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A study of exactness for discrete groups

by

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ABSTRACT

FACULTY OF ENGINEERING, SCIENCE AND MATHEMATICS

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A STUDY OF EXACTNESS FOR DISCRETE GROUPS

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We recall the concepts of exactness for both C^*-algebras and groups. We explore some new properties linked or equivalent to exactness, including Property A, a second property we term Property O, and Hilbert space compression [GK2, O, Yu]. We use geometric methods to show that a variety of groups satisfy these properties. We then deduce that those groups are exact.

In particular we show that Properties O and A are equivalent. We show that the integers, groups of subexponential growth, amenable groups and free groups satisfy Property O by constructing a family of Ozawa kernels for each case. To construct these families we exploit growth properties of the integers and groups of subexponential growth, Følner's criterion for amenable groups and geometric properties of the Cayley graph for free groups. For each of these groups we deduce that they are exact and have Property A. Finally we turn to Hilbert space compression to prove our main theorem that groups acting properly and cocompactly on CAT(0) cube complexes are exact and have Property A.

Contents

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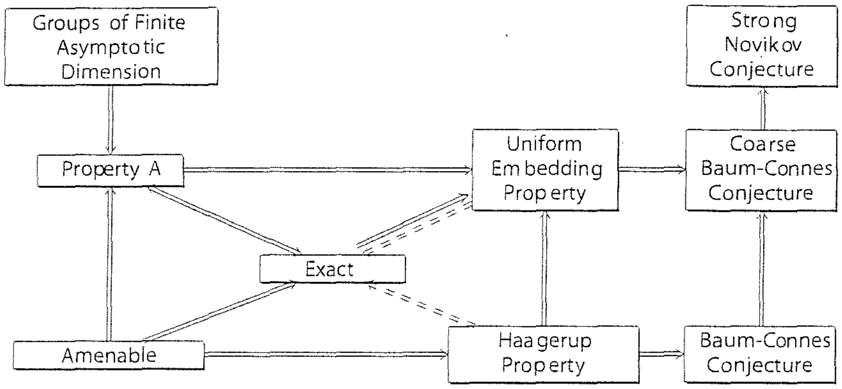
Chapter 1

Introduction

1.1 Introduction

A considerable part of recent pure mathematics rsearch has been centred on the Baum Connes conjecture. The Baum Connes conjecture was formulated in the early 80s by Paul Baum and Alain Connes. It conjectures a link between the K-theory of the reduced C^*-algebra of a group and the K-homology of the corresponding space of proper actions of that group. It has provoked a lot of interest as it implies some other famous conjectures such as the Novikov Conjecture. Furthermore it ties together ideas from seemingly different disciplines such as geometric group theory and analysis. Through work inspired by the Baum Connes Conjecture, links have been made between many other different properties and ideas both analytic and geometric. This is shown in more detail on the diagram on the following page. In particular, this includes work by Higson, Kasparov [HK], Yu [Yu] and others.

The aim of this thesis is to explore a property which strongly links many of these ideas: exactness of a group. The importance of the property of exactness can be visualised in the following diagram. Full arrows represent proven facts. Dotted arrows represent conjectures.



The top line of this diagram represents more geometric ideas which were linked together in [Yu]. Yu introduced an equivalent property to exactness called Property A. This property can be thought of as a weaker from of Følner's condition which characterises amenable groups. It implies that the group is uniformly embeddable in a Hilbert space which in turn implies that the group satisfies the coarse Baum Connes Conjecture.

It is worth noting that although we know that exactness also directly implies Hilbert Space embeddability, the converse is not known to be true. However in [GK2], Guentner and Kaminker have introduced a new invariant called the Hilbert space compression of a group. Roughly speaking, this measures the amount of distortion that necessarily occurs when embedding the group in a Hilbert space. This invariant is linked to exactness since they showed that if it is strictly greater than 1/2, then the group is exact, i.e. its reduced C^*-algebra is exact. However the converse is not true, since there exist exact groups for which the Hilbert space compression is less than 1/2. For example the wreath product of $\mathbb{Z}$ with its own wreath product on itself, denoted $\mathbb{Z}\, wr\, (\mathbb{Z}\, wr\, \mathbb{Z})$, is amenable and hence exact yet its Hilbert space compression is less than 1/2 [AGS, Cor 1.10].

The bottom row represents more analytic ideas. For example, amenable groups have an invariant mean and are known to be exact. They also satisfy the Haagerup property, namely that they admit a proper isometric action on some affine Hilbert space. The Haagerup property links back to both the Uniform embedding property and the Baum Connes Conjecture.

Exactness of a group was first defined as a property of the group's C^*-algebra. However more recent research has shown that exactness can be characterised by other criteria[GK][O][Yu] which allow a more geometric approach.

1.2 Aims and results of this thesis

Proving that groups are exact by analytic methods is often a lengthy and complex process. For example, even proving that a simple group such as the integers is exact is non-trivial. Exactness for the integers was first proved by Kirchberg and Wassermann by showing that the integers admit an amenable action on their Stone-Čech compactification. Our aim is to use more geometric methods to show that a selection of groups satisfy one of several possible properties which have recently been shown to be equivalent to exactness [GK2][O][Yu].

This thesis first explores some of these properties [GK2, O, Yu]. Yu was the first to introduce a geometric property equivalent to exactness. He introduced the now well known Property A which can be thought of as a weaker form of Følner's criterion. In the same paper [Yu], he showed that this property is equivalent to uniform embedding into a Hilbert space and that it implies both the Coarse Baum Connes and the strong Novikov conjectures.

The next property we study is due to Ozawa [O] and we term it Property O. This property is also equivalent to exactness. It asserts the existence of a family of real valued functions (Ozawa kernels) on the group with certain properties. The existence of families of Ozawa kernels has been used by Guentner and Kaminker to prove a theorem relating asymptotic compression and exactness [GK2, Thm 3.2]. However no explicit examples of families of Ozawa kernels can be found in the literature.

Another approach to exactness was recently developed by Guentner and Kaminker. In [GK2] they studied the Hilbert space compression of a discrete group, a real valued invariant which measures the distortion necessary to embed the group into a Hilbert space. They showed that for a finitely generated discrete group if the Hilbert space compression is strictly greater than 1/2 then the group is exact, ie its reduced C^*-algebra is exact. They illustrated their approach by proving that the Hilbert space compression for the free group of rank 2 is 1 thus giving a new proof that this group is exact.

In this thesis, after first studying these three properties we go on to show that Properties A and O are equivalent. Although both are equivalent to exactness and therefore equivalent to each other, it was unclear as to how they were directly related. We prove the following theorem by showing that the existence of a Property A type function implies the existence of a family of Ozawa kernels and vice versa:

Theorem 1. *Property A is equivalent to Property O.*

We then go on to use some of these properties in order to show that a selection of groups are exact.

We start with an easy group to define and understand: the integers. This is known to be an exact group, however proving this is non trivial and requires the use of analytic concepts such as amenable actions and the Stone-Čech compactification of a group. We will adopt a different strategy and use purely geometric features of the group to construct a family of Ozawa kernels. By doing this we prove the following theorem:

Theorem 2. *The family of kernels u_N constructed in section 5.1 forms a family of Ozawa kernels for the integers $\mathbb{Z}$ and so they satisfiy property O.*

As a corollary of this theorem, we can deduce

Corollary 1.2.1. *$\mathbb{Z}$ is an exact group and therefore satisfies property A.*

Our construction of a family of Ozawa kernels for the integers relies solely on a growth property of the group. We are thus able to extend our result to a far larger class of groups which have a similar characteristic: those with subexponential growth. We prove the following:

Theorem 3. *The family of kernels u_N constructed in section 5.2 forms a family of Ozawa kernels for groups of subexponential growth and so they satisfy property O.*

The following corollary immediately follows:

Corollary 1.2.2. *Groups of subexponential growth are exact and therefore satisfy property A.*

The previous two examples rely on growth properties: a relationship between the volume and the surface of a ball of some radius n. However, in both cases the sequence of balls of radius n forms a Følner sequence. Roughly speaking, Følner sequences satisfy a property which relates the size of a set to that of its intersection with another set. The existence of a Følner sequence is a characteristic of a large class of groups called amenable groups.

We adapt the previous constructions on balls of radius n to a construction on Følner sets. This allows us to prove the following theorem:

Theorem 4. [C, Thm 2] *The family of kernels u_N constructed in section 5.3 forms a family of Ozawa kernels for amenable groups and so they satisfy property O.*

We can also deduce the following corollary

Corollary 1.2.3. [C, Cor 3.4] *Amenable groups are exact and therefore have Yu's Property A.*

Another well known example of exact groups is the class of free groups. Although easy groups to define, showing that they are exact is complex. In [KW, Cor 3.5], Kirchberg and Wassermann prove that these groups are exact by showing that the reduced C^*-algebra of the free group of rank 2 can be embedded in that of $Z_2 * Z_3$, which in turn is embeddable in the Cuntz algebra $\mathcal{O}_\epsilon$. This algebra is nuclear [Wa, p18, 2.1] and any nuclear algebra is exact [Wa, Property 2.5.1.]. Since a subalgebra of an exact algebra is exact [Wa, Prop 2.6], this implies that the reduced C^*-algebra of the $\mathbb{F}_2$ is exact, and hence the group itself is exact. They then show that $C^*(\mathbb{F}_n) \subseteq \mathbb{F}_2$, which implies that free groups of rank n are exact. The reasoning is the same for $\mathbb{F}_\infty$ with countably many generators. Finally they show that $\mathbb{F}_\infty$ with infinitely many free generators is exact by showing that subgroups generated by countably many generators are exact and that the collection of these forms a lattice under inclusion [KW, Thm 2.5]. This proof is very analytic in nature. Our aim is to instead use purely geometric properties of the Cayley graph to show that free groups satisfy Property O. This allows us to deduce that these groups are exact.

Our previous method of exploiting growth properties or subset properties of the group does not work here, since free groups have rapid growth. However we are able to use some other geometric properties of the Cayley graph of a free group to obtain the following theorem:

Theorem 5. [C, Thm 1] *The family of kernels u_N constructed in section 5.4 forms a family of Ozawa kernels for free groups and so they satisfiy property O.*

We obtain the following corollary:

Corollary 1.2.4. [C, Cor 2.4] *Free groups are exact and therefore have Yu's Property A.*

1.2.1 Main result:

We next turn to a very different way of proving that a group is exact. Guentner and Kaminker have shown that the free group of rank 2 is exact by using Hilbert space compression which they introduced in [GK2]. They also use geometric properties of the Cayley graph of the free group, in particular that of unique edge paths and the median property in order to show that the

Hilbert space compression of a tree is 1. This shows that the free group of rank 2 is exact.

We now come to the main result of this thesis. A tree is a CAT(0) cube complex of dimension 1. Our aim is to extend Guentner and Kaminker's method to any finite dimensional CAT(0) cube complex. The class of groups acting properly and cocompactly on CAT(0) cube complexes is large, and includes free groups, finitely generated Coxeter groups, finitely generated right angled Artin groups, finitely presented groups satisfying the B(4)-T(4) cancellation properties and all those word-hyperbolic groups satisfying the B(6) condition. Others are the infinite simple groups constructed by Burger and Mozes.

Unlike free groups, a general CAT(0) cube complex does not have the property of unique edge paths between two points, a property which was used in Guentner and Kaminker's proof involving Hilbert space compression. Nonetheless we are able to construct an embedding with Hilbert space compression 1 which generalises Guentner and Kaminker's theorem as follows:

Theorem 6. [CN, Thm 12]

If G is a group acting properly and cocompactly on a $CAT(0)$ cube complex then G is exact and therefore has Yu's Property A.

Chapter 2

C^*-algebras , groups and exactness

In this chapter we will introduce the notions of C^*-algebras and group C^*-algebras. We will explain what it means for each of them to be exact.

2.1 C^*-algebras

We start by defining two different spaces which are characterised by the existence of an inner product and a norm respectively.

Definition 2.1.1 (Hilbert space). *A Hilbert space is an inner product space which is a complete metric space with respect to the metric induced by its inner product.*

Definition 2.1.2 (Banach space). *A Banach space is a normed space* *which is a complete metric space with respect to the metric induced by its norm.*

Note that every Hilbert space is a Banach space but the converse is not true. Every inner product defines a norm (by taking $\|x\| =< x, x >^{1/2}$), but not every norm can define an inner product.

We will use the definition of a Banach space in a moment in order to define a Banach algebra and from there a C^*-algebra .

We will next define some algebras and relations which characterise C^*-algebras and then give a couple of well known examples.

Definition 2.1.3 ($\mathbb{C}$-algebra). *A $\mathbb{C}$-algebra is a $\mathbb{C}$ vector space A with a multiplication satisfying $\forall a, b, c \in A$ and any constant $\lambda \in \mathbb{C}$:*

$$\begin{aligned} a(b+c) &= ab + ac \\ (a+b)c &= ac + bc \\ a(\lambda b) &= (\lambda a)b \\ &= \lambda(ab) \end{aligned}$$

In other words, a $\mathbb{C}$-algebra is one which is both left and right distributive with respect to its elements and associative with respect to any constant in $\mathbb{C}$. The simplest example of a $\mathbb{C}$-algebra would be the real numbers.

Definition 2.1.4 (Banach algebra). *A Banach algebra is an algebra A*

over a field F that has a norm relative to which A is a Banach space and such that $\|ab\| \leq \|a\| \; \|b\|$ for all $a, b \in A$.

An algebra over a field is a Banach algebra if there exists a norm for which the algebra is a complete metric space with respect to the metric induced by that norm. In addition the norm must satisfy the above condition $\|ab\| \leq \|a\| \; \|b\|$ for all $a, b \in A$. The field is usually taken to be the complex numbers.

Definition 2.1.5 (Star algebra). *A star algebra is an algebra A with a $*$ operator such that for all $a, b \in A$:*

$$\begin{aligned} (a^*)^* &= a \\ (\lambda a + \mu b)^* &= \overline{\lambda} a^* + \overline{\mu} b^* \\ (ab)^* &= b^* a^* \end{aligned}$$

A star algebra is one for which we can find an operator satisfying the above conditions. Such an operator is also denoted as a $*$ operator.

Definition 2.1.6 (C^*-algebra). *A C^*-algebra A is a Banach algebra with $*$-operation $\forall a \in A$ such that $\| a \|^2 = \| a^* a \|$.*

A couple of examples are as follows:

Example 1. *If $\mathcal{H}$ is a Hilbert Space and $\mathrm{B}(\mathcal{H})$ is the algebra of bounded functions on $\mathcal{H}$, then any closed star algebra of $\mathrm{B}(\mathcal{H})$ is a C^*-algebra.*

In fact any C^*-algebra is a C^*-subalgebra of $B(\mathcal{H})$.

Example 2. *If X is a compact Hausdorff space (a space such that any two points have disjoint neighborhoods), then $C(X) = \{f : X \to \mathbb{C} | f \text{ continuous}\}$ is an abelian C^*-algebra where:*

$$
\begin{aligned}
(fg)(x) &= f(x)g(x) \\
f^*(x) &= \overline{f(x)} \\
\| f \| &= sup|f(x)|
\end{aligned}
$$

Dually, by the Gelfand-Naimark theorem, any unital abelian C^*-algebra is isometrically isomorphic to $C(X)$ for some uniquely determined (up to homeomorphism) compact Hausdorff space X.

2.2 Group C^*-algebras and exactness

Given a group G, it is possible to construct a C^*-algebra associated to it and a reduced C^*-algebra which we will define in a moment. The C^*-algebra of a group is sometimes referred to as its full C^*-algebra . Some properties of the group are carried through to its associated full and/or reduced C^*-algebra . This is a powerful tool which allows us to use both analytic and group theory methods to prove facts about the group and its full or reduced C^*-algebra .

2.2.1 The reduced C^*-algebra of a group

We first need a few definitions.

First recall that the Haar measure is a non-zero measure μ on a sigma ring S, generated by compact subsets of a topological, locally compact group such that the measure is left or right invariant ($\mu(xA) = \mu(A)$ for all $x \in G$ and $A \in S$ or $\mu(Ax) = \mu(A)$ for all $x \in G$ and $A \in S$).

All the following statements refer to a locally compact group G equipped with the Haar measure.

Definition 2.2.1. $\mathrm{L}^1(G)$ *is the space of integrable functions on* G.

$\mathrm{L}^1(G) = \{f : G \to \mathbb{C} \mid \int \mid f(g) \mid dg < \infty\}$.

Definition 2.2.2. $\mathrm{L}^2(G)$ *is the space of square integrable functions on* G.

$\mathrm{L}^2(G) = \{f : G \to \mathbb{C} \mid \int \mid f(g) \mid^2 dg < \infty\}$.

If G is a discrete countable group Γ with a counting measure, then these integrals can be represented by a sum. In this case we write $\ell^1(\Gamma)$ in place of $\mathrm{L}^1(G)$ as shown in the next two definitions.

Definition 2.2.3. $\ell^1(\Gamma)$ *is the space of summable functions on* Γ.

$$\ell^1(\Gamma) \;=\; \{f : \Gamma \to \mathbb{C} \mid \sum_{g \in \Gamma} \mid f(g) \mid < \infty\}$$

Definition 2.2.4. *$\ell^2(\Gamma)$ is the space of square summable functions on Γ.*

$$\ell^2(\Gamma) \;=\; \{f : \Gamma \to \mathbb{C} \mid \sum_{g\in\Gamma} \mid f(g) \mid^2 < \infty\}$$

Remark 2.2.5. *$\mathrm{B}(l^2(\Gamma))$ is the space of bounded operators on the space $l^2(\Gamma)$.*

We can now define the reduced C^*-algebra.

Consider the left regular representation of $\ell^1(\Gamma)$ on the Hilbert space $\ell^2(\Gamma)$ which is defined by $(\lambda(g)\eta)(h) = \eta(g^{-1}h)$ where $g, h \in \Gamma$, $\eta \in l^2(\Gamma)$. This makes each element of $\ell^1(\Gamma)$ into a bounded operator on $\ell^2(\Gamma)$ and defines a homomorphism between $\ell^1(\Gamma)$ and the bounded operators on $\ell^2(\Gamma)$.

Definition 2.2.6 (Reduced C^*-algebra of Γ). *The closure of this representation is the reduced C^*-algebra of Γ.*

2.2.2 The full C^*-algebra of a group

We now need to define an important norm on elements of $\ell^1(\Gamma)$ called the supremum norm. The C^*-algebra of a group Γ is obtained by completing the space $\ell^1(\Gamma)$ in this norm.

Take any representation π of $\ell^1(\Gamma)$ on any Hilbert space. Then define the norm of an element of $\ell^1(\Gamma)$ as the norm of its image in $\mathcal{H}$. It can be shown that the supremum of all cyclic norms over all cyclic representations is finite

[D, p184]. This is a norm, also referred to as the maximal norm.

$$\| f \| = sup\{\| \pi(f) \| : \pi \text{ is a } *\text{-representation of } \ell^1(\Gamma)\}$$

Definition 2.2.7 (**C^*-algebra of Γ**). *The completion of $\ell^1(\Gamma)$ in this norm forms the full C^*-algebra of Γ.*

In some cases the reduced C^*-algebra of Γ is the same as the full C^*-algebra of Γ. For example, if G is an abelian group, then the reduced C^*-algebra of G is the same as the full C^*-algebra of G.

We now need to define the property of exactness. This requires some more technical concepts such as tensor products and C^* norms. We will go through them in the next couple of sections and then define exactness for both C^*-algebras and group C^*-algebras.

2.2.3 The minimal norm on C^*-algebras and the spatial tensor product

Tensor products provide a way of combining C^*-algebras . The minimal norm allows us to define the spatial tensor product which is used in the definition of exactness.

We will first define the algebraic tensor product which is the combination of two vector spaces A, B to form a linear vector space with an additional bilinear structure.

To obtain bilinearity, consider the vector space

$$\mathbb{C}^{(A\times B)} := \left\{\sum \lambda(e,f)|\lambda \in \mathbb{C}, e \in E, f \in F, \text{only finitely many } \lambda \neq 0\right\}$$

Any element is uniquely determined by its finite set of non-zero complex coefficients λ.

Now construct a null space $\mathcal{N}$ consisting of finite linear combinations of elements of the form

$$(a_1 + a_2, b) - (a_1, b) - (a_2, b)$$
$$(a, b_1 + b_2) - (a, b_1) - (a, b_2)$$
$$\lambda(a,b) - (\lambda a, b) \text{ and } \lambda(a,b) - (a, \lambda b)$$

where $a, a_1, a_2 \in A$, $b, b_1, b_2 \in B$ and $\lambda \in \mathbb{C}$.

Definition 2.2.8 (Algebraic tensor product). *The algebraic tensor product is the quotient vector space* $A \odot B := \mathbb{C}^{(A\times B)}/\mathcal{N}$

The elements $\pi(a,b) \in A\odot B$, $a \in A$, $b \in B$ (where π denotes the quotient map of $\mathbb{C}^{(A\times B)}$ onto $A\odot B$) are called elementary tensors,denoted $a\otimes b$. They have the following properties:

$$\begin{aligned}(a_1 + a_2) \otimes b &= a_1 \otimes b + a_2 \otimes b \\ a \otimes (b_1 + b_2) &= a \otimes b_1 + a \otimes b_2 \\ \lambda(a \otimes b) = \lambda a \otimes b &= a \otimes \lambda b\end{aligned}$$

The elements of $A \odot B$ are finite sums of elementary tensors $\sum_{k=1}^{n} \lambda_k a_k \otimes b_k$ where $\lambda \in \mathbb{C}$.

Definition 2.2.9 (Algebraic representation). *An algebraic representation of $A \odot B$ where A,B are C^*-algebras, is a linear, multiplicative and $*$-preserving map from $A \odot B$ to $\mathrm{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.*

To each pair of representations π_1, π_2 of C^*-algebras A_1, A_2 on Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$, there is a unique algebraic representation [WO, Prop T.5.1.]:

$$\pi_1 \odot \pi_2 : \quad A_1 \odot A_2 \to \mathrm{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$$

of the algebraic tensor product $A_1 \odot A_2$ as operators of the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ satisfying:

$$(\pi_1 \odot \pi_2)(a_1 \otimes a_2) = \pi_1(a_1) \otimes \pi_2(a_2) \quad \in \mathrm{B}(\mathcal{H}_1) \odot \mathrm{B}(\mathcal{H}_2) \subset \mathrm{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$$

If both π_1 and π_2 are injective, then $\pi_1 \odot \pi_2$ is injective.

Definition 2.2.10 (Faithful). *A faithful representation is one that has kernel 0.*

Lemma 2.2.11. *The tensor product representation $\pi_1 \odot \pi_2$ of two faithful representations π_1 and π_2 is faithful.*

Suppose π_1 and π_2 are faithful representations of A_1 and A_2. Since $\pi_1 \odot \pi_2$ is injective, the norm on $\mathrm{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ can be pulled back to define a norm on $A_1 \odot A_2$. This norm is independent of the representations used [WO, Theorem T.5.15].

Definition 2.2.12 (Minimum norm). *This is called the minimum or spatial norm:* $\| x \|_{\pi_1 \odot \pi_2} = \| (\pi_1 \odot \pi_2)(x) \|_{\mathrm{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)} = \| x \|_{min} = \| x \|_{\sigma}$.

Definition 2.2.13 (Spatial tensor product). *The spatial or minimal tensor product of C^*-algebras A, B is the completion of $A \odot B$ in $\| \ \|_{min}$. It is usually denoted $A \otimes B$ or $A \otimes_{min} B$.*

It can be shown [WO, Thm T.6.21.], that for any C^* norm $\| \ \|_\beta$ on $A \odot B$, $\| x \|_{min} \leq \| x \|_\beta$

We will use this norm to define exactness for C^*-algebras as follows.

2.2.4 Exactness for C^*-algebras

If A, B, C are C^*-algebras, α, β are $*$-homomorphisms, then the sequence

$$A \rightarrow_\alpha B \rightarrow_\beta C$$

is exact if im(α)=ker(β)

A C^*-algebra C is said to be exact if for any exact sequence

$$0 \rightarrow J \rightarrow B \rightarrow B/J \rightarrow 0$$

the operation of taking the cross product preserves exactness, in other words the sequence

$$0 \rightarrow J \otimes_{min} C \rightarrow B \otimes_{min} C \rightarrow B/J \otimes_{min} C \rightarrow 0$$

is also exact.

We can now move on to exactness for groups.

2.2.5 Exactness for groups

We say that a group Γ is exact if the operation of taking the reduced crossed product with Γ preserves exactness of short exact sequences of Γ-C^*-algebras.

In other words, Γ is exact if and only if for every exact sequence of Γ-C^*-algebras

$$0 \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow 0$$

the sequence

$$0 \longrightarrow C^*_r(\Gamma, B) \longrightarrow C^*_r(\Gamma, C) \longrightarrow C^*_r(\Gamma, D) \longrightarrow 0$$

of crossed product algebras is exact.

