The Ordinary Weight Conjecture and Dade's Projective Conjecture for p-Blocks with an Extra-Special Defect Group

by

Ahmad M. Alghamdi

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Abstract

Let p be a rational odd prime number, G be a finite group such that $|G| = p^a m$, with $p \nmid m$. Let B be a p-block of G with a defect group E which is an extra-special p-group of order p^3 and exponent p. Consider a fixed maximal (G, B)-subpair (E, b_E). Let b be the Brauer correspondent of B for $N_G(E, b_E)$. For a non-negative integer d, let $k_d(B)$ denote the number of irreducible characters χ in B which have $\chi(1)_p = p^{a-d}$ and let $k_d(b)$ be the corresponding number of b. Various generalizations of Alperin's Weight Conjecture and McKay's Conjecture are due to Reinhard Knörr, Geoffrey R. Robinson and Everett C. Dade. We follow Geoffrey R. Robinson's approach to consider the Ordinary Weight Conjecture, and Dade's Projective Conjecture. The general question is whether it follows from either of the latter two conjectures that $k_d(B) = k_d(b)$ for all d for the p-block B. The objective of this thesis is to show that these conjectures predict that $k_d(B) = k_d(b)$, for all non-negative integers d. It is well known that $N_G(E, b_E)/EC_G(E)$ is a p'-subgroup of the automorphism group of E. Hence, we have considered some special cases of the above question. The unique largest normal p-subgroup of G, $O_p(G)$ is the central focus of our attention. We consider the case that $O_p(G)$ is a central p-subgroup of G, as well as the case that $O_p(G)$ is not central. In both cases, the common factor is that $O_p(G)$ is strictly contained in the defect group of B.

Dedication

To my parents, my wife, and to my children.

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Contents

List of Tables

G	A finite group
p	A rational prime number
$GF(p)$	The field of p-elements
n_p	The highest power of p dividing the non-zero integer n
R	A complete discrete valuation ring of characteristic zero
$\mathbb{K}$	The field of fractions of R
$J(R)$	The Jacobson radical of the ring R
$F = R/J(R)$	An algebraically closed field of characteristic p
FG	The group algebra of the finite group G
$Z(FG)$	The centre of the group algebra FG
$H \leq G$	H is a subgroup of the finite group G.
$H < G$	H is a proper subgroup of the finite group G.
$U' = [U, U]$	The commutator subgroup of U
$[G : H]$	The index of the subgroup H in G
$[G/H]$	The set of the right coset representatives of H in G
$\|g\|$	The order of an element $g \in G$
$O_p(G)$	The largest normal p-subgroup of the finite group G
H^x	The conjugation subgroup $= \{h^x = x^{-1}hx : h \in H\}$
$H \leq_G K$	There is $x \in G$ such that $H^x \leq K$
$Res^G_H(-)$	The restriction from G to its subgroup H
$Ind^G_H(-)$	The induction from the subgroup H to G
$N_G(H)$	The normalizer of the subgroup H in G
$C_G(x)$	The set of all elements in G which commute with x
$C_G(H)$	The centralizer of the subgroup H in G
RG^Q	The set of all fixed elements in RG under Q-action
B	A p-block of the group algebra FG
b^G	A p-block of G which is associated with the p-block b
Br_Q	The Brauer map with respect to the p-group Q
$\mathscr{P}(G)$	The collection of chains of which all members are p-subgroups
$\mathscr{R}(G)$	The radical p-chains
$\mathscr{E}(G)$	Chains of which all members are elementary abelian p-subgroups
$\mathscr{N}(G)$	Chains in which all members are normal in the final p-subgroup
$\mathscr{P}(G, B)$	The representatives of chains σ of (G, B)-subpairs
$\mathscr{R}(G\|(Q, b_Q))$	Radical (G, B)-chains which start with the (G, B)-subpair (Q, b_Q)
σ	A chain in $\mathscr{P}(G)$ or a chain of (G, B)-subpairs
Q_σ	The initial p-subgroup of the chain σ
Q^σ	The final p-subgroup of the chain σ
$N_G(\sigma)$	The stabilizer subgroup of the chain σ
$B(\sigma)$	The stabilizer p-block of the chain σ
$[x]$	A class under certain action with x as a representative

Table 1: Notation (1)

σ_i	The initial sub-chain of the chain σ which ends with Q_i
σ^i	The sub-chain of the chain σ which starts with Q_i
$\|\sigma\|$	The number of non-trivial p-subgroups in σ
$\Phi(G)$	The Frattini subgroup of the finite group G
$\mathscr{P}(G)^Q$	Chains σ such that $\sigma^x = \sigma \ \forall x \in Q$
$\ell(B)$	The number of simple FG-modules in the p-block B
$k(B)$	The number of simple $\mathbb{K}G$-modules in the p-block B
$k_d(B)$	The number of irreducible $\mathbb{K}$-characters in B with defect d
$k_d(B, \lambda)$	$k_d(B)$, but lying over the linear character λ of a central p-group
$f_0^{(B)}(-)$	The number of p-blocks of defect zero of $(-)$ corresponding to B
$I_G(-)$	The inertial subgroup of $(-)$ in G
$h(\chi)$	The height of an irreducible character χ
χ	An irreducible ordinary character of the finite group G
$Irr(G)$	The set of ordinary irreducible characters of G
$\langle , \rangle_G$	The inner product in the space of the class functions of G
$\bar{\chi}$	The conjugation character such that $\bar{\chi}(x) = \chi(x^{-1})$
g_p	The p-part of the element g
$g_{p'}$	The p'-part of the element g
$B_0(G)$	The principal p-block of the finite group G
$\langle x \rangle$	The cyclic subgroup of the finite group G which is generated by x
E	An extra-special p-group of order p^3 and exponent p
M	A maximal subgroup of E
$d(\chi)$	The defect of an irreducible character χ
$Irr_d(G)$	The set of irreducible characters of G with defect d
(U, b_U)	A (G, B)-subpair
$Mat(n, -)$	The set of all $n \times n$ matrices
$GL(2, p)$	The general linear group of dimension 2 over $GF(p)$
$SL(2, p)$	The special linear group of dimension 2 over $GF(p)$
$Aut(-)$	The automorphism group of $(-)$
$\alpha \otimes \beta$	The product of characters
$Fix_X(G)$	Elements of the set X which are fixed under the G-action
AWC	Alperin's Weight Conjecture
DPC	Dade's Projective Conjecture
OWC	Ordinary Weight Conjecture
$\lessapprox$	Isomorphic to a subgroup of
$\lesssim$	Isomorphic to a proper subgroup of
Max	Maximal element of a set
$\rtimes$	Semidirect product
$=_G$	equals up to G-conjugacy
# Orbits	The number of the orbits under certain action

Table 2: Notation (2)

Chapter 1

INTRODUCTION AND THE MAIN RESULTS

Let p be a rational prime number and $(\mathbb{K}, R, F)$ be a p-modular system. Let G be a finite group. The p-local subgroups of G are the normalizers of the non-identity p-subgroups of G. It is well-known that there is a bijection between certain p-blocks of G and certain p-blocks of a fixed p-local subgroup of G. It is natural to seek the properties which are shared and other connections between the p-blocks which are preserved under this bijection.

However, the notion of (G, B)-subpairs in p-block theory is a generalization of p-subgroups in group theory. Therefore, we still have a bijection between certain p-blocks of G and certain p-blocks of a fixed (G, B)-local subgroup of G, the normalizer of a certain (G, B)-subpair.

For each irreducible character χ of G, the defect of χ is the non-negative integer, say $d(\chi)$, which satisfies $\chi(1)_p \cdot p^{d(\chi)} = |G|_p$. Let B be a p-block of G and write $d(B)$ for the maximum of the defect of irreducible characters of B. We call $d(B)$ the defect of the p-block B. Then $\chi(1)_p \cdot p^{d(B)} = |G|_p \cdot p^{h(\chi)}$, where $h(\chi) = d(B) - d(\chi)$ is called the height of χ. It is well-known that p-blocks of defect zero of the finite group G are in a one to one correspondence with irreducible ordinary characters χ of G in such a way that $\chi(1)_p = |G|_p = [G : 1_G]_p$. The simple G-modules which afford such irreducible characters are the projective simple G-modules.

Alperin's Weight Conjecture (AWC) is concerned with the equality of the number of simple FG-modules of a p-block B of G with the totality of the number of projective simple modules in certain Brauer correspondents of B.

At the same time, the Alperin-McKay Conjecture predicts an equality between the number of irreducible ordinary characters of B of height zero and the corresponding number of the single Brauer correspondent b of B.

Nowadays, the generalizations of these conjectures, which are due to Reinhard Knörr, Geoffrey R. Robinson [45], [61] and Everett C. Dade [22], have various types. However, the idea behind these conjectures is to get p-global information for certain invariant of G and its p-block B from the p-local information of p-local subgroups of G.

Contributions to the development of this theory have been made using different approaches. We shall deal with Geoffrey R. Robinson's approach, which culminates in the Ordinary Weight Conjecture (OWC). This conjecture expresses the number of ordinary irreducible characters of defect d in a p-block B in terms of an alternating sum whose terms can all be calculated p-locally.

The ultimate aim of our project is to be able to say that the OWC holds for p-blocks of finite groups with a defect group which is an extra special p-group, say E, of order p^3 and exponent p for an odd prime number p. However, we have in this thesis investigated whether the OWC is equivalent to attaining a bijection which preserves the defect between the ordinary irreducible characters of the p-block B and the ordinary irreducible characters of the associated Brauer correspondent of a p-local subgroup of G.

To reach this aim, we use character theoretic methods. Clifford theory, which exploits the action of a finite group on irreducible characters of a normal subgroup of G, is an essential tool in our work. Cancellation processes using chains of (G, B)-subpairs form another important tool in the present work.

This thesis is divided into six chapters. The first chapter contains the introduction and the main results. Chapter Two contains background materials and conjectures. After specifying an appropriate p-modular system, we start with the notion of the defect of a character and the defect of a p-block. The concept of a p-block of a finite group can be defined in many ways. We look at a p-block of the finite group G as a collection of ordinary irreducible characters of G, as well as a two sided ideal in the group algebras RG or FG. Then, we study the methods of the cancellation processes. Since (G, B)-subpairs are refinements of p-subgroups, we may use (G, B)-chains which consist of (G, B)-subpairs. The idea behind these methods is to reduce the number of terms in the alternating sums which we are dealing with.

In Section 2.4, Clifford theory is introduced. There are two levels on which to apply Clifford theory. The first one is that the inertial subgroup of the relevant irreducible character is a proper subgroup. The second is that the irreducible character in question is invariant, which leads to the existence of a central extension and a projective representation.

In Section 2.5, we survey the conjectures which we shall deal with. Although we start by considering Dade's Projective Conjecture (DPC), the main work will be for the Ordinary Weight Conjecture (OWC). This is because these two conjectures are equivalent at least, in the case that $O_p(G) \leq Z(G)$ and under our assumption about the defect group of B. Since the defect group of any p-block of G which we consider is E where E is an extra-special p-group of order p^3 and exponent p (as above), Section 2.6 in Chapter Two contains the characters of E and some properties of its automorphism group.

Chapter Three is concerned with the fusion patterns and the orbit structure of characters. The notion of a nilpotent p-block is introduced. Then the action of the inertial quotient on the Frattini quotient is studied. We discuss the natural action of $SL(2,p)$ on a maximal subgroup of the defect group of B. The final section in Chapter Three contains a discussion of the fusion patterns. We define the notion of control fusion of (G,B)-subpairs and we find that $N_G(E,b_E)$ controls the fusion of (G,B)-subpairs under the condition that $SL(2,p) \not\lessapprox N_G(M,b_M)/C_G(M)$, where M is a maximal subgroup of E, which is a radical p-subgroup of G.

In Chapter Four , we calculate the inertial subgroups of the irreducible characters of the initial p-subgroup of a (G,B)-chain in its stabilizer. We start with the inertial group of an irreducible linear character of E in $N_G(E,b_E)$, where (E,b_E) is a maximal (G,B)-subpair. Then we study the inertial group of an irreducible non-linear character of E in $N_G(E,b_E)$. For M, a maximal subgroup of E, which is a radical p-subgroup of G, we calculate the inertial subgroup of an irreducible character of M in $N_G(M,b_M)$. It turns out that if $SL(2,p) \lessapprox N_G(M,b_M)/C_G(M)$ then, $N_G(M,b_M)$ acts transitively on the set of non-trivial irreducible characters of M.

Chapter Five is devoted to the predictions of the conjectures for p-blocks with an extra-special defect group of order p^3 and exponent p for an odd prime number p. The main idea is to exploit the fact that for each maximal subgroup M of E, the associated p-block of B in $C_G(M)$, say b_M, is a nilpotent p-block with abelian defect group M. Thus, it contains exactly p^2 irreducible ordinary characters, each of which corresponds to a unique irreducible ordinary character of M. Furthermore, b_M has a unique modular simple $FC_G(M)$-module. It turns out that there is a unique irreducible ordinary character of b_M which lies over a given irreducible character of M. This enables us to examine the irreducible characters of $N_G(\sigma)$ which are M-projective, where σ is a chain of (G,B)-subpairs.

It is well known that the inertial quotient $N_G(E,b_E)/EC_G(E)$ is a p'-subgroup of the automorphism group of E. On the one hand, this inertial quotient might contain no copy of the automorphism group of the centre of E. The situation in this case is that no (G, B)-subpair (M, b_M) is an Alperin-Goldschmidt (G, B)-subpair for each maximal subgroup M of E, which is a radical p-subgroup of G. Thus, (E, b_E) is the unique Alperin-Goldschmidt (G, B)-subpair. On the other hand, if this inertial quotient has a copy of the automorphism group of the centre of E, then using the fact that for large p, the special linear group of dimension two over a field of p elements has a trivial Schur multiplier together with Clifford theory can be used to attain the desired result.

However, the largest normal p-subgroup of G, $O_p(G)$ plays a central part in our consideration. We should emphasise that Dade's Projective Conjecture is formulated only in the case that $O_p(G)$ is a central p-subgroup of G and the defect group of the p-block B is greater than $O_p(G)$. Nevertheless, the Ordinary Weight Conjecture (OWC) can be formulated without assuming that $O_p(G)$ is a central p-subgroup of G.

In Section 5.2, we study the character correspondence for a nilpotent p-block. In Section 5.3, we discuss the case that $O_p(G)$ is trivial and $N_G(E, b_E)/EC_G(E)$ has no element of order $p-1$. The conclusion in this section is that the contribution from chains starting with the (G, B)-subpair (M, b_M), where M is a radical p-subgroup of G which is a maximal subgroup of E, is zero.

Section 5.4 is concerned with the case that $O_p(G)$ is a central p-subgroup of G and equal to the centre of E. The conclusion in this section is that $SL(2, p)$ cannot be a section of $N_G(M, b_M)/C_G(M)$, and, hence, $N_G(E, b_E)$ controls the fusion of (G, B)-subpairs.

The case that $O_p(G)$ is not a central p-subgroup of G is introduced in Section 5.5. It turns out that either $O_p(G)$ is the unique maximal subgroup of E, which is a radical p-subgroup of G or $O_p(G)$ is isomorphic to the centre of E. We deal only with the OWC in this case and we use Clifford theory to obtain the desired result. In Section 5.6, we study the case that $O_p(G)$ is trivial and $N_G(E, b_E)/EC_G(E)$ is isomorphic to $C_2 \times C_{p-1}$.

The final chapter is devoted to the conclusion and suggestion for further research.

Our notation is standard. However, Tables 1 and 2 contain the main symbols which we have used.

THE MAIN RESULTS

The main results in this thesis are the following:

Theorem A. *Let G be a finite group, B be a p-block of G with defect group E, which is an extra-special p-group of order p^3 and exponent p for an odd prime number p. Consider (E, b_E) as a maximal (G, B)-subpair. Let (M, b_M) be an arbitrary (G, B)-subpair which is contained in (E, b_E), where M is a maximal subgroup of E which is a radical p-subgroup of G. Assume that the OWC holds for the p-block B. If $SL(2, p) \not\lesssim N_G(M, b_M)/C_G(M)$, then, for each non-negative integer d, $k_d(B) = k_d(b)$, where b is the unique p-block of $N_G(E, b_E)$ such that $b^G = B$.*

Theorem B. *Let G be a finite group, B be a p-block of G with defect group E, which is an extra-special p-group of order p^3 and exponent p for an odd prime number p. Consider (E, b_E) as a maximal (G, B)-subpair. Let (M, b_M) be an arbitrary (G, B)-subpair which is contained in (E, b_E), where M is a maximal subgroup of E, which is a radical p-subgroup of G. Write $\mathscr{R}(G|(M, b_M))$ for radical chains which start with (M, b_M). Then, whenever $SL(2, p) \lnapprox N_G(M, b_M)/C_G(M)$,*

$$\sum_{\sigma \in \mathscr{R}(G|(M,b_M))} (-1)^{|\sigma|+1} \sum_{\mu \in Irr(M)/N_G(\sigma)} f_0^{(B)}(I_{N_G(\sigma)}(\mu)/M) = 0.$$

Theorem C. *Let G be a finite group, B be a p-block of G with defect group E, which is an extra-special p-group of order p^3 and exponent p for an odd prime number p. Consider (E, b_E) as a maximal (G, B)-subpair. Then the OWC holds if, and only if, for each non-negative integer d, $k_d(B) = k_d(b)$, where b is the unique p-block of $N_G(E, b_E)$ such that $b^G = B$.*

Theorem D. *Let G be a finite group, B be a p-block of G with defect group E, which is an extra-special p-group of order p^3 and exponent p for an odd prime number p. Consider (E, b_E) as a maximal (G, B)-subpair. Assume that $O_p(G) \leq Z(G)$. Then the DPC holds* *if, and only if, for each positive integer d,* $k_d(B) = k_d(b)$*, where* b *is the unique* p*-block of* $N_G(E, b_E)$ *such that* $b^G = B$*.*

Theorem E. *Let* G *be a finite group,* B *be a* p*-block of* G *with defect group* E *which is an extra-special* p*-group of order* p^3 *and exponent* p *for an odd prime number* p*. Assume that* $O_p(G) = 1$ *and the section* $N_G(E, b_E)/EC_G(E) \cong C_2 \times C_{p-1}$*. If* B *satisfies OWC then* $k(B) - k(E) = p$*.*

Chapter 2

BASIC FACTS AND THE CONJECTURES

2.1 Introduction

In this chapter, we shall discuss some basic facts. In Section 2.2, the defect of an ordinary irreducible character and the defect of a p-block are discussed. The main tools in this thesis are the cancellation processes and Clifford theory which are introduced in Section 2.3 and Section 2.4 respectively. In Section 2.5, we outline the conjectures which we shall deal with. Section 2.6 in this chapter contains the characters of an extra-special p-group of order p^3 and exponent p for an odd prime number p and some properties of its automorphism group.

2.2 The defect of a character and of a p-block

In this section, we state some background concepts which are related to the concept of the defect of an irreducible ordinary character, as well as the defect of a p-block.

We shall use a p-modular system. This system enables us to move between ordinary representation and p-modular representation. It consists of the following:

- A complete discrete valuation ring R of characteristic 0 and a unique maximal ideal $J(R)$; its Jacobson radical. We assume that R contains enough p-power roots of unity. For our purposes, this means that for each group H under consideration, R contains a primitive $|H|_p$-th root of unity.

- The field, say $\mathbb{K}$, of fractions of R. We assume that $\mathbb{K}$ contains a primitive $|H|_p$-th root of unity for any finite group H we consider.

- An algebraically closed field $F := R/J(R)$ of prime characteristic p.

Note that the hypotheses of F and R imply that $\mathbb{K}$ contains p'-roots of unity of all orders. One can assume that $\mathbb{K}$ has the field of rational numbers $\mathbb{Q}$ as its prime subfield. Hence, we may suppose that every root of unity in $\mathbb{K}$ is contained in the complex field $\mathbb{C}$. These processes enable us to identify the complex characters of any finite group under consideration with the $\mathbb{K}$-valued characters of that group. For more details, we refer to [69, Part One] and [57]. However, for our purposes, we state the following definition:

Definition 2.2.1. *With the notation as above, the triple* $(\mathbb{K}, R, F)$ *is called a p-modular system.*

There are many ways to deal with the concept of a p-block of a finite group G. We refer to the following [5, Chapter IV], [20, Chapter 7], [32, Chapter III & IV] and [53, Chapter 5] to see the approaches of this concept and related matters. However, we state here some definitions which can be used as notation.

Definition 2.2.2. *Let G be a finite group and B be a p-block of RG. We write*

$$k(B) = |Irr(B)| = |\{\chi \in Irr(G) \ : \ \ \chi \in B\}|, \quad and \quad l(B) = |IBr(B)|,$$

where $IBr(B)$ means the set of irreducible Brauer characters (for the definition of Brauer characters see [54, Chapter 2]).

Definition 2.2.3. *Let G be a finite group and d be an arbitrary non-negative integer. Then $\chi \in Irr(B)$ is said to be of defect d if $p^d\chi(1)_p = |G|_p$. We write $d = d(\chi)$.*

Now if χ is an irreducible character of G and η is an irreducible character of a normal subgroup H of G, then we have the following definition:

Definition 2.2.4. *Let B be a p-block of RG and let d be an arbitrary non-negative integer. We write*

$$k_d(B, \eta) = |Irr_d(B, \eta)| = |\{\chi \in Irr(G) \ : \ \chi \in B, \ 0 \neq \langle Res^G_H(\chi), \eta\rangle_H \ \ \& \ \ \chi(1)_p p^d = |G|_p\}|.$$

We say that $k_d(B, \eta)$ is the number of irreducible characters of B with defect d which lie over η.

Now we assign to each p-block B of G a unique non-negative integer, also called the defect of the p-block B.

Definition 2.2.5. *Let B be a p-block of the finite group G. The defect of the p-block B is $d(B)$, where $d(B) = Max\{d(\chi) \ : \ \chi \in Irr(B)\}$.*

Definition 2.2.6. *Let B be a p-block of the finite group G, and χ be an irreducible character of B. The height of χ, say $h(\chi)$, is the difference $d(B) - d(\chi)$.*

Let us characterize the p-blocks of G which contain a character of defect zero.

Lemma 2.2.7. *$Irr(B)$ contains an irreducible character of defect zero if, and only if, $B \cong Mat(n, R)$, for some positive integer n. This happens if, and only if, $k(B) = 1 = l(B)$.*

Proof: See [53, Chapter 3, Theorem 6.29]. $\square$

Now let Q be an arbitrary p-subgroup of G. Probably the most important map in modular representation theory is the Brauer map, $Br_Q : RG^Q \to FC_G(Q)$. For more details about this map, see [71, Chapter 2, §11] or [14]. Let us use the Brauer map to assign for each p-block B of G a certain p-subgroup of G which is a unique p-subgroup up to G-conjugacy.

Definition 2.2.8. *A defect group of the p-block B is a p-subgroup D of G which is maximal subject to $Br_D(1_B) \neq 0$.*

Note that we regard B as a two sided ideal in the group algebra FG and 1_B is the central primitive idempotent in FG for which $1_B \cdot FG = B$. It turns out that $|D| = p^{d(B)}$. James A. Green has shown that the defect group of an arbitrary p-block of G is always an intersection of at most 2 Sylow p-subgroups. In particular, the defect group of any p-block contains the unique largest normal p-subgroup of G (see [5, Chapter IV, Theorem 6]). As a result, if the finite group G has a non-trivial normal p-subgroup, then it has no p-block of defect zero. We will use this fact without further reference.

Let us now state the Brauer First Main Theorem on p-Blocks:

Theorem 2.2.9. [Brauer First Main Theorem on p-Blocks]
For a p-subgroup Q of G, Br_Q gives a bijection between p-blocks of G with defect group Q and p-blocks of $N_G(Q)$ with defect group Q.

Proof: The proof can be found in [32, Chapter III, Theorem 9.7]. $\square$

It is then natural to seek the invariants of the corresponding p-blocks and the conditions for which they are preserved under this bijection.

2.3 Some cancellation methods

In this section, we study some cancellation processes. In [45], [61] and [63], these methods are used as tools for formulating Alperin's Weight Conjecture and the companion conjectures. We discuss the types of chains of p-subgroups, as well as the chains of (G, B)-subpairs. We begin by defining the chain in G and its stabilizer under the conjugation action of G.

Definition 2.3.1. *A chain σ for G is a strictly increasing chain $\sigma := Q_0 < Q_1 < \cdots < Q_n$ of p-subgroups of G. We define the stabilizer of σ under the conjugation action of G to be $N_G(\sigma) = \cap_{i=0}^{n} N_G(Q_i)$, the length of σ to be the number of nontrivial p-subgroups which are involved in σ. We denote this length, the initial subgroup and the final subgroup of σ by $|\sigma|$, Q_σ and Q^σ respectively.*

Definition 2.3.2. *Let G be a finite group and let $\sigma : Q_\sigma < \cdots < Q_n = Q^\sigma$ be a chain for G. We define the i-th initial subchain of σ to be $\sigma_i := Q_\sigma < \cdots < Q_i$ and the i-th final subchain of σ to be $\sigma^i := Q_i < \cdots < Q^\sigma$.*

Since we have various types of p-subgroups of G, we have also various types of chain.

1. The set $\mathscr{P}(G) = \{\sigma | \sigma := Q_0 < Q_1 < \cdots < Q_n\}$, where the inclusion is strict and Q_i is a p-group for each $i \in \{0, 1, 2, \cdots, n\}$. This set contains all possible chains of p-subgroups that we can obtain for a finite group G.

2. The set $\mathscr{E}(G) = \{\sigma | \sigma := Q_0 < Q_1 < \cdots < Q_n\}$, where the inclusion is strict and Q_i is an elementary abelian p-group for each $i \in \{0, 1, 2, \cdots, n\}$. Note that a finite p-group Q is said to be an elementary abelian p-group if $g^p = 1$ for every $g \in Q$.

3. The set $\mathscr{N}(G) = \{\sigma | \sigma := Q_0 < Q_1 < \cdots < Q_n\}$, where the inclusion is strict and Q_i is a normal p-subgroup of Q_n for each $i \in \{0, 1, 2, \cdots, n\}$.

4. The set $\mathscr{R}(G) = \{\sigma | \sigma := Q_0 < Q_1 < \cdots < Q_n\}$, where the inclusion is strict and $Q_i = O_p(\cap_{j=1}^{i} N_G(Q_i)) = O_p(N_G(\sigma_i))$ for each $i \in \{0, 1, 2, \cdots, n\}$.

5. The set $\mathscr{U}(G) = \{\sigma | \sigma := Q_0 < Q_1 < \cdots < Q_n\}$, where the inclusion is strict and $Q_i = O_p(N_G(Q_i))$ for each $i \in \{0, 1, 2, \cdots, n\}$.

Definition 2.3.3. *A radical p-subgroup of the finite group G is a p-subgroup Q of G which satisfies $Q = O_p(N_G(Q))$.*

It is clear that $O_p(G)$ and a Sylow p-subgroup of G are examples of radical p-subgroups. In fact, any defect group is a radical p-subgroup. Furthermore, if Q is a radical p-subgroup of G, then $Q \cap H$ is a radical p-subgroup of H for each normal subgroup H of G.

Lemma 2.3.4. *Let G be a finite group, and let U be a radical p-subgroup of G. Then whenever $N_G(U) \leq N_G(Q)$ for a p-subgroup Q of G, we have $Q \leq U$.*

Proof: The proof is standard and we omit the details. □

Let us now discuss some cancellation procedures. As in [63], we consider an abelian group A and a G-stable function $f : \mathscr{P}(G) \to A$. We assume that the function depends only on the chain stabilizer $N_G(\sigma)$ and the initial subgroup Q_σ of the chain $\sigma := Q_\sigma < Q_1 < \cdots < Q^\sigma$. The most remarkable observation for dealing with chains of p-subgroups is the following lemma:

Lemma 2.3.5. *Let G be a finite group. Write $\mathscr{P}(G)/G$ for the set of the representatives of the orbits of the action of G on $\mathscr{P}(G)$. With the notation above,*

$$\sum_{\sigma \in \mathscr{P}(G)/G} (-1)^{|\sigma|} f(\sigma) = \sum_{\sigma \in \mathscr{U}(G)/G} (-1)^{|\sigma|} f(\sigma) = \sum_{\sigma \in \mathscr{N}(G)/G} (-1)^{|\sigma|} f(\sigma) = \sum_{\sigma \in \mathscr{R}(G)/G} (-1)^{|\sigma|} f(\sigma).$$

Proof: This is [63, Lemma 1.1]. $\square$

We use the following lemma for the cancellation processes in many places.

