Robust Optimisation and its Application to Portfolio Planning

by

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Abstract

Decision making under uncertainty presents major challenges from both modelling and solution methods perspectives. The need for stochastic optimisation methods is widely recognised; however, compromises typically have to be made in order to develop computationally tractable models. Robust optimisation is a practical alternative to stochastic optimisation approaches, particularly suited for problems in which parameter values are unknown and variable. In this thesis, we review robust optimisation, in which parameter uncertainty is defined by budgeted polyhedral uncertainty sets as opposed to ellipsoidal sets, and consider its application to portfolio selection. The modelling of parameter uncertainty within a robust optimisation framework, in terms of structure and scale, and the use of uncertainty sets is examined in detail. We investigate the effect of different definitions of the bounds on the uncertainty sets

An interpretation of the robust counterpart from a min-max perspective, as applied to portfolio selection, is given. We propose an extension of the robust portfolio selection model, which includes a buy-in threshold and an upper limit on cardinality. We investigate the application of robust optimisation to portfolio selection through an extensive empirical investigation of cost, robustness and performance with respect to risk-adjusted return measures and worst case portfolio returns.

We present new insights into modelling uncertainty and the properties of robust optimal decisions and model parameters. Our experimental results, in the application of portfolio selection, show that robust solutions come at a cost, but in exchange for a guaranteed probability of optimality on the objective function value, significantly greater achieved robustness, and generally better realisations under worst case scenarios.

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Chapter 1

An Introduction to Decision Making under Uncertainty

Optimisation is fundamental to decision making. Each time we make a decision we take in the available information (past, present and future), process it, and then take what we believe is the best action. In fact, most of the time, the information we use to make our decisions is not known exactly, we may only have estimates. Essentially, we are optimising under uncertainty, although, we may not think of it that way. Consider the following typical domestic situation: You've hired an electrician to install a new fuse box at your flat tomorrow, but you don't know exactly what time he will arrive, only that it will be between 8am and 1pm. You have taken the day off work because you need to be home during the installation. In addition, you have a list of errands to run, to make the most of your day off, but you know they will not all get done if the electrician does not show up before 12pm. You also know, from past experience, that he most likely will not arrive before 10am; however, if you pop out in the morning and he arrives earlier than expected he will not wait around – you will have to schedule another appointment. What should you do?

This problem is actually multi-objective, the aim is to minimise the risk of having to reschedule whilst maximising the number of errands that get done. As input there are known parameters (i.e. number of total hours available in your day, number of errands to run, the upper and lower bounds for the arrival time of the electrician) and uncertain parameters (i.e. the actual arrival time of the electrician). In addition, there may be particular errands that have to be done while others can wait for another day. Thus, they are ranked in order of importance; some may even have a penalty associated with not getting them done on the day. Lastly, there is past experience (i.e. available historical data) which enables you to estimate the time period in which the electrician is most likely to arrive. In essence, you are optimising under uncertainty. Thus, there is a natural intuition to consider uncertainty in a decision making process; that is, to incorporate the uncertainties into our assessment of a situation such that they have an impact on our decisions.

Throughout the remainder of this chapter, we show that just as we are naturally inclined to consider uncertainty in a decision making process, when uncertainties are present, so should uncertainty be considered in the modelling stage of a real-world mathematical programming problem. This necessitates the development and application of techniques such as robust optimisation, which incorporate parameter uncertainties into the modelling process

1.1 Background and Motivation

Many of the great contributions to research arise from practical situations which demand better ways of getting a job done. The birth of mathematical programming and the developments henceforth are no different. The advent of linear programming and the implementation of the simplex method as a general solution method during the late 1940s were spurred by George Dantzig's work in the Pentagon during World War II. It began for Dantzig with a need to mechanise the planning process and ended with a novel problem formulation and solution method for which even Dantzig didn't initially recognise the potential (Dantzig, 2002). His planning problem formulation implemented an objective function in place of ad-hoc ground rules, which were common place, and expressed the problem constraints as a system of linear equalities and inequalities (Dantzig, 2002). Later that same year he proposed the simplex method, which, with its subsequent improvements over the years, continues to be one of the most applied methods for solving linear programs.

Why is it that it was not until Dantzig's time that those breakthroughs in mathematical programming were made? It wasn't that the mathematical tools did not exist or that the knowledge prior to 1947 to create linear programming and its solution methods was lacking. As far back as 1826, traces of the idea of linear programming and the simplex method can be found in works by Fourier (Schrijver, 1986). Dantzig (2002) suggests that before 1947, there was simply "a lack of interest in trying to optimise" which he corresponds to a lack in computing power. We see through history that Dantzig's assessment is probably correct as the increases in computing power (or the promises thereof) are closely correlated with more advanced optimisation models and solution methods for more complicated problem formulations.

Interestingly, although Dantzig's new techniques could be used to solve large complex systems, it was a deterministic formulation. However, his initial problem was planning dynamically under uncertainty; thus, his initial problem remained unsolved. It wasn't until 1955 that Dantzig proposed a method for solving linear programs under uncertainty, which marked the beginning of stochastic programming, and essentially the beginning of optimisation under uncertainty (a similar work was also published the same year independently by Beale (1955)). Subsequent methodologies for planning dynamically under uncertainty include, but are not restricted to, chance constrained programming (Charnes and Cooper, 1959), stochastic dynamic programming (Greenberg, 1968) and robust optimisation (Soyster, 1973).

So, why did Dantzig consider his initial problem unsolved? Is not a deterministic formulation sufficient? Does optimisation under uncertainty really demand our attention? In many real-world problems, the data are not known exactly. Ben-Tal, El Ghaoui and Nemirovski (2009) differentiate between two types of uncertain data: high precision data and stochastic data. An example of high precision data is what the authors term "ugly reals", data which is given to say five or more decimal places. They argue that this type of precise data is rarely, if ever, known with 100% certainty. However, the range within which the true value lies is typically known. Stochastic data is not only unknown, but variable. Variability, commonly expressed as a statistical measure (e.g. standard deviation, variance), is the "naturally occurring, unpredictable change" (Burgman, 2005) of a parameter and is not reducible by the acquisition of more knowledge (Vose, 2000). That is, no matter how much information is acquired about the parameter it is not going to change its behaviour. Uncertainty, however, reflects a lack of knowledge about a future event and can be reduced (but not necessarily eliminated) by gathering more information (Vose, 2000); for example, by collecting more data, parameter distributions can be more precisely estimated. To illustrate the difference between variability and uncertainty, consider the random walk of a stock price, which depicts its "naturally occurring, unpredictable change". It is not possible to change the random walk, i.e. make it less volatile, no matter how much data is collected or knowledge acquired regarding its past behaviour – this is variability. However, if we estimate the distribution of a random walk, acquiring more information about its past behaviour will increase the precision of that estimate – this is uncertainty.

Regardless of the type of data uncertainty, high precision or stochastic, not incorporating information about uncertainty, such as the interval within which the true value is likely to fall or characteristics about the variability and uncertainty of the data, can be problematic. To illustrate the potential impact of using a deterministic model when data is uncertain, consider the following example given by Ben-Tal, El Ghaoui and Nemirovski (2009), using PILOT4 from the NETLIB library. This LP has 1000 variables and 410 constraints. In constraint 372, 28 of the variable's coefficients are non zero and range in value from -122.163055 to 717.562256, seven of which are between -1 and 1. Let's say that these coefficients are uncertain, but within 0.1% of their "true" values. The authors show that in the worst case, the constraint will be violated by as much as 450%. They also show that even if the uncertain coefficients do not take their worst case, but only assume a random value within 0.1% of the nominal value (assuming they are uniformly distributed within that range), on average the constraint will be violated by 125%. This example is not an exception either, nor was it the worst; similar results were obtained using other examples from the NETLIB library (see Ben-Tal et. al. (2009) for further detail).

To answer our previous questions, Dantzig considered his initial problem unsolved because of the answers to the second and third question: no, deterministic formulations are not sufficient, and yes, data uncertainty does demand our attention, as was illustrated by the preceding example. We also hold to this view, in conjunction with many other authors, and argue that a deterministic formulation of a model, which has parameters that are not known exactly, is not sufficient. In fact, assuming that it is sufficient and implementing such solutions can be misleading, and in many cases costly, depending upon the area of application. Therefore, there is a need for the development and application of methodologies for decision making under uncertainty.

1.2 Methodologies for Decision making under Uncertainty

There are three common approaches to decision making when a problem's parameters are uncertain: 1) simply ignoring it and applying a deterministic model, 2) post-optimal analysis, such as sensitivity analysis, or 3) treating uncertainty in the modelling stage, such as stochastic optimisation and robust optimisation. A deterministic approach assigns static estimates to unknown parameter values which may yield unreliable and unusable decisions. If the realised parameter values deviate too much from their estimates, constraints are violated and decisions become infeasible. A post-optimal analysis, such as sensitivity analysis, assesses the sensitivity of the solution to changes in parameter values by changing one parameter at a time. That is, it asks the question "How much can the actual parameter value differ from its estimate before the solution loses optimality or feasibility?" However, it is only a means of studying the effects of variability and uncertainty on the optimal decision, but does not protect against them.

Alternatively, methodologies such as stochastic optimisation and robust optimisation treat variability and uncertainty in the modelling stage. Under the umbrella of stochastic optimisation are a range of approaches. Three of the most widely known and applied approaches are stochastic programming, stochastic dynamic programming and chance-constrained programming. In a stochastic optimisation approach, parameter distributions, or a model that accurately creates them, are assumed to be known. Thus, it is dependent upon precise estimates of the probability distributions of uncertain parameters which are used to generate scenarios over which the problem is optimised. In this paradigm, constraints are allowed to be violated, but with a specified penalty. Therefore, the solution may not be feasible for all scenarios. The idea is to hedge against the risk of unfavourable scenarios that may occur in the future. A difficulty with stochastic optimisation approaches is that as the number of scenarios increases, the computational demands increase significantly. In addition, the quality of the solution is determined by the validity of the assumptions governing the stochastic process used to generate scenarios. In contrast, robust optimisation re-formulates the uncertain model so that the stochastic element is removed and the problem is deterministic. Under this approach, knowledge of the underlying probability distributions is not required and scenarios are not needed; instead, uncertain parameters are bounded by convex sets (known as uncertainty sets) and the problem is optimised under the restriction that decisions must be feasible no matter what value each parameter takes within its defined set.

1.2.1 Stochastic Optimisation

Stochastic Programming. The basic stochastic programming formulation of an uncertain problem is to minimise expected costs, $f(\boldsymbol{x}, \xi)$. It is assumed that the random vector ξ has a 'known' distribution. The objective seeks an optimal decision which will be best 'on average' and feasible for all possible realisations of the random vector ξ. This basic formulation can be extended to a two-stage (Beale, 1955 and Dantzig, 1955) or a multi-stage problem. Two-stage stochastic programs are the most common type of formulation; in many situations they provide better solutions than single stage problems and are more readily solved than multi-stage problems (Shapiro and Philpott, 2007).

The goal of a two-stage problem is to determine the best decision now (at the first stage) given all possible scenarios of what could happen after the decision is made. The optimal solution is not only a set of first stage decisions, but also a set of second stage decisions to be made in response to the random scenarios. Essentially, a two-stage problem seeks to minimise the cost of the decisions made now and the expected cost of the second stage decisions.

Stochastic programming is known to suffer from "the curse of dimensionality" because relatively small problems can be computationally intractable. In fact, Dyer and Stougie (2006) provide theoretical evidence that stochastic programming problems are generally hard to solve, even more so than a majority of well known combinatorial optimisation problems. In order to solve a two-stage problem numerically, it is assumed that the number of possible scenarios of $\boldsymbol{\xi}$ is finite and that each scenario can occur with probability $\boldsymbol{P}$ (Shapiro and Philpott, 2007). However, due to computational complexity, it is necessary to restrict the cardinality of the set of possible scenarios of $\boldsymbol{\xi}$ by restricting either the number of random parameters or the number of potential values each parameter can take, or both. To demonstrate how easily a stochastic programming formulation can become too computationally costly, consider the following example: say $\boldsymbol{\xi}$ is a random vector containing 20 random variables with known distributions, and each ξ_i will take one of 2 possible values at each stage, with probability 1. With a small problem of this type, taking the expectation of f requires summing over $2^{(20 \text{ x } 2 \text{ stages})}$ or 10^{12} realisations of the vector ξ. With modern computing power 10^{12} may not seem like a large number, but as the number of possible values of each random variable and/or the number of random variables increases, and the problem size increases exponentially. For example, if $\boldsymbol{\xi}$ contains 40 random variables, each with 5 possible values at each stage, the expectation requires summing $5^{(40 \text{ x } 2 \text{ stages})}$ or $8 \text{x} 10^{55}$ realisations of the vector $\boldsymbol{\xi}$, which results in an impractically large scale optimisation problem. Despite its "curse of dimensionality", stochastic programming is a powerful optimisation methodology. However, when solving large, real world problems, approximation algorithms are necessary (Sen, 2004).

Stochastic Dynamic Programming. The "*essence*" of dynamic programming, as Bellman (1966) puts it, is to "*do the best we can starting from where we are*". Bellman demonstrates this principle using an example of calculating the optimal trajectory of a space vehicle. After leaving its initial position, the space vehicle is likely to veer off course, at which point, the optimal trajectory from its current position is no longer the initial optimal trajectory. Thus, instead of trying to get back on course, the space vehicle should follow the optimal trajectory calculated from its current position. In other words, the optimal decision of a problem, at any given point in time does not depend on previous decisions (i.e. the initial optimal trajectory) or the previous state (i.e. the previous position of the space vehicle); it only depends on the current state (i.e. current position in space), the available resources (i.e. time, fuel, etc…) and the target (i.e. the destination of the space vehicle).

Essentially, dynamic programming formulates a problem as a multistage decision process and finds the optimal decision by starting at the final stage and working backwards, solving the problem recursively. That is, an optimal solution is found at each stage, but is only dependent upon the current stage and state as well as the remaining stages. Dynamic programming has proven to be a powerful tool, particularly for deterministic problems. Challenges arise, however, when the states at each stage are uncertain as a result of uncertain parameters – this is a *stochastic* dynamic programming problem (Greenberg, 1968). Uncertain parameters are characterised by probability distributions which are used to generate scenarios at each stage. As with stochastic programming, stochastic dynamic programming suffers from the "curse of dimensionality", thus limiting the number of stages, states and scenarios that can be handled in practice.

Chance Constrained Programming. Stochastic and stochastic dynamic programming approaches allow for the possibility of constraints to be violated, although typically at some cost. In certain situations, it is not practical to permit such violations. Instead, it is more appropriate to use an approach which provides a strong guarantee on the feasibility of the solution by introducing probabilistic or chance constraints (Charnes and Cooper, 1959), which guarantee the feasibility of every constraint with high probability.

The basic chance constrained programming model reformulates an optimisation problem with uncertain parameters in the constraints, by introducing the condition that the constraints, either individually or jointly, must be feasible with some guarantee of confidence α. That is, either the probability of each individual constraint being feasible is greater than or equal to α or the probability of any constraint being feasible (considered jointly) is greater than or equal to α. In this approach, not only is it assumed that the distributions of the uncertain parameters are known or can be estimated, but that the probability distributions defining the individual or joint chance constraints are also known or can be estimated. Recently, however, the application of ambiguous chance constraints has been introduced, eliminating the second assumption. This approach considers the case when the probability distributions defining the chance constraints are unknown, but are known to belong to an uncertainty set.

Chance constrained problems, first considered 50 years ago, are still very difficult to solve exactly. Shapiro and Nemirovski (2005) argue that in general, chance constraints are not 'practical' because even if the constraint is a simple function, say $f(\boldsymbol{x}, \xi)$, where ξ belongs to a simple distribution, the chance constraint defines a feasible set of decision variables which is nonconvex, making it potentially intractable. The authors note that there are only two generic cases in which this nonconvexity does not exist. Typically, either the chance constraints are replaced by tractable deterministic approximations or approximation algorithms are applied.

1.2.2 Robust Optimisation

Model robustness can be defined in many different ways. When we use the term *robust optimisation* we are not referring to models which can be called robust strictly by the definition of the word given by a dictionary. We are referring to a specific branch of optimisation under uncertainty known as robust optimisation, whose roots can be found in the field of robust control and in the work of A. L. Soyster (1973) as well as later works by Ben-Tal and Nemirovski (1997, 1998 and 1999) and independently by El Ghaoui and Lebret (1997) and El Ghaoui, Oustry and Lebret (1998). Robust optimisation is a min-regret modelling methodology that seeks to minimise the negative impact of future events when the values of model parameters are high precision or stochastic. Consequently, we define a model as robust if it guarantees, with high probability, that the optimal objective will be achieved or exceeded and that the solution will be feasible for all possible realisations of each unknown parameter, contained within the bounds of an uncertainty set, even if the assumed distributions and estimates of the parameters are imprecise. Although parameter values are unknown, historical data (if available) may be used to estimate the uncertainty set, which does not need to encapsulate every possible realisation of the parameter, but only the "most likely" values, the specification of which is partially a subjective decision. The two most common ways of defining the geometry of uncertainty sets are polyhedral and ellipsoidal sets, discussed further in Chapter 2.

During the last ten years, robust optimisation has gained support as a tractable alternative to stochastic optimisation. The robust counterpart is a deterministic formulation, which optimises an objective such that all constraints are satisfied for all possible values of each uncertain parameter defined within a set. Unlike stochastic optimisation approaches, it does not rely on knowing the exact distributions of parameters – which are rarely known in practice and typically estimated. In addition, the robust counterpart does not suffer from the "curse of dimensionality"; it does not require scenario generation. Thus, increasing the number of uncertain parameters does not have an exponential effect on the problem size. However, this does not guarantee that the robust counterpart will not be difficult to solve – it will be at least as complex as the nominal problem, in terms of appropriate solution methods for the problem structure. For example, if the nominal problem is linear then the robust counterpart will remain linear if uncertainty sets are polyhedral, but will be a second order cone problem if uncertainty sets are ellipsoidal, which is more difficult to solve. In addition, the problem complexity is dependent upon the geometry of the uncertainty sets defining the uncertain parameters. As long as the uncertainty sets U are convex and computationally tractable, the robust counterpart will be tractable (Ben-Tal, Nemirovski and El Ghaoui, 2009).

Lastly, underpinning robust optimisation is a desire for mathematical models producing solutions insensitive to changes in uncertain parameters such that a) it is computationally manageable, b) decisions are useable – if input data changes, the solution is near optimal with high probability and c) the robustness of the solution is worth the sacrifice of optimality.

1.3 Thesis Outline

The research presented in this thesis focuses on the computational evaluation of the robust optimisation methodology and its application to the portfolio selection problem. In particular, the budgeted robust counterpart of the Expected value – Variance model (*E*-*V*) for portfolio selection, which models the unknown and variable return of an asset by budgeted polyhedral uncertainty sets (introduced by Bertsimas and Sim, 2004), is presented. We aim to evaluate this methodology and contribute insight into defining uncertainty sets, the properties of robust decisions and model parameters. We also aim to establish whether the robust models investigated form a suitable foundation upon which to build real-world portfolio selection models. We do this through an extensive empirical investigation which examines the trade-off between the robustness of robust portfolios and the sacrifice in optimality and the properties of robust portfolios from a practical perspective; that is, do robust portfolios make investment sense?

Within the area of robust optimisation are a handful of methods for modelling parameter uncertainty (both in structure and scale) which lead to different robust formulations. In Chapter 2, we detail how parameter uncertainty is modelled, by considering different structures for the uncertainty set U (defining uncertain parameters), with particular focus on budgeted polyhedral structures and how they relate to ellipsoidal and polyhedral structures. In addition, we clearly define the different aspects relating to the scale of U and highlight recent work in this area.

In Chapter 3, we discuss the portfolio selection problem. We argue that the *E*-*V* model is problematic because it is assumed that the distribution of asset returns is known, or at least estimated to a high degree of accuracy. That is, precise estimates of the expected return and variance of each asset can be obtained. In order to address this difficulty, a robust portfolio selection model, which treats the distribution of asset returns as uncertain, is presented as an alternative approach. In the remainder of the chapter we review the derivation of the budgeted robust counterpart by duality, first given by Bertsimas and Sim (2004), followed by an interpretation of that model from a min-max perspective. Lastly, we present the linear robust portfolio selection model resulting from the budgeted polyhedral uncertainty sets proposed by Bertsimas and Sim (2004) and then propose an extended model which includes a buy-in threshold constraint and cardinality constraint.

In Chapter 4, we investigate the cost and robustness of the budgeted robust counterpart resulting from different definitions of the uncertainty set U. In each instance, the structure of U remains a budgeted polyhedral uncertainty set, but the scale of that structure changes. With respect to scale, we consider different definitions of the parameters which specify how the bounds of the uncertainty set are defined, as well as different values of the scaling factor c (which determines the magnitude of the structure of U). In addition, we investigate the effect of changing the size of the historical dataset from which the specific value of each parameter is estimated. The purpose of this chapter is to evaluate the cost and robustness of the proposed models corresponding to these changes, both in the scale of U and in the size of the historical dataset. We do this by introducing several measures of cost and robustness. Results suggest that certain definitions of the parameters which specify how the bounds of the uncertainty set are defined result in portfolios with a better trade-off between cost and robustness. The investigations presented in Chapter 5 and Chapter 7 make use of the concepts and results reported in this chapter

In Chapter 5, we compare the cost and robustness of the budgeted robust counterpart to that of the *E-V* model and to the budgeted robust counterpart with added constraints (a buy-in threshold, an upper limit on cardinality or both). For both robust models, we choose fixed definitions for the parameters specifying how the bounds of the uncertainty set are defined, which were established in Chapter 4. Numerical results show that robust models do come at a cost, but in exchange for significantly greater robustness. In addition, portfolios constrained by a buy-in threshold and/or cardinality yielded solutions that were at least as Robust, but at the same time decisions that were at least as costly, as the solutions and decisions of the unconstrained robust portfolios.

In Chapter 6, we discuss the properties of robust models with respect to diversification, asset selection and the distribution of asset weights amongst selected assets, based upon the total number of assets available, the size of the historical dataset (or number of observations) and the desired level of the guaranteed probability of optimality. In addition, we examine whether these properties hold when threshold and/or cardinality constraints are imposed.

In Chapter 7, we compare the performance of the unconstrained robust portfolio, in terms of portfolio return, to that of *E-V* portfolios and an Index portfolio and to robust portfolios constrained by a buy-in threshold and/or cardinality. We back-test these portfolios over the out-of-sample period as well as two bootstrap samples which were generated using the out-of-sample period as the original sample from which to draw. We report the performance of these portfolios using two risk-adjusted return measures (the Sharpe and Sortino ratio) as well as downside risk and reward statistics. In addition, we evaluate worst-case performance under four worst-case scenarios estimated using the out-of-sample period and both bootstrap samples.

Lastly, in Chapter 8, we present our conclusions and outline future directions for research.

Chapter 2

Uncertain Mathematical Programs and Robust Decisions

Robust optimisation is a modelling methodology that seeks to minimise the negative impact of future events when model parameters are high precision or stochastic. Hence, the robust counterpart is a deterministic worst case formulation of an uncertain mathematical program. Within this framework, model parameters are assumed to be uncertain, but symmetrically distributed over a bounded interval. Instead of nominal values, the uncertain parameters can potentially take any value within a bounded and symmetric set, known as an uncertainty set U. The structure and scale of U is specified by the modeller, typically based upon statistical estimates. The structure refers to the geometry or shape of the constraint set U, such as ellipsoidal or polyhedral. It is important that the structure of U be convex in order for the robust counterpart to be computationally tractable (Ben-Tal, Nemirovski and El Ghaoui, 2009). The scale refers to the magnitude of the deviations of the uncertain parameters from their nominal values; the scale can be thought of as the size of the structure defining U. For example, in three dimensions, the scale refers to the width, height and depth of the structure. As a result, the structure and scale directly affect the computational complexity of the robust counterpart, conservativeness of the solution and the probability of feasibility/optimality (feasibility if the uncertainty is in the constraints and optimality if the uncertainty is in the objective). We will frequently use the phrase "guaranteed robustness" when referring to the probability of feasibility/optimality in a general sense (i.e. when we are not discussing specific numerical values of either probability).

An uncertain linear program (LP) can be expressed in the following general form (Ben-Tal and Nemirovski, 1998):

$$\begin{aligned} &\text{Max} && c^T x . \\ &\text{Subject to} && Ax \le b , \end{aligned} \tag{2.1}$$

where *A*, *b* and *c* are uncertain parameters belonging to an uncertainty set *U*. A general form of the robust counterpart of (2.1) is given as

$$\begin{aligned} &\underset{x}{\text{Max}} && [\underset{c \in U}{\text{Min}} \ (c^T x)]. \\ &\text{Subject to} && Ax \le b , \qquad \forall (A,b,c) \in U . \end{aligned} \tag{2.2}$$

The difference between the constraints in (2.1) and (2.2) is that the former simply states that *A*, *b* and *c* are uncertain but known to belong to the set *U*, whereas the latter stipulates that the solution must be feasible for every possible value of *A*, *b* and *c* within the set *U*. In addition, the objective in (2.2) seeks the best worst case solution by minimising the function with respect to the uncertain parameter and maximising with respect to *x*. The robust counterpart cannot be solved directly using (2.2). Typically, duality is applied to express it in a tractable form.

As defined by Ben-Tal and Nemirovski (1998), feasible solutions to Equation (2.2) are *robust feasible solutions* and the optimal solution to Equation (2.2) is a *robust optimal solution*. Bertsimas and Sim (2004) introduced the concept "price of robustness" which considers how "heavily" the objective function value is penalised when we are guarded against objective underperformance and/or constraint violation. Implicitly, this is the difference between the *robust optimal solution* and the objective function value of the nominal problem. In Chapter 4, we explicitly define a similar measure called the cost of robustness.

Bertsimas and Sim (2004) introduced an alternative robust counterpart with budgeted uncertainty, which is referred to as a budgeted robust counterpart. They relaxed the condition that the solution must be feasible *for all* $(A,b,c) \in U$ under the assumption that not every parameter will take its worst case value. Thus, the solution must be feasible only *for some* $(A,b,c) \in U$. The numerical value of '*for some*' is represented by a user defined parameter Γ, which can take any real value between 1 and $|J|$, where *J* is set of uncertain parameters; hence, $|J|$ is the cardinality of *J*. The value of Γ affects the structure of *U* and thus, the guaranteed robustness of the solution. To illustrate this, consider the uncertainty in a 1 x *N* vector $\boldsymbol{A}$. In the robust optimisation framework, the true value a_i, of an uncertain parameter, is given by the following equation:

$$a_i = \overline{a}_i + \hat{a}_i \eta_i , \qquad \forall i , \qquad (2.3)$$

where $\overline{a}_i$ is a statistical estimate of the expected value of a_i (commonly referred to as a point estimate), $\hat{a}_i$ is a statistical estimate of the maximum distance that a_i is "likely" to deviate from the point estimate $\overline{a}_i$ and η_i is a random variable which is bounded by and symmetrically distributed within the interval $[-1,1]$. The nature of the uncertain parameters will determine how η_i is distributed over this interval (for example, η_i may be stochastic in nature, uniformly distributed, etc…). Recall that Γ is chosen by the user as the maximum number of uncertain parameters that can take their worst case value; in our example, Γ is the maximum number of a_i that will take a value of $\overline{a}_i - \hat{a}_i$. Bertsimas and Sim express this by the following constraint:

$$\sum_{i=1}^{J} |\eta_i| \leq \Gamma , \qquad |J| \in [1, N], \qquad (2.4)$$

where $|J|$ is the number of uncertain parameters and *N* is the total number of parameters. Using our very simple example, in which uncertainty is only in the 1 x

N vector $\boldsymbol{A}$, a general form of the budgeted robust counterpart of Bertsimas and Sim can be written as follows:

$$\underset{x}{\text{Max}} \quad [\underset{c \in U}{\text{Min}} \ (c^T x)]. \tag{2.5}$$

$$\text{Subject to} \qquad Ax \le b, \qquad \text{where } A \in U,$$

$$\sum_{i=1}^{J} |\eta_i| \le \Gamma, \qquad J \in [1, N],$$

Clearly, when $\Gamma = 0$ none of the uncertain parameters take their worst case value; thus, the dimension of the structure of U is zero and the budgeted robust counterpart is similar to (2.1), the non-robust nominal problem. When $\Gamma = |J|$, all of the uncertain parameters take their worst case value; thus, the dimension of the structure of U is $|J|$ and the budgeted robust counterpart is similar to (2.2) – the solution must be feasible *for all* $\mathbf{A} \in U$. In Chapter 3, we show the derivation and interpretation of a budgeted robust counterpart of the portfolio selection problem in which duality has been applied.

Soyster (1973) was the first to show that uncertain linear programs could be formulated as robust convex linear programs such that feasibility was preserved for all possible values of the uncertain parameters defined within a set. Many authors, including Soyster, agreed that his formulation was overly conservative – too much of the optimal objective value was lost in exchange for the preservation of feasibility. In other words, robustness cost too much. Soyster's approach lay dormant until the early 1990s when Ben-Tal and Nemirovski reformulated a less conservative model, one more realistic for application.

Consider an uncertain linear program in which the uncertainty is in the constraints. In Soyster’s model, the worst case solution is guaranteed to be feasible *for all* $A \in U$, where the uncertainty set U includes every possible realisation of the uncertain parameters such that the probability of violating the ith constraint is zero (Table 2.1). In Ben-Tal and Nemirovski’s model (1998), it is possible to scale down the uncertainty set U such that it only includes the "most likely" values of the uncertain parameters instead of every possible value. The "most likely" values are determined by the user, based upon statistical estimates. Since scaling down U opens up the possibility for the true value of any uncertain parameter to be outside the bounds of U, for which the worst case solution has not accounted, feasibility is no longer 100% guaranteed. Ben-Tal and Nemirovski prove that the probability of violating the ith constraint, given by a confidence term γ, is greater than 0 and bounded above by an exponential term that is a function of the scale of U (Table 2.1).

The budgeted robust counterpart of Bertsimas and Sim (2004) builds on similar principles to those of Ben-Tal and Nemirovski's model. In their model, it is also possible to adjust U such that it only includes "most likely" values, and feasibility is not 100% guaranteed. In addition, the probability of constraint violation is bounded above by an exponential term, however, the exponential term does not incorporate information about the scale of U, but the structure of U (Table 2.1). We will expand upon this difference in the following section.

2.1 Uncertainty Sets

The structure and scale of an uncertainty set U determines the computational complexity of the robust counterpart as well as the conservativeness and guaranteed robustness of the solution. In Ben-Tal and Nemirovski's model (1998) U is structured as ellipsoids and intersections of ellipsoids and the scale of U, that is the size of the ellipsoidal structure of U, is adjusted by a user defined parameter Ω. Within this framework, they show that the probability of constraint violation, given by the confidence term γ, is bounded below by zero and above by $e^{-\Omega^2/2}$ (Table 2.1). As mentioned previously, the scale of U determines the guaranteed robustness and conservativeness of the solution. Therefore, the parameter Ω, determined by the user based upon the desired confidence term γ, adjusts the trade-off between robustness and the optimality of the solution with respect to that of the nominal problem. It follows then, that if $\Omega = 0$, there is no robustness and the formulation is simply the nominal problem. As Ω increases, the probability of constraint violation decreases and the optimal objective value deteriorates. A drawback to their approach is that the robust counterpart is more difficult to solve than the nominal problem. For example, LPs become second order cone programs (SOCPs), SOCPs become semi-definite programs (SDPs) and SDPs are NP-hard (Bertsimas et al., 2008).

Bertsimas and Sim (2004) suggested budgeted polyhedral uncertainty as an alternative structure of U, which retains the degree of the original linear problem[1]. Similar to Ben-Tal and Nemirovski, Bertsimas and Sim's model was also less conservative than Soyster's model and allowed the user to control the following: 1) the level of guaranteed robustness and 2) the optimal value of the solution with respect to the nominal problem. Unlike Soyster and Ben-Tal & Nemirovski, they did not guarantee feasibility *for all* uncertain parameters in U; instead they guaranteed feasibility given that no more than Γ uncertain parameters changed (recall that $\Gamma \in [1, |J|]$ is a user defined parameter). However, Bertsimas and Sim show that there is a probabilistic guarantee that decisions will remain feasible and that the robust optimal objective will be achieved or exceeded, even if more than Γ parameters change. This probabilistic guarantee (i.e. the probability of constraint violation), given by the confidence term γ, is greater than 0 and bounded above by $e^{-\Gamma^2/2|J|}$ (Table 2.1), which is a function of the user defined parameter Γ and the number of uncertain parameters $|J|$. The difference between the upper bound of γ in Ben-Tal and Nemirovski, and Bertsimas and Sim's formulation are due to the differences in how uncertainty sets were defined, both in structure and scale, as well as in how their models allowed for the relationship between the guaranteed robustness of the solution and optimality of the objective (with respect to the optimal objective value of the nominal problem) to be adjusted. Table 2.1 provides a comparison of the guaranteed robustness of the models given by Soyster, Ben-Tal and Nemirovski, and Bertsimas and Sim. In the following section we consider ellipsoidal and polyhedral uncertainty sets in more detail.

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[1] In many cases the same is true for SOCPs, although exceptions exist, but Bertsimas, Brown and Caramanis (2008) highlight a work by Nemirovski (1993) which shows that even with polyhedral uncertainty, the robust counterpart of an SDP is NP-hard.

Soyster ('73)	Ben-Tal & Nemirovski ('98)	Bertsimas & Sim ('04)
$\mathbf{Pr}(a'x^* > b) = 0$ For all possible a.	$\mathbf{Pr}(a'x^* > b) < \gamma$ For all possible a, where $0 < \gamma < e^{-\Omega^2/2}$	$\mathbf{Pr}(a'x^* > b) < \gamma$ For up to Γ a_i values, where $0 < \gamma < e^{-\Gamma^2/2\lvert J\rvert}$ Where $\lvert J\rvert$ is the cardinality of the subset of uncertain parameters a_i.

Table 2.1. Comparison of robust optimisation models; x* is the optimal solution vector.

2.1.1 The structure of Uncertainty Sets

In the robust optimisation framework, the structure of U determines the complexity of the robust counterpart of an uncertain LP, and must be convex in order to preserve computational tractability (the underlying model must also be convex). Throughout the literature there are a handful of possible convex representations of U: the general conic representation, in which U is given by a closed convex pointed cone with a nonempty interior; and more specific representations of U such as ellipsoidal, polyhedral, or budgeted uncertainty sets (Ben-Tal, Nemirovski and El Ghaoui and Bertsimas, Brown and Caramanis, 2008)[2]. In this section, we are mainly interested in the similarities and differences between ellipsoidal and polyhedral/budgeted polyhedral uncertainty sets and the relationship between these structures and the corresponding upper bound on the confidence term γ, which is the probability of constraint violation.

Recall from Section 2.1 that Ben-Tal and Nemirovski (1998) modelled uncertainty sets by ellipsoids and intersections of ellipsoids, which increases the complexity of the problem, while Bertsimas and Sim modelled uncertainty by polyhedral sets which preserves the degree of the problem – the robust counterpart of an LP remains an LP. To illustrate the relationship between these two types of structures consider the example given previously in Section 2.1 in which the uncertainty is in vector $\mathbf{A} \in U$. The uncertain parameters $a_i \in \mathbf{A}$, $\forall i$, are unknown, mutually independent

[2] We refer the interested reader to Ben-Tal, Nemirovski and El Ghaoui (2009) and Bertsimas, Brown and Caramanis (2008) for a detailed comparison of uncertainty sets and their corresponding robust counterparts of uncertain LPs as well as uncertain SOCPs, SDPs, QCQPs (quadratically constrained quadratic programs), etc….

parameters given by (2.3) and symmetrically distributed with respect to $\bar{a}_i$ on the interval $[\bar{a}_i - \hat{a}_i, \bar{a}_i + \hat{a}_i]$.

It follows that ellipsoidal uncertainty sets (Ben-Tal and Nemirovski, 1998) are given by the following:

$$U^{\Omega} = \{a \in R^n : \sum \frac{(a_i - \bar{a}_i)^2}{\hat{a}_i^{\,2}} \leq \Omega^2, \|\boldsymbol{\eta}\|_\infty \leq 1\}, \tag{2.6}$$

where Ω is a user defined parameter and adjusts the trade-off between robustness and optimality. As Ω increases, the area of the ellipsoid defining the uncertainty set also increases (see Figure 2.1). Hence, the upper bound on the probability of constraint violation, $e^{-\Omega^2/2}$ (Table 2.1), decreases (i.e. the model is more robust). Therefore, the scale of the uncertainty sets (or ellipsoids) and the probability of constraint violation are determined by the parameter Ω, which is dependent upon the user's risk preference. Ben-Tal and Nemirovski (1998) suggested ellipsoidal uncertainty sets because ellipsoids can approximate more complicated uncertainty sets well, they can represent stochastic uncertainty sets deterministically and they have a convenient mathematical structure. However, ellipsoidal uncertainty means that the robust counterpart of an LP becomes a SOCP, but the authors argue that this is not a problem because large SOCPs can be solved efficiently[3] in polynomial time (Ben-Tal and Nemirovski, 1998). Typically, SOCPs are solved via interior-point methods which are polynomial time iterative solution procedures.

A simpler structure of U is given by polyhedral sets, sometimes referred to as interval or box uncertainty:

$$U = \{a \in R^n : |a_i - \bar{a}_i| \leq \hat{a}_i, \|\boldsymbol{\eta}\|_\infty \leq 1\}. \tag{2.7}$$

[3] We refer the interested reader to a text book by Boyd and Vandenberghe (2004) on convex optimisation as well as a textbook containing a series of lectures on modern convex optimisation by Ben-Tal and Nemirovski (2001) for more information on efficient solution methods for linear and nonlinear convex robust counterparts.

Bertsimas and Sim (2004) modelled uncertainty by budgeted polyhedral uncertainty sets given by the following modification of (2.7):

$$U = \{a \in R^n : |a_i - \bar{a}_i| \leq \hat{a}_i, \|\boldsymbol{\eta}\|_\infty \leq 1, \|\boldsymbol{\eta}\|_1 \leq \Gamma\}, \tag{2.8}$$

where $\|\boldsymbol{\eta}\|_1 \leq \Gamma$ is equivalent to (2.4) and Γ is a user defined parameter interpreted as the maximum number of uncertain parameters allowed to take their worst case value. We can see from (2.8) that changing Γ will change the number of bounds, which define the polyhedron, thus changing the structure of *U* and adjusting the robustness of the model. Bertsimas and Sim prove that the probability of constraint violation is bounded above by $e^{-\Gamma^2/2|J|}$ (Table 2.1). Thus, as Γ increases, more protection is given and the solution is more robust. In contrast to the parameter Ω, introduced by Ben-Tal and Nemirovski (1997), Γ does not affect the scale of the uncertainty set, shown in (2.6) and (2.8).

Figure 2.1 illustrates the relationship between ellipsoidal and box uncertainty sets with a simple two-dimensional example. Let $a_1, a_2 \in \mathbf{A}$, be the uncertain parameters which are symmetrically distributed with respect to $\bar{a}_i$ and bounded by the interval $[\bar{a}_i - \hat{a}_i, \bar{a}_i + \hat{a}_i]$, where i = 1, 2. Values for the parameters $\bar{a}_i$ and $\hat{a}_i$, for i = 1, 2 and n as well as the equations for the ellipse and box uncertainty set are given in Figure 2.1; there are only two uncertain parameters, thus $n = 2$. The bounds of the box uncertainty sets are shown by dashed lines. Two ellipsoidal uncertainty sets, corresponding to two values of Ω are plotted with the box. The inner ellipse, which fits just inside the box, defines the bounds of the uncertainty set when Ω = 1. The outer ellipse, which just contains the box, defines the bounds of the uncertainty set when $\Omega = \sqrt{n}$; in this case $\Omega = \sqrt{2}$. The relationship between polyhedral uncertainty sets and the inner and outer ellipsoidal sets, for any dimension, was mentioned in a 1999 paper by Ben-Tal and Nemirovski.

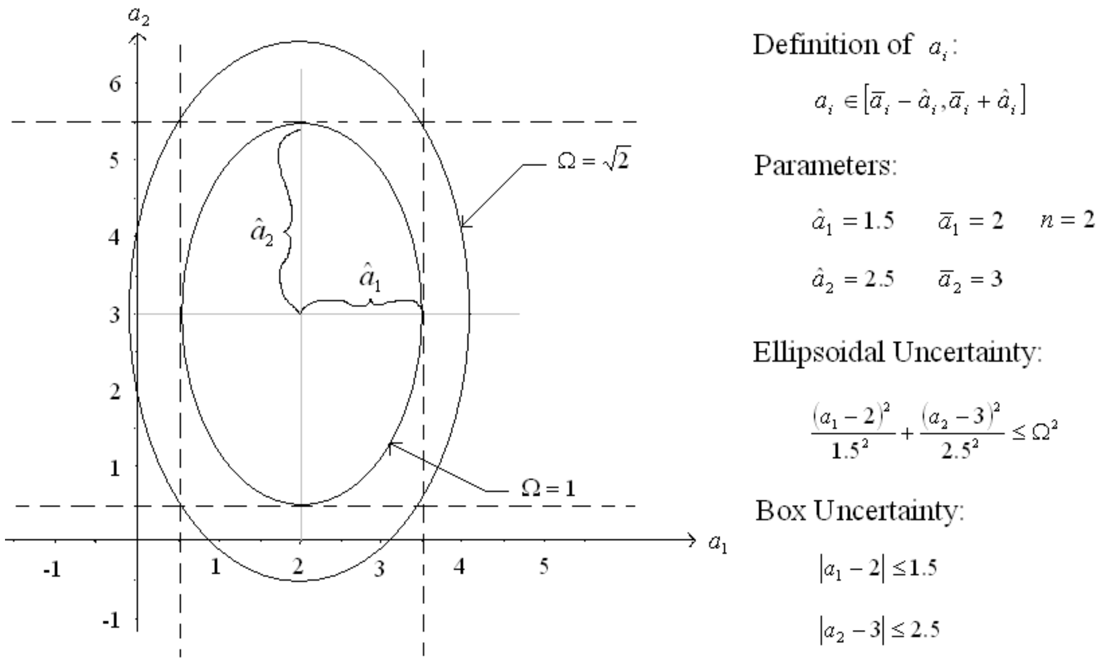

Figure 2.1. Example of the relationship between box and ellipsoid uncertainty sets in 2 dimensions.

Within the last five years, several approaches have been proposed in an effort to address the two main criticisms of how uncertainty sets are constructed in a linear robust optimisation framework. Chen, Sim and Pang (2007) state that the two main criticisms are: 1) any available distributional information of uncertain parameters is not utilised as much as it could be and 2) uncertain parameters are rarely symmetrically distributed. Chen, Sim and Pang (2007), Bienstock (2007) and Bertsimas and Brown (2008) propose different approaches for determining the structure of *U* which address both criticisms.

In 2007 Chen, Sim and Pang proposed a generalised form of the approaches of Ben-Tal and Nemirovski (1997), and Bertsimas and Sim (2004). Recall the random variable η_i which is bounded by and symmetrically distributed within the interval $[-1,1]$. In the generalised form, η_i is bounded by the interval $[\underline{\eta}_i, \bar{\eta}_i]$, where $\underline{\eta}_i$ and $\bar{\eta}_i$ are forward and backward deviation measures which capture the distributional asymmetry of uncertain parameters. In addition, instead of the l_∞ and the l_1 norm, used to construct ellipsoidal and polyhedral uncertainty sets, respectively, they introduce the use of the *regular norm*. They prove that their generalised uncertainty set, using the regular norm in the context of their general framework of deviation measures, which capture asymmetry, preserves convexity and tractability as well as provides an upper bound on the probability of constraint violation of $e^{-\Omega^2/2}$, equivalent to that of Ben-Tal and Nemirovski.

Recent work by Bertsimas and Brown (2008) proposed a data-driven approach for the construction of an uncertainty set. By data-driven they mean that only the historical observations of the data are used to construct the uncertainty set. No assumptions were made about the probability distributions, such as symmetric but bounded, as is common in a robust optimisation framework; thus, they do not choose a structure first, i.e. ellipsoidal or polyhedral. Alternatively, they first choose a coherent risk measure reflecting the decision-maker's desired level of guaranteed robustness (i.e. 1-γ). The authors show that coherent risk measures provide some level of control over the robustness (and conservativeness) of the solution while retaining the convexity of the problem. The authors argue that uncertainty sets constructed using a data-driven approach yield more accurate structures representing the uncertain parameters; hence, they result in better robust optimal solutions. Lastly, Bienstock (2007) also proposed a data-driven approach for constructing *U*, within which are two types of uncertainty sets: 1) a histogram model which constructs a distribution of *shortfalls* obtained by taking the difference between each data point in the available time series and the expectation of that series; 2) also a histogram model of deviations, but incorporates additional information about the correlation among those deviations.

2.1.2 The scale of Uncertainty Sets

There are two main aspects of uncertainty sets: structure and scale. In Section 2.1.1 we discussed the most common structures, ellipsoidal and polyhedral. In this section we address the question of scale. For the purpose of clarity, we redefine the uncertain parameter a_i, which was given by (2.3) earlier in this chapter. Introducing a scaling factor *c*, we redefine a_i by the following equation:

$$a_i = \overline{a}_i + c\hat{a}_i\eta_i, \qquad \forall i. \tag{2.9}$$

In this form we have factored out the coefficient *c* from $\hat{a}_i$ to distinguish between the deviation measure $\hat{a}_i$ from its scaling factor *c*. Intuitively, $\hat{a}_i$ tells us how a_i deviates from the point estimate $\overline{a}_i$, while *c* tells us by how much. Therefore, a_i lies on the interval $[\overline{a}_i - c\hat{a}_i, \overline{a}_i + c\hat{a}_i]$. For example, if we determine that the distribution of a_i is best represented by defining $\hat{a}_i$ as the standard deviation of a_i, then *c* would represent the number of standard deviations a_i can deviate from $\overline{a}_i$. Thus, a_i would lie on the interval $[\overline{a}_i - c\sigma_i, \overline{a}_i + c\sigma_i]$.

