Conservation laws and symmetries of difference equations

Olexandr G. Rasin, MSc Hons.

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Supervisor: Prof. P.E. Hydon

Co-supervisor: Prof. I. Roulstone

Department of Mathematics

School of Electronics and Physical Sciences

University of Surrey

Guildford, GU2 7XH

United Kingdom

0.1 Abstract

This thesis deals with conservation laws and symmetries of difference equations.

The main new results in the field of conservation laws are:

- We have improved the effectiveness of Hydon's direct method for constructing conservation laws;
- A classification of all three-point conservation laws for a large class of integrable difference equations that has been described by Nijhoff, Quispel and Capel is presented. We show that every nonlinear equation from this class has at least two nontrivial conservation laws.
- We deal with conservation laws for all integrable difference equations that belong to the famous Adler-Bobenko-Suris classification. All inequivalent three-point conservation laws are found, as are three five-point conservation laws for each equation.
- We describe a method of generating conservation laws from known ones; this method can be used to generate higher-order conservation laws from those that are listed here.
- An example of conservation laws for a Toda type system is presented. The connection between these conservation laws and symmetries is shown.
- Conservation laws for nonautonomous quad-graph equations are found.
- We include a Maple program for deriving three-point conservation laws for quad-graph equations.

The main new results in the field of symmetries are:

- Symmetries of all integrable difference equations that belong to the Adler-Bobenko-Suris classification are described. For each equation, the characteristics of symmetries satisfy a functional equation, which we solve by reducing it to a system of partial differential equations. In this way, all five-point symmetries of integrable

equations on the quad-graph are found. These include mastersymmetries, which allow one to construct infinite hierarchies of local symmetries.

- We demonstrate a connection between the symmetries of quad-graph equations and those of the corresponding Toda type difference equations.

- A program for deriving five-point symmetries for quad-graph equations is presented.

0.2 Acknowledgements

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Contents

Chapter 1

Introduction

This chapter describes a brief history of the topic and gives a motivation of the thesis. The two main topics of study in the thesis are: conservation laws of partial difference equations (PΔE's) and symmetries of partial difference equations.

1.1 Equations on the quad-graph

Partial difference equations on the quad-graph have recently attracted much interest, especially from the integrable systems community. The first quad-graphs were derived in the works of Hirota [30, 31]. Since then, Lax pairs have been derived for these and many other quad-graph equations [14, 15, 47, 49].

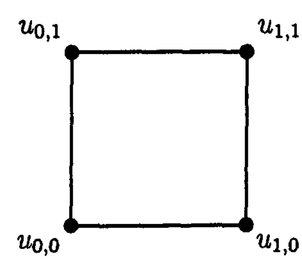

Figure 1.1: Quad-graph

Quad-graphs have a scalar dependent variable u that is defined on the domain $\mathbb{Z}^2$; we shall use the coordinates (k, l) as the independent variables. For brevity we denote the values of u at the vertices of the quad-graph by $u_{0,0} = u(k,l)$, $u_{1,0} = u(k+1,l)$, $u_{0,1} = u(k,l+1)$, $u_{1,1} = u(k+1,l+1)$, as shown. More generally $u_{i,j}$ denotes $u(k+i,l+j)$.

At present, there are various criteria for PΔEs to be integrable [5, 14, 27, 47, 59, 61, 69]. Attention has largely focused on quad-graphs (Fig. 1.1). The easiest definition of integrability for quad-graphs is consistency on a cube. This implies the existence of a Lax pair [14], so that the system is integrable via a spectral problem. Adler, Bobenko & Suris(ABS) have classified all integrable, scalar equations on the quad-graph that are consistent on a cube, linear in each variable, invariant under the symmetries of a square, and possess the tetrahedron property [5]. Hietarinta discovered a quad-graph that lacks the tetrahedron property but has a Lax pair [28]. However, this has recently been shown to be linearizable [63], so it can be integrated without recourse to the spectral problem. It is not yet known whether all quad-graphs that have a Lax pair but lack the tetrahedron property can be linearized.

Classification does not tell us whether a given quad-graph (that does not satisfy the assumptions of any known classification) is integrable. Various tests have been developed which indicate (but do not prove) integrability; the most notable of these is the method of singularity confinement [27]. Another possibility is to seek symmetries or conservation laws. In particular, one definition of integrability is the possession of infinitely many higher symmetries or conservation laws [22, 45].

1.1.1 Classification of integrable equations on quad-graphs. Consistency approach.

The complete classification of integrable equations on the quad-graphs (Figure 1.1) was done by Adler *et al.* in [5]. Here we briefly show the method of classification and its result.

The general form of a scalar difference equation on the quad-graph is

$$P(u_{0,0}, u_{1,0}, u_{0,1}, u_{1,1}; \alpha, \beta) = 0. \tag{1.1}$$

In [5] the authors classified equations which satisfy the following conditions.

1) Linearity. The function $P(u_{0,0}, u_{1,0}, u_{0,1}, u_{1,1}; \alpha, \beta)$ is linear in each argument (affine linear):

$$P(u_{0,0}, u_{1,0}, u_{0,1}, u_{1,1}; \alpha, \beta) = a_1 u_{0,0} u_{1,0} u_{0,1} u_{1,1} + \cdots + a_{16},. \tag{1.2}$$

where coefficients a_i depend on α, β.

2) Symmetry. For quad-graphs there are no distinguished coordinate directions with the understanding that indices are used locally (within one quadrilateral), and do not stand for shifts into the globally defined coordinate directions. So, $u_{0,0}, u_{1,0}, u_{1,1}, u_{0,1}$ can be any cyclic enumeration of the vertices of an elementary quadrilateral. Equation (1.1) should not depend on the enumeration of vertices, therefore the following assumption is natural when considering equations on general quad-graphs. The equation (1.1) is invariant under the group D_4 of the square symmetries, namely, the function P satisfies the symmetry properties

$$\begin{aligned} P(u_{0,0}, u_{1,0}, u_{0,1}, u_{1,1}; \alpha, \beta) &= \epsilon P(u_{0,0}, u_{0,1}, u_{1,0}, u_{1,1}; \beta, \alpha) \\ &= \sigma P(u_{1,0}, u_{0,0}, u_{1,1}, u_{0,1}; \alpha, \beta) \end{aligned} \tag{1.3}$$

with $\epsilon, \sigma = \pm 1$ (see Figure 1.2). Of course, due to the symmetries (1.3) not all coefficients a_i in (1.2) are independent.

Integrability can be detected in an algorithmic manner starting with no more information than the equation itself: a sufficient condition for integrability of an equation is its **three-dimensional consistency**. This property means that the equation (1.1) may

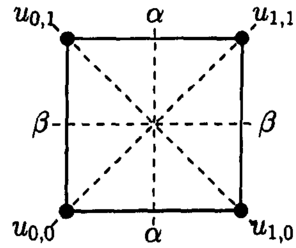

Figure 1.2: D_4 symmetry

be consistently embedded in a three-dimensional lattice, so that similar equations hold for all six faces of any elementary cube, as on Figure 1.3 (it is supposed that the values of the parameters α_j assigned to the opposite edges of any face are equal to one another, so that, for instance, all edges (x_2, x_{12}), (x_3, x_{31}), and (x_{23}, x_{123}) carry the label α_1):

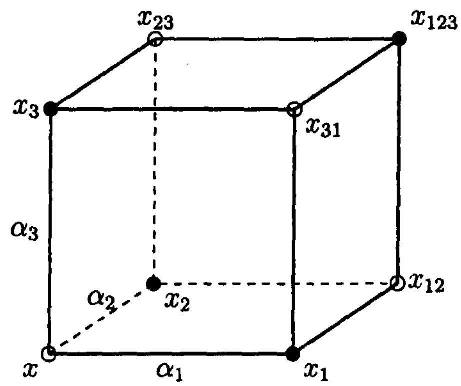

Figure 1.3: Three-dimensional consistency

To describe more precisely what is meant by three-dimensional consistency, consider the Cauchy problem with the initial data x, x_1, x_2, x_3. The equations

$$P(x, x_i, x_j, x_{ij}; \alpha_i, \alpha_j) = 0 \tag{1.4}$$

allow one to determine uniquely the values x_{12}, x_{23}, x_{31}. After that one has three different equations for x_{123}, coming from the faces $(x_1, x_{12}, x_{31}, x_{123})$, $(x_2, x_{23}, x_{12}, x_{123})$, and $(x_3, x_{31}, x_{23}, x_{123})$. Consistency means that all three values thus obtained for x_{123} coincide.

3) Tetrahedron property. The function $x_{123} = z(x, x_1, x_2, x_3; \alpha_1, \alpha_2, \alpha_3)$, existing due to the three-dimensional consistency, actually does not depend on the variable x, that is, $z_x = 0$.

Under the tetrahedron condition we can paint the vertices of the cube in black and white, as on Figure 1.3, and the vertices of each of the two tetrahedra satisfy an equation of the form

$$\widehat{P}(x_1, x_2, x_3, x_{123}; \alpha_1, \alpha_2, \alpha_3) = 0; \tag{1.5}$$

it is easy to see that under the assumption 1) (linearity) the function $\widehat{P}$ may also be taken as linear in each argument.

The tetrahedron condition is closely related to another property of (1.1), namely to the existence of a *three-leg form* of this equation [14]:

$$\psi(u_{0,0}, u_{1,0}; \alpha) - \psi(u_{0,0}, u_{0,1}; \beta) = \phi(u_{0,0}, u_{1,1}; \alpha, \beta). \tag{1.6}$$

The three terms in this equation correspond to three legs: two short ones, $(u_{0,0}, u_{1,0})$ and $(u_{0,0}, u_{0,1})$, and a long one, $(u_{0,0}, u_{1,1})$. The short legs are edges of the original quad-graph, while the long one is not. In some cases it is more convenient to write the three-leg equation (1.1) in the multiplicative form

$$\frac{\Psi(u_{0,0}, u_{1,0}; \alpha)}{\Psi(u_{0,0}, u_{0,1}; \beta)} = \Phi(u_{0,0}, u_{1,1}; \alpha, \beta). \tag{1.7}$$

The three-leg form gives an explanation for the equation for the black tetrahedron from Fig. 1.3. Consider the three faces adjacent to the vertex x_{123} on this figure, namely the quadrilaterals $(x_1, x_{12}, x_{31}, x_{123})$, $(x_2, x_{23}, x_{12}, x_{123})$, and $(x_3, x_{31}, x_{23}, x_{123})$. A summation of three-leg forms (centered at x_{123}) of equations corresponding to these three faces leads to the equation

$$\phi(x_{123}, x_1; \alpha_2, \alpha_3) + \phi(x_{123}, x_2; \alpha_3, \alpha_1) + \phi(x_{123}, x_3; \alpha_1, \alpha_2) = 0. \tag{1.8}$$

This equation, in any event, relates the fields in the black vertices of the cube only, i.e. has the form of (1.5).

So the tetrahedron property is a necessary condition for the existence of a three-leg form. On the other hand, a verification of the tetrahedron property is much more straightforward than finding the three-leg form, since the latter contains two a priori unknown functions ψ, ϕ.

It remains to mention that, as demonstrated in [14], the existence of the three-leg form allows one to immediately establish a relation to discrete systems of the Toda type [4]. Indeed, if x is a common vertex of n adjacent quadrilaterals with faces

$$(x, x_k, x_{k,k+1}, x_{k+1}), \qquad k = 1, 2, ..., n,$$

with the parameters k assigned to the edges (x, x_k), then the fields at the point x and at the black vertices of the adjacent faces satisfy the following equation:

$$\sum_{k=1}^{n} \phi(x, x_{k,k+1}, \alpha_k, \alpha_{k+1}) = 0. \tag{1.9}$$

This is a discrete Toda type equation.

Theorem 1.1. (Adler, Bobenko and Suris [5, 15]) *Up to common Möbius transformations of the variables $u_{0,0}$ and point transformations of the parameters α, the three-dimensionally consistent quad-graph equations (1.4) with the properties 1),2),3) (linearity, symmetry, tetrahedron property) are exhausted by the following three lists* $\mathbf{Q}, \mathbf{H}, \mathbf{A}$ *($u_{1,0} = x_1$, $u_{0,1} = x_2$, $u_{1,1} = x_{12}$, $\alpha = \alpha_1$, $\beta = \alpha_2$).*

$$\mathbf{Q1}: \quad \alpha(u_{0,0} - u_{0,1})(u_{1,0} - u_{1,1}) - \beta(u_{0,0} - u_{1,0})(u_{0,1} - u_{1,1}) + \delta^2\alpha\beta(\alpha - \beta) = 0,$$

$$\mathbf{Q2}: \quad \alpha(u_{0,0} - u_{0,1})(u_{1,0} - u_{1,1}) - \beta(u_{0,0} - u_{1,0})(u_{0,1} - u_{1,1}) + \alpha\beta(\alpha - \beta)(u_{0,0} + u_{1,0} + u_{0,1} + u_{1,1}) - \alpha\beta(\alpha - \beta)(\alpha^2 - \alpha\beta + \beta^2) = 0,$$

$$\mathbf{Q3}: \quad (\beta^2 - \alpha^2)(u_{0,0}u_{1,1} + u_{1,0}u_{0,1}) + \beta(\alpha^2 - 1)(u_{0,0}u_{1,0} + u_{0,1}u_{1,1}) - \alpha(\beta^2 - 1)(u_{0,0}u_{0,1} + u_{1,0}u_{1,1}) - \delta^2(\alpha^2 - \beta^2)(\alpha^2 - 1)(\beta^2 - 1)/(4\alpha\beta) = 0,$$

$$\mathbf{Q4}: \quad \mathrm{sn}(\alpha)(u_{0,0}u_{1,0} + u_{0,1}u_{1,1}) - \mathrm{sn}(\beta)(u_{0,0}u_{0,1} + u_{1,0}u_{1,1}) - \mathrm{sn}(\alpha - \beta)(u_{0,0}u_{1,1} + u_{1,0}u_{0,1}) + \mathrm{sn}(\alpha - \beta)\mathrm{sn}(\alpha)\mathrm{sn}(\beta)(1 + K^2 u_{0,0}u_{1,0}u_{0,1}u_{1,1}) = 0,$$

$$\mathbf{H1}: \quad (u_{0,0} - u_{1,1})(u_{1,0} - u_{0,1}) + \beta - \alpha = 0,$$

$$\mathbf{H2}: \quad (u_{0,0} - u_{1,1})(u_{1,0} - u_{0,1}) + (\beta - \alpha)(u_{0,0} + u_{1,0} + u_{0,1} + u_{1,1}) + \beta^2 - \alpha^2 = 0,$$

$$\mathbf{H3}: \quad \alpha(u_{0,0}u_{1,0} + u_{0,1}u_{1,1}) - \beta(u_{0,0}u_{0,1} + u_{1,0}u_{1,1}) + \delta^2(\alpha^2 - \beta^2) = 0,$$

$$\mathbf{A1}: \quad \alpha(u_{0,0} + u_{0,1})(u_{1,0} + u_{1,1}) - \beta(u_{0,0} + u_{1,0})(u_{0,1} + u_{1,1}) - \delta^2\alpha\beta(\alpha - \beta) = 0,$$

$$\mathbf{A2}: \quad (\beta^2 - \alpha^2)(u_{0,0}u_{1,0}u_{0,1}u_{1,1} + 1) + \beta(\alpha^2 - 1)(u_{0,0}u_{0,1} + u_{1,0}u_{1,1}) - \alpha(\beta^2 - 1)(u_{0,0}u_{1,0} + u_{0,1}u_{1,1}) = 0.$$

Here α, β are constants and $\mathrm{sn}(\alpha) = \mathrm{sn}(\alpha; K)$ is a Jacobi elliptic function with modulus K. The ABS equations depend on two arbitrary functions $\alpha = \alpha(k)$ and $\beta = \beta(l)$. Without loss of generality, the parameter δ is restricted to the values 0 and 1. For convenience, we have used the form of **Q4** *that was discovered by Hietarinta [29].*

The oldest equations in this list are **H1** and $\mathbf{H3}_{\delta=0}$, which can be found in the work of Hirota [31] (of course not on general quad-graphs but only on the standard square lattice with the labels α constant in each of the two lattice directions). Equations **Q1** and **Q3**$_{\delta=0}$ go back to [61], see also a review in [47]. Equation **Q4** was found in [3] in the Weierstrass normalization of an elliptic curve; in [15] was shown that the formulas become much nicer in the Jacobi normalization. The discovery of this equation is a very important achievement, because other equations from the list can be obtained from **Q4** by appropriate limits of the parameters.

In our research we deal with two wide classes of integrable quad-graphs. The first class is the ABS equations which are listed above. The second class was introduced by Nijhoff, Quispel and Capel [51]:

$$\frac{(p+s)u_{1,0}-(p-r)u_{0,0}-1}{(q+s)u_{0,1}-(q-r)u_{0,0}-1}=\frac{(q+r)u_{1,1}-(q-s)u_{1,0}-1}{(p+r)u_{1,1}-(p-s)u_{0,1}-1}, \tag{1.10}$$

where r, s are free constant parameters. The parameters p and q can be interpreted as the corresponding constant lattice parameters.

When $s^2 \neq r^2$, $(p+r)(p-s)(q+r)(q-s) \neq 0$ equation (1.10) can be transformed to **Q3**$_{\delta=0}$ by the following map

$$u(k,l)\mapsto\left(\alpha\frac{p-s}{p+r}\right)^k\left(\beta\frac{q-s}{q+r}\right)^l u(k,l)+\frac{1}{r+s},\quad p\mapsto\sqrt{\frac{s^2\alpha^2-r^2}{\alpha^2-1}},\quad q\mapsto\sqrt{\frac{s^2\beta^2-r^2}{\beta^2-1}}.$$

1.2 Conservation laws

Conservation laws are ubiquitous in applied mathematics. In some cases, they express conservation of physical quantities. Even when they do not, they are usually of mathematical interest. Much attention has been given to integrable systems that have infinite hierarchies of conservation laws, which are related to generalized symmetries by Noether's theorem. Conservation laws of integrable and nonintegrable systems can be used in many ways, such as to prove existence and uniqueness theorems, to derive shock conditions, and to check that numerical methods are not producing spurious results (at least qualitatively). If a differential equation is to be approximated using a finite difference method, it seems desirable that the discretized equation should retain as much of the original structure as possible, including discrete analogues of the conservation laws. Thus it would be useful to have a systematic method for constructing conservation laws of a given difference equation that does not require the equation to be integrable.

Noether's theorem provides the best-known method of constructing conservation laws of any partial differential equation (PDE) that is the Euler-Lagrange equation for a variational problem [54]. This method uses variational symmetries, which form a subset of the set of generalized (or Lie-Bäcklund) symmetries of the PDE. Generalized symmetries of a particular order can be found systematically from the symmetry condition, which amounts to an overdetermined system of PDEs (see [12, 57] for a modern introduction). Noether's theorem has been extended to Hamiltonian PDEs [57] and to multisymplectic PDEs [16, 35]. However, Noether's theorem does not apply to all PDEs, but only to those that have at least one of the special structures listed above. Alternatively, conservation laws of all types of PDEs may be found by a direct method which does not use symmetries [8, 10, 57].

The theory of conservation laws for partial difference equations mirrors that for PDEs. The first conservation laws for the fully discrete sine-Gordon equation were presented by Orphanidis in [58]. Hydon and Mansfield [36] have formulated the basic theory for the realization of conservation laws in a discrete space. Just as for PDEs, the discrete analogue of Noether's Theorem [21, 37, 42, 75] is applicable only to equations with a known variational, Hamiltonian or multisymplectic structure [16]. Until recently, this condition greatly restricted the class of PΔEs for which conservation laws could be found. This is partly because symmetry analysis of difference equations was not introduced until Maeda's 1987 paper [43], and it has since been developed along several different lines [32, 41, 62]. Furthermore, symmetry calculations are typically lengthy, and only a few of the symmetries that have been found so far are variational symmetries.

Generally speaking, it is much harder to calculate conservation laws for difference equations than for differential equations, because one has to solve a complicated functional equation rather than a system of overdetermined partial differential equations. The first systematic technique for obtaining all conservation laws of a given type was introduced by Hydon [34], who found all three-point conservation laws of the discrete potential modified

Korteweg-de Vries (dpmKdV) equation.

1.2.1 Useful results

A conservation law for a partial difference equation on $\mathbb{Z}^2$ is an expression of the form

$$(S_k - id)F + (S_l - id)G = 0 \tag{1.11}$$

that is satisfied on all solutions of the equation. Here id is the identity mapping and S_k, S_l are forward shifts of the coordinates k and l respectively:

$$S_k : (k, l, u(k,l)) \to (k+1, l, u(k+1,l)), \qquad S_l : (k, l, u(k,l)) \to (k, l+1, u(k, l+1)).$$

This is the general form of the kernel of the Euler equation [36]. Other forms are possible, e. g.

$$(S_kF)(S_lG) = FG.$$

Analytic properties may not be preserved by going from one form to another.

Just as a local conservation law for a PDE can be integrated to obtain global conserved quantities, so (1.11) can be summed over m or n to obtain such quantities for difference equations.

The conservation laws for the dpmKdV equation

$$u_{1,1} = u_{0,0}\frac{\nu(k,l)u_{1,0} - u_{0,1}}{\nu(k,l)u_{0,1} - u_{1,0}}, \tag{1.12}$$

were considered in [34]. Here $\nu(k,l)$ is an arbitrary function. The functions F and G were chosen in the form

$$F = F(k,\ l,\ u_{0,0},\ u_{0,1}),$$
$$G = G(k,\ l,\ u_{0,0},\ u_{1,0}).$$

In [34] it was found that conservation laws for (1.12) exist only in the case when

$$\nu(k,l) = \alpha(k)\beta(l),$$

where $\alpha(k)$ and $\beta(l)$ are arbitrary functions. The conservation laws for the dpmKdV equation are

$$F_1 = -\frac{u_{0,0}u_{0,1}}{\beta(l)}, \qquad G_1 = \alpha(k)u_{0,0}u_{1,0},$$

$$F_2 = -\frac{1}{\beta(l)u_{0,0}u_{0,1}}, \qquad G_2 = \frac{\alpha(k)}{u_{0,0}u_{1,0}},$$

$$F_3 = -\beta(l)\left(\frac{u_{0,1}}{u_{0,0}} + \frac{u_{0,0}}{u_{0,1}}\right), \qquad G_3 = \frac{1}{\alpha(k)}\left(\frac{u_{1,0}}{u_{0,0}} + \frac{u_{0,0}}{u_{1,0}}\right),$$

$$F_4 = (-1)^{k+l}\beta(l)\left(\frac{u_{0,1}}{u_{0,0}} - \frac{u_{0,0}}{u_{0,1}}\right), \qquad G_4 = -\frac{(-1)^{k+l}}{\alpha(k)}\left(\frac{u_{1,0}}{u_{0,0}} - \frac{u_{0,0}}{u_{1,0}}\right).$$

1.3 Symmetries

Symmetries of PΔE's first appeared as similarity constraints for integrable lattices. In [50], it is shown that discrete analogues of the Painlevé equations arise from similarity constraints. Similarity constraints and reductions to discrete Painlevé equations for the cross-ratio, discrete Korteweg-de Vries (dKdV) and discrete potential modified Korteweg-de Vries equations were later considered in [26, 46, 47, 48, 52, 53]. One notable feature of similarity constraints for quad-graphs is that circle patterns can be formed for certain initial conditions [6, 7, 13, 15]. Recently, Tongas *et al.* pointed out that the similarity constraints for quad-graph equations obtained previously are equivalent to characteristics of symmetries [73]. They used an indirect method to discover mastersymmetries and higher-order symmetries for the dKdV equation. Symmetries of several other quad-graph equations have been found in [38, 39, 60, 72].

An alternative approach to symmetries of difference equations (not only integrable ones) is to try to construct discretizations of differential equations that retain all Lie point symmetries of the original system [17, 19, 20, 40, 41]. This requires a non-constant grid; in effect, the original continuous independent variables become extra dependent variables in the discretized system.

For a given difference equation, whether it is integrable or not, the main problem in finding symmetries is solving the linearized symmetry condition, which is a functional equation. Hydon developed a direct method of solving such functional equations by creating an associated system of differential equations that can be solved [32, 34].

1.4 Overview of the thesis

The goal of the thesis is to describe methods for calculating conservation laws and symmetries of partial difference equations and to find them for parameter-rich classes of integrable, scalar equations on the quad-graph.