Lemma 2.3.6. *Let G be a finite group, A be an abelian group, $f : \mathscr{P}(G) \to A$ be a G-stable function such that $f(\sigma)$ depends only on the chain stabilizer $N_G(\sigma)$ and the initial subgroup of the chain σ. Let U be a non-trivial p-subgroup of G which is not radical. Then chains which begin with (conjugates of) U make zero contribution to the alternating sum $\sum_{\sigma\in\mathscr{P}(G)/G}(-1)^{|\sigma|}f(\sigma)$.*

Proof: This is [63, Corollary 1.2]. $\square$

In the following lemma, we can eliminate chains from $\mathscr{E}(G)$ in the same manner, but we may affect the initial subgroup in the chain σ. Therefore, we assume that the function f does not depend on the initial p-subgroup of the chain σ.

Lemma 2.3.7. *With the notation above, $\sum_{\sigma\in\mathscr{P}(G)/G}(-1)^{|\sigma|}f(\sigma) = \sum_{\sigma\in\mathscr{E}(G)/G}(-1)^{|\sigma|}f(\sigma)$.*

Proof: The proof can be found in [45], [61] and [63]. $\square$

We begin, as in [45], by considering a p-block of $N_G(\sigma)$, where σ is a chain in $\mathscr{P}(G)$. We call this p-block the stabilizer p-block of σ.

Definition 2.3.8. *Let $\sigma \in \mathscr{P}(G)$ and let B be a p-block of RG. The stabilizer p-block of $N_G(\sigma)$ is $B(\sigma) := Br_{Q^\sigma}(1_B)\cdot RN_G(\sigma)$.*

The following lemma is one of the main discoveries in [45], which guarantees the link between the irreducible characters of G and those in the Brauer correspondents in $N_G(\sigma)$. For more details about how Brauer correspondence is defined, the reader can see [53, Chapter 5, Lemma 3.1] or [32, Chapter III, Section 9].

Lemma 2.3.9. *Let $\sigma : Q_\sigma < Q_1 < \cdots < Q^\sigma$ be a chain in $\mathscr{P}(G)$. Then*

1. *All central idempotents of $RN_G(\sigma)$ lie in $RC_G(Q^\sigma)$.*

2. *$N_G(\sigma)$ and $N_G(\sigma)Q^\sigma$ have the same central idempotents. In addition, the stabilizer p-block $B(\sigma)$ depends only on the subgroup $N_G(\sigma)$.*

3. *Whenever b is a p-block of $N_G(\sigma)$, b^G is defined. Furthermore, b is a summand of $B(\sigma)$ if, and only if, $b^G = B$.*

Proof: For the proof, see [45, Lemma 3.1 & Lemma 3.2]. □

The notion of (G, B)-subpairs was introduced in p-block theory by J. Alperin and M. Broué in [7]. It is a generalization of the concept of p-groups. For our purposes, (G, B)-subpairs can be used for the cancellation processes especially when we are dealing with a non-principal p-block. Let us define the (G, B)-subpair.

Definition 2.3.10. *A (G, B)-subpair is a pair (Q, b_Q), where Q is a p-subgroup of G and b_Q is a p-block of $C_G(Q)$ such that b_Q^G is defined and $B = b_Q^G$.*

We observe that if $g \in G$, then $(Q, b_Q)^g := (Q^g, b_Q^g)$ is a $(G, B)^g = (G, B)$-subpair. This means that G acts by conjugation on the set of all (G, B)-subpairs.

Definition 2.3.11. *Let (Q, b_Q) and (P, b_P) be two (G, B)-subpairs. We say that (Q, b_Q) is normal (G, B)-subpair of (P, b_P) if Q is a normal subgroup of P, the p-block b_Q of $C_G(Q)$ is stable under P-conjugation and the p-block b_P of $C_G(P)$ appears in the decomposition of the image of b_Q under the Brauer map Br_P. We shall write $(Q, b_Q) \triangleleft (P, b_P)$ to indicate that (Q, b_Q) is a normal (G, B)-subpair of (P, b_P)*

Now we define the inclusion of (G, B)-subpairs.

Definition 2.3.12. *Let (Q, b_Q) and (P, b_P) be two (G, B)-subpairs. We say that (Q, b_Q) is contained in (P, b_P) if there is an integer $n \geq 1$ and a series of (G, B)-subpairs $(R_0, b_{R_0}), \cdots, (R_n, b_{R_n})$ such that $(Q, b_Q) = (R_0, b_{R_0})$, $(R_n, b_{R_n}) = (P, b_P)$ and (R_i, b_{R_i})*

is a normal (G,B)-subpair of $(R_i, b_{R_{i+1}})$, for each $i \in \{0,1,\cdots,n\}$. As a notation, we shall write $(Q,b_Q) \leq (P,b_P)$ to indicate the containment between (G,B)-subpairs.

It is clear that the action of G on the set of (G,B)-subpairs respects the above inclusion. As usual, the stabilizer of a (G,B)-subpair (Q,b_Q) is $N_G(Q) \cap I_G(b_Q) := N_G(Q,b_Q)$. Note that in the case that Q is a defect group of the p-block B in G, we say that the (G,B)-subpair (Q,b_Q) is a maximal (G,B)-subpair. If $Z(Q)$ is a defect group of the p-block b_Q in $C_G(Q)$, we say that (Q,b_Q) is a centric (G,B)-subpair. We mention the following facts, analogous to Sylow's theorems for p-subgroups.

Lemma 2.3.13.

1. *All maximal (G,B)-subpairs are G-conjugate.*
2. *Let (Q,b_Q) be an arbitrary (G,B)-subpair. Then there is a maximal (G,B)-subpair (D,b_D) such that $(Q,b_Q) \leq (D,b_D)$.*
3. *If (P,b_P) is a (G,B)-subpair and Q is a subgroup of P then there exists one, and only one, p-block b_Q of $C_G(Q)$ such that $(Q,b_Q) \leq (P,b_P)$.*

Proof: These are [7, Theorem 3.4 and Theorem 3.10]. □

Because of Lemma 2.3.13, we can define the (G,B)-chain σ to be the strict inclusion $(Q_\sigma, b_{Q_\sigma}) < (Q_2, b_{Q_2}) < \cdots < (Q^\sigma, b_{Q^\sigma})$, where $\{(Q_i, b_{Q_i})\}_{i=1}^{i=n}$ are (G,B)-subpairs which are uniquely determined by the defect group of the p-block B. It turns out that the stabilizer of such σ is $I_{N_G(\sigma)}(b_{Q^\sigma})$, which is the inertial subgroup of the p-block b_{Q^σ} in the stabilizer of σ.

The following result shows that when B is the principal p-block of G, (G,B)-subpairs correspond canonically to p-subgroups of G.

Lemma 2.3.14. *Let P and Q be p-subgroups of G.*

1. *$P < Q$ if, and only if, $(P, B_0(C_G(P))) < (Q, B_0(C_G(Q)))$.*

2. *P is a normal subgroup of Q if, and only if, $(P, B_0(C_G(P)))$ is a normal (G, B)-subpair of $(Q, B_0(C_G(Q)))$.*

3. *If $x \in G$, then $P = Q^x$ if, and only if, $(P, B_0(C_G(P))) = (Q, B_0(C_G(Q)))^x$. Furthermore, the normalizer of the (G, B)-subpair $(Q, B_0(C_G(Q)))$ is $N_G(Q)$.*

Proof: This is [7, Corollary 3.14]. $\square$

2.4 Clifford theory

Clifford theory can be used to obtain a bijection between the set of irreducible characters of a finite group which lie over an irreducible character of a normal subgroup and the set of irreducible characters of the inertial subgroup of this character lying over it.

Theorem 2.4.1. [A. H. Clifford 1937]
Let H be a normal subgroup of the finite group G. Let μ be an irreducible character of H. Choose $\chi \in Irr(G, \mu)$. For each $g \in G$, define the the conjugate character of μ to be μ^g such that $\mu^g(h) = \mu(ghg^{-1})$ for each $h \in H$. Write $I_G(\mu)$ for the inertial subgroup of μ in G. Then

1. *There is a non-negative integer e such that $Res^G_H(\chi) = e(\sum_{t \in [G/I_G(\mu)]} \mu^t)$.*

2. *There exists a unique $\eta \in Irr(I_G(\mu), \mu)$ such that $\chi = Ind^G_{I_G(\mu)}(\eta)$ and $Res^{I_G(\mu)}_H(\eta) = e\mu$.*

3. *If $I_G(\mu) \leq T \leq G$, then the map $Ind^G_T : Irr(T, \mu) \to Irr(G, \mu)$ $(\eta \mapsto Ind^G_T(\eta))$ is a bijection. In particular, $H = I_G(\mu)$ if, and only if, $Ind^G_H(\mu)$ is an irreducible character of G.*

Proof: The proof can be found in many text-books, such as [53, Chapter 3, Theorem 3.8], [37, Theorem 3.5], [42, Theorem 6.11] and [40, Theorem 19.6]. □

It is clear that the bijection in Theorem 2.4.1 preserves the defect of the irreducible characters.

Now what is the situation if that character is stable under the action of the finite group under consideration? This case is more subtle. It leads to new methods for investigating the relationships between the irreducible characters of a finite group and the irreducible characters of a normal subgroup of this group.

The first observation for the stable case is the following lemma which is originally due to Gallagher, see [36].

Lemma 2.4.2. *Let G be a finite group with normal subgroup N. Write the full set of irreducible characters of the quotient group G/N as the set $\{\Psi_1, \Psi_2, \cdots, \Psi_t\}$. Let χ be an irreducible character of G which lies over an irreducible character μ of N. Assume that $Res_N^G(\chi) = \mu$. Then, $Ind_N^G(\mu) = \Psi_1(1)(\chi \otimes \Psi_1) + \Psi_2(1)(\chi \otimes \Psi_2) + \cdots + \Psi_t(1)(\chi \otimes \Psi_t)$.*

Proof: The proof can be found in [40, Theorem 19.5]. □

However, the existence of a finite group which is a central extension of G and a projective representation of G which arises from such μ above was discussed earlier, (see [18, Definition 53.6, p. 361]).

Lemma 2.4.3. *Let G be a finite group with a normal subgroup N. Let μ be an irreducible character of N such that μ is G-invariant. Then there is a finite group $\tilde{G}$ with a normal subgroup $\tilde{Z}$ such that $\tilde{G}/\tilde{Z} \cong G/N$ such that there is a bijection between $Irr(G, \mu)$ and $Irr(\tilde{G}, \tilde{\lambda})$ for some linear character $\tilde{\lambda}$ of $\tilde{Z}$.*

Proof: The proof of this result is standard and we omit the details. □

The compatibility of the bijection in Lemma 2.4.3 with p-block theory is investigated in [29] and [28]. The reader can see also, [58], [48] and [35].

From now on, we assume that N to be V_σ, where σ is the radical chain $(V_\sigma, b_{V_\sigma}) < (V_2, b_{V_2}) < \cdots < (V^\sigma, b_{V^\sigma})$. Then N is a normal p-subgroup of $N_G(\sigma)$. Let $B(\sigma)$ be the sum of Brauer correspondents of B in $RN_G(\sigma)$. Recall that $B(\sigma) = Br_{V^\sigma}(1_B)\cdot RN_G(\sigma)$.

Let $k_d(B(\sigma))$ be the number of irreducible characters in $N_G(\sigma)$, which belong to $B(\sigma)$ and have p-parts of their degrees equal to p^{a-d}, where p^a is the order of a Sylow p-subgroup of $N_G(\sigma)$.

Our target is to compute the alternating sum $\sum_{\sigma\in\mathscr{R}(G,N)/G}(-1)^{|\sigma|}k_d(B(\sigma))$.

Now if μ is an irreducible character of N, then any irreducible character of $N_G(\sigma)$ which lies over μ lies also over a unique $N_G(\sigma)$-conjugacy class of irreducible characters of N (namely the $N_G(\sigma)$-conjugates of μ). Theorem 2.4.1 ensures that we can write the above alternating sum as

$$\sum_{\sigma\in\mathscr{R}(G,N)/G}(-1)^{|\sigma|}k_d(B(\sigma)) = \sum_{\sigma\in\mathscr{R}(G,N)/G}(-1)^{|\sigma|}\sum_{\mu\in Irr_d(N)/N_G(\sigma)}k_d(B(\sigma),\mu), \qquad (2.4.1)$$

where $k_d(B(\sigma),\mu)$ is the number of irreducible characters of $B(\sigma)$ which lie over $N_G(\sigma)$-conjugates of μ and have defect d.

The order of the summation may be changed to write 2.4.1 into

$$\sum_{\sigma\in\mathscr{R}(G,N)/G}(-1)^{|\sigma|}k_d(B(\sigma)) = \sum_{\mu\in Irr_d(N)/G}\ \sum_{\sigma\in\mathscr{R}(I_G(\mu),N)/I_G(\mu)}(-1)^{|\sigma|}k_d(B(\sigma),\mu). \qquad (2.4.2)$$

We see that the bijection between irreducible characters of $N_G(\sigma)$ which lie over μ and those of $I_{N_G(\sigma)}(\mu)$ which also lie over μ gives us the opportunity to deduce that there is a set of p-blocks, say $b(\sigma)$, of $I_{N_G(\sigma)}(\mu)$ such that $b(\sigma)^{N_G(\sigma)} = B(\sigma)$. Hence, using the transitivity of induction of p-blocks (see [32, Chapter III, Lemma 9.2]), we have $b(\sigma)^G = B$. Therefore, we denote the collection of such p-blocks of $I_{N_G(\sigma)}(\mu)$ by $B((\sigma,\mu),\ \mu)$.

Consequently, the alternating sum under consideration can be written as follows

$$\sum_{\sigma\in\mathscr{R}(G,N)/G}(-1)^{|\sigma|}k_d(B(\sigma))=\sum_{\mu\in Irr_d(N)/G}\ \sum_{\sigma\in\mathscr{R}(I_G(\mu),N)/I_G(\mu)}(-1)^{|\sigma|}k_d(B(\sigma,\mu),\ \mu),$$

where $B((\sigma,\mu),\ \mu)$ is the set of p-blocks $b(\sigma)$ of $I_{N_G(\sigma)}(\mu)$ such that $b(\sigma)^G=B$. Note that $b(\sigma)^G=B$ is defined by [45, Lemma 3.2].

On the other hand, the stabilizer of $(V_\sigma,b_{V_\sigma})<(V_2,b_{V_2})<\cdots<(V^\sigma,b_{V^\sigma})$ is

$$I_{N_G(\sigma)}(b_{V^\sigma}):=N_G(\sigma,b_{V^\sigma}).$$

Then we denote the p-block $1_{b_{V^\sigma}}\cdot RN_G(\sigma,b_{V^\sigma})$ by $B(\sigma,b_{V^\sigma})$. Now if μ is an irreducible character of V_σ then $I_{N_G(\sigma)}(b_{V^\sigma},\mu):=N_G(\sigma,b_{V^\sigma},\mu)$ is the inertial subgroup of μ and we define the p-block $B(\sigma,b_{V^\sigma},\mu)$ in a similar way. Therefore, we consider the number of irreducible characters in $B(\sigma,b_{V^\sigma},\mu)$ which contain V_σ in their kernels and lie in p-blocks of defect zero of $N_G(\sigma,b_{V^\sigma},\mu)/V_\sigma$.

However, before we proceed to discuss the conjectures we are dealing with, we state the following lemma:

Lemma 2.4.4. *Dade's Projective Conjecture is equivalent to the following:*

$$k_d(B)=\sum_{\sigma\in\mathscr{R}(G)/G}(-1)^{|\sigma|+1}\sum_{\mu\in Irr_d(V_\sigma)/N_G(\sigma)}f_0^{(B)}(I_{N_G(\sigma)}(\mu)/V_\sigma),\qquad(2.4.3)$$

where $f_0^{(B)}(-)$ means the number of p-blocks of defect zero of $(-)$ which are in Brauer's correspondent with B.

Proof: For the proof of this lemma see, [28], [61, Conjecture 4.1], [65, p. 216] and [22, 17.10]. □

2.5 Survey of conjectures in p-block theory

For our purpose, we shall start from 1986. In this year, J. L. Alperin announced the following conjecture [6] which is called Alperin's Weight Conjecture (AWC).

Conjecture 2.5.1. *Let G be a finite group and fix p to be a prime number. Denote the normalizer of each p-subgroup P of G by $N_G(P)$. Consider the group algebra FG, where F is a field of characteristic p. Write the number of simple FG-modules as $l(FG)$. Then we should have $l(FG) = \sum_{(P)} f_0(FN_G(P)/P)$, where the sum is taken over all representatives (up to G-conjugacy) of p-subgroups of the finite group G and $f_0(FN_G(P)/P)$ denotes the number of p-blocks of defect zero of $FN_G(P)/P$.*

We see immediately that Conjecture 2.5.1 predicts p-global information from p-local information. The strategy for attaining p-global information from p-local information underlies most of the work following what was done by R. Brauer and James A. Green.

Alperin suggested in [6] that to get an affirmative answer for his conjecture, two approaches have to be followed. The first one is the filtration of the endomorphism ring of modules and the other is to use Geoffrey R. Robinson's simplicial methods approach. It is worth saying that the present author, following L. Barker in [10] in his attempt to generalize Alperin's Weight Conjecture, studied the p-blocks of the endomorphism ring of a module in his Master's dissertation. See [2] and [1].

We observe that there is no mention of defect groups in Conjecture 2.5.1. This is because Alperin's Weight Conjecture has a non-block-wise form. One can consider the following definition:

Definition 2.5.2. *Let G be a finite group, B be a p-block of G. A weight of B is a pair (Q, S), where Q is a p-subgroup of G and S is a projective simple $F[N_G(Q, b_Q)/Q]$-module which lies in a Brauer correspondence with B.*

Now we state block-wise form of Alperin's Weight Conjecture.

Conjecture 2.5.3. *Let B be a p-block of G. Then*

$$l(B) = \sum_{(Q,b_Q)} f_0^{(B)}(F[N_G(Q,b_Q)/Q]),$$

where $f_0^{(B)}(F[N_G(Q,b_Q)/Q])$ means the number of isomorphism types of projective simple $F[N_G(Q,b_Q)/Q]$-modules which are not annihilated by b_Q and (Q,b_Q) runs over a set of representatives for the conjugacy classes of (G,B)-subpairs.

Alperin's Weight Conjecture is known to hold for the following kind of groups:

- Symmetric groups and general linear groups, see [4].
- Finite groups of Lie type in natural characteristics, see [16].
- p-solvable groups, see [59].
- For nilpotent p-blocks, see [45].

The concept of simplicial complexes is purely topological. The subtle idea for dealing with Alperin's Weight Conjecture is the work of Reinhard Knörr and Geoffrey R. Robinson [45]. They used some properties of the partial ordered set of the nontrivial p-subgroups of the finite group G to convert this conjecture to a new environment which enables both group theorists and representation theorists, as well as people working in the homotopy and simplicial complexes, to study the applications and consequences of this conjecture.

Perhaps the most important reason for using p-subgroup complexes is that they give precision to p-local theory. This is because chain stabilizers in the action of G on this complex are intersections of p-local subgroups. Note that we mean by p-local subgroups of G, the normalizer of non-identity p-subgroups of G.

In 1990, Geoffrey R. Robinson and R. Staszewski [59] generalized this conjecture to a situation which gave the opportunity to deal with a group G which has a non-identity normal subgroup. This places the problem in the environment of Clifford theory.

On the other hand, McKay's Conjecture is concerned with counting the number of irreducible ordinary characters of G whose degrees are not divisible by p.

Recall that for a p-subgroup Q of G, Br_Q gives a bijection between p-blocks of G with defect group Q and p-blocks of $N_G(Q)$ with defect group Q. Furthermore, if Q is the defect group of B and b is the unique p-block of $N_G(Q)$ such that $b^G = B$ then b is called the Brauer correspondent of B in $N_G(Q)$. Thus, a refinement of McKay's Conjecture in the form of (G, B)-subpairs is the following conjecture which is the Alperin-McKay Conjecture.

Conjecture 2.5.4. *Let B be a p-block of G, with defect group Q, and let b be the Brauer correspondent of B in $N_G(Q, b_Q)$. Then B and b contain the same number of irreducible characters of height zero.*

It is clear that the Alperin-McKay Conjecture implies the McKay's Conjecture. However, the converse is not always true. For more details, the reader can see [50, Section 4, p. 21] and [52, Chapter IV, §15]

Now, a natural generalization of both Alperin-McKay conjecture and Alperin's Conjecture is Dade's conjectures (see [62, 22, 21, 23, 24] for more details).

The simplest of Dade's Conjectures is Dade's Ordinary Conjecture, which can be stated as follows:

Conjecture 2.5.5. *Let G be a finite group, B be a p-block of G with nontrivial defect group. Assume that the unique largest normal p-subgroup of G is trivial. Then for each positive integer d, we have $\sum_{\sigma \in \mathscr{P}(G)/G} (-1)^{|\sigma|} k_d(B(\sigma)) = 0$.*

Dade's Ordinary Conjecture holds for the following kind of groups:

- Unipotent p-blocks with an abelian defect group, according to the work of Broué, Malle, and Michel (see [12]).

- Principal 2-block with abelian defect group, according to the work of Fong and Harris (see [33]).

- Symmetric groups by the work of Olsson, Uno, and An (see [56] and [8]).

- Blocks of alternating groups with abelian defect groups by the work of Fong and Harris (see [34]).

- The principal p-block with an abelian defect group of sporadic groups, by the work of Rouquier (see [68]).

- Some finite Chevalley groups.

- p-Blocks with cyclic defect groups by the work of Dade (see [23]).

- Some of sporadic simple groups.

In 1996, the paper "Local structure, vertices and Alperin's conjecture" by Geoffrey R. Robinson [61], contains, among other things, new variants and interpretations of these conjectures. In fact, this paper reveals the main difference between Geoffrey R. Robinson's approach and Everett C. Dade's approach. The former relies on the theory of characters of relatively projective modules, while the latter depends on the classification of finite simple groups and blocks of twisted group algebras. We have seen also in this paper the professional use of radical chains, computing the number of irreducible characters in a p-block in terms of the fusion of p-subgroups and p-elements, and the subtle idea is the consideration of the case that the unique largest normal p-subgroup of G is non-central. Although Robinson's conjectures in this paper and Dade's conjectures are related, we have to distinguish between the strategies which are involved in each approach. In the following we state Dade's Projective Conjecture (DPC):

Conjecture 2.5.6. *Let G be a finite group, B be a p-block of G with nontrivial defect group which is not central in G. Assume that $O_p(G) \leq Z(G)$ and the defect group of B contains $O_p(G)$ strictly. Then for each positive integer d, and for each linear character λ of $O_p(G)$, we have $\sum_{\sigma \in \mathscr{P}(G)/G} (-1)^{|\sigma|+1} k_d(B(\sigma), \lambda) = 0$.*

Now, in attempting to consider a conjecture, many ways are possible. One is to try to apply the conjecture to a specific type of groups. A second way is to determine a p-block with a certain defect group and attempt to satisfy the conjecture directly. A third way is to assume that the conjecture holds for certain p-blocks and, accordingly, try to obtain some theoretical consequences. A fourth approach is to convert the conjecture into another environment to perform some sort of reduction theorem.

Using the theory of (G, B)-subpairs, Geoffrey R. Robinson has formulated this type of conjecture in the sense that the stabilizers of the (G, B)-chains under consideration are the inertial subgroups of the (G, B)-subpairs which are involved in such chains (see [63]). It turns out that the stabilizer is the inertial group of a p-block of the centralizer of a maximal p-group (usually, the defect group of the p-block B) in the original stabilizer $N_G(\sigma)$.

Dade's Projective Conjecture holds for the following kind of groups:

- p-solvable groups, according to the work of Geoffrey R. Robinson, (see [64]).
- In the case that the unique largest normal p-subgroup is cyclic and the corresponding quotient has a trivial intersection Sylow p-subgroups, according to the work of Charles Eaton, (see [30]).
- p-blocks with cyclic defect group, by the work of Everett C. Dade, see [22].
- Some sporadic simple groups.

In our case, we shall use the (G, B)-subpairs formulation of Dade's Projective Conjecture which was explicitly mentioned in [63].

Conjecture 2.5.7. *Let G be a finite group, B be a p-block of G with non-trivial defect group D and let (D, b_D) be a maximal (G, B)-subpair. Let $\mathscr{P}(G, B)$ be a full set of representatives for the distinct G-conjugacy classes of chains σ of (G, B)-subpairs $(V_\sigma, b_\sigma) < (V_1, b_1) < \cdots < (V^\sigma, b^\sigma)$, with $(V^\sigma, b^\sigma) \leq (D, b_D)$, $C_D(V_\sigma) = Z(V_\sigma)$ and*

(D, b_D) contains a maximal $(N_G(\sigma), B(\sigma)) -$ subpair. Assume that $O_p(G) \leq Z(G)$ and that D contains $O_p(G)$ strictly. Then, whenever d is a positive integer and λ is a linear character of $O_p(G)$, we should have $k_d(B, \lambda) = \sum_{\sigma \in \mathscr{P}(G,B)} (-1)^{|\sigma|+1} k_d(B(\sigma), \lambda)$.

However, we shall deal with the Ordinary Weight Conjecture (OWC) which can be formulated for any finite group and without the assumption that the unique largest normal p-subgroup is central (see [67] for more details).

Conjecture 2.5.8. *Let G be a finite group, B be a p-block of G with non-trivial defect group D and let (D, b_D) be a maximal (G, B)-subpair. Let $\mathscr{N}(G, B)$ be a full set of representatives for the distinct G-conjugacy classes of chains σ of (G, B)-subpairs $(V_\sigma, b_\sigma) < (V_1, b_1) < \cdots < (V^\sigma, b^\sigma)$ with $(V^\sigma, b^\sigma) \leq (D, b_D)$, $C_D(V_\sigma) = Z(V_\sigma)$ and (D, b_D) contains a maximal $(N_G(\sigma), B(\sigma)) -$ subpair. Then, whenever d is a non-negative integer, we should have*

$$k_d(B) = \sum_{\sigma \in \mathscr{N}(G,B)} (-1)^{|\sigma|+1} \sum_{\mu \in Irr_d(V_\sigma)/N_G(\sigma)} f_0^{(B)}(I_{N_G(\sigma)}(\mu)/V_\sigma),$$

where $f_0^{(B)}(I_{N_G(\sigma)}(\mu)/V_\sigma)$ denotes the number of p-blocks of defect zero of $I_{N_G(\sigma)}(\mu)/V_\sigma$.

In this thesis, we shall choose that D, in Conjecture 2.5.7 and Conjecture 2.5.8, be an extra-special p-group of order p^3 and exponent p for an odd prime number p. We prefer to use the letter E instead of D.

Lemma 2.5.9. *Let G be a finite group, B be a p-block of G with a defect group which is an extra-special p-group of order p^3 and exponent p for an odd prime number p. Assume that $O_p(G) \leq Z(G)$. Then the OWC and the DPC are equivalent.*

Proof: This is an especial case of the results in [31]. □

We shall study the predictions of both the DPC and the OWC. Hence, this choice of the defect group justifies the following section.

2.6 The characters of extra-special p-groups of order p^3

In this section, we are concerned with some properties of an extra-special p-group, say E, of order p^3 and exponent p, where p is an odd prime number. To determine the orbit structure of characters, we have studied some properties of the automorphism group of E and its maximal subgroups.

It is well known that, up to isomorphism, there are two types of non-abelian groups of the order p^3, (see [9, 23.13 & 23.14]). Now E is a non-abelian p-group since by definition, $Z(E) = [E, E] = \Phi(E) \cong C_p$. If E has an exponent p, then E can be represented as

$$E = \langle x, y, z, | x^p = y^p = z^p = [x, z] = [y, z] = 1 \ \& \ [x, y] = z \rangle. \tag{2.6.1}$$

While if E has an exponent p^2 then

$$E = \langle x, y, z, | x^{p^2} = y^p = z^p = [x, z] = [y, z] = 1 \ \& \ [x, y] = z \rangle. \tag{2.6.2}$$

Through-out this thesis, we fix that E to be an extra-special p-group of order p^3 and exponent p for an odd prime number p. Hence, the presentation of E will be that in 2.6.1. We fix that $Z(E) = \langle z \rangle$. For the character table of E, we have the following:

Lemma 2.6.1. *E has $p^2 + p - 1$ ordinary irreducible characters, where p^2 of them are linear irreducible characters and the remaining $p - 1$ are non-linear irreducible characters, each of which is of degree p and each of which vanishes off $Z(E)$.*

Proof: The number of linear irreducible characters of E is the index of the commutator subgroup of E. Hence, this number is p^2. Also, since the degree of every irreducible character of E must be 1 or p (p^2 is already too large) and the number of linear irreducible characters of E is p^2, we have $p^2+p^2r = p^3$, where r is the number of non-linear irreducible characters of E. It follows that r is $p-1$.

