The Micro-Evolution and Transfer of Conceptual Knowledge about Negative Numbers

by

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Declaration

Previous relevant publications:

Pratt, D. & Simpson, A. (2004a). McDonald's vs Father Christmas. *Australian Primary Mathematics Classroom*. **9** (3). pp 4-10.

Pratt, D. & Simpson, A. (2004b) Numbers and Maps: The Dynamic Interaction of Internal Meanings and External Resources in Use. http://www.merga.net.au/documents/RP562004.pdf [Accessed 20.09.08]

Stringer, M., Rode, J. A., Toye, E., Blackwell, A. & Simpson, A. (2005) "Webkit: A Case Study of Iterative Prototyping of a Tangible User Interface." *IEEE Pervasive Computing.* Special issue on Rapid Prototyping for Ubiquitous Computing - Oct-Dec 2005, **4** (4), pp 35-41

The above publications reported on a study carried out in 2002-2004, a European project "Webkit: Intuitive physical interfaces to the WWW" (IST-2001-341 171). I was involved as one of a team of researchers working across Europe. I contributed (as second and fifth author respectively) to the above papers. I acknowledge (and briefly describe) Webkit within this thesis as an experience which prompted my wish to undertake deep study into children's conceptual development. However, the research focus for Webkit was not the same as for my thesis and Webkit data is not included as data for this thesis. Any ideas for which Webkit may have been a resource are acknowledged within the thesis.

This thesis is all my own work and has not been submitted elsewhere for examination or publication.

Abstract

Children's failure to re-use knowledge will continue to be problematic until processes that contribute to conceptual growth are better understood. The notion that conceptual knowledge, soundly constructed and reinforced, forms the basis of future learning, as the learner uses it unproblematically to make sense of new situations in related areas, is appealing. This thesis will show this to be an overly simplistic view of learning, failing to take sufficient account of fine-grained processes that contribute to the micro-evolution of knowledge and of connections between cognition and other factors.

Much previous research focused on abstraction as key to learning. This thesis examines the role of abstraction in the development of mathematics concepts by children aged 8-9 years, using negative numbers as a window on their development of knowledge in a new domain. The assumption, prevalent in the literature, that abstraction is a requirement for transfer of knowledge is questioned.

Three research questions are explored:

1. What resources shape the nature of transfer and the growth of knowledge about negative numbers?
2. What is the role of the interplay of resources in the micro-transfer of knowledge about negative numbers?
3. What is the relationship between abstracting and transferring knowledge about negative numbers?

Methodology is based on a case study approach, initially recording the work of 3 small groups of children throughout a series of tasks and using progressive focusing techniques to create two case studies which are analysed in depth.

The thesis reports how the extent of conceptual development about negative numbers was influenced by interpersonal and intrapersonal learner characteristics, and describes a complex interplay between cognitive and affective factors. Micro-transfer and intermediate abstractions, and reinforcement of the connections that these construct, are found to be crucial for conceptual growth, though abstraction is not a condition for transfer at the micro-level.

Chapter 1: Introduction

1.1 Motivation

My motivation for conducting research in this area arose from my experience as a primary teacher. I like to think that I was an effective teacher; one who strived to develop my pedagogical and subject knowledge in order to maximise the potential for children to “learn” what I want them to learn. And yet, even where careful assessments informed me that “success criteria” had been met, and that children had learned what I had wanted them to learn, it often became clear, later, that they did not apply that learning in other relevant situations. For example, when I was a classteacher, Shaun had successfully measured a number of pencils and had been able to record their lengths in order; he knew that 21.4cm was longer than 21.25cm. However, in another lesson later in the same week, he was ordering a list of numbers and stated that 13.65 was bigger than 13.7. This exemplifies a phenomenon commonly described by teachers; that when faced with similar problems in other situations, children do not realise that they can use knowledge that had been effective in another situation previously to solve the problem: the new problem is regarded as novel, rather than a variation of one already encountered.