Consequently, when we speak of scale, we are concerned with how $\overline{a}_i$ and $\hat{a}_i$ are defined, how their values are estimated and how $\hat{a}_i$ is scaled (the value of *c*). There is very little research published that addresses these questions. Almost all of the literature in robust optimisation only mentions the structure, but not the scale of *U*. The only work we are aware of that discusses scale is by Tütüncü and Koenig (2004), in which $\overline{a}_i$ and $\hat{a}_i$ are defined as the 50^{th} percentile and estimated using a bootstrapped sample. The deviation measure $\hat{a}_i$ was scaled by *c* = 47.5%. Thus, a_i lies between the 2.5 and 97.5 percentile values estimated from the bootstrap sample. Moreover, the authors conclude that the scale of *U* should be dependent upon the risk preferences of the decision-maker. Our empirical investigation of cost and robustness (further in Chapter 4) suggest that defining both $\overline{a}_i$ and $\hat{a}_i$ as the 50^{th} percentile yield portfolios which are counterintuitive with respect to the relationship between *c* and the trade-off between cost and robustness; such portfolios increased in cost and decreased in robustness when the scale of uncertainty set *U* was increased.

As a consequence of the lack of literature regarding the scale of *U*, we investigate different definitions of $\overline{a}_i$ and $\hat{a}_i$, estimated from a historical dataset, combined with different values of *c*, further in Chapter 4. We evaluate the corresponding uncertainty sets by evaluating the cost and robustness of a budgeted robust counterpart of the portfolio selection problem, where the structure of U is polyhedral. Our results suggest, that for this problem, the most appropriate definitions of $\overline{a}_i$ and $\hat{a}_i$ are a measure of central tendency and a measure of spread, respectively, while c depends on the risk preferences of the investor.

Chapter 3

Robust Optimisation and Portfolio Selection

In the portfolio selection problem an investor chooses the proportion of capital to be invested in each of N assets such that a desired set of goals is achieved. For example, an investor will want to maximise return and minimise risk, but may also have views on the number of assets in a portfolio. One significant reason why portfolio selection continues to be a challenging mathematical problem is due to the variability of asset returns and the uncertainty of their distributions. In other words, not only is the future value of each asset unknown, but it is very difficult to estimate. Capturing variability and uncertainty within a portfolio optimisation model has been shown to be very difficult ever since portfolio selection was first considered as a mathematical problem. Up until the 1990s, when stochastic programming approaches were applied, variability was not explicitly addressed in the optimisation model. Instead the focus was on uncertainty which was defined by a measure of risk, typically in the model's objective function. A tremendous amount of research has been done in this area. The most common ways of defining risk include variance, standard deviation, value-at-risk, utility functions and lower partial moments. A particular hindrance to the progress of research in this area has been computing power. Although technology has greatly improved since the turn of the 21st century, there are still limitations on the formulation and size of the problem that can be considered, which means there are limitations on how risk is defined and to what extent variability is addressed.

3.1 Expected Value - Variance Models

The first notable work to consider risk in portfolio optimisation was published in 1952, when Markowitz presented the well-known Expected value - Variance (*E-V*) model for portfolio optimisation, in which a portfolio that achieves a specified expected return at minimum risk (defined by portfolio variance) is determined. Typically, short selling is not permitted, hence the proportions of capital invested in each asset must be greater than or equal to zero and sum to one. Consider the following model which minimises risk, as measured by the variance of the portfolio's return, subject to achieving a specified expected return:

$$\text{Minimise} \qquad \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_{ij}$$

$$\text{Subject to} \qquad \sum_{i=1}^{N} w_i \mu_i \geq \textit{Target Return},$$

$$\sum_{i=1}^{N} w_i = 1,$$

$$w_i \geq 0, \qquad \forall i = 1..N,$$

where μ_i is the expected return on asset i, σ_{ij} is the covariance of assets i and j and w_i is the fraction of total wealth invested in asset i.

Alternatively, we may maximise expected return given a specified upper limit on risk. Now, let E be expected portfolio return, V portfolio variance and S represent the set of all possible (E,V) combinations. Markowitz's *E-V* model assumes that an investor would only consider the subset of portfolios which he termed "efficient." An efficient portfolio is one that yields the highest expected return for a specific variance or has the least variance for a specific expected return. The above model only yields "efficient" portfolios. When solved for all possible values of *Target*

Return it produces an efficient frontier, from which an investor can select an optimal portfolio according to their risk and return preferences.

The portfolio selection problem is essentially multi-objective. According to Markowitz's *E-V* model, which has formed the foundations of Modern Portfolio Theory (MPT), an investor would ideally wish to maximise expected return whilst minimising risk. However, the two objectives conflict and it is not generally possible to simultaneously optimise on both. Maximising expected return would tend to result in a high risk portfolio whilst minimising risk would tend to produce a low expected return. In addition, Markowitz highlights the attractiveness of diversity, with respect to cardinality. Again, there is a trade off: the greater the number of assets in a portfolio (i.e. the more diverse), the lower the risk; however, a lower risk typically corresponds to lower expected return. In practice, compromise is required.

An underlying assumption of Markowitz's model is that precise estimates of μ_i and σ_{ij} have been obtained. Hence, *E-V* optimisation is only concerned with the optimisation of a portfolio and does not address the issue of how to obtain estimates for μ_i and σ_{ij} nor does it consider the possibility of those estimates being imprecise. As seen by the *E-V* model discussed above, μ_i and σ_{ij} are treated as known constants; however, asset returns are variable. Therefore, as Bienstock (2007) suggested, μ_i and σ_{ij} are, in practice, not known constants, but soft quantities. It is reasonable to conclude that a model which treats returns as known constants will produce a portfolio whose realised return is different from the optimal portfolio return given by the objective function value. In particular, when the realised asset returns are less than the estimates used to optimise the model, the realised portfolio return will be less than the optimal portfolio return given by the objective.

E-V optimisation has encountered other criticisms as well, particularly with respect to the composition of efficient portfolios. Michaud (1989) suggested that *E-V* portfolios "don't make investment sense" because they maximise estimation error. He states that efficient portfolios give too much weight to assets whose μ_i and σ_{ij} estimates are more likely to be furthest from their true values. Ceria and Stubbs (2006) agreed, suggesting that *E-V* optimisation is "counterintuitive" and too sensitive to fluctuations in the first and second moments of asset returns. That is, small changes in μ_i and σ_{ij} yield very different efficient (E,V) combinations. Ceria and Stubbs also suggested that there are two approaches to overcome estimation error in *E-V* optimisation: better estimation techniques or better techniques for optimisation under uncertainty (2006). The authors argued that even though there are estimation techniques for μ_i and σ_{ij} that may produce a more stable *E-V* portfolio, statistical methods are driven by underlying distributions, and this is problematic. Therefore, in addition to uncertainty, variability should be incorporated into the optimisation process, creating a need for methods such as robust optimisation, which treat uncertain parameters as soft quantities in the optimisation process, i.e. instead of using a single value such as μ_i, asset returns can take any value within a defined set of possible outcomes.

3.2 Alternative Measures of Risk in Portfolio Selection Models

What is risk and how should it be measured in portfolio selection? The terms *risk*, *volatility*, *uncertainty* and *variance* are frequently used interchangeably in the context of portfolio selection, but are these terms truly synonymous? What is actually meant by *minimising the risk* of a portfolio? Throughout the literature, it is evident that there is no universally agreed correct answer as to how to measure risk in portfolio selection. Thus, the term *minimise risk* is a bit fuzzy.

As shown in the *E-V* model, Markowitz (1952) defined *risk* as the *variance* or standard deviation of a portfolio. Because standard deviation measures the spread of a distribution with respect to its mean, risk in this context can be thought of as an indicator of how frequently and by how much the true portfolio return is likely to deviate from its mean. An obvious difficulty is that one must make three assumptions: 1) the expected return of the portfolio is the true mean of the distribution of portfolio returns, 2) the distribution is symmetric and close to normal and 3) there is no distinction between the spread of returns above the mean versus the spread of returns below the mean. This last assumption has caused the most concern, particularly when the second assumption does not hold. Practitioners argue that portfolio return distributions are rarely normal and that risk should only be associated with "bad" returns (Sortino and Price, 1994), (Rom and Ferguson, 2003); thus, standard deviation is not an adequate measure of risk (Sortino, 2003).

There is a wide body of literature for alternatives to standard deviation, commonly known as downside risk measures. To cover the breadth of research in this area is beyond the scope of interest here, hence we will briefly mention three measures, just to provide an idea of how researchers and practitioners incorporate risk into the optimisation stage of portfolio selection: Value-at-Risk (VaR), Conditional Value-at-Risk (CVaR) and Downside Deviations (DD).

Value-at-Risk and Conditional Value-at-Risk consider the tail of the portfolio returns distribution – how much of an investors wealth is "at risk"? The VaR is defined as the minimum amount of wealth an investor is at risk of losing with a probability of α. An obvious limitation of VaR is that it only considers one point (α) on the distribution. VaR tells us nothing about the rest of the tail, such as the shape or the spread of the distribution of losses associated with smaller values of α. An alternative to VaR is CVaR, which provides more insight into the tail of the distribution. The CVaR is defined as the expected amount of wealth an investor is at risk of losing with a probability of α. In other words, it is the expectation of losses in the tail of the distribution to the left of the VaR. While both measure potential losses and are indicators of risk, they only focus on a certain type of risk – the risk of extreme events (generally $\alpha \leq 5\%$). Extremely bad portfolio returns are only part of the risk picture. Recent research suggests that whilst extreme events are a concern, they are not the primary concern – investors are more interested in whether or not they will achieve their investment goals (Sortino, 2003). The risk measure downside deviation (*DD*), introduced by Sortino and van der Meer (1991), attempts to convey this type of risk.

Downside deviation (*DD*) is the portfolio semivariance below a target return. The target return separates what Sortino and Price call the "good" returns from the "bad" returns (1994). They argue that risk should only be associated with "bad" returns, thus, the *DD* only measures the variance of returns below an investors target return. Several authors, including Sortino (2003), Balzer (2003) and Nawrocki (1999), point out that in later works Markowitz discussed below-mean semivariance and below-target semivariance as alternatives to variance. They comment that computing power was a key factor influencing his choice of variance to define and measure a portfolio's risk.

3.3 Robust Portfolio Selection

As previously mentioned, the main reason for applying a robust optimisation framework to the portfolio selection problem is because asset returns are unknown and variable. Although the distributions of asset returns are uncertain, we may assert that μ or σ, or both, belong to an uncertainty set, the bounds of which we can define. Most robust portfolio models describe asset returns by ellipsoidal uncertainty sets, based on the methodology of Ben-Tal and Nemirovski (1998) and El Ghaoui and Lebret (1997), in which the user defined parameter Ω adjusts the guaranteed and achieved robustness of the portfolio. Previously, robustness has been evaluated based upon performance, particularly worst case performance, then compared to the worst case performance of a non-robust model such as the *E-V* model. In addition to worst case performance, we suggest that it is also important to evaluate robustness based upon whether a model yields portfolios that achieve their guaranteed robustness in practice (see Chapters 4 and 5).

In 2000, Lobo & Boyd presented two robust portfolio models: the first gave an upper-bound on the risk associated with a portfolio, given a set of decisions; the second, minimised the upper-bound on risk. They presented several different methods for modelling the uncertainty sets for the expected returns vector and covariance matrices, such as box or ellipsoidal sets. Each robust model was a semi-definite program solved via interior point methods. Their results focused on the performance of the solution method rather than on the robustness of the optimal portfolios.

Goldfarb and Iyengar (2003) defined asset returns by robust factor models in which uncertainty was modelled by ellipsoidal sets. The robustness of a robust Sharpe Ratio problem was evaluated based on performance, particularly in worst case scenarios, and compared to the *E-V* portfolio model. Results showed the worst case performance of the robust model was approximately 200% better than the non-robust model; thus, they concluded that robust portfolios were more apt to withstand noisy data.

El Ghaoui, Oks and Oustry (2003), introduced and evaluated the robustness of a worst case VaR model in which the uncertainty (of both μ and σ) was modelled by ellipsoidal sets. Results showed that the non-robust model 'wins' if there is no uncertainty, but the robust model 'wins' in the worst case scenario, as one would expect.

Tütüncü and Koenig (2004) describe μ and σ by uncertainty sets in order to optimise a model which seeks to find the "best worst case" portfolio and compare its performance to *E-V* portfolios. Results showed that robust portfolios are only "marginally inefficient" when returns take their expected value but *E-V* portfolios are "severely inefficient" when returns take their worst case values (as defined by their uncertainty set). In addition, robust portfolios tended to concentrate on a small set of asset classes, having chosen mostly large capital value stocks.

Ceria and Stubbs (2006) presented a model which minimised the difference between the estimated and actual efficient frontiers while maximising portfolio return. Typically the true frontier lies between the estimated and actual frontiers, hence, minimising their distance will bring them closer to the true frontier. Results showed that the robust model yielded greater "realised returns" in most cases.

Kim and Boyd (2007) presented the robust efficient frontier analysis method to address the problem of poor performance by *E-V* optimisation resulting from the use of estimates of μ and σ. Their main focus was to construct a worst case efficient frontier representing the optimal trade-off between worst case risk-return pairs in which the uncertainty in μ and σ are independent. They analyse the basic properties of a worst case robust efficient frontier and present several models of uncertainty which are computationally tractable.

Pflug and Wozabal (2007) took a slightly different perspective on modelling uncertainty by considering the probability model of an asset to be unknown. They constructed a robust portfolio selection problem in which a 'confidence set' described the probability distribution of asset returns. In addition, they evaluated the trade-off between risk, robustness and portfolio return. Their results showed that as robustness increased, risk and portfolio return decrease, and portfolios were more diversified.

Robust optimisation techniques have been criticised for giving equal weight to all possible values of the uncertain parameters, specified within their respective uncertainty sets, which may not be a realistic assumption (Bienstock, 2007). Bienstock (2007) addresses this criticism by defining two types of uncertainty sets that give higher weight to more significant data realisations. The first type defines returns based on a histogram of shortfalls from a point estimate, and the second type models correlations among deviations. Cutting-plane algorithms were introduced to solve the robust models resulting from both types of uncertainty sets.

More recently, Bertsimas and Pachamonova (2008) suggested a multi-period portfolio optimisation model built upon the approach of Ben-Tal, Margalit and Nemirovski (1999), but with polyhedral, instead of ellipsoidal, uncertainty sets. They compared the computational performance of their linear robust models with a single period mean-variance model using simulations of future returns of 3 assets. Results suggested that a robust multi-period approach should be considered as an alternative to single period *E-V* models.

As is evident from the literature, nearly all robust portfolio models construct uncertainty sets as ellipsoids, based on the work of Ben-Tal and Nemirovski (1998) and El Ghaoui and Lebret (1997). Typically, solution robustness is evaluated by comparing the worst case performance of the robust model with that of a non-robust model. In addition, there is not an explicit evaluation of the cost of robustness. In this next section, we consider the robust portfolio model of Bertsimas and Sim (2004), which results from modelling uncertainty by budgeted polyhedral uncertainty sets. In Chapters 4 and 5 we investigate the guaranteed and achieved robustness of the solution as well as the cost of robustness. Furthermore, by altering model parameters, we evaluate the stability of the model itself. In other words, as model parameters change, how much do our decisions change?

3.4 Linear Robust Counterpart to Portfolio Optimisation

In this section, we discuss the formulation of the linear robust counterpart to the portfolio optimisation problem, first from a duality perspective (introduced by Bertsimas and Sim (2004)) and then we explain the rationale for the model.

3.4.1 Robust Counterpart by Duality

The basic portfolio optimisation problem is defined as follows:

$$\begin{aligned} &\text{Maximise} && \textstyle\sum_i r_i w_i \,. && (3.1)\\ &\text{Subject to} && \textstyle\sum_i w_i \le 1\,, \\ &&& w_i \ge 0\,, \qquad \forall i\,. \end{aligned}$$

Asset returns, r_i, are uncertain parameters with unknown distributions defined as bounded and symmetric with respect to a point estimate $\bar{r}_i$:

$$r_i \in [\bar{r}_i - \hat{r}_i, \bar{r}_i + \hat{r}_i]\,. \tag{3.2}$$

Even though the true distribution of r_i is unknown, historical data can be used to estimate the mean log return of asset i, which is substituted for the point estimate $\bar{r}_i$.

A new stochastic variable η_i (Bertsimas and Thiele, 2006) measures the deviation of parameter r_i from $\bar{r}_i$ and takes values in $[-1,1]$,

$$\eta_i = \frac{r_i - \bar{r}_i}{\hat{r}_i}.$$

By rearranging this equation, r_i can be expressed as:

$$r_i = \bar{r}_i + \hat{r}_i \eta_i. \tag{3.3}$$

Let $|J|$ be the number of parameters, r_i, that are uncertain; then for Soyster's and Ben-Tal & Nemirovski's model

$$\sum_i \frac{|r_i - \bar{r}_i|}{\hat{r}_i} = |J| \qquad \text{or} \qquad \sum_i |\eta_i| = |J|.$$

Bertsimas and Sim (2004) relaxed this condition by defining a new parameter Γ (the *budget of uncertainty*) as the number of uncertain parameters that take their worst case value $\bar{r}_i - \hat{r}_i$. Therefore,

$$\sum_i |\eta_i| \le \Gamma \text{, such that } \Gamma \in [0, |J|].$$

Rewrite the initial portfolio optimisation problem, given by (3.1), by substituting (3.3) for r_i:

$$\max_{w_i} \quad \sum_i (\bar{r}_i + \hat{r}_i \eta_i) w_i. \tag{3.4}$$

$$\text{Subject to} \quad \sum_i w_i \le 1$$

$$w_i \geq 0\,, \qquad \forall i\,,$$

$$-1 \leq \eta_i \leq 1\,, \quad \forall i\,.$$

We wish to minimise the worst case portfolio return, therefore, (3.4) is rewritten as follows:

$$\underset{w_i}{\text{Max}} \qquad (\sum_i \bar{r}_i w_i + \underset{\eta_i}{Min} \sum_i \hat{r}_i \eta_i w_i\,). \tag{3.5}$$

Subject to $\quad \sum_i w_i \leq 1\,,$

$$\sum_i |\eta_i| \leq \Gamma\,,$$

$$w_i \geq 0\,,\ -1 \leq \eta_i \leq 1\,, \quad \forall i\,.$$

Expressing η_i as a non-negative variable, we rewrite (3.5) as follows:

$$\underset{w_i}{\text{Max}} \qquad (\sum_i \bar{r}_i w_i - \underset{\eta_i}{Max} \sum_i \hat{r}_i \eta_i^+ w_i)\,. \tag{3.6}$$

Subject to $\quad \sum_i w_i \leq 1\,,$

$$\sum_i \eta_i^+ \leq \Gamma\,,$$

$$w_i \geq 0\,, \quad 0 \leq \eta_i^+ \leq 1\,, \qquad \forall i\,.$$

By duality (Bertsimas and Sim, 2004), the inner maximisation problem subject to the stochastic constraints becomes:

$$\underset{p,q_i}{\text{Min}} \qquad \Gamma p + \sum_i q_i\ .$$

Subject to $\quad p + q_i \geq \hat{r}_i w_i\,, \qquad \forall i\,,$

$p \geq 0,$

$q_i \geq 0, \qquad \forall i .$

Substituting this result back into (3.6), we obtain the following robust counterpart:

$$\underset{w_i}{\text{Max}} \quad (\textstyle\sum_i \bar{r}_i w_i - \underset{p,q_i}{\text{Min}}(\Gamma p + \textstyle\sum_i q_i)) \quad \equiv \quad \underset{w_i}{\text{Max}} \; (\textstyle\sum_i \bar{r}_i w_i - \Gamma p - \textstyle\sum_i q_i).$$

Subject to $\quad \sum_i w_i \leq 1,$

$p + q_i \geq \hat{r}_i w_i, \quad \forall i ,$

$p \geq 0,$

$w_i, q_i \geq 0, \qquad \forall i .$

3.4.2 Interpretation of Robust Counterpart

The robust counterpart of an uncertain linear problem is a max-min or min-max model; the objective is to optimise the worst case performance. Soyster's and Ben-Tal & Nemirovski's model stipulate that every constraint be feasible for every uncertain parameter defined within a bounded symmetric set (an uncertainty set). That is, their models are optimised for *every* uncertain parameter taking its worst case value.

Bertsimas and Sim (2004) introduced a budgeted robust counterpart that assumes at most Γ uncertain parameters will take their worst case values, not *every* parameter. Applying this concept, consider the basic portfolio model stated in Section 3.1, but using the definition of r_i in (3.3).

We desire the portfolio with the best worst case return given that Γ assets take their worst case values, $\bar{r}_i - \hat{r}_i$. Therefore,

$$\text{Max} \quad (\text{Min} \sum_i \bar{r}_i w_i - \sum_{t \in T} \hat{r}_t w_t). \quad \equiv \quad \text{Max} \sum_i \bar{r}_i w_i - \text{Min}(\text{Max} \sum_{t \in T} \hat{r}_t w_t).$$

$$\text{Subject to} \quad \sum_i w_i \le 1,$$

where $I = \{i | 1 \le i \le N\}$, $T \subseteq I$, $|T| = \Gamma$, that is, T is the subset of Γ assets that take their worst case values, $\bar{r}_t - \hat{r}_t$, $t \in T$. The min-max term in the objective seeks to minimise the worst case. The inner maximisation term seeks to choose Γ assets with the largest $\hat{r}_i w_i$ as the subset T whilst the outer minimisation term seeks to make the sum as small as possible with respect to w_i. For the moment, assume that the quantity $\hat{r}_t w_t$ is the same for all t, and denoted by p. Then the term $\sum_{t \in T} \hat{r}_t w_t$ becomes Γp.

Now consider the possibility that $p_t \ne p$ for some t, where $p_t = \hat{r}_t w_t$ for all t. Clearly, at optimality, the term $\sum_{t \in T} \hat{r}_t w_t$ will be greater than or equal to zero, hence, we will only consider the case when $p_t \ge p$. Therefore, the difference $p_t - p$, for all t, needs to be added onto Γp and the quantity $\text{Min}(\text{Max} \sum_{t \in T} \hat{r}_t w_t)$ becomes:

$$\text{Min}[\Gamma p + \sum_t (p_t - p)], \qquad \forall t = \{t | (p_t - p) \ge 0\}. \tag{3.7}$$

We can restrict the difference $p_t - p$ to be greater than or equal to zero if we introduce a new variable q_t given by the following equation:

$$q_t = \max(0, p_t - p). \tag{3.8}$$

Thus, q_t is defined by the following constraints:

$$q_t \ge p_t - p, \qquad \forall t,$$

$q_t \geq 0$, $\forall t$.

Therefore (3.7) can be rewritten as:

$$\text{Min}(\Gamma p + \sum_t q_t), \qquad \forall t. \tag{3.9}$$

The question now is: Which p_t is chosen as our p value? Recall that the min-max term in the objective seeks to maximise the quantity $\sum_{t\in T} \hat{r}_t w_t$ by choosing the Γ largest $\hat{r}_i w_i$ as the subset T and that the quantity $\sum_t q_t$ is greater than zero only when $p_t > p$. Thus, p is chosen as the smallest $\hat{r}_t w_t$, over all t, which means it is the Γ^{th} largest $\hat{r}_i w_i$, over all i.

The final portfolio optimisation model becomes:

$$\text{Max} \qquad (\sum_i \bar{r}_i w_i - \Gamma p - \sum_i q_i). \tag{3.10}$$

$$\text{Subject to} \qquad \sum_i w_i \leq 1.$$

$$q_i \geq \hat{r}_i w_i - p, \qquad \forall i,$$

$$q_i \geq 0, \; w_i \geq 0, \qquad \forall i,$$

$$p \geq 0.$$

Remark: $\sum_{i\in I} q_i$ can be substituted for $\sum_{t\in T} q_t$ and $\hat{r}_i w_i$ substituted for $\hat{r}_t w_t$, because every $\hat{r}_i w_i$, where $i \notin T$, will be less than p. Therefore, $p_i - p$ will be less than zero, and the corresponding q_i will be zero.

3.5 Robust Portfolio Model for Uncorrelated Asset Returns

3.5.1 The Linear Robust Portfolio Model

Bertsimas and Sim (2004) reformulated a maximum expected return portfolio model as a linear robust optimisation problem, as shown in Section 3.3:

$$\text{Max} \qquad \sum_{i=1}^{n} \bar{r}_i w_i - p\Gamma - \sum_{i=1}^{n} q_i \tag{3.11}$$

$$\text{Subject to} \qquad \sum_{i=1}^{n} w_i \leq 1,$$

$$q_i \geq c\hat{r}_i w_i - p, \qquad \forall i,$$

$$w_i, q_i \geq 0, \qquad \forall i,$$

$$p \geq 0,$$

where $\bar{r}_i$ is the point estimate for the log return on asset i (e.g. the median or mean log return), w_i is the proportion of total wealth invested in asset i and r_i is the true log return of asset i. The true log return of asset i, r_i, belongs to the interval $[\bar{r}_i - c\hat{r}_i, \bar{r}_i + c\hat{r}_i]$, where $\hat{r}_i$ is chosen by the user and determines how the uncertainty set defines r_i and $c \in \Re^+$ defines the magnitude of the range of the set U, as discussed in Section 2.1.2. For example, if $\hat{r}_i$ is the standard deviation of asset i, then c would determine the width of the interval in terms of the number of standard deviations. Alternatively, if $\hat{r}_i = \bar{r}_i$, where $\bar{r}_i$ is the mean log return, then c would determine the width of the interval in terms of the percentage of $\bar{r}_i$ that the true log return deviates from $\bar{r}_i$. The user defined parameter Γ is given a value between 0 and n. As Γ increases, the probability of underperforming the robust optimal objective decreases. At optimality, p is the Γ^{th} largest $c\hat{r}_i w_i$ and $q_i = \max(0, c\hat{r}_i w_i - p)$ for each asset i. The focus of the Bertsimas and Sim's paper was to present their robust approach and not portfolio optimisation per se; their experimental results were for a set of 150 stocks with $\bar{r}_i$ and $\hat{r}_i$ generated by arithmetic progressions.

3.5.2 Extending the Linear Robust Portfolio Model

We extend the simple portfolio selection model in Section 3.5.1 to a more comprehensive model that includes conditions imposed by investors such as threshold and cardinality constraints. Threshold constraints specify the minimum proportion of total capital to be invested in an asset if it is selected for the portfolio. Cardinality constraints specify the maximum number of assets selected for the portfolio. We consider the following linear optimisation problem:

$$\text{Max} \quad \sum_{i=1}^{n} \bar{r}_i w_i - p\Gamma - \sum_{i=1}^{n} q_i \tag{3.12}$$

$$\begin{aligned}
\text{Subject to} \quad & \sum_{i=1}^{n} w_i \le 1, & \\
& \sum_{i=1}^{n} \delta_i \le k, & \\
& q_i \ge c\hat{r}_i w_i - p, & \forall i, \\
& \delta_i = \{0,1\}, & \forall i, \\
& w_i \ge \alpha\delta_i, & \forall i, \\
& w_i, q_i \ge 0, & \forall i, \\
& w_i \le \delta_i, & \forall i, \\
& p \ge 0, &
\end{aligned}$$

where $\bar{r}_i$, w_i, $\hat{r}_i$, p and q_i are defined as before, α is the buy-in threshold on asset weights and k is the maximum number of assets selected.

3.6 Computational Platform

We chose to optimise this particular linear robust portfolio model using the solver CPLEX version 10.1, a common computational platform, within the modelling language AMPL. Other specialised modelling languages, such as CVX (Grant, Boyd and Ye, 2008) which is implemented in Matlab, may be chosen for solving convex problems such as (3.11). However, CVX does not have discrete features and (3.12) is a mixed-integer program, and thus not convex.

Chapter 4

The Cost of Robustness

Robustness, viewed as a performance guarantee, comes at a cost. In the case of portfolio optimisation, it is the probability guarantee that the portfolio return will be at least equal to that of the optimal robust solution. One would expect that in order to achieve robustness, a sacrifice, in terms of optimal objective value, will occur. But how much does this sacrifice cost? And is it worth it? There is a two-fold motivation to the investigation detailed in this section. Firstly, to provide a measure for the cost of robustness and determine if the robust methodology in Chapter 3 is, in reality, robust. Secondly, to examine how the cost and robustness (both guaranteed and achieved robustness) are affected when the following are changed: *i*) the point estimate of asset *i* ($\bar{r}_i$), the deviation parameter ($\hat{r}_i$) which indicates the maximum amount the true return of asset *i* may deviate from its point estimate, and scaling factor *c* defining the uncertainty set *U*, of r_i, given by $r_i \in [\bar{r}_i - c\hat{r}_i, \bar{r}_i + c\hat{r}_i]$ and/or *ii*) the size of the set of historical data used to estimate $\bar{r}_i$ and $\hat{r}_i$.

Recall from Chapter 2 the two aspects of an uncertainty set *U*: structure and scale. Throughout this chapter, the structure remains constant in that it is polyhedral. However, because we allow the possibility for Γ to be different, the dimension of the polyhedral set can change. With respect to the scale of *U*, $\bar{r}_i$ and $\hat{r}_i$ are estimated from the historical data set in the same manner for each model, but the definition of $\bar{r}_i$ and $\hat{r}_i$, and the scaling factor *c* do change (see Table 4.1).

The data set used in this investigation consists of 120 monthly log returns of 68 assets from the FTSE 100 starting 1 February 1996 through to 1 January 2007. A set of $m \in M$ months, where M = {20, 40, 60}, was randomly selected and used to generate 10 robust models, R_j, where j = 1..10 (Table 4.1), each with a different uncertainty set defining r_i, the true log return of asset i, which is unknown and variable. For each model, 100 instances (henceforth referred to as trials, t) were generated by randomly selecting m months from the sample period, in order to obtain a distribution for both measures of cost and both measures of robustness detailed in Sections 4.1 and 4.2. Thus for each t, from 1 to 100, a set of m randomly selected months (which was different for each trial) was considered as the set of available historical data and used to optimise 10 robust models, for a total of 1000 optimal portfolios. The set of m months was also used for in-sample back-testing in the evaluation of robustness.

Model R_j	$\bar{r}_i$	$\hat{r}_i$	c
j = 1..3	Mean Log Return	Standard Deviation	1, 2, 3
j = 4..7	Mean Log Return	Mean Log Return	0.90, 0.95, 0.98, 1
j = 8..10	Median Log Return	Standard Deviation	1, 2,3

Table 4.1. Summary of robust models.

For each trial t, the m randomly selected months were used to estimate the value $\bar{r}_i$ and $\hat{r}_i$; each robust optimal solution, $Z^{Opt}_{j,t,m}$, was obtained using (3.11) from Section 3.5. The Γ value that yielded the most robust diversified portfolio (i.e. the portfolio with the smallest probability of underperformance and consisting of more than one asset) was chosen as the optimal robust portfolio for model R_j, for each trial t and each set of $m \in M$ months. A characteristic of the robust models, which we discuss further in Chapter 6, is that as Γ increases from an initial value of 0, the number of assets in each corresponding portfolio also increases until all but one asset are suddenly dropped, which corresponds to $\Gamma = \Gamma_{drop}$. Results show that when $\Gamma = \Gamma_{drop} - 1$, the resulting portfolio is the most robust diversified portfolio consisting of at least as many assets as each portfolio corresponding to all other values of Γ. This value of Γ and hence, the probability of underperformance, given by (4.1), may not be the same for all of the 1000 portfolios, for each set m. However, since there are only N = 68 possible values of Γ_{drop}, many of the portfolios will have the same probability of underperformance. Bertsimas and Thiele (2006) define the probability of underperformance as follows:

$$\Pr(Z_{j,t,m,l}^{true} \leq Z_{j,t,m}^{Opt}) \leq \varepsilon, \qquad (4.1)$$

where,

$$\varepsilon = 1 - \Phi((\Gamma - 1)/\sqrt{N}).$$

$Z_{j,t,m,l}^{true}$ is the realised portfolio return for model R_j, during trial t, given the set of m months, evaluated at month l such that l = 1..m. $Z_{j,t,m}^{Opt}$ is the robust optimal objective function value for Γ and N is the total number of assets.

4.1 Measures of Cost

For each trial t and set m, robust model R_j yields a n-vector of optimal asset weights, $\mathbf{w}^*_{j,t,m}$. Therefore, the total portfolio return of R_j for each t and m, is given by $P_{j,t,m}^{Total}$ in (4.2):

$$P_{j,t,m}^{Total} = \sum_{i=1}^{n} \bar{r}_{i,t,m} w^*_{i,j,t,m}, \qquad \forall j,t,m \qquad (4.2)$$

where $\bar{r}_{i,t,m}$ is the mean log return of asset i over a set of m months for trial t. We introduce two measures for the cost of robustness. Let $r_{t,m}^{MMax}$ denote the return of the asset with the largest mean log return for trial t and set of m months. *Cost*1 and *Cost*2 measure the cost of the robust optimal portfolio, $P_{j,t,m}^{Total}$, with respect to $r_{t,m}^{MMax}$. *Cost*1 measures the deviation between the value of the non-robust solution (i.e. with just a single asset) and the value of the robust solutions, whereas *Cost*2 measures the deviation as a ratio with respect to $r_{t,m}^{MMax}$. In other words, *Cost*2 can be thought of as a cost-to-maximum potential reward ratio.

$$Cost1_{j,t,m} = r_{t,m}^{MMax} - P_{j,t,m}^{Total}, \qquad \forall j,t,m. \tag{4.3}$$

$$Cost2_{j,t,m} = (r_{t,m}^{MMax} - P_{j,t,m}^{Total}) / r_{t,m}^{MMax}, \qquad \forall j,t,m. \tag{4.4}$$

Clearly, $P_{j,t,m}^{Total}$ will be in the interval $[-1, r_{t,m}^{MMax}]$, therefore, *Cost*1 and *Cost*2 will be in the interval $[0, \infty]$.

Bertsimas and Sim (2004) introduced the concept "price of robustness" as the difference between the robust optimal objective value and that of the nominal problem. The measures of cost given by (4.3) and (4.4) differ from the price of robustness in that they measure the difference between the optimal objective of the nominal problem and the objective function value of the nominal problem evaluated at the robust optimal solutions.

4.1.1 Measures of Cost: Results

The motivation for the cost analysis is to determine 1) the distributions of the cost of $P_{j,t,m}^{Total}$ for each model and 2) observe how changing the definitions of $\bar{r}_i$, $\hat{r}_i$ and the scaling factor *c* and/or the size of the data set $m \in M$, where *M* = {20, 40, 60}, affects cost.

First, consider the distribution of *Cost*1 for each model, R_j, when $m = 20$ months (Figure 4.1) and $m = 60$ months (Figure 4.2). For each model, the minimum, maximum, median and mean costs are plotted; each set of piecewise linear functions corresponds to a different scaling factor *c*, for fixed definitions of $\bar{r}_i$ and $\hat{r}_i$ (detailed in Table 4.1). Observe the effect of changing *c* on each statistic and distribution (Figures 4.1 and 4.2). For example, consider models R_1, R_2 and R_3 when *m* = 20 months (Figure 4.1). As *c* increased from 1 to 2 (R_1 to R_2), the value of each statistic of *Cost*1 also increased; as *c* increased from 2 to 3 (R_2 to R_3), the value of each statistic of *Cost*1 did not change. Similar behaviour was observed between models R_8, R_9 and R_{10}. Models R_4 to R_7, showed an increase in the value of each statistic of *Cost1* corresponding to an increase in *c*, particularly noticeable in maximum cost. Moreover, the histograms of *Cost*1, for m = 20, show that the distribution of each R_j was close to Normal, but with slightly higher peaks and positively skewed, with one outlier (the maximum). An example of these characteristics is given in Figure 4.3, which shows the histograms of *Cost*1 for models R_1, R_6 and R_{10}. Lastly, as *c* increased, the distribution of *Cost*1 maintained a similar shape but was shifted up (Figure 4.1). In other words, as the uncertainty set increased in scale, the cost of robustness also tended to increase.

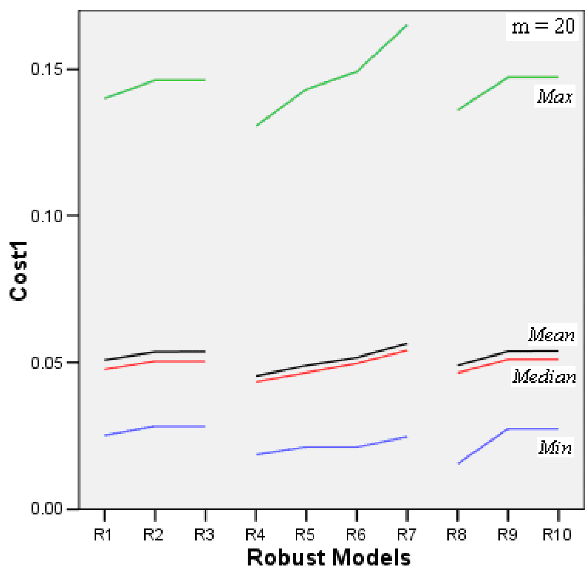

Figure 4.1. Distribution of *Cost*1 for each model *Rj*, when $m = 20$ months.

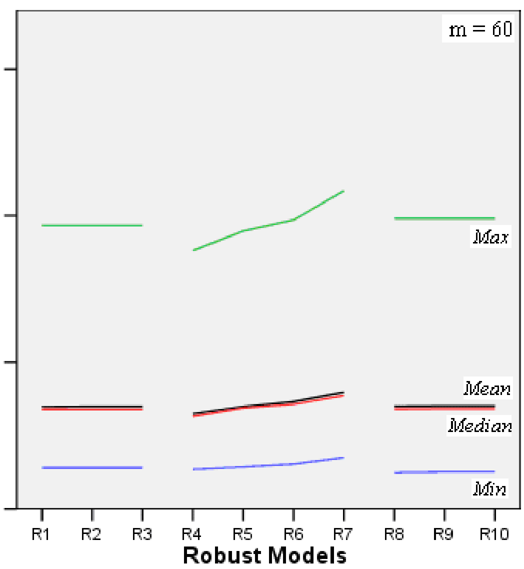

Figure 4.2. Distribution of *Cost*1 for each model *Rj*, when $m = 60$ months.

Consider when m = 60 months (Figure 4.2). An increase in *c* did not affect the min, max, median, or mean of *Cost*1 for R_1 to R_3 or R_8 to R_{10}; however, as when m = 20, models R_4 to R_7 showed an increase in the value of each statistic of *Cost1* corresponding to an increase in *c*, and again, this was particularly noticeable for the maximum cost. In addition, the distributions of *Cost*1 were close to Normal, but with higher peaks and positively skewed, with either 2 or 3 outliers (the 2 or 3 largest values); this is observed in Figure 4.4, which shows the histograms of *Cost*1 for models R_1, R_6 and R_{10}. Finally, Figure 4.2 shows that when m = 60, in contrast to

$m = 20$, an increase in c had a minuscule effect on cost for models R_1 to R_3 and R_8 to R_{10}. Therefore, increasing the range of the uncertainty set did not significantly increase the cost of robustness for these 6 models. However, the cost of robustness for models R_4 to R_7 did tend to increase.

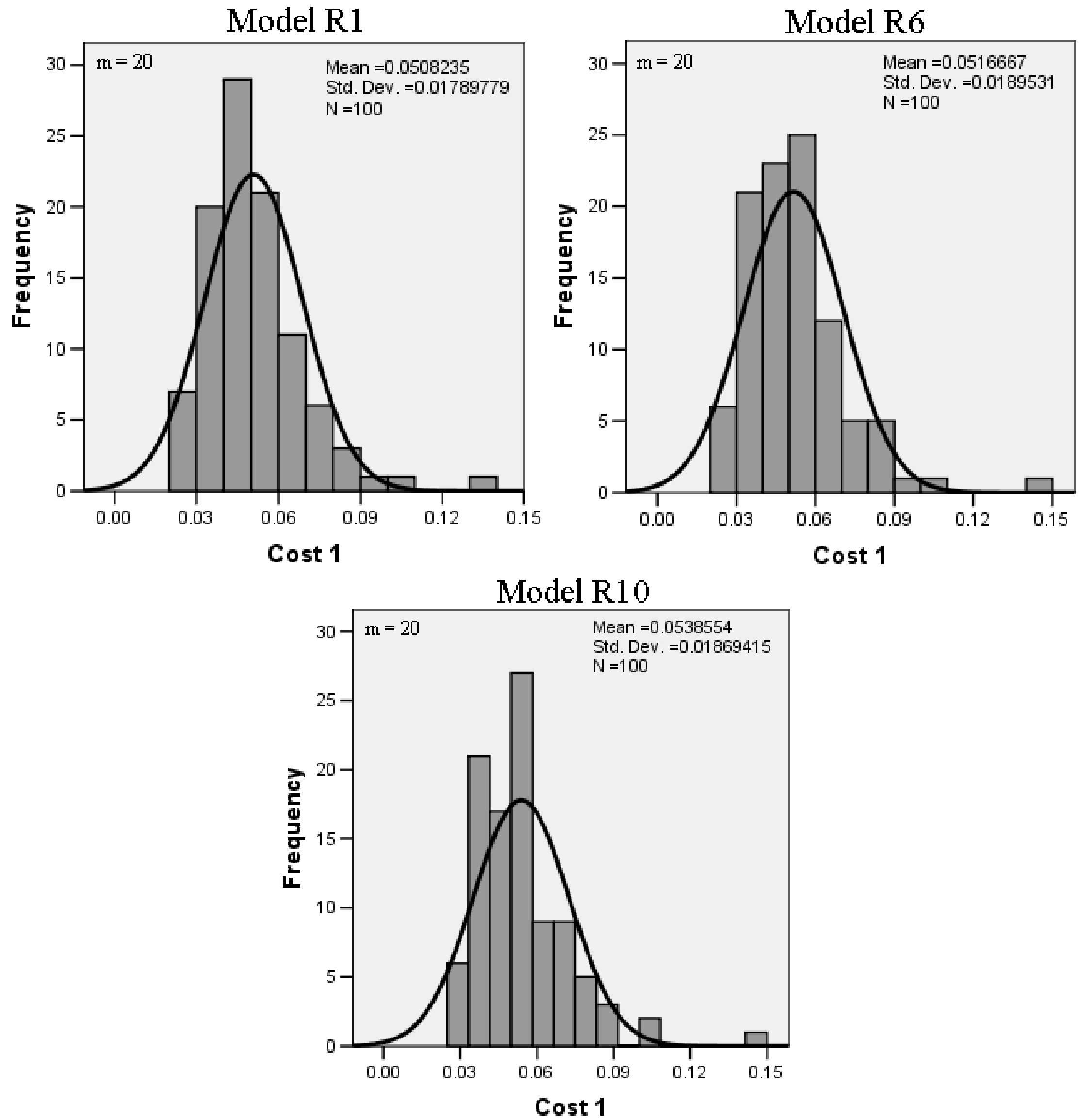

Figure 4.3. Histograms of *Cost*1 for Models R_1, R_6 and R_{10}, where $m = 20$ months.

As observed previously, there are instances in which an increase in c does not appear to affect the distribution of *Cost*1 (Figures 4.1 and 4.2). For m = 20 months, the composition of portfolios for c = 2 and c = 3 reveal that the strong similarity between the distributions of the corresponding portfolios, R_2 & R_3 and R_9 & R_{10}, is because their optimal weights were either identical or very similar in many of the trials. For example, both R_2 and R_3 yielded the same decisions for trial 1. The same is true when $m = 60$ months, but for $c = 1$, 2 and 3.

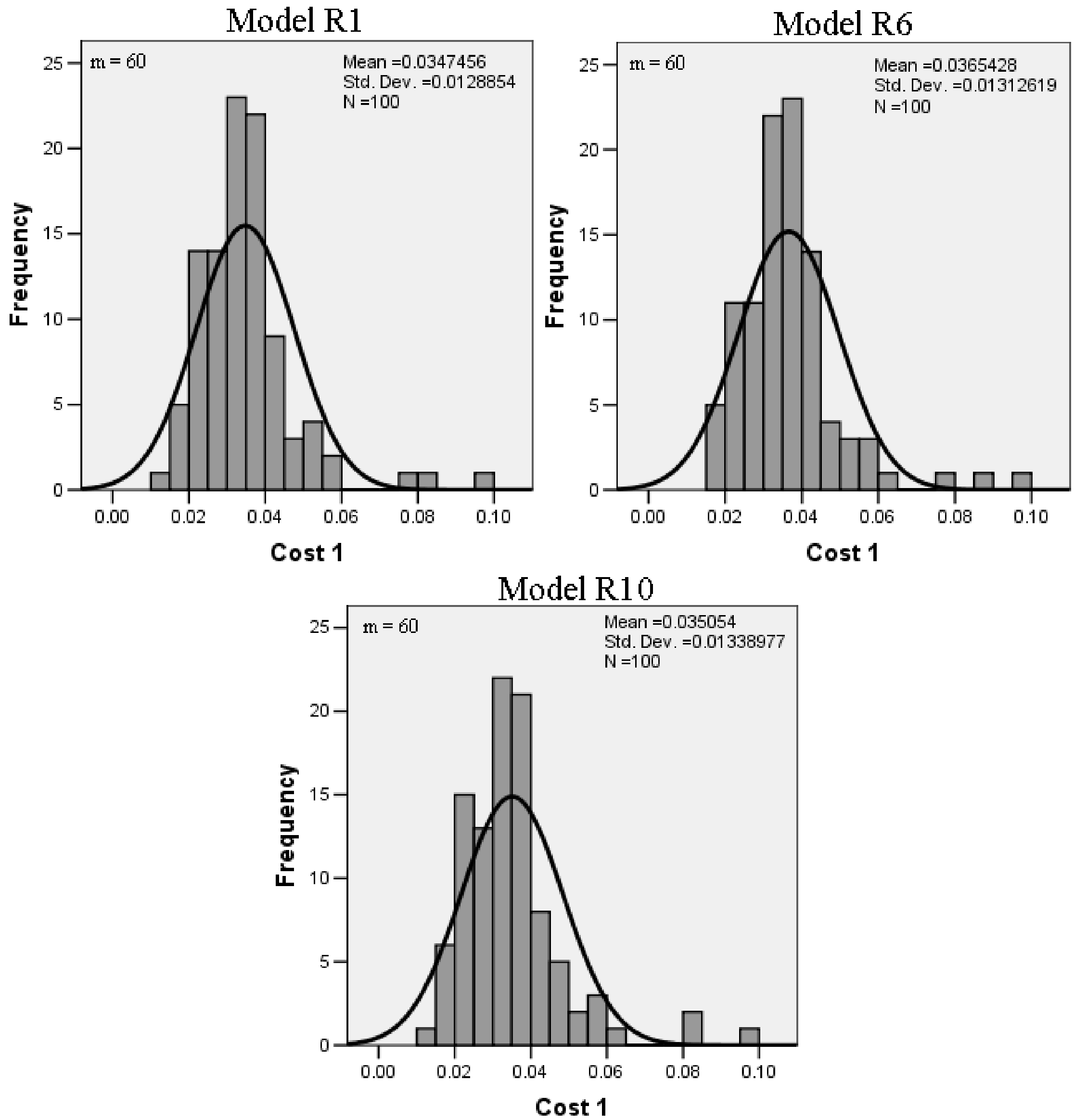

Figure 4.4. Histograms of *Cost*1 for Models R_1, R_6 and R_{10}, where $m = 60$ months.