Kirchberg and Wasserman have shown that a discrete group Γ is exact if and only if the reduced C^*-algebra $C^*_r(\Gamma)$ is exact [KW, Thm 5.2].

Known examples of exact groups:

One large class of exact groups is that of amenable groups which amongst others includes all abelian groups, groups of subexponential growth, Grigorchuk's group and nilpotent groups. Other classes of exact groups include free groups, word hyperbolic groups and Coxeter groups [GK, DJ, KW].

Chapter 3

Properties related to exactness

Proving that a specific group is exact using the definitions from the previous chapter may require some powerful and complex analytic tools and theorems. Further more recent work [O, GK2, Yu] has used them to produce alternative more geometric properties of groups which are equivalent to or linked to exactness. The three we will study in this chapter are Property A, Property O and Hilbert space compression.

The first alternative and most famous property was introduced in [Yu] and is known as Property A.

3.1 Property A

3.1.1 Yu's Property A

In [Yu], Yu introduced a property for discrete metric spaces based on Følner type properties. He called it Property A and showed that it implies the Hilbert space embeddability property [Yu, Thm 2.2] and the Coarse Baum Connes conjecture [Yu, Thm 1.1].

Higson and Roe later proved that discrete groups whose underlying metric space satisfies Property A act amenably on their Stone-Čech compactification [HR, Thm 1.1]. In conjunction with a result of Anantharaman-Delaroche-Renault [ADR, Chapter 6], this proves that Property A is also equivalent to exactness.

The following definition of Property A is Definition 2.1 from [Yu]. We will use the following notation: for a given metric space and any $R > 0$, then Δ_R denotes $\{(x, y) \in X \times X | d(x, y) < R\}$.

Definition 3.1.1 (Property A). *A discrete metric space X is said to have property A if for any $R > 0, \epsilon > 0$ there exist $S > 0$ and a family $(A_x)_{x \in X}$ of finite, nonempty subsets of $X \times \mathbb{N}$ such that:*

- $(y, n) \in A_x$ *implies* $(x, y) \in \Delta_S$
- *for all* $(x, y) \in \Delta_R$,
$$\frac{\sharp(A_x \triangle A_y)}{\sharp(A_x \cap A_y)} \leq \epsilon$$

If a group acts properly and cocompactly on a space which has Property A, then we say that the group has Property A and this implies that the group is exact. Since it can be considered a metric space property, it allows us to exploit geometric properties of the Cayley graph of the group. As an example, in [DJ, Prop 1], Dranishnikov and Januszkiewicz construct a Property A function for trees by exploiting some of their geometric characteristics such as unique edge paths between any two vertices and the median property. They thus show that trees have property A. Since the Cayley graph of any free group is a tree, and any group acts properly and cocompactly on its Cayley graph, we can deduce that free groups are exact.

3.1.2 Alternative definitions

As stated in the introduction, Property A is linked to many important conjectures such as the coarse Baum Connes conjecture and other geometric properties such as uniform embeddability into a Hilbert space. As a result it has been extensively studied and has been shown to have many equivalent formulations.

Higson and Roe's definition of Property A

One such alternative definition is found in [HR, Lemma 3.5] and concerns discrete metric spaces with bounded geometry. Recall that we say a metric space X has bounded geometry if for every $C > 0$ there exists N such that every ball of radius C in X contains at most N elements.

Definition 3.1.2 (Property A, Higson-Roe). *A discrete metric space Z with bounded geometry has Property A if and only if there is a sequence of maps $a^n : Z \to P(Z)$ such that*

1. *for every n there is some $R > 0$ with the property that for every $z \in Z$, supp $(a^n_z) \subset \{z' \in Z \mid d(z, z') < R\}$ and*

2. *for every $K > 0$, $\lim_{n\to\infty} \sup_{d(z,w)<K} \|a^n_z - a^n_w\|_1 = 0$*

The map a^n_z belongs to a sequence of probability maps over Z associated to an element z and indexed by n. For any n, z we have $\sum_{z' \in Z} a^n_z(z') = 1$.

The set supp(a^n_z) is the set of points z' in Z such that $a^n_z(z') \neq 0$. If Z has property A, then all such points must belong to the ball of radius R around z. Note that since Z has bounded geometry there is a finite number of such points.

We consider all pairs of points (z,w) in Z less than some distance K apart. We calculate the norm of the function $a^n_z - a^n_w$ and find its supremum over all such pairs as n tends to infinity. If Z has Property A, then this must be equal to 0 for any choice of K.

Equivalence of Yu's Property A and Higson's and Roe's definition

The equivalence of these two definitions is proved in [HR, lemma 3.5].

Proof. They first show that the above property implies that Z satisfies Yu's Property A.

First assume there exist probability maps a_z^n as above. Since Z has bounded geometry there are a finite number of values $a_z^n(z'), z' \in Z$, each of which lies between 0 and 1. So for each n we can assume by an approximation argument that they only take values in the range $\frac{0}{M}, \frac{1}{M}, \frac{2}{M}, \ldots, \frac{M}{M}$ for some natural number M.

Next define $A_n(z) \subset Z \times \mathbb{N}$ by $(z', j) \in A_n(z) \Leftrightarrow \frac{j}{M} \leq a_z^n(z')$.

The finite support condition from Yu's definition is satisfied since the support of a_z^n is finite.

Given $z' \in Z$, let $j_{z'}$ be the largest j such that $\frac{j}{M} \leq a_z^n(z')$. Then the size of $A_n(z)$ is $\sum_{z' \in Z} j_{z'}$.

Since $a_z^n(z')$ only takes values in $\frac{0}{M}, \frac{1}{M}, \frac{2}{M}, \ldots, \frac{M}{M}$, we also have $\sum_{z' \in Z} \frac{j_{z'}}{M} = \sum_{z' \in Z} a_z^n(z') = 1$. And so we have that $\frac{1}{M} \sum_{z' \in Z} j_{z'} = 1$ and $|A_n(z)| = M$.

We also have that $|A_n(z) \Delta A_n(w)| = M \|a_z^n - a_w^n\|_1 = |A_n(z)| \, \|a_z^n - a_w^n\|_1$.

Using the second hypothesis of Higson and Roe's definition, we get

$$\lim_{n \longrightarrow \infty} \sup_{d(z,w) < K} \frac{|A_n(z) \Delta A_n(w)|}{|A_n(z)|} = \lim_{n \longrightarrow \infty} \sup_{d(z,w) < K} \|a_z^n - a_w^n\|_1 = 0$$

Higson and Roe had previously shown [HR, Lemma 3.4] that

$$\lim_{n \longrightarrow \infty} \sup_{d(z,w) < K} \frac{|A_n(z) \Delta A_n(w)|}{|A_n(z) \cap A_n(w)|} = 0 \Leftrightarrow \lim_{n \longrightarrow \infty} \sup_{d(z,w) < K} \frac{|A_n(z) \Delta A_n(w)|}{|A_n(z)|} = 0$$

Thus we have

$$\lim_{n\longrightarrow\infty}\sup_{d(z,w)<K}\frac{|A_n(z)\Delta A_n(w)|}{A_n(z)\cap A_n(w)|}=0$$

Yu's second condition is satisfied and Z has Yu's Property A.

They then show that Yu's Property A implies that their definition holds.

We set

$$a_z^n(z')=\frac{|\{j|(z',j)\in A_n(z)\}|}{|A_n(z)|}$$

The first support condition is satisfied since $A_n(z)$ are finite sets.

Consider $|A_n(z)\Delta A_n(w)|$. We have

$$\begin{aligned}|A_n(z)\Delta A_n(w)| &= \sum_{z'\in Z}|\{j|(z',j)\in A_n(z)\}|+\sum_{z'\in Z}|\{j|(z',j)\in A_n(w)\}|\\ &- 2\sum_{z'\in Z}|\{j|(z',j)\in A_n(z)\cap A_n(w)\}|\end{aligned}$$

Now consider $\|\ a_z^n\ |A_n(z)|-a_w^n\ |A_n(w)|\ \|_1$. We have

$$\|a_z^n\ |A_n(z)|-a_w^n\ |A_n(w)|\ \|_1=\sum_{z'\in Z}|\ |\{j_z|(z',j_z)\in A_n(z)\}|-|\{j_w|(z',j_w)\in A_n(w)\}|\ |$$

In both cases an element (z',k) must belong to at least one of $A_n(z)$ or $A_n(w)$ to affect the calculation. So we can disregard all elements which belong to neither set. Elements belonging to both sets also do not affect the calculation and can be disregarded.

Next we consider elements which belong to only one of the sets $A_n(z), A_n(w)$. Every such element contributes one to the size of $|A_n(z)\Delta A_n(w)|$. However, when calculating $\|a_z^n\,|A_n(z)| - a_w^n\,|A_n(w)|\,\|_1$, the calculation is based on the total number of elements in $A_n(z)$ and $A_n(w)$. They need not be the same element. Thus an element (z', k) could belong to just one of the two sets yet not contribute to the sum if there exists some other different element $(z', k)'$ belonging to the other set.

So we have

$$\|a_z^n\,|A_n(z)| - a_w^n\,|A_n(w)|\,\|_1 \leq |A_n(z)\Delta A_n(w)|$$

and thus with the second condition of Yu's Property A, we get

$$\lim_{n\longrightarrow\infty}\sup_{d(z,w)<K}\left\|a_z^n - a_w^n\frac{|A_n(w)|}{|A_n(z)|}\right\|_1 \leq \lim_{n\longrightarrow\infty}\sup_{d(z,w)<K}\frac{|A_n(z)\Delta A_n(w)|}{|A_n(z)|} = 0$$

By [HR, Lemma 3.4], $\lim_{n\longrightarrow\infty}\sup_{d(z,w)<K}\frac{|A_n(w)|}{|A_n(z)|} = 1$ and so

$$\lim_{n\longrightarrow\infty}\sup_{d(z,w)<K}\|a_z^n - a_w^n\| = 0$$

as required. This concludes the proof. □

Higson and Roe's definition is used in [DJ, Prop 1] to show that trees have Property A. We adapt their construction later in this thesis to construct a family of Ozawa kernels and show that free groups have Property O.

Other equivalent versions of Property A

Many different versions of Property A, including the two already mentioned are cited in Proposition 3.2 in [Tu]. In total, six equivalent versions of Property A are introduced.

We will cite them here and briefly state how to show they are equivalent. A full proof can be found in [Tu, Prop 3.2]. Recall that Δ_S denotes the set of all pairs of points which are at distance at most S apart.

Proposition 3.1.3. *Let X be a discrete metric space with bounded geometry. The following are equivalent:*

1. *X has Property A (as expressed in [Yu]): For any $R > 0, \epsilon > 0$ there exist $S > 0$ and a family $(A_x)_{x \in X}$ of finite, nonempty subsets of $X \times \mathbb{N}$ such that:*

    - *$(y, n) \in A_x$ implies $(x, y) \in \Delta_S$*
    - *for all $(x, y) \in \Delta_R$,*
    $$\frac{\sharp(A_x \vartriangle A_y)}{\sharp(A_x \cap A_y)} \leq \epsilon$$

2. *$\forall R > 0, \forall \epsilon > 0, \exists S > 0, \exists (\xi_x)_{x \in X}$ such that*

    - *$supp(\xi_x) \subset B(x, S)$*
    - *$\| \xi_x \|_{l^1(X)} = 1$*
    - *$\| \xi_x - \xi_y \|_{l^1(X)} \leq \epsilon$ whenever $d(x, y) \leq R$*

Definition 2 is actually equivalent to the definition used by Higson and Roe [HR, Lemma 3.5].

Their definition uses sequences of probability functions. However these are l^1 functions [HR, Def 2.1] as required in Tu's definition.

We also need ζ_x to be to a probability function on X. By definition 2, we know that $\|\zeta_x\| = 1$. We can suppose that ζ_x are non negative. Thus we have that $\|\zeta_x\| = \sum_{x \in X} |\zeta_x| = \sum_{x \in X} \zeta_x = 1$ and hence ζ_x is a probability function as required.

The support condition is immediately satisfied. All that remains is to check that the final conditions concerning the norm of the differences of the functions at two distinct points are equivalent. To do this, we index ζ_x by ϵ and set $\zeta_x^\epsilon = a_x^n$. We then set $\epsilon = \frac{1}{n}$ and the result follows. The converse can be seen in a similar fashion where we set $n = \frac{1}{\epsilon}$.

And so as seen earlier in [HR, Lemma 3.5], the two definitions are equivalent.

3. $\forall R > 0, \forall \epsilon > 0, \exists S > 0, \exists (\chi_x)_{x \in X}, \chi_x \in l^1(X)$, *such that*

- *supp* $(\chi_x) \subset B(x, S)$
- $\frac{\|\chi_x - \chi_y\|_{l^1(X)}}{\|\chi_x\|_{l^1(X)}} \leq \epsilon$ *whenever* $d(x, y) \leq R$.

Definitions 2 and 3 are equivalent. The fact that 2 implies 3 is obvious since we can simply take $\chi_x = \xi_x$. To see that 3 implies 2, take $\xi_x =$

$\frac{\chi_x}{\|\chi_x\|_l^1(X)}$. The support condition is immediately satisfied. We also have:

$$\begin{aligned}
\|\xi_x - \xi_y\|_1 &= \left\| \frac{\chi_x}{\|\chi_x\|_1} - \frac{\chi_y}{\|\chi_y\|_1} + \frac{\chi_y}{\|\chi_x\|_1} - \frac{\chi_y}{\|\chi_x\|_1} \right\|_1 \\
&= \left\| \frac{\chi_x - \chi_y}{\|\chi_x\|_1} + \chi_y \left(\frac{1}{\|\chi_x\|_1} - \frac{1}{\|\chi_y\|_1} \right) \right\|_1 \\
&\leq \frac{\|\chi_x - \chi_y\|_1}{\|\chi_x\|_1} + \|\chi_y\|_1 \left| \frac{1}{\|\chi_x\|_1} - \frac{1}{\|\chi_y\|_1} \right| \\
&= \frac{\|\chi_x - \chi_y\|_1}{\|\chi_x\|_1} + \|\chi_y\|_1 \left| \frac{1}{\|\chi_x\|_1} - \frac{\|\chi_x\|_1}{\|\chi_x\|_1 \|\chi_y\|_1} \right| \\
&= \frac{\|\chi_x - \chi_y\|_1}{\|\chi_x\|_1} + \frac{|\, \|\chi_y\|_1 - \|\chi_x\|_1 \,|}{\|\chi_x\|_1} \\
&\leq \frac{2\|\chi_x - \chi_y\|_1}{\|\chi_x\|_1}
\end{aligned}$$

Remark 3.1.4. *The last step of the above calculation is obtained via the triangle inequality.* $\|\chi_x\|_1 = \|\chi_x + \chi_y - \chi_y\|_1 \leq \|\chi_x - \chi_y\|_1 + \|\chi_y\|_1$. *This gives us* $|\, \|\chi_x\|_1 - \|\chi_y\|_1 \,| \leq \|\chi_x - \chi_y\|_1$.

And so the required inequality is satisfied.

4. $\forall R > 0, \forall \epsilon > 0, \exists S > 0, \exists (\eta_x)_{x \in X}, \eta_x \in l^2(X)$ *such that*

- *supp* $(\eta_x) \subset B(x, S)$
- $\| \eta_x \|_{l^2(X)} = 1$
- $\| \eta_x - \eta_y \|_{l^2(X)} \leq \epsilon$ *whenever* $d(x, y) \leq R$

Definitions 2 and 4 are equivalent. To see that definition 2 implies 4, define $\eta_x = |\xi_x|^{1/2}$ and denote by $\int_X$ the integral with counting measure on X. Then the support condition is immediately satisfied and we also have:

$$\begin{aligned} \|\eta_x - \eta_y\|^2_{l^2(X)} &= \int_X |\eta_x - \eta_y|^2 \leq \int_X |\eta_x - \eta_y|\ |\eta_x + \eta_y| \\ &= \int_X |\eta_x^2 - \eta_y^2| = \|\ |\xi_x| - |\xi_y|\ \|_{l^1(X)} \leq \|\xi_x - \xi_y\|_{l^1(X)} \end{aligned}$$

To see the converse, assume that η_x is positive and define $\xi_x = \eta_x^2$. The support condition is immediately satisfied. We get the desired inequality from definition 2 as follows:

$$\begin{aligned} \|\xi_x - \xi_y\| &= \int_X |\eta_x^2 - \eta_y^2| = \int_X |\eta_x - \eta_y|(\eta_x + \eta_y) \\ &\leq \|\eta_x - \eta_y\|_{l^2(X)}\ \|\eta_x + \eta_y\|_{l^2(X)} \\ &= \|\eta_x - \eta_y\|_{l^2(X)}(\|\eta_x\|_{l^2(X)} + \|\eta_y\|_{l^2(X)}) \\ &\leq 2\|\eta_x - \eta_y\|_{l^2(X)} \end{aligned}$$

Step 1 to 2 is obtained by using the Cauchy-Schwarz inequality $|\langle f, g\rangle| \leq \|f\|_2\|g\|_2$. We can rewrite $|\langle f, g\rangle|$ as $\int_X f * g$. We then take f to be $|\eta_x - \eta_y|$ and g to be $(\eta_x + \eta_y)$ to get the required inequality $\int_X |\eta_x - \eta_y|(\eta_x + \eta_y) \leq \|\eta_x - \eta_y\|_{l^2(X)}\ \|\eta_x + \eta_y\|_{l^2(X)}$. Finally, the last inequality holds because $\|\eta_x\|_{l^2(X)} = 1$.

5. $\forall R > 0, \forall \epsilon > 0, \exists S > 0, \exists (\zeta_x)_{x \in X}, \zeta_x \in l^2(X \times \mathbb{N})$ *such that*

- *supp* $(\zeta_x) \subset B(x,S) \times \mathbb{N}$

- $\| \zeta_x \|_{l^2(X\times\mathbb{N})} = 1$

- $\| \zeta_x - \zeta_y \|_{l^2(X\times\mathbb{N})} \leq \epsilon$ *whenever* $d(x,y) \leq R$

Definition 4 is equivalent to definition 5. Definition 4 implies 5 since we can take $\zeta_x(z,.)$ to be $\eta_x(z)$ for any $n \in \mathbb{N}$ and $z \in X$.

To see that 5 implies 4, let $\eta_x(z) = \|\zeta_x(z,.)\|_{l^2(\mathbb{N})}$. The support condition is satisfied. We can check the required inequality of definition 4 as follows:

$$\begin{aligned}
\|\eta_x - \eta_y\|^2_{l^2(X)} &= \sum_{z\in X} | \, \|\zeta_x(z,.)\|_{l^2(\mathbb{N})} - \|\zeta(z,.)\|_{l^2(\mathbb{N})} |^2 \\
&\leq \sum_{z\in X} \|\zeta(z,.) - \zeta(z,.)\|^2_{l^2(\mathbb{N})} \\
&= \|\zeta_x - \zeta_y\|^2_{l^2(X\times\mathbb{N})}
\end{aligned}$$

6. $\forall R > 0, \forall \epsilon > 0, \exists S > 0, \exists \varphi : X \times X \to \mathbb{R}$ *such that*

   - φ *is of positive type*

   - *supp* $\varphi \subset \Delta_S$

   - $|1 - \varphi(x,y)| \leq \epsilon$ *whenever* $d(x,y) \leq R$

Definitions 4 and 6 are equivalent. To show that 4 implies 6, let $\varphi(x,y) = \langle \eta_x, \eta_y \rangle$. In this case the support of φ is contained in Δ_{2S}.

In addition, when two points are less than distance R apart then $1 - \varphi(x, y) = \frac{1}{2}\|\eta_x - \eta_y\|^2_{l^2(X)} \leq \frac{1}{2}\epsilon^2$

The proof to show that 6 implies 4 is fairly long, so we will just give a very brief outline of the method. The full proof can be found in [Tu, p.120].

Given φ and ϵ, we define an operator dependent on φ: $(T_\varphi \eta)(x) = \sum_{y \in X} \varphi(x, y)\eta(y)$. This operator can be shown to be bounded and positive as well as satisfying several inequalities involving S and R from definition 6.

We then construct an operator η_x via convolution products on φ involving polynomials satisfying specific conditions. These conditions together with the properties of the operator $(T_\varphi \eta)(x)$ allow us to deduce that η_x satisfies all the required properties of definition 4.

3.1.3 Comments

Although these properties are expressed in different ways, they all share some characteristics. They each admit a finite support condition: the set of points for which the function is non zero must be finite. They also each admit some condition on the norm of the difference of the function evaluated at a pair of points less than some distance K apart.

It is worth noting that these definitions are all made on the assumption that we are working with a discrete metric space with bounded geometry.

However current research is expanding these concepts to non-discrete metric spaces. In [DG], Dadarlat and Guentner introduce an idea of exactness for general (not necessarily discrete) metric spaces. Any group which acts properly and cocompactly on such a space is exact and has property A. As a specific example, unpublished work by Claire Vatcher has shown that unbounded $\mathbb{R}$-trees which admit a geodesic ray have Property A.

We will now move on to another property introduced in [O] which we term Property O. In the same paper, Ozawa shows that groups satisfying Property O are exact.

3.2 Property O

3.2.1 Uniform Roe Algebra

Ozawa defines Property O by using operators belonging to the Uniform Roe Algebra which we will now introduce.

We start by defining finite width operators. To do this, consider the set of $A : \Gamma \times \Gamma \to \mathbb{C}$ satisfying:

1. $\exists M > 0$ such that $\mid A(s,t) \mid \leq M \ \forall s,t \in \Gamma$
2. $\exists R > 0$ such that $A(s,t) = 0$ if $d(s,t) > R$

These are simply functions which have a finite upper bound for all pairs of elements of Γ and which take value zero if the two elements are at distance greater than some finite R apart. Note that these may be useful characteristics when trying to construct operators or functions satisfying Property A, since these require finite support.

Each such A defines a bounded operator on $\ell^2(\Gamma)$ via matrix multiplication. The bounded operator evaluated at s is the sum over all possible $r \in \Gamma$ of the product of the finite width operator evaluated at (s, r) with an operator from $\ell^2(\Gamma)$ evaluated at r. Here we take $\xi \in \ell^2(\Gamma)$.

$$A\xi(s) = \sum_{r \in \Gamma} A(s,r)\xi(r)$$

These bounded operators are referred to as finite width operators.

Remark 3.2.1. *The collection of finite width operators is a $*$-subalgebra of* $\mathrm{B}(\ell^2(\Gamma))$ *[GK, p6].*

We are now in a position to define the Uniform Roe Algebra [GK, p6]:

Definition 3.2.2 (Uniform Roe algebra). *The Uniform Roe Algebra of Γ, $UC^*(\Gamma)$ is the closure of the $*$-algebra of finite width operators. It is a C^*-algebra.*

The reduced C^*-algebra of a group is contained in the group's Uniform Roe Algebra. This can be seen as follows:

An element $t \in \Gamma$ acts on $\ell^2(\Gamma)$ by the left regular representation. The action of $t \in \Gamma$ on $\ell^2(\Gamma)$ can be represented by the matrix A defined by $A(s,r) = 1$ iff $s = tr$.

To see this, consider δ_t, the characteristic function of t. It takes value 1 if the element is t and 0 otherwise. We associate every element $t \in \Gamma$ to $\delta_t \in \ell^2(\Gamma)$ which acts by convolution on $\ell^2(\Gamma)$.

Take δ_t and δ_p and consider the convolution product.

$$\delta_t * \delta_p(s) = \sum_{r \in \Gamma} \delta_t(sr^{-1})\delta_p(r)$$

This is equal to 1 if $t = sr^{-1}$ (or equivalently $s = tr$) and $p = r$ and 0 otherwise.

Now consider $A\delta_p(s) = \sum_{r\in\Gamma} A(s,r)\delta_p(r)$ where A is the matrix defined above as $A(s,r) = 1$ iff $s = tr$.

Comparing $\delta_t(sr^{-1})$ and $A(s,r)$, both are equal to 1 if $s = tr$ and $p = r$ and 0 otherwise. Hence the two are equivalent and the action of t on $\ell^2(\Gamma)$ is represented by the matrix $A(s,r)$.

This means that $t \in \Gamma$ is acting as a finite width operator which belongs to the Uniform Roe algebra of Γ. Any element of the group ring is a linear function of delta functions of group elements. Hence we have that the group ring $\mathbb{C}[\Gamma] \subseteq UC^*(\Gamma)$. The reduced C^*-algebra , which is the closure of the group ring will also be contained in the closure of the algebra of finite width operators which is by definition $UC^*(\Gamma)$. Hence we have $C_r^*(\Gamma) \subseteq UC^*(\Gamma)$.

3.2.2 Property O

We will need the following definition in order to define Property O:

Definition 3.2.3 (Positive kernel). *A positive kernel is a function $u: G \times G \to \mathbb{R}$ which has the property that for any set of $\lambda_i, \lambda_j \in \mathbb{R}$,*

$$\sum_{i,j}^{n} \lambda_i \lambda_j u(g_i, g_j) \geq 0$$

In addition, $u(g_i, g_j) = u(g_j, g_i)$. Equivalently, this means that the matrix $[u(g_i, g_j)] \in \mathbb{M}_n$ is positive for any n and $g_1, g_2, \dots g_n \in G$.