Now if $e \in E - Z(E)$, we see that $|C_E(e)| = p^2$. Therefore, the orthogonality relations give us that each of non-linear irreducible characters of E vanishes off $Z(E)$. $\square$

Now we shall establish a number of properties of the automorphism group of E.

Lemma 2.6.2. *The automorphism group of E is $(C_p \times C_p) \rtimes GL(2,p)$.*

Proof: The proof of this fact can be found in [25, Theorem 8.20]. $\square$

Suppose that E occurs as a defect group of a p-block B of some finite group G, and that (E, b_E) is a maximal (G,B)-subpair. Since $N_G(E, b_E)/C_G(E)$ is a subgroup of the automorphism group of E, the inertial subgroup of each irreducible character of E in $N_G(E, b_E)$ is a homomorphic image of a subgroup of $Aut(E)$. According to [49, Chapter 13, Lemma 13.1] or [53, Chapter 5, Theorem 5.16 (b)], $I_{N_G(E,b_E)}(\eta)/EC_G(E)$ is a $p^{'}$-subgroup of $Aut(E)$ for each irreducible character η of E. On the other hand, each maximal subgroup of E can be regarded as a vector space over $GF(p)$. Let M be a maximal subgroup of E. Then $M = \langle e, z \mid e^p = z^p = [x,z] = 1\rangle$, where $e \in E - Z(E)$. It is well-known that $GL(2,p)$ is the automorphism group of M, (see for instance, [3, Chapter 2, Section 4, Proposition 1]).

It is clear that $SL(2,p)$ is a normal subgroup of the automorphism groups of both E and its maximal subgroup M. Now let η be an element of $GL(2,p)$, we can then represent η as the following matrix:

$$\eta = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \text{ where, } a_{ij} \in GF(p),\ i,j \in \{1,2\} \text{ and } det(\eta) \neq 0. \tag{2.6.3}$$

We can define the action of η on E as follows:

$$x^{\eta} = x^{a_{11}} y^{a_{12}}. \tag{2.6.4}$$

$$y^{\eta} = x^{a_{21}} y^{a_{22}}. \tag{2.6.5}$$

Now it is clear that if g_1, g_2 and g_3 are arbitrary elements of a group G, then

$$[g_1, g_2 g_3] = [g_1, g_3][g_1, g_2]^{g_3}.$$

Similarly

$$[g_1 g_2, g_3] = [g_1, g_3]^{g_2}[g_2, g_3].$$

Now in our group E, $[g_1, g_2] \in Z(E)$ for all g_1 and g_2 in E. Then the above identities reduce to

$$[g_1, g_2 g_3] = [g_1, g_3][g_1, g_2] = [g_1, g_2][g_1, g_3]$$

and

$$[g_1 g_2, g_3] = [g_1, g_3][g_2, g_3].$$

Therefore, using 2.6.4 and 2.6.5 and repeating the arguments above for a suitable number of times, we obtain

$$z^{\eta} = [x, y]^{\eta} = [x^{\eta}, y^{\eta}] = [x^{a_{11}} y^{a_{12}}, x^{a_{21}} y^{a_{22}}] = [x, y]^{a_{11}a_{22} - a_{12}a_{21}} = z^{det(\eta)}. \tag{2.6.6}$$

Lemma 2.6.3. *$SL(2, p)$ acts trivially on the centre of E. Furthermore, no larger subgroup of $GL(2, p)$ than $SL(2, p)$ acts trivially on the centre of E.*

Proof: Since $Z(E) = \langle z \rangle$ and every element in $SL(2, p)$ has the determinant 1, the result follows from 2.6.6. $\square$

Consider now the group $C_2 \times C_2$ as a subgroup of the automorphism group of E. Then we can represent this group as follows:

$$\langle a, b : a^2 = b^2 = (ab)^2 = c^2 = 1 \rangle \tag{2.6.7}$$

where

$$1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ a = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ b = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \text{ and } c = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{2.6.8}$$

Then, we see the action of $C_2 \times C_2$ on E as follows:

$$x^a = x, \ y^a = y^{-1}, \ z^a = [x, y]^a = [x^a, \ y^a] = [x, y^{-1}]. \tag{2.6.9}$$

It is clear that

$$z^a = z^{-1}, \tag{2.6.10}$$

as $det(a) = -1$. This means that a inverts the centre of E. Similarly, b inverts $Z(E)$ and c centralizes $Z(E)$.

The following lemma is useful for counting the number of extensions of the characters.

Lemma 2.6.4. *We have,* $1 = EC_G(E) \cap \langle a \rangle = EC_G(E) \cap \langle b \rangle = EC_G(E) \cap \langle c \rangle$.

Proof: This is clear from the relations above. □

Now let us fix P to be an arbitrary Sylow p-subgroup of $SL(2, p)$. It is well-known that $SL(2, p)$ has $p + 1$ Sylow p-subgroups. One can see that $N_{SL(2,p)}(P)/P \cong C_{p-1}$. Furthermore $C_{SL(2,p)}(P)/P \cong Z(SL(2, p)) \cong C_2$. Let us record the following lemma:

Lemma 2.6.5. *With the notation above,* $N_{SL(2,p)}(P)/C_{SL(2,p)}(P) \cong C_{\frac{p-1}{2}}$.

Proof: This is clear from the relations above. $\square$

Lemma 2.6.6. *$SL(2, p)$ is generated by its p-elements and has a trivial Schur multiplier for $p \geq 5$.*

Proof: The proof for the first statement can be found in [38, Chapter 2, Theorem 8.4], while the other half can be found in [43, Section 7.1, Theorem 7.1.1 (i)] or [70]. $\square$

The following theorem is very useful for us in certain places in this thesis.

Theorem 2.6.7. *Every proper subgroup of $SL(2, p)$ of order divisible by p has a unique Sylow p-subgroup. This means that the largest normal p-subgroup of such a group is non-trivial.*

Proof: Since, on the one hand, $SL(2, p)$ has order $(p-1)p(p+1)$, it has $p+1$ Sylow p-subgroups, each of which is a cyclic p-subgroup of order p. Now let H be a proper subgroup of $SL(2, p)$ with order divisible by p. Thus, H has at most $p+1$ Sylow p-subgroups. On the other hand, the number of Sylow p-subgroups of H is congruent to $1 \ mod\ (p)$ and divides $p+1$. Therefore, H has a unique Sylow p-subgroup which is a necessarily normal p-subgroup of H or it has $p+1$ Sylow p-subgroups. However, if H has $p+1$ Sylow p-subgroups then H contains all Sylow p-subgroups of $SL(2, p)$. In this case, the first part of Lemma 2.6.6 implies that $H = SL(2, p)$, which is not the case, as H is a proper subgroup of $SL(2, p)$. $\square$

Chapter 3

FUSION PATTERNS AND THE ORBIT STRUCTURE ON CHARACTERS

3.1 Introduction

In this chapter, our main concerns are fusion patterns and the orbit structure of irreducible characters. Thus, we start in Section 3.2 to discuss the notion of nilpotent p-blocks. Next, we study the action of the inertial quotient on a certain Frattini quotient in Section 3.3. The action of $SL(2, p)$ on a maximal subgroup of the defect group E is introduced in Section 3.4. Section 3.5 is devoted to the fusion patterns and cancellation theorems.

3.2 Nilpotent p-blocks and p-nilpotent groups

In this section, we shall discuss the notions of nilpotent p-blocks and p-nilpotent groups. We shall follow [13] and [46] for the definition of nilpotent p-blocks.

Definition 3.2.1. *Let G be a finite group, B be a p-block of G with a defect group D. Let (D, b_D) be a maximal (G, B)-subpair. We say that B is a nilpotent p-block if the quotient group $N_G(Q, b_Q)/C_G(Q)$ is a p-group, whenever $(Q, b_Q) \leq (D, b_D)$.*

It is clear that p-blocks of defect zero and p-blocks with central defect groups are examples of nilpotent p-blocks.

Recall that the subgroup of G which is generated by all p-regular elements is a normal subgroup, which we denoted by $O^p(G)$. Note that $O^p(G)$ is the minimal normal subgroup of G such that $G/O^p(G)$ is a p-group. Therefore, Definition 3.2.1 says that B is a nilpotent p-block when $O^p(N_G(Q, b_Q)) \leq C_G(Q)$, for each (G, B)-subpair $(Q, b_Q) \leq (D, b_D)$. However, in general, $O^p(G)$ need not be a p'-group.

The following definition appears in [67].

Definition 3.2.2. *Let B be a p-block of G with defect group D. Fix a maximal (G, B)-subpair (D, b_D). A (G, B)-subpair (Q, b_Q) which is contained in (D, b_D) is called a p-radical (G, B)-subpair if, $O_p(N_G(Q, b_Q)/QC_G(Q)) = 1$.*

A p-nilpotent group is one which has a normal p-complement. Let us record the definition of a p-nilpotent group.

Definition 3.2.3. *Let X be a finite group. We say that X is a p-nilpotent group if $X = O_{p'}(X) \rtimes P$, where P is a Sylow p-subgroup of X and $O_{p'}(X)$ is the unique largest normal p'-subgroup of X.*

We are in the situation that $O_{p'}(X) = O^p(X)$. Hence, p does not divide the order of the group $O^p(X)$ in this case.

Let us now state the following theorem which links nilpotent p-blocks with p-nilpotent groups. In fact, Theorem 3.2.4 below is the criterion for p-nilpotency which is due to Frobenius, see [49, Chapter 14] and [71, Chapter 7, §49] for more details.

Theorem 3.2.4. *Let G be a finite group and denote its principal p-block by $B_0(G)$. Then G is a p-nilpotent group if, and only if, $B_0(G)$ is a nilpotent p-block.*

Proof: Let P be a Sylow p-subgroup of G. Then P is a defect group of the principal p-block $B_0(G)$. Now if G is a p-nilpotent group then $N_G(Q)/C_G(Q)$ is a p-group for each $Q \leq P$. Hence, $B_0(G)$ is a nilpotent p-block.

Conversely, assume that $B_0(G)$ is a nilpotent p-block. Then $N_G(Q)/C_G(Q)$ is a p-group, for each $Q \leq P$. This is equivalent to saying that G is a p-nilpotent group. $\square$

Lemma 3.2.5. *Assume that E is a Sylow p-subgroup of the finite group G. Then, the centralizer $C_G(M)$ is a p-nilpotent group. In particular, $C_G(M) = M \times O_{p'}(N_G(M, b_M))$.*

Proof: It is clear that M and $O_{p'}(N_G(M, b_M))$ are normal subgroups of $C_G(M)$ and $M \cap O_{p'}(N_G(M, b_M)) = 1$. Moreover, $C_G(M) = M \cdot O_{p'}(N_G(M, b_M))$. However, $([C_G(M) : M], |M|) = 1$. Thus, the Schur-Zassenhaus Theorem (see [9, 18.1, p. 70]) completes the proof of Lemma 3.2.5. $\square$

Corollary 3.2.6. *For the principal p-block $B_0(G)$ with defect group E which is an extra-special p-group of order p^3 and exponent p, we have $C_G(E) = Z(E) \times O_{p'}(N_G(E, b_E))$.*

Proof: It is clear that $Z(E)$ is an abelian normal Sylow p-subgroup of $C_G(E)$. Therefore, the result follows from the Schur-Zassenhaus Theorem (see [9, 18.1, p. 70]). $\square$

Lemma 3.2.7. *Let B be a nilpotent p-block of the finite group G with defect group D. Then, B has a unique irreducible Brauer character. Furthermore, if D is abelian then B has $|D|$ irreducible ordinary characters.*

Proof: The proof can be found in [13]. See also [11]. $\square$

Lemma 3.2.8. *Let H be a normal subgroup of the finite group G such that G/H is a p-group. Let b be a G-stable nilpotent p-block of H. Then b is nilpotent as a p-block of G.*

Proof: This is [15, Theroem 2] $\square$

Let us now try to exploit these facts about M, E and their centralizers.

Corollary 3.2.9. *Let G be a finite group with a Sylow p-subgroup E, which is an extra-special p-group of order p^3 and exponent p for an odd prime p. Let M be an arbitrary maximal subgroup of E. Then, each p-block of $C_G(M)$ and of $C_G(E)$ is nilpotent.*

Proof: This is clear because M and $Z(E)$ are central p-subgroups of $C_G(M)$ and of $C_G(E)$, respectively. $\square$

3.3 The faithful action of the inertial quotient on the Frattini quotient

In this section, we shall be concerned with the action of the inertial quotient of an arbitrary (G, B)-subpair on a certain elementary abelian p-group.

Let U be an arbitrary p-subgroup of G. Consider a p-block idempotent b_U in $C_G(U)$ such that $b_U^G = B$. Then (U, b_U) is a (G, B)-subpair. The normalizer subgroup of (U, b_U) is $N_G(U, b_U) = \{g \in G \mid U^g = U \quad \& \quad b_U^g = b_U\}$.

The Frattini subgroup of U which is denoted by $\Phi(U)$ is the intersection of all maximal subgroups of U. Since U is a p-group, each maximal subgroup of U is a normal subgroup of index p. It follows that, if N is a maximal subgroup of U, then $U/N \cong C_p$. On the one hand, U/N is an abelian group for each maximal subgroup N of U implies that $U' = [U, U] \leq \Phi(U)$, where $U' = [U, U]$ is the commutator subgroup of U. Consequently, $U/\Phi(U)$ is an abelian group, as it is a subgroup of the abelian group U/U'.

On the other hand, if $u \in U$, then $N = (uN)^p = u^p N$. So, for each $u \in U$, $u^p \in N$. This situation recurs for each maximal subgroup of U. Hence, $u^p \in \Phi(U)$ for each $u \in U$. This means that the Frattini quotient $\bar{U} := U/\Phi(U)$ is an elementary abelian p-group. From now on, $\bar{u}$ will be written instead of $u\Phi(U)$ for any element of $\bar{U}$.

Now if $g \in N_G(U, b_U)$, then $u^g \in U$ for each $u \in U$. Thus, $N_G(U, b_U)$ acts on $\bar{U}$ by the rule $\bar{u}^g = \overline{u^g}$.

Lemma 3.3.1. *Let G be a finite group, let B be a p-block of G, let (U, b_U) be a (G, B)-subpair. Then the inertial subgroup $N_G(U, b_U)$ acts on the Frattini quotient $\bar{U}$.*

Proof: This is clear from the discussion above. □

Now U is a normal subgroup of $N_G(U, b_U)$. Then, we consider the restriction of the action which has been defined above to U. We claim that U acts trivially on $\bar{U}$. To show this, let $\bar{x} \in \bar{U}$ and $u \in U$. Then

$$\bar{x}^u = \overline{x^u}. \tag{3.3.1}$$

We observe that

$$x^u = 1_G \cdot u^{-1}xu = x(x^{-1}u^{-1}xu) = x \cdot [x, u]. \tag{3.3.2}$$

However, $[U, U] \leq \Phi(U)$, and hence, $[x, u] \in \Phi(U)$. So, we conclude that

$$\bar{x}^u = \overline{x^u} = \bar{x}\overline{[x, u]} = \bar{x}. \tag{3.3.3}$$

This means that U acts trivially on the Frattini quotient $\bar{U}$. Similarly, we can consider the restriction of the action above from $N_G(U, b_U)$ to its normal subgroup $C_G(U)$. Again, we claim that $C_G(U)$ acts trivially on $\bar{U}$. To see this, observe that if $g \in C_G(U)$ then $u^g = u$ for each $u \in U$. Thus, obviously, $\bar{u}^g = \overline{u^g} = \bar{u}$. As a result, the normal subgroup $UC_G(U)$ of $N_G(U, b_U)$ acts trivially on the Frattini quotient $\bar{U}$.

Lemma 3.3.2. *With the notation as above, $UC_G(U)$ acts trivially on $\bar{U}$.*

Proof: The proof is clear from the discussion above. □

Now let us consider $N_G(U, b_U)/UC_G(U)$. It is a subgroup of the automorphism group of U. We define the action of this group on $\bar{U}$ as follows:

Definition 3.3.3. *We define the action of the inertial quotient $N_G(U, b_U)/UC_G(U)$ on the Frattini quotient by the rule: for each $\bar{g} \in N_G(U, b_U)/UC_G(U)$, and for each $\bar{u} \in \bar{U}$,*

$$\bar{u}^{\bar{g}} := \overline{u^g}. \tag{3.3.4}$$

Note that Definition 3.3.3 makes sense because $UC_G(U)$ acts trivially on $\bar{U}$. Accordingly, there is a group homomorphism

$$\rho : N_G(U, b_U)/UC_G(U) \to Aut(\bar{U});$$

such that for all $\bar{g} \in N_G(U, b_U)/UC_G(U)$,

$$\rho(\bar{g}) = \rho_{\bar{g}} \ : \bar{U} \to \bar{U}; \tag{3.3.5}$$

where

$$\rho_{\bar{g}}(\bar{x}) = \overline{x^g}, \tag{3.3.6}$$

for each $\bar{x} \in \bar{U}$.

What is the kernel of this action? It is the kernel of the homomorphism ρ. Thus, it is a normal subgroup of $N_G(U, b_U)/UC_G(U)$.

The following proposition guarantees that this normal subgroup of the inertial quotient $N_G(U, b_U)/UC_G(U)$ is a p-group. Note that if ψ is an automorphism of a group and g is an element in this group, then the commutator element $[g, \psi]$ is the element $g^{-1} \cdot g^{\psi}$, where g^{ψ} is the image of g under ψ.

Proposition 3.3.4. *The kernel of the action of the inertial quotient $N_G(U, b_U)/UC_G(U)$ on $\bar{U}$ is a p-group.*

Proof: Let $\bar{g}$ be an arbitrary p-regular element in this kernel. Then $\bar{g} = g \cdot UC_G(U)$ for some p-regular element $g \in N_G(U, b_U)$ and

$$\rho_{\bar{g}}(\bar{x}) = id_{\bar{U}}(\bar{x}) = \bar{x}, \tag{3.3.7}$$

for each $\bar{x} \in \bar{U}$.

Consequently,

$$\overline{x^g} = \bar{x}, \tag{3.3.8}$$

for each $\bar{x} \in \bar{U}$. So,

$$\bar{x}^{-1} \cdot \overline{x^g} = \bar{1}, \tag{3.3.9}$$

for each $\bar{x} \in \bar{U}$. Therefore,

$$\bar{x}^{-1} \cdot \overline{x}^g = \bar{x}^{-1} \cdot \overline{x}^{\bar{g}} = [\bar{x}, \bar{g}] = \bar{1}, \tag{3.3.10}$$

for each $\bar{x} \in \bar{U}$. Hence, $[\bar{U}, \bar{g}] = \bar{1}$.

Now $\bar{g}$ is a p-regular element of the automorphism group of U, which induces the identity automorphism of $\bar{U}$. Hence, by a theorem of W. Burnside (see [38, Chapter 5, Theorem 1.4]), we deduce that

$$[x, g] = 1, \tag{3.3.11}$$

for each $x \in U$. We can read 3.3.11 as that $g \in C_G(U)$. Consequently, $g \in UC_G(U)$. Therefore, $\bar{g} = \bar{1}$.

Now we are in the situation that each p-regular element of $N_G(U, b_U)$ which acts trivially on the Frattini quotient $\bar{U}$ is a member of $UC_G(U)$. As a result, the kernel of this action is a normal p-subgroup in the inertial quotient $N_G(U, b_U)/UC_G(U)$. This completes the proof. $\square$

The first benefit from Proposition 3.3.4, is the following corollary:

Corollary 3.3.5. *Let G be a finite group, B be a p-block of G with a defect group D. Then, for each maximal (G, B)-subpair (D, b_D), $N_G(D, b_D)/DC_G(D)$ acts faithfully on the Frattini quotient $D/\Phi(D)$.*

Proof: By Proposition 3.3.4, the kernel of the action of $N_G(D, b_D)/DC_G(D)$ on the Frattini quotient $D/\Phi(D)$ is a normal p-subgroup of $N_G(D, b_D)/DC_G(D)$. But, according to [49, Chapter 13, Lemma 13.1] or [53, Chapter 5, Theorem 5.16 (b)], $N_G(D, b_D)/DC_G(D)$ is a p'-group. Hence, $N_G(D, b_D)/DC_G(D)$ acts faithfully on the Frattini quotient $D/\Phi(D)$.
$\square$

Another consequence of Proposition 3.3.4 is the following corollary:

Corollary 3.3.6. *Let G be a finite group, B be a p-block of G. Then for each maximal (G, B)-subpair (D, b_D), $N_G(D, b_D)/DC_G(D)$ is isomorphic to a p'-subgroup of the automorphism group of $D/\Phi(D)$.*

Proof: This is clear from Proposition 3.3.4. $\square$

Note that Corollary 3.3.5 and Corollary 3.3.6 are true for each maximal (G, B)-subpair (D, b_D) without any further assumption on D. However, if D is an elementary abelian p-group, then $\Phi(D) = 1$. Hence, we are in the old situation.

Now let us return back to the case that D is an extra-special p-group of order p^3 and exponent p for an odd prime number p, which we prefer to denote by E. Assume that E is generated by x, y and z, where z is the generator of the centre of E. Let M be a maximal subgroup of E. It is clear that $M = \langle g, z \rangle$, where g is a non-central element of E. Let us assume that $g = x$.

According to Corollary 3.3.6, $N_G(E, b_E)/EC_G(E)$ is isomorphic to a p'-subgroup of the automorphism group of $E/\Phi(E)$. On the one hand, $E/\Phi(E) \cong Inner(E)$, where $Inner(E)$ is the inner automorphism group of E. But, we know that $Aut(E) = Inner(E) \rtimes GL(2, p)$. On the other hand, $E/\Phi(E) = \langle x\Phi(E), y\Phi(E) \rangle \cong C_p \times C_p$. Write $\bar{E}$ for $E/\Phi(E)$. So, $Aut(\bar{E}) \cong GL(2, p)$. Also $Aut(M) \cong GL(2, p)$. Therefore, in any case, we have to work in $GL(2, p)$. Note that $\langle x, y \rangle$ is not a maximal subgroup of E. In fact, $E/\Phi(E) = \langle x\Phi(E), y\Phi(E) \rangle = [\langle x, y \rangle \Phi(E)]/\Phi(E)$. So, $E = \langle x, y \rangle \Phi(E)$ and, hence, $E = \langle x, y \rangle$. Now $N_G(M, b_M)/C_G(M)$ is a subgroup of $GL(2, p)$. Then, there are three cases which should be distinguished:

1. $N_G(M, b_M)/C_G(M)$ is a p'-group. But this case does not occur, because it would lead us to the situation that $E \leq C_G(M)$, which is not the case.

2. $N_G(M, b_M)/C_G(M)$ has a unique Sylow p-subgroup, which is necessarily a normal p-subgroup. According to Proposition 3.3.4, it is the kernel of the action of $N_G(M, b_M)/C_G(M)$ on M. Thus, its inverse image $EC_G(M)$ will be a normal subgroup of $N_G(M, b_M)$.

3. $N_G(M, b_M)/C_G(M)$ has $p + 1$ Sylow p-subgroups, and, hence contains $SL(2, p)$.

Hence, $O_p(N_G(M, b_M)/C_G(M) = 1$. It follows that each such (G, B)-subpair is a p-radical.

Now the set of maximal subgroups of E can be regarded as an $N_G(E)$-set. This is because, if M is a maximal subgroup of E, then M^g is also a maximal subgroup of E for each $g \in N_G(E)$. Therefore, under this action, this set is partitioned into $N_G(E)$-conjugacy classes. It is important to know exactly what the G-conjugacy classes are of the maximal subgroups of E which are radical p-subgroups of G. Note that the stabilizer of M under this action is $N_G(M) \cap N_G(E)$. Similarly, the set of (G, B)-subpairs which are contained in the maximal (G, B)-subpair (E, b_E) is an $N_G(E, b_E)$-set. In this case, the stabilizer of (M, b_M) is the subgroup $N_G(M, b_M) \cap N_G(E, b_E)$ of G.

Lemma 3.3.7. *There is a one to one correspondence between maximal subgroups of E and the maximal subgroups of $E/Z(E)$.*

Proof: This is clear because $\Phi(E) = Z(E)$. $\square$

Now Lemma 3.3.7, gives us the opportunity to deal with maximal subgroups of $E/Z(E)$. We shall consider the action of $N_G(E, b_E)/EC_G(E)$ on the maximal subgroups of $E/Z(E)$.

Lemma 3.3.8. *E has $p + 1$ maximal subgroups, each of which contains the centre of E.*

Proof: Let $\overline{M}$ be a maximal subgroup of $E/Z(E)$. Then, $\overline{M}$ has order p, and hence, $\overline{M}$ has $p - 1$ nontrivial elements. Since $E/Z(E)$ has order p^2, we must have $\frac{p^2-1}{p-1} = p + 1$ maximal subgroups of $E/Z(E)$. Hence, Lemma 3.3.7 completes the first part of the lemma. Now, if there is a maximal subgroup of E which does not contain $Z(E)$, then this is a contradiction with the definition of the Frattini subgroup of E. Therefore, each maximal subgroup of E contains the centre of E. $\square$

Corollary 3.3.9. *Let G be a finite group with an extra-special p-subgroup E of order p^3 and exponent p, for an odd prime number p. Assume that $N_G(E, b_E)/EC_G(E) \cong C_2 \times C_2$. Then, the action of $N_G(E, b_E)$ on the maximal subgroups of E, has $\frac{p+3}{2}$ orbits.*

Proof: According to Lemma 3.3.7, it is enough to study the action on the corresponding maximal subgroups of $\bar{E} := E/Z(E)$. Since $EC_G(E)$ acts trivially on the maximal subgroups of $\bar{E}$, we have to consider the action of $N_G(E, b_E)/EC_G(E)$ on this set. Now, counting fixed points, by a familiar counting argument which is attributed to Burnside [55, Theorem 9.1, p. 100], we have the number of orbits under this action. Note that we are seeking for a set-wise fixed point, as we are dealing with maximal subspaces. Consequently, using the notation in Section 2.6, the automorphisms a and b fix only 2 maximal subspaces, while the automorphisms 1 and c fix all maximal subspaces. As a result, the number of orbits is $\frac{p+3}{2}$. $\square$

3.4 The natural action of $SL(2, p)$ on a maximal subgroup of E.

Let p be an odd prime number. Let M be an arbitrary maximal subgroup of E. Regarding $M = \langle x, z \rangle$ as a vector space over $GF(p)$, $SL(2, p)$ acts on M in a natural way. We shall discuss this action of $SL(2, p)$ on M. Consider $H \cong M \rtimes SL(2, p)$. Now H is a finite group of order $(p-1)p^3(p+1)$. This group has a Sylow p-subgroup, say S, of the form $M \rtimes \langle \eta \rangle$ where η is the linear map $\eta : M \to M$, such that $\eta(x) = x + z$ and $\eta(z) = z$. We observe that η is of order p. Consequently, S is an extra-special p-group of order p^3 and exponent p. We claim that H has a unique p-block, namely the principal p-block $B_0(H)$. To show this, it is enough (see [32, Corollary 3.11, Chapter V, p. 200]), to find a normal p-subgroup of H which is self-centralizing in H. On the basis of the construction above, M is a normal p-subgroup of H. So, we have to prove that $C_H(M) \cong M$. However,

$C_H(M) = M \rtimes C_{SL(2,p)}(M)$. The following proposition completes the claim that M is self-centralizing in H.