We have observed how the scaling factor c affected the distribution of *Cost*1. Second, observe the effect of changing either the definition of $\bar{r}_i$, $\hat{r}_i$ or the number of months m (Figures 4.1 and 4.2). For example, compare the distributions of R_1, R_2 and R_3 with those of R_4, R_5, R_6 and R_7. The latter set of models, in which both $\bar{r}_i$ and $\hat{r}_i$ are defined as the mean log return of asset i, tended to cost less than the former (with the exception of R_7) as their distributions are shifted down; however, they tended to have larger outliers (maxima). A similar comparison can be made between the distributions of R_8 to R_{10} and R_4 to R_7. Next, compare the distributions of *Cost*1 when m = 20 and m = 60 months, for R_1, R_2 and R_3; an increase in the number of months of historical data resulted in lower costs. There are several possible causes: 1) increasing the size of the dataset results in more precise estimates of $\bar{r}_i$ and $\hat{r}_i$, which yield more cost effective solutions, 2) $r_{t,m}^{MMax}$ may be significantly larger for each trial when m = 20, or 3) a combination of both 1) and 2). Results for m = 40 months (not shown) suggest that it is the first cause, because the distributions for *Cost*1 fell between those for 20 and 60 months. *Cost*2 provides a more accurate indication of the effect of changing m (Figures 4.5 and 4.6)

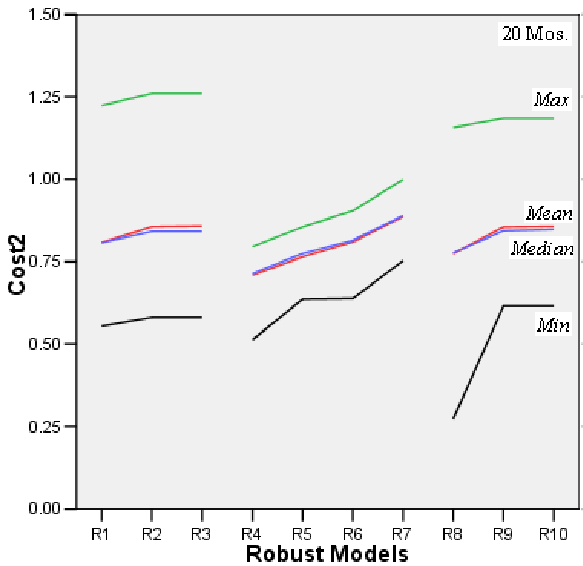

Figure 4.5. Distributions of *Cost*2 for each model *Rj*, when $m = 20$ months.

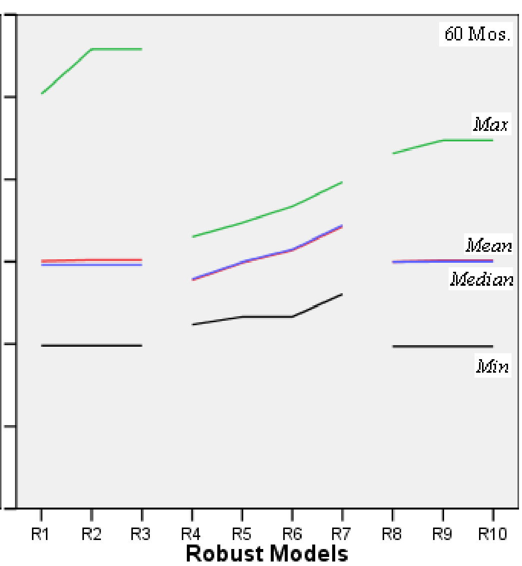

Figure 4.6. Distributions of *Cost*2 for each model *Rj*, when $m = 60$ months.

The distribution of *Cost*2 for each model, R_j, is shown in Figure 4.5 (m = 20 months) and Figure 4.6 (m = 60 months). As with *Cost*1, the minimum, maximum, median and mean are plotted for each model; each set of piecewise linear functions correspond to a different scaling factor c, for fixed definitions of $\bar{r}_i$ and $\hat{r}_i$. Observe the effect of changing c on each statistic and distribution (Figures 4.5 and 4.6). As with *Cost*1, an increase in c tended to correspond to an increase in *Cost*2, and, with the exception of the maximum for R_1 to R_3, each model tended to cost less when $m = 60$ than when $m = 20$, although only slightly.

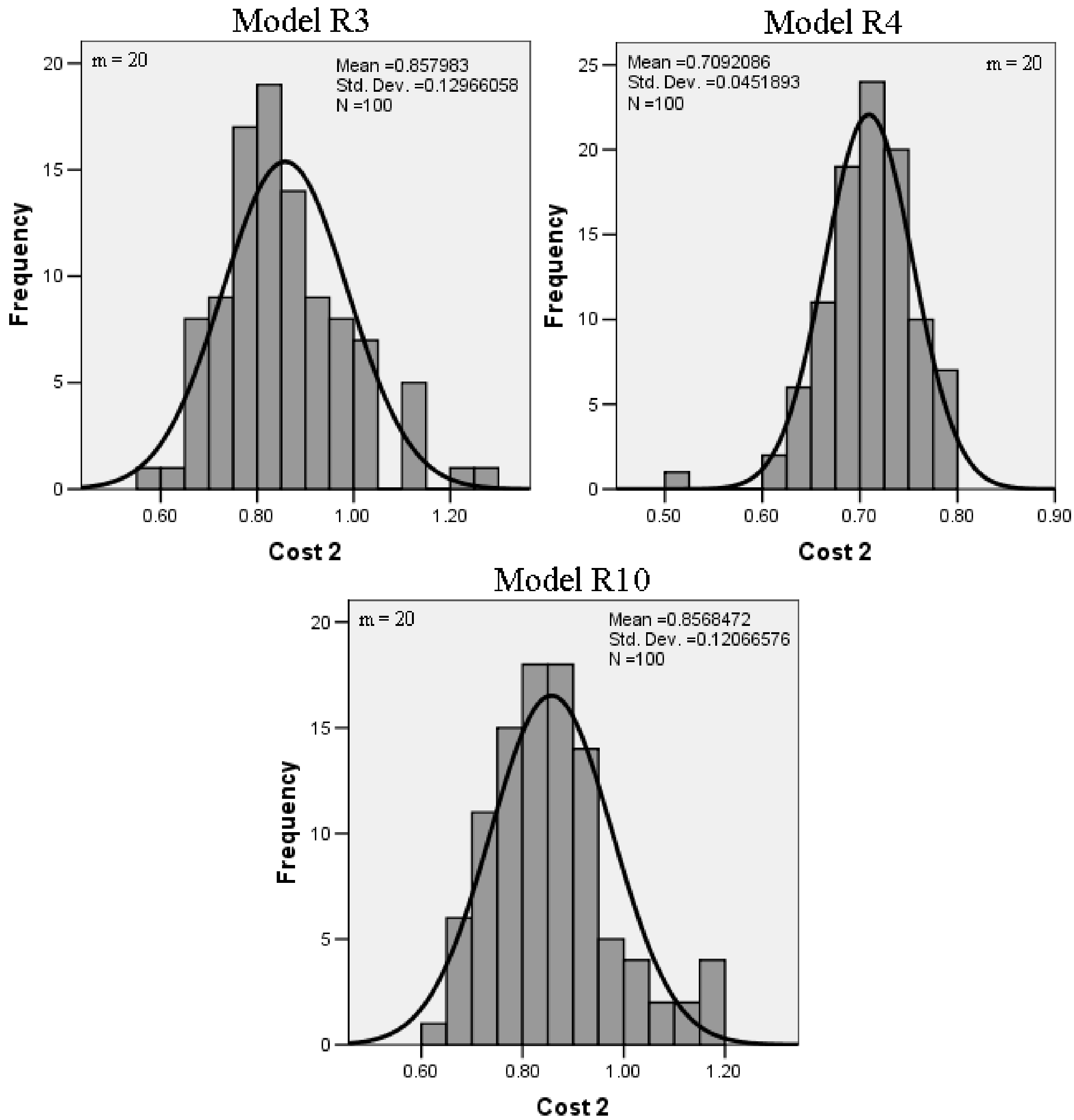

Figure 4.7. Histograms of *Cost*2 for Models R_3, R_4 and R_{10}, where $m = 20$ months.

One distinction from *Cost*1 is that models R_4 to R_7 tended to cost less than the other six models. Observed by their distributions and histograms, R_4 to R_7 had a much smaller spread and were closer to being Normally distributed, with means close to the other six models, but without any outliers. The histograms of *Cost*2 (m = 20, 60) for each R_j, j = 1..3, 8..10, were close to Normal, but with slightly higher peaks and positively skewed (Figure 4.7 shows the histograms of *Cost*2 for models R_3, R_4 and R_{10}, when m = 20 months and Figure 4.8 shows the histograms of *Cost*2 for models R_3, R_6 and R_{10}, when m = 60 months). Models R_1 to R_3 had 1 or 2 outliers which were maxima whereas R_8 had an outlier that was a minimum, but only for $m = 20$

months. From an analysis of Figures 4.5 and 4.6, we conclude that one can expect models R_1 to R_3 and R_8 to R_{10} to cost approximately 75-83% on average, and models R_4 to R_7 to cost approximately 70-85% on average, when the number of months in the historical data set is between 20 and 60 months. It is possible that further increasing the number of months in the historical data set would result in decreased costs, however, further investigations not presented here suggest that the mean of *Cost*2 would not decrease significantly.

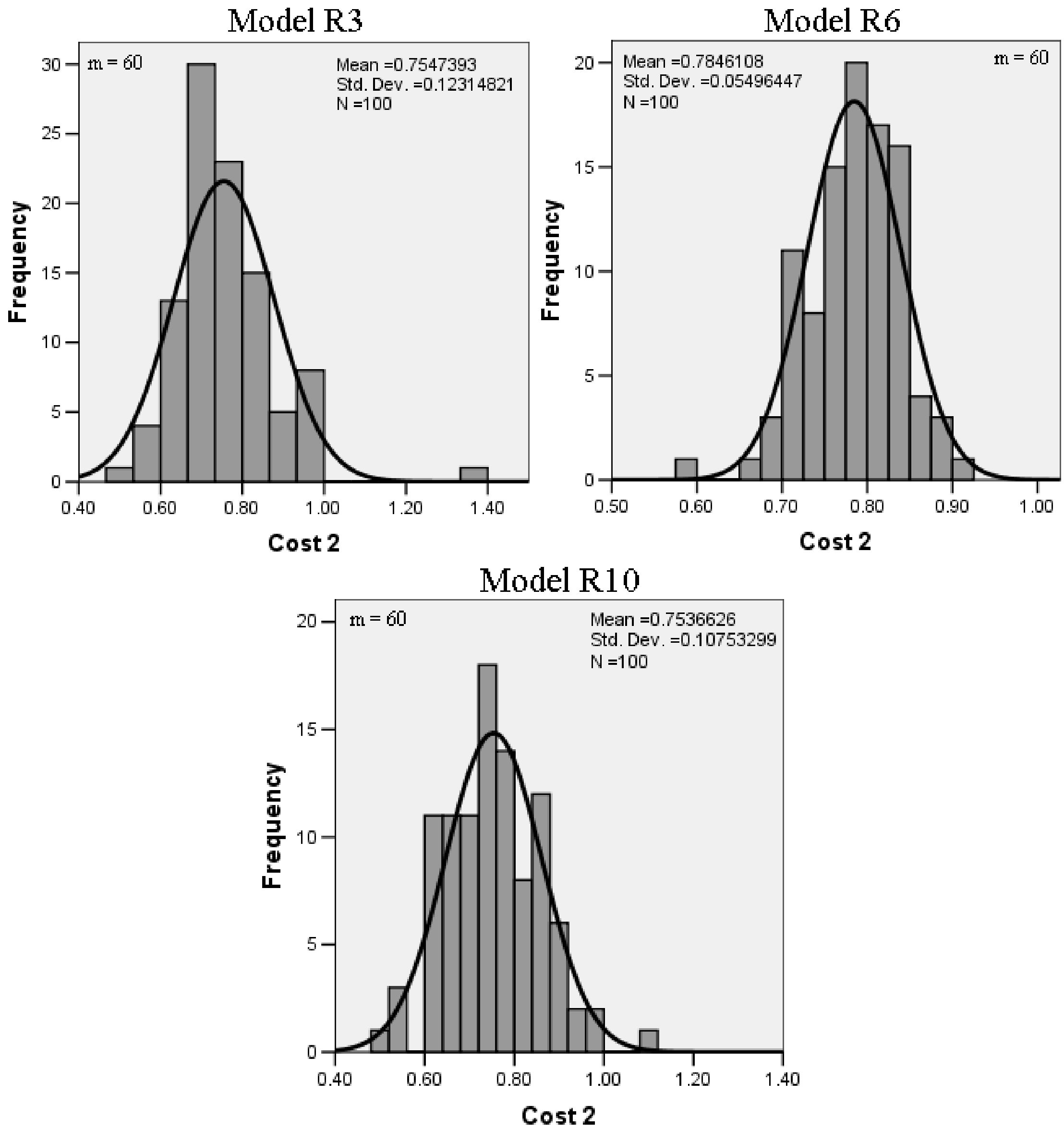

Figure 4.8. Histograms of *Cost*2 for Models R_3, R_6 and R_{10}, where $m = 60$ months.

To summarise, the distribution of the loss in portfolio return with respect to $r_{t,m}^{MMax}$ for all ten models was very similar, with none having a significant advantage over the others, particularly with respect to the mean and median of both *Cost*1 and *Cost*2. The mean percentage loss for each model, regardless of $\bar{r}_i$, $\hat{r}_i$, m or c, was approximately 70-85%. However, the distributions of the percentage loss in portfolio return with respect to $r_{t,m}^{MMax}$ showed that models R_4 to R_7 had a more consistent percentage loss and were closer to a Normal distribution without outliers. In addition, results suggest that increasing the scaling factor c, i.e. increasing the scale (or size) of the structure the uncertainty set U defining r_i, tended to increase costs whereas increasing the number of months in the set of historical data tended to decrease costs.

4.2 Measures of Robustness

Robustness is measured by 1) the probability of underperformance, which is dependent upon Γ, and 2) the proportion of evaluated portfolios that underperform the robust optimal objective (*PLO*: Proportion of portfolios Less than Objective), given by (4.5) and (4.6) respectively.

$$PLO_{j,t,m}^{Max} = (1-\Pr(Z_{j,t,m,l}^{true} \geq Z_{j,t,m}^{Opt})) = (1-\Phi((\Gamma-1)/\sqrt{N})), \quad \forall j,t,m. \quad (4.5)$$

$$PLO_{j,t,m}^{Eval} = \sum_l \delta_{j,t,m,l} / m, \quad \forall j,t,m, \quad (4.6)$$

where $\delta_{j,t,m,l}$ is 1 if $Z_{j,t,m,l}^{true} < Z_{j,t,m}^{Opt}$ and 0 otherwise. Because both (4.5) and (4.6) are measures of underperformance, as they decrease, robustness increases and as they increase, robustness decreases. Comparing the distributions of $PLO_{j,t,m}^{Max}$ and $PLO_{j,t,m}^{Eval}$, $\forall j,m$, we can evaluate the robustness of this methodology for the stated definitions of r_i, c and m.

4.2.1 Measures of Robustness: Results

The motivation of the robust analysis is to determine 1) if the robustness guaranteed by each model is actually achieved and 2) how changing the definition of $\bar{r}_i$, $\hat{r}_i$, the scaling factor c and/or the size of the data set $m \in M$, affects the robustness of the solution (guaranteed and achieved).

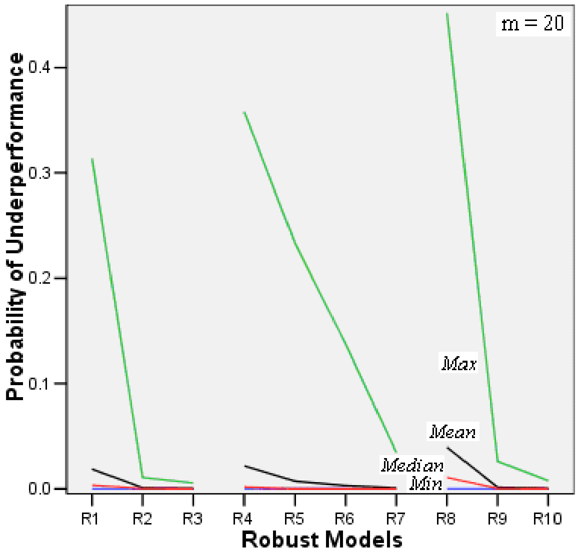

Figure 4.9. Distributions of $PLO_{j,t,m}^{Max}$ for each model R_j, when $m = 20$ months.

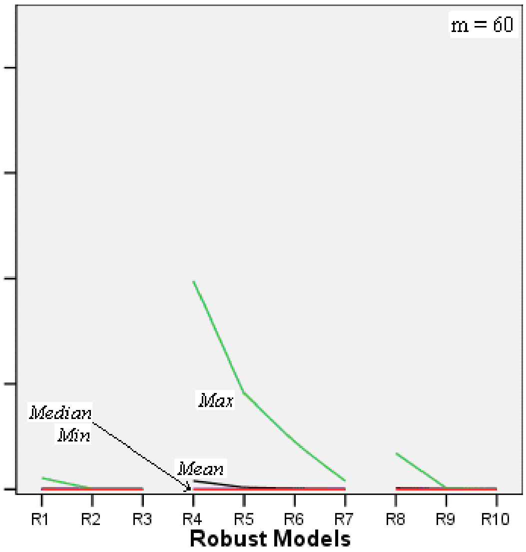

Figure 4.10. Distributions of $PLO_{j,t,m}^{Max}$ for each model R_j, when $m = 60$ months.

The distribution of the guaranteed robustness, $PLO_{j,t,m}^{Max}$, is shown in Figure 4.9 ($m = 20$ months) and Figure 4.10 ($m = 60$ months). For each model, the minimum, maximum, median and mean probability of underperformance is plotted; each set of piecewise linear functions corresponds to different values of c for fixed values of $\bar{r}_i$ and $\hat{r}_i$. For both 20 months and 60 months, an increase in c corresponded to a decrease in the probability of underperformance; the distributions became tighter and means and medians were closer to 0 (Figures 4.9 and 4.10). In addition, the maximum for each model when $m = 20$ (Figure 4.9) was much higher than the maximum when $m = 60$ (Figure 4.10) which suggests that an increase in m also tended to result in a decrease in the probability of underperformance, thus, greater guaranteed robustness. For example, R_1 had a max close to 0.30 when $m = 20$ but a max of about 0.02 when m = 60. In addition, when $m = 60$ months, the distribution of each model had a much smaller spread and a mean and median closer to 0. Recall that the probability of underperformance, $PLO^{Max}_{j,t,m}$, is dependent upon $\Gamma_{drop} - 1$, as larger $\Gamma_{drop} - 1$ yield a smaller $PLO^{Max}_{j,t,m}$. Therefore, a larger data set of m months yielded diversified portfolios for larger values of Γ (i.e. portfolios with greater guaranteed robustness). Lastly, although there is not one type of model that guaranteed significantly more robustness, those which define $\bar{r}_i$ as the mean log return and $\hat{r}_i$ as the standard deviation of asset i (R_1 to R_3 and R_8 to R_{10}), were characterised by tighter distributions and smaller maximum values (Figures 4.9 and 4.10), suggesting that these models are slightly more advantageous (for both 20 and 60 months).

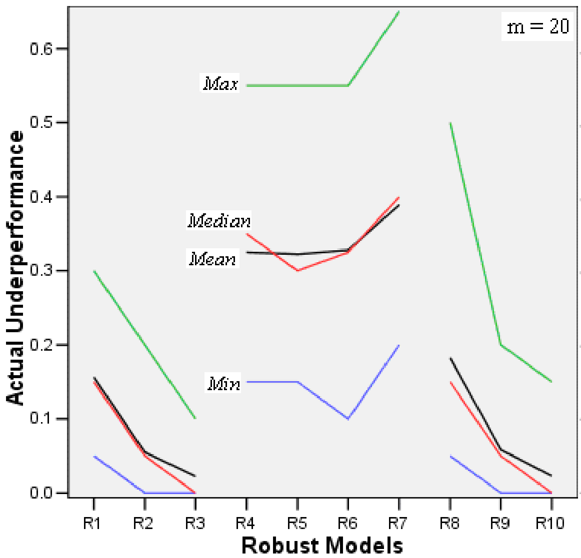

Figure 4.11. Distribution of $PLO^{Eval}_{j,t,m}$ for each model R_j, when $m = 20$ months.

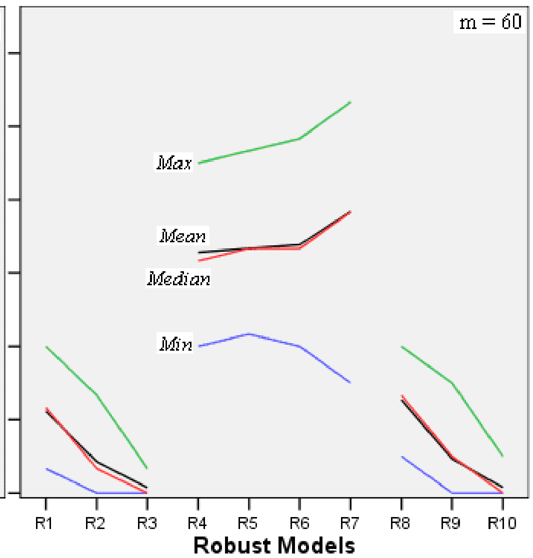

Figure 4.12. Distribution of $PLO^{Eval}_{j,t,m}$ for each model R_j, when $m = 60$ months.

The distribution of the proportion of portfolios that actually underperform the robust optimal objective ($PLO^{Eval}_{j,t,m}$), is shown for model R_j, for m = 20 months (Figures 4.11) and m = 60 months (Figure 4.12). For each R_j, the minimum, maximum, median and mean underperformance are plotted; each set of piecewise linear functions correspond to a different scaling factor c for fixed definitions of $\bar{r}_i$ and $\hat{r}_i$. When $\bar{r}_i$ was defined as the mean or median log return and $\hat{r}_i$ as the standard deviation of asset i, increasing c also increased robustness (Figure 4.11, Figure 4.12 and Table 4.2). However, when both $\bar{r}_i$ and $\hat{r}_i$ were defined as the mean log return (R_4 to R_7), increasing c tended to decrease robustness, shown by an increase in the mean and median. Recall that an increase in c resulted in higher costs for models R_4 to R_7, thus, as these models increased in cost, they were also less robust.

Table 4.2 shows the number of trials in which the proportion of portfolios that underperformed the optimal objective was less than the probability of underperformance (i.e. the number of instances in which $PLO^{Eval}_{j,t,m}$ was less than $PLO^{Max}_{j,t,m}$). Models R_3 and R_{10}, in which c = 3, had the largest percentage of trials that achieved or exceeded the guaranteed level of robustness, i.e. in over 60% of the trials the percentage of portfolios that underperformed the robust optimal objective was less than the probability of underperformance. Conversely, for models R_4 to R_7, not one trial achieved or exceeded the guaranteed level of robustness for any set m.

How often is the guaranteed robustness achieved?

m	R1	R2	R3	R4	R5	R6	R7	R8	R9	R10
20	1	29	63	**0**	**0**	**0**	**0**	2	27	63
40	1	11	73	**0**	**0**	**0**	**0**	1	9	67
60	**0**	9	64	**0**	**0**	**0**	**0**	**0**	8	62

Table 4.2. The number of trials (out of 100) in which the guaranteed level of robustness was achieved or exceeded, for each model R_j for a given set of m months.

Lastly, increasing m tended to decrease the probability of underperformance, shown by decreased max values and tighter distributions for the same model (Figures 4.11 and 4.12). However, this did not necessarily correspond to an increase in the number of trials that achieved or exceeded the guaranteed robustness (Table 4.2). In addition, an increase in m also decreased the probability of underperformance (Figures 4.9 and 4.10), which in most cases was less than 1%. In order for the actual percentage of portfolios that underperform to be less than 1%, for any given trial, the portfolio return for every month l (l = 1..m) must be greater than the robust optimal objective – not one out of the m months can underperform. For models R_2, R_3, R_9 and R_{10}, when m = 40 or 60 months, many trials did not achieve guaranteed robustness because only one portfolio, out of the 40 or 60, underperformed the robust optimal objective.

In summary, the distributions of the probability of underperformance suggest that increasing the scale of the structure of uncertainty set U, defining r_i, decreases the probability that the actual portfolio return will underperform the robust optimal objective ($PLO_{j,t,m}^{Max}$). Likewise, when $\bar{r}_i$ was the mean or median log return and $\hat{r}_i$ was the standard deviation of asset i, the actual proportion of portfolios that underperform ($PLO_{j,t,m}^{Eval}$) also decreases, i.e. they were more robust. However, when both $\bar{r}_i$ and $\hat{r}_i$ were the mean log return of asset i, the actual proportion of portfolios that underperform tended to increase, which means portfolios were less robust than they were guaranteed to be; these models were also much less robust than the other six models.

4.3 Discussion

Given that the mean log return of asset i is uncertain and lies on the interval $[\bar{r}_i - c\hat{r}_i, \bar{r}_i + c\hat{r}_i]$, and applying the robust portfolio model given in (3.11), we have provided measures to assess the cost of robustness and examined how the cost and robustness (both guaranteed and achieved robustness) are affected when the following are changed: *i*) $\bar{r}_i$, $\hat{r}_i$ and scaling factor c, defining the uncertainty set of r_i, and/or *ii*) the size of the set of historical data used to estimate $\bar{r}_i$ and $\hat{r}_i$. When $\bar{r}_i$ and $\hat{r}_i$ were both specified as the mean log return of asset i (R_4 to R_7), portfolios were slightly less costly, with respect to *Cost*1 (difference between the non-robust and robust total portfolio return) and *Cost*2 (cost-to-maximum potential reward ratio), but also less robust than the other models, particularly with respect to achieved robustness. In addition, when $\bar{r}_i$ and $\hat{r}_i$ were both specified as the mean log return of asset i, an increase in the scale of uncertainty set U not only increased cost but decreased robustness, which is counterintuitive. One would expect that in exchange for a sacrifice in portfolio return there would be an increase in achieved robustness, i.e. fewer portfolios would underperform the optimal objective function value. For the other six models, increasing the range of the uncertainty set also increased cost, but that was in exchange for increased robustness. The results suggest that models which define $\bar{r}_i$ as the mean or median log return and $\hat{r}_i$ as the standard deviation of asset i have the most desirable trade-off between cost and achieved robustness as well as guaranteed and achieved robustness.

Chapter 5

Computational Investigation

The purpose of this investigation is to evaluate the cost and robustness of robust portfolios compared to *E-V* portfolios. One would expect that robust portfolios will cost more, but that is in exchange for greater achieved robustness. We investigate whether this expectation is true for the robust portfolio models detailed in Chapter 3, by applying the measures of cost and robustness introduced in Chapter 4 to the robust and *E-V* solutions, both in-sample and out-of-sample. The motivations of our investigation are as follows:

1. Analysis of cost: How costly is robustness? What is the relationship between c and the conservativeness of the total portfolio return evaluated using the optimal weights of the robust solution (recall that $r_i \in [\bar{r}_i - c\hat{r}_i, \bar{r}_i + c\hat{r}_i]$)?
2. Analysis of robustness: Is the guaranteed probability of optimality achieved? What is the relationship between c and the robustness of the optimal objective value?

The dataset consists of the monthly log returns of 30 stocks selected at random from the FTSE 100 index beginning 1 January 1992 through to 1 December 2006. This time interval includes 2001-02, a period of economic stress and increased stock market volatility; therefore, two sets of experiments were carried out, referred to as Case 1 and Case 2. In Case 1, of the 180 time periods, the first 132 (1 January 1992 – 1 December 2002) were used to construct the optimal portfolio for each model tested. The last 48 time periods (1 January 2003 – 1 December 2006) were reserved for the out-of-sample analysis. In Case 2, of the 180 time periods, the first 108 (1

January 1992 – 1 December 2000) were used to construct the optimal portfolio for each model tested. The last 72 time periods (1 January 2001 – 1 December 2006) were reserved for the out-of-sample analysis. For each month, logarithmic returns were used. Each model was solved using the CPLEX 10.1 solver.

In our experiments, only 30 assets, randomly selected from the FTSE 100 were used. The rationale for this choice is as follows. The threshold and cardinality constrained models are Mixed Integer Programs, and, for more than 30 assets, these models take too long to solve. Since the purpose of our investigation is not computational efficiency, or to improve the MIP solution technology, we have simply chosen this restricted number of assets in order to obtain computationally tractable models for our empirical study.

5.1 Model Descriptions

In each set of experiments nine portfolio optimisation models were tested: eight robust models and one Expected value – Variance model (Table 5.1). The *E-V* optimisation model minimised portfolio variance subject to the expected portfolio return achieving a target return, as described in Chapter 3. The first robust model (R_1) is given by (3.11) in Section 3.5. The seven remaining robust models, given by the extended model (3.12) in Section 3.6, were subject to a buy-in threshold, an upper limit on cardinality or both.

Model R_j	Description	Threshold (α)	Cardinality (k)
R_1	Robust	None	None
R_2	Robust	None	≤ 20
R_3	Robust	0.02	None
R_4	Robust	0.02	≤ 20
R_5	Robust	0.03	None
R_6	Robust	0.03	≤ 20
R_7	Robust	0.04	None
R_8	Robust	0.04	≤ 20
R_{EV}	Expected value -Variance	None	None

Table 5.1. Summary of models and model constraints, for each R_j, where $j = 1..8, EV$.

An upper limit of $k = 20$ was small enough to impose a restriction on cardinality because robust models tend to diversify both in terms of the number of assets and in the distribution of weights. In addition, the buy-in threshold α belongs to the set $\{0.02, 0.03, 0.04\}$ because an α less than 0.01 did not affect the optimal decisions (with a pool of 30 assets, those selected were given a weight greater than 0.01) and an α greater than 0.05 imposed a cardinality restriction of 20 assets.

Each robust model was optimised for integer values of Γ from 0 to *N*, where *N* = 30. The *E-V* model was optimised for 31 equidistant points (with respect to target return) between the minimum variance portfolio and maximum total return portfolio. In addition, for each model it was assumed that the covariance matrix was known but asset returns were unknown and variable. The sensitivity of the robust model to changes in the uncertainty set defining asset returns was investigated by optimising each robust model for five uncertainty sets defined by $r_i \in [\bar{r}_i - c\hat{r}_i, \bar{r}_i + c\hat{r}_i]$. Where $\bar{r}_i$ was the mean log return of asset *i*, $\hat{r}_i$ was the standard deviation of asset *i* and the scaling factor *c* was an integer between one and five, inclusive. That is, we considered the true log return of asset *i*, r_i, to be within one to five standard deviations of its mean. We chose to investigate beyond three standard deviations in order to observe robust optimal solutions when the scaling factor *c* results in a very conservative uncertainty set *U*.

For both Case 1 and Case 2 a total of 1240 robust portfolios (5 uncertainty sets, 8 models and 31 values of Γ) and 31 *E-V* portfolios were optimised. The number of portfolios evaluated was reduced in order to analyse cost and robustness: for each robust model, portfolios resulting from Γ = 8, 10, 18 and $\Gamma_{drop} - 1$ were chosen. These Γ values correspond, respectively, to the $\Pr(Z^{true}_{j,m,c} \geq Z^{Opt}_{j,c})$ ≈ 0.90, 0.95, 0.999 and the maximum guaranteed probability of optimality. We have not specified a numerical value for the maximum guaranteed probability of optimality because $\Gamma_{drop} - 1$ was not the same for all models. In addition, for several models, specifically those with cardinality constraints, $\Gamma_{drop} - 1$ was less than 18, thus, Γ = 18 resulted in a portfolio consisting of 1 asset. For those models we only considered three values of Γ: 8, 10 and $\Gamma_{drop} - 1$.

One of the motivations of this section is to assess the robustness of the robust models compared with the *E-V* models, therefore, we have chosen five portfolios from the $E - V$ efficient frontier from a possible 31. Starting with the minimum variance portfolio, every fourth efficient point was selected, resulting in the set of points *P* = {EV.31, EV.27, EV.23, EV.19, EV.15} (see Figure 5.1). Points EV.14 through to EV.1 represent portfolios consisting of less than 7 assets; although these portfolios would have cost less than those in set *P*, in general, they also would have been less robust. Thus, to compare the robust methodology with the most robust *E-V* portfolios we have limited our set of evaluated portfolios to that of set *P*.

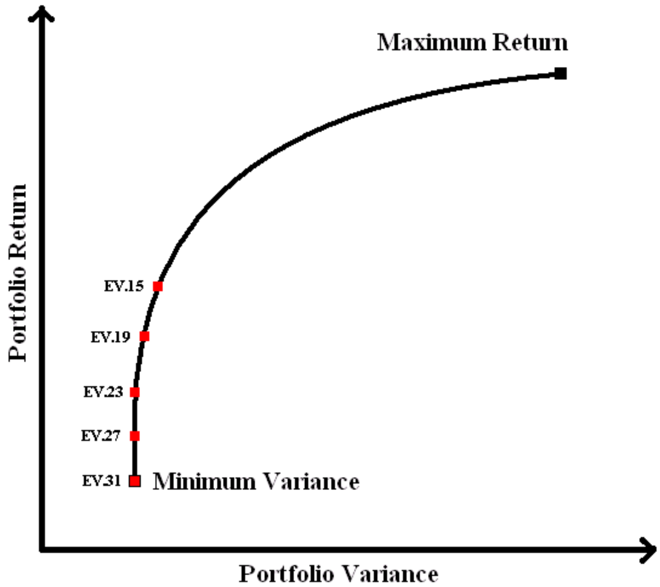

Figure 5.1. Representation of the 5 *E-V* portfolios selected as the set *P*.

The measures of cost introduced in Chapter 4 have been adjusted and are given in (5.1) & (5.2). Likewise, the measures of robustness introduced in Chapter 4 were adjusted and are given in (5.3) & (5.4).

$$Cost1_{j,c} = r^{MMax} - P^{Total}_{j,c}, \qquad \forall j, c. \tag{5.1}$$

$$Cost2_{j,c} = (r^{MMax} - P^{Total}_{j,c}) / r^{MMax} , \qquad \forall j, c . \qquad (5.2)$$

$$PLO^{Max}_{j,c} = (1 - \Pr(Z^{true}_{j,m,c} \geq Z^{Opt}_{j,c}))$$

$$= (1 - \Phi((\Gamma - 1)/\sqrt{N})) , \qquad \forall j : j \neq EV, m, c \qquad (5.3)$$

$$PLO^{Eval}_{j,c} = \sum\nolimits_{m=1}^{M} \delta_{j,m,c} / M, \qquad \forall j, c , \qquad (5.4)$$

where parameter c determines the magnitude of the range of the uncertainty set defining r_i, j is the set of models and $m = 1..M$, where M is the number of months in the respective in-sample or out-of-sample period. Recall that r^{MMax} denotes the return of the asset with the largest mean log return over a given set of M months, $P^{Total}_{j,c}$ is the total portfolio return of model R_j for each value of c, $Z^{true}_{j,m,c}$ is the true portfolio return of model R_j, for each value of c, at month m, $Z^{Opt}_{j,c}$ is the optimal objective function value of model R_j for each value of c and $\delta_{j,m,c}$ is a 0-1 variable which takes a value of 1 if $Z^{true}_{j,m,c} < Z^{Opt}_{j,c}$ and 0 otherwise. For the evaluation of cost and robustness for the *E-V* models, (5.1), (5.2) and (5.4) are similar, only the subscript c is removed, and (5.3) is not applicable because *E-V* models do not guarantee robustness.

5.2 Case 1

For Case 1, the in-sample dataset, consisting of 132 monthly log returns for 30 assets from the FTSE 100 from 1 January 1992 through to 1 December 2002, was used to estimate $\bar{r}_i$ (the mean log return of each asset), $\hat{r}_i$ (the standard deviation of asset i) and the covariance matrix for the *E-V* model. Case 1 captures 2001 through to 2002 within the in-sample set; thus we observe how each model responds to periods of loss in the modelling stage. The out-of-sample dataset consisted of 48 monthly log returns from 1 January 2003 through to 1 December 2006. We optimised 1240 robust portfolios and 31 *E-V* portfolios, and evaluated 120 robust portfolios and 5 *E-V* portfolios.

5.2.1 Analysis of Cost

*Cost*1 (Figure 5.2) and *Cost*2 (Figures 5.3 and 5.4) were calculated for each R_j for Γ values of 8, 10, 18 and $\Gamma_{drop}-1$; the scale markers along the x-axis at every R_j and between successive R_j represent each value of Γ. As mentioned in Section 5.1, $\Gamma_{drop}-1$ was less than 18 for robust models with cardinality constraints; thus, for R_2, R_4, R_6 and R_8 only three values of Γ (8, 10 and $\Gamma_{drop}-1$) are shown for *Cost*1 and *Cost*2. *Cost*1, both in-sample and out-of-sample, is shown in Figure 5.2 for each value of *c*. *Cost*2 is shown in Figure 5.3 (in-sample) and Figure 5.4 (out-of-sample), for each value of *c*. In each Figure, if either respective cost is the same for two or more values of *c*, or two or more values of R_j and Γ, it indicates that the robust optimal decisions were the same. For example, *Cost*1 for R_1 is the same for all *c* and all Γ, which indicates that the optimal decision vector $\boldsymbol{w}^*$ was the same for each of those 20 portfolios (4 values of Γ and 5 values of *c*).

First, we consider the effect that *c* has on cost. An increase in the magnitude of the range of the uncertainty set for each asset *i* means that the worst case value of each asset will decrease and the robust optimal objective will deteriorate. Thus, as *c* increases, we would expect the total portfolio return to be more conservative, corresponding to greater costs. Observe that for a given R_j, an increase in *c* corresponded to either the same costs or an increase in costs (Figures 5.2, 5.3 and 5.4). More specifically,

$$Cost1_{j,c} \le Cost1_{j,c+1}, \qquad \forall j = 1..8, c = 1..4, \tag{5.5}$$

$$Cost2_{j,c} \le Cost2_{j,c+1}, \qquad \forall j = 1..8, c = 1..4. \tag{5.6}$$

Second, we compare the cost of model R_1 to robust portfolios with cardinality and/or buy-in threshold constraints. In comparison to R_1, cardinality constraints tended to decrease costs for values of $c \leq 2$, but increase costs for values of $c \geq 3$ (Figure 5.2). Threshold constraints that did not constrain the cardinality of the portfolio (i.e. α = 0.02, 0.03, corresponding to models R_1 and R_5, respectively) tended to result in very similar costs as R_1 (Figures 5.2, 5.3 and 5.4). In addition, because the plot of *Cost*1 out-of-sample has the same shape as that of in-sample (Figure 5.2), it is a very good indication of which models will be more costly in the future, and an investor can act accordingly. Thus, with this particular set of data, in-sample results for *Cost*1 (Figure 5.2) and *Cost*2 (Figure 5.3) indicate that a more risk seeking investor, one who restricts the true log return of asset *i* to lie within a smaller interval (indicated by smaller values of *c*), should include cardinality constraints in order to reduce costs out-of-sample. However, a risk-averse investor, one who defines the true log return of asset *i* to lie within a larger interval (indicated by larger values of *c*), should either use model R_1, which is unconstrained, or only include threshold constraints which do not impose cardinality restrictions in order to avoid increasing costs out-of-sample.

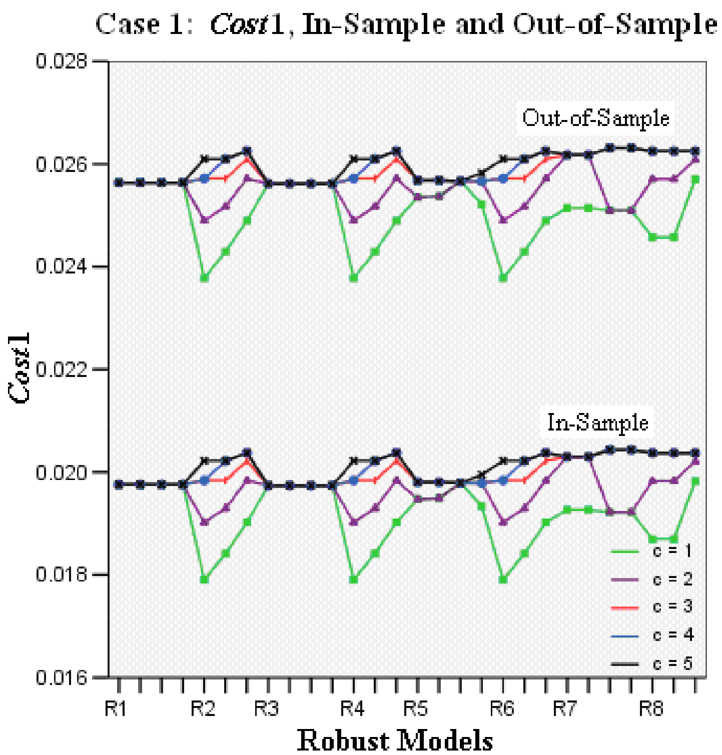

Figure 5.2. *Cost1* for all robust models at specific values of Γ, for all *c*, both in-sample and out-of-sample.

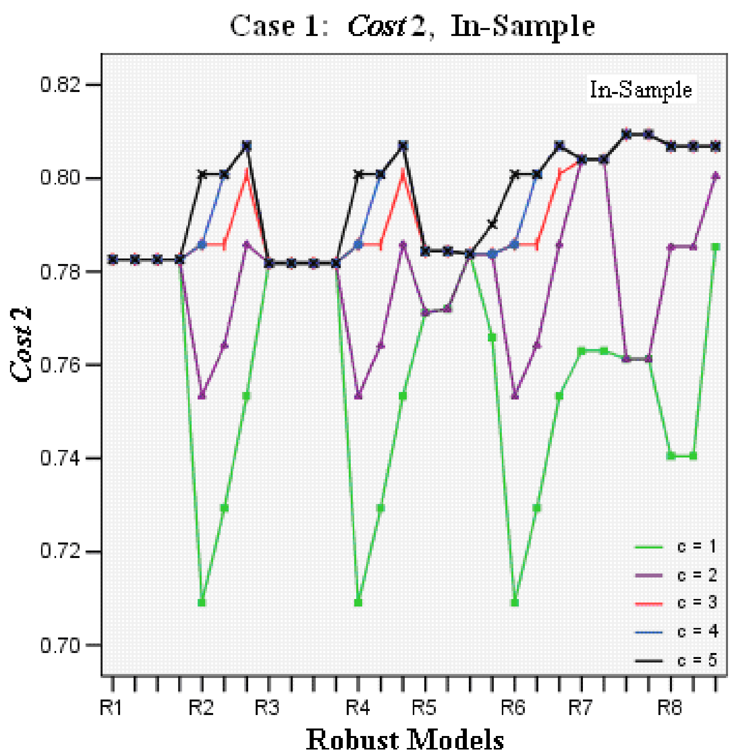

Figure 5.3. *Cost*2 for all robust models at specific values of Γ, for all *c*, in-sample.

