers. The table 7–5 below summarizes all students' responses and their explanations:

	Function or not	Explanation
Ali	Function	Colloquial definition
Arif	Function	Colloquial definition
Ahmet	Function	Colloquial definition
Aysel	Function	Colloquial definition followed by vertical line test
Belgin	Function	Colloquial definition wrongly used considering y axis as the domain and x axis as the range
Cem	Function	Finding the corresponding values of the numbers in the domain
Demet	Function	Drawing a straight line through the graph
Deniz	Function	Drawing a straight line through the graph
Belma	Function	Drawing a straight line through the graph

Table 7-5. A summary of students' responses to the points on $y = x$ with the domain of projected points.

All of the students considered this graph as a graph of a function. However only four of them used the colloquial definition correctly to consider the graph as a function:

> 'This is a function….because, this time the domain is those mentioned places. All of them has an image, therefore it's a function' (Ali) (See Figure 7–.25 below).

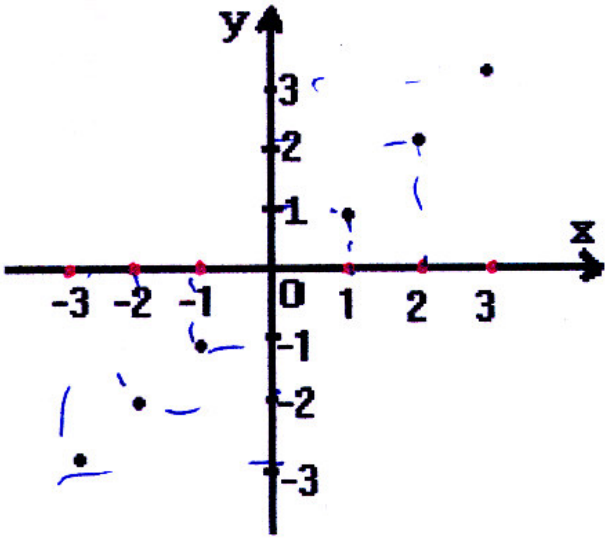


Figure 7-25. Ali's written explanation for the points on $y = x$ with the domain of projected points.

> 'The corresponding value of 1 in the domain is 1, 2 for 2, 3 for 3. It doesn't take two values. One element in the domain is not assigned to two elements in the range, -1 to -1' (Arif).

Although Ahmet's focus of attention is not the colloquial definition at first, he referred to the definitional properties in his explanation:

> 'From 1 to 1, from 2 to 2… x is element of integers… x and y are elements of integers. What if we say $y = x$. We'll show the domain. Is this conditional function?…for 1, 2, 3 it's again. 1, 2, 3…for 1 there aren't two different values in the range…is this a constant function? No it's not, because everything goes to same thing for constant function…is this onto function? I think yes, it's a function' (Ahmet) (See Figure 7–26).

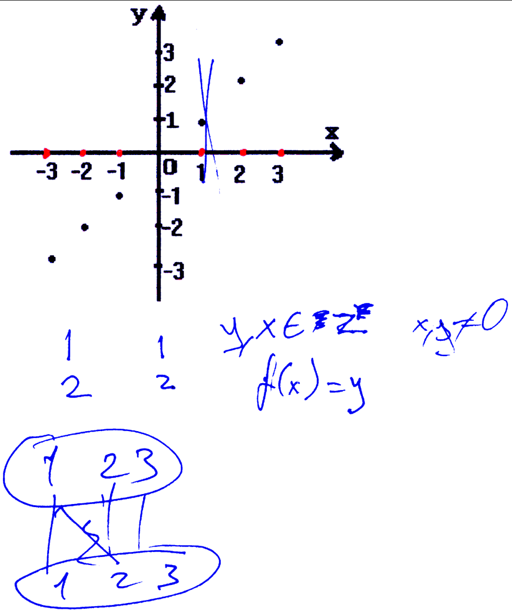


Figure 7-26. Ahmet's written explanation for the points on $y = x$ with the domain of projected points.

As seen in his explanation, he did not only assign the numbers on x axis to the numbers on y axis, but also mentioned that one value can not be assigned to two different values.

Aysel used the colloquial definition followed by an explanation with the vertical line test as shown in Figure 7–27 below:

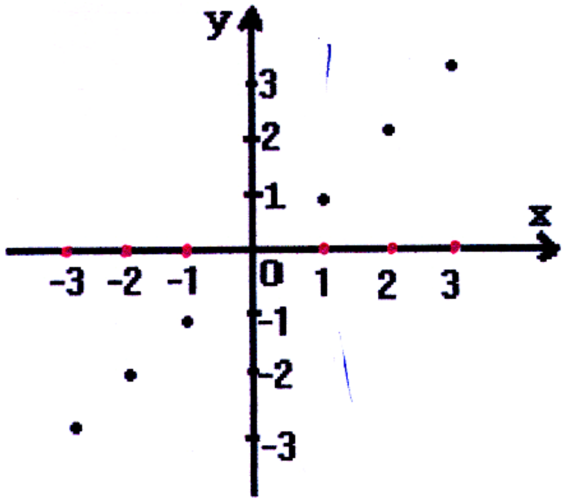


Figure 7-27. Aysel's written explanation for the points on $y = x$ with the domain of projected points.

> 'Function. Because…there aren't any elements left in the domain. For every element in the domain, there are elements. For instance 2 for $f(2)$, 1 for $f(1)$, it does not go to more than one place. If we draw lines, it passes through the function once. I think it's a function' (Aysel).

One student (Belgin) used the colloquial definition wrongly:

> 'This is definitely a function since all elements in the domain are assigned to elements in the range' (Belgin).

However when she was asked which one is the domain, she said that y axis is the domain and x axis is the range by assigning each element on y axis with elements on x axis.

One student (Cem) assigned the numbers on x axis to the numbers on y axis without any reference to the definitional properties.

The other three students (Demet, Deniz, Belma) considered the graph as a function by drawing a straight line through the graph. They related this graph to their earlier experiences of graph drawing. For instance, Demet drew a straight line as shown in Figure 7–28 below:

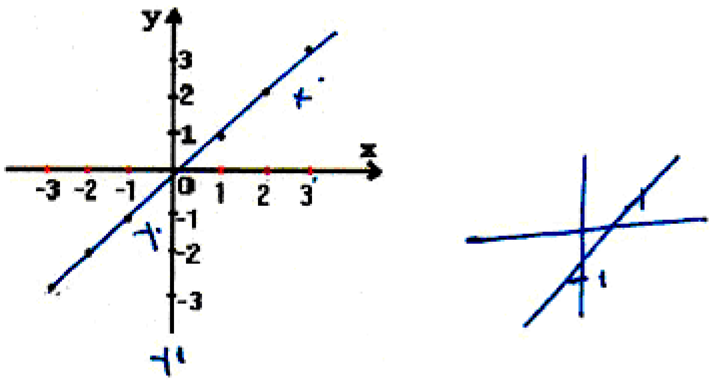


Figure 7-28. Demet's written explanation for the points on $y = x$ with the domain of projected points.

Referring to first quadrant (which she calles 'x region') and third quadrant (which she called 'y region'), she said that it passed through these two places:

> 'If it passed through one side, it wouldn't be a function' (Demet).

Deniz drew a straight line through the graph as shown in Figure 7–29 below:

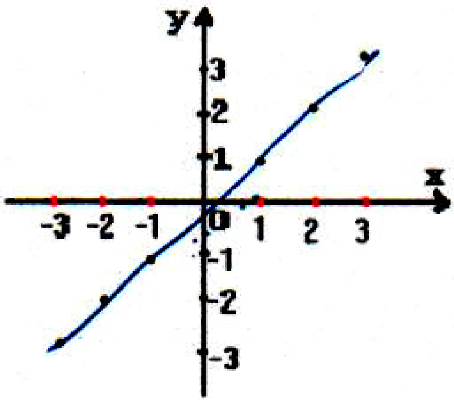


Figure 7-29. Deniz's written explanation for the points on $y = x$ with the domain of projected points.

> 'It's a function since x and y intersect each other. This line (straight line she drew passing through graph) is passing through from this and this (probably referring to the points on the graph)' (Deniz).

Belma did not draw a straight line. She still considered the graph as a function since she considered it a straight line graph:

> 'Shall I consider it as a straight line?…assigning (1 with 1, 2 with 2) we have a straight line…it's a function. $y = x$, it's passing through the origin' (Belma).

7.1.6 Points on a line

In the interview, all students were shown a graph as shown below in Figure 7–30:

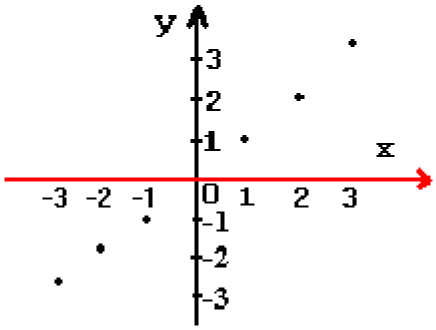


Figure 7-30. Points on a straight line with the domain of R.

Students were told that the x axis coloured as red was the domain. Then they were asked whether it was a function or not. They were then asked to explain the reasons for their answers. The table 7–6 below summarizes all students' responses and their explanations:

	Function or not	Explanation
Ali	Not a function	Colloquial definition
Aysel	Not a function	Colloquial definition
Ahmet	Function	Colloquial definition wrongly used
Belgin	Function	Colloquial definition wrongly used
Demet	Function	Drawing a straight line
Arif	Function	Drawing a straight line
Deniz	Function	Considering the graph the same as the earlier graph which has a different domain
Cem	Function	Considering the graph the same as the earlier graph which has a different domain
Belma	Could not decide first, then changed to function	Looking for a formula

Table 7-6. A summary of students' responses to the points on a line.

Two out of nine students (Ali and Aysel) used the colloquial definition and correctly did not consider the graph as a function:

> 'It's not a function, because every element in the domain does not have a corresponding value' (Ali).

Referring to the numbers which do not have a corresponding value as he sketched in Figure 7–31 below, he said that 'only 1, 2, 3, -1, -2, -3 have (corresponding values)'.

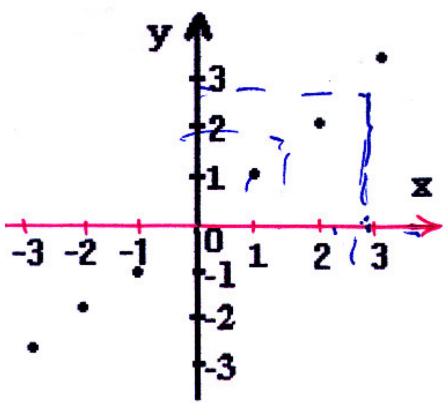


Figure 7-31. Ali's written explanations for the points on a straight line with the domain of R

Aysel used the colloquial definition as follows:

> 'I think it's not, there are elements left here, between 1 and 2, here is domain, all real numbers are here, between 1 and 2, it does not go to anywhere, between 2 and 3 too, in other words it's not a function' (Aysel).

One student (Belgin) considered the graph as a function by using the colloquial definition wrongly. She first assumed that the domain is between -3 and 3. She was then reminded that it is the whole x axis. She then explained as follows:

> 'It's also a function…this is like the other one before. All the elements in the domain are assigned to elements in the range' (Belgin).

To focus her attention to the elements which are left unassigned in the domain, she was then asked to find the corresponding value for $3/2$. However, she could not focus on the graph and found the value as $3/2$ as shown in Figure 7–32:

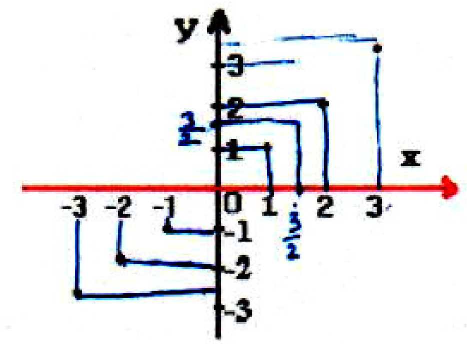


Figure 7-32. Belgin's written explanations for the points on a straight line with the domain of R

Two students (Demet and Arif) drew straight lines through the graph and considered the graph as a function. Demet drew a straight line as shown in Figure 7–33 below:

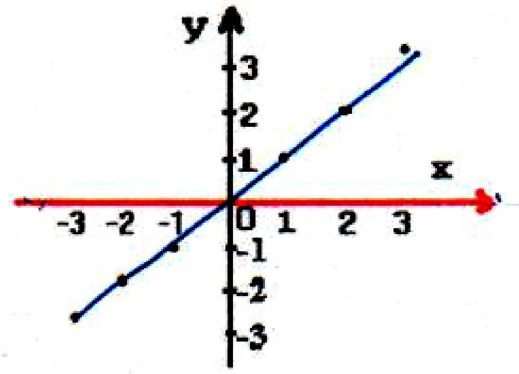


Figure 7-33. Demet's written explanations for the points on a straight line with the domain of R

> 'As I said before (for the same graph with the domain of $\{-3,-2,-1,0,1,2,3\}$) this is also a function' (Demet).

When she was asked whether there is a difference between these two graphs, she could not respond.

Arif drew a straight line through the given graph in the first quadrant as shown in Figure 7–34 below:

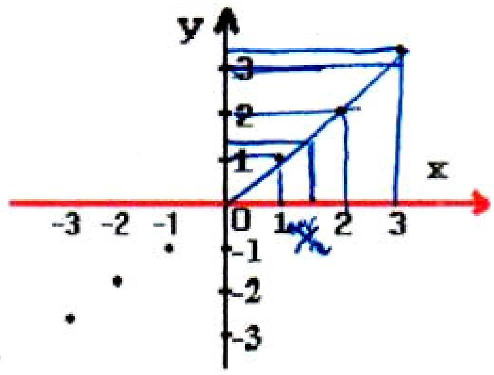


Figure 7-34. Arif's written explanations for the points on a straight line with the domain of R

He was reminded that the graph is the given points. He then said:

> 'If I don't draw the line then I can't find the values…if I draw the straight line, it's like the formula for a graph. There is one formula for a line. If here is $3/2$ then I can find its (corresponding) value. If it wasn't a straight line then I couldn't find it' (Arif).

Two students (Deniz and Cem) considered the graph as a function seeing no difference between the graph with the previous graph which has a different domain.

Belma first could not decide about the graph. She was then asked to find the corresponding value for 1. She said that it could be 1 or -1. When she was asked how she found it, she said that 'it depends on the given formula'. She was told to decide by considering the graph. She said:

> 'It could be itself (1)…for 3, it could be 4 or itself…it's a function' (Belma).

Explaining on the graph as shown in Figure 7–35 below, Ahmet said the following:

> 'For $f(1)$, it's almost 1…are all of the elements between 0 and -1, and 0 and 1 assigned to 1?...this is also function, conditional function. No it's not, because it says that x is an element of reals…1 for 1, 2 for 2…I can't see anything for . I think it's still a function…actually we can deduce it from the definition. Our teacher noted down two details about it…we study it for the exam' (Ahmet).

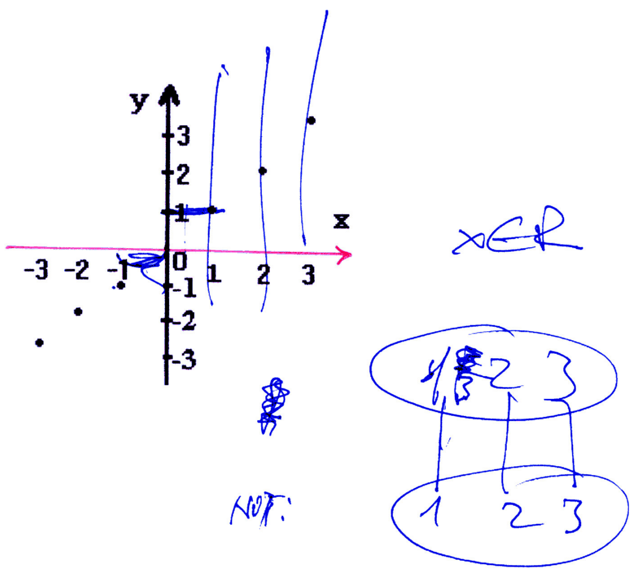


Figure 7-35. Ahmet's written explanations for the points on a straight line with the domain of R

7.1.7 *Graph of smiley face*

In the interview, all students were shown a graph as shown Figure 7–36 below:

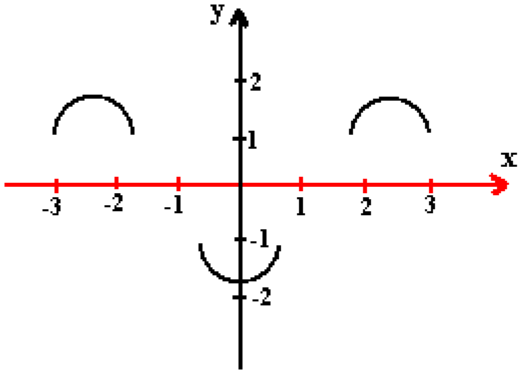


Figure 7-36. The graph of smiley face.

They were told that the x axis which was coloured as red was the domain. They were then asked whether it was a function or not. They were then asked to explain the reasons for their answers. Table 7–7 summarizes all students' responses and explanations:

	Function or not	Explanation
Aysel	Not a function	Colloquial definition
Ali	Function/changed to not a function	Colloquial definition wrongly used/ignoring elements left in the domain/when mentioned 1 in the domain changed his mind
Arif	Function/changed to not a function	Used colloquial definition when reminded of -1 on x axis.
Ahmet	Not sure	Vertical line test/drawing of set-correspondence diagrams
Demet	Not sure	Focused on x axis under the areas of three pieces of the graph/no further explanation
Deniz	Not a function/changed to function	The numbers on y axis is not the same as the numbers on x axis
Belma	Function	Exemplar based response (the graph is like a parabola)
Cem	Not a function	The shape is different
Belgin	Not sure	The shape is diferent

Table 7-7. A summary of students' responses to the graph of smiley face.

Aysel used the colloquial definition to consider the graph as a function:

> 'I think this is not a function…like in the other function. There should not be elements left in the domain, but 1 does not take any value in *y*. I think it's not (a function)' (Aysel).

She seemed to change her response to consider the graph as a function. Applying vertical line test, she did not change her response:

> ‘There are elements left in the domain even I apply verticals…this is a graph of a function. Here for instance, I drew one through, 2 takes only one value, it intersects once, but at the end 1 also should have taken a value. I think it’s not a function…but can there be elements that make function undefined?...no no it’s not (a function)’ (Aysel).

Ali first considered it as a function since he thought of the domain as the line segments under the graph. He explained by drawing a set diagram as shown in Figure 7–37 below:

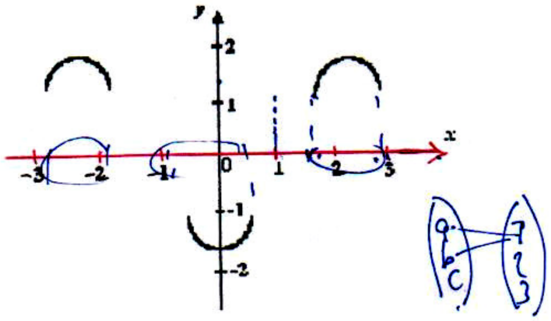


Figure 7-37. Ali’s written explanations for the graph of smiley face.

> ‘This is a function…because …2 in y can take the same value, two different values, two elements of the domain’ (Ali).

When his attention was drawn to corresponding value for 1, he changed his mind and did not consider it as a function.

Arif first considered it as a function from *R* to *R*. He assigned a few numbers on the x axis with the numbers on y axis as shown in Figure 7–38 below:

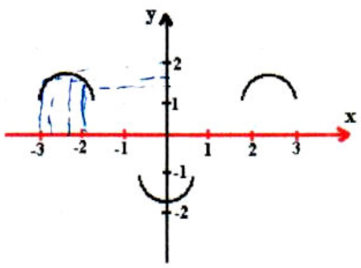


Figure 7-38. Arif’s written explanations for the graph of smiley face.

He was then asked to find the corresponding value for -1 on axis. He said that he could not find it. When he was asked whether this affected it being a function, he said that 'there can't be elements left in the domain, there can be in the range. It's not a function'.

Ahmet considered it as a function by applying the vertical line test to one piece of the graph on the right, as shown in Figure 7–39 below:

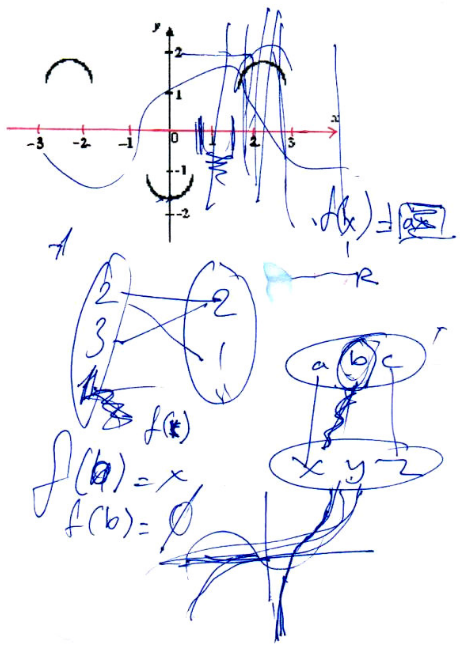


Figure 7-39. Ahmet's written explanations for the graph of smiley face.

He then drew set correspondence pictures leaving one element in the domain unassigned (See Figure 7–39 above):

> 'This (b) is in the domain but it does not go to anywhere. (Focusing on the graph) For instance, 0 is in the domain but it does not go to anywhere in y …there are no corresponding values for -1 and 1' (Ahmet).

When asked for 0, he said that its corresponding value is -1.5. Referring to the second set diagram, he said that $f(a)=x$ but $f(b)$ is empty. He then focused on 1 on the x axis, but he could not decide whether or not the fact that 1 does not have any corresponding value affects the graph to be a function.

Demet could not decide about the graph. She focused on the numbers on the x-axis, -2 and -3, 2 and 3 (under the areas of two pieces of the graphs) and -1 and -2 (on the y axis) as shown in Figure 7–40 below:

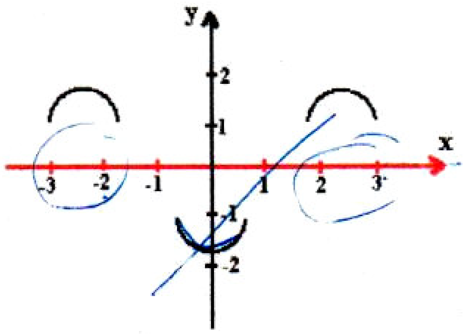


Figure 7-40. Demet's written explanations for the graph of smiley face.

She could not decide whether it is a function or not.

Deniz did not consider it as a function because the y axis is labelled between -2 and 2 while the x axis is labelled between -3 and 3. When he was told that he could put -3 and 3 on the y axis, he considered it as a function (See Figure 7–41 below).

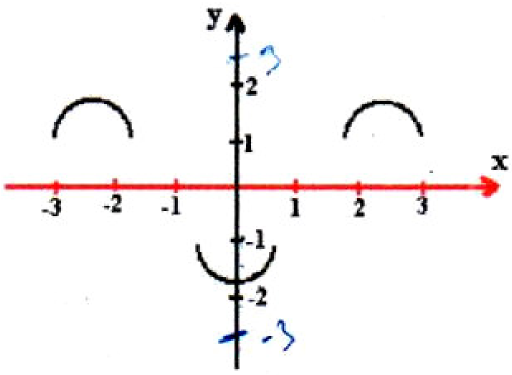


Figure 7-41. Deniz's written explanations for the graph of smiley face.

Belma considered it as a function since the graph is similar to a parabola. She tried to join the three pieces as shown in Figure 7–42 below:

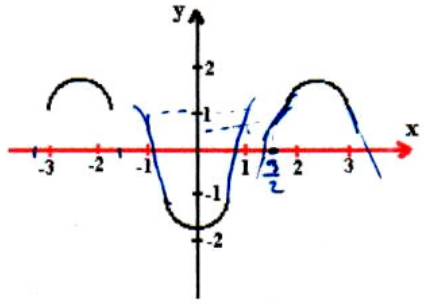


Figure 7-42. Belma's written explanations for the graph of smiley face.

> 'These are like parabolas, however I can't say anything more…1 (is assigned) with 1, −1 with −1…function if we join them they are like parabolas, increasing and decreasing, sine and cosine…' (Belma).

Cem did not consider it as a function since he is seeing such a thing like this for the first time.

Belgin was not sure about the graph. She said the following:

> 'I don't wanna do this (question), the shapes are very different' (Belgin).

7.1.8 Non exemplar graph 1

In the interview, all students were presented with a graph as shown in Figure7–43 below:

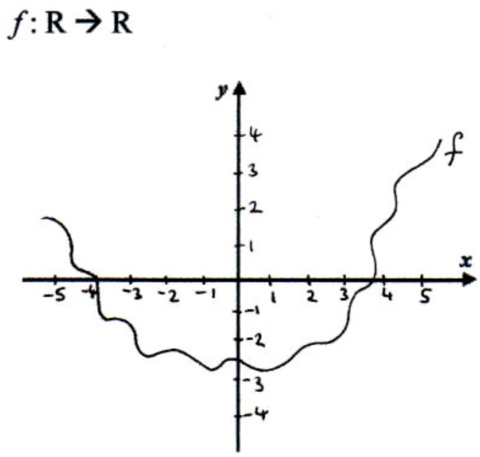


Figure 7-43. Non-exemplar graph 1.

Students were asked whether it was a function or not. To realize that it is not a function, one should focus on it very carefully since the graph bends onto itself. Students were then asked to explain the reasons for their answers. The table 7–8 below summarizes all students' responses and explanations:

	Function or not	Explanation
Ali	Not a function	Vertical line test/colloquial definition
Aysel	Not a function	Vertical line test/colloquial definition
Ahmet	Not a function	Vertical line test/use of set-correspondence diagram/colloquial definition
Belma	Not a function	Numbers on axes are irrational.
Belgin	Function	Finding corresponding values of numbers on x axis
Arif	Not a function/ change to function	Finding corresponding values of numbers on x axis
Cem	Function	Visual hints. Numbers on x axis (-3, -2, -1, 1, 2, 3) are inside the graph
Deniz	Not a function	General appearance of the graph
Demet	Not a function	General appearance of the graph

Table 7-8. A summary of students' responses to the non-exemplar graph 1.

Ali did not consider the graph as a function using the vertical line test as shown in Figure 7–44 below:

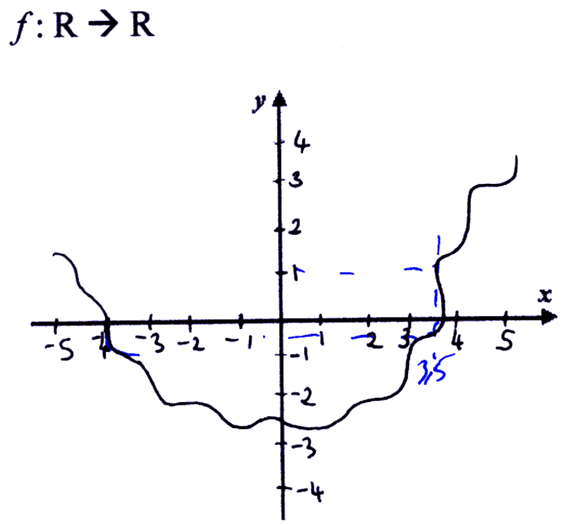


Figure 7-44. Ali's written explanations for the non-exemplar graph 1.

He first wanted to know whether or not the part of the graph between x values of 3.5 and 4 has a slope:

'Isn't it passing through the same point, is it? In other words, it's not vertical, is it? Does it have a slope?' (Ali).

When he was told that it has a slope, he did not consider the graph as a function:

'For instance, here, for two x values, there are different values of y. For instance, for 3.5…it is 1, ½, –½…it's not a function' (Ali).

Aysel did not consider the graph as a function using the vertical line test and the colloquial definition as shown in Figure 7–45 below:

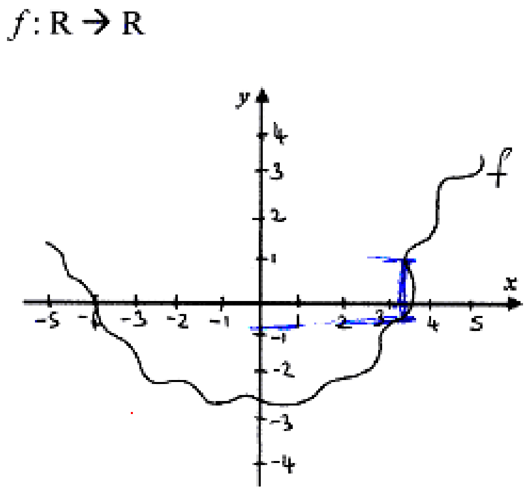


Figure 7-45. Aysel's written explanations for the non-exemplar graph 1.

'This isn't a function, because …it's the rule of a function. In the domain, it can't go to more than one in the range…' (Aysel).

Ahmet did not consider the graph as a function using the vertical line test. He first asked whether or not the graph bends onto itself. When he was told that it did bend onto itself, he did not consider it as a function by using the vertical line test as shown in Figure 7–46 below:

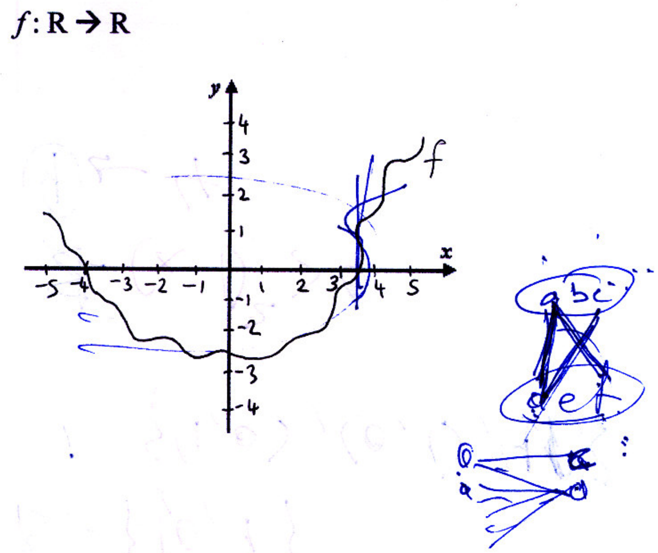


Figure 7-46. Ahmet's written explanations for the non-exemplar graph 1.

When he was asked the reason why, he drew a set diagram picture as shown in Figure 7–46 above and used the colloquial definition. He first said that two elements cannot be assigned to one element in the range. He then changed his mind and said:

> 'one element in the domain is not given to two different values in the range. When I draw verticals here, x gives two different y values, like a parabola, pardon opposite parabola. It's not a function' (Ahmet).

Belma did not consider the graph as a function since she considered the numbers on the axes are irrational:

> 'From R to R…the values between this and this are rational values. I mean between -2 and -3 (referring to y-axis). From rational numbers to rational numbers. This isn't a function…because there isn't an integer between -2 and -3, not 2 or 1 for instance. There are normally irrational numbers between these numbers. That's why (it's not a function)' (Belma).

Belgin considered the graph as a function since she could find corresponding values of some values of x as shown in Figure 7– 47 below:

> 'I have looked at the numbers. They have certain values, therefore it's a function' (Belgin).

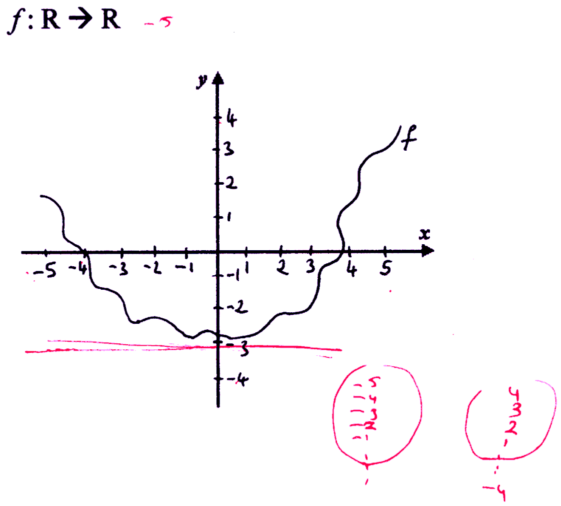


Figure 7-47. Belgin's written explanations for the non-exemplar graph 1.

Demet did not consider the graph as a function due to the general appearance of the graph:

> 'It's impossible, this can't be a function…I can't think of a function like this…function can be on the same plane, and can be proportional, but it starts here then goes wavy' (Demet).

Deniz did not consider the graph as a function due to the general appearance of the graph:

> 'This is not a function. First of all, the lines didn't go straight. It goes shape by shape. To be able to intersect exactly, it shouldn't be like this shape' (Deniz).

He was then asked to draw how the graph should have been drawn. He then drew straight lines close to the graph as shown in Figure 7–48 below:

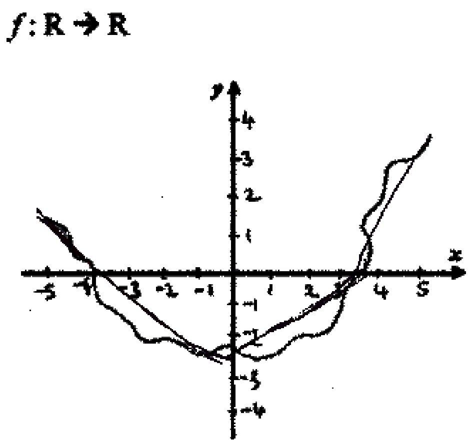


Figure 7-48. Deniz's written explanations for the non-exemplar graph 1.

Cem focused on some irrelevant visual hints. He considered the graph as a function since -1, -2, -3, 1, 2, 3, (on x axis) are inside the graph (between the x intercepts).

Arif did not at first consider the graph as a function since he could not find corresponding values of *x:*

> 'For x value, it's passing through 3 and 4, nearly 3.5 (on the x axis). In *y*, it's passing through -2 and -3…I think this isn't a function…I don't think I can find values by looking at this shape…when I look at the graph, I should be able to find the corresponding values for some *x*. For instance $f(x)$…I should be able to find the image of $f(x)$, but here I can't find' (Arif).

He considered the graph as a function by finding the corresponding values for a few numbers as shown in Figure 7–49 below:

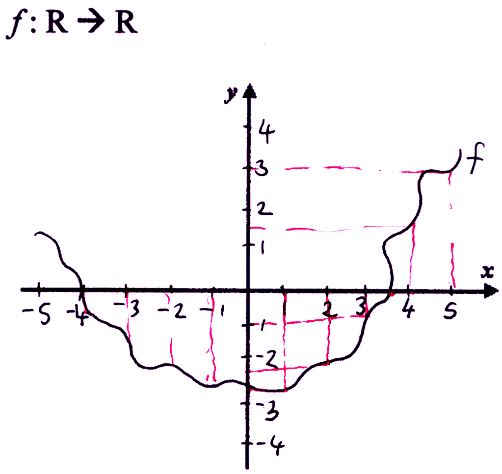


Figure 7-49. Arif's written explanations for the non-exemplar graph 1.

> 'Before, I said that it's not a function…because I couldn't find integer values. But then I realized that it shouldn't be integer values. Because it says that it's from reals to reals…I didn't take this into account. Since it's from reals to reals, it's a function' (Arif).

7.1.9 Non exemplar graph 2

In the interview, all students were shown a graph as shown in Figure 7–50 below:

$f: \mathbf{R} \rightarrow \mathbf{R}$

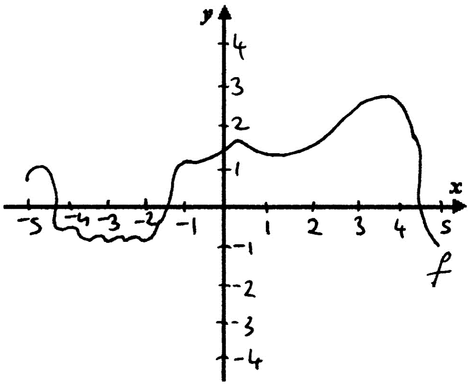


Figure 7-50. Non-exemplar graph 2.

Students were asked whether it was a function or not. To realize that it is not a function, one should focus on it very carefully since the graph bends onto itself. Students were then asked to explain the reasons for their answers. Table 7–9 below summarizes all students' responses and their explanations:

	Function or not	Explanation
Ali	Function	Colloquial definition
Aysel	Not a function	Colloquial definition
Ahmet	Not a function	Coloquail definition/Vertical line test
Belma	Not a function	There are two x-intercepts and they are rational numbers
Arif	Not a function	No formula to find corresponding values/ relating x and y values without any particular direction
Cem	Not a function	General appearance of the graph
Deniz	Not a function	General appearance of the graph
Demet	Not a function	General appearance of the graph
Belgin	Function	Graph has a formula/Could not tell the formula

Table 7-9. A summary of students' responses to the non-exemplar graph 2.

Ali considered the graph as a function using the colloquial definition:

> 'This is a function…because this coincides with the function definition…I can't say the definition now…every element in the domain has only one image. Two things don't meet' (Ali).

To see whether he decided by using the vertical line test, he was asked whether he was taught the vertical line test. Although he said he has not seen such a test, when he was given some explanation about it he used it as follows as seen in Figure 7–51:

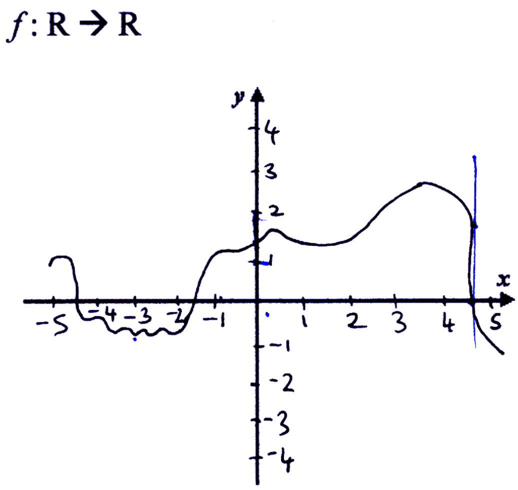


Figure 7-51. Ali's written explanations for the non-exemplar graph 2.

He explained the meaning of the vertical line test by referring to the colloquial definition. He then applied the vertical line test and said that an x value takes two different values.

Aysel did not consider the graph as a function using the colloquial definition.

> 'I think this isn't a function. Because here it intersects the axis between 4 and 5 (referring to negative x axis) and the function takes the value of 0. If we look at the curve under here between 4 and 5 (again referring to negative x axis), it takes one value from here. A value between 4 and 5, one value in the domain takes two values in the range. This can't be a function. Function can't have two values…here 4.5, it takes 0 on the function and a y value between -1 and 0 (in negative y-axis). Therefore this isn't a function' (Aysel) (See Figure 7–52 below).

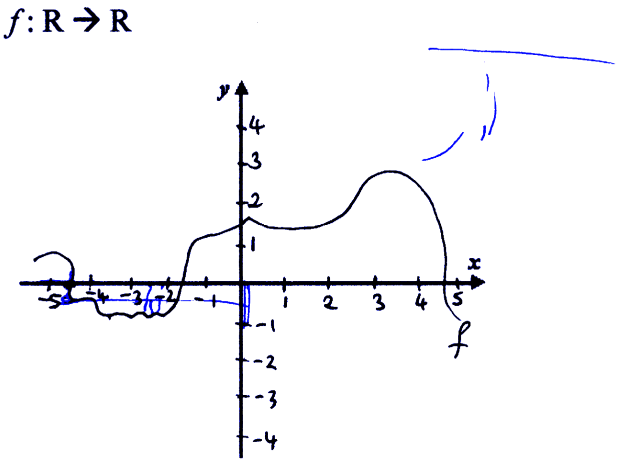


Figure 7-52. Aysel's written explanations for the non-exemplar graph 2.

Ahmet did not consider it as a function using the colloquial definition with the vertical line test as shown in Figure 7–53:

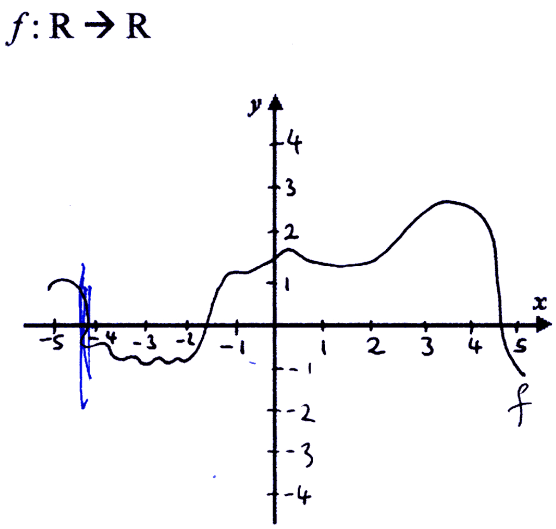


Figure 7-53. Ahmet's written explanations for the non-exemplar graph 2.