Another example from my experience as a primary teacher is the case of Sophie who had demonstrated (what appeared to be) secure knowledge about acute, obtuse and right angles. In a subsequent lesson, she was learning how to use a protractor to measure angles and, when faced with the decision of which scale around the protractor to use, seemed to choose one or the other quite arbitrarily. She therefore measured a 75° angle as 115°. Had she thought to use the knowledge that she had about acute and obtuse angles, she would have been able to work out that the angle she was measuring could not be 115° because it was clearly less than 90°. It is apparent that children are often not able to “apply” their knowledge – that is, they fail to transfer knowledge developed in one situation, to a new situation. Of course, such a description is a simplification of what we are actually asking children to do, as I shall show.

The mathematics curriculum in UK primary schools is presented as a spiral curriculum, whereby pupils are exposed to the same ideas many times, each time in a slightly different guise, intended to facilitate understanding that grows in depth and complexity. Prevalent within the curriculum are Piagetian principles relating to learning through experience, starting with what pupils already know, and to knowledge being a product of the way that individuals respond to and reflect on their experience. It is something of a paradox, therefore, that "The Primary Framework for Literacy and Mathematics" (DfES 2008) sets out a programme of teaching that is provided in each year of primary schooling. Mathematics education in UK primary schools is, therefore, currently based on a set of assumptions about what children need to learn according to their age.

I believe that the UK primary mathematics curriculum presents an overly simplistic view of the development of mathematical knowledge. It is portrayed as a stage process in which attainment of one level prepares the learner for the next and in which development occurs in a particular sequence. This model suggests that knowledge exists in different forms at different stages, and that it is perceived and experienced passively, as something that is possessed by the learner. This assumption – that knowledge is something to be possessed; that it can be given or acquired or lost – is, I believe, highly questionable; I question whether the learning pathway for individuals is so predictable. I also am unconvinced that children are able to revisit ideas often enough or in appropriate ways to facilitate effective conceptual growth and change.

In my view, the development of mathematical knowledge is far more complex than the "Primary Framework" suggests, both within each developing knowledge "thread" and in relation to other threads, which co-exist in various states of development.

Previous research in education has led to the development of a variety of theories and perspectives on learning. The roots of the curriculum are in research and theory. It is interesting to relate some of the dominant theories to concept development in mathematics and to consider whether the model for learning described above, that dominates our classrooms and curricula, actually coincides with, and is appropriate to, the development of mathematical concepts. If not, it may not be an appropriate way to be teaching primary children.

Constructivist principles underpin many of the ideals implicit in our modern curriculum. Although there are many variations of constructivism, a common focus, on learning by doing, and on building from existing knowledge, is apparent. von Glaserfeld (1983, cited in Lerman 1993) states that,

> *"We come to see knowledge and competence as products of the individual's conceptual organisation of the individual's experience." (p66)*

Confrey (1999) points out that Piagetian learning theory leads to a focus on the operational aspects of mathematical concepts. She notes that symmetry is therefore understandable through the action of folding and a circle is *"defined in relation to the action by which they are made"* (p6)

There are many examples in the UK primary curriculum of the facilitation of concept development from operational foundations such as: young children begin to learn about position and direction through physical activities involving movement and rotation of their own bodies; counting and calculating begins with movements on a number track.

However, in my view, constructivism and Piagetian learning theory fail to describe how learners manage the vast number of links and connections that must form networks or webs of knowledge; or how it is possible for knowledge structures (which are already multi-dimensional and in varying states of development) to remain stable under additional pressure from the huge cognitive load which must be caused by the processes of continuously modifying those, as well as "new", knowledge structures.

The constructivist view does, however, emphasise the role of the learner and of the learner's experience as being crucial in shaping the learning that occurs. I agree that this is central to any understanding about the cognitive mechanisms which are in play during learning. Constructivism does not satisfactorily illuminate the processes that enable boundaries between contexts to be overcome when similarities are not clear.