Chapter 2 deals with conservation laws of difference equations. The structure of this chapter is as follows: §2.1 describes the direct method for calculating conservation laws of quad-graph equations. §2.2 is a classification of all three-point conservation laws for the NQC equation. In §2.3 there is a list of all three-point conservation laws for each equation in the ABS classification; three five-point conservation laws for each ABS equation are given in §2.4.

Chapter 3 deals with symmetries of difference equations. The chapter begins with an introduction to the theory that is the basis for our calculations. In §3.2, we explain the method of finding local symmetries. §3.3 lists all symmetries on the 3×3 square for equations from the ABS classification; mastersymmetries are given in §3.4.

Chapter 4 shows an example of symmetries and conservation laws for Toda type equations. In §4.1, we explain the connection between symmetries of equations from the ABS classification and those of the corresponding Toda type equations. In §4.2, conservation laws for a Toda type system are given.

Applications of symmetries and conservation laws are presented in Chapter 5. A method that enables one to generate a new law from a known one is described in §5.1. In §5.2 we use conservation laws as an indicator of integrability for generalized nonautonomous dKdV and dpmKdV equations. To illustrate one of the most common applications of symmetries, §5.3 describes the construction of a group-invariant solution.

In Chapter 6, we draw conclusions and describe some open problems.

Programs which help to find three-point conservation laws and five-point symmetries for quad-graph equations are presented in the Appendix.

Chapter 2

Conservation laws

2.1 The Method

The general form of a scalar PΔE on the quad-graph is:

$$P(k, l, u_{0,0}, u_{1,0}, u_{0,1}, u_{1,1}) = 0. \tag{2.1}$$

A conservation law for any quad-graph equation (2.1) is an expression of the form

$$(S_k - \mathrm{id})F + (S_l - \mathrm{id})G = 0 \tag{2.2}$$

that is satisfied by all solutions of the equation. Here the functions F and G are the components of the conservation law and id is the identity mapping.

A conservation law is trivial if it holds identically (not just on solutions of the PΔE), or if F and G both vanish on all solutions of (2.1).

We consider conservation laws that lie on the quad-graph. This means that the functions F, G, S_kF and S_lG must depend upon only k, l, $u_{0,0}$, $u_{1,0}$, $u_{0,1}$ and $u_{1,1}$. Consequently the most general form of F and G is:

$$F = F(k,\ l,\ u_{0,0},\ u_{0,1}), \tag{2.3}$$

$$G = G(k,\ l,\ u_{0,0},\ u_{1,0}). \tag{2.4}$$

The dependence of F and G upon the continuous variables $u_{i,j}$ is illustrated in Figure 2.1; together, these functions lie on three points of the quad-graph. For this reason, we call such conservation laws *three-point conservation laws.*

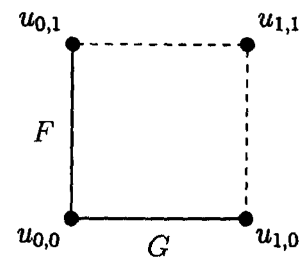

Figure 2.1: Three-point conservation law

The three-point conservation laws can be determined directly by substituting (2.1) into

$$F(k+1,l,u_{1,0},u_{1,1})-F(k,l,u_{0,0},u_{0,1})+G(k,l+1,u_{0,1},u_{1,1})-G(k,l,u_{0,0},u_{1,0})=0, \quad (2.5)$$

and solving the resulting functional equation. Suppose that (2.1) can be solved for $u_{1,1}$ as follows:

$$u_{1,1}=\omega(k,l,u_{0,0},u_{1,0},u_{0,1}). \quad (2.6)$$

(This means that (2.6) and (2.1) are entirely equivalent.) Then (2.5) amounts to

$$F(k+1,l,u_{1,0},\omega)-F(k,l,u_{0,0},u_{0,1})+G(k,l+1,u_{0,1},\omega)-G(k,l,u_{0,0},u_{1,0})=0. \quad (2.7)$$

In order to solve this functional equation we have to reduce it to a system of partial differential equations. To do this, first eliminate functional terms $F(k+1,l,u_{1,0},\omega)$ and $G(k,l+1,u_{0,1},\omega)$ by applying each of the following (commuting) differential operators to (2.7):

$$L_1=\frac{\partial}{\partial u_{0,1}}-\frac{\omega_{u_{0,1}}}{\omega_{u_{0,0}}}\frac{\partial}{\partial u_{0,0}}, \qquad L_2=\frac{\partial}{\partial u_{1,0}}-\frac{\omega_{u_{1,0}}}{\omega_{u_{0,0}}}\frac{\partial}{\partial u_{0,0}},$$

where $\omega_{u_{i,j}}$ denotes $\frac{\partial\omega}{\partial u_{i,j}}$. The operators L_1 and L_2 differentiate with respect to $u_{0,1}$ and $u_{1,0}$ respectively, keeping ω fixed, so

$$L_1\left(F(k+1,l,u_{1,0},\omega)\right)=0, \qquad L_2\left(G(k,l+1,u_{0,1},\omega)\right)=0.$$

This procedure does not depend upon the form of ω; it can be applied equally to any quad-graph equation. Applying L_1 and L_2 to (2.7) yields

$$L_1L_2\Big(F(k,l,u_{0,0},u_{0,1})+G(k,l,u_{0,0},u_{1,0})\Big)=0. \quad (2.8)$$

If (2.8) is divided by the factor that multiplies a particular derivative of $G(k, l, u_{0,0}, u_{1,0})$ and is then differentiated with respect to $u_{0,1}$, we obtain a functional differential equation which is independent of that derivative. This process is repeated for each derivative of $G(k, l, u_{0,0}, u_{1,0})$ and finally for $G(k, l, u_{0,0}, u_{1,0})$ itself; this produces a PDE for $F(k, l, u_{0,0}, u_{0,1})$. If the coefficients involve $u_{1,0}$, the PDE can be split into a system of PDEs.

Further information about F may be found by substituting

$$u_{1,0} = \Omega(k, l, u_{0,0}, u_{0,1}, u_{1,1})$$

into (2.5). Here $u_{1,0} = \Omega$ is another representation of (2.1). Then (2.5) amounts to

$$F(k+1, l, \Omega, u_{1,1}) - F(k, l, u_{0,0}, u_{0,1}) + G(k, l+1, u_{0,1}, u_{1,1}) - G(k, l, u_{0,0}, \Omega) = 0. \quad (2.9)$$

We eliminate the terms $F(k+1, l, \Omega, u_{1,1})$ and $G(k, l, u_{0,0}, \Omega)$ by applying each of the following (commuting) differential operators to (2.9):

$$L_3 = \frac{\partial}{\partial u_{0,0}} - \frac{\Omega_{u_{0,0}}}{\Omega_{u_{0,1}}}\frac{\partial}{\partial u_{0,1}}, \qquad L_4 = \frac{\partial}{\partial u_{1,1}} - \frac{\Omega_{u_{1,1}}}{\Omega_{u_{0,1}}}\frac{\partial}{\partial u_{0,1}}.$$

This yields

$$L_3 L_4\Big(- F(k, l, u_{0,0}, u_{0,1}) + G(k, l+1, u_{0,1}, u_{1,1})\Big) = 0.$$

This equation can also be reduced to a system of partial differential equations for $F(k, l, u_{0,0}, u_{0,1})$ which (typically) is different from obtained previously.

Having differentiated the determining equation for a conservation law several times, we have created a hierarchy of functional differential equations that every three-point conservation law must satisfy. The functions F and G can be determined completely by going up the hierarchy, a step at a time, to determine the constraints that these equations place on the unknown functions. As the constraints are solved sequentially, more and more information is gained about the functions. At the highest stage, the determining equation (2.7) is satisfied, and the only remaining unknowns are the constants that multiply each conservation law.

Mansfield and Szanto [44] proved that the algorithm for finding conservation laws and symmetries is well-founded. In other words, it will yield all conservation laws and symmetries of a given class whenever the equation can be put in the form (2.6).

In essence, the above approach is a generalization of a method introduced in 1823 by Abel, who solved functional equations for which each function depends on a single continuous variable [1]. (For a modern description of Abel's method, see [2].) Surprisingly, this method was not applied to the problem of finding symmetries and first integrals of difference equations until Hydon used it in 1999. Until then, symmetries had been found by series techniques and other Ansätze (see [32] for a discussion). In 2000, Hydon generalized this approach to deal with functions that depend upon several continuous variables, enabling conservation laws to be found systematically [34]. However, as the number of variables increases, the length of the functional differential equations grows exponentially. This 'expression swell' limits the complexity of the conservation laws that can be found by the direct method. Therefore great care is needed to ensure that differentiations are applied in an order that minimizes expression swell. At present, this is still something of an art.

2.1.1 Example of conservation laws of the discrete KdV equation

The Korteweg-de Vries equation is [30]:

$$(p+q+v_{1,1}-v_{0,0})(q-p+v_{1,0}-v_{0,1})=q^2-p^2, \tag{2.10}$$

which is an integrable quad-graph equation. Here p, q are parameters and $p^2 \neq q^2$. To simply matters, we use the transformation

$$v(k,l)=u(k,l)\sqrt{q^2-p^2}-qk-pl$$

to reduce (2.10) to

$$(u_{1,1}-u_{0,0})(u_{1,0}-u_{0,1})=1. \tag{2.11}$$

We shall call this equation dKdV. As with all integrable quad-graph equations, the dKdV equation may be solved to write any one of $u_{0,0}, u_{1,0}, u_{0,1}, u_{1,1}$ in terms of the other three. In particular, we will write the dKdV equation as either

$$u_{1,1}=\omega, \qquad \text{where} \quad \omega=\frac{1}{u_{1,0}-u_{0,1}}+u_{0,0},$$

or

$$u_{1,0} = \Omega, \qquad \text{where} \quad \Omega = \frac{1}{u_{1,1} - u_{0,0}} + u_{0,1}.$$

With help of L_1, L_2 the determining equation (2.7) for the dKdV equation is reduced to

$$F_{u_{0,0}u_{0,0}} + G_{u_{0,0}u_{0,0}} - (u_{1,0} - u_{0,1})^2 (F_{u_{0,0}u_{0,1}} - G_{u_{0,0}u_{1,0}}) - 2(u_{1,0} - u_{0,1})(F_{u_{0,0}} + G_{u_{0,0}}) = 0, \quad (2.12)$$

where $F = F(k, l, u_{0,0}, u_{0,1})$ and $G = G(k, l, u_{0,0}, u_{1,0})$. Differentiating (2.12) three times with respect to $u_{0,1}$ eliminates G and its derivatives, leaving the necessary condition

$$F_{u_{0,0}{}^2 u_{0,1}{}^3} - (u_{1,0} - u_{0,1})^2 F_{u_{0,0}u_{0,1}{}^4} + 4(u_{1,0} - u_{0,1}) F_{u_{0,0}u_{0,1}{}^3} = 0. \quad (2.13)$$

This equation can be split into an overdetermined system by equating powers of $u_{1,0}$. Further information about F may be found by substituting $u_{1,0} = \Omega$ into (2.5). Differentiating

$$F(k+1, l, \Omega, u_{1,1}) - F(k, l, u_{0,0}, u_{0,1}) + G(k, l+1, u_{0,1}, u_{1,1}) - G(k, l, u_{0,0}, \Omega) = 0$$

with respect to $u_{0,0}$, $u_{1,1}$ and keeping Ω fixed yields

$$F_{u_{0,1}u_{0,1}} - \tilde{G}_{u_{0,1}u_{0,1}} - (u_{1,1} - u_{0,0})^2 (F_{u_{0,0}u_{0,1}} + \tilde{G}_{u_{0,1}u_{1,1}}) - 2(u_{1,1} - u_{0,0})(F_{u_{0,0}} - \tilde{G}_{u_{0,1}}) = 0, \quad (2.14)$$

where $\tilde{G} = G(k, l+1, u_{0,0}, u_{1,1})$. The function $\tilde{G}$ and its derivatives are eliminated by differentiating three times with respect to $u_{0,1}$, which yields

$$F_{u_{0,0}{}^3 u_{0,1}{}^2} - (u_{1,1} - u_{0,0})^2 F_{u_{0,0}{}^4 u_{0,1}} + 4(u_{1,1} - u_{0,0}) F_{u_{0,0}{}^3 u_{0,1}} = 0. \quad (2.15)$$

The overdetermined system of partial differential equations (2.13), (2.15) is easily solved to obtain

$$F = C_1 u_{0,0} u_{0,1} + C_2 {u_{0,0}}^2 u_{0,1} + C_3 u_{0,0} {u_{0,1}}^2 + C_4 {u_{0,0}}^2 {u_{0,1}}^2 + F_1 + F_2, \quad (2.16)$$

where each C_i is an arbitrary function of k, l, and $F_1 = F_1(k, l, u_{0,1})$, $F_2 = F_2(k, l, u_{0,0})$ are arbitrary functions. The term $F_2(k, l, u_{0,0})$ can be removed (without loss of generality) by adding the trivial conservation law

$$\begin{aligned} F_T &= (S_l - id) F_2, \\ G_T &= -(S_k - id) F_2, \end{aligned}$$

to F and G respectively.

So far, we have differentiated the determining equations for a conservation law five times; this has created a hierarchy of functional differential equations that every three-point conservation law must satisfy. The unknown functions C_i, F_1 and G are found by going up the hierarchy, a step at a time, to determine the constraints these equations place on the unknown functions.

By this technique we have found all independent nontrivial three-point conservation laws for the dKdV equation; they are as follows:

1. $F = u_{0,0}\,(u_{0,1})^2 - (u_{0,0})^2\,u_{0,1} + u_{0,0} - u_{0,1},$
   $G = (u_{0,0})^2\,u_{1,0} - u_{0,0}\,(u_{1,0})^2,$

2. $F = (-1)^{k+l+1}\left\{u_{0,0}\,(u_{0,1})^2 + (u_{0,0})^2\,u_{0,1} - u_{0,0} - u_{0,1}\right\},$
   $G = (-1)^{k+l}\left\{(u_{0,0})^2\,u_{1,0} + u_{0,0}\,(u_{1,0})^2\right\},$

3. $F = (-1)^{k+l+1}\left\{(u_{0,0}u_{0,1})^2 - 2u_{0,0}u_{0,1} + \frac{1}{2}\right\},$
   $G = (-1)^{k+l}\left\{(u_{0,0}u_{1,0})^2\right\},$

4. $F = (-1)^{k+l+1}\left\{u_{0,0}u_{0,1} - \frac{1}{2}\right\},$
   $G = (-1)^{k+l}\left\{u_{0,0}u_{1,0}\right\}.$

Three of these conservation laws depend on k and l explicitly. If we had chosen functions F and G that depended only upon $u_{i,j}$ on the quad-graph (and not also upon k and l), we would have found only the first of the four three-point conservation laws.

2.2 Conservation laws for NQC-type equations.

The purpose of the current section is to use the direct method to classify the three-point conservation laws of a wide class of integrable quad-graphs that were introduced by Nijhoff, Quispel and Capel [51]. These are of the form

$$\frac{(p+s)u_{1,0} - (p-r)u_{0,0} - 1}{(q+s)u_{0,1} - (q-r)u_{0,0} - 1} = \frac{(q+r)u_{1,1} - (q-s)u_{1,0} - 1}{(p+r)u_{1,1} - (p-s)u_{0,1} - 1}, \tag{2.17}$$

where r, s are free constant parameters. The parameters p and q can be interpreted as the corresponding constant lattice parameters. We shall refer to (2.17) as the NQC equation; the factors $(p \pm r)$, $(p \pm s)$, $(q \pm r)$ and $(q \pm s)$ will be called the *coefficients* of the NQC equation.

2.2.1 Simplification of the NQC equation

We begin by showing that the NQC equation is mapped to another NQC equation under the group of equivalence transformations D_4 generated by rotations and reflections.

First consider rotations. Let

$$\hat{k} = -l, \quad \hat{l} = k, \quad \hat{u}(\hat{k}, \hat{l}) = u(\hat{l}, -\hat{k}) = u(k, l).$$

Then

$$\begin{aligned} \hat{u}(\hat{k}-1, \hat{l}) &= u(\hat{l}, 1-\hat{k}) = u(k, l+1), \\ \hat{u}(\hat{k}-1, \hat{l}+1) &= u(\hat{l}+1, 1-\hat{k}) = u(k+1, l+1), \\ \hat{u}(\hat{k}, \hat{l}+1) &= u(\hat{l}+1, -\hat{k}) = u(k+1, l). \end{aligned}$$

Note that $S_{\hat{k}}^{-1}\hat{u}(\hat{k}, \hat{l}) = S_l u(k, l)$ and $S_{\hat{l}}\hat{u}(\hat{k}, \hat{l}) = S_k u(k, l)$. The NQC equation (2.17) can be rewritten as

$$\frac{(p+s)\hat{u}_{0,1} - (p-r)\hat{u}_{0,0} - 1}{(q+s)\hat{u}_{-1,0} - (q-r)\hat{u}_{0,0} - 1} = \frac{(q+r)\hat{u}_{-1,1} - (q-s)\hat{u}_{0,1} - 1}{(p+r)u_{-1,1} - (p-s)\hat{u}_{-1,0} - 1}, \tag{2.18}$$

where $\hat{u}_{i,j} = \hat{u}(\hat{k}+i, \hat{l}+j)$. Apply $S_{\hat{k}}$ to (2.18) and then rearrange to obtain

$$\frac{(-q+r)\hat{u}_{1,0} - (-q-s)\hat{u}_{0,0} - 1}{(p+r)\hat{u}_{0,1} - (p-s)\hat{u}_{0,0} - 1} = \frac{(p+s)\hat{u}_{1,1} - (p-r)\hat{u}_{1,0} - 1}{(-q+s)\hat{u}_{1,1} - (-q-r)\hat{u}_{0,1} - 1}.$$

This is of the form

$$\frac{(\hat{p}+\hat{s})\hat{u}_{1,0} - (\hat{p}-\hat{r})\hat{u}_{0,0} - 1}{(\hat{q}+\hat{s})\hat{u}_{0,1} - (\hat{q}-\hat{r})\hat{u}_{0,0} - 1} = \frac{(\hat{q}+\hat{r})\hat{u}_{1,1} - (\hat{q}-\hat{s})\hat{u}_{1,0} - 1}{(\hat{p}+\hat{r})\hat{u}_{1,1} - (\hat{p}-\hat{s})\hat{u}_{0,1} - 1} \tag{2.19}$$

where

$$(\hat{p}, \hat{q}, \hat{r}, \hat{s}) = (-q, p, s, r). \tag{2.20}$$

Therefore this rotation generates the equivalence transformation

$$\Gamma_a : (p, q, r, s) \mapsto (-q, p, s, r). \tag{2.21}$$

Suppose that for a particular choice of parameters $(\hat{p}, \hat{q}, \hat{r}, \hat{s})$ equation (2.19) has a conservation law

$$(S_{\hat{k}} - \mathrm{id})\hat{F}(\hat{k}, \hat{l}, \hat{u}_{0,0}, \hat{u}_{0,1}) + (S_{\hat{l}} - \mathrm{id})\hat{G}(\hat{k}, \hat{l}, \hat{u}_{0,0}, \hat{u}_{1,0}) = 0. \tag{2.22}$$

Then, applying $S_{\hat{k}}^{-1}$ to this and expanding, we obtain

$$\begin{aligned}\hat{F}(\hat{k}, \hat{l}, \hat{u}_{0,0}, \hat{u}_{0,1}) - \hat{F}(\hat{k} - 1, l, \hat{u}_{-1,0}, \hat{u}_{-1,1}) + \hat{G}(\hat{k} - 1, l + 1, \hat{u}_{-1,1}, \hat{u}_{0,1})& \\ -\hat{G}(\hat{k} - 1, \hat{l}, \hat{u}_{-1,0}, \hat{u}_{0,0}) = 0.&\end{aligned}$$

In terms of the original variables, this amounts to

$$\begin{aligned}\hat{F}(-l, k, u_{0,0}, u_{1,0}) - \hat{F}(-l - 1, k, u_{0,1}, u_{1,1}) + \hat{G}(-l - 1, k + 1, u_{1,1}, u_{1,0})& \\ -\hat{G}(-l - 1, k, u_{0,1}, u_{0,0}) = 0,&\end{aligned}$$

which can be written as

$$(S_k - \mathrm{id})\hat{G}(-l - 1, k, u_{0,1}, u_{0,0}) + (S_l - \mathrm{id})[-\hat{F}(-l, k, u_{0,0}, u_{1,0})] = 0. \tag{2.23}$$

This is a conservation law for the original NQC equation (2.17) with the parameters $(p, q, r, s) = (\hat{q}, -\hat{p}, \hat{s}, \hat{r})$.

In the same way, we can examine the effect of reflections upon conservation laws. Let

$$\hat{k} = -k, \quad \hat{l} = l, \quad \hat{u}(\hat{k}, \hat{l}) = u(-\hat{k}, \hat{l}) = u(k, l).$$

Then

$$\begin{aligned}\hat{u}(\hat{k} - 1, \hat{l}) &= u(1 - \hat{k}, \hat{l}) = u(k + 1, l), \\ \hat{u}(\hat{k} - 1, \hat{l} + 1) &= u(1 - \hat{k}, \hat{l} + 1) = u(k + 1, l + 1), \\ \hat{u}(\hat{k}, \hat{l} + 1) &= u(-\hat{k}, \hat{l} + 1) = u(k, l + 1).\end{aligned}$$

So $S_{\hat{k}}^{-1}\hat{u}(\hat{k},\hat{l}) = S_k u(k,l)$ and $S_{\hat{l}}\hat{u}(\hat{k},\hat{l}) = S_l u(k,l)$. As before, write the NQC equation (2.17) in terms of the new variable as follows:

$$\frac{(p+s)\hat{u}_{-1,0}-(p-r)\hat{u}_{0,0}-1}{(q+s)\hat{u}_{0,1}-(q-r)\hat{u}_{0,0}-1} = \frac{(q+r)\hat{u}_{-1,1}-(q-s)\hat{u}_{-1,0}-1}{(p+r)\hat{u}_{-1,1}-(p-s)\hat{u}_{0,1}-1}.$$

Now apply $S_{\hat{k}}$ and rearrange to obtain

$$\frac{(-p+r)\hat{u}_{1,0}-(-p-s)\hat{u}_{0,0}-1}{(q+r)\hat{u}_{0,1}-(q-s)\hat{u}_{0,0}-1} = \frac{(q+s)\hat{u}_{1,1}-(q-r)\hat{u}_{1,0}-1}{(-p+s)\hat{u}_{1,1}-(-p-r)\hat{u}_{0,1}-1}.$$

This is of the form (2.19) with

$$(\hat{p},\hat{q},\hat{r},\hat{s}) = (-p,q,s,r). \tag{2.24}$$

Therefore the above reflection amounts to the equivalence transformation

$$\Gamma_b : (p,q,r,s) \mapsto (-p,q,s,r). \tag{2.25}$$

The conservation law (2.22) amounts (after applying $S_{\hat{k}^{-1}}$) to

$$(S_k - \mathrm{id})[-\hat{F}(-k,l,u_{0,0},u_{0,1})] + (S_l - \mathrm{id})\hat{G}(-k-1,l,u_{1,0},u_{0,0}) = 0. \tag{2.26}$$

Note that $r+s$ and $|p^2-q^2|$ are invariants of the group generated by Γ_a and Γ_b.

For certain parameter values, the NQC equation is degenerate in one of two senses: either it can be factorized into a pair of linear PDEs, or it can be integrated to yield an OΔE. These degenerate cases are classified by the following two lemmas.

Lemma 2.1. *If $p^2 = q^2$ then the NQC equation is factorizable.*

Proof. Substituting $p = q$ into (2.17) and rearranging this as a polynomial equation gives the factorization

$$(u_{1,0}-u_{0,1})\left[(q+r)(q+s)u_{1,1}-(q-r)(q-s)u_{0,0}-2q\right] = 0.$$

Substituting $p = -q$ into (2.17) gives

$$(u_{0,0}-u_{1,1})\left[(q+r)(q+s)u_{0,1}-(q-r)(q-s)u_{1,0}-2q\right] = 0.$$

□

Lemma 2.2. *If at least one of p, q and at least one of r, s are zero then the NQC equation may be reduced to an OΔE.*

Proof. Without loss of generality, we may restrict attention to the case $q = r = 0$. All other cases may be obtained from this one by using Γ_a and Γ_b. Note that if $p = q = r = 0$ then (2.17) is a trivial equation, so assume that $p \neq 0$.