> 'I think it passes through…I mean it touches…I mean when I draw vertically, at two points. Two y values for x. A point between -4 and -5 has two different values of y. One is in minus y values, and one in positive y values. I think it's not (a function)' (Ahmet).

Belma did not consider it as a function since the x-intercepts are rational numbers:

> 'Same as before. It's not a function since it's passing through two numbers (referring to x intercepts)…(it's not a function) because they are rational numbers' (Belma).

She was reminded that the function is defined from *R* to *R* and asked what a real number is. She said that minus numbers are real numbers. When she was asked whether $\frac{2}{3}$ is a real number, she said that it is not a real number but it is a rational number and she added that they are different.

Belgin considered it as a function since it has a formula:

> ‘This is a function…if we take a function and put some values then we find this graph…our teacher told us to think of a formula that is appropriate for this graph, for instance $x+x$. We then put some values’ (Belgin).

However, when she was asked the formula for this graph, she could not say anything about it.

Demet did not consider it as a function since the shape of the graph is different. Sharpening one part of the graph as a straight line as shown in Figure 7–54 below, she said that only this part is a function:

$f: \mathbf{R} \rightarrow \mathbf{R}$

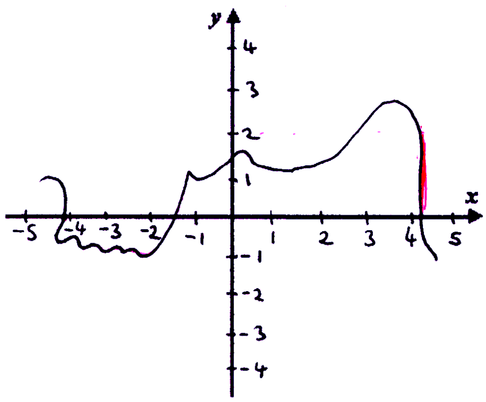


Figure 7-54. Demet’s written explanations for the non-exemplar graph 2.

She said that the graph should be like the part where she marked with red. Clearly, she has a strong focus on the shape of the graph and she is trying to make the graph look like a familiar exemplar she has, namely the straight line.

Deniz did not consider the graph as a function since the general appearance of the graph is different:

> ‘This can’t be. It’s like the Arabic letters, shapes are different. They don’t intersect each other…if it was straight, it might be…but it’s not’ (Deniz) (See Figure 7–55).

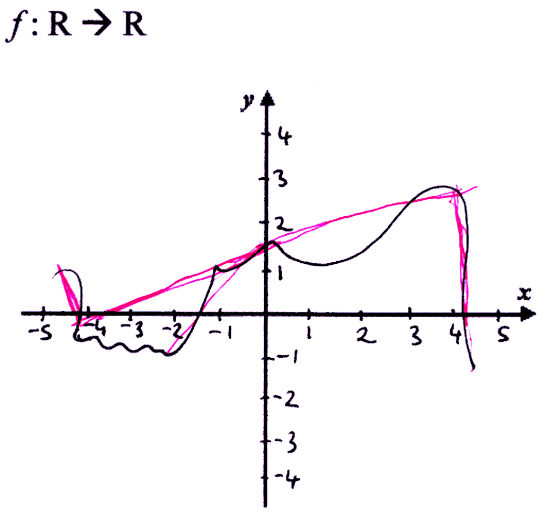


Figure 7-55. Deniz’s written explanations for the non-exemplar graph 2.

Cem did not consider the graph as a function because of the general appearance of the graph. He said that it could be a function if it was like a straight line as he drew below in Figure 7–56:

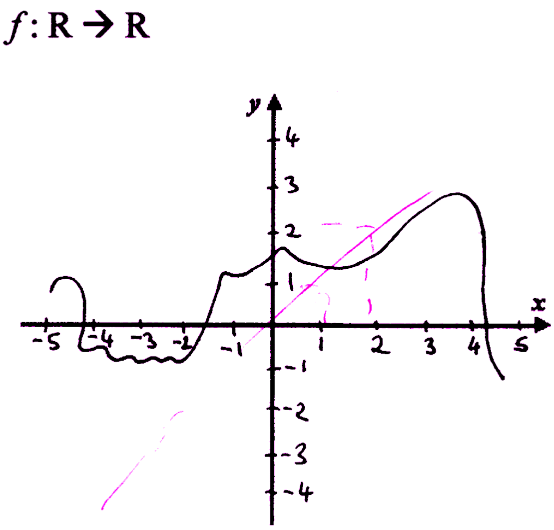


Figure 7-56. Cem’s written explanations for the non-exemplar graph 2.

Arif did not consider it as a function since there was not a formula for the graph. He said that if there was a formula then he would consider it as a function. He was then told to focus on the graph without thinking of a formula. He then found the corresponding values for y. He was then told to do the opposite (finding corresponding values for x). He wrongly found the corresponding value 2 for 2. He then changed his mind and found the corresponding values for 1, 2, 3 and 4 correctly as shown in Figure 7–57 below:

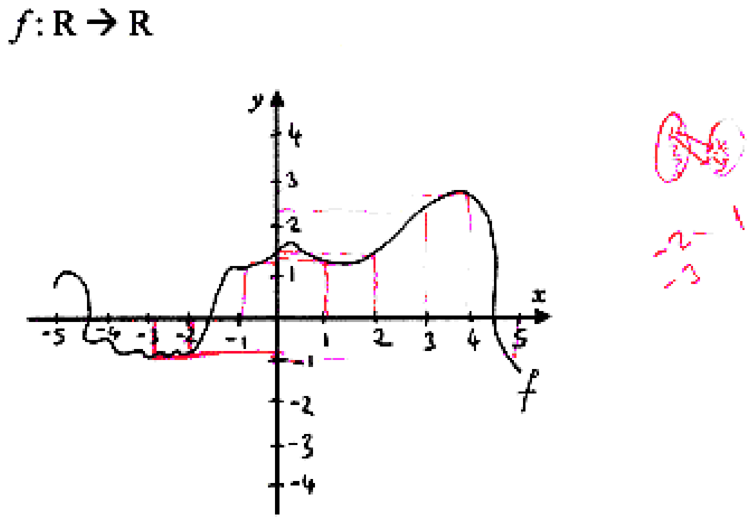


Figure 7-57. Arif's written explanations for the non-exemplar graph 2.

Although he found the corresponding values for x, he drew a set-correspondence diagram assigning 1 in the first set (from y-axis) to two different values in the second set. His explanations reveal that he thinks of a function as some kind of relation between x and y without it having a particular direction.

7.1.10 Graph of $f(x) = -\sin x$

In the interview, all students were presented with a graph as shown in Figure 7–58 below, which is the graph of $f(x) = -\sin x$:

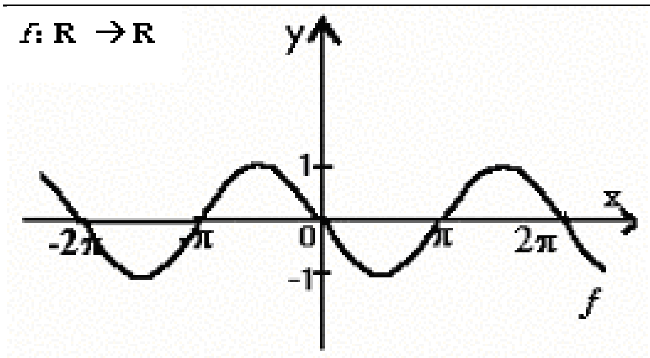


Figure 7-58. The graph of $f(x) = -\sin x$.

Students were asked whether it was a function or not. They were then asked to explain the reasons for their answers. Table 7–10 below summarizes all students' responses and their explanations:

	Function or not	Explanation
Ahmet	Function	Vertical line test followed by colloquial definition explained by set-correspondence diagram
Aysel	Function	Exemplar based focus/definitional properties/action on the graph (assigning values of x to the graph, but not to the y-axis).
Ali	Function	Exemplar based focus (recognizing as a sine function) followed by action on the graph.
Belma	Function	Exemplar based focus (recognizing as a sine function because of □).
Belgin	Function	Exemplar based focus (recognizing as a sine function because of □).
Arif	Function	Exemplar based focus/familiarity to parabolas
Deniz	Not a function	Visual hints irrelevant to definitional properties.
Demet	Not a function	General appearance unfamiliar
Cem	Not sure	General appearance unfamiliar

Table 7-10. A summary of students' responses to the graph of $f(x) = -\sin x$.

Six out of nine students considered this graph as a function. Only two of them (Ahmet and Aysel) refered to the colloquial definition. However, they did not directly use the colloquial definition. For instance, Ahmet first used the vertical line test saying that all lines intersect once. He then explained the definitional properties by using a set-correspondence diagram as shown in Figure 7–59:

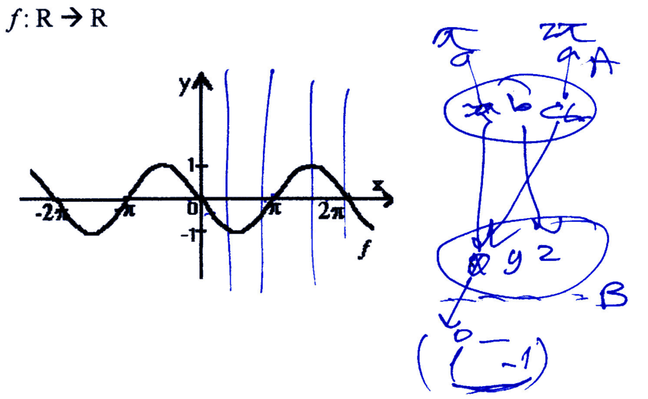


Figure 7-59. Ahmet's written explanation for the graph of $f(x) = -\sin x$.

> "Here set diagrams come to my mind' (Ahmet).

He drew a set-correspondence diagram by putting a, b and c in the first set (writing π for a and 2π for c) and x, y, z in the second set:

> 'The domain is between 1 and -1. Sorry, the range, for all values of f. In fact, yes for all values (of f) it's in that interval. I think it's a function" (Ahmet).

Aysel first referred to the exemplar cluster of trigonometric functions:

> 'It's a function, and it's a trigonometric function' (Aysel)

Although she said that 'every value goes to one value, not many values' which can be interpreted as the use of the coloquial definition, when she was asked to give a few examples, she joined $\pi/2$ and $-\pi/2$ to the graph not to the corresponding values on y-axis as shown in Figure 7–60 below:

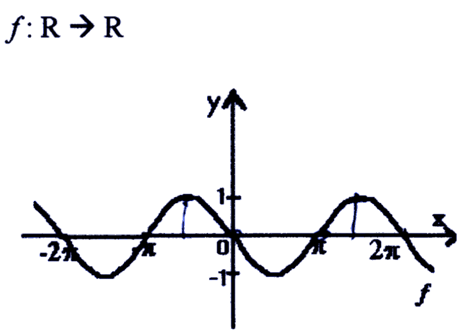


Figure 7-60. Aysel's written explanation for the graph of $f(x) = -\sin x$.

Four students (Ali, Belma, Belgin, Arif) gave exemplar-based explanations without any reference to definitional properties. Three of these students recognized it as a sine function while one of them found it similar to combinations of parabolas:

> 'I see sine function here, or cosine…From its shape the sine function comes into my mind. If $\pi/2$ takes value of 1, at 90 it takes 1, sine function' (Ali).

> 'I think this is a kind of sine function…I recognize it from π, but I can't explain why it's a function or not…if it's the graph of $f(x) = \sin x$ then it's a function' (Belma).

> 'Is this π…I think this is a function of sine…I remember it from last year…I understand from these π numbers. I can't explain why it's a function or not, because I don't remember…if it's the graph of sine function then it's a function' (Belgin).

> 'I say it's a function…and there is a shape of a parabola here. Different parabolas, here one parabola, here another' (Arif).

Arif found the corresponding values of 0, π, $-\pi$, 2π, $3\pi/2$, $-3\pi/2$. Although he found these corresponding values, he did not seem to refer to the definitional properties. When he was asked what $\pi/2$ and $-\pi/2$ correspond to, he could not find them correctly. He said that $\pi/2$ corresponds to $-1/2$, and $-\pi/2$ corresponds to $1/2$ (See Figure 7–61 below).

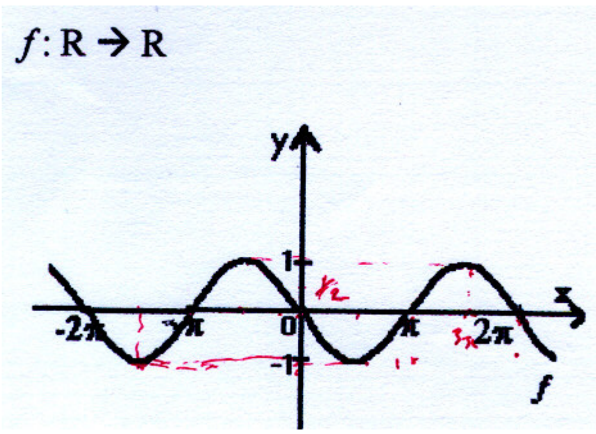


Figure 7-61. Arif's written explanation for the graph of $f(x) = -\sin x$.

Two students (Deniz and Demet) did not consider this graph as a function. Deniz did not consider it as a function since the numbers on y axis are not the same as the numbers on x axis:

> 'I think it's not (a function)…numbers are marked up to 2 (probably referring to 2π on the x axis), and here there is 1 (referring to y-axis)' (Deniz).

When he was asked to explain how we can make it be a function, he said that if the numbers on y-axis were the same as the numbers in x axis then it would be a function.

Demet did not consider the graph as a function since she found the shape of the graph unfamiliar:

> 'This is not a function…because it's not going on the same plane. It's going very wavy, as long as I know a function can go on the same plane, none of these are on the same plane' (Demet).

To understand what she meant by 'a function on a same plane', she was asked to draw a function which is on the same plane. She drew a coordinate system with the same numbers on the x and y-axes. She then plotted $(-\pi,0)$ and $(-\pi,-1)$ and joined them to the origin as shown in Figure 7–62 below and said:

> 'I think it's like this, we can join them like this' (Demet).

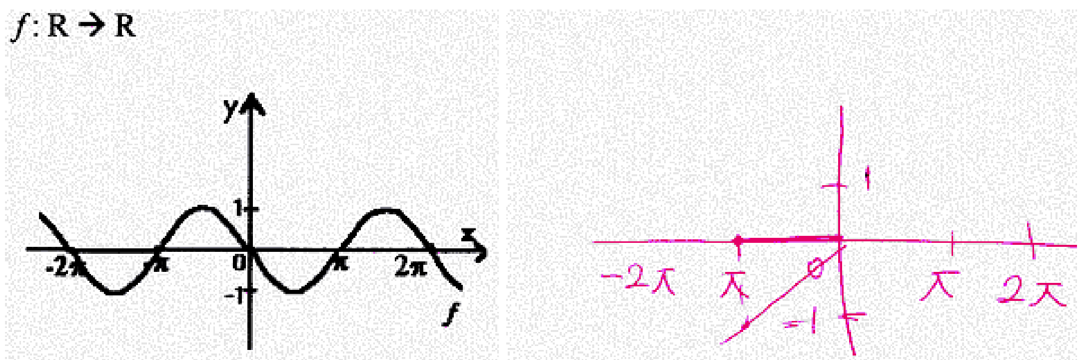


Figure 7-62. Demet's written explanation for the graph of $f(x) = -\sin x$.

One student (Cem) could not decide whether it is a function or not:

'I don't know very well. It's different and strange' (Cem) (See Figure 7–63 below).

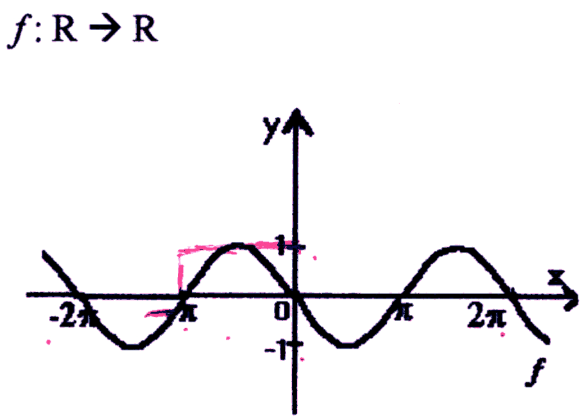


Figure 7-63. Cem's written explanation for the graph of $f(x) = -\sin x$.

Two of the students (Demet and Deniz) did not give any reason for their answers. Demet considered the straight line as a function. Deniz could not decide and said that he had no idea.

7.1.11 Graph of $f(x) = \sin x - 2$

In the interview, all students were shown a graph as shown in Figure 7–64 below:

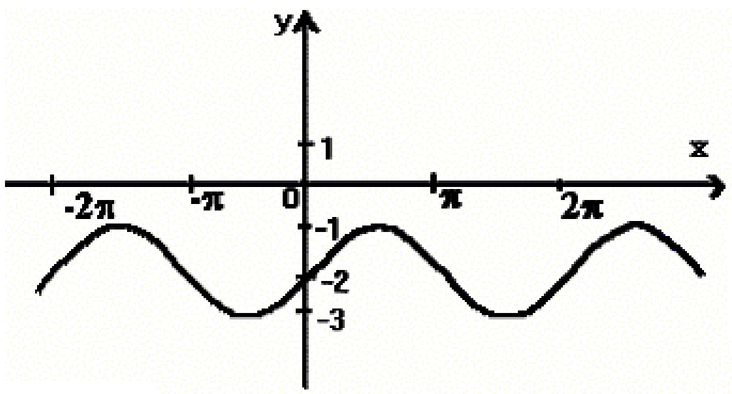


Figure 7-64. The graph of $f(x) = \sin x - 2$.

Students were asked whether it was a function or not. They were then asked to explain the reasons for their answers. Table 7–11 summarizes all students' responses and explanations:

	Function or not	Explanation
Ali	Function	Colloquial definition
Aysel	Function	Colloquial definition

Ahmet	Function	Colloquial definition by applying vertical line test and drawing a set diagram
Arif	Function	Assigning x and y values to each oher.
Belgin	Function	General shape of the graph (increases and decreases).
Belma	Not a function	The graph passes through y-axis only.
Cem	Not a function	The graph passes through y-axis only.
Deniz	Function/not a function	The graph passes through y-axis only.
Demet	Not a function	The graph is below x-axis.

Table 7-11. A summary of students' responses to the graph of $f(x)=\sin x-2$.

Three out of nine students considered this graph as a function by using the colloquial definition:

Ali used the colloquial definition as follows:

> 'I will consider the definition of a function. I will consider whether a point on x is defined on y and whether a point on x is defined for two values on y… π takes -2…each value of x has a corresponding value and only one corresponding value' (Ali) (See Figure 7–65 below).

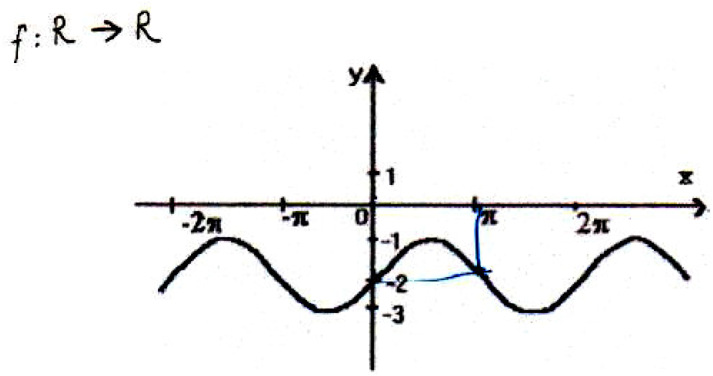


Figure 7-65. Ali's written explanation for the graph of $f(x)=\sin x-2$.

Aysel did not consider it as a function at first. She then used the colloquial definition focusing on the graph as continuing along the whole x-axis:

> 'I first thought that the graph was between 2π and -2π. Then I realized that two ends of the graph go on. Each element is assigned to an element, because these (two

ends of the graph) continues, and every element with only one element, not more than one element. Therefore it's a function' (Aysel)

Ahmet used the colloquial definition by the vertical line test followed by an explanation with the set-correspondence diagram as shown in Figure 7–66 below:

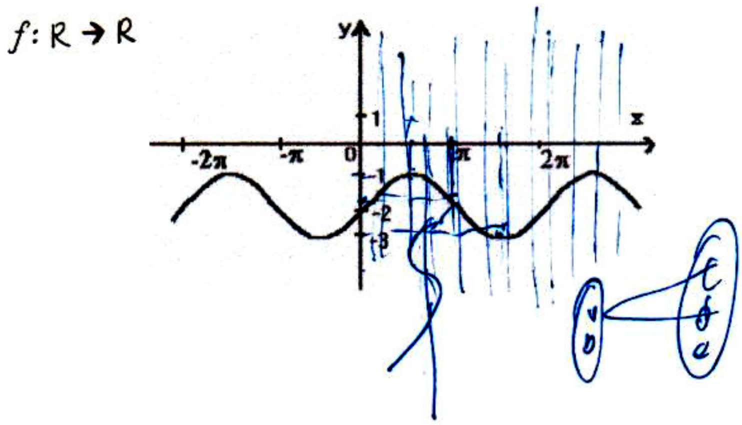


Figure 7-66. Ahmet's written explanation for the graph of $f(x) = \sin x - 2$.

'π has only one value, $\pi/2$ has one value on y …I think this is a function…in such cases I generally draw vertical lines, vertical to x, parallel to y It's to understand whether there are more than one value for a value in the domain…. a can not have two values, c and d …if it (vertical lines) intersect at one point then it's a function. If it (the graph) was like a letter S, then the vertical line would intersect at three points'(Ahmet).

Arif considered it as a function since the graph passes through 2 (referring to -2 on y-axis) and there are elements corresponding to values of x and y. When he was asked to give an example for these values, he drew the following:

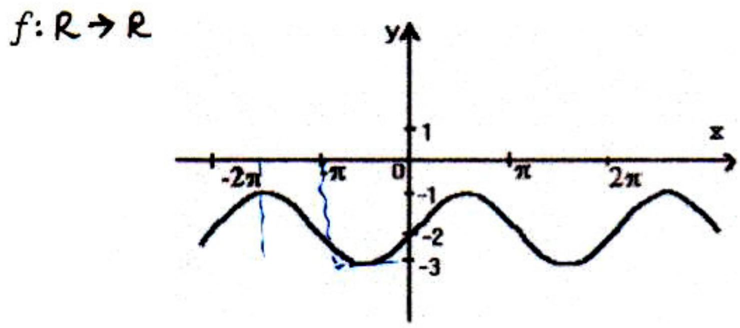


Figure 7-67. Arif's written explanation for the graph of $f(x) = \sin x - 2$.

Although he said that there is a value for each element, he was confused with domain and range:

> 'When I take -3, it corresponds to π. There is definitely a value for each value. It's from reals to reals' (Arif).

Belgin considered the graph as a function by focusing on its general appearance. She said that it is a function since the graph decreases and increases.

Three students did not consider the graph as a function since it only passes through the y-axis. Belma did not consider the graph as a function since the graph passed through one point only, -2. She was then given a similar graph passing through x and y axes as follows:

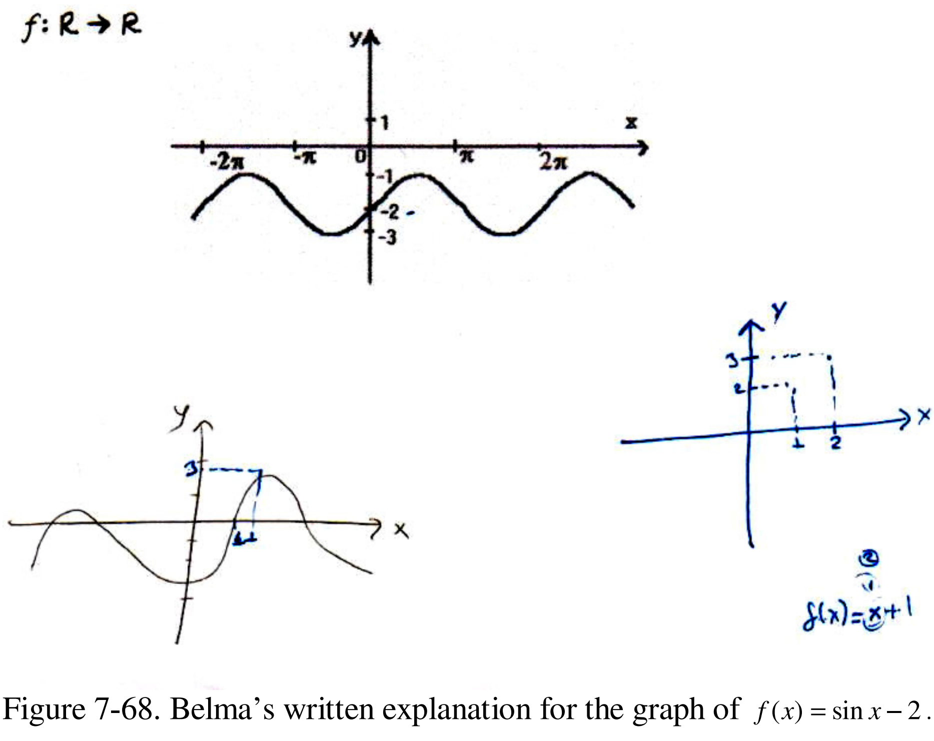


Figure 7-68. Belma's written explanation for the graph of $f(x) = \sin x - 2$.

She considered this as a function:

> 'This may be a function because, there is a corresponding value when it's assigned…when we give a value for x we can find its corresponding value…corresponding value for 1' (Belma).

Cem did not consider it as a function since it passes through only -2:

> 'This is not a function…because of -2 (on y-axis)…the graph doesn't intersect any other points…it should also pass through π and 2π' (Cem).

Deniz first considered it as a function and changed his mind since it only passes through y axis:

> 'Generally, there are numbers (on the axes), lines (axes), shapes (the graphs)' (Deniz).

When he was asked the properties of a function, he said that he did not know. He was then asked whether or not the fact that the graph did not touch the x axis had an effect. He then changed his mind:

> 'no this is better, more sensible. It does not pass through the x line. It only passes through the y line…it's not a function' (Deniz).

Demet did not consider the graph as a function since it is below x axis. She was then asked to draw a graph that can be a function. She drew the following as shown in Figure 7–69:

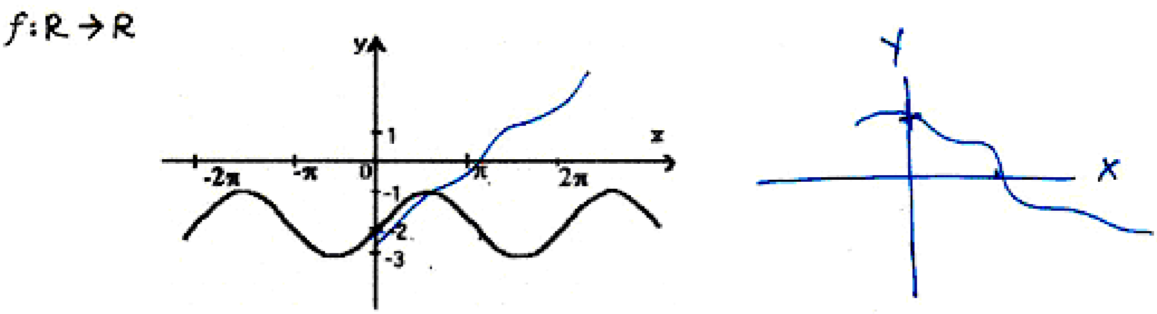


Figure 7-69. Demet's written explanation for the graph of $f(x) = \sin x - 2$.

7.1.12 Expression of the split-domain function

In the interview, all students were shown an expression as shown below:

$f : R \rightarrow R$

$$f(x) = \begin{cases} 1, \text{ if } x^2 - 2x + 1 > 0 \\ 0, \text{ if } x^2 - 2x + 1 = 0 \\ -1, \text{ if } x^2 - 2x + 1 < 0 \end{cases}$$

Students were asked whether it was a function or not. They were then asked to explain the reasons for their answers. Table 7–12 summarizes all students' responses and explanations:

	Function or not	Explanation
Ali	Function	Recognizing as signum function. Correct graph. Set-correspondence diagram.
Ahmet	Function	Recognising as signum function. Although confused about the domain, he drew a correct graph applying vertical line test.
Aysel	Function	Recognizing as condinitional function. Wrong graph. Set-correspondence diagram assigning values less than 1 to -1, 1 to 0, values greater than 1 to 1.
Belma	Function	Recognizing as split-domain function.
Belgin	Not a function	Substituted -1, 0, 1 in $x^2 - 2x + 1$.
Arif	Function	Recognizing as signum function
Cem	Function	Notational hint: $f(x)$.
Deniz	Function	Relating the numbers on the right hand side of the expressions $x^2 - 2x + 1 > 0$, $x^2 - 2x + 1 = 0$, $x^2 - 2x + 1 < 0$ to the numbers of the range, 1, 0, -1
Demet	Not a function	Specific hints. 'we can't take a square of a function'.

Table 7-12. A summary of students' responses to the expression of the split-domain function in the interview.

Ali considered the expression as a function. Although he did not use the colloquial definition, he gave an explanation with its graph and set diagram. He drew a set diagram assigning x and y to 1 (noting that $x < 1$ and $y > 1$) and 1 to 0, and leaving -1 in the range unassigned (See Figure 7–70 below):

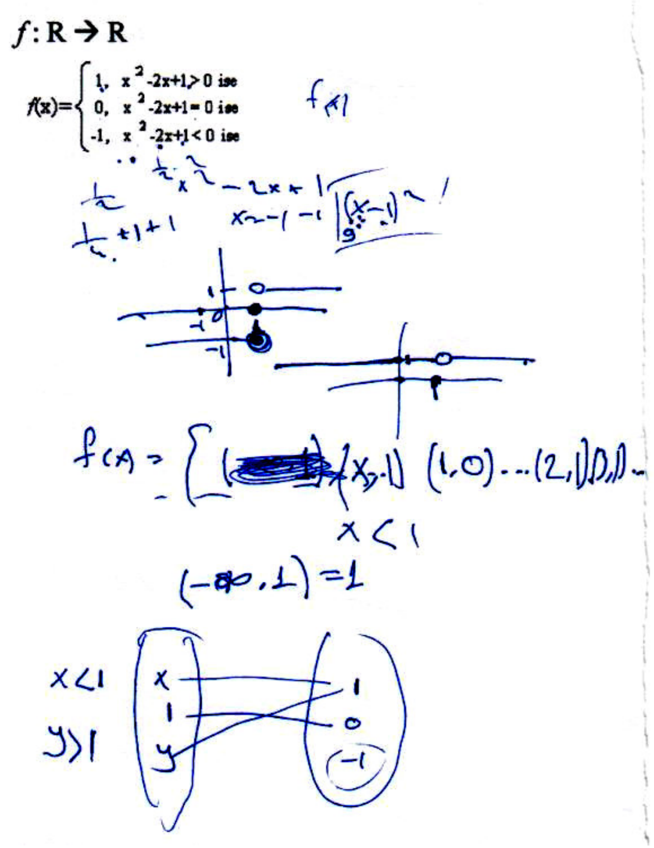


Figure 7-70. Ali's written explanation for the split-domain function expression.

Belma considered the expression as a split-domain function. She substituted a few numbers in the expression, $x^2 - 2x + 1$. She could not assign values in the domain to 1, 0, -1 in the range since she focused on them as the elements of the domain instead of the elements of the range. She substituted 1 in the expression of the condition, $x^2 - 2x + 1 > 0$ (See Figure 7–71 below).

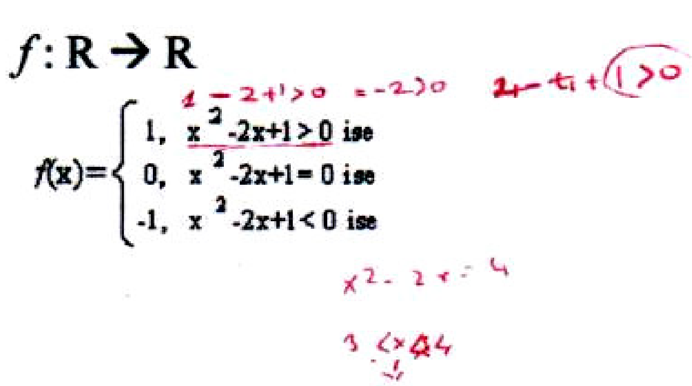


Figure 7-71. Belma's written explanation for the split-domain function expression.

She then ignored $x^2 - 2x + 1$, and said that the function takes -1, for x values less than 0, 0 for 0, and 1 for values less than 1.

Belgin did not consider it as a function. When she substituted 2 in $x^2 - 2x + 1$, she found -4 and said that it is not a function since -4 can not be greater than 0. When she was asked how she decided what to substitute in $x^2 - 2x + 1$, she said that her teacher told them to substitute -1, 0, 1 when the function is defined from real numbers to real numbers.

Deniz considered it as a function by relating the numbers on the right hand side of the expressions $x^2 - 2x + 1 > 0$, $x^2 - 2x + 1 = 0$, $x^2 - 2x + 1 < 0$ to the numbers of the range, 1, 0, -1:

> 'It is a function…the one at the top is greater than 0. Since the one at the bottom is minus it is less than 0. Therefore it is a function. In other words it is directly proportional' (Deniz).

Cem considered it as a function because of $f(x)$ notation. He was confused about the domain.

7.1.13 $y = 5$

In the interview, all students were given the following expression as shown below:

$$y = 5$$

Students were asked whether it was a function or not. They were then asked to explain the reasons for their answers. Table 7–13 below summarizes all students' responses and explanations:

	Function or not	Explanation
Ali	Function	Drawing the graph/constant function
Ahmet	Function	Drawing the graph/constant function
Aysel	Not a function/ function	Specifying the domain as R/Drawing the graph/constant function
Arif	Function	Drawing a set-correspondence diagram

Demet	Function	Marking $(-5,0)$ as 5 and joining it to 5
Deniz	Function	No explanation
Cem	Function	y equals to 5
Belma	Not sure	Drawing $y=5$ /putting values for y
Belgin	Not sure	Looked for $f(x)$

Table 7-13. A summary of students' responses to $y=5$.

7.1.14 $y=5$ (for $x \leq 2$)

In the interview, all students were given the following expression as shown below:

$y=5$ (for $x \leq 2$)

Students were asked whether it was a function or not. They were then asked to explain the reasons for their answers. Table 7–14 summarizes all students' responses and explanations:

	Function or not	Explanation
Arif	Function	Colloquial definition / Assigning values less than or equal to 2 to 5.
Ali	Function	Recognising as a constant function /Drawing the graph
Aysel	Function	First asking the domain, drew the graph correctly.
Ahmet	Function	Assigning values less than 2 to 5 and drawing the graph correctly/Drawing a set-correspondence diagram.
Belma	Not sure	Drawing the graph for all values of x.
Cem	Function	Considering (for $x<2$) as a condition with no reference to definitional properties
Belgin	Not sure	Looked for *f* notation. Could not respond.
Deniz	Could not decide	There is no relation between $y=5$ and '$y=5$ (for $x \leq 2$)'.
Demet	Function/not a function	5 is not less than two

Table 7-14. A summary of students' responses to $y=5$ (for $x \leq 2$).

Ali used the colloquial definition as follows:

> 'For x less than 2 y is equal to 5, this is also a function. Is it a conditional function?... $y=5$, for every value in the domain which is less than 2…there is something called a constant function, $f(x)$ is equal to…a, x changes, a does not change, this is constant function, since $f(x)$ is equal to y…erm I'm confused…because, for two different values of *x*, no this is a function…the range is only 5, can there be a function like this? No there can't be a function and graph like this' (Ali) (See Figure 7–72 below).

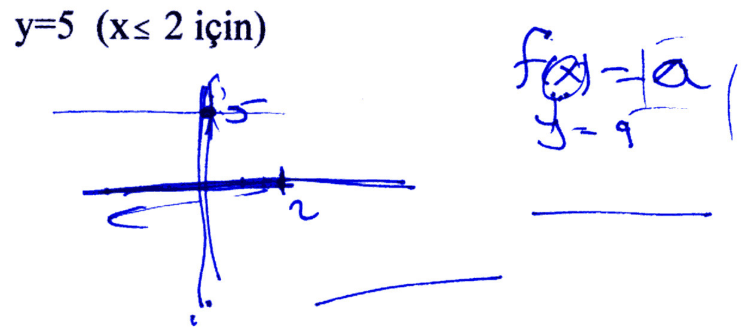


Figure 7-72. Ali's written explanation for $y = 5$ (for $x \le 2$).

Aysel considered the expression as a function. She first wanted to know the domain. When she was pointed to 'for $x \le 2$' in brackets she said the following:

> 'for values of x less than or equal to 2, y is always 5, including 2, here it's graph (drawing the graph correctly as shown below), this is a function, since there is a domain' (Aysel).

Ahmet considered the expression as a function assigning values less than 2 to 5. He drew the graph correctly and sketched a set diagram picture as shown in Figure 7–73 below:

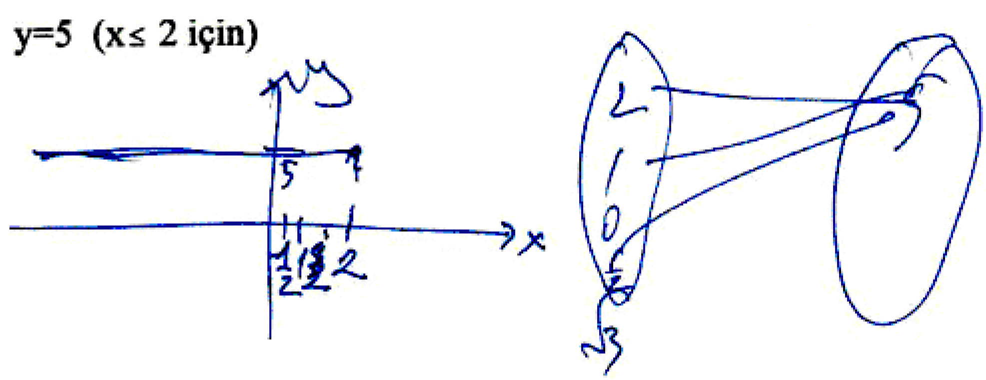


Figure 7-73. Ahmet's written explanation for $y = 5$ (for $x \le 2$).

Arif considered it as a function assigning values less than or equal to 2 to 5:

> 'Including 2, all of them, including x, 2, 1, 0. 2 and all values less than 2 take the value of 5. This is function' (Arif).

Belma was not sure about the expression. Although she said that values less than or equal to 2 are assigned to 5, she drew the graph for all real numbers as shown in Figure 7–74 below:

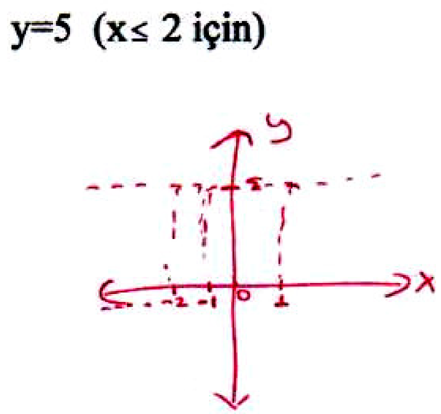


Figure 7-74. Belma's written explanation for $y = 5$ (for $x \leq 2$).

Cem considered 'for $x \leq 2$' as a condition without any reference to the colloquial definition:

> 'Yes this is also a function, *y* is equal to 5. It says what is required for x less than 2' (Cem).

Belgin could not decide about this expression. She wrote *f(x)*=*x*+2 and substituted 3 in the expression. She said that she had chosen 3 since she considered it as the smallest number (probably referring to *integer*) greater than 2. She could not give any more explanations.

Demet first considered it as a function then changed her mind:

> 'because it says that 5 is both less than and equal to 2, we can take its function, we can't draw it, 5 is not less than 2' (Demet).