Proposition 3.4.1. *Regarding M as a vector space over $GF(p)$,*

$$C_{SL(2,p)}(M) = \{s \in SL(2,p) : \quad s \cdot m = m, \quad \forall\ m \in M\} = 1. \tag{3.4.1}$$

Proof: Let us write $M = \langle x, z \rangle$, where $x := \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $z := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. One can identify maximal subgroups of M with maximal subspaces of M. Then M has $p+1$ maximal subspaces, namely,

$$\mathscr{M}(M) := \{H_\infty, \quad H_0, \quad H_\lambda : \ \lambda \in GF(p) - \{0\}\}, \tag{3.4.2}$$

where,

$$H_\infty := \{ \begin{bmatrix} i \\ 0 \end{bmatrix} : \ i \in GF(p)\}, \tag{3.4.3}$$

$$H_0 := \{ \begin{bmatrix} 0 \\ i \end{bmatrix} : \ i \in GF(p)\} \tag{3.4.4}$$

and

$$H_\lambda := \{ \begin{bmatrix} i \\ i\lambda \end{bmatrix} : \ i \in GF(p) \ \& \ \lambda \in GF(p) - \{0\}\}. \tag{3.4.5}$$

Now an easy matrix computation shows that

$$C_{SL(2,p)}(H_\infty) = \{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} : b \in GF(p) \} \cong C_p,$$

$$C_{SL(2,p)}(H_0) = \{ \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} : b \in GF(p) \} \cong C_p,$$

and

$$C_{SL(2,p)}(H_\lambda) = \{ \begin{bmatrix} 1-\lambda b & b \\ -\lambda^2 b & 1+\lambda b \end{bmatrix} : b \in GF(p) \} \cong C_p.$$

Therefore, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the unique element of $SL(2,p)$, which fixes each element of M. This completes the proof of the proposition. $\square$

Corollary 3.4.2. *There is a one to one correspondence between the maximal subspaces of M and Sylow p-subgroups of $SL(2,p)$.*

Proof: By the proof of Proposition 3.4.1, we have seen that each maximal subspace of M has a Sylow p-subgroup as a stabilizer group. Then the map sending a maximal subspace to its stabilizer is a one to one map between two sets with the same cardinality. Thus each Sylow p-subgroup of $SL(2,p)$ is the stabilizer of a unique maximal subspace of M. This establishes the required correspondence. $\square$

Now by Proposition 3.4.1, $C_{SL(2,p)}(M) = 1$, and, hence, $C_H(M) \cong M$. Thus, it is implied [32, Corollary 3.11, Chapter V, p. 200] that H has a unique p-block.

With the same argument, one can show also that the normalizer of the Sylow p-subgroup S in H has only one p-block. Therefore, they correspond to each other by Brauer's Third Main Theorem on p-blocks [20, Theorem 61.16, Chapter 7, §61 C].

On the one hand, $N_H(M) = H$, because M is a normal subgroup of H. Thus, $N_H(M)/C_H(M) = H/M \cong SL(2,p)$. On the other hand, it is clear that $N_H(S)/SC_H(S) \cong C_{p-1}$.

Lemma 3.4.3. *With the notation above, $O_p(H) = M \not\leq Z(H)$.*

Proof: This is clear from the construction above. $\square$

Now let X denote the set of non-zero vectors of M. Then the cardinality of X is p^2-1. Consequently, $SL(2,p)$ acts on X. We claim that this action is transitive. To see this, let a and b be arbitrary distinct elements in X. Then there are $i\ ,\ j,\ k,\ l \in \{0,1,2,\cdots,p-1\}$ such that

$$a = \begin{bmatrix} i \\ j \end{bmatrix} \neq \begin{bmatrix} k \\ l \end{bmatrix} = b.$$

Now if $i = 0 = k$, then $0 \neq j \neq l \neq 0$, and we have,

$$\begin{bmatrix} l^{-1}j & 0 \\ 0 & j^{-1}l \end{bmatrix} \cdot \begin{bmatrix} 0 \\ j \end{bmatrix} = \begin{bmatrix} 0 \\ l \end{bmatrix}.$$

Hence, we may assume that $i \neq 0 \neq k$. Then the matrix

$$s := \begin{bmatrix} i^{-1}(k-jb) & b \\ i^{-1}(l-jk^{-1}(i+lb)) & k^{-1}(i+lb) \end{bmatrix}$$

satisfies

$$s \cdot a = \begin{bmatrix} i^{-1}(k-jb) & b \\ i^{-1}(l-jk^{-1}(i+lb)) & k^{-1}(i+lb) \end{bmatrix} \cdot \begin{bmatrix} i \\ j \end{bmatrix} = \begin{bmatrix} k \\ l \end{bmatrix} = b.$$

The above calculations enable us to exploit the isomorphism as an $SL(2,p)$-set between the set of non-trivial irreducible ordinary characters of M and the set of non-zero vectors of M to deduce Theorem 3.4.4 below. This gives us the opportunity to reduce some alternating sums under consideration.

Theorem 3.4.4. *If $SL(2,p) \lneqq N_G(M,b_M)/C_G(M)$ then, $N_G(M,b_M)$ acts transitively on the set of non-trivial irreducible characters of M.*

Proof: This is clear from the discussion above and the fact that $C_G(M)$ acts trivially on the set of irreducible characters of M. □

Corollary 3.4.5. *With the assumption on Theorem 3.4.4, $N_G(M,b_M)$ has two orbits when it acts on the set of irreducible characters of M, namely, the trivial character and the other p^2-1 non-trivial irreducible characters of M.*

Proof: This is clear from the discussion above. □

Let us consider the restriction of this action to an arbitrary Sylow p-subgroup, say P, of $SL(2,p)$. In this situation, we have a non-trivial p-group acting on a set of cardinality p^2-1. So, the set of fixed points of X under this action is non-empty because $0 \not\equiv |X| \equiv |Fix_X(P)|\ (mod\ p)$, where $Fix_X(P)$ refers to the set of fixed points of X under the action by P . So, if $x \in Fix_X(P)$, then x generates a maximal subspace of M, which is invariant under the action by P. However, by Corollary 3.4.2, P fixes a unique maximal subspace of M. The conclusion from this discussion is the following theorem.

Theorem 3.4.6. *With the above notation, a Sylow p-subgroup of $SL(2,p)$ fixes $p-1$ non-zero vectors of M.*

Proof: The result follows from Corollary 3.4.2. □

Now let us consider the Sylow normalizer group, $N_{SL(2,p)}(P)$, where P is an arbitrary Sylow p-subgroup of $SL(2,p)$. It acts on the set of non-zero vectors of M. According to Corollary 3.4.2, its unique Sylow p-subgroup fixes $p-1$ non-zero vectors. As a result, we have the following theorem:

Theorem 3.4.7. *The Sylow normalizer $N_{SL(2,p)}(P)$ has two orbits when it acts on the set of non-zero vectors of M. Furthermore, one of these orbits has a stabilizer subgroup which is isomorphic to the unique cyclic p-group of order p and the other has the trivial subgroup as a stabilizer group.*

Proof: The typical element in $N_{SL(2,p)}(P)$ has the form $\begin{bmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{bmatrix}$, where $\alpha \in GF(p)-\{0\}$ and $\beta \in GF(p)$. It has $p^2 - p$ elements, the identity element, $p-1$ p-elements and a further p^2-2p elements. Obviously, the action is not transitive as x cannot be transformed into z, otherwise, α would be zero, which is not the case. Hence, by Burnside's Counting Theorem [27, Chapter XIII, Section 52]) we deduce that

$$\# \ \text{Orbits} = \frac{(p^2-1)+(p-1)(p-1)}{p^2-p} = 2. \tag{3.4.6}$$

Now, it is clear that the representatives for these two orbits are x and z. Further matrix calculation shows that $|[x]| = p-1$ and $|[z]| = p^2-p$. Thus, the Orbit-Stabilizer Theorem yields that each element of $[x]$ has a stabilizer subgroup which is a cyclic p-subgroup of order p, namely, the Sylow p-subgroup and each element of $[z]$ has the trivial subgroup as a stabilizer group. □

3.5 An attempt to understand the fusion patterns

Let G be a finite group, H be a subgroup of G. Assume that G acts on a non-empty set X. In our case, X may be a collection of p-subgroups or a collection of (G, B)-subpairs. One can consider the restriction of this action to the subgroup H of G. But many of the properties of the former action cannot be inherited to this restriction. In the following, we shall investigate under which condition we might find an action with an element of

G which could be performed by an element from H. This kind of phenomenon is called fusion. Since G acts on the collection of all (G, B)-subpairs, we shall consider that X above is a collection of (G, B)-subpairs which are contained in a maximal (G, B)-subpair. We start with the definition of fusion.

Definition 3.5.1. *Let G be a finite group, H be a subgroup of G, and let X be a collection of (G, B)-subpairs. We say that H controls the fusion of X if, whenever (Q, b_Q) and (P, b_P) are two arbitrary elements in X with $(Q, b_Q)^g = (P, b_P)$ for some $g \in G$ then $g = ch$ for some $c \in C_G(Q)$ and $h \in H$.*

Let us go back to considering E as an extra-special p-group of order p^3 and exponent p, for an odd prime number p. In the following, we discuss the possibilities of chains within E. Note that M_i refers to a maximal normal subgroup of E of order p^2 and exponent p.

1. $\sigma_1 : 1_G$.
2. $\sigma_2 : \langle e \rangle$, $e \notin Z(E)$
3. $\sigma_3 : \langle e \rangle < M_i$, $e \notin Z(E)$, where $i = 1, \cdots, p+1$.
4. $\sigma_4 : \langle e \rangle < M_i < E$, $e \notin Z(E)$, where $i = 1, \cdots, p+1$.
5. $\sigma_5 : M_i$, $i = 1, \cdots, p+1$.
6. $\sigma_6 : Z(E)$.
7. $\sigma_7 : Z(E) < M_i$, $i = 1, \cdots, p+1$.
8. $\sigma_8 : M_i < E$, $i = 1, \cdots, p+1$.
9. $\sigma_9 : Z(E) < M_i < E$, $i = 1, \cdots, p+1$.
10. $\sigma_{10} : E$.

Lemma 3.5.2. *Let E be an extra-special p-group of order p^3 and exponent p for an odd prime number p in a finite group G. Let M be a maximal subgroup of E which is a radical p-subgroup of G. Then, $N_G(M) \not\leq N_G(E)$.*

Proof: Suppose that $N_G(M) \leq N_G(E)$. Then, Lemma 2.3.4 implies that $E \leq M$, which is not the case. $\square$

The contribution from chains which start with a non-radical p-subgroups is zero (see [63, Corollary 1.2]).

Recall that an indecomposable G-module, say N, is called an H-projective G-module if there exists an H-module L such that N is a direct summand of the induced G-module $Ind_H^G(L)$. For more details for this account see [19, Chapter 2, §19]. There is an analogous character theory. So, we give the following definition:

Definition 3.5.3. *Let G be a finite group with normal p-subgroup U. Let χ be an irreducible character of G which is afforded by a U-projective RG-module. Then, we say that χ is a U-projective irreducible character of G with respect to the p-modular system $(\mathbb{K}, R, F)$.*

For more details of this direction, see [47] or [60]. However, in the following, we state the theorems which are the main tools for counting the numbers of p-blocks of defect zero lying over certain irreducible characters of a normal p-subgroup of the finite group G.

It is well-known that p-blocks of defect zero of the finite group G are in a one to one correspondence with irreducible ordinary characters χ of G in such a way that $\chi(1)_p = |G|_p = [G : 1_G]_p$, (see [51, Chapter I, Proposition 16.1]). A generalization of this fact for U-projective characters where U is a normal p-subgroup of the finite group G is due to Burkhard Külshammer and Geoffrey R. Robinson, see [47] or [61].

Lemma 3.5.4. *Let U be a normal p-subgroup of the finite group G. Then, there is a one to one correspondence between U-projective irreducible characters of G which lie over an* *irreducible character η of U and the p-blocks of defect zero of $I_G(\eta)/U$.*

Proof: This is [47, Proposition 3.2 (ii) & (iii)]. $\square$

Lemma 3.5.5. *Let U be a normal p-subgroup of the finite group G and let χ be an irreducible ordinary character of G, which belongs to a p-block with a defect group D. Assume that $C_D(U) = Z(U)$. Then, χ is U-projective if, and only if, $(\frac{\chi(1)}{\eta(1)})_p = |G/U|_p$, for each irreducible character η of U which appears in $Res_U^G(\chi)$.*

Proof: This is [61, Lemma 4.4]. $\square$

It is clear that if χ is a U-projective irreducible character of the finite group G, then χ is also $UC_G(U)$-projective. However, the converse holds for normal p-subgroups U of G which are self-centralizing in the defect group of the p-block which contains χ.

Lemma 3.5.6. *Let U be a normal p-subgroup of the finite group G and let χ be an irreducible ordinary character of G. Assume that $C_D(U) = Z(U)$, where D is a defect group for a p-block of G which contains χ. Then, χ is U-projective if, and only if, χ is $UC_G(U)$-projective.*

Proof: The proof can be found in [67]. $\square$

Lemma 3.5.7. [Reinhard Knörr] *Let G be a finite group with normal p-subgroup U. Let B be a p-block of G with defect group D. If B has an irreducible U-projective RG-module, then $C_D(U) = Z(U)$.*

Proof: The proof can be found in [61, Proposition 3.2]. $\square$

We return now to discuss a general setup for dealing with the cancellation problem. Let us consider an arbitrary p-subgroup of G, say Q. Let σ be the chain $V_\sigma < \cdots < V^\sigma$.

Then either $Q =_G V_\sigma$ or not and $Q =_G V^\sigma$ or not. In the following, we see that, for any σ which has a proper contribution, V_σ is self-centralizing in the defect group of B and V^σ must be a subgroup of the defect group of the p-block B of the finite group G.

Lemma 3.5.8. *Let B be a p-block of the finite group G with defect group D. If $k_d(B(\sigma)) \neq 0$, then, $V^\sigma \leq_G D$, where d is any non-negative integer.*

Proof: This is clear because $1_{B(\sigma)} = Br_{V^\sigma}(1_B)$ and D is a maximal p-subgroup (up to G-conjugacy) of G such that $Br_D(1_B) \neq 0$. □

The following theorem which is due Geoffrey R. Robinson will be very useful for us.

Theorem 3.5.9. [61, Geoffrey R. Robinson, Theorem 5.1] *Let G be a finite group, B be a p-block of G with defect group D. Then, if there is an irreducible character μ of V_σ such that $f_0^{(B)}(I_{N_G(\sigma)}(\mu)/V_\sigma) \neq 0$ then, $C_D(V_\sigma) = Z(V_\sigma)$.*

The benefits from Theorem 3.5.9 are the following corollaries.

Corollary 3.5.10. *Let G be a finite group, B be a p-block of G with defect group D. Then the only chains (up to G-conjugacy) which contribute to any alternating sum of the OWC and a minimal counter-example to DPC are those whose initial subgroup V_σ satisfies $C_D(V_\sigma) \subseteq V_\sigma$.*

Proof: The result follows from Theorem 3.5.9. □

Corollary 3.5.11. *Let G be a finite group, B be a p-block of G with a defect group E which is an extra-special p-group of order p^3 and an exponent p for an odd prime number p. Then, chains which have nontrivial contribution to the alternating sum of the OWC or the DPC satisfy $Z(E) < V_\sigma$.*

Proof: Assume that $V_\sigma = Z(E)$. Then by Theorem 3.5.9, $C_E(Z(E)) \subseteq Z(E)$. However, we observe that if $e \in E$ then $ez = ze$ for all $z \in Z(E)$, so $e \in C_E(Z(E))$. Thus,

$e \in Z(E)$. Hence, $E \leq Z(E)$. This contradicts with the fact that E is an extra-special p-group of order p^3. $\square$

As usual, E is an extra-special p-group of order p^3 and exponent p, for an odd prime p which can be written as $E = \langle x, y, z\rangle$ where z is the generator of the centre of E. Each element of E is of order p as E has exponent p. The following corollary indicates that we have to consider and deal only with the maximal normal subgroups of E with order p^2.

Lemma 3.5.12. *Let H be any cyclic subgroup of E with order p. Then $H \neq V_\sigma$, where σ is an arbitrary chain which has a proper contribution to the alternating sum under consideration.*

Proof: If $H = Z(E)$, then the result is the same as in Corollary 3.5.11. Thus, we may assume that $H = \langle x\rangle$, where x is one of the non central generators of E. If $H = V_\sigma$, then by Theorem 3.5.9, we face the following situation $\langle x, z\rangle = C_E(x) = C_E(\langle x\rangle) \subseteq H = \langle x\rangle$. But this is not the case, because $\langle x, z\rangle$ is an elementary abelian p-subgroup of E with order p^2. $\square$

Consequently, and in the language of (G, B)-subpair, we are left with the empty chain and the following (G, B)-subpairs: (M_i, b_{M_i}), $(M_i, b_{M_i}) < (E, b_E)$, $i = 1, 2, \cdots, n$, with $n \leq p + 1$, and (E, b_E). Note that b_{M_i} is a p-block of $C_G(M_i)$ and (E, b_E) satisfies the definition of maximal (G, B)-subpairs because E is the defect group of the p-block B. The following definition appears in [67].

Definition 3.5.13. *Let (U, b_U) be an arbitrary (G, B)-subpair. We say that (U, b_U) is an Alperin-Goldschmidt (G, B)-subpair if it is centric and a p-radical (G, B)-subpair.*

Lemma 3.5.14. *Let (U, b_U) be an arbitrary (G, B)-subpair which is not an Alperin-Goldschmidt (G, B)-subpair. Then the chains starting with (U, b_U) has zero contribution to the alternating sum under consideration.*

Proof: This is [44, Theorem 1.10]. $\square$

The following lemma of W. Burnside is the starting point for studying the fusion patterns of the above chains.

Lemma 3.5.15. *Let G be a finite group, S be a $Syl_p(G)$. Suppose that X and Y are normal subgroups of S, then X and Y are G-conjugate if, and only if, they are $N_G(S)$-conjugate.*

Proof: The proof can be found in [42, Lemma 12.30]. □

Let us now give a generalization of Burnside's Theorem in the case of a maximal (G, B)-subpair (D, b_D).

Lemma 3.5.16. *Let G be a finite group, B be a p-block of G with defect group D, and let (X, b_X), (Y, b_Y) be normal (G, B)-subpairs of the maximal (G, B)-subpair (D, b_D). Then $(X, b_X) =_G (Y, b_Y)$ if, and only if, $(X, b_X) =_{N_G(D,b_D)} (Y, b_Y)$.*

Proof: Since $DC_G(D) \leq N_G(X, b_X) \leq G$, for a p-block of $N_G(X, b_X)$, say b, b^G is defined and $b^G = B$. Now D is a defect group of b in $N_G(X, b_X)$. Hence, (D, b_D) is a maximal $(N_G(X, b_X), b)$-subpair. So, the result follows from a similar way as in Lemma 3.5.15. □

Let us return to the defect group E of a p-block B of G, where E is an extra-special p-group of order p^3 and exponent p for an odd prime number p. Then E acts trivially on the maximal subgroups of $E/\Phi(E)$ and $C_G(E)$ also acts trivially on the maximal subgroups of $E/\Phi(E)$. So, $EC_G(E)$ acts trivially on the maximal subgroups of $E/\Phi(E)$. Therefore, if one studies the action of $N_G(E, b_E)/EC_G(E)$ on the maximal subgroups of $E/\Phi(E)$ and determines the orbits under this action, then by Lemma 3.5.16, these orbits are precisely the distinct conjugacy classes of the maximal subgroups of $E/\Phi(E)$ under the G-conjugation. But by Lemma 3.3.7, we know that there is a one to one correspondence between the maximal subgroups of E and the maximal subgroups of $E/\Phi(E)$, so to determine the G-conjugacy classes of maximal subgroups of E, we shall study the action of $N_G(E, b_E)/EC_G(E)$ on the maximal subgroups of the elementary abelian p-group $E/\Phi(E)$.

Proposition 3.5.17. *Write $\bar{N} = N_G(M, b_M)/C_G(M)$ and $\bar{E} := EC_G(M)/C_G(M) \cong E/M$. Then, we have $N_{\bar{N}}(\bar{E}) \cong N_G(M, b_M) \cap N_G(E, b_E)/C_G(M) \cap N_G(E, b_E)$.*

Proof: On the one hand,

$$N_G(M, b_M) \cap N_G(E, b_E)/C_G(M) \cap N_G(E, b_E)$$

$$\cong N_G(M, b_M) \cap N_G(E, b_E)/(C_G(M) \cap [N_G(M, b_M \cap N_G(E, b_E)])$$

$$\cong (N_G(M, b_M) \cap N_G(E, b_E))C_G(M)/C_G(M) \leq \bar{N}.$$

Also, if $gC_G(M) = \bar{g} \in \bar{N}$ with $g \in (N_G(E, b_E) \cap N_G(M, b_M))C_G(M)$. Then $(EC_G(M))^g = E^g C_G(M^g) = EC_G(M)$. This is true because both $N_G(M, b_M) \cap N_G(E, b_E)$ and $C_G(M)$ are subgroups of $N_G(EC_G(M))$. Hence, $\bar{g} \in N_{\bar{N}}(\bar{E})$. We conclude that

$$N_G(E, b_E) \cap N_G(M, b_M)/N_G(E, b_E) \cap C_G(M) \leq N_{\bar{N}}(\bar{E}).$$

On the other hand, $\bar{g} \in N_{\bar{N}}(\bar{E})$ if, and only if, $\bar{E}^{\bar{g}} = \bar{E}$, which happens if, and only if, $\bar{g} = gC_G(M)$, with $g \in N_G(M, b_M) \cap N_G(E, b_E)$. This completes the proof of the proposition. □

The following corollary is useful for counting the number of p-blocks of defect zero.

Corollary 3.5.18. *If the section, $N_G(M, b_M)/C_G(M) \cong SL(2, p)$, then the number of orbits of $N_G(M, b_M) \cap N_G(E, b_E)$ on the set of irreducible ordinary characters of M is three. Moreover, if 1_M, μ and λ are the representatives of such orbits, then we have*

$$I_{N_G(M,b_M) \cap N_G(E,b_E)}(1_M) = N_G(M, b_M) \cap N_G(E, b_E),$$

$$I_{N_G(M,b_M) \cap N_G(E,b_E)}(\mu) = (C_G(M) \cap N_G(E, b_E)) \cdot \bar{E},$$

and

$$I_{N_G(M,b_M)\cap N_G(E,b_E)}(\lambda) = C_G(M) \cap N_G(E, b_E).$$

Proof: We have seen that $C_G(M)$ acts trivially on the set of non-trivial characters of M. As a result, one can exploit the action of $SL(2,p)$ on the set of non-zero vectors of M. Then Theorem 3.4.7 implies that we have three such orbits after adding the trivial character. The rest of the result also follows from Theorem 3.4.7. $\square$

Theorem 3.5.19. *With the notation as in Proposition 3.5.17, $\bar{E} \in Syl_p(\bar{N})$ if, and only if, $EC_G(M)$ contains a Sylow p-subgroup of $N_G(M, b_M)$.*

Proof: If $\bar{E} \in Syl_p(\bar{N})$, then the index $[\bar{N} : \bar{E}]$ is co-prime to p. However, $[\bar{N} : \bar{E}] = [N_G(M, b_M) : EC_G(M)]$. So, $EC_G(M)$ contains a Sylow p-subgroup of $N_G(M, b_M)$. The converse is clear, since the argument is reversible. $\square$

Proposition 3.5.20. *Let G be a finite group, let B be a p-block of G with a defect group E which is an extra-special p-group of order p^3 and exponent p for an odd prime number p. Then, if the section $N_G(E, b_E)/EC_G(E)$ has no subgroup which is isomorphic to C_{p-1}, then $O_p(N_G(M, b_M)/C_G(M)) \neq 1$.*

Proof: Let us assume that $O_p(\bar{N}) = O_p(N_G(M, b_M)/C_G(M)) = 1$. Then $SL(2,p) \leq \bar{N} \leq GL(2,p)$. Regarding $\bar{E}$ as a Sylow p-subgroup of $SL(2,p)$, $\bar{N}$ and $GL(2,p)$, we know that $N_{SL(2,p)}(\bar{E}) \leq N_{\bar{N}}(\bar{E}) \leq N_{GL(2,p)}(\bar{E})$. However, the Sylow p-normalizer of $SL(2,p)$ has the form $N_{SL(2,p)}(\bar{E}) = \{\begin{bmatrix} \lambda & b \\ 0 & \lambda^{-1} \end{bmatrix} : 0 \neq \lambda \in GF(p) \ \ \& \ \ b \in GF(p)\}$. So, this normalizer has order $p(p-1)$. Therefore, $p(p-1) = |N_{SL(2,p)}(\bar{E})| \leq |N_{\bar{N}}(\bar{E})| \leq |N_{GL(2,p)}(\bar{E})| = p(p-1)^2$. The centralizers of $\bar{E}$ in these groups satisfy the following inclusion up to isomorphism $C_{SL(2,p)}(\bar{E}) \leq C_{\bar{N}}(\bar{E}) \leq C_{GL(2,p)}(\bar{E})$. Consequently,

$$1 \neq C_{\frac{p-1}{2}} \cong N_{SL(2,p)}(\bar{E})/C_{SL(2,p)}(\bar{E}) \leq N_{\bar{N}}(\bar{E})/C_{\bar{N}}(\bar{E}) \leq N_{GL(2,p)}(\bar{E})/C_{GL(2,p)}(\bar{E}) \lessapprox C_{p-1}.$$

But on the one hand, $C_{SL(2,p)}(\bar{E})$ contains $\bar{E}$. Hence, p divides the order of $C_{SL(2,p)}(\bar{E})$. Hence, either,

$$C_{\frac{p-1}{2}} \cong N_{SL(2,p)}(\bar{E})/C_{SL(2,p)}(\bar{E}) \cong N_{\bar{N}}(\bar{E})/C_{\bar{N}}(\bar{E})$$

or

$$N_{\bar{N}}(\bar{E})/C_{\bar{N}}(\bar{E}) \cong N_{GL(2,p)}(\bar{E})/C_{GL(2,p)}(\bar{E}) \cong C_{p-1}.$$

On the other hand,

$$\frac{N_G(E,b_E) \cap N_G(M,b_M)}{N_G(E,b_E) \cap C_G(M)} \cong \frac{(N_G(E,b_E) \cap N_G(M,b_M))/C_G(E)}{(N_G(E,b_E) \cap C_G(M))/C_G(E)}.$$

Using Proposition 3.5.17, it follows that

$$N_{\bar{N}}(\bar{E})/C_{\bar{N}}(\bar{E}) \lessapprox \frac{(N_G(E,b_E) \cap N_G(M,b_M))/C_G(E)}{(N_G(E,b_E) \cap C_G(M))/C_G(E)}.$$

However, $\frac{(N_G(E,b_E) \cap N_G(M,b_M))/C_G(E)}{(N_G(E,b_E) \cap C_G(M))/C_G(E)}$ is a section of the quotient group $N_G(E,b_E)/EC_G(E)$. Hence, we conclude that $C_{p-1} \leq N_G(E,b_E)/EC_G(E)$. □

Remark 3.5.21. *Assume that $p = 3$. Then $N_G(E,b_E)/EC_G(E)$ is isomorphic to a subgroup of $GL(2,3)$ with order is not divisible by 3. Therefore, in the case that $p = 3$ and $N_G(E,b_E)/EC_G(E)$ has no subgroup of order 2, we have $N_G(E,b_E) = EC_G(E)$. It follows that for each maximal subgroup M of E, which is a radical 3-subgroup of G, $N_G(E,b_E) \leq N_G(M,b_M)$. Hence, the contribution from chains starting with the (G,B)-subpair (M,b_M) is zero.*

As a consequence of Proposition 3.5.20, we have the following corollary:

Corollary 3.5.22. *Let G be a finite group, let B be a p-block of G with defect group E which is an extra-special p-group of order p^3 and exponent p for an odd prime number p. If $N_G(E,b_E)/EC_G(E)$ has no subgroup which is isomorphic to C_{p-1}, then $N_G(M,b_M)/C_G(M)$ has no p-block of defect zero.*

Proof: Suppose that $N_G(E, b_E)/EC_G(E)$ has no subgroup which is isomorphic to C_{p-1}. Then Proposition 3.5.20 implies that $O_p(N_G(M, b_M)/C_G(M)) \neq 1$. Therefore, $N_G(M, b_M)/C_G(M)$ has no p-block of defect zero. $\square$

Remark 3.5.23. *Assume that $p = 3$. Then $N_G(M, b_M)/C_G(M)$ is isomorphic to a subgroup of $GL(2, 3)$ with order which is divisible by 3. Therefore, in the case that $p = 3$ and $N_G(M, b_M)/C_G(M)$ is a proper subgroup of $SL(2, 3)$, we have $N_G(M, b_M)/C_G(M) \cong C_3 \times C_2$. It follows that $C_3 \cong O_3(N_G(M, b_M)/C_G(M)) \neq 1$, and, hence, $N_G(M, b_M)/C_G(M)$ has no 3-block of defect zero. However, in the case that $SL(2, 3) \lessapprox N_G(M, b_M)/C_G(M) \lessapprox GL(2, 3)$, we see that either $SL(2, 3) \cong N_G(M, b_M)/C_G(M)$, and, hence, it has a unique 3-block of defect zero or $N_G(M, b_M)/C_G(M) \cong GL(2, 3)$ which contains two 3-blocks of defect zero.*

Chapter 4

THE INERTIAL GROUPS OF IRREDUCIBLE CHARACTERS OF THE INITIAL p-SUBGROUP OF A (G, B)-CHAIN IN ITS STABILIZER

4.1 Introduction

Let G, B, E, M and p be as before. In this chapter, we shall compute the inertial groups of certain irreducible characters of the initial p-subgroup of a (G, B)-chain in its stabilizer. Using Lemma 3.5.12, we have only a few cases to consider. The first one is the inertial subgroups of the irreducible characters of E, which will be investigated in Section 4.2. Then the inertial subgroup of an irreducible character of M is calculated in Section 4.3.