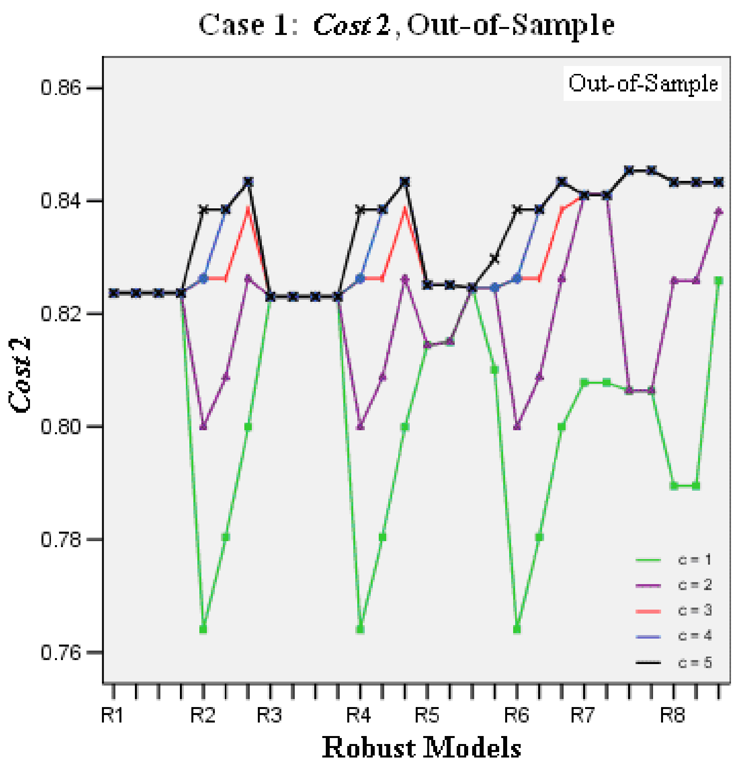

Figure 5.4. *Cost*2 for all robust models at specific values of Γ, for all *c*, out-of-sample.

Third, we compare in-sample and out-of-sample costs. Both *Cost*1 and *Cost*2 were greater out-of-sample than they were in-sample (Figures 5.2, 5.3 and 5.4). At first, this may appear to suggest that all R_j performed poorly out-of-sample; however further investigation shows differently. Recall that the in-sample dataset included a period of higher volatility and poorer returns for many assets compared to the out-of-sample dataset. In addition, both costs were measured with respect to r^{MMax}, which was greater for the out-of-sample period, and both in-sample and out-of-sample costs were calculated using the same $P_{j,c}^{Total}$, $\forall j,c$; thus, it follows that all R_j would cost more out-of-sample. Further results (Section 5.2.2) strongly suggest that R_j are more robust out-of-sample, in exchange for higher costs.

Lastly, we compare the costs of the unconstrained robust model (R_1) to the costs of *E-V* portfolios. Recall that portfolio EV.31 corresponds to the minimum variance portfolio which also has the smallest portfolio return. Each subsequent portfolio, EV.27-EV.15 from the efficient frontier (Figure 5.5), has a greater portfolio return, greater variance and less cost. In-sample and out-of-sample results show that all five *E-V* portfolios cost less than R_1 (Table 5.2). However, R_1 only costs approximately .008 (*Cost*1) and 3% (*Cost*2) more than EV.31; in a robust framework, we are willing to accept higher costs, if increased robustness is achieved, discussed further in the following section.

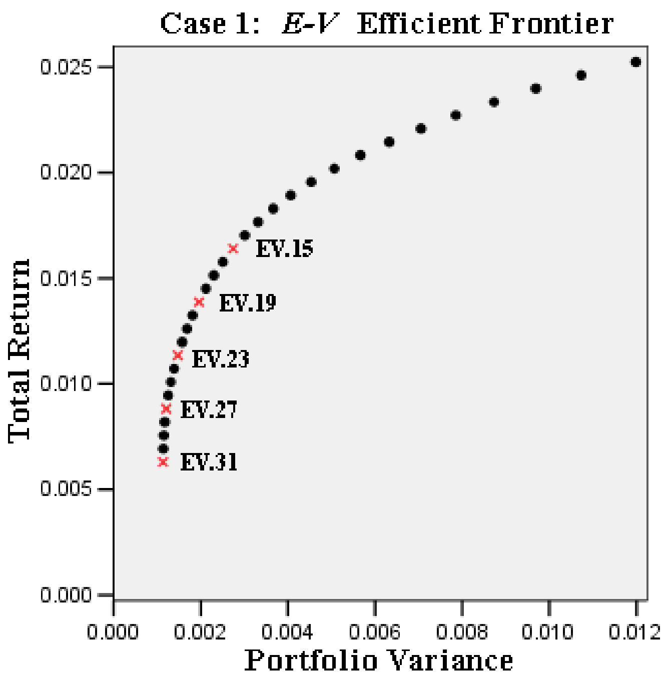

Figure 5.5. *E-V* portfolios selected from the efficient frontier.

	*Cost*1		*Cost*2	
	In-Sample	**Out-of-Sample**	**In-Sample**	**Out-of-Sample**
R_1	0.0198	0.0256	0.7826	0.8237
EV.31	0.0190	0.0248	0.7507	0.7978
EV.27	0.0164	0.0223	0.6506	0.7166
EV.23	0.0139	0.0198	0.5505	0.6354
EV.19	0.0114	0.0173	0.4504	0.5542
EV.15	0.0089	0.0147	0.3503	0.4730

Table 5.2. *Cost*1 & *Cost*2 in-sample and out-of-sample for all *E-V* portfolios and robust model R_1.

5.2.2 Analysis of Robustness

First, we evaluate whether guaranteed robustness was achieved and how robustness is affected by the scaling factor *c*. The realised portfolio return of R_1 ($Z_{1,m,c}^{true}$), for all *c*, was the same for each $\Gamma \in \{8, 10, 18, 22\ (\Gamma_{drop} - 1)\}$, both in-sample (Figure 5.6) and out-of-sample (Figure 5.7). In addition, for a given Γ and scaling factor *c*, the robust optimal objective value of R_1 ($Z_{1,c}^{Opt}$) was held constant over a sample period; thus only one horizontal line is plotted for each *c* (Figures 5.6 and 5.7). The same scale was used for all four in-sample time-series (Figure 5.6), likewise for all four out-of-sample time-series (Figure 5.7).

Recall that larger Γ values correspond to greater probabilities of optimality, hence, portfolios should be more robust. The actual robustness of each portfolio was measured by comparing the percentage of returns that dipped below the line $Z_{1,c}^{Opt}$ for each *c*, with the probability of underperformance (1 – [Probability of optimality]). Observe that an increase in *c*, Γ or both, increased achieved robustness (Figures 5.6 and 5.7 and Tables 5.3 and 5.4)). In addition, more robust portfolios achieved their guaranteed probability of optimality out-of-sample.

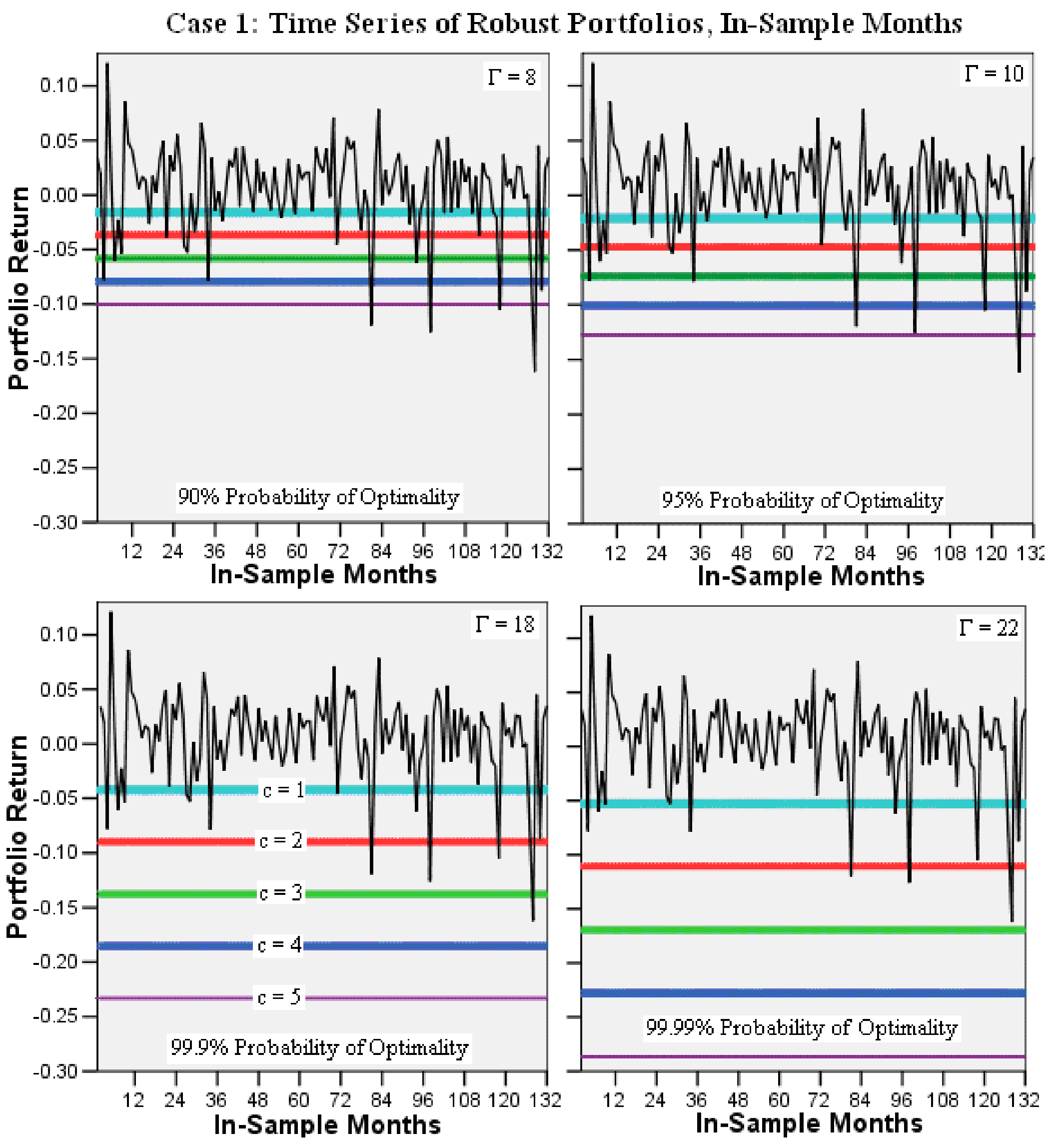

Figure 5.6. R_1 in-sample plots of $Z_{1,m,c}^{true}$ and $Z_{1,c}^{Opt}$ for Γ = 8, 10, 18 and 22, $\forall c$.

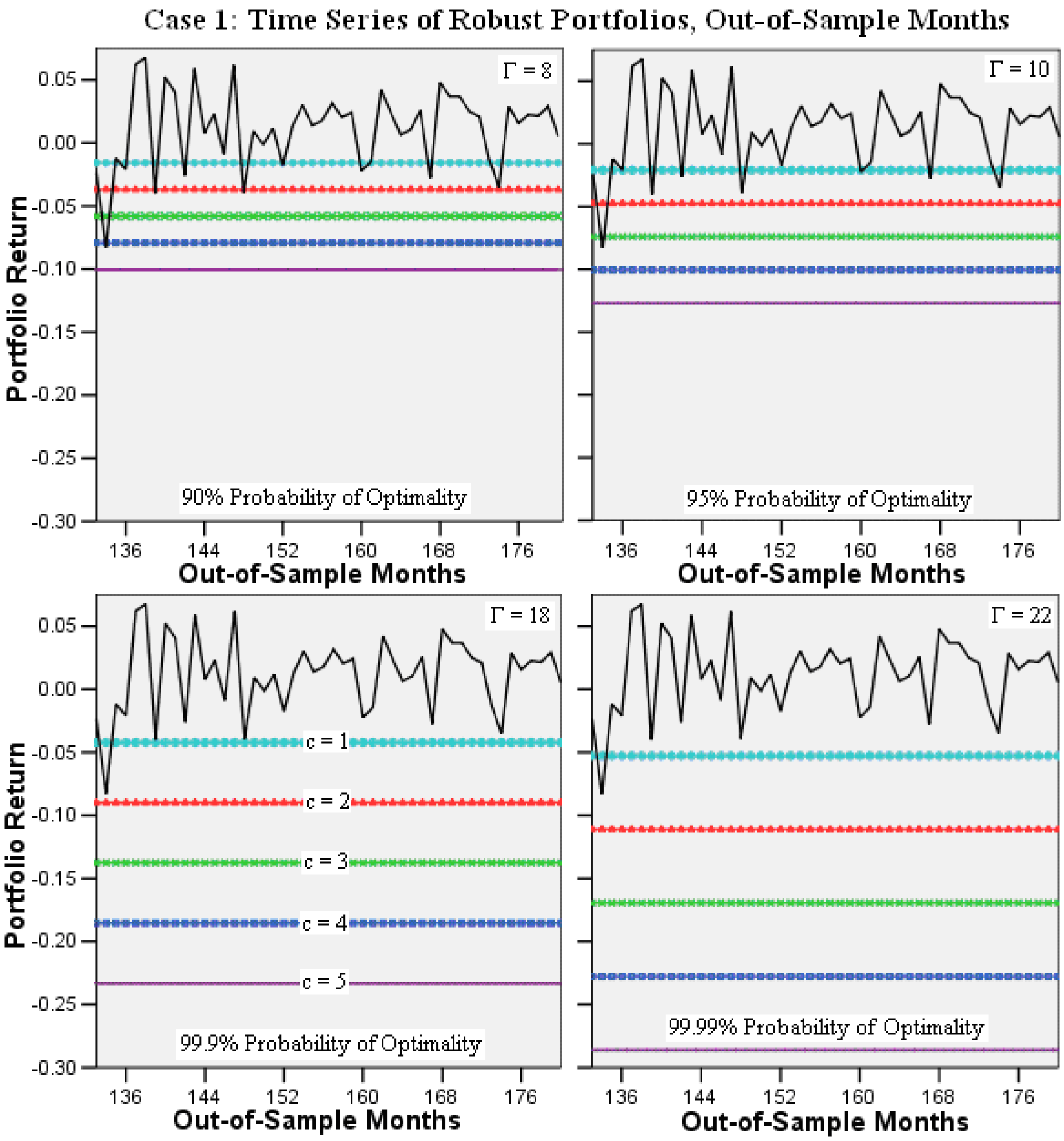

Figure 5.7. R_1 out-of-sample plot of $Z^{true}_{1,m,c}$ and $Z^{Opt}_{1,c}$ for Γ = 8, 10, 18 and 22, $\forall c$.

The guaranteed robustness and achieved robustness of model R_1 as well as the achieved robustness of the *E-V* portfolios are given in Table 5.3 (in-sample) and Table 5.4 (out-of-sample). Numerical figures shaded grey denote instances in which the percentage of portfolios that underperformed their respective robust optimal objective function value ($PLO^{Eval}_{1,c}$) was less than their guaranteed probability of underperformance ($PctLO^{Max}_{1,c}$), for corresponding c and Γ. For Case 1, a greater number of portfolios achieved or exceeded their guaranteed robustness out-of-sample than in-sample (recall that the in-sample period includes a period of poorer returns). Moreover, for every c and Γ, the actual percentage of underperformance out-of-sample was less than or equal to that of the corresponding in-sample portfolio. For example, for $\Gamma = 8$ and $c = 3$, the actual percentage of portfolios that underperformed out-of-sample was 2.08% (Table 5.4) whereas that of in-sample is 7.63% (Table 5.3).

	$PLO_{1,c}^{Max}$	$PLO_{1,c}^{Eval}$					PLO_{EV}^{Eval}	
Γ	**c = 1..5**	**c = 1**	**c = 2**	**c = 3**	**c = 4**	**c = 5**	**EV.31**	43.51%
8	10.06%	21.37%	12.21%	**7.63%**	**3.82%**	**3.05%**	**EV.27**	45.80%
10	5.02%	16.79%	9.16%	6.11%	**3.05%**	**0.76%**	**EV.23**	48.85%
18	0.10%	10.69%	3.05%	0.76%	**0.00%**	**0.00%**	**EV.19**	49.62%
22	0.01%	8.40%	2.29%	**0.00%**	**0.00%**	**0.00%**	**EV.15**	51.91%

Table 5.3. Guaranteed and achieved robustness for model R_1 and achieved robustness for *E-V* models, in-sample.

	$PLO_{1,c}^{Max}$	$PLO_{1,c}^{Eval}$					PLO_{EV}^{Eval}	
Γ	**c = 1..5**	**c = 1**	**c = 2**	**c = 3**	**c = 4**	**c = 5**	**EV.31**	35.42%
8	10.06%	20.83%	**6.25%**	**2.08%**	**2.08%**	**0.00%**	**EV.27**	39.58%
10	5.02%	16.67%	**2.08%**	**2.08%**	**0.00%**	**0.00%**	**EV.23**	39.58%
18	0.10%	2.08%	**0.00%**	**0.00%**	**0.00%**	**0.00%**	**EV.19**	39.58%
22	0.01%	2.08%	**0.00%**	**0.00%**	**0.00%**	**0.00%**	**EV.15**	43.75%

Table 5.4. Guaranteed and achieved robustness for model R_1 and achieved robustness for *E-V* models, out-of-sample.

Second, we consider whether larger *c* yielded portfolios that were *too robust*. By *too robust* we mean that $Z_{j,c}^{Opt}$ was too far below $Z_{j,m,c}^{true}$ for a given Γ. For example, consider the time series for $\Gamma = 22$. The worst realised portfolio return out-of-sample (Figure 5.7) was just above -0.09 and $Z_{1,m,c}^{true}$ for the remaining months was never below -0.05; in-sample (Figure 5.6), $Z_{1,m,c}^{true}$ was never below -0.16. However, when $c = 5$, $Z_{1,5}^{Opt} \approx -0.28$. Is an investor interested in being protected against (with a high degree of probability) realised returns 12% less than the worst $Z_{1,m,c}^{true}$? Whilst investors do seek downside protection, many do not require protection against the absolute worst return that could ever happen, i.e. a rare and extreme event. Alternatively, they want protection against worst case returns that are more likely to occur during the investment period. The robust methodology allows an investor to scale the uncertainty sets such that there is protection against what they consider to be worst case returns. It follows then, that robust portfolios in which $c \geq 4$ may have been *too robust* when $\Gamma \geq 18$, particularly out-of-sample. On the other hand, when c was less than or equal to one, robust portfolios were not robust enough (Tables 5.3 and 5.4). Thus, the uncertainty set defining r_i should be symmetric with respect to $\bar{r}_i$ by more than one standard deviation, in order to achieve the robustness guaranteed by the model out-of-sample; results suggest that an investor may wish to choose a value of c greater than or equal to 2. A risk-averse investor may well wish to choose c to equal a value of at least 3; regardless, the investor may choose to optimise the portfolio for $\Gamma \geq 18$. Although one can deliberate over the value of c to be chosen in order to yield portfolios out-of-sample that are robust, but not *too robust*, in the end, the decisions were the same for all c and $8 \leq \Gamma \leq 22$. In other words, selecting c and Γ (on the interval [8, 22]) is more a matter of trying to accurately assess risk, than it is trying to select the most robust portfolio.

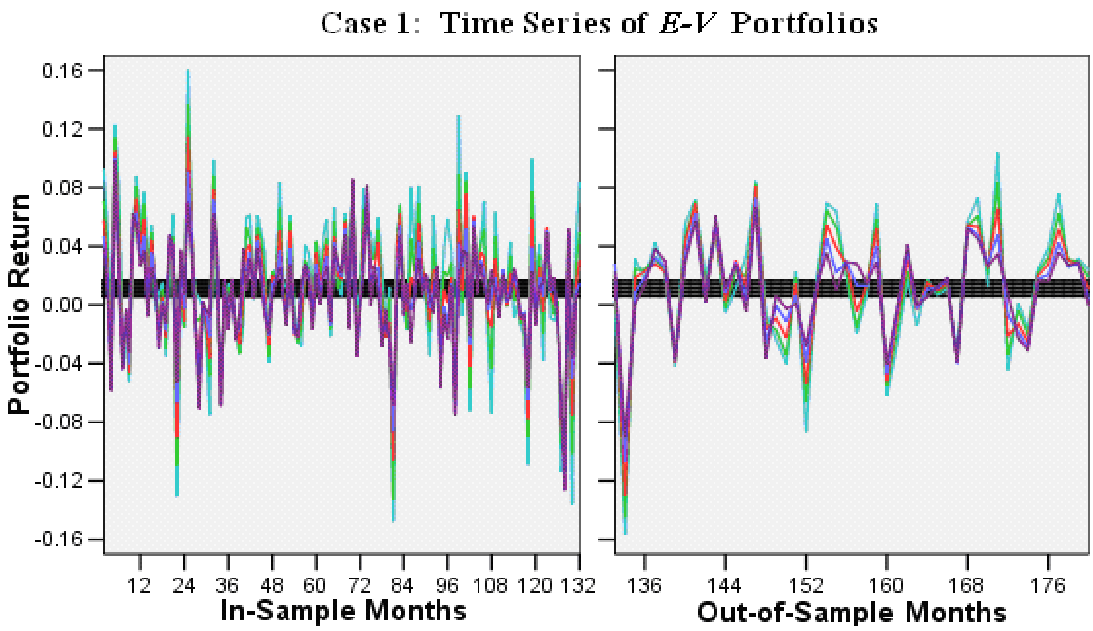

Figure 5.8. Time series of in-sample returns for all 5 E-V portfolios versus optimal objective function value (horizontal black lines).

Figure 5.9. Time series of out-of-sample returns for all 5 E-V portfolios versus optimal objective function value (horizontal black lines).

Third, we compare the robustness of robust model R_1 and *E-V* portfolios. A time series of *E-V* portfolio returns shows that the actual portfolio return, in-sample (Figure 5.8) and out-of-sample (Figure 5.9), was less than the optimal objective function value (horizontal black lines in each figure) much more often than for robust model R_1. Moreover, every *E-V* portfolio underperformed its optimal objective value over 40% of the time in-sample (Table 5.3) and over 35% of the time out-of-sample (Table 5.4), while the *least robust* robust portfolio underperformed no more than 22% of the time ($c = 1, \Gamma = 8$), in both samples.

Analysis of Robustness, In-Sample

		$PLO_{j,c}^{Max}$	$PLO_{j,c}^{Eval}$				
Model	**Γ**	**c = 1..5**	**c = 1**	**c = 2**	**c = 3**	**c = 4**	**c = 5**
***R*₂**	**8**	10.06%	15.27%	**8.40%**	**3.05%**	**1.53%**	**0.76%**
	10	5.02%	13.74%	6.11%	**1.53%**	**0.76%**	**0.00%**
	Γ_{drop} - 1	0.30%	16.79%	3.05%	**0.00%**	**0.00%**	**0.00%**
***R*₃**	**8**	10.06%	21.37%	12.21%	**7.63%**	**3.82%**	**3.05%**
	10	5.02%	16.79%	9.16%	6.11%	**3.05%**	**0.76%**
	18	0.10%	10.69%	3.05%	0.76%	**0.00%**	**0.00%**
	Γ_{drop} - 1	0.01%	8.40%	2.29%	**0.00%**	**0.00%**	**0.00%**
***R*₄**	**8**	10.06%	15.27%	**8.40%**	**3.05%**	**1.53%**	**0.76%**
	10	5.02%	13.74%	6.11%	**1.53%**	**0.76%**	**0.00%**
	Γ_{drop} - 1	0.30%	16.79%	3.05%	**0.00%**	**0.00%**	**0.00%**
***R*₅**	**8**	10.06%	19.08%	**9.92%**	**5.34%**	**2.29%**	**1.53%**
	10	5.02%	16.03%	7.63%	**3.05%**	**1.53%**	**0.76%**
	18	0.10%	9.92%	3.05%	0.76%	**0.00%**	**0.00%**
	Γ_{drop} - 1	0.01%	9.16%	2.29%	**0.00%**	**0.00%**	**0.00%**
***R*₆**	**8**	10.06%	15.27%	**8.40%**	**3.05%**	**1.53%**	**0.76%**
	10	5.02%	13.74%	6.11%	**1.53%**	**0.76%**	**0.00%**
	Γ_{drop} - 1	0.30%	16.79%	3.05%	**0.00%**	**0.00%**	**0.00%**
***R*₇**	**8**	10.06%	18.32%	**9.16%**	**3.05%**	**1.53%**	**1.53%**
	10	5.02%	15.27%	6.87%	**1.53%**	**1.53%**	**0.00%**
	18	0.10%	8.40%	1.53%	**0.00%**	**0.00%**	**0.00%**
	Γ_{drop} - 1	0.01%	8.40%	1.53%	**0.00%**	**0.00%**	**0.00%**
***R*₈**	**8**	10.06%	17.56%	**8.40%**	**3.05%**	**1.53%**	**0.00%**
	10	5.02%	14.50%	**3.82%**	**1.53%**	**0.76%**	**0.00%**
	Γ_{drop} - 1	0.30%	16.79%	3.05%	**0.00%**	**0.00%**	**0.00%**

Table 5.5. Guaranteed and achieved robustness for model R_j, for $j = 2..8$, in-sample.

Analysis of Robustness, Out-of-Sample							
		$PLO^{Max}_{j,c}$	$PLO^{Eval}_{j,c}$				
Model	**Γ**	**c = 1..5**	**c = 1**	**c = 2**	**c = 3**	**c = 4**	**c = 5**
R_2	**8**	10.06%	14.58%	**2.08%**	**2.08%**	**0.00%**	**0.00%**
	10	5.02%	8.33%	**2.08%**	**0.00%**	**0.00%**	**0.00%**
	Γ_{drop} **- 1**	0.10%	6.25%	**0.00%**	**0.00%**	**0.00%**	**0.00%**
R_3	**8**	10.06%	20.83%	**6.25%**	**2.08%**	**2.08%**	**0.00%**
	10	5.02%	16.67%	**2.08%**	**2.08%**	**0.00%**	**0.00%**
	18	0.10%	2.08%	**0.00%**	**0.00%**	**0.00%**	**0.00%**
	Γ_{drop} **- 1**	0.01%	2.08%	**0.00%**	**0.00%**	**0.00%**	**0.00%**
R_4	**8**	10.06%	14.58%	**2.08%**	**2.08%**	**0.00%**	**0.00%**
	10	5.02%	8.33%	**2.08%**	**0.00%**	**0.00%**	**0.00%**
	Γ_{drop} **- 1**	0.10%	6.25%	**0.00%**	**0.00%**	**0.00%**	**0.00%**
R_5	**8**	10.06%	14.58%	**2.08%**	**2.08%**	**0.00%**	**0.00%**
	10	5.02%	10.42%	**2.08%**	**2.08%**	**0.00%**	**0.00%**
	18	0.10%	2.08%	**0.00%**	**0.00%**	**0.00%**	**0.00%**
	Γ_{drop} **- 1**	0.01%	2.08%	**0.00%**	**0.00%**	**0.00%**	**0.00%**
R_6	**8**	10.06%	14.58%	**2.08%**	**2.08%**	**0.00%**	**0.00%**
	10	5.02%	8.33%	**2.08%**	**0.00%**	**0.00%**	**0.00%**
	Γ_{drop} **- 1**	0.10%	6.25%	**0.00%**	**0.00%**	**0.00%**	**0.00%**
R_7	**8**	10.06%	12.50%	**2.08%**	**2.08%**	**0.00%**	**0.00%**
	10	5.02%	8.33%	**2.08%**	**0.00%**	**0.00%**	**0.00%**
	18	0.10%	2.08%	**0.00%**	**0.00%**	**0.00%**	**0.00%**
	Γ_{drop} **- 1**	0.01%	2.08%	**0.00%**	**0.00%**	**0.00%**	**0.00%**
R_8	**8**	10.06%	12.50%	**2.08%**	**2.08%**	**0.00%**	**0.00%**
	10	5.02%	6.25%	**2.08%**	**0.00%**	**0.00%**	**0.00%**
	Γ_{drop} **- 1**	0.10%	6.25%	**0.00%**	**0.00%**	**0.00%**	**0.00%**

Table 5.6. Guaranteed and achieved robustness for model R_j, for $j = 2..8$, out-of-sample.

Finally, we compare the robustness of R_1 with robust portfolios constrained by cardinality and/or a buy-in threshold (R_2 to R_8). For non-cardinality constrained models (R_3, R_5 and R_7) we considered portfolios corresponding to Γ values of 8, 10, 18 and $\Gamma_{drop} - 1$. For cardinality constrained models (R_2, R_4, R_6 and R_8), $\Gamma_{drop} - 1$ was less than 18; thus, we only considered portfolios corresponding to Γ values of 8, 10 and $\Gamma_{drop} - 1$. Numerical figures shaded grey in Table 5.5 (in-sample) and Table 5.6 (out-of-sample) indicate instances in which the percentage of portfolios that underperformed the optimal objective ($PLO_{j,c}^{Eval}$) was less than the probability of underperformance ($PLO_{j,c}^{Max}$).

Observe that in-sample (Table 5.5), cardinality constrained models were more robust than those without cardinality constraints. On the other hand, out-of-sample (Table 5.6), there was not much difference between the achieved robustness of each model for corresponding values of *c*. A comparison with R_1 in-sample (Tables 5.3) and out-of-sample (Table 5.4) shows that the inclusion of cardinality constraints improved achieved robustness ($PLO_{1,c}^{Eval}$), for all values of *c*. Similarly, the inclusion of threshold constraints improved achieved robustness in all instances when $c \geq 3$, but only a few instances when $c \leq 2$.

5.3 Case 2

For Case 2, the in-sample dataset, consisting of 108 monthly log returns for 30 assets from the FTSE 100 from 1 January 1992 through to 1 December 2000, was used to estimate $\bar{r}_i$ (the mean log return of each asset), $\hat{r}_i$ (the standard deviation of asset *i*) and the covariance matrix for the *E-V* model. The out-of-sample dataset consisted of 72 monthly log returns from 1 January 2001 through to 1 December 2006. Case 2 reserves the time period spanning from 2001 through to 2002 for the out-of-sample analysis. We optimised 1240 robust portfolios and 31 *E-V* portfolios, and evaluated 120 robust portfolios and 5 *E-V* portfolios.

5.3.1 Analysis of Cost

*Cost*1 (Figure 5.10) and *Cost*2 (Figure 5.11) were calculated for each R_j for Γ values of 8, 10, 18 and $\Gamma_{drop} - 1$; the scale markers along the x-axis at every R_j and between successive R_j represent each value of Γ. As in Case 1, $\Gamma_{drop} - 1$ was less than 18 for robust models with cardinality constraints; thus for R_2, R_4, R_6 and R_8 only three values of Γ (8, 10 and $\Gamma_{drop} - 1$) are shown for *Cost*1 and *Cost*2. *Cost*1, both in-sample and out-of-sample, is shown in Figure 5.10 for each value of *c*. *Cost*2 is shown in Figure 5.3 (in-sample) and Figure 5.4 (out-of-sample), for each value of *c*. In each Figure, if either respective cost is the same for two or more values of *c*, or for two or more values of R_j and Γ, it indicates that the robust optimal decisions were the same. For example, *Cost*1 for R_3 is the same for all *c* and all Γ, which indicates that the optimal decision vector $\boldsymbol{w}^*$ was the same for each of those 20 portfolios (4 values of Γ and 5 values of *c*).

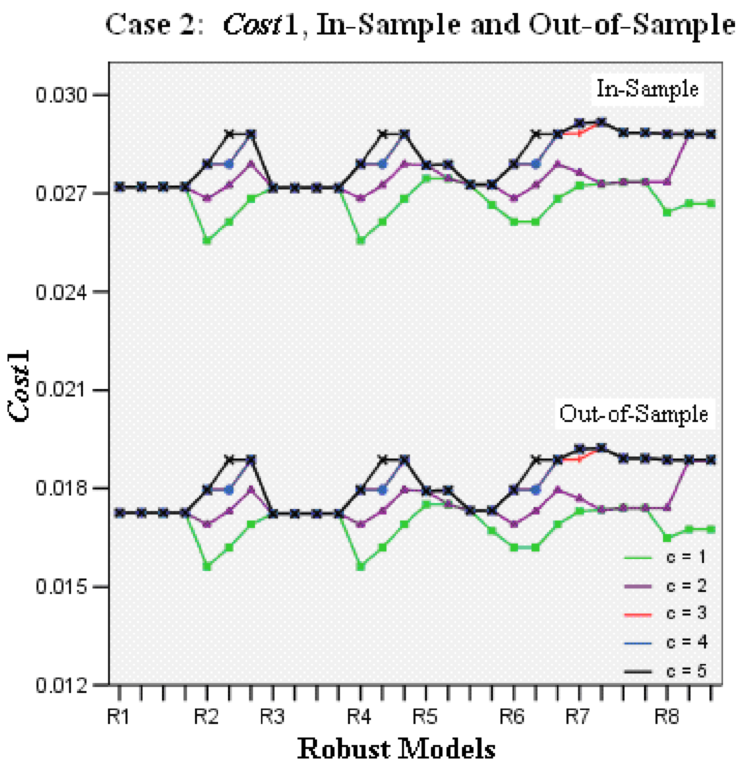

Figure 5.10. *Cost*1 for all robust models at specific values of Γ, for all c, both in-sample and out-of-sample.

First, we consider the effect that *c* has on cost. As stated in Case 1, an increase in the magnitude of the range of the uncertainty set for each asset *i* means that the worst case value of each asset will decrease and the robust optimal objective will deteriorate. Thus, as *c* increases, we would expect the total portfolio return to be more conservative, corresponding to greater costs. Observe that for a given R_j, an increase in *c* corresponded to either the same costs or an increase in costs (Figures

5.10, 5.11, and 5.12). This is a similar relationship to that observed in Case 1, given by (5.5) and (5.6) in Section 5.2.1.

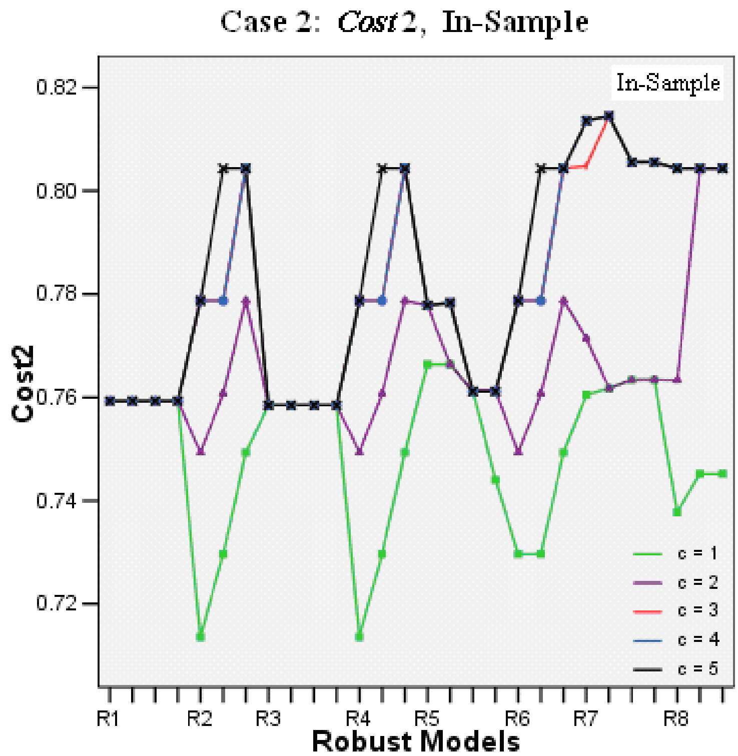

Figure 5.11. *Cost*2 for all robust models at specific values of Γ, for all *c*, in-sample.

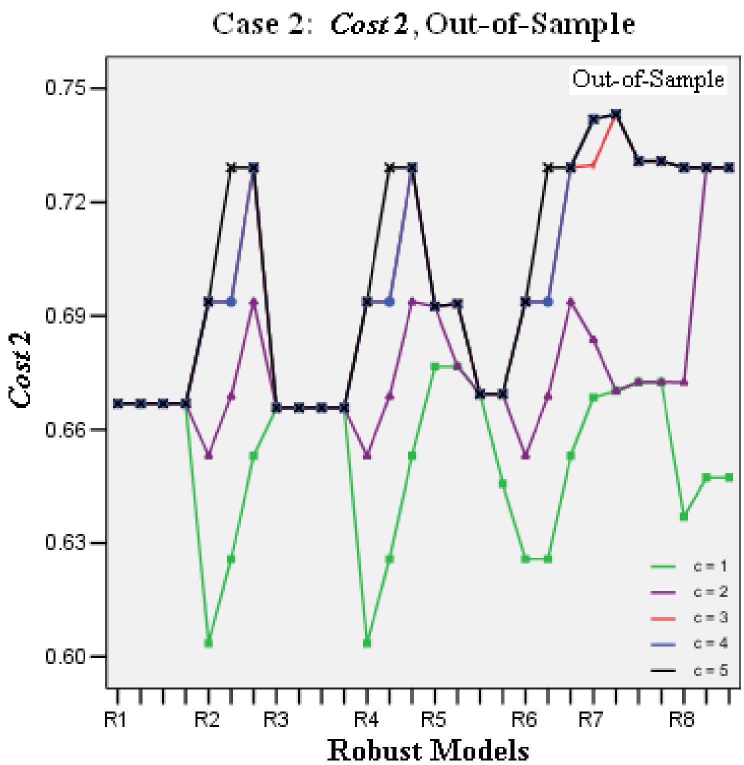

Figure 5.12. *Cost*2 for all robust models at specific values of Γ, for all *c*, out-of-sample.

Second, we compare the cost of model R_1 with robust models constrained by cardinality and/or a buy-in threshold. In comparison to R_1, cardinality constraints tended to decrease costs for values of $c \leq 1$, but increase costs for values of $c \geq 2$ (Figure 5.10). Threshold constraints that do not constrain the cardinality of the portfolio (i.e. $\alpha = 0.02, 0.03$, corresponding to models R_3 and R_5 respectively) tended to result in very similar costs as obtained for model R_1 (Figures 5.10, 5.11 and 5.12). In addition, because the out-of-sample plot of *Cost*1 has the same shape as that of *Cost*1 in-sample (Figure 5.10), it is a very good indication of which models will be more costly in the future, and an investor can act accordingly. Thus, with this particular set of data, in-sample results for *Cost*1 (Figure 5.10) and *Cost*2 (Figure 5.11) indicate that in order to reduce costs out-of-sample, a more risk seeking investor, one who restricts the true log return of asset i to lie within a smaller interval (indicated by smaller values of c), should only include cardinality constraints if $c = 1$. However, a risk-averse investor, one who defines the true log return of asset i to lie within a larger interval (indicated by larger values of c), should either use the unconstrained model (R_1), or only include threshold constraints which do not impose cardinality restrictions, in order to avoid increasing costs out-of-sample.

Third, we compare in-sample and out-of-sample costs. Both *Cost*1 and *Cost*2 were greater in-sample than they were out-of-sample (Figures 5.10, 5.11 and 5.12). Recall that the out-of-sample dataset included a period of higher volatility and poorer returns for many assets compared to the in-sample dataset. In addition, both costs are measured with respect to r^{MMax}, which was greater for the in-sample period, and both in-sample and out-of-sample costs were calculated using the same $P_{j,c}^{Total}$, $\forall j, c$; thus, it follows that all R_j would cost more in-sample.

Finally, we compare the costs of the unconstrained robust model (R_1) to the costs of *E-V* portfolios. Recall that portfolio EV.31 corresponds to the minimum variance portfolio, which also has the smallest portfolio return. Each subsequent portfolio, EV.27-EV.15 from the efficient frontier (Figure 5.13), has a greater portfolio return, greater variance and less cost. In addition, these four portfolios cost less than R_1, whereas the minimum variance portfolio, EV.31, costs more (although by a very small margin), both in-sample and out-of-sample (Table 5.7). The analysis of robustness in Section 5.3.2, will help to determine whether incurring greater costs (associated with R_1) has been in exchange for greater achieved robustness.

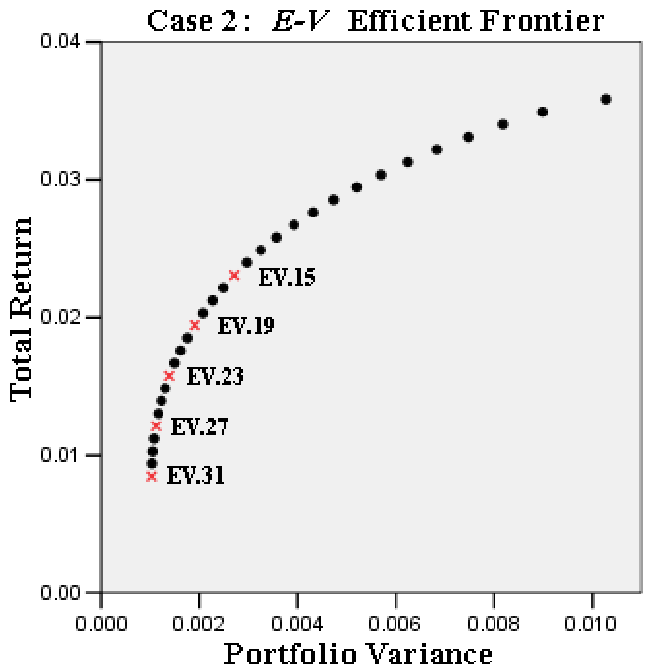

Figure 5.13. *E-V* portfolios selected from the efficient frontier.

	Cost1		*Cost2*	
	In-Sample	**Out-of-Sample**	**In-Sample**	**Out-of-Sample**
R_1	0.0272	0.0173	0.7593	0.6668
EV.31	0.0274	0.0174	0.7642	0.6737
EV.27	0.0237	0.0138	0.6623	0.5327
EV.23	0.0201	0.0101	0.5604	0.3916
EV.19	0.0164	0.0065	0.4586	0.2506
EV.15	0.0128	0.0028	0.3567	0.1096

Table 5.7. *Cost*1 and *Cost*2, in-sample and out-of-sample, for all 5 *E-V* portfolios and model R_1.

5.3.2 Analysis of Robustness

First, we evaluate whether guaranteed robustness was achieved as well as how robustness is affected by the scaling factor *c*. As in Case 1, the realised portfolio return of R_1 ($Z_{1,m,c}^{true}$), for all *c*, was the same for each $\Gamma \in \{8, 10, 18, 22\ (\Gamma_{drop} - 1)\}$, both in-sample (Figure 5.14) and out-of-sample (Figure 5.15). In addition, for a given Γ and scaling factor c, the robust optimal objective value of R_1 ($Z_{1,c}^{Opt}$) was held constant over a sample period; thus only one horizontal line is plotted for each c (Figures 5.14 and 5.15). The same scale is used for all four in-sample time-series (Figure 5.14), likewise for all four out-of-sample time-series (Figure 5.15).

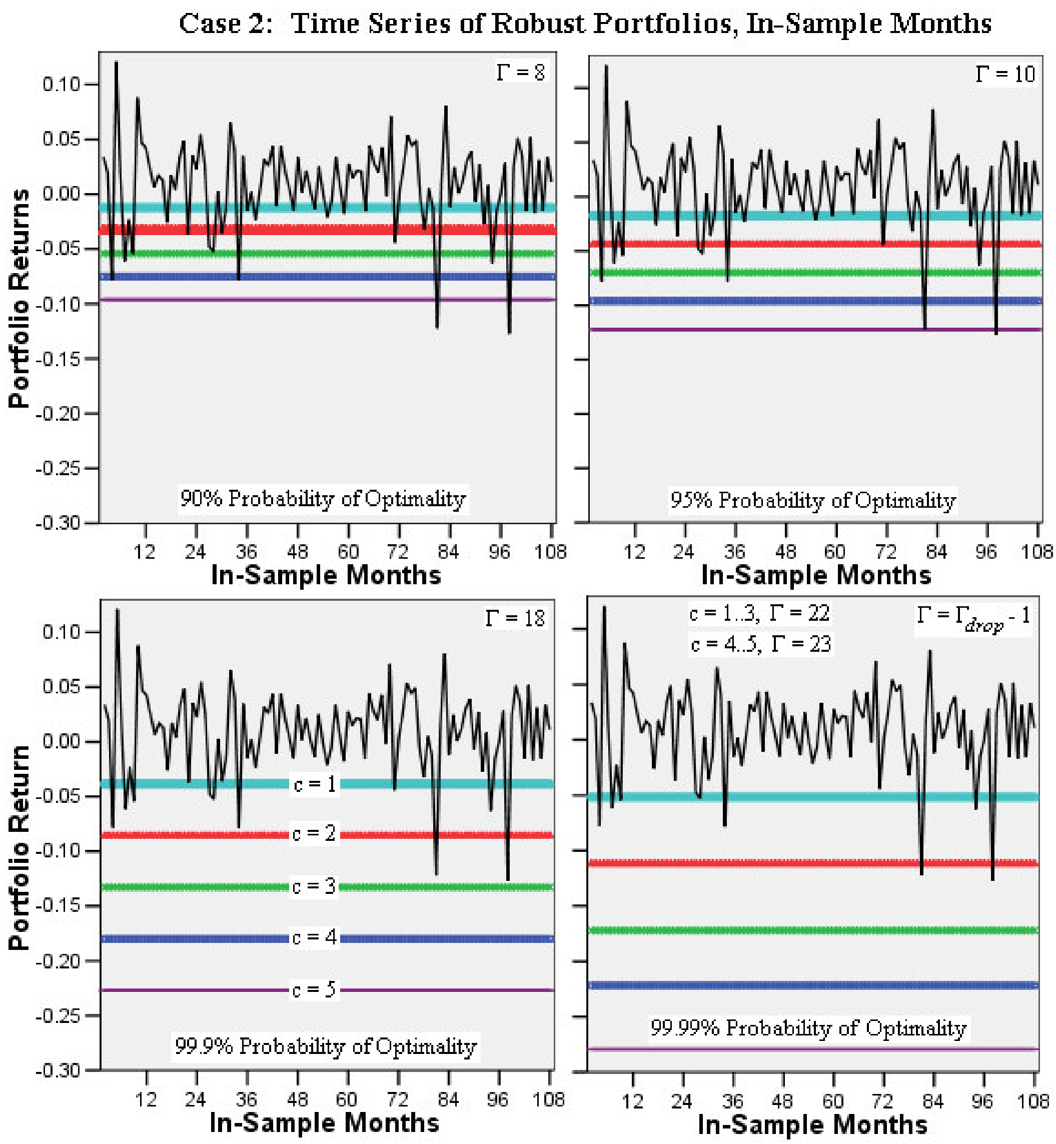

Figure 5.14. R_1 in-sample plots of $Z_{1,m,c}^{true}$ and $Z_{1,c}^{Opt}$ for Γ = 8, 10, 18 and 22, $\forall c$.