Deniz was not sure about this expression. He said that $y = 5$ is a function and he could not see any relation between these two.

7.1.15 $y = 5$ (for all values of x)

In the interview, all students were shown an expression as shown below:

$y = 5$ (for all values of x)

Students were asked whether it was a function or not. They were then asked to explain the reasons for their answers. Table 7–15 summarizes all students' responses and explanations:

	Function or not	Explanation
Ali	Function	Assigning all values of x to 5. Drawing the graph.
Aysel	Function	Assigning all real numbers to 5. Drawing the graph.
Ahmet	Function	Constant function. Assigning all numbers to 5. Confused by the domain.
Belma	Function	Recognising it as the same as the other two. Drawing the graph.
Belgin	Not sure	Confused by the domain and range. Looking for a formula to substitute numbers to get 5.
Arif	Function	Assigning all values of x to 5.
Cem	Function	Looking for specific values of x.
Deniz	No answer	No explanation
Demet	Not sure	Giving values for y.

Table 7-15. A summary of students' responses to "$y = 5$ (for all values of x)" in the interview.

Ali considered the expression as a function mentioning that x refers to real numbers. By referring to the graph he drew for $y = 5$ he said that he considered for all values of *x.*

Aysel assigned all real numbers to 5 by explaining it with the graph she drew as shown in Figure 7–75 below:

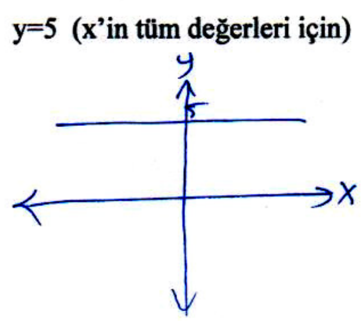


Figure 7-75. Aysel's written explanation for "$y = 5$ (for all values of x)".

'for all values of *x*, real numbers, complex numbers, all of them' (Aysel).

Ahmet considered it as a constant function assigning all numbers to 5:

'this is a function, because this is also a constant function, for -1, for $\sqrt{3}$ or $-\frac{1}{2}$, all come to 5, there is one element in the domain' (Ahmet).

However, he was confused by the domain and said that all elements are assigned to 5 in the domain instead of the range.

Cem considered it as a function but could not explain it successfully. He looked for specific values for *x*:

'if *y* is equal to 5, what is required for all values for *x*?' (Cem).

Belgin could not decide about the expression. First she said that *y*, in the range, is equal to 5 when we give any value for *x*. However, she was confused about the domain and range. She drew the following as shown in Figure 7–76:

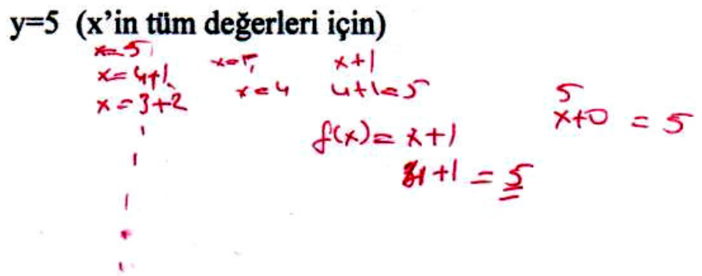


Figure 7-76. Belgin's written explanation for "$y = 5$ (for all values of *x*)".

She looked for a formula such as $f(x) = x + 1$ and substituted numbers for x to get 5, e.g. 4+1, 5+0.

Demet was not sure about the expression and she gave values for *y*.

7.1.16 $f(x) = \sin x - 2$

In the interview, all students were given the following expression as shown below:

$f(x) = \sin x - 2$

Students were asked whether it was a function or not. They were then asked to explain the reasons for their answers. Table 7–16 summarizes all students' responses and their explanations:

	Function or not	Explanation
Ali	Function	Colloquial definition
Aysel	Function	Colloquial definition
Ahmet	Function	Exemplar based focus/colloquial definition
Belgin	Function	Exemplar based focus (trigonometric function)
Arif	Function	Finding *f*(0)
Deniz	Function	Notational hints
Demet	Not a function	Drawing a wrong graph
Cem	Not a function	No explanation
Belma	Not sure	I don't know very well

Table 7-16. A summary of students' responses to $f(x) = \sin x - 2$.

Six out of nine students considered $f(x) = \sin x - 2$ as a function.

Three students (Ali, Aysel and Ahmet) considered the expression as a function by referring to the colloquial definition:

> 'If we put a real number for *x*, we find a value, a value between -3 and -1, since *sine* is between -1 and 1…the co-domain…to be a function, there should be a corresponding value for every value we put for *f(x)* (probably meaning x in *f(x)*), and there is' (Ali).

One student (Aysel) focused on the uniqueness of the assignment as well as the assignment of each element in the domain. She found the value of *sin*0 as 0 with the help of a unit circle and used the colloquial definition as follows:

> 'There shouldn't be elements left…and there should be only one' (Aysel).

First considering the expression as a trigonometric function, Ahmet found the values $f(0)$, $f(90)$, $f(270)$ and $f(360)$ as -2, -1, -3, -2 by drawing a unit circle:

> 'For any real value that I put for x in $\sin x$, it's again a real number. If I subtract a real number from a real number, then it's again a real number' (Ahmet).

The way he substituted values of x in the expression focusing on the domain and range (which are real numbers) is considered as using the colloquial definition.

Two students (Belgin, Deniz) considered $f(x) = \sin x - 2$ as a function without any reasonable explanation. Belgin considered it as a function since she recognized it as a trigonometric function. Deniz considered it as a function since he could see the notational features such as '*f*', '$R \rightarrow R$' and the formula '$f(x) = \sin x - 2$'.

One student (Arif) found only one value of the function, $f(0) = -2$. He was reminded that the function is from R to R. He then said that the value of the function can be an integer, rational or square root of any number. However, when he was asked to tell the definitional properties, he could not respond.

Two out of nine students (Demet and Cem) did not consider it as a function. Demet tried to draw its graph (see Figure 7–77 below). She labeled -2 on the x axis and said that it is not a function since it is only one point (-2). She also said that if it was *sinx*, then it would be on the other side (referring to 2 on the positive x axis).

f: R → R f(x)=sinx–2

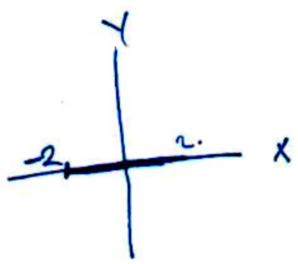


Figure 7-77. Demet's written explanation for "$f(x) = \sin x - 2$".

Cem could not give any explanation as to why he did not consider it a function. One student (Belma) could not decide about the expression.

7.1.17 *Drawing the graph of* "$f : R \to R$, $f(x) = 5$"

In the interview, all students were given the following question:

> Draw the graph of "$f : R \to R$, $f(x) = 5$".

Table 7–17 below summarizes all students' responses and their explanations:

	Drawing
Ali	Correct graph
Aysel	Correct graph
Ahmet	Correct graph
Arif	Draws the graph between -2≤x≤2
Belma	Draws the graph of x=5
Belgin	Marking 5 on positive x and y axes
Demet	Marking 0 and 5 on x axis and joining them
Deniz	Draws a straight line through (5,0) and (0,5)
Cem	Labeling x and y axes and trying to plot points

Table 7-17. A summary of students' responses to the transformation of $f : R \to R$, $f(x) = 5$ to its graph.

Three out of nine students drew the graph correctly. Ali, referring to questions about constant function, drew the following as shown in Figure 7–78 below:

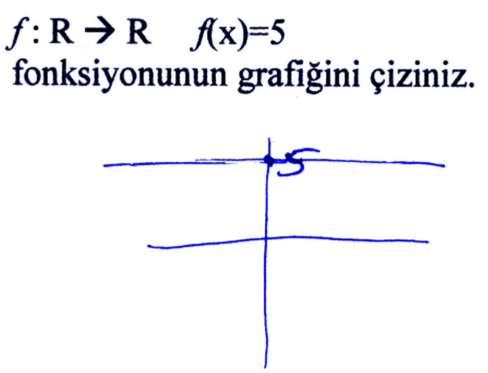


Figure 7-78. Ali's written explanations for the graph of "$f : R \to R$, $f(x) = 5$".

'$f(x) = 5$, here y indicates 5. For these values, that's the graph' (Ali).

Aysel drew the following as seen in Figure 7–79:

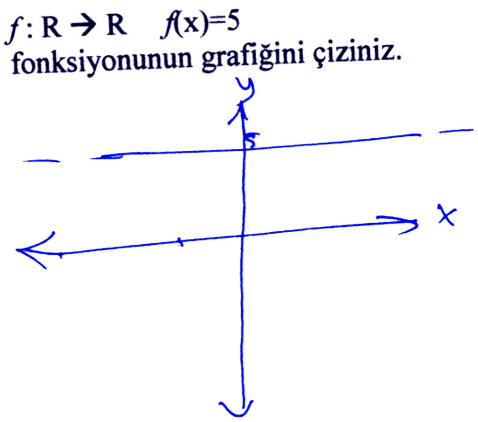


Figure 7-79. Aysel's written explanations for the graph of "$f : R \rightarrow R , f(x) = 5$".

'From real numbers, here is the domain. $y = 5$, it goes on like this' (Aysel).

Ahmet drew the graph as shown in Figure 7–80 below:

'For all values of *x*, $y = 5$' (Ahmet).

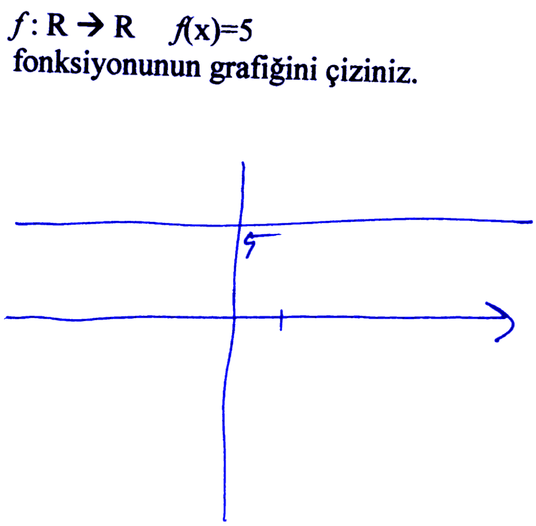


Figure 7-80. Ahmet's written explanations for the graph of "$f : R \rightarrow R , f(x) = 5$".

Arif drew the graph between -2≤*x*≤2 as shown in Figure 7–81 below:

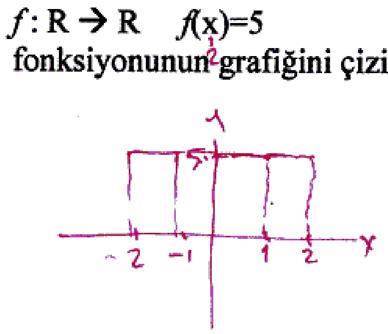


Figure 7-81. Arif's written explanations for the graph of "$f: R \rightarrow R, f(x) = 5$".

Belma drew the graph as shown in Figure 7–82 below:

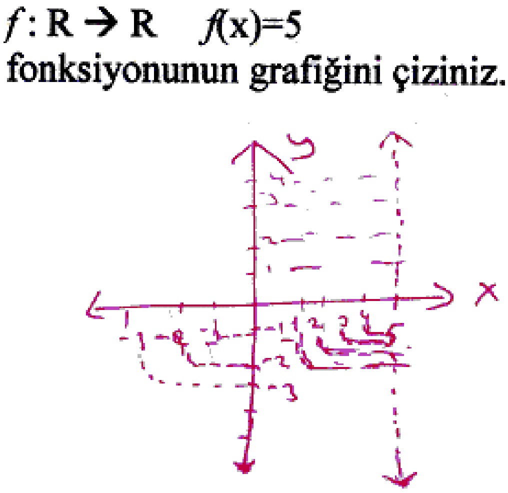


Figure 7-82. Belma's written explanations for the graph of "$f: R \rightarrow R, f(x) = 5$".

Her response reveals that she was confused by the aspects of domain and range. By drawing vertical lines to *x*=5 line, she said:

> 'the only values that we give to y are equal to 5…actually the values that we give to x are also equal to 5 but negatives values of x are equal to the values which are just opposite to them (negative y values). They're not equal to 5, –1 with –1, –2 with –2…in other words, the negative values of x and y are equal to each other. Only positive values of y and values of x are equal to 5. That's what I think' (Belma).

Belgin could not draw the graph. She said that x and y take the value of 5 and marked 5 on positive x and y axes as shown in Figure 7–83 below:

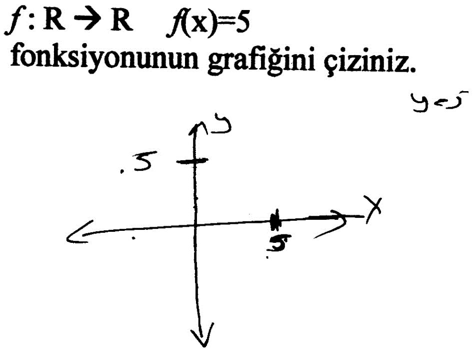


Figure 7-83. Belgin's written explanations for the graph of "$f: R \rightarrow R, f(x)=5$".

Demet could not draw the graph. She marked 0 and 5 on the x axis and joined them together as shown in Figure 7–84 below:

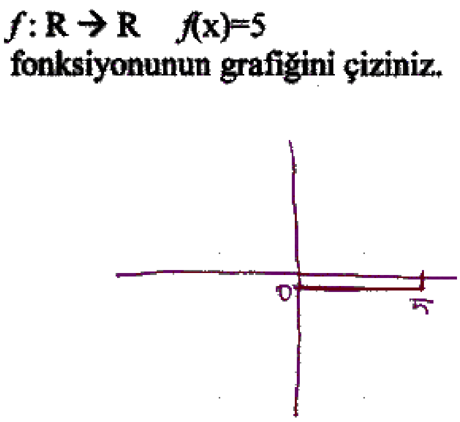


Figure 7-84. Demet's written explanations for the graph of "$f: R \rightarrow R, f(x)=5$".

Deniz could not draw the graph. He then drew a straight line through (5,0) and (0,5) as shown in Figure 7–85 below:

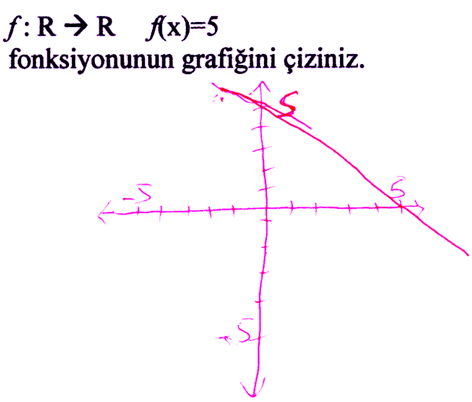


Figure 7-85. Deniz's written explanations for the graph of "$f: R \rightarrow R, f(x)=5$".

Probably 5 in the expression prompted him to mark 5 on the axes and joined them by relating to his early experiences of graph drawing.

Cem could not draw the graph. He first labeled positive and negative y axis and negative x axis with integer numbers until 5. He tried to plot a few points as shown in Figure 7–86 below:

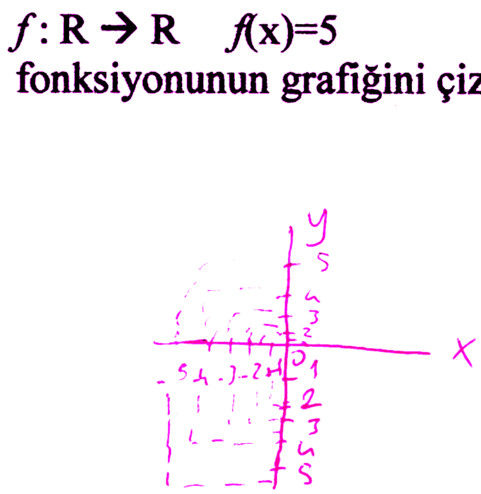


Figure 7-86. Cem's written explanations for the graph of "$f : R \to R$, $f(x) = 5$".

7.1.18 Drawing the set-correspondence diagram of "$f : R \to R$, $f(x) = 5$"

In the interview, all students were given the following question:

> Draw the set-correspondence diagram of "$f : R \to R$, $f(x) = 5$"

Table 7–18 summarizes all students' responses and explanations:

	Drawing
Ali	Correct diagram. Saying that there are infinite number of elements in the first set assigned $-\infty$ and $+\infty$ in the first set to 5 in the second set.
Aysel	Correct diagram. Assigning x_1, x_2, x_3, x_4 (which represents all reals) in the first set to 5 in the second set.
Ahmet	Correct diagram. Assigning -1, 1, $\sqrt{3}$, $\sqrt{2}$ (which represents all reals) in the first set to 5 in the second set.
Arif	Correct diagram.
Belma	Could not draw.

Belgin	Could not draw. Confusion between domain and range.
Demet	Wrong diagram.
Deniz	Wrong diagram. Assinging 1 to 1, 2 to 2 and so on up to 5. Changed his mind and assigned on 1, 2, 3, 4, 5 to 5.
Cem	Wrong diagram.

Table 7-18. A summary of students' responses to the transformation of $f: R \rightarrow R$, $f(x) = 5$ to the set-correspondence diagram.

Four students drew the correct diagram. Arif's diagram (see Figure 7–87 below) is considered correct even though it only shows x values in Z:

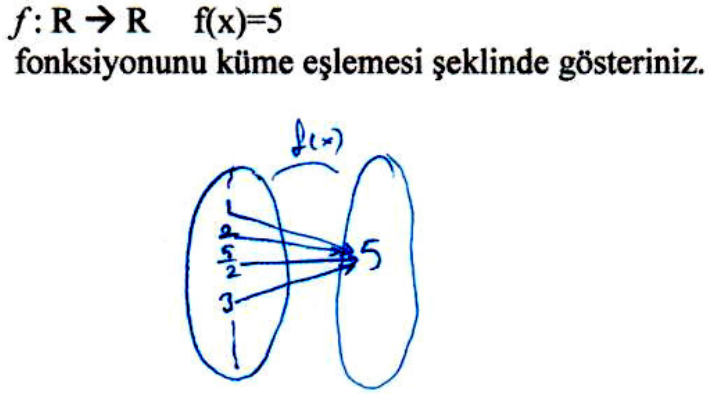


Figure 7-87. Arif's written explanation for the set-correspondence diagram of "$f: R \rightarrow R$, $f(x) = 5$".

He said that there could be 11, 100 in the domain and all of them are assigned to 5. He also said that there are rationals and irrationals since the function is from *R* to *R*.

Aysel, Ahmet drew the following diagrams as shown in Figure 7–88, Figure 7–89 below:

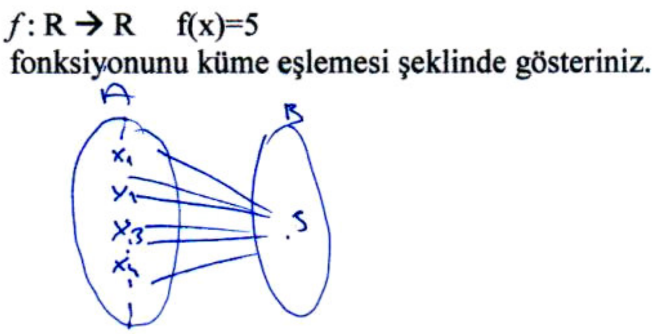


Figure 7-88. Aysel's written explanation for the set-correspondence diagram of "$f: R \rightarrow R$, $f(x) = 5$".

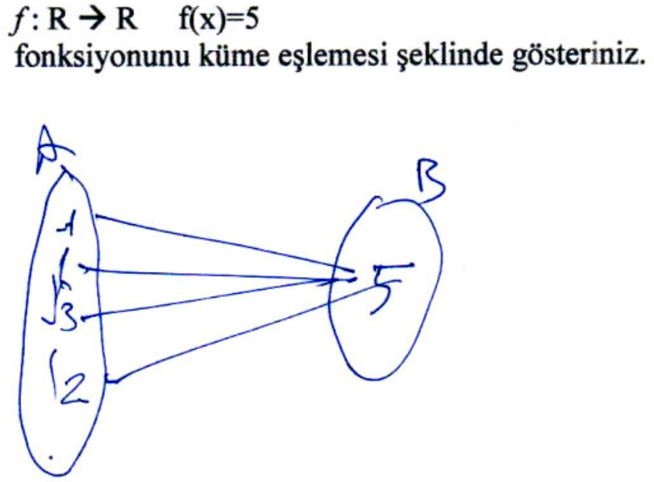


Figure 7-89. Ahmet's written explanation for the set-correspondence diagram of "$f : R \to R, f(x) = 5$".

Belma could not draw the set-correspondence diagram correctly. She was confused by the notation. For $f(x) = 5$, she put a few numbers in the first set and no numbers in the second set. She said that x is equal to values in its own set since $f(x) = 5$:

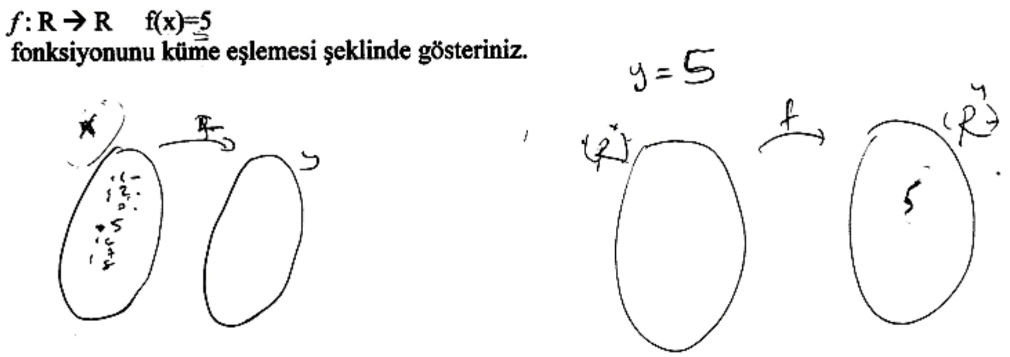


Figure 7-90. Belma's written explanation for the set-correspondence diagram of "$f : R \to R, f(x) = 5$".

She then draw another diagram considering $y = 5$ as shown in Figure 7–90 above:

Belgin could not draw the set-correspondence diagram. She wrote a few formulas for $f(x)$ ($f(x) = x + 1$ and $f(x) = x + 3$). She then tried to substitute a few values in $f(x)$:

> However, since it's from *R* to *R* we can give values up to 5…if we start from negatives (she wrote –1, 0, 1, 2, 3, 4)' (Belgin).

Demet drew a wrong diagram as shown in Figure 7–91 below:

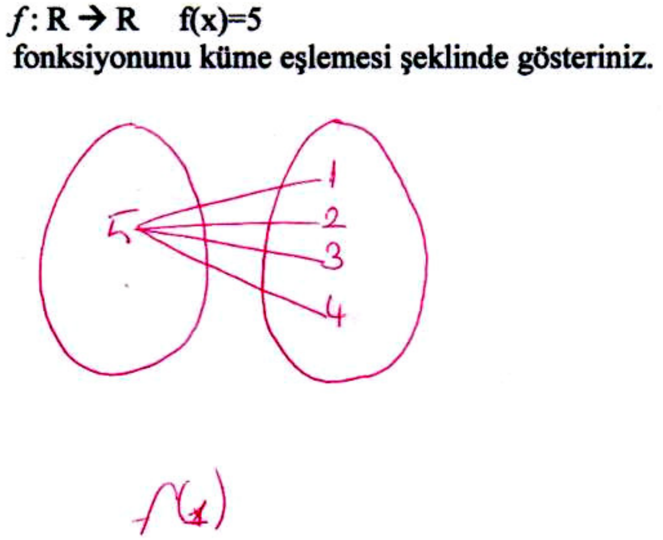


Figure 7-91. Demet's written explanation for the set-correspondence diagram of "$f : R \rightarrow R, f(x) = 5$".

She said she put 1, 2, 3, 4 in the second set since $f(x) = 5$ expresses the values of x up to 5. She then said that 5 may be included. When she was asked the value of $f(1)$ she could not find it and said that it is an empty set.

Deniz drew a wrong diagram as shown in Figure 7–92 below:

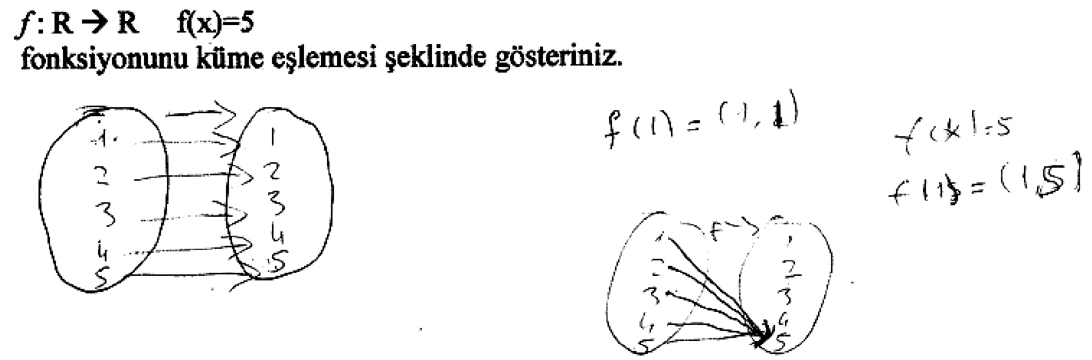


Figure 7-92. Deniz's written explanation for the set-correspondence diagram of "$f : R \rightarrow R, f(x) = 5$".

His use of notation was also wrong. He wrote $f(1)=(1,1)$. He then changed his mind to $f(1)=(1,5)$. He then drew the set diagram again by assigning 1, 2, 3, 4, 5 to only 5 as shown in Figure 7–92 above:

When he was asked the value of $f(10)$, he first said that 10 cannot be put for x. He then changed his mind and said that 'whatever we put for x, it's equal to 5'.

Cem drew a wrong diagram as shown in Figure 7–93 below:

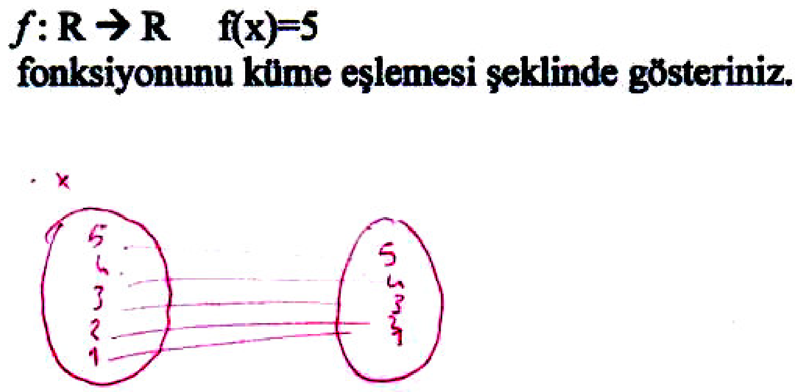


Figure 7-93. Cem's written explanation for the set-correspondence diagram of "$f: R \rightarrow R$, $f(x)=5$".

When he was asked to find $f(2)$ in the function, he said that it is 5 by referring to $f(x)$.

7.1.19 The set of ordered pairs for "$f: R \rightarrow R$, $f(x)=5$"

In the interview, all students were given the following question:

Write the set of ordered pairs for "$f: R \rightarrow R$, $f(x)=5$"

Table 7–19 summarizes all students' responses and explanations:

Aysel	$f(x)=\{...(x_1,5)...(x_2,5)...\}$
Ali	$\{...(1,5)..(2,5)..(3,5)....\}$
Ahmet	{(1,5), (0,5), (½,5),…}
Arif	$f(x)$=(-1,5),(1,5),(2,5),(3,5),(4,5)…
Belma	f={(5,1) (5,2) (5,3)…} followed by f={(1,5), (2,5)…}
Deniz	$f(x)=5$ $([x,1)(x,2)(x,3)(x,4)(x,5)]$

Cem	$f(x) = (1,1),(1,2),(1,3),(1,4),(1,5)$
Belgin	Could not write
Demet	Could not write

Table 7-19. A summary of students' responses to the transformation of $f: R \to R$, $f(x) = 5$ to the set of ordered pairs.

Three out of nine students (Aysel, Ali and Ahmet) responded by writing the ordered pairs in the brackets with dots in between. Two of them (Aysel and Ali) put dots in both directions (negative and positive) to refer to the infinite numbers of ordered pairs. They could focus on not only the integers but all real numbers:

> 'I can't write the set of ordered pairs because I can't write all values…they can take any value, 1, 2, $\sqrt{5}$, complex numbers, all of them but I can't write all of them' (Aysel).

> 'It would last too long…there are loads of *x*'s in real numbers…shall I put dots' (Ali).

Ahmet put dots in positive direction writing the pairs in different order:

> '$A = \{...1,0,\frac{1}{2},1....\}$, $B = \{5\}$, $f: A \to B$, B is the range and it takes 5 for all values' (Ahmet).

One student (Arif) wrote the set of ordered pairs for integer numbers putting them together without brackets.

Three students (Belma, Deniz and Cem) varied the second coordinate. Belma confused domain and range and varied y instead of x. She said she put 5 as the first coordinate because 5 is an element of x. When she was asked about the second coordinate, she said that they are also elements of x. Since she was confused with '$f(x)$' notation, she was told to consider $y = 5$ instead of $f(x) = 5$. She then changed her response to '$f = \{(1,5),(2,5)...\}$'.

Deniz varied the second coordinate. He said that he put integer numbers 1, 2, 3, 4, 5 since the function is given as "$f(x)=5$" and put "x" in the first coordinate since he did not know x and x is an unknown.

Cem put integers 1, 2, 3, 4, 5 for the second coordinate and put 1 for the first coordinate. When he was asked fo find $f(1)$ he said that all values take the same value, 5.

Two (Belgin and Demet) students could not write the set of ordered pairs:

> '(Writing $\{1,2,3,4,5\}$) We draw the set of values that 5 can take, form 1 to 5'. She was asked which values it can take. She replied: 'Function of x' (Demet).

7.2 A summary of chapter 7

Results presented in this chapter revealed that nine students in the interview dealt with different aspects of functions in different ways. More students could focus on definitional properties for the set-correspondence diagram and the set of ordered pairs. On the other hand, the graphs and the expressions caused more difficulties in terms of focusing on the definitional properties. They evoked concept images which need not to be a coherent whole.

In the next chapter, the analysis focuses on individual students to categorize their responses over a spectrum of performance. To do this categorization, the coherency of each student's overall responses is investigated.

CHAPTER 8 – CATEGORIZATION OF STUDENTS' RESPONSES

8.1 An overview

The aim of this chapter is to focus on each student and to categorize their performances. To do the categorization, students' focus on the core concept of function is investigated. As discussed in the theoretical framework chapter, the coherency in recognizing different aspects of functions with a strong focus on the definitional properties is considered as an indication of the ability of focusing on the core concept of function. Therefore, responses from nine students are categorized considering the research questions below:

- How is a student's overall response to different aspects of functions affected by the subtle differences among different aspects?
- How coherent is a student's response as s/he move from one aspect to the other?
- How do students who give coherent responses to different aspects of functions cope with this?

To do the categorization, a grid is prepared by summarizing the results as discussed in chapter 6 and chapter 7.

8.2 The grid

Students gave various reasons as they responded to different questions as discussed in chapter 6 and chapter 7. Therefore, different categories emerged from their responses to different aspects of functions. Summarizing these results to prepare the grid, these categories are refined by considering the two criteria of a category system: internal homogeneity (responses considered in the same category share common properties as much as possible) and external heterogeneity (each category of responses differ from the other categories) as described by Guba (1978) in Patton (1990). Therefore, a new set of overall categories is determined. The categories of "Visual hints/ Notational hints" and "First impression/General appearance" are considered under the same category called "exemplar-based focus". This has a purpose based on the theoretical framework. As discussed earlier, in chapter 6 and chapter 7, the visual hints come from the external representations of different aspects of functions. Responses like "numbers on the axes are given in equal distances" for a graph, "an expression cannot include a square root" or "there is not an *f*" for an expression, "arrows intresect each other" for a set-correspondence diagram" are all considered in the category of "exemplar-based focus". The reason is that those hints are established from students' earlier experiences. In other words, students reject or accept an item as a function because the item does or does not have those hints as the previous examples they have experienced.

Similarly, students' reliance on the first impressions and general appearances of the items are considered as in the category of "exemplar-based focus". Because the first impressions have come from the examples they have experienced so far. They simply accept an item since the general appearance of it resonates with those exemplars which were stored earlier.

The new set of categories are presented below. They are labelled with an abbreviation to be put in a cell in the grid:

Colloquial definition (CD): The use of the colloquial definition. Making statements to check the definitional properties.

Colloquial definition wrongly used (CDW): Either recalling the colloquial definition wrongly (e.g. saying that two elements in the domain can be assigned to the same element in the range) or using it in a wrong way (missing out that one element in the domain is not assigned to any element in the range).

Exemplar-based focus (EBF): Recalling specific examples e.g. recognising

$$f(x)=\begin{cases} 1, & \text{if } x^2-2x+1>0 \\ 0, & \text{if } x^2-2x+1=0 \\ -1, & \text{if } x^2-2x+1<0 \end{cases}$$

as a signum function. Responses that focus on the visual hints, the notational hints, the first impressions and the general appearances are also considered in this category because of the reasons as discussed above.

Vertical line test (VLT): Drawing vertical lines through the graph.

Set diagram (SD): Drawing a set diagram to decide whether or not the given item is a function.

Graph (GR): Drawing the graph of the given item.

Wrong graph (WGR): Drawing the wrong graph for the given item.

Constant function (CF): Recognizing the given item as a constant function without any other explanation.

Domain-range confusion (DRC): Considering the domain as the range of the function or vice versa.

✓:Correct answer for transformation of functions.

✕: Wrong answer for transformation of functions.

Other (OTH): Other

No response (---)

A detailed account of how each student's responses are labeled with the above categories is given in Appendix B3.

The cells in the grid are coloured as follows:

Colloquial definition (CD)

Colloquial definition together with the other categories e.g. CD-SD

Colloquial definition wrongly used (CDW)

Exemplar-based focus (EBF)

A spectrum of colour grey reflects a spectrum of performance of the students since as the grey colour becomes bolder the focus on the core concept of function gets stronger. Therefore, a spectrum of colours reveals a categorization of students.

The grid is presented in Table 8.1 below:

	Ali	Ahmet	Aysel	Arif	Belma	Belgin	Cem	Deniz	Demet
SET-CORRESPONDENCE DIAGRAMS	CD	CD	CD	CD	CD	CD	CDW	EBF	EBF
	CD EBF	CD	---	CD EBF	CD	CD EBF	---	EBF	---
SETS OF ORDERED PAIRS	CDW CD	CD SD	CD	CD SD	CD	---	CDW	EBF	OTH
	CD EBF	CD CDW		SD CD	SDW	CD EBF	---	EBF	---
GRAPHS – Straight line	EBF CD	VLT CD	EBF CD	DRC	OTH	OTH	EBF	---	---
Straight line in three pieces	CD	CD SD	CD	CD	EBF	---	EBF	EBF	EBF
Points graph (D=R)	CD	CDW	CD	EBF	OTH	CDW	OTH	EBF	EBF
Points graph (D=points)	CD	CD	CD VLT	CD	EBF	CDW	OTH	EBF	EBF
Smiley graph (D=R)	CD	VLT SD	CD	CD	EBF	EBF	EBF	EBF	EBF
Non-exemplar 1	VLT CD	VLT SD CD	VLT CD	OTH	EBF	OTH	EBF	EBF	EBF
Non-exemplar 2	CD	CD VLT	CD	DRC	EBF	OTH	EBF	EBF	EBF
$f(x)=-\sin x$ graph	EBF	VLT CD SD	EBF	EBF	EBF	EBF	EBF	EBF	EBF
$f(x)=\sin x-2$ graph	CD	CD VLT SD	CD	OTH	EBF	EBF	EBF	EBF	EBF
Graphs in the questionnaire.	CD EBF	CD VLT	EBF CD	EBF CD	---	---	EBF	EBF ---	EBF
EXPRESSIONS – Signum function	EBF GR SD	EBF GR VLT	EBF WGR	EBF	EBF	DRC	EBF	OTH	OTH
$y=5$	GR CF	GR CF	GR CF	SD	WGR	EBF	OTH	---	WGR
$y=5$ (for $x\le 2$)	CF GR	GR SD	GR	CD	WGR	EBF	OTH	---	OTH
$y=5$ (for all values of x)	CD GR	CF CD	CD GR	CD	GR	DRC	OTH	---	WGR
$f:R\to R$ $f(x)=\sin x-2$	CD	EBF CD	CD	OTH	---	EBF	---	EBF	WGR
Expressions in questionnaire.	EBF	EBF CD		EBF	---	EBF ---	OTH	--- EBF	---
$f(x)=5$ to its graph	✓	✓	✓	✗	✗	✗	✗	✗	✗
$f(x)=5$ to its set diagram	✓	✓	✓	✓	---	✗	✗	✗	✗
$f(x)=5$ to the set of ordered pair	✓	✓	✓	✗	✗	---	✗	✗	---

Table 8-1: A grid for a summary of students' responses.Abbrevations: CD: Colloquial Definition; CDW: Colloquial definition wrongly used; EBF: Exemplar-Based Focus; SD: Set Diagram; CF: Constant function; VLT: Vertical Line Test; GR: Graph; WGR: Wrong graph; OTH: Other; √: Correct transformation; ---: No Response

A few more remarks should be made about the grid. Firstly, one cell is assigned to more than one label where necessary. The order of labels is also important. For instance, SD-CD means that this particular student first draws a set diagram then uses the colloquial definition making an explanation on the diagram.

Secondly, the meaning of colours are not absolute. Rather, the colouring is made to help see how overall categories differentiate between each other. In the grid, students are presented from the left to the right. Students who have a stronger focus on the core concept of function are on the left. As it goes to the right, they are less likely to focus on the definitional properties.

As discussed in the theoretical framework in chapter 4, the coherency in recognizing different aspects of functions by referring to the definitional properties is considered as an indication of an understanding of the core concept of function. Therefore, the categorization is made by focusing on the grid vertically to investigate this coherency. When doing the categorization, cells that represent the responses from the interviews are given priority.

8.3 A note on triangulation

As discussed in the methodology chapter, one of the reasons for combining the qualitative and quantitative approaches is for integrative purpose to do the triangulation. The data from the questionnaire and the interview is triangulated to increase the validity and reliability of the findings. Hammersley & Atkinson (1983) state that triangulation is valuable because of the increased quality control achieved by combining methods, observers and data sources. However, it does not mean that merely combining different kinds of data will unproblematically add up to produce a more complete picture. Multiple methods may also serve to magnify error. In other words, each method has some errors associated with it, and some methods have more than others. This has a multiplying effect. The different methods must be weighed and considered in terms of their relative biases and limitations. Thus, an important aspect of triangulation is to consider the relationships of the different kinds of data to counteract the threat to validity of each. (Hammersley & Atkinson, 1983). Therefore, nine students' responses in the questionnaires which are summarized in the grid in table 8.1 are considered as a secondary source of data. In other words, the cells in the grid which contain data from the questionnaires (the last row for each aspect of function as seen in table 8.1) are not given the same importance as the other cells which contains the qualitative data.

8.4 A categorization of the responses of students

The colours in the grid reveals four different categories. Students from each category are named starting with a different letter, A, B, C, D. Grey colours spread across all aspects of functions for four students (Ali, Aysel, Ahmet, Arif) as seen in table 8.1. Therefore, these four students are considered in the first category. They could focus on the definitional properties not only for the set-correspondence diagrams and the sets of ordered pairs but also for the graphs and expressions. In the second category, there are two students (Belma and Belgin) who could focus on the definitional properties for the set-correspondence diagrams and the sets of ordered pairs but not for the graphs or expressions. They gave complicated responses for the graphs and expressions. In the third category, there is one student (Cem) who could focus on the definitional properties but could not check the definitional properties correctly. In the fourth category, there are two students (Deniz and Demet) who could not focus on the definitional properties for any aspect of the function concept. In other words, they gave very complicated explanations which did not act as a coherent whole.

Categorization of students revealed that the categories are not homogeneous. Therefore there might be alternative ways of categorization. In particular, Arif could be considered in the B category, and Cem could be placed in the last category with Deniz and Demet.