4.2 The inertial group of an irreducible character of E in $N_G(E, b_E)$.

Although the objective is to compute the inertial subgroup of an irreducible character of E in $N_G(E, b_E)$ without any restriction, we will first restrict ourselves to the case that

$$N_G(E, b_E)/EC_G(E) \cong C_2 \times C_2. \tag{4.2.1}$$

As usual, E is an extra-special p-group of order p^3 and exponent p for an odd prime p. Write $E = \langle x, y, z \rangle$, where $\langle z \rangle = Z(E)$. Now E is a normal p-subgroup of $N_G(E, b_E)$. It is clear that $N_G(E, b_E)$ acts on the set of irreducible characters of E. Using Lemma 2.6.1, we have to distinguish between two cases, first, the case that $N_G(E, b_E)$ acts on the set of linear irreducible ordinary characters of E and second, the case that $N_G(E, b_E)$ acts on the set of non-linear irreducible ordinary characters of E. In both cases, the common factor is the assumption made in 4.2.1.

Since part of the alternating sum under consideration is over the conjugacy classes of irreducible characters of E under the action of $N_G(E, b_E)$, we will start with the following lemma.

Lemma 4.2.1. *With the notation and the assumption above, the number of orbits of the set of ordinary irreducible linear characters of E under the action by $N_G(E, b_E)$ is $(\frac{p+1}{2})^2$.*

Proof: Let η be an irreducible linear character of E. Then $\langle z \rangle \leq ker(\eta)$, because η is linear character and E is an extra-special p-group. Hence, we can regard η as an irreducible character of $E/\Phi(E) = \langle x\Phi(E), y\Phi(E) \rangle \cong \langle x \rangle \times \langle y \rangle$. It is clear that $EC_G(E)$ acts trivially on the set under consideration. Hence, the action of $N_G(E, b_E)/EC_G(E)$ on the set of non-zero vectors of the Frattini quotient $E/\Phi(E) := \bar{E}$ can be used to deduce the result for the non-trivial irreducible linear characters of E.

However, using the assumption in 4.2.1, we can compute the fixed points of the set of non-zero vectors by the relations 2.6.9, 2.6.10, and all the others in Section 2.6. Therefore, the identity element has $p^2 - 1$ fixed points, a has $p-1$ fixed points and b also has $p-1$ fixed points, but ab has no fixed points. Accordingly, the Burnside's Counting argument implies that

$$\#\text{Orbits} = \frac{1}{4}(p^2 - 1 + p - 1 + p - 1 + 0) = \frac{1}{4}(p^2 + 2p - 3) = \frac{(p-1)(p+3)}{4}.$$

Since the trivial character of E forms an orbit under the action of $N_G(E, b_E)$, the number of orbits of the set of irreducible ordinary linear characters of E under the action by $N_G(E, b_E)$ is $1 + \frac{1}{4}(p^2 + 2p - 3) = (\frac{p+1}{2})^2$. This completes the proof. $\square$

Similarly, we have to count the number of orbits of the action of $N_G(E, b_E)$ on the set of irreducible ordinary non-linear characters of E.

Lemma 4.2.2. *With the notation and the assumption above, the number of orbits of the set of ordinary irreducible non-linear characters of E under the action by $N_G(E, b_E)$ is $\frac{p-1}{2}$.*

Proof: By Lemma 2.6.1, E has $p-1$ ordinary irreducible non-linear characters, each of which vanishes outside $Z(E)$. Therefore, it is enough to study the action of $N_G(E, b_E)$ on $Z(E)$. However, $EC_G(E)$ acts trivially on $Z(E)$, and hence we can exploit the action of $N_G(E, b_E)/EC_G(E)$ on $Z(E)$. Using the assumption in 4.2.1, the relations in Section 2.6 and the Burnside's Counting Theorem, we deduce the result. $\square$

4.2.1 The inertial group of an irreducible linear character of E in $N_G(E, b_E)$.

Let η be an irreducible linear character of E. Then η has one of the following forms:

1. $\eta_1 = 1_{\langle x \rangle} \otimes 1_{\langle y \rangle}$. Only one irreducible character of E has this form. Obviously, E is the kernel of η_1.

2. $\eta_2 = 1_{\langle x \rangle} \otimes \beta$, for some nontrivial irreducible character β of $\langle y \rangle$. Since $\langle y \rangle$ is a cyclic group of order p, it has $p-1$ nontrivial irreducible characters. Hence, we have $p-1$ linear irreducible characters in E, each of which has the form of η_2. It is clear that $\langle x, z \rangle$ is the kernel of η_2.

3. $\eta_3 = \alpha \otimes 1_{\langle y \rangle}$, for some nontrivial irreducible character α of $\langle x \rangle$. We observe that $\langle y, z \rangle$ is the kernel of η_3. We have $p-1$ linear irreducible characters of E of this type.

4. $\eta_4 = \alpha \otimes \beta$, where both α and β are nontrivial irreducible characters of $\langle x \rangle$ and $\langle y \rangle$ respectively. Note that we have $p^2 - 2p + 1$ irreducible characters of this form. The kernel of η_4 is $Z(E)$.

The inertial subgroup of η in $N_G(E, b_E)$ is $I_{N_G(E,b_E)}(\eta) = \{g \in N_G(E, b_E) |\ \eta^g = \eta\}$. Hence, E and $C_G(E)$ are certainly contained in this group. We write $EC_G(E) \leq I_{N_G(E,b_E)}(\eta) \leq N_G(E, b_E)$. So, $\bar{1} \leq I_{N_G(E,b_E)}(\eta)/EC_G(E) \leq N_G(E, b_E)/EC_G(E)$.

However, the assumption in 4.2.1 enables us to write $I_{N_G(E,b_E)}(\eta)/EC_G(E) \lessapprox C_2 \times C_2$. Hence, if g is an element in $N_G(E, b_E) - EC_G(E)$ then $gEC_G(E) := \bar{g}$ is a nontrivial element in the group $N_G(E, b_E)/EC_G(E)$. Therefore, $\bar{g} \in \{a, b, c\}$, using the notation in Section 2.6. Thus, for the inertial group in this case, we have the possibilities that $I_{N_G(E,b_E)}(\eta) = EC_G(E)\langle \bar{g} \rangle$, where $\bar{g} \in \{1, a, b, c\}$.

From the discussion above, for η_1, the inertial subgroup is $N_G(E, b_E)$. For η_2, we have $I_{N_G(E,b_E)}(\eta_2) = EC_G(E)\langle b\rangle$. This is because, for each $e \in E$, and for $\bar{g} = b$, then

$$\eta_2^g(e) = \eta_2^{\bar{g}}(e) = \eta_2(e^{\bar{g}}) = \eta_2(e^b) = 1_{\langle x\rangle}(e_1^b) \otimes \beta(e_2^b) = 1_{\langle x\rangle}(e_1^{-1}) \otimes \beta(e_2) =$$

$$1_{\langle x\rangle}(e_1) \otimes \beta(e_2) = \eta_2(e).$$

Note that we have identified $e \in E$ by $(e_1, e_2) \in \langle x\rangle \times \langle y\rangle$ as $Z(E) \leq ker(\eta)$, and $E/Z(E) \cong \langle x\rangle \times \langle y\rangle$. Similarly, if η has the form η_3, then, for each $e \in E$, we have

$$\eta_3^g(e) = \eta_3^{\bar{g}}(e) = \eta_3(e^{\bar{g}}) = \eta_3(e^a) = \alpha(e_1^a) \otimes 1_{\langle y\rangle}(e_2^a) = \alpha(e_1) \otimes 1_{\langle y\rangle}(e_2^{-1}) =$$

$$\alpha(e_1) \otimes 1_{\langle y\rangle}(e_2) = \eta_3(e).$$

Hence, in this case, we have $I_{N_G(E,b_E)}(\eta_3) = EC_G(E)\langle a\rangle$.

Now if $\eta = \eta_4$, then $I_{N_G(E,b_E)}(\eta) = EC_G(E)$. This is because at least one of $\langle x\rangle$ or $\langle y\rangle$ is inverted by a, b or c.

By Lemma 4.2.1, $N_G(E, b_E)$ has $(\frac{p+1}{2})^2$ orbits when it acts on the set of linear irreducible ordinary characters of E. So, there are p^2 irreducible linear characters of E which distribute to $(\frac{p+1}{2})^2$ orbits of different sizes. Obviously, the first orbit is $\mathscr{O}_1 = \{1_{\langle x\rangle} \otimes 1_{\langle y\rangle}\}$. An easy computation shows that there are $\frac{p-1}{2}$ orbits, each of which contains two irreducible characters of the form $\alpha \otimes 1_{\langle y\rangle}$ and $\bar{\alpha} \otimes 1_{\langle y\rangle}$, where $\bar{\alpha}$ is the conjugate character of α and α is a non-trivial irreducible character of $\langle x\rangle$. Let $1 \leq i \leq p-1$, then $\mathscr{O}_i = \{\alpha^i \otimes 1_{\langle y\rangle}, \bar{\alpha}^i \otimes 1_{\langle y\rangle}\}$. Therefore, the inertial subgroup of each orbit in this form is $EC_G(E)\langle a\rangle$.

Similarly, on the one hand, there are $\frac{p-1}{2}$ orbits each of which contains two irreducible linear characters of E of the form $1_{\langle x\rangle} \otimes \beta$ and $1_{\langle x\rangle} \otimes \bar{\beta}$, where β is a non-trivial irreducible character of $\langle y\rangle$. Using the argument above, one can write $\mathscr{O}_j = \{1_{\langle x\rangle} \otimes \beta^j, 1_{\langle x\rangle} \otimes \bar{\beta}^j\}$ where $1 \leq j \leq \frac{p-1}{2}$. Thus, the inertial subgroup of each orbit in this form is $EC_G(E)\langle b\rangle$.

On the other hand, there are $(\frac{p-1}{2})^2$ orbits, each of which contains 4 irreducible linear ordinary characters of E of the form $\{\alpha \otimes \beta,\ \alpha \otimes \bar{\beta},\ \bar{\alpha} \otimes \beta,\ \bar{\alpha} \otimes \bar{\beta}\}$, where α and β are both non-trivial irreducible characters of $\langle x \rangle$ and $\langle y \rangle$ respectively. We label these orbits as follows: $\mathscr{O}_{i,\ j} = \{\alpha^i \otimes \beta^j, \alpha^i \otimes \bar{\beta}^j, \bar{\alpha}^i \otimes \beta^j, \bar{\alpha}^i \otimes \bar{\beta}^j\}$, where $\{1 \leq i,\ j \leq \frac{p-1}{2}\}$. The inertial subgroup of any representative of $\mathscr{O}_{i,\ j}$ in $N_G(E, b_E)$ is $EC_G(E)$.

Therefore, we have the following tables which summarise the above discussion:

$\eta \in Irr_3(E)$	$\#(\eta)$	$I_{N_G(E,b_E)}(\eta)$	$[I_{N_G(E,b_E)}(\eta) : EC_G(E)]$
$1_{\langle x \rangle} \otimes 1_{\langle y \rangle}$	1	$N_G(E, b_E)$	1
$1_{\langle x \rangle} \otimes \beta^j$	$\frac{p-1}{2}$	$EC_G(E)\langle b \rangle$	2
$\alpha^i \otimes 1_{\langle y \rangle}$	$\frac{p-1}{2}$	$EC_G(E)\langle a \rangle$	2
$\alpha^i \otimes \beta^j$	$(\frac{p-1}{2})^2$	$EC_G(E)$	4

Table 4.1: The inertial subgroups and the orbit structures of $Irr_3(E)$

$\eta \in Irr_3(E)$	$\#(\eta)$	$I_{C_2 \times C_2}(\eta)$	$[C_2 \times C_2 : I_{C_2 \times C_2}(\eta))]$
$1_{\langle x \rangle} \otimes 1_{\langle y \rangle}$	1	$C_2 \times C_2$	1
$1_{\langle x \rangle} \otimes \beta^j$	$\frac{p-1}{2}$	$\langle b \rangle$	2
$\alpha^i \otimes 1_{\langle y \rangle}$	$\frac{p-1}{2}$	$\langle a \rangle$	2
$\alpha^i \otimes \beta^j$	$(\frac{p-1}{2})^2$	1	4

Table 4.2: The action of $C_2 \times C_2$ and the orbit structures of $Irr_3(E)$

Note that by Lemma 2.6.4, $EC_G(E) \cap \langle a \rangle = EC_G(E) \cap \langle b \rangle = EC_G(E) \cap \langle c \rangle = 1$.

4.2.2 The inertial group of an irreducible non-linear character of E in $N_G(E, b_E)$.

Now let η be a nonlinear irreducible character of E. We have $p - 1$ such character, each of which has degree p. η has the form $\eta = Ind_M^E(\mu)$ for a character μ of a maximal subgroup M of E.

By Lemma 4.2.2, the action of $N_G(E, b_E)$ on the set of ordinary irreducible non-linear characters of E has $\frac{p-1}{2}$ orbits. It follows that each orbit contains two irreducible characters. So, $[\eta] = \{\eta, \bar{\eta}\}$, where $\bar{\eta}$ is the conjugate character of η. The commutator subgroup of E is not contained in the kernel of η, since η is nonlinear. Hence, the centre of E is not contained in this kernel, because $[E, E] = Z(E)$. It turns out that $\eta(e) = 0$, whenever $e \in E - Z(E)$. In particular, $\eta(x) = 0 = \eta(y)$.

Let us try to compute the inertial subgroup of η in $N_G(E, b_E)$. Since η vanishes outside the centre of E, we tackle only the elements in the centre of E. In particular, we consider the action on the generator of $Z(E)$, namely, the element z, because $Z(E)$ is a cyclic group of order p. Now let $g \in N_G(E, b_E)$ be a nontrivial element in which $gEC_G(E) = \bar{g}$ is a nontrivial element of the group $N_G(E, b_E)/EC_G(E)$. Consequently, $\bar{g} \in \{a, b, c\}$. We have to use the notation and the relations shown in Section 2.6. Then, in the case that $\bar{g} = a$, we see that $\eta^g(z) = \eta^{\bar{g}}(z) = \eta(z^{\bar{g}}) = \eta(z^a) = \eta(z^{-1}) \neq \eta(z)$. Thus, $g \notin I_{N_G(E,b_E)}(\eta)$. Similarly, for the case that $\bar{g} = b$, we observe that $\eta^g(z) = \eta^{\bar{g}}(z) = \eta(z^{\bar{g}}) = \eta(z^b) = \eta(z^{-1}) \neq \eta(z)$. Thus, $g \notin I_{N_G(E,b_E)}(\eta)$. However, in the case where $\bar{g} = ab = c$, we obtain $\eta^{\bar{g}}(z) = \eta^{\bar{g}}(z) = \eta(z^{\bar{g}}) = \eta(z^c) = \eta(z)$. Thus, $g \in I_{N_G(E,b_E)}(\eta)$. We conclude that $I_{N_G(E,b_E)}(\eta) = EC_G(E)\langle c \rangle$.

Now the observation that $I_{N_G(E,b_E)}(\eta)/EC_G(E)$ is a p'-group for each irreducible linear character of E yields the following theorem:

Theorem 4.2.3. *With the notation above, and denoting the number of p-blocks of defect zero for each section H/K which lie in Brauer correspondent with B by $f_0^{(B)}(H/K)$, we have*

$$f_0^{(B)}(I_{N_G(E,b_E)}(\eta)/EC_G(E)) = f_0^{(B)}(I_{N_G(E,b_E)}(\eta)/E) = k^{(B)}(I_{N_G(E,b_E)}(\eta)/EC_G(E)),$$

for all irreducible characters, η of E.

Proof: The first equality follows from Lemma 3.5.4 and Lemma 3.5.5. Since (E, b_E) is a maximal (G, B)-subpair, $I_{N_G(E,b_E)}(\eta)/EC_G(E)$ is a p'-group. Then each Sylow p-subgroup of $I_{N_G(E,b_E)}(\eta)$ is contained in $EC_G(E)$. As a result, each irreducible ordinary character of $I_{N_G(E,b_E)}(\eta)/EC_G(E)$ has defect zero. □

Using the Clifford correspondence, we have the following theorem:

Theorem 4.2.4. *Let G be a finite group, B be a p-block of G with defect group E, which is an extra-special p-group of order p^3 and exponent p, for an odd prime number p. Consider that (E, b_E) as a maximal (G, B)-subpair. Let b be the unique p-block of $N_G(E, b_E)$ which covers b_E and satisfies $b^G = B$. Then, $k(b, \eta) = f_0^{(B)}(I_{N_G(E,b_E)}(\eta)/E)$.*

Proof: Clifford Theorem 2.4.1, yields a bijection between $Irr(I_{N_G(E,b_E)}(\eta), \eta)$ and $Irr(N_G(E, b_E), \eta)$, for each ordinary irreducible character of E. Furthermore, this bijection preserves the defect. Now, using Theorem 4.2.3 and the assumption that b is the unique p-block of $N_G(E, b_E)$ which covers b_E and satisfies $b^G = B$ allows us to conclude that $k(b, \eta) = f_0^{(B)}(I_{N_G(E,b_E)}(\eta)/E)$. □

4.2.3 $N_G(E, b_E)/EC_G(E)$ with an element of order $p - 1$

Let us consider the case that $N_G(E, b_E)/EC_G(E)$ has an element of order $p - 1$, for an odd prime number p. We know that $N_G(E, b_E)/EC_G(E)$ is a p'-subgroup of the automorphism group of E. Fix λ to be the generator of $GF(p) - \{0\}$. Let us for the moment assume that

$$N_G(E, b_E)/EC_G(E) \cong \langle \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix} \rangle. \tag{4.2.2}$$

Then, $N_G(E, b_E)/EC_G(E) \cong C_2 \times C_{p-1}$. Now, on the one hand, $N_G(E, b_E)$ acts on the set of irreducible characters of E. Let us consider the action of $N_G(E, b_E)$ on the set of linear irreducible characters of E. On the other hand, using the observation that $EC_G(E)$ acts trivially on the set of irreducible characters of E, we exploit the action of $N_G(E, b_E)/EC_G(E)$ on the Frattini quotient $E/\Phi(E) = \langle x\Phi(E), y\Phi(E)\rangle := \bar{E} = \langle \bar{x}, \bar{y}\rangle$ to study the action of $N_G(E, b_E)$ on $Irr_3(E)$. We use such correspondence interchangeably.

We see that $\bar{E}$ has $p^2 - 1$ non-zero vectors. Let us write $\bar{x} := \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\bar{y} := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

It is clear that the element $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ does not fix any non-zero vector of $\bar{E}$. However, $\begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix}$ fixes $\bar{y}$, and, hence, it has $p-1$ fixed points in the set of non-zero vectors of $\bar{E}$. Accordingly, we have the following lemma:

Lemma 4.2.5. *Assume 4.2.2. Then, under the action of $N_G(E, b_E)$, $Irr_3(E)$ is partitioned into $p+1$ orbits.*

Proof: There is no new idea in the proof. It is a straightforward proof, using the Burnside's Counting Theorem and the correspondence mentioned above. Thus,

$$\#\text{Orbits} = \frac{1}{2(p-1)}(p^2 - 1 + (p-1)(p-1)) = p.$$

Adding the trivial character, we conclude that $N_G(E, b_E)$ has $p+1$ orbits when it acts on the set of linear irreducible characters of E. $\square$

Similarly, one can compute the orbits structure of $Irr_2(E)$ under the action of $N_G(E, b_E)$, with the assumption that $N_G(E, b_E)/EC_G(E) \cong C_2 \times C_{p-1}$.

Lemma 4.2.6. *With the assumption in 4.2.2, $N_G(E, b_E)$ acts transitively on the set of irreducible non-linear ordinary characters of E.*

Proof: The action of $N_G(E, b_E)/EC_G(E)$ on $Z(E)$ has a unique orbit. This is because an irreducible non-linear ordinary character of E is zero outside $Z(E)$ and only the identity element and the unique element of order 2 have fixed points. However, $EC_G(E)$ acts trivially. Hence, $N_G(E, b_E)$ acts transitively on the set of irreducible non-linear ordinary characters of E. □

Now, in general, and with the assumption that $N_G(E, b_E)/EC_G(E)$ contains a subgroup which is isomorphic to C_{p-1}, non-identity elements of $SL(2,p) \cap (N_G(E, b_E)/EC_G(E))$ have determinant 1, and, hence, act trivially on the centre of E. Also, these elements act as fixed points freely on the nontrivial irreducible linear characters of E. This means that for all $1 \neq x \in SL(2,p) \cap N_G(E, b_E)/EC_G(E)$, and for all $\eta \in Irr_3(E)$, $\eta^x \neq \eta$, while, for $\eta \in Irr_2(E)$, $\eta^x = \eta$. However, only elements from $GL(2,p)$ fixing a linear irreducible character of E are conjugate to $\begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix}$ and such elements fix no irreducible characters of $Irr_2(E)$.

4.3 The inertial group of an irreducible character of M in $N_G(M, b_M)$.

Let M be a maximal subgroup of E which is a radical p-subgroup of G. Our main concern is to establish a bijection which preserves the defect between the unique p-block b of $N_G(M, b_M)$ which covers b_M and the unique p-block $B(\sigma)$ of $N_G(\sigma)$ which satisfies $B(\sigma)^G = B$. Here, σ is the chain $(M, b_M) < (E, b_E)$. This means that, for each non-negative integer d, we are seeking to prove $k_d^{(B)}(N_G(M, b_M)) = k_d^{(B)}(N_G(M, b_M) \cap N_G(E, b_E))$.

Let us take an arbitrary irreducible character μ of M. We may write $M = \langle x, z \rangle =$

$\langle x \rangle \times \langle z \rangle$ where z is the generator of the centre of E and x is the other generator of E which is not central in E. Since M is the direct product of 2 cyclic groups, namely $\langle x \rangle$ and $\langle z \rangle$, each of which is of order p. Then $Aut(M) \cong GL(2,p)$.

Now M has p^2 such μ, each of which is a linear irreducible character. Furthermore, since the character of the direct product is the product of the characters, we have the following possibilities:

1. $\mu := \mu_1 = 1_{\langle x \rangle} \otimes 1_{\langle z \rangle}$. Note that we have only one irreducible character of this form and the kernel of μ is M.

2. $\mu := \mu_2 = 1_{\langle x \rangle} \otimes \beta$, where β is a nontrivial irreducible character of $\langle z \rangle$. There are $p-1$ irreducible characters of this form, each of which has a kernel which coincides with $\langle x \rangle$.

3. $\mu := \mu_3 = \alpha \otimes 1_{\langle z \rangle}$, where α is a nontrivial irreducible character of $\langle x \rangle$. We have $p-1$ irreducible characters of this form and the kernel of such μ is $\langle z \rangle$.

4. $\mu := \mu_4 = \alpha \otimes \beta$, where both α and β are nontrivial irreducible characters of $\langle x \rangle$ and $\langle z \rangle$ respectively. There are $(p-1)^2 = p^2 - 2p + 1$ irreducible characters of this form and the kernel of μ is a cyclic group of order p.

Now our task is to compute the inertial subgroup of μ in the normalizer of the (G,B)-subpair (M, b_M). For an arbitrary element $g \in G$, $\mu^g = \mu$ if, and only if, $\mu^g(x) = \mu(x)$ and $\mu^g(z) = \mu(z)$, which means that $\mu(gxg^{-1}) = \mu(x)$ and $\mu(gzg^{-1}) = \mu(z)$.

Consider the quotient group $N_G(M, b_M)/C_G(M)$. It is a section of the automorphism group of M. Therefore, we study the relationship with the normal subgroup $SL(2,p)$ of $Aut(M)$. It follows that either $N_G(M, b_M)/C_G(M) < SL(2,p)$, $N_G(M, b_M)/C_G(M) \cong SL(2,p)$, $SL(2,p) < N_G(M, b_M)/C_G(M) < GL(2,p)$ or $N_G(M, b_M)/C_G(M) \cong GL(2,p)$.

4.3.1 The section $N_G(M, b_M)/C_G(M)$ of G with a non-identity normal p-subgroup

Let us assume that the section $N_G(M, b_M)/C_G(M)$ of G is a proper subgroup of the special linear group $SL(2, p)$. Then $\bar{E} := EC_G(M)/C_G(M)$ is a cyclic p-subgroup of $N_G(M, b_M)/C_G(M)$ because $EC_G(M)/C_G(M) \cong E/E \cap C_G(M) \cong \langle y \rangle$.

The first observation is that $\bar{E}$ is a Sylow p-subgroup of $N_G(M, b_M)/C_G(M)$, and, hence, $N_G(M, b_M)/C_G(M)$ cannot be a p'-group. According to this observation, we have the following corollary:

Corollary 4.3.1. *$EC_G(M)$ has $p - 1$ fixed points when it acts on the set of ordinary irreducible non-trivial characters of M.*

Proof: The result follows because $C_G(M)$ acts trivially on the set of ordinary irreducible characters of M, Theorem 3.4.6 and the observation above. □

The second observation is that $O_p(N_G(M, b_M)/C_G(M)) \neq 1$. This is because nontrivial proper subgroups of $SL(2, p)$ with order divisible by p have a unique Sylow p-subgroup, as we have already seen in Theorem 2.6.7. In this case, we may exploit the fact that chains which start with such (M, b_M) make no contribution to the alternating sum under consideration, see [65] and [66].

Corollary 4.3.2. *In this case, no (G, B)-subpair (M, b_M) is an Alperin-Goldschmidt (G, B)-subpair.*

Proof: This is clear, because $O_p(N_G(M, b_M)/C_G(M)) \neq 1$. □

However, finite groups with non-identity normal p-subgroup have no p-blocks of defect zero. So, the corollary below follows:

Corollary 4.3.3. *In this case, the section $N_G(M, b_M)/C_G(M)$ has no p-blocks of defect zero.*

Proof: This is clear from Corollary 4.3.2. $\square$

Theorem 4.3.4. *In this case, there is no $C_G(M)$-projective irreducible character in $N_G(M, b_M)$ in a Brauer correspondent with B which lies over an $N_G(M, b_M)$-stable irreducible character μ of $C_G(M)$.*

Proof: In this case, $O_p(N_G(M, b_M)/C_G(M) > 1$. Then $f_0^{(B)}(N_G(M, b_M)/C_G(M)) = 0$. Therefore, using the correspondence between p-blocks of defect zero of $N_G(M, b_M)/C_G(M)$ and $C_G(M)$-projective irreducible characters in $N_G(M, b_M)$ which are $N_G(M, b_M)$-stable, the result follows. $\square$

The following lemma summarises the situation in this case.