Remark 3.2.4. *In [GK, GK2, O], the authors call such a kernel positive definite rather than positive. However since the general convention in the literature is to call this type of kernel positive, that is the notation we will adopt in this thesis.*

In [O], Ozawa introduces the following property which we will call Property O:

Definition 3.2.5 (Ozawa's Property O). *A discrete group G is said to have Ozawa's Property O if for any finite subset $E \subset G$ and any $\epsilon > 0$, there are a finite subset $F \subset G$ and $u: G \times G \to \mathbb{C}$ such that*

- *u is a positive kernel*
- *$u(s,t) \neq 0$ only if $st^{-1} \in F$*
- *$|1 - u(s,t)| < \epsilon$ if $st^{-1} \in E$*

The second condition means that u is in fact a finite width kernel. Together with the other two conditions, this means that the identity can be approximated by finite width positive kernels.

We will use the following notation:

For any given pair (E, ϵ) we will call a kernel satisfying the conditions of Property O an Ozawa kernel.

A family of Ozawa kernels is one for which there exists an Ozawa kernel for any pair (E, ϵ).

Thus a group has Property O if there exists a family of Ozawa kernels on the group.

We now need the following definition:

Definition 3.2.6 (Nuclear). *A C^*-algebra A is nuclear iff for any C^*-algebra B, $\| \quad \|_{max} = \| \quad \|_{min}$ on the algebraic tensor product $A \odot B$.*

Ozawa proves the following theorem [O, Thm 3]:

Theorem 7. *The following three statements are equivalent:*

1. *The reduced group C^*-algebra $C^*_r(G)$ is exact.*

2. *G has Property O.*

3. *The uniform Roe algebra $UC^*(G)$ is nuclear.*

A brief outline of the proof is as follows:

To show the first equivalence, Ozawa starts by showing that given a Hilbert space H and any exact C^*-algebra A of $\mathbf{B}(H)$ there exists a unital, positive, finite rank operator θ satisfying $||\theta(x) - x|| \leq \epsilon$ for all $\epsilon > 0$ and $x \in E$ where $E \subseteq A$. He defines $u(s,t) = \langle \delta_s, \theta(\delta_{st^{-1}})\delta_t \rangle$. Using properties from [GK, Thm 3.1], he shows that this function is an Ozawa kernel. Such a function exists for any choice of E and ϵ. Thus there exists a family of Ozawa kernels and G has Property O.

To show the second equivalence, Ozawa assumes that there exists a net of functions u_i and sets E_i which form a family of Ozawa kernels. He constructs a set of Schur multipliers θ_i associated to each u_i, and uses the fact that they are positive contractions to prove that given a unital C^*-algebra B,

$$UC^*(G) \otimes_{min} B = UC^*(G) \otimes_{max} B$$

This shows that whenever there exists a family of Ozawa kernels, the Roe algebra $UC^*(G)$ is nuclear.

Finally, to show the final equivalence, we need the following theorem:

Theorem 8. *Any nuclear algebra is also exact* [Wa, Property 2.5.1.].

Since any subalgebra of an exact algebra is exact[Wa, Prop 2.6], we have that any subalgebra of a nuclear algebra must be exact. As seen earlier, the reduced C^*-algebra of the group is a closed subalgebra of the Roe algebra [GK, p6] and so must be exact. This concludes the proof.

Since Kirchberg and Wasserman had previously shown that a discrete group is exact if and only if its reduced C_r^* algebra is exact [KW, Thm 5.2], this shows that a discrete group Γ is exact if and only if it satisfies Property O.

Alternative definition

We can also think of Property O in an alternative way. Instead of considering different sets E and ϵ and finding a function for each case, we can construct a sequence of positive definite functions u_n with the necessary support condition which tends to 1 uniformly as n tends to infinity.

This is the version used by Guentner and Kaminker in [GK2, Prop 3.3] in which they prove a theorem relating the property of Hilbert space compression to exactness. It is formally defined below:

Proposition 3.2.7. *Let Γ be a finitely generated discrete group equipped with word length and metric associated to a finite symmetric set of generators. Then Γ is exact iff there exists a sequence of positive functions $u_n : \Gamma \times \Gamma \longrightarrow \mathbb{R}$ satisfying:*

1. *For all $C > 0$, $u_n \longrightarrow 1$ uniformly on the strip $s, t : d(s,t) < C$.*

2. *For all n there exists R such that $u_n(s,t) = 0$ if $d(s,t) \geq R$.*

This is equivalent to our previous definition of Property O. The main difference is that instead of letting E and F from Ozawa's definition to be any finite set, they only consider balls of radius C and R.

Consider their first condition. Given any finite set E, they choose C so that E is contained in the ball of radius C. By the definition of uniform convergence, for all $\epsilon > 0$ there exists some N such that if $n > N$, $|1 - u_n| < \epsilon$ when pairs of points are distance less than C apart. We can pick any one of these u_n to be our Ozawa kernel.

The second condition is equivalent to the condition of the existence of a finite set F outside which the function has value zero. In this case F is the ball of radius R which is a finite set as required.

Left or right equivariance

In the above section, all the definitions and proofs consider a right handed Cayley graph and right equivariance. Since by convention most literature uses left handed Cayley graphs and left equivariance, we will adapt the definition accordingly for the remainder of this thesis. We consider a finitely presented group with a length function l determined by a finite, symmetric set of generators which determines an invariant metric. Instead of considering that the length function l determines a right invariant metric via $d(s,t) = l(st^{-1})$ we will use the left invariant metric via $d(s,t) = l(s^{-1}t)$.

Ozawa's definition becomes:

Definition 3.2.8 (Ozawa's Property O). *A discrete group G is said to have Ozawa's Property O if for any finite subset $E \subset G$ and any $\epsilon > 0$, there are a finite subset $F \subset G$ and $u\colon G \times G \to \mathbb{C}$ such that*

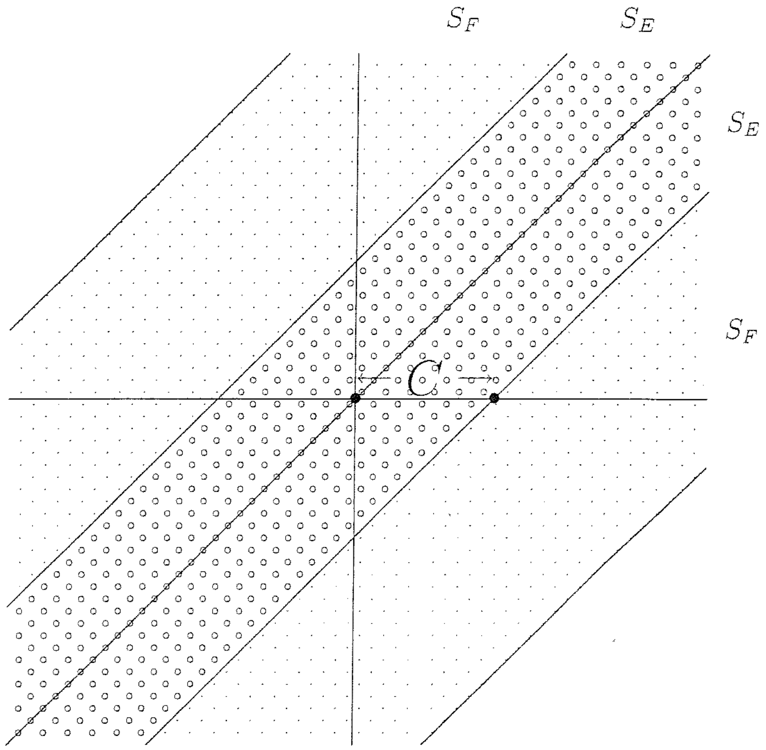
- *u is a positive kernel*
- *$u(s,t) \neq 0$ only if $s^{-1}t \in F$*
- *$|1 - u(s,t)| < \epsilon$ if $s^{-1}t \in E$*

Example with the Integers

This part provides an example to help us visualise what Ozawa kernels represent. The following diagram of $\mathbb{Z} \times \mathbb{Z}$ allows us to visualize the requirements of Property O.

Assume that we are given some $\epsilon > 0$, and that we have a set $E \in \mathbb{Z}$ such that the largest absolute value of an element of E is C. We have another set F larger than E such that the largest absolute value of an element of F is R.

Now consider the following diagram of $\mathbb{Z} \times \mathbb{Z}$:



The central diagonal line represents pairs of elements of $\mathbb{Z}$ which are distance 0 apart. The inner shaded area S_E represents points which are distance less than some C apart. The outer shaded area S_F (which contains S_E) represents points which are distance less than some $R > C$ apart. Finally, the non shaded area represents points which are distance greater than R apart.

Then S_F contains all pairs s, t such that $s^{-1}t \in F$. Similarly, S_E contains all pairs s, t such that $s^{-1}t \in E$.

Any property O function $u(s,t) : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ must have the following properties. In the shaded area S_E, $u(s,t)$ must be ϵ-close to 1. Outside S_F, $u(s,t)$ must be 0. In the area belonging to S_F but not S_E, $u(s,t)$ may take any value.

Finally we look at Hilbert space compression, introduced by Guentner and Kaminker in [GK2, Defn 2.2] and which is an invariant of the group taking a value between 0 and 1. If this invariant is strictly greater than 1/2 then the group is exact. However the converse is not true.

3.3 Hilbert space compression

3.3.1 Original definition

In [GK2, Defn 2.2] Guentner and Kaminker introduce the concept of Hilbert space compression. This concept is linked to that of uniform embeddability. By embedding a space into a Hilbert space we may be able to deduce properties of that space based on properties of the embedding.

In this case Guentner and Kaminker restrict their attention to embeddings via Large-scale Lipschitz functions which we now define:

Definition 3.3.1 (Large-scale Lipschitz). *A function $f : X \to Y$ is* *large-scale Lipschitz if there exist $C > 0$ and $D \geq 0$ such that*

$$d_Y(f(x), f(y)) \leq C d_X(x, y) + D$$

These functions give us some control over the distortion of the embedding. The distance of the images in the target space Y is linked to that of the elements in the original space X. This ensures that points which are close together in the original space are not mapped to points which are too far apart in the target space.

We now define the compression of a large-scale Lipschitz function as follows:

Definition 3.3.2 (Compression). *Following Gromov, the compression $\rho(f)$ of $f \in Lip^{ls}(X, Y)$ is*

$$\rho_f(r) = \inf_{d_X(x,y) \geq r} d_Y(f(x), f(y))$$

This gives us information on how much distances in the original space are compressed by the large scale Lipschitz functions. It looks at pairs of points at least some distance r apart in X and finds the minimum distance of their images in Y.

We are now ready to introduce Guentner and Kaminker's notion of asymptotic and Hilbert space compression. The asymptotic compression is the result of a calculation involving the compression of a general large scale Lipschitz map between any two metric spaces X and Y. The supremum of this value over all possible large scale Lipschitz maps is termed the compression of X in Y. When the target space Y is a Hilbert space, then it is called Hilbert space compression. This is an invariant taking a value between 0 and 1 which roughly speaking measures the necessary distortion which occurs when embedding the group into a Hilbert space via Large Scale Lipschitz Maps. The formal definition is as follows [GK2, Defn 2.2]:

Definition 3.3.3 (Asymptotic and Hilbert space compression). *Let X be a metric space with an unbounded metric.*

1. *The asymptotic compression R_f of a large scale Lipschitz map $f \in Lip^{ls}(X, Y)$ is*
$$R_f = \liminf_{r\to\infty} \frac{\log \rho_f^*(r)}{\log r}$$
where $\rho_f^*(r) = \max\{\rho_f(r), 1\}$.

2. *The compression of X in Y is*
$$R(X, Y) = \sup\{R_f : f \in Lip^{ls}(X, Y)\}.$$

3. *If Y is a Hilbert space, then the Hilbert space compression of X is*
$$R(X) = R(X, Y).$$

This invariant is linked to the property of uniform embeddability. In particular, groups which are not uniformly embeddable in a Hilbert space have Hilbert space compression 0 [GK2, Prop 3.1]. But this property is also linked to exactness in the following way. Guentner and Kaminker show that if the Hilbert space compression of a finitely generated discrete group is strictly greater than 1/2, then there exists a family of Ozawa kernels for this group and hence it has Property O and is exact. This is summarised in the following theorem, [GK2, thm 3.2]:

Theorem 9. *Let Γ be a finitely generated discrete group regarded as a metric space via the word metric. If the Hilbert space compression of Γ is strictly greater than 1/2 then Γ is exact.*

To show that a group Γ is exact, it suffices to construct a large scale Lipschitz embedding of the group into a Hilbert space such that its Hilbert space compression is strictly greater than 1/2. Guentner and Kaminker illustrate this approach with the free group of rank 2, proving that its Hilbert space compression is 1 [GK2, Prop 4.2]. We will look at this in more detail in a later chapter and will adapt their construction to groups acting properly and cocompactly on CAT(0) cube complexes.

It is important to note that having Hilbert space compression less than or equal to 1/2 does not necessarily mean that the group is not exact. For example, the group $\mathbb{Z}\ wr\ (\mathbb{Z}\ wr\ \mathbb{Z})$, which is the wreath product of $\mathbb{Z}$ with its own wreath product on itself, is amenable and hence exact, but its Hilbert space compression lies between 0 and 1/2 [AGS, Cor 1.10].

3.3.2 Alternative definition

The definition of Hilbert space compression for a finitely generated discrete group is expressed slightly differently in [AGS, Defn 1.2]. Their definition relies more noticeably on the concept of uniform embeddings which we will now define:

Definition 3.3.4 (uniform embedding). *Let (Γ, d) be a metric space. Let $\mathcal{H}$ be a separable Hilbert space. A map $f : \Gamma \longrightarrow \mathcal{H}$ is said to be a uniform embedding if there exist non-decreasing functions $\rho_1, \rho_2 : \mathbb{R}_+ \longrightarrow \mathbb{R}$ such that*

1. $\rho_1(d(x,y)) \leq \|f(x) - f(y)\|_{\mathcal{H}} \leq \rho_2(d(x,y))$ *for all* $x, y \in \mathbb{R}$
2. $\lim_{r \longrightarrow +\infty} \rho_i(r) = +\infty$ *for* $i = 1, 2$

The definition of Hilbert space compression in [AGS, Defn 1.2] is expressed as follows:

Definition 3.3.5 (Hilbert space compression). *The Hilbert space compression of a finitely generated discrete group G is the number $R(G)$ which is the supremum of all $\alpha \geq 0$ for which there exists a uniform embedding of G into a Hilbert space with $\rho_1(r) = Cr^\alpha$ with a constant $C > 0$ and linear ρ_2.*

It may not be immediately apparent that this definition is equivalent to the one originally stated in [GK2, Defn 2.2].

Consider the case when $\rho_1(r) = r^\alpha$. Since this is the lower bound for the embedding, we can substitute this into the definition for asymptotic compression. We get

$$R_f = \liminf_{r \longrightarrow \infty} \frac{\log(r^\alpha)}{\log(r)} = \liminf_{r \longrightarrow \infty} \alpha \frac{\log(r)}{\log(r)} = \alpha$$

And so by definition, the Hilbert space compression is the supremum of all such α as required.

Chapter 4

Equivalence of Property A and Property O

Since both Property O and Property A are equivalent to exactness, they are also equivalent to each other. However, it is possible to establish their equivalence directly without needing to refer to exactness. We will now prove the following theorem.

Theorem (1). *Property A is equivalent to Property O.*

4.0.3 Equivalent definitions of Property A

We first need some alternative definitions of Property A. As previously noted, in [Tu, Prop 3.2], six equivalent versions of Property A are introduced. For our purposes we will concentrate on definition 6 which we recall here.

Proposition 4.0.6. *Let X be a discrete metric space with bounded geometry. The following are equivalent:*

- *X has Property A (as expressed in [Yu])*
- *$\forall R > 0, \forall \epsilon > 0, \exists S > 0, \exists \varphi : X \times X \to \mathbb{R}$ such that*
  - *φ is of positive type*
  - *supp $\varphi \subset \Delta_S$*
  - *$|1 - \varphi(x, y)| \leq \epsilon$ whenever $d(x, y) \leq R$*

This definition of Property A is in a form very reminiscent of Property O. It concerns a positive function, the support of which is contained in a strip around the diagonal, and whose value is close to 1 on balls of radius R.

We will first show that the existence of a family of Ozawa kernels implies the existence of a function satisfying the above definition. We will then prove the converse, thus showing that the two properties are equivalent.

4.0.4 Equivalence of Property A and Property O

We will now prove the following theorem:

Theorem (1). *Property A is equivalent to Property O.*

Proof.

Lemma 4.0.7. *Property O implies Property A.*

Proof. Assume that a space X has property O. So for any finite subset $E \subset X$ and any $\epsilon > 0$, there are a finite subset $F \subset X$ and $u \colon X \times X \to \mathbb{R}$ such that

- u is a positive kernel
- $u(x, y) \neq 0$ only if $x^{-1}y \in F$
- $|1 - u(x, y)| < \epsilon$ if $x^{-1}y \in E$

We will show that this implies the existence of a function φ which satisfies Tu's definition of Property A.

We have that for any finite subset $E \subset X$ and any $\epsilon > 0$, $|1 - u(x, y)| < \epsilon$ if $x^{-1}y \in E$. So in particular this is true for any ball of radius R and any $\epsilon > 0$. Define $u_{R,\epsilon}$ to be the Ozawa kernel associated to the ball of radius R and some $\epsilon > 0$. Then given any R and ϵ, take $\varphi(x, y) = u_{R,\epsilon}(x, y)$. This is a positive kernel. We have as required that $\forall R$, $\forall \epsilon$, $\exists u_{R,\epsilon}$ (and hence $\varphi(x, y)$) such that if $d(x, y) < R$, $|1 - \varphi(x, y)| < \epsilon$. Finally, if we take S to be the radius of the ball containing F, then $\mathrm{supp}(\varphi) \subset \Delta_S$ as required. □

Lemma 4.0.8. *Property A implies Property O.*

Proof. Assume that a space X has property A. So $\forall R > 0, \forall \epsilon > 0, \exists S > 0, \exists \varphi : X \times X \to \mathbb{R}$ such that

- φ is of positive type
- supp $\varphi \subset \Delta_S$

- $|1-\varphi(x,y)| \leq \epsilon$ whenever $d(x,y) \leq R$

Let $\varphi_{R,\epsilon}$ be the function satisfying the above conditions for a particular pair R, ϵ. Given any finite subset E, there exists R such that E is contained in the ball of radius R. Now assume we want to find an Ozawa kernel for some pair (E, ϵ). We enlarge E and consider the ball of radius R containing it. Since the space has property A we know that there exists a $\varphi_{R,\epsilon}$.

Given any finite subset E and ϵ, we take $u(x,y) = \varphi_{R,\epsilon}(x,y)$.

This is of positive type. By the definition of $\varphi_{R,\epsilon}(x,y)$, for any $x^{-1}y \in B_R$, $|1-\varphi(x,y)| \leq \epsilon$ and so for any $x^{-1}y \in E \subseteq B_R$, $|1-u(x,y)| \leq \epsilon$.

We take F to be the ball of radius S.

Hence $u(x,y)$ is an Ozawa kernel. Such a function exists for any pair (E, ϵ) and thus there exists a family of Ozawa kernels and X has Property O. □

Hence Property A also implies Property O and these properties are equivalent for discrete metric spaces. □

Chapter 5

Exactness and Property O

Although the existence of families of Ozawa kernels is well known and has been used to prove the theorem relating Hilbert space compression and exactness [GK2, Thm 3.2], there exists no explicit example of a family of Ozawa kernels in the literature.

In this chapter we will construct an explicit family of Ozawa kernels for the integers and then extend the construction to groups of subexponential growth and on to the class of amenable groups. We will then give a construction for free groups. In each case the Ozawa kernels constructed can be viewed as weighted mass functions.

Constructing these families of Ozawa kernels shows that all these groups satisfy Property O. As corollaries we deduce that these groups are both exact and have Yu's Property A.

Parts of this chapter are to appear in the Bulletin of the London Mathematical Society [C]. The paper is included in the appendix of this thesis.

5.1 The Integers $\mathbb{Z}$

We choose this group since it is simple to define and has a lot of properties which we might be able to exploit. For instance it is an abelian group, has low growth rate and is also amenable.

We will first recall Ozawa's formal definition and then show that there exists a family of Ozawa kernels for this group.

Definition 3.2.8 *A discrete group G is said to have Ozawa's Property O if for any finite subset $E \subset G$ and any $\epsilon > 0$, there are a finite subset $F \subset G$ and $u\colon G \times G \to \mathbb{C}$ such that*

- *u is a positive kernel*
- *$u(x, y) \neq 0$ only if $x^{-1}y \in F$*
- *$|1 - u(x, y)| < \epsilon$ if $x^{-1}y \in E$*

Properties of Ozawa kernels

Finding a family of Ozawa kernels is complex. We will now briefly discuss some ideas which may allow us to construct such a function.

First of all an Ozawa kernel must be positive. The function must satisfy for any set of $\lambda_i, \lambda_j \in \mathbb{R}$,

$$\sum_{i,j}^{n} \lambda_i \lambda_j u(g_i, g_j) \geq 0$$

Showing that a function is positive is non trivial and is the subject of much research in the area of operational research. One type of positive function is one which can be separated into a product of two functions, one dependent only on i and the other only on j. In this case we have a sum of squares, which is positive. This method is used in [NR, Technical Lemma] to prove that a particular distance function in CAT(0) cube complexes is negative definite.

Secondly there must exist a finite set F outside which the function has value 0. One possible solution might be to define a function which uses the characteristic function $\chi(x)$ on some set. $\chi(x) = 1$ if x belongs to the set and 0 otherwise. Given a function on two variables x, y we then obtain that it takes the value 0 when neither x nor y belong to the set.

Finally there must be a finite set E in which the function is ϵ close to 1. This may require some scaling of the function we choose. Note that the set E must be contained in F. The main difficulty is obtaining a function which will work for any possible pair (E, ϵ).

Ozawa kernels for the integers

Take $\mathbb{Z}$ and consider the function u_n which counts the number of balls of a given radius n containing both points x and y, scaled by the size of a ball of radius n. The pair (E, ϵ) as given in the definition of Property O will determine the value of n. We will now prove that each u_n is an Ozawa kernel and that the set of all u_n forms a family of Ozawa kernels.

We will denote a ball of radius n by B_n.

Let $u_n(x, y) : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ be the number of balls of radius n which contain both x and y, scaled by the size of a ball of radius n. There are two other ways of defining this function which will come in useful.

A ball of radius n contains both x and y if and only if its centre is within distance n of both x and y. Hence the function $u_n(x, y)$ is also the number of vertices which are such centres. This is precisely the size of the intersection of the ball of radius n centred at x and the one centred at y. Hence

$$u_n(x, y) = \frac{|xB_n \cap yB_n|}{|B_n|}$$

Now consider the characteristic function of balls of radius n on $\mathbb{Z}$:

$$\chi_{k,n}(x) = \begin{cases} 1 & \text{if } x \in \text{the ball of radius } n \text{ centred at } k \\ 0 & \text{if } x \notin \text{the ball of radius } n \text{ centred at } k \end{cases}$$

The size of a ball of radius n in $\mathbb{Z}$ is $2n + 1$. We can rewrite $u_n(x, y)$ as follows

$$u_n(x,y) = \frac{1}{2n+1}\sum_{k\in\mathbb{Z}} \chi_{k,n}(x)\chi_{k,n}(y)$$

An element $k \in \mathbb{Z}$ contributes to this sum if and only if both x and y are contained in the ball of radius n centred at k. (Otherwise one or both of $\chi_{k,n}(x)$ or $\chi_{k,n}(y)$ must be equal to 0 and thus the product would be 0.) Hence this sum is equal to the total number of elements $k \in \mathbb{Z}$ which are the centres of a ball of radius n containing both x and y. This is precisely $u_n(x,y)$ as previously described.

Remark 5.1.1. *This sum is finite since although there are infinitely many balls of radius n, there are only finitely many which contain both x and y.*

We will now show that the set of functions u_n forms a family of Ozawa kernels.