If $s = 0$ then the NQC equation reduces to

$$[1 + p(u_{0,1} - u_{1,1})]\,[1 + p(u_{0,0} - u_{1,0})] = 1.$$

Hence

$$(S_l + \mathrm{id}) \ln [1 + p(u_{0,0} - u_{1,0})] = 0,$$

which reduces to the OΔE

$$\ln [1 + p(u_{0,0} - u_{1,0})] = (-1)^l f(k),$$

where $f(k)$ is an arbitrary function.

If $p \neq s \neq 0$ then the substitution $u(k,l) \mapsto (1 - s/p)^k u(k,l) + 1/s$ reduces the NQC equation to

$$\frac{(p^2 - s^2)u_{1,0} - p^2 u_{0,0}}{s^2 u_{1,0}} = \frac{u_{0,1}}{u_{1,1} - u_{0,1}},$$

which can be rearranged as

$$(S_l + \mathrm{id}) \ln \left(1 - \frac{u_{0,0}}{u_{1,0}}\right) = \ln \left(\frac{s^2}{p^2}\right).$$

This may be integrated to yield the OΔE

$$\ln \left(1 - \frac{u_{0,0}}{u_{1,0}}\right) = \frac{1}{2} \ln \left(\frac{s^2}{p^2}\right) + (-1)^l f(k).$$

Finally, if $p = s \neq 0$ then $u(k,l) \mapsto u(k,l) + 1/s$ reduces the NQC equation to

$$(S_l + \mathrm{id}) \left(\frac{u_{0,0}}{u_{1,0}}\right) = 2,$$

which yields the OΔE

$$\frac{u_{0,0}}{u_{1,0}} = 1 + (-1)^l f(k).$$

□

We now seek to to simplify the NQC equation by using equivalence transformations. All choices of p, q, r, s are considered, subject only to the two *nondegeneracy constraints*:

1. $p^2 \neq q^2$;

2. at most one element in each of the pairs $\{p, r\}$, $\{p, s\}$, $\{q, r\}$, $\{q, s\}$ is zero.

These constraints will be assumed to hold from here on. We will use point transformations of the form

$$u(k,l) \mapsto \lambda^k \mu^l u(k,l) + f(k,l),$$

where λ, μ are nonzero constants, to simplify the coefficients of the NQC equation.

Case I: $s = r, \quad (p^2 - r^2)(q^2 - r^2) \neq 0$

In this case, the NQC equation can be transformed into the *cross-ratio equation* [47],

$$\frac{\alpha(u_{1,0} - u_{0,0})}{\beta(u_{0,1} - u_{0,0})} = \frac{u_{1,1} - u_{1,0}}{u_{1,1} - u_{0,1}}, \qquad \alpha \neq 0, \quad \beta \neq 0, \quad \alpha \neq \beta. \tag{2.27}$$

Here $\alpha = p^2 - r^2$ and $\beta = q^2 - r^2$. If $r = s = 0$, the required transformation is

$$u(k,l) \mapsto u(k,l) + \frac{k}{p} + \frac{l}{q}, \tag{2.28}$$

which gives $\alpha = p^2$ and $\beta = q^2$. Otherwise, the transformation is

$$u(k,l) \mapsto \left(\frac{p-r}{p+r}\right)^k \left(\frac{q-r}{q+r}\right)^l u(k,l) + \frac{1}{2r}; \tag{2.29}$$

Case II: $s = r, \quad (p^2 - r^2)(q^2 - r^2) = 0$

Here at least one of the coefficients $(p \pm r)$ and $(q \pm r)$ is zero. Furthermore, the nondegeneracy constraints are only satisfied if exactly one such coefficient is zero. Consequently r is nonzero. By using the equivalence transformations generated by Γ_a, we can set $q + r = 0$ without loss of generality; then the NQC equation amounts to

$$\frac{(p+r)u_{1,0} - (p-r)u_{0,0} - 1}{2ru_{0,0} - 1} = \frac{2ru_{1,0} - 1}{(p+r)u_{1,1} - (p-r)u_{0,1} - 1}.$$

The point transformation

$$u(k,l) \mapsto \left(\frac{p-r}{p+r}\right)^k \left(\frac{4r^2}{p^2 - r^2}\right)^l u(k,l) + \frac{1}{2r} \tag{2.30}$$

reduces the NQC equation to

$$\frac{u_{1,0}-u_{0,0}}{u_{0,0}}=\frac{u_{1,0}}{u_{1,1}-u_{0,1}}. \tag{2.31}$$

Case III: $s=-r\neq 0$

Here the NQC equation amounts to

$$\frac{(p-r)(u_{1,0}-u_{0,0})-1}{(q-r)(u_{0,1}-u_{0,0})-1}=\frac{(q+r)(u_{1,1}-u_{1,0})-1}{(p+r)(u_{1,1}-u_{0,1})-1}.$$

At least three of the four coefficients must be nonzero, for otherwise the nondegeneracy constraints are violated. We can use Γ_a and Γ_b to set $(p+r)(q+r)\neq 0$. Then the transformation

$$u(k,l)\mapsto u(k,l)+\frac{k}{p+r}+\frac{l}{q+r} \tag{2.32}$$

simplifies the NQC equation to

$$\frac{(p^2-r^2)(u_{1,0}-u_{0,0})-2r}{(q^2-r^2)(u_{0,1}-u_{0,0})-2r}=\frac{u_{1,1}-u_{1,0}}{u_{1,1}-u_{0,1}},\qquad p^2\neq q^2,\quad r\neq 0. \tag{2.33}$$

Case IV: $s^2\neq r^2$

First note that the transformation

$$u(k,l)\mapsto u(k,l)+\frac{1}{r+s} \tag{2.34}$$

reduces the NQC equation to

$$\frac{(p+s)u_{1,0}-(p-r)u_{0,0}}{(q+s)u_{0,1}-(q-r)u_{0,0}}=\frac{(q+r)u_{1,1}-(q-s)u_{1,0}}{(p+r)u_{1,1}-(p-s)u_{0,1}}. \tag{2.35}$$

At least one of the coefficients in each numerator and denominator in (2.35) must be nonzero, for otherwise (2.35) reduces to an OΔE or a trivial identity. Therefore at least four of the eight coefficients are nonzero. Suppose that (2.35) can be transformed by an element of the group generated by Γ_a, Γ_b into a form for which

$$(p+r)(p-s)(q+r)(q-s)\neq 0. \tag{2.36}$$

Then the transformation

$$u(k,l) \mapsto \left(\frac{p-s}{p+r}\right)^k \left(\frac{q-s}{q+r}\right)^l u(k,l) \tag{2.37}$$

reduces (2.35) to

$$\frac{(p^2-s^2)u_{1,0}-(p^2-r^2)u_{0,0}}{(q^2-s^2)u_{0,1}-(q^2-r^2)u_{0,0}} = \frac{u_{1,1}-u_{1,0}}{u_{1,1}-u_{0,1}}, \qquad p^2 \neq q^2 \quad s^2 \neq r^2. \tag{2.38}$$

We have already seen that this equation can be reduced to $\mathbf{Q3}_{\delta=0}$, which has two essential parameters.

All that remains is to consider whether there are any circumstances in which the coefficients cannot be transformed to satisfy (2.36). If only one coefficient is zero, it can be transformed into $q+s=0$, which does not violate (2.36). Suppose then, that $q+s=0$ and that at least one of the factors in (2.36) is also zero. Clearly, $p-s$ and $q-s$ must be nonzero to satisfy the nondegeneracy constraints. The case $q+r=0$ does not need to be considered, as this violates $s^2 \neq r^2$. Finally, if $p+r=q+s=0$ then the rotation Γ_a transforms these conditions to $q+s=-p+r=0$, and so (2.36) is satisfied.

This completes the classification of the simplified forms of the NQC equation. We have shown that, up to equivalence transformations, the only nondegenerate cases of NQC may be mapped to one of (2.27), (2.31), (2.33) and (2.38).

2.2.2 Three-point conservation laws of the simplified equations

In this section we present a complete classification of the three-point conservation laws of (2.27), (2.31), (2.33) and (2.38). These conservation laws have been obtained by the method described in Section 2.1. The computer algebra system MAPLE [74](see Appendix A.1) was used to carry out the calculations, details of which are omitted.

Case I A complete set of three-point conservation laws for the cross-ratio equation (2.27) is

1) $F = \dfrac{\alpha}{u_{0,1}-u_{0,0}},$

$G = -\dfrac{\beta}{u_{1,0}-u_{0,0}},$

2) $F = \frac{\alpha u_{0,0}}{u_{0,1} - u_{0,0}},$
$G = -\frac{\beta u_{1,0}}{u_{1,0} - u_{0,0}},$

3) $F = \frac{\alpha u_{0,0} u_{0,1}}{u_{0,1} - u_{0,0}},$
$G = -\frac{\beta u_{0,0} u_{1,0}}{u_{1,0} - u_{0,0}},$

4) $F = (-1)^{k+l}(2\ln[u_{0,1} - u_{0,0}] - \ln[\alpha]),$
$G = -(-1)^{k+l}(2\ln[u_{1,0} - u_{0,0}] - \ln[\beta]).$

Case II A complete set of three-point conservation laws for equation (2.31) is

1) $F = \frac{u_{0,1}}{u_{0,0}},$
$G = \frac{u_{1,0}}{u_{1,0} - u_{0,0}},$

2) $F = (-1)^{k+l}\ln[u_{0,0}],$
$G = -(-1)^{k+l}\ln[u_{1,0} - u_{0,0}].$

Case III A complete set of three-point conservation laws for equation (2.33) is

1) $F = \ln\left[\frac{(q^2 - r^2)(u_{0,1} - u_{0,0}) - 2r}{u_{0,1} - u_{0,0}}\right],$
$G = -\ln\left[\frac{(p^2 - r^2)(u_{1,0} - u_{0,0}) - 2r}{u_{1,0} - u_{0,0}}\right],$

2) $F = (-1)^{k+l}\ln[(u_{0,1} - u_{0,0})((q^2 - r^2)(u_{0,1} - u_{0,0}) - 2r)],$
$G = -(-1)^{k+l}\ln[(u_{1,0} - u_{0,0})((p^2 - r^2)(u_{1,0} - u_{0,0}) - 2r)].$

Case IV A complete set of three-point conservation laws for equation (2.38), except for the cases $p = r, q = -s$ and $p = -s, q = r$, is

1) $F = \ln\left[\frac{u_{0,1} - u_{0,0}}{(q^2 - s^2)u_{0,1} - (q^2 - r^2)u_{0,0}}\right],$
$G = -\ln\left[\frac{u_{1,0} - u_{0,0}}{(p^2 - s^2)u_{1,0} - (p^2 - r^2)u_{0,0}}\right],$

2) $F = (-1)^{k+l}\ln[(u_{0,1} - u_{0,0})((q^2 - s^2)u_{0,1} - (q^2 - r^2)u_{0,0})],$
$G = -(-1)^{k+l}\ln[(u_{1,0} - u_{0,0})((p^2 - s^2)u_{1,0} - (p^2 - r^2)u_{0,0})].$

If $p = r, q = -s$ in (2.38) then the three-point conservation laws are

1) $F = \ln\left[\frac{u_{0,1} - u_{0,0}}{u_{0,0}}\right],$

$G = -\ln\left[\frac{u_{1,0} - u_{0,0}}{u_{1,0}}\right],$

2) $F = (-1)^{k+l}\ln[u_{0,0}(u_{0,1} - u_{0,0})],$

$G = -(-1)^{k+l}\ln[u_{1,0}(u_{1,0} - u_{0,0})],$

3) $F = (k+l)\ln\left[\frac{u_{0,1} - u_{0,0}}{u_{0,1}}\right] - \ln[u_{0,1}u_{0,0}],$

$G = -(k+l)\ln\left[\frac{u_{0,0}(u_{1,0} - u_{0,0})}{{u_{1,0}}^2}\right].$

Finally, if $p = -s, q = r$ then the three-point conservation laws are

1) $F = \ln\left[\frac{u_{0,1} - u_{0,0}}{u_{0,1}}\right],$

$G = -\ln\left[\frac{u_{1,0} - u_{0,0}}{u_{0,0}}\right],$

2) $F = (-1)^{k+l}\ln[u_{0,1}(u_{0,1} - u_{0,0})],$

$G = -(-1)^{k+l}\ln[u_{0,0}(u_{1,0} - u_{0,0})],$

3) $F = (k+l)\ln\left[\frac{u_{0,0}(u_{0,1} - u_{0,0})}{u_{0,1}^2}\right],$

$G = -(k+l)\ln\left[\frac{u_{1,0} - u_{0,0}}{u_{1,0}}\right] + \ln[u_{1,0}u_{0,0}].$

2.2.3 Constructing the conservation laws of the NQC equation

Having classified the conservation laws of the simplified representatives of each equivalence class, we now show how to obtain conservation laws of the original NQC equation for a particular example. If

$$(p, q, r, s) = (-1, 2, 1, 1) \tag{2.39}$$

then the NQC equation amounts to

$$\frac{2u_{0,0} - 1}{3u_{0,1} - u_{0,0} - 1} = \frac{3u_{1,1} - u_{1,0} - 1}{2u_{0,1} - 1}. \tag{2.40}$$

This case satisfies the condition $r = s,\ p + r = 0$, so (2.40) is a subcase of Case II. From the discussion in §2.2.1, all equations in Case II can be obtained by applying one of the rotations generated by Γ_a to

$$\frac{(\hat{p}+\hat{r})\hat{u}_{1,0}-(\hat{p}-\hat{r})\hat{u}_{0,0}-1}{2\hat{r}\hat{u}_{0,0}-1}=\frac{2\hat{r}\hat{u}_{1,0}-1}{(\hat{p}+\hat{r})\hat{u}_{1,1}-(\hat{p}-\hat{r})\hat{u}_{0,1}-1}. \tag{2.41}$$

To obtain the NQC equation with coefficients (2.39) we need to apply Γ_a to (2.41) once. Therefore, from (2.20), the choice of coefficients in (2.41) that corresponds to (2.40) is

$$\frac{3\hat{u}_{0,0}-\hat{u}_{1,0}-1}{2\hat{u}_{0,0}-1}=\frac{2\hat{u}_{1,0}-1}{3\hat{u}_{0,1}-\hat{u}_{1,1}-1}. \tag{2.42}$$

Equation (2.42) has two conservation laws which can be obtained from the conservation laws for (2.31) by inverting the transformation (2.30). After rescaling (for convenience), these are

$$\begin{aligned}
1)\ \ &\hat{F}(\hat{k},\hat{l},\hat{u}_{0,0},\hat{u}_{0,1})=-\frac{3(2\hat{u}_{0,1}-1)}{2(2\hat{u}_{0,0}-1)},\\
&\hat{G}(\hat{k},\hat{l},\hat{u}_{0,0},\hat{u}_{1,0})=\frac{2\hat{u}_{1,0}-1}{3\hat{u}_{0,0}-\hat{u}_{1,0}-1},\\
2)\ \ &\hat{F}(\hat{k},\hat{l},\hat{u}_{0,0},\hat{u}_{0,1})=(-1)^{\hat{k}+\hat{l}+1}\ln[1-2\hat{u}_{0,0}],\\
&\hat{G}(\hat{k},\hat{l},\hat{u}_{0,0},\hat{u}_{1,0})=(-1)^{\hat{k}+\hat{l}}\ln[3\hat{u}_{0,0}-\hat{u}_{1,0}-1].
\end{aligned}$$

So (2.23) tells us that the NQC equation with $(p,q,r,s)=(-1,2,1,1)$ has the two conservation laws

$$\begin{aligned}
1)\ \ &F(k,l,u_{0,0},u_{0,1})=\hat{G}(-l-1,k,u_{0,1},u_{0,0})=\frac{2u_{0,0}-1}{3u_{0,1}-u_{0,0}-1},\\
&G(k,l,u_{0,0},u_{1,0})=-\hat{F}(-l,k,u_{0,0},u_{1,0})=\frac{3(2u_{1,0}-1)}{2(2u_{0,0}-1)},\\
2)\ \ &F(k,l,u_{0,0},u_{0,1})=(-1)^{k-l}\ln[3u_{0,1}-u_{0,0}-1],\\
&G(k,l,u_{0,0},u_{1,0})=-(-1)^{k-l}\ln[1-2u_{0,0}].
\end{aligned}$$

2.3 Three-point conservation laws for ABS equations

In this section we present all three-point conservation laws for integrable equations on the quad-graph that are listed in [5]; these were found by the method described in §2.1.

All three-point conservation laws for these equations are listed in Table 2.1. We omit the details of our calculations, which were carried out using the computer algebra system

MAPLE [74](see Appendix A.1); they are very complex and it is impossible to present them in any suitable form. Three-point conservation laws for **H1**, $\mathbf{H3}_{\delta=0}$ and $\mathbf{Q1}_{\delta=0}$ have already appeared in [34] and previous sections. One conservation law for **Q4** involves the following Jacobi elliptic functions with modulus K:

$$\mathrm{cn}(\alpha) = \mathrm{cn}(\alpha; K), \qquad \mathrm{ns}(\alpha) = \mathrm{ns}(\alpha; K), \qquad \mathrm{dn}(\alpha) = \mathrm{dn}(\alpha; K).$$

This conservation law can be derived from a Bäcklund transformation, as described in [3, 9].

Note that in all three-point conservation laws for ABS equations, the component F does not depend upon α and G does not depend upon β. Therefore the conservation laws from Table 2.1 are valid for all functions $\alpha = \alpha(k)$ and $\beta = \beta(l)$.

For the simplest quad-graph equations $\mathbf{Q1}_{\delta=0}, \mathbf{H1}, \mathbf{H3}_{\delta=0}$, there are 4 three-point conservation laws. Orphanidis also found 4 conservation laws for the discrete sine-Gordon equation [58]. This led to an opinion that perhaps this number of conservation laws is associated with integrability. However Table 2.1 shows that many integrable quad-graph equations have fewer than 4 three-point conservation laws. It seems likely that every integrable quad-graph has infinitely many conservation laws (see also Orphanidis' comment [58] about generating these in a nonlocal form by repeated Bäcklund transformations).

Table 2.1: **Three-point conservation laws for equations from the ABS classification**

Eq.	Generators
$\mathbf{Q1}_{\delta=0}$	$F_1 = -\beta(u_{0,0}-u_{0,1})^{-1}, \quad F_2 = -\beta u_{0,0}(u_{0,0}-u_{0,1})^{-1}, \quad F_3 = -\beta u_{0,0}u_{0,1}(u_{0,0}-u_{0,1})^{-1}, \quad F_4 = -(-1)^{k+l}\left(2\ln(u_{0,0}-u_{0,1})-\ln(\beta)\right),$ $G_1 = \alpha(u_{0,0}-u_{1,0})^{-1}, \quad G_2 = \alpha u_{1,0}(u_{0,0}-u_{1,0})^{-1}, \quad G_3 = \alpha u_{0,0}u_{1,0}(u_{0,0}-u_{1,0})^{-1}, \quad G_4 = (-1)^{k+l}\left(2\ln(u_{0,0}-u_{1,0})-\ln(\alpha)\right),$
$\mathbf{Q1}_{\delta=1}$	$F_1 = \ln(\beta-u_{0,1}+u_{0,0})-\ln(\beta-u_{0,0}+u_{0,1}), \quad F_2 = -(-1)^{k+l}\left(\ln(\beta-u_{0,1}+u_{0,0})+\ln(\beta-u_{0,0}+u_{0,1})-\ln(\beta)\right),$ $G_1 = \ln(\alpha-u_{0,0}+u_{1,0})-\ln(\alpha-u_{1,0}+u_{0,0}), \quad G_2 = (-1)^{k+l}\left(\ln(\alpha-u_{0,0}+u_{1,0})+\ln(\alpha+u_{0,0}-u_{1,0})-\ln(\alpha)\right),$
$\mathbf{Q2}$	$F_1 = -(-1)^{k+l}\left(\ln\left(u_{0,0}^2+(\beta^2-u_{0,1})^2-2u_{0,0}(\beta^2+u_{0,1})\right)-\ln(\beta)\right),$ $G_1 = (-1)^{k+l}\left(\ln\left(u_{0,0}^2+(\alpha^2-u_{1,0})^2-2u_{0,0}(\alpha^2+u_{1,0})\right)-\ln(\alpha)\right),$
$\mathbf{Q3}_{\delta=0}$	$F_1 = \ln(u_{0,1}-\beta u_{0,0})-\ln(\beta u_{0,1}-u_{0,0}), \quad F_2 = -(-1)^{k+l}\left(\ln(u_{0,1}-\beta u_{0,0})+\ln(\beta u_{0,1}-u_{0,0})-\ln(\beta^2-1)\right),$ $G_1 = \ln(\alpha u_{1,0}-u_{0,0})-\ln(u_{1,0}-\alpha u_{0,0}), \quad G_2 = (-1)^{k+l}\left(\ln(\alpha u_{1,0}-u_{0,0})+\ln(u_{1,0}-\alpha u_{0,0})-\ln(\alpha^2-1)\right),$
$\mathbf{Q3}_{\delta=1}$	$F_1 = -(-1)^{k+l}\left(\ln\left((1-\beta^2)^2+4\beta(u_{0,1}-\beta u_{0,0})(\beta u_{0,1}-u_{0,0})\right)-\ln\left(\beta(\beta^2-1)\right)\right),$ $G_1 = (-1)^{k+l}\left(\ln\left((1-\alpha^2)^2+4\alpha(u_{1,0}-\alpha u_{0,0})(\alpha u_{1,0}-u_{0,0})\right)-\ln\left(\alpha(\alpha^2-1)\right)\right),$

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Eq.	Generators
Q4	$F_1 = -(-1)^{k+l}\ln\left(\mathrm{ns}(\beta)\,(u_{0,1}^2+u_{0,0}^2) - K^2\mathrm{sn}(\beta)\,u_{0,1}^2u_{0,0}^2 - 2\mathrm{cn}(\beta)\mathrm{dn}(\beta)\mathrm{ns}(\beta)\,u_{0,1}u_{0,0} - \mathrm{sn}(\beta)\right)$,
	$G_1 = (-1)^{k+l}\ln\left(\mathrm{ns}(\alpha)\,(u_{1,0}^2+u_{0,0}^2) - K^2\mathrm{sn}(\alpha)\,u_{1,0}^2u_{0,0}^2 - 2\mathrm{cn}(\alpha)\mathrm{dn}(\alpha)\mathrm{ns}(\alpha)\,u_{1,0}u_{0,0} - \mathrm{sn}(\alpha)\right)$,
$K=\pm 1$	$F_2 = -\ln(\tanh(\beta)u_{0,0}u_{0,1} + u_{0,0} - u_{0,1} - \tanh(\beta)) + \ln(\tanh(\beta)u_{0,0}u_{0,1} - u_{0,0} + u_{0,1} - \tanh(\beta))$,
	$G_2 = \ln(\tanh(\alpha)u_{0,0}u_{1,0} + u_{0,0} - u_{1,0} - \tanh(\alpha)) - \ln(\tanh(\alpha)u_{0,0}u_{1,0} - u_{0,0} + u_{1,0} - \tanh(\alpha))$,
H1	$F_1 = -(-1)^{k+l}(2u_{0,0}u_{0,1} - \beta)$, $F_2 = -(u_{0,0} - u_{0,1})(u_{0,0}u_{0,1} - \beta)$, $F_3 = -(-1)^{k+l}(u_{0,0} + u_{0,1})(u_{0,0}u_{0,1} - \beta)$,
	$G_1 = (-1)^{k+l}(2u_{0,0}u_{1,0} - \alpha)$, $G_2 = (u_{0,0} - u_{1,0})(u_{0,0}u_{1,0} - \alpha)$, $G_3 = (-1)^{k+l}(u_{0,0} + u_{1,0})(u_{0,0}u_{1,0} - \alpha)$,
	$F_4 = -(-1)^{k+l}(2{u_{0,0}}^2{u_{0,1}}^2 - 4\beta u_{0,0}u_{0,1} + \beta^2)$,
	$G_4 = (-1)^{k+l}(2{u_{0,0}}^2{u_{1,0}}^2 - 4\alpha u_{0,0}u_{1,0} + \alpha^2)$,
H2	$F_1 = -(-1)^{k+l}(2u_{0,0}u_{0,1} - \beta^2 - 2\beta u_{0,0} - 2\beta u_{0,1})$, $F_2 = -(-1)^{k+l}\ln(\beta + u_{0,0} + u_{0,1})$,
	$G_1 = (-1)^{k+l}(2u_{0,0}u_{1,0} - \alpha^2 - 2\alpha u_{0,0} - 2\alpha u_{1,0})$, $G_2 = (-1)^{k+l}\ln(\alpha + u_{0,0} + u_{1,0})$,
$\mathbf{H3}_{\delta=0}$	$F_1 = -(-1)^{k+l}\beta u_{0,0}u_{0,1}$, $F_2 = -(-1)^{k+l}\beta(u_{0,0}u_{0,1})^{-1}$, $F_3 = ({u_{0,0}}^2 - {u_{0,1}}^2)(\beta u_{0,0}u_{0,1})^{-1}$, $F_4 = -(-1)^{k+l}({u_{0,0}}^2 + {u_{0,1}}^2)(\beta u_{0,0}u_{0,1})^{-1}$,
	$G_1 = (-1)^{k+l}\alpha u_{0,0}u_{1,0}$, $G_2 = (-1)^{k+l}\alpha(u_{0,0}u_{1,0})^{-1}$, $G_3 = ({u_{1,0}}^2 - {u_{0,0}}^2)(\alpha u_{0,0}u_{1,0})^{-1}$, $G_4 = (-1)^{k+l}({u_{0,0}}^2 + {u_{1,0}}^2)(\alpha u_{0,0}u_{1,0})^{-1}$,