Having noted a few examples from my experience as a primary class teacher of children failing to re-use knowledge appropriately, it is also interesting to consider that in my current role as a senior lecturer in initial teacher training I now work with young adults who frequently demonstrate that they too have developed poor understanding of many mathematical concepts and skills. They are often not able to solve problems or even to perform elementary mathematics processes because they are unable to remember procedures and routines on which they had depended for their previous success in examinations at the end of their secondary education. One example is Rosemary who gave the answer 380 when asked to multiply 26 by 45. She explained that she didn't know how to "do long multiplication" and showed that she worked it out this way:

$$\begin{array}{r} 26 \\ \times \ \underline{{}_{3}45} \\ 380 \end{array}$$

She explained, "I did 6 times 5, that's 30 so put zero here and a little 3 up there. Then, 2 times 4, that's 8. Bring the 3 back in so it's three hundred and eighty? It's probably wrong 'coz I can't remember how to do it." Rosemary not only failed to use an appropriate algorithm, she also failed to realise that the answer should be a much larger number.

I see many examples of students' difficulties when I ask them to add and subtract negative numbers. In every group of students that I teach, I find several who are very uncertain about what to do when asked to carry out simple operations with negative numbers. Some find it difficult to add a negative number; even more are unable to subtract a negative number. My students often tell me that they did "do" negative numbers at school but never understood it; that they just guessed the answers and sometimes got them right. Some students remember rules such as “ two minuses make a plus” but can’t explain what that means; why the rule “works”. The young adults who I teach are the outputs of the education system and the mathematics curriculum in the UK.

It would appear that it is not only my students who have difficulties with negative numbers; in November 2007, a National Lottery scratchcard was withdrawn because the public did not understand how to order negative numbers. A report in the Manchester Evening News (Leeming 2007) explained:

> *“The Cool Cash game - launched on Monday - was taken out of shops yesterday after some players failed to grasp whether or not they had won.*
>
> *To qualify for a prize, users had to scratch away a window to reveal a temperature lower than the figure displayed on each card. As the game had a winter theme, the temperature was usually below freezing.*
>
> *But the concept of comparing negative numbers proved too difficult for some. Camelot received dozens of complaints on the first day from players who could not understand how, for example, -5 is higher than -6.*
>
> *Tina Farrell, from Levenshulme, called Camelot after failing to win with several cards.*
>
> *The 23-year-old, who said she had left school without a maths GCSE, said: "On one of my cards it said I had to find temperatures lower than -8. The numbers I uncovered were -6 and -7 so I thought I had won, and so did the woman in the shop. But when she scanned the card the machine said I hadn't.*

"I phoned Camelot and they fobbed me off with some story that -6 is higher - not lower - than -8 but I'm not having it.

"I think Camelot are giving people the wrong impression - the card doesn't say to look for a colder or warmer temperature, it says to look for a higher or lower number. Six is a lower number than 8. Imagine how many people have been misled."

A Camelot spokeswoman said the game was withdrawn after reports that some players had not understood the concept.

She said: "The instructions for playing the Cool Cash scratchcard are clear - and are printed on each individual card and in the game procedures available at each retailer. However, because of the potential for player confusion we have decided to withdraw the game."

It would seem that many young adults have been failed by the mathematics curriculum in the UK since, for many, it does not appear to enable the development of good conceptual knowledge.

As well as teaching young adults at university and children in primary schools I have also worked with children in schools where I was not employed as a teacher but worked as an education researcher. For two years I worked with children in primary and secondary schools and encountered there, too, many examples of poor conceptual knowledge. In one school, Gavin, aged 8 was explaining to me that he could work out ¼ of a number by halving it and then halving again. He told me that he was annoyed with himself, however, when calculating the length of each side of a square with perimeter 48cm, that he couldn't do that because "I don't know my 4 times table that far, I can only go 4, 8, 12, 16, 20, 24, 28, 32, 36 … up to 40cm." Clearly, Gavin knew what "perimeter" means and knew that squares have 4 equal sides but he did not know that the halving strategy he had learned in another context would help him with the square problem.