Lemma 5.1.2. *For any n, u_n is a positive kernel.*

Proof. To see this, note that:

$$\begin{aligned}
\sum_{i,j}^{n} \lambda_i \lambda_j u_n(x_i, x_j) &= \frac{1}{2n+1} \sum_{i,j}^{n} \lambda_i \lambda_j \sum_{k \in \mathbb{Z}} \chi_{k,n}(x_i) \chi_{k,n}(x_j) \\
&= \frac{1}{2n+1} \sum_{k \in \mathbb{Z}} \left(\sum_{i}^{n} \lambda_i \chi_{k,n}(x_i) \sum_{j}^{n} \lambda_j \chi_{k,n}(x_j) \right) \\
&= \frac{1}{2n+1} \sum_{k \in \mathbb{Z}} \left(\sum_{i}^{n} \lambda_i \chi_{k,n}(x_i) \right)^2 \\
&\geq 0
\end{aligned}$$

□

Lemma 5.1.3. *For any n, there exists a finite subset F such that $u_n(x, y) \neq 0$ only if $x^{-1}y \in F$.*

Proof. Take F to be the ball of radius n around the origin. If the distance between two points x and y is strictly greater than $2n$, (i.e. $x^{-1}y$ is outside F), then no ball of radius n can contain both points x and y. So $u_n(x, y) = 0$ as required. □

Lemma 5.1.4. *Given any finite set E and $\epsilon > 0$, we can choose n so that $u_n(x, y)$ is $\epsilon-$close to 1 on E.*

Proof. We need to obtain a function $u_n(x, y)$ such that on E, $1 - \epsilon < u_n(x, y) < 1 + \epsilon$.

E is a finite subset. So there exists some finite m such that if $x^{-1}y \in E$ then $0 \leq d(x, y) \leq m$.

In $\mathbb{Z}$, the size of the intersection of two balls of radius n around two points x and y is either 0 or, if $d(x,y) \leq 2n+1$, it is equal to $2n+1-d(x,y)$.

So on E, we have $2n+1-m \leq |xB_n \cap yB_n| \leq 2n+1$.

So we get

$$1 - \frac{m}{2n+1} \leq u_n(x,y) \leq 1$$

So to ensure that $u_n(x,y) > 1-\epsilon$, we must choose n such that

$$\frac{m}{2n+1} < \epsilon$$

So for any pair (E,ϵ), we choose some $n = N$ such that

$$N > \frac{m}{2\epsilon} - \frac{1}{2}$$

And we take u_N to be:

$$u_N(x,y) = \frac{1}{2N+1} \sum_{k \in \mathbb{Z}} \chi_{k,N}(x)\chi_{k,N}(y)$$

□

This u_N is an Ozawa kernel and such a u_N exists for any pair (E,ϵ). Thus we have a new geometric proof of the following theorem:

Theorem (2). *The family of kernels u_N forms a family of Ozawa kernels for the group $\mathbb{Z}$ and so $\mathbb{Z}$ satisfies property O.*

The following corollary immediately follows:

Corollary (1.2.1). *$\mathbb{Z}$ is an exact group and therefore satisfies Property A.*

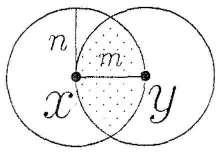
5.2 Groups of Subexponential Growth

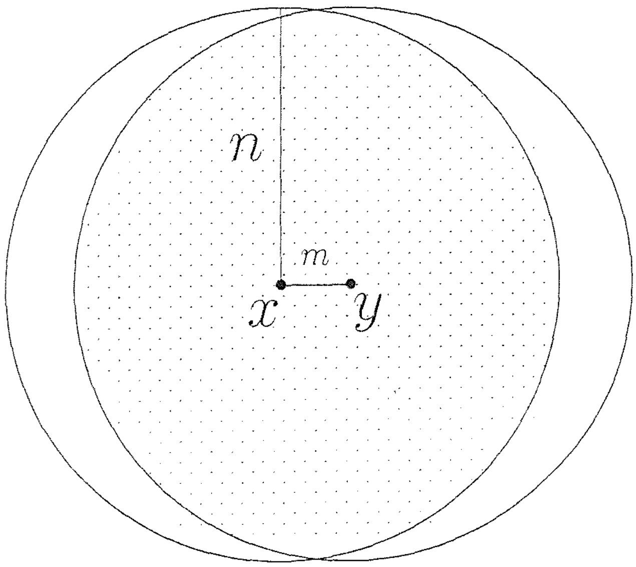
5.2.1 Expansion to groups of subexponential growth

Our previous set of functions for the integers forms a family of Ozawa kernels because of its slow growth rate. As n increases, the size of the boundary of a ball of radius n grows slowly compared to its volume.

Each function u_n looks at the size of the intersection of two balls of radius n which are a given distance m apart. As we increase n, this overlap gets larger and larger. More importantly, it gets closer and closer to the size of B_n

This is represented in the following diagram. We take two points x and y distance m apart and consider the balls of radius n centred at each point. Our function takes the size of the intersection, which is the shaded area of the diagram, and scales it by the size of a ball of radius n.


n
m
x
y


n
m
x
y

The difference between the size of the overlap and the size of a ball of radius n is roughly represented by the non shaded area of the diagram. Since the boundary is growing much more slowly than the volume, the size of the non shaded area becomes negligeable in comparision to that of the shaded area. Hence as n tends to infinity, the size of the intersection tends to the size of B_n. Thus our family of functions u_n forms a sequence of functions which tends to 1. This is what allowed us to prove that the set of functions u_n is a family of Ozawa kernels.

Since the growth property of the group $\mathbb{Z}$ is the only property our method relies on, we now turn our attention to the more general case of groups of subexponential growth.

A former method of proving that this class of groups is exact was to show that the sequence of balls of radius n form a Følner sequence which means that groups of subexponential growth are amenable [BHV, Cor G.5.5]. (We will say more about Følner sequences in the next section regarding amenable groups). It has been shown by Kirchberg and Wasserman that amenable groups are exact [KW, p.174].

We will adopt a different method and exploit their growth property to construct a family of Ozawa kernels, thus proving that they have Property O. As a direct corollary of this theorem we can deduce that they are exact and satisfy Yu's Property A.

5.2.2 Background and definitions

We will first review the definition of groups of subexponential growth.

Definition 5.2.1 (Growth function). *Let Γ be a group with generating set A. Let $\beta_A(n)$ be the number of vertices in the closed ball of radius n about 1 in the Cayley graph of the group generated by A. The growth function of Γ with respect to A is $n \to \beta_A(n)$.*

Definition 5.2.2 (Subexponential growth). *Γ has subexponential growth if $\beta_A(n) \leq e^{\sqrt{n}}$ for all $n \in \mathbb{N}$.*

Examples of groups of subexponential growth include finite groups, abelian groups, nilpotent groups and Grigorchuk's group.

5.2.3 Construction of a family of Ozawa kernels

We will now construct a family of Ozawa kernels to show that groups of subexponential growth satisfy Property O.

We denote a ball of radius n by B_n and the intersection of two balls of radius n centred at x and y by $xB_n \cap yB_n$.

We first need the following lemma:

Lemma 5.2.3. *If d is the distance between x and y, then $|B_{n-d}| \leq |xB_n \cap yB_n|$*

Proof. An element $h \in G$ belongs to the intersection of the balls of radius n centred at x and y only if $d(x,h)$ and $d(y,h) \leq n$.

Now consider the ball of radius $n-d$ centred at y. An element $g \in G$ belongs to this ball only if $d(y, g) \leq n-d$. So any such g also belongs to the ball of radius n centred at y.

We also have that

$$d(x, g) \leq d(x, y) + d(y, g) \leq d + n - d = n$$

And so g also belongs to the ball of radius n centred at x.

Hence the ball of radius $n-d$ centred at y is contained in the intersection of the balls of radius n centred at x and y.

Since $|B_{n-d}| = |yB_{n-d}|$, we have that $|B_{n-d}| \leq |xB_n \cap yB_n|$

$\square$

Now let G be a group of subexponential growth.

Consider the function

$$u_n(x, y) = \frac{|xB_n \cap yB_n|}{|B_n|}$$

This is the scaled number of points in the intersection between two balls of radius n centred at x and y respectively. Alternatively it can be regarded as the scaled number of balls of radius n which contain both points x and y.

Lemma 5.2.4. *For any n, u_n is a positive kernel.*

Proof. An element $g \in G$ belongs to the intersection $xB_n \cap yB_n$ only if

$g \in xB_n$ and $g \in yB_n$. This is equivalent to $x^{-1}g \in B_n$ and $y^{-1}g \in B_n$. Again this is equivalent to $x^{-1} \in B_n g^{-1}$ and $y^{-1} \in B_n g^{-1}$. Since B_n is symmetric, $B_n = B_n^{-1}$. And so the condition becomes $x \in gB_n$ and $y \in gB_n$.

So u_n can be rewritten as

$$u_n(x,y) = \frac{1}{|B_n|} \sum_{g \in G} \chi_{g,n}(x)\chi_{g,n}(y)$$

where for any $k \in G$, $\chi_{g,n}(k)$ is 1 when k belongs to the ball of radius n centred at g and 0 otherwise. Thus we have:

$$\begin{aligned}
\sum_{i,j}^{n} \lambda_i \lambda_j u_n(x_i, x_j) &= \sum_{i,j}^{n} \lambda_i \lambda_j \frac{1}{|B_n|} \sum_{g \in G} \chi_{g,n}(x_i)\chi_{g,n}(x_j) \\
&= \frac{1}{|B_n|} \sum_{g \in G} \left(\sum_{i}^{n} \lambda_i \chi_{g,n}(x_i) \sum_{j}^{n} \lambda_j \chi_{g,n}(x_j) \right) \\
&= \frac{1}{|B_n|} \sum_{g \in G} \left(\sum_{i}^{n} \lambda_i \chi_{g,n}(x_i) \right)^2 \\
&\geq 0
\end{aligned}$$

□

Lemma 5.2.5. *For any n, there exists a finite subset F such that $u_n(x,y) \neq 0$ only if $x^{-1}y \in F$.*

Proof. Take F to be the ball of radius n around the origin. If the distance between two points x and y is strictly greater than $2n$,(i.e. $x^{-1}y$ is outside F), no ball of radius n can contain both points x and y. So $u_n(x,y) = 0$ as required. $\square$

Lemma 5.2.6. *Given any finite set E and $\epsilon > 0$ we can choose n such that u_n is $\epsilon-$close to 1 on E.*

Proof. We showed earlier that $|B_{n-d(x,y)}| \leq |xB_n \cap yB_n|$. In addition $|xB_n \cap yB_n| \leq |B_n|$.

Hence

$$\frac{|B_{n-d(x,y)}|}{|B_n|} \leq u_n(x,y) \leq \frac{|B_n|}{|B_n|}$$

E is a finite subset of G. So there exists a maximum distance m such that if $x^{-1}y \in E$ then $0 \leq d(x,y) \leq m$. This means that if $x^{-1}y \in E$, then $|B_{n-m}| \leq |B_{n-d(x,y)}| \leq |B_n|$. And so on E,

$$\frac{|B_{n-m}|}{|B_n|} \leq u_n(x,y) \leq 1$$

Since G is a group of subexponential growth, we have

$$\frac{e^{\sqrt{n-m}}}{e^{\sqrt{n}}} \leq u_n(x,y) \leq 1$$

Since $\lim_{n\to\infty} \frac{e^{\sqrt{n-m}}}{e^{\sqrt{n}}} = 1$, for any pair (ϵ, m) and (and therefore any pair (E,ϵ)), there exists $n = N$ such that $|1 - u_N(x,y)| \leq \epsilon$ as required. $\square$

The function $u_N(x,y) = \frac{|xB_N \cap yB_N|}{|B_N|}$ is an Ozawa kernel. Such a u_N exists for any pair (E, ϵ) and we have a direct geometric proof of the following theorem:

Theorem (3). *The family of kernels u_N forms a family of Ozawa kernels for groups of subexponential growth and so they satisfy property O.*

The following corollary immediately follows:

Corollary (1.2.2). *Groups of subexponential growth are exact and therefore have Yu's Property A.*

5.3 Amenable groups

The integers and groups of subexponential growth are both examples of amenable groups. Amenable groups form a well known example of exact groups [KW, p.174].

One of the definitions of an amenable group is the existence of Følner sets which we will explain in further detail in the next section. These can be viewed as a sequence of subsets of the group with a particular relationship between the size of the intersection of two sets and the size of the set itself. In the case of the integers or groups of subexponential growth, balls of radius n actually form Følner sets, and this fact can be used to deduce that these groups are amenable and hence exact. We will now expand our previous function and adapt it to general Følner sets rather than balls of radius n. In doing so we will show that amenable groups admit a family of Ozawa kernels and so satisfy Property O. We deduce as a corollary that amenable groups are exact and have Yu's Property A .

5.3.1 Definitions and properties of amenable groups

Amenability was first described as a measure theoretic property as follows:

Let Γ be a locally compact group and $\mathbf{L}^\infty(\Gamma)$ be the Banach space of all essentially bounded functions $\Gamma \longrightarrow \mathbb{R}$ with respect to the Haar measure.

Definition 5.3.1 (Mean). *A linear functional on* $\mathbf{L}^\infty(\Gamma)$ *is called a mean if it maps the constant function* $f(g) = 1$ *to 1 and non-negative functions to non-negative numbers.*

Definition 5.3.2. *Let* $\mathbf{L}_g$ *be the left action of* $g \in \Gamma$ *on* $f \in \mathbf{L}^\infty(\Gamma)$. $(\mathbf{L}_g f)(h) = f(gh)$.

A mean is said to be left invariant if $\mu(\mathbf{L}_g f) = \mu(f)$ for all $g \in \Gamma$ and $f \in \mathbf{L}^\infty(\Gamma)$.

Similarly, a mean is right invariant if $\mu(\mathbf{R}_g f) = \mu(f)$ for all $g \in \Gamma$ and $f \in \mathbf{L}^\infty(\Gamma)$, where $\mathbf{R}_g$ is the right action .

Definition 5.3.3 (Amenable). *A locally compact group is amenable if there is a left (or right) invariant mean on* $\mathbf{L}^\infty(\Gamma)$.

Example 3. *All finite groups and all abelian groups are amenable. Compact groups are amenable as the Haar measure is an (unique) invariant mean.*

One source of interest in the notion of amenability is using it as a way to classify groups. A large class of groups is that of groups which have Kazhdan's property T. A group G is said to have this property if every representation of G which almost has invariants also has a non trivial invariant vector. For countable groups this is equivalent to property (FH) where every isometric action of G on a Hilbert space has a fixed point [HV]. Discrete infinite groups which are amenable do not have Kazhdan's property T and discrete infinite groups which have Kazhdan's property T are not amenable [Wa, Rem 3.9]. These form two very large disparate classes of groups. Groups which have neither property are the subject of much study.

The notion of amenability has been extended to many equivalent conditions which are summarised in the following theorem [W, Thm 10.11(AC)].

Theorem 10. *For a group G, the following are equivalent:*

1. *G is amenable.*

2. *There is a left-invariant mean on G.*

3. *G is not paradoxical.*

4. *G satisfies the Invariant Extension Theorem: A G- invariant measure on a subring of a Boolean algebra may be extended to a G-invariant measure on the entire algebra.*

5. *G satisfies the Hahn-Banach Extension Property: Suppose*

    - *G is a group of linear operators on a real vector space V;*

- *F is a G-invariant linear functional on V_o, a G-invariant subspace of V;*
- *$F(v) \leq p(v)$ for all $v \in V_o$, where p is some real-valued function on V such that $p(v_1 + v_2) \leq p(v_1) + p(v_2)$ for $v_1, v_2 \in V_o$, $p(\alpha v) \leq \alpha p(v)$ for $\alpha \geq 0$, $v \in V_o$, and $p(g(v)) \leq p(v)$ for $g \in G, v \in V$.*

Then there is a G-invariant linear functional $\overline{F}$ on V that extends F and is dominated by p.

6. *G satisfies Følner's Condition: For any finite subset E of G and every $\epsilon > 0$, there is another finite subset W of G such that for any $g \in E$, $\frac{|gW \triangle W|}{|W|} \leq \epsilon$*

7. *G satisfies Dixmier's Condition: If $f_1, ..., f_n \in B(G)$ and $g_1, ..., g_n \in G$, then for some $h \in G$, $\sum f_i(h) - f_i(g_i^{-1}h) \leq 0$*

8. *G satisfies the Markov-Kakutani Fixed Point Theorem: Let K be a compact convex subset of a locally convex linear topological space X, and suppose G acts on K in such a way that each transformation $g : K \to K$ is continuous and affine $g(\alpha x + (1-\alpha)y) = \alpha g(x) + (1-\alpha)g(y)$ whenever $x, y \in K$ and $0 \leq \alpha \leq 1$). Then there is some x in K that is fixed by each $g \in G$.*

For our purposes we will only consider definition 6 of amenablity: Folner's criterion. We will expand our previous function on balls of radius n to more general Følner sets and show that amenable groups have Property O.

5.3.2 A family of Ozawa kernels for amenable groups

In this proof we will use the following version of Følner's criterion:

Følner's condition

Definition 5.3.4 (Følner's condition). *An amenable group satisfies Følner's condition [W, Thm10.11(AC)]: for any finite subset E of G and every $\epsilon > 0$, there is another finite subset W of G such that for any $g \in E$, $\frac{|gW \triangle W|}{|W|} \leq \epsilon$*

This can be rewritten as follows:

$$\begin{aligned} \frac{|gW \triangle W|}{|W|} &= \frac{|gW \cup W - gW \cap W|}{|W|} \\ &= \frac{|gW \cup W|}{|W|} - \frac{|gW \cap W|}{|W|} \end{aligned}$$

The maximum possible value of $|gW \cup W|$ is $2|W|$ and the minimum is $|W|$. Similarly, the maximum possible value of $|gW \cap W|$ is $|W|$ and the minimum is 0.

So $\frac{|gW \cup W|}{|W|}$ lies between 1 and 2, while $\frac{|gW \cap W|}{|W|}$ lies between 0 and 1. Since the difference between them is less than ϵ and we are dealing with bounded sets of real numbers, we have that $\left|1 - \frac{|gW \cap W|}{|W|}\right| < \epsilon$.

A family of Ozawa kernels

Consider the function

$$u_W(x,y) = \frac{|xW \cap yW|}{|W|}$$

We will show that the set of all such functions forms a family of Ozawa kernels.

Lemma 5.3.5. *For any finite subset W, $u_W(x,y)$ is a positive kernel*

Proof. An element $g \in G$ belongs to the intersection $xW \cap yW$ only if $g \in xW$ and $g \in yW$. This is equivalent to $x^{-1}g \in W$ and $y^{-1}g \in W$. Again this is equivalent to $x^{-1} \in Wg^{-1}$ and $y^{-1} \in Wg^{-1}$. And so the condition becomes $x \in gW^{-1}$ and $y \in gW^{-1}$.

So u can be rewritten as

$$u(x,y) = \frac{1}{|W|}\sum_{g \in G} \chi_{gW^{-1}}(x)\chi_{gW^{-1}}(y)$$

Thus:

$$\begin{aligned}
\sum_{i,j}^{n} \lambda_i \lambda_j u(x_i, x_j) &= \sum_{i,j}^{n} \lambda_i \lambda_j \frac{1}{|W|} \sum_{g \in G} \chi_{gW^{-1}}(x_i) \chi_{gW^{-1}}(x_j) \\
&= \frac{1}{|W|} \sum_{g \in G} \left(\sum_{i}^{n} \lambda_i \chi_{gW^{-1}}(x_i) \sum_{j}^{n} \lambda_j \chi_{gW^{-1}}(x_j) \right) \\
&= \frac{1}{|W|} \sum_{g \in G} \left(\sum_{i}^{n} \lambda_i \chi_{gW^{-1}}(x_i) \right)^2 \\
&\geq 0
\end{aligned}$$

□

Lemma 5.3.6. *Given any finite subset E and $\epsilon > 0$, there exists a finite set W and associated function $u_W(x, y)$ such that $|1 - u_W(x, y)| < \epsilon$ if $x^{-1}y \in E$*

Proof. From Følners Condition we have that for any $g \in E$ there exists W such that $\left|1 - \frac{|gW \cap W|}{|W|}\right| < \epsilon$. So now let g be equal to $x^{-1}y$ and the condition becomes $\left|1 - \frac{|x^{-1}yW \cap W|}{|W|}\right| = \left|1 - \frac{|xW \cap yW|}{|W|}\right| = |1 - u_W(x, y)| \leq \epsilon$. And so we have as required that if $x^{-1}y \in E$, then $|1 - u_W(x, y)| \leq \epsilon$. □

Lemma 5.3.7. *Given any pair (E, ϵ) and associated $W, u_W(x, y)$, there exists a finite set F such that $u_W(x, y) \neq 0$ only if $x^{-1}y \in F$*

Proof. Since W is finite, it is contained within a ball of some diameter r. Let F be the ball of radius r around the origin. If $d(x, y) > 2r$, ie $x^{-1}y \notin F$ there is no intersection between xW and yW, so $u_W(x, y) = 0$. □

Every u_W is an Ozawa kernel and such a function exists for any pair (E, ϵ). Thus we have directly proved the following theorem:

Theorem (4). [C, Thm 2] *The family of kernels u_N forms a family of Ozawa kernels for amenable groups and so they satisfy property O.*

The following corollary immediately follows:

Corollary (1.2.3). [C, Cor 3.4] *Amenable groups are exact and therefore have Yu's Property A.*

5.3.3 A variation on the family of Ozawa kernels for amenable groups

Følner's condition can also be expressed in the following way which allows us to create a variation on our previous family of functions:

Theorem 11. *A group G is amenable if there exists a sequence $\{G_n\}$ of subsets of G such that $\forall g \in G$,*

$$\lim_{n \to \infty} \frac{|gG_n \triangle G_n|}{|G_n|} = 0$$

This can be rewritten as follows:

$$\begin{aligned} \frac{|gG_n \triangle G_n|}{|G_n|} &= \frac{|gG_n \cup G_n - gG_n \cap G_n|}{|G_n|} \\ &= \frac{|gG_n \cup G_n|}{|G_n|} - \frac{|gG_n \cap G_n|}{|G_n|} \end{aligned}$$

The maximum possible value of $|gG_n \cup G_n|$ is $2|G_n|$ and the minimum is $|G_n|$. Similarly, the maximum possible value of of $|gG_n \cap G_n|$ is $|G_n|$ and the minimum is 0.

So $\frac{|gG_n \cup G_n|}{|G_n|}$ lies between 1 and 2, while $\frac{|gG_n \cap G_n|}{|G_n|}$ lies between 0 and 1. Since the difference between them tends to 0 and we are dealing with bounded sets of real numbers, we have that $\lim_{n\to\infty} \frac{|gG_n \cap G_n|}{|G_n|} = 1$.

Consider the sequence of functions

$$u_n(x, y) = \frac{|xG_n \cap yG_n|}{|G_n|}$$

By a very similar method to the previous case, we can show that this sequence forms a family of Ozawa kernels by proving the following three lemmas:

Lemma 5.3.8. *For each n, u_n is a positive kernel.*

To do this we adopt the same method as in all previous constructions. We rewrite u_n as a sum over all elements of the group of a product of characteristic functions on x and y and show that this is a sum of squares.

Lemma 5.3.9. *Given any finite subset E and $\epsilon > 0$ there exists a function u_N such that $|1 - u_N(x, y)| < \epsilon$ if $x^{-1}y \in E$.*

In this case we adopt a similar method to the case for groups of subexponential growth since in both cases the limit of the sequence of functions as n tends to infinity is 1.

Lemma 5.3.10. *For any N, there exists a finite set F such that $u_N(x, y) \neq 0$ only if $x^{-1}y \in F$*

We comment that any finite set G_N is contained in a ball of some finite radius R and take F to be this ball.

Every u_N is an Ozawa kernal and such a function exists for any pair (E, ϵ). Thus we have directly proved the following theorem:

Theorem (4). [C, Thm 2] *The family of kernels u_N forms a family of Ozawa kernels for amenable groups and so they satisfy property O.*

The following corollary immediately follows:

Corollary (1.2.3). [C, Cor 3.4] *Amenable groups are exact and therefore have Yu's Property A.*

In this proof and those for the integers, groups of subexponential groups and as we will see in a moment, free groups, it is necessary at some point to expand the set E to a ball of finite radius which contains it. As we saw previously, Guentner and Kaminkner's alternative definition of Property O only considers balls of finite radius, as does Tu's equivalent definition of Property A.

However this was not the case for the construction of the family of Ozawa kernels $\{u_W\}$ for amenable groups. It is interesting to note that it is possible to construct an Ozawa kernel where this enlarging of E is not necessary.

5.4 Free groups

5.4.1 Definitions and properties

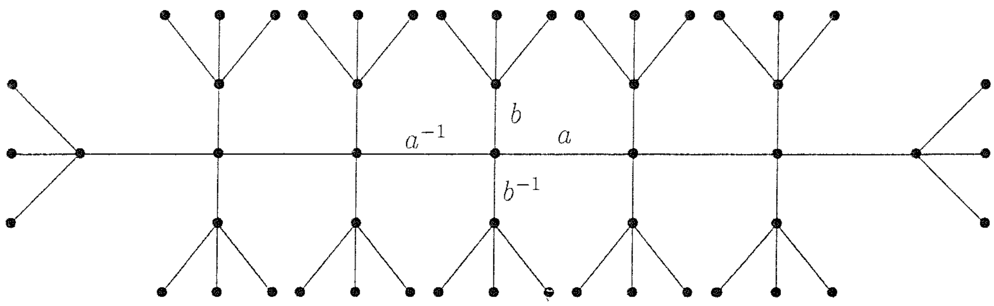
We will now move on to another well known class of exact groups: free groups.