Continued on next page

Eq.	Generators
$\mathbf{H3}_{\delta=1}$	$F_1 = -(-1)^{k+l}\ln(\beta + u_{0,0}u_{0,1})\,, \quad F_2 = -(-1)^{k+l}\beta(\beta + 2u_{0,0}u_{0,1})\,,$ $G_1 = (-1)^{k+l}\ln(\alpha + u_{0,0}u_{1,0})\,, \quad G_2 = (-1)^{k+l}\alpha(\alpha + 2u_{0,0}u_{1,0})\,,$
$\mathbf{A1}_{\delta=0}$	$F_1 = -(-1)^{k+l}\beta(u_{0,0} + u_{0,1})^{-1}, \quad F_2 = \beta u_{0,0}(u_{0,0} + u_{0,1})^{-1}, \quad F_3 = -(-1)^{k+l}\beta u_{0,0}u_{0,1}(u_{0,0} + u_{0,1})^{-1},$ $G_1 = (-1)^{k+l}\alpha(u_{0,0} + u_{1,0})^{-1}, \quad G_2 = \alpha u_{1,0}(u_{0,0} + u_{1,0})^{-1}, \quad G_3 = (-1)^{k+l}\alpha u_{0,0}u_{1,0}(u_{0,0} + u_{1,0})^{-1},$ $F_4 = -(-1)^{k+l}(2\ln(u_{0,0} + u_{0,1}) - \ln(\beta))\,,$ $G_4 = (-1)^{k+l}(2\ln(u_{0,0} + u_{1,0}) - \ln(\alpha))\,,$
$\mathbf{A1}_{\delta=1}$	$F_1 = -(-1)^{k+l}(2\ln(u_{0,0} + u_{0,1} + \beta) - \ln(\beta))\,, \quad F_2 = -(-1)^{k+l}(2\ln(u_{0,0} + u_{0,1} - \beta) - \ln(\beta))\,,$ $G_1 = (-1)^{k+l}(2\ln(u_{0,0} + u_{1,0} + \alpha) - \ln(\alpha))\,, \quad G_2 = (-1)^{k+l}(2\ln(u_{0,0} + u_{1,0} - \alpha) - \ln(\alpha))\,,$
$\mathbf{A2}$	$F_1 = -(-1)^{k+l}\left(2\ln(u_{0,0}u_{0,1} - \beta) - \ln(\beta^2 - 1)\right), \quad F_2 = -(-1)^{k+l}\left(2\ln(\beta u_{0,0}u_{0,1} - 1) - \ln(\beta^2 - 1)\right),$ $G_1 = (-1)^{k+l}\left(2\ln(u_{0,0}u_{1,0} - \alpha) - \ln(\alpha^2 - 1)\right), \quad G_2 = (-1)^{k+l}\left(2\ln(\alpha u_{0,0}u_{1,0} - 1) - \ln(\alpha^2 - 1)\right).$

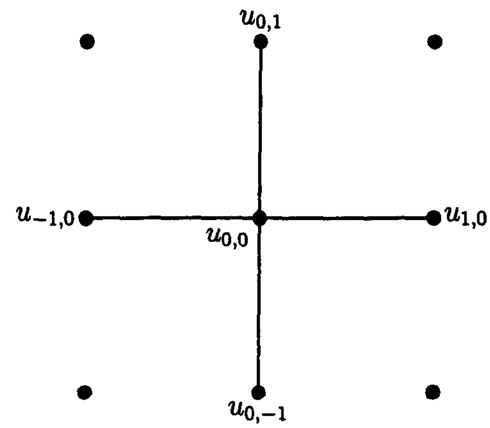

Figure 2.2: Form of a five-point conservation law

2.4 Five-point conservation laws for ABS equations

The simplest higher conservation laws are defined on five points. These can be arranged in various configurations, but for each of the ABS equations the one shown in Figure 2.2 gives the conservation laws in their most concise form. The functions F and G are of the form

$$F = F(k, l, u_{0,-1}, u_{-1,0}, u_{0,0}, u_{0,1}), \qquad G = G(k, l, u_{0,-1}, u_{-1,0}, u_{0,0}, u_{1,0}). \tag{2.43}$$

Therefore the determining equation for the five-point conservation laws is

$$\begin{aligned} &F(k+1, l, u_{1,-1}, u_{0,0}, u_{1,0}, u_{1,1}) - F(k, l, u_{0,-1}, u_{-1,0}, u_{0,0}, u_{0,1}) + \\ &G(k, l+1, u_{0,0}, u_{-1,1}, u_{0,1}, u_{1,1}) - G(k, l, u_{0,-1}, u_{-1,0}, u_{0,0}, u_{1,0}) = 0. \end{aligned} \tag{2.44}$$

Shifted versions of each quad-graph equation are used to eliminate $u_{-1,1}, u_{1,-1}$ and $u_{1,1}$.

The direct method yields three five-point conservation laws for **H1** and **H3**$_{\delta=0}$. For the other equations, the complexity of the calculations has prevented us from solving the determining equation (2.44) directly when F and G are of the form (2.43). However, for each of the equations **H1** and **H3**$_{\delta=0}$, the three five-point conservation laws can be written in the form

$$F_1 = F(k, l, u_{-1,0}, u_{0,0}, u_{0,1}), \qquad G_1 = G(k, l, u_{-1,0}, u_{0,0}, u_{1,0}),$$
$$F_2 = F(l, k, u_{0,-1}, u_{0,0}, u_{1,0}), \qquad G_2 = G(l, k, u_{0,-1}, u_{0,0}, u_{0,1}),$$
$$F_3 = kF_1 + lF_2, \qquad G_3 = kG_1 + lG_2.$$

This suggests that, for each of the remaining ABS equations, we should seek a five-point conservation law of the form

$$F = F(k, l, u_{-1,0}, u_{0,0}, u_{0,1}), \quad G = G(k, l, u_{-1,0}, u_{0,0}, u_{1,0}). \tag{2.45}$$

By substituting (2.45) into (2.11) we obtain a determining equation that is simpler than (2.44):

$$F(k+1, l, u_{0,0}, u_{1,0}, u_{1,1}) - F(k, l, u_{-1,0}, u_{0,0}, u_{0,1}) +$$
$$G(k, l+1, u_{-1,1}, u_{0,1}, u_{1,1}) - G(k, l, u_{-1,0}, u_{0,0}, u_{1,0}) = 0. \tag{2.46}$$

Shifted versions of the quad-graph equation are used to eliminate $u_{-1,1}$ and $u_{1,1}$.

By using the direct method to solve (2.46) we found one five-point conservation law for $\mathbf{Q1}_{\delta=0,1}$, $\mathbf{Q3}_{\delta=0}$, **H2**, $\mathbf{H3}_{\delta=1}$, $\mathbf{A1}_{\delta=0,1}$, **A2**. Let

$$F_1 = F_s(k, l, u_{-1,0}, u_{0,0}, u_{0,1}), \quad G_1 = G_s(k, l, u_{-1,0}, u_{0,0}, u_{1,0}) \tag{2.47}$$

be the solution of (2.46) for an ABS equation. All ABS equations are invariant under the transformation

$$k \to \tilde{l}, \quad l \to \tilde{k}.$$

Therefore each of the above equations has a second five-point conservation law,

$$F_2 = F_s(l, k, u_{0,-1}, u_{0,0}, u_{0,1}), \quad G_2 = G_s(l, k, u_{0,-1}, u_{0,0}, u_{1,0}). \tag{2.48}$$

For equations **Q2**, $\mathbf{Q3}_{\delta=1}$ and **Q4**, we could not solve the simplified determining equation (2.46) directly. However, we observed that each of the other ABS equations has two five-point conservation laws of the form:

$$\begin{aligned}
F_1 &= -S_k^{-1}(-1)^{k+l}\tilde{G} + f(u_{-1,0}, u_{0,0}, u_{0,1}),\\
G_1 &= S_k^{-1}(-1)^{k+l}\tilde{G} - a\ln(u_{1,0} - u_{-1,0}),\\
F_2 &= S_l^{-1}(-1)^{k+l}\tilde{F} + a\ln(u_{0,1} - u_{0,-1}),\\
G_2 &= -S_l^{-1}(-1)^{k+l}\tilde{F} - f(u_{0,-1}, u_{0,0}, u_{1,0}).
\end{aligned} \tag{2.49}$$

Here f is a function and a is a constant; furthermore, $\tilde{F}$ and $\tilde{G}$ are components of a three-point conservation law of the same equation, of the form

$$\tilde{F} = (-1)^{k+l}\ln(\dots), \qquad \tilde{G} = (-1)^{k+l}\ln(\dots). \tag{2.50}$$

Table 2.1 shows that most equations from the ABS classification have a three-point conservation law of the form (2.50); the only exceptions are **H1**, $\mathbf{H3}_{\delta=0}$, whose five-point conservation laws we have already found. Therefore we have sought two five-point conservation laws that can be written in the form (2.49) for each of the remaining equations **Q2**, $\mathbf{Q3}_{\delta=1}$ and **Q4**. By substituting F_1, G_1 from (2.49) into (2.11) we obtain the determining equation for f and a. For each of **Q2**, $\mathbf{Q3}_{\delta=1}$ and **Q4**, this determining equation can be solved by the direct method.

So far we have described how to find two five-point conservation laws for all ABS equations. Our results for **H1** and $\mathbf{H3}_{\delta=0}$ suggest that other equations from the ABS classification may have a third conservation law that is related to the other two as follows:

$$F_3 = kF_1 + lF_2, \qquad G_3 = kG_1 + lG_2. \tag{2.51}$$

We find that in (2.51) the two conservation laws must be written in the form

$$\begin{aligned}
F_1 &= -S_lS_k^{-1}(-1)^{k+l}\tilde{G} - S_k^{-1}(-1)^{k+l}\tilde{G} + a_1 f(u_{-1,0}, u_{0,0}, u_{0,1}) + a_2,\\
G_1 &= (-1)^{k+l}\tilde{G} + S_k^{-1}(-1)^{k+l}\tilde{G} - a_3\ln(u_{1,0} - u_{-1,0}),\\
F_2 &= (-1)^{k+l}\tilde{F} + S_l^{-1}(-1)^{k+l}\tilde{F} + a_3\ln(u_{0,1} - u_{0,-1}),\\
G_2 &= -S_kS_l^{-1}(-1)^{k+l}\tilde{F} - S_l^{-1}(-1)^{k+l}\tilde{F} - a_1 f(u_{0,-1}, u_{0,0}, u_{1,0}) + a_2.
\end{aligned} \tag{2.52}$$

Here $\tilde{F}$, $\tilde{G}$ and f are the same as in (2.49), and the constants a_i can be found by substituting (2.51) into (2.11). The terms $S_lS_k^{-1}(-1)^{k+l}\tilde{G}$ and $S_kS_l^{-1}(-1)^{k+l}\tilde{F}$ depend on

$u_{-1,1}, u_{1,-1}$, which do not lie on the cross (2.43); these variables can be eliminated by shifted versions of the quad-graph equation.

The results of the above are summarized in Table 2.2, in which we list the generators $\bar{F}_i$ and $\bar{G}_i$ of the five-point conservation laws for each of the ABS equations. The corresponding conservation laws are

$$\begin{aligned}
(F_1, G_1) &= (\bar{F}_1, \bar{G}_1),\\
(F_2, G_2) &= (\bar{F}_2, \bar{G}_2),\\
(F_3, G_3) &= (k\bar{F}_1 + l\bar{F}_2, k\bar{G}_1 + l\bar{G}_2).
\end{aligned}$$

In Table 2.2 we use F_n and G_n to denote the components of n^{th} three-point conservation law for the same equation as given in Table 2.1. For **Q4** alone, we have presented the result without eliminating $u_{-1,1}, u_{1,-1}$, as this is far shorter than the result after elimination.

We have checked that all five-point conservation laws cannot be reduced to lower-order conservation laws; in other words, they are independent of those which we have found so far.

All of the three-point conservation laws apply to nonautonomous equations, for which α and β are not constants. However, each equation from the ABS classification has only one five-point conservation law whose component G does not depend upon α and one five-point conservation law for which F does not depend upon β. Consequently, if exactly one of α and β is constant then only one of the five-point conservation laws survives. If neither α nor β is constant, none of the above conservation laws hold.

Table 2.2: **Generators for five-point conservation laws for equations from the ABS classification**

Eq.	Generators
$\mathbf{Q1}_{\delta=0}$	$\bar{F}_1 = (-1)^{k+l}F_4 - S_k^{-1}(-1)^{k+l}G_4 + 2\ln((u_{0,1} - u_{-1,0})(\alpha(u_{0,1} - u_{0,0}) + \beta(u_{0,0} - u_{-1,0}))),$
	$\bar{G}_1 = (-1)^{k+l}G_4 + S_k^{-1}(-1)^{k+l}G_4 - 4\ln(u_{1,0} - u_{-1,0}),$
	$\bar{F}_2 = (-1)^{k+l}F_4 + S_l^{-1}(-1)^{k+l}F_4 + 4\ln(u_{0,1} - u_{0,-1}),$
	$\bar{G}_2 = (-1)^{k+l}G_4 - S_l^{-1}(-1)^{k+l}F_4 - 2\ln((u_{1,0} - u_{0,-1})(\alpha(u_{0,0} - u_{0,-1}) + \beta(u_{1,0} - u_{0,0}))),$
$\mathbf{Q1}_{\delta=1}$	$\bar{F}_1 = -\ln((u_{0,1} - u_{0,0} + \beta)(u_{0,0} - u_{-1,0} - \alpha)(u_{0,1} - u_{-1,0} - \alpha + \beta)^{-1}(\alpha(u_{0,1} - u_{0,0}) + \beta(u_{0,0} - u_{-1,0}))^{-1}),$
	$\bar{G}_1 = \ln((u_{1,0} - u_{0,0} + \alpha)(u_{0,0} - u_{-1,0} - \alpha)(u_{1,0} - u_{-1,0})^{-2}),$
	$\bar{F}_2 = -\ln((u_{0,0} - u_{0,-1} - \beta)(u_{0,1} - u_{0,0} + \beta)(u_{0,1} - u_{0,-1})^{-2}),$
	$\bar{G}_2 = \ln((u_{0,0} - u_{0,-1} - \beta)(u_{1,0} - u_{0,0} + \alpha)(u_{1,0} - u_{0,-1} + \alpha - \beta)^{-1}(\beta(u_{1,0} - u_{0,0}) + \alpha(u_{0,0} - u_{0,-1}))^{-1}),$
$\mathbf{Q2}$	$\bar{F}_1 = (-1)^{k+l}F_1 - S_k^{-1}(-1)^{k+l}G_1 + \ln((u_{-1,0}^2 + ((\alpha-\beta)^2 - u_{0,1})^2 - 2u_{-1,0}((\alpha-\beta)^2 + u_{0,1}))(\alpha u_{0,1} + (\alpha-\beta)(\alpha\beta - u_{0,0}) - \beta u_{-1,0})^2),$
	$\bar{G}_1 = (-1)^{k+l}G_1 + S_k^{-1}(-1)^{k+l}G_1 - 4\ln(u_{1,0} - u_{-1,0}),$
	$\bar{F}_2 = (-1)^{k+l}F_1 + S_l^{-1}(-1)^{k+l}F_1 + 4\ln(u_{0,1} - u_{0,-1}),$
	$\bar{G}_2 = (-1)^{k+l}G_1 - S_l^{-1}(-1)^{k+l}F_1 - \ln((u_{0,-1}^2 + ((\alpha-\beta)^2 - u_{1,0})^2 - 2u_{0,-1}((\alpha-\beta)^2 + u_{1,0}))(\alpha u_{0,-1} + (\alpha-\beta)(\alpha\beta - u_{0,0}) - \beta u_{1,0})^2),$
$\mathbf{Q3}_{\delta=0}$	$\bar{F}_1 = -\ln\left((\alpha u_{-1,0} - u_{0,0})(\beta u_{0,1} - u_{0,0})(\beta u_{0,1} - \alpha u_{-1,0})^{-1}(\beta(1-\alpha^2)u_{0,1} + (\alpha^2-\beta^2)u_{0,0} - \alpha(1-\beta^2)u_{-1,0})^{-1}\right),$
	$\bar{G}_1 = \ln\left((\alpha u_{1,0} - u_{0,0})(u_{0,0} - \alpha u_{-1,0})(u_{1,0} - u_{-1,0})^{-2}\right),$
	$\bar{F}_2 = -\ln\left((\beta u_{0,1} - u_{0,0})(\beta u_{0,-1} - u_{0,0})(u_{0,1} - u_{0,-1})^{-2}\right),$
	$\bar{G}_2 = \ln\left((\alpha u_{1,0} - u_{0,0})(u_{0,0} - \beta u_{0,-1})(\alpha u_{1,0} - \beta u_{0,-1})^{-1}(\beta(1-\alpha^2)u_{0,-1} + (\alpha^2-\beta^2)u_{0,0} - \alpha(1-\beta^2)u_{1,0})^{-1}\right),$

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Eq.	Generators
$Q3_{\delta=1}$	$\bar{F}_1 = (-1)^{k+l}F_1 - S_k^{-1}(-1)^{k+l}G_1 + \ln(((\alpha^2-\beta^2)^2 + 4\alpha\beta(\alpha u_{0,1} - \beta u_{-1,0})(\beta u_{0,1} - \alpha u_{-1,0}))(\beta(1-\alpha^2)u_{0,1} + (\alpha^2-\beta^2)u_{0,0} - \alpha(1-\beta^2)u_{-1,0})^2),$
	$\bar{G}_1 = (-1)^{k+l}G_1 + S_k^{-1}(-1)^{k+l}G_1 - 4\ln(u_{1,0} - u_{-1,0}),$
	$\bar{F}_2 = (-1)^{k+l}F_1 + S_l^{-1}(-1)^{k+l}F_1 + 4\ln(u_{0,1} - u_{0,-1}),$
	$\bar{G}_2 = (-1)^{k+l}G_1 - S_l^{-1}(-1)^{k+l}F_1 - \ln(((\alpha^2-\beta^2)^2 + 4\alpha\beta(\alpha u_{1,0} - \beta u_{0,-1})(\beta u_{1,0} - \alpha u_{0,-1}))(\beta(1-\alpha^2)u_{0,-1} + (\alpha^2-\beta^2)u_{0,0} - \alpha(1-\beta^2)u_{1,0})^2),$
Q4	$\bar{F}_1 = -S_l S_k^{-1}(-1)^{k+l}G_1 - S_k^{-1}(-1)^{k+l}G_1 +$
	$2\ln((\mathrm{sn}(\alpha-\beta)^2(1+K^2\mathrm{sn}(\beta)^2\mathrm{sn}(\alpha)^2) - \mathrm{sn}(\beta)^2 - \mathrm{sn}(\alpha)^2)u_{0,1}u_{-1,0} + \mathrm{sn}(\alpha)\mathrm{sn}(\beta)(u_{0,1}^2 + u_{-1,0}^2 - \mathrm{sn}(\alpha-\beta)^2(1+K^2u_{0,1}^2u_{-1,0}^2))),$
	$\bar{G}_1 = (-1)^{k+l}G_1 + S_k^{-1}(-1)^{k+l}G_1 - 4\ln(u_{1,0} - u_{-1,0}),$
	$\bar{F}_2 = (-1)^{k+l}F_1 + S_l^{-1}(-1)^{k+l}F_1 + 4\ln(u_{0,1} - u_{0,-1}),$
	$\bar{G}_2 = -S_k S_l^{-1}(-1)^{k+l}F_1 - S_l^{-1}(-1)^{k+l}F_1 -$
	$2\ln((\mathrm{sn}(\alpha-\beta)^2(1+K^2\mathrm{sn}(\beta)^2\mathrm{sn}(\alpha)^2) - \mathrm{sn}(\beta)^2 - \mathrm{sn}(\alpha)^2)u_{1,0}u_{0,-1} + \mathrm{sn}(\alpha)\mathrm{sn}(\beta)(u_{1,0}^2 + u_{0,-1}^2 - \mathrm{sn}(\alpha-\beta)^2(1+K^2u_{1,0}^2u_{0,-1}^2))),$
H1	$\bar{F}_1 = -\ln(u_{0,1} - u_{-1,0}), \quad \bar{F}_2 = -\ln(u_{0,1} - u_{0,-1}),$
	$\bar{G}_1 = \ln(u_{1,0} - u_{-1,0}), \quad \bar{G}_2 = \ln(u_{1,0} - u_{0,-1}),$
H2	$\bar{F}_1 = (-1)^{k+l}F_2 - S_k^{-1}(-1)^{k+l}G_2 + \ln((\beta-\alpha-u_{0,1}+u_{-1,0})(\beta-\alpha+u_{0,1}-u_{-1,0})^3),$
	$\bar{G}_1 = (-1)^{k+l}G_2 + S_k^{-1}(-1)^{k+l}G_2 - 4\ln(u_{1,0} - u_{-1,0}),$
	$\bar{F}_2 = (-1)^{k+l}F_2 + S_l^{-1}(-1)^{k+l}F_2 + 4\ln(u_{0,1} - u_{0,-1}),$
	$\bar{G}_2 = (-1)^{k+l}G_2 - S_l^{-1}(-1)^{k+l}F_2 - \ln((\alpha-\beta-u_{1,0}+u_{0,-1})(\alpha-\beta+u_{1,0}-u_{0,-1})^3),$
$H3_{\delta=0}$	$\bar{F}_1 = -\ln\left((\alpha u_{0,1} - \beta u_{-1,0})u_{0,0}^{-1}\right), \quad \bar{F}_2 = -\ln\left((u_{0,1} - u_{0,-1})u_{0,0}^{-1}\right),$
	$\bar{G}_1 = \ln\left((u_{1,0} - u_{-1,0})u_{0,0}^{-1}\right), \quad \bar{G}_2 = \ln\left((\beta u_{1,0} - \alpha u_{0,-1})u_{0,0}^{-1}\right),$