My research focus, on the finest grain processes that are involved in learning, has therefore evolved out of my experience as a teacher, manager and researcher in different contexts. The “Webkit” project [1] was a 2 year study which explored a potential application for novel technological user interfaces (“TUI”s) in schools. It considered affordances of real-world and virtual environments and evaluated the effect of using tangible interfaces across the 2 settings on the likelihood of knowledge learned, through one task or experience, being re-used in another (i.e. application or transfer). In the lifespan of the project it was not possible to evaluate re-usability with any confidence. However, what did emerge, that was of interest to me, was evidence of robust and flexible learning that related to some difficult concepts (using a TUI). This was an interesting outcome because the quality of children’s learning evident on completion of their tasks was higher than I would “normally” have expected to see, based on my own primary mathematics teaching experience. This prompted me to consider features of the research sessions that might have contributed to improved effectiveness of teaching and learning.

An aspect that inspired particular interest for me was the insight that the research trials provided into the ways children developed understanding and new knowledge by linking it with experiences and “old” knowledge and with other new knowledge. Analysis of trials data provided insight into ways that children used a wide range of resources available to them. I was intrigued to note that children’s existing knowledge included knowledge in many different stages of construction. Pre-existing pieces of knowledge had been processed in some way and were available as resources to help children make sense of their task.

It was also possible to infer different ways in which the children were able to make links “in-action” – i.e. whilst actually engaged with the task given, they were becoming aware of connections with other aspects of the task and of their own thinking. Such thinking-in-change was also evident in

[1] This is a European project “Webkit: Intuitive physical interfaces to the WWW” (IST-2001-341 171).

children's responses to each others' contributions. Marie, for example, whilst subtracting using a number line, suddenly remarked, "So, adding a minus 3 is the same as taking away 3. Hang on ... that's 2 lots of minus 3 is minus 6, 2 times minus 3 is minus 6! Hey! When Father Christmas goes somewhere 3 degrees colder its like taking away 3 degrees and its like adding minus degrees, adding coldness. So they're the same??!!"

So, the focus for my research, emerging out of a longstanding interest in the pedagogy of mathematics, and out of my Webkit experience, is on learning, and particularly on factors and processes (both real-world and cognitive) that affect the re-usability of knowledge. I want to understand more about the mechanisms by which children process knowledge and experience and understanding in order to create new knowledge; changing and reshaping old knowledge and incorporating new knowledge. I want to understand better what it is that enables or facilitates (and, by implication, inhibits or limits) the re-use of knowledge in new situations.

In order to be able to learn about the way children use and re-use resources (to develop knowledge) and knowledge (as a resource) we must create the conditions where this might occur and can be observed. These, and other key methodological and research design issues will be fully addressed in Chapter 4.

Learning is complex; in acknowledging that complexity, I imply that complexity is also required in order to observe and understand what is involved. I would argue that, until more is known about re-use of knowledge in a range of tasks and environments, the potential for those tasks that are used in empirical research studies to illuminate the complex cognitive processes (that are involved in learning) is inevitably limited by the extent of our understanding of those processes. Therefore, in a broad sense, research must, necessarily, be iterative if research methods are to converge in a way that means that the finer processes are fully observable. My own research methodology has been devised in the light of current understanding and is therefore a tentative dipping of a toe into the water. I would, therefore, expect my research to generate more questions than answers.

What I have set out to do is to shed some light on cognitive processes - relating to learning, to conceptual growth, and to the application or transfer of knowledge - that will contribute to a deeper understanding of those processes. A valuable outcome of my research would be that it informs an increasingly appropriate and relevant methodology for subsequent studies. Those studies might, thereby, be enabled to discern and describe cognitive processes leading to conceptual development more precisely and more certainly than I am able to. It is only when we know more about how children learn mathematics that studies can be designed to optimise tasks and conditions in order to maximise the scope to be able to see their learning.

In the next chapter, I shall explore a range of literature in the field of learning in mathematics, particularly about conceptual growth and change and the re-use or “transfer” of knowledge. Through an analysis of relevant literature, I shall state my own position and develop and clarify my research questions.