We will first recall the definition of a free group.

Definition 5.4.1 (Free group). *A group is free if it has a set of generators such that the only product of generators and their inverses that equal identity are of the form* aa^{-1} *or* $a^{-1}a$.

Remark 5.4.2. *The rank of a free group is its number of generators. The Cayley graph of a free group is a tree.*

As an example, the following diagram represents part of the free group on two generators a, b and their inverses a^{-1}, b^{-1}.



The growth rate of a free group is not sub-exponential. Given any free group, increasing the radius of a ball of radius n by a small amount induces a very large increase in the number of vertices on its boundary. As an example, the number of vertices on the edges of the above diagram for the free group of rank 2 increases rapidly as the distance from the central vertex increases.

Our functions up to this point relied on the fact that the boundary of the set increased far more slowly than the volume. Since this is not the case here, we will have to adopt a different approach.

In [DJ, Prop 1], it was shown that trees have Property A. To do so they constructed a measure which relied on Dirac functions. We will now adapt this construction to obtain a family of Ozawa kernels and prove that free groups satisfy Property O. Instead of looking at the size of the intersection of balls of radius n as we did for groups of subexponential growth, we will consider the size of the overlap between rays of length n.

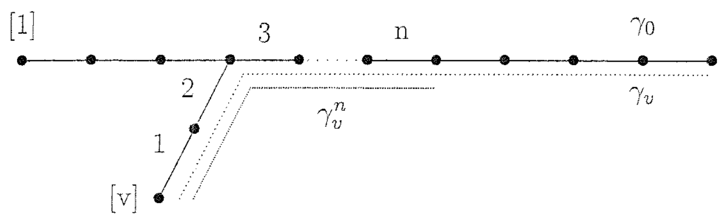
5.4.2 Construction of a family of Ozawa kernels for free groups

Let T be the Cayley graph of a free group (a tree) and V its set of vertices.

Let $\gamma_0 : \mathbb{R} \to T$ be a geodesic ray in T.

Let γ_v be the unique geodesic ray issuing from v and intersecting γ_0 along a geodesic ray.

Let γ_v^n be the initial ray of γ_v of length n as represented in the following diagram:



We will define our function $u_n(x, y)$ where $x, y \in V$ to be the size of the overlap of the two n-length rays γ_x^n and γ_y^n, scaled by $n + 1$, which is the number of vertices on a ray of length n.

$$u_n(x, y) = \frac{|\gamma_x^n \cap \gamma_y^n|}{n + 1}$$

We will show that the set of such functions forms a family of Ozawa kernels.

Lemma 5.4.3. *Given any n, u_n is a positive kernel.*

Proof. Define $f_v(x)$ and $\chi_{v,n}(x)$ as follows:

$$f_v(x) = \begin{cases} 1 & \text{if } v \text{ separates } x \text{ from } \gamma_0 \\ 0 & \text{otherwise} \end{cases}$$

And

$$\chi_{v,n}(x) = \begin{cases} 1 & \text{if } x \text{ is contained in the ball of radius } n \text{ around } v \\ 0 & \text{otherwise} \end{cases}$$

Then we have

$$u_n(x,y) = \frac{1}{n+1} \sum_{v \in V} f_v(x) f_v(y) \chi_{v,n}(x) \chi_{v,n}(y)$$

The only vertices contributing to this sum are whose which are within distance n of both x and y and which separate both x and y from γ_0. This is precisely the size of the intersection of the n-length rays γ_x^n and γ_y^n.

We can now rearrange u_n to show that this is a positive kernel.

$$\begin{aligned}
\sum_{i,j}^{k} \lambda_i \lambda_j u_n(x_i, x_j) &= \sum_{i,j}^{k} \lambda_i \lambda_j \sum_{v \in V} f_v(x_i) f_v(x_j) \chi_{v,n}(x_i) \chi_{v,n}(x_j) \\
&= \sum_{v \in V} \left(\sum_{i}^{k} \lambda_i f_v(x_i) \chi_{v,n}(x_i) \sum_{j}^{k} \lambda_j f_v(x_j) \chi_{v,n}(x_j) \right) \\
&= \sum_{v \in V} \left(\sum_{i}^{k} \lambda_i f_v(x_i) \chi_{v,n}(x_i) \right)^2 \\
&\geq 0
\end{aligned}$$

□

Lemma 5.4.4. *For any n, there exists a finite set F such that $u_n(x,y) \neq 0$ only if $x^{-1}y \in F$.*

Proof. Let F be the ball of radius n around the origin. If $d(x,y) > 2n$, ie $x^{-1}y \notin F$, there is no overlap between the n-length rays γ_x^n and γ_y^n and so $u_n(x,y) = 0$. □

Lemma 5.4.5. *Given any finite subset E and $\epsilon > 0$, there exists N such that $|1 - u_N(x,y)| < \epsilon$ if $x^{-1}y \in E$.*

Proof. Since E is a finite subset, there exists some finite number m such that if $x^{-1}y \in E$ then $d(x,y) < m$.

Now if $d(x,y) < m$, then the minimum size of the overlap of γ_x^n and γ_y^n is $n - 2m$.

And so we have

$$\frac{n-2m}{n+1} \leq u_n(x,y) \leq \frac{n+1}{n+1}$$

Hence for all pairs $x, y \in E$, $\lim_{n \to \infty} u_n(x,y) = 1$.

And so $\forall \epsilon > 0$, $\exists N$ such that $\forall x^{-1}y \in E$ we have $|1 - u_N(x,y)| < \epsilon$. □

Every u_N is an Ozawa kernel and there exists such a function for any pair (E, ϵ). Thus we have directly proved the following theorem:

Theorem (5). [C, Thm 1] *The family of kernels u_N forms a family of Ozawa kernels for free groups and so they satisfy property O.*

The following corollary immediately follows:

Corollary (1.2.4). [C, Cor 2.4] *Free groups are exact and therefore have Yu's Property A.*

Chapter 6

Hilbert space compression and CAT(0) cube complexes

We will now move on to a different approach to the problem of proving that groups are exact. In this section we will use properties of Hilbert space compression to prove that groups acting properly and cocompactly on CAT(0) cube complexes are exact. The method relies on the following theorem from Guentner and Kaminker [GK2, Thm 3.2].

Theorem (9). *Let Γ be a finitely generated discrete group regarded as a metric space via the word metric. If the Hilbert space compression of Γ is greater than 1/2 then Γ is exact.*

Recall that Yu showed that Property A implies that the group is uniformly embeddable in a Hilbert space. Since exactness is equivalent to Property A, we know that if a group is exact then it is uniformly embeddable. The converse is not known to be true. However there are no known examples of groups which are uniformly embeddable but not exact. Hilbert space compression goes some way to linking uniform embeddability back to exactness. It measures how much a space needs to be distorted in order to obtain a uniform embedding into a Hilbert space. If this distortion is not too large (i.e. the Hilbert space compression is strictly greater than $1/2$), then the group is exact. It is important to note that this is quite a strong condition and that there do exist exact groups which do not have Hilbert space compression greater than $1/2$. For example, $\mathbb{Z}\, wr\, (\mathbb{Z}\, wr\, \mathbb{Z})$, the wreath product of $\mathbb{Z}$ with its own wreath product on itself, is amenable and hence exact, but its Hilbert space compression lies between 0 and $1/2$ [AGS, Cor 1.10].

Guentner and Kaminker illustrate the Hilbert space compression approach with the example of the free group of rank 2. They construct a family of functions on its Cayley graph and show that the Hilbert space compression is 1. They deduce that free groups are exact. We will first explain their approach and then expand this method to obtain our main result:

Theorem (6). *[CN, Thm 12.] If G is a group acting properly and cocompactly on a $CAT(0)$ cube complex then G is exact and therefore has Yu's Property A.*

Parts of this section were used in a joint paper with Dr G. Niblo [CN].

6.1 Example: Free Groups

In [GK2, Proposition 4.2], Guentner and Kaminker show that the Free Group of rank 2 is exact by constructing a family of large scale Lipschitz functions from a tree to a Hilbert space whose asymptotic compression tends to 1. This section will explain their approach.

They start off by considering the Cayley graph of the free group of rank 2, which is a tree. Call it $X = (V, E)$ where V is the set of vertices and E is the set of edges.

For any vertex s, consider the edges on the unique path from s to 1. Starting from s, label these e_1 to e_k where k is the distance from s to 1. Let $\delta_{e_i}(s)$ be the characteristic function on the set of edges. This means that $\delta_{e_i}(s)(e) = 1$ if the edge is e_i and 0 otherwise.

The following diagram illustrates the labeling from s:

The first function considered is $f : \mathbb{F}_2 \to \mathcal{H}$ with

$$f(s) = \delta_{e_1}(s) + \ldots + \delta_{e_k}(s)$$

Then $\|f(s)\| = \langle s, 1\rangle^{1/2} = \sqrt{d(s,1)}$

For two points s, t we have $\|f(s) - f(t)\| = \sqrt{d(s,t)}$

Recall that the Gromov compression of a function f is

$$\rho_f(r) = \inf_{d(s,t)\geq r} \|f(s) - f(t)\|$$

So in this case, since $\|f(s) - f(t)\| = \sqrt{d(s,t)}$ and $r \in \mathbb{R}$, we have

$$\sqrt{r} \leq \rho_f(r) \leq \sqrt{r+1}$$

Now recall that the asymptotic compression of a function f is

$$R_f = \lim_{r\to\infty} \inf \frac{\log \rho_f(r)}{\log r}$$

And so here we have:

$$\frac{1}{2} \leq \lim_{r\to\infty} \inf \frac{\log(r^{1/2})}{\log r} \leq R_f \leq \lim_{r\to\infty} \inf \frac{\log(r+1)^{1/2}}{\log r} \leq \frac{1}{2}$$

And hence $R_f = 1/2$. Recall that the Hilbert space compression $R(X) = sup\{R_f : f \in Lip^{ls}(X, H)\}$. Thus they can conclude that when X is a tree, $R(X) \geq 1/2$. However this is not enough to conclude that free groups are exact since the Hilbert space compression needs to be strictly greater than 1/2.

They then weight their function to obtain a family of functions labeled by $0 < \epsilon \leq 1/2$ which have the desired asymptotic compression.

The new function is $f_\epsilon : \mathbb{F}_2 \to l^2(E)$ with

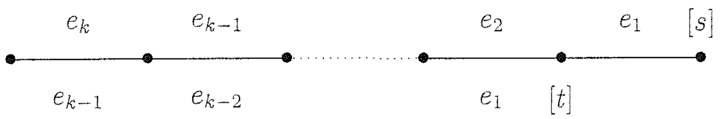
$$f_\epsilon(s) = 1^\epsilon \delta_{e_1}(s) + \ldots + i^\epsilon \delta_{e_i}(s) + \ldots + k^\epsilon \delta_{e_k}(s)$$

They need to show that for each $0 < \epsilon \leq 1/2$, f_ϵ is both large scale Lipschitz and satisfies $R_{f_\epsilon} \geq 1/2 + \epsilon$.

Large scale Lipschitz

To show that it is large scale Lipschitz, it suffices to show that $d(s,t) = 1 \Rightarrow \|f_\epsilon(s) - f_\epsilon(t)\|^2 \leq C \ \forall s, t \in \mathbb{F}_2$.

Proof. Note that if $d(s,t) = 1$, then we have the following situation.



And so we have:

$$\begin{aligned}
\|f_\epsilon(s) - f_\epsilon(t)\|^2 &= 1^{2\epsilon} + (2^\epsilon - 1^\epsilon)^2 + \ldots + [i^\epsilon - (i-1)^\epsilon]^2 + \ldots + [k^\epsilon - (k-1)^\epsilon]^2 \\
&= 1 + \sum_{i=2}^{k} [i^\epsilon - (i-1)^\epsilon]^2
\end{aligned}$$

They then show that $\sum_i^{\infty}[i^{\epsilon}-(i-1)^{\epsilon}]^2$ is finite by the following calculation:

$$\begin{aligned}\sum_i^{\infty}[i^{\epsilon}-(i-1)^{\epsilon}]^2 &= \sum_i^{\infty}\left(\int_{i-1}^{i}(x^{\epsilon})'dx\right)^2 \\ &\leq \sum_i^{\infty}\int_{i-1}^{i}\left[(x^{\epsilon})'\right]^2 dx \\ &= \int_1^{\infty}\epsilon^2 x^{2\epsilon-2}dx = \frac{\epsilon^2}{1-2\epsilon}\end{aligned}$$

To understand the second step of this calculation we must first recall the general Cauchy Schwarz inequality:

$$|\langle\psi_1,\psi_2\rangle|^2 \leq \langle\psi_1|\psi_1\rangle\langle\psi_2|\psi_2\rangle$$

It can be written explicitly in integral form:

$$\left[\int_a^b \psi_1(x)\psi_2(x)dx\right]^2 \leq \int_a^b [\psi_1(x)]^2 dx \int_a^b [\psi_2(x)]^2 dx$$

In this proof we need to deal with the expression $\left(\int_{i-1}^{i}(x^{\epsilon})'dx\right)^2$. By setting $\psi_1(x) = (x^{\epsilon})'$ and $\psi_2(x) = 1$ we get $\int_a^b[\psi_2(x)]^2dx = \int_{i-1}^{i}1^2dx = 1$. And so $\left(\int_{i-1}^{i}(x^{\epsilon})'dx\right)^2 \leq \int_{i-1}^{i}\left[(x^{\epsilon})'\right]^2 dx$ as required.

Since $\sum_i^{\infty}[i^{\epsilon}-(i-1)^{\epsilon}]^2$ is finite, then so is $1+\sum_{i=2}^{k}[i^{\epsilon}-(i-1)^{\epsilon}]^2$ and f_{ϵ} is large scale Lipschitz as required. $\square$

Hilbert space compression $\geq 1/2 + \epsilon$

To show that $R_{f_\epsilon} \geq 1/2 + \epsilon$, it is sufficient to show that

$$\|f_\epsilon(s) - f_\epsilon(t)\|^2 \geq C_\epsilon r^{1+2\epsilon}$$

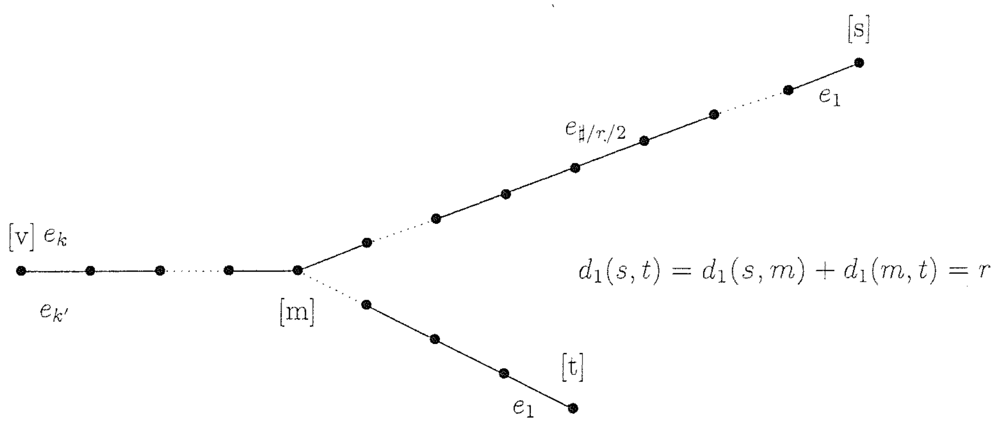
where C_ϵ is some constant dependent only on ϵ.

In that case we have

$$R_{f_\epsilon} \geq \frac{\log(\sqrt{C_\epsilon} r^{1/2+\epsilon})}{\log r} = 1/2 + \epsilon$$

Proof. Take 2 points (s,t) such that $d(s,t) \geq r$. Assume that $d(1,s) \leq d(1,t)$. Now let $\sharp(r/2)$ be the smallest integer such that $\sharp(r/2) \geq r/2$.

From the structure of the tree, the edge $e_{\sharp(r/2)}$ must belong to the section $[m,t]$ where m is the intersection of the path $[s,1]$ with the path $[t,1]$. An example of this is given in the following diagram:



And so

$$\begin{aligned}
\|f_\epsilon(s) - f_\epsilon(t)\|^2 &\geq 1^{2\epsilon} + \ldots + \sharp(r/2)^{2\epsilon} \\
&= \int_0^{r/2} x^{2\epsilon} \\
&= \left[\frac{x^{2\epsilon+1}}{2\epsilon+1}\right]_0^{r/2} \\
&= \frac{r^{2\epsilon+1}}{(2\epsilon+1)2^{2\epsilon+1}}
\end{aligned}$$

□

And so they obtain a family of large scale Lipschitz functions f_ϵ where $R_f \geq 1/2 + \epsilon$ and $0 < \epsilon < 1/2$. Hence the Hilbert space compression of $\mathbb{F}_2$ is 1 and they have a new proof of the following theorem:

Theorem 12. *The free group of rank 2 is exact.*

6.2 Groups acting properly and cocompactly on $CAT(0)$ cube complexes

We will now adopt a similar method and construct a family of large scale Lipschitz functions on a finite dimensional $CAT(0)$ cube complex. We will show that the asymptotic compression of this family of functions tends to 1, thus proving the Hilbert space compression of a $CAT(0)$ cube complex is 1.

By using another theorem [GK2, Cor 2.13] which states that Hilbert space compression is a quasi-isometry invariant, we will prove our main theorem:

Theorem (6). *If G is a group acting properly and cocompactly on a finite dimensional $CAT(0)$ cube complex then G is exact and satisfies Yu's Property A.*

In this section we will first give some background information about $CAT(0)$ cube complexes and then give a proof of the main theorem. Although this was joint work, the proof that the function we define is large scale Lipschitz is principally the work of Dr Niblo, while the proof that its compression is strictly greater than $1/2$ is that of the author. The paper coauthored by the author and Dr G.Niblo [CN] can be found in the appendix of this thesis.

6.2.1 $CAT(0)$ cube complexes

We will review the definitions and properties of a CAT(0) cube complex. Some of the following defintions can be found in [NR].

The cube complex

Definition 6.2.1 (Cube complex). *A cube complex is a polyhedral complex of cells isometric to a Euclidean cell. The gluing of these cells is by isometries. The dimension of a cube complex is the highest dimension of one of its cells.*

Definition 6.2.2 **($CAT(0)$ cube complex).** *A $CAT(0)$ cube complex is a cube complex which is non positively curved.*

This means that the following conditions on the link of C, lk$C$, are satisfied:

1. (*no bigons*) For each pair of vertices in lk$C$ there is at most one edge containing them.

2. (*no triangles*) Every edge cycle of length three in lk$C$ is contained in a 2-simplex of lk$C$.

Example 4. *Any tree is a 1-dimensional $CAT(0)$ cube complex.*

Example 5. *Euclidean space with vertices at the integer lattice points has the structure of a $CAT(0)$ cube complex.*

The dimension n of a $CAT(0)$ cube complex is that of its highest dimension cube.

Hyperplanes

Definition 6.2.3 **(Midplane).** *The midplane of a cube is its intersection with a codimension 1 coordinate hyperplane.*

So every n-cube contains n midplanes each of which is an $(n-1)$-cube, and any m of which intersect in a $(n-m)$-cube.

Given any edge e in the $CAT(0)$ cube complex, there is a unique hyperplane which cuts e transversly in its midpoint. The hyperplane is obtained by developing the midplanes in the cubes containing e.

Example 6. *In the case of a tree, the the hyperplane is the midpoint of the edge e.*

Example 7. *In the case of euclidean space it is a geometric hyperplane.*

Any hyperplane is isometrically embedded in the $CAT(0)$ cube complex. In addition, each hyperplane separates the $CAT(0)$ cube complex into two half spaces.

Lengths

The set of vertices of a CAT(0) cube complex X can be viewed as a discrete metric space, where the metric $d_1(s,t)$ is given by the length of a shortest edge path between the vertices s and t. We will refer to this as the ℓ^1 metric on the vertices.

Alternatively we can measure the distance by restricting the path metric on X to obtain the ℓ^2 metric on the vertices.

If X is finite dimensional these metrics are quasi-isometric, and we have $d(s,t) \leq d_1(s,t) \leq \sqrt{n}d(s,t)$ where d denotes the CAT(0) (geodesic) metric on X and n is the dimension of the complex.

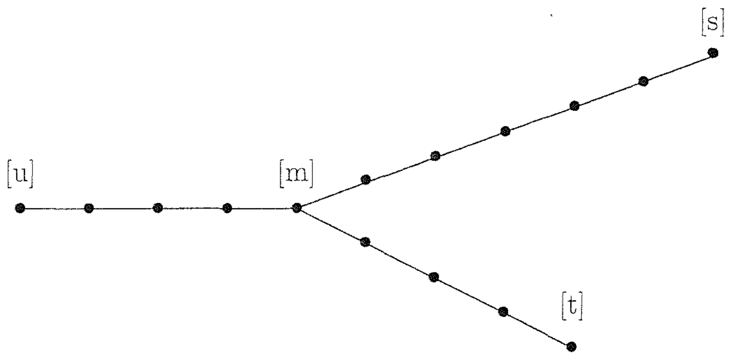
Sageev [S] observed that the shortest path in the 1-skeleton crosses any hyperplane at most once, and since every edge crosses exactly one hyperplane, the ℓ^1 distance between two vertices is the number of hyperplanes separating them.

Intervals and Medians

Definition 6.2.4 (Interval). *An interval between two vertices s, t, denoted $[s, t]$, is the set of all vertices which lie on an edge geodesic from s to t.*

Given any three vertices s, t, u, there exist three intervals $[s, t], [s, u], [t, u]$. These allow us to define the median.

Definition 6.2.5 (Median). *The median of a triple s, t, u is a single point m which is the intersection of the intervals $[s, t], [s, u], [t, u]$.*



The median has the following important property: the intersection of the hyperplanes which separate s from t and those which separate s from u are precisely the hyperplanes which separate s from the median m.

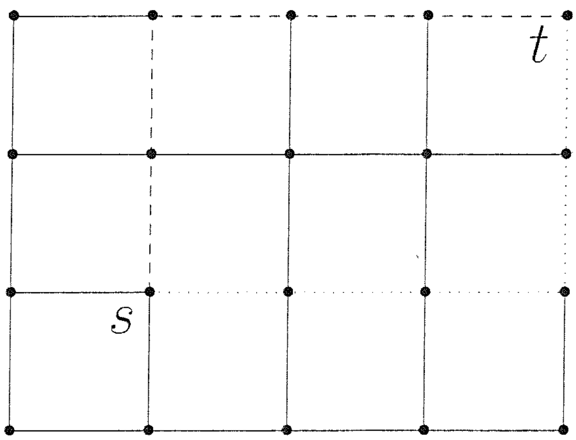
In addition, the hyperplanes which separate t from u are precisely those which separate t from m and those which separate u from m.

This induces the following distance property:

$$d_1(t, u) = d_1(t, m) + d_1(u, m) = d_1(s, t) + d_1(s, u) - 2d_1(s, m)$$

Normal Cube Paths

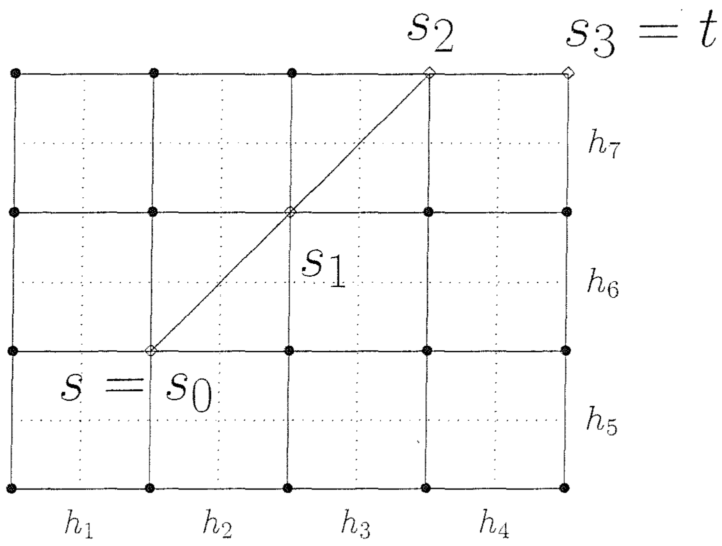
In a $CAT(0)$ cube complex, a geodesic edge path may not be unique. For instance consider the Euclidean plane with vertices at the integer lattice points. In the following diagram, the dotted and dashed paths represent two separate paths from s to t, both of which have the same edge length.



The only $CAT(0)$ cube complex where geodesic edge paths are unique is a tree. We will now introduce the notion of a normal cube path which allows us to mimic some of the properties of geodesic edge paths in trees.