Continued on next page

Eq.	Generators
$\mathbf{H3}_{\delta=1}$	$\bar{F}_1 = (-1)^{k+l}F_1 - S_k^{-1}(-1)^{k+l}G_1 + \ln((\beta u_{0,1} - \alpha u_{-1,0})(\alpha u_{0,1} - \beta u_{-1,0})^3),$
	$\bar{G}_1 = (-1)^{k+l}G_1 + S_k^{-1}(-1)^{k+l}G_1 - 4\ln(u_{1,0} - u_{-1,0}),$
	$\bar{F}_2 = (-1)^{k+l}F_1 + S_l^{-1}(-1)^{k+l}F_1 + 4\ln(u_{0,1} - u_{0,-1}),$
	$\bar{G}_2 = (-1)^{k+l}G_1 - S_l^{-1}(-1)^{k+l}F_1 - \ln((\alpha u_{1,0} - \beta u_{0,-1})(\beta u_{1,0} - \alpha u_{0,-1})^3),$
$\mathbf{A1}_{\delta=0}$	$\bar{F}_1 = (-1)^{k+l}F_4 - S_k^{-1}(-1)^{k+l}G_4 + 2\ln((u_{0,1} - u_{-1,0})(\alpha(u_{0,1} + u_{0,0}) - \beta(u_{0,0} + u_{-1,0}))),$
	$\bar{G}_1 = (-1)^{k+l}G_4 + S_k^{-1}(-1)^{k+l}G_4 - 4\ln(u_{1,0} - u_{-1,0}),$
	$\bar{F}_2 = (-1)^{k+l}F_4 + S_l^{-1}(-1)^{k+l}F_4 + 4\ln(u_{0,1} - u_{0,-1}),$
	$\bar{G}_2 = (-1)^{k+l}G_4 - S_l^{-1}(-1)^{k+l}F_4 - 2\ln((u_{1,0} - u_{0,-1})(\beta(u_{1,0} + u_{0,0}) - \alpha(u_{0,0} + u_{0,-1}))),$
$\mathbf{A1}_{\delta=1}$	$\bar{F}_1 = (-1)^{k+l}(F_1 + F_2) - S_k^{-1}(-1)^{k+l}(G_1 + G_2) + 2\ln(((\alpha-\beta)^2 - (u_{0,1} - u_{-1,0})^2)(\alpha(u_{0,1} + u_{0,0}) - \beta(u_{0,0} + u_{-1,0}))^2),$
	$\bar{G}_1 = (-1)^{k+l}(G_1 + G_2) + S_k^{-1}(-1)^{k+l}(G_1 + G_2) - 8\ln(u_{1,0} - u_{-1,0}),$
	$\bar{F}_2 = (-1)^{k+l}(F_1 + F_2) + S_l^{-1}(-1)^{k+l}(F_1 + F_2) + 8\ln(u_{0,1} - u_{0,-1}),$
	$\bar{G}_2 = (-1)^{k+l}(G_1 + G_2) - S_l^{-1}(-1)^{k+l}(F_1 + F_2) - 2\ln(((\alpha-\beta)^2 - (u_{1,0} - u_{0,-1})^2)(\beta(u_{1,0} + u_{0,0}) - \alpha(u_{0,0} + u_{0,-1}))^2),$
$\mathbf{A2}$	$\bar{F}_1 = (-1)^{k+l}(F_1 + F_2) - S_k^{-1}(-1)^{k+l}(G_1 + G_2) + 2\ln((\alpha u_{0,1} - \beta u_{-1,0})(\beta u_{0,1} - \alpha u_{-1,0})(\alpha(1-\beta^2)u_{0,1} - u_{-1,0}(\beta(1-\alpha^2) + (\alpha^2 - \beta^2)u_{0,0}u_{0,1}))^2),$
	$\bar{G}_1 = (-1)^{k+l}(G_1 + G_2) + S_k^{-1}(-1)^{k+l}(G_1 + G_2) - 8\ln(u_{1,0} - u_{-1,0}),$
	$\bar{F}_2 = (-1)^{k+l}(F_1 + F_2) + S_l^{-1}(-1)^{k+l}(F_1 + F_2) + 8\ln(u_{0,1} - u_{0,-1}),$
	$\bar{G}_2 = (-1)^{k+l}(G_1 + G_2) - S_l^{-1}(-1)^{k+l}(F_1 + F_2) - 2\ln((\beta u_{1,0} - \alpha u_{0,-1})(\alpha u_{1,0} - \beta u_{0,-1})(\beta(1-\alpha^2)u_{1,0} - u_{0,-1}(\alpha(1-\beta^2) + (\beta^2 - \alpha^2)u_{0,0}u_{1,0}))^2),$

Chapter 3

Symmetries

3.1 Symmetries of quad-graph equations

The general form of the ABS equations on the quad-graph is

$$P(k, l, u_{0,0}, u_{1,0}, u_{0,1}, u_{1,1}, \alpha_k, \beta_l) = 0. \tag{3.1}$$

The transformation

$$\Gamma : \ (k, l, u_{0,0}, u_{1,0}, u_{0,1}, u_{1,1}, \alpha_k, \beta_l) \mapsto (k, l, \hat{u}_{0,0}, \hat{u}_{1,0}, \hat{u}_{0,1}, \hat{u}_{1,1}, \hat{\alpha}_k, \hat{\beta}_l)$$

is a symmetry for (3.1) if

$$P(k, l, \hat{u}_{0,0}, \hat{u}_{1,0}, \hat{u}_{0,1}, \hat{u}_{1,1}, \hat{\alpha}_k, \hat{\beta}_l) = 0, \tag{3.2}$$

whenever (3.1) holds. Lie symmetries are obtained by linearizing the symmetry condition about the identity, as follows. We seek one-parameter (local) Lie groups of symmetries of the form

$$\begin{aligned} \hat{u}_{0,0} &= u_{0,0} + \epsilon\eta + O(\epsilon^2), \\ \hat{\alpha}_k &= \alpha_k + \epsilon\xi_1(k, \alpha_k) + O(\epsilon^2), \\ \hat{\beta}_l &= \beta_l + \epsilon\xi_2(l, \beta_l) + O(\epsilon^2). \end{aligned} \tag{3.3}$$

The functions η, ξ_1 and ξ_2 are components of the characteristic $\mathbf{Q}$ of the one-parameter group. The function η depends on finitely many shifts of $u_{0,0}$; this is discussed in the next section. By shifting (3.3) in the k and l directions we obtain

$$\hat{u}_{i,j} = u_{i,j} + \epsilon S_k^i S_l^j \eta + O(\epsilon^2),$$
$$\hat{\alpha}_{k+i} = \alpha_{k+i} + \epsilon\xi_1(k+i, \alpha_{k+i}) + O(\epsilon^2),$$
$$\hat{\beta}_{l+j} = \beta_{l+j} + \epsilon\xi_2(l+j, \beta_{l+j}) + O(\epsilon^2),$$

for every $i, j \in \mathbb{Z}$. Expanding (3.2) to first order in ϵ yields the linearized symmetry condition

$$XP = 0 \qquad \text{whenever (3.1) holds,}$$

where

$$X = \eta\frac{\partial}{\partial u_{0,0}} + (S_k\eta)\frac{\partial}{\partial u_{1,0}} + (S_l\eta)\frac{\partial}{\partial u_{0,1}} + (S_kS_l\eta)\frac{\partial}{\partial u_{1,1}} + \xi_1\frac{\partial}{\partial \alpha_k} + \xi_2\frac{\partial}{\partial \beta_l}. \tag{3.4}$$

3.2 The method

If we were to seek only Lie point symmetries, then η would be of the form

$$\eta = \eta(k, l, u_{0,0}, \alpha_k, \beta_l).$$

However, we shall consider higher symmetries that depend upon the values of the dependent variable on a 3×3 square that is centred on (k, l). By using the quad-graph equation to eliminate the corner nodes (Figure 3.1), we simplify η to the following form:

$$\eta = \eta(k, l, u_{-1,0}, u_{0,-1}, u_{0,0}, u_{1,0}, u_{0,1}, \alpha_k, \beta_l). \tag{3.5}$$

As η depends on five values of the dependent variable, we call such symmetries *five-point symmetries*.

In fact η can be simplified still further. To show this, we apply a symmetry generator (3.4) to (3.1) and obtain the linearized symmetry condition:

$$\eta P_{u_{0,0}} + S_k\eta P_{u_{1,0}} + S_l\eta P_{u_{0,1}} + S_kS_l\eta P_{u_{1,1}} + \xi_1 P_{\alpha_k} + \xi_2 P_{\beta_l} = 0. \tag{3.6}$$

This expression has to be satisfied by all solutions of (3.1). Let

$$\bar{u}_{0,0}(u_{1,0}, u_{0,1}, u_{1,1}), \quad \bar{u}_{1,0}(u_{0,0}, u_{0,1}, u_{1,1}), \quad \bar{u}_{0,1}(u_{0,0}, u_{1,0}, u_{1,1}), \quad \bar{u}_{1,1}(u_{0,0}, u_{1,0}, u_{0,1}),$$

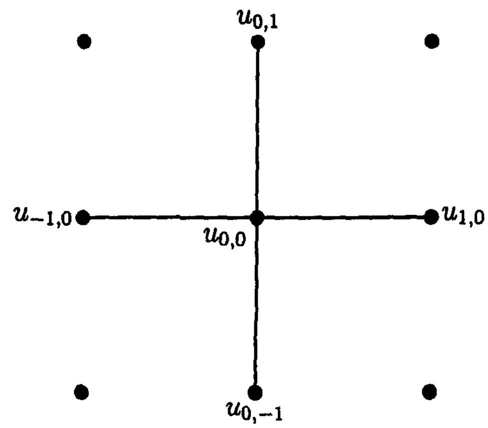

Figure 3.1: Form of a five-point symmetry

denote the result of solving (3.1) for $u_{0,0}, u_{1,0}, u_{0,1}$ and $u_{1,1}$ respectively. In the following, to save space, we suppress the dependence on k, l, α_k and β_l, and we use $\tilde{u}_{1,1}$ to denote $\bar{u}_{1,1}(u_{0,0}, u_{1,0}, u_{0,1})$. To write out the linearized symmetry condition explicitly, we substitute

$$\begin{aligned} u_{-1,0} &= \bar{u}_{0,0}(u_{0,0}, u_{-1,1}, u_{0,1}), \\ u_{0,-1} &= \bar{u}_{0,0}(u_{1,-1}, u_{0,0}, u_{1,0}), \\ u_{2,1} &= \bar{u}_{1,1}(u_{1,0}, u_{2,0}, \tilde{u}_{1,1}), \\ u_{1,2} &= \bar{u}_{1,1}(u_{0,1}, \tilde{u}_{1,1}, u_{0,2}), \\ u_{1,1} &= \tilde{u}_{1,1}, \end{aligned}$$

into (3.6), to obtain

$$\begin{aligned} &\eta(\bar{u}_{0,0}(u_{0,0}, u_{-1,1}, u_{0,1}), \bar{u}_{0,0}(u_{1,-1}, u_{0,0}, u_{1,0}), u_{0,0}, u_{1,0}, u_{0,1})P_{u_{0,0}} \\ &\qquad + \eta(u_{0,0}, u_{1,-1}, u_{1,0}, u_{2,0}, \tilde{u}_{1,1})P_{u_{1,0}} + \eta(u_{-1,1}, u_{0,0}, u_{0,1}, \tilde{u}_{1,1}, u_{0,2})P_{u_{0,1}} \\ &\qquad + \eta(u_{0,1}, u_{1,0}, \tilde{u}_{1,1}, \bar{u}_{1,1}(u_{1,0}, u_{2,0}, \tilde{u}_{1,1}), \bar{u}_{1,1}(u_{0,1}, \tilde{u}_{1,1}, u_{0,2}))P_{u_{1,1}} + \xi_1 P_{\alpha_k} + \xi_2 P_{\beta_l} = 0. \end{aligned} \tag{3.7}$$

By differentiating (3.7) with respect to $u_{-1,1}$ and $u_{1,-1}$, we obtain the necessary condition

$$P_{u_{0,0}} \frac{\partial^2}{\partial u_{-1,1} \partial u_{1,-1}} \eta(\bar{u}_{0,0}(u_{0,0}, u_{-1,1}, u_{0,1}), \bar{u}_{0,0}(u_{1,-1}, u_{0,0}, u_{1,0}), u_{0,0}, u_{1,0}, u_{0,1}) = 0. \tag{3.8}$$

The coefficient of η is nonzero, so the solution of (3.8) shows that η can be split into the sum of two functions which have a simpler form than (3.5). New conditions for η can be obtained, for instance, by differentiating (3.7) with respect to $u_{-1,0}$ and $u_{0,-1}$. Taken together, all such conditions give a system of PDE's with the following solution

$$\eta_{cross}(k,l,u_{-1,0},u_{0,-1},u_{0,0},u_{1,0},u_{0,1},\alpha_k,\beta_l) = \eta_k(k,l,u_{-1,0},u_{0,0},u_{1,0},\alpha_k,\beta_l) + \eta_l(k,l,u_{0,-1},u_{0,0},u_{0,1},\alpha_k,\beta_l),$$

where η_k and η_l are functions which have to be found. Therefore we have demonstrated that η is of the form η_{cross}, which is the sum of the terms in the k and l direction separately. Similarly it is possible to show that for any higher-order symmetry generator, if η is simplified to depend only on values on a cross, it consists of two separate terms in the k and l direction respectively.

We now explain the method for calculating the characteristics $\mathbf{Q}$ for a given quad-graph equation. By substituting η_{cross} into (3.7) we obtain the following determining equation for η_k, η_l, ξ_1 and ξ_2 (again, we suppress k,l,α_k,β_l for brevity).

$$\begin{aligned}(\eta_k(\tilde{u}_{0,0}(u_{0,0},u_{-1,1},u_{0,1}),u_{0,0},u_{1,0}) + \eta_l(\tilde{u}_{0,0}(u_{1,-1},u_{0,0},u_{1,0}),u_{0,0},u_{0,1}))P_{u_{0,0}} \\ + (\eta_k(u_{0,0},u_{1,0},u_{2,0}) + \eta_l(u_{1,-1},u_{1,0},\tilde{u}_{1,1}))\,P_{u_{1,0}} \\ + (\eta_k(u_{-1,1},u_{0,1},\tilde{u}_{1,1}) + \eta_l(u_{0,0},u_{0,1},u_{0,2})\,P_{u_{0,1}} + (\eta_k(u_{0,1},\tilde{u}_{1,1},\tilde{u}_{1,1}(u_{1,0},u_{2,0},\tilde{u}_{1,1})) \\ + \eta_l(u_{1,0},\tilde{u}_{1,1},\tilde{u}_{1,1}(u_{0,1},\tilde{u}_{1,1},u_{0,2})))P_{u_{1,1}} + \xi_1 P_{\alpha_k} + \xi_2 P_{\beta_l} = 0. \quad (3.9)\end{aligned}$$

To solve this functional equation we use an idea which is described in Section 2.1, namely we reduce it to a PDE. By differentiating (3.9) with respect to $u_{2,0}$ we obtain

$$P_{u_{1,0}}\frac{\partial}{\partial u_{2,0}}\eta_k(u_{0,0},u_{1,0},u_{2,0}) + P_{u_{1,1}}\frac{\partial}{\partial u_{2,0}}\eta_k(u_{0,1},\tilde{u}_{1,1},\tilde{u}_{1,1}(u_{1,0},u_{2,0},\tilde{u}_{1,1})) = 0. \quad (3.10)$$

This is a functional-differential equation, but it contains fewer sets of arguments than (3.9) does. The first term can be eliminated by dividing by $P_{u_{1,0}}$ and then differentiating with respect to $u_{0,1}$, to obtain

$$\frac{\partial}{\partial u_{0,1}}\left(\frac{P_{u_{1,1}}}{P_{u_{1,0}}}\frac{\partial}{\partial u_{2,0}}\eta_k(u_{0,1},\tilde{u}_{1,1},\tilde{u}_{1,1}(u_{1,0},u_{2,0},\tilde{u}_{1,1}))\right) = 0.$$

After making the substitution

$$u_{0,0} = \tilde{u}_{0,0}, \quad u_{2,0} = S_k\tilde{u}_{1,0},$$

we get a PDE for the function η_k and solve it. The constraints for the function η_l can be found in a similar way.

So far, we have differentiated the determining equations (3.9) twice; this has created a hierarchy of functional-differential equations that every five-point symmetry must satisfy. The unknown functions η_k, η_l, ξ_1 and ξ_2 can be found completely by going up the hierarchy, a step at a time, to determine more constraints. As the constraints are solved sequentially, more and more information is gained about the functions. At the highest stage, the determining equation is satisfied.

3.3 Five-point symmetries of integrable equations on the quad-graph

In this section we present all five-point symmetries for integrable equations on the quad-graph that are listed in [5]; these were found by the method described in the previous section.

All five-point symmetries for these equation are listed in Table 3.1. In these tables $\mathrm{cn}(\alpha) = \mathrm{cn}(\alpha; K)$ and $\mathrm{dn}(\alpha) = \mathrm{dn}(\alpha; K)$ are Jacobi elliptic functions with modulus K. In [15] it was shown that when $K = 0$, equation **Q4** is equivalent to the case $\mathbf{Q3}_{\delta=1}$. When $K = 0$, all symmetries for equation **Q4** are equivalent to the symmetries for equation $\mathbf{Q3}_{\delta=1}$. We omit the details of our calculations, which were carried out using the computer algebra system MAPLE [74](see Appendix A.2); they are very massive and it is impossible to present them in any suitable form. Five-point symmetries for **H1**, $\mathbf{H3}_{\delta=0}$ and $\mathbf{Q1}_{\delta=0}$ have already appeared in [26, 46, 47, 48, 50, 52, 53, 60, 72, 73].

Note that each equation from the ABS classification has two nonpoint symmetries in the k direction and two nonpoint symmetries in the l direction. In each case, one of these symmetries in the k direction depends explicitly on k; in the next section, we denote this symmetry by X_{km}. The other symmetry in the k direction does not depend on k; we will denote it by X_k. Similarly, we will denote the nonpoint symmetries in the l direction by X_{lm} and X_l.

So far we have seen symmetries only for autonomous equations, for which α and β are constants. The same point symmetries occur even when α and β are not constant. However, there are no other five-point symmetries in the k (respectively l) direction if α (respectively β) is not constant.

Table 3.1: **Symmetry generators for equations from the ABS classification**

Equations	**Generators**
$\mathbf{Q1}_{\delta=0}$	$X_1 = \alpha\partial_\alpha + \beta\partial_\beta,\quad X_2 = \partial_{u_{0,0}},\quad X_3 = u_{0,0}\partial_{u_{0,0}},\quad X_4 = (u_{0,0})^2\partial_{u_{0,0}},\quad X_5 = (u_{1,0}-u_{0,0})(u_{0,0}-u_{-1,0})(u_{1,0}-u_{-1,0})^{-1}\partial_{u_{0,0}},$
	$X_6 = (u_{0,1}-u_{0,0})(u_{0,0}-u_{0,-1})(u_{0,1}-u_{0,-1})^{-1}\partial_{u_{0,0}},\quad X_7 = k(u_{1,0}-u_{0,0})(u_{0,0}-u_{-1,0})(u_{1,0}-u_{-1,0})^{-1}\partial_{u_{0,0}} + \alpha\partial_\alpha,$
	$X_8 = l(u_{0,1}-u_{0,0})(u_{0,0}-u_{0,-1})(u_{0,1}-u_{0,-1})^{-1}\partial_{u_{0,0}} + \beta\partial_\beta,$
$\mathbf{Q1}_{\delta=1}$	$X_1 = \partial_{u_{0,0}},\quad X_2 = u_{0,0}\partial_{u_{0,0}} + \alpha\partial_\alpha + \beta\partial_\beta,\quad X_3 = \{\alpha^2 + (u_{1,0}-u_{0,0})(u_{0,0}-u_{-1,0})\}(u_{1,0}-u_{-1,0})^{-1}\partial_{u_{0,0}},$
	$X_4 = \{\beta^2 + (u_{0,1}-u_{0,0})(u_{0,0}-u_{0,-1})\}(u_{0,1}-u_{0,-1})^{-1}\partial_{u_{0,0}},\quad X_5 = k\{\alpha^2 + (u_{1,0}-u_{0,0})(u_{0,0}-u_{-1,0})\}(u_{1,0}-u_{-1,0})^{-1}\partial_{u_{0,0}} + \alpha\partial_\alpha,$
	$X_6 = l\{\beta^2 + (u_{0,1}-u_{0,0})(u_{0,0}-u_{0,-1})\}(u_{0,1}-u_{0,-1})^{-1}\partial_{u_{0,0}} + \beta\partial_\beta,$
Q2	$X_1 = 2u_{0,0}\partial_{u_{0,0}} + \alpha\partial_\alpha + \beta\partial_\beta,\quad X_2 = \{(u_{0,0}-u_{1,0})(u_{0,0}-u_{-1,0}) - \alpha^2(2u_{0,0}+u_{1,0}+u_{-1,0}) + \alpha^4\}(u_{1,0}-u_{-1,0})^{-1}\partial_{u_{0,0}},$
	$X_3 = \{(u_{0,0}-u_{0,1})(u_{0,0}-u_{0,-1}) - \beta^2(2u_{0,0}+u_{0,1}+u_{0,-1}) + \beta^4\}(u_{0,1}-u_{0,-1})^{-1}\partial_{u_{0,0}},$
	$X_4 = k\{(u_{0,0}-u_{1,0})(u_{0,0}-u_{-1,0}) - \alpha^2(2u_{0,0}+u_{1,0}+u_{-1,0}) + \alpha^4\}(u_{1,0}-u_{-1,0})^{-1}\partial_{u_{0,0}} - \alpha\partial_\alpha,$
	$X_5 = l\{(u_{0,0}-u_{0,1})(u_{0,0}-u_{0,-1}) - \beta^2(2u_{0,0}+u_{0,1}+u_{0,-1}) + \beta^4\}(u_{0,1}-u_{0,-1})^{-1}\partial_{u_{0,0}} - \beta\partial_\beta,$

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Equations	Generators
Q3$_{\delta=0}$	$X_1 = u_{0,0}\partial_{u_{0,0}}$, $\quad X_2 = \{(1+\alpha^2)u_{0,0}(u_{1,0}+u_{-1,0}) - 2\alpha((u_{0,0})^2 + u_{1,0}u_{-1,0})\}(u_{1,0}-u_{-1,0})^{-1}\partial_{u_{0,0}}$,
	$X_3 = \{(1+\beta^2)u_{0,0}(u_{0,1}+u_{0,-1}) - 2\beta((u_{0,0})^2 + u_{0,1}u_{0,-1})\}(u_{0,1}-u_{0,-1})^{-1}\partial_{u_{0,0}}$,
	$X_4 = k\{(1+\alpha^2)u_{0,0}(u_{1,0}+u_{-1,0}) - 2\alpha((u_{0,0})^2 + u_{1,0}u_{-1,0})\}(u_{1,0}-u_{-1,0})^{-1}\partial_{u_{0,0}} + \alpha(\alpha^2-1)\partial_\alpha$,
	$X_5 = l\{(1+\beta^2)u_{0,0}(u_{0,1}+u_{0,-1}) - 2\beta((u_{0,0})^2 + u_{0,1}u_{0,-1})\}(u_{0,1}-u_{0,-1})^{-1}\partial_{u_{0,0}} + \beta(\beta^2-1)\partial_\beta$,
Q3$_{\delta=1}$	$X_1 = \{2\alpha(1+\alpha^2)u_{0,0}(u_{1,0}+u_{-1,0}) - 4\alpha^2(u_{1,0}u_{-1,0} + (u_{0,0})^2) - (1-\alpha^2)^2\}(u_{1,0}-u_{-1,0})^{-1}\partial_{u_{0,0}}$,
	$X_2 = \{2\beta(1+\beta^2)u_{0,0}(u_{0,1}+u_{0,-1}) - 4\beta^2(u_{0,1}u_{0,-1} + (u_{0,0})^2) - (1-\beta^2)^2\}(u_{0,1}-u_{0,-1})^{-1}\partial_{u_{0,0}}$,
	$X_3 = k\{2\alpha(1+\alpha^2)u_{0,0}(u_{1,0}+u_{-1,0}) - 4\alpha^2(u_{1,0}u_{-1,0} + (u_{0,0})^2) - (1-\alpha^2)^2\}(u_{1,0}-u_{-1,0})^{-1}\partial_{u_{0,0}} + 2\alpha^2(\alpha^2-1)\partial_\alpha$,
	$X_4 = l\{2\beta(1+\beta^2)u_{0,0}(u_{0,1}+u_{0,-1}) - 4\beta^2(u_{0,1}u_{0,-1} + (u_{0,0})^2) - (1-\beta^2)^2\}(u_{0,1}-u_{0,-1})^{-1}\partial_{u_{0,0}} + 2\beta^2(\beta^2-1)\partial_\beta$,
Q4	$X_1 = \{\mathrm{cn}(\alpha)\mathrm{dn}(\alpha)u_{0,0}(u_{1,0}+u_{-1,0}) + \mathrm{sn}^2(\alpha)(1+K^2u_{-1,0}(u_{0,0})^2u_{1,0}) - u_{-1,0}u_{1,0} - (u_{0,0})^2\}(u_{1,0}-u_{-1,0})^{-1}\partial_{u_{0,0}}$,
	$X_2 = \{\mathrm{cn}(\beta)\mathrm{dn}(\beta)u_{0,0}(u_{0,1}+u_{0,-1}) + \mathrm{sn}^2(\beta)(1+K^2u_{0,-1}(u_{0,0})^2u_{0,1}) - u_{0,-1}u_{0,1} - (u_{0,0})^2\}(u_{0,1}-u_{0,-1})^{-1}\partial_{u_{0,0}}$,
	$X_3 = k\{\mathrm{cn}(\alpha)\mathrm{dn}(\alpha)u_{0,0}(u_{1,0}+u_{-1,0}) + \mathrm{sn}^2(\alpha)(1+K^2u_{-1,0}(u_{0,0})^2u_{1,0}) - u_{-1,0}u_{1,0} - (u_{0,0})^2\}(u_{1,0}-u_{-1,0})^{-1}\partial_{u_{0,0}} + \mathrm{sn}(\alpha)\partial_\alpha$,
	$X_4 = l\{\mathrm{cn}(\beta)\mathrm{dn}(\beta)u_{0,0}(u_{0,1}+u_{0,-1}) + \mathrm{sn}^2(\beta)(1+K^2u_{0,-1}(u_{0,0})^2u_{0,1}) - u_{0,-1}u_{0,1} - (u_{0,0})^2\}(u_{0,1}-u_{0,-1})^{-1}\partial_{u_{0,0}} + \mathrm{sn}(\beta)\partial_\beta$,
	when $K=\pm 1$ $\quad X_5 = \{1-(u_{0,0})^2\}\partial_{u_{0,0}}$,