My research focus is on learning and conceptual change and transfer; I need to observe those processes in some detail as they occur if I am to be able to elucidate them. Therefore, I need to select an area within mathematics where these processes are likely to be invoked. I have already identified negative numbers as an area within mathematics that my pupils and students find difficult. I shall therefore, in Chapter 3, review research findings about learning about negative numbers and consider why this might be an appropriate domain to use as a window through which I might observe learning processes and sub-processes.

Chapter 4 will describe and rationalise my methodology which is based on a series of teaching sessions with children in a primary school. I shall describe the tasks that I devise with the intention of introducing new knowledge and then extending the range of contexts in which children may or may not re-use their new knowledge.

In my analysis of my findings, in Chapter 5, I shall focus on the work of two individual children and detail the cognitive changes that I am able to infer from the data. In Chapter 6, I shall discuss my findings in a more thematic way, considering factors, identified in my review of the literature, that might influence learning and reflecting on patterns and peculiarities that emerged.

In the concluding chapter, Chapter 7, I shall draw together all the facets of my research interests and my data analysis in order to re-draw core relationships between learning, transfer and conceptual change. I shall conclude that conceptual change is linked to many factors, only one of which is cognition. I shall conjecture that cognition is so deeply connected to other factors that it is not possible to understand learning unless the nature of those connections is understood and taken into consideration. Moreover, I shall suggest that, if the only changes considered worthy of investigation in the field of mathematics education research are cognitive changes, it is unlikely that our knowledge about learning can progress.

Chapter 2: Literature Review

2.1 Disparity in the focus of previous research

I must consider the theoretical frameworks and perspectives that underpin research in this field and try to highlight the commonalities and shared understandings that exist, even though the labels that are used for them may vary. I will go on to present my view that apparent confusion and disagreement within the literature is not actually that; rather, it is simply the result of a differential focus – brought about by differences in perspective, leading to differences in the grain size of the processes selected for investigation in the development of a range of theories about learning and knowledge transfer. There are at least 3 different levels at which development can be observed and described; 2 at what might be described a macro-level, and a third at a micro-level:

Level 1: Grand Theory

“Grand theory” is a term used to describe broad theoretical frameworks that describe some aspect of the world or human experience in its most general sense. They are abstract and normative and, in the social sciences, seek to explain the nature of vast populations. In the physical sciences contributors to grand theory include Einstein and Newton – responsible for such well-known contributions such as Einstein’s Theory of General Relativity and Newton’s Laws of Motion. In education, Skinner, Piaget and Vygotsky have contributed and I shall briefly consider ways in which their contributions relate to my interests. In the physical sciences, such theories serve to unify and bring together other theories and rules and can be used to explain phenomena at very specific, as well as general, levels. The appeal of grand theory in education is, I believe, the way it offers logical, well-argued analysis which illuminates many of the issues which concern those of us who have an interest in education: it provides hope and purpose and direction. However, at the classroom level, the experience of teaching and learning does not always follow the rules that educational grand theory would predict. For teachers and pupils, there are variations and deviations from the expected path that are often complex and convoluted. Very often, the learning pathway diverges from that which grand theory would have predicted and never re-joins the original expected route or attains the “normal” outcome. Educational grand theory, has, thus far, been unable to unify the diverse theories that have been developed to describe aspects of learning. It does not accommodate the messiness and noise that characterises the real world. This messiness and noise often leads to distortion and disturbance of the directions and outcomes of learners’ experience. Grand theory cannot and should not, therefore, be used to predict the learning pathway of individuals and small groups within the whole population.

diSessa & Cobb (2004) describe what might be seen as a taxonomy of theories. Their focus of interest is design research; in my opinion, that is not to say that their observations are not applicable to a wider audience than those who consider themselves design researchers. diSessa & Cobb begin by acknowledging the importance of research being related to theory but go on to state that, unlike theories in the physical sciences,