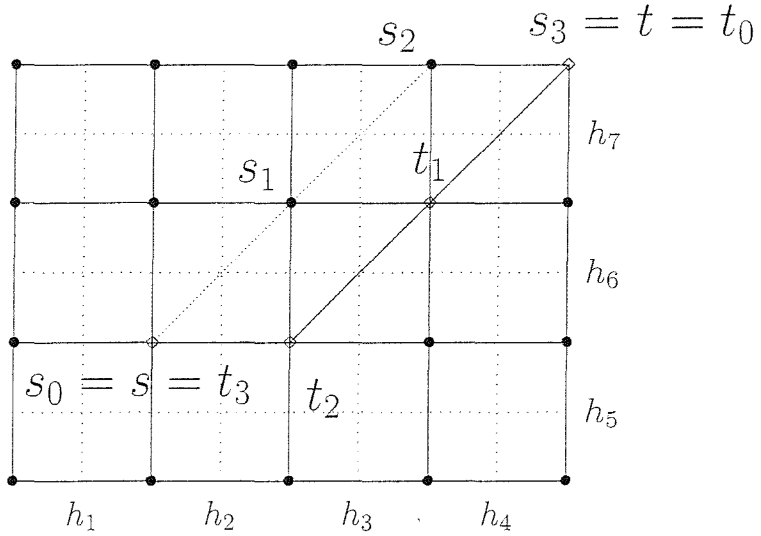
Let X be a cube complex and $\mathcal{H}$ be the set of hyperplanes. Consider two vertices s and t and some path from s to t. The path starts at the vertex s and then defines a sequence of vertices s_i until it reaches the vertex t. There are a finite number of hyperplanes separating s and t. A path of minimal length between the two vertices must cross each hyperplane just once. A normal cube path follows a greedy path. At each step i, as many hyperplanes separating s and t as possible must be crossed.

This is illustrated in the next diagram. We consider here the normal cube path from s to t. We have labelled the hyperplanes from 1 to 7. Only those hyperplanes which separate s from t will be crossed: h_2, h_3, h_4, h_6 and h_7.



Our starting point is at the vertex s which we denote by s_0. The first step on the normal cube path will take us to the vertex s_1. We need to cross as many of the hyperplanes separating s from t as possible. We can cross hyperplanes 2 and 6, giving us the only possible vertex s_1. We now move on to step two on the normal cube path. Starting from s_1 we need to cross as many of the hyperplanes separating s_1 from t. We can cross hyperplanes 3 and 7, giving us the only possible vertex s_2. Finally in step 3 we cross the remaining hyperplane number 4, thus giving us the only possible vertex s_3. This vertex is also t and marks the end of the normal cube path.

Note that the normal cube path from s to t may not be the same as that from t to s. For example, the following diagram illustrates the normal cube path from t to s. We have left in the normal cube path from s to t for comparision.



The hyperplanes are crossed in a different order and at different stages of the path. In this case the vertices defined by the normal cube path from t to s differ to those defined by the normal cube path from s to t.

Note also that if a hyperplane to be crossed is adjacent to a vertex on the normal cube path, then it must by definition be crossed at the next step. The rigorous definition of a normal cube path is as follows [CN, Def 4]:

Definition 6.2.6 (Normal cube path). *A cube path is a sequence of cubes $\mathcal{C} = \{C_0, \dots C_n\}$, each of dimension at least 1, such that each cube meets its successor in a single vertex, $v_i = C_{i-1} \cap C_i$ and such that for $1 \le i \le n-1$, C_i is the (unique) cube of minimal dimension containing v_i and v_{i+1}. Note that v_i and v_{i+1} are diagonally opposite vertices of C_i. We define v_0 to be the vertex of C_0 which is diagonally opposite v_1, and v_n to be the vertex of C_n diagonally opposite v_{n-1}. We call the v_i vertices of the cube-path, with v_0 the initial vertex and v_n the terminal vertex. Given a cube path from s to t we can construct edge paths from s to t which travel via the edges of the cubes C_i so every hyperplane separating s from t must intersect at least one of the cubes C_i. We say the cube path is normal if $C_{i+1} \cap u(C_i) = v_i$ for each i, where $u(C_i)$ is the union of all cubes which contain C_i as a face (including C_i itself).*

It has been shown [NR2, Prop 3.3] that given two vertices s and t, there is a unique normal cube path from s to t.

An important lemma, also from [NR2, Prop 5.2] which will be used later is as follows:

Lemma 6.2.7. *Let s, t, v_0 be vertices of a CAT(0) cube complex with s and t diagonally adjacent across some cube E_0. Let $s = s_0, s_1, \dots, s_m = v$, $t = t_0, t_1, \dots, t_n = u$ be the vertices of the (unique) normal cube paths from s to*

v_0 and from t to v_0 respectively. Let $\{C_i \mid i = 1, \ldots m\}$ be the cubes on the normal cube path from s to v_0 and $\{D_j \mid j = 1, \ldots n\}$ be the cubes on the normal cube path from t to v_0. Then:

1. *Each hyperplane separating s from v_0 intersects exactly one of the cubes C_i and each hyperplane separating t from v_0 intersects exactly one of the cubes D_j.*

2. *For each $i \leq \min\{m, n\}$ there is a cube E_i such that s_i is diagonally adjacent to t_i across E_i .*

6.2.2 Proof of the main theorem

In the case of the free group of rank 2, the function used to find the Hilbert space compression was defined according to the edges on the unique edge path between s and 1. However this will not work in a $CAT(0)$ cube complex as edge paths between points are not unique. Instead, we will define our function by using the hyperplanes crossed in the normal cube paths between two vertices.

Let V be the set of vertices of a finite dimensional cube complex X and $\mathcal{H}$ be the set of hyperplanes. Given a vertex s, we will consider the normal cube path from s to a base point v. A finite number of hyperplanes are crossed along this path. We will weight all hyperplanes according to the stage at which they are crossed. If $C = \{C_0, C_1, \ldots, C_n\}$ is the unique normal cube path from s to v, then the weighting of a hyperplane h relative to s is:

$$\omega_s(h) = i + 1$$

where h intersects the cube C_i. If h does not intersect any of the cubes (ie does not separate s from v), then $\omega_s(h) = 0$. Note that several hyperplanes can have the same weighting. Note also that when n is the dimension of the CAT(0) cube complex, no more than n hyperplanes can share any given weighting, since by definition any cube contains at most n hyperplanes.

Now for each $0 < \epsilon \leq 1/2$ we define

$$f_\epsilon(h) = \sum_{h \in \mathcal{H}} (\omega_s(h))^\epsilon$$

The support of this function is finite since the only hyperplanes contributing to the sum are those which separate s from v.

As before, we need to show that for each $0 < \epsilon < 1/2$, f_ϵ is both large scale Lipschitz and satisfies $R_{f_\epsilon} \geq 1/2 + \epsilon$.

Large Scale Lipschitz

To show that f_ϵ is large scale Lipschitz it suffices to show that $d(s,t) = 1 \Rightarrow \|f_\epsilon(s) - f_\epsilon(t)\|^2 \leq C \ \forall s, t \in \mathbb{F}_2$.

We now need the following technical lemma:

Lemma 6.2.8. *Let s, t, v be vertices of the CAT(0) cube complex X with s*

and t diagonally opposite across some cube E_0. Let $s = s_0, s_1, \ldots, s_m = v$, $t = t_0, t_1, \ldots, t_n = v$ be the vertices of the (unique) normal cube paths from s to v and from t to v respectively. Let $\{C_i \mid i = 1, \ldots m\}$ be the cubes on the normal cube path from s to v and $\{D_j \mid j = 1, \ldots n\}$ be the cubes on the normal cube path from t to v. If h is a hyperplane in X which separates both s and t from v and which intersects the cube C_i then h also intersects one of the cubes D_{i-1}, D_i, D_{i+1}.

Proof. Consider 2 vertices s, t distance 1 apart (or in other words at most diagonally opposite some cube E_0) and the normal cube paths from s and t to the base point v. By lemma 6.2.7 the hyperplane h must intersect the normal cube path from t to v in one of the cubes D_j.

In addition, since $s = s_0$ and $t = t_0$ are diagonally opposite across the cube E_0 this means that for each $i \leq \min\{m, n\}$ s_i is diagonally opposite to t_i across some cube E_i.

Now h separates s_{i-1}, s_i and also separates t_{j-1}, t_j. We want to show that $|i - j| \leq 1$.

Let $k = \min\{i, j\}$. Assume first that h separates s_{k-1} and s_k so $i = k \leq j$. If h also separates t_{k-1} and t_k then h crosses $D_k = D_i$ as required.

Assume h does not separate t_{k-1} from t_k. By the minimality of k, h does not separate t from t_{k-1}. But it must separate t from v and so it must also separate t_k from v.

Now we construct an edge path from t_k to v as follows. First cross over the cube E_k from t_k to s_k then follow the path through the cubes

$C_{k+1}, C_{k+2}, \ldots, C_m$ to v. This gives an edge path from t_k to v so it must cross h.

However since h was crossed in the kth cube on the normal cube path from s to v, none of the cubes $C_{k+1}, \ldots, C_m$ intersect h. Hence h must cross E_k and so h is adjacent to t_k. But as h separates t_k from v and is adjacent to t_k it must cross the first cube (D_{k+1}) on the normal cube path from t_{k+1} to v as required.

The case when h separates t_{k-1} and t_k so $i = k$ but does not separate s_{k-1} and s_k is argued in exactly the same way reversing the roles of s and t, C and D and so on. $\square$

Thus when $d(s,t) = 1$, if a hyperplane h is crossed at stage i from s, then it is crossed at one of the stages i or $i \pm 1$ from t. It follows that either $\omega_s(h) = \omega_t(h)$ or $\omega_s(h) = \omega_t(h) \pm 1$.

Hence considering the set $\{h_1, \ldots, h_m\}$ of hyperplanes which separate s from v, we have:

$$\begin{aligned} \|f_\epsilon(s) - f_\epsilon(t)\|^2 &= \sum_{i=0}^{m} [\omega_s(h_i)^\epsilon - \omega_t(h_i)^\epsilon]^2 \\ &\leq 1^{2\epsilon} + \sum_{i=1}^{m} [\omega_s(h_i)^\epsilon - (\omega_s(h_i) + 1)^\epsilon]^2 \end{aligned}$$

Since X is a $CAT(0)$ cube complex of dimension n, at most n hyperplanes can be crossed at each stage of the normal cube path from s to v. Hence at most n hyperplanes can have a given weight $\omega_s(h) = j$. And so we have:

$$\|f_\epsilon(s) - f_\epsilon(t)\|^2 \leq 1^{2\epsilon} + n\sum_{j}^{\infty}[j^\epsilon - (j-1)^\epsilon]^2$$

As noted in [GK2], $\sum_j^\infty [j^\epsilon - (j-1)^\epsilon]^2$ is finite and hence f_ϵ is large scale Lipschitz as required.

Hilbert space compression $\geq 1/2 + \epsilon$

We will now show that the asymptotic compression R_{f_ϵ} is greater than 1/2. As before, it suffices to show that

$$\|f_\epsilon(s) - f_\epsilon(t)\|^2 \geq C_\epsilon r^{1+2\epsilon}$$

where C_ϵ is some constant dependant only on ϵ.

Consider two points s, t and let $d(s,t) \geq r$. Assume that $d(s,1) > d(t,1)$. As before, let $\sharp(r/2)$ be the smallest integer greater than $r/2$. There are at least $\sharp(r/2)$ hyperplanes which separate s from 1 but not t from 1. We can label these hyperplanes $h_1, h_2, \ldots, h_{\sharp(r/2)}$.

The weight of each of these hyperplanes lies between 1 and $\sharp(r/2)$. Recall that the weight $\omega_s(h)$ of a hyperplane is determined by which cube it intersects on the normal cube path $\{C_1, C_2, \ldots, C_m\}$ from s to 1. The dimension of any cube must be less or equal to n. Hence there are at most n hyperplanes which can intersect any one cube and thus at most n hyperplanes can share any particular weight.

Now write $\sharp(r/2)$ as $kn + m$ for some integers m, k such that $0 \leq m < n$ and $k \geq 0$.

Then we have:

$$\|f_\epsilon(s)-f_\epsilon(t)\|^2 \geq \omega_s(h_1)^{2\epsilon}+\omega_s(h_2)^{2\epsilon}+\ldots+\omega_s(h_{\sharp(r/2)})^{2\epsilon} \geq n(1^{2\epsilon}+2^{2\epsilon}+\ldots+k^{2\epsilon})+m(k+1)^{2\epsilon}$$

We will now show that

$$n(1^{2\epsilon} + 2^{2\epsilon} + \ldots + k^{2\epsilon}) + m(k+1)^{2\epsilon} \geq \frac{1}{n}[1^{2\epsilon} + 2^{2\epsilon} + \ldots + \sharp(r/2)^{2\epsilon}]$$

Lemma 6.2.9. *For any $i \geq 1$,*

$$ni^{2\epsilon} > \frac{1}{n}[((i-1)n+1)^{2\epsilon} + \ldots + (in)^{2\epsilon}]$$

Proof. Since $\epsilon < \frac{1}{2}$ and $n \geq 1$ we have $ni^{2\epsilon} > n^{2\epsilon}i^{2\epsilon} = (in)^{2\epsilon}$

On the other hand, since $\epsilon > 0$ and $in > ik$ for all $k < n$ we have

$$\frac{1}{n}[((i-1)n+1)^{2\epsilon} + \ldots + (in)^{2\epsilon}] < \frac{1}{n}(n(in)^{2\epsilon}) = (in)^{2\epsilon}$$

So

$$ni^{2\epsilon} > (in)^{2\epsilon} > \frac{1}{n}[((i-1)n+1)^{2\epsilon} + \ldots + in^{2\epsilon}]$$

□

Lemma 6.2.10.

$$m(k+1)^{2\epsilon} > \frac{1}{n}\left((kn+1)^{2\epsilon} + \ldots + (kn+m)^{2\epsilon}\right)$$

Proof. We have that

$$m(k+1)^{2\epsilon} \geq m^{2\epsilon}(k+1)^{2\epsilon} = (mk+m)^{2\epsilon}$$

Looking at the RHS of the statement of the claim we have:

$$\begin{aligned}
\frac{1}{n}\left((kn+1)^{2\epsilon} + \ldots + (kn+m)^{2\epsilon}\right) &< \frac{m}{n}(kn+m)^{2\epsilon} \quad \text{(since (kn + m) is the biggest term)} \\
&< \left(\frac{m}{n}\right)^{2\epsilon}(kn+m)^{2\epsilon} \quad \left(\text{since } \frac{m}{n} < 1\right) \\
&= (mk + \frac{m^2}{n})^{2\epsilon} \\
&< (mk+m)^{2\epsilon} \quad \left(\text{since } \frac{m}{n} < 1\right)
\end{aligned}$$

And so

$$m(k+1)^{2\epsilon} \geq (mk+m)^{2\epsilon} > \frac{1}{n}\left((kn+1)^{2\epsilon} + \ldots + (kn+m)^{2\epsilon}\right)$$

□

Putting both claims together, we have that:

$$n * 1^{2\epsilon} > \frac{1}{n}\left(1^{2\epsilon} + \ldots + n^{2\epsilon}\right)$$

$$n * 2^{2\epsilon} > \frac{1}{n}\left((n+1)^{2\epsilon} + \ldots + (2n)^{2\epsilon}\right)$$

$$\vdots$$

$$n * k^{2\epsilon} > \frac{1}{n}\left(((k-1)n+1)^{2\epsilon} + \ldots + (kn)^{2\epsilon}\right)$$

$$m * (k+1)^{2\epsilon} \geq \frac{1}{n}\left((kn+1)^{2\epsilon} + \ldots + (kn+m)^{2\epsilon}\right)$$

And so

$$n * 1^{2\epsilon} + n * 2^{2\epsilon} + \ldots + n * k^{2\epsilon} + m(k+1)^{2\epsilon} > \frac{1}{n}\left(1^{2\epsilon} + 2^{2\epsilon} + \ldots + \sharp(\frac{r}{2})^{2\epsilon}\right)$$

And as proved in the case of the free group of rank 2 [GK2, Prop 4.2],

$$[1^{2\epsilon} + 2^{2\epsilon} + \ldots + \sharp(r/2)^{2\epsilon}] \geq \frac{r^{2\epsilon+1}}{(2\epsilon+1)2^{2\epsilon+1}}$$

Hence, we have, as needed:

$$\|f_\epsilon(s) - f_\epsilon(t)\|^2 \geq \frac{r^{2\epsilon+1}}{n(2\epsilon+1)2^{2\epsilon+1}}$$

6.2.3 Conclusions

For each $0 < \epsilon < 1/2$ we can construct a large scale Lipschitz embedding f_ϵ of the finite dimensional CAT(0) cube complex whose asymptotic compression is at least $1/2+\epsilon$. Since Hilbert space compression is the supremum of the asymptotic compression over all possible large scale Lipschitz functions, we have proved the following theorem:

Theorem 13. *The Hilbert space compression of a finite dimensional CAT(0) cube complex is 1.*

Now let G be a group acting properly and cocompactly on a finite dimensional $CAT(0)$ cube complex X. By fixing a generating set of G, we can regard G as a metric space via the edge metric on the Cayley graph. Thus G is quasi isometric to (X, d). Since Hilbert space compression is a quasi-isometry invariant [GK2], we have

Corollary 6.2.11. *Let G be regarded as a metric space via the word metric with respect to some finite generating set. If G acts properly and cocompactly on a $CAT(0)$ cube complex then G has Hilbert space compression 1.*

And so we have:

Theorem (6). *If G is a group acting properly and cocompactly on a $CAT(0)$ cube complex then G is exact and therefore has Yu's Property A.*

The class of groups acting properly and cocompactly on CAT(0) cube complexes is large, and includes free groups, finitely generated Coxeter groups, finitely generated right angled Artin groups, finitely presented groups satisfying the B(4)-T(4) cancellation properties and all those word-hyperbolic groups satisfying the B(6) condition. Others are the infinite simple groups constructed by Burger and Mozes.

Bibliography

[AGS] G.N.Arzhantseva, V.S.Guba, M.V. Sapir, Metrics on diagram groups and uniform embeddings in a Hilbert space, preprint maths.GR/0411605v2, 2004.

[ADR] C.Anantharaman-Delaroche, J.Renault, Amenable groupoids, Monographies de L'Enseignement Mathematique, 36., Geneva, 2000.

[B] M.Bridson, Geodesics and curvature in metric simplicial complexes, in: "Group Theory from a Geometrical Viewpoint", E. Ghys et al. (eds), World Scientific (1991) pp 373-463.

[BHV] B.Bekka, P.De la Harpe, A.Valette, Projet de livre sur la Propriété (T), http://name.math.univ-rennes1.fr/bachir.bekka/, 2003.

[C] Sarah Campbell, Exactness for free and amenable groups by construction of Ozawa kernels, Bull. London Math. Soc.,(to appear).

[CN] Sarah Campbell, Graham Niblo, Hilbert space compression and exactness of discrete groups, Journal of functional analysis 222, 2005, pp 292-305.

[D] Kenneth R.Davidson, C^*-Algebras by Example, Fields Institute Monographs, 1996.

[DG] Marius Dadarlat, Jerome Guentner, Uniform embeddability of relatively hyperbolic groups, preprint math.GR/0501495v1, Jan 2005.

[DJ] A.Dranishnikov, T.Januszkiewicz, Every Coxeter group acts amenably on a compact space, Proceedings of the 1999 Topology and Dynamics Conference (Salt Lake City, UT). Topology Proc. 24 (1999), Spring, pp 135–141.

[G] M.Gromov: Asymptotic invariants of infinite groups, Geometric group theory, Vol.2 (Sussex, 1991), 1-295, London Math. Soc. Lecture Note Ser., 182, Cambridge Univ.Press, Cambridge, 1993.

[G2] M.Gromov: Hyperbolic Groups, in Essays in Group Theory, S. M. Gersten, Ed, M.S.R.I Publ., vol8, Springer-Verlag, 1987, pp 75-267.

[GK] Erik Guentner, Jerome Kaminker, Exactness and the Novikov conjecture, Topology 41 (2002), no. 2, pp 419–420.

[GK2] Erik Guentner, Jerome Kaminker, Exactness and uniform embeddability of Discrete Groups, preprint math.OA/0309166 2003.

[HK] N. Higson, G.G. Kasparov, Operator K-theory for groups which act properly isometrically on Hilbert space, Invent. Math. 144 (2001) pp 23-74.

[HR] N.Higson, J.Roe, Amenable actions and the Novikov Conjecture,J. Reine Angew. Math. 519 (2000), pp 143–153.

[HV] Pierre de la Harpe, Alain Valette, La propriété (T) de Kazhdan pour les groupes localement compactes, Astérisque, vol175, Société Mathématique de France.

[KW] Kirchberg, Wasserman, Exact groups and continuous bundles of C^*-algebras, Mathematische Annalen, Volume 315, Number 2, 1999, pp 169 - 203.

[NR] Graham Niblo, Lawrence Reeves, Groups acting on CAT(0) cube complexes, Geometry & Topology, Vol1, 1997.

[NR2] G.A.Niblo, L.D.Reeves. The geometry of cube complexes and the complexity of their fundamental groups, Topology, Vol. 37, No 3, pp 621-633, 1998.

[NR3] G.A.Niblo, L.D.Reeves. Coxeter groups act on CAT(0) cube complexes, Journal of Group Theory, 6, (2003), pp 309-413.

[O] N. Ozawa, Amenable actions and exactness for discrete groups, CR Acad. Sci. Paris Sr. I Math., 330 (2000), pp 691-695.

[S] M. Sageev, Ends of group pairs and non-positively curved cube complexes. Proc. London Math. Soc. (3), 71(3) (1995), pp 585–617.

[Tu] Remarks on Yu's "Property A" for discrete metric spaces and groups, Jean-Louis Tu, Bulletin de la societe mathematique de France 129 (1), 2001, pp 115-139.

[W] Stan Wagon, The Banach-Tarski Paradox, Cambridge University Press, 1985.

[Wa] Simon Wassermann, Exact C^*-algebras and related topics, Lecture notes series Number 19, Seoul National University, Research Institute of mathematics global analysis research center, 1994.

[WO] N.E. Wegge-Olsen, K-Theory and C^*-algebras, Oxford University Press, 1993.

[Yu] G.Yu, The Coarse Baum Connes conjecture for spaces, which admit a uniform embedding into a hilbert space, Inventiones Math. 139 (2000), pp 201-240.

Appendix A

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Hilbert space compression and exactness of discrete groups

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Abstract

We show that the Hilbert Space compression of any (unbounded) finite dimensional CAT(0) cube complex is 1 and deduce that any finitely generated group acting properly, co-compactly on a CAT(0) cube complex is exact, and hence has Yu's Property A. The class of groups covered by this theorem includes free groups, finitely generated Coxeter groups, finitely generated right angled Artin groups, finitely presented groups satisfying the B(4)-T(4) small cancellation condition and all those word-hyperbolic groups satisfying the B(6) condition. Another family of examples is provided by certain canonical surgeries defined by link diagrams.

Introduction

We say that a group Γ is exact if the operation of taking the reduced crossed product with Γ preserves exactness of short exact sequences of Γ-C^*-algebras. In other words, Γ is exact if and only if for every exact sequence of Γ-C^*-algebras

$$0 \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow 0$$

the sequence

$$0 \longrightarrow C^*_r(\Gamma, B) \longrightarrow C^*_r(\Gamma, C) \longrightarrow C^*_r(\Gamma, D) \longrightarrow 0$$

of crossed product algebras is exact. Kirchberg and Wassermann [9] proved that when Γ is discrete, it is exact if and only if its reduced C^*-algebra $C^*_r(\Gamma)$ is exact. This means that the functor $B \mapsto C^*_r(\Gamma) \otimes_{\min} B$ is exact, i.e. preserves exactness of sequences of C^*-algebras.

In [19] Yu introduced Property A, analagous to Fölner's criterion for amenability, which for a finitely generated group is equivalent to exactness.

Definition A.0.12. *A discrete metric space Γ is said to have Property A if for any $r > 0, \epsilon > 0$, there exist a family of finite subsets $\{A\gamma\}_{\gamma\in\Gamma}$ of $\Gamma \times \mathbb{N}$ such that*

- $(\gamma, 1) \in A_\gamma$ *for all* $\gamma \in \Gamma$*;*
- $\frac{|A_\gamma \setminus A_{\gamma'}| + |(A_{\gamma'} \setminus A_\gamma|}{|A_\gamma \cap A_{\gamma'}|} < \epsilon$ *for all γ and $\gamma' \in \Gamma$ satisfying $d(\gamma, \gamma') \leq r$, where, for each finite set $A, |A|$ is the number of elements in A;*

- $\exists R > 0$ *such that if* $(x, m) \in A_\gamma$, $(y, n) \in A_\gamma$ *for some* $\gamma \in \Gamma$, *then* $d(x, y) \leq R$.