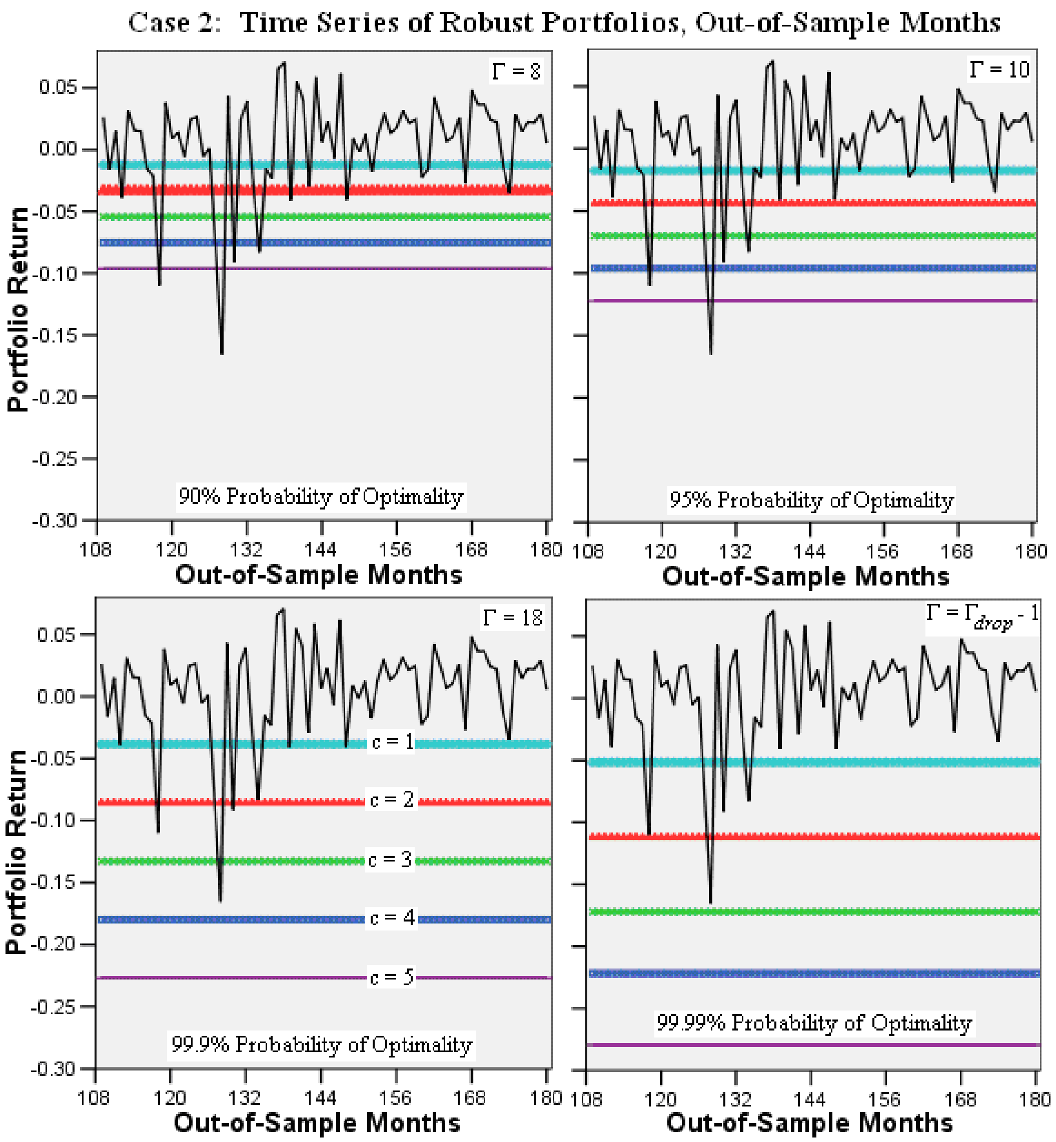

Figure 5.15. R_1 out-of-sample plot of $Z_{1,m,c}^{true}$ and $Z_{1,c}^{Opt}$ for $\Gamma = 8$, 10, 18 and $\Gamma_{drop} - 1$, $\forall c$.

As larger Γ correspond to greater probabilities of optimality, it was expected that larger Γ would yield more robust portfolios. The actual robustness of each portfolio was measured by comparing the percentage of returns that dipped below the line $Z_{1,c}^{Opt}$ for each c, with the probability of underperformance (1 – [Probability of optimality]). Observe that an increase in c, Γ or both, increased achieved robustness (Figures 5.14 and 5.15 and Tables 5.8 and 5.9). In addition, more robust portfolios achieved their guaranteed probability of optimality in-sample.

The guaranteed robustness and achieved robustness of model R_1 as well as the achieved robustness of the *E*-*V* portfolios are given in Table 5.8 (in-sample) and

Table 5.9 (out-of-sample). Numerical figures shaded grey denote instances in which the percentage of portfolios that underperformed their respective robust optimal objective function value ($PLO_{1,c}^{Eval}$) was less than their guaranteed probability of underperformance ($PctLO_{1,c}^{Max}$), for corresponding c and Γ. For Case 2, a greater number of portfolios achieved or exceeded their guaranteed robustness in-sample than out-of-sample. Moreover, for every c and Γ, the actual percentage of underperformance in-sample was less than or equal to that of the corresponding out-of-sample portfolio. For example, for Γ = 10 and c = 4, the actual percentage of portfolios that underperformed in-sample was 1.87% (Table 5.8) whereas that of out-of-sample was 2.78% (Table 5.9).

	$PLO_{1,c}^{Max}$	$PLO_{1,c}^{Eval}$					PLO_{EV}^{Eval}	
Γ	**c = 1..5**	**c = 1**	**c = 2**	**c = 3**	**c = 4**	**c = 5**		
8	10.062%	27.10%	11.21%	**6.54%**	**3.74%**	**1.87%**	**EV.31**	46.73%
10	5.017%	17.76%	9.35%	**3.74%**	**1.87%**	**0.93%**	**EV.27**	46.73%
18	0.096%	9.35%	1.87%	**0.00%**	**0.00%**	**0.00%**	**EV.23**	44.86%
22	0.003%	7.48%	1.87%	**0.00%**			**EV.19**	49.53%
23	0.006%				**0.00%**	**0.00%**	**EV.15**	49.53%

Table 5.8. Guaranteed versus achieved robustness for model R_1 and achieved robustness for *E-V* models, in-sample. Note $\Gamma_{drop} - 1$ is different for c = 1..3 versus c = 4..5.

	$PLO_{1,c}^{Max}$	$PLO_{1,c}^{Eval}$					PLO_{MV}^{Eval}	
Γ	**c = 1..5**	**c = 1**	**c = 2**	**c = 3**	**c = 4**	**c = 5**		
8	10.062%	29.17%	12.50%	**6.94%**	**6.94%**	**2.78%**	**EV.31**	47.22%
10	5.017%	22.22%	6.94%	6.94%	**2.78%**	**1.39%**	**EV.27**	52.78%
18	0.096%	11.11%	4.17%	1.39%	**0.00%**	**0.00%**	**EV.23**	55.56%
22	0.003%	6.94%	1.39%	**0.00%**			**EV.19**	59.72%
23	0.006%				**0.00%**	**0.00%**	**EV.15**	68.06%

Table 5.9. Guaranteed versus achieved robustness for model R_1 and achieved robustness for *E-V* models, out-of-sample. Note $\Gamma_{drop} - 1$ is different for c = 1..3 and c = 4..5.

Second, we consider whether larger c yield portfolios that were *too robust*. As discussed in Section 5.2.2, by *too robust* we mean that $Z_{j,c}^{Opt}$ was too far below $Z_{j,m,c}^{true}$ for a given Γ. It follows that robust portfolios in which $c \geq 4$ may have been *too robust* when $\Gamma \geq 18$. On the other hand, when $c \leq 2$, robust portfolios were not robust enough (Tables 5.8 and 5.8). Thus, the uncertainty set defining r_i should be symmetric with respect to $\bar{r}_i$ by at least three standard deviations, in order to achieve the robustness guaranteed by the model out-of-sample. Results suggest that an investor would wish to choose a value of c greater than 2. A risk-averse investor may well wish to choose c to equal a value of at least 2; regardless, the investor may choose to optimise the portfolio for $\Gamma \geq 18$. Just as in Case 1, although one can deliberate over the value of c to be chosen in order to yield portfolios out-of-sample that are robust, but not *too robust*, in the end, the decisions were the same for all c and $8 \leq \Gamma \leq 22$. In other words, selecting c and Γ (on the interval [8, 22]) is more a matter of trying to accurately assess risk, than it is trying to select the most robust portfolio.

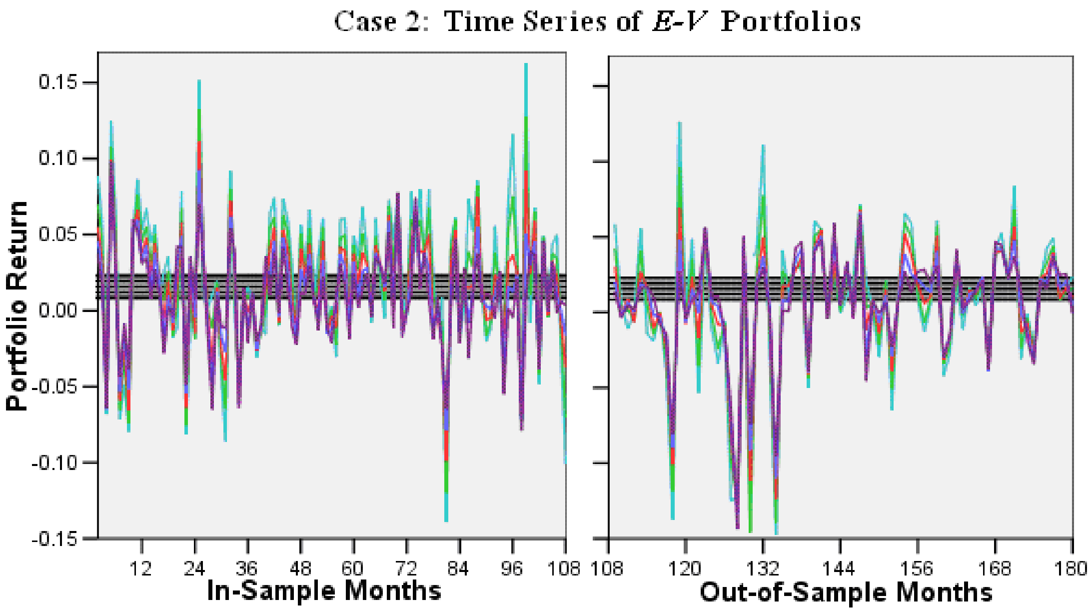

Figure 5.16. Time series of in-sample returns for all 5 *E-V* portfolios versus optimal objective function value (horizontal black lines).

Figure 5.17. Time series of out-of-sample returns for all 5 *E-V* portfolios versus optimal objective function value (horizontal black lines).

Third, we compare the robustness of robust model R_1 and *E-V* portfolios. A time series of *E-V* portfolio returns shows that the actual portfolio return, in-sample (Figure 5.16) and out-of-sample (Figure 5.17), was less than the optimal objective function value (horizontal black lines in either figure) much more often than for robust model R_1. Every *E-V* portfolio underperformed its optimal objective value over 46% of the time in-sample (Table 5.16) and over 37% of the time out-of-sample

(Table 5.17), while the *least robust* robust portfolio underperformed no more than 30% of the time, in both samples.

Analysis of Robustness, In-Sample

		$PLO_{j,c}^{Max}$	$PLO_{j,c}^{Eval}$				
Model	**Γ**	**c = 1..5**	**c = 1**	**c = 2**	**c = 3**	**c = 4**	**c = 5**
R_2	**8**	10.06%	16.82%	**7.48%**	**2.80%**	**0.93%**	**0.00%**
	10	5.02%	14.95%	**4.67%**	**0.93%**	**0.00%**	**0.00%**
	Γ_{drop} **- 1**	0.10%	14.02%	1.87%	**0.00%**	**0.00%**	**0.00%**
R_3	**8**	10.06%	27.10%	11.21%	**5.61%**	**3.74%**	**1.87%**
	10	5.02%	16.82%	9.35%	**3.74%**	**1.87%**	**0.93%**
	18	0.10%	9.35%	1.87%	**0.00%**	**0.00%**	**0.00%**
	Γ_{drop} **- 1**	0.01%	7.48%	1.87%	**0.00%**	**0.00%**	**0.00%**
R_4	**8**	10.06%	16.82%	**7.48%**	**2.80%**	**0.93%**	**0.00%**
	10	5.02%	14.95%	**4.67%**	**0.93%**	**0.00%**	**0.00%**
	Γ_{drop} **- 1**	0.10%	14.02%	1.87%	**0.00%**	**0.00%**	**0.00%**
R_5	**8**	10.06%	19.63%	**9.35%**	**2.80%**	**1.87%**	**1.87%**
	10	5.02%	18.69%	7.48%	**1.87%**	**1.87%**	**0.00%**
	18	0.10%	9.35%	1.87%	**0.00%**	**0.00%**	**0.00%**
	Γ_{drop} **- 1**	0.01%	7.48%	1.87%	**0.00%**	**0.00%**	**0.00%**
R_6	**8**	10.06%	17.76%	**7.48%**	**2.80%**	**0.93%**	**0.00%**
	10	5.02%	14.95%	**4.67%**	**0.93%**	**0.00%**	**0.00%**
	Γ_{drop} **- 1**	0.10%	14.02%	1.87%	**0.00%**	**0.00%**	**0.00%**
R_7	**8**	10.06%	17.76%	**7.48%**	**2.80%**	**0.93%**	**0.93%**
	10	5.02%	15.89%	**4.67%**	**1.87%**	**0.93%**	**0.00%**
	18	0.10%	6.54%	0.93%	**0.00%**	**0.00%**	**0.00%**
	Γ_{drop} **- 1**	0.01%	6.54%	0.93%	**0.00%**	**0.00%**	**0.00%**
R_8	**8**	10.06%	17.76%	**6.54%**	**2.80%**	**0.93%**	**0.00%**
	10	5.02%	14.02%	**4.67%**	**0.93%**	**0.00%**	**0.00%**
	Γ_{drop} **- 1**	0.10%	14.02%	1.87%	**0.00%**	**0.00%**	**0.00%**

Table 5.10. Guaranteed and achieved robustness for model R_j, for j = 2..8, in-sample.

Finally, we compare the robustness of R_1 to that of robust portfolios constrained by cardinality and/or a buy-in threshold (R_2 to R_8). For non-cardinality constrained models (R_3, R_5 and R_7) we considered portfolios corresponding to Γ values of 8, 10, 18 and $\Gamma_{drop} - 1$. For cardinality constrained models (R_2, R_4, R_6 and R_8), $\Gamma_{drop} - 1$ was less than 18, thus, we only considered portfolios corresponding to Γ values of 8, 10 and $\Gamma_{drop} - 1$. Numerical figures shaded grey in Table 5.10 (in-sample) and Table 5.11 (out-of-sample) indicate instances in which the percentage of portfolios that underperformed the optimal objective ($PLO_{j,c}^{Eval}$) was less than the probability of underperformance ($PLO_{j,c}^{Max}$).

Analysis of Robustness, Out-of-Sample							
		$PLO_{j,c}^{Max}$	$PLO_{j,c}^{Eval}$				
Model	**Γ**	**c = 1..5**	**c = 1**	**c = 2**	**c = 3**	**c = 4**	**c = 5**
R_2	**8**	10.06%	19.44%	**6.94%**	**5.56%**	**1.39%**	**1.39%**
	10	5.02%	11.11%	6.94%	**1.39%**	**1.39%**	**0.00%**
	Γ_{drop} **- 1**	0.10%	9.72%	1.39%	**0.00%**	**0.00%**	**0.00%**
R_3	**8**	10.06%	29.17%	12.50%	**6.94%**	**6.94%**	**2.78%**
	10	5.02%	22.22%	6.94%	6.94%	**2.78%**	**1.39%**
	18	0.10%	11.11%	4.17%	1.39%	**0.00%**	**0.00%**
	Γ_{drop} **- 1**	0.01%	6.94%	1.39%	**0.00%**	**0.00%**	**0.00%**
R_4	**8**	10.06%	19.44%	**6.94%**	**5.56%**	**1.39%**	**1.39%**
	10	5.02%	11.11%	6.94%	**1.39%**	**1.39%**	**0.00%**
	Γ_{drop} **- 1**	0.10%	9.72%	1.39%	**0.00%**	**0.00%**	**0.00%**
R_5	**8**	10.06%	22.22%	**8.33%**	**6.94%**	**4.17%**	**1.39%**
	10	5.02%	16.67%	6.94%	5.56%	**1.39%**	**1.39%**
	18	0.10%	6.94%	4.17%	1.39%	**0.00%**	**0.00%**
	Γ_{drop} **- 1**	0.01%	6.94%	1.39%	1.39%	**0.00%**	**0.00%**
R_6	**8**	10.06%	16.67%	**6.94%**	**5.56%**	**1.39%**	**1.39%**
	10	5.02%	11.11%	6.94%	**1.39%**	**1.39%**	**0.00%**
	Γ_{drop} **- 1**	0.10%	9.72%	1.39%	**0.00%**	**0.00%**	**0.00%**
R_7	**8**	10.06%	19.44%	**6.94%**	**5.56%**	**1.39%**	**1.39%**
	10	5.02%	13.89%	6.94%	**2.78%**	**1.39%**	**1.39%**
	18	0.10%	6.94%	1.39%	**0.00%**	**0.00%**	**0.00%**
	Γ_{drop} **- 1**	0.01%	6.94%	1.39%	**0.00%**	**0.00%**	**0.00%**
R_8	**8**	10.06%	15.28%	**6.94%**	**2.78%**	**1.39%**	**1.39%**
	10	5.02%	9.72%	6.94%	**1.39%**	**1.39%**	**0.00%**
	Γ_{drop} **- 1**	0.10%	9.72%	1.39%	**0.00%**	**0.00%**	**0.00%**

Table 5.11. Guaranteed and achieved robustness for model R_j, for j = 2..8, out-of-sample.

Observe that in-sample (Table 5.10) and out-of-sample (Table 5.11), cardinality constrained portfolios were more robust than models without cardinality constraints. A comparison with R_1 in-sample (Table 5.8) and out-of-sample (Table 5.9) shows that the inclusion of cardinality constraints improved achieved robustness ($PLO_{j,c}^{Eval}$), for all values of *c*. Similarly, the inclusion of buy-in threshold constraints improved achieved robustness in all instances when $c \geq 3$, but in only a few instances when $c \leq 2$.

5.4 Discussion

Increasing *c* increases the magnitude of the range of the uncertainty set; thus, decreasing the worst case log return of each asset. It makes sense then that an increase in *c* corresponded to a decrease in the robust optimal objective function value, resulting in increased costs (measures of cost were introduced in Section 5.2.1 in (5.5) and (5.6)). In addition, an increase in *c* decreased the actual probability of underperformance. Therefore, the likelihood that a portfolio would achieve the robustness guaranteed by the model increased as the magnitude of the range of the uncertainty set was increased.

While a change in *c* affects achieved robustness, a change in Γ affects both guaranteed and achieved robustness. Results show that as Γ increased, given that $\Gamma \leq \Gamma_{drop} - 1$, the probability of underperformance decreased, as well as the actual proportion of portfolios which underperformed the robust optimal objective function value. Results also showed that smaller Γ (e.g. 8 or 10) required larger values of *c* in order to achieve guaranteed robustness.

The inclusion of threshold and cardinality constraints proved advantageous with respect to model R_1, in terms of robustness. Both in-sample and out-of-sample results suggest that models constrained by cardinality, a buy-in threshold, or both, were at least as robust as the unconstrained model R_1. In other words, the probability of underperformance was equal to or less than that of model R_1, for corresponding values of *c* and Γ. With respect to cost, we observed that threshold constraints which did not limit cardinality had costs similar to those of model R_1; but if the threshold constraint did limit cardinality, then the costs were similar to those of the cardinality constrained models. In addition, unlike model R_1, the costs of cardinality constrained models were greatly affected by changing *c*; in-sample and out-of-sample, the two or three smallest values of *c* resulted in lower costs (except when $\Gamma = \Gamma_{drop} - 1$) and the two or three largest values of *c* resulted in higher costs.

Although the relationship between the costs of each model can be somewhat difficult to explain in brief, a plot of *Cost*1 and *Cost*2 effectively depicts their relationship. Because *Cost*1 is measured with respect to $P_{j,c}^{Total}$, which is the same in-sample and out-of-sample, the shape of the plot will be the same for both samples; only the position of the plot will change out-of-sample (seen by a parallel shift up or down). Similarly, *Cost*2 has a similar shape in-sample and out-of-sample, but because it is a proportion measured with respect to r^{MMax}, which is different for each sample, the out-of-sample plot will not only be shifted to be parallel, but also will be vertically skewed with respect to the in-sample plot.

Results suggest that the unconstrained robust model (R_1) has costs similar to those of the minimum variance *E-V* portfolio (in-sample and out-of-sample). In Case 1, *Cost*2 of the robust model was only approximately 3% more, and in Case 2 approximately 0.5% less, than the minimum variance portfolio. Although the costs of model R_1 and EV.31 were similar, the robust model was much more robust, both in-sample and out-of-sample. Results showed that, in-sample, the *most robust E-V* portfolio underperformed its optimal objective function value 40% of the time (Case 1) and 46% of the time (Case 2). Likewise, out-of-sample, the *most robust E-V* portfolio underperformed 35% of the time (Case 1) and 37% of the time (Case 2). Compare these results with the *least robust* of the robust portfolios (Γ = 8, *c* = 1), which underperformed its robust optimal objective function value less than 22% of the time (Case 1) and 30% of the time (Case 2) in both samples. Our results strongly suggest that the unconstrained robust model is both cost-effective and much more robust than *E-V* portfolios.

Back-testing for each Case provided insight into how robust models perform given different sets of historical data for both the modelling and evaluation time periods. In Case 1, 2001 through to 2002 was captured within the in-sample period and we observed that this resulted in portfolios which were more costly, but also more robust (with respect to $PLO_{j,c}^{Eval}$) out-of-sample. In Case 2, 2001 through to 2002 was reserved for the out-of-sample period. We observed that this resulted in portfolios which were less costly and also slightly less robust (with respect to $PLO_{j,c}^{Eval}$) out-of-sample, however, all portfolios achieved their guaranteed robustness when $c \geq 3$.

Lastly, we consider whether robust portfolios are *too robust* for any particular values of Γ and c. That is, is there too much of a difference between the optimal objective function value and the actual portfolio returns within a sample period? Results suggest that a value of c greater than one and a value of Γ corresponding to a probability of optimality of at least 99% should be chosen. Results also suggest that if $c \geq 4$ the portfolio will likely be too robust, particularly if the in-sample period used to optimise the model includes an economic downturn. We also observed that when Γ was on the interval [8,22], regardless of the choice of c, every portfolio was the same. Therefore, if decisions are the same for corresponding Γ for different values of c, increasing c may not affect the composition of the portfolios, but will result in a more robust solution. This is an important property of this robust methodology. It suggests that for both a risk-averse and more risk seeking investor, it is likely that the selected portfolio will be the same. It is only the robust optimal objective function value, which is the standard by which portfolio robustness is measured, that is affected by an investor's disposition towards risk.

Chapter 6

Properties of Robust Portfolios

In this chapter we discuss the properties of robust portfolios. We investigate the components of a portfolio and its return with respect to diversity based on the total number of assets, the size of the historical dataset (or number of observations) and desired level of guaranteed probability of optimality. We also examine whether these properties hold when threshold or cardinality constraints are included.

In Sections 6.1, 6.2 and 6.3 we discuss the properties of unconstrained robust models and in Section 6.4 we discuss the differences and similarities in these properties when threshold and/or cardinality constraints are introduced. Five different sets of assets, taken from four indices, were used to investigate the properties of unconstrained robust models (Table 6.1). From each stock market index, only assets that had prices available for the entire time period were chosen; hence, the number of assets in each dataset is less than the size of the corresponding index. In one case, dataset 1, we consider a subset of 30 out of a possible 68 assets from the FTSE 100. This dataset was used to investigate threshold and cardinality constrained robust models, since these models are mixed integer programs, which are difficult to solve with larger sets of assets. In addition, for the sake of simplicity we have primarily used dataset 1 to illustrate the properties of robust models when their properties are the same for all five datasets.

Dataset	1	2	3	4	5
Index	FTSE 100	FTSE 100	FTSE250	FTSE350	S&P500
No. Assets	30	68	169	248	441
Time Period	1 Jan 1992- 1 Dec 2002	1 Feb 1996- 1 Jan 2007	1 Oct 1998- 1 Sept 2008	1 Oct 1998- 1 Sept 2008	1 Oct 1998- 1 Sept 2008

Table 6.1. Summary of datasets.

6.1 Portfolio Composition

Within the framework of robust optimisation, asset returns are bounded by an uncertainty set *U*. Throughout this chapter the interval for the true return of asset *i*, r_i, was given as $r_i \in [\bar{r}_i - c\hat{r}_i, \bar{r}_i + c\hat{r}_i]$. In addition, we defined $\bar{r}_i$ as the mean log return of asset *i* and $\hat{r}_i$ as the standard deviation of asset *i*. In some instances, the scaling factor *c*, for which the model was optimised, is specified. However, in many instances it is not, because the particular value of *c* used is not relevant.

Diversification. Consider *N* + 1 consecutive portfolios corresponding to integer values of Γ from 0 to *N*. When $\Gamma = 0$, the portfolio consisted of 1 asset; this is simply the maximum return problem with no robustness. As Γ increased, the number of assets increased until a maximum number of assets was reached, which in most cases was *N*. From this point, the composition of portfolios for successive values of Γ remained constant until all but 1 asset were dropped, corresponding to $\Gamma = \Gamma_{drop}$. Lastly, for $\Gamma \geq \Gamma_{drop}$ the optimal portfolio consisted of the asset with the largest risk-adjusted return, $\bar{r}_i - c\hat{r}_i$. This behaviour is shown for dataset 1 in Figure 6.1 ($N = 30$ assets from the FTSE 100), and dataset 4 in Figure 6.2 ($N = 248$ assets from the FTSE 350).

In Figure 6.2, a plot of the number of assets held at each Γ is shown for three different models (corresponding to *c* = 2, 3 and 4) which were optimised using dataset 4. As *c* increased from 2 to 4, the number of assets selected converged to *N* (248) sooner (i.e. at a smaller value of Γ) and the portfolio with *N* assets was held for larger values of $\Gamma_{drop} - 1$. Thus, when *c* was larger, more portfolios consist of all *N* assets. These results suggest that when assets are defined by smaller uncertainty sets a wider variety of portfolios are offered, particularly for smaller values of Γ. In addition, for all three values of c, the portfolios which consisted of all 248 assets had exactly the same composition.

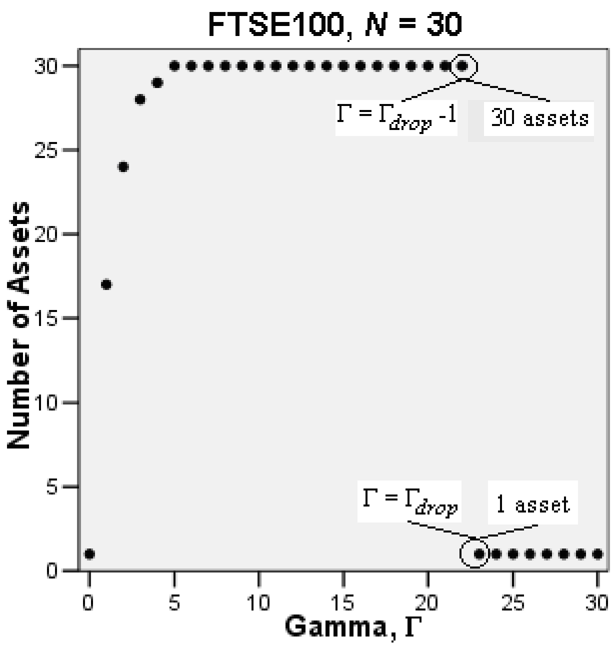

Figure 6.1. Number of assets selected for the portfolio at each Γ, when $N = 30$.

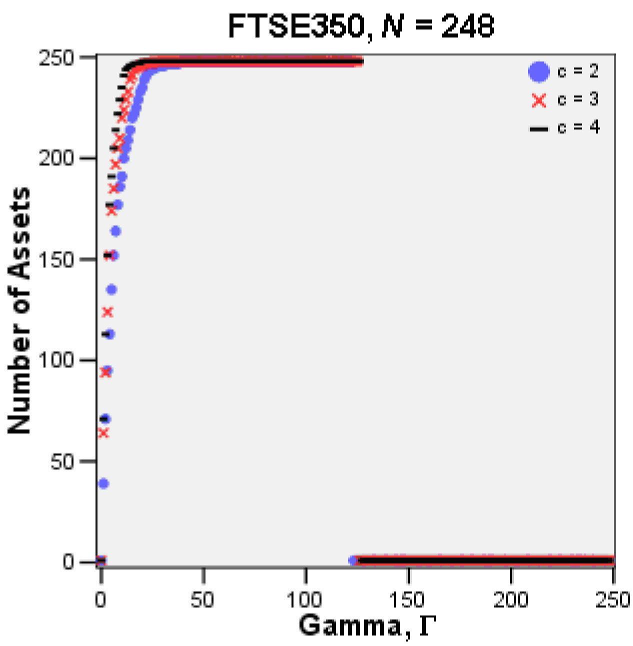

Figure 6.2. Number of assets selected for the portfolio at each Γ (dataset 4, $N = 248$, $c = 2$, 3 & 4).

Lastly, in Chapter 4, having used dataset 2, we observed that for trials consisting of larger sets of historical data (i.e. 60 months), the maximum number of assets held in a portfolio was almost always N. In contrast, when the dataset was smaller (composed of 20 or 30 months) the maximum number of assets held in a portfolio rarely reached N, although in many trials the number of assets held was within five of N.

Selection & Weights. In Figure 6.3, the assets along the x-axis are in descending order by $\bar{r}_i$ (mean log return); asset 1 has the largest $\bar{r}_i$ and asset 30 has the smallest $\bar{r}_i$. Figure 6.3 has two purposes: Firstly, to show the assets held in each portfolio when $\Gamma = 0$ through to $\Gamma = 5$ and secondly, to show how the weight of each asset changed when Γ increased to $\Gamma + 1$. Again, consider $N + 1$ consecutive optimal portfolios corresponding to integer values of Γ from 0 to N. As Γ increased, the number of assets held also increased until a maximum number of assets (in this case N) was reached. In addition, those with a larger $\bar{r}_i$ were selected first. For example, when $\Gamma = 0$, the portfolio consisted of the asset with the largest $\bar{r}_i$ (Figure 6.3). When $\Gamma = 1$, the 17 assets with the largest $\bar{r}_i$ were held and when $\Gamma = 2$, seven more assets are added to the portfolio (the seven with the next largest $\bar{r}_i$).

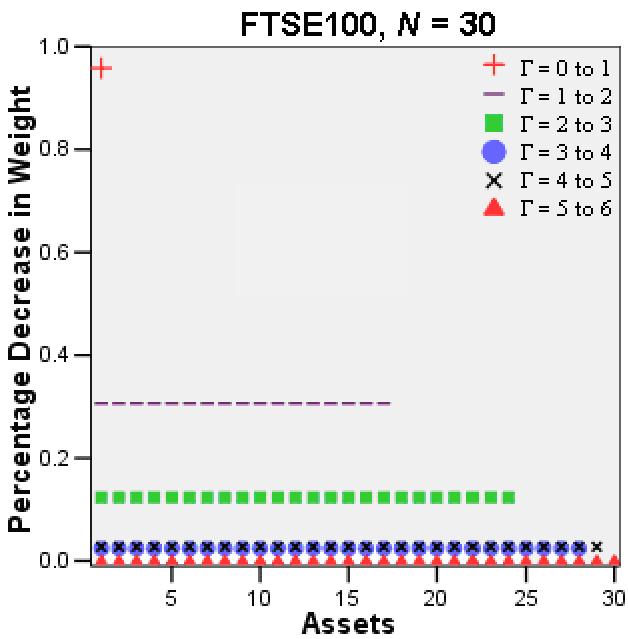

Figure 6.3. Assets in descending order by $\bar{r}_i$. An example of how assets are selected and how weights change as more assets are included in the portfolio.

An interesting relationship exists between the weights of each asset held in successive portfolios, i.e. from Γ to $\Gamma + 1$ (Figure 6.3). Let $NumA_{\Gamma}$ be the number of assets selected in a portfolio, for $\Gamma = 0..N$. If $NumA_{\Gamma+1} \geq NumA_{\Gamma}$ then every asset held at Γ decreased by the same percentage in order to include additional assets at $\Gamma + 1$ (Figure 6.3). For example, all 17 assets held at $\Gamma = 1$ decreased in weight by the same percentage, approximately 31%, so that 7 more assets could be added to the portfolio at $\Gamma = 2$. This is seen when Γ increased from two to three, three to four and four to five (Figure 6.3). When Γ increased from 5 to 6, the percentage decrease in asset weights was zero because all N assets were held in the same proportions when $\Gamma = 5$ and $\Gamma = 6$.

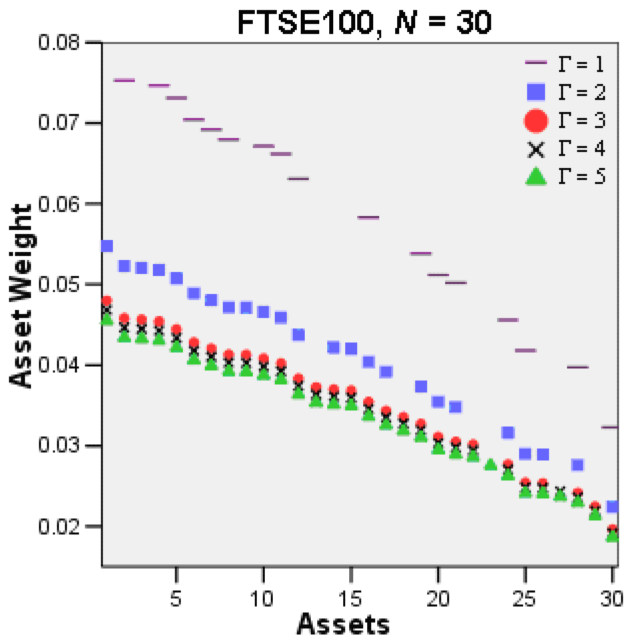

Figure 6.4. Assets in ascending order by $\hat{r}_i$. An example of how robust models weight assets, when $N = 30$. Portfolio weights shown for $\Gamma = 1..5$.

We have shown that assets with a larger $\bar{r}_i$ are the first to be added to a robust portfolio, but how is weight distributed amongst the chosen assets? In Figures 6.4 and 6.5, the assets along the x-axis are in ascending order by $\hat{r}_i$ (standard deviation); asset 1 has the smallest $\hat{r}_i$ and asset 30 has the largest $\hat{r}_i$. Observe that once selected, the assets with the smallest $\hat{r}_i$ were given the most weight, hence, the plot in Figure

6.4 and 6.5 are monotonically decreasing[4]. Asset weights are only shown for values of Γ from one to five because at $\Gamma = 0$ only one asset is held, at $\Gamma = 6$ through to $\Gamma = \Gamma_{drop} - 1$ the portfolio composition and weights are the same as at $\Gamma = 5$ and at $\Gamma = \Gamma_{drop}$ through to $\Gamma = N$ where only the asset with the largest $\bar{r}_i - c\hat{r}_i$ is held.

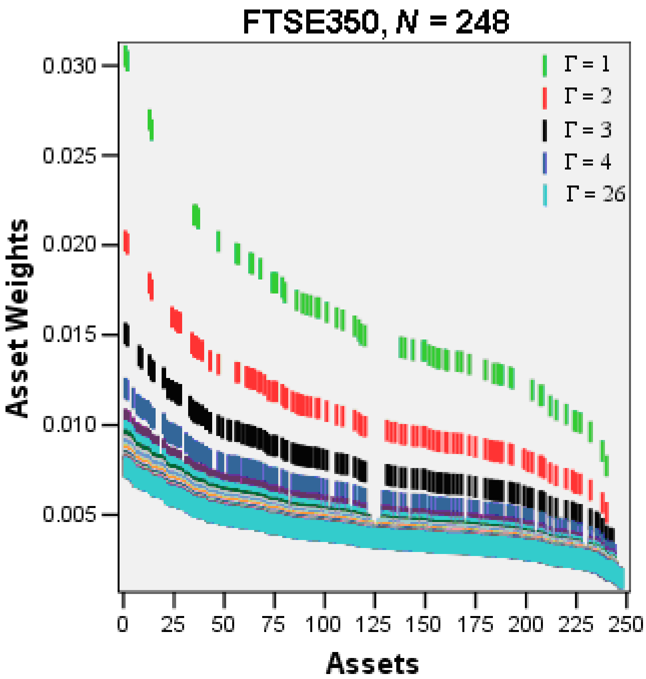

Figure 6.5. Assets in ascending order by $\hat{r}_i$. An example of how robust models weight assets, when $N = 248$. Portfolio weights shown for Γ = 1..21, 26.

6.2 Model Parameters

As discussed in Chapter 5, robust optimal decisions are more sensitive to the definitions of parameters $\bar{r}_i$ and $\hat{r}_i$ than the scaling factor c, which define the scale of the uncertainty set for r_i. However, in addition to $\bar{r}_i$ and $\hat{r}_i$, the robust optimal objective function value is sensitive to c. For example, consider the total portfolio return, $\sum_{i=1}^{n} \bar{r}_i w_i^*$, where w_i^* is the optimal decision vector, and the robust optimal objective, given by (3.11) in Chapter 3, for c = 2 and c = 5 (Figure 6.6). The total return of the two portfolios was the same at almost every Γ; in those instances, the

[4] In Figure 6.5 it may appear that the plots for each Γ are not monotonically decreasing over the interval [1, 248], however, it is due to the resolution of the figure and not the numerical results.

weight of each asset was also the same. Thus, the optimal decisions (asset weights) were fairly insensitive to the value of *c*. However, the robust optimal objective function value was worse when $c = 5$ and decreased at a faster rate than when $c = 2$, as expected (Figure 6.6).

There were observable factors which may explain the observed insensitivity of the total portfolio return to changes in *c*. Let $\bar{r}_i - c\hat{r}_i$ define the risk-adjusted return of asset *i* (Bertsimas and Sim, 2004), this is simply the worst case return. Now consider *N* assets in descending order by their risk-adjusted return; thus, the asset ranked 1^{st} has the *best* worst case return and the asset ranked N^{th} has the *worst* worst case return. We observed that the optimal decisions for models optimised using different values of *c* were the same if the descending order of the assets by risk-adjusted return was the same. For example, consider the optimal decisions when $c = 2$ and $c = 3$, respectively. The optimal decisions when $c = 3$ was the same for corresponding values of Γ when $c = 2$ if the rankings of assets in descending order by $\bar{r}_i - c\hat{r}_i$ are the same; likewise for any other two values of *c*. In addition, when the order of assets was different, decisions were only different for the first several values of Γ and/or a few values of Γ around Γ_{drop}. For example, consider the composition of portfolios corresponding to $c = 2$ and $c = 5$, respectively, whose total portfolio return and robust optimal objective value are plotted in Figure 6.6. For values of $\Gamma \geq 4$, the optimal decisions were the same for both $c = 2$ and $c = 5$: for $4 \leq \Gamma \leq 22$ each portfolio consisted of all 30 assets and when $\Gamma \geq 23$ each portfolio consisted of a single asset. It was only values of Γ less than 4 that yielded different portfolios.

These results suggest that increasing the scaling factor of the uncertainty set *U* will likely yield very similar decisions (if not the same decisions), particularly for values of Γ yielding fully diversified portfolios (i.e. the max number of assets is selected), which typically correspond to a probability of optimality of 90%-99.9%. However, the robust optimal objective function value will deteriorate.

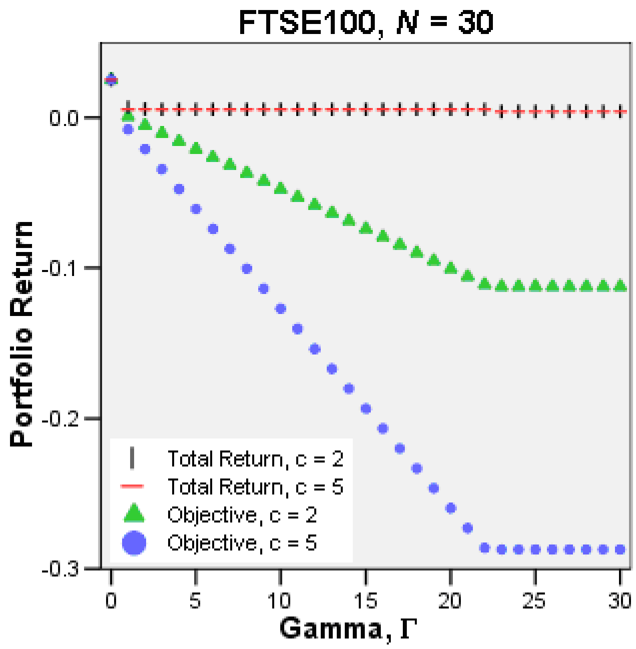

Figure 6.6. Total portfolio return and robust optimal objective when c = 2 and c = 5.

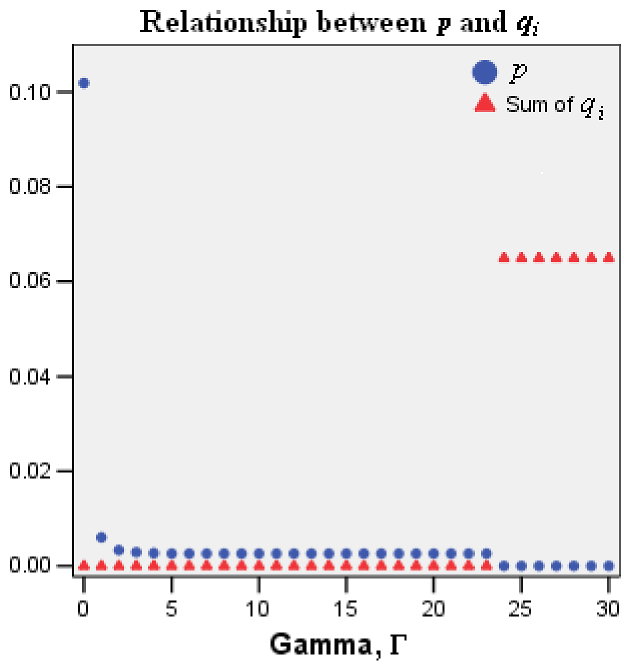

Figure 6.7. Plot of p ($\hat{r}_i w_i = p$, $\forall i$) with $\sum_i q_i$ for $\Gamma = 1..N$.

Note on p and q_i's: Recall from Section 3.3 we deduced that p is chosen as the Γ^{th} largest $\hat{r}_i w_i$, over all i. In our empirical results, not only was p the Γ^{th} largest $\hat{r}_i w_i$, but $\hat{r}_i w_i = p$ for all i, given that $w_i \geq 0$ and $\Gamma \leq \Gamma_{drop} - 1$. Consequently, $q_i = 0$ for all i. When $\Gamma \geq \Gamma_{drop}$, each portfolio held only 1 asset, thus, p = 0 and $q_i = \hat{r}_i w_i$

(which is 0 for all but one i). The relationship between p and q_i is illustrated in Figure 6.7 using dataset 1. Only p and $\sum_i q_i$ are plotted for all Γ instead of plotting p_i and q_i for all i because $\hat{r}_i w_i = p$ (for all i) and when $\sum_i q_i \neq 0$, q_i for the asset with the largest risk-adjusted return is the only q_i that does not equal zero; thus, $\sum_i q_i$ equals $\hat{r}_i w_i$ of the asset with the largest worst case return.

6.3 Robust Efficient Frontier

As described in Markowitz's model (Section 3.1), if S represents the set of all possible E-V combinations, then an investor is interested in the subset of efficient portfolios. When plotted, this subset of E-V combinations is known as the efficient frontier. A similar plot can be obtained for the robust model by adding the following constraint to (3.11) in Section 3.5:

$$\sum_{i=1}^{N} w_i \mu_i = \mathit{Target\ Return},$$

where *Target Return* takes C equidistant values between the minimum and maximum mean log return. A robust E-V efficient frontier (red dots) is plotted with a Markowitz E-V efficient frontier (blue dots) in Figure 6.8. For every Γ from 0 to N the robust model was optimised for each of $C = 31$ *Target Return* values. As in the Markowitz model, an investor would be interested in the subset of efficient points. Therefore, for each *Target Return*, the desired portfolio was the one with the smallest variance over all Γ (denoted by the red dots in Figure 6.8).

The robust E-V efficient frontier is useful for comparison with the Markowitz model. However, it is not the best representation of the robust model because many of the efficient portfolios will not be generated without the target return constraint. Without this constraint, the robust optimal portfolios for almost all Γ are clustered within a small interval of portfolio variance and total portfolio return (denoted by black **x**'s in Figure 6.8). Note that the robust optimal portfolio consists of the asset with the largest mean log return when $\Gamma = 0$ and the asset with the largest risk-adjusted return, $\bar{r}_i - c\hat{r}_i$, when $\Gamma = \Gamma_{drop}..N$ (see Figure 6.8).

The Markowitz model measures risk by portfolio variance whereas the robust model measures risk as the probability of the actual portfolio return being less than the robust optimal objective, also known as the probability of underperformance. Therefore, it is more accurate to represent the robust efficient frontier as the probability of underperformance versus the robust optimal objective (Figure 6.9). Notice that the sacrifice, in terms of optimal objective function value, becomes significantly larger when the probability of underperformance is less than 1.5%. By this representation of the robust efficient frontier, an investor can clearly see the trade-off between guaranteed robustness and the optimal objective function value.