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Equations	Generators
H1	$X_1 = \partial_\alpha + \partial_\beta,\quad X_2 = \partial_{u_{0,0}},\quad X_3 = (-1)^{k+l}\partial_{u_{0,0}},\quad X_4 = u_{0,0}\partial_{u_{0,0}} + 2\alpha\partial_\alpha + 2\beta\partial_\beta,\quad X_5 = (-1)^{k+l}u_{0,0}\partial_{u_{0,0}},$ $X_6 = (u_{1,0} - u_{-1,0})^{-1}\partial_{u_{0,0}},\quad X_7 = (u_{0,1} - u_{0,-1})^{-1}\partial_{u_{0,0}},\quad X_8 = k(u_{1,0} - u_{-1,0})^{-1}\partial_{u_{0,0}} - \partial_\alpha,\quad X_9 = l(u_{0,1} - u_{0,-1})^{-1}\partial_{u_{0,0}} - \partial_\beta,$
H2	$X_1 = \partial_{u_{0,0}} - 2\partial_\alpha - 2\partial_\beta,\quad X_2 = (-1)^{k+l}\partial_{u_{0,0}},\quad X_3 = u_{0,0}\partial_{u_{0,0}} + \alpha\partial_\alpha + \beta\partial_\beta,\quad X_4 = (u_{1,0} + u_{-1,0} + 2u_{0,0} + 2\alpha)(u_{1,0} - u_{-1,0})^{-1}\partial_{u_{0,0}},$ $X_5 = (u_{0,1} + u_{0,-1} + 2u_{0,0} + 2\beta)(u_{0,1} - u_{0,-1})^{-1}\partial_{u_{0,0}},\quad X_6 = k\,(u_{1,0} + u_{-1,0} + 2u_{0,0} + 2\alpha)(u_{1,0} - u_{-1,0})^{-1}\partial_{u_{0,0}} - \partial_\alpha,$ $X_7 = l\,(u_{0,1} + u_{0,-1} + 2u_{0,0} + 2\beta)(u_{0,1} - u_{0,-1})^{-1}\partial_{u_{0,0}} - \partial_\beta,$
H3 $_{\delta=0}$	$X_1 = \alpha\partial_\alpha + \beta\partial_\beta,\quad X_2 = u_{0,0}\partial_{u_{0,0}},\quad X_3 = (-1)^{k+l}u_{0,0}\partial_{u_{0,0}},\quad X_4 = u_{0,0}(u_{1,0} + u_{-1,0})(u_{1,0} - u_{-1,0})^{-1}\partial_{u_{0,0}},$ $X_5 = u_{0,0}(u_{0,1} + u_{0,-1})(u_{0,1} - u_{0,-1})^{-1}\partial_{u_{0,0}},\quad X_6 = ku_{0,0}(u_{1,0} + u_{-1,0})(u_{1,0} - u_{-1,0})^{-1}\partial_{u_{0,0}} + \alpha\partial_\alpha,$ $X_7 = lu_{0,0}(u_{0,1} + u_{0,-1})(u_{0,1} - u_{0,-1})^{-1}\partial_{u_{0,0}} + \beta\partial_\beta,$
H3 $_{\delta=1}$	$X_1 = u_{0,0}\partial_{u_{0,0}} + 2\alpha\partial_\alpha + 2\beta\partial_\beta,\quad X_2 = (-1)^{k+l}u_{0,0}\partial_{u_{0,0}},\quad X_3 = (2\alpha + u_{0,0}u_{1,0} + u_{0,0}u_{-1,0})(u_{1,0} - u_{-1,0})^{-1}\partial_{u_{0,0}},$ $X_4 = (2\beta + u_{0,0}u_{0,1} + u_{0,0}u_{0,-1})(u_{0,1} - u_{0,-1})^{-1}\partial_{u_{0,0}},\quad X_5 = k(2\alpha + u_{0,0}u_{1,0} + u_{0,0}u_{-1,0})(u_{1,0} - u_{-1,0})^{-1}\partial_{u_{0,0}} + \alpha\partial_\alpha,$ $X_6 = l(2\beta + u_{0,0}u_{0,1} + u_{0,0}u_{0,-1})(u_{0,1} - u_{0,-1})^{-1}\partial_{u_{0,0}} + \beta\partial_\beta,$

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Equations	Generators
$\mathbf{A1}_{\delta=0}$	$X_1 = \alpha\partial_\alpha + \beta\partial_\beta,\quad X_2 = (-1)^{k+l}\partial_{u_{0,0}},\quad X_3 = u_{0,0}\partial_{u_{0,0}},\quad X_4 = (-1)^{k+l}(u_{0,0})^2\partial_{u_{0,0}},$
	$X_5 = (u_{0,0}+u_{1,0})(u_{0,0}+u_{-1,0})(u_{1,0}-u_{-1,0})^{-1}\partial_{u_{0,0}},\quad X_6 = (u_{0,0}+u_{0,1})(u_{0,0}+u_{0,-1})(u_{0,1}-u_{0,-1})^{-1}\partial_{u_{0,0}},$
	$X_7 = k(u_{0,0}+u_{1,0})(u_{0,0}+u_{-1,0})(u_{1,0}-u_{-1,0})^{-1}\partial_{u_{0,0}} + \alpha\partial_\alpha,\quad X_8 = l(u_{0,0}+u_{0,1})(u_{0,0}+u_{0,-1})(u_{0,1}-u_{0,-1})^{-1}\partial_{u_{0,0}} + \beta\partial_\beta,$
$\mathbf{A1}_{\delta=1}$	$X_1 = (-1)^{k+l}\partial_{u_{0,0}},\quad X_2 = u_{0,0}\partial_{u_{0,0}} + \alpha\partial_\alpha + \beta\partial_\beta,\quad X_3 = \{\alpha^2 - (u_{0,0}+u_{1,0})(u_{0,0}+u_{-1,0})\}(u_{1,0}-u_{-1,0})^{-1}\partial_{u_{0,0}},$
	$X_4 = \{\beta^2 - (u_{0,0}+u_{0,1})(u_{0,0}+u_{0,-1})\}(u_{0,1}-u_{0,-1})^{-1}\partial_{u_{0,0}},\quad X_5 = k\{\alpha^2 - (u_{0,0}+u_{1,0})(u_{0,0}+u_{-1,0})\}(u_{1,0}-u_{-1,0})^{-1}\partial_{u_{0,0}} - \alpha\partial_\alpha,$
	$X_6 = l\{\beta^2 - (u_{0,0}+u_{0,1})(u_{0,0}+u_{0,-1})\}(u_{0,1}-u_{0,-1})^{-1}\partial_{u_{0,0}} - \beta\partial_\beta,$
$\mathbf{A2}$	$X_1 = (-1)^{k+l}u_{0,0}\partial_{u_{0,0}},\quad X_2 = \{(1+\alpha^2)u_{0,0}(u_{1,0}+u_{-1,0}) - 2\alpha(1+(u_{0,0})^2u_{1,0}u_{-1,0})\}(u_{1,0}-u_{-1,0})^{-1}\partial_{u_{0,0}},$
	$X_3 = \{(1+\beta^2)u_{0,0}(u_{0,1}+u_{0,-1}) - 2\beta(1+(u_{0,0})^2u_{0,1}u_{0,-1})\}(u_{0,1}-u_{0,-1})^{-1}\partial_{u_{0,0}},$
	$X_4 = k\{(1+\alpha^2)u_{0,0}(u_{1,0}+u_{-1,0}) - 2\alpha(1+(u_{0,0})^2u_{1,0}u_{-1,0})\}(u_{1,0}-u_{-1,0})^{-1}\partial_{u_{0,0}} + \alpha(1-\alpha^2)\partial_\alpha,$
	$X_5 = l\{(1+\beta^2)u_{0,0}(u_{0,1}+u_{0,-1}) - 2\beta(1+(u_{0,0})^2u_{0,1}u_{0,-1})\}(u_{0,1}-u_{0,-1})^{-1}\partial_{u_{0,0}} + \beta(1-\beta^2)\partial_\beta.$

3.4 Mastersymmetries

Definition 1. *A mastersymmetry for a symmetry generator X is a symmetry generator X_m such that*

$$[X_m, X], \quad [X_m, [X_m, X]], \dots$$

is an infinite set of linearly independent symmetry generators.

For continuous integrable systems, Fuchssteiner [24, 55, 56] has explained the link between mastersymmetries and symmetries that are linear in the independent variables. Furthermore, [73] showed that the dKdV equation also has mastersymmetries that are linear in the independent variables. For each of the ABS equations with constant α and β, the generators X_{km} and X_{lm} have this property, which suggests that they may be mastersymmetries. An algebraic approach to mastersymmetries gives the following criterion [23, 64, 70].

Theorem 3.1. *A symmetry X_m is a mastersymmetry for the symmetry X if it satisfies*

$$[X_m, X] \neq 0, \qquad [[X_m, X], X] = 0. \tag{3.11}$$

Here $[\cdot,\cdot]$ denotes the commutator.

By checking these properties for all symmetries from Table 3.1 we find that X_{km} is a mastersymmetry for X_k and X_{lm} is a mastersymmetry for X_l for each equation in the ABS classification. Therefore we can obtain a hierarchy of symmetries in the k direction:

$$X_{k_1} = [X_{km}, X_k], \quad X_{k_2} = [X_{km}, X_{k_1}], \quad \dots, \quad X_{k_{n+1}} = [X_{km}, X_{k_n}].$$

Similarly, there is a hierarchy of symmetries in the l direction:

$$X_{l_1} = [X_{lm}, X_l], \quad X_{l_2} = [X_{lm}, X_{l_1}], \quad \dots, \quad X_{l_{n+1}} = [X_{lm}, X_{l_n}].$$

As an example, consider the autonomous equation $\mathbf{Q1}_{\delta=0}$. The commutator of symmetries X_7 and X_5 gives us a new symmetry:

$$X_9 = \frac{(u_{1,0} - u_{0,0})^2 (u_{0,0} - u_{-1,0})^2}{(u_{1,0} - u_{-1,0})^2} \left(\frac{1}{u_{2,0} - u_{0,0}} + \frac{1}{u_{0,0} - u_{-2,0}} \right) \partial_{u_{0,0}}. \tag{3.12}$$

This symmetry cannot be reduced to any lower-order symmetry, for its characteristic depends on $u_{2,0}, u_{-2,0}$. The symmetry (3.12) lies on a line of five points; if we apply the mastersymmetry a second time we will obtain an expression which lies on a seven-point line, and so on. The same situation occurs for each of the ABS equations, namely the order of a symmetry increases by two each time one applies a mastersymmetry, creating hierarchies with the following dependencies:

$$X_{k_n} = \eta_{k_n}(u_{-n,0}, u_{-n+1,0}, \ldots, u_{n-1,0}, u_{n,0})\partial_{u_{0,0}},$$
$$X_{l_n} = \eta_{l_n}(u_{0,-n}, u_{0,-n+1}, \ldots, u_{0,n-1}, u_{0,n})\partial_{u_{0,0}}.$$

Chapter 4

Toda type equations

In this chapter we compute conservation laws and symmetries for an example of the Toda type system that corresponds to the autonomous equations **H1** and $\mathbf{Q1}_{\delta=0}$:

$$\frac{1}{u_{1,1}-u_{0,0}}-\frac{1}{u_{-1,1}-u_{0,0}}-\frac{1}{u_{1,-1}-u_{0,0}}+\frac{1}{u_{-1,-1}-u_{0,0}}=0. \tag{4.1}$$

This equation is the so-called *missing identity of Frobenius* [25, 76].

4.1 Symmetries of Toda type equations

We can use the connection between integrable quad-graph equations and Toda type systems to transform symmetries of quad-graph equations into symmetries of the corresponding Toda type systems. A Toda system can be obtained from any equation in the ABS classification

$$P(k,l,u_{0,0},u_{1,0},u_{0,1},u_{1,1},\alpha,\beta)=0 \tag{4.2}$$

by the substitution

$$u_{0,1}=S_k^{-1}\tilde{u}_{1,1},\quad u_{1,0}=S_l^{-1}\tilde{u}_{1,1},\quad u_{-1,0}=\bar{u}_{0,1}(u_{-1,-1},u_{0,-1},u_{0,0}). \tag{4.3}$$

Here, we are using the notation introduced in §3.2. Note: it is necessary to make the substitution $u_{-1,0}=\bar{u}_{0,1}(u_{-1,-1},u_{0,-1},u_{0,0})$ after the substitution $u_{0,1}=S_k^{-1}\tilde{u}_{1,1}$ because equation (4.2) does not depend on $u_{-1,0}$.

We have verified that each of listed symmetries for the ABS classification can be transformed to a symmetry for the corresponding Toda type system by the substitution (4.3).

The characteristics of symmetries for (4.1) can be obtained by transformation of the characteristics of the symmetries for **H1** and $\mathbf{Q1}_{\delta=0}$ by (4.3). Note that for **H1** and $\mathbf{Q1}_{\delta=0}$ the substitutions (4.3) are different.

The point symmetries stay the same after substitution (4.3) for both **H1** and $\mathbf{Q1}_{\delta=0}$; they are

$$\begin{aligned} &X_1 = \partial_{u_{0,0}}, \quad X_2 = (-1)^{k+l}\partial_{u_{0,0}}, \quad X_3 = u_{0,0}\partial_{u_{0,0}}, \\ &X_4 = (-1)^{k+l}u_{0,0}\partial_{u_{0,0}}, \quad X_5 = {u_{0,0}}^2\partial_{u_{0,0}}. \end{aligned} \tag{4.4}$$

(We have omitted the components ξ_1 and ξ_2, because (4.1) does not depend on α or β.) The commutators of (4.4) yield one more symmetry generator:

$$X_6 = (-1)^{k+l}{u_{0,0}}^2\partial_{u_{0,0}}.$$

The rescaled remaining five-point symmetries of **H1** transform by (4.3) to

$$\begin{aligned} X_7 &= \frac{(u_{0,0} - u_{-1,-1})(u_{1,-1} - u_{0,0})}{u_{1,-1} - u_{-1,-1}}\partial_{u_{0,0}}, \\ X_8 &= \frac{(u_{0,0} - u_{-1,-1})(u_{-1,1} - u_{0,0})}{u_{-1,1} - u_{-1,-1}}\partial_{u_{0,0}}, \\ X_9 &= \frac{k(u_{0,0} - u_{-1,-1})(u_{1,-1} - u_{0,0})}{u_{1,-1} - u_{-1,-1}}\partial_{u_{0,0}}, \\ X_{10} &= \frac{l(u_{0,0} - u_{-1,-1})(u_{-1,1} - u_{0,0})}{u_{-1,1} - u_{-1,-1}}\partial_{u_{0,0}}. \end{aligned}$$

The same result is obtained from the symmetries for $\mathbf{Q1}_{\delta=0}$. All these symmetries were found in the previous chapter. The Toda system (4.1) also has mastersymmetries. As expected, X_9 is the mastersymmetry for X_7 and X_{10} is the mastersymmetry for X_8. Two hierarchies of the local symmetries therefore can be constructed.

In the same way, each Toda system for the other quad-graph equations has mastersymmetries that can be obtained from the mastersymmetries of the corresponding quad-graph equations.

Note that the five-point symmetries for (4.1) lie on the same five-point cross on which the Toda system is defined (Figure 4.1), not on the one which is in Figure 3.1.

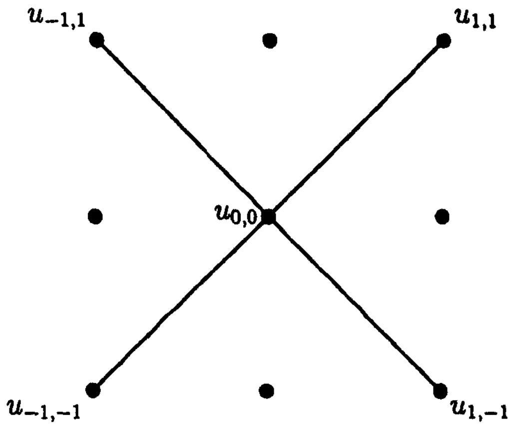

Figure 4.1: Five-point symmetries for (4.1)

4.2 Conservation laws of Toda type equations

The connection between integrable quad-graph equations and Toda type systems is described in the Introduction. It follows that a conservation law for a quad-graph equation is a conservation law for the corresponding Toda type system.

Some of the conservation laws for (4.1) are

- $F_1 = -(u_{0,1} - u_{0,-1})(u_{0,1} - u_{-1,0})^{-1}(u_{0,-1} - u_{-1,0})^{-1}$,
  $G_1 = (u_{0,0} + u_{0,-1} - u_{-1,0} - u_{-1,-1})(u_{0,0} - u_{-1,-1})^{-1}(u_{0,-1} - u_{-1,0})^{-1}$,
- $F_2 = (-1)^{k+l}(u_{0,1} - u_{0,-1})(u_{0,1} - u_{-1,0})^{-1}(u_{0,-1} - u_{-1,0})^{-1}$,
  $G_2 = (-1)^{k+l}(u_{0,0} + u_{-1,0} - u_{0,-1} - u_{-1,-1})(u_{0,0} - u_{-1,-1})^{-1}(u_{0,-1} - u_{-1,0})^{-1}$,
- $F_3 = -u_{-1,0}(u_{0,1} - u_{0,-1})(u_{0,1} - u_{-1,0})^{-1}(u_{0,-1} - u_{-1,0})^{-1}$,
  $G_3 = (u_{0,0}u_{0,-1} - u_{-1,0}u_{-1,-1})(u_{0,0} - u_{-1,-1})^{-1}(u_{0,-1} - u_{-1,0})^{-1}$,
- $F_4 = -(-1)^{k+l}u_{-1,0}(u_{0,1} - u_{0,-1})(u_{0,1} - u_{-1,0})^{-1}(u_{0,-1} - u_{-1,0})^{-1}$,
  $G_4 = (-1)^{k+l}(u_{0,0}u_{-1,0} - u_{0,-1}u_{-1,-1})(u_{0,0} - u_{-1,-1})^{-1}(u_{0,-1} - u_{-1,0})^{-1}$,
- $F_5 = -{u_{-1,0}}^2(u_{0,1} - u_{0,-1})(u_{0,1} - u_{-1,0})^{-1}(u_{0,-1} - u_{-1,0})^{-1}$,
  $G_5 = \dfrac{u_{0,0}u_{0,-1}u_{-1,0} - u_{0,0}u_{-1,0}u_{-1,-1} + u_{0,0}u_{0,-1}u_{-1,-1} - u_{0,-1}u_{-1,0}u_{-1,-1}}{(u_{0,0} - u_{-1,-1})(u_{0,-1} - u_{-1,0})}$,
- $F_6 = -(-1)^{k+l}{u_{-1,0}}^2(u_{0,1} - u_{0,-1})(u_{0,1} - u_{-1,0})^{-1}(u_{0,-1} - u_{-1,0})^{-1}$,
  $G_6 = \dfrac{(-1)^{k+l}(u_{0,0}u_{0,-1}u_{-1,0} + u_{0,0}u_{-1,0}u_{-1,-1} - u_{0,0}u_{0,-1}u_{-1,-1} - u_{0,-1}u_{-1,0}u_{-1,-1})}{(u_{0,0} - u_{-1,-1})(u_{0,-1} - u_{-1,0})}$,
- $F_7 = -2\ln(u_{-1,1} - u_{0,0}) + \ln(u_{-1,1} - u_{-1,-1}) + \ln(u_{0,1} - u_{0,-1})$,
  $G_7 = 2\ln(u_{0,0} - u_{1,-1}) - \ln(u_{1,0} - u_{-1,0}) - \ln(u_{1,-1} - u_{-1,-1})$,

- $F_8 = -(-1)^{k+l}(2\ln(u_{-1,1} - u_{0,0}) - \ln(u_{-1,1} - u_{-1,-1}) + \ln(u_{0,1} - u_{0,-1}))$,
  $G_8 = (-1)^{k+l}(2\ln(u_{0,0} - u_{1,-1}) + \ln(u_{1,0} - u_{-1,0}) - \ln(u_{1,-1} - u_{-1,-1}))$.

The first six were obtained by transforming the conservation laws for **H1** and $\mathbf{Q1}_{\delta=0}$. The other two were found directly. It is known that (4.1) has a variational formulation [5, 14, 15, 18] with the Lagrangian

$$L = \ln(u_{1,1} - u_{0,0}) - \ln(u_{1,-1} - u_{0,0}). \tag{4.5}$$

Noether's Theorem for difference equations [21, 37, 42, 75] produces conservation laws from variational symmetries, it is not known whether the converse is true. However we can obtain symmetries for (4.1) from the above conservation laws. If we divide each expression

$$(S_k - \mathrm{id})F_i + (S_l - \mathrm{id})G_i, \quad i = 1,8 \tag{4.6}$$

by (4.1) we obtain symmetry characteristics. Symmetries which correspond to the first six conservation laws are

$$\begin{aligned} &X_1 = \partial_{u_{0,0}}, \quad X_2 = (-1)^{k+l}\partial_{u_{0,0}}, \quad X_3 = u_{0,0}\partial_{u_{0,0}}, \\ &X_4 = (-1)^{k+l}u_{0,0}\partial_{u_{0,0}}, \quad X_5 = {u_{0,0}}^2\partial_{u_{0,0}}, \quad X_6 = (-1)^{k+l}{u_{0,0}}^2\partial_{u_{0,0}}. \end{aligned} \tag{4.7}$$

The 7^{th} and 8^{th} conservation laws have logarithmic form and it is not clear how to factor out equation (4.1). However from (4.6) we obtain

$$Q_7 = \frac{\ln\left(\frac{(u_{-1,1}-u_{-1,-1})(u_{0,0}-u_{1,-1})^2(u_{1,1}-u_{-1,1})}{(u_{-1,1}-u_{0,0})^2(u_{1,1}-u_{1,-1})(u_{1,-1}-u_{-1,-1})}\right)}{\frac{1}{u_{1,1}-u_{0,0}} - \frac{1}{u_{-1,1}-u_{0,0}} - \frac{1}{u_{1,-1}-u_{0,0}} + \frac{1}{u_{-1,-1}-u_{0,0}}}, \tag{4.8}$$

$$Q_8 = (-1)^{k+l}\frac{\ln\left(\frac{(u_{-1,1}-u_{-1,-1})(u_{0,0}-u_{1,-1})^2(u_{1,1}-u_{-1,1})}{(u_{-1,1}-u_{0,0})^2(u_{1,1}-u_{1,-1})(u_{1,-1}-u_{-1,-1})}\right)}{\frac{1}{u_{1,1}-u_{0,0}} - \frac{1}{u_{-1,1}-u_{0,0}} - \frac{1}{u_{1,-1}-u_{0,0}} + \frac{1}{u_{-1,-1}-u_{0,0}}}. \tag{4.9}$$

It is easy to check that $X_7 = Q_7\partial_{u_{0,0}}$ and $X_8 = Q_8\partial_{u_{0,0}}$ are variational symmetries [57] for (4.5). On solutions of (4.1) Q_7 and Q_8 are undefined expressions of the form $\left\{\frac{0}{0}\right\}$. This problem can be solved by expanding the characteristics as Taylor series and taking into account relation (4.1). So we obtain equivalent not nonvariational symmetries for (4.1):

$$\begin{aligned} &\bar{X}_7 = \left(\frac{(u_{0,0} - u_{-1,-1})(u_{-1,1} - u_{0,0})}{u_{-1,1} - u_{-1,-1}} - \frac{(u_{0,0} - u_{-1,-1})(u_{1,-1} - u_{0,0})}{u_{1,-1} - u_{-1,-1}}\right)\partial_{u_{0,0}}, \\ &\bar{X}_8 = (-1)^{k+l}\left(\frac{(u_{0,0} - u_{-1,-1})(u_{-1,1} - u_{0,0})}{u_{-1,1} - u_{-1,-1}} - \frac{(u_{0,0} - u_{-1,-1})(u_{1,-1} - u_{0,0})}{u_{1,-1} - u_{-1,-1}}\right)\partial_{u_{0,0}}. \end{aligned} \tag{4.10}$$

In the previous section we show that (4.1) has two mastersymmetries for $\bar{X}_7$ and therefore these are mastersymmetries for X_7:

$$X_{m1} = \frac{k(u_{0,0} - u_{-1,-1})(u_{1,-1} - u_{0,0})}{u_{1,-1} - u_{-1,-1}} \partial_{u_{0,0}},$$
$$X_{m2} = \frac{l(u_{0,0} - u_{-1,-1})(u_{-1,1} - u_{0,0})}{u_{-1,1} - u_{-1,-1}} \partial_{u_{0,0}}.$$

Since X_{m1} and X_{m2} depend just upon $u_{-1,-1}, u_{1,-1}, u_{-1,1}, u_{0,0}$, X_{m1} and X_{m2} are also mastersymmetries for $\bar{X}_8$. So infinite hierarchies of local symmetries can be obtained. Then we can check which are variational symmetries. If there is a variational symmetry one can construct a conservation law from it with the help of the discrete version of Noether's Theorem.