The authors would like to thank Jacek Brodzki who explained to us the equivalence of Property A with exactness as follows: according to Higson and Roe, [8], a finitely generated group has Property A if and only if it acts amenably on its Stone-Cech compactification. By the theorem of Ozawa in [14] this is equivalent to exactness for the group. Exactness should be thought of as a weak form of amenability; the property was first made prominent by the work of Kirchberg and Wassermann [9], and studied by several authors [1, 6, 8, 14, 18, 19]. Examples of exact groups include groups of finite asymptotic dimension, for example Gromovs word hyperbolic groups, discrete subgroups of connected Lie groups and amenable groups. The class is closed under the semi-direct product, [19].

By way of further motivation for the study of exactness we should point out that groups with property A admit a uniform embedding into Hilbert Space and satisfy the strong Novikov conjecture, and the coarse Baum Connes conjecture [19]. In [7] Guentner and Kaminker introduced a numerical quasi-isometry invariant of a finitely generated group, the values of which parametrize the difference between the group being uniformly embeddable in a Hilbert Space and the reduced C*-algebra of the group being exact.

Theorem (Guentner and Kaminker, see [6]) *Let G be a discrete group. If the Hilbert Space compression of G is strictly greater than $1/2$ then G is exact.*

We will define Hilbert Space compression later, but note here that it is a measure of the amount of distortion that is necessary when trying to embed the group in a Hilbert Space via a large scale Lipschitz map. Guentner and Kaminker illustrated their theorem by showing that the Hilbert Space compression of a finite rank free group is 1 thus giving a new proof of exactness for free groups. It should be noted that they did not construct an embedding of the free group in a Hilbert Space with (asymptotic) compression 1, but rather, thinking of the group as a tree via its Cayley graph, produced a family of large scale Lipschitz embeddings with asymptotic compression arbitrarily close to 1. Those familiar with CAT(0) complexes would recognize that the first of their embeddings (with asymptotic compression $1/2$) can be used without change to embed the vertex set of a CAT(0) cube complex into a Hilbert Space with asymptotic compression $1/2$ though this is not in itself enough to establish exactness for a group acting on the cube complex. Guentner and Kaminker showed that in the case of a tree the embedding can be modified to obtain new embeddings with asymptotic compression arbitrarily close to 1.

The main purpose of this note is to show how to adapt the construction from [7] to the class of unbounded, finite dimensional CAT(0) cube complexes. In the case of a tree one uses the fact that there is a unique edge geodesic joining any two points in the tree; the same is of course not true for CAT(0) cube complexes of dimension at least 2 so the embedding and the argument need to be modified appropriately. In place of unique edge geodesics we will use the normal cube paths originally introduced in [11] to establish biautomaticity for groups acting freely and properly discontinuously on CAT(0) cube complexes.

Theorem 10 *Let X be an unbounded finite dimensional CAT(0) cube complex. The Hilbert space compression of X is* 1.

In [6] it is shown that Hilbert Space compression is a quasi-isometry invariant so if a discrete group G acts freely and co-compactly on an unbounded CAT(0) cube complex it follows that the group (regarded as a metric space via the word length metric) has Hilbert Space compression 1. Since $1 > 1/2$ we obtain:

Theorem 12 *If G is a group acting properly and co-compactly on a CAT(0) cube complex then G is exact and therefore has Yu's Property A.*

Note that if G acts properly on a bounded CAT(0) cube complex then G is finite and therefore exact, so the hypothesis that the cube complex should be unbounded (which is only inserted in Theorem 10 in order to ensure that asymptotic compression can be defined) is not needed in Theorem 12.

The paper is organised as follows: In section A.1 we recall the definition of a CAT(0) cube complex and, stating the definitions, show how to construct a large scale Lipschitz embedding of such a complex in an associated Hilbert Space, with asymptotic compression $1/2$. In section A.2 we outline some preliminary results concerning the existence and properties of normal cube paths in a CAT(0) cube complex. The results in this section are taken from [11]. In section A.3 we define a family of embeddings $\{f_\epsilon \mid 0 < \epsilon < 1/2\}$ of the vertices of a cube complex X into the Hilbert Space of square summable real valued functions on the set of hyperplanes of X. We also show that these embeddings are large-scale Lipschitz. In section A.4 we show that the compression of each map f_ϵ is $1/2+\epsilon$ and deduce that the Hilbert Space compression of the metric space $(X^{(0)}, d_1)$ is 1, where $X^{(0)}$ denotes the vertex set of X and d_1 is the edge metric. In section A.5 we deduce the exactness of groups acting properly and co-compactly on a CAT(0) cube complex.

The class of groups covered by this theorem includes free groups, finitely generated Coxeter groups [13], and finitely generated right angled Artin groups (for which the Salvetti complex is a CAT(0) cube complex). A rich class of interesting examples is furnished by Wise, [17], in which it is shown that many small cancellation groups act properly and co-compactly on CAT(0) cube complexes. The examples include every finitely presented group satisfying the B(4)-T(4) small cancellation condition and all those word-hyperbolic groups satisfying the B(6) condition. Finally many 3-manifolds admit decompositions as CAT(0) cube complexes, so their fundamental groups are also covered by the theorem, a family of examples is provided by certain canonical surgeries defined by link diagrams (see [2] and [3]). Classical examples are furnished by groups acting simply transitively on buildings with the structure of a product of trees.

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A.1 CAT(0) cube complexes

A *cube complex* X is a metric polyhedral complex in which each cell is isometric to the Euclidean cube $[-1/2, 1/2]^n$, and the gluing maps are isometries. If there is a bound on the dimension of the cubes then the complex carries a complete geodesic metric, [4].

A cube complex is *non-positively curved* if for any cube C the following conditions on the link of C, $\mathrm{lk}C$, are satisfied:

1. (*no bigons*) For each pair of vertices in $\mathrm{lk}C$ there is at most one edge containing them.

2. (*no triangles*) Every edge cycle of length three in $\mathrm{lk}C$ is contained in a 2-simplex of $\mathrm{lk}C$.

The following theorem of Gromov relates the combinatorics and the geometry of the complex.

Lemma A.1.1. (Gromov, [5]) *A cube complex X is locally $CAT(0)$ if and only if it is non-positively curved, and it is CAT(0) if and only if it is non-positively curved and simply connected.*

Any graph may be regarded as a 1-dimensional cube complex, and the curvature conditions on the links are trivially satisfied. The graph is CAT(0) if and only if it is a tree. Euclidean space also has the structure of a CAT(0) cube complex with its vertices at the integer lattice points.

A *midplane* of a cube $[-1/2, 1/2]^n$ is its intersection with a codimension 1 coordinate hyperplane. So every n-cube contains n midplanes each of which is an $(n-1)$-cube, and any m of which intersect in a $(n-m)$-cube. Given an edge in a non-positively curved cube complex, there is a unique codimension 1 hyperplane in the complex which cuts the edge transversely in its midpoint. This is obtained by developing the midplanes in the cubes containing the edge. In the case of a tree the hyperplane is the midpoint of the edge, and in the case of Euclidean space it is a geometric (codimension-1) hyperplane.

In general a hyperplane is analogous to an immersed codimension 1 submanifold in a Riemannian manifold and in a CAT(0) cube complex one can show that the immersion is a local isometry. An application of the Cartan-Hadamard theorem then shows that the hyperplane is isometrically embedded. Furthermore any hyperplane in a CAT(0) cube complex separates it into two components referred to as the half spaces associated with the hyperplane. This is a consequence of the fact that the complex is simply connected. The hyperplane gives rise to 1-cocycle which is necessarily trivial, and hence the hyperplane separates the space.

The set of vertices of a CAT(0) cube complex X can be viewed as a discrete metric space, where the metric $d_1(u, v)$ is given by the length of a shortest edge path between the vertices u and v. We will refer to this as the ℓ^1 metric on the vertices. Alternatively we can measure the distance by restricting the path metric on X to obtain the ℓ^2 metric on the vertices. If X is finite dimensional these metrics are quasi-isometric, and we have

$d(u,v) \leq d_1(u,v) \leq \sqrt{n}d(u,v)$ where d denotes the CAT(0) (geodesic) metric on X and n is the dimension of the complex.

Sageev [16] observed that the shortest path in the 1-skeleton crosses any hyperplane at most once, and since every edge crosses exactly one hyperplane, the ℓ^1 distance between two vertices is the number of hyperplanes separating them.

Finally we will need the concept of a median. In any CAT(0) cube complex there is a well defined notion of an interval; given any two vertices u, v the interval between them, denoted $[u, v]$ consists of all the vertices which lie on an edge geodesic from u to v. Given any three vertices u, v, w there are three intervals $[u, v], [v, w], [w, u]$ and the intersection of these three intervals is always a single point m known as the median of the triple u, v, w (see [15] for details). It has the following important property: If we consider the hyperplanes which separate the pair u, v and those which separate the pair u, w the intersection of these two families consists of precisely the hyperplanes which separate u and the median m. Furthermore the hyperplanes which separate v from w are precisely those hyperplanes which separate m from v together with those which separate m from w so we have $d_1(v,w) = d_1(v,m) + d_1(m,w) = d_1(v,u) + d_1(w,u) - 2d_1(m,u)$. We will use this fact in section A.4.

In [10] it was shown how to use the hyperplane structure of a CAT(0) cube complex X to obtain an ℓ^1 embedding of the cube complex in the Hilbert Space $\ell^2(H, \mathbb{R})$ of square summable (real valued) functions on the set H of hyperplanes in X. An alternative description of the embedding, based on the one used in [7] in the context of a tree, is as follows:

Choose a basepoint v in $X^{(0)}$ and for each vertex $w \in X^{(0)}$ set $H_w = \{h \in H \mid h \text{ separates } v \text{ and } w\}$. Define $f_w : H \longrightarrow \mathbb{R}$ by $f_w = \sum_{h \in H_w} \delta_h$ where δ_h denotes the characteristic function of the singleton $\{h\} \subset H$.

It is easy to see that the function f_w is ℓ^1 and therefore ℓ^2 and since the Hilbert Space is contractible (in fact uniquely geodesic) the map extends to an embedding of X in $\ell^2(H, \mathbb{R})$. If X is a cube then this embedding is isometric, however in the case of a tree (consisting of more than a single edge) then it is not. For example let T be the tree consisting of two edges e_s, e_t both adjacent to a vertex v, and with the other two vertices labelled s, t. The tree has two hyperplanes, corresponding to the midpoints of the two edges, so that $\ell^2(T, \mathbb{R}) \sim \mathbb{R}e_s \bigoplus \mathbb{R}e_t$. The vertex v is not separated from itself by either of the hyperplanes so we have $f_v = 0$. The vertex s is only separated from v by the hyperplane s so we have $f_s = \delta_{e_s}$ and similarly $f_t = \delta_{e_t}$. Now in the tree we have $d_1(s,t) = d_2(s,t) = 2$ however in the Hilbert Space we have $d_1(f_s, f_t) = 2 \neq \sqrt{2} = d(f_s, f_t)$, where we have used d_1 to denote the ℓ^1 metric and d to denote the Hilbert metric.

Although the embedding defined above is not necessarily an isometry it is relatively easy to show that it is a large scale Lipshcitz map, and we can measure the distortion of such a map in terms of its compression:

Definition A.1.2. *A function $f : X \to Y$ is large-scale Lipschitz if there exist $C > 0$ and $D \geq 0$ such that $d_Y(f(x), f(y)) \leq C d_X(x,y) + D$. Following*

Gromov, the compression $\rho(f)$ of f is given by $\rho_f(r) = \inf_{d_X(x,y)\geq r} d_Y(f(x), f(y))$. Assuming that X is unbounded the asymptotic compression R_f is given by

$$R_f = \liminf_{r\to\infty} \frac{\log \rho_f^*(r)}{\log r}$$

where $\rho_f^*(r) = \max\{\rho_f(r), 1\}$.

In the case of the embedding of the vertices described above the map is large scale Lipschitz with $C = 1, D = 0$. The argument used by Guentner and Kaminker [7] to compute the asymptotic compression of the embedding of a tree goes through without change to our more general context to show that the asymptotic compression is $1/2$. (It should be noted here that we are regarding the cube complex as a metric space via the ℓ^1 metric not the (geodesic) ℓ^2 metric.)

In order to obtain large scale Lipschitz embeddings with asymptotic compression close to 1 we need to adapt the embedding described above. The idea, taken from [7] is to weight the functions δ_h according to how far the hyperplane is from the basepoint. Whereas in the case of a tree the hyperplanes which separate two vertices are linearly ordered in a higher dimensional cube complex they are not and there are several partial orders one could use in modifying the argument. It turns out that the appropriate ordering is furnished by the normal cube paths introduced in [11] and we describe these next.

A.2 Normal cube paths

Definition A.2.1. *A cube path is a sequence of cubes $\mathcal{C} = \{C_0, \ldots C_n\}$, each of dimension at least 1, such that each cube meets its successor in a single vertex, $v_i = C_{i-1} \cap C_i$ and such that for $1 \leq i \leq n-1$, C_i is the (unique) cube of minimal dimension containing v_i and v_{i+1}. Note that v_i and v_{i+1} are diagonally opposite vertices of C_i. We define v_0 to be the vertex of C_0 which is diagonally opposite v_1, and v_n to be the vertex of C_n diagonally opposite v_{n-1}. We call the v_i, vertices of the cube-path, with v_0 the initial vertex and v_n the terminal vertex. Given a cube path from u to v we can construct edge paths from u to v which travel via the edges of the cubes C_i so every hyperplane separating u from v must intersect at least one of the cubes C_i. We say the cube path is normal if $C_{i+1} \cap st(C_i) = v_i$ for each i, where $st(C_i)$ is the union of all cubes which contain C_i as a face (including C_i itself).*

In [11] it was shown that given any two vertices u, v there is a unique normal cube path $\mathcal{C} = \{C_0, \ldots, C_n\}$ from u to v. We will need the following key facts about normal cube paths all of which may be found in [11].

Lemma A.2.2. *Let s, t, v_0 be vertices of a CAT(0) cube complex with s and t diagonally adjacent across some cube E_0. Let $s = s_0, s_1, \ldots, s_m = v$, $t = t_0, t_1, \ldots, t_n = u$ be the vertices of the (unique) normal cube paths from s to v_0 and from t to v_0 respectively. Let $\{C_i \mid i = 1, \ldots m\}$ be the cubes on the normal cube path from s to v_0 and $\{D_j \mid j = 1, \ldots n\}$ be the cubes on the normal cube path from t to v_0. Then:*

1. *Each hyperplane separating s from v_0 intersects exactly one of the cubes C_i and each hyperplane separating t from v_0 intersects exactly one of the cubes D_j.*

2. *For each $i \leq \min\{m, n\}$ there is a cube E_i such that s_i is diagonally adjacent to t_i across E_i .*

We will need the following technical lemma:

Lemma A.2.3. *Let s, t, v be vertices of the CAT(0) cube complex X with s and t diagonally opposite across some cube E_0. Let $s = s_0, s_1, \ldots, s_m = v$, $t = t_0, t_1, \ldots, t_n = v$ be the vertices of the (unique) normal cube paths from s to v and from t to v respectively. Let $\{C_i \mid i = 1, \ldots m\}$ be the cubes on the normal cube path from s to v and $\{D_j \mid j = 1, \ldots n\}$ be the cubes on the normal cube path from t to v. If h is a hyperplane in X which separates both s and t from v and which intersects the cube C_i then h also intersects one of the cubes D_{i-1}, D_i, D_{i+1}.*

Proof. By lemma A.2.2 the hyperplane h can only (and must) intersect the normal cube path from t to v in one of the cubes D_j, and the hypothesis that $s = s_0$ and $t = t_0$ are diagonally opposite across the cube E_0 ensures that for each $i \leq \min\{m, n\}$ s_i is diagonally opposite to t_i across some cube E_i.

Now h separates s_{i-1}, s_i and also separates t_{j-1}, t_j. let $k = \min\{i, j\}$. Assume first that h separates s_{k-1} and s_k so $i = k \leq j$; if h also separates t_{k-1} and t_k then h crosses $D_k = D_i$ as required. If on the other hand h does not separate t_{k-1} from t_k then, since it does not separate t from t_{k-1}

by the minimality of k, but does separate t from v, it must also separate t_k from v. Now we construct an edge path from t_k to v as follows. First cross over the cube E_k from s_k to t_k then follow a path through the cubes $C_{k+1}, C_{k+2}, \ldots, C_m$ to v. This gives an edge path from t_k to v so it must cross h. However none of the cubes $C_{k+1}, \ldots, C_m$ intersect h so h must cross E_k and hence h is adjacent to t_k. But as h separates t_k from v and is adjacent to t_k it must cross the first cube (D_{k+1}) on the normal cube path from t_{k+1} to v as required. The case when h separates t_{k-1} and t_k so $i = k$ but does not separate s_{k-1} and s_k is argued in exactly the same way reversing the roles of s and t, C and D and so on. $\square$

From now on we fix a vertex v as a basepoint and for each vertex s we define an integer-valued weight function w_s on the set of hyperplanes as follows. Let $\mathcal{C} = \{C_0, \ldots, C_n\}$ be the unique normal cube path from s to v. If the hyperplane h separates s and v then set $w_s(h) = i + 1$ where h intersects the cube C_i, otherwise set $w(h) = 0$. Hence w_s has finite support. From Lemma A.2.3 we get:

Corollary A.2.4. *If s and t are adjacent in X and h separates both s and t from v then $|w_t(h) - w_s(h)| \leq 1$.*

Proof. If the normal cube path from s to v is denoted by the cubes C_i as above and the normal cube path from t to v is denoted by D_j then h intersects precisely the cubes $C_{w_s(h)}$ and $D_{w_t(h)}$ so by the lemma $D_{w_t(h)} = D_{w_s(h)\pm 1}$, and $w_t(h) = w_s(h)$ or $w_t(h) = w_s(h) \pm 1$ as required. $\square$

Note that in the statement of the corollary "adjacent" may be taken to mean adjacent across the diagonal of any cube, however in our application we will only need it to mean that s and t are vertices of a common edge.

A.3 The large scale Lipschitz embeddings

As in the last section we fix a CAT(0) cube complex X (not necessarily finite dimensional) and a base vertex v. We will show how to construct a family (indexed by the interval $(0, 1/2)$) of large scale Lipschitz embeddings of the vertex set $X^{(0)}$ into the Hilbert Space of ℓ^2 functions from the set H of hyperplanes in X to $\mathbb{R}$.

For each $\epsilon \in (0, 1/2)$ define $f_\epsilon(s) = \sum_{h \in h} w_s(h)^\epsilon \delta_h$. As noted before since the s-weight of a hyperplane is 0 unless the hyperplane is one of the finitely many separating s from the basepoint v, this sum is always finite and therefore is an element of $\ell^2(H, \mathbb{R})$.

In order to show that f_ϵ is a large scale Lipschitz map it suffices to show that there is a constant C such that whenever $d_1(s,t) = 1$, $\| f_\epsilon(s) - f_\epsilon(t) \|^2 \leq C$.

Lemma A.3.1. *For each $\epsilon \in (0, 1/2)$ there is a constant C such that for any vertices $s, t \in X^{(0)}$ with $d_1(s,t) = 1$ we have $\| f_\epsilon(s) - f_\epsilon(t) \|^2 \leq C$.*

Proof. Let h_0 be the hyperplane cutting the edge joining s, t. Assume, without loss of generality that h separates t from v but not s from v so that $d_1(s,v) + 1 = d_1(t,v)$ and the set of hyperplanes separating t from v is the union of the set $\{h_1, \ldots, h_m\}$ of the hyperplanes separating s from v together with h.

We need to compute

$$\| f_\epsilon(s) - f_\epsilon(t) \|^2 = \sum_{i=0}^{m} [w_s(h_i)^\epsilon - w_t(h_i)^\epsilon]^2 = 1^{2\epsilon} + \sum_{i=1}^{m} [w_s(h_i)^\epsilon - w_t(h_i)^\epsilon]^2 \quad \text{(A.1)}$$

Now according to corollary A.2.4 we have $|w_t(h_i) - w_s(h_i)| \leq 1$. Suppose that for a particular hyperplane h_i we have $w_s(h_i) = k$ so that $w_t(h_i)$ takes one of the values $k-1, k, k+1$ and $[w_s(h_i)^\epsilon - w_t(h_i)^\epsilon]^2$ takes one of the values $[k^\epsilon - (k+1)^\epsilon]^2, [k^\epsilon - k^\epsilon]^2, [k^\epsilon - (k-1)^\epsilon]^2$

An elementary calculation of the first derivative shows that the function $X \mapsto [X^\epsilon - (X+1)^\epsilon]^2$. is strictly increasing so we have $[k^\epsilon - (k+1)^\epsilon]^2 > [(k-1)^\epsilon - k^\epsilon]^2 = [k^\epsilon - (k-1)^\epsilon]^2 > 0 = [k^\epsilon - k^\epsilon]^2$ hence we have $[w_s(h_i)^\epsilon - w_t(h_i)^\epsilon]^2 \leq [w_s(h_i)^\epsilon - (w_s(h_i)+1)^\epsilon]^2$ and so

$$\sum_{i=1}^{m} [w_s(h_i)^\epsilon - w_t(h_i)^\epsilon]^2 \leq \sum_{i=1}^{m} [w_s(h_i)^\epsilon - (w_s(h_i)+1)^\epsilon]^2. \qquad \text{(A.2)}$$

We can split the final sum as a double sum taken over all hyperplanes with a given s-weight. Let J denote the set of all s-weights.

$$\sum_{i=1}^{m}[w_s(h_i)^\epsilon - (w_s(h_i)+1)^\epsilon]^2 = \sum_{j\in J}\sum_{w_s(h_i)=j}[w_s(h_i)^\epsilon - (w_s(h_i)+1)^\epsilon]^2$$
$$= \sum_{j\in J}\sum_{w_s(h_i)=j}[j^\epsilon - (j+1)^\epsilon]^2. \tag{A.3}$$

Since the cube complex has dimension n we can cross at most n hyperplanes in any given cube so the number of hyperplanes with $w_s(h) = j$ is at most n for any j and

$$\sum_{j}\sum_{w_s(h_i)=j}[j^\epsilon - (j+1)^\epsilon]^2 \leq \sum_{j} n[j^\epsilon - (j+1)^\epsilon]^2.$$

Putting $w_j = j^\epsilon$ and adding additional positive terms we see that

$$\| f_\epsilon(s) - f_\epsilon(t) \|^2 \leq n\sum_{j=0}^{\infty}[w_j - w_{j+1}]^2.$$

The series $\sum_{j=0}^{\infty}[w_j - w_{j+1}]^2$ converges so putting $C = n\sum_{j=0}^{\infty}[w_j - w_{j+1}]^2$ we get $\| f_\epsilon(s) - f_\epsilon(t) \|^2 \leq C$ and f_ϵ is large scale Lipschitz as required. □

A.4 Hilbert Space compression

While establishing that the map is large scale Lipschitz required us to show that $\| f_\epsilon(s) - f_\epsilon(t) \|^2$ is small for nearby vertices, to establish that the embedding has large asymptotic compression requires us to show that $\|$

$f_\epsilon(s) - f_\epsilon(t) \|^2$ is relatively large for points s, t which are sufficiently far apart.