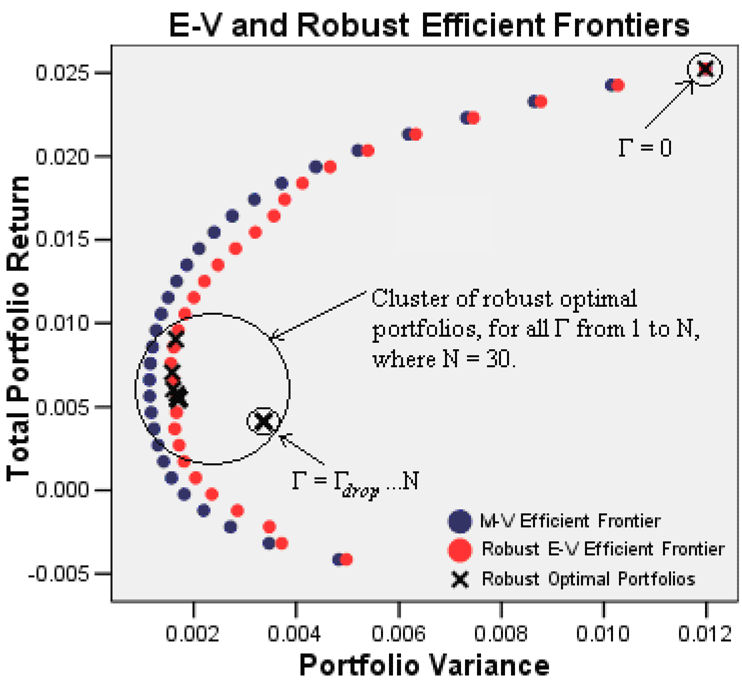

Figure 6.8. Plot of the *E-V* efficient frontier, robust *E-V* efficient frontier and optimal robust portfolios using dataset 1.

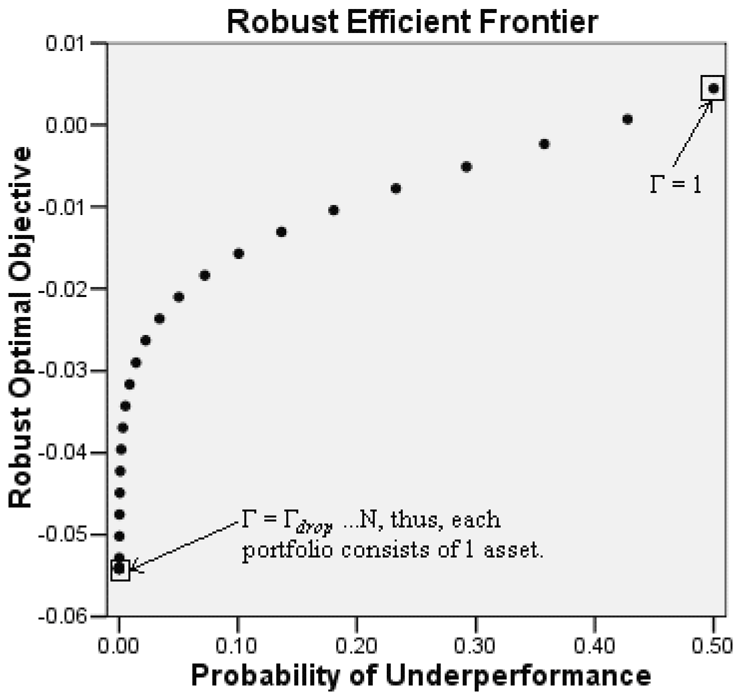

Figure 6.9. Robust Efficient Frontier: Probability of Underperformance versus Robust Portfolio Return. Constructed using dataset 1, $N = 30$, $\Gamma = 1..N$.

6.4 Threshold and Cardinality Constrained Robust Models

In this section, we discuss the similarities and differences in the properties of robust portfolios when threshold and/or cardinality constraints are introduced. The results in this section are based upon the observations of the models studied in Chapter 5. For a summary of the threshold and cardinality constrained robust models, please refer to Table 5.1 in Section 5.1.

The following is a brief summary of the properties of an unconstrained robust model, which were given in Sections 6.1 and 6.2:

1. Assets with the largest $\bar{r}_i$ were chosen first.
2. Of those chosen, assets with the smallest $\hat{r}_i$ were given the most weight.
3. Assets were added to successive portfolios (in terms of incrementing Γ), until a maximum number of assets was reached. The same number of assets was held until Γ_{drop}, when all but one asset were dropped.

4. Robust optimal decisions were sensitive to *c*, whereas total portfolio return tended to be insensitive.
5. All assets included in the portfolio at Γ decreased in weight by the same percentage if more assets are added to the portfolio at Γ + 1.
6. Given a value of Γ on the interval [1, Γ_{drop} - 1], then the following is true:

   if $w_i > 0$, then $\hat{r}_i w_i = p$ (hence, $\sum_i q_i = 0$).

We state that some properties of threshold and/or cardinality constrained models hold "without exception", however, we recognise that they may be so only for dataset 1. It is possible that for larger datasets, these properties may not hold "without exception" but may only be observed as a "general rule".

Threshold constraints. We observed that properties 1, 3 and 4 held without exception, whereas properties 2, 5 and 6 held most of the time (some models had one or two exceptions); there were instances when an asset with a larger $\hat{r}_i$ was given more weight than an asset with a smaller $\hat{r}_i$, and/or an asset did not decrease by the same percentage as all other assets when Γ increased to $\Gamma + 1$, or and/or $\hat{r}_i w_i \neq p$ for an asset *i*. Typically, those exceptions corresponded to assets whose weight equalled the threshold (i.e. $w_i = \alpha$).

Cardinality Constraints. We observed that properties 2, 3, 4 and 6 held without exception. Property 1 tended to hold, but only for the first few values of Γ. For example, consider a list of assets in descending order by $\bar{r}_i$; say 20 assets were selected at Γ = 1, then it is possible that the 23rd ranked asset is selected instead of an asset whose $\bar{r}_i$ ranked in the top 20.

As a result of the cardinality restriction, when Γ ≥ 3, portfolios tended to drop assets previously held in favour of those with larger risk-adjusted returns. In addition, although property 3 held, unlike the unconstrained models, assets selected at Γ may not be held at $\Gamma + 1$. Thus, once a value of Γ was reached that held *k* assets (the max number allowed), the number of assets remained constant (as with the unconstrained models), but the composition tended to change. Consequently, an amendment to property 5 must be made: all assets included in the portfolio at Γ decreased in weight by the same percentage at Γ + 1, unless an asset was dropped altogether.

Both Threshold and Cardinality Constraints. We observed that only properties 3 and 4 held without exception; property 4 is illustrated in Figure 6.10 ($\alpha = 0.03$ and $k = 20$). Similar to portfolios with only cardinality constraints, property 1 only held for $\Gamma = 1$. When $\Gamma \geq 2$, previously held assets may be dropped in order to include those with larger risk-adjusted returns. In addition, similar to cardinality constrained portfolios, property 3 held, although the composition of portfolios from Γ to Γ + 1 tended to change, even if the same number of assets was held. Finally, properties 2, 6 and the amended version of property 5 tended to hold with only a few exceptions at each Γ, which typically corresponded to assets whose weight equalled the threshold (i.e. $w_i = \alpha$).

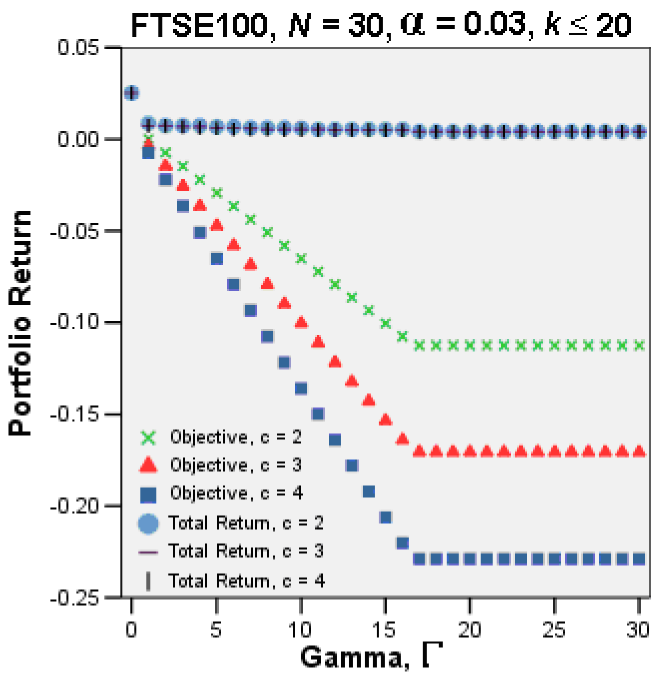

Figure 6.10. Total portfolio return and robust optimal objective when $c = 2$ and $c = 5$.

These results suggest that threshold and cardinality constraints do not significantly change the properties of robust portfolios. There are instances in which certain properties will not hold and there are certain instances in which a new property exists, however, the properties of unconstrained models give insight into the properties of threshold and cardinality constrained models.

Chapter 7

Evaluation of Portfolio Performance

In this chapter we analyse the performance of robust and *E-V* portfolios by evaluating the actual return these portfolios achieved over the out-of-sample period. Whilst robust portfolios may be more robust in terms of achieved and guaranteed robustness, examined previously, in Chapter 5, here we are interested in their performance in terms of portfolio return and the associated risk, compared to non-robust *E-V* portfolios. Each portfolio is evaluated using performance statistics, such as risk-adjusted return measures and reward and downside risk statistics, applied to two subsets of data taken from the two datasets (Case 1 and Case 2) described in Chapter 5: 1) the out-of-sample period and 2) a non-parametric bootstrap sample drawn from the out-of-sample period.

Non-parametric bootstrapping is a common statistical tool for generating an approximate sampling distribution of a statistic from one sample, in order to estimate a parameter. Bootstrapping is particularly helpful when the sampling distribution of the desired statistic is unknown. Since the distribution of an asset's return is uncertain, and in particular, its mean log return is unknown and variable, we applied the bootstrap method to generate a sampling distribution of the mean log return of each asset, which was used as a set of asset return scenarios for back-testing. The bootstrap distributions were generated using the out-of-sample returns of each asset as the original sample.

Risk-adjusted return measures are composite measures of portfolio performance which combine both risk (which differs depending on the measure) and portfolio return. Typically, risk-adjusted return measures are expressed as a ratio of excess return to risk and interpreted as the amount of excess return received per unit of risk. We have chosen to use the Sharpe ratio (Sharpe, 1966) and Sortino ratio (Sortino and Price, 1994). Furthermore, we present statistics for reward and downside risk which can provide additional insight into the sources of risk and reward, especially when the distributions of portfolio returns are asymmetric.

The robust and *E-V* portfolios evaluated in this chapter were a subset of the optimal portfolios resulting from the robust and *E-V* models detailed in Chapter 5. Recall, there were eight robust models and one *E-V* model, referred to as models R_j, where $j = \{1..8, EV\}$. Robust models were optimised based on a specified uncertainty set defining the true log return of each asset, $r_i \in [\bar{r}_i - c\hat{r}_i, \bar{r}_i + c\hat{r}_i]$. Throughout this chapter, $c = 3$, $\bar{r}_i$ is the mean log return of asset i and $\hat{r}_i$ is the standard deviation of asset i. Thus, the robust portfolios are those resulting from models R_j, for $j = \{1..8\}$, being optimised for the following uncertainty set $r_i \in [\bar{r}_i - 3\sigma_i, \bar{r}_i + 3\sigma_i]$, i.e. the true log return of asset i, r_i, lies within three standard deviations of its mean log return. The *E-V* portfolios are those resulting from model R_{EV} being optimised for a fixed target return whilst minimising portfolio variance. Of the 31 efficient portfolios generated, we selected the following five: portfolios 31, 27, 23, 19 and 15, as detailed in Chapter 5 (starting with portfolio 31, which simply minimised portfolio variance with no constraint on total return, we selected every 4th efficient portfolio). As mentioned before, we chose portfolios from the efficient frontier associated with lower variance and lower total return because they are more robust than those associated with higher variance and higher total return.

In addition to the robust and *E-V* portfolios, we consider an Index portfolio, which was obtained using the same 30 assets from the FTSE 100 used to optimise the robust and *E-V* portfolios. We estimated the weights for our Index portfolio by normalising the market capitalisation weights of each asset for the first month of the out-of-sample period. Due to the gap between the time we first acquired our historical data and when we acquired the market capitalisation weights, five of the assets in our original dataset were no longer in the FTSE 100. However, we were able to individually acquire the weights for these five dead assets from Datastream.

To distinguish the difference between the specific portfolio optimised using each model and the model itself, we refer to the portfolios being evaluated as portfolio P_l, where $l = \{1..8\} \cup \{EV.15, EV.19, EV.23, EV.27, EV.31\} \cup \{Index\}$. The set $\{1..8\}$ represents the robust portfolios obtained by model R_j, $j = \{1..8\}$, where r_i is defined as set out above. The set $\{EV.15, EV.19, EV.23, EV.27, EV.31\}$ are the *E-V* portfolios obtained by model R_{EV}. Lastly, the set {Index} is the Index portfolio.

The dataset is the same as that used in Chapter 5. It consists of the monthly logarithmic returns of 30 stocks selected at random from the FTSE 100 index beginning 1 January 1992 through to 1 December 2006. In this chapter, we are only concerned with the out-of-sample periods for Case 1 and Case 2. For Case 1, of the 180 time periods, the last 48 months (1 January 2003 – 1 December 2006) were reserved for the out-of-sample analysis. For Case 2, of the 180 time periods, the last 72 months (1 January 2001 – 1 December 2006) were reserved for the out-of-sample analysis.

In Section 7.1 we describe the non-parametric bootstrap back-test and evaluate the robust and *E-V* portfolios by generating a sampling distribution consisting of 1000 scenarios for the mean log return of each asset. In Section 7.2 we compare risk-adjusted return measures for the out-of-sample periods and bootstrap samples. Lastly, in Section 7.3 we carry out a worst case analysis using four worst case scenarios.

7.1 Bootstrap Procedure

The Sharpe and Sortino ratios require two different bootstrap samples, best obtained by two different bootstrap sampling procedures: 1) Bootstrapping a sample of 1000 monthly log returns and 2) Bootstrapping a sample of 1000 annual log returns. Both use the out-of-sample monthly log returns as the original sample.

Sharpe (1994) in his paper *The Sharpe Ratio* suggested that the returns used to calculate the Sharpe ratio should be taken over shorter time periods, such as month long periods, in order to "maximise information content". In addition, Sortino and Forsey (1996) suggested that to generate a bootstrap sample, the original sample should consist of at least 5 years of monthly or 10 years of quarterly data. Following with these and other such suggestions, a sample of 1000 monthly log return scenarios were bootstrapped and used to evaluate each portfolio from which the Sharpe ratio is calculated. The procedure for bootstrapping monthly log returns is described further in Section 7.1.1.

Calculating downside risk and the Sortino ratio require annualised portfolio returns. To generate annualised returns Riddles (2003) suggests bootstrapping annual log returns, using a sample of monthly log returns, by repeatedly summing 12 randomly selected months. Applying this technique, a sample of 1000 annual return scenarios were bootstrapped and used to evaluate each portfolio; portfolio returns were annualised before calculating downside risk and the Sortino ratio. The procedure for bootstrapping annual log returns is described further in Section 7.1.2.

Through the evaluation of portfolios using the bootstrap samples of monthly and annual asset returns, we observed that the bootstrap sample of monthly asset returns yielded portfolio returns that were less volatile than the bootstrap sample of annual asset returns. The effect of the sample consisting of higher returns is evidenced by the risk-adjusted return ratios and reward and downside risk statistics in Section 7.2. In addition, the robust portfolios were less affected by the volatility of either bootstrap sample compared to *E-V* portfolios. We discuss further the observed differences between the two samples in Appendix A.

7.1.1 Bootstrap Sampling of Monthly Returns

The purpose of generating the bootstrap sample of monthly log returns is to obtain S estimates of the mean monthly log return for each asset i, in order to calculate S scenarios of the total return of each portfolio. First, S bootstrap samples, $\mathrm{B}_{is} = \{x_1^*,...,x_M^*\}_{is}$, were generated for each asset i, where x_m^* was randomly chosen with replacement from the out-of-sample dataset $\{x_1,...,x_M\}$, m = 1..M and M is the number of out-of-sample months. Second, estimates of the mean monthly log return for each asset i, $\hat{\theta}_{is}$, were calculated for scenario s:

$$\hat{\theta}_{is} = \frac{1}{M}\sum_{m=1}^{M} x_{m,i,s}^* , \qquad \forall i = 1..N, s = 1..S , \tag{7.1}$$

where M is the number of out-of-sample months, N is the number of assets and S is the number of scenarios. Lastly, the total return was calculated for each portfolio P_l and scenario s:

$$TotalReturn_{sl}^{B} = \sum_{i=1}^{N} \hat{\theta}_{is} w_{il}^* , \tag{7.2}$$

$$\forall s = 1..S, l = \{1..8\} \cup \{EV.15, EV19, EV.23, EV.27, EV.31\} \cup \{Index\}$$

where w_{il}^* was the optimal weight of asset i, in portfolio P_l.

Using a discrete distribution, in which all M out-of-sample monthly log returns were given an equal probability ($\frac{1}{M}$) of occurring, s random scenarios, each with M randomly selected monthly log returns, were generated for each asset i. Thus, for asset i, each column s of M random observations comprised a bootstrap sample B_{is}. Using (7.1), $\hat{\theta}_{is}$ was calculated by taking the average of all M random observations in column s. Once the log return of each asset for every scenario s had been estimated, the total return, $TotalReturn_{sl}^{B}$, was calculated for every portfolio P_l.

7.1.2 Bootstrap Sampling of Annual Returns

The purpose of generating the bootstrap sample of annual log returns was to obtain S estimates of the annual log return for each asset i, in order to calculate S scenarios of the total return of each portfolio P_j. First, S bootstrap samples, $\mathrm{A}_{is} = \{x_1^*,...x_K^*\}_{is}$, were generated for each asset i, where x_k^* was randomly chosen with replacement from the out-of-sample dataset $\{x_1,...x_M\}$, $m = 1..M$, M is the number of out-of-sample months), $k = 1..\mathrm{K}$ and $\mathrm{K} = 12$ (the sum of 12 monthly log returns is one annual log return). Second, estimates of the annual log return of each asset i, $\hat{\Theta}_{is}$, were calculated for scenario s:

$$\hat{\Theta}_{is} = \sum\nolimits_{k=1}^{K} x_{k,i,s}^* , \qquad \forall i = 1..N, s = 1..S , K = 12, \tag{7.3}$$

where N is the number of assets and S is the number of scenarios. Lastly, the total return for each portfolio P_l under scenario s was calculated:

$$TotalReturn_{sl}^{A} = \sum\nolimits_{i=1}^{N} \hat{\Theta}_{is} w_{il}^* , \tag{7.4}$$

$$\forall s = 1..S, l = \{1..8\} \cup \{EV.15, EV19, EV.23, EV.27, EV.31\} \cup \{Index\},$$

where w_{il}^* was the optimal weight of asset i, in portfolio P_l.

Again, using a discrete distribution, in which all M out-of-sample monthly log returns were given an equal probability ($\frac{1}{M}$) of occurring, s random scenarios, each with K randomly selected returns, were generated for each asset i. Thus, for asset i, each column s of K random observations comprised a bootstrap sample A_{is}. Estimates of the annual log return of each asset, $\hat{\Theta}_{is}$ in (7.3), were calculated by summing all K random observations in column s. Once the annual log return of each asset for every scenario s had been estimated, the total return, $TotalReturn_{sl}^{A}$, was calculated for every portfolio P_l.

7.2 Analysis of Performance Statistics

In this section we evaluate the total portfolio return of the robust and *E-V* models by applying two risk-adjusted return measures as well as additional statistics which provide further insight into the downside risk and reward of each portfolio's return using the out-of-sample periods for Case 1 (48 months) and Case 2 (72 months).

Jones, in his book *Investments: Analysis and Management* (2007), states that there are three aspects of portfolio performance to be considered: the adequacy of the portfolio's return, the 'riskiness' of the portfolio (via a risk measure), and the expectations of the investor with respect to the risks taken? In previous chapters, we have considered both the riskiness of the decisions and investor's expectations, from a robust optimisation perspective. The riskiness of a portfolio and whether or not the investor's expectations of realised return were met, were evaluated based upon a portfolio's guaranteed (only applicable for robust models) and achieved robustness. In this section we consider the adequacy of the portfolio's return. This requires more than simply looking at a time series of returns. It requires an evaluation of the portfolio's return which has been adjusted such that it accounts for the risk associated with not meeting a specified level of return, or benchmark; this is known as a risk-adjusted return[5]. The definitions of the risk and benchmark are what differentiate risk-adjusted return measures. For example, we have chosen to apply the Sharpe ratio (Sharpe, 1966) and Sortino ratio (Sortino and Price, 1994). The Sharpe ratio defines the benchmark as the risk-free rate (Sharpe, 1994) and risk as the portfolio's

[5] Risk-adjusted return, as defined in this chapter, is different from the term "risk-adjusted return", given by Bertsimas and Sim (2004), used in previous chapters. The former refers to a performance measure, whereas the later refers to the worst-case return of an asset.

standard deviation. Typically, a benchmark is a reference portfolio comprising assets from the same asset classes, in the same proportions, and tracked over the same time period as the invested portfolio. This naturally leads to the selection of an index as the benchmark. Sharpe (1994) originally specified the benchmark in the Sharpe ratio to be the risk-free rate, but noted that, by the early 1990s, works were being published which used a passive benchmark portfolio instead. The Sortino ratio replaces the benchmark with a minimal accepted return (MAR), which is simply the minimum return necessary for an investor to achieve their goals, and defines risk as the semi-standard deviation of portfolio returns below the MAR. The MAR is similar to a benchmark in that it is a target return, but it differs in that it is not necessarily a benchmark portfolio nor is it determined in the same fashion. The MAR is more dependent upon the goals of the investor; it is the minimum return an investor is willing to accept in order to achieve their objectives.

The Sharpe ratio, introduced by William Sharpe (1966) as the reward-to-variability ratio (*R/V*), is defined as follows:

$$\text{Ex Post Sharpe Ratio} = \frac{\overline{R}_p - RF}{\sigma_p},$$

where $\overline{R}_p$ is the average portfolio return, *RF* is the risk-free rate and σ_p is the standard deviation of the difference between the portfolio return at time *t* and *RF* (Sharpe calls this difference the differential return). The Sharpe ratio is interpreted as the amount of excess return received, above the benchmark, per unit of risk. It was derived from the capital market line (CML) which is a line tangent to the *E-V* efficient frontier passing through the point (0, *RF*). Thus, the CML consists of all possible combinations of the optimal *E-V* portfolio (*E-V*, σ) and the risk-free rate. If the assumptions of the Capital Asset Pricing Model (CAPM) hold, and if portfolio standard deviation is an appropriate measure of risk, then clearly, as Sharpe stated, the "best" portfolio is that which has the largest Sharpe ratio (1966).

However, if portfolio standard deviation is not an appropriate measure of risk and return distributions are not symmetric, then, as Sortino (2003) observed, the Sharpe ratio is likely to rank portfolios incorrectly. Sortino and Price (1994) argued that the measure of risk should not include both good and bad portfolio returns, but only those below the MAR (minimal accepted return). Thus, instead of using standard deviation, they suggested a measure of risk known as downside deviation (*DD*), which is the semi-standard deviation of portfolio returns below the MAR. In addition, Sortino (2003) stated that in the case of asymmetric return distributions, *DD* more accurately ranks portfolios than does variance. The Sortino ratio is defined as follows:

$$\text{Sortino Ratio} = \frac{\tilde{R}_p - MAR}{DD_p},$$

where $\tilde{R}_p$ is the portfolio's annualised rate of return, the MAR is the minimum return necessary for an investor to achieve their goals and DD_p is the below-MAR semi-deviation of portfolio *p*. The Sortino ratio can be interpreted as the amount of excess return received above the MAR per unit of risk associated with not achieving the MAR. The DD_p can be calculated using either a continuous or discrete formulation. A continuous formulation estimates the true distribution of the total portfolio return by fitting a continuous probability distribution to an approximate distribution of the total portfolio return generated using a large bootstrap sample of asset returns. The DD_p is then calculated using the continuous probability distribution which has been fitted to the sample distribution.

Alternatively, a discrete formulation simply calculates DD_p as the below-MAR semi-standard deviation of the total portfolio return using either a historical dataset or a large bootstrap sample of asset returns (Riddles, 2003). For practical purposes, we have chosen to calculate DD_p using a discrete formulation. We are aware of the work by Sortino and Forsey (1996) entitled *On The Use and Misuse of Downside Risk* in which they strongly discourage the use of a discrete formulation because it only captures what did happen, whereas a continuous formulation captures what could have happened. However, we are only concerned with an ex post evaluation and are not optimising our portfolios with respect to the Sortino ratio, therefore we believe a discrete formulation is sufficient.

One of the advantages of the Sortino ratio is that it gives rise to several reward and downside risk statistics; we will consider four: volatility skewness, downside deviation (*DD*), average upside return and the downside frequency. Volatility skewness is the ratio of the upside deviation (*UD*) to the downside deviation (*DD*). Similar to *DD*, *UD* is the semi-standard deviation of the portfolio returns above the MAR. The average downside return is the average portfolio return below the MAR and the downside frequency is the number of portfolio returns less than the MAR. If we were solely interested in ranking portfolio performance we may only be interested in the risk-adjusted return ratios. However, we are also interested in the distribution of each portfolio's total return which provides additional insight into the stability and characteristics of returns over time; therefore, we have included additional statistics. We are cautious of including an excess of statistics, but feel that those provided are relevant and beneficial to the analysis of each portfolio.

It is important to recognise that the choice of the benchmark and MAR will have an impact on the rankings of both the Sharpe and Sortino ratio, respectively. As a result, Sharpe ratios calculated using different benchmarks are not comparable and likewise Sortino ratios calculated using different MARs are not comparable (Riddles, 2003). For the Sharpe ratio, changing the benchmark does not change the measure of variation σ_p, i.e. the shape of the distribution of differential returns, but it does shift that distribution either to the right (benchmark decreases) or to the left (benchmark increases). For the Sortino ratio, changing the MAR moves the reference point from which the *DD* is calculated, but does not change the shape or shift the distribution of portfolio returns. That is, increasing the MAR, increases the size of the tail of the distribution, which corresponds to an increase in *DD*. Likewise, decreasing the MAR decreases the size of the tail of the distribution, which corresponds to a decrease in *DD*. The nature of the relationship between the MAR and *DD* make it possible to have a *DD* of zero, which corresponds to the MAR being less than the entire distribution of portfolio returns. Instances such as this occurred in the evaluation of the bootstrapped annual returns, which will be discussed in the following two sections (7.2.1 and 7.2.2).

The Sharpe ratio and Sortino ratio require a risk-free rate and a MAR, respectively. The risk-free rate was estimated using the UK 3 month Treasury Bill and we specify two instances for the MAR: 1) MAR = the risk-free rate and 2) MAR = 0. We have chosen a MAR of zero because we are primarily evaluating robust portfolios, in which our objective is to minimise the worst case performance and not outperform a benchmark. We have associated the worst case with negative portfolio returns; thus, our main interest is in the distribution of and risk associated with negative portfolio returns.

Lastly, the Sharpe ratio was only calculated for portfolio returns generated using the out-of-sample monthly returns and the bootstrap sample of monthly returns; the Sortino ratio was only calculated for portfolio returns generated using the out-of-sample monthly asset returns and the bootstrap sample of annual asset returns; and the reward and downside risk statistics were calculated for portfolio returns generated using all three samples, respectively. In addition, calculating the Sharpe ratio, Sortino ratio and reward and downside risk statistics required different forms of a portfolio's return. The Sharpe ratio was calculated using the same form of the portfolio's return as the data set over which it was calculated; i.e. using monthly returns for the out-of-sample and bootstrap sample of monthly asset returns. Conversely, the Sortino ratio and the reward and downside risk statistics were calculated after first annualising the portfolio's return.

7.2.1 Case 1: Results

In this section, the results of the analysis of performance are given for Case 1. With respect to Case 2, Case 1 is characterised by a larger in-sample period (132 months) used to optimise the portfolios being evaluated, and a smaller out-of-sample period (48 months) used to analyse portfolio performance. Recall that the in-sample data set for Case 1 included monthly returns from 2001-02, a time period characterised by lower asset returns and increased stock market volatility, hence, the out-of-sample mean log return of each asset tended to be greater than that of the in-sample.

The results in this section are presented in five tables. The Sharpe and Sortino ratios for portfolio P_l are given in Table 7.2 (out-of-sample) and Table 7.4 (bootstrap samples); in each table the robust portfolios are shaded grey and the Index portfolio is italicised. In each table portfolios are ranked in descending order for each ratio. In Table 7.4, the Sharpe ratio has been calculated using $TotalReturn_{sl}^{B}$ and $TotalReturn_{sl}^{A}$ was used to calculate the Sortino ratios (MAR = Risk-free rate and MAR = Zero). The reward and downside risk statistics for portfolio P_l are given in Table 7.3 (out-of-sample), Table 7.5 (bootstrap of annual asset returns), and Table 7.6 (bootstrap of monthly asset returns). A legend for Tables 7.3, 7.5, and 7.6 is given in Table 7.1.

Legend for Tables of Reward and Downside Risk Statistics	
N < MAR	Downside Frequency (Number of returns less than the MAR)
DD	Downside Deviation
ADR	Average Downside Return (Avg. of returns less than the MAR)
VS	Volatility Skewness

Table 7.1. Legend for reward and downside risk statistics Tables 7.3, 7.5, 7.6, 7.8, 7.10 and 7.11.

Out-of-sample evaluation. The results show that all portfolios have a positive Sharpe ratio and positive Sortino ratio (Table 7.2), which indicates that the average portfolio return over the out-of-sample period was greater than zero and greater than the risk-free rate; thus, each portfolio yielded a positive excess return per unit of risk. Contrary to expectation, robust portfolios were ranked highest by the Sharpe ratio as opposed to the Sortino ratios. Additional statistics (Table 7.3) show that when the MAR was the risk-free rate, the Index had the smallest *DD* and highest average downside return (ADR) and that robust portfolios tended to have a smaller *DD* and higher ADR than the *E-V* portfolios. Although *E-V* portfolios tended to have a larger *DD* and smaller ADR, they have a larger volatility skewness (VS), an indication that more of the variation in portfolio returns is due to returns above the risk-free rate. These additional statistics indicate that the *E-V* portfolios are ranked highest by the

Sortino ratio because of larger average portfolio returns (to counter larger downside deviations) and that the distribution of returns above the risk-free rate is more varied. In addition, the VS tends to be closer to 1 for robust portfolios which indicates that the variation of returns above the risk-free rate is similar to (although slightly more than) the variation of returns below the risk-free rate. Similar conclusions are reached when the MAR is zero. The difference between the two sets of results is that the Index does not have the smallest *DD* nor the largest ADR when the MAR is zero.

Ranking of portfolio P_l by Sharpe and Sortino Ratio, Bootstrap Sample					
Benchmark = Risk-free rate		***MAR = Risk-free rate***		***MAR = Zero***	
P_l	Sharpe Ratio	P_l	Sortino Ratio	P_l	Sortino Ratio
7	0.299	**EV.15**	0.763	**EV.15**	0.869
8	0.297	**EV.19**	0.686	**EV.19**	0.855
2	0.291	**EV.23**	0.657	**EV.23**	0.807
4	0.291	**5**	0.591	**5**	0.796
6	0.291	**EV.31**	0.588	**7**	0.794
5	0.288	**1**	0.585	**1**	0.788
EV.31	0.279	**3**	0.584	**3**	0.787
EV.27	0.278	**7**	0.581	**EV.27**	0.738
1	0.275	**EV.27**	0.564	**EV.31**	0.708
3	0.274	**2**	0.528	**2**	0.686
EV.23	0.263	**4**	0.528	**4**	0.686
EV.19	0.245	**6**	0.528	**6**	0.686
Index	*0.228*	***Index***	*0.511*	**8**	0.633
EV.15	0.223	**8**	0.498	***Index***	*0.628*

Table 7.2. Case 1 out-of-sample portfolio rankings for the Sharpe ratio (using monthly portfolio returns) and Sortino Ratios (using annualised monthly portfolio returns).

Reward and Downside Risk Statistics for Portfolio P_l, Out-of-Sample								
	MAR = Risk-free rate (RF)				**MAR = 0**			
P_l	**N < RF**	**DD**	**ADR**	**VS**	**N < 0**	**DD**	**ADR**	**VS**
1	15	0.320	-0.249	1.602	15	0.286	-0.249	1.899
2	15	0.290	-0.195	1.369	13	0.279	-0.225	1.494
3	15	0.321	-0.249	1.603	15	0.287	-0.249	1.899
4	15	0.290	-0.195	1.369	13	0.279	-0.225	1.494
5	15	0.315	-0.237	1.553	15	0.282	-0.237	1.846
6	15	0.290	-0.195	1.369	13	0.279	-0.225	1.494
7	15	0.302	-0.222	1.503	15	0.270	-0.222	1.803
8	13	0.317	-0.230	1.232	11	0.311	-0.271	1.324
EV.15	16	0.405	-0.293	2.109	14	0.400	-0.334	2.142
EV.19	15	0.387	-0.286	1.800	15	0.355	-0.286	2.044
EV.23	15	0.358	-0.253	1.701	14	0.338	-0.271	1.859
EV.27	13	0.354	-0.256	1.416	13	0.323	-0.256	1.646
EV.31	16	0.306	-0.192	1.574	13	0.308	-0.237	1.590
Index	19	0.271	-0.171	1.610	14	0.281	-0.233	1.537

Table 7.3. Case 1 reward and downside risk statistics for the out-of-sample period. All calculations made having annualised the monthly portfolio returns.

Lastly, considering robust portfolios only, those with a cardinality constraint (l = 2, 4, 6 and 8) are ranked higher by the Sharpe ratio than those without a cardinality constraint, whereas the Sortino ratio (for both MARs) ranks them lower (Table 7.2). In addition, cardinality constrained robust portfolios had smaller *DD*s and ADRs below the benchmark as well as smaller *DD*s and ADRs below zero (Table 7.3). Their VS was also smaller, which shows that in addition to less variation of returns below the MAR, they also had less variation of returns above the MAR.

Bootstrap Evaluation. As in the out-of-sample results, the Sharpe ratio and Sortino ratios (Table 7.4) were positive for all portfolios, which indicates that the average return of each portfolio over both bootstrap samples was greater than zero and the risk-free rate. However, unlike the out-of-sample results, the Sharpe and Sortino ratio ranks the portfolios very similarly, with all eight robust portfolios ranked higher than all *E-V* portfolios and the Index. In addition, the ratios were rather large, particularly for the Sortino ratios of the robust portfolios, which were ranked higher than all *E-V* portfolios and the Index. The Sortino ratios were calculated using portfolio returns generated from $TotalReturn_{sl}^{A}$; when the MAR was the risk-free rate (Table 7.5), the downside frequency was considerably less for robust portfolios (<10) than for *E-V* portfolios (52-160) or the Index (169). Likewise, robust portfolios had smaller *DD*s and larger ADRs; in other words, there was less variation of robust returns below the risk-free rate and those returns tended to be larger than for *E-V* or Index returns. In addition, the VS indicates that there was more variation in robust returns above the risk-free rate than below; this is also true for *E-V* portfolios and the Index, but more so for robust portfolios. Thus, the Sortino ratio is quite large for robust portfolios resulting from very small *DD*s and larger average returns over all $TotalReturn_{sl}^{A}$ (l = 1..8), particularly for P_1, P_3 and P_5 (non-cardinality constrained portfolios).

Consider the Sortino ratio when the MAR is zero (Table 7.4); three robust portfolios have a ratio of 44.38 and the remaining robust portfolios do not have a ratio, while the *E-V* portfolios and the Index are between two and six. The additional statistics (Table 7.5) show that the *DD*, ADR and VS do not exist for the top five ranked robust portfolios, thus, the Sortino ratio does not exist. The remaining three portfolios only had one return less than zero, resulting in a very small *DD*, and thus, a very large Sortino ratio. The *E-V* portfolios and the Index have much smaller ratios because more of their returns were below zero, hence, a larger *DD* and smaller Sortino ratio.

Ranking of portfolio P_l by Sharpe and Sortino Ratio, Bootstrap Sample					
Benchmark = Risk-free rate		***MAR = Risk-free rate***		***MAR = Zero***	
P_l	Sharpe Ratio	P_l	Sortino Ratio	P_l	Sortino Ratio
5	5.251	**5**	13.453	**5**	–
1	5.242	**1**	11.916	**1**	–
3	5.236	**3**	11.352	**3**	–
7	4.892	**8**	6.984	**8**	–
8	4.687	**7**	6.376	**7**	–
2	4.558	**2**	6.150	**2**	44.383
4	4.558	**4**	6.150	**4**	44.383
6	4.558	**6**	6.150	**6**	44.383
EV.31	3.251	**EV.31**	3.447	**EV.31**	5.313
EV.27	3.059	**EV.27**	3.095	**EV.27**	4.886
EV.23	2.676	**EV.23**	2.336	**EV.23**	3.586
EV.19	2.288	**EV.19**	1.871	***Index***	*3.155*
Index	*2.076*	**EV.15**	1.699	**EV.19**	2.814
EV.15	2.002	***Index***	*1.549*	**EV.15**	2.392

Table 7.4. Case 1 portfolio rankings for the Sharpe ratio (using monthly portfolio returns, from bootstrap of monthly asset returns) and Sortino Ratios (using annualised annual portfolio returns, from bootstrap of annual asset returns).

Reward and Downside Risk Statistics for Portfolio P_l, Bootstrap of Annual Returns								
	MAR = Risk-free rate (RF)				**MAR = 0**			
P_l	**N < RF**	**DD**	**ADR**	**VS**	**N < 0**	**DD**	**ADR**	**VS**
1	7	0.010	0.030	12.866	0	–	–	–
2	9	0.017	0.027	6.736	1	0.003	-0.003	46.569
3	7	0.010	0.030	12.258	0	–	–	–
4	9	0.017	0.027	6.736	1	0.003	-0.003	46.569
5	7	0.009	0.032	14.530	0	–	–	–
6	9	0.017	0.027	6.736	1	0.003	-0.003	46.569
7	8	0.018	0.024	6.939	0	–	–	–
8	9	0.015	0.025	7.650	0	–	–	–
EV.15	160	0.088	-0.029	2.603	98	0.078	-0.061	3.218
EV.19	123	0.076	-0.021	2.664	73	0.064	-0.049	3.573
EV.23	92	0.058	-0.007	3.085	42	0.048	-0.037	4.274
EV.27	68	0.040	0.009	3.845	21	0.033	-0.024	5.540
EV.31	52	0.033	0.014	4.144	12	0.029	-0.017	5.876
Index	169	0.045	0.003	2.113	64	0.034	-0.024	3.592

Table 7.5. Case 1 reward and downside risk statistics for portfolios evaluated using the bootstrap sample of 1000 annual asset returns. All calculations made having annualised the annual portfolio returns.

Reward and Downside Risk Statistics for Portfolio P_l, Bootstrap of Monthly Returns								
	MAR = Risk-free rate (RF)				MAR = 0			
P_l	N < RF	DD	ADR	VS	N < 0	DD	ADR	VS
1	0	–	–	–	0	–	–	–
2	0	–	–	–	0	–	–	–
3	0	–	–	–	0	–	–	–
4	0	–	–	–	0	–	–	–
5	0	–	–	–	0	–	–	–
6	0	–	–	–	0	–	–	–
7	0	–	–	–	0	–	–	–
8	0	–	–	–	0	–	–	–
EV.15	25	0.029	0.016	5.740	5	0.014	-0.013	14.562
EV.19	13	0.021	0.021	7.400	1	0.006	-0.006	28.890
EV.23	3	0.022	0.019	6.518	0	–	–	–
EV.27	3	0.012	0.029	10.337	0	–	–	–
EV.31	2	0.005	0.035	22.498	0	–	–	–
Index	37	0.013	0.028	4.979	0	–	–	–

Table 7.6. Case 1 reward and downside risk statistics for portfolios evaluated using the bootstrap sample of 1000 monthly asset returns. All calculations made having annualised the monthly portfolio returns.

When portfolios were evaluated using the bootstrap sample of monthly returns, Table 7.6 shows that the returns of all robust portfolios was greater than the risk-free rate (and zero) for every scenario. This is partially due to the composition of robust portfolios and partially due to the bootstrap sample. As illustrated further in Appendix A, the bootstrap sample of monthly returns generally consisted of larger returns than did the out-of-sample period or the bootstrap sample of annual returns; this resulted in larger portfolio returns ($TotalReturn_{sl}^{B}$) for all P_l, most of which were above the MAR (Table 7.6). Since the Sharpe ratio (Table 7.4) was calculated using $TotalReturn_{sl}^{B}$, and $TotalReturn_{sl}^{B}$ was never less than the risk-free rate for the robust portfolios, P_1 to P_8 were ranked above the *E-V* portfolios and the Index.

Comparing only robust portfolios, in contrast to the out-of-sample results, cardinality constrained portfolios (l = 2, 4, 6 and 8) were ranked lower than portfolios not constrained by cardinality for the Sharpe ratio and both Sortino ratios (Table 7.4); when the MAR is zero, portfolios P_2, P_4 and P_6 are the only robust portfolios with a Sortino ratio (due to a downside frequency of one). Also contrary to the out-of-sample results, cardinality constrained robust portfolios tend to have larger *DD*s and smaller ADRs (Table 7.5). However, similar to the case with out-of-sample results, their VS was smaller than robust portfolios not constrained by cardinality, which, considering the large difference in VS, indicates, that in addition to less variation of returns below the MAR, they also had less variation of returns above the MAR.

7.2.2 Case 2: Results

In this section, the results of the analysis of performance are given for Case 2. With respect to Case 1, Case 2 is characterised by a smaller in-sample period (108 months) used to optimise the portfolios being evaluated, and a larger out-of-sample period (72 months) used to analyse portfolio performance. Recall that the out-of-sample data set for Case 2 included monthly returns from 2001-02, a time period characterised by lower asset returns and increased stock market volatility, hence, the out-of-sample mean log return of each asset tended to be less than that of the in-sample and also less than that of Case 1.

The results in this section are presented in five tables. The Sharpe and Sortino ratios for portfolio P_l are given in Table 7.7 (out-of-sample) and Table 7.8 (bootstrap samples); the robust portfolios are shaded grey and the Index portfolio is italicised. In each table portfolios are ranked in descending order for each ratio. In Table 7.9, the Sharpe ratio has been calculated using $TotalReturn_{sl}^{B}$, while $TotalReturn_{sl}^{A}$ was used to calculate the Sortino ratios (MAR = Risk-free rate and MAR = Zero). The reward and downside risk statistics for portfolio P_l are given in Table 7.8 (out-of-sample), Table 7.10 (bootstrap of annual asset returns), and Table 7.11 (bootstrap of monthly asset returns).

Out-of-sample Evaluation. First, consider the Sharpe ratio (Table 7.7). The bottom five portfolios have negative ratios suggesting that on average those portfolios underperformed the benchmark. In addition, the Index and the three *E-V* portfolios corresponding to the bottom of the efficient frontier (smaller portfolio variance) are ranked the highest; their Sharpe ratio suggests they provide more excess return per unit of risk than the other portfolios and 4 to 7 times more excess return per unit of risk than the highest ranked robust portfolio. The statistics in Table 7.8 indicate that these portfolios are ranked higher due larger average returns, and not lower standard deviations of returns, over the out-of-sample period.

Ranking of portfolio P_l by Sharpe and Sortino Ratio, Out-of-Sample

Benchmark = Risk-free rate		***MAR = Risk-free rate***		***MAR = Zero***	
P_l	Sharpe Ratio	P_l	Sortino Ratio	P_l	Sortino Ratio
EV.31	0.035	**EV.15**	0.451	**EV.15**	0.613
EV.27	0.027	**EV.19**	0.346	**EV.19**	0.500
Index	*0.023*	**EV.23**	0.288	***Index***	*0.478*
EV.23	0.017	***Index***	*0.285*	**EV.23**	0.477
2	0.004	**EV.31**	0.283	**EV.27**	0.453
4	0.004	**EV.27**	0.274	**EV.31**	0.433
6	0.004	**5**	0.257	**5**	0.433
8	0.004	**3**	0.248	**3**	0.424
5	0.002	**1**	0.248	**1**	0.424
1	-0.002	**7**	0.223	**2**	0.376
3	-0.002	**2**	0.195	**4**	0.376
EV.19	-0.004	**4**	0.195	**6**	0.376
EV.15	-0.008	**6**	0.195	**8**	0.376
7	-0.012	**8**	0.195	**7**	0.373

Table 7.7. Case 2 out-of-sample portfolio rankings for the Sharpe ratio (using monthly portfolio returns) and Sortino Ratios (using annualised monthly portfolio returns).