Theorem 4.3.5. *Let G be a finite group, B be a p-block of G with defect group E which is an extra-special p-group of order p^3 and exponent p, for an odd prime number p. If $N_G(E, b_E))/EC_G(E)$ has no element of order $p-1$, then $N_G(M, b_M)/C_G(M) < SL(2, p)$.*

Proof: We conclude from Proposition 3.5.20 and the condition that $N_G(E, b_E)/EC_G(E)$ has no element of order $p-1$, that $O_p(N_G(M, b_M)/C_G(M)) \neq 1$. Thus, in this case, we must have $N_G(M, b_M)/C_G(M) < SL(2, p)$, using Theorem 2.6.7. $\square$

The following theorem holds in this case:

Theorem 4.3.6. *Let G be a finite group, B be a p-block of G with defect group E, which is an extra-special p-group of order p^3 and exponent p, for an odd prime number p. Assume that the OWC holds for the p-block B. If for each maximal subgroup M of E, $N_G(M, b_M)/C_G(M) < SL(2, p)$. Then, for each non-negative integer d, $k_d(B) = k_d(b)$, where b is the unique p-block of $N_G(E, b_E)$ such that $b^G = B$.*

Proof: Since $EC_G(M)$ is a normal subgroup of $N_G(M, b_M)$,

$$N_G(M, b_M) = N_{N_G(M,b_M)}(EC_G(M)) = (N_G(M, b_M) \cap N_G(E, b_E))C_G(M).$$

Thus, the action of $N_G(M, b_M)$ on the set of irreducible characters of M is the same as the action of $N_G(M, b_M) \cap N_G(E, b_E)$ on the same set. Thus, the contribution from the chains $\{(M, b_M)\}$ cancels the contribution from the chains $\{(M, b_M) < (E, b_E)\}$. $\square$

Now to complete that case that $SL(2, p) \not\lesssim N_G(M, b_M)/C_G)(M)$, we assume that $N_G(M, b_M)/C_G(M)$ is a cyclic p-group of order p. Therefore, $N_G(M, b_M)/C_G(M) \cong EC_G(M)/C_G(M)$, which holds if, and only if, $N_G(M, b_M) = EC_G(M)$.

In this case, we consider the subgroup $(N_G(E, b_E) \cap N_G(M, b_M))C_G(M)$. Using the observation above and the modular Law, we have

$$(N_G(E, b_E) \cap N_G(M, b_M))C_G(M) = N_G(E, b_E)(EC_G(M)) \cap EC_G(M) = N_G(M, b_M).$$

Therefore, $N_G(E, b_E)$ controls the fusion of (G, B)-subpairs in this case. Let us record the main result in this section.

Theorem 4.3.7. *Let G be a finite group, B be a p-block of G with defect group E, which is an extra-special p-group of order p^3 and exponent p, for an odd prime number p. Assume that the OWC holds for the p-block B. If for each maximal subgroup M of E which is radical p-subgroup of G, $SL(2, p) \not\lesssim N_G(M, b_M)/C_G(M)$. Then, for each non-negative integer d, $k_d(B) = k_d(b)$, where b is the unique p-block of $N_G(E, b_E)$ such that $b^G = B$.*

4.3.2 The section $N_G(M, b_M)/C_G(M)$ of G containing the special linear group $SL(2, p)$.

In this subsection, we consider the case that the quotient group $N_G(M, b_M)/C_G(M)$ contains $SL(2, p)$. Let us start by investigating the case that $SL(2, p) \cong N_G(M, b_M)/C_G(M)$.

Since our aim is to count the p-blocks of defect zero in certain sections, we state the following lemma:

Lemma 4.3.8. *The special linear group $SL(2, p)$ has a unique p-block of defect zero.*

Proof: The proof can be found in [26, Theorem 71.3]. $\square$

Write $\bar{E}$ for $EC_G(M)/C_G(M)$. By Lemma 2.6.5, $[N_{SL(2,p)}(\bar{E}) : C_{SL(2,p)}(\bar{E})] = \frac{p-1}{2}$. Write $M = \langle x, z \rangle$, where $x := \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $z := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Note that $SL(2, p)$ is generated by its p-elements.

Since $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C_{SL(2,p)}(x) = \langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \rangle$ and $C_{SL(2,p)}(z) = \langle \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \rangle$ respectively.

In this case, $N_G(M, b_M)/C_G(M)$ has $p+1$ Sylow p-subgroups, each of which is isomorphic to the image of E under the natural map. Therefore, we can identify $EC_G(M)/C_G(M)$ with a Sylow p-subgroup of $SL(2, p)$. For instance, this Sylow p-subgroup can be generated by the automorphism $\eta : M \to M$, such that $\eta(x) = x + z$ and $\eta(z) = z$. Indeed, η has order p and can be represented in the matrix form, as follows: $\eta = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Now let us write $N_G(M, b_M) = C_G(M) \cdot SL(2, p)$. Recall that there is a one to one correspondence between the irreducible characters of M and the set of vectors of M, when we regard M as a vector space over $GF(p)$. Then $I_{N_G(M,b_M)}(\mu) = C_G(M)C_{SL(2,p)}(x_\mu)$, because $C_G(M)$ acts trivially on each irreducible character of M, where x_μ is the vector of M which corresponds to μ.

Our goal is to know the inertial subgroup of the irreducible characters of M in $N_G(M, b_M)$. Of course, $\mu = \mu_1 = 1_{\langle x \rangle} \otimes 1_{\langle z \rangle}$ has $N_G(M, b_M)$ as an inertial subgroup. For $\mu = \mu_2$, we see that $I_{N_G(M,b_M)}(\mu_2) = I_{N_G(M,b_M)}(1_{\langle x \rangle} \otimes \beta) = I_{N_G(M,b_M)}(\beta) = C_G(M)\langle \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \rangle$. For the irreducible characters of the forms $\mu_3 = \alpha \otimes 1_{\langle z \rangle}$, similarly, $I_{N_G(M,b_M)}(\mu_3) = I_{N_G(M,b_M)}(\alpha \otimes 1_{\langle z \rangle}) = I_{N_G(M,b_M)}(\alpha) = C_G(M)\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \rangle$. However, if m is an arbitrary non-trivial element of M, then there are $i \in \{0, 1, \cdots, p-1\}$ and $j \in \{0, 1, \cdots, p-1\}$ (not both equalling zero) such that $m = x^i z^j := \begin{bmatrix} i \\ j \end{bmatrix}$. Now M has $p+1$ maximal subspaces, namely, $H_\infty,\ H_0\ \ \&\ H_\lambda;\ \lambda \in GF(p) - \{0\}$. We have observed in Section 3.4 that

$$C_{SL(2,p)}(H_\infty) = \{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} : b \in GF(p)\ \} \cong C_p,$$

$$C_{SL(2,p)}(H_0) = \{ \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} : b \in GF(p)\ \} \cong C_p$$

and

$$C_{SL(2,p)}(H_\lambda) = \{ \begin{bmatrix} 1-\lambda b & b \\ -\lambda^2 b & 1+\lambda b \end{bmatrix} : b \in GF(p)\ \} \cong C_p.$$

For $\mu \neq 1_M$, $I_{N_G(M,b_M)}(\mu)/C_G(M)$ is a cyclic p-subgroup of $SL(2,p)$ of order p. Therefore, there is no p-block of defect zero in these sections in the Brauer correspondence with B. So, $f_0^{(B)}(I_{N_G(M,b_M)}(\mu)/M) = 0$, for each non-trivial irreducible character in M. Therefore, the conclusion in this case is that there is a unique irreducible character of $N_G(M, b_M)$ which is M-projective. Hence, $\sum_{\mu \in Irr(M)/N_G(M,b_M)} f_0^{(B)}(I_{N_G(M,b_M)}(\mu)/M) = 1$.

Let us write s_λ for the matrix $\begin{bmatrix} 1-\lambda b & b \\ -\lambda^2 b & 1+\lambda b \end{bmatrix}$, noting that $N_G(M, b_M)$ acts transitively on the set of non-trivial irreducible characters of M. We summarize the above discussion in Table 4.3.

$\mu \in Irr(M)$	$\#(\mu)$	$I_{N_G(M,b_M)}(\mu)$	$[I_{N_G(M,b_M)}(\mu) : C_G(M)]$
1_M	1	$N_G(M, b_M)$	$(p-1)p(p+1)$
μ	p^2-1	$C_G(M)\langle s_\lambda\rangle$	p

Table 4.3: The action of $SL(2,p)$ and the orbit structure of $Irr(M)$.

Consider now the case that $SL(2,p) < N_G(M,b_M)/C_G(M) < GL(2,p)$. We know that $GL(2,p)/SL(2,p) \cong C_{p-1}$, where C_{p-1} is the cyclic group of order $p-1$. Consequently, in this case, $N_G(M,b_M)/C_G(M) \cong SL(2,p) \rtimes A$, where A is a cyclic p'-group of the form $\{\begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix} : \alpha \in GF(p) - \{0\}\}$ where α is of order r and $1 < r < p-1$. Note that a cyclic group has a unique subgroup for each divisor of its order which is again a cyclic subgroup. Then there is a cyclic p'-subgroup, say A, of C_{p-1} such that $N_G(M,b_M)/C_G(M) \cong SL(2,p) \rtimes A$. Let $|A| = r$. Therefore, each irreducible character of M has an inertial subgroup of $N_G(M,b_M)$ equal to $C_G(M) \rtimes I_{SL(2,p)}(\mu) \rtimes I_A(\mu)$. Consequently, mod $C_G(M)$, we are seeking for certain subgroups of $GL(2,p)$ which coincide with $I_{SL(2,p)}(\mu) \rtimes I_A(\mu)$. It turns out that we have to study the relationship between a subgroup of the automorphism group of M, namely A, which is a p'-group, and M in this case.

We see that any (G,B)-subpair (M,b_M) is p-radical. Let us record this result here.

Theorem 4.3.9. *In this case, each (G,B)-subpair, (M,b_M) is a p-radical (G,B)-subpair.*

Proof: $N_G(M,b_M)/C_G(M)$ has $p+1$ Sylow p-subgroups, each of which has order p. The conclusion is that $N_G(M,b_M)/C_G(M)$ has no non-trivial normal p-subgroup. This completes the proof, since (M,b_M) is an arbitrary (G,B)-subpair. □

We have seen that $N_G(M, b_M)$ acts transitively on the set of non-trivial ordinary irreducible characters of M. We represent such an action in Table 4.4.

$\mu \in Irr(M)$	$\#(\mu)$	$I_{N_G(M,b_M)}(\mu)$	$[I_{N_G(M,b_M)}(\mu) : C_G(M)]$
1_M	1	$N_G(M, b_M)$	$(p-1)p(p+1)$
μ	$p^2 - 1$	$C_G(M)\langle s_\lambda \rangle \cdot A$	pr

Table 4.4: The action of $SL(2,p) \rtimes C_r$ and the orbit structure of $Irr(M)$.

Note that $s_\lambda = \begin{bmatrix} 1-\lambda b & b \\ -\lambda^2 b & 1+\lambda b \end{bmatrix}$, $\lambda \in GF(p) - \{0\}$ and $I_{N_G(M,b_M)}(\mu)/C_G(M) \cong \langle s_\lambda \rangle \times A$ is a cyclic group of order pr. Since we are interested in counting the number of p-blocks of defect zero in the section $I_{N_G(M,b_M)}(\mu)/M$, we have to consider the section $I_{N_G(M,b_M)}(\mu)/C_G(M)$.

Now assume that $N_G(M, b_M)/C_G(M) \cong GL(2,p)$. It follows that $N_G(M, b_M) = N_G(M)$. Considering the action of $GL(2,p)$ on M and using the matrix form, we can compute the centralizer of x and z in $GL(2,p)$. Thus,

$$C_{GL(2,p)}(x) = \{ \begin{bmatrix} 1 & b \\ 0 & d \end{bmatrix} : \ b \in GF(p) \ \ \& \ \ d \in GF(p) - \{0\}\} \cong C_{p(p-1)}$$

and

$$C_{SL(2,p)}(z) = \{ \begin{bmatrix} a & 0 \\ c & 1 \end{bmatrix} : \ c \in GF(p) \ \ \& \ \ a \in GF(p) - \{0\}\} \cong C_{p(p-1)}.$$

We observe that $GL(2,p)$ acts transitively on the set of non-zero vectors, because the linear transformation $\begin{bmatrix} 0 & b \\ 1 & d \end{bmatrix}$, for $b \neq 0$, carries x to z. As a result, we can deduce that the action of $N_G(M, b_M)$ on the set of non-trivial characters of M is transitive. Let us denote the representative of the unique orbit under this action by μ. It is clear that the inertial subgroup of such μ in $N_G(M, b_M)$ is the subgroup $C_G(M) \cdot \langle s_\lambda \rangle$ where $\langle s_\lambda \rangle$ is a cyclic subgroup of $GL(2, p)$. It turns out that $\langle s_\lambda \rangle$ is the stabilizer subgroup of a non-zero vector of M. The conclusion from the discussion above is that for each nontrivial irreducible character of M, $I_{N_G(M,b_M)}(\mu)/C_G(M) \cong C_{p(p-1)}$.

Chapter 5

THE PREDICTIONS OF THE CONJECTURES FOR p-BLOCK WITH AN EXTRA-SPECIAL DEFECT GROUP

5.1 Introduction

Let G be a finite group and B be a p-block of G with defect group E, which is an extra-special p-group of order p^3 and exponent p, for an odd prime number p. Our main concern is to show that Conjecture 2.5.6 and Conjecture 2.5.8 for B predict an equality between the number of irreducible characters of B with defect d, for each d, and the corresponding number in the unique Brauer correspondent with B in $N_G(E, b_E)$, say b, where (E, b_E) is the maximal (G, B)-subpair which is associated with B.

After the discussion of the general set up, we study several cases which depend upon certain conditions for both $O_p(G)$ and the inertial quotient $N_G(E, b_E)/EC_G(E)$. In Section 5.3, we discuss the case that $O_p(G)$ is trivial and $N_G(E, b_E)/EC_G(E)$ has no element of order $p-1$. Section 5.4 concerns the case that $O_p(G)$ is a central p-subgroup of G and equal to centre E. In Section 5.5, we study the case that $O_p(G)$ is not a central p-subgroup of G. Section 5.6 is devoted to the case that $O_p(G)$ is trivial and $N_G(E, b_E)/EC_G(E)$ is isomorphic to $C_2 \times C_{p-1}$.

5.2 Correspondence with characters in nilpotent p-blocks

Let G be a finite group and B a p-block of G with defect group E which is an extra-special p-group of order p^3 and exponent p for an odd prime number p. For each subgroup H of G denote the number of irreducible characters of H with defect d which lie in the Brauer correspondent with B by $k_d^{(B)}(H)$. Of course, $k_d^{(B)}(G)$ is the number of irreducible characters of G with defect d and belong to B.

We have to show that $k_d^{(B)}(G) = k_d^{(B)}(N_G(E, b_E)) = k_d^{(B)}(N_G(M, b_M))$, where M is an arbitrary maximal subgroup of E, which is a radical p-subgroup of G. This happens if, and only if, we have established the following bijection: $Irr_d^{(B)}(G) \longleftrightarrow Irr_d^{(B)}(N_G(E, b_E)) \longleftrightarrow Irr_d^{(B)}(N_G(M, b_M))$, which happens if and only if $Irr^{(B)}(G) \longleftrightarrow Irr^{(B)}(N_G(E, b_E)) \longleftrightarrow Irr^{(B)}(N_G(M, b_M))$. We claim that there is a one to one correspondence between the set of (G, B)-subpairs $\{(E, b_E)|b_E \in \ block\ C_G(E)\}$ and the set of irreducible characters of $O_{p'}(C_G(E))$, namely, the set $\{\mu|\mu \in Irr(O_{p'}(C_G(E)))\}$, in the case of the principal p-block. We see that the element $e_\mu = m \sum_{g \in O_{p'}(C_G(E))} \mu(g^{-1})g$, where $m = |O_{p'}(C_G(E))|^{-1}$, μ is an irreducible character of $O_{p'}(C_G(E))$ and e_μ is a central primitive idempotent of $C_G(E)$. Conversely, if b_E is a p-block of $C_G(E)$, then certainly b_E comes from an irreducible character of $O_{p'}(C_G(E))$. Let us write $Irr_d(C_G(E)|\ b_E)$ to indicate the set of irreducible characters in the p-block b_E as a character of $C_G(E)$ with defect d. Likewise, $Irr_d(C_G(M)|\ b_M)$ is the set of irreducible characters in the p-block b_M view as a character of $C_G(M)$ which has the defect d.

Lemma 5.2.1. *Let G be a finite group. Let B be a p-block of G with defect group E, where E is an extra-special p-group of order p^3, exponent p, for an odd prime number p. Let (E, b_E) be a maximal (G, B)-subpair. Then (E, b_E) is a maximal $(N_G(M, b_M), B^*)$-subpair, where M is a maximal subgroup of E, b_M is the unique p-block of $C_G(M)$ in which $(M, b_M) \leq (E, b_E)$ and B^* is the Brauer correspondent of B in $N_G(M, b_M)$.*

Proof: Let us write $\delta(B^*)$ for the defect group of B^*. Then it is clear that E is a p-subgroup of $N_G(M, b_M)$, b_E is a p-block of $C_{N_G(M,b_M)}(E)$ and $Br_E(1_{B^*})b_E = b_E$. In addition, $\delta(B^*) =_{N_G(M,b_M)} E$. This is clear because E is a p-subgroup of $N_G(M, b_M)$ since M is a normal subgroup of E and b_M is an E-invariant p-block of $C_G(M)$. Also, $C_G(E) \subseteq C_G(M) \subseteq N_G(M, b_M)$, which is equivalent to the equality $N_G(M, b_M) \cap C_G(E) = C_{N_G(M,b_M)}(E) = C_G(E)$. Since $EC_G(E) \subseteq EC_G(M) \subseteq N_G(M, b_M)$, $B^* := b_E^{N_G(M,b_M)}$ is defined, and hence, it is a p-block of $N_G(M, b_M)$ with the defect group E. Since $b_E^G = B$, $B = (b_E^{N_G(M,b_M)})^G = B^{*G}$. Consequently B^* is the Brauer correspondent of B in $N_G(M, b_M)$. Working with the Brauer map $Br_E : RN_G(M, b_M)^E \to FC_{N_G(M,b_M)}(E)$ and using the observation that (M, b_M) is a normal $(N_G(M, b_M), B^*)$-subpair of (E, b_E), we conclude that $Br_E(1_{B^*})b_E = b_E$. Finally, $\delta(B^*) =_{N_G(M,b_M)} E$, as B^* is the Brauer correspondent of B in $N_G(M, b_M)$. □

If P is an arbitrary Sylow p-subgroup of G and D is an arbitrary subgroup of P, then there exists another subgroup Q of P such that $Q =_G D$ and $N_P(Q)$ is a Sylow p-subgroup of $N_G(Q)$. The generalization of this fact, for the (G, B)-subpair and the extra-special case, is the following lemma:

Lemma 5.2.2. *Let E be an extra-special p-group of order p^3 and exponent p for an* *odd prime number p, M be a maximal subgroup of E which is a radical p-subgroup of G and (E, b_E) be a maximal (G, B)-subpair. Then (E, b_E) is a maximal $(N_G(M, b_M) \cap N_G(E, b_E), b)$-subpair, where b is the Brauer correspondent of B in $N_G(M, b_M) \cap N_G(E, b_E)$.*

Proof: It is clear that $N_E(M) = E$ and b_M is E-invariant. Hence, (M, b_M) is a normal (G, B)-subpair of (E, b_E). Now (E, b_E) is a maximal $(N_G(E, b_E), b_1)$-subpair, where b_1 is the Brauer correspondent of B in $N_G(E, b_E)$. It follows that there is an element $g \in N_G(E, b_E)$ such that $(M, b_M)^g \lhd (E, b_E)$ and (E, b_E) is a maximal $(N_G(M, b_M) \cap N_G(E, b_E), b)$-subpair, using the observation that $N_G((M, b_M)^g) \cap N_G(E, b_E) = N_G(M, b_M) \cap N_G(E, b_E)$. □

Lemma 5.2.3. *M is a defect group of b_M as a p-block of $C_G(M)$.*

Proof: Let b be the p-block of $N_G(M, b_M)$ which covers b_M. Then $bb_M = b_M$. Thus, b can be decomposed into primitive central idempotents in $C_G(M)$. Since $C_G(M)$ is a normal subgroup of $N_G(M, b_M)$ which is the inertial subgroup of b_M, b is the unique p-block of $N_G(M, b_M)$ which covers b_M. Hence, we identify b with b_M. It follows that b_M has a defect group in $C_G(M)$ which is the intersection of the defect group of b in $N_G(M, b_M)$ and $C_G(M)$. Since $EC_G(E) \subseteq EC_G(M) \subseteq N_G(M, b_M)$, E is a defect group of b in $N_G(M, b_M)$. Therefore, $\delta(b_M) = E \cap C_G(M) = C_E(M) = M$. This completes the proof of the lemma. □

Corollary 5.2.4. *For the maximal (G, B)-subpair (E, b_E), b_E has a defect group, as a p-block of $C_G(E)$. Moreover, this defect group coincides with $Z(E)$.*

Proof: The result follows by a similar argument to that in Lemma 5.2.3. □

Corollary 5.2.5. *The p-blocks b_E and b_M are nilpotent p-blocks in $C_G(E)$ and $C_G(M)$ respectively.*

Proof: This is clear because b_M and b_E have central defect groups in $C_G(M)$ and in $C_G(E)$ respectively. $\square$

Corollary 5.2.6. *The sets $Irr_2(C_G(M)\ |b_M)$ and $Irr_2(C_G(E)\ |b_E)$ contain p^2 and p irreducible characters of $C_G(M)$ and of $C_G(E)$ respectively.*

Proof: By Corollary 5.2.5, b_E and b_M are nilpotent p-blocks in $C_G(E)$ and $C_G(M)$ with abelian defect groups $Z(E)$ and M respectively. Then the result follows from the fact that the number of ordinary irreducible characters of a nilpotent p-block is the same as the order of the defect group of that p-block. See Lemma 3.2.7. $\square$

Lemma 5.2.7. *The section $N_G(E, b_E)/C_G(E)$ has a Sylow p-subgroup which is isomorphic to $C_p \times C_p$.*

Proof: Since $N_G(E, b_E)/EC_G(E)$ is a p'-group, $\frac{N_G(E,b_E)/C_G(E)}{EC_G(E)/C_G(E)}$ is a p'-group. Then $[N_G(E, b_E)/C_G(E) : EC_G(E)/C_G(E)]$ is co-prime to p. However, $EC_G(E)/C_G(E) \cong E/E \cap C_G(E) = E/Z(E) \cong C_p \times C_p$. This completes the proof of the lemma. $\square$

Now let M be an arbitrary maximal subgroup of E. Suppose that M is not a radical p-subgroup of G. Then $E = O_p(N_G(M, b_M))$. This means that E is a normal subgroup of $N_G(M, b_M)$. Hence, $N_G(M, b_M) \leq N_G(E, b_E)$. Thus, the contributions from chains starting with (M, b_M) cancel each other.

Let us assume that M is a radical p-subgroup of G. We consider $N_G(M, b_M) \cap N_G(E, b_E)$. It is not a non-identity group, because it contains $EC_G(E)$. Note that $N_G(E, b_E) \cap N_G(M, b_M)$ is an important group because it is the stabilizer of the chain $\sigma :\ (M, b_M) < (E, b_E)$ of the (G, B)-subpairs. We can denote this group by $N_G(\sigma)$. It is clear that $EC_G(E)$ and $C_G(M) \cap N_G(E, b_E)$ are normal subgroups of $N_G(\sigma)$. Note that $EC_G(M)$ need not be a subgroup of $N_G(\sigma)$. But the stabilizer $N_G(\sigma)$ normalizes $EC_G(M)$ as $(EC_G(M))^g = EC_G(M)$, whenever $g \in N_G(\sigma)$.

Remark 5.2.8. *We define $O_{p',p}(N_G(M,b_M))$ to be the subgroup, say H, of $N_G(M,b_M)$ for which $H/O_{p'}(N_G(M,b_M)) = O_p(N_G(M,b_M)/O_{p'}(N_G(M,b_M)))$. If M is a Sylow p-subgroup of $O_{p',p}(N_G(M,b_M))$, then $O_{p',p}(N_G(M,b_M)) = M \times O_{p'}(N_G(M,b_M))$. Yet,*

$$O_{p'}(N_G(M,b_M)) \leq C_G(M).$$

So,

$$\bar{1} \neq H/C_G(M) \leq H/O_{p'}(N_G(M,b_M)) \leq N_G(M,b_M)/O_{p'}(N_G(M,b_M)).$$

We conclude that $H/C_G(M)$ is a normal p-subgroup of $N_G(M,b_M)/C_G(M)$, and, hence, $H/C_G(M) = E/M = EC_G(M)/C_G(M)$. Therefore, $H = EC_G(M)$.

Lemma 5.2.9. *With the notation above, $N_G(E,b_E) \cap C_G(M) = MC_G(E)$.*

Proof: This is a special case of [44, Lemma 2.6]. □

Theorem 5.2.10. *There is a bijection between the irreducible characters of $Z(E)$ and the set of irreducible characters $Irr_2(C_G(E)|\ b_E)$ which cover them.*

Proof: We observe that $Res^G_H(1_G) = 1_H$, for each subgroup H of G. Now choose $\eta \in Irr(E)$ and $\theta \in Irr(EC_G(E))$ such that $Res^E_{Z(E)}(\eta) = 1_{Z(E)}$ and $\langle Res^{EC_G(E)}_E(\theta), \eta\rangle \neq 0$. Then $\theta = \eta \otimes \beta$ where β is an irreducible character of $C_G(E)$ such that $Res^{C_G(E)}_{Z(E)}(\beta) = Res^E_{Z(E)}(\eta) = 1_{Z(E)}$, which is equivalent to writing $\langle Res^{C_G(E)}_{Z(E)}(\beta), 1_{Z(E)}\rangle \neq 0$. Let us for the moment assume that $EC_G(E)\langle \bar{g}\rangle = I_{N_G(E,b_E)}(\eta)$, where $\langle \bar{g}\rangle$ is a cyclic $p^{'}$-group of order r and $1 \leq r < p$. We need to compute the inertial subgroup of $\eta \otimes \beta$ in $X := I_{N_G(E,b_E)}(\eta)$. Since η is X-stable, $I_X(\eta \otimes \beta) = I_X(\beta)$.

Now $EC_G(E)$ acts trivially on β. Consequently, $I_X(\beta) = EC_G(E)I_{\langle \bar{g}\rangle}(\beta)$. This means that we have to know the inertial subgroup of the irreducible character β of $C_G(E)$ in $\langle \bar{g}\rangle$. Obviously, this is C_n where $n \leq r$. Now we see that $I_X(\eta \otimes \beta)/EC_G(E) = C_n$ or $I_X(\eta \otimes \beta)/EC_G(E) = 1$. In the first case, $\eta \otimes \beta$ extends to X, as this factor group is cyclic (see [42, 11.22]). This means that there are irreducible characters $\xi_1, \xi_2, \cdots, \xi_n$ of X such that $Res^X_{EC_G(E)}(\xi_i) = \eta \otimes \beta$, $1 \leq i \leq n$. In the second case, we see immediately that $Ind^X_{EC_G(E)}(\mu \otimes \beta)$, is an irreducible character of X. By Clifford's Theorem 2.4.1, we pass to $N_G(E, b_E)$ by induction, which gives us a bijection between the irreducible characters of $N_G(E, b_E)$ which lie over η and the irreducible characters of $EC_G(E)$ which lie over η. The result follows as p-blocks of $EC_G(E)$ and of $C_G(E)$ are in one to one correspondence, since the central primitive idempotents of both $EC_G(E)$ and of $C_G(E)$ have the same support of p-regular elements. $\square$

Recall that if H/K is a section of the finite group G and B is a p-block of G, then $f_0^{(B)}(H/K)$ refers to the number of p-blocks of defect zero in the Brauer correspondent with B.

Corollary 5.2.11. *With the notation above,* $f_0^{(B)}((C_G(M) \cap N_G(E, b_E))/M) = 1$.

Proof: Using Lemma 5.2.9, we see that

$$f_0^{(B)}((C_G(M) \cap N_G(E, b_E))/M) = f_0^{(B)}(MC_G(E)/M) = f_0^{(B)}(C_G(E)/Z(E)) = 1.$$

$\square$

Now, the set of irreducible characters of $Irr_2(C_G(M) \mid b_M)$ can be related to the irreducible characters of M.