Specifically we will prove:

Lemma A.4.1. *For any positive r and any $\epsilon \in (0, 1/2)$ there is a constant C_ϵ such that $\| f_\epsilon(s) - f_\epsilon(t) \|^2 \geq C_\epsilon r^{1+2\epsilon}$. Hènce $\rho_{f_\epsilon}(r) \geq \sqrt{C_\epsilon} r^{1/2+\epsilon}$*

Proof. Let $D = d_1(s,t) \geq r$ and assume $d(1,s) \leq d(1,t)$ so that, letting m denote the median of the triple $1, s, t$, we have $d(m,t) \geq d(m,s)$. It follows that $d(m,t) \geq \sharp(\frac{D}{2}) \geq \sharp(\frac{r}{2})$ where $\sharp(n)$ denotes the smallest integer greater than n. Hence there are at least $\sharp(\frac{r}{2})$ hyperplanes which separate t from 1 but which do not separate s from 1. We will denote these hyperplanes $h_1, h_2, \ldots, h_{\sharp(\frac{r}{2})}$. Now consider the normal cube path $C_0, C_1, \ldots C_n$ from t to 1. As noted in lemma A.2.2 each of the hyperplanes h_i must intersect exactly one of the cubes C_j, and by definition $w_t(h_i) = (j+1)$. By relabelling if necessary we may assume that the t-weight increases (not necessarily strictly) with the index i of the hyperplane, and given that the cube complex has dimension n at most n of the hyperplanes can have the same t-weight, i.e., at most n of the hyperplanes have weight 1^ϵ and the others have weight at least 2^ϵ; at most n of the remaining hyperplanes can have weight 2^ϵ and the others have to have weight at least 3^ϵ and so on. Recall that for each of these hyperplanes $w_s(h_i) = 0$ by hypothesis so, writing $\sharp(\frac{r}{2}) = kn + m$ for some integer $0 \leq m < n$ we have

$$\| f_\epsilon(s)-f_\epsilon(t) \|^2 \geq w_t(h_1)^{2\epsilon}+\ldots+w_t(h_{\sharp(\frac{r}{2})})^{2\epsilon} \geq n(1^{2\epsilon}+2^{2\epsilon}+\ldots+k^{2\epsilon})+m(k+1)^{2\epsilon}.$$

We will now show that the RHS of this equation is greater than the expression

$$\begin{aligned}
& \frac{1}{n}\left(1^{2\epsilon}+2^{2\epsilon}+\ldots+\sharp(\frac{r}{2})^{2\epsilon}\right) \\
= & \frac{1}{n}(1^{2\epsilon}+\ldots+n^{2\epsilon}+(n+1)^{2\epsilon}+\ldots+(2n)^{2\epsilon}+(2n+1)^{2\epsilon}+\ldots \\
+ & (kn)^{2\epsilon}+(kn+1)^{2\epsilon}+\ldots+(kn+m))^{2\epsilon})
\end{aligned}$$

Claim: For any $i \geq 1$,

$$ni^{2\epsilon} > \frac{1}{n}[((i-1)n+1)^{2\epsilon}+\ldots+(in)^{2\epsilon}]$$

Since $\epsilon < \frac{1}{2}$ and $n \geq 1$ we have $ni^{2\epsilon} > n^{2\epsilon}i^{2\epsilon} = (in)^{2\epsilon}$

On the other hand, since $\epsilon > 0$ and $in > ik$ for all $k < n$ we have

$$\frac{1}{n}[((i-1)n+1)^{2\epsilon}+\ldots+(in)^{2\epsilon}] < \frac{1}{n}(n(in)^{2\epsilon}) = (in)^{2\epsilon}$$

So

$$ni^{2\epsilon} > (in)^{2\epsilon} > \frac{1}{n}[((i-1)n+1)^{2\epsilon}+\ldots+in^{2\epsilon}]$$

Claim:

$$m(k+1)^{2\epsilon} > \frac{1}{n}\left((kn+1)^{2\epsilon} + \ldots + (kn+m)^{2\epsilon}\right)$$

We have that

$$m(k+1)^{2\epsilon} \geq m^{2\epsilon}(k+1)^{2\epsilon} = (mk+m)^{2\epsilon}$$

Looking at the RHS of the statement of the claim we have:

$$\begin{aligned}
\frac{1}{n}\left((kn+1)^{2\epsilon} + \ldots + (kn+m)^{2\epsilon}\right) &< \frac{m}{n}(kn+m)^{2\epsilon} \quad \text{(since (kn + m) is the biggest term)} \\
&< \left(\frac{m}{n}\right)^{2\epsilon}(kn+m)^{2\epsilon} \quad (\text{since}\frac{m}{n} < 1) \\
&= (mk + \frac{m^2}{n})^{2\epsilon} \\
&< (mk+m)^{2\epsilon} \quad (\text{since}\frac{m}{n} < 1)
\end{aligned}$$

And so

$$m(k+1)^{2\epsilon} \geq (mk+m)^{2\epsilon} > \frac{1}{n}\left((kn+1)^{2\epsilon} + \ldots + (kn+m)^{2\epsilon}\right)$$

Putting both claims together, we have that:

$$
\begin{aligned}
n * 1^{2\epsilon} &> \frac{1}{n}\left(1^{2\epsilon} + \ldots + n^{2\epsilon}\right) \\
n * 2^{2\epsilon} &> \frac{1}{n}\left((n+1)^{2\epsilon} + \ldots + (2n)^{2\epsilon}\right) \\
&\vdots \\
n * k^{2\epsilon} &> \frac{1}{n}\left(((k-1)n+1)^{2\epsilon} + \ldots + (kn)^{2\epsilon}\right) \\
m * (k+1)^{2\epsilon} &\geq \frac{1}{n}\left((kn+1)^{2\epsilon} + \ldots + (kn+m)^{2\epsilon}\right)
\end{aligned}
$$

And so

$$
n * 1^{2\epsilon} + n * 2^{2\epsilon} + \ldots + n * k^{2\epsilon} + m(k+1)^{2\epsilon} > \frac{1}{n}\left(1^{2\epsilon} + 2^{2\epsilon} + \ldots + \sharp(\frac{r}{2})^{2\epsilon}\right)
$$

Hence,

$$
\begin{aligned}
\| f_\epsilon(s) - f_\epsilon(t) \|^2 &\geq w_t(h_1)^{2\epsilon} + \ldots + w_t(h_{\sharp(\frac{r}{2})})^{2\epsilon} \\
&\geq n * 1^{2\epsilon} + n * 2^{2\epsilon} + \ldots + n * k^{2\epsilon} + m(k+1)^{2\epsilon} \\
&\geq \frac{1}{n}(1^{2\epsilon} + 2^{2\epsilon} + \ldots + \sharp(\frac{r}{2})^{2\epsilon})
\end{aligned}
$$

In [6] Guentner and Kaminker showed that $(1^{2\epsilon} + 2^{2\epsilon} + \ldots + \sharp(\frac{r}{2})^{2\epsilon}) \geq \frac{r^{2\epsilon+1}}{(2^{2\epsilon+1})(2\epsilon+1)}$ so putting $C_\epsilon = \frac{1}{n(2^{2\epsilon+1})(2\epsilon+1)}$ we obtain, as required,

$$\| f_\epsilon(s) - f_\epsilon(t) \|^2 \geq C_\epsilon r^{2\epsilon+1}$$

□

Now we obtain:

Lemma A.4.2. *For each ϵ the asymptotic compression of the map f_ϵ is at least $1/2 + \epsilon$.*

Proof. We have

$$R_{f_\epsilon} = \liminf_{r\to\infty} \frac{\log \rho_{f_\epsilon}(r)}{\log r} \geq \liminf_{r\to\infty} \frac{\log \sqrt{C_\epsilon} r^{1/2+\epsilon}}{\log r} = 1/2 + \epsilon$$

□

A.5 Exactness for groups acting properly and co-compactly on a CAT(0) cube complex

The Hilbert Space compression of an unbounded metric space is defined to be the supremum of the asymptotic compression of all possible large scale Lipschitz maps from the metric space to a Hilbert Space so putting together the results of sections A.3 and A.4 we get

Theorem 14. *The Hilbert Space compression of an unbounded finite dimensional CAT(0) cube complex (X, d) is 1.*

Proof. For each $\epsilon \in (0, 1/2)$ we have constructed a large scale Lipschitz embedding f_ϵ of the metric space $(X^{(0)}, d_1)$ into the Hilbert Space $\ell^2(H, \mathbb{R})$ with compression at least $1/2 + \epsilon$. Hence the Hilbert Space compression of $(X^{(0)}, d_1)$ is 1. Since X is finite dimensional, of dimension n say, we have $d(s,t) \leq d_1(s,t) \leq \sqrt{n}d(s,t)$ so $(X^{(0)}, d_1)$ is quasi-isometric to (X, d), and since Hilbert Space compression is a quasi-isometry invariant we obtain the result. □

Now suppose G s a group acting properly and co-compactly on an unbounded CAT(0) cube complex X. Choose a finite generating set for G and regard G as a metric space via the edge metric on the Cayley graph. Then G is quasi-isometric to (X, d). Again by quasi-isometry invariance we obtain

Corollary A.5.1. *Let G be a finitely generated group regarded as a metric space via the word metric with respect to some finite generating set. If G acts properly and co-compactly on an unbounded CAT(0) cube complex then G has Hilbert Space compression* 1.

Finally since Guentner and Kaminker showed that a discrete group with Hilbert Space compression strictly greater than $1/2$ is exact we obtain:

Theorem 15. *If G is a group acting properly and co-compactly on a CAT(0) cube complex then G is exact and therefore has Yu's Property A.*

As noted in the introduction, if G acts properly on a bounded CAT(0) cube complex then G is finite and therefore exact, so the hypothesis that the cube complex should be unbounded (which was only inserted in the supporting results in order to ensure that asymptotic compression can be defined) would be superfluous here.

Bibliography

[1] C. Anantharaman-Delaroche, J. Renault, Monographies de L'Enseignement Mathématique, 36. L'Enseignement Mathmatique, Geneva, 2000.

[2] I. R. Aitchison, J. H. Rubinstein, An introduction to polyhedral metrics of non-positive curvature on 3-manifolds in "Geometry of Low-dimensional Manifolds: 2 Symplectic Manifolds and Jones-Witten Theory", London Mathematical Society Lecture Notes Series 151, Cambridge University Press (1990) 127–161.

[3] I. R. Aitchison, J. H. Rubinstein, Canonical surgery on alternating link diagrams in Knots 90, (1992) 543–558.

[4] M.R. Bridson and A. Haefliger, Metric spaces of non-positive curvature, Grundlehren der mathematischen Wissenschaften Volume 319, Springer, 1999.

[5] M. Gromov, Hyperbolic groups in S.M. Gersten (ed.) Essays on Group Theory (Mathematical Sciences Research Institute Publications) Springer-Verlag 1987.

[6] E. Guentner, J. Kaminker, Exactness and the Novikov conjecture. Topology 41 (2002), no. 2, 411–418; Addendum: Topology 41 (2002) no. 2, 419–420.

[7] E. Guentner, J. Kaminker, Exactness and Uniform Embeddability of Discrete Groups, ArXiv preprint math.OA/0309166.

[8] N. Higson, J. Roe, Amenable group actions and the Novikov conjecture. J. Reine Angew. Math. 519 (2000) 143–153.

[9] Kirchberg, Wasserman, Exact Groups and Continuous Bundles of C^*-algebras, Mathematische Annalen, 1999.

[10] G. Niblo, M. Roller, Groups acting on cubes and Kazhdan's property (T). Proc. Amer. Math. Soc. **126** (1998), no. 3, 693–699.

[11] G.A .Niblo, L.D. Reeves. The geometry of cube complexes and the complexity of their fundamental groups,Topology, Vol. 37,No 3 (1998) 621–633.

[12] G.A.Niblo, L.D.Reeves. Groups acting on CAT(0) cube complexes, Geometry and Topology, Vol. 1 (1997) 1–7.

[13] G.A.Niblo, L.D.Reeves. Coxeter groups act on CAT(0) cube complexes, Journal of Group Theory, 6, (2003) 309–413.

[14] N. Ozawa, Amenable actions and exactness for discrete groups, *Comptes Rendus Acad. Sci. Paris* **330** (2000) 691–695.

[15] M. A. Roller, Poc Sets, Median Algebras and Group Actions. An extended study of Dunwoody's construction and Sageev's theorem *Habilitationeschrift, Regensberg* 1998.

[16] M. Sageev, Ends of group pairs and non-positively curved cube complexes. Proc. London Math. Soc. (3), 71(3) (1995) 585–617.

[17] D.T. Wise, Cubulating Small Cancellation Groups, Preprint, http://www.gidon.com/dani/tl.cgi?athe=pspapers/SmallCanCube.ps.

[18] G. Yu, The Novikov conjecture for groups with finite asymptotic dimension. Ann. of Math. (2) 147 (1998), no. 2, 325–355.

[19] G. Yu, The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert Space. Invent. Math. 139 (2000), no. 1, 201–240.

Appendix B

Paper 2, to appear in the Bulletin of the London Mathematical Society

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Exactness of free and amenable groups by the construction of Ozawa kernels

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Abstract

Using properties of their Cayley graphs, specific examples of Ozawa kernels are constructed for both free and amenable groups, thus showing that these groups satisfy Property O. It is deduced both that these groups are exact and satisfy Yu's Property A.

B.1 Introduction

The property of exactness was first introduced as an analytic property of C^*-algebras. We say that a C^*-algebra C is exact if the operation of taking the cross product with C preserves exactness of short exact sequences. In other words, for any short exact sequence

$$0 \to J \to B \to B/J \to 0$$

the sequence

$$0 \to J \otimes_{min} C \to B \otimes_{min} C \to B/J \otimes_{min} C \to 0$$

is also exact.

We say that a group Γ is exact if the operation of taking the reduced crossed product with Γ preserves exactness of short exact sequences of Γ-C^*-algebras. In other words, Γ is exact if and only if for every exact sequence of Γ-C^*-algebras

$$0 \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow 0$$

the sequence

$$0 \longrightarrow C_r^*(\Gamma, B) \longrightarrow C_r^*(\Gamma, C) \longrightarrow C_r^*(\Gamma, D) \longrightarrow 0$$

of crossed product algebras is exact. Following work by Kirchberg and

Wassermann, [KW], a discrete group is said to be exact if and only if its reduced C^*-algebra is exact. Exactness is linked to conjectures such as the Novikov Conjecture and the Baum Connes conjecture and exact groups satisfy the Coarse Baum Connes Conjecture. Examples of exact groups include hyperbolic groups, groups with finite asymptotic dimension and groups acting on finite dimensional CAT(0) cube complexes [CN].

More recent work has shown that when we regard the group as a metric space in the word metric, the property of exactness can be defined by more geometric means. In particular, Yu showed that exactness of a group is equivalent to Property A, a measure theoretic property, and implies the Uniform Embedding Property [Yu]. In [O], Ozawa introduced the following property which we will call Property O and proved that for a discrete group it is equivalent to exactness of its reduced C^*-algebra:

Definition B.1.1. *A discrete group G is said to have Ozawa's Property O if for any finite subset $E \subset G$ and any $\epsilon > 0$, there are a finite subset $F \subset G$ and $u\colon G \times G \to \mathbb{R}$ such that*

1. *$u(x, y)$ is a positive definite kernel*
2. *$u(x, y) \neq 0$ only if $x^{-1}y \in F$*
3. *$|1 - u(x, y)| < \epsilon$ if $x^{-1}y \in E$*

We will call functions satisfying the conditions of Property O Ozawa kernels.

This property has been used by Guentner and Kaminker to prove their theorem relating asymptotic compression and exactness [GK]. No explicit examples of Ozawa kernels can be found in the literature. The aim of this note is to construct explicit Ozawa kernels for two classical cases of exact groups, amenable groups and free groups by using geometric properties of their Cayley graphs. In both cases the functions can be viewed as weighted mass functions. This shows that amenable and free groups satisfy Property O. This is sufficient for us to deduce both that their reduced C^*-algebra is exact and that they satisfy Yu's property A.

B.2 Free groups

Theorem 16. *Free groups admit an Ozawa kernel and thus satisfy Property O.*

Proof. This construction is based on the proof that trees have Property A [DJ].

Let T be the Cayley graph of a free group (a tree) and V its set of vertices. Let $\gamma_0 : \mathbb{R} \to T$ be a geodesic ray in T. Let γ_v be the unique geodesic ray issuing from $v \in V$ and intersecting γ_0 along a geodesic ray. Let γ_v^n be the initial segment of γ_v of length n.

For any $x, y \in V$, we define our function $u_n(x, y)$ to be the size of the overlap of the n-length rays γ_x^n and γ_y^n, scaled by $n+1$.

$$u_n(x,y) = \frac{|\gamma_x^n \cap \gamma_y^n|}{n+1}$$

We will now show that this family of functions can be used to define an Ozawa kernel.

Lemma B.2.1. *For any n, u_n is a positive definite kernel.*

Proof. Define $f_v(x)$ and $\chi_{v,n}(x)$ as follows:

$$f_v(x) = \begin{cases} 1 & \text{if } v \text{ separates } x \text{ from the end of } \gamma_0 \\ 0 & \text{otherwise} \end{cases}$$

And

$$\chi_{v,n}(x) = \begin{cases} 1 & \text{if } x \text{ is contained in the ball of radius } n \text{ around } v \\ 0 & \text{otherwise} \end{cases}$$

Then we have

$$u_n(x,y) = \frac{1}{n+1} \sum_{v \in V} f_v(x) f_v(y) \chi_{v,n}(x) \chi_{v.n}(y)$$

The only vertices contributing to this sum are whose which are within distance n of both x and y and which separate both x and y from the end of γ_0. This is precisely the size of the intersection of the n-length rays γ_x^n and

γ_y^n.

We can now rearrange u_n to show that this is a positive definite kernel.

$$\begin{aligned}
\sum_{i,j}^{k} \lambda_i \lambda_j u_n(x_i, x_j) &= \sum_{i,j}^{k} \lambda_i \lambda_j \sum_{v \in V} f_v(x_i) f_v(x_j) \chi_{v,n}(x_i) \chi_{v_{B_n}}(x_j) \\
&= \sum_{v \in V} \left(\sum_{i}^{k} \lambda_i f_v(x_i) \chi_{v,n}(x_i) \sum_{j}^{k} \lambda_j f_v(x_j) \chi_{v,n}(x_j) \right) \\
&= \sum_{v \in V} \left(\sum_{i}^{k} \lambda_i f_v(x_i) \chi_{v,n}(x_i) \right)^2 \geq 0
\end{aligned}$$

□

Lemma B.2.2. *For each n there exists a finite set F such that $u_n(x,y) \neq 0$ only if $x^{-1}y \in F$.*

Proof. Let F be the ball of radius n around the origin. If $d(x,y) > 2n$, ie $x^{-1}y \notin F$ there is no overlap between the n-length rays γ_x^n and γ_y^n and so $u_n(x,y) = 0$. □

Lemma B.2.3. *Given any finite subset E and $\epsilon > 0$ there exists N such that $|1 - u_N(x,y)| < \epsilon$ if $x^{-1}y \in E$.*

Proof. Since E is a finite subset, there exists some number m such that if $x^{-1}y \in E$ then $d(x,y) < m$.

Now if $d(x,y) < m$, then the minimum size of the overlap of γ_x^n and γ_y^n is $n-m$ whilst the maximum is $n+1$.

And so we have

$$\frac{n-m}{n+1} \leq u_n(x,y) \leq \frac{n+1}{n+1}$$

Hence $\lim_{n\to\infty} u_n(x,y) = 1$

And so $\forall \epsilon > 0$, $\exists N$ such that $\forall x^{-1}y \in E$ we have as required $|1 - u_N(x,y)| < \epsilon$. $\square$

Hence u_N is an Ozawa type kernel and free groups satisfy Property O. $\square$

The following corollary immediately follows:

Corollary B.2.4. *Free groups are exact and satisfy Yu's property A.*

B.3 Amenable groups

We will first review the definition of an amenable group via Folner's condition:

Definition B.3.1. *If a group G is amenable, then there exists a Folner sequence G_n of finite subsets of G such that $\forall g \in G$,*

$$\lim_{n\to\infty} \frac{|gG_n \triangle G_n|}{|G_n|} = 0$$

This can be rewritten as follows:

$$\begin{aligned}\frac{|gG_n \triangle G_n|}{|G_n|} &= \frac{|gG_n \cup G_n - gG_n \cap G_n|}{|G_n|} \\ &= \frac{|gG_n \cup G_n|}{|G_n|} - \frac{|gG_n \cap G_n|}{|G_n|}\end{aligned}$$

The maximum possible value of $|gG_n \cup G_n|$ is $2|G_n|$ and the minimum is $|G_n|$. Similarly, the maximum possible value of of $|gG_n \cap G_n|$ is $|G_n|$ and the minimum is 0.

So $\frac{|gG_n \cup G_n|}{|G_n|}$ lies between 1 and 2, while $\frac{|gG_n \cap G_n|}{|G_n|}$ lies between 0 and 1. Since the difference between them tends to 0 and we are dealing with bounded sets of real numbers, we have that $\lim_{n\to\infty} \frac{|gG_n \cap G_n|}{|G_n|} = 1$.

We can now prove our next theorem:

Theorem 17. *Amenable groups admit an Ozawa kernel and thus satisfy Property O.*

Proof. Consider the family of functions $u_n(x,y) = \frac{|xG_n \cap yG_n|}{|G_n|}$.

Lemma B.3.2. *For each n, u_n is a positive definite kernel.*

Proof. An element $g \in G$ belongs to the intersection $xG_n \cap yG_n$ only if $g \in xG_n$ and $g \in yG_n$. This is equivalent to $x^{-1}g \in G_n$ and $y^{-1}g \in G_n$. Again this is equivalent to $x^{-1} \in G_n g^{-1}$ and $y^{-1} \in G_n g^{-1}$. And so the condition becomes $x \in gG_n^{-1}$ and $y \in gG_n^{-1}$.

So taking $\chi(x)$ to be the characteristic function, u_n can be rewritten as

$$u_n(x,y) = \frac{1}{|G_n|}\sum_{g\in G}\chi_{gG_n^{-1}}(x)\chi_{gG_n^{-1}}(y)$$

Thus:

$$\begin{aligned}
\sum_{i,j}^{n}\lambda_i\lambda_j u(x_i,x_j) &= \sum_{i,j}^{n}\lambda_i\lambda_j\frac{1}{|G_n|}\sum_{g\in G}\chi_{gG_n^{-1}}(x_i)\chi_{gG_n^{-1}}(x_j)\\
&= \frac{1}{|G_n|}\sum_{g\in G}\left(\sum_{i}^{n}\lambda_i\chi_{gG_n^{-1}}(x_i)\sum_{j}^{n}\lambda_j\chi_{gG_n^{-1}}(x_j)\right)\\
&= \frac{1}{|G_n|}\sum_{g\in G}\left(\sum_{i}^{n}\lambda_i\chi_{gG_n^{-1}}(x_i)\right)^2 \geq 0
\end{aligned}$$

□

Lemma B.3.3. *For each n, there exists a finite set F such that $u_n(x,y) \neq 0$ only if $x^{-1}y \in F$.*

Proof. Consider the Cayley graph of G. Since G_n is finite it is contained within a ball of diameter r. Let F be the ball of radius r around the origin.

If $d(x,y) > 2r$, ie $x^{-1}y \notin F$, there is no intersection between xG_n and yG_n and so $u_n = 0$ as required. □

Lemma B.3.4. *Given any finite subset E and $\epsilon > 0$ there exists a function u_N such that $|1 - u_N(x,y)| < \epsilon$ if $x^{-1}y \in E$.*

Proof. $\lim_{n\to\infty} \frac{|gG_n \cap G_n|}{|G_n|} = 1$. So for a given g, $\forall \epsilon > 0$, there exists R such that if $n > R$, $|1 - \frac{|gG_n \cap G_n|}{|G_n|}| < \epsilon$. Since this holds for any $g \in G$, it holds in particular for $x^{-1}y$. Since E is a finite subset, there exists $N = \max\{R | g = x^{-1}y \in E\}$ such that $\forall x^{-1}y \in E$, $|1 - \frac{|x^{-1}yG_N \cap G_N|}{|G_N|}| = |1 - \frac{|xG_N \cap yG_N|}{|G_N|}| = |1 - u_N(x,y)| < \epsilon$. □

This function u_N is an Ozawa kernel and thus amenable groups satisfy Property O. □

The following corollary immediately follows:

Corollary B.3.5. *Amenable groups are exact and satisfy Yu's property A.*

B.3.1 Example: groups of subexponential growth

A good example of the above construction which clearly shows the importance of the geometry of the Cayley graph is that of groups of subexponential growth which we define as follows:

Definition B.3.6. *Let G be a group with generating set A. Let $\beta_A(n)$ be the number of vertices in the closed ball of radius n about 1 in the Cayley graph* *of the group generated by A. The growth function of G with respect to A is $n \to \beta_A(n)$.*

Definition B.3.7. *G has subexponential growth if $\beta_A(n) \leq e^{\sqrt{n}}$ for all $n \in \mathbb{N}$.*

Examples of groups of subexponential growth include finite groups, abelian groups and nilpotent groups. All groups of subexponential growth are amenable and so satisfy Folners condition. In fact, it can be shown that balls of radius n in the Cayley graph of G are Folner sets [BHV].

In this case, the Ozawa kernel $u(x, y)$ is simply the size of the intersection of the balls of radius n centred at x and y, scaled by the size of B_n. We choose the radius n according to the given ϵ and finite set E.

$$u_n(x, y) = \frac{|xB_n \cap yB_n|}{|B_n|}$$

Alternatively we can regard the function as the number of balls of radius n which contain both x and y, scaled by the size of a ball of radius n.

$$u_n(x, y) = \frac{|\{B_n | B_n \text{ contains both } x \text{ and } y.\}|}{|B_n|}$$

Bibliography

[BHV] B.Bekka, P.De la Harpe, A.Valette, Kazhdan's Property (T), 2003.

[CN] Sarah Campbell, Graham Niblo, Hilbert space compression and exactness of discrete groups, Journal of functional analysis 222, 2005, 292-305.

[DJ] A.Dranishnikov, T.Januszkiewicz, Every Coxeter group acts amenably on a compact space, Proceedings of the 1999 Topology and Dynamics Conference (Salt Lake City, UT). Topology Proc. 24 (1999), Spring, 135–141.

[GK] Erik Guentner, Jerome Kaminker, Exactness and uniform embeddability of Discrete Groups, preprint math.OA/0309166 2003.

[KW] Kirchberg, Wasserman, Exact groups and continuous bundles of C^*-algebras, Mathematische Annalen, Volume 315, Number 2, 1999, 169 - 203.

[O] N. Ozawa, Amenable actions and exactness for discrete groups, CR Acad. Sci. Paris Sr. I Math., 330 (2000), 691-695.

[Yu] G.Yu, The Coarse Baum Connes conjecture for spaces, which admit a uniform embedding into a hilbert space, Inventiones Math. 139 (2000), 201-240.