Chapter 5

Application of conservation laws and symmetries

5.1 High-order conservation laws

New conservation laws can be obtained by applying the generator of a five-point symmetry repeatedly to a known conservation law [10, 11, 57]. For instance, let us consider equation **H1**. By applying the infinitesimal generator (which is found in the previous chapter)

$$X = \frac{k}{u_{1,0} - u_{-1,0}} \partial_{u_{0,0}} - \partial_{\alpha}$$

to the conservation law

$$F = -\ln\left(u_{0,1} - u_{-1,0}\right), \quad G = \ln\left(u_{1,0} - u_{-1,0}\right),$$

then adding a trivial conservation law, we obtain

$$F_n = -\{(u_{0,0} - u_{-2,0})(u_{0,1} - u_{-1,0})\}^{-1}, \quad G_n = \{(u_{0,0} - u_{-2,0})(u_{1,0} - u_{-1,0})\}^{-1}.$$

At present, there is no proof that this method will always yield a new conservation law (that cannot be reduced to a known or trivial one); however, we do not know of any counterexamples.

We know from the previous chapter that each equation from the ABS classification has symmetries in the k and l directions. By applying symmetries in the k direction to where $Q(k,l)$ is a solution of

$$Q(k+1,l) - Q(k,l+1) + A(k,l) = 0,$$

and where

$$\bar{B}(k,l) = B(k,l) + Q(k+1,l+1) - Q(k,l).$$

This simplification greatly speeds up the computations, without affecting the number of independent conservation laws that exist. When the direct method is used we find that

$$\begin{aligned} F &= 8\,u_{0,0}^2u_{0,1}\zeta(k,l) \\ &+ 4\,u_{0,1}\left(u_{0,1}u_{0,0}^2\nu(k,l) + 2\,u_{0,1}u_{0,0}\xi(k,l) + 4\,\mu(k,l)u_{0,0} - 2\,\zeta(k,l+1)C(k,l) + 2\,C(k,l+1)\zeta(k,l+1)\right) \\ &+2\,\frac{u_{0,1}T\left(\nu(k,l)C(k,l+1) - \nu(k,l)C(k,l) + 2\,\zeta(k,l+1)u_{0,1} + 2\,\mu(k,l+1) - 2\,\mu(k,l)\right)}{\zeta(k,l)} + \frac{\nu(k,l)u_{0,1}^2T^2}{\zeta(k,l)^2}, \end{aligned}$$

$$\begin{aligned} G &= -8\,u_{0,0}\left(u_{1,0}u_{0,0} - C(k,l)\right)\zeta(k,l) \\ &- 4\,u_{1,0}\left(2\,\zeta(k,l+1)C(k,l) + 4\,\mu(k,l)u_{0,0} + u_{1,0}\nu(k,l)u_{0,0}^2 - 2\,\nu(k,l)C(k,l)u_{0,0} + 2\,u_{1,0}\xi(k,l)u_{0,0}\right) \\ &-2\,\frac{u_{1,0}T\left(\nu(k,l)C(k,l) + \nu(k,l)C(k,l+1) - 2\,\mu(k,l) + 2\,u_{1,0}\zeta(k,l+1) + 2\,\mu(k,l+1)\right)}{\zeta(k,l)} - \frac{\nu(k,l)u_{1,0}^2T^2}{\zeta(k,l)^2}, \end{aligned}$$

where $T = \nu(k,l)C(k,l) - \nu(k,l)C(k,l+1) - 2\,\mu(k,l) - 2\,\mu(k,l+1)$. Here $\xi(k,l), \mu(k,l), \nu(k,l)$ and $\zeta(k,l)$ are functions which satisfy the following constraints:

$$\xi(k+1,l) = -\zeta(k,l), \quad \zeta(k+1,l) = \zeta(k,l+1), \tag{5.3}$$

$$\nu(k+1,l) = -\nu(k,l), \quad \nu(k,l+1) = -\nu(k,l), \tag{5.4}$$

$$2\xi(k,l)\zeta(k,l) + 2\zeta(k,l+1)\zeta(k,l) = -4\nu(k,l)\mu(k,l) - \nu(k,l)^2C(k,l), \tag{5.5}$$

$$\bar{B}(k,l) = -\frac{4\mu(k,l) + \nu(k,l)C(k,l)}{2\zeta(k,l)}. \tag{5.6}$$

Note: at this stage, we have not completed the direct method calculation of the conservation laws, but the above necessary conditions lead to a substantial further simplification of the problem.

The general solution of the system (5.3), (5.4) is

$$\zeta(k,l) = H(k+l), \quad \xi(k,l) = -H(k+l-1), \quad \nu(k,l) = c_1(-1)^{k+l}, \tag{5.7}$$

a conservation law with component F in the k direction we also obtain a conservation law with component F in the k direction. In this way we might construct an infinite hierarchy of conservation laws with component F in the k direction and another hierarchy of conservation laws with component G in the l direction.

5.2 Conservation laws for generalized dKdV and dpmKdV equations

In this section, we use the existence of three-point conservation laws as an indicator for integrability for the generalized nonautonomous dKdV equation

$$(u_{1,1} - u_{0,0} + B(k,l))(u_{1,0} - u_{0,1} + A(k,l)) = C(k,l),$$

and the generalized nonautonomous dpmKdV equation

$$u_{1,1} = u_{0,0}\frac{[A(k,l)\ u_{1,0} - B(k,l)u_{0,1}]}{[u_{0,1} - C(k,l)\ u_{1,0}]}.$$

Namely, we are looking for conservation laws for the generalized nonautonomous dKdV and dpmKdV equations by the direct method, which is explained in Section 2.1. During the computation we obtain constraints on coefficients of equations which must be satisfied if nontrivial three-point conservation laws exist.

5.2.1 The generalized nonautonomous dKdV equation

The generalization of the nonautonomous dKdV equation

$$(u_{1,1} - u_{0,0} + B(k,l))(u_{1,0} - u_{0,1} + A(k,l)) = C(k,l), \tag{5.1}$$

where $B(k,l), A(k,l)$ and $C(k,l)$ are arbitrary functions, can be transformed to

$$(u_{1,1} - u_{0,0} + \bar{B}(k,l))(u_{0,1} - u_{1,0}) = C(k,l), \tag{5.2}$$

by the point transformation

$$u(k,l) \mapsto u(k,l) + Q(k,l),$$

where $H(k+l)$ is an arbitrary nonzero function and c_1 is an arbitrary nonzero constant. Combining these results with (5.5), and (5.6), we obtain

$$\bar{B}(k,l) = \frac{2}{c_1}(-1)^{k+l}(H(k+l+1) - H(k+l-1)).$$

Therefore three-point conservation laws exist only if the nonautonomous dKdV equation is of the form

$$\left(u_{1,1} - u_{0,0} + \frac{2}{c_1}(-1)^{k+l}[H(k+l+1) - H(k+l-1)]\right)(u_{0,1} - u_{1,0}) = C(k,l). \quad (5.8)$$

This equation is mapped by the point transformation

$$u(k,l) \mapsto u(k,l) - \frac{2}{c_1}(-1)^{k+l}H(k+l-1)$$

to

$$(u_{1,1} - u_{0,0})(u_{0,1} - u_{1,0}) = C(k,l). \quad (5.9)$$

Therefore it is enough to seek conservation laws of (5.9). Applying the full direct method to (5.9) gives us one further condition on $C(k,l)$:

$$C(k+1,l+1) - C(k+1,l) - C(k,l+1) + C(k,l) = 0.$$

Consequently all nonautonomous dKdV equations that have nontrivial three-point conservation laws can be mapped to

$$(u_{1,1} - u_{0,0})(u_{1,0} - u_{0,1}) = f(k) - g(l). \quad (5.10)$$

5.2.2 The generalized nonautonomous dpmKdV equation

In the same way, we apply the direct method to the generalization of the nonautonomous dpmKdV equation

$$u_{1,1} = u_{0,0}\frac{[A(k,l)\ u_{1,0} - B(k,l)u_{0,1}]}{[u_{0,1} - C(k,l)\ u_{1,0}]}, \quad (5.11)$$

where $A(k,l)$, $B(k,l)$ and $C(k,l)$ are arbitrary functions. Then the components F and G are of the form

$$F = \nu(k,l)u_{0,0}u_{0,1} - \xi(k,l)\frac{u_{0,1}}{u_{0,0}} - \zeta(k,l)\frac{u_{0,0}}{u_{0,1}} + \frac{\mu(k,l)}{u_{0,0}u_{0,1}},$$

$$G = \xi(k,l)C(k,l)\frac{u_{1,0}}{u_{0,0}} - \nu(k,l)\frac{A(k,l)}{B(k,l)}u_{0,0}u_{1,0} + \zeta(k,l)\frac{B(k,l)}{A(k,l)}\frac{u_{0,0}}{u_{1,0}} - \frac{\mu(k,l)}{C(k,l)u_{0,0}u_{1,0}},$$

where the functions ξ, μ, ν and ζ satisfy the constraints

$$\xi(k+1,l)\zeta(k+1,l) = \xi(k,l)\zeta(k,l), \tag{5.12}$$

$$\mu(k+1,l)\nu(k+1,l) = \mu(k,l)\nu(k,l), \tag{5.13}$$

$$C(k,l+1)/C(k,l) = \xi(k+1,l)/\xi(k,l+1), \tag{5.14}$$

$$A(k,l) = \zeta(k+1,l)/\xi(k,l), \tag{5.15}$$

$$\xi(k,l+1)\mu(k,l+1) = \mu(k,l)\zeta(k,l), \tag{5.16}$$

$$\nu(k,l+1)\zeta(k,l+1) = \xi(k,l)\nu(k,l), \tag{5.17}$$

$$\nu(k+1,l)B(k,l) = \nu(k,l)C(k,l). \tag{5.18}$$

From (5.12), (5.13) and (5.16), (5.17) we have

$$\begin{cases} \xi(k+1,l)\mu(k+1,l)\nu(k+1,l)\zeta(k+1,l) = \xi(k,l)\mu(k,l)\nu(k,l)\zeta(k,l), \\ \xi(k,l+1)\mu(k,l+1)\nu(k,l+1)\zeta(k,l+1) = \xi(k,l)\mu(k,l)\nu(k,l)\zeta(k,l), \end{cases}$$

$$\Rightarrow \ \xi(k,l)\mu(k,l)\nu(k,l)\zeta(k,l) = c, \tag{5.19}$$

where c is a constant. Also the conditions (5.12) gives

$$\xi(k,l)\zeta(k,l) = H(l). \tag{5.20}$$

Here $H(l)$ is an arbitrary function. So, from (5.19) we have

$$\mu(k,l)\nu(k,l) = \frac{c}{H(l)}. \tag{5.21}$$

Define a new function $\psi(k,l)$ by

$$\xi(k,l) = \frac{\psi(k,l+1)}{\psi(k,l)}. \tag{5.22}$$

Note that $\psi(k,l)$ is defined up to arbitrary factor that is a function of k only. Then (5.14) gives

$$C(k,l) = \frac{\psi(k+1,l)}{\psi(k,l+1)}. \tag{5.23}$$

From (5.15), (5.20) and (5.22) we have

$$A(k,l) = \frac{H(l)}{\xi(k,l)\xi(k+1,l)} = H(l)\frac{\psi(k,l)\psi(k+1,l)}{\psi(k,l+1)\psi(k+1,l+1)}, \tag{5.24}$$

and from (5.17), (5.21) and (5.22)

$$H(m+1)\nu(k,l+1) = \xi(k,l)\xi(k,l+1)\nu(k,l) = \frac{\nu(k,l)\psi(k,l+2)}{\psi(k,l)}. \tag{5.25}$$

Let $G(l)$ be a solution of

$$\frac{G(l+1)}{G(l)} = H(l+1),$$

so $G(l)$ is defined up to an arbitrary constant factor. Then (5.25) yields

$$\frac{G(l+1)\nu(k,l+1)}{\psi(k,l+1)\psi(k,l+2)} = \frac{G(l)\nu(k,l)}{\psi(k,l)\psi(k,l+1)} \Rightarrow \nu(k,l) = \frac{\psi(k,l)\psi(k,l+1)}{G(l)F(k)}. \tag{5.26}$$

Here $F(k)$ is an arbitrary function. From (5.21) and (5.20) we have

$$\mu(k,l) = c\frac{F(k)G(m-1)}{\psi(k,l)\psi(k,l+1)}, \quad \zeta(k,l) = \frac{G(l)\psi(k,l)}{G(l-1)\psi(k,l+1)}.$$

So far, we have written $\xi(k,l)$, $\mu(k,l)$, $\nu(k,l)$, $\zeta(k,l)$ in terms of $\psi(k+i,l+j)$ and the arbitrary functions $F(k)$, $G(l)$. The condition (5.23) gives $C(k,l)$ in terms of $\psi(k+i,l+j)$. The identity (5.24) can written as

$$A(k,l) = \frac{G(l)\psi(k,l)\psi(k+1,l)}{G(l-1)\psi(k,l+1)\psi(k+1,l+1)}.$$

From (5.18), (5.23) and (5.26), we obtain

$$B(k,l) = \frac{\nu(k,l)}{\nu(k+1,l)}C(k,l) = \frac{F(k+1)\psi(k,l)}{F(k)\psi(k+1,l+1)}.$$

Therefore, the only equations with nonzero A, B and C that admit three-point conservation laws are of the form:

$$u_{1,1} = u_{0,0}\frac{\psi(k,l)}{\psi(k+1,l+1)}\left(\frac{g(l)^2\psi(k+1,l)u_{1,0} - f(k)^2\psi(k,l+1)u_{0,1}}{\psi(k,l+1)u_{0,1} - \psi(k+1,l)u_{1,0}}\right), \tag{5.27}$$

where $g(l)^2 = \frac{G(l)}{G(l-1)}$ and $f(k)^2 = \frac{F(k+1)}{F(k)}$. The transformation

$$u_{0,0} \mapsto u_{0,0}\frac{\delta(l)\gamma(k)}{\psi(k,l)},$$

where $\frac{\delta(l+1)}{\delta(l)} = g(l)$ and $\frac{\gamma(k+1)}{\gamma(k)} = f(k)$, reduces (5.27) to the standard dpmKdV equation

$$u_{1,1} = u_{0,0}\frac{[g(l)u_{1,0} - f(k)u_{0,1}]}{[g(l)u_{0,1} - f(k)u_{1,0}]}.$$

5.3 Similarity solutions

One way of finding similarity solutions of quad-graph equations was considered in [48, 50]. In [6, 7, 13, 15] geometrical aspects are discussed; these papers provide a clear introduction to reduction. The authors define three initial points, from which they construct symmetric initial conditions for a given quad-graph equation. This approach typically yields an integrable map; discrete Painlevé equations can be generated in this way.

Another approach is to reduce the number of variables by requiring that the solution is invariant under the symmetries generated by a characteristic. This method is widely used for continuous systems [33, 57], and has been applied to the dKdV equation in [73].

We shall illustrate the method by seeking nonzero solutions of $\mathbf{H3}_{\delta=0}$ that are invariant under the symmetries generated by

$$Q = Q_4 - aQ_2 = \frac{u_{0,0}(u_{1,0} + u_{-1,0})}{u_{1,0} - u_{-1,0}} - au_{0,0}, \qquad a > 1.$$

The general solution of the invariance condition $Q = 0$ (with $u_{0,0} \neq 0$) is

$$u_{0,0} = (f_1(l) + (-1)^k f_2(l)) \left(\sqrt{\frac{a+1}{a-1}}\right)^k, \tag{5.28}$$

where f_1 and f_2 are arbitrary functions. By substituting (5.28) into $\mathbf{H3}_{\delta=0}$ we find that

$$f_1(l) = c_1(-1)^l f_2(l),$$

where c_1 is an arbitrary constant. If $c_1^2 \neq 1$ then f_2 satisfies the following ordinary difference equation:

$$\alpha\sqrt{a^2-1}\left(\frac{f_2(l+1)}{f_2(l)}\right)^2 + 2a\beta\frac{f_2(l+1)}{f_2(l)} + \alpha\sqrt{a^2-1} = 0, \tag{5.29}$$

which yields a large family of exact solutions, including

$$u_{0,0} = (\bar{c}_1 + c_2(-1)^{k+l}) \left(\sqrt{\frac{a+1}{a-1}}\right)^k \left(\frac{a\beta + \sqrt{\alpha^2 + a^2\beta^2 - a^2\alpha^2}}{\alpha\sqrt{a^2-1}}\right)^l,$$

where $\bar{c}_1 = c_1 c_2$.

If $c_1 = \pm 1$ then there are no further constraints, so there are two families of invariant solutions

$$u_{0,0} = f_2(l)\left((-1)^l \pm (-1)^k\right)\left(\sqrt{\frac{a+1}{a-1}}\right)^k. \tag{5.30}$$

These belong to the following degenerate class of solutions of $\mathbf{H3}_{\delta=0}$:

$$u_{0,0} = F(k,l)\left((-1)^l \pm (-1)^k\right), \tag{5.31}$$

where F is an arbitrary function.

Chapter 6

Conclusion

6.1 Conservation laws

All presented results are obtained with the help of Hydon's method [34]. This method is a practical way of determining the conservation laws of a given form. The method of invariant differentiation enables the user to obtain closed form solutions of the determining equations. Once these solutions have been found, the reconstruction of the conservation law is usually easy. The most complicated part of the technique is the derivation of PDEs by invariant differentiation, but this is not difficult if a reliable computer algebra system is used.

All three-point conservation laws for all equations from NQC equation and the ABS classification have been found. For each of the equations from the ABS classification we found three five-point conservation laws. The effectiveness of Hydon's direct method for constructing conservation laws [34] has been improved by using two reductions to PDEs instead of only one, and by using commuting differential operators to begin the reduction. We have used this method as far as possible to calculate conservation laws, as this guarantees that all conservation laws of a particular type have been found. However, for all but two of the ABS equations, it was necessary to supplement the direct method with extra hypotheses, based on the results that we had obtained so far. This hybrid approach led to the discovery that each of the ABS equations (for constant α and β) has three five-point conservation laws. If only one of α and β is constant then we can find only one five-point conservation law. It seems likely that these are the only five-point conservation laws, but we cannot yet be certain that this is so.

A technique which generates a conservation law from a known one was shown. So far it is the only technique which may give an infinite number of conservation laws. This technique is easy to use, but it does not guarantee that new conservation law cannot be reduced to a known or trivial one. Therefore we cannot say that ABS equations have an infinite number of conservation laws.

An example of conservation laws for a Toda type system is presented. The connection between these conservation laws and symmetries is shown.

We have shown that construction of conservation laws provides an efficient tool to investigate integrable nonautonomous nonlinear partial difference equations. We used these methods to find integrable nonautonomous versions of the dKdV and dpmKdV equations.

ABS equations that are nonautonomous do not admit five-point conservation laws with G in the k direction when α is not constant; similarly there are no five-point conservation laws with F in the l direction when β is not constant. A similar situation arises for symmetries: there are no five-point symmetries in the k (respectively l) direction when α (respectively β) is not constant. Recently Levi and coworkers used isospectral and nonisospectral deformations of the Lax pair to find symmetries for two of the ABS equations [38, 39]. They found that the local mastersymmetries disappear if the equations are nonautonomous. If the same is true for the other nonautonomous ABS equations, this may well be the reason why there are no five-point conservation laws.

We show the program for deriving three-point conservation laws for quad-graph equations.

6.2 Symmetries

The result of this work is the derivation of the complete set of symmetries on a five-point cross for equations from the ABS classification. We found all symmetries by a generalization of the method which is described in [32]. This confirms that this method can be used in a systematic way without making restrictive assumptions about the form of symmetries.

The symmetries that we have found have various applications. For instance, symmetries can be used to obtain group-invariant reductions that lead to exact solutions of the quad-graph equations. We have only considered a single example of such a reduction (for $\mathbf{H3}_{\delta=0}$). However we have shown that all ABS equations have infinitely many symmetries, any of which could be used to construct invariant solutions. Five-point and other higher symmetries can also be used for the generation of new conservation laws. It is notable that all equations from [5] have four five-point symmetries that have similar forms.

Mastersymmetries for integrable equations on the quad-graph have been derived. These mastersymmetries allow us to construct infinite hierarchies of local symmetries. It is important to allow mastersymmetries to act on α and β; otherwise the mastersymmetries for **Q3** and **Q4** would not have been found. The existence of mastersymmetries shows the similarity of structures for continuous and difference equations.

We have discussed the relation between the symmetries of quad-graph equations and symmetries for Toda type systems. We have also verified that for each symmetry of the integrable quad-graph equation there is a corresponding symmetry of the related Toda type system. It is not yet known whether this relationship is true for all symmetries of integrable quad-graph equations.

A program for deriving five-point symmetries for quad-graph equations is presented.

Our results show that the set of symmetries of integrable quad-graphs has similar features to the continuous integrable case. It is likely that the same is true for conservation laws.

A general question that arises from Chapters 2 and 3 is how conservation laws and symmetries for **Q4** coalesce to those of the other ABS equations in appropriate limits of the parameters.

There are large classes of difference equations which have not been considered in this thesis, such as multicomponent quad-graph equations or equations that involve more points (for instance the Boussinesq lattice equation which involves 9 points). Looking for conservation laws and symmetries for these classes of equations would be a good topic for future research.

The results presented in this thesis have been published in [65, 66, 67, 68, 71].

Appendix A

Maple programs

A.1 Program for deriving three-point conservation laws for quad-graph equations.

This program helps the user to find three-point conservation laws for quad-graph equations. These conservation laws satisfy a functional equation, which we solve by reducing it to a system of partial differential equations. The theoretical part of the program was explained before.