Theorem 5.2.12. *There is a bijection between the irreducible characters of M and the irreducible characters in $Irr_2(C_G(M) \mid b_M)$ which lie over them.*

Proof: The idea is that b_M is a nilpotent p-block of $C_G(M)$ with the defect group M which is an abelian p-group of order p^2. Thus, b_M has p^2 irreducible characters in $C_G(M)$. At the same time, M has p^2 irreducible characters too. As a result, the induction process which sends μ of $Irr(M)$ to an irreducible character of $Irr_2(C_G(M))$ which lies over μ and belongs to b_M is a one to one map and thus a surjective map. Therefore, we have established a one to one correspondence between the irreducible characters of M and the irreducible characters in the p-block b_M of $C_G(M)$ with defect 2. This completes the proof of the theorem. □

An immediate result of the bijection mentioned above is the following corollary:

Corollary 5.2.13. *With the notation above, an irreducible character χ of $N_G(M, b_M)$ is M-projective if, and only if, χ is $C_G(M)$-projective.*

Proof: Let χ be an irreducible character of $N_G(M, b_M)$ lying over an irreducible character μ of M. It is clear that if χ is M-projective, then χ is $C_G(M)$-projective. Conversely, if χ is $C_G(M)$-projective, then there is an irreducible character η of $C_G(M)$ such that $\chi = Ind_{C_G(M)}^{N_G(M,b_M)}(\eta)$. By the bijection in Theorem 5.2.12, we may choose η to be the unique extension of μ. Thus, η is an M-projective irreducible character of $C_G(M)$ because M is the defect group of b_M. We conclude that χ is an M-projective too. □

Now E stabilises the nilpotent p-block b_M of $C_G(M)$. Then b_M is $EC_G(M)$-stable nilpotent p-block. Since $EC_G(M)/C_G(M)$ is a p-group, Lemma 3.2.8 implies that b_M extends to a unique nilpotent p-block, say $\tilde{b_M}$, of $EC_G(M)$. Note that all p-regular elements of $EC_G(M)$ lie in $C_G(M)$, and, hence, $1_{b_M} \in RC_G(M)$. It follows that $\tilde{b_M}$ is the unique p-block of $EC_G(M)$ which covers the nilpotent p-block b_M of $C_G(M)$. Then $1_{\tilde{b_M}}$ is a central idempotent of $RC_G(M)$ and

$$1_{\tilde{b_M}} \cdot 1_{b_M} \neq 0.$$

But b_M is E-stable, so it lies in $Z(REC_G(M))$. Therefore, 1_{b_M} is a primitive idempotent in $Z(REC_G(M))$ and $\tilde{b_M}$ has defect group E in $EC_G(M)$.

Lemma 5.2.14. *With the notation above, (E, b_E) is a maximal $(EC_G(M), \tilde{b_M})$-subpair.*

Recall that (M, b_M) is a (G, B)-subpair, means that b_M is a p-block of $C_G(M)$ with $b_M^G = B$. Hence, Theorem 5.2.12 implies that B has p^2 irreducible characters, each of which has the same defect as those of M. This means that B also has p^2 irreducible characters each of which has defect 2. So, $k_2(B \mid b_M) = p^2$, where $k_d(B \mid b_M)$ refers to the number of irreducible ordinary characters of B which lie over the characters of b_M with the defect d. Similarly, $k_3(B \mid b_E) = p$.

Under these circumstances, and for $d = 2$, we have reduced the conjectures with which we are dealing to show that $p^2 = \sum_{\sigma \in \mathscr{R}(G)^{\#}/G} (-1)^{|\sigma|+1} \sum_{\mu \in Irr2(V_\sigma)/N_G(\sigma)} f_0^{(B)}(I_{N_G(\sigma)}(\mu)/V_\sigma)$. In the principal p-block case, we can replace $N_G(E, b_E)$ by the inertial subgroup of the irreducible character of $O_{p'}(C_G(E))$ which corresponds to b_E. So, the bijection that we seek to satisfy is $Irr^{(B)}(G) \longleftrightarrow Irr^{(B)}(I_{N_G(E,b_E)}(\beta)) \longleftrightarrow Irr^{(B)}(I_{N_G(M,b_M)}(\mu))$, where β and μ are irreducible characters of $O_{p'}(C_G(E))$ and $O_{p'}(C_G(M))$, respectively.

Now let us consider the case that $N_G(M, b_M)/C_G(M) \cong SL(2, p)$. By Theorem 5.2.12, each irreducible character of M extends uniquely to an irreducible character of b_M as a character of $C_G(M)$. It follows that the action of $N_G(M, b_M)$ on $Irr(M)$ extends to an action on the unique extension of b_M, because $C_G(M)$ acts trivially. By Theorem 3.4.4, $N_G(M, b_M)$ acts transitively on the set of non-trivial irreducible characters of M. Hence, $N_G(M, b_M)$ transitively permutes $p^2 - 1$ irreducible characters in $Irr_2(C_G(M)|b_M)$. Let μ be an irreducible character of M which is non-trivial. We have already computed the inertial subgroup of such μ in $N_G(M, b_M)$. It follows that $I_{N_G(M,b_M)}(\mu)/C_G(M) \cong C_p$.

According to the bijection mentioned in Theorem 5.2.12, let μ^* be the corresponding irreducible character of $Irr_2(C_G(M)|b_M)$ to μ. It is clear that $I_{N_G(M,b_M)}(\mu) \leq I_{N_G(M,b_M)}(\mu^*)$. Using the bijection $Irr_2(I_{N_G(M,b_M)}(\mu^*), \mu^*) \longleftrightarrow Irr_2(N_G(M, b_M), \mu^*)$ and Lemma 2.4.2, for each irreducible character $\theta \in Irr(N_G(M, b_M), \mu^*)$, there is an irreducible character $\beta \in I_{N_G(M,b_M)}(\mu)/C_G(M)$, such that $\theta = \mu^* \otimes \beta$. As a result, $\theta(1) = \mu^*(1)\beta(1)$. But β is a linear character, since it is a character of the cyclic group $I_{N_G(M,b_M)}(\mu)/C_G(M) \cong C_p$. Therefore, $\theta(1) = \mu^*(1)$. On the other hand, $Res_M^{C_G(M)}(\mu^*) = Res_M^{C_G(M)}(Ind_M^{C_G(M)}(\mu)) =$

$[C_G(M):M]\mu$. Therefore, $\mu^*(1) = [C_G(M):M]\mu(1) = [C_G(M):M]$, because μ is an irreducible character of the abelian group M. Hence,

$$\theta(1) = [C_G(M):M]. \tag{5.2.1}$$

Now, by Lemma 3.5.5, θ is M-projective if, and only if, $\theta(1)_p = [N_G(M,b_M):M]_p$. Hence, using 5.2.1, θ is M-projective if, and only if, $[N_G(M,b_M):M]_p = [C_G(M):M]_p$, which happens if, and only if, $[N_G(M,b_M):C_G(M)]_p = 1$. However, this cannot occur under the assumption that $N_G(M,b_M)/C_G(M) \cong SL(2,p)$. Hence, there are no M-projective irreducible characters of $N_G(M,b_M)$ which lie over the non-trivial irreducible character of M. So, by Lemma 3.5.4, there is no p-block of defect zero of the section $I_{N_G(M,b_M)}(\mu)/M$ in the Brauer correspondence with B, for any irreducible non-trivial character of M. Hence, only the trivial character of M yields a proper contribution. Therefore, using the assumption and Theorem 4.3.8, we conclude that $\sum_{\mu\in Irr(M)/N_G(M,b_M)} f_0^{(B)}(I_{N_G(M,b_M)}(\mu)/M) = 1$. Let us record the above discussion as a proposition:

Proposition 5.2.15. *Assume that $N_G(M,b_M)/C_G(M) \cong SL(2,p)$, for each (G,B)-subpair (M,b_M). Then, we have $\sum_{\mu\in Irr(M)/N_G(M,b_M)} f_0^{(B)}(I_{N_G(M,b_M)}(\mu)/M) = 1$.*

Now, we have to compute the contribution from the chain $(M,b_M) < (E,b_E)$. It has $N_G(M,b_M)\cap N_G(E,b_E)$ as a stabilizer group.

Theorem 5.2.16. *Assume that $N_G(M,b_M)/C_G(M) \cong SL(2,p)$, for each (G,B)-subpair (M,b_M). Then, $\sum_{\mu\in Irr(M)/N_G(M,b_M)\cap N_G(E,b_E)} f_0^{(B)}(I_{N_G(M,b_M)\cap N_G(E,b_E)}(\mu)/M) = 1$.*

Proof: By Corollary 3.5.18, $N_G(M,b_M)\cap N_G(E,b_E)$ has three orbits when it acts on the set of ordinary irreducible characters of M. Let 1_M, μ and λ be the representatives of these three orbits. The bijection in Theorem 5.2.12 gives rise to a bijection between $Irr(M)$ and $Irr_2(C_G(M)\cap N_G(E,b_E)|b_M)$. We can write $1_M \longleftrightarrow 1_M^* \in Irr_2(C_G(M)\cap$

$N_G(E, b_E)|b_M)$, $\mu \longleftrightarrow \mu^* \in Irr_2(C_G(M) \cap N_G(E, b_E)|b_M)$, and $\lambda \longleftrightarrow \lambda^* \in Irr_2(C_G(M) \cap N_G(E, b_E)|b_M)$. We have seen the inertial groups of μ and of λ in Corollary 3.5.18. Then the alternating sum under consideration becomes

$$\sum_{\mu \in Irr(M)/N_G(M,b_M) \cap N_G(E,b_E)} f_0^{(B)}(I_{N_G(M,b_M) \cap N_G(E,b_E)}(\mu)/M) = I_1 + I_2 + I_3,$$

where

$$I_1 = f_0^{(B)}(I_{N_G(M,b_M) \cap N_G(E,b_E)}(1_M)/M) = f_0^{(B)}(N_G(M, b_M) \cap N_G(E, b_E)/M),$$

$$I_2 = f_0^{(B)}(I_{N_G(M,b_M) \cap N_G(E,b_E)}(\mu)/M) = f_0^{(B)}(C_G(M) \cap N_G(E, b_E)) \cdot \bar{E})/M),$$

$$I_3 = f_0^{(B)}(I_{N_G(M,b_M) \cap N_G(E,b_E)}(\lambda)/M) = f_0^{(B)}(C_G(M) \cap N_G(E, b_E))/M).$$

We claim that $I_1 = 0 = I_2$, and $I_3 = 1$.

Let us start by showing that $I_3 = 1$. We observe that λ^* is an irreducible character of $C_G(M) \cap N_G(E, b_E)$ with $I_{N_G(M,b_M) \cap N_G(E,b_E))}(\lambda^*) = C_G(M) \cap N_G(E, b_E)$. Then

$$Ind_{C_G(M) \cap N_G(E,b_E))}^{N_G(M,b_M) \cap N_G(E,b_E)}(\lambda^*)$$

is an irreducible character of $N_G(M, b_M) \cap N_G(E, b_E)$. However, $\lambda^* = Ind_M^{C_G(M) \cap N_G(E,b_E)}(\lambda)$. Hence,

$$Ind_{C_G(M) \cap N_G(E,b_E))}^{N_G(M,b_M) \cap N_G(E,b_E)}(\lambda^*) = Ind_{C_G(M) \cap N_G(E,b_E))}^{N_G(M,b_M) \cap N_G(E,b_E)}(Ind_M^{C_G(M) \cap N_G(E,b_E)}(\lambda)).$$

Using the transitivity of the induction processes, $\theta := Ind_M^{N_G(M,b_M) \cap N_G(E,b_E)}(\lambda)$ is an irreducible character of $N_G(M, b_M) \cap N_G(E, b_E)$. Therefore, each member of the orbit $[\lambda]$ induces to the irreducible character θ of $N_G(M, b_M) \cap N_G(E, b_E)$ which is M-projective. Furthermore, θ belongs to the unique p-block b of $N_G(M, b_M) \cap N_G(E, b_E)$ which covers b_M. Therefore, $I_3 = f_0^{(B)}((C_G(M) \cap N_G(E, b_E))/M) = 1$.

For I_2, a similar argument to the proof of Proposition 5.2.15 can be used to show that $I_2 = 0$. Now for I_1, using Lemma 3.5.6, it is enough to compute $f_0^{(B)}(N_G(M, b_M) \cap N_G(E, b_E))/C_G(M) \cap N_G(E, b_E))$. But, by Proposition 3.5.17,

$$N_G(M, b_M) \cap N_G(E, b_E))/C_G(M) \cap N_G(E, b_E) \cong N_{SL(2,p)}(\bar{E}).$$

Thus, $I_3 = f_0^{(B)}(N_G(M, b_M) \cap N_G(E, b_E)/C_G(M) \cap N_G(E, b_E)) = 0$, because it has a non-trivial normal p-subgroup. The result is that

$$\sum_{\mu \in Irr(M)/N_G(M,b_M) \cap N_G(E,b_E)} f_0^{(B)}(I_{N_G(M,b_M) \cap N_G(E,b_E)}(\mu)/M) = 1.$$

□

Now we have to gather Proposition 5.2.15 and Theorem 5.2.16 into the following corollary:

Corollary 5.2.17. *Let G be a finite group, B be a p-block of G with a defect group which is an extra-special p-group of order p^3 and exponent p for an odd prime number p. Consider (E, b_E) as a maximal (G, B)-subpair. Let (M, b_M) be an arbitrary subpair which is contained in (E, b_E). Write $\mathscr{R}(G|(M, b_M))$ for radical chains which start with (M, b_M). Then if $N_G(M, b_M)/C_G(M) \cong SL(2, p)$,*

$$\sum_{\sigma \in \mathscr{R}(G|(M,b_M))} (-1)^{|\sigma|+1} \sum_{\mu \in Irr(M)/N_G(\sigma)} f_0^{(B)}(I_{N_G(\sigma)}(\mu)/V_\sigma) = 0.$$

Proof: The result follows from Proposition 5.2.15, Theorem 5.2.16 and the observation of the dimension of σ. □

However, the conclusion of the Corollary 5.2.17 can be attained with the assumption that $SL(2, p) \lnapprox N_G(M, b_M)/C_G(M)$. To see this, we first have the following theorem:

Theorem 5.2.18. *With the notation above, $I_{N_G(M,b_M)}(\mu^*)/C_G(M) \cong SL(2, p)$.*

Proof: Let us assume that $SL(2,p) < N_G(M,b_M)/C_G(M) < GL(2,p)$. Let μ be a non-trivial irreducible character of M. By Theorem 5.2.12, μ extends uniquely to an irreducible character μ^* of $(C_G(M)|b_M)$. One can consider the inertial subgroup of μ^* in $N_G(M,b_M)$, namely $X := I_{N_G(M,b_M)}(\mu^*)$. Thus, μ^* is an irreducible character of the normal subgroup $C_G(M)$, which is X-stable. Then, using Lemma 2.4.3, there is a finite central extension group $\tilde{X}$ with normal subgroup $\tilde{C}$ such that $\tilde{X}/\tilde{C} \cong X/C_G(M)$. However, if p is large, $SL(2,p)$ is a perfect group with no central extension (see [43, Section 7.1, Theorem 7.1.1 (i)] or [70]) and this forces us to conclude that $X/C_G(M) \cong SL(2,p)$. This proves Theorem 5.2.18. □

Corollary 5.2.19. *In the case that $SL(2,p) \lneqq N_G(M,b_M)/C_G(M)$, with the notation above, $N_G(M,b_M)/I_{N_G(M,b_M)}(\mu^*) \cong C_r$.*

Proof: Using Theorem 5.2.18, the second isomorphism theorem, and the assumption

$$N_G(M,b_M)/I_{N_G(M,b_M)}(\mu^*) \cong [N_G(M,b_M)/C_G(M)]/[I_{N_G(M,b_M)}(\mu^*)/C_G(M)]$$

$$\cong [N_G(M,b_M)/C_G(M)]/SL(2,p) \cong C_r.$$

□

Now, if θ is an irreducible character of (X,μ^*) which is $N_G(M,b_M)$-stable, then we have to count the number of irreducible characters of $N_G(M,b_M)$ which are extensions of such θ, and hence, we compute the number of M-projective irreducible characters of $N_G(M,b_M)$ which belong to the unique p-block of $N_G(M,b_M)$ which covers b_M. To reach this aim, we exploit Corollary 5.2.19. Therefore, $N_G(M,b_M)/X \cong C_r$, where r is determined by the assumption that $N_G(M,b_M)/C_G(M) \cong SL(2,p) \rtimes C_r$, with $1 \leq r \leq p-1$. As a result, θ, and hence, μ have r distinct extensions, each of which is an irreducible character of $N_G(M,b_M)$ and each of which is an M-projective.

On the other hand, the proof that $N_G(M, b_M) \cap N_G(E, b_E)$ has r distinct irreducible characters, each of which is M-projective, can be made in a similar manner. To see this, we observe that each irreducible character μ of M extends uniquely to an irreducible character, say μ^* of $(C_G(M) \cap N_G(E, b_E)|b_M)$. One can consider the inertial subgroup of μ^*, say Y, in $N_G(M, b_M) \cap N_G(E, b_E)$. Clifford's Theorem 2.4.1 gives us the opportunity to say that there is a bijection between the irreducible characters of Y which lie over μ^* and the irreducible characters of $N_G(M, b_M) \cap N_G(E, b_E)$ which lie over μ. Now $C_G(M) \cap N_G(E, b_E)$ is a normal subgroup of the finite group $N_G(M, b_M) \cap N_G(E, b_E)$ and μ^* is an $N_G(M, b_M) \cap N_G(E, b_E)$-stable irreducible character of $C_G(M) \cap N_G(E, b_E)$. Then there is a central extension of $\frac{N_G(M,b_M) \cap N_G(E,b_E)}{C_G(M) \cap N_G(E,b_E)}$. But the section $\frac{N_G(M,b_M) \cap N_G(E,b_E)}{C_G(M) \cap N_G(E,b_E)}$ is isomorphic to $N_{\bar{N}}(\bar{E})$, where $\bar{N} = N_G(M, b_M)/C_G(M)$ and $\bar{E} = EC_G(M)/C_G(M)$. Using the observation that

$$N_G(M, b_M) \cap N_G(E, b_E)/Y \cong \frac{N_G(M, b_M) \cap N_G(E, b_E)}{C_G(M) \cap N_G(E, b_E)} / \frac{Y}{C_G(M) \cap N_G(E, b_E)} \cong C_r,$$

we conclude that each $N_G(M, b_M) \cap N_G(E, b_E)$ has r irreducible characters which are M-projective. Therefore, for $d = 2$, the contribution from the chain which start with (M, b_M) cancels that which comes from $(M, b_M) < (E, b_E)$. Hence, whenever, $SL(2, p) \lneqq N_G(M, b_M)/C_G(M)$, the OWC is equivalent to the equality between $k_d(B)$ and $k_d(b)$, for each d, where b is the unique p-block of $N_G(E, b_E)$ which is in Brauer correspondence with B.

5.3 The case that $O_p(G)$ is trivial and $N_G(E, b_E)/EC_G(E)$ has no element of order $p - 1$.

We shall start this section with the following lemma which is a generalization of the

Frattini argument in the (G, B)-subpairs:

Lemma 5.3.1. *Let H be a normal subgroup of the finite group G and b be a p-block of H which is G-stable with defect group Q. Then for each maximal (H, b)-subpair (Q, b_Q), we have $G = HN_G(Q, b_Q)$.*

It seems to us that $EC_G(M)$ will play the central part of our discussion, in this section. This is because if $O_p(G)$ is trivial and $N_G(E, b_E)/EC_G(E)$ has no element of order $p-1$ then, by Proposition 3.5.20, $O_p(N_G(M, b_M)/C_G(M)) \neq 1$. Therefore, the inverse image of this group is a normal subgroup of $N_G(M, b_M)$. However, $EC_G(M)$ is the inverse image of $O_p(N_G(M, b_M)/C_G(M)) \neq 1$. The first observation is to make the following corollary of Lemma 5.3.1.

Corollary 5.3.2. *Let B be a p-block of G with defect group E which is an extra-special p-group of order p^3 and exponent p for an odd prime number p. With the assumption in this section, we have $N_G(M, b_M) = (N_G(M, b_M) \cap N_G(E, b_E))C_G(M)$, for each p-radical (G, B)-subpair (M, b_M) which is contained in (E, b_E).*

Theorem 5.3.3. *If B is the principal p-block with an extra-special defect group, E, which is of order p^3 and exponent p, for an odd prime number p, then $[N_G(M, b_M) : EC_G(M)] \not\equiv 0$ mod (p), for each maximal subgroup M of E.*

Proof: We see that $E \leq EC_G(M) \leq N_G(M, b_M)$. Consequently, we have

$$[N_G(M, b_M) : E] = [N_G(M, b_M) : EC_G(M)][EC_G(M) : E].$$

However, E is a Sylow p-subgroup of $N_G(M, b_M)$ and

$$[EC_G(M) : E] = \frac{|EC_G(M)|}{|E|} = \frac{|C_G(M)|}{|C_E(M)|} = \frac{|C_G(M)|}{|M|} = |O_{p'}(N_G(M, b_M))|,$$

as the assumption in this result that we are dealing with the principal p-block,

$$C_G(M) = M \times O_{p'}(N_G(M, b_M)).$$

Hence, p does not divide the index $[N_G(M, b_M) : EC_G(M)]$. $\square$

The following result is a kind of generalization of the Frattini argument see [9, 6.2, & 6.3 p. 20].

Proposition 5.3.4. *With the above notation,* $N_G(M, b_M) = (N_G(M, b_M) \cap N_G(E, b_E)C_G(M)$ *if, and only if,* $\{E^g \mid g \in N_G(M, b_M)\} = \{E^c \mid c \in C_G(M)\}$.

Proof: Assume that $N_G(M, b_M) = (N_G(M, b_M) \cap N_G(E, b_E))C_G(M)$. Then for each $g \in N_G(M, b_M)$, we have $g = nc$ for some $n \in N_G(E, b_E) \cap N_G(M, b_M)$ and $c \in C_G(M)$. Thus, $E^g = E^{nc} = E^c$.

Conversely, suppose that $\{E^g \mid g \in N_G(M, b_M)\} = \{E^c \mid c \in C_G(M)\}$. Clearly, $N_G(M, b_M) \supseteq (N_G(M, b_M) \cap N_G(E, b_E))C_G(M)$. Now the Dedekind Modular Law (see [9, 1.14, p. 6] implies that $N_G(M, b_M) \supseteq N_G(M, b_M) \cap N_G(E, b_E) \cdot C_G(M)$ Now, let $g \in N_G(M, b_M)$. Then either $g \in C_G(M)$ or $g \notin C_G(M)$. So, if $g \in C_G(M)$ then clearly $g \in N_G(M, b_M) \cdot C_G(M)$ and hence, $g \in (N_G(M, b_M) \cap N_G(E, b_E))C_G(M)$. Thus, we have to consider the case that $g \in N_G(M, b_M) - C_G(M)$. This means that $gC_G(M)$ is a non-trivial element of $N_G(M, b_M)/C_G(M)$. $\square$

Proposition 5.3.5. *With the assumption in this section, for each maximal subgroup M of E which is a radical p-subgroup of G, we have* $N_G(EC_G(M)) = N_G(E, b_E)C_G(M)$.

Proof: Let $g \in N_G(E, b_E)C_G(M)$. Then $g = nc$ for some $n \in N_G(E, b_E)$ and some $c \in C_G(M)$. The idea here is that Sylow p-subgroups of G form a single G-conjugacy class of maximal p-subgroups and that M runs through maximal subgroups of E which are radical p-subgroups of G. Also, we know that $N_G(E, b_E)$ acts on such Ms. Hence, $EC_G(M)^g = E^gC_G(M^g) = E^cC_G(M^{nc})$. However, E^c is still a Sylow p-subgroup of G and also M^n is a maximal subgroup of E. It follows from this categorical perspective that we may assume that $g \in N_G(EC_G(M))$.

Conversely, let g be an element of g for which $EC_G(M)^g = EC_G(M)$. Then $E^gC_G(M^g) = EC_G(M)$. Again since this holds for each Sylow p-subgroup of G and for each maximal subgroup of E which is radical p-subgroup of G, we can conclude that $g \in N_G(E, b_E)C_G(M)$. $\square$

Corollary 5.3.6. *$N_G(E, b_E)$ controls the fusion of (G, B)-subpairs G if, and only if, $N_G(M, b_M)$ normalizes $EC_G(M)$ for each maximal subgroup M of E which is a radical p-subgroup of G. In particular, the assumption in this section implies that $N_G(E, b_E)$ controls the fusion of (G, B)-subpairs.*

Proof: Assume that $N_G(M, b_M)$ normalizes $EC_G(M)$. However, $EC_G(M)$ is a subgroup of $N_G(M, b_M)$, since M is a normal subgroup of E. Hence, $EC_G(M)$ is a normal subgroup of $N_G(M, b_M)$. Now we are in the situation that (E, b_E) is a maximal $(EC_G(M), b_1)$-subpair where b_1 is the unique nilpotent p-block of $EC_G(M)$ which covers b_M and $N_G(M, b_M)$-stable. Thus the Frattini argument gives us that $N_G(M, b_M) = N_{N_G(M,b_M)}(E) \cdot EC_G(M)$. But E is contained in $N_G(M, b_M)$ because M is a normal subgroup of E. Hence, E is contained in $N_{N_G(M,b_M)}(E)$. Thus,

$$N_G(M, b_M) = N_{N_G(M,b_M)}(E) \cdot C_G(M) = (N_G(E, b_E) \cap N_G(M, b_M))C_G(M).$$

This means that $N_G(E, b_E)$ controls the fusion of (G, B)-subpairs.

Conversely, suppose that $N_G(E, b_E)$ controls the fusion of (G, B)-subpairs. This means that $N_G(M, b_M) = (N_G(E, b_E) \cap N_G(M, b_M))C_G(M)$, for all maximal subgroups of E which are radical p-subgroups of G. However, $C_G(M)$ is contained in $N_G(M, b_M)$. Thus, using the Dedekind Modular Law (see [9, 1.14, p. 6]), we have $N_G(M, b_M) = N_G(E, b_E)C_G(M) \cap N_G(M, b_M)$. Now, using Proposition 5.3.5, $N_G(M, b_M) = N_G(EC_G(M)) \cap N_G(M, b_M)$. So, $N_G(M, b_M) = N_{N_G(M,b_M)}(EC_G(M))$. This completes the proof that $EC_G(M)$ is a normal subgroup of $N_G(M, b_M)$. □

As usual, let M be a maximal subgroup of E which is a radical p-subgroup of E. Again, we shall consider the automorphism group of M, namely $GL(2, p)$, because M is an elementary abelian group of order p^2. We assume that E is an extra-special group of order p^3 and exponent p for an odd prime number p. The following proposition and the assumption in this section show that $N_G(E, b_E)$ controls the fusion of $G, B)$-subpairs.

Proposition 5.3.7. *If $N_G(M, b_M)/C_G(M)$ has a unique Sylow p-subgroup, then we have $N_G(M, b_M) \leq N_G(E, b_E)C_G(M)$.*

Proof: If $N_G(M, b_M)/C_G(M)$ has a unique Sylow p-subgroup then it is a normal p-subgroup, and, hence, $O_p(N_G(M, b_M)/C_G(M) \neq 1$. It follows that $EC_G(M)$ is a normal subgroup of $N_G(M, b_M)$. Hence,

$$N_G(M, b_M) = (N_G(E, b_E) \cap N_G(M, b_M))C_G(M) = N_G(E, b_E)C_G(M) \cap N_G(M, b_M).$$

Hence, $N_G(M, b_M) \leq N_G(E, b_E)C_G(M)$, which is the required conclusion. □

Now we summarize the above discussion into the following proposition. Note that $N_G(M, b_M)/C_G(M)$ has a unique Sylow p-subgroup, which means that the quotient group is a proper subgroup of $SL(2, p)$.