Arif's responses for the graphs are not as coherent as the responses of Ali, Aysel and Ahmet. Therefore he might be considered in the second category. However, he could use the colloquial definition for some of the expressions and graphs while students in the second category could not.

Cem the single student in category C could also be considered in category D since he could not successfully use the colloquial definition for any aspects of functions. Having alerted the reader to this alternative categorization, the initial four categories are kept throughout the discussion below.

8.4.1 First category: Getting closer to the core concept of function

Ali, Aysel, Ahmet, Arif are considered in this category since they used the colloquial definition for all different aspects of functions. Their responses are less likely to be influenced by the subtle differences among different items compared to the responses from the other students. Although they are the most succesful students among others, some of their responses to the graphs, and especially expressions, were *complicated* in the sense that they did not act as a coherent whole to apply in different contexts.

8.4.1.1 The case for Ali

Ali's overall responses for the different aspects of functions were mostly coherent. He mostly used the colloquial definition to recognize a function. Although in the questionnaire he gave the definition of a function as "a relation with a range which has no elements left", when he referred to the definition he could focus on the definitional properties. Although his overall responses were coherent, his focus on the colloquial definition is disrupted by the subtle differences in the way functions are presented as expressions. As seen in the grid in table 8.1, his focus on the colloquial definition is stronger for the set-correspondence diagrams, the set of ordered pairs and the graphs. In the interview he directly used the colloquial definition for the set-correspondence diagram and the set of ordered pairs. In the questionnaire, his responses to these aspects were exemplar based, considering a set of ordered pairs in 7c as an identity function and a set-correspondence diagram in 8d as a constant function. Although he recognized familiar graphs (straight line graph and the graph of $f(x) = -\sin x$) as exemplars, he could use the colloquial definition when he was asked to. For non-exemplar graphs, he directly used the colloquial definition. His focus on the colloquial definition was not strong for the expressions. He did not use the colloquial definition for most of the expressions. He instead used graphs as a stepping stone to checking the definitional properties. He drew the correct graphs for some of the expressions. For instance, he drew the graph of the split-domain function followed by its set-correspondence diagram. He assigned x and y to 1 (noting that $x < 1$ and $y > 1$) and 1 to 0 leaving -1 in the range unassigned. This is obviously a use of the colloquial definition with an explanation on the set-correspondence diagram. For all three cases for constant function, he drew the graphs. Although he used the colloquial definition for "$y = 5$ (for all values of x)", he did not use it for the two constant functions ("$y = 5$" and "$y = 5$ (for $x \leq 2$)"). His overall responses to expressions indicated that his focus on the core concept is affected when the context changes to expressions. His responses to the transformations of the constant function "$f : R \to R$, $f(x) = 5$" to the other aspects of functions reveal that he focused on the constant function as the core concept of constant function.

In summary, Ali could focus on different aspects of functions using the colloquial definition. His use of the colloquial definition is disrupted for some of the expressions. However, he could still successfully decide about the expressions by focusing on their graphs as a stepping stone to the colloquial definition.

8.4.1.2 The case for Ahmet

Ahmet gave coherent responses for all aspects of functions. He was more successful with the set-correspondence diagrams and the sets of ordered pairs. He could not focus on all definitional properties when giving his own personal concept definition in the questionnaire. He wrote it as 'a relation which has a value for any element in the domain'. However, he could successfully use the colloquial definition for the set-correspondence diagrams and the sets of ordered pairs in the interview. In the questionnaire, he used the colloquial definition wrongly for some of the set of ordered pairs.

His focus on the definitional properties was not always direct for the graphs. In other words, he relied on the vertical line test and drawing the set-correspondence diagrams to explain his responses. As seen in the grid in table 8.1, he either used one of them or used both of them. For most of the graphs, he could successfully check the definitional properties with these methods. For instance, for the two non-exemplar graphs as discussed in section 7.1.8 and 7.1.9, using the vertical line test he could strongly focus on the definitional properties. He could point out the elements in the domain which have more than one corresponding values in the range. The graphs only caused a few complications. For instance, for the points on a line graph, as discussed in section 7.1.6, he could not focus on the elements on the x axis which are not assigned to any element. He did not remember the colloquial definition wrongly but applied it to the graph in a wrong way. He said that the numbers between 0 and -1 and 0 and 1 on the x axis are assigned to 1. However, in another context, he could focus on the elements which are not assigned to any elements and rejected the graph as a function as discussed in section 7.1.7. His overall responses to the graphs revealed that he not only relied on the vertical line test but also the set-correspondence diagrams of the graphs. More importantly, he used the set-correspondence diagrams for the graphs in a prototypical way to focus on the definitional properties.

He also successfully dealt with the expressions. However, it was hard work for him to decide about the expressions. For all of the expressions except for "$f(x) = \sin x - 2$", he first drew the graphs of the expressions. For instance, for the split-domain function he first considered it as an exemplar, namely the *signum function*. He then drew the graph of it and applied the vertical line test to the graph. He considered all three forms of the constant function as a function by drawing their graphs correctly. His responses to the transformation of "$f : R \to R,\ f(x) = 5$" to other aspects of functions reveal that he could focus on the definitional properties of the constant function.

Ahmet's overall responses revealed that his responses to different aspects of functions are hardly affected by the subtle differences among different items. As summarized in the grid in table 8.1, he was more successful with the set-correspondence diagrams and the sets of ordered pairs compared to the graphs and expressions. The complexities of graphs and expressions still caused a few complications. However, he overcame these by using different aspects of functions in a prototypical way, using the set-correspondence diagrams for the graphs and drawing the graphs for the expressions.

8.4.1.3 The case for Aysel

Aysel's overall responses were mostly coherent. She used the colloquial definition for four different aspects of functions. She was more successful with the set-correspondence diagram and the set of ordered pairs. In the questionnaire she focused on all properties of the definition when giving her personal concept definition:

> '$A \neq \emptyset$ and $B \neq \emptyset$, for relations in ($A \times B$), every element in the domain is assigned to one and only one element in the range and if there is no elements left in the domain then this relation is a function'.

She correctly used the colloquial definition for the set-correspondence diagram and the set of ordered pairs. She also used the colloquial definition for most of the graphs and for some of the expressions. However, the interview results as discussed in chapter 7 revealed that her responses to the graphs and expressions were not always coherent.

For most of the graphs, she used the colloquial definition. Otherwise, she either used the vertical line test (with the colloquial definition) or focused on the graphs (straight line graph and graph of $f(x) = -\sin x$) as exemplars. Although she mostly used the colloquial definition for the graphs, her responses revealed a few complications. For instance, when finding the corresponding values for the numbers on the x axis for the graph of $f(x) = -\sin x$, she focused on the graph rather than the corresponding values on the y axis (as discussed in section 7.1.10). Apart from a few complications, she could successfully decide about the given graphs. Her explanations to the non-exemplar graphs (as discussed in section 7.1.8 and section 7.1.9) reveal that she has a strong focus on the definitional properties when the graphs are unfamiliar. She focused on the uniqueness of the assignment of each element in the domain. Her responses to the straight line in pieces as discussed in section 7.1.4 indicated that she could focus on the elements in the domain which are not assigned to any elements in the range therefore she rejected it as a function.

When the context changes to the expressions, her responses reveal more complications. When dealing with the expressions, she referred to the other aspects, graphs and set diagrams. However, she was not always successful in that. For instance, she dealt with the split-domain function by trying different methods. However, she could not check the definitional properties correctly. She first considered it as a specific example, namely the *conditional function*. She then tried to draw its graph but she could not draw it correctly. She then tried to draw the set-correspondence diagram. However, she still could not find which elements are assigned to -1, 0, 1 in the range. She incorrectly assigned values less than 1 to -1, 1 to 0 and values greater than 1 to 1.

Her responses to the three forms of constant function were less complicated. She drew graphs for these expressions. She did not use the colloquial definition explicitly. For instance, for $y = 5$ she first asked what the domain is. She was told to decide about it. She then drew the graph of the constant function by considering all real numbers as the domain. She finally said that it is a constant function. For the other two constant functions, for "$y = 5$ (for $x \leq 2$)" and "$y = 5$ (for all values of x)", she first specified the domain then said that all elements are assigned to 5 by drawing the graphs. For the last expression, $f(x) = \sin x - 2$, she used the colloquial definition. She focused on the uniqueness of the assignment as well as the assignment of each element in the domain. To do that she used the unit circle to decide the value of $\sin 0$ (section 7.1.16).

Aysel's overall reponses revealed that she successfully dealt with most aspects of functions. For the set-correspondence diagrams and the set of ordered pairs, she confidently used the colloquial definition. When the context changes to the graphs and expressions, she could still respond successfully but by applying different methods. She is successful with carrying out procedures in various contexts rather than focusing on the essential ideas which applies in all contexts. In other words, she could not focus on the simplicity of the function concept in every context.

8.4.1.4 The case for Arif

Similar to the other students in this category, the use of the colloquial definition for Arif spread across all aspects of functions as seen in the grid in table 8.1. However, Arif gave more complicated explanations for the graphs and expressions compared to the other students in this category.

When writing his personal concept definition in the questionnaire, Arif could not focus on all properties of the definition. He said the following:

> 'By a function, we mean, we can find the corresponding value of an element in the domain and it will be in the range'.

He used the colloquial definition for the set-correspondence diagram and the set of ordered pairs. For his response to the set of ordered pairs both in the interview and the questionnaire, he used the set-correspondence diagram in a prototypical way. He focused on the definitional properties by drawing a set-correspondence diagram. Although, he used the colloquial definition for three of the graphs, some of the graphs (straight line graph and the non exemplar graph 2) caused a few complications. For these graphs, he focused on the assignment of the elements without having a particular direction. For the other two graphs, points on a line and the graph of $f(x) = -\sin x$ as discussed in section 7.1.6 and section 7.1.10, his focus was exemplar-based.

His responses to the expressions were also complicated. He could not coherently focus on the definitional properties for all expressions. He used the colloquial definition for the two expressions "$y = 5$ (for $x \le 2$)" and "$y = 5$ (for all values of x)". On the other hand, he focused on the split-domain function as an exemplar. He considered the split-domain function as a signum function without any reference to the definitional properties. He could not focus on the definitional properties for $f(x) = \sin x - 2$ either. He only found $f(0)$

as -2. Although he said that the value of the function could be any number, he could not focus on the definitional properties when he was asked to do so.

Overall, Arif seemed to focus on the definitional properties for the set-correspondence diagram and the set of ordered pairs while he is less succesful with the graphs and the expressions. He responded in more complicated ways to the graphs and expressions compared to the other aspects of functions. This indicates that he is less focused on the core concept of function compared to the other three students in this category.

8.4.1.5 An overview of the first category

Responses from four students (Ali, Ahmet, Aysel and Arif) are considered within the same category, a category in which students could focus on the definitional properties in different contexts where they recognize various aspects of functions. They successfully used the colloquial definition for the set-correspondence diagrams and the sets of ordered pairs.

Although students in the other categories gave complicated responses for the graphs and expressions as will be discussed in the following sections, students in this category could focus on the definitional properties in these contexts by responding in various ways. For instance, for the graphs they used the vertical line test as a conceptual tool or used the set-correspondence diagrams in a prototypical way to focus on the definitional properties. For the expressions, they used the graphs of the expressions to decide about them.

Although these four students successfully dealt with graphs and expressions as well as the set-correspondence diagrams and the set of ordered pairs, even their responses were complicated in a few occasions where they could not focus on the definitional properties.

Arif is considered as the least successful in this category since his responses to the graphs are less focused on the core concept of function.

8.4.2 Second category

In the second category, there are two students (Belma and Belgin) who could focus on the definitional properties only for the set-correspondence diagrams and the sets of ordered pairs but not for the graphs and expressions. These two students gave complicated responses for the graphs and expressions. They focused on the properties of the graphs and expressions which are irrelevant to the core concept of function.

8.4.2.1 The case for Belma

Overall responses from Belma indicated that she could not focus on the core concept of function. She could focus on the definitional properties for only two aspects of functions, the set-correspondence diagram and the set of ordered pairs. In her responses, the complexity of the function concept reveals itself as complications in the context of graphs and expressions.

In the questionnaire Belma gave her personal concept definition as follows:

> 'Let $f : A \rightarrow B$. If every element in A is assigned to B then this is called a function' (Belma).

Although her personal concept definition does not focus on all properties of the definition, Belma used the colloquial definition correctly for the set-correspondence diagram and for the set of ordered pairs. She could focus on the elements in the domain which are not assigned to any elements in the range and the elements in the domain which are assigned to two elements in the range. However, she could not focus on the definitional properties for the graphs and expressions. She instead focused on some other properties which are irrelevant to the core concept of function. She relied on the appearances of the graphs or the specific hints from the graphs. Therefore, her evoked concept images were not connected to the core concept of function which, in the end, revealed itself through complicated explanations. For instance, she rejected the graph of $f(x)=\sin x-2$ since it has only one intercept of the axes as discussed in section 7.1.11 or she simply considered the smiley face graph (as discussed in 7.1.7) as a function since it looked like a graph of a parabola.

Belma's responses to the expressions were very complicated. Her focus of attention was not the definitional properties. Mainly speaking, she approached the expressions in two different ways; either substituting a few values in the expressions or trying to draw the graphs of the expressions. However, she was not successful in doing these. For instance, for the split-domain function she focused on the numbers -1, 0, 1 as the elements of the domain, and then substituted them in x^2-2x+1, the expression of the condition on the domain (section 7.1.12). She was not successful to draw the graphs of the expressions either. The subtle differences between the two notations for the constant functions, $y=5$ and $f(x)=5$ caused a lot of complications. She drew two different graphs for them. Although she drew the graphs for the three cases of constant function along the line $y=5$, when transforming "$f(x)=5$" to its graph she drew the graph as the line $x=5$ as shown in figure 8.1 below:

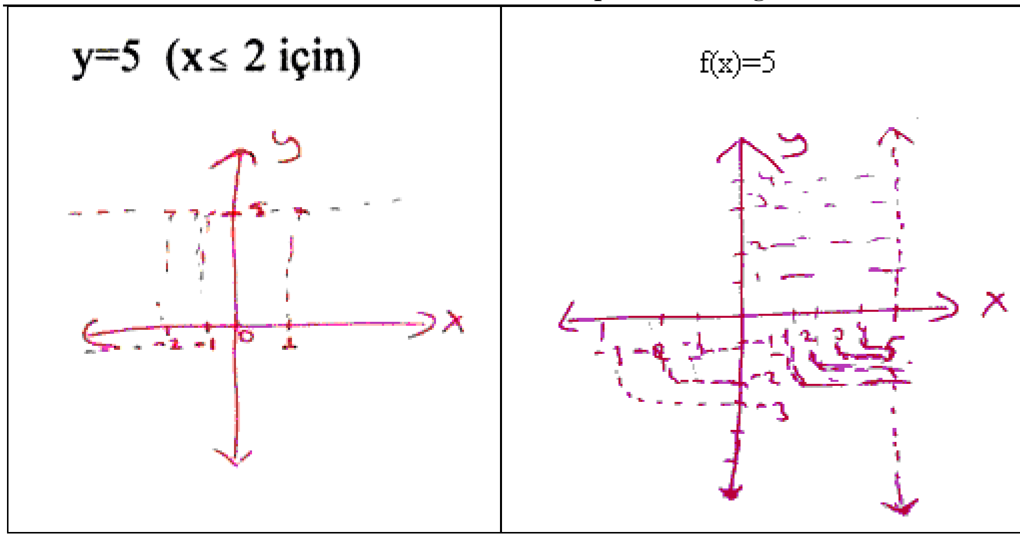


Figure 8-1. Belma's drawings for the two constant functions.

When the constant function is given in the form of $f(x)=5$, she could not focus on x as a variable although she tried to assign the numbers on the x axis to the $x=5$ line.

8.4.2.2 The case for Belgin

Similar to Belma, Belgin was more successful with the set-correspondence diagrams and the set of ordered pairs and could not focus on the definitional properties when dealing with the graphs and expressions.

In the questionnaire Belgin did not write anything for the definition of a function. However, she used the colloquial definition for the set-correspondence diagram (both in the interview and some items for the questionnaire) and set of ordered pairs (in the questionnaire). However, she did not give any explanations for the set of ordered pairs in the interview. For the graphs, she either used the colloquial definition wrongly or focused on the graphs as exemplars (see the grid in table 8.1). For instance, she rejected the graph of $f(x)=\sin x-2$ since the general appearance of it was different and considered the graph of $f(x)=-\sin x$ as a function because of the visual hint, π on the x axis. She could not focus on the definitional properties for the expressions either. As seen in the grid in table 8.1, she either focused on expressions as exemplars or confused the domain with the range. For instance, she considered $f(x) = \sin x - 2$ as a trigonometric function without referring to the definitional properties. For '$y = 5$' and '$y = 5$ for $x \leq 2$' she did not find the symbol $f(x)$ in the expression therefore did not consider them as functions.

In summary, Belgin's overall responses revealed that she gave more complicated responses to the graphs and expressions.

8.4.3 Third category

In the third category, there is only one student (Cem) who could not use the colloquial definition correctly for any aspects of functions. He was considered in a different category from the other three students in the fourth category who could not focus on the definitional properties for any aspects of functions since he referred to the colloquial definition although without success.

8.4.3.1 The case for Cem

Cem used the colloquial definition wrongly for the set-correspondence diagram and the set of ordered pairs, not because he could not apply it but he remembered the colloquial definition incorrectly. For instance, for the set-correspondence diagram he said that one element in the domain (6) can be assigned to two elements but not three elements. For the set of ordered pairs he said that it can not contain (3,3) and (4,3) together, instead the set of ordered pairs should only contain (3,3). For graphs and expressions he could not focus on the definitional properties at all. Both for the graphs and expressions, his responses were exemplar-based. For the graphs, he relied on specific hints and general appearances of the graphs. For instance, he rejected the straight line in three pieces as a function since it has three parts. He could not focus on the elements in the domain which were not assigned to any element in the range. He rejected the graph of $f(x) = \sin x - 2$ as a function since the shape of the graph is strange. Similarly, he could not focus on the definitional properties for the expressions. Instead, he focused on some other properties of the expressions. He considered the split-domain function as a function since there is $f(x)$ in the expression. His responses to the transformations of the constant function "$f : R \rightarrow R$, $f(x) = 5$" to the other aspects of functions reveal that he could not focus on the definitional properties of the expression, "$f : R \rightarrow R$, $f(x) = 5$". For instance, he wrote the set of ordered pairs as $f(x) = (1,1), (1,2), (1,3), (1,4), (1,5)$ by varying the second coordinate of the ordered pairs.

In summary, Cem's overall responses were complicated. He focused on different properties of the given items in different contexts. He could not focus on the definitional properties. Although he referred to the definitional properties for the set-correspondence diagram and the set of ordered pairs, he could not use the colloquial definition correctly.

8.4.4 Fourth category

In the fourth category there are two students (Deniz and Demet) who could not focus on the definitional properties for any aspects of the function concept at all. They could not use the colloquial definition even for the two aspects, set-correspondence diagram and the set of ordered pairs, as the students in the other categories. Their explanations to the questions were mostly exemplar-based for all different aspects of functions.

8.4.4.1 The case for Deniz

In the questionnaire, Deniz said that he did not know the definition of a function. Most of his responses were exemplar-based. For the set-correspondence diagram, he focused on the visual hints from the diagram which are irrelevant to the core concept of function. His concept image of a set-correspondence diagram is a diagram in which arrows from the first set to the second set do not intersect each other. Similarly, he focused on the irrelevant properties of the set of ordered pairs. Without focusing on the definitional properties, he said that the number of elements in the set of ordered pairs should be equal to the number of elements in the domain. His response to the graphs were completely exemplar-based. For instance, he rejected graphs in three pieces without focusing on the elements in the domain which were not assigned to any elements in the range. He rejected the graph of $f(x) = \sin x - 2$ since the graph intersected the axes at only one point. He was more reluctant with the expressions. He could not give any reason for his answers to the three forms of constant function. He considered "$f : R \to R$, $f(x) = \sin x - 2$" as a function without focusing on the definitional properties. He said that it was a function because of the notational features such as f, $R \to R$ and the formula itself.

8.4.4.2 The case for Demet

In the questionnaire Demet said that she did not know the definition of a function. As Deniz, she also could not focus on the definitional properties for any aspects of the functions. For the set-correspondence diagram, she focused on the properties of the diagram which are irrelevant to the core concept of a function. She said that the set diagram is not a function since the arrows intersect each other. For the set of ordered pairs, she tried to plot a few points and joined them to the origin. All of her responses to the graphs were exemplar-based. She did not consider some of the graphs (e.g. the graph of $f(x) = -\sin x$, or the non-exemplar graphs 1 and 2) as a function since their general apprearances were unfamiliar. Or she rejected some graphs as a function because of the visual hints from the graphs. For instance, she rejected the graph of $f(x) = \sin x - 2$ as a function since it was below the x-axis. For most of the expressions ('$y = 5$', '$y = 5$ (for all values of x)' and '$f(x)=\sin x-2$'), she drew wrong graphs. For the split-domain function, although she related the three conditions to 1, 0, -1 respectively, she could not focus on the definitional properties.

8.4.5 Final remarks on the categorization of students' responses

Students' responses vary from those which focused on the simplicity of the core concept of function to a range of complicated responses which focus on various properties irrelevant to the core concept of function. As discussed above, four categories of student performance were distinguished.

Students in the first category gave coherent responses with a focus on the definitional properties as the context changed from one to the other. As we move to the second category, responses started to be less focused on the definitional properties and got complicated. Students in the second category could not focus on the definitional properties for the graphs and expressions In the third category there is one student who used the colloquial definition wrong for the set-correspondence diagram and the sets of ordered pairs and could not focus on the definitional properties for the graphs and expressions. In the fourth category, there are responses which were very complicated. Students who gave these complicated responses focused on the contextual properties which were irrelevant to the core concept of function.

These four categories reveal that being able to deal with graphs and expressions successfully, distinguished the top group from the other students. They are considered as having a strong focus on the core concept of function. However, it should be also mentioned that even the responses of the students who have a strong focus on the core concept of function might be affected by the subtle differences in the way expressions are given.

Students' focus on the core concept of function should be evaluated with its own limitations. Students in this study have a limited experience with different aspects of functions. For instance, functions such as functions with two or more variables or functions which define an isomorphism between two mathematical structures have not been experienced by these students. However, with the limited experience of functions very few students (in the first category) could focus on the essential properties of different aspects of functions which are relevant to the core concept. For other students, as their experience with different aspects of functions grow, they add on irrelevant properties of these aspects into isolated compartments. Therefore, in the end, the simple notion of the core concept of function can not be abstracted.

Students' responses to transformation from $f(x)=5$ to its graph, set-correspondence diagram and set of ordered pairs reveal that students who have a stronger focus on the core concept of function are more likely to make the links between different aspects.

8.5 A summary of the chapter 8

In this chapter, we attempted to categorize students' responses to various aspects of functions. The analysis of the responses from the nine students in the interview revealed a spectrum of performances. As discussed above, four categories were distinguished in this spectrum. In the first category, there are students who could focus on the core concept of function by focusing on the definitional properties for all different aspects of functions in a coherent way. In the second category, there are two students who could focus on the definitional properties only for the set-correspondence diagram and the set of ordered pairs. They gave complicated responses for the graphs and expressions. In the third category, there is one student who used the colloquial definition wrongly for the set-correspondence diagram and the set of ordered pairs and focused on the irrelevant properties of the graphs and expressions. In the fourth category, there are two students who gave very complicated responses to all different aspects of functions which were mostly exemplar-based.

CHAPTER 9 – DISCUSSION

9.1 Going back to the departure point: the core concept of function

The departure point of the theoretical framework was the notion of the *core concept of function*. Thompson (1994) makes a distinction between different aspects of functions and the core concept of function which cannot be represented by what is commonly called the multiple representations of functions. Considering this distinction, the theoretical framework makes a parallel distinction between the *simplicity* of the core concept of function and the *complexity* (richness) of the function concept. When analyzing students' responses, it was aimed to investigate the cognitive *complications* of the function concept which is mathematically both simple and complex. Successful students are the ones who could develop cognitive structures that can handle the flexibility of the mathematical simplicity and complexity of the function concept.

It was attempted to investigate students' focus on the definitional properties as they respond to various aspects of functions, set-correspondence diagrams, sets of ordered pairs, graphs and expressions. The data indicated that students focused on the definitional properties by using the colloquial definition. The coherency in using the colloquial definition is considered as an indication of an understanding of the core concept of function.

As discussed in chapter 8, the data obtained from the interviews with nine students revealed a spectrum of performances. A categorization of students' responses addressed the research questions below:

How is a student's overall response to different aspects of functions affected by the subtle differences among various aspects? (3d).

How coherent is a student's response as s/he move from one aspect to the other? (4a).

As discussed in chapter 8, the analysis of responses from nine students in the interview revealed four categories. This categorization indicates that very few students strongly focused on the core concept of function. Ali is considered as a student who has a strong focus on the core concept of function. Two of the other successful students in the first category (Ahmet and Aysel) seemed to overcome the possible complications by different methods such as applying the vertical line test to graphs. Although Arif used the colloquial definition for all different aspects of functions, he gave more complicated responses compared to the other three successful students in the first category.

9.1.1 A limitation of the theoretical framework

Basically speaking the theoretical framework has two limitations. The first limitation is related to *what* is investigated, namely students' understanding of the core concept of function. The core concept of function is unattainable since students' experiences with functions are limited. For instance, students in this study who were in grade 3 of high school (17 year-old students) have not studied functions with two variables, or implicit functions or derivatives as functions. Therefore, students' understanding of the core concept of function can only be assessed through concept images which are not rich enough for the core concept of function. In other words, the complexity of the function concept is not complete since students have not studied functions at a higher level.

The second limitation is related to *how* we assess students' understanding of the core concept of function. The theoretical framework considers the coherency in recognizing different aspects of functions correctly with a strong focus on the definitional properties as an indication of the ability of focusing on the core concept of function. However, any aspect presented to students cannot represent the core concept of function. Especially, the drawings of the graphs have some limitations. Carvalho *et al.* (2002) define a theoretical-computational conflict as the situations in which representations of a concept are contradictory to the formal definition of a concept. In that sense the drawings of the graphs have conflicts with the function it is supposed to represent. Some of the students asked questions about these conflicts. For instance, Aysel asked whether or not the graph of $f(x) = \sin x - 2$ continued. She then said that every element is assigned to an element since the two ends of the graph continue. Students' responses to the transformations of "$f: R \to R$, $f(x) = 5$" to the set-correspondence diagram revealed that some of the students were aware of the conflicts between the diagram and the core concept of function. Obviously, an infinite number of elements cannot be listed in a set diagram by using the set-container metaphor as Lakoff & Núñez (2000) use the term. Successful students, Ali, Ahmet, Aysel and Arif were aware of the limitation of the set-container metaphor. They focused on the infinite number of elements in the domain. They said that they cannot put all the elements in a set so they represented them by a a few numbers or symbols e.g. "x_1, x_2, x_3, x_4" and "$-\infty, +\infty$". Although some students were aware of the conflict, the physical drawings have similar potential conflicts for students in general (see also Aspinwall et al., 1997).

9.2 Prototypes and exemplars of functions

In the Turkish context, the set-correspondence digram is used in a prototypical way to explain the colloquial definition as presented in Figure 1.1 in section 1.2.2. On the other hand, graphs and expressions are taught in clusters of exemplars in various stages in the curriculum. For instance, students study linear functions then trigonometric functions and then move onto logarithmic functions and so on (table 1.2). The data in this study indicated that students' cognitive development of learning the function concept followed a similar structure. As seen in the grid in table 8.1, the colloquial definition is mostly used for the set-correspondence diagram and the sets of ordered pairs. In other words, in these contexts, more students focused on the definitional properties. The successful students are the ones who could focus on the definitional properties even for the exemplars of functions, namely the graphs and expressions. Less successful students such as students in the second category, could use the colloquial definition for the set-correspondence diagrams and the sets of ordered pairs but not for the graphs and expressions.

The least successful students, such as students in the third and fourth category, could not focus on the definitional properties for any aspects of functions. In other words, the set-correspondence diagram and set of ordered pairs were also exemplars for them as well as the graphs and expressions.

So, as one of the research questions tried to find out, how do these students who could coherently use the colloquial definition both for the prototypes and exemplars achieve this? Responses from successful students (Ali, Ahmet, Aysel and Arif) in the first category revealed that they used particular aspects of functions in a prototypical way to check the definitional properties for the other aspects of functions. They directly used the colloquial definition for the prototypes of functions, the set-correspondence diagram. For the exemplars of functions, they did not use the colloquial definition directly as summarized in the grid in table 8.1. For instance, they used the graphs in a prototypical way to focus on the definitional properties when dealing with the expressions. While less successful students gave complicated responses for the expressions, those successful students mostly overcame these complications by drawing the graphs of the given expressions and then focusing on the definitional properties on those graphs. One student, Ahmet used the set-correspondence diagram in a prototypical way. He drew the set diagram and put a few elements in the sets and checked whether every element in the domain was assigned to a unique element in the range.

Lakoff (1987a) states that according to the classical view of categorization, category members only have definitional properties and all category members have those properties. However, as discussed in the literature review, human beings do not categorize in such a way that all category members share the same definitional properties. In other words, people do not treat categories as clear-cut entities. Lakoff (1987a) distinguishes between *essential* and *incidental (accidental) properties*. Essential properties are 'those properties that make the thing what it is, and without which it would not be that *kind* of thing' (p. 161). Other properties are incidental. They are the properties that things happen to have but not the ones that capture the essence of the thing. The function concept, being in a well-defined category, has essential properties called definitional properties, that is given two non-empty sets each element in the first set is assigned to a unique element in the second. The analysis of the data indicated that, while successful students focused on these essential properties, less successful students focused on the incidental properties such as the visual hints from the graphs and diagrams etc. In other words, exemplar based responses are the ones which focused on the incidental properties of different aspects of functions. It is claimed that the graphs and expressions being introduced in clusters carry more incidental properties. Therefore, they caused more complications especially for the students, as indicated by the data in this study. Even the successful students responded in exemplar-based ways for a few questions. However their overall responses indicated that they focused on the essential properties of different aspects of functions.

The data indicated not only that the students developed exemplars of graphs and expressions, but also that they developed exemplars in separate clusters. Since graphs and expressions have different incidental properties, students developed exemplars in separate symbolic and graphical clusters. In other words, students might reject a graph as a function yet accept it as a function in symbolic form. The results reveal that graphically $f(x) = -\sin x$ acted like an exemplar while the graph of $f(x) = \sin x - 2$ acted as a non-exemplar. There is a big difference between the number of students who accept the former and latter graphs (table 6.3, section 7.10 and section 7.1.11). On the other hand, symbolically this is not the case. More students tend to accept the expression $f(x) = \sin x - 2$ as a function compared to its graph. This concludes that a function can be a non-exemplar as one aspect and an exemplar as another aspect.

9.3 Cognitive loads and cognitive economy

Prototype exemplar distinction has implications for the simplicity and complications of the function concept. It is claimed that prototypes cause less complications while exemplars cause much more complications. To explain the attributes of prototypes and exemplars to the possible cognitive complications, two terms will be introduced; cognitive loads and cognitive economy. An aspect of a concept is said to have *cognitive load*, if it has a lot of contextual properties which are not necessarily relevant to the core concept. An aspect of a concept is said to have *cognitive economy*, if it does not carry a lot of properties which are not relevant to the core concept. It is believed that prototypes provide cognitive economy. They are cognitively economic since they can act like a cognitive unit to extract the definitional properties. Students do not need to deal with the complexities of the situation.

As seen in the grid in table 8.1, not only the successful students in the first category but also the less successful students in the second category focused on the definitional properties for the set-correspondence diagrams and the sets of ordered pairs. Even the most successful students in the first category were much more well-focused on the definitional properties for the set-correspondence and the sets of ordered pairs. As discussed in chapter 8, successful students in the first category directly used the colloquial definition for the set-correspondence diagram while they applied different methods for checking the definitional properties when dealing with graphs and expressions. For instance, they applied vertical line test to the graphs, drew the set-correspondence diagrams of the given items and the graphs of the given expressions. On the other hand, exemplars do not act like a cognitive unit rather they caused cognitive loads due to incidental properties. In the curriculum, graphs and expressions are given as clusters in different contexts. These clusters carrying out various incidental properties in different contexts are accumulated together. When there are so many clusters it becomes difficult, for less successful students, to decide whether or not it is a function since they can only decide by relying on their previous experiences but not the definitional properties. That is because, for graphs and expressions students focus on the properties irrelevant to the core concept. These properties may be helpful in one particular context but become an extra load for other contexts Therefore, the complexity of the function concept reveals itself as complications for students as they attempt to deal with those loads. Weaker students were overwhelmed by those cognitive loads, therefore gave very complicated responses. For weaker students, even the prototypes caused cognitive complications. For instance, Deniz and Demet in the fourth category, focused on the visual properties (which are incidental properties in Lakoff's (1987a) words) of the set diagrams. They did not consider the set-correspondences as functions since the arrows intersect each other. For these students, the set-correspondence diagram is not helpful to abstract the definitional properties. They considered the arrows as physical objects but not as part of the diagram carrying out the meaning of the assignment between two elements.

The fact that different properties of function concept in different contexts forms concept images conflicting to the core concept of function is also found in the previous research by DeMarois, McGowen & Tall (2000a). They claim that since students deal with functions in different contexts such as function of one variable with domain and ranges as numbers, they assign some extra properties to the concept in every different context. DeMarois, McGowen & Tall (2000a) assert that it is not the function concept itself which is studied, but rather it is a special kind of function such as linear, quadratic, trigonometric, given by a formula, differentiable etc. Instead of the term "function", they use the term "function plus", where "plus" refers to the additional properties which change the nature of the function concept. A linear function, for instance, is uniquely determined by two pairs of input-output. In other words, the whole set of ordered pairs can be determined by the two ordered pairs. They mention that the "plus" is extremely subtle if the graph of a function in R is considered. In that case it is assumed that the elements of the domain and range, the real numbers, are ordered. This is an extra property that a function may carry. In other words, the concept imagery is gained from the examples of "function plus". Thus, students may have conflicting concept images with any arbitrary function.

9.4 Limitations of the study

Any study should be evaluated with its limitations (Cohen & Manion, 1994). As well as the theoretical limitations which were discussed in section 9.1.1, this study has some other limitations. The first limitation is concerned with the representation of the whole population. This study focused on Turkish students' understanding of the core concept of function. It was aimed to chose a sample of the students from three different subject groups (mathematics & science, social subjects, Turkish & mathematics) to represent the variation in the whole population. This was achieved by selecting nearly the same number of students from each subject group (table 5.1). On the other hand, this study is restricted to only two schools, and therefore has a limitation to represent the whole population.

Secondly, the results from the questionnaires are not as strong as the results from the preliminary study. Including an "I don't know" choice for the answers made the number of correct answers decline. However, the interviews revealed a similar picture in both studies, a spectrum of performance with a few students who could strongly focus on the core concept of function.

The third limitation is concerned with the methodology. Although features of clinical interviewing such as immediate interpretation of the subject's response and on-the-spot hypothesis making and testing are considered in the interviews, according to Piaget, a year's training in the method is required to achieve expertise (Ginsburg, 2000). In that sense, interviews may lack validity due to lack of experience in clinical intervewing technique.

CHAPTER 10 - CONCLUSION

This thesis has attempted to address the research questions which were defined in section 5.2 in the methodology chapter. The main research questions were defined as follows:

1. Do students use the core concept of function to recognize a function?
2. Whatever the response is, what do they do to recognize a function?
3. How do the various aspects of a function play their part?
4. What do these three research questions imply for students' understanding of the core concept of function?

A categorization of students' responses as discussed in chapter 8 suggested answers to the subquestions below:

- How is a student's overall response to different aspects of functions affected by the subtle differences among different aspects?
- How coherent is a student's response as s/he move from one aspect to the other?
- How do students who give coherent responses to different aspects of functions cope with this?

As an attempt to answer the research questions, mainly the following findings emerged from this thesis:

- There is a spectrum of performance of students when dealing with various aspects of functions. In this spectrum, a few successful students could handle the flexibility of *the mathematical simplicity and complexity of the core concept of function*. For most of the students in the interviews, the mathematical complexity of the function concept revealed itself as *cognitive complications*.

- Students in the interviews treated various aspects of functions in cognitively different ways. They dealt with set-correspondence diagrams and sets of ordered pairs as *prototypes* and with graphs and expressions as *exemplars*. Prototypes caused less complications while exemplars caused much more complication. Only successful students coped with the complexity of the function concept in different contexts and could handle the possible cognitive complications.

10.1 Implications

In this study the way students deal with the complexity of different aspects of functions is investigated. The data indicated that most of the students dealt with different aspects of functions without focusing on the simplicity of the core concept of function. It can be claimed that the core concept of function requires a long term dissemination. Because, as Bakar & Tall (1992) state, the main obstable is that:

> The learner cannot construct the abstract concept of function without experiencing examples of the function concept in action, and they cannot study examples of the function concept in action without developing prototype examples having built-in limitations that do not apply to the abstract concept. (Bakar & Tall, 1992, p. 13)

This has an implication about the curriculum design and is illustrated by Figure 10.1 below:

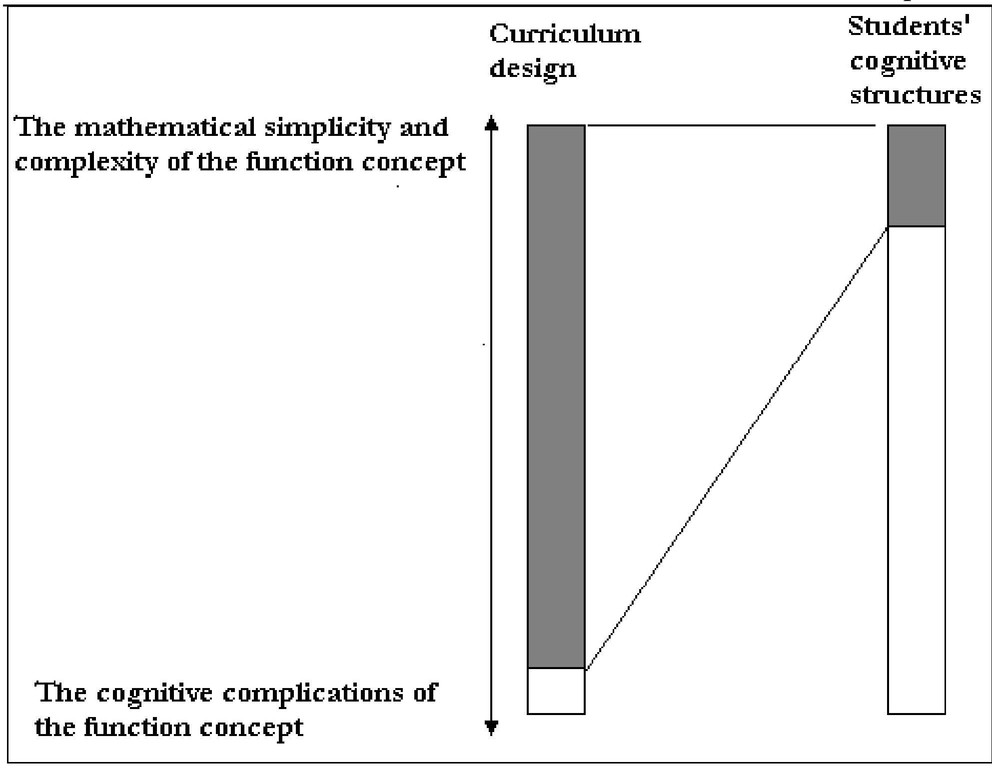


Figure 10-1. Curriculum design and students' cognitive structures.

As seen in Figure 10.1 above, there is a contradiction between the way the curriculum is designed and the cognitive structures of the students. The Turkish curriculum is designed in such a way that function is a foundational concept and an organizing principle. What is desired in the curriculum is that the majority of the students would handle the simplicity and complexity of the function concept in a flexible way. However, as the data indicated in this study, very few students focus on the simplicity of the function concept. On the contrary, the way the function concept is taught in the curriculum causes cognitive complications for most of the students. Especially, the way graphs and expressions is presented, as clusters of exemplars, affects students to focus on the incidental properties of different items but not the essential properties which are determined by the definition.