> *“Theories concerning educational matters seem to replace one another, rather than subsume, extend, or complement other theories. While the state of the art constantly changes, it is often difficult to tell that progress is being made.”(p79)*

diSessa & Cobb point out that the difficulties with grand theory are typically due to,

> *“ … some combination of being, as yet, immature (e.g. false as categorical prescriptions of cognitive or social processes), imprecise (so that implications at the level of design decisions are unsure), or simply too high-level to inform the vast majority of consequential decisions in creating good instruction. To take a specific case, Piaget developed his theory to address*

> *epistemological issues that concern the nature and growth of knowledge. Nonetheless, his work had a strong educational influence from the 1960s at least through the early 90s. We feel, as many others do, that Piaget's ideas were overextended into education." (p80)*

Burkhardt & Schoenfield (2003) agree,

> *"Most of the theories that have been applied to education are quite broad. They lack what might be called "engineering power". To put it a different way, they lack the specificity that helps to guide design, to take good ideas and make sure that they work in practice. ….. Education lags far behind in the range and reliability of its theories. By overestimating theories' strength (or perhaps better, by not constraining their application appropriately) damage has been done. ….. The harm comes from overestimating their generality and power, and underestimating the need to specify the contexts in which they are effective and the steps necessary to implement them successfully." (p10).*

This concurs with my own view, previously stated, that grand theory should not be expected to explain or predict the ways that learning occurs at the level of the individual or classroom.

Level 2: Subject-focused theories

Other theories, whilst emerging from a subject-specific base have, nonetheless, described understanding about knowledge and learning that is more generally applicable. It might be argued that these, too, constitute grand theory. These include Sfard (1991), Dubinsky (1991) and Lave (1988). These authors have developed theories describing learning, including explanations of aspects of the learning process. These contributors offer sensible, rational and interesting characterisations of key issues and processes; yet, these too, at the classroom level, fail to explain the processes and mechanisms involved in learning in sufficient detail for me to evaluate, or sometimes even recognise, them in my classroom. Sfard, for example, describes the journey towards achievement of what she calls “abstract objects” through three steps: interiorisation, condensation and reification. She emphasises that this is a difficult process and that reification is a complex phenomenon. Dubinsky also considers the transition from processes and actions to the conceptualisation of those processes as mathematical objects; he calls this “encapsulation”, part of the process of reflective abstraction. Lave was concerned that what is learned in school does not appear to be utilised in the real world; that knowledge did not transfer across school-real world boundaries. She concluded that knowledge was situated in the context in which it had been learned and that educators need to acknowledge the weakness of a system for education that assumes relatively unproblematic transfer of knowledge across different contexts and settings.

Any attempt by me to apply such theory to individual children in my primary classes was largely unsuccessful. I discovered that I didn’t actually know what reification or encapsulation looked like: I didn’t understand what the indicators might be that would show that it had, or had not, occurred. The notion that knowledge was unlikely to be used in situations away from school simply because they were outside school was a prospect that I, as a classteacher, found frustrating and unhelpful. It is apparent that these theories, though finer-grained than grand theory are still too “grand” to be useful at the classroom and individual level.

Later in this chapter, I shall briefly review theories of learning that **are** helpful, in some way, at the macro level and I shall also go on to consider those that have focused on more subject-specific learning, particularly on the development of concepts relating to mathematics and science.

Level 3: Micro-level theories

There are some ideas and theories that are emerging from even more finely grained analyses of learning processes and it is these that offer the most promise for developing my own understanding about learning. It is the work of researchers such as diSessa (1993); diSessa & Wagner (2005); Pratt & Noss (2002) that provides greatest insight and scope for development of a deeper understanding. I shall therefore consider the work of these contributors in more depth.

Initial research questions

So far, I have described a taxonomy of types of theories and ways in which they vary in terms of their generality. I have suggested that the level of generality is directly associated with their lack of applicability and usefulness for teachers and others working with individual learners. I have explained that I shall review theories at 3 levels of generality: grand theory crossing all subjects; subject-focused theories that aim to describe and predict, in some detail, general learning pathways and behaviours for the whole population; and fine-grained theories that focus on understanding particular learning experiences for individuals, i.e. "micro-level" theories.