Reward and Downside Risk Statistics for Portfolio P_l, Out-of-Sample

	MAR = Risk-free rate (RF)				**MAR = 0**			
P_l	**N < RF**	**DD**	**ADR**	**VS**	**N < 0**	**DD**	**ADR**	**VS**
1	26	0.423	-0.300	1.119	25	0.383	-0.312	1.341
2	26	0.383	-0.247	0.941	25	0.352	-0.279	1.133
3	26	0.423	-0.300	1.120	25	0.383	-0.312	1.342
4	26	0.383	-0.247	0.941	25	0.352	-0.279	1.133
5	26	0.423	-0.299	1.137	25	0.383	-0.311	1.360
6	26	0.383	-0.247	0.941	25	0.352	-0.279	1.133
7	26	0.388	-0.283	1.124	25	0.386	-0.320	1.176
8	26	0.383	-0.247	0.941	25	0.352	-0.279	1.133
EV.15	26	0.443	-0.356	2.132	25	0.419	-0.319	2.263
EV.19	26	0.405	-0.305	1.641	25	0.396	-0.294	1.688
EV.23	26	0.397	-0.271	1.251	25	0.360	-0.261	1.488
EV.27	26	0.376	-0.240	1.178	25	0.355	-0.250	1.335
EV.31	26	0.361	-0.231	1.239	25	0.368	-0.273	1.243
Index	26	0.377	-0.271	1.348	25	0.345	-0.261	1.567

Table 7.8. Case 2 reward and downside risk statistics for the out-of-sample period. All calculations made having annualised the monthly portfolio returns.

The Sortino ratio (MAR is the risk-free rate), in contrast to the Sharpe ratio, suggests that *every* portfolio yields a positive return per unit of risk and has inverted the rankings of the *E-V* portfolios and the Index (with one exception) as well as ranked all six of them above the eight robust portfolios (Table 7.7). When the MAR was equal to zero or the risk-free rate, the Sortino ratio ranked portfolios very similarly. Table 7.8 shows that the downside frequencies, *DD*s and ADRs are similar for all portfolios when the MAR was the risk-free rate and when the MAR was zero. However, for both MARs, the *E-V* portfolios and the Index almost always had a larger VS, indicating that they had more variation of returns above the MAR than the robust portfolios. In addition, when the MAR was the risk-free rate, these portfolios had more variation of returns above the MAR, than below, as opposed to the cardinality constrained robust portfolios which had more variation of returns below the MAR.

Comparing only robust portfolios, just as in Case 1, the Sharpe ratio ranked cardinality constrained portfolios highest whereas the Sortino ratio ranked non-cardinality constrained portfolios highest (Table 7.7). The *DD*s, ADRs and VS (Table 7.8) indicate why conflicting ranks were given by the Sharpe and Sortino ratio (when the MAR was the risk-free rate). Smaller VS combined with *DD*s less than one indicate that the returns of cardinality constrained portfolios were less varied out-of-sample; thus, the Sharpe ratios were larger because their average monthly portfolio returns were larger. In addition, the smaller *DD*s of the cardinality constrained portfolios indicate that the Sortino ratio ranked them at the bottom because their average annualised portfolio return was less than that of the non-cardinality constrained portfolios.

Bootstrap Evaluation. The Sharpe ratio, which corresponds to the bootstrap sample of monthly asset returns (Table 7.9), ranked portfolios similarly to the out-of-sample Sharpe ratio rankings (Table 7.7): the Index and the *E-V* portfolios with the least variance were highest and their ratios gave an excess return 3 to 7 times larger per unit of risk than the highest ranked robust portfolio. In addition, the Sharpe ratio of portfolios P_1, P_3, P_5 and $P_{EV.15}$ were negative, indicating that their average return over all scenarios was less than the risk-free rate, resulting in a loss per unit of risk. The additional statistics (Table 7.11) show that the robust portfolios performed poorly compared to *E-V* portfolios and the index. Although their *DD*s and ADRs were smaller, indicating less variance of returns and higher returns, on average, below the risk-free rate, they had larger downside frequencies, i.e. fatter tails.

Ranking of portfolio P_l by Sharpe and Sortino Ratio, Bootstrap Samples

Benchmark = Risk-free rate		***MAR = Risk-free rate***		***MAR = Zero***	
P_l	Sharpe Ratio	P_l	Sortino Ratio	P_l	Sortino Ratio
Index	*0.615*	***Index***	*0.330*	**5**	2.241
EV.31	0.593	**EV.31**	0.279	**1**	2.227
EV.27	0.469	**EV.27**	0.174	**3**	2.214
EV.23	0.286	**5**	0.089	**2**	1.931
2	0.089	**EV.23**	0.076	**4**	1.931
4	0.089	**2**	0.064	**6**	1.931
6	0.089	**4**	0.064	**8**	1.931
8	0.089	**6**	0.064	**EV.31**	1.769
5	0.063	**8**	0.064	**7**	1.645
EV.19	0.016	**1**	0.018	***Index***	*1.510*
EV.15	-0.021	**3**	0.014	**EV.27**	1.471
1	-0.042	**EV.19**	-0.038	**EV.23**	1.027
3	-0.048	**EV.15**	-0.040	**EV.19**	0.569
7	-0.291	**7**	-0.074	**EV.15**	0.426

Table 7.9. Case 2 portfolio rankings for the Sharpe ratio (using monthly portfolio returns, from bootstrap of monthly asset returns) and Sortino Ratios (using annualised annual portfolio returns, from bootstrap of annual asset returns).

Reward and Downside Risk Statistics for Portfolio P_l, Bootstrap of Annual Returns

	MAR = Risk-free rate (RF)				**MAR = 0**			
P_l	**N < RF**	**DD**	**ADR**	**VS**	**N < 0**	**DD**	**ADR**	**VS**
1	529	0.044	0.022	3.959	112	0.026	-0.019	1.169
2	479	0.047	0.020	3.276	101	0.031	-0.024	1.075
3	529	0.044	0.022	3.940	112	0.026	-0.019	1.162
4	479	0.047	0.020	3.276	101	0.031	-0.024	1.075
5	490	0.046	0.021	3.863	108	0.027	-0.021	1.197
6	479	0.047	0.020	3.276	101	0.031	-0.024	1.075
7	541	0.052	0.015	3.194	145	0.033	-0.026	1.009
8	479	0.047	0.020	3.276	101	0.031	-0.024	1.075
EV.15	520	0.147	-0.064	1.654	380	0.121	-0.098	1.033
EV.19	522	0.116	-0.036	1.786	326	0.093	-0.076	1.016
EV.23	474	0.085	-0.011	2.302	252	0.062	-0.049	1.095
EV.27	460	0.064	0.007	2.777	166	0.047	-0.035	1.236
EV.31	413	0.057	0.012	3.006	121	0.042	-0.033	1.292
Index	374	0.071	0.000	2.828	165	0.051	-0.039	1.304

Table 7.10. Case 2 reward and downside risk statistics for portfolios evaluated using the bootstrap sample of 1000 annual asset returns. All calculations made having annualised the annual portfolio returns.

The rankings given by the Sortino ratio, which corresponds to the bootstrap sample of annual asset returns, were similar to those given by the Sharpe ratio, when the MAR was the risk-free rate. The Index and *E-V* portfolios $P_{EV.31}$ and $P_{EV.27}$ gave an excess return 2 to 4 times larger, per unit of downside risk, than the top ranked robust portfolio (Table 7.9). In addition, the Sortino ratios of the bottom three ranked portfolios, $P_{EV.19}$, $P_{EV.15}$ and P_7, were negative, indicating that their average return over all scenarios was less than the risk-free rate. Although the robust portfolios were outperformed by at least two *E-V* portfolios and the Index, their *DD*s were between 1% and 10% less and their ADRs were larger, and all positive (Table 7.10). These statistics indicate that robust portfolios had less variation of returns and greater returns, on average, below the risk-free rate. In addition, their VS was larger (Table 7.10); thus, most of the variation of robust returns was attributed to returns above, than below, the risk-free rate, even more so than for the other portfolios.

The Sortino Ratio, for a MAR of zero, suggests that robust portfolios were more advantageous if the investor's goal was simply to have nonnegative returns (Table 7.9). In addition to greater excess return per unit of downside risk, robust portfolios had less downside frequency, smaller *DD*s and greater ADRs (Table 7.10). In other words, robust portfolios had smaller tails, less variation of returns and greater returns, on average, below zero. Lastly, the VS indicates that the *UD*s were only slightly higher than the *DD*s for all portfolios, thus, all P_l had close to the same variability above zero as they did below zero. Since the *DD*s were smaller for robust portfolios and the VS was close to one for all portfolios, we can conclude that the *E-V* portfolios and the Index had higher *UD*s and thus, more variation in returns both above and below zero.

Comparing only robust portfolios, cardinality constrained portfolios were ranked highest by the Sharpe ratio (as they were in the out-of-sample analysis) and by the Sortino ratio when the MAR was the risk-free rate (with the exception of P_5), but were ranked lowest by the Sortino ratio when the MAR was zero (with the exception of P_7). The additional statistics in Table 7.10 show mixed results for both MARs: cardinality constrained portfolios have less downside frequency, but more variation and smaller returns, on average, below the MAR, and all robust portfolios had a VS close to one.

Reward and Downside Risk Statistics for Portfolio P_l, Bootstrap of Monthly Returns								
	MAR = Risk-free rate (RF)				MAR = 0			
P_l	N < RF	DD	ADR	VS	N < 0	DD	ADR	VS
1	528	0.019	0.042	1.029	0	–	–	–
2	454	0.020	0.042	1.068	0	–	–	–
3	530	0.020	0.042	1.023	0	–	–	–
4	454	0.020	0.042	1.068	0	–	–	–
5	479	0.019	0.042	1.113	0	–	–	–
6	454	0.020	0.042	1.068	0	–	–	–
7	611	0.024	0.038	0.805	2	0.011	-0.008	5.128
8	454	0.020	0.042	1.068	0	–	–	–
EV.15	501	0.060	0.010	1.008	162	0.046	-0.035	1.959
EV.19	492	0.047	0.021	1.025	95	0.037	-0.029	2.159
EV.23	371	0.034	0.031	1.199	35	0.022	-0.017	3.592
EV.27	303	0.026	0.037	1.380	7	0.013	-0.009	6.104
EV.31	273	0.022	0.040	1.604	5	0.007	-0.006	11.801
Index	246	0.023	0.039	1.912	5	0.011	-0.010	8.095

Table 7.11. Case 2 reward and downside risk statistics for portfolios evaluated using the bootstrap sample of 1000 monthly asset returns. All calculations made having annualised the monthly portfolio returns.

7.2.3 Discussion

We set out to determine the adequacy of the total return of a robust portfolio. We evaluated the risk-adjusted return measures (the Sharpe and Sortino ratio) of robust portfolios and ranked them against *E-V* portfolios and an Index portfolio. The same evaluation was carried out on two datasets, Case 1 and Case 2 (Table 7.12); the in-sample period was used to optimise the portfolios and the out-of-sample period was used for back-testing and served as the 'original sample' from which the bootstrap samples were generated.

Dataset	In-Sample	Out-of-Sample	Sample containing 2001-2002
Case 1	132 months	48 months	In-sample
Case 2	108 months	72 months	Out-of-sample

Table 7.12. Summary of Case 1 and Case 2.

Case 1 Summary of Results. Out-of-sample, the Sharpe ratio ranked the robust portfolios higher than *E-V* portfolios, in general, with the Index and the cardinality constrained robust portfolios tending to be ranked highest. The Sortino ratio (for both MARs) tended to rank *E-V* portfolios highest and cardinality constrained robust portfolios lowest. With respect to the bootstrap samples, the Sharpe ratio and both Sortino ratios ranked all eight robust portfolios higher than *E-V* portfolios and the Index, with non-cardinality constrained robust portfolios ranking highest. In addition, the bootstrap sample of monthly asset returns resulted in robust portfolios which had no returns below zero or the risk-free rate and the bootstrap sample of annual asset returns yielded robust portfolios which had less than 10 returns below the risk-free rate and one or no returns below zero. This is in contrast to *E-V* portfolios and the Index which had between 52 and 169 returns each below the risk-free rate and between 12 and 98 returns each below zero.

Case 2 Summary of Results. Out-of-sample, the Sharpe ratio ranked three of the five *E-V* portfolios and the Index highest with the cardinality constrained robust portfolios ranking higher than those without cardinality constraints. The Sortino ratio (for both MARs) ranked all *E-V* portfolios and the Index highest and the cardinality constrained robust portfolios lowest. With respect to the bootstrap samples, both the Sharpe ratio and Sortino ratio (MAR = risk-free rate) tended to rank the Index and *E-V* portfolios highest, with non-cardinality constrained robust portfolios tending to rank lowest, whilst the Sortino ratio when the MAR was zero ranked all but one robust portfolio above all *E-V* portfolios and the Index, with the non-cardinality constrained robust portfolios ranking highest. In addition, bootstrap samples resulted in seven robust portfolios having no returns below zero and one robust portfolio with two returns below zero. In contrast, two *E-V* portfolios and the Index had less than seven returns below zero, whilst the remaining three *E-V* portfolios had between 35 and 16.

Comments on the portfolio rankings given by the Sharpe and Sortino Ratio. According to Sortino and Price (1994) and Nawrocki (1999), if the distribution of asset returns is asymmetric, then the Sortino ratio will provide a more accurate ranking than the Sharpe ratio. In addition, Sortino (2003) observed that not only is it likely that the Sharpe ratio will rank portfolios incorrectly, but they will be ranked nearly the reverse to the ranking given by the Sortino ratio (when the MAR is the benchmark). We observed this also for Case 1 in the out-of-sample back-test, but not in the bootstrap sample back-test. For Case 2, we observed this to be partially true for the out-of-sample back-test, but not for the bootstrap sample back-test. By partially true, we mean that the rankings given by the Sortino ratio of robust portfolios tended to reverse the rankings of the robust portfolios given by the Sharpe ratio. Likewise, the rankings given by the Sortino ratio of *E-V* portfolios tended to reverse the rankings of *E-V* portfolios given by the Sharpe ratio. The combined rankings of *E-V* and robust portfolios given by the Sortino ratio did not, however, reverse the combined rankings given by the Sharpe ratio. In addition, the portfolio rankings given by the Sharpe and Sortino ratio in the bootstrap back-test, for Case 1 and Case2, were almost identical.

***Evaluation of Performance*.** With respect to the Sharpe and Sortino ratios, results show that in general, *E-V* portfolios and the Index performed better than robust portfolios in Case 2, in both out-of-sample and bootstrap back-testing. However, the robust portfolios significantly outperformed *E-V* portfolios and the Index in Case 1 bootstrap back-testing. Case 1 out-of-sample back-testing showed mixed results; the Sharpe and Sortino ratio gave conflicting rankings. In the instances when the ratios give different rankings, we consider the arguments of Sortino and Price (1994) and Nawrocki (1999), and chose the Sortino ratio because the distribution of asset returns was asymmetric. Thus, we conclude that *E-V* portfolios and the Index outperformed the robust portfolios out-of-sample in Case 1 and Case 2. Likewise, the non-cardinality constrained robust portfolios out-performed those with cardinality constraints. Lastly, we observed that in both Case 1 and Case 2, nearly all portfolios' risk-adjusted returns were nonnegative for each ratio. Thus, those portfolios (with nonnegative ratios) yielded a positive excess return above the benchmark/MAR per unit of risk (standard deviation/below-MAR semi-deviation).

Was the return of the robust portfolios 'adequate'? Results suggest they were. Although the robust portfolios were out-performed out-of-sample, their positive Sharpe and Sortino ratios suggest that they yield an excess return per unit of downside risk (with a couple of exceptions). In addition, they significantly outperformed *E-V* portfolios and the Index in the bootstrap back-test for all ratios in Case 1 and the Sortino ratio, when the MAR was zero, in Case 2.

7.3 Worst case Analysis

In the analysis of worst case events we evaluate the total return of each portfolio assuming that the realised return of each asset is its worst case value; we have estimated four worst case scenarios based upon the out-of-sample data and the monthly and annual bootstrap samples. In the robust portfolio literature, back-testing has not been carried out very frequently, but the majority of authors who have published back-testing results have included a comparison of the worst case performance of their robust model(s) with that of an *E-V* based model (see El Ghaoui, Oks and Oustry (2003), Tütüncü and Koenig (2004), Kim and Boyd (2007) and Gülpinar and Rustem (2007)). We have adopted the same approach.

The motivation for our analysis is as follows:

1. Is a robust portfolio beneficial in a worst case event? That is, how much, in terms of portfolio return, is gained or lost with respect to *E-V* portfolios?
2. How close is the worst case total portfolio return to its optimal objective function value? How does that compare to *E-V* portfolios?
3. How do threshold and cardinality constrained robust models compare to the unconstrained robust model in a worst case scenario?

The worst case value of each asset was estimated in two ways using three datasets, leading to four worst case scenarios. The first scenario, called *Out_Min*, was determined by taking the worst return of each asset over the entire out-of-sample period. The remaining three scenarios were determined by taking the lower bound of each asset given by $\bar{r}_i - 3\hat{r}_i$, where $\bar{r}_i$ and $\hat{r}_i$ are the mean log return and standard deviation, respectively, of the dataset. We have used a lower bound of $\bar{r}_i - 3\hat{r}_i$

because the robust models evaluated were optimised using $r_i \in [\bar{r}_i - 3\sigma_i, \bar{r}_i + 3\sigma_i]$ as the bounds for the uncertainty set where $\bar{r}_i$ and $\hat{r}_i$ are the mean log return and standard deviation, respectively, of the in-sample period. The second scenario, called *Out_LB*, was determined by the lower bound of each asset, where $\bar{r}_i$ and $\hat{r}_i$ were estimated from the out-of-sample period. The third scenario, called *Annual_LB*, is determined by the lower bound of each asset, where $\bar{r}_i$ and $\hat{r}_i$ were estimated from the bootstrap sample of 1000 annual returns. The fourth scenario, called *Monthly_LB*, was determined by the lower bound of each asset, where $\bar{r}_i$ and $\hat{r}_i$ were estimated from the bootstrap sample of 1000 monthly returns. We evaluated the worst case total portfolio return for every portfolio within each model, R_j ($j = 1..8, EV$), by comparing the optimal and worst case efficient frontiers of the robust and *E-V* models. A summary of models R_j can be found at the beginning of Chapter 5 (Table 5.1).

7.3.1 Case 1

First, we consider the benefits (if any) of a robust portfolio in a worst case event with respect to *E-V* portfolios by comparing the returns of each worst case scenario given by the robust and *E-V* models. The range of portfolio return for each model, under each worst case scenario, is given in Table 7.13. The maximum return *E-V* portfolio (EV.01) and the non-robust portfolio corresponding to $\Gamma = 0$ are not taken into account in determining these ranges. Both portfolios were the same, consisting of the asset with the largest mean log return. Neither portfolio would be chosen by an investor.

Range of Worst Case Portfolio Returns for *E-V* and R_1 Portfolios						
Worst Case Scenarios	***E-V* portfolios**			**R_1 portfolios**		
Out_Min	-19%	to	-12%	-15%	to	-6%
Out_LB	-31%	to	-20%	-23.5%	to	-17%
Annual_LB	-56%	to	-41%	-47%	to	-17%
Monthly_LB	-1.7%	to	-0.1%	-1.4%	to	0.3%

Table 7.13. Case 1: Range in portfolio return for *E-V* portfolios (efficient points 2 to 31) and R_1 portfolios (Γ=1..30) under each worst case scenario. Returns are expressed as percentages.

Results show that robust portfolios from model R_1 resulted in a range of higher returns under every worst case scenario (Table 7.13). For example, under scenario *Out_Min*, the *worst* worst case robust portfolio return resulted in a 15% loss whilst the *worst* worst case *E-V* portfolio return resulted in a 19% loss. Likewise, the *best* worst case robust portfolio return resulted in a loss of 6% whilst the *best* worst case *E-V* portfolio return resulted in a loss of 12%. Similar results were observed for the remaining worst case scenarios (Table 7.13). Thus, the robust portfolios were generally more beneficial than *E-V* portfolios in a *worst case* event with respect to the range of portfolio returns.

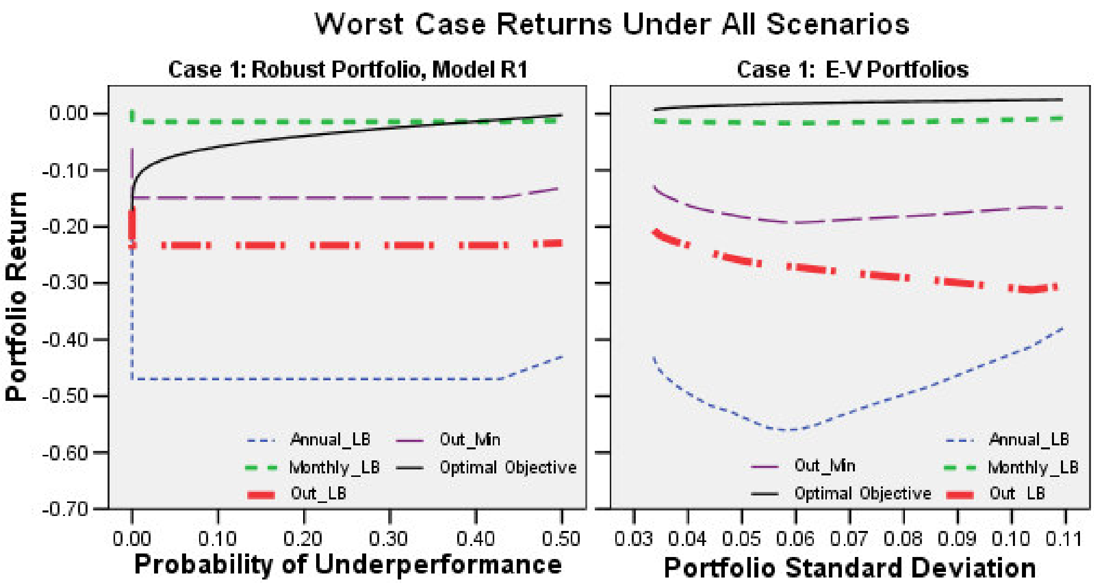

Figure 7.1. Case 1, the robust optimal objective value plotted with four worst case return scenarios for model R_1, for $\Gamma = 0..30$.

Figure 7.2. Case 1, the robust optimal objective value and four worst case return scenarios for the *E-V* model, for all 31 efficient points.

Second, we consider the distance between the optimal objective function value and each worst case scenario for the robust portfolios (Figures 7.1 and 7.3) and *E-V* portfolios (Figures 7.2 and 7.4). Plots of the optimal objective function value with each scenario show that the worst case returns of all *E-V* portfolios were less than the corresponding optimal objective under every scenario (Figure 7.1), whilst a number of robust portfolios resulted in worst case returns greater than the corresponding optimal objective under scenario *Monthly_LB*; similarly for a handful of portfolios under scenario *Out_Min* (Figure 7.2). In addition, plots showing the difference between the optimal objective and worst case for each portfolio, under every scenario, show that robust portfolios tended to be closer to their optimal objective function value (Figure 7.3) compared to *E-V* portfolios (Figure 7.4). Thus, losses resulting from a worst case event, with respect to the optimal objective function value, were generally less for robust portfolios than for *E-V* portfolios.

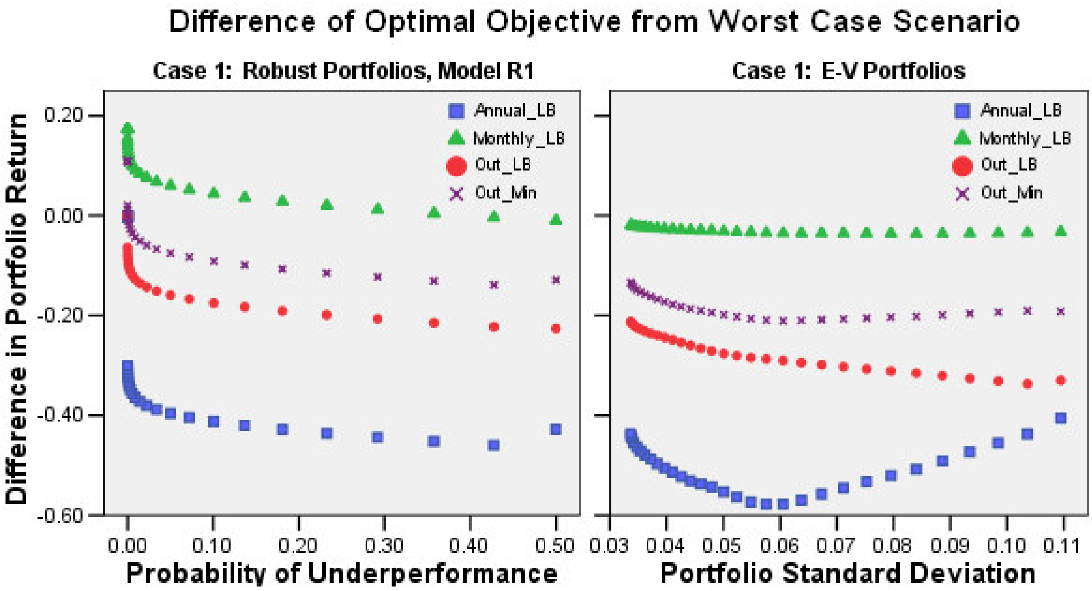

Figure 7.3. Case 1, the worst case robust return minus the robust optimal objective value for model R_1, for $\Gamma = 0..30$. This difference is plotted for all four worst case scenarios.

Figure 7.4. Case 1, the worst case return minus the optimal objective value for all 31 efficient points optimised using the *E-V* model. This difference is plotted for all four worst case scenarios.

Lastly, under scenario *Out_LB*, we compare the worst case returns of robust portfolios constrained by cardinality, a buy-in threshold or both to the worst case returns of unconstrained robust portfolios (Figures 7.5 and 7.6). Observe that models with the *worst* worst case returns also had the greatest optimal objective function value (Figure 7.5); thus, they also had the greatest difference in portfolio return (Figure 7.6). In addition, models R_1 and R_3 had the greatest difference whilst model R_8 had the least difference in portfolio return with respect to their optimal objective function values (Figure 7.6). Thus, portfolios constrained by cardinality, a buy-in threshold ($\alpha \neq 0.03$) or both, tended to have better worst case returns (Figure 7.5) and those worst case returns tended to be closer to their optimal objective function value (Figure 7.6). Moreover, the model with the highest buy-in threshold ($\alpha = 0.04$) in conjunction with a cardinality constraint of 20 assets had the *best* worst case return and was the closest to its optimal objective function value over all Γ. Similar results were observed under scenarios *Out_Min*, *Annual_LB* and *Monthly_LB*, and are given in Appendix B.

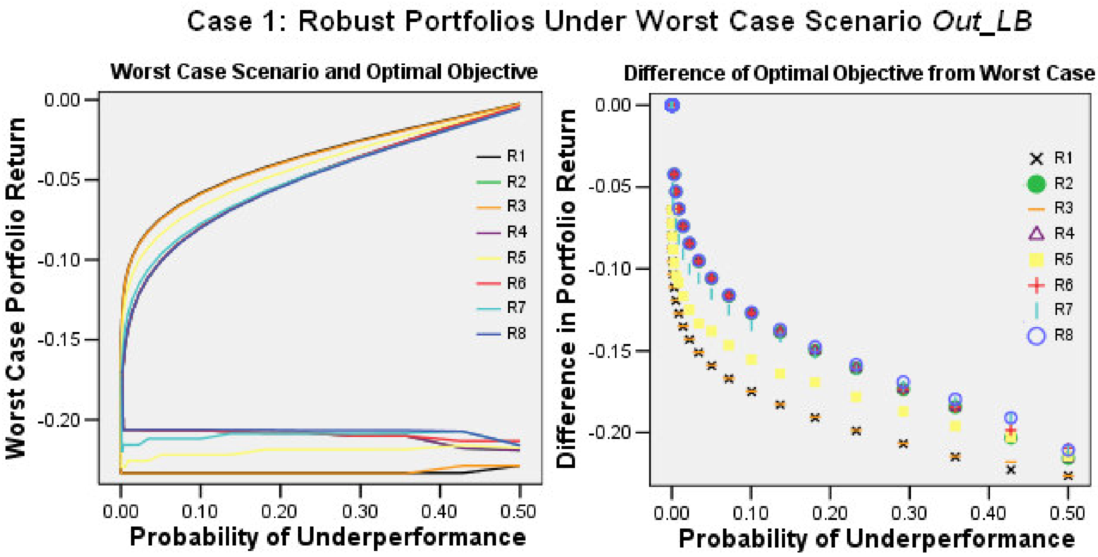

Figure 7.5. Case 1, the robust optimal objective value and worst case return for scenario *Out_LB* for model R_j, $j = 1..8$, and $\Gamma = 0..30$.

Figure 7.6. Case 1, the worst case robust return, under scenario *Out_LB*, minus the robust optimal objective value for models R_j, $j = 1..8$, and $\Gamma = 0..30$.

7.3.2 Case 2

As before, we first consider the benefits (if any) of a robust portfolio in a worst case event with respect to *E-V* portfolios by comparing the returns of each worst case scenario given by the robust and *E-V* models. The range in portfolio return for each model, under each worst case scenario, is given in Table 7.14. As in Case 1, these ranges do not include the maximum return *E-V* portfolio (EV.01) nor the non-robust portfolio corresponding to $\Gamma = 0$. Both portfolios consisted of the asset with the largest mean log return; thus, including their worst case return in the ranges may cause the results to be misleading.

Range of Worst Case Portfolio Returns for *E-V* and R_1 Portfolios						
Worst Case Scenarios	***E-V* portfolios**			**R_1 portfolios**		
Out_Min	-39%	to	-18%	-21%	to	-12%
Out_LB	-29%	to	-21%	-23%	to	-17%
Annual_LB	-52%	to	-41%	-65%	to	-45%
Monthly_LB	-1.7%	to	-0.1%	-2.4%	to	-1.3%

Table 7.14. Case 2: Range in portfolio return for *E-V* portfolios (efficient points 2 to 31) and R_1 portfolios $(\Gamma = 1..30)$ under each worst case scenario. Returns are expressed as percentages.

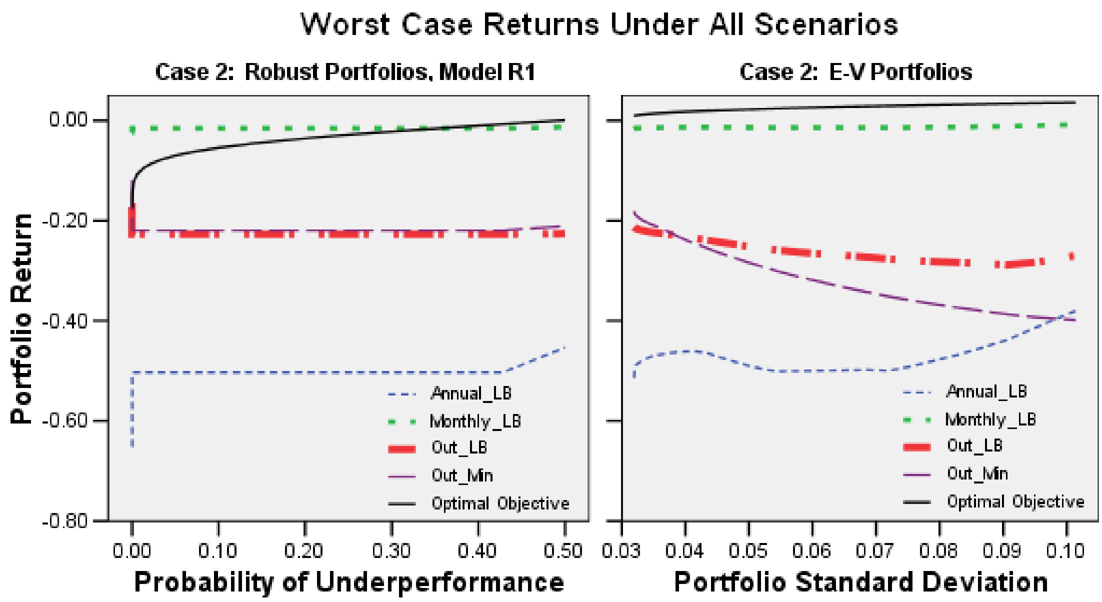

Figure 7.7. Case 2, the robust optimal objective value and four worst case return scenarios for model R_1, all $\Gamma = 0..30$.

Figure 7.8. Case 2, the robust optimal objective value and four worst case return scenarios for the *E-V* model, for all 31 efficient points.

Second, we consider the distance between the optimal objective function value and each worst case scenario for the robust portfolios (Figures 7.7 and 7.9) and *E-V* portfolios (Figures 7.8 and 7.10). As in Case 1, plots of the optimal objective function value with each scenario show that the worst case returns of all *E-V* portfolios were less than the corresponding optimal objective under every scenario (Figure 7.8), whilst a number of robust portfolios resulted in worst case returns greater than the corresponding optimal objective under scenario *Monthly_LB*; similarly for a handful of portfolios under scenario *Out_Min* (Figure 7.7). In addition, plots showing the difference between the optimal objective and worst case for each portfolio, under every scenario, show that robust portfolios tended to be closer to their optimal objective function value (Figure 7.9) compared to *E-V* portfolios (Figure 7.10). Thus, losses resulting from a worst case event, with respect to the optimal objective function value, were generally less for robust portfolios than for *E-V* portfolios.

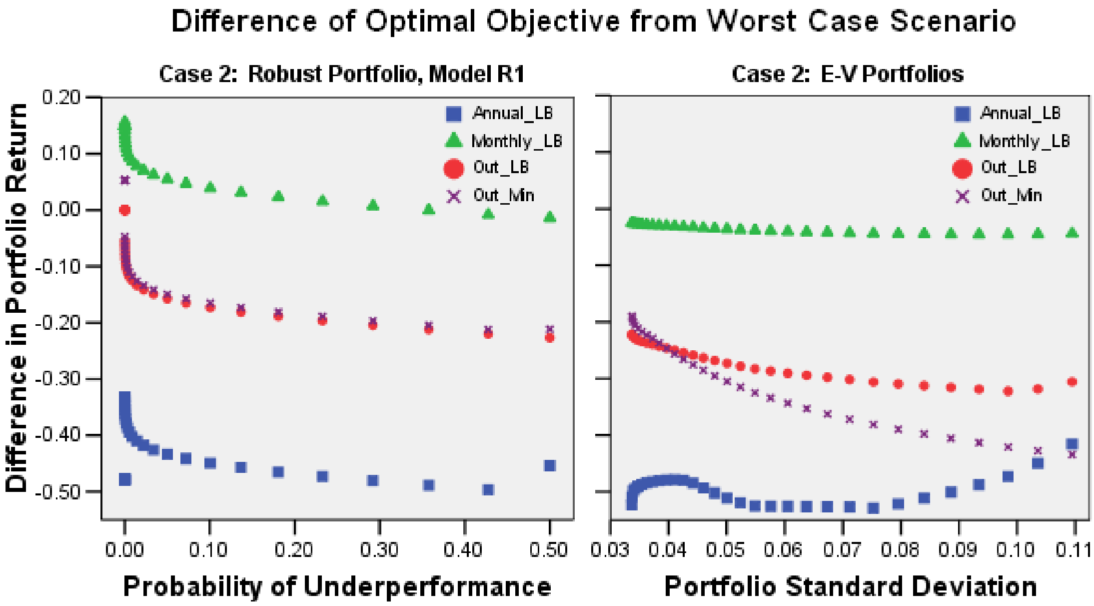

Figure 7.9. Case 2, the worst case robust return minus the robust optimal objective value for model R_1, for $\Gamma = 0..30$. This difference is plotted for all four worst case scenarios.

Figure 7.10. Case 2, the worst case return minus the optimal objective value for all 31 efficient points optimised using the *E-V* model. This difference is plotted for all four worst case scenarios.

Lastly, under scenario *Out_LB*, we compare the worst case returns of robust portfolios constrained by cardinality, a buy-in threshold or both to the worst case returns of unconstrained robust portfolios (Figures 7.11 and 7.12). Observe that models with the *worst* worst case returns also had the greatest optimal objective function value (Figure 7.11); thus, they also had the greatest difference in portfolio return (Figure 7.12). In addition, model R_1 had the greatest difference whilst model R_8 had the least difference in portfolio return (Figure 7.12). Thus, portfolios constrained by cardinality, a buy-in threshold or both, tended to have better worst case returns (Figure 7.11) and those worst case returns tended to be closer to their optimal objective function value (Figure 7.12). Moreover, the model with the highest buy-in threshold ($\alpha = 0.04$) in conjunction with a cardinality constraint of 20 assets had the *best* worst case return and was the closest to its optimal objective function value over all Γ. Similar results were observed under scenarios *Out_Min*, *Annual_LB* and *Monthly_LB*, and are given in Appendix B.

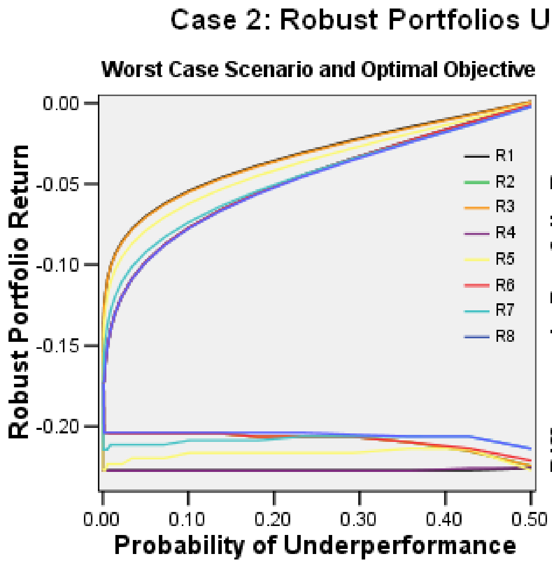

Figure 7.11. Case 2, the robust optimal objective value and worst case return for scenario *Out_LB* for model R_j, $j = 1..8$, and $\Gamma = 0..30$.

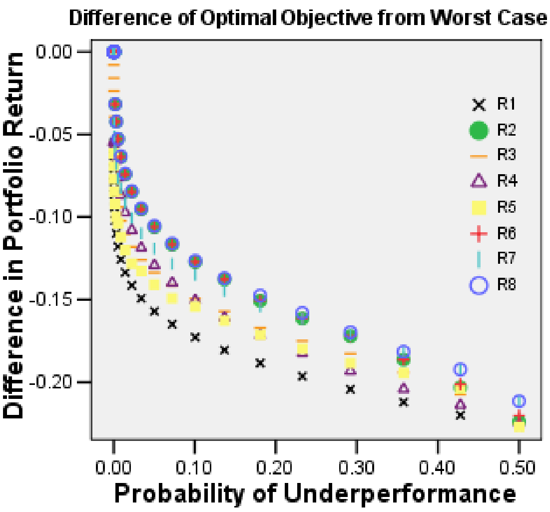

Figure 7.12. Case 2, the worst case robust return, under scenario *Out_LB*, minus the robust optimal objective value for models R_j, $j = 1..8$, and $\Gamma = 0..30$.

7.3.3 Discussion

The purpose of the worst case analysis was to evaluate and compare the total portfolio return of each model assuming that the realised return of each asset was its worst case value. Thus, for each Case, we generated four worst case scenarios: two from the out-of-sample period (*Out_Min* and *Out_LB*), one from the bootstrap sample of monthly returns (*Monthly_LB*) and one from the bootstrap sample of annual returns (*Annual_LB*). Case 1 and Case 2 yielded similar results, from which the same conclusions can be drawn. Their results showed that under every scenario, the robust portfolios (optimised using model R_1) tended to be closer to their optimal objective function value compared to *E-V* portfolios. Thus, losses, with respect to the optimal objective function value, resulting from a worst case event, were less for robust portfolios. In addition, compared to *E-V* portfolio returns, Case 1 robust portfolios (using R_1) resulted in a range of higher returns under all scenarios, whilst Case 2 robust portfolios (using R_1) resulted in a range of higher returns under the out-of-sample based scenarios (*Out_Min* and *Out_LB*), but a range of lower returns under the scenarios generated from the bootstrap samples. We showed that, although they resulted in a range of lower returns under scenarios *Monthly_LB* and *Annual_LB*, the difference from the range of *E-V* portfolio returns under scenario *Monthly_LB* was within 2% and the difference under scenario *Annual_LB* was in large part due to robust portfolios which would not be chosen as an investment, as they consist of only one asset.

Lastly, results from Case 1 and Case 2 showed that robust portfolios optimised by model R_1 had *worse* worst case returns and the greatest losses with respect to their optimal objective function values than did the other seven robust portfolios. Thus, portfolios constrained by cardinality, a buy-in threshold or both, tended to have better worst case returns and those worst case returns tended to be closer to their optimal objective function value compared to unconstrained portfolios. Moreover, the model with the highest buy-in threshold ($\alpha = 0.04$) in conjunction with a cardinality constraint of 20 assets had the *best* worst case return and was the closest to its optimal objective function value over all Γ.

Chapter 8

Conclusions and Future Research

8.1 Thesis Summary

In many real-world decision problems, a deterministic formulation is not sufficient since data may be characterised as either high precision or stochastic and thus, not known exactly. Consequently, not incorporating information about uncertainty, such as the interval within which the true value is likely to fall or characteristics about the variability and uncertainty of the data, can be misleading, and in many cases costly, depending upon the area of application. Therefore, there is a need for the development and application of methodologies for decision making under uncertainty. In Chapter 1, we discussed two of the most common methodologies, which treat variability and uncertainty in the modelling stage: stochastic optimisation and robust optimisation. While stochastic optimisation approaches have the potential to generate quality solutions, they are restricted by the computational demands of optimising over a set of scenarios for each uncertain parameter; thus limiting the size of the problem. In addition, they assume that the distributions of the uncertain parameters are either known or can be estimated with high precision; these distributions are then used to estimate the value of model parameters as well as generate scenarios. In contrast, robust optimisation makes very few assumptions regarding the distribution of uncertain parameters and does not require knowing the point estimate for any parameter value. Consequently, robust optimisation is widely considered as a practical alternative to stochastic optimisation approaches in the area of portfolio selection.

In this thesis we have investigated a specific robust optimisation approach to the portfolio selection problem in which the unknown and variable return of an asset is modelled by budgeted polyhedral uncertainty sets (introduced by Bertsimas and Sim, 2004). In particular, we have evaluated the corresponding budgeted robust counterpart of the Expected value – Variance portfolio selection model (*E*-*V*). Our aim was to determine whether or not this methodology forms a suitable foundation upon which to build real-world portfolio selection models. We did this through an extensive empirical investigation examining the trade-off between the robustness of robust portfolios and the sacrifice in optimality as well as the properties of robust portfolios from a practical perspective; that is, we wanted to assess whether robust portfolios make investment sense.

In Chapter 2, we established a basic understanding of how parameter uncertainty is modelled (both in structure and scale) in a robust optimisation framework. With respect to structure, our main focus was on a budgeted polyhedral representation of the uncertainty set U, and how it relates to ellipsoidal and polyhedral structures. In addition, we clearly defined the different aspects relating to the scale of U, highlighting recent work in this area.

In Chapter 3, we presented the portfolio selection problem. We argued that the assumptions of the *E*-*V* model are problematic because asset returns are not known constants. Robust portfolio selection, which treats the distribution of asset returns as uncertain, was presented as an alternative approach. In particular, the budgeted robust counterpart, resulting from budgeted polyhedral uncertainty sets, was presented in detail. The main contributions of this chapter included an interpretation of the robust portfolio selection model and the extension of that model to include a buy-in threshold constraint and cardinality constraint.

In Chapter 4, we investigated the cost and robustness of the unconstrained robust portfolio selection model given in Chapter 3, and we computed optimal solutions of the model for different descriptions of the uncertainty set U. In each instance, the structure of U remained constant (a budgeted polyhedral uncertainty set), but the scale of that structure changed. With respect to scale, we considered different definitions of the parameters specifying how the bounds of the uncertainty set are defined, as well as different values of the scaling factor *c* (which determines the magnitude of the structure of *U*). That is, we considered different definitions of the point estimate ($\bar{r}_i$) and deviation parameter ($\hat{r}_i$) of asset *i* defining the interval on which the true return (r_i) of asset *i* lies, i.e. $r_i \in [\bar{r}_i - c\hat{r}_i, \bar{r}_i + c\hat{r}_i]$. In addition, we investigated the effect of changing the size of the historical dataset from which the specific value of each parameter was estimated. The main objective of this chapter was to evaluate the cost and robustness of the robust models corresponding to these changes, both in the scale of *U* and in the size of the historical dataset.

In Chapter 5, we compared the cost and robustness of the unconstrained robust portfolio selection model to that of the *E-V* model and to the constrained robust portfolio models which included either a buy-in threshold, an upper limit on cardinality or both. For both the unconstrained and constrained robust models, we choose fixed definitions for the parameters $\bar{r}_i$ and $\hat{r}_i$, which were established in Chapter 4.

In Chapter 6, we discussed the properties of robust models with respect to diversification, asset selection and the distribution of asset weights amongst selected assets, based upon the total number of assets available, the size of the historical dataset (or number of observations) and the desired level of guaranteed probability of optimality. In addition, we examined whether these properties held when threshold and/or cardinality constraints are included.

Lastly, in Chapter 7, we compared the performance of the unconstrained robust portfolio, in terms of portfolio return, to that of *E-V* portfolios and an Index portfolio and to robust portfolios constrained by a buy-in threshold and/or cardinality. For two sets of data (Case 1 and Case 2), we back-tested these portfolios over the out-of-sample period as well as over two bootstrap samples which were generated using the out-of-sample period as the original sample, and evaluated their returns based upon two risk-adjusted return measures (the Sharpe and Sortino ratio) as well as downside risk and reward statistics (downside deviation (*DD*), volatility skewness (VS), downside frequency and average downside return (ADR)). In addition, we evaluated the worst-case performance of each model under four worst-case scenarios which were estimated using the out-of-sample period and both the monthly and annual bootstrap samples.