10.2 Future directions

It is believed that this study suggests further research in two aspects:

The first is concerned with students' understanding of the core concept of function at a more advanced level. As discussed in section 9.1.1., as a theoretical limitation, potentially for students in high school by their levels, the core concept of function cannot be achieved since it is applied to various ideas in advanced mathematics. Therefore, it is a possibility for further research to look at students' understanding of the core concept of function at a more advanced level to see whether they handle the mathematical simplicity and complexity of the function concept in a more flexible way.

The second possibility for further research is concerned with the teaching of the function concept to achieve the core concept of function. It is my belief that teaching functions by introducing it with the notion of function box could be beneficial in the sense of reducing the cognitive complications. However, this needs to be linked to the subsequent development of the concept of function. In addition, this use of function box with its implicit meaning both as a process (as input-output) and an object (the box) needs to be well-integrated with the subsequent development of the function concept.

In the Turkish context, the function box is not used. It can be a cognitive root by helping the long term dissemination of the core concept of function. It might act as a prototype in the similar way the set-correspondence diagram does.

The function box as a cognitive root is also suggested by DeMarois, McGowen & Tall (2000a and 2000b). They claim that the function box, as a generic image, can act as a cognitive root for the function concept. A cognitive root is 'an anchoring concept which the learner finds easy to comprehend, yet forms a basis on which a theory may be built' (Tall, 1992, p.497). DeMarois, McGowen & Tall (2000a) give a refined definition of a cognitive root. They state that a cognitive root:

(i) is a meaningful cognitive unit of core knowledge for the student at the beginning of the learning sequence,

(ii) allows initial development through a strategy of cognitive expansion rather than significant cognitive reconstruction,

(iii) contains the possibility of long-term meaning in later developments,

(iv) is robust enough to remain useful as more sophisticated understanding develops (p. 3).

It is believed that cognitive complications caused by incidental properties of specific exemplars can be lessened by focusing on various aspects which represents the same function. The following suggestion, including various aspects in the function box, by DeMarois, McGowen & Tall (2000a) has potential to do this:

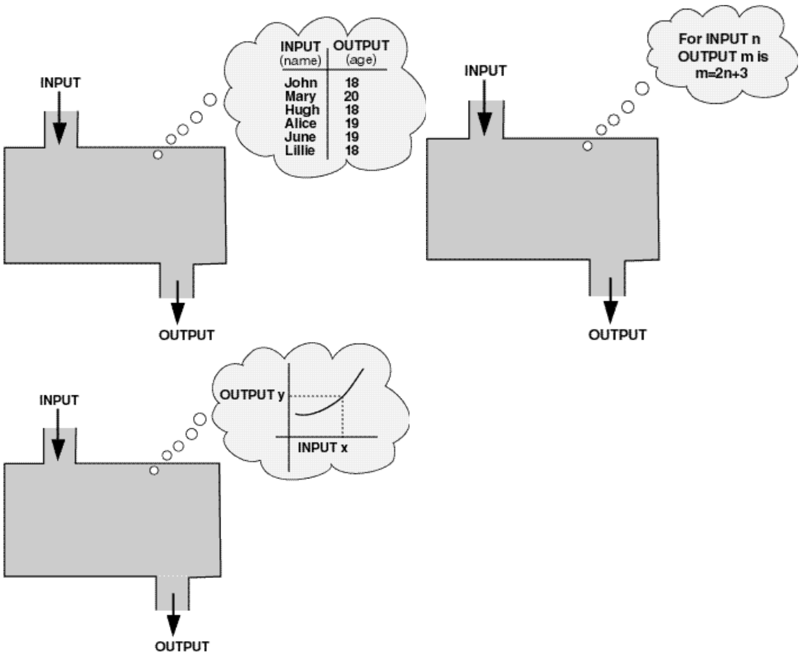


Figure 10-2. Function box (DeMarois, McGowen & Tall, 2000a, p. 4)

The study of DeMarois, McGowen & Tall (2000b) indicates that the function box had improved the students' (students who experience difficulty in mathematics and take remedial college algebra courses) flexibility in moving between various representations of function. It is my belief that function box may act as a cognitive root to the core concept of function. As Thompson (1994) emphasis students should see something, *the core concept of function*, remains the same as they move from one aspect to the other.

Reference

Akkoç, H. & Tall, D.O. (2002). 'The Simplicity, Complexity and Complications of the Function Concept', in *Proceedings of the 26th International Conference on the Psychology of Mathematics Education*, Norwich, UK, Vol. 2, pp. 25-32.

Akkoç, H. (2000). 'The Function Concept: A Synthesis of Different Theoretical Frameworks', *Unpublieshed EdD Project 1*, University of Warwick.

Akkoç, H. (2001). 'The Function Concept: A Preliminary Study', *Unpublieshed EdD Project 2*, University of Warwick.

Anderson, G. & Arsenault, N. (1998). *Fundamentals of Educational Research* - London : Falmer Press.

Arcavi, A. & Schoenfeld, A. H. (1992). 'Mathematics Tutoring Through a Constructivist Lens: The Challenges of Sense-Making', *Journal of Mathematical Behaviour*, Vol. 11, pp. 321-335.

Aspinwall, L., Shaw, K. L. & Presmeg, N. C. (1997). 'Uncontrollable Mental Imagery: Graphical Connections Between a Function and its Derivative', *Educational Studies in Mathematics*, Vol. 33, pp. 301-317.

Bakar, M. N. & Tall, D. O. (1992). 'Students' Mental Prototypes for Functions and Graphs', *International Journal of Mathematics Education in Science and Technology*, Vol. 23, No. 1, pp. 39-50.

Beineke, J., Hawks, J., & Schwingendorf, K. (1992). 'Horizontal and Vertical Growth of Students' Conception of Function', in G. Harel, and E. Dubinsky, (Eds) *The Concept of Function: Aspects of Epistemology and Pedagogy*, MAA, pp. 133-149.

Bell, J. (1999). *Doing Your Research Project: A Guide for First-time Researchers in Education.*. Buckingham: Open University Press.

Bills, L. & Tall, D. O. (1998). 'Operable Definitions in Advanced Mathematics: The Case of the Least Upper Bound', *Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education*, Stellenbosch, South Africa, Vol.2, pp. 104-111.

Boyer, C. (1968). *A History of Mathematics*, New York: Wiley & Sons.

Breidenbach, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992). 'Development of the Process Conception of Function', *Educational Studies in Mathematics*, Vol.23, No.3, pp. 247-285.

Brenner, M. E., Mayer, R. E., Moseley, B., Brar, T., Durán, R., Reed, B. S. & Webb, D. (1997). 'Learning by Understanding: The Role of Multiple Representations in Learning Algebra', *American Educational Research Journal*, Vol. 34, No. 4, pp. 663-689.

Bruckheimer, M., Eylon, B., & Markovits, Z. (1986). 'Functions Today and Yesterday', *For the Learning of Mathematics*, Vol. 6, No. 2, pp. 18-24.

Cajori, F. (1980). *A History of Mathematics*, Chelsea: USA

Carvalho, L. M., Giraldo, V. & Tall, D. O. (2002). 'Theoretical-computational Conflicts and the Concept Image of Derivative', *Proceedings of British Society of Research in Learning Mathematics Day Conference*, University of Bristol, UK, pp. 37-42.

Cohen, L. & Manion, L. (1994). *Research Methods in Education*, 4th Edition, London: Routledge.

Confrey, J. (1994). 'Six Approaches to Transformation of Function Using Multi-Representational Software', *Proceedings of the 18th Conference of the International Group for the Psychology of Mathematics Education*, University of Lisbon, Portugal, Vol. 2, pp. 217-224.

Cottrill, J., Dubinsky, E., Nichols, D., Schwingendorf, K., Thomas, K., & Vidakovic, D. (1996). 'Understanding the Limit Concept: Beginning with a Co-ordinated Process Schema', *Journal of Mathematical Behaviour*, Vol.15, pp. 167-192.

Crowley, L. & Tall, D. O. (2001). 'Attainment and Potential: Procedures, Cognitive Kit-Bags and Cognitive Units', *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education*, Utrecht, The Netherlands, Vol. 1, p. 300.

Czarnocha, B., Dubinsky, E., Prabhu, V., Vidakovic, D. (1999). 'One Theoretical Perspective in Undergraduate Mathematics Education Research', in O. Zaslavsky (Ed.), *Proceedings of the 23rd Conference of the International Group for the Psychology of Mathematics Education*, Haifa, Israel, Vol.1, pp. 95-110.

DeMarois, P. & Tall, D.O. (1996). 'Facets and Layers of the Function Concept', *Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education*, Valencia, Vol. 2, pp. 297-304.

DeMarois, P. & Tall, D.O. (1999). 'Function: Organizing Principle or Cognitive Root?', in O. Zaslavsky (Ed.), *Proceedings of the 23th Conference of the International Group for the Psychology of Mathematics Education*, Haifa, Israel, Vol.2, pp. 257-264.

DeMarois, P. McGowen, M.A., & Tall, D.O. (2000b). 'Using the Function Machine as a Cognitive Root', in *Proceedings of the Conference of the International Group for the Psychology of Mathematics Education* NA.

DeMarois, P., McGowen, M.A., & Tall, D.O. (2000a). 'The Function Machine as a Cognitive Root for the Function Concept', in *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education* , NA..

Demiralp, A., Gürkan, M. & Pelit, T. (2000). *Matematik Lise 1 – Ders Kitabı*. Ankara: Başarı Yayınları.

Denscombe, M. (1998). *The Good Research Guide for Small-scale Social Research Projects*, Buckingham: Open University Press.

Denzin, N.K. & Lincoln, Y.S. (1994). 'Strategies of Inquiry' in N.K. Denzin and Y.S. Lincoln (Eds.), *Handbook of Qualitative Research*. Thousands Oaks, CA: Sage, pp. 199-208.

Dubinsky, E. & Harel, G. (1992). 'The nature of the process conception of function', in G. Harel, & E. Dubinsky, (Eds) *The Concept of Function: Aspects of Epistemology and Pedagogy*, MAA, pp. 85-106.

Dubinsky, E. (1991). 'Reflexive Abstraction in Advanced Mathematical Thinking' in D. O. Tall (Ed), *Advanced Mathematical Thinking*, Dordrecht: Kluwer Academic Publishers, pp. 95-123.

Eisenberg, T., (1991). 'Functions and Associated Learning Difficulties', In D.O. Tall (Ed.), *Advanced Mathematical Thinking*, Dordrecht: Kluwer, pp. 3-21.

Even, R. (1998). 'Factors Involved in Linking Representations of Functions', *Journal of Mathematical Behaviour*, Vol.17, No.1, pp. 105-121.

Ginsburg, P. H. (1997). *Entering the Child's Mind: The Clinical Interview in Psychological Research and Practice*, Cambridge University Press.

Goldenberg, P. (1988). 'Mathematics, Metaphors, and Human Factors: Mathematical, Technical, and Pedagogical Challenges in the Educational Use of Graphical Representations of functions', *Journal of Mathematical Behaviour*, Vol.7, No.3, pp. 135-173.

Gray, E. M. & Tall, D. O. (1994). 'Duality, Ambiguity and Flexibility: A Proceptual View of Simple Arithmetics', *Journal of Research in Matematics Education*, Vol. 25, No. 2, pp. 115-141.

Hammersley, M. & Atkinson, P. (1983). *Ethnography : Principles in Practice* - London : Routledge.

Hardy, G. H. (1967). '*A Mathematican's Apology*', Cambridge University Press.

Hitt, F. (1998). 'Difficulties in the Articulation of Different Representations Linked to the Concept of Function', *Journal of Mathematical Behaviour*, Vol. 17, No. 1, pp. 123-134.

Kaput, J. J. (1989). 'Linking Representations in the Symbol Systems of Algebra', in C. Kieran, and S. Wagner (Eds), *Research Agenda for Mathematics Education: Research Issues in the Learning and Teaching of Algebra*, Hillsdale, NJ: Lewrence Erlbaum Publishers, pp. 167-194.

Kaput, J.J. (1992). 'Technology and Mathematics Education', in D. A. Grouws (Ed) *NCTM Handbook of Research on Mathematics Teaching and Learning*, pp. 515-556.

Keller, B.A. & Hirsch, C. R. (1998). 'Student Preferences for Representations of Functions', *International Journal of Mathematics Education in Science and Technology,* Vol. 29, No.1, pp. 1-17.

Kieran, C. (1994). 'A Functional Approach to the Introduction of Algebra – Some Pros and Cons', in *Proceedings of the 18th International Conference on the Psychology of Mathematics Education*, Vol. 1, pp. 157-175.

Lakoff, G. & Núñez, R. E. (2000). 'Where Do Mathematics Come From: How the Embodied Mind Brings Mathematics into Being', Basic Books: New York.

Lakoff, G. (1987a). *Women, Fire and Dangerous Things – What Categories Reveal About the Mind*, Chicago: University of Chicago Press.

Lakoff, G. (1987b). 'Cognitive Models and Prototype Theory', in U. Neisser (Ed.), *Concepts and Conceptual Development*, Cambridge: Cambridge University Press.

Lansing, J. B., Ginsberg, G. P. & Braaten, K. (1961). *An Investigation of Response Error* (Bureau of Economic and Business Research, University of Illinois).

Leinhardt, G., Stein, M.K., & Zaslavsky, O. (1990). 'Functions, Graphs, and Graphing: Tasks, Learning and Teaching', *Review of Educational Research*, Vol. 60, No. 1, pp. 1-64.

Malik, M A. (1980). 'Historical and Pedagogical Aspects of the Definition of Function', *International Journal of Mathematical Education in Science & Technology*, Vol. 11, No. 4, pp. 489-92.

Mason, J. (1996). *Qualitative Researching*. London: Sage.

Ministry of National Education, Turkey (T.C. Milli Eğitim Bakanlığı). http://www.meb.gov.tr

Monk, S. (1992). 'Students' Understanding of a Function Given by a Physical Model', in G. Harel, & E. Dubinsky, (Eds.) *The Concept of Function: Aspects of Epistemology and Pedagogy*, MAA, pp. 175-193.

O'Callaghan, B. R. (1998). 'Computer-Intensive Algebra and Students' Conceptual Knowledge of Functions', *Journal for Research in Mathematics Education*, Vol. 29, No. 1, pp. 21-40.

Patton, M. Q. (1990). *Qualitative Evaluation and Research Methods*, Newbury Park, CA; London: Sage.

Rosch, E. (1975). 'Cognitive Representations of Semantic Categories', *Journal of Experimental Psychology: General*, Vol. 104, No. 3, pp. 192-233.

Rosch, E. (1978). 'Principles of Categorization' in E. Rosch & B. B. Lloyd (Eds.) *Cognition and Categorization*, Hillsdale: Lawrrence Erlbaum Associates.

Ross, H.B & Makin, V.S. (1999). 'Prototype versus Exemplar Models in Cognition' in R.J. Sternberg (Ed) *The Nature of Cognition*, Massachusetts Institute of Technology, pp. 205-241.

Selden, A. & Selden, J. (1992). 'Research Perspectives on Conceptions of Function Summary and Overview', in G. Harel, & E. Dubinsky, (Eds) *The Concept of Function: Aspects of Epistemology and Pedagogy*, MAA, pp.1-16.

Sfard, A. (1991). 'On the Dual Nature of Mathematical Conceptions', *Educational Studies in Mathematics*, Vol.22, pp.1-36.

Sfard, A. (1992). 'Operational Origins of Mathematical Objects and the Quandary of Reification – The Case of Function ', in G. Harel, & E. Dubinsky, (Eds) *The Concept of Function: Aspects of Epistemology and Pedagogy*, MAA, pp. 59-84.

Sierpinska, A. (1992). 'On Understanding the Notion of Function Concept', in Harel, G. & Dubinsky, E. (Eds.), *The Concept of Function: Aspects of Epistemology and Pedagogy*, MAA, pp. 25-58.

Silverman, D. (2000). *Doing Qualitative Research – A Practical Handbook*, London: Sage.

Tall, D. O. (1992). 'The Transition to Advanced Mathematical Thinking: Functions, Limits, Infinity, and Proof', in Grouws, D. A. (Ed), *Handbook of Research on Mathematics Teaching and Learning*, New York: Macmillan Publishing Company, pp. 495-511.

Tall, D. O. (1999). 'Reflections on APOS theory in Elementary and Advanced Mathematical Thinking', in O. Zaslavsky (Ed) *Proceedings of the 23rd Conference of the International Group for the Psychology of Mathematics Education*, Haifa, Israel, Vol. 1, pp. 111-118.

Tall, D.O. & Vinner, S. (1981). 'Concept Image and Concept Definition in Mathematics with Particular Reference to Limit and Continuity', *Educational Studies in Mathematics*, Vol. 12, pp. 151-169.

Thompson, P. W. (1994). Students, Functions, and the Undergraduate Curriculum. In E. Dubinsky, A. Schoenfeld, & J. Kaput (Eds.), *Research in Collegiate Mathematics Education, I, CBMS Issues in Mathematics Education*, 4, pp. 21–44.

Vinner, S. (1983). 'Concept Definition Concept Image and the Notion of Function', *International Journal for Mathematics Education in Science and Technology*, Vol. 14, No. 3, pp. 293-305.

Vinner, S. (1992). 'The Function Concept as a Prototype for Problems in Mathematics Learning', in G. Harel, & E. Dubinsky, (Eds) *The Concept of Function: Aspects of Epistemology and Pedagogy*, MAA, pp. 195-213.

Yerushalmy, M. (1991). 'Student Perceptions of Aspects of Algebraic Function Using Multiple Representation Software', *Journal of Computer Assisted Learning*, Vol. 7, pp. 42-57.

APPENDIX

Appendix A – Questionnaire

A1 – Questionnaire

FUNCTION TEST

This is a questionnaire based on a doctoral thesis in the University of Warwick, UK. The purpose of this questionnaire is not to assess your correct and incorrect answers. It is important to see your own explanations, your thinking and what is in your mind in the answers. Thanks for completing this questionnaire and good luck…

First, a couple of questions about yourself:

Name: _______________
Surname: _______________
School: _______________
Class: _______________

Questions

Question1: Give a couple of examples of functions.

Question2: Think of a graph of a function in your mind.

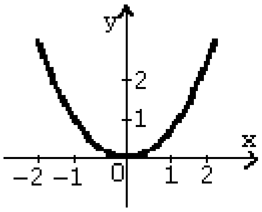
Can you see it?

- Yes
- No

Now draw a sketch of the function here:

Question 3: Below various graphs are given. Which of the following graphs are graphs of a function of x from R to R? Tick as appropriate. Give reasons for your answers.

Example:

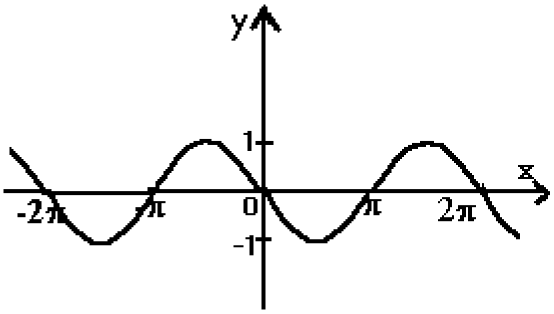


- ☑ Function
- ☐ Not a function
- ☐ I don't know

Reason:

---

a)

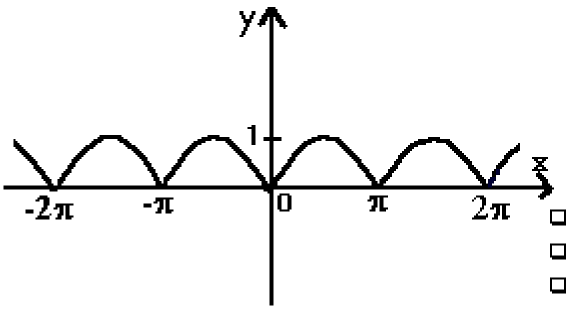


- ☐ Function
- ☐ Not a function
- ☐ I don't know

Reason:

---

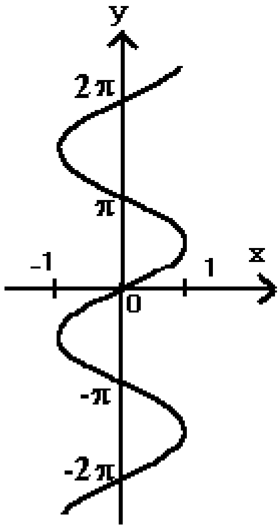
b)



- ☐ Function
- ☐ Not a function
- ☐ I don't know

Reason:

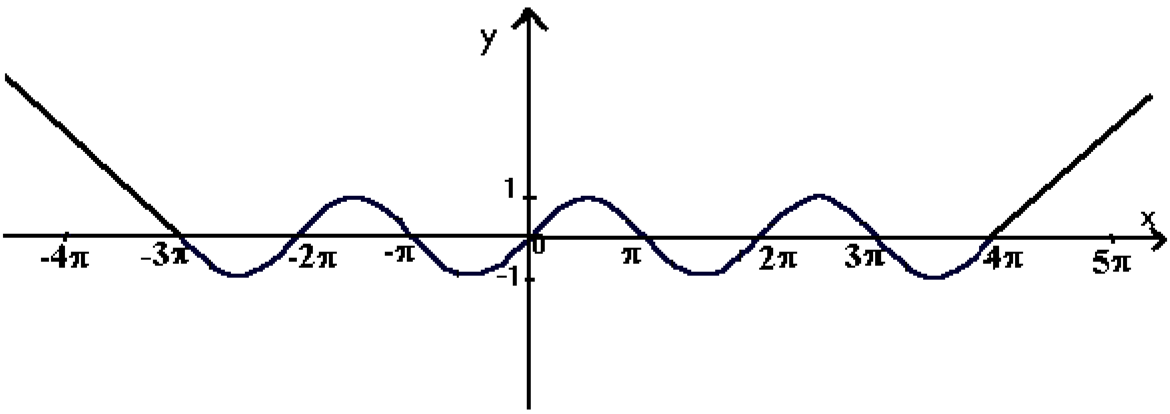
c)



- Function
- Not a function
- I don't know

Reason:

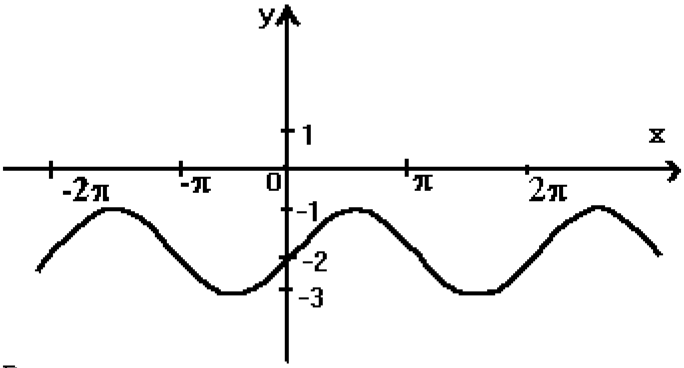
d)



- Function
- Not a function
- I don't know

Reason:

e)

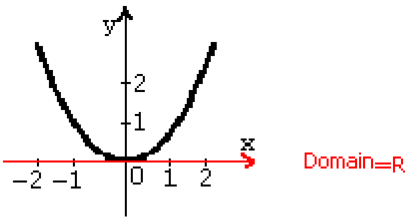


- ☐ Function
- ☐ Not a function
- ☐ I don't know

Reason:

Question4: Below various graphs are given. The domain is coloured as red. Which of the following graphs are graphs of a function of *x*? Tick as appropriate. Give the reason for your answer.

Example:

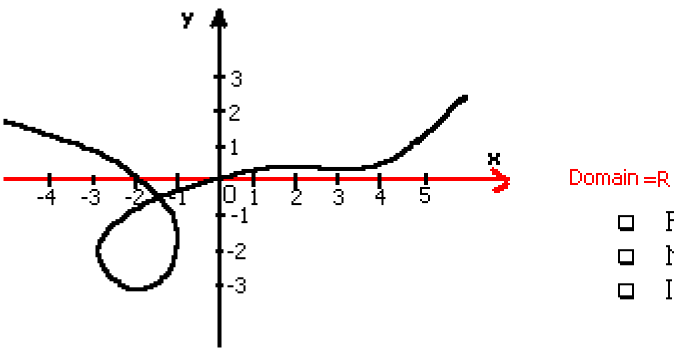


- ☑ Function
- ☐ Not a function
- ☐ I don't know

Reason:

---

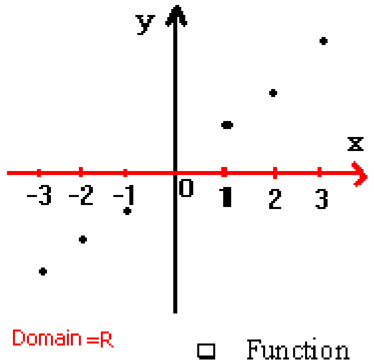
a)



- ☐ Function
- ☐ Not a function
- ☐ I don't know

Reason:

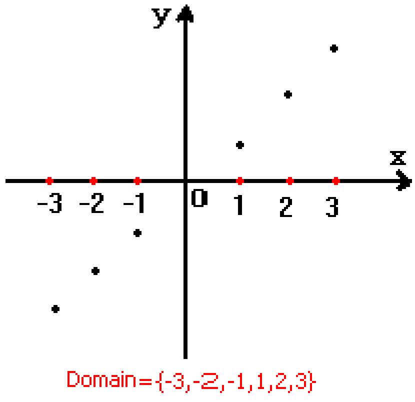
b)



- ❑ Function
- ❑ Not a function
- ❑ I don't know

Reason:

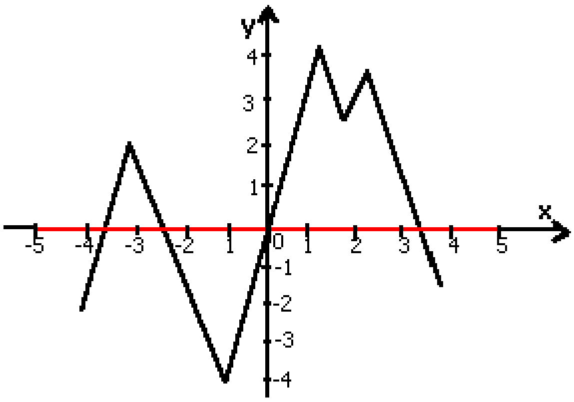
c)



- ❑ Function
- ❑ Not a function
- ❑ I don't know

Reason:

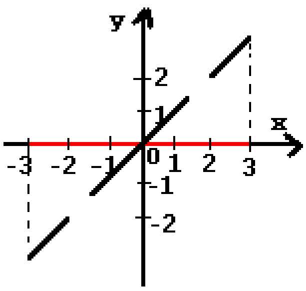
d)



- ❑ Function
- ❑ Not a function
- ❑ I don't know

Reason:

e)



- ❑ Function
- ❑ Not a function
- ❑ I don't know

Reason:

Question 5: Write down a function equation which comes into your mind immediately.

Question 6: Below various equations are given. Which of the following equations represent a function of *x*? Tick as appropriate. Give the reasons for your answers.

Example:

$f: \mathrm{R} \rightarrow \mathrm{R}, \quad f(\mathrm{x})=\mathrm{x}^2$

- ☑ Function
- ☐ Not a function
- ☐ I don't know

Reason:

a)

$f: \mathrm{R} \rightarrow \mathrm{R}, \quad f(\mathrm{x})=\sqrt{x^2-16}$

- ☐ Function
- ☐ Not a function
- ☐ I don't know

Reason:

b)

$f: \mathrm{R} \rightarrow \mathrm{R}, \quad \mathrm{x}^2+\mathrm{y}^2=1$

- ☐ Function
- ☐ Not a function
- ☐ I don't know

Reason:

c)

$y=5$

- ☐ Function
- ☐ Not a function
- ☐ I don't know

Reason:

d)

y= 5 (for $x \geq 2$)

- ☐ Function
- ☐ Not a function
- ☐ I don't know

Reason:

e)

$y=5$ (for all values of x)

- ☐ Function
- ☐ Not a function
- ☐ I don't know

Reason:

f) $f: R^+ \rightarrow R$, $f(x)=|x^2-4|$

- Function
- Not a function
- I don't know

Reason:

g) $f: R \rightarrow R$,

$$f(x)=\begin{cases} 1, & x^2-2x+1>0 \text{ ise} \\ 0, & x^2-2x+1=0 \text{ ise} \\ -1, & x^2-2x+1<0 \text{ ise} \end{cases}$$

- Function
- Not a function
- I don't know

Reason:

Question 7: A={1,2,3,4} B={1,2,3}are given.

Which of the set of ordered pairs are functions from A to B? Tick as appropriate. Give reasons for your answers?

a) f : A → B f ={(1,1), (2,1), (3,2), (4,2)}

- Function
- Not a function
- I don't know

Reason:

b) g : A → B g ={(1,1), (1,2), (2,2), (3,3), (4,3)}

- Function
- Not a function
- I don't know

Reason:

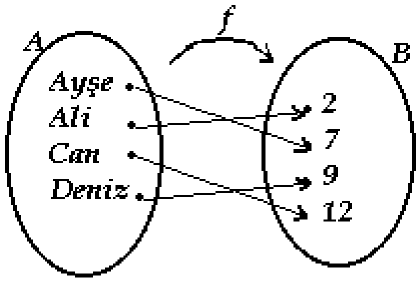
c) h : A → B h = {(1,1),(2,2)}

- Function
- Not a function
- I don't know

Reason:

Question 8: Which of the following are functions? Tick as appropriate.Give reasons to your answers.

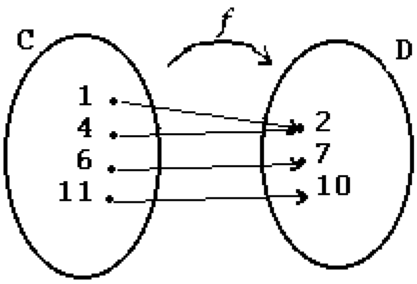
a)



- ☐ Function
- ☐ Not a function
- ☐ I don't know

Reason:

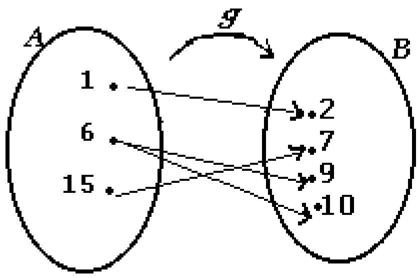
b)



- ☐ Function
- ☐ Not a function
- ☐ I don't know

Reason:

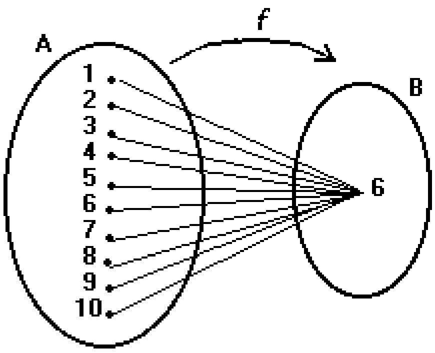
c)



- ☐ Function
- ☐ Not a function
- ☐ I don't know

Reason:

d)



- ☐ Function
- ☐ Not a function
- ☐ I don't know

Reason:

Question9: Give the definition of a function.

A2 – Frequencies from the questionnaire

A2.1 – Reasons for responses to 3a, 3b, 3c, 3d, 3e

Frequencies

Statistics

		Reasons for the response to 3a	Reasons for the response to 3b	Reasons for the response to 3c	Reasons for the response to 3d	Reasons for the response to 3e
N	Valid	114	114	113	113	114
	Missing	0	0	1	1	0

Table A - 1

Frequency Tables

Reasons for the response to 3a

		Frequency	Valid Percent
Valid	Colloquial definition	7	6.1
	First impression/General appearance	10	8.8
	Specific visual hints	18	15.8
	Other	24	21.1
	No response	55	48.2
	Total	114	100.0

Table A - 2

Reasons for the response to 3b

		Frequency	Valid Percent
Valid	Colloquial definition	4	3.5
	First impression/General appearance	11	9.6
	Specific visual hints	6	5.3
	Other	15	13.2
	No response	78	68.4
	Total	114	100.0

Table A - 3

Reasons for the response to 3c

		Frequency	Valid Percent
Valid	Colloquial definition	6	5.3
	First impression/General appearance	8	7.1
	Specific visual hints	6	5.3
	Other	14	12.4
	No response	79	69.9
	Total	113	100.0
Missing	System	1	
Total		114	

Table A - 4

Reasons for the response to 3d

		Frequency	Valid Percent
Valid	Colloquial definition	5	4.4
	Colloquial definition wrongly used	1	.9
	First impression/General appearance	10	8.8
	Specific visual hints	3	2.7
	Other	15	13.3
	No response	79	69.9
	Total	113	100.0
Missing	System	1	
Total		114	

Table A - 5

Reasons for the response to 3e

		Frequency	Valid Percent
Valid	Colloquial definition	2	1.8
	First impression/General appearance	3	2.6
	Specific visual hints	20	17.5
	Other	13	11.4
	No response	76	66.7
	Total	114	100.0

Table A - 6

The percentages are summarized as a bar chart as shown in the table below:

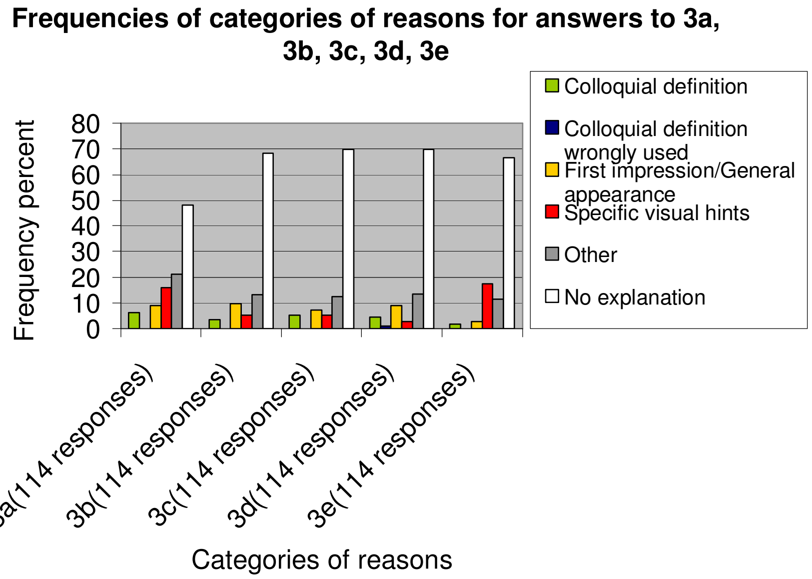


Table A - 7

To be able to find the percentages of categories for correct and incorrect answers, responses for each item are crosstabulated with categories of reasons for each item. These crosstabulations are presented below:

Case Processing Summary

	Cases					
	Valid		Missing		Total	
	N	Percent	N	Percent	N	Percent
Reasons for the response to 3a * Response to question 3a	114	100.0%	0	.0%	114	100.0%

Table A - 8

Reasons for the response to 3a * Response to question 3a Crosstabulation

			Response to question 3a				Total
			Function	Not a function	I don't know	NR	
Reasons for the response to 3a	Colloquial definition	Count	7				7
		% within Response to question 3a	13.0%				6.1%
	First impression/General appearance	Count	5	2	3		10
		% within Response to question 3a	9.3%	12.5%	7.0%		8.8%
	Specific visual hints	Count	9	9			18
		% within Response to question 3a	16.7%	56.3%			15.8%
	Other	Count	14	2	8		24
		% within Response to question 3a	25.9%	12.5%	18.6%		21.1%
	No response	Count	19	3	32	1	55
		% within Response to question 3a	35.2%	18.8%	74.4%	100.0%	48.2%
Total		Count	54	16	43	1	114
		% within Response to question 3a	100.0%	100.0%	100.0%	100.0%	100.0%

Table A - 9

Case Processing Summary

	Cases					
	Valid		Missing		Total	
	N	Percent	N	Percent	N	Percent
Reasons for the response to 3b * Response to question 3b	114	100.0%	0	.0%	114	100.0%

Table A - 10

Reasons for the response to 3b * Response to question 3b Crosstabulation

			Response to question 3b				Total
			Function	Not a function	I don't know	NR	
Reasons for the response to 3b	Colloquial definition	Count	4				4
		% within Response to question 3b	15.4%				3.5%
	First impression/General appearance	Count		10	1		11
		% within Response to question 3b		34.5%	1.8%		9.6%
	Specific visual hints	Count	2	3	1		6
		% within Response to question 3b	7.7%	10.3%	1.8%		5.3%
	Other	Count	7	5	3		15
		% within Response to question 3b	26.9%	17.2%	5.3%		13.2%
	No response	Count	13	11	52	2	78
		% within Response to question 3b	50.0%	37.9%	91.2%	100.0%	68.4%
Total		Count	26	29	57	2	114
		% within Response to question 3b	100.0%	100.0%	100.0%	100.0%	100.0%

Table A - 11

Case Processing Summary

	Cases					
	Valid		Missing		Total	
	N	Percent	N	Percent	N	Percent
Reasons for the response to 3c * Response to question 3c	113	99.1%	1	.9%	114	100.0%

Table A - 12

Reasons for the response to 3c * Response to question 3c Crosstabulation

			Response to question 3c				Total
			Function	Not a function	I don't know	NR	
Reasons for the response to 3c	Colloquial definition	Count		6			6
		% within Response to question 3c		19.4%			5.3%
	First impression/General appearance	Count		8			8
		% within Response to question 3c		25.8%			7.1%
	Specific visual hints	Count	3	2	1		6
		% within Response to question 3c	12.5%	6.5%	1.8%		5.3%
	Other	Count	6	5	3		14
		% within Response to question 3c	25.0%	16.1%	5.3%		12.4%
	No response	Count	15	10	53	1	79
		% within Response to question 3c	62.5%	32.3%	93.0%	100.0%	69.9%
Total		Count	24	31	57	1	113
		% within Response to question 3c	100.0%	100.0%	100.0%	100.0%	100.0%

Table A - 13

Case Processing Summary

	Cases					
	Valid		Missing		Total	
	N	Percent	N	Percent	N	Percent
Reasons for the response to 3d * Response to question 3d	113	99.1%	1	.9%	114	100.0%

Table A - 14

Reasons for the response to 3d * Response to question 3d Crosstabulation

			Response to question 3d				Total
			Function	Not a function	I don't know	NR	
Reasons for the response to 3d	Colloquial definition	Count	5				5
		% within Response to question 3d	14.7%				4.4%
	Colloquial definition wrongly used	Count		1			1
		% within Response to question 3d		5.0%			.9%
	First impression/General appearance	Count	2	6	2		10
		% within Response to question 3d	5.9%	30.0%	3.6%		8.8%
	Specific visual hints	Count	1	1	1		3
		% within Response to question 3d	2.9%	5.0%	1.8%		2.7%
	Other	Count	7	5	2	1	15
		% within Response to question 3d	20.6%	25.0%	3.6%	25.0%	13.3%
	No response	Count	19	7	50	3	79
		% within Response to question 3d	55.9%	35.0%	90.9%	75.0%	69.9%
Total		Count	34	20	55	4	113
		% within Response to question 3d	100.0%	100.0%	100.0%	100.0%	100.0%

Table A - 15

Case Processing Summary

	Cases					
	Valid		Missing		Total	
	N	Percent	N	Percent	N	Percent
Reasons for the response to 3e * Response to question 3e	114	100.0%	0	.0%	114	100.0%

Table A - 16

Reasons for the response to 3e * Response to question 3e Crosstabulation

			Response to question 3e				
			Function	Not a function	I don't know	NR	Total
Reasons for the response to 3e	Colloquial definition	Count	2				2
		% within Response to question 3e	13.3%				1.8%
	First impression/General appearance	Count		3			3
		% within Response to question 3e		6.7%			2.6%
	Specific visual hints	Count		20			20
		% within Response to question 3e		44.4%			17.5%
	Other	Count	3	6	3	1	13
		% within Response to question 3e	20.0%	13.3%	6.0%	25.0%	11.4%
	No response	Count	10	16	47	3	76
		% within Response to question 3e	66.7%	35.6%	94.0%	75.0%	66.7%
Total		Count	15	45	50	4	114
		% within Response to question 3e	100.0%	100.0%	100.0%	100.0%	100.0%

Table A - 17

The results from these crosstabulations are summarized in Table A – 18 and A – 19 below. Students' verbal explanations for each category are also given:

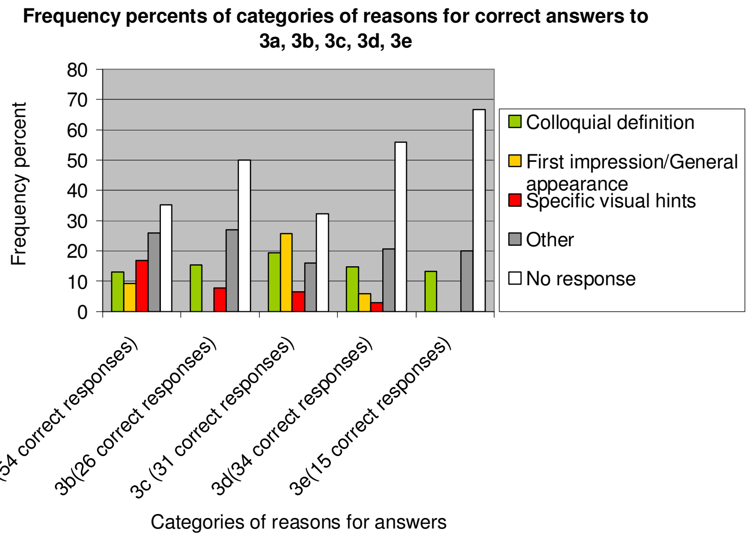


Table A - 18

Colloquial definition: 3a: "No element is assigned to more than one element", "elements in the domain are not assigned to more than one element in the range", "for $x \in R$, y takes a value between 1 and -1".