Corollary 5.3.8. *Let G be a finite group, let B be a p-block of G with a defect group E which is an extra-special p-group of order p^3, and exponent p, for an odd prime number p. Let M be an arbitrary maximal subgroup of E which is a radical p-subgroup of G. Then we have, if $N_G(M, b_M)/C_G(M) < SL(2, p)$, we infer that $N_G(E, b_E)$ controls the fusion* *of* (G, B)*-subpairs.*

Proof: Suppose that $N_G(M, b_M)/C_G(M) < SL(2, p)$. Then, Theorem 2.6.7 implies that $O_p(N_G(M, b_M)/C_G(M)) \neq 1$. Hence, Proposition 5.3.7 implies that $N_G(M, b_M) \leq N_G(E, b_E)C_G(M)$. Therefore, $N_G(M, b_M) = N_G(E, b_E))C_G(M) \cap N_G(M, b_M)$. Using the Dedekind Modular Law, (see [9, 1.14, p. 6]), $N_G(M, b_M) = (N_G(E, b_E) \cap N_G(M, b_M))C_G(M)$. This means that $N_G(E, b_E))$ controls the fusion of (G, B)-subpairs. Since M was an arbitrary maximal subgroup of E which is a radical p-subgroup of G, we conclude that the situation above holds for all Ms. $\square$

Theorem 5.3.9. *Let G be a finite group with E, a p-subgroup which is an extra-special group of order p^3, with exponent p, for an odd prime number p. Let B be a p-block of G with the defect group E. Then we have, if Dade's Projective Conjecture (DPC) holds for B, then $k_d(B) = k_d(b)$, for all non-negative integer d, where b is the unique p-block of $N_G(E, b_E)$ such that $b^G = B$.*

Proof: By Corollary 5.3.8, $N_G(E, b_E)$ controls the fusion of (G, B)-subpairs. Then the action of the stabilizer of (M, b_M) and of the stabilizer of $(M, b_M) < (E, b_E)$ is the same and the contributions cancel each other. We are left only with the contribution from the trivial chain and the chain (E, b_E). $\square$

Finally, in this section we have the following corollary:

Corollary 5.3.10. *Let G be a finite group and let B be a p-block of G with defect group E, which is isomorphic to an extra-special group of order p^3 and exponent p. Then the DPC projective conjecture holds for B if, and only, if, for all non-negative integer d, $k_d(B) = k_d(b)$, where b is the unique p-block of $N_G(E, b_E)$ such that $b^G = B$.*

5.4 The case where $O_p(G)$ is a central p-subgroup of G and equals $Z(E)$.

In this section, we shall study the case that $O_p(G)$ is a central p-subgroup of G. Since E is a defect group of B, $O_p(G) \leq E$. Hence, $O_p(G) \leq E \cap Z(G) \leq Z(E)$. Thus $O_p(G) \in \{1,\ Z(E)\}$. Furthermore, in this case, we can deal with both the OWC and the DPC.

Let us consider that $O_p(G) = Z(E)$. Now, on the one hand, $O_p(G)$ is a cyclic normal p-subgroup of G of order p. On the other hand, for each maximal subgroup $M = \langle x, z\rangle$ of $E = \langle x, y, z\rangle$, we have $E/O_p(G) = \langle xO_p(G), yO_p(G)\rangle$ and $M/O_p(G) = \langle xO_p(G)\rangle \cong C_p$.

We record the following observation in this case:

Theorem 5.4.1. *In this case, $O_p(G)$ cannot be V_σ for any chain σ in the alternating sum under consideration.*

Proof: Since $O_p(G) = Z(E)$, and E is a non-abelian defect group of the p-block B, Theorem 3.5.9 implies that the only chains (up to G-conjugacy) which contribute properly to the alternating sum under consideration are those whose initial subgroups V_σ satisfy $C_E(V_\sigma) \subseteq V_\sigma$. So, the result follows because $Z(E) < E$. $\square$

Let M be an arbitrary maximal subgroup of E which is a radical p-subgroup of G.

Theorem 5.4.2. *In this case, $SL(2,p)$ cannot be a section in $N_G(M, b_M)/C_G(M)$.*

Proof: Assume that $SL(2,p) \lessapprox N_G(M, b_M)/C_G(M)$. Then $N_G(M, b_M)/C_G(M)$ has $p+1$ Sylow p-subgroups, each of which is cyclic of order p. To prove the theorem, we shall construct a non-trivial normal p-subgroup of $N_G(M, b_M)/C_G(M)$. Note that $E = \langle x, y, z\rangle$, $M = \langle x, z\rangle$ and $\langle z\rangle = Z(E) = O_p(G) \leq Z(G)$. So, $M/O_p(G) \cong \langle x\rangle$.

Consider

$$C_G(M/O_p(G)) := \{g \in G \mid [M, \langle g \rangle] \leq O_p(G)\}. \tag{5.4.1}$$

The first observation is that $C_G(M)$ is a proper subgroup of $C_G(M/O_p(G))$. This is because $y \notin C_G(M)$, but $y \in C_G(M/O_p(G))$ as $[M, y] \ni [x, y] = z \in O_p(G) = Z(E)$. We claim that $C_G(M/O_p(G))$ is a normal subgroup of $N_G(M, b_M)$, since we see that if $g \in N_G(M, b_M)$ then $(C_G(M/O_p(G)))^g = C_G(M^g/O_p(G)^g) = C_G(M/O_p(G))$.

Now $C_G(M/O_p(G))$ contains $C_G(M)$ properly. So, $C_G(M/O_p(G))/C_G(M)$ is a non-trivial normal subgroup of $N_G(M, b_M)/C_G(M)$. We must show that $C_G(M/O_p(G))/C_G(M)$ is a p-group. Let g be a p-regular element in $C_G(M/O_p(G))$. Then, by 5.4.1, $[M, \langle g \rangle] \leq O_p(G)$. Note that $\langle g \rangle$ is the cyclic $p^{'}$-group generated by g. Since $O_p(G)$ is a cyclic p-subgroup of order p, we have either that $[M, \langle g \rangle] = 1$ or $[M, \langle g \rangle] = O_p(G)$. However, if $[M, \langle g \rangle] = O_p(G)$, we observe that $[M, \langle g \rangle, \langle g \rangle] = [\ [M, \langle g \rangle]\ , \langle g \rangle] = [O_p(G), \langle g \rangle] = 1$, because $O_p(G)$ is central. Now the co-prime action [9, 24.5, p. 113] implies that $[M, \langle g \rangle] = 1$. Thus, $g \in C_G(M)$. Since g is an arbitrary p-regular element in $C_G(M/O_p(G))$, we conclude that each p-regular element in $C_G(M/O_p(G))$ belongs to $C_G(M)$. This suffices to show that $C_G(M/O_p(G))/C_G(M)$ is a non-trivial normal p-subgroup of $N_G(M, b_M)/C_G(M)$ which is a contradiction with the assumption that $SL(2, p) \lessapprox N_G(M, b_M)/C_G(M)$. Hence, in this case, $SL(2, p)$ cannot be a section in the quotient group $N_G(M, b_M)/C_G(M)$. This completes the proof. $\square$

Now we are in the situation that $N_G(M, b_M)/C_G(M)$ is not a $p^{'}$-group in $GL(2, p) \cong SL(2, p) \rtimes C_{p-1}$ and $SL(2, p) \not\lessapprox N_G(M, b_M)/C_G(M)$. Therefore, we are left with the following corollary:

Corollary 5.4.3. *In this case, $N_G(M, b_M)/C_G(M) < SL(2, p)$.*

Proof: $SL(2, p) \not\lessapprox N_G(M, b_M)/C_G(M)$ and $N_G(M, b_M)/C_G(M)$ is not a $p^{'}$-group, yielding that $N_G(M, b_M)/C_G(M) < SL(2, p)$. In addition, the map $\eta : M \to M$, such that $\eta(x) = x + z$ and $\eta(z) = z$ gives us the required inclusion. Indeed, η is an automorphism of M of order p, which does not centralise M. Hence, η can be regarded as an element of $N_G(M, b_M)/C_G(M)$. On the other hand, η can be represented in the matrix form as follows: $\eta = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. We see that η is an element in $SL(2, p)$, as it has the determinant 1. So, in fact, $N_G(M, b_M)/C_G(M) < SL(2, p)$. □

Now Theorem 2.6.7 implies the following important result:

Corollary 5.4.4. *In this case, $O_p(N_G(M, b_M)/C_G(M)) \neq 1$.*

Proof: This is clear from Theorem 2.6.7 and Corollary 5.4.3. □

The main result of this section is the following theorem.

Theorem 5.4.5. *In this case, $N_G(E, b_E)$ controls the fusion of (G, B)-subpairs.*

Proof: In this case, $EC_G(M)$ is a normal subgroup of $N_G(M, b_M)$. Hence, the result follows from Theorem 5.3.6. □

The main result in this section is the following theorem:

Theorem 5.4.6. *Let G be a finite group. Let B be a p-block of G with a defect group which is isomorphic to an extra-special p-group, say E, of order p^3 and exponent p for an odd prime number p. Let b be the unique p-block of $N_G(E, b_E)$ such that $b^G = B$. Assume that $Z(E) = O_p(G) \leq Z(G)$. Then the DPC or the OWC hold for B if, and only if, $k_d(B) = k_d(b)$, for all non-negative integers d.*

Proof: It is clear that if $d = 3$ then DPC is equivalent to $k_3(B) = k_3(b)$. Assume that $d = 2$. Let M be a maximal subgroup of E which is a radical p-subgroup of G. The assumption and Theorem 5.4.5 enable us to conclude that the action of $N_G(E, b_E) \cap N_G(M, b_M)$ on $Irr(M)$ is the same action as that of $N_G(M, b_M)$ on $Irr(M)$.

Thus, $\sum_{\sigma \in \mathscr{R}(G,B)} (-1)^{|\sigma|+1} \sum_{\mu \in Irr_2(V_\sigma)/N_G(\sigma)} f_0^{(B)}(I_{N_G(\sigma)}(\mu)/V_\sigma) = k_2(B) - k_2(b)$. It follows that the conjecture under consideration holds for B if, and only if, $k_d(B) = k_d(b)$, for all non-negative integers d. $\square$

5.5 The situation that $O_p(G)$ is not a central p-subgroup of G.

We devote this section to the OWC. The assumption that $O_p(G)$ is not a central p-subgroup of G does not allow us to deal with the DPC. So, we have to tackle the OWC. Now, $O_p(G)$ is contained in E. Thus, we have two options in this case. The first one is that $O_p(G)$ is a maximal subgroup of E. The second is that $O_p(G)$ is the centre of E. Observe that, in this case, $O_p(G) \neq 1$, as 1_G is a central p-subgroup of G. As a result, G has no p-blocks of defect zero in this case.

5.5.1 The case that $O_p(G)$ is a maximal subgroup of E

Now let us discuss the case that $O_p(G) = M$, for a maximal subgroup M of E. Of course, E cannot be a normal p-subgroup in this case.

First of all, we claim that M is the unique radical p-subgroup of G of order p^2 which is contained in E. Let us record this observation as a lemma.

Lemma 5.5.1. *Let M be an arbitrary radical p-subgroup of G which is a maximal subgroup of E. Then if $M = O_p(G)$ then M is the unique radical p-subgroup of G of order p^2 which is contained in E.*

Proof: If N is a radical p-subgroup of G which is contained in E, then $O_p(G) \leq N$. However, $p^2 = |O_p(G)| = |N|$ implies that $N = O_p(G) = M$. $\square$

We observe that $O_p(G) \leq C_G(O_p(G))$, as $O_p(G)$ is abelian. Now $C_G(O_p(G))$ is a normal subgroup of G. Consequently, $G/C_G(O_p(G)) \lesssim Aut(O_p(G)) \cong GL(2,p)$.

Lemma 5.5.2. *In this case, either $O_p(G) = C_G(O_p(G))$, or $O_p(G) < C_G(O_p(G))$.*

Proof. This is clear, because $O_p(G)$ is an abelian subgroup of G.

Now let us discuss the case that $O_p(G)$ is self-centralizing in G. We mean that in the following we shall assume that $O_p(G) = C_G(O_p(G))$. Hence, using [32, Corollary 3.11, Chapter V, p. 200], G has a unique p-block, namely, the principal p-block.

Corollary 5.5.3. *In this case, $G/O_p(G) \lesssim GL(2,p)$.*

Now, three cases have to be distinguished; that $G/O_p(G)$ is a $p^{'}$-group, that $G/O_p(G)$ has a unique Sylow p-subgroup and that $G/O_p(G)$ has $p+1$ Sylow p-subgroups. However, the following lemmas exclude two of these cases.

Lemma 5.5.4. *In this case, $G/O_p(G)$ cannot be a $p^{'}$-group.*

Proof: We observe that $O_p(G) < E < G$. Consequently, $[G : O_p(G)] = [G : E][E : O_p(G)]$. Hence, $[G : O_p(G)]_p = p$, which means that $G/O_p(G)$ cannot be a $p^{'}$-group. $\square$

Lemma 5.5.5. *In this case, $G/O_p(G)$ has more than one Sylow p-subgroup.*

Proof: If $G/O_p(G)$ has a unique Sylow p-subgroup then it is a normal p-subgroup. However, a Sylow p-subgroup of $G/O_p(G)$ has the form $P/O_p(G)$, for some Sylow p-subgroup of G. This implies that P is a normal p-subgroup of G, which is not the case, using Lemma 5.5.4. Thus, in this case, $G/O_p(G)$ has $p+1$ Sylow p-subgroups. $\square$

Again, three cases have to be distinguished:

- $G/O_p(G) \cong SL(2,p)$, and $G \cong O_p(G) \rtimes SL(2,p)$.

- $G/O_p(G) \cong X$, where $SL(2,p) < X < GL(2,p)$. Hence, $G \cong O_p(G) \rtimes X$.

- $G/O_p(G) \cong GL(2,p)$, and hence, $G \cong O_p(G) \rtimes GL(2,p)$.

Now let us assume that $O_p(G) < C_G(O_p(G))$. Consequently, we have the following corollary:

Corollary 5.5.6. *In this case, $O_{p'}(G) \neq 1$.*

Proof: This is true, because $C_G(O_p(G)) = O_p(G) \times O_{p'}(G)$. □

Using Corollary 5.2.17, we obtain the following result.

Theorem 5.5.7. *Consider the notation and the assumption in this subsection. Then*

$$\sum_{\sigma \in \mathscr{R}(G|(M,b_M))} (-1)^{|\sigma|+1} \sum_{\mu \in Irr(M)/N_G(\sigma)} f_0^{(B)}(I_{N_G(\sigma)}(\mu)/M) = 0.$$

Proof: This is a special case of Corollary 5.2.17. □

Corollary 5.5.8. *Let B be a p-block of G with defect group E. Let b be the unique p-block of $N_G(E, b_E)$ such that $b^G = B$. Consider the notation and the assumption in this subsection. Then the OWC holds if, and only if $k_d(B) = k_d(b)$ for all non-negative integer d.*

Proof: This is clear, in the light of Theorem 5.5.7. □

5.5.2 The case that $Z(E) = O_p(G)$

Now let us continue to deal with the OWC and the case that $O_p(G) = Z(E)$. The first observation in this case is that $O_p(G)$ cannot be the initial p-subgroup of any chain under consideration. This is because E is a non-abelian defect group of the p-block B and we know from Theorem 3.5.9 that the only chains (up to G-conjugacy) which contribute properly to the alternating sum under consideration are those whose initial subgroup V_σ satisfies $C_E(V_\sigma) \subseteq V_\sigma$. Let us record this observation as a lemma.

Lemma 5.5.9. *In this case, $O_p(G) \neq V_\sigma$, for any chain σ under consideration.*

The second observation is that G has no p-block of defect zero as $1 \neq O_p(G)$.

Theorem 5.5.10. *In this case, G has no p-block of defect zero.*

We see that E is contained in $C_G(O_p(G))$ because $O_p(G) = Z(E)$. However, the following lemma tells us that $O_p(G)$ cannot be self-centralising.

Lemma 5.5.11. *In this case, $O_p(G) \neq C_G(O_p(G))$.*

Theorem 5.5.12. *We have $O_p(G) = O_p(C_G(O_p(G)))$.*

Proof: Since $O_p(G)$ is a normal p-subgroup of $C_G(O_p(G))$, $O_p(G) \leq O_p(C_G(O_p(G)))$. Conversely, that $O_p(C_G(O_p(G)))$ is a characteristic p-subgroup of $C_G(O_p(G))$ and $C_G(O_p(G))$ is a normal subgroup of G enable us to conclude that $O_p(C_G(O_p(G)))$ is a normal p-subgroup of G. Hence, $O_p(C_G(O_p(G))) \leq O_p(G)$. This completes the theorem. $\square$

Now, the assumption that $O_p(G)$ is not central in G means that the normal subgroup $C_G(O_p(G))$ is a proper subgroup of G. In fact, we have the following chain: $1 < O_p(G) < C_G(O_p(G)) < G$. Consequently, we have $G/C_G(O_p(G)) \lesssim Aut(O_p(G))$. However, $O_p(G)$ is the centre of E, so $Aut(O_p(G)) \cong C_{p-1}$. This justifies the following theorem:

Theorem 5.5.13. *In this case, $G/C_G(O_p(G)) \lesssim C_{p-1}$.*

Let b be the unique p-block of $C_G(O_p(G))$ which is the Brauer correspondent with B. Now, since $C_G(O_p(G))$ is a normal subgroup of G, and (E, b_E) is a maximal $(C_G(O_p(G)), b)$-subpair, $G = N_G(E, b_E)C_G(O_p(G))$. Note that elements of G which centralise $Z(E)$ need not normalize E. In fact, $N_G(E, b_E) \neq C_G(O_p(G))$. Let us record this observation as a theorem.

Theorem 5.5.14. *In this case, we have $G = N_G(E, b_E)C_G(O_p(G))$.*

Now, Theorem 5.5.13 implies that $G/C_G(O_p(G))$ is a cyclic p'-group. Consequently, we have the following corollary:

Corollary 5.5.15. *In this case, each irreducible character of $C_G(O_p(G))$ which is G-stable extends to an irreducible character of G.*

Proof: This is true, since we have this fact in the general case, the proof of which can be found in [42, 11.22]. □

It is well-known that if M is an indecomposable G-module then M is a direct summand of the induced G-module of the restriction of M to a Sylow p-subgroup of G. As a result, M is an H-projective G-module for each subgroup H of G which contains a Sylow p-subgroup of G.

Now, $G/C_G(O_p(G)) \lessapprox C_{p-1}$ means that each Sylow p-subgroup of G is contained in $C_G(O_p(G))$, and, hence, each indecomposable G-module, as well as each irreducible G-module, is $C_G(O_p(G))$-projective. Therefore, we conclude in this case that each irreducible character of G is a $C_G(O_p(G))$-projective. However, if M is a maximal subgroup of E, which is a radical p-subgroup of G, then this M is contained in $C_G(O_p(G))$. So, we conclude that each irreducible character of G which lies over an irreducible character of M is M-projective.

On the other hand, the isomorphism $G/C_G(O_p(G)) \lessapprox C_{p-1}$ gives us the opportunity to deduce that each Sylow q-subgroup of $G/C_G(O_p(G))$ for each prime number q which divides $p-1$ is cyclic. Thus, for each irreducible character μ of M, $I_G(\mu)/C_G(O_p(G))$ has no non-trivial central extension. This is because all its Sylow q-subgroups are cyclic, and, hence, have trivial Schur Multipliers, (see [43, Section 7.1] or [70]).

In this case, we have to resort to Clifford theory. This means we have to fix $\zeta \in Irr(O_p(G))$ and try to compute: $k_d(B, \zeta) = I_1 - I_2 + I_3$, where

$$I_1 = \sum_{\mu \in Irr_d(M,\zeta)/N_G(M,b_M)} f_0^{(B)}(I_{N_G(M,b_M)}(\mu)/M),$$

$$I_2 = \sum_{\mu \in Irr_d(M,\zeta)/N_G(M,b_M)\cap N_G(E,b_E)} f_0^{(B)}(I_{N_G(M,b_M)\cap N_G(E,b_E)}(\mu)/M)$$

and

$$I_3 = \sum_{\eta \in Irr_d(E,\zeta)/N_G(E,b_E)} f_0^{(B)}(I_{N_G(E,b_E)}(\eta)/E).$$

If $d = 3$, then $Irr_3(M)$ is the empty set and hence, $I_1 = I_2 = 0$. So, $k_3(B, \zeta) = I_3$, in this case. Note that this is true for each $\zeta \in Irr(O_p(G))$. In the case that $d = 2$, Corollary 5.2.17, with its assumption, implies that the contributions from the chain starting with (M, b_M) cancel each other. So, we are left in this case also with the conclusion that $k_2(B, \zeta) = I_3$. Therefore, for each positive integer d,

$$k_d(B) = \sum_{\eta \in Irr(O_p(G))} k_d(B, \eta) = k_d(b),$$

where b is the unique p-block of $N_G(E, b_E)$ such that $b^G = B$.

5.6 The case that $O_p(G) = 1$ and the inertial quotient is $C_2 \times C_{p-1}$

In this section, we shall consider the case that $O_p(G)$ is trivial and $N_G(E, b_E)/EC_G(E) \cong C_2 \times C_{p-1}$. We have already computed the orbit structure for this case in Subsection 4.2.3.

The DPC and the OWC can be discussed under the hypothesis in this section. We shall continue to assume that the p-block B with defect group E satisfies OWC.

Of course, there is no problem in the cases where $0 = d$ and $1 = d$, either. Let us examine the case that $d = 3$. So, $Irr_3(M)$ is the empty set and by Lemma 4.2.5, the set $Irr_3(E)$ is partitioned into $p + 1$ orbits. Clearly 1_E forms an orbit and the second orbit has a representative of the form $1_{\langle x \rangle} \otimes \beta$ where β is a non-trivial irreducible character of $\langle y \rangle$. The other $p^2 - p$ linear irreducible characters of E distribute to $p - 1$ orbits, each of which contains p irreducible linear characters of the form $\alpha \otimes \beta^i$ where α is a non-trivial irreducible character of $\langle x \rangle$ and $0 \leq i \leq p$. For the inertial subgroups in $N_G(E, b_E)$, $1_{\langle x \rangle} \otimes \beta$ has C_2, while $\alpha \otimes \beta^i$ has C_{p-1} as an inertial subgroup.

Now the alternating sum under consideration for $d = 3$ reduces to the following sum

$$k_3(B) = \sum_{\eta \in Irr_3(E)/N_G(E,b_E)} f_0^{(B)}(I_{N_G(E,b_E)}(\eta)/E).$$

Using the orbits structure above and Theorem 4.2.3, we have to show that

$$k_3(B) = I_1 + I_2 + I_3,$$

where

$$I_1 = f_0^{(B)}(N_G(E, b_E)/EC_G(E)) = 2p - 2,$$

$$I_2 = f_0^{(B)}(EC_G(E)C_2/EC_G(E)) = 2$$

$$I_3 = (p - 1)f_0^{(B)}(EC_G(E)C_{p-1}/EC_G(E)) = p^2 - 2p + 1.$$

Hence, $k_3(B) = p^2 + 1$. Now for $d = 2$, we want to compute the sum

$$\sum_{\mu \in Irr_2(V_\sigma)/N_G(\sigma)} f_0^{(B)}(I_{N_G(\sigma)}(\mu)/V_\sigma).$$

In this case, M cannot be a normal p-subgroup of G. However, $N_G(M, b_M)$ acts transitively on the set of non-trivial irreducible characters of M and $N_G(E, b_E)$ acts transitively on the set of non-linear irreducible characters of E. Using Corollary 5.2.17 and its hypothesis, for chains starting with a radical (G, B)-subpair (M, b_M), has no proper contribution to the sum above. As a result, $k_2(B) = f_0^{(B)}(N_G(E, b_E)/EC_G(E)) = 2p-2 = k_2(b)$, where b is the unique p-block of $N_G(E, b_E)$ such that $b^G = B$. Hence, $k(B) = p^2 + 1 + 2p - 2 = p^2 + 2p - 1$. Therefore, $k(B) - k(E) = p$.

Now, we summarize the above into the following theorem.

Theorem 5.6.1. *Let G be a finite group, B be a p-block of G with defect group E which is an extra-special p-group of order p^3 and exponent p for an odd prime number p. Assume that $O_p(G) = 1$ and the section $N_G(E, b_E)/EC_G(E) \cong C_2 \times C_{p-1}$. If B satisfies OWC then $k(B) - k(E) = p$.*

Chapter 6

CONCLUSION AND SUGGESTIONS FOR FURTHER RESEARCH

We fix an odd rational prime number p. We have studied the Ordinary Weight Conjecture and Dade's Projective Conjecture for a p-block of a finite group which has a defect group isomorphic to an extra-special p-group of order p^3 and exponent p. The idea behind these conjectures is to satisfy the p-local theory. We have employed the subpair formulation of these conjectures. Hence, we are in a position to deal with both the approach of characters and the p-block theory approach, as well as the module approach.

The methods which we have used are the cancellation process, which originated in the work of Geoffrey R. Robinson. This kind of technique enables us to reduce the calculation under consideration to consider chains which start with an Alperin-Goldschmidt subpair.

After fixing a suitable p-modular system, the theory of a relative projective is translated for irreducible ordinary characters. Thus, we exploit Geoffrey R. Robinson's contribution towards this theory to count and compare the number of irreducible characters which are projective relative to a certain normal p-subgroup.

The inertial quotient of a certain subpair is used to measure the fusion. The case that this inertial group is trivial gives us the opportunity to say that each associated p-block is nilpotent. On the other hand, using the fact that this inertial quotient is a p'-group for the maximal subpairs, we study the action of this p'-group on a certain elementary abelian p-group which is called the Frattini quotient.

We have exploited the fact that the outer automorphism group of the defect group under consideration and also the automorphism group of an arbitrary maximal subgroup of this defect group is the general linear group of dimension two over a field of p elements. Thus, we are working in the general linear group and its subgroups. This situation enables us to study the action of the inertial quotient to determine the orbit structure of the irreducible characters of the initial subgroups of the chain of the alternating sum. However, using the cancellation theorems, we have to consider only the irreducible characters of the defect group and the characters of the maximal subgroups of such a defect p-group which are the radical p-groups.

The main dichotomy is that the inertial quotient of a maximal subpair either has or has not an element of order $p-1$. It follows that we are dealing with the case that the inertial quotient of a certain subpair has a non-trivial normal p-subgroup, and, hence, has no p-block of defect zero. However, in the other case, the situation is that, for large p, the special linear group of dimension two over a field of p elements has a trivial Schur multiplier together with Clifford theory ensure that the alternating sums under consideration cancel each other, except for those from the empty chain and the associated maximal subpair. Therefore, the conjectures predict a bijection between the irreducible characters of a p-block of a finite group and the irreducible characters of the Brauer correspondent of the stabilizer of the associated maximal subpair.

Special attention is drawn to the case that the unique largest normal p-subgroup is non-trivial. Thus, according to the set-up of the conjectures which we are tackling, two cases have to be distinguished. The first one is that this normal p-subgroup is central.

In this case, it follows that the special linear group of dimension two over a field of p elements does not involve the inertial quotient of non-maximal subpairs. As a result, the maximal subpair is the unique Alperin-Goldschmidt subpair and the desired conclusion follows.

However, we are concerned with the case that the unique largest normal p-subgroup is not central. In this case, Clifford theory is the main tool to obtain the result. Now, according to our choice of the defect group, we have two cases. The first one is that this normal p-subgroup is an elementary abelian p-group of order p^2 and the second is a cyclic p-group of order p. In the former case, it follows that we have a unique radical p-subgroup, while in the latter case, we see that each irreducible character of the finite group under consideration is projective relative to the initial subgroup of the required term of the alternating sum. In both cases, we see that the contribution from the chain which starts with a non-maximal subpair cancel each other, and we are left only with the contributions from the empty chain and the singleton chain of the maximal subpair. We have been concerned with the case that the unique largest normal p-subgroup is trivial and the inertial quotient of the associated maximal subpair has an element of order $p-1$. This result is proved in a similar fashion by exploiting the fact that each p-block of the centralizers of both the defect group and its maximal subgroups are nilpotent p-blocks. Then, we use the covering of nilpotent p-blocks and the extension of a certain action to count and equate the number of irreducible characters which are projective relative to the initial subgroup of the chain which we are considering.

SUGGESTIONS FOR FURTHER RESEARCH

Here, we shall draw attention to questions which have arisen in the course of the present research.

- The general case needs to be made whether it true that the OWC holds for p-blocks with an extra-special defect p-group.

- We ask under which conditions the results of this thesis hold in a case where E has order p^{2n+1}, as well as the generalised extra-special p-groups.

- We consider these results from the point of view of the Külshammer-Puig result and the Morita type which uses the Harris Knörr correspondence.

- It seems that there is a connection with some other conjectures and this connection merits further study. See [39].

- We might use the classification of finite simple groups to tackle this problem. We know that the principal 13-block of the Monster group has a defect group which is isomorphic to an extra-special 13-group of order 13^3 and exponent 13. The same is true for $p = 11$ in the Janko group J_4. The defect group of the principal 7-block of the Held group is an extra-special 7-group of order 7^3. However, for $p = 3, 5$ there are many simple groups having Sylow p-subgroups which are extra-special p-groups of order p^3 and exponent p. See[17].

- We might generalise these conjectures for arbitrary finite dimensional algebras, using the notion of pointed groups and source algebras.

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