Packages that we use in this worksheet

```
> restart;
> with(DEtools,rifsimp):
> with(LREtools):
```

Input the equation which must depend upon all four values $u[k,l], u[k+1,l], u[k,l+1], u[k+1,l+1]$. The equation should be linear with respect to $u[k+1,l], u[k+1,l+1]$ and polynomial with respect to $u[k,l], u[k,l+1]$. The equation may involve constants or functions, which should be written $\alpha, \beta,$... or $\alpha[k,l], \beta[k,l]$,..., as appropriate.

```
> eq:=(u[k+1,l+1]-u[k,l])*(u[k+1,l]-u[k,l+1])+(beta-alpha)*(u[k+1,l+1]+u[k+1,l]+u[k
,l+1]+u[k,l])+beta^2-alpha^2;
> U[k+1,l]:=u[k+1,l]=solve(eq,u[k+1,l]):
> U[k+1,l+1]:=u[k+1,l+1]=solve(eq,u[k+1,l+1]):
```

$$eq := (u_{k+1,l+1} - u_{k,l})(u_{k+1,l} - u_{k,l+1}) + (\beta - \alpha)(u_{k+1,l+1} + u_{k+1,l} + u_{k,l+1} + u_{k,l}) + \beta^2 - \alpha^2$$

Components of three-point conservation laws

```
> f:=F(u[k,l],u[k,l+1]);
> g:=G(u[k,l],u[k+1,l]);
```

$$f := F(u_{k,l}, u_{k,l+1})$$

$$g := G(u_{k,l}, u_{k+1,l})$$

We provide a procedure which differentiates A with respect to C until A is not a polynomial in B. For instance if an expression A is a polynomial in two functions $P1(C)$, $P2(B)$ then an expression that does not involve B can be obtained by repeated differentiation with respect to C.

```
> SSplit:=proc(A,B,C)
> local POL;
> POL:=A:
> while(type(POL, polynom(anything,B))=false) do
> POL:=numer(simplify(diff(POL,C)));
> end do;
> return(POL);
> end proc:
```

The three-point conservation laws can be determined directly by substituting the quad-graph equation eq into $R = 0$, where

```
> R:=f-shift(f,k)+g-shift(g,l);
```

$$R := F(u_{k,l}, u_{k,l+1}) - F(u_{k+1,l}, u_{k+1,l+1}) + G(u_{k,l}, u_{k+1,l}) - G(u_{k,l+1}, u_{k+1,l+1})$$

and solving the resulting functional equation. In order to solve this functional equation we have to reduce it to a system of partial differential equations. To do this, first eliminate functional terms by applying (commuting) differential operators $L[1], L[2]$. After that we obtain equation which has two unknown functions with different arguments. In order to eliminate one of the functions we use the procedure *SSplit*. We repeat all this process two times, first when we solve *eq* with respect to $u[k+1,l]$ and plug it into R and second when we solve *eq* with respect to $u[k+1,l+1]$ and plug it into R. We derive a system of PDE's on $F(u[k,l], u[k,l+1])$

```
> for j from 0 by 1 to 1 do
> assign(U[k+1,l+j]);
> L[1]:=A->diff(A,u[k,l+j])-diff(u[k+1,l+j],u[k,l+j])/diff(u[k+1,l+j],u[k,l+1-j])*
diff(A,u[k,l+1-j]):
> L[2]:=A->diff(A,u[k+1,l+1-j])-diff(u[k+1,l+j],u[k+1,l+1-j])/diff(u[k+1,l+j],u[k,
l+1-j])*diff(A,u[k,l+1-j]):
> R2:=L[2](L[1](R)):
> unassign('u[k+1,l+j]'):
> R3:=numer(simplify(R2)):
> R4:=SSplit(R3,u[k+1,l+1-j],u[k,l+j]);
> u[k,l]:=a:u[k,l+1]:=b:
> R5:=numer(primpart(R4,[D[1,2](F)(a,b),D[1,2,2](F)(a,b),D[1,1,2](F)(a,b)])):
> sys[j]:=coeffs(R5,u[k+1,l+1-j]);
> unassign('u[k,l]','u[k,l+1]');
> end do:
> sys1:=convert({sys[0],sys[1]},diff):
```

The order of this system can be decreased by two with help of the substitution

```
> F(a,b):=int(int(F1(a,b),a),b):
> sys2:=simplify(sys1):
```

The resulting equations are easy to solve

```
> RS:=rifsimp(convert(sys2,diff)):
> sol:=simplify(convert(pdsolve(RS['Solved']),int));
> assign(sol):
```

$$sol := \left\{ F1\,(a,b) = \frac{\_C2\,\alpha^2 + 2\,\_C2\,\alpha\,a + 2\,\_C2\,\alpha\,b + \_C2\,a^2 + 2\,\_C2\,ab + \_C2\,b^2 + \_C1}{(\alpha+a+b)^2} \right\}$$

It can cause some problems if this solution has arbitrary functions; do not expect the program to find conservation laws in this case. However the preliminary form of $F1(a,b)$ can be used for further calculations which can be done with input from the user. By integrating back we obtain two arbitrary functions $FA(a)$ and $FB(b)$. The function $FA(a)$ can be removed (without loss of generality) by adding the trivial conservation law

```
> F(a,b):=int(int(F1(a,b),a),b)+FB(b);
> F:=unapply(F(a,b),a,b):
```

$$F(a,b) := \_C2\,ab - \_C1\,\ln(\alpha + a + b) + FB(b)$$

So far we did not use the fact that f explicitly depends on k, l. Now we redefine constants to make them depend upon k, l.

```
> f:=subs(_C1=C1[k,l],_C2=C2[k,l],_C3=C3[k,l],_C4=C4[k,l],(F(u[k,l],u[
k,l+1]))):
```

Now we derive and solve the system of PDE's for $G(u[k,l], u[k,l+1])$. The expression for $F(u[k,l], u[k,l+1])$ which we just obtained is used to derive an equation for $G(u[k,l], u[k,l+1])$. That is why the following procedure is not so difficult as for $F(u[k,l], u[k,l+1])$. We use just one differential operator $L[1]$ in order to obtain a PDE

```
> R:=f-shift(f,k)+g-shift(g,l):
> assign(U[k+1,l+1]):
> L[1]:=A->diff(A,u[k+1,l])-diff(u[k+1,l+1],u[k+1,l])/diff(u[k+1,l+1],u[k,l])*diff
(A,u[k,l]):
> R1:=simplify(L[1](R)):
> unassign('u[k+1,l+1]'):
> R2:=primpart(SSplit(numer(R1),u[k,l+1],u[k+1,l]),D[2](G)(u[k,l],u[k+1,l])):
> sys1:=coeffs(R2,u[k,l+1]):
> RS2:=rifsimp(convert({sys1},diff)):
> so:=simplify(convert(pdsolve(select(has,RS2['Solved'],G(u[k,l],u[k+1,l]))),int));
> assign(so):
```

$$sol2 := \{G(u_{k,l}, u_{k+1,l}) = -C2_{k,l}\alpha\, u_{k,l} - u_{k,l}C2_{k,l}u_{k+1,l} + \beta\, C2_{k,l}u_{k,l} + C1_{k,l}\ln(u_{k+1,l} + u_{k,l} + \beta) + C2_{k,l}\beta\, u_{k+1,l} - C2_{k,l}\alpha\, u_{k+1,l} + \_C1\}$$

So far we did not use the fact that g explicitly depends on k, l. Again we redefine constants to make them depend upon k, l.

```
> g:=subs(_C1=C5[k,l],_C2=C6[k,l],_C3=C7[k,l],_C4=C8[k,l],(G(u[k,l],u[k+1,l]))):
```

Now we derive equations for $Ci[k,l], i = 1..8$ and for $FB(u[k,l+1])$

```
> for j from 0 by 1 to 1 do
> R:=f-shift(f,k)+g-shift(g,l);
> assign(U[k+1,l+1]);
> L[1]:=A->diff(A,u[k+j,l+1-j])-diff(u[k+1,l+1],u[k+j,l+1-j])/diff(u[k+1,l+1],u[k,
l])*diff(A,u[k,l]):
> R1:=numer(simplify(L[1](R))):
> unassign('u[k+1,l+1]'):
> R2:=numer(simplify(primpart(R1,{C1[k,l],C2[k,l],C3[k,l]}))):
> sys[j]:=coeffs(R2,{u[k,l],u[k+1,l+1],u[k+1-j,l+j],u[k+1,l]});
> end do:
> sys3:=convert({sys[0],sys[1]},diff):
```

These equations can be simplified to

```
> RS3:=rifsimp(sys3);
> sl:=subs(_C1=0,_C2=0,dsolve(select(has,RS3['Solved'],FB(u[k,l+1])))):
> assign(sl):
```

$$RS3 := TABLE([Solved = [\frac{d}{du_{k,l+1}} FB(u_{k,l+1}) = 0, C1_{k+1,l} = -C1_{k,l}, C1_{k,l+1} = -C1_{k,l}, C2_{k+1,l} = -C2_{k,l}, C2_{k,l+1} = -C2_{k,l}]])$$

The final form of f and g is

```
> f:=simplify(combine(f,ln,symbolic));
> g:=simplify(combine(g,ln,symbolic));
```

$$f := C2_{k,l} u_{k,l} u_{k,l+1} - C1_{k,l} \ln(\alpha + u_{k,l} + u_{k,l+1})$$

$$g := -C2_{k,l} \alpha u_{k,l} - u_{k,l} C2_{k,l} u_{k+1,l} + \beta C2_{k,l} u_{k,l} + C1_{k,l} \ln(u_{k+1,l} + u_{k,l} + \beta) + C2_{k,l} \beta u_{k+1,l} - C2_{k,l} \alpha u_{k+1,l} + C5_{k,l}$$

We obtained expressions for functions f and g. $Ci[k,l], i = 1..8$ can be derived from $RS3$ and by direct substitution for f and g in $f - shift(f,k) + g - shift(g,l) =$

0. When this equation is satisfied the only remaining unknowns are the constants that multiply each conservation law. Pivots in $RS3$ should be taken into account because they can lead to special cases.

A.2 Program for deriving five-point symmetries for quad-graph equations.

This program helps the user to find five-point symmetries for quad-graph equations. Symmetry characteristics satisfy a functional equation, which we solve by reducing it to a system of partial differential equations. The theoretical part of the program was explained before.

Packages that we use in this worksheet:

```
> restart;
> with(LREtools):
> with(DEtools,rifsimp):
```

Input the equation which must depend upon all four values $u[k,l], u[k+1,l], u[k,l+1], u[k+1,l+1]$. The equation should be linear with respect to $u[k+1,l], u[k+1,l+1]$ and polynomial with respect to $u[k,l], u[k,l+1]$. The equation may involve constants or functions, which should be written $\alpha, \beta,...$ or $\alpha[k,l], \beta[k,l],...$, as appropriate.

```
> eq:=alpha*(u[k,l]*u[k+1,l]+u[k,l+1]*u[k+1,l+1])-beta*(u[k,l]*u[k,l+1]+u[k+1,l]*u[
k+1,l+1])+(alpha^2-beta^2);
> U[k+1,l+1]:=u[k+1,l+1]=solve(eq,u[k+1,l+1]):
> U[k,l]:=u[k,l]=solve(eq,u[k,l]):
> U[k+2,l+1]:=shift(u[k+1,l+1]=solve(eq,u[k+1,l+1]),k):
> U[k+1,l+2]:=shift(u[k+1,l+1]=solve(eq,u[k+1,l+1]),l):
> U[k-1,l]:=shift(u[k,l]=solve(eq,u[k,l]),k,-1):
> U[k,l-1]:=shift(u[k,l]=solve(eq,u[k,l]),l,-1):
> U[k+2,l]:=shift(u[k+1,l]=solve(eq,u[k+1,l]),k):
> U[k,l+2]:=shift(u[k,l+1]=solve(eq,u[k,l+1]),l):
> U[k-1,l+1]:=shift(u[k,l+1]=solve(eq,u[k,l+1]),k,-1):
> U[k+1,l-1]:=shift(u[k+1,l]=solve(eq,u[k+1,l]),l,-1):
```

$$eq := \alpha\,(u_{k,l}u_{k+1,l} + u_{k,l+1}u_{k+1,l+1}) - \beta\,(u_{k,l}u_{k,l+1} + u_{k+1,l}u_{k+1,l+1}) + \alpha^2 - \beta^2$$

Five-point symmetry characteristic:

```
> q:=Q1(u[k-1,l],u[k,l],u[k+1,l])+Q2(u[k,l-1],u[k,l],u[k,l+1]);
```

$$q := Q1\,(u_{k-1,l}, u_{k,l}, u_{k+1,l}) + Q2\,(u_{k,l-1}, u_{k,l}, u_{k,l+1})$$

The infinitesimal generator is

```
> X:=A->q*diff(A,u[k,l])+shift(q,k)*diff(A,u[k+1,l])+shift(q,l)*diff(A,u[k,l+1])+sh
ift(shift(q,l),k)*diff(A,u[k+1,l+1])+xi[2](beta)*diff(A,beta):
```

The linearized symmetry condition can be reduced to four PDE's.

```
> assign(U[k+1,l+1],U[k+2,l+1],U[k+1,l+2],U[k+1,l-1],U[k-1,l+1]):
> R21:=diff(-1/diff(u[k+1,l+1],u[k+1,l])*diff(Q1(u[k,l+1],u[k+1,l+1],u[k+2,l+1]),u[
k+2,l]),u[k,l+1]):
> R22:=diff(-1/diff(u[k+1,l+1],u[k,l+1])*diff(Q2(u[k+1,l],u[k+1,l+1],u[k+1,l+2]),u[
k,l+2]),u[k+1,l]):
> R23:=diff(diff(u[k+1,l+1],u[k+1,l])/diff(u[k+1,l+1],u[k,l])*diff(Q2(u[k+1,l-1],u[
k+1,l],u[k+1,l+1]),u[k,l-1]),u[k+1,l]):
> R24:=diff(diff(u[k+1,l+1],u[k,l+1])/diff(u[k+1,l+1],u[k,l])*diff(Q1(u[k-1,l+1],u[
k,l+1],u[k+1,l+1]),u[k-1,l]),u[k,l+1]):
> unassign('u[k+1,l+1]'):unassign('u[k+2,l+1]'):unassign('u[k+1,l+2]'):unassign('u[
k-1,l+1]'):unassign('u[k+1,l-1]'):
> assign(U[k,l],U[k+2,l],U[k,l+2],U[k-1,l],U[k,l-1]):
> R31:=primpart(numer(simplify(R21)),D[3](Q1)(u[k,l+1],u[k+1,l+1],u[k+2,l+1])):
> R32:=primpart(numer(simplify(R22)),D[3](Q2)(u[k+1,l],u[k+1,l+1],u[k+1,l+2])):
> R33:=primpart(numer(simplify(R23)),D[1](Q2)(u[k+1,l-1],u[k+1,l],u[k+1,l+1])):
> R34:=primpart(numer(simplify(R24)),D[1](Q1)(u[k-1,l+1],u[k,l+1],u[k+1,l+1])):
> unassign('u[k,l]'):unassign('u[k+2,l]'):unassign('u[k,l+2]'):unassign('u[k-1,l]')
:unassign('u[k,l-1]'):
```

Each of these equations can be split into a system of PDE's. After splitting of all the equations we obtain

```
> sys1:=coeffs(shift(shift(R31,k,-1),l,-1),{u[k,l+1],u[k-1,l+1],u[k+1,l-1],u[k+1,l+
1]}):
```

```
> sys2:=coeffs(shift(shift(R32,k,-1),l,-1),{u[k+1,l],u[k-1,l+1],u[k+1,l-1],u[k+1,l+
1]}):
> sys3:=coeffs(shift(R33,k,-1),{u[k+1,l],u[k-1,l+1],u[k+1,l-1],u[k+1,l+1]}):
> sys4:=coeffs(shift(R34,l,-1),{u[k,l+1],u[k-1,l+1],u[k+1,l-1],u[k+1,l+1]}):
> SYS:={sys1,sys2,sys3,sys4}:
```

The solution of this system is obtained after simplification with *rifsimp*.

```
> RS1:=rifsimp(convert(SYS,diff)):
> sol1:=convert(pdsolve(RS1['Solved']),list);
```

$$sol1 := [Q2\left(u_{k,l-1}, u_{k,l}, u_{k,l+1}\right) = \frac{\left(-u_{k,l-1} + u_{k,l+1}\right) \_F6\left(u_{k,l}\right) + \_C2\ \left(\beta + u_{k,l-1}u_{k,l}\right)}{-u_{k,l-1} + u_{k,l+1}},$$
$$Q1\left(u_{k-1,l}, u_{k,l}, u_{k+1,l}\right) = \frac{\left(-u_{k+1,l} + u_{k-1,l}\right) \_F3\left(u_{k,l}\right) - \_C1\ \left(\alpha + u_{k-1,l}u_{k,l}\right)}{-u_{k+1,l} + u_{k-1,l}}]$$

Further calculations very much depend on $Q1$ and $Q2$. The next procedure helps us to find unknown functions in $Q1$ and $Q2$. It works in the case when each of $Q1$ and $Q2$ depend upon one unknown function of $u[k,l]$. So far we did not use the fact that q explicitly depends on k, l. Now we redefine constants to make it depend upon k, l. Without loss of generality one of unknown functions in $Q1$ and $Q2$ is zero and another is denoted as $F1$

```
> assign(subs(_C1=C1[k,l],_C2=C2[k,l],_C3=C3[k,l],_C4=C4[k,l],_C5=C5[k,l],_C6=C6[k,
l],_F1(u[k,l])=0,_F2(u[k,l])=0,_F3(u[k,l])=0,_F4(u[k,l])=0,_F5(u[k,l])=0,_F6(u[k,l])
=0,_F7(u[k,l])=0,_F8(u[k,l])=0,_F9(u[k,l])=0,sol1[1]));
> assign(subs(_C1=C1[k,l],_C2=C2[k,l],_C3=C3[k,l],_C4=C4[k,l],_C5=C5[k,l],_C6=C6[k,
l],_F1(u[k,l])=F1(u[k,l]),_F2(u[k,l])=F1(u[k,l]),_F3(u[k,l])=F1(u[k,l]),_F4(u[k,l])=
F1(u[k,l]),_F5(u[k,l])=F1(u[k,l]),_F6(u[k,l])=F1(u[k,l]),_F7(u[k,l])=F1(u[k,l]),_F8(
u[k,l])=F1(u[k,l]),_F9(u[k,l])=F1(u[k,l]),sol1[2]));
> 'q'=q;
```

$$q = \frac{\left(-u_{k+1,l} + u_{k-1,l}\right) F1\left(u_{k,l}\right) - C1_{k,l}\left(\alpha + u_{k-1,l}u_{k,l}\right)}{-u_{k+1,l} + u_{k-1,l}} + \frac{C2_{k,l}\left(\beta + u_{k,l-1}u_{k,l}\right)}{-u_{k,l-1} + u_{k,l+1}}$$

By differentiating the linearized symmetry condition we obtain intermediate conditions which simplify our calculations

```
> R:=X(u[k+1,l+1]-solve(eq,u[k+1,l+1])):
> assign(U[k+1,l+1],U[k+2,l+1],U[k+1,l+2],U[k+1,l-1],U[k-1,l+1]):
> RR:=simplify(R):
> R1:=primpart(numer(simplify(diff(R,u[k+2,l])))),{C1[k+1,l],C2[k+1,l],C3[k+1,l],C4[
k+1,l],C5[k+1,l],C6[k+1,l]});
> R2:=primpart(numer(simplify(diff(R,u[k,l+2])))),{C1[k,l+1],C2[k,l+1],C3[k,l+1],C4[
k,l+1],C5[k,l+1],C6[k,l+1]}):
> unassign('u[k+1,l+1]'):unassign('u[k+2,l+1]'):unassign('u[k+1,l+2]'):unassign('u[
k-1,l+1]'):unassign('u[k+1,l-1]'):
> sys:={coeffs(R2,{u[k,l],u[k,l+1],u[k+1,l],u[k-1,l],u[k,l-1],u[k,l+2]}),coeffs(R1,
{u[k,l],u[k,l+1],u[k+1,l],u[k-1,l],u[k,l-1],u[k+2,l]})};
> RS1:=rifsimp(convert({sys},diff));
> assign(RS1['Solved']):
```

$$R1 := -C1_{k+1,l} + C1_{k+1,l+1}$$

$$RS1 := TABLE([Solved = [C1_{k+1,l+1} = C1_{k+1,l},\, C2_{k+1,l+1} = C2_{k,l+1}]])$$

We find the function $F1$

```
> R11:=simplify('if'(has(RR,F1(u[k,l]))=true,diff(RR/coeff(RR,F1(u[k,l])),u[k,l+1])
,RR)):
> R12:=simplify('if'(has(R11,F1(u[k+1,l]))=true,diff(R11/coeff(R11,F1(u[k+1,l])),u[
k,l]),R11)):
> R13:='if'(has(R12,F1(u[k, l+1]))=true,diff(R12/coeff(R12,F1(u[k,l+1])),u[k,l]),R1
2):
> assign(U[k,l]):
> R2:=primpart(numer(simplify(R13)),[F1(u[k+1,l+1]),D(F1)(u[k+1,l+1]),'@@'(D, 2)(F1
)(u[k+1,l+1])]):
> unassign('u[k,l]'):
> sys3:=coeffs(shift(shift(R2,k,-1),l,-1),{u[k+1,l+1],u[k,l+1],u[k+1,l],u[k-1,l],u[
k,l-1]}):
> RS2:=rifsimp(convert({sys3},diff));
> sol:=subs(_C1=C7[k,l],_C2=C8[k,l],_C3=C9[k,l],_C4=C10[k,l],dsolve(RS2['Solved']));
> assign(sol):
```

$$RS2 := TABLE([Solved = [\frac{d^2}{du_{k,l}^2} F1\,(u_{k,l}) = 0],$$
$$Case = [[\beta \neq 0, \frac{d^2}{du_{k,l}^2} F1\,(u_{k,l})]], Pivots = [\beta \neq 0]])$$
$$sol := \{F1\,(u_{k,l}) = C7_{k,l} u_{k,l} + C8_{k,l}\}$$

Now we derive the final conditions on the unknown functions. This procedure works if q is a polynomial. All other cases should be considered individually.

```
> simplify(Q1(u[k-1,l],u[k,l],u[k+1,l])+Q2(u[k,l-1],u[k,l],u[k,l+1])):
> q:=simplify(%):
> R:=X(u[k+1,l+1]-solve(eq,u[k+1,l+1])):
> assign(U[k+1,l+1],U[k+2,l+1],U[k+1,l+2],U[k+1,l-1],U[k-1,l+1]):
> R1:=primpart(numer(simplify(R)),u[k,l]):
> sys1:=coeffs(R1,{u[k,l],u[k,l+1],u[k+1,l],u[k-1,l],u[k,l-1],u[k+2,l],u[k,l+2]}):
> RS3:=rifsimp(convert({sys1},diff)):
```

The final form of q with all conditions is

```
> 'q'=convert(convert(q,parfrac,u[k+1,l]),parfrac,u[k,l+1]);
> cond:=[RS1['Solved'],RS3['Solved']];
```

$$q = C7_{k,l} u_{k,l} + C8_{k,l} + \frac{C1_{k,l}\,(\alpha + u_{k-1,l} u_{k,l})}{u_{k+1,l} - u_{k-1,l}} + \frac{C2_{k,l}\,(\beta + u_{k,l-1} u_{k,l})}{-u_{k,l-1} + u_{k,l+1}}$$

$$cond := [[C1_{k+1,l} = C1_{k+1,l}, C2_{k,l+1} = C2_{k,l+1}],$$
$$[\xi_1\,(\alpha) = -\alpha\,\left(C2_{k,l} + C1_{k,l} - C7_{k,l} - C7_{k+1,l}\right),$$
$$\xi_2\,(\beta) = -\beta\, C2_{k,l} - \beta\, C1_{k,l} + \beta\, C7_{k,l} + \beta\, C7_{k,l+1},$$
$$C1_{k+1,l} = -C7_{k,l+1} + C7_{k+1,l+1} + C7_{k+1,l} + C1_{k,l} - C7_{k,l},$$
$$C1_{k,l+1} = C1_{k,l}, C2_{k+1,l} = C2_{k,l}, C8_{k+1,l+1} = 0, C8_{k+1,l} = 0, C8_{k,l+1} = 0,$$
$$C8_{k,l} = 0, C2_{k,l+1} = -C7_{k,l} - C7_{k+1,l} + C7_{k+1,l+1} + C7_{k,l+1} + C2_{k,l}]]$$

We obtained expressions for the function q. $Ci[k,l], i = 1..9$ can be derived from *cond*. Pivots in $RS3$ should be taken into account because they can lead to special cases.

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