3b: "For all values of x, y takes a value between 0 and 1", "every element in x is assigned to only one element in y".

3c: "Because value of x=1 is assigned to more than one element in y axis", "for values other than -1 and 1, there may not be $y \in R$".

3d: "For any value of x, $y \in R$", "every element in x is assigned to only one element in y".

3e: "For every value of x, $y \in [1,3]$ (probably meaning [-1,-3])', "Images of all of them is between -1 and -3".

First impression/General appearance:

3a: "there may be a function like this".

3c: "this is a wrong graph", "it's not symmetrical", "that's a stupid drawing".

3d: "it has symmetry property".

Specific visual hints:

3a: "(it's a function) since the numbers are given in equal length", "it's passing through the origin".

3b: "because of the numbers on the axes".

3c: "it intersects all of the numbers given on the axis (y axis)", "it's passing through x and y axes".

3d: "the graph increases and decreases towards + and – ".

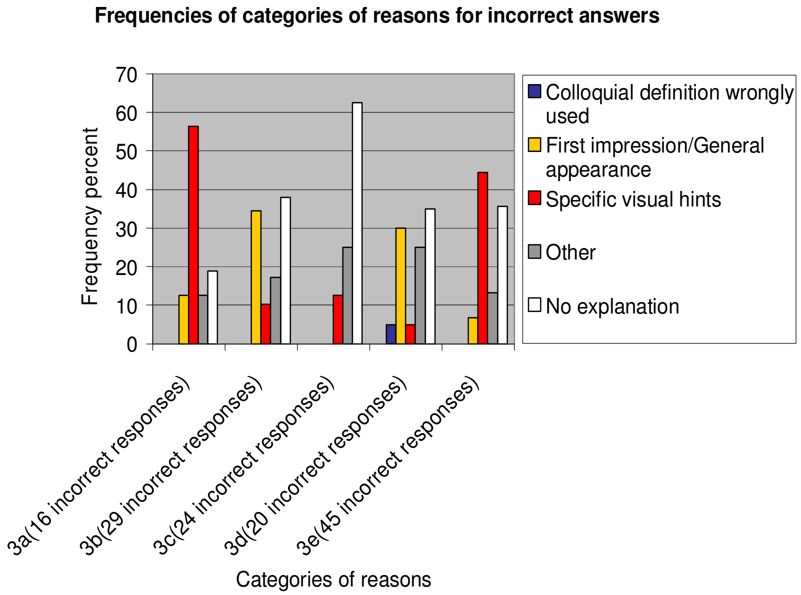


Table A - 19

Colloquial definition wrongly used: "Same values takes different values. For instance, it should be, $f(5\pi)=f(2\pi)$".

First impression/General appearance:

3a: "this shape doesn't look like a function", "a graph can not be like this".

(3b) "I've never seen such a function graph in my life before. I can't believe that it's a function", "a graph can't be like this".

3d: "that's a wrong graph", "I haven't seen such a graph like this before, like mountains in a row, like Taurus Mountains".

(3e) "it doesn't look like a function".

Specific visual hints:

3a: "it's continuously on the same surface (probably referring to x axis)", "it intersects x axis at various places", "function can not be negative on y axis".

3b: "I don't know what π is for", "A function can't go only upwards".

3c "it's (the graph) on the same surface (probably referring to y axis)".

3d: "because, under the graph, it's empty. Where do the lines go? It's not clear".

3e: "it's out of the domain", "it's not a function since it has nothing related to x axis", "it doesn't intersect x axis", "it doesn't touch to x axis", "it only passes through y axis", "it does not pass through neither x nor y".

A2.2 –Reasons for responses to 4a, 4b, 4c, 4d, 4e

Frequencies

Statistics

		Reason for answer to 4a	Reason for answer to 4b	Reason for answer to 4c	Reason for answer to 4d	Reason for answer to 4e
N	Valid	114	114	114	114	114
	Missing	0	0	0	0	0

Table A - 20

Frequency Tables

Reasons for answer to 4a

		Frequency	Valid Percent
Valid	Colloquial definition	3	2.6
	First impression/General appearance	29	25.4
	Specific visual hints	4	3.5
	Other	5	4.4
	No explanation	73	64.0
	Total	114	100.0

Table A - 21

Reasons for answer to 4b

		Frequency	Valid Percent
Valid	Colloquial definition	2	1.8
	Colloquial defintiton wrongly used	5	4.4
	First impression/General appearance	9	7.9
	Specific visual hints	15	13.2
	Other	16	14.0
	No explanation	67	58.8
	Total	114	100.0

Table A - 22

Reasons for answer to 4c

		Frequency	Valid Percent
Valid	Colloquial definition	6	5.3
	Colloquial defintiton wrongly used	1	.9
	First impression/General appearance	4	3.5
	Specific visual hints	12	10.5
	Other	14	12.3
	No explanation	77	67.5
	Total	114	100.0

Table A - 23

Reasons for answer to 4d

		Frequency	Valid Percent
Valid	Colloquial definition	1	.9
	Colloquial definititon wrongly used	4	3.5
	First impression/General appearance	15	13.2
	Specific visual hints	4	3.5
	Other	9	7.9
	No explanation	81	71.1
	Total	114	100.0

Table A - 24

Reasons for answer to 4e

		Frequency	Valid Percent
Valid	Colloquial definition	9	7.9
	First impression/General appearance	11	9.6
	Specific visual hints	10	8.8
	Other	9	7.9
	No explanation	75	65.8
	Total	114	100.0

Table A - 25

The percentages are summarized as a bar chart as shown in the table below:

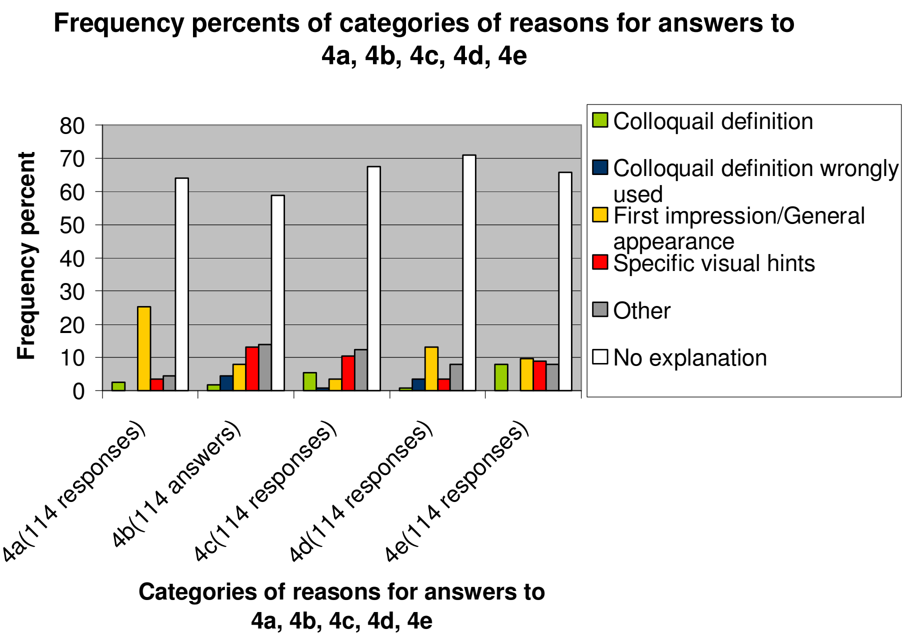


Table A - 26

To be able to find the percentages of categories for correct and incorrect answers, responses for each item are crosstabulated with categories of reasons for each item. Crosstabulations are given below:

Case Processing Summary

	Cases					
	Valid		Missing		Total	
	N	Percent	N	Percent	N	Percent
Reason for answer to 4a * Response to question 4a	114	100.0%	0	.0%	114	100.0%

Table A - 27

Reason for answer to 4a * Response to question 4a Crosstabulation

			Response to question 4a				Total
			Function	Not a function	I don't know	NR	
Reason for answer to 4a	Colloquial definition	Count		3			3
		% within Response to question 4a		5.2%			2.6%
	First impression/General appearance	Count		24	5		29
		% within Response to question 4a		41.4%	11.1%		25.4%
	Specific visual hints	Count		3	1		4
		% within Response to question 4a		5.2%	2.2%		3.5%
	Other	Count		5			5
		% within Response to question 4a		8.6%			4.4%
	No explanation	Count	9	23	39	2	73
		% within Response to question 4a	100.0%	39.7%	86.7%	100.0%	64.0%
Total		Count	9	58	45	2	114
		% within Response to question 4a	100.0%	100.0%	100.0%	100.0%	100.0%

Table A - 28

Case Processing Summary

	Cases					
	Valid		Missing		Total	
	N	Percent	N	Percent	N	Percent
Reason for answer to 4b * Response to question 4b	114	100.0%	0	.0%	114	100.0%

Table A - 29

Reason for answer to 4b * Response to question 4b Crosstabulation

			Response to question 4b				Total
			Function	Not a function	I don't know	NR	
Reason for answer to 4b	Colloquial definition	Count		2			2
		% within Response to question 4b		8.7%			1.8%
	Colloquial definition wrongly used	Count	3	2			5
		% within Response to question 4b	6.3%	8.7%			4.4%
	First impression/General appearance	Count	4	5			9
		% within Response to question 4b	8.3%	21.7%			7.9%
	Specific visual hints	Count	11	3	1		15
		% within Response to question 4b	22.9%	13.0%	2.5%		13.2%
	Other	Count	14	2			16
		% within Response to question 4b	29.2%	8.7%			14.0%
	No explanation	Count	16	9	39	3	67
		% within Response to question 4b	33.3%	39.1%	97.5%	100.0%	58.8%
Total		Count	48	23	40	3	114
		% within Response to question 4b	100.0%	100.0%	100.0%	100.0%	100.0%

Table A - 30

Case Processing Summary

	Cases					
	Valid		Missing		Total	
	N	Percent	N	Percent	N	Percent
Reason for answer to 4c * Response to question 4c	114	100.0%	0	.0%	114	100.0%

Table A - 31

Reason for answer to 4c * Response to question 4c Crosstabulation

			Response to question 4c				Total
			Function	Not a function	I don't know	NR	
Reason for answer to 4c	Colloquial definition	Count	3				3
		% within Response to question 4c	6.7%				2.6%
	Colloquial definition wrongly used	Count	3	1			4
		% within Response to question 4c	6.7%	7.1%			3.5%
	First impression/General appearance	Count	2	2			4
		% within Response to question 4c	4.4%	14.3%			3.5%
	Specific visual hints	Count	7	4		1	12
		% within Response to question 4c	15.6%	28.6%		10.0%	10.5%
	Other	Count	11	1	1	1	14
		% within Response to question 4c	24.4%	7.1%	2.2%	10.0%	12.3%
	No explanation	Count	19	6	44	8	77
		% within Response to question 4c	42.2%	42.9%	97.8%	80.0%	67.5%
Total		Count	45	14	45	10	114
		% within Response to question 4c	100.0%	100.0%	100.0%	100.0%	100.0%

Table A - 32

Case Processing Summary

	Cases					
	Valid		Missing		Total	
	N	Percent	N	Percent	N	Percent
Reason for answer to 4d * Response to question 4d	113	99.1%	1	.9%	114	100.0%

Table A - 33

Reason for answer to 4d * Response to question 4d Crosstabulation

			Response to question 4d				Total
			Function	Not a function	I don't know	NR	
Reason for answer to 4d	Colloquial definition	Count		1			1
		% within Response to question 4d		2.7%			.9%
	Colloquial definition wrongly used	Count	2	2			4
		% within Response to question 4d	11.8%	5.4%			3.5%
	First impression/General appearance	Count		8	7		15
		% within Response to question 4d		21.6%	12.5%		13.3%
	Specific visual hints	Count		4			4
		% within Response to question 4d		10.8%			3.5%
	Other	Count	4	4	1		9
		% within Response to question 4d	23.5%	10.8%	1.8%		8.0%
	No explanation	Count	11	18	48	3	80
		% within Response to question 4d	64.7%	48.6%	85.7%	100.0%	70.8%
Total		Count	17	37	56	3	113
		% within Response to question 4d	100.0%	100.0%	100.0%	100.0%	100.0%

Table A - 34

Case Processing Summary

	Cases					
	Valid		Missing		Total	
	N	Percent	N	Percent	N	Percent
Reason for answer to 4e * Response to question 4e	114	100.0%	0	.0%	114	100.0%

Table A - 35

Reason for answer to 4e * Response to question 4e Crosstabulation

			Response to question 4e				
			Function	Not a function	I don't know	NR	Total
Reason for answer to 4e	Colloquial definition	Count		9			9
		% within Response to question 4e		34.6%			7.9%
	First impression/General appearance	Count	5	6			11
		% within Response to question 4e	13.2%	23.1%			9.6%
	Specific visual hints	Count	6	2	2		10
		% within Response to question 4e	15.8%	7.7%	4.1%		8.8%
	Other	Count	6	2	1		9
		% within Response to question 4e	15.8%	7.7%	2.0%		7.9%
	No explanation	Count	21	7	46	1	75
		% within Response to question 4e	55.3%	26.9%	93.9%	100.0%	65.8%
Total		Count	38	26	49	1	114
		% within Response to question 4e	100.0%	100.0%	100.0%	100.0%	100.0%

Table A - 36

The results from these crosstabulations are summarized in Table A – 37 and A – 38 below. Students' verbal explanations for each category are also given:

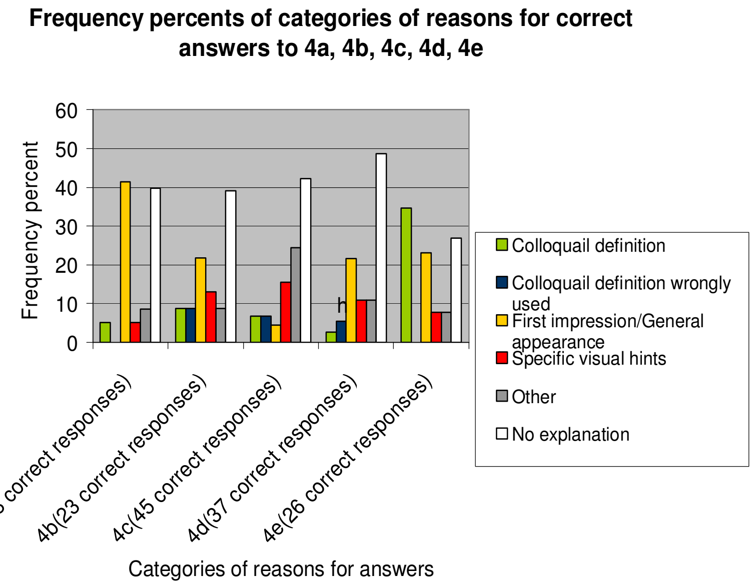


Table A - 37

Colloquial definition:

4a: "Same element of the domain can not be assigned to two different values", "an element in the domain can not be assigned to more than one element in the range".

4b: "(it's not a function) since most of the elements of the domain are left", "there can't be elements left in the domain".

4c: "all elements are assigned and there isn't any element left".

4d: "there are elements left in the domain".

4e: "some elements of the domain do not have corresponding values", "values of x, 2 and -2 are not assigned".

Colloquial definition wrongly used:

4b: "when x, the domain, is R, the range should have been R", "(it's not a function) since the domain is not R".

4c: "because, in a relation, elements in the domain are not left unassigned".

4d: "3/2 is undefined".

First impression/General appearance:

4a: "it doesn't look like a function", "it can't be like this, it's stupid".

4b: "(it's not a function) it's rather like a straight line".

4c: "it looks like familiar".

4d: "there can't be a function like this, like a graph of a beating heart".

4e: "(it's not a function) it makes triangles".

Specific visual hints:

4a: "there can't be such values in a function graph".

4b: "(it's not a function) since there is no drawing".

4c: "it passes through x and y axes".

4d: "the graph doesn't pass through from integers on the x axis such as 3 or 4".

4e: "(it's not a function). There are gaps on the line".

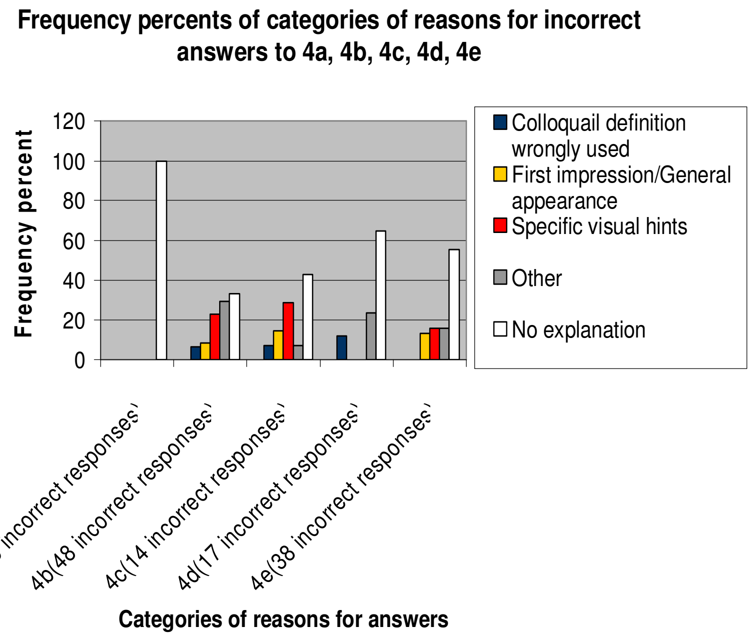


Table A - 38

Colloquial definition wrongly used:

4b: "It's a function since all elements are assigned to each other".

4c: "it's not a function since elements in the domain are assigned to more than one element".

4d: "all of the elements in the domain are assigned to an element".

First impression/General appearance:

4b: "a bisector line is a graph of a function".

4c: "(it's not a function) it's rather like a straight line".

4e: "I've seen such a drawing before".

Specific visual hints:

4b: "(it's a function) since the numbers on the graph pass through the origin".

4c: "(it's not a function) since there is no drawing".

4e: "(it's a function) since the line are joint on x and y", "(it's a function) since it passes through a specific point".

A2.3 – Reasons for responses to 6a, 6b, 6c, 6d, 6e, 6f, 6g

Frequencies for reasons are presented below. Frequency tables are also summarized as bar charts.

Frequencies

Statistics

		Reason for answer to 6a	Reason for answer to 6b	Reason for answer to 6c	Reason for answer to 6d	Reason for answer to 6e	Reason for answer to 6f	Reason for answer to 6g
N	Valid	113	113	113	113	113	114	114
	Missing	1	1	1	1	1	0	0

Table A - 39

6a) $f : R \to R \qquad f(x) = \sqrt{x^2 - 16}$

Frequency Tables

Reasons for answer to 6a

		Frequency	Valid Percent
Valid	Colloquial definition	6	5.3
	Specific visual hints	6	5.3
	Finding a value for x and/or f(x)	8	7.1
	Other	11	9.7
	No explanation	82	72.6
	Total	113	100.0
Missing	System	1	
Total		114	

Table A - 40

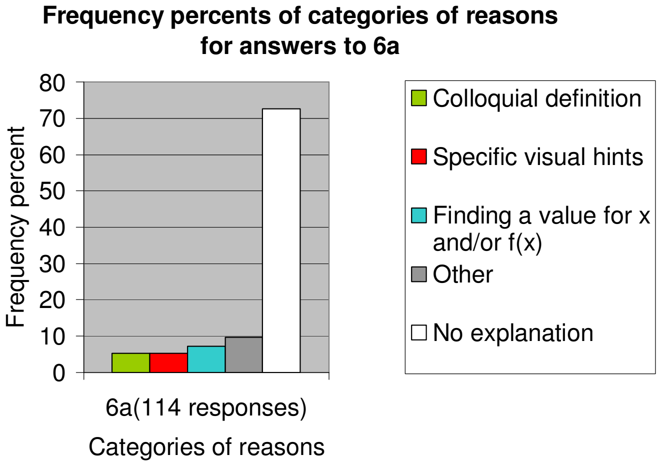


Table A - 41

6b) $f: R \to R$, $x^2 + y^2 = 1$

Reasons for answer to 6b

		Frequency	Valid Percent
Valid	Colloquial definition	2	1.8
	Specific visual hints	6	5.3
	Equation/unknown	14	12.4
	Other	4	3.5
	No explanation	87	77.0
	Total	113	100.0
Missing	System	1	
Total		114	

Table A - 42

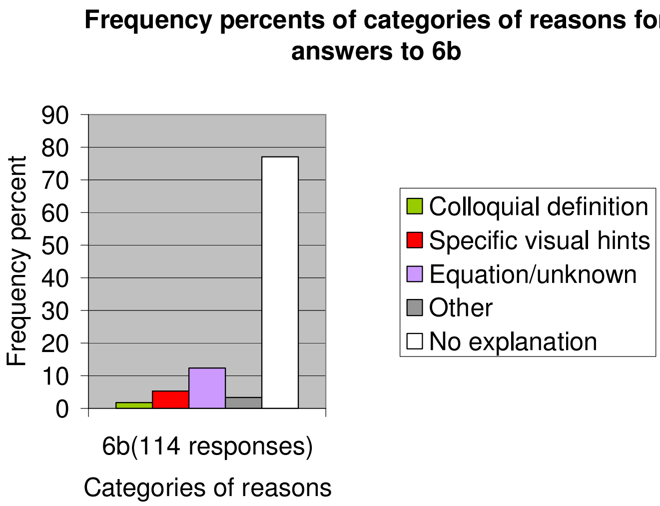


Table A - 43

6c) $y = 5$

Reasons for answer to 6c

		Frequency	Percent	Valid Percent
Valid	Colloquial definition	1	.9	.9
	Expression has a value/result	7	6.1	6.2
	Specific visual hints	5	4.4	4.4
	Constant function	7	6.1	6.2
	Other	16	14.0	14.2
	No explanation	77	67.5	68.1
	Total	113	99.1	100.0
Missing	System	1	.9	
Total		114	100.0	

Table A - 44

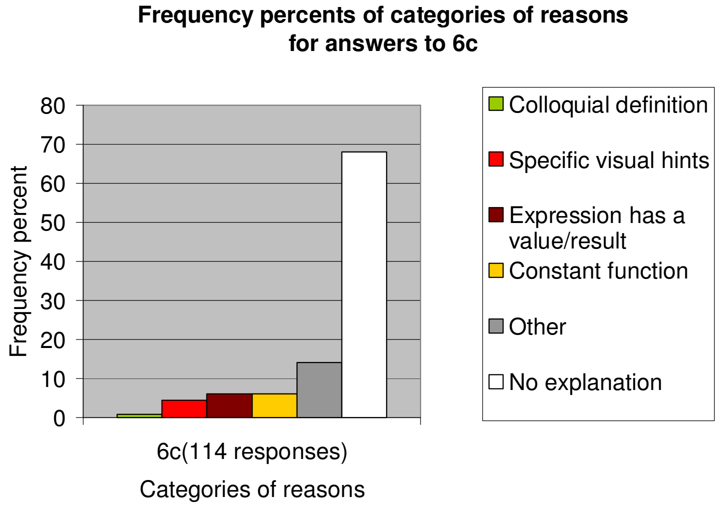


Table A - 45

6d) $y = 5$ (for $x \geq 2$)

Reasons for answer to 6d

		Frequency	Valid Percent
Valid	Colloquial definition	1	.9
	Specific visual hints	2	1.8
	Constant function	2	1.8
	Other	19	16.8
	No explanation	89	78.8
	Total	113	100.0
Missing	System	1	
Total		114	

Table A - 46

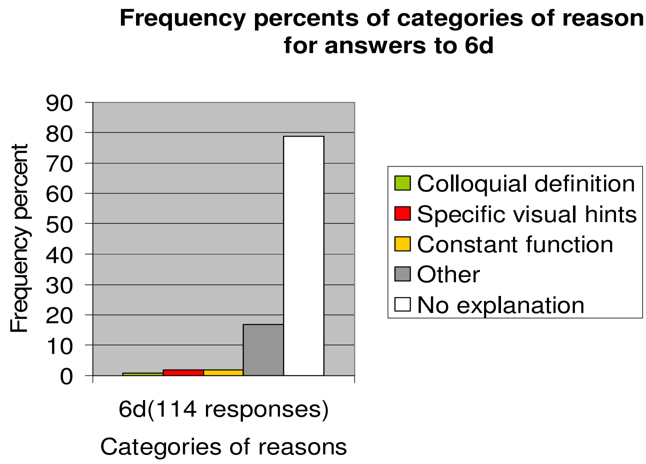


Table A - 47

6e) $y = 5$ (for all values of x)

Reasons for answer to 6e

		Frequency	Percent	Valid Percent
Valid	Colloquial definition	1	.9	.9
	Colloquial definititon wrongly used	1	.9	.9
	Specific visual hints	3	2.6	2.7
	Constant function	7	6.1	6.2
	Other	11	9.6	9.7
	No explanation	90	78.9	79.6
	Total	113	99.1	100.0
Missing	System	1	.9	
Total		114	100.0	

Table A - 48

Frequency percents of categories of reasons for answers to 6e

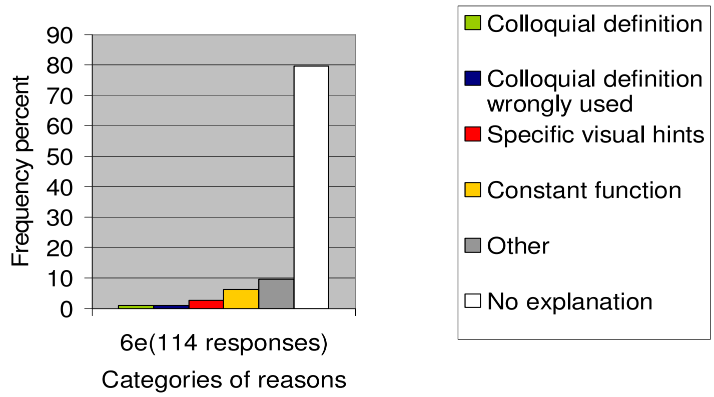


Table A - 49

6f) $f : R^{+} \rightarrow R,\ f(x) = | x^2 - 4 |$

Reasons for answer to 6f

		Frequency	Valid Percent
Valid	Colloquial definition	6	5.3
	Solving f(x)=0 for x	5	4.4
	Absolute value function	4	3.5
	f(x) has a value	3	2.6
	Other	12	10.5
	No explanation	84	73.7
	Total	114	100.0

Table A - 50

Frequency percents of categories of reasons for answers to 6f

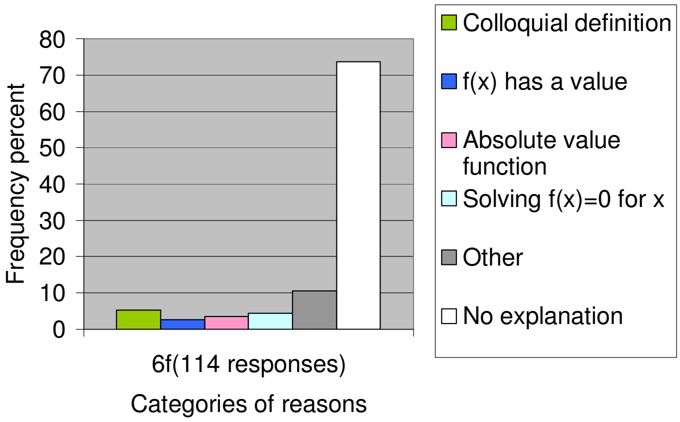


Table A - 51

Reasons for answer to 6g

		Frequency	Valid Percent
Valid	Colloquial definition wrongly used	1	.9
	Split domain function	4	3.5
	Signum function	9	7.9
	Other	22	19.3
	No explanation	78	68.4
	Total	114	100.0

Table A - 52

6g) $f : R \to R$

$$f(x)=\begin{cases} 1, & \text{if } x^2-2x+1>0 \\ 0, & \text{if } x^2-2x+1=0 \\ -1, & \text{if } x^2-2x+1<0 \end{cases}$$

Frequency percents of categories of reasons for answers to 6g

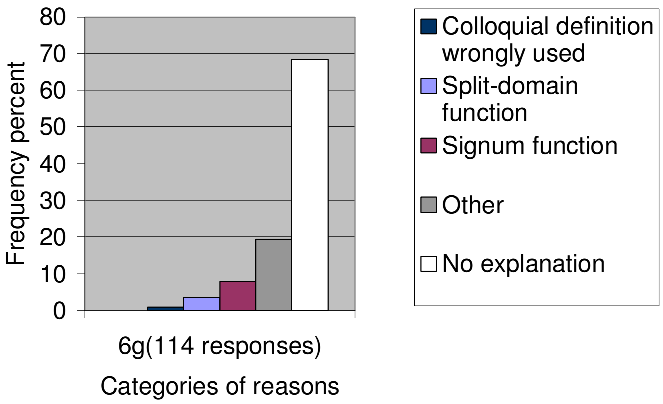


Table A - 53

To be able to find the percentages of categories for correct and incorrect answers, responses for each item are crosstabulated with categories of reasons for each item. These crosstabulations are presented below. Crosstabulations are also summarized as bar charts. Students' verbal explanations for each category are given after each bar chart:

Case Processing Summary

	Cases					
	Valid		Missing		Total	
	N	Percent	N	Percent	N	Percent
Reason for answer to 6a * Response to question 6a	112	98.2%	2	1.8%	114	100.0%

Table A - 54

Reason for answer to 6a * Response to question 6a Crosstabulation

			Response to question 6a				Total
			Function	Not a function	I don't know	NR	
Reason for answer to 6a	Colloquial definition	Count	1	5			6
		% within Response to question 6a	2.0%	41.7%			5.4%
	Specific visual hints	Count	4	2			6
		% within Response to question 6a	8.0%	16.7%			5.4%
	Finding a value for x and/or f(x)	Count	8				8
		% within Response to question 6a	16.0%				7.1%
	Other	Count	9	2			11
		% within Response to question 6a	18.0%	16.7%			9.8%
	No explanation	Count	28	3	47	3	81
		% within Response to question 6a	56.0%	25.0%	100.0%	100.0%	72.3%
Total		Count	50	12	47	3	112
		% within Response to question 6a	100.0%	100.0%	100.0%	100.0%	100.0%

Table A - 55

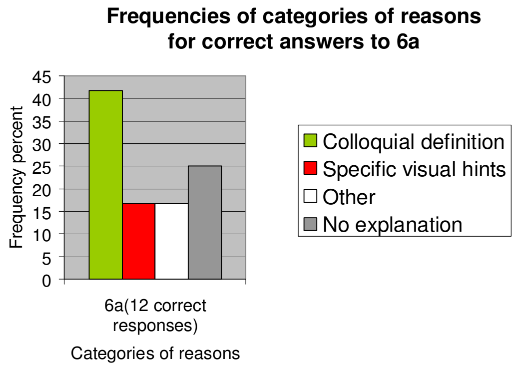


Table A - 56

Colloquial definition: 41.7% (5) of the students who responded correctly to 6a used the colloquial definition for their explanations:

"If we put 0 for *x*, then there is a negative number in the square root".

"It doesn't satisfy for the interval -4<*x*<4".

"For x=0, $f(x) \notin R$".

"If I give 1, then square root is minus. It can't be minus".

"We can't take square root of negative numbers".

Specific visual hints: 16.7% (2) students who responded correctly to 6a gave responses based on specific visual hints. They said that a function can not include an expression with a square root.

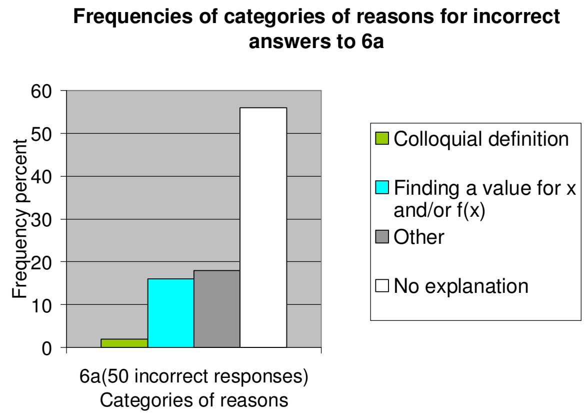


Table A - 57

Colloquial definition wrongly used: 2% (1) of the students considered 6a as a function by using colloquial definition wrongly:

"for every value of x, we can find another value. It says $R \rightarrow R$".

Specific visual hints: 8% (4) of the students who incorrectly consider 6a as a function gave explanations based on specific visual hints:

"It's a function...with a square root expression, defined on $R \rightarrow R$".

"it's a function of a square root expression".

The other two students considered it as a function due to hints like "$f: R \rightarrow R$" and "f".

Finding a value for x and/or $f(x)$: 16% (8) of the students who incorrectly considered 6a as a function, explained their answers by finding a value for $f(x)$ and/or x :

Case Processing Summary

	Cases					
	Valid		Missing		Total	
	N	Percent	N	Percent	N	Percent
Reason for answer to 6b * Response to question 6b	113	99.1%	1	.9%	114	100.0%

Table A - 58

Reason for answer to 6b * Response to question 6b Crosstabulation

			Response to question 6b				Total
			Function	Not a function	I don't know	NR	
Reason for answer to 6b	Colloquial definition	Count		2			2
		% within Response to question 6b		9.5%			1.8%
	Specific visual hints	Count	4	2			6
		% within Response to question 6b	10.3%	9.5%			5.3%
	Equation/unknown	Count	10	4			14
		% within Response to question 6b	25.6%	19.0%			12.4%
	Other	Count	3	1			4
		% within Response to question 6b	7.7%	4.8%			3.5%
	No explanation	Count	22	12	47	6	87
		% within Response to question 6b	56.4%	57.1%	100.0%	100.0%	77.0%
Total		Count	39	21	47	6	113
		% within Response to question 6b	100.0%	100.0%	100.0%	100.0%	100.0%

Table A - 59

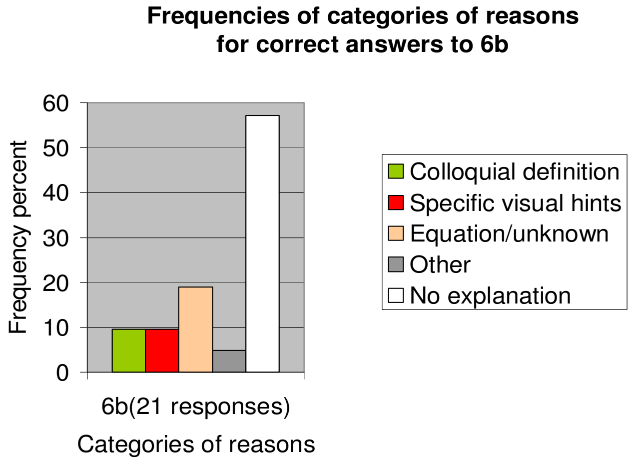


Table A - 60

Colloquial definition: 9.5% (2) of the students who correctly did not consider 6b as a function, explained their answers by using the colloquial definition:

“Some of the negative numbers are in the root, therefore they don’t have corresponding values”.

“for x=5, y^2=-24 and this is not possible”.

Equation/unknown: 19% (4) of the students who correctly did not consider 6b as a function with wrong reasons related to the expression being an equation or contains an unknown:

“The equation is not satisfied”.

“it’s not a function, it’s an equation”.

“whatever we substitute for x and y, the result is 1”.

“$x^2+y^2=1$, $(x+y)(x-y)=1$. This is an equation”.

Specific visual hints: 9.5% (2) of the students who correctly did not consider 6b as a function, gave wrong reasons related to specific hints such as the absence of f at the front.

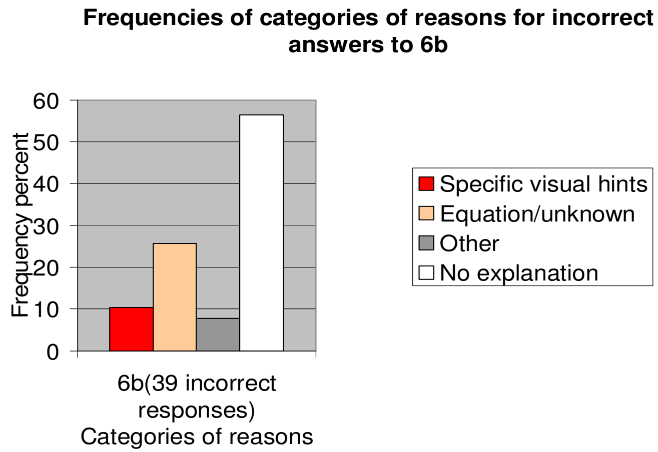


Table A - 61

Equation/unknown: 25.6% (10) of the students who incorrectly considered 6b as a function thought that it was a function since it is an equation and numbers can be substituted for the unknowns:

“we can substitute values for x and y”.

Specific visual hints: 10.3% (4) of the students who incorrectly considered 6b as a function gave explanations related to specific hints such as the existence of “$f: R \rightarrow R$”.

Case Processing Summary

	Cases					
	Valid		Missing		Total	
	N	Percent	N	Percent	N	Percent
Reason for answer to 6c * Response to question 6c	112	98.2%	2	1.8%	114	100.0%

Table A - 62

Reason for answer to 6c * Response to question 6c Crosstabulation

			Response to question 6c				Total
			Function	Not a function	I don't know	NR	
Reason for answer to 6c	Colloquial definition	Count	1				1
		% within Response to question 6c	3.6%				.9%
	Expression has a value/result	Count	7				7
		% within Response to question 6c	25.0%				6.3%
	Specific visual hints	Count		5			5
		% within Response to question 6c		12.5%			4.5%
	Constant function	Count	7				7
		% within Response to question 6c	25.0%				6.3%
	Other	Count	4	11	1		16
		% within Response to question 6c	14.3%	27.5%	2.4%		14.3%
	No explanation	Count	9	24	40	3	76
		% within Response to question 6c	32.1%	60.0%	97.6%	100.0%	67.9%
Total		Count	28	40	41	3	112
		% within Response to question 6c	100.0%	100.0%	100.0%	100.0%	100.0%

Table A - 63

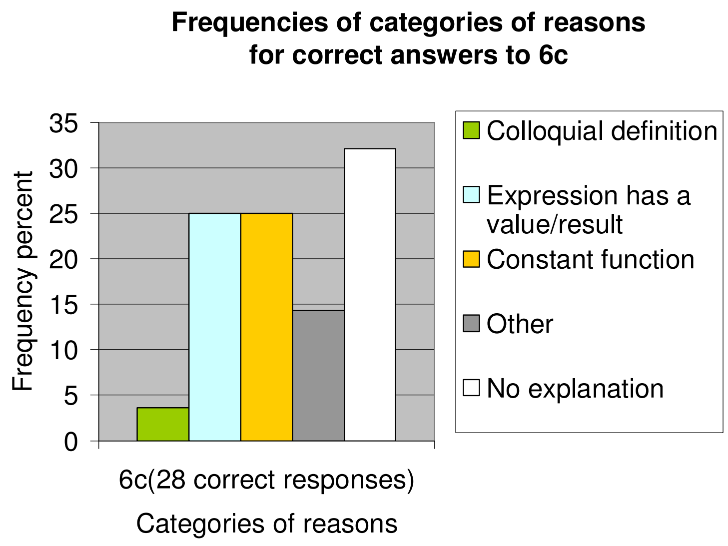


Table A - 64

Colloquial definition: 3.6 % (1) of the students who considered this expression as a function used the colloquial definition for his/her response:

"It's in the form *y=f(x)*. $y = 5$ is the corresponding value of f(x)" (81).