8.2 New Insights and Contributions

In this thesis we first provide insights into the robust formulation of an uncertain linear program, in which parameter uncertainty is defined by budgeted polyhedral uncertainty sets. We consider the robust counterpart as applied to portfolio selection, originally derived by duality, and show that it can be formulated by approaching an uncertain LP as a min-max problem. We further explain the properties of robust optimal decisions and model parameters, and examine the distinction between the structure and scale of an uncertainty set, with particular focus on scale. We suggest that the scale of the uncertainty set U, defining an uncertain parameter, has three aspects: first, how the bounds of the uncertainty set are defined (i.e. the point estimate $\bar{r}_i$ and the deviation parameter $\hat{r}_i$); second, once defined, how these bounds are estimated (i.e. historical data, bootstrapping, etc…), and third, the scaling factor c, which determines the magnitude of the structure of U. This again, provides further insights into the properties of this robust optimisation decision model. We have presented an empirical investigation of the cost and robustness of the robust counterpart to the portfolio selection problem, optimised for various definitions of the scale of U. We have re-defined the scale of U by changing how the bounds of the uncertainty set were defined and by changing the scaling factor c. Our results suggest that, of the definitions of $\bar{r}_i$ and $\hat{r}_i$ considered, the portfolios with the best trade-off between cost and robustness resulted from defining $\bar{r}_i$ as a measure of central tendency and $\hat{r}_i$ as a measure of spread, with respect to the distribution of the ith uncertain parameter. Results also suggest that the value of c is dependent upon the risk preferences of the modeller, as larger values of c may not affect the decisions, but will result in a more robust solution.

We have reported the application of robust optimisation to portfolio selection through an extensive empirical investigation of cost, robustness and performance with respect to risk-adjusted return measures and worst case portfolio returns. Furthermore, we proposed an extension of the robust portfolio selection model, which included a buy-in threshold and an upper limit on cardinality. We compared the unconstrained robust models to *E-V* models and to robust models constrained by a buy-in threshold, an upper limit on cardinality or both. The findings of our empirical study reinforces the intuitive view that robust solutions do come at a cost, but that is in exchange for a guaranteed probability of optimality on the objective function value and significantly greater achieved robustness. In addition, robust decisions generally yielded better realisations under worst case scenarios. Robust models constrained by a buy-in threshold and/or cardinality yielded solutions that were at least as robust, but at the same time decisions that were at least as costly, as the solutions and decisions of unconstrained robust models. In addition, the decisions of constrained robust models almost always yielded better realisations under worst case scenarios.

The research reported in this thesis offers new insights into the properties and behaviour of robust formulations of uncertain linear programmes. In particular, the role and construction of bounded convex sets to describe parameter uncertainty and the expected trade-off between cost and robustness are examined in detail. These insights enable more accurate descriptions of uncertainty with respect to budgeted polyhedral uncertainty to be constructed and, as a result, a better application of the robust methodology.

8.3 Future Directions

Through an extensive empirical investigation, we have established that a robust portfolio selection model formulated by modelling asset returns by budgeted polyhedral uncertainty sets is a suitable foundation on which to further build real-world portfolio selection models. By ‘suitable’ we mean that they provide an attractive trade-off between cost and robustness, the optimal solution achieves the robustness guaranteed by the model when the scaling factor c is chosen appropriately and their composition makes "investment sense". In a real-world portfolio selection problem, additional considerations, beyond the scope of this thesis, need to be made. First, we consider how the scale of the uncertainty set U may be defined in further investigations. In terms of defining the scale of U, we suggest the use of downside deviation, instead of standard deviation, as the deviation parameter $\hat{r}_i$. In addition, it would be interesting to consider the point estimate $\bar{r}_i$ as the MAR (minimum accepted return) of asset i, by which the value of the downside deviation ($\hat{r}_i$) was calculated. In this capacity, the MAR would be less influenced by an investor's goals as it would be by the distribution of returns for each asset. That is, the MAR of asset i would be in large part dependent upon the minimum return an investor is willing to accept from that asset rather than the minimum return an investor is willing to accept from the optimal portfolio. Based upon the evidence given throughout the literature, in support of below-target semi-variance as a more accurate measure of downside risk as well as the Sortino ratio as a more appropriate risk-adjusted return measure when distributions are asymmetric, we suggest that defining $\bar{r}_i$ and $\hat{r}_i$ as the downside deviation and the MAR of asset i, respectively, may yield more precise estimates of the worst case return of each asset. As assets are weighted by their worst case return, more precise estimates would correspond to an improved portfolio composition.

We consider how the parameters $\bar{r}_i$ and $\hat{r}_i$, which define the bounds of the uncertainty set U, are estimated. Throughout our empirical investigations we used raw market data, in the sense that, we did not apply any techniques to improve the estimates of these parameters. In practice, an investor would want the best estimates possible for $\bar{r}_i$ and $\hat{r}_i$; thus, they may wish to apply estimation techniques such as bootstrapping, moving averages, simulation, or forecasting.

We consider the weaknesses of using such a simplistic model. One critique is that the model is based on the assumption that asset returns are uncorrelated. Thus, one could incorporate asset correlations into the model; Bertsimas and Sim (2004) give one possible way of doing this. In addition, the model only considers one time-period and no recourse. Current research in the area of portfolio selection is moving toward two-stage and multi-stage problems, which leads to the inclusion of additional constraints on the costs incurred from recourse decisions. Furthermore, we only treated asset returns as uncertain, but not the covariance matrix of those returns. This did not have an effect on our solutions as the covariance between assets was not included in the optimisation of the model. However, if in further research the covariance of returns was included in the model's objective or constraints, then one would want to consider including uncertain parameters representing the variance/covariance of assets and modelling these parameters by uncertainty sets.

Lastly, our mixed-integer robust portfolio selection models, constrained by a buy-in threshold and/or an upper limit on cardinality, were limited in terms of the number of decision variables. Since the results showed these models to yield better portfolios than the unconstrained robust model, it would be interesting to further investigate more efficient solution methods for mixed-integer robust portfolio selection models such that a larger pool of assets can be used in the optimisation of the portfolio.

References

BEALE, E.M.L., 1955. On Minimizing A Convex Function Subject to Linear Inequalities. Journal of the Royal Statistical Society. Series B (Methodological), 17(2), pp. 173-184.

BELLMAN, R., 1966. Dynamic Programming. Science, 153(3731), pp. 34-37, American Association for the Advancement of Science.

BEN-TAL, A., EL GHAOUI, L. and NEMIROVSKI, A., 2009. Robust Optimization. Princeton, New Jersey: Princeton University Press. Princeton Series in Applied Mathematics.

BEN-TAL, A., MARGALIT, T. and NEMIROVSKI, A., 1999. Robust Modelling of Multi-Stage Portfolio Problems. In: H. FRENK, K. ROOS, T. TERLAKY and S. ZHANG, eds, High Performance Optimization. 1st edn. The Netherlands: Kluwer Academic Publishers, pp. 303-328.

BEN-TAL, A. and NEMIROVSKI, A., 1997. Robust Truss Topology Design via Semidefinite Programming. SIAM Journal on Optimization, 7(4), pp. 991-1016.

BEN-TAL, A. and NEMIROVSKI, A., 2001. Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications. Philadelphia, Pennsylvania, USA: Society for Industrial and Applied Mathematics (SIAM).

BEN-TAL, A. and NEMIROVSKI, A., 2000. Robust solutions of Linear Programming problems contaminated with uncertain data. Mathematical Programming, 88(3), pp. 411-424.

BEN-TAL, A. and NEMIROVSKI, A., 1999/8. Robust solutions of uncertain linear programs. Operations Research Letters, 25(1), pp. 1-13.

BEN-TAL, A. and NEMIROVSKI, A., 1998. Robust convex optimization. Mathematics of Operations Research, 23(4), pp. 769-805.

BERTSIMAS, D.J., BROWN, D. and CARAMANIS, C., 2008. Theory and Applications of Robust Optimization. Submitted.

BERTSIMAS, D.J. and BROWN, D.B., 2007. Constructing uncertainty sets for robust linear optimization. Available: http://faculty.fuqua.duke.edu/%7Edbbrown/bio/papers/bertsimas_brown_coherent_08.pdf.

BERTSIMAS, D.J. and PACHAMANOVA, D., 2008/1. Robust multiperiod portfolio management in the presence of transaction costs. Computers & Operations Research, 35(1), pp. 3-17.

BERTSIMAS, D.J. and SIM, M., 2006. Tractable approximations to robust conic optimization problems. Mathematical Programming, 107(1-2), pp. 5-36.

BERTSIMAS, D.J. and SIM, M., 2004. The Price of Robustness. Operations research, 52(1), pp. 35-53.

BERTSIMAS, D.J. and THIELE, A., 2006. "Robust and data driven optimization: modern decision making under uncertainty", in M. Johnson, B. Norman and N. Secomandi, editors, *Tutorials on Operations Research*, Chapter 4, pp.95-22, INFORMS, Hanover, MD.

BIENSTOCK, D., 2007. Experiments in Robust Portfolio Optimization. Available: http://www.columbia.edu/~dano/papers.html.

BLAZER, L., 2003. Chapter 8: Investment risk: a unified approach to upside and downside returns. In: F. SORTINO and S. SATCHELL, eds, Managing

Downside Risk in Financial Markets. Oxford, UK: Elsevier Butterworth-Heinemann, pp. 103-155.

BOYD, S. and VANDENBERGHE, L., 2004. Convex Optimization. Cambridge, UK: Cambridge University Press.

BURGMAN, M., 2005. Risks and Decisions for Conservation and Environmental Management. Cambridge, UK: Cambridge University Press.

CERIA, S. and STUBBS, R.A., 2006. Incorporating estimation errors into portfolio selection: Robust portfolio construction. J Asset Manag, 7(2), pp. 109-127.

CHARNES, A. and COOPER, W.W., 1959. Chance-Constrained Programming. Management Science, 6(1), pp. 73-79.

CHEN, X., SIM, M. and SUN, P., 2007. A Robust Optimization Perspective on Stochastic Programming. Operations research, 55(6), pp. 1058-1071.

DANTZIG, G.B., 2002. Linear Programming. Operations research, 50(1), pp. 42-47.

DANTZIG, G.B., 1955. Linear Programming Under Uncertaity. Management Science, 1(3 and 4), pp. 197-198-206.

DYER, M. and STOUGIE, L., 2006. Computational complexity of stochastic programming problems. Mathematical Programming, 106(3), pp. 423-432.

EL GHAOUI, L. and LEBRET, H., 1997. Robust Solutions to Least-Squares Problems with Uncertain Data. SIAM Journal on Matrix Analysis and Applications, 18(4), pp. 1035-1064.

EL GHAOUI, L., OKS, M. and OUSTRY, F., 2003. Worst-case Value-at-Risk and robust portfolio optimization: A conic programming approach. Operations research, 51(4), pp. 543-556.

EL GHAOUI, L., OUSTRY, F. and LEBRET, H., 1998. Robust Solutions to Uncertain Semidefinite Programs. SIAM Journal on Optimization, 9(1), pp. 33-52.

FABOZZI, F.J., 1999. Investment Management. Second Edition edn. Upper Saddle River, New Jersey: Prentice Hall.

FABOZZI, F.J., KOLM, P.N., PACHAMANOVA, D.A. and FOCARDI, S.M., eds, 2007. Robust Portfolio Optimization and Management. Hoboken, New Jersey: John Wiley & Sons, Inc.

GOLDFARB, D. and IYENGAR, G., 2003. Robust portfolio selection problems. Mathematics of Operations Research, 28(1), pp. 1.

GRANT, M., BOYD, S. and YE, Y., 2008. cvx Users' Guide for cvx version 1.2. Available: http://www.stanford.edu/~boyd/cvx/cvx_usrguide.pdf.

GREENBERG, H.J., 1968. Dynamic Programming with Linear Uncertainty. Operations research, 16(3), pp. 675-678.

GÜLPINAR, N. and RUSTEM, B., 2007. Worst-case robust decisions for multi-period mean–variance portfolio optimization. European Journal of Operational Research, 183(3), pp. 981-1000.

HENRION, R., 04 September 2004, 2004-last update, Introduction to chance constrained programming [Homepage of The Committee on Stochastic Programming], [Online]. Available: http:\\stoprog.org/SPIntro/intro2ccp.html [07/11, 2007].

IYENGAR, G., 2007. Lectures in Robust Optimisation. Lecture Notes edn. London Business School.

JONES, C., 2007. Investments: Analysis and Management. Tenth Edition edn. Hoboken, New Jersey: John Wiley & Sons, Inc.

KALL, P. and WALLACE, S.W., 1994. Stochastic Programming. Second Edition edn. Chichester: John Wiley & Sons.

KIM, S. and BOYD, S., 2007. Robust Efficient Frontier Analysis with a Separable Uncertainty Model [Stanford University]. Available: http://www.stanford.edu/~boyd/papers/pdf/rob_ef_sep.pdf.

LOBO, M.S. and BOYD, S., 2000. The worst-case risk of a portfolio [Stanford University]. Available: http://faculty.fuqua.duke.edu/%7Emlobo/bio/research-optimization.htm.

MARKOWITZ, H., 1952. Portfolio Selection. The Journal of Finance, 7(1), pp. 77-91.

MICHAUD, R.O., 1989. The Markowitz Optimization Enigma: Is 'Optimized' Optimal? Financial Analysts Journal, 45(1), pp. 31-42.

NAWROCKI, D., 1999. A Brief History of Downside Risk Measures. Journal of Investing, 8(3), pp. 9.

PFLUG, G. and WOZABAL, D., 2007. Ambiguity in portfolio selection. Quantitative Finance, 7(4), pp. 435-442.

RIDDLES, N., 2003. Chapter 7: A portfolio manager's view of downside risk. In: F. SORTINO and S. SATCHELL, eds, Managing Downside Risk in Financial Markets. Oxford, UK: Elsevier Butterworth-Heinemann, pp. 93-100.

ROM, B.M. and FERGUSON, K.W., 2003. Chapter 5: A software developer's view: using Post-Modern Portfolio Theory to improve investment performance measurement. Managing Downside Risk in Financial Markets. Oxford, UK: Elsevier Butterworth-Heinemann, pp. 59-73.

SCHRIJVER, A., 1986. Theory of linear and integer programming. Chichester, UK: Wiley.

SEN, S., 2004. Stochastic Programming: Computational Issues and Challenges. In: S. GASS and C. HARRIS, eds, Encyclopedia of Operations Research and Management Science. Dordrecht, The Netherlands: Kluwer Academic Publishers, pp. 784-789.

SHAPIRO, A. and NEMIROVSKI, A., 2005. On Complexity of Stochastic Programming Problems. In: V. JEYAKUMAR and A.M. RUBINOV, eds, Continuous Optimization: Current Trends and Applications. New York: Springer-Verlag, pp. 111-144.

SHAPIRO, A. and PHILPOTT, A., 2007-last update, A tutorial on stochastic programming [Homepage of The Committee on Stochastic Programming (COSP)], [Online]. Available: http://stoprog.org/SPTutorial/SPTutorial.html [07/11, 2007].

SHARPE, W.F., 1994. The Sharpe ratio. Journal of Portfolio Management, 21(1), pp. 49.

SHARPE, W.F., 1975. Adjusting for risk in portfolio performance measurement. Journal of Portfolio Management, 1(2), pp. 29-34.

SHARPE, W.F., 1966. Mutual Fund Performance. Journal of Business, 39(1), pp. 119.

SHARPE, W.F., ALEXANDER, G.J. and BAILEY, J.V., 1999. Investments. Sixth Edition edn. Upper Saddle River, New Jersey: Prentice-Hall International, Inc.

SORTINO, F., 2003. Chapter 1: From alpha to omega. In: F. SORTINO and S. SATCHELL, eds, Managing Downside Risk in Financial Markets. Oxford, UK: Elsevier Butterworth-Heinemann, pp. 3-25.

SORTINO, F. and FORSEY, H., 1996. On the use and misuse of downside risk. Journal of Portfolio Management, 22(2), pp. 35.

SORTINO, F. and KUAN, B., 25 February 2008, 2008-last update, the U-P strategy: A paradigm shift in performance measurement [Homepage of Pension Research Institute], [Online]. Available: http://www.sortino.com/htm/U-P%20Strategy.htm [February/4, 2009].

SORTINO, F. and PRICE, L.N., 1994. Performance Measurement in a Downside Risk Framework. The Journal of Investing, 3(3), pp. 59-64.

SORTINO, F.A. and VAN DER MEER, R., 1991. Downside risk. Journal of Portfolio Management, 17(4), pp. 27-31.

SOYSTER, A.L., 1973. Convex Programming with Set-Inclusive Constraints and Applications to Inexact Linear Programming. Operations Research, 21(5), pp. 1154-1157.

TUTUNCU, R.H. and KOENIG, M., 2004. Robust Asset Allocation. Annals of Operations Research, V132(1), pp. 157-187.

VOSE, D., 2000. Risk Analysis: A Quantitative Guide. 2nd edn. West Sussex, England: John Wiley & Sons Ltd., pp. 18-27.

Appendix A

Illustration of the Behaviour of Bootstrap Samples

In Section 7.1 we discussed the bootstrap sampling procedures used to generate two samples, which were used for back-testing portfolio returns in Sections 7.2 and 7.3. A bootstrap sample of 1000 monthly log returns was used in the evaluation of the Sharpe ratio and a sample of 1000 annual log returns was used in the evaluation of the Sortino ratio. In addition, the following downside risk and reward statistics were given for each sample: downside deviation (*DD*), volatility skewness (VS), downside frequency and average downside return (ADR). In Section 7.1 we mentioned that the bootstrap sample of monthly asset returns yielded portfolio returns that were less volatile than the bootstrap sample of annual asset returns, the effect of which was evidenced, in Section 7.2, by the risk-adjusted return ratios and reward and downside risk statistics. In addition, observe that, compared to the time series of *E*-*V* portfolio returns, the time series of robust portfolio returns is much less volatile. We illustrate this phenomenon for Case 1, in Section A.1, and Case 2, in Section A.2, through a time series of portfolio returns (for each sample) for robust models R_1 to R_8 ($\Gamma = \Gamma_{drop} - 1$ for each model) and the *E*-*V* model for efficient points EV.31, EV.27, EV.23, EV.19 and EV.15.

A.1 Case 1

In Figures A1.1 and A1.2 the time series of annualised monthly portfolio returns and annual portfolio returns, respectively, are shown using the same scale in order to show the marked difference between the returns resulting from the two bootstrapped samples. In addition to illustrating the difference between portfolio returns resulting from each bootstrap sample, Figures A1.1 and A1.2 show that the robust model resulted in more stable portfolio returns, for both bootstrapped samples.

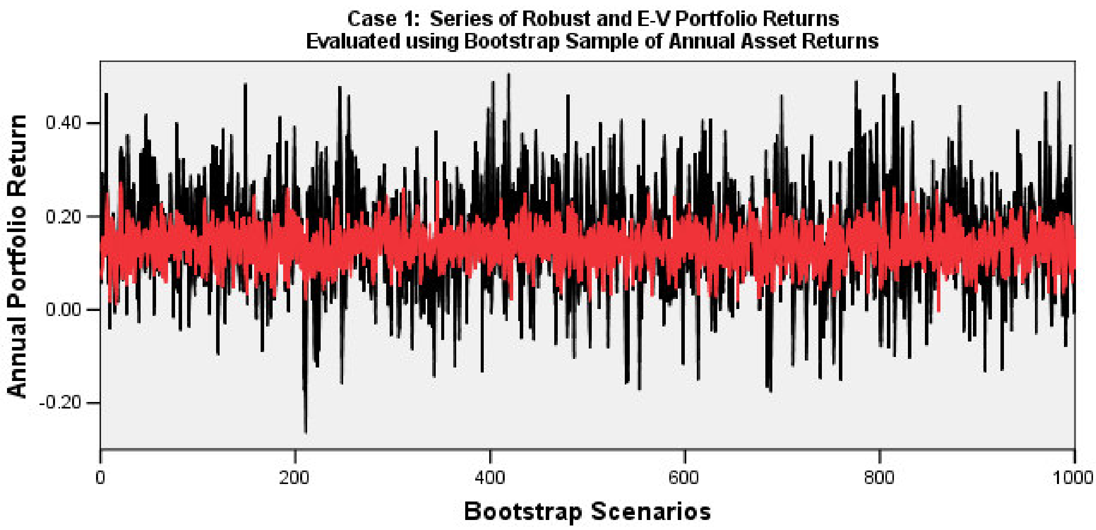

Figure A1.1. Case 1: Time series of portfolio returns from models R_1 to R_8 ($\Gamma = \Gamma_{drop} - 1$) and model R_{EV} (for efficient points EV.31, EV.27, EV.23, EV.19 and EV.15) evaluated using the bootstrap sample of annual asset returns. Robust time series are in *red*, *E-V* time series are in *black*.

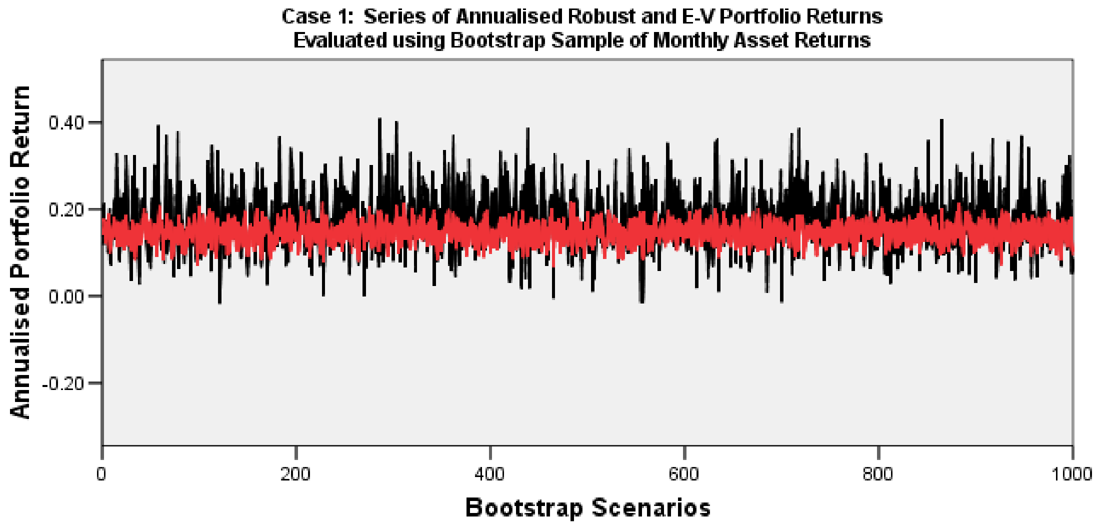

Figure A1.2. Case 1: Time series of annualised portfolio returns from models R_1 to R_8 ($\Gamma = \Gamma_{drop} - 1$) and model R_{EV} (for efficient points EV.31, EV.27, EV.23, EV.19 and EV.15) evaluated using the bootstrap sample of monthly asset returns. Robust time series are in *red*, *E-V* time series are in *black*.

A.2 Case 2

In Figures A2.1 and A2.2 the time series of annualised monthly portfolio returns and annual portfolio returns, respectively, are shown using the same scale in order to show the marked difference between the returns resulting from the two bootstrapped samples. In addition to illustrating the difference between portfolio returns resulting from each bootstrap sample, Figures A2.1 and A2.2 show that the robust model resulted in more stable portfolio returns, for both bootstrapped samples.

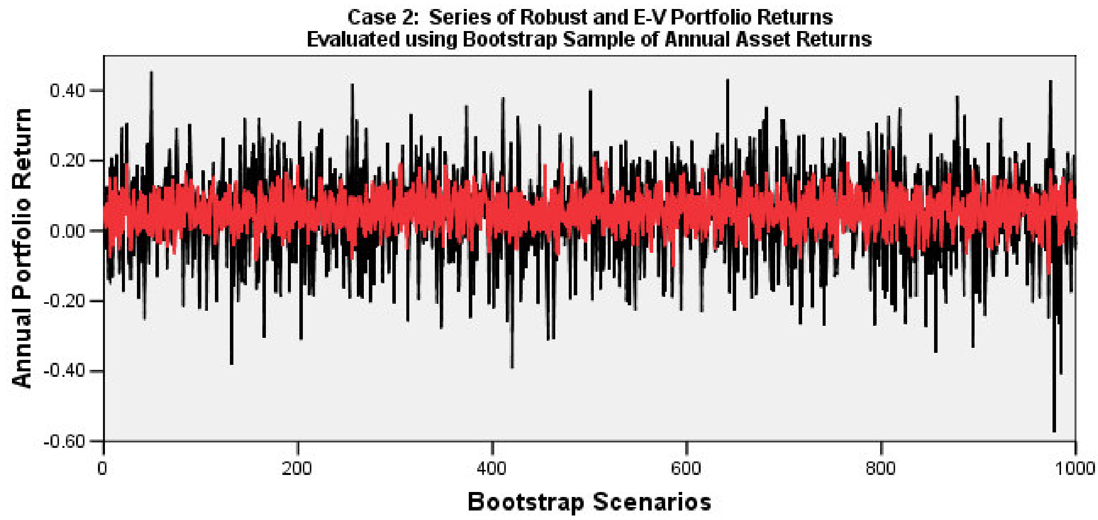

Figure A2.1. Case 2: Time series of portfolio returns from models R_1 to R_8 $(\Gamma = \Gamma_{drop} - 1)$ and model R_{EV} (for efficient points EV.31, EV.27, EV.23, EV.19 and EV.15) evaluated using the bootstrap sample of annual asset returns. Robust time series are in *red*, *E-V* time series are in *black*.

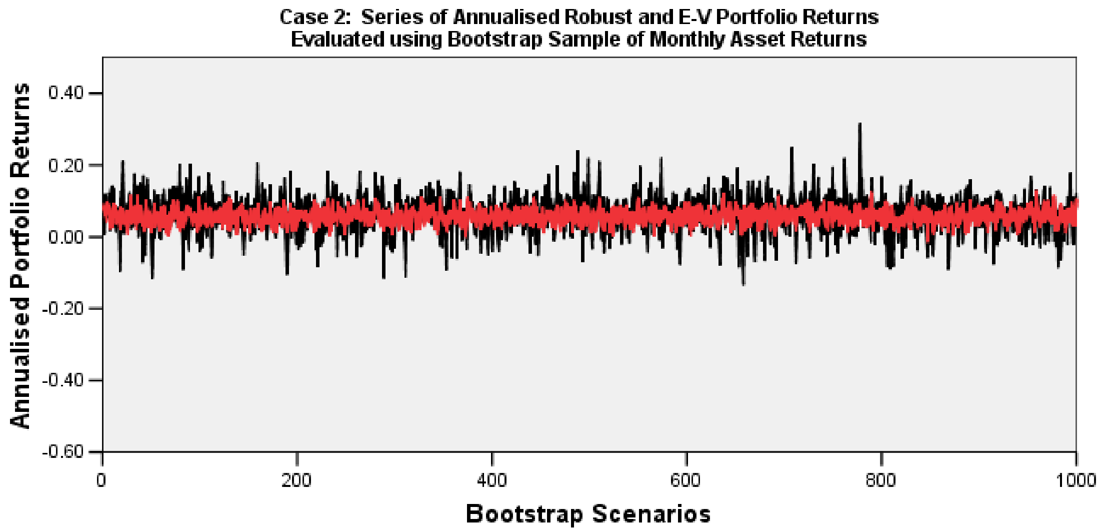

Figure A2.2. Case 2: Time series of annualised portfolio returns from models R_1 to R_8 $(\Gamma = \Gamma_{drop} - 1)$ and model R_{EV} (for efficient points EV.31, EV.27, EV.23, EV.19 and EV.15) evaluated using the bootstrap sample of monthly asset returns. Robust time series are in *red*, *E-V* time series are in *black*.

A.3 Discussion

Similar observations are made for Case 1 and Case 2. Recall that the period of returns from 2001 through to 2002 was included in the out-of-sample period for Case

2; thus, the time series of returns generated using the monthly and annual bootstrapped samples tended to be less in Case 2 than in Case 1.

The bootstrap methodology selected should be carefully considered, as vastly different samples can result. Ideally, one would want to generate a sample that is representative of the possible behaviour of asset returns. A comparison between the out-of-sample returns of robust and *E-V* portfolios for both data sets (Chapter 5), shows that whilst portfolio returns generated using the monthly bootstrap sample captured the positive returns, it did not adequately capture the negative returns observed out-of sample. This undesirable feature did not arise with the returns generated using the bootstrap of annual returns, which captured both positive and negative returns beyond those observed out-of-sample. This suggests that the bootstrap of annual returns may yield a more desirable description of the distribution of asset returns as it captured the possible behaviour.

Appendix B

Further Worst Case Analysis of Robust Models

B.1 Case 1

In Section 7.3.1 we investigated the returns of robust models R_1 to R_8 and the Expected value – Variance model (R_{EV}), under four worst case scenarios: *Out_Min*, *Out_LB*, *Annual_LB*, and *Monthly_LB*, for Case 1. More specifically we compared the optimal objective for the portfolios within each model to the corresponding worst case returns, under every scenario. We illustrated results for the comparison of robust model R_1 to *E-V* model R_{EV} for all four scenarios. In addition, we illustrated results for the comparison of robust model R_1 to robust models R_2 to R_8 for one scenario (*Out_LB*). In this section, we illustrate the results for the comparison of robust model R_1 to robust models R_2 to R_8 for the remaining three scenarios, *Annual_LB*, *Out_Min* and *Monthly_LB*, for Case 1.

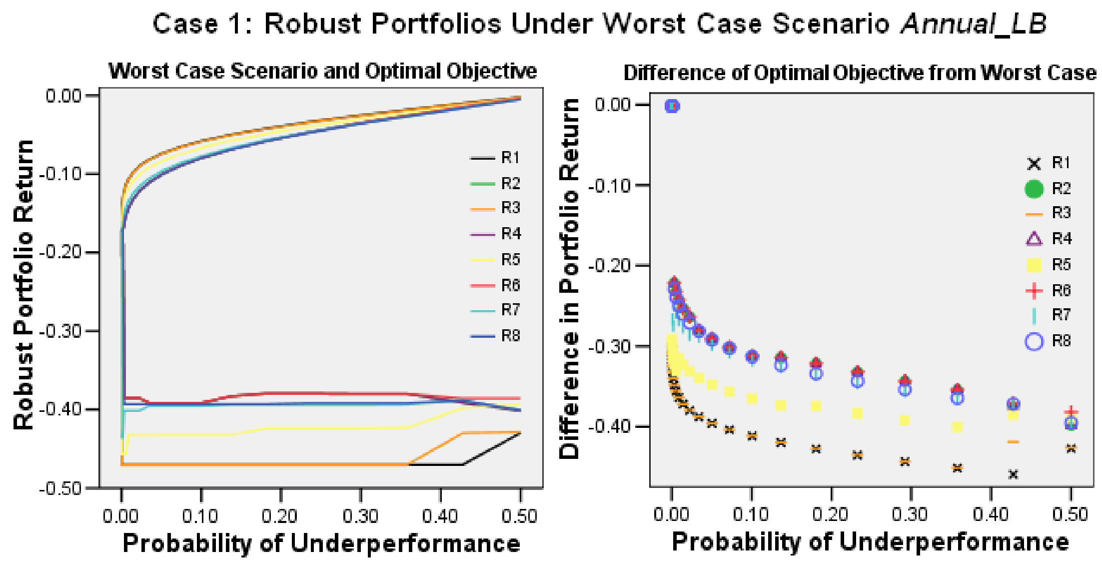

Figure B1.1. Case 1, the robust optimal objective value and worst case return for scenario *Annual_LB* for model R_j, $j = 1..8$, and $\Gamma = 0..30$.

Figure B1.2. Case 1, the worst case robust return, under scenario *Annual _LB*, minus the robust optimal objective value for models R_j, $j = 1..8$, and $\Gamma = 0..30$.

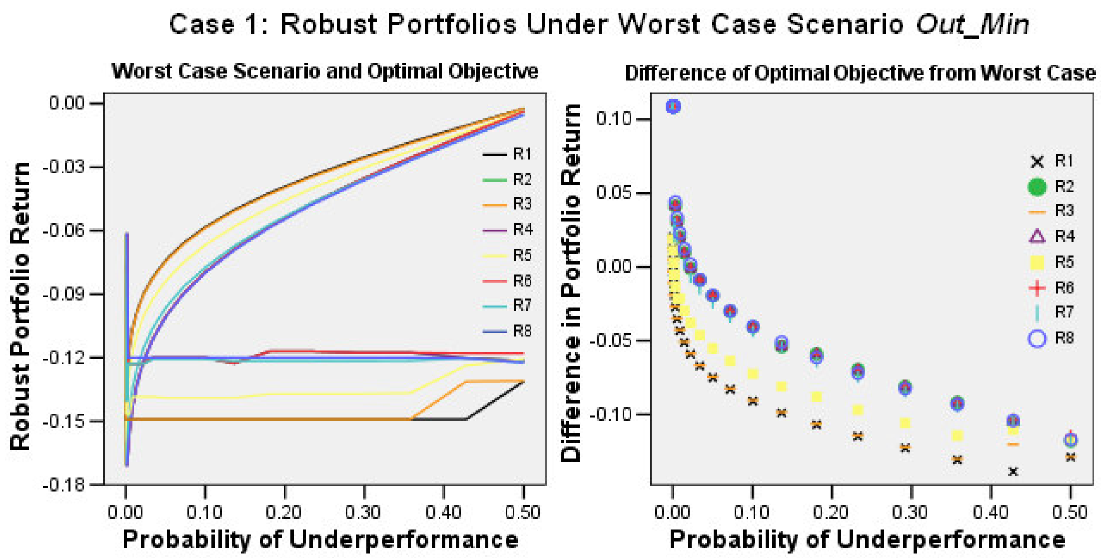

Figure B1.3. Case 1, the robust optimal objective value and worst case return for scenario *Out_Min* for model R_j, $j = 1..8$, and $\Gamma = 0..30$.

Figure B1.4. Case 1, the worst case robust return, under scenario *Out_ Min*, minus the robust optimal objective value for models R_j, $j = 1..8$, and $\Gamma = 0..30$.

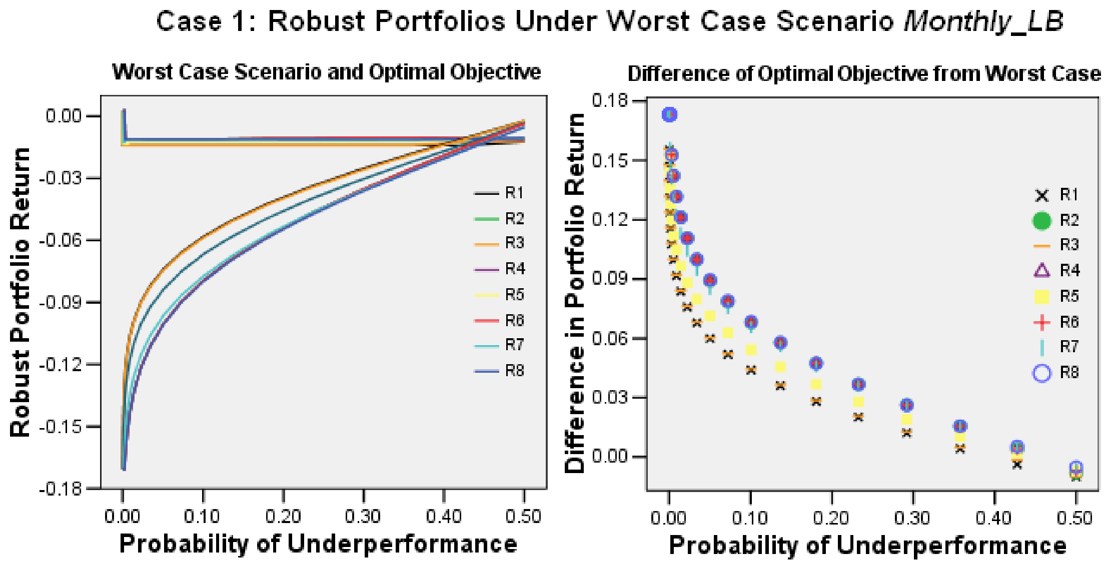

Figure B1.5. Case 1, the robust optimal objective value and worst case return for scenario *Monthly_LB* for model R_j, $j = 1..8$, and $\Gamma = 0..30$.

Figure B1.6. Case 1, the worst case robust return, under scenario *Monthly_LB*, minus the robust optimal objective value for models R_j, $j = 1..8$, and $\Gamma = 0..30$.

Under scenarios Annual_LB (Figures B1.1 and B1.2), Out_Min (Figures B1.3 and B1.4) and Monthly_LB (Figures B1.5 and B1.6) we compare the worst case returns of robust portfolios constrained by cardinality, a buy-in threshold or both to the worst case returns of unconstrained robust portfolios. Observe that under all three scenarios, models with the *worst* worst case returns also had the greatest optimal objective function value (Figures B1.1, B1.3 and B1.5); thus, they also had the greatest difference in portfolio return (Figures B1.2, B1.4 and B1.6). In addition, models R_1 and R_3 had the greatest difference whilst models R_2, R_4, R_6, R_7 and R_8 had the least difference in portfolio return, with respect to their optimal objective function values (Figures B1.2, B1.4 and B1.6). Thus, portfolios constrained by cardinality, a buy-in threshold ($\alpha \neq 0.03$) or both, tended to have better worst case returns (Figures B1.1, B1.3 and B1.5) and those worst case returns tended to be closer to their optimal objective function value (Figures B1.2, B1.4 and B1.6).

B.2 Case 2

In Section 7.3.2 we investigated the returns of robust models R_1 to R_8 and the Expected value – Variance model (R_{EV}), under four worst case scenarios: *Out_Min*, *Out_LB*, *Annual_LB*, and *Monthly_LB*, for Case 2. More specifically we compared the optimal objective for the portfolios within each model to the corresponding worst case returns, under every scenario. We illustrated results for the comparison of robust model R_1 to *E-V* model R_{EV} for all four scenarios. In addition, we illustrated results for the comparison of robust model R_1 to robust models R_2 to R_8 for one scenario (*Out_LB*). In this section, we illustrate the results for the comparison of robust model R_1 to robust models R_2 to R_8 for the remaining three scenarios, *Annual_LB* (Figures B2.1 and B2.2), *Out_Min* (Figures B2.3 and B2.4) and *Monthly_LB* (Figures B2.5 and B2.6), for Case 2.

Case 2: Robust Portfolios Under Worst Case Scenario *Annual_LB*

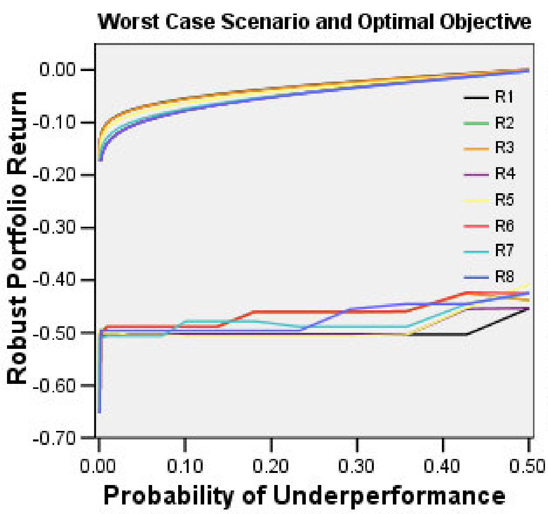

Figure B2.1. Case 2, the robust optimal objective value and worst case return for scenario *Annual_LB* for model R_j, $j = 1..8$, and $\Gamma = 0..30$.

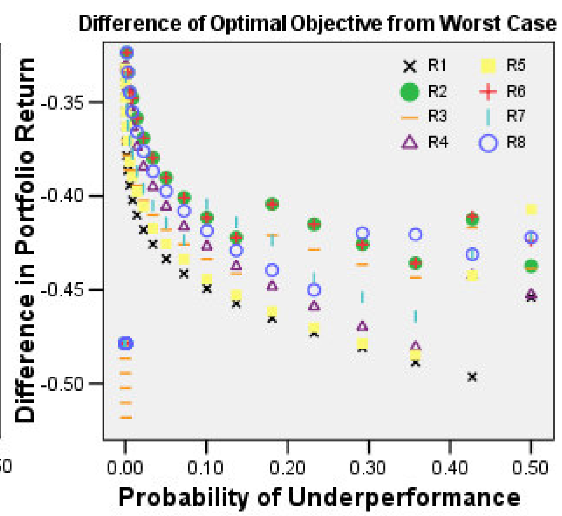

Figure B2.2. Case 2, the worst case robust return, under scenario *Annual_LB*, minus the robust optimal objective value for models R_j, $j = 1..8$, and $\Gamma = 0..30$.

Case 2: Robust Portfolios Under Worst Case Scenario *Out_Min*

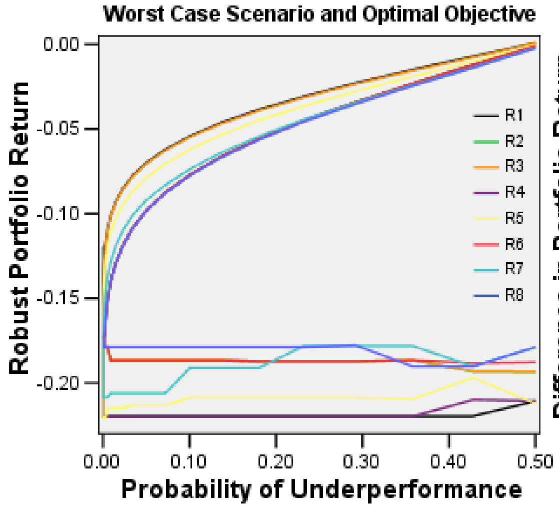

Figure B2.3. Case 2, the robust optimal objective value and worst case return for scenario *Out_Min* for model R_j, $j = 1..8$, and $\Gamma = 0..30$.

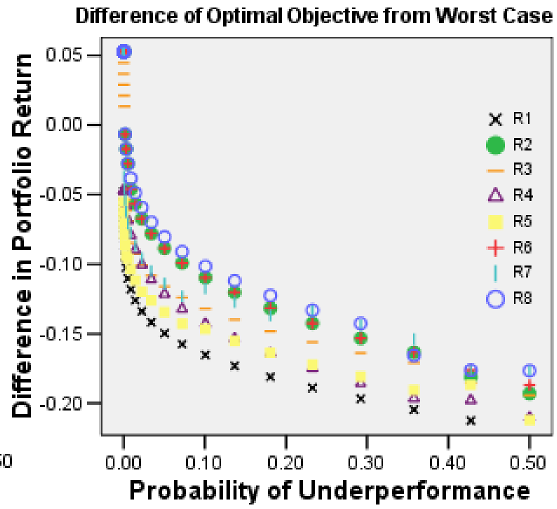

Figure B2.4. Case 2, the worst case robust return, under scenario *Out_Min*, minus the robust optimal objective value for models R_j, $j = 1..8$, and $\Gamma = 0..30$.

Case 2: Robust Portfolios Under Worst Case Scenario *Monthly_LB*

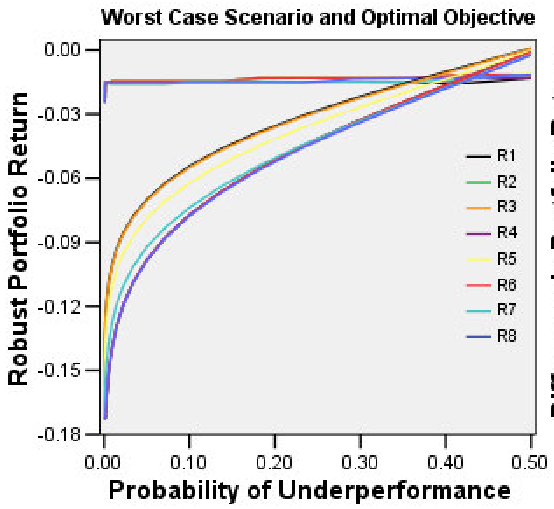

Figure B2.5. Case 2, the robust optimal objective value and worst case return for scenario *Monthly_LB* for model R_j, $j = 1..8$, and $\Gamma = 0..30$.

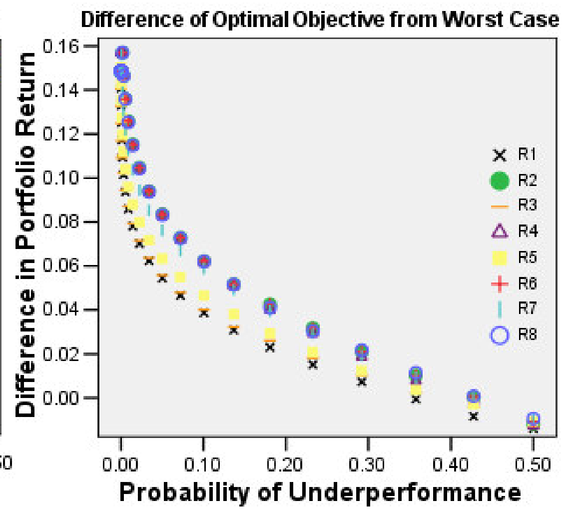

Figure B2.6. Case 2, the worst case robust return, under scenario *Monthly_LB*, minus the robust optimal objective value for models R_j, $j = 1..8$, and $\Gamma = 0..30$.

Under scenarios Annual_LB (Figures B2.1 and B2.2), Out_Min (Figures B2.3 and B2.4) and Monthly_LB (Figures B2.5 and B2.6) we compare the worst case returns of robust portfolios constrained by cardinality, a buy-in threshold or both to the worst case returns of unconstrained robust portfolios. Observe that under all three scenarios, models with the *worst* worst case returns tended to also have the greatest optimal objective function values (Figures B2.1, B2.3 and B2.5); thus, they tended to have the greatest difference in portfolio return (Figures B2.2, B2.4 and B2.6). In addition, models R_1 consistently had the greatest difference whilst models R_2 and R_6 tended to have the least difference in portfolio return, with respect to their optimal objective function values (Figures B2.2, B2.4 and B2.6). Moreover, portfolios constrained by cardinality, a buy-in threshold ($\alpha \neq 0.03$) or both, tended to have better worst case returns (Figures B2.1, B2.3 and B2.5) and those worst case returns tended to be closer to their optimal objective function value (Figures B2.2, B2.4 and B2.6).