Constant function: 25% (7) of the students who considered $y = 5$ as a function said that it is a constant function therefore it is a function.

Expression has a value/result: 25% (7) of the students who considered $y = 5$ as a function focused on it as an expression which has a value or a result:

"It says that *y* is 5".

"here the result of the function *y* is equal to 5. *f*(*x*)=*y*".

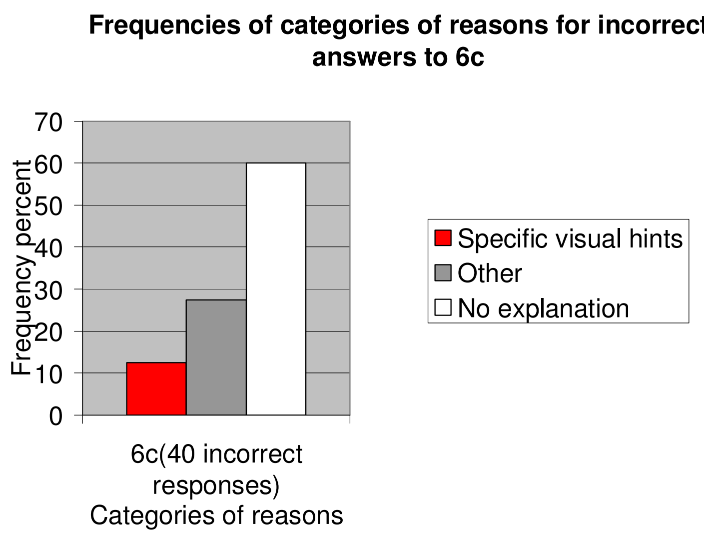


Table A - 65

Specific visual hints: 12.5% (5) of the students who did not consider $y = 5$ as a function responded based on specific hints such as the absence of the notation *f*.

Case Processing Summary

	Cases					
	Valid		Missing		Total	
	N	Percent	N	Percent	N	Percent
Reason for answer to 6d * Response to question 6d	113	99.1%	1	.9%	114	100.0%

Table A - 66

Reason for answer to 6d * Response to question 6d Crosstabulation

			Response to question 6d				Total
			Function	Not a function	I don't know	NR	
Reason for answer to 6d	Colloquial definition	Count	1				1
		% within Response to question 6d	4.0%				.9%
	Specific visual hints	Count		2			2
		% within Response to question 6d		7.1%			1.8%
	Constant function	Count	2				2
		% within Response to question 6d	8.0%				1.8%
	Other	Count	8	10	1		19
		% within Response to question 6d	32.0%	35.7%	1.8%		16.8%
	No explanation	Count	14	16	56	3	89
		% within Response to question 6d	56.0%	57.1%	98.2%	100.0%	78.8%
Total		Count	25	28	57	3	113
		% within Response to question 6d	100.0%	100.0%	100.0%	100.0%	100.0%

Table A - 67

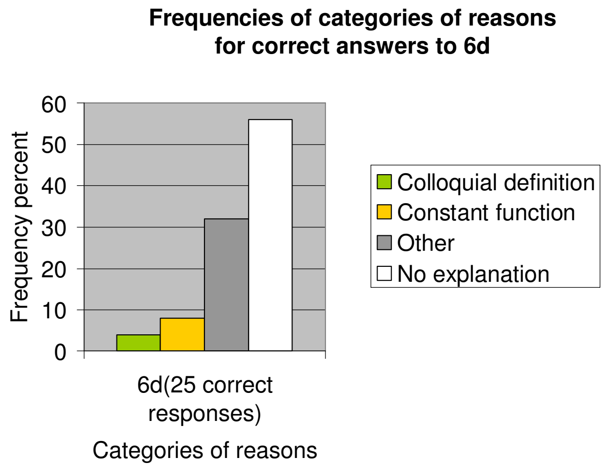


Table A - 68

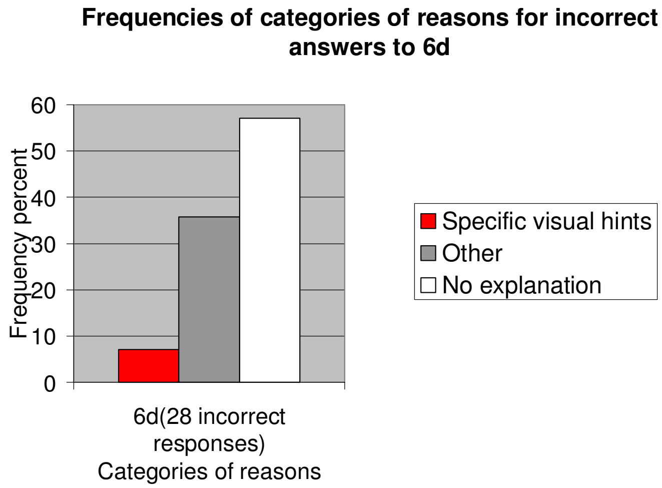


Table A - 69

Colloquial definition: 4% (1) of the students who considered "$y= 5$ (for $x\geq 2$)" as a function used the colloquial definition:

"The elements of the domain are the ones which are equal to or greater than 2. Elements of the domain are assigned to only one element of the range".

Constant function: 8% (2) of the students who considered "$y= 5$ (for $x\geq 2$)" as a function said that it is a constant function.

Other: 32% (8) of the students who considered "$y= 5$ (for $x\geq 2$)" as a function gave various other explanations which can not form further categorizations:

"(It's a function) since there is an x in the expression".

"there are specific values for x and y. Its graph can be drawn".

Specific visual hints: 7.1% (2) of the students who did not consider "$y= 5$ (for $x\geq 2$)" as a function explained their answers focusing on specific hints such as absence of f.

Other: 35.7% (10) of the students who did not consider "$y= 5$ (for $x\geq 2$)" as a function gave various other explanations which can not form further categorizations.

Case Processing Summary

	Cases					
	Valid		Missing		Total	
	N	Percent	N	Percent	N	Percent
Reason for answer to 6e * Response to question 6e	113	99.1%	1	.9%	114	100.0%

Table A - 70

Reason for answer to 6e * Response to question 6e Crosstabulation

			Response to question 6e				Total
			Function	Not a function	I don't know	NR	
Reason for answer to 6e	Colloquial definition	Count	1				1
		% within Response to question 6e	4.0%				.9%
	Colloquial definition wrongly used	Count		1			1
		% within Response to question 6e		4.3%			.9%
	Specific visual hints	Count		3			3
		% within Response to question 6e		13.0%			2.7%
	Constant function	Count	7				7
		% within Response to question 6e	28.0%				6.2%
	Other	Count	5	5		1	11
		% within Response to question 6e	20.0%	21.7%		16.7%	9.7%
	No explanation	Count	12	14	59	5	90
		% within Response to question 6e	48.0%	60.9%	100.0%	83.3%	79.6%
Total		Count	25	23	59	6	113
		% within Response to question 6e	100.0%	100.0%	100.0%	100.0%	100.0%

Table A - 71

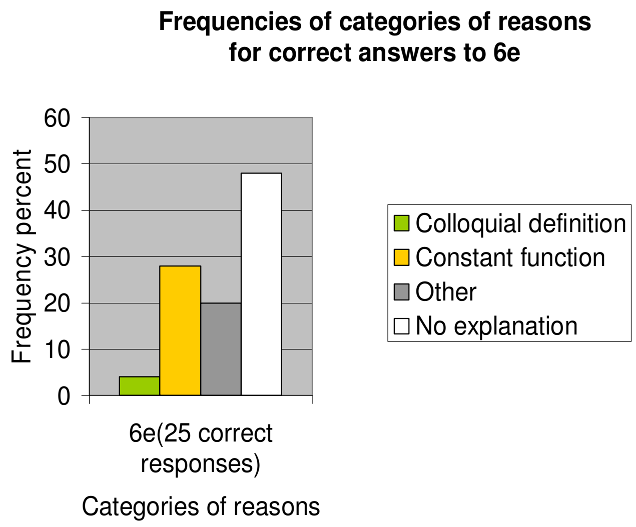


Table A - 72

Colloquial definition: 4% (1) of the students who consider "$y = 5$ (for all values of x)" as a function used the colloquial definition correctly:

"Elements of the domain can be assigned to one element in the range".

Constant function: 28% (7) of the students who considered "$y = 5$ (for all values of x)" as a function said that it is a constant function.

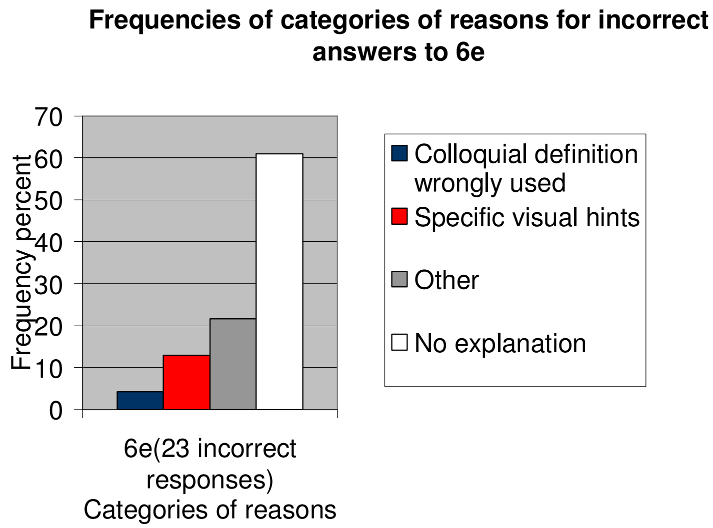


Table A - 73

Colloquial definition wrongly used: 4.3% (1) of the students who did not consider "$y = 5$ (for all values of x)" as a function used the colloquial definition wrongly:

"The corresponding values of 5 can not be at more than one place".

Specific visual hints: 13% (3) of the students who did not consider "$y = 5$ (for all values of x)" as a function gave explanations based on specific hints. All these three students did not consider this as a function since there is not an f in the expression.

Case Processing Summary

	Cases					
	Valid		Missing		Total	
	N	Percent	N	Percent	N	Percent
Reason for answer to 6f * Response to question 6f	114	100.0%	0	.0%	114	100.0%

Table A - 74

Reason for answer to 6f * Response to question 6f Crosstabulation

			Response to question 6f				Total
			Function	Not a function	I don't know	NR	
Reason for answer to 6f	Colloquial definition	Count	6				6
		% within Response to question 6f	11.3%				5.3%
	Solving f(x)=0 for x	Count	5				5
		% within Response to question 6f	9.4%				4.4%
	Absolute value function	Count	4				4
		% within Response to question 6f	7.5%				3.5%
	f(x) has a value	Count	3				3
		% within Response to question 6f	5.7%				2.6%
	Other	Count	9	1	2		12
		% within Response to question 6f	17.0%	16.7%	4.1%		10.5%
	No explanation	Count	26	5	47	6	84
		% within Response to question 6f	49.1%	83.3%	95.9%	100.0%	73.7%
Total		Count	53	6	49	6	114
		% within Response to question 6f	100.0%	100.0%	100.0%	100.0%	100.0%

Table A - 75

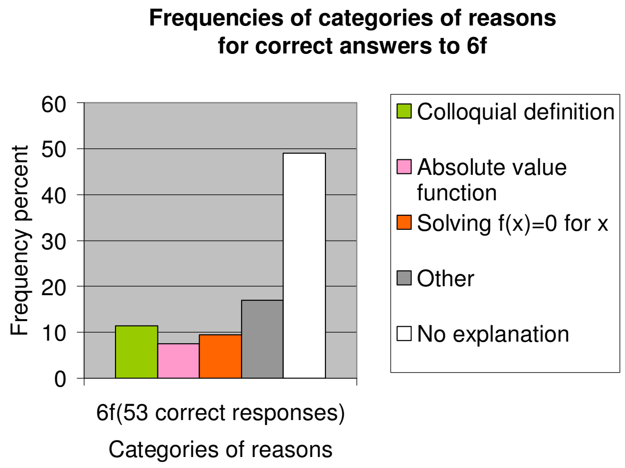


Table A - 76

Colloquial definition: 11.3% (6) of the students who considered 6f as a function used the colloquial definition correctly in their explanations:

"For every x value, there is a value in the range"

"$f(1)=|1^2-4|=|-3|=+3$, for all values of x, there is a value"

"non of the elements in the domain is left unassigned".

"every x value finds its value".

Solving f(x)=0 for x: 9.4% (5) of the students who considered 6f as a function solved $f(x)=0$ for x.

Absolute value function: 7.5% (4) of the students who considered 6f as a function said that it is an absolute value function.

f(x) has a value: 5.7% (3) of the students who considered 6f as a function said that it is a function since $f(x)$ has a value.

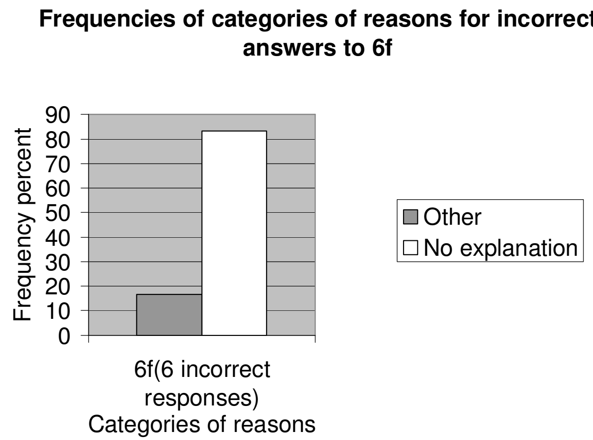


Table A - 77

Case Processing Summary

	Cases					
	Valid		Missing		Total	
	N	Percent	N	Percent	N	Percent
Reason for answer to 6g * Response to question 6g	114	100.0%	0	.0%	114	100.0%

Table A - 78

Reason for answer to 6g * Response to question 6g Crosstabulation

			Response to question 6g				Total
			Function	Not a function	I don't know	NR	
Reason for answer to 6g	Colloquial definition wrongly used	Count		1			1
		% within Response to question 6g		20.0%			.9%
	Split domain function	Count	4				4
		% within Response to question 6g	7.4%				3.5%
	Signum function	Count	9				9
		% within Response to question 6g	16.7%				7.9%
	Other	Count	19	1	1	1	22
		% within Response to question 6g	35.2%	20.0%	2.3%	9.1%	19.3%
	No explanation	Count	22	3	43	10	78
		% within Response to question 6g	40.7%	60.0%	97.7%	90.9%	68.4%
Total		Count	54	5	44	11	114
		% within Response to question 6g	100.0%	100.0%	100.0%	100.0%	100.0%

Table A - 79

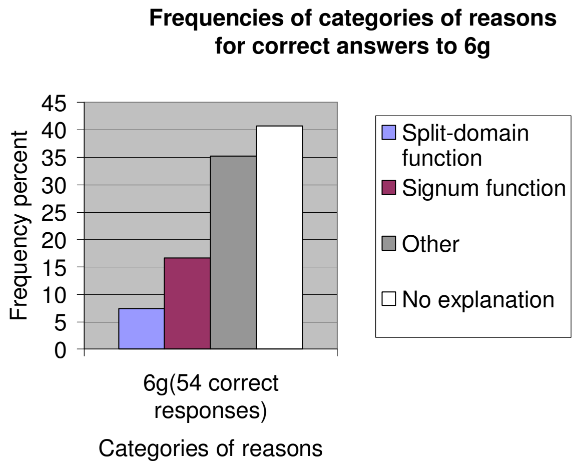


Table A - 80

Colloquial definition: None of the students used the colloquial definition for their explanations.

Split-domain function: 7.4% (4) of the students who considered 6g as a function said that it is a split-domain function.

Signum function: 16.7% (9) of the students who considered 6g as a function said that it is a signum function.

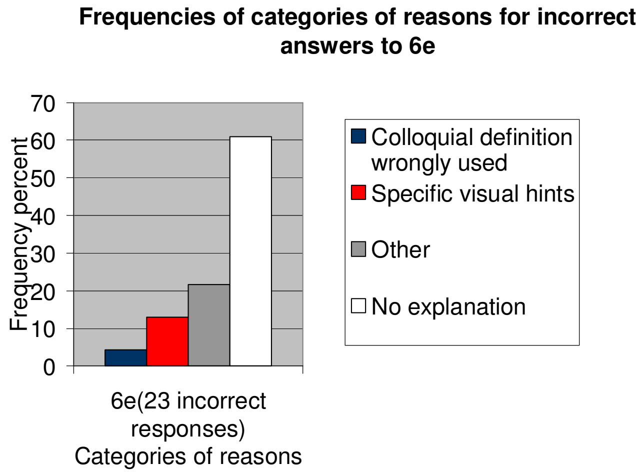


Table A - 81

Colloquial definition wrongly used: One student did not consider this as a function by using the colloquial definition wrongly:

"For every value of *x*, there are three values".

A2.4 –Reasons for responses to 7a, 7b, 7c

Reasons for response to 7a

		Frequency	Valid Percent
Valid	Colloquial definition	15	13.2
	Colloquial definition wrongly used	5	4.4
	Specific visual hints	3	2.6
	Drawing a set diagram	7	6.1
	Other	21	18.4
	No explanation	63	55.3
	Total	114	100.0

Table A - 82

Reasons for response to 7b

		Frequency	Valid Percent
Valid	Colloquial definition	19	16.7
	Colloquial definition wrongly used	8	7.0
	Specific visual hints	4	3.5
	Drawing a set diagram	3	2.6
	Other	17	14.9
	No explanation	63	55.3
	Total	114	100.0

Table A - 83

Reasons for response to 7c

		Frequency	Valid Percent
Valid	Colloquial definition	7	6.1
	Colloquial definition wrongly used	7	6.1
	Specific visual hints	4	3.5
	One to one	3	2.6
	Drawing a set diagram	5	4.4
	Other	20	17.5
	No explanation	68	59.6
	Total	114	100.0

Table A - 84

The percentages are summarized as a bar chart as shown in the table below:

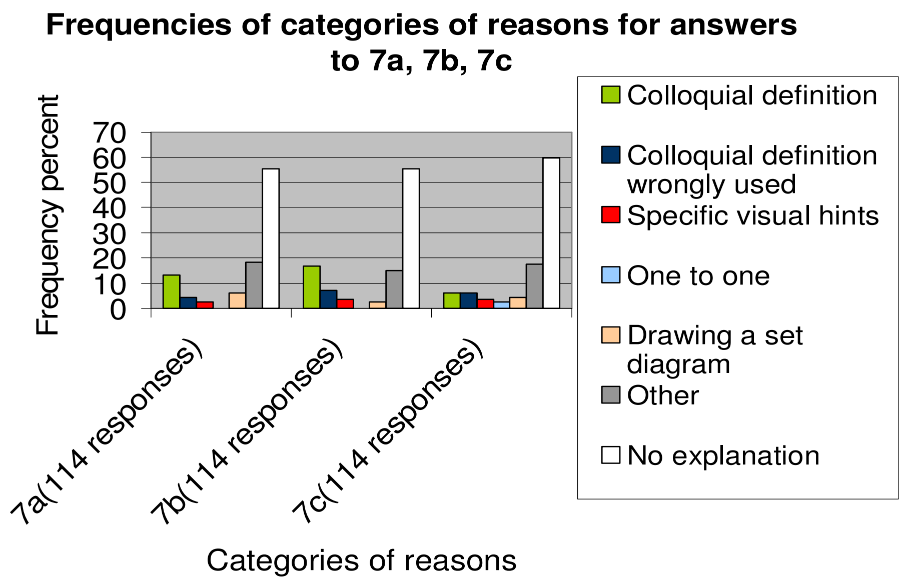


Table A - 85

To be able to find the percentages of categories for correct and incorrect answers, responses for each item are crosstabulated with categories of reasons for each item. These crosstabulations are presented below:

Case Processing Summary

	Cases					
	Valid		Missing		Total	
	N	Percent	N	Percent	N	Percent
Reasons for response to 7a * Response to question 7a	114	100.0%	0	.0%	114	100.0%

Table A - 86

Reasons for response to 7a * Response to question 7a Crosstabulation

			Response to question 7a				Total
			Function	Not a function	I don't know	NR	
Reasons for response to 7a	Colloquial definition	Count	15				15
		% within Response to question 7a	27.8%				13.2%
	Colloquial definition wrongly used	Count	4	1			5
		% within Response to question 7a	7.4%	8.3%			4.4%
	Specific visual hints	Count	3				3
		% within Response to question 7a	5.6%				2.6%
	Drawing a set diagram	Count	6			1	7
		% within Response to question 7a	11.1%			20.0%	6.1%
	Other	Count	15	3	3		21
		% within Response to question 7a	27.8%	25.0%	7.0%		18.4%
	No explanation	Count	11	8	40	4	63
		% within Response to question 7a	20.4%	66.7%	93.0%	80.0%	55.3%
Total		Count	54	12	43	5	114
		% within Response to question 7a	100.0%	100.0%	100.0%	100.0%	100.0%

Table A - 87

Case Processing Summary

	Cases					
	Valid		Missing		Total	
	N	Percent	N	Percent	N	Percent
Reasons for response to 7b * Response to question 7b	114	100.0%	0	.0%	114	100.0%

Table A - 88

Reasons for response to 7b * Response to question 7b Crosstabulation

			Response to question 7b				Total
			Function	Not a function	I don't know	NR	
Reasons for response to 7b	Colloquial definition	Count		19			19
		% within Response to question 7b		61.3%			16.7%
	Colloquial definition wrongly used	Count	7	1			8
		% within Response to question 7b	19.4%	3.2%			7.0%
	Specific visual hints	Count	3	1			4
		% within Response to question 7b	8.3%	3.2%			3.5%
	Drawing a set diagram	Count		3			3
		% within Response to question 7b		9.7%			2.6%
	Other	Count	12	2	3		17
		% within Response to question 7b	33.3%	6.5%	6.8%		14.9%
	No explanation	Count	14	5	41	3	63
		% within Response to question 7b	38.9%	16.1%	93.2%	100.0%	55.3%
Total		Count	36	31	44	3	114
		% within Response to question 7b	100.0%	100.0%	100.0%	100.0%	100.0%

Table A - 89

Case Processing Summary

	Cases					
	Valid		Missing		Total	
	N	Percent	N	Percent	N	Percent
Reasons for response to 7c * Response to question 7c	114	100.0%	0	.0%	114	100.0%

Table A - 90

Reasons for response to 7c * Response to question 7c Crosstabulation

			Response to question 7c				
			Function	Not a function	I don't know	NR	Total
Reasons for response to 7c	Colloquial definition	Count		7			7
		% within Response to question 7c		43.8%			6.1%
	Colloquial definition wrongly used	Count	6	1			7
		% within Response to question 7c	12.0%	6.3%			6.1%
	Specific visual hints	Count	3	1			4
		% within Response to question 7c	6.0%	6.3%			3.5%
	One to one	Count	3				3
		% within Response to question 7c	6.0%				2.6%
	Drawing a set diagram	Count	5				5
		% within Response to question 7c	10.0%				4.4%
	Other	Count	17		3		20
		% within Response to question 7c	34.0%		7.0%		17.5%
	No explanation	Count	16	7	40	5	68
		% within Response to question 7c	32.0%	43.8%	93.0%	100.0%	59.6%
Total		Count	50	16	43	5	114
		% within Response to question 7c	100.0%	100.0%	100.0%	100.0%	100.0%

Table A - 91

The results from these crosstabulations are summarized in Table A – 92 and A – 93 below. Students' verbal explanations for each category are also given:

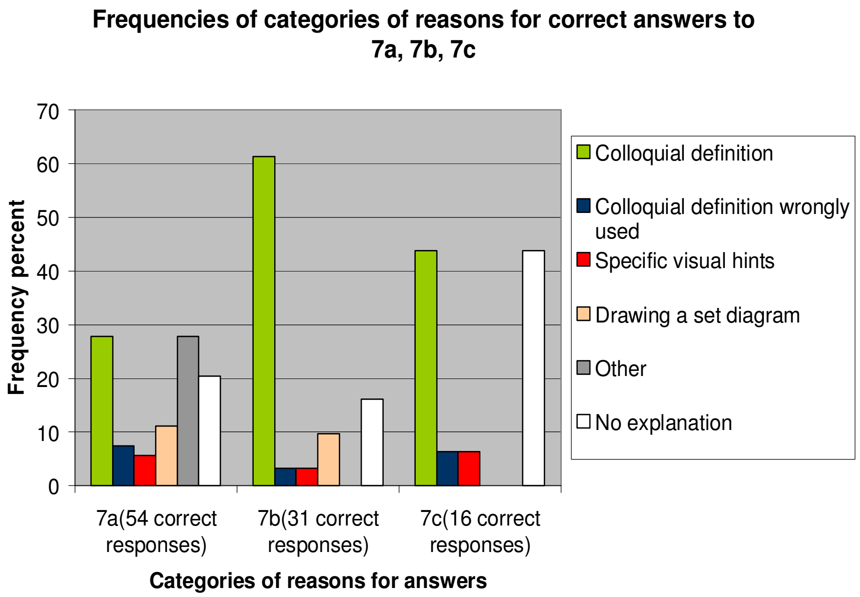


Table A - 92

Colloquial definition: 22.2% (12) of the students who responded correctly to 7a explained their answers using the colloquial definition correctly:

7a: "there isn't any elements left in the domain", "all elements of the domain have their corresponding values" (49), "all elements in the domain are assigned", "every element of A is assigned to elements of B. There aren't any element left in A".

7b: "it's not a function, it's a relation, 1 has two different values", "an element in the domain can't be assigned to more than one element", "this is a relation".

7c: "an element in the domain is left".

Specific visual hints:

7a: "it's a function, since it says A →B".

7b: "it doesn't behave in a regular way".

7c: "there shouldn't be height *h* in a function".

Drawing a set diagram: In this category, there are students who drew the set diagram pictures for the given sets of ordered pairs to answer 7a and 7b correctly.

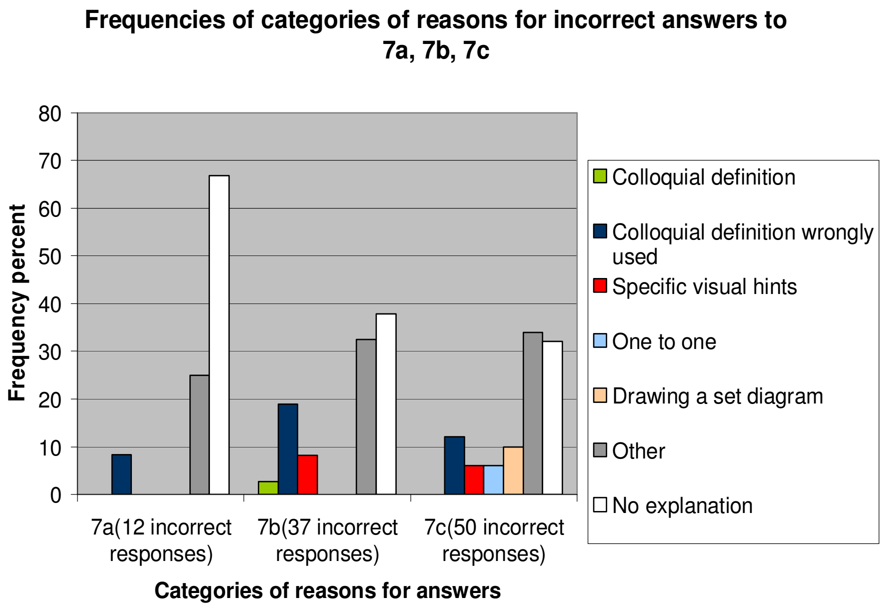


Table A - 93

Colloquial definition wrongly used: 7a: "first element should come from *A*, and second element should come from *B*".

7b: "there is not an element left in the domain".

7c: "for a value of x in *A*, there is a value in *B*. There aren't two elements for the same value".

Specific visual hints: 7b: "it's a function, since it says $A \rightarrow B$".

7c: "it's a function, since it says $A \rightarrow B$".

One to one:

7c: "The set of ordered pair is a one to one function".

Drawing a set diagram: In this category, there are students who drew the set diagram pictures for the given sets of ordered pairs to answer 7c incorrectly.

A2.5 –Reasons for responses to 8a,8b,8c, 8d

Frequencies

Statistics

		Reasons for response to 8a	Reasons for response to 8b	Reasons for response to 8c	Reasons for response to 8d
N	Valid	114	114	114	114
	Missing	0	0	0	0

Table A - 94

Frequency Tables

Reasons for response to 8a

		Frequency	Percent	Valid Percent
Valid	Colloquial definition	24	21.1	21.1
	Colloquial definition wrongly used	2	1.8	1.8
	Specific visual hints	8	7.0	7.0
	One to one and onto-ness	17	14.9	14.9
	Other	12	10.5	10.5
	No explanation	51	44.7	44.7
	Total	114	100.0	100.0

Table A - 95

Reasons for response to 8b

		Frequency	Percent	Valid Percent
Valid	Colloquial definition	17	14.9	14.9
	Colloquial definition wrongly used	3	2.6	2.6
	Specific visual hints	5	4.4	4.4
	One to one and onto-ness	15	13.2	13.2
	Other	8	7.0	7.0
	No explanation	66	57.9	57.9
	Total	114	100.0	100.0

Table A - 96

Reasons for response to 8c

		Frequency	Percent	Valid Percent
Valid	Colloquial definition	27	23.7	23.7
	Colloquial definition wrongly used	7	6.1	6.1
	Specific visual hints	5	4.4	4.4
	One to one and onto-ness	6	5.3	5.3
	Other	10	8.8	8.8
	No explanation	59	51.8	51.8
	Total	114	100.0	100.0

Table A - 97

Reasons for response to 8d

		Frequency	Percent	Valid Percent
Valid	Colloquial definition	15	13.2	13.2
	Colloquial definition wrongly used	2	1.8	1.8
	Specific visual hints	6	5.3	5.3
	One to one and onto-ness	5	4.4	4.4
	Constant function	24	21.1	21.1
	Other	8	7.0	7.0
	No explanation	54	47.4	47.4
	Total	114	100.0	100.0

Table A - 98

The percentages are summarized as a bar chart as shown in the table below:

Frequencies of categories of reasons for 8a, 8b, 8c, 8d

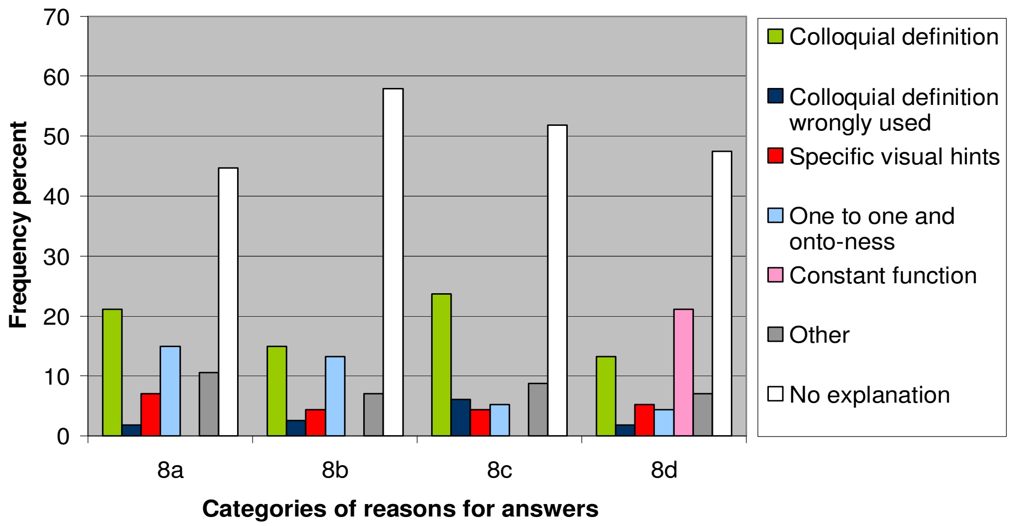


Table A - 99

To be able to find the percentages of categories for correct and incorrect answers, responses for each item are crosstabulated with categories of reasons for each item. These crosstabulations are presented below:

Case Processing Summary

	Cases					
	Valid		Missing		Total	
	N	Percent	N	Percent	N	Percent
Reasons for response to 8a * Response to question 8a	114	100.0%	0	.0%	114	100.0%

Table A - 100

Reasons for response to 8a * Response to question 8a Crosstabulation

			Response to question 8a				Total
			Function	Not a function	I don't know	NR	
Reasons for response to 8a	Colloquial definition	Count	24				24
		% within Response to question 8a	31.2%				21.1%
	Colloquial definition wrongly used	Count	1	1			2
		% within Response to question 8a	1.3%	11.1%			1.8%
	Specific visual hints	Count	6	2			8
		% within Response to question 8a	7.8%	22.2%			7.0%
	One to one and onto ness	Count	17				17
		% within Response to question 8a	22.1%				14.9%
	Other	Count	10	2			12
		% within Response to question 8a	13.0%	22.2%			10.5%
	No explanation	Count	19	4	25	3	51
		% within Response to question 8a	24.7%	44.4%	100.0%	100.0%	44.7%
Total		Count	77	9	25	3	114
		% within Response to question 8a	100.0%	100.0%	100.0%	100.0%	100.0%

Table A - 101

Case Processing Summary

	Cases					
	Valid		Missing		Total	
	N	Percent	N	Percent	N	Percent
Reasons for response to 8b * Response to question 8b	114	100.0%	0	.0%	114	100.0%

Table A - 102

Reasons for response to 8b * Response to question 8b Crosstabulation

			Response to question 8b				Total
			Function	Not a function	I don't know	NR	
Reasons for response to 8b	Colloquial definition	Count	17				17
		% within Response to question 8b	23.3%				14.9%
	Colloquial definition wrongly used	Count	2	1			3
		% within Response to question 8b	2.7%	25.0%			2.6%
	Specific visual hints	Count	4	1			5
		% within Response to question 8b	5.5%	25.0%			4.4%
	One to one and onto ness	Count	15				15
		% within Response to question 8b	20.5%				13.2%
	Other	Count	7		1		8
		% within Response to question 8b	9.6%		3.3%		7.0%
	No explanation	Count	28	2	29	7	66
		% within Response to question 8b	38.4%	50.0%	96.7%	100.0%	57.9%
Total		Count	73	4	30	7	114
		% within Response to question 8b	100.0%	100.0%	100.0%	100.0%	100.0%

Table A - 103

Case Processing Summary

	Cases					
	Valid		Missing		Total	
	N	Percent	N	Percent	N	Percent
Reasons for response to 8c * Response to question 8c	114	100.0%	0	.0%	114	100.0%

Table A - 104

Reasons for response to 8c * Response to question 8c Crosstabulation

			Response to question 8c				Total
			Function	Not a function	I don't know	NR	
Reasons for response to 8c	Colloquial definition	Count		27			27
		% within Response to question 8c		60.0%			23.7%
	Colloquial definition wrongly used	Count	7				7
		% within Response to question 8c	22.6%				6.1%
	Specific visual hints	Count	3	2			5
		% within Response to question 8c	9.7%	4.4%			4.4%
	One to one and onto ness	Count	5	1			6
		% within Response to question 8c	16.1%	2.2%			5.3%
	Other	Count	8	2			10
		% within Response to question 8c	25.8%	4.4%			8.8%
	No explanation	Count	8	13	32	6	59
		% within Response to question 8c	25.8%	28.9%	100.0%	100.0%	51.8%
Total		Count	31	45	32	6	114
		% within Response to question 8c	100.0%	100.0%	100.0%	100.0%	100.0%

Table A - 105

Case Processing Summary

	Cases					
	Valid		Missing		Total	
	N	Percent	N	Percent	N	Percent
Reasons for response to 8d * Response to question 8d	114	100.0%	0	.0%	114	100.0%

Table A - 106

Reasons for response to 8d * Response to question 8d Crosstabulation

			Response to question 8d				
			Function	Not a function	I don't know	NR	Total
Reasons for response to 8d	Colloquial definition	Count	15				15
		% within Response to question 8d	19.7%				13.2%
	Colloquial definition wrongly used	Count		2			2
		% within Response to question 8d		28.6%			1.8%
	Specific visual hints	Count	5	1			6
		% within Response to question 8d	6.6%	14.3%			5.3%
	One to one and onto ness	Count	5				5
		% within Response to question 8d	6.6%				4.4%
	Constant function	Count	23			1	24
		% within Response to question 8d	30.3%			25.0%	21.1%
	Other	Count	6	1	1		8
		% within Response to question 8d	7.9%	14.3%	3.7%		7.0%
	No explanation	Count	22	3	26	3	54
		% within Response to question 8d	28.9%	42.9%	96.3%	75.0%	47.4%
Total		Count	76	7	27	4	114
		% within Response to question 8d	100.0%	100.0%	100.0%	100.0%	100.0%

Table A - 107

The results from these crosstabulations are summarized in Table A – 108 and A – 109 below. Students' verbal explanations for each category are also given:

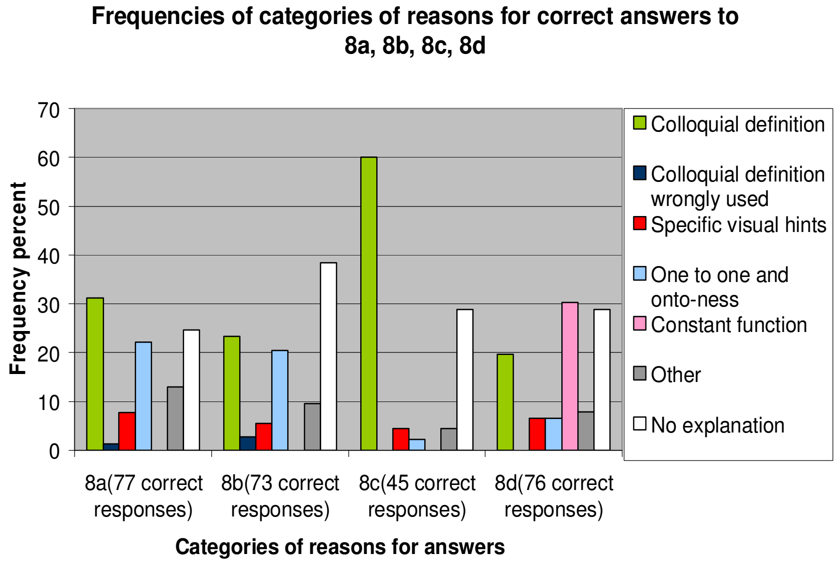


Table A - 108

Colloquial definition:

8a: "there aren't elements left in the domain".

8b: "every element in *C* has a value in *D*".

8c: "an element in *A* can not be assigned to more than one element in *B*".

8d: "every element in *A* takes the same element, 6, in *B*".

Colloquial definition wrongly used:

8a: "elements in the domain and range, all of them are assigned".

8b: "(it's a function), 1 and 4 are joined to 2".

Specific visual hints:

8a: "names are connected to numbers".

8b: "(it's a function) since two lines can intersect with each other".

8c: "(it's a function) since the notations are correct".

8d: "(it's a function) since the notations are correct".

One-to-one and onto-ness:

Students considered 8a, 8b, 8c as a function since they are one-to-one, onto or one-to-one and onto. One student did not consider 8c as a function since it is not one-to-one:

"it's not one-to-one. One element should be used once".

Constant function: In this category, there are students who considered the given items as functions since they are constant functions.

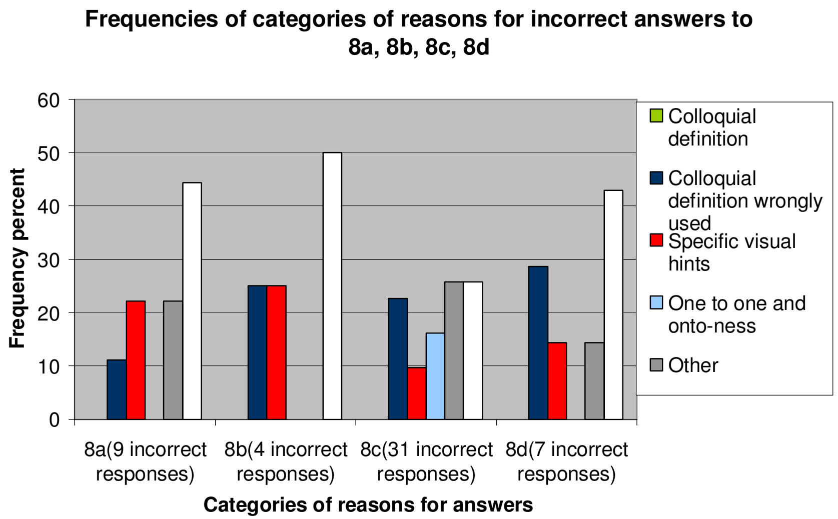


Table A - 109

Colloquial definition wrongly used:

8a: “this is not a function, because to be a function every element should go to a unique element”.

8b: “(it’s not a function) since there is an element in *D*”.

8c: “(it’s a function) an element in the domain can be assigned to more than one element in the range”.

8d: “(it’s not a function) since an element in the range is assigned to the domain”.

Specific visual hints:

8a: “(it’s not a function). No element intersects one another”.

8b: “(it’s not a function). 1, 4, 2 intersect”.

8c: “(it’s not a function) since the numbers are not equal”.

8d: “(it’s not a function) since the sign at the top goes to the reverse direction”.

One-to-one and onto-ness:

8c: “one-to-one function. Every element of *A* is assigned to an element of *B*. It’s also an onto function”.

“it’s an onto function”.

“it is a one-to-one and onto function”.

A2.6 – Frequencies for the number of yes responses to the three forms of constant function

Frequencies

Statistics

Number of yes responses to three constant functions

N	Valid	113
	Missing	1

Table A - 110

Number of yes responses to three forms of constant functions

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	0	72	63.2	63.7	63.7
	1	15	13.2	13.3	77.0
	2	16	14.0	14.2	91.2
	3	10	8.8	8.8	100.0
	Total	113	99.1	100.0	
Missing	System	1	.9		
Total		114	100.0		

Table A - 111

A2.7 – Frequencies of total number of correct answers to questions 3, 4, 6, 7, 8:

Frequencies

Statistics

		Number of correct answers to question 3	Number of correct answers to question 4	Number of correct answers to question 3 and 4	Number of correct answers to question 6	Number of correct answers to question 7	Number of correct answers to question 8	Number of correct answers to all closed questions
N	Valid	114	113	113	113	114	114	112
	Missing	0	1	1	1	0	0	2

Table A - 112

Frequency Tables

Number of correct answers to question 3 (a total of five items)

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	No correct answers	43	37.7	37.7	37.7
	1 Correct answer	27	23.7	23.7	61.4
	2 Correct answers	19	16.7	16.7	78.1
	3 Correct answers	13	11.4	11.4	89.5
	4 Correct answers	10	8.8	8.8	98.2
	5 Correct answers	2	1.8	1.8	100.0
	Total	114	100.0	100.0	

Table A - 113

Number of correct answers to question 4 (a total of five items)

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	No correct answers	32	28.1	28.3	28.3
	1 Correct answer	27	23.7	23.9	52.2
	2 Correct answers	39	34.2	34.5	86.7
	3 Correct answers	11	9.6	9.7	96.5
	4 Correct answers	4	3.5	3.5	100.0
	Total	113	99.1	100.0	
Missing	System	1	.9		
Total		114	100.0		

Table A - 114

Number of correct answers to question 3 and 4 (a total of ten items)

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	No correct answers	26	22.8	23.0	23.0
	1 Correct answer	8	7.0	7.1	30.1
	2 Correct answers	18	15.8	15.9	46.0
	3 Correct answers	24	21.1	21.2	67.3
	4 Correct answers	12	10.5	10.6	77.9
	5 Correct answers	15	13.2	13.3	91.2
	6 Correct answers	4	3.5	3.5	94.7
	7 Correct answers	4	3.5	3.5	98.2
	8 Correct answers	2	1.8	1.8	100.0
	Total	113	99.1	100.0	
Missing	System	1	.9		
Total		114	100.0		

Table A - 115

Number of correct answers to question 6 (a total of seven items)

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	No correct answers	31	27.2	27.4	27.4
	1 Correct answer	17	14.9	15.0	42.5
	2 Correct answers	11	9.6	9.7	52.2
	3 Correct answers	13	11.4	11.5	63.7
	4 Correct answers	21	18.4	18.6	82.3
	5 Correct answers	12	10.5	10.6	92.9
	6 Correct answers	6	5.3	5.3	98.2
	7 Correct answers	2	1.8	1.8	100.0
	Total	113	99.1	100.0	
Missing	System	1	.9		
Total		114	100.0		

Table A - 116

Number of correct answers to question 7c (a total of three items)

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	No correct answers	42	36.8	36.8	36.8
	1 Correct answer	23	20.2	20.2	57.0
	2 Correct answers	28	24.6	24.6	81.6
	3 Correct answers	21	18.4	18.4	100.0
	Total	114	100.0	100.0	

Table A - 117

Number of correct answers to question 8 (a total of four items)

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	No correct answers	28	24.6	24.6	24.6
	1 Correct answer	8	7.0	7.0	31.6
	2 Correct answers	9	7.9	7.9	39.5
	3 Correct answers	47	41.2	41.2	80.7
	4 Correct answers	22	19.3	19.3	100.0
	Total	114	100.0	100.0	

Table A - 118

Number of correct answers to all closed questions (a total of twenty four items)

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	0	13	11.4	11.6	11.6
	1	2	1.8	1.8	13.4
	2	5	4.4	4.5	17.9
	3	8	7.0	7.1	25.0
	4	4	3.5	3.6	28.6
	5	4	3.5	3.6	32.1
	6	5	4.4	4.5	36.6
	7	6	5.3	5.4	42.0
	8	7	6.1	6.3	48.2
	9	5	4.4	4.5	52.7
	10	4	3.5	3.6	56.3
	11	5	4.4	4.5	60.7
	12	9	7.9	8.0	68.8
	13	6	5.3	5.4	74.1
	14	16	14.0	14.3	88.4
	15	5	4.4	4.5	92.9
	16	3	2.6	2.7	95.5
	17	2	1.8	1.8	97.3
	18	2	1.8	1.8	99.1
	20	1	.9	.9	100.0
	Total	112	98.2	100.0	
Missing	System	2	1.8		
Total		114	100.0		

Table A - 119

Appendix B - Interview

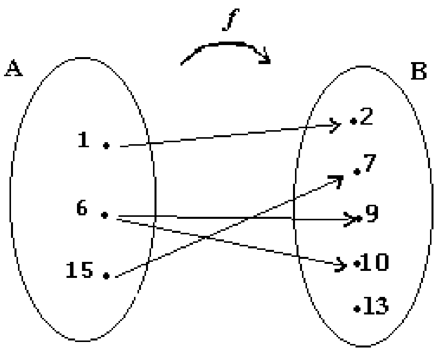
B1 – Interview schedule

I will give you various questions on these papers. You can use this pen if you need to. I want you to tell me what is on your mind. It is not important that you are right or wrong. I want to know what is in your mind so think aloud.

Then I show them the following items and asked them:

Is this a function?...Can you explain why?

1 – Set-correspondence diagram



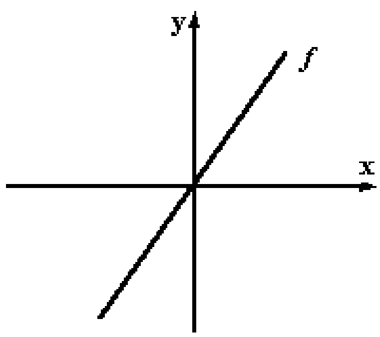
2 – Set of ordered pairs

$A = \{1,2,3,4\}$

$f : A \to R$, $f = \{(1,1), (1,2), (2,2), (3,3), (4,3)\}$

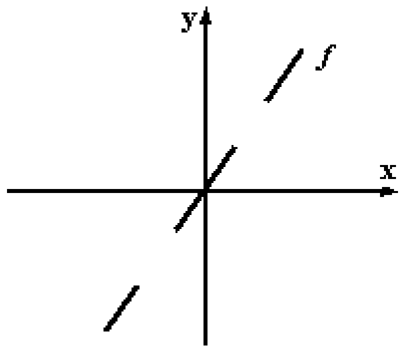
3 – Graph 1

$f : R \to R$

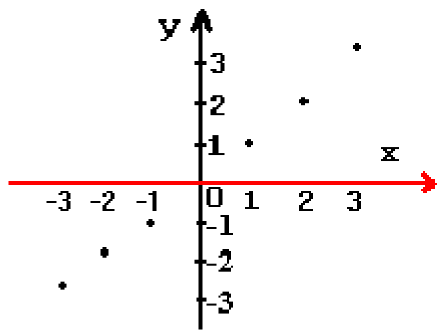


4 – Graph 2

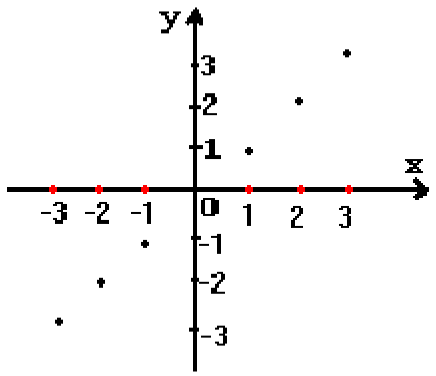
$f : R \to R$



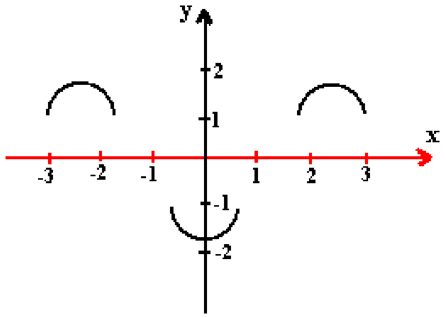
5 – Graph 3



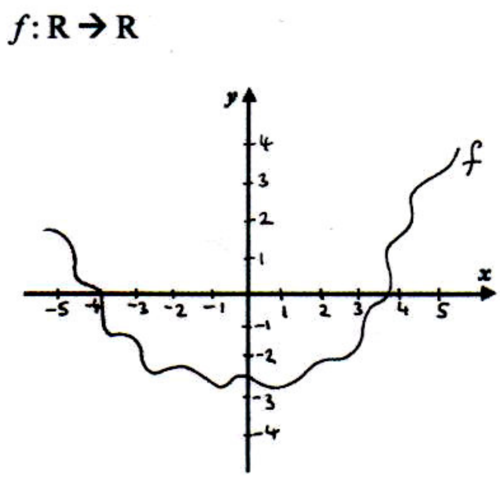
6 – Graph 4



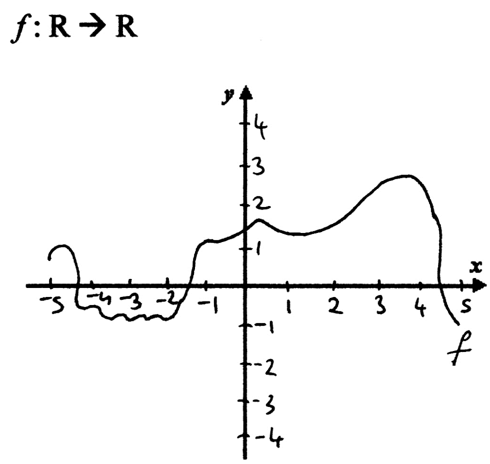
7 – Graph 5



8 – Graph 6 (Non exemplar graph1)

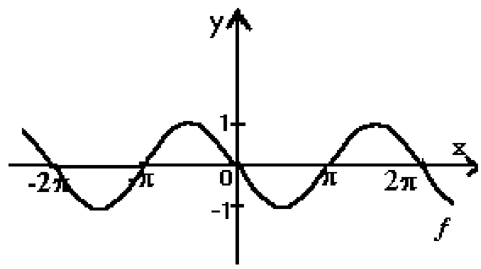


9 – Graph 7 (Non exemplar graph 2)



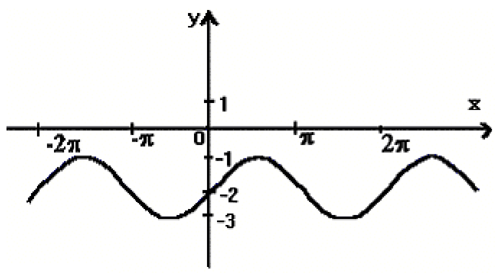
10 – Graph 8

$f : R \to R$



11 – Graph9

$f : R \to R$



12 – Expression 1

$f : R \to R$

$$f(x)=\begin{cases} 1, \text{ if } & x^2-2x+1>0 \\ 0, \text{ if } & x^2-2x+1=0 \\ -1, \text{ if } & x^2-2x+1<0 \end{cases}$$

13 – Expression 2	15 – Expression 4
$y = 5$	$y = 5$ (for all values of x)
14 – Expression 3	16 – Expression 5
$y = 5$ (for $x \le 2$)	$f : R \to R \quad f(x) = \sin x - 2$

Taking account of individual explanations, follow up questions are asked to reveal students' reasoning about each item.

Then students are given the constant function "$f : R \to R$, $f(x) = 5$" and are asked to transform it to a graph, a set-correspondence diagram and a set of ordered pairs:

17 – Transformation 1

Draw the graph of "$f : R \to R$, $f(x) = 5$".

18 – Transformation 2

Draw the set-correspondence diagram of "$f : R \to R$, $f(x) = 5$".

19 – Transformation 3

Write the set of ordered pais for "$f : R \to R$, $f(x) = 5$".

B2 – Interviewees' answers and explanations in the questionnaires

Ali

Definition: 'Is a relation with a range which have no elements left'

Graphs:

3a	3b	3c	3d	3e
Function	Function	Function	Not a function	Function

Table B2-1

When working with graphs in question 3, he found the formulas for each graph. He first found the formula for the first graph, y=sinx. Based on that formula he found the formulas for the other graphs as follows: $f(x) = |\sin x|$ for 3b, $f(\sin x) = x$ for 3c, $f(x) = \sin x - 2$ for 3e. He did not consider item 3d as a function since he could not find the formula for it. He wrote the following for the reason:

'same elements take different values. For example it should be f(5□)=f(2□)'

4a	4b	4c	4d	4e
Not a function	Not a function	Function	Function	Not a function

Table B2-2

When dealing with graphs with coloured domains in question 4, he used the definitional properties. For instance, he did not consider 4a as a function and wrote:

'an element in the domain can not be assigned two different values'

He could also distinguished the elements in the domain which are not assigned to any elements, therefore he did not consider 4e as a function.

Expression:

6a	6b	6c	6d	6e	6f	6g
Function	Function	Function	Function	Function	Function	Function

Table B2-3

When dealing with expressions he could not give any reasons for his answers except the cases for constant functions (6c, 6d, 6e) and item 6g. He considered 6c, 6d, 6e as a function because they are all constant functions. He wrote that he could not see any difference between '$y = 5$' and '$y = 5$ (for all values of x)'. As a reason for '$y = 5$ (for x≥2)' he wrote 'according to the domain, it is a constant function'. He considered 6g as a absolute value function.

Ordered pairs:

7a	7b	7c
Function	Not a function	Function

Table B2-4

In question 7, he correctly answered 7a and 7b by using the definitional properties. However, he incorrectly consider 7c as a function and wrote that it is an identity function.

Set diagrams:

8a	8b	8c	8d
Function	Function	Not a function	Function

Table B2-5

In question 8, he considered 8a as a one to one and onto function and 8b as an onto function. He correctly did not consider 8c as a function since 6 has two different values. He considered 8d as a function since it is a constant function.

Aysel

Definition: 'A≠Ø and B≠ Ø, for relations in (A×B), every element in the domain is assigned to one and only one element in the range and if there is no elements left in the domain then this relation is a function'.

Graphs:

3a	3b	3c	3d	3e
Function	Function	Not a function	Function	Not a function

Table B2-6

For 3a she wrote that elements in the domain are not assigned to more than one element in the range. She could not give any reason for 3b. She did not consider 3c as a function since x=1 has more than one value on the y-axis. As a reason for 3d she referred to 3a.

4a	4b	4c	4d	4e
Not a function	Function	Function	Function	Function

Table B2-7

She did not give any reason for her answers except 4b. She considered 4b as a function since it is the constant function f(x)=0.

Expression:

6a	6b	6c	6d	6e	6f	6g
Function	Not a function	Function	Function	Function	Not a function	Function

Table B2-8

She did not give any reasons for her answers.

Ordered pairs:

7a	7b	7c
Function	Not a function	Function

Table B2-9

She did not give any reason for her answers.

Set diagrams:

8a	8b	8c	8d
Function	Function	Not a function	Function

Table B2-10

Again, she did not give any reasons for her answers.

Ahmet

Definition: 'It is a relation which has a value for any elements in the domain'.

Graphs:

3a	3b	3c	3d	3e
Function	Function	Not a function	Function	Function

Table B2-11

Ahmet could give reasons for his answers by using definitional properties e.g. for 3a he wrote that y takes values between 1 and -1 for $x \in R$. He gave similar reasons for 3b, 3e. For 3d he wrote that for any value of x, $y \in R$. He did not consider 3c as a function since for values of x other than -1 and 1, there may not be any elements $y \in R$.

4a	4b	4c	4d	4e
Not a function	I don't know	I don't know	Function	I don't know

Table B2-12

Ahmet used vertical line test for 4a and 4d. He did not consider 4a as a function and wrote that if lines are drawn parallel to y axis it intersects twice. He considered 4d as a function because lines parallel to y axis intersect once.

Expression:

6a	6b	6c	6d	6e	6f	6g
Not a function	Not a function	Function	Function	Function	Function	Not a function

Table B2-13

He did not consider 6a and 6b by giving examples of values of x where the function is undefined. He considered 6c, 6d, 6e as a function because they are all functions. He did consider 6f as a function because it is absolute value function and takes value of Z^+. He did not consider 6g as a function because for every element of x there are three different values

Ordered pairs:

7a	7b	7c
Function	Not a function	Function

Table B2-14

Ahmet used the definitional properties correctly for 7a and 7b. However, he considered 7c as a function since he could only focus on the uniqueness of the elements in the domain but not the elements left in the domain.

Set diagrams:

8a	8b	8c	8d
Function	Function	Not a function	Function

Table B2-15

For all set diagram, items he could successfully use the definitional properties. He could also focus on 8a as a one to one and onto function and 8d as a constant function.

Arif

Definition: 'By a function, we mean, we can find the corresponding value of an element in the domain and it will be in the range'.

Graphs:

3a	3b	3c	3d	3e
I don't know	Function	I don't know	No response	I don't know

Table B2-16

He could not give any reasons for his answers except 3b which he considered as a constant function.

4a	4b	4c	4d	4e
I don't know	Function	I don't know	Not a function	Not a function

Table B2-17

He did not give any reason for 4a and 4c. He considered 4b as the graph of y=x. for 4d, he wrote that it is not defined at 3/2 (probably thinking of 3.2). For 4e, he could focus on the domain and wrote that there is no corresponding value for 2 and the function is not defined at 2.

Expression:

6a	6b	6c	6d	6e	6f	6g
Function	Function	Function	Not a function	Function	Function	Function

Table B2-18

He incorrectly considered 6a as a function and wrote that for every value the corresponding value can be found because of R $\rightarrow$ R. 6b was considered as signum function. He wrote that it takes the values 1, 0, -1 probably referring to maximum values of x and y. He also wrote that the values of x and y are 1 and 0 and the sum is 1. He considered '$y = 5$' and '$y = 5$ (for all values of x)' as a function but not '$y = 5$ (for x≥2)'. He wrote that the y value has the value 5 and note that he did not know the equation for it. For '$y = 5$ (for all values of x)' he only wrote that it is constant function. The reason why he considered '$y = 5$ (for x≥2)' is that y ca not be 3 or 4. For 6f he found f(1)=3 and wrote that for every value there is another value. He considered 6g as a function because it is a signum function.

Ordered pairs:

7a	7b	7c
Function	Not a function	Function

Table B2-19

Arif draw set diagrams for all sets of ordered pairs. He did not consider 7b as a function because one element in the domain can not have two values. However, he could not focus on the elements left in the domain.

Set diagrams:

8a	8b	8c	8d
Function	Function	Not a function	Function

Table B2-20

Arif used definitional properties successfully for the first three set diagrams. He considered 6d as a function since it is constant function.

Belma

Definition: 'Let $f : A \rightarrow B$. If every element in A is assigned to B then this is called function'.

Graphs:

3a	3b	3c	3d	3e
Function	I don't know	I don't know	I don't know	No response

Table B2-21

Belma could not give any reasons for her answers in question3. She noted that she did not understand or could not explain.

4a	4b	4c	4d	4e
I don't know	Function	Function	I don't know	I don't know

Table B2-22

She could not give explanations for her answers except 4b and 4c. For 4b and 4c she wrote that elements are assigned to each other.

Expression:

6a	6b	6c	6d	6e	6f	6g
I don't know	I don't know	Not a function	Not a function	I don't know	Functio n	Not a function

Table B2-23

Belma could not give any reasons for her answers in question6. For 6g she put 1, 0, -1 in succession for $x^2 - 2x + 1 > 0$, $x^2 - 2x + 1 = 0$, $x^2 - 2x + 1 < 0$. She did not consider this as a function since the three conditions on the domain do not satisfy them.

Ordered pairs:

7a	7b	7c
Function	Not a function	Function

Table B2-24

Belma drew set diagrams for each item in question7. She could not focus on the elements left in the domain for 7c.

Set diagrams:

8a	8b	8c	8d
No response	Not a function	Not a function	Function

Table B2-25

For 8b, 8c and 8d, she correctly used the definitional properties.

Belgin

Definition: Belgin did not write anything for the definition of a function.

Graphs:

3a	3b	3c	3d	3e
No response	No response	I don't know	No response	I don't know

Table B2-26

Belgin did not give any explanations for question 3.

4a	4b	4c	4d	4e
No response	Function	Function	No response	I don't know

Table B2-27

She only gave explanations for 4b and 4c. She wrote that all elements are assigned to each other.

Expression:

6a	6b	6c	6d	6e	6f	6g
I don't know	Not a function	I don't know	I don't know	I don't know	Function	I don't know

Table B2-28

Belgin only gave explanations for 6b and 6f. She did not consider 6b as a function since whatever is given to x and y the result is 1. She considered 6f as a function since it is absolute value function. She wrote that when values are given to x, one positive one negative value are found, therefore every number is definitely assigned to a number.

Ordered pairs:

7a	7b	7c
Function	Not a function	Function

Table B2-29

Belgin successfully used the definitional properties for 7a and 7b. However, she did not consider the elements left in the domain and considered this as a one to one and onto function.

Set diagrams:

8a	8b	8c	8d
Function	Function	Not a function	Function

Table B2-30

Belgin used definitional properties for items in question8. As well as the definitional properties she considered 8d as constant function.

Cem

Definition: 'functions are line segments that are constructed by the intersection of two line segments. It is a group of the elements of two sets'.

Graphs:

3a	3b	3c	3d	3e
Not a function	Function	Function	I don't know	Not a function

Table B2-31

Cem did not consider 3a as a function since the graph intersects the x axis more than once. He did not give any explanations for 3b, 3c and 3d. He did not consider 3e as a function since the graph is below the x axis.

4a	4b	4c	4d	4e
Not a function	Function	Not a function	Not a function	Function

Table B2-32

Cem could not use the definitional properties in question 4.

Expression:

6a	6b	6c	6d	6e	6f	6g
Not a function	Not a function	Not a function	Not a function	Function	Function	Function

Table B2-33

Among three forms of constant functions he only considered '$y = 5$ (for all values of x)' as a function since 'for all values of x' is mentioned and wrote that the line is drawn according to the values of x.

Ordered pairs:

7a	7b	7c
Function	Function	Not a function

Table B2-34

Cem could not use the definitional properties for sets of ordered pairs. He considered 7a and 7b as a function since he could see the values in the domain and range in the sets of ordered pairs. He did not give any reason for 7c.

Set diagrams:

8a	8b	8c	8d
Function	Function	Not a function	Function

Table B2-35

Cem could not use the definitional properties for set diagrams. For 8a and 8b he only wrote that elements in two sets are joined together.

Demet

Definition: 'I don't know'

Graphs:

3a	3b	3c	3d	3e
Not a function	I don't know	Not a function	Not a function	I don't know

Table B2-36

For 3a Demet wrote that it is drawn wrong and it is continuously on the same surface probably meant that the graph repeats itself. She gave the same reason for 3c. She did not explain the other items.

4a	4b	4c	4d	4e
I don't know	I don't know	I don't know	I don't know	I don't know

Table B2-37

Expression:

6a	6b	6c	6d	6e	6f	6g
I don't know	I don't know	I don't know	I don't know	I don't know	I don't know	I don't know

Table B2-38

Ordered pairs:

7a	7b	7c
I don't know	I don't know	I don't know

Table B2-39

Set diagrams:

8a	8b	8c	8d
I don't know	I don't know	I don't know	I don't know

Table B2-40

Deniz

Definition: ' I don't know'

Graphs:

3a	3b	3c	3d	3e
I don't know	Function	Not a function	Not a function	Not a function

Table B2-41

Deniz did not give any explanations for 3a and 3b. He did not consider 3c and 3d as a function since the shape of the graph is very different. He did not consider 3e as a function since it does not pass through the x axis.

4a	4b	4c	4d	4e
I don't know	Not a function	Function	Not a function	Not a function

Table B2-42

Deniz did not consider 4b as a function since x axis is a different line. That is probably because of the way x axis is drawn in a different colour. He considered 4c as a function and wrote that there is a good intersection. He did not give a clear explanation for 4d. He did not consider 4e as a function since there are gaps in the intersection.

Expression:

6a	6b	6c	6d	6e	6f	6g
Not a function	Function	I don't know	Function	I don't know	I don't know	I don't know

Table B2-43

Deniz did not consider 6a as a function since a function can not include root. He considered 6b as a function since there is a result for it. Among three cases of constant functions, he only considered $y = 5$ (for x≥2) as a function since there is an explanation for it probably referring to explanation in the brackets. He did not gave any explanation for 6f. For 6g he wrote that it was too long.

Ordered pairs:

7a	7b	7c
Function	Not a function	Not a function

Table B2-44

Deniz could not use definitional properties for sets of ordered pairs. He did not give any clear explanations for 7a and 7b. He did not consider 7c as a function since there can not be height in a function.

Set diagrams:

8a	8b	8c	8d
I don't know	Not a function	I don't know	Not a function

Table B2-45

Deniz did not use the definitional properties for set diagrams. For 8b he used the uniqueness property in the opposite direction and wrote that 2 is intersects with 1 and 4.

B3 – Labeling students' responses to prepare the grid in table 8.1

1. Set-correspondence diagram

	Function or not	Explanation	Label
Ali	Not a function	Colloquial definition	CD
Aysel	Not a function	Colloquial definition	CD
Ahmet	Not a function	Colloquial definition	CD
Belma	Not a function	Colloquial definition	CD
Belgin	Not a function	Colloquial definition	CD
Arif	Not a function	Colloquial definition	CD
Cem	Function	Colloquial definition wrongly used	CDW
Deniz	Not a function	Visual hints	EBF
Demet	Not a function	Visual hints	EBF

Table B3 - 1

2. Set of ordered pairs

	Function or not	Explanation	Label
Belma	Not a function	Colloquial definition	CD
Aysel	Not a function	Colloquial definition with an explanation with vertical line test	CD
Ahmet	Not a function	Colloquial definition using a set-correspondence diagram	CD-SD
Arif	Not a function	Colloquial definition using a set-correspondence diagram	CD-SD
Ali	First considered as a function then changed his mind	Colloquial definition wrongly used. When reminded of 1 having two different values, he correctly used the colloquial definition.	CDW-CD
Cem	Not a function	Colloquial definition wrongly used	CDW
Deniz	Not a function	Numbers of elements of ordered pairs is not equal to numbers of elements of the domain.	EBF
Belgin	Not a function	No explanation	---
Demet	Function	Ploting a point and joining it to the origin.	OTH

Table B3 - 2

3. Straight line graph

	Function or not	Explanation	Label
Ali	Function	Exemplar based focus followed by colloquial definition/ use of set diagram	EBF-CD
Aysel	Function	Exemplar based focus followed by colloquial definition	EBF-CD
Ahmet	Function	Vertical line test with reference to the colloquial definition	VLT-CD
Belma	Function	Action on the graph (assigning numbers on x to the numbers on y)	OTH
Belgin	Function	Action on the graph (assigning numbers on x to the numbers on y)	OTH
Arif	Function	Action on the graph (confused with the domain and range / assigning numbers on x and y with each	DRC

		other)	
Cem	Function	Visual hints	EBF
Demet	Function	No explanation	---
Deniz	Could not decide	No explanation	---

Table B3 - 3

4. Straight line in three pieces

	Function or not	Explanation	Label
Ali	Not a function	Colloquial definition	CD
Aysel	Not a function	Colloquial definition	CD
Arif	Not a function	Colloquial definition	CD
Ahmet	Function (on a restricted domain)	Colloquial definition by explaining it with a set-correspondence diagram	CD-SD
Belma	Not sure	Visual hints	EBF
Demet	Not a function	Visual hints	EBF
Deniz	Not a function	Visual hints	EBF
Cem	Not a function	Visual hints	EBF
Belgin	Function	No clear explanation	---

Table B3 - 4

5. Points on a line with the domain of projected points

	Function or not	Explanation	Label
Ali	Function	Colloquial definition	CD
Arif	Function	Colloquial definition	CD
Ahmet	Function	Colloquial definition	CD
Aysel	Function	Colloquial definition followed by vertical line test	CD-VLT
Belgin	Function	Colloquial definition wrongly used considering y axis as the domain and x axis as the range	CDW
Cem	Function	Finding the corresponding values of the numbers in the domain	OTH
Demet	Function	Drawing a straight line through the graph	EBF
Deniz	Function	Drawing a straight line through the graph	EBF
Belma	Function	Drawing a straight line through the graph	EBF

Table B3 - 5

6. Points on a line

	Function or not	Explanation	Label
Ali	Not a function	Colloquial definition	CD
Aysel	Not a function	Colloquial definition	CD
Ahmet	Function	Colloquial definition wrongly used	CDW
Belgin	Function	Colloquial definition wrongly used	CDW
Demet	Function	Drawing a straight line	EBF
Arif	Function	Drawing a straight line	EBF
Deniz	Function	Considering the graph the same as the earlier graph which has a different domain	EBF
Cem	Function	Considering the graph the same as the earlier graph which has a different domain	OTH

Belma	Function	Looking for a formula	OTH

Table B3 - 6

7. Graph of smiley face

	Function or not	Explanation	Label
Aysel	Not a function	Colloquial definition	CD
Ali	Function/not a function	Colloquial definition wrongly used/ignoring elements left in the domain/when mentioned 1 in the domain changed his mind	CD
Arif	Function/not a function	Used colloquial definition when reminded of -1 on x axis.	CD
Ahmet	Not sure	Vertical line test/drawing of set-correspondence diagrams	VLT-SD
Demet	Not sure	Focused on x axis under the areas of three pieces of the graph/no further explanation	EBF
Deniz	Not a function/ function	The numbers on y axis is not the same as the numbers on x axis	EBF
Belma	Function	Exemplar based response (the graph is like a parabola)	EBF
Cem	Not a function	The shape is different	EBF
Belgin	Not sure	The shape is diferent	EBF

Table B3 - 7

8. Non-exemplar graph 1

	Function or not	Explanation	Label
Ali	Not a function	Vertical line test/colloquial definition	VLT-CD
Aysel	Not a function	Vertical line test/colloquial definition	VLT-CD
Ahmet	Not a function	Vertical line test/use of set-correspondence diagram/colloquial definition	VLT-SD-CD
Belma	Not a function	Numbers on axes are irrational.	EBF
Belgin	Function	Finding corresponding values of numbers on x axis	OTH
Arif	Not a function/ change to function	Finding corresponding values of numbers on x axis	OTH
Cem	Function	Visual hints. Numbers on x axis (-3, -2, -1, 1, 2, 3) are inside the graph	EBF
Deniz	Not a function	General appearance of the graph	EBF
Demet	Not a function	General appearance of the graph	EBF

Table B3 - 8

9. Non-exemplar graph 2

	Function or not	Explanation	Label
Ali	Function	Colloquial definition	CD
Aysel	Not a function	Colloquial definition	CD
Ahmet	Not a function	Coloquail definition/Vertical line test	CD-VLT
Belma	Not a function	There are two x- intercepts and they are rational numbers	OTH

Arif	Not a function	No formula to find corresponding values/relating x and y values without any particular direction	DRC
Cem	Not a function	General appearance of the graph	EBF
Deniz	Not a function	General appearance of the graph	EBF
Demet	Not a function	General appearance of the graph	EBF
Belgin	Function	Graph has a formula/Could not tell the formula	EBF

Table B3 - 9

10. Graph of $f(x) = -\sin x$

	Function or not	Explanation	Label
Ahmet	Function	Vertical line test followed by colloquial definition explained by set-correspondence diagram	VLT-CD-SD
Aysel	Function	Exemplar based focus/definitional properties/action on the graph (assigning values of x to the graph, but not to the y axis).	EBF
Ali	Function	Exemplar based focus (recognizing as a sine function) followed by action on the graph.	EBF
Belma	Function	Exemplar based focus (recognizing as a sine function because of □).	EBF
Belgin	Function	Exemplar based focus (recognizing as a sine function because of □).	EBF
Arif	Function	Exemplar based focus/familiarity to parabolas	EBF
Deniz	Not a function	Visual hints irrelevant to definitional properties.	EBF
Demet	Not a function	General appearance unfamiliar	EBF
Cem	Not sure	General appearance unfamiliar	EBF

Table B3 - 10

11. Graph of $f(x) = \sin x - 2$

	Function or not	Explanation	Label
Ali	Function	Colloquial definition	CD
Aysel	Function	Colloquial definition	CD
Ahmet	Function	Colloquial definition by applying vertical line test and drawing a set diagram	CD-VLT
Arif	Function	Assigning x and y values to each oher.	OTH
Belgin	Function	General shape of the graph (increases and decreases).	EBF
Belma	Not a function	The graph passes through y axis only.	EBF
Cem	Not a function	The graph passes through y axis only.	EBF
Deniz	Function/not a function	The graph passes through y axis only.	EBF
Demet	Not a function	The graph is below x axis.	EBF

Table B3 - 11

12. Split domain function

	Function or not	Explanation	Label
Ali	Function	Recognizing as signum function. Correct graph. Set-correspondence diagram.	EBF-GR-SD
Ahmet	Function	Recognising as signum function. Although confused about the domain, he drew a correct graph applying vertical line test.	EBF-GR-VLT
Aysel	Function	Recognizing as condinitional function. Wrong graph. Set-correspondence diagram assigning values less than 1 to -1, 1 to 0, values greater than 1 to 1.	EBF-WGR
Belma	Function	Recognizing as split-domain function.	EBF
Belgin	Not a function	Substituted -1, 0, 1 in x^2-2x+1.	DRC
Arif	Function	Recognizing as signum function	EBF
Cem	Function	Notational hint: f(x).	EBF
Deniz	Function	Relating the numbers on the right hand side of the expressions $x^2-2x+1>0$, $x^2-2x+1=0$, $x^2-2x+1<0$ to the numbers of the range, 1, 0, -1	OTH
Demet	Not a function	Specific hints. 'we can't take a square of a function'.	OTH

Table B3 - 12

13. $y=5$

	Function or not	Explanation	Label
Ali	Function	Drawing the graph/constant function	GR-CF
Ahmet	Function	Drawing the graph/constant function	GR-CF
Aysel	Not a function/ function	Specifying the domain as R/Drawing the graph/constant function	GR-CF
Arif	Function	Drawing a set-correspondence diagram	SD
Demet	Function	Marking (-5,0) as 5 and joining it to 5	WGR
Deniz	Function	No explanation	---
Cem	Function	y equals to 5	OTH
Belma	Not sure	Drawing $y=5$ /putting values for y	WGR
Belgin	Not sure	Looked for $f(x)$	EBF

Table B3 - 13

14. $y=5(for x\leq 2)$

	Function or not	Explanation	Label
Arif	Function	Colloquial definition / Assigning values less than or equal to 2 to 5.	CD
Ali	Function	Recognising as a constant function /Drawing the graph	CF-GR
Aysel	Function	First asking the domain, drew the graph correctly.	GR
Ahmet	Function	Assigning values less than 2 to 5 and drawing the graph correctly/Drawing a set-correspondence diagram.	GR
Belma	Not sure	Drawing the graph for all values of x.	WGR

Cem	Function	Considering (for x<2) as a condition with no reference to definitional properties	OTH
Belgin	Not sure	Looked for *f* notation. Could not respond.	EBF
Deniz	Could not decide	There is no relation between $y=5$ and $y=5$ (x≤2).	---
Demet	Function/not a function	5 is not less than two	OTH

Table B3 - 14

15. $y=5$ (for all values of x)

	Function or not	Explanation	Label
Ali	Function	Assigning all values of x to 5. Drawing the graph.	CD-GR
Aysel	Function	Assigning all real numbers to 5. Drawing the graph.	CD-GR
Ahmet	Function	Constant function. Assigning all numbers to 5. Confused by the domain.	CF-CD
Belma	Function	Recognising it as the same as the other two. Drawing the graph.	GR
Belgin	Not sure	Confused by the domain and range. Looking for a formula to substitute numbers to get 5.	EBF
Arif	Function	Assigning all values of x to 5.	CD
Cem	Function	Looking for specific values of x.	OTH
Deniz	No answer	No explanation	---
Demet	Not sure	Giving values for y.	OTH

Table B3 - 15

16. $f(x)=\sin x-2$

	Function or not	Explanation	Label
Ali	Function	Colloquial definition	CD
Aysel	Function	Colloquial definition	CD
Ahmet	Function	Exemplar based focus/colloquial definition	EBF-CD
Belgin	Function	Exemplar based focus (trigonometric function)	EBF
Arif	Function	Finding *f*(0)	OTH
Deniz	Function	Notational hints	EBF
Demet	Not a function	Drawing a wrong graph	WGR
Cem	Not a function	No explanation	---
Belma	Not sure	I don't know very well	---

Table B3 - 16

17. Drawing the graph of $f:R\rightarrow R$, $f(x)=5$

	Drawing	Labe

		1
Ali	Correct graph	✓
Aysel	Correct graph	✓
Ahmet	Correct graph	✓
Arif	Draws the graph between $-2 \leq x \leq 2$	✗
Belma	Draws the graph of $x=5$	✗
Belgin	Marking 5 on positive x and y axes	✗
Demet	Marking 0 and 5 on x axis and joining them	✗
Deniz	Draws a straight line through (5,0) and (0,5)	✗
Cem	Labeling x and y axes and trying to plot points	✗

Table B3 - 17

18. Drawing the set-correspondence diagram of $f: R \rightarrow R$, $f(x) = 5$

	Drawing	Label
Ali	Correct diagram. Saying that there are infinite number of elements in the first set assigned -∞ and +∞ in the first set to 5 in the second set.	✓
Aysel	Correct diagram. Assigning x_1, x_2, x_3, x_4 (which represents all reals) in the first set to 5 in the second set.	✓
Ahmet	Correct diagram. Assigning –1, 1, $\sqrt{3}$, $\sqrt{2}$ (which represents all reals) in the first set to 5 in the second set.	✓
Arif	Correct diagram.	✓
Belma	Could not draw.	✗
Belgin	Could not draw. Confusion between domain and range.	✗
Demet	Wrong diagram.	✗
Deniz	Wrong diagram. Assinging 1 to 1, 2 to 2 and so on up to 5. Changed his mind and assigned on 1, 2, 3, 4, 5 to 5.	✗
Cem	Wrong diagram.	✗

Table B3 - 18

19. Write the set of ordered pairs for $f: R \rightarrow R$, $f(x) = 5$

	Drawing	Label
Aysel	$f(x)$={…(x_1,5)…(x_2,5)…}	✓
Ali	{…(1,5)..(2,5)..(3,5)….}	✓
Ahmet	{(1,5), (0,5), (½,5),…}	✓
Arif	$f(x)$=(-1,5),(1,5),(2,5),(3,5),(4,5)…	✗
Belma	f={(5,1) (5,2) (5,3)…} followed by f={(1,5), (2,5)…}	✗
Deniz	$f(x) = 5$ ([x,1) (x,2) (x,3) (x,4) (x,5)]	✗
Cem	$f(x)$=(1,1), (1,2), (1,3), (1,4), (1,5)	✗
Belgin	Could not write	---
Demet	Could not write	---

TableB3-19