My initial research questions are fairly general:

- How do children learn?
- How do new concepts develop?
- Do all children construct new knowledge in the same way?
- Why do children so often fail to transfer knowledge from one setting to another?

Why should I focus on concept development?

It is not possible to consider or observe transfer unless the conceptual development from which it arises is also considered. Moreover, it is not enough to only describe or identify the underlying conceptual change (which is, itself, learning); it needs to be deeply understood if transfer arising from it, or within it, is to be understood. The focus for my review of the development of theory is not, therefore, transfer itself, but must be the broader concept of learning, i.e. of concept development or conceptual change.

It is interesting, at this point, to consider the impact that research in education has had on policy and practice. Commitment to pedagogical transformation is notoriously difficult to achieve and even more difficult to sustain at the classroom level (Bishop et al 1993; Harries & Spooner, 2000). So, has educational research, historically, succeeded in illuminating the flaws and needs within the system? Furthermore, has it succeeded in developing theories that ultimately inform and improve practice? It might be argued that the lack of clarity and consensus that characterises research in this field to date means that the impact of research on policy and its influence on children's learning outcomes must be questionable.

Sfard (1998) contrasts 2 metaphors for learning which are evident in contemporary education research: acquisition and participation. She begins by acknowledging a consensus put forward by *"all theoreticians of intellectual development"* , that *"new knowledge germinates in old knowledge" (p4)*. She implies that there is a conflict or tension within education systems generally, relating to knowledge about what actually constitutes learning and what learning requires. She believes that the "acquisition metaphor" and the "participation metaphor" are implicit or explicit in a wide range of educational research. She points out that the acquisition metaphor is a more old-fashioned focus than the participation metaphor which (she finds) dominates more recent studies. It is, after all, a more modern idea to emphasise the social, apprenticeship, activity-based aspects of learning that have formed the basis of whole fields of study within the education arena. Nevertheless, I would point out that, in the prevailing statutory and non-statutory requirements and recommendations that drive primary mathematics education (DfEE/QCA 1999; DfES 2008) in England and Wales today, it is evident that the notion of acquisition is prevalent in policy makers' conceptions relating to learning mathematics. For example, the teaching programme relating to calculation strategies is highly prescriptive: the “Primary Framework” (DfES 2008) sets out a portfolio of mental and written strategies that should be taught in each year group. There is little scope within this for children to discover or invent strategies for themselves. Also, although group work is a feature of one segment of mathematics lessons, such work is usually highly structured and adult- (or text book-) led. It could be argued that the UK primary mathematics curriculum is, almost literally, “delivered” by a prescriptive programme of teaching to a pupil population who are expected to take on board concepts and skills as possessions that are acquired, rather than as understandings that are developed through participatory experience of them.

Moreover, I would suggest that, in developing the notion of apprenticeship, it is important to realise that, if authenticity of the environment, (including the task and its purpose) is crucial, opportunities for learning are limited to those experiences in the real world that offer authentic purposes, contexts and outcomes. However, such opportunities are not usually available to teachers and learners in educational settings. As Sfard (1998) points out, *“ … real-life situations which would be rich enough in mathematical content to become for mathematics students what craftsman’s workshop is for the apprentice are extremely difficult to find “ (p10)* (I would add that Sfard’s reference to “rich”-ness of mathematical content is an interesting notion in itself since in the real world, content would lead to open-ended and unpredictable outcomes. It is extremely difficult, if at all possible, to plan (as good teachers often strive to) for the unpredictable. I would argue that authenticity is not achievable in classrooms since any activity that is conducted in the classroom is not authentic practice from any other setting. The metaphor of the learner as apprentice is not, in my opinion, reconcilable with the school setting.

Sfard describes a shift in the language used by *“the new researcher”* and concludes that, *“The talk about states has been replaced with attention to activities …. the permanence of having gives way to the constant flux of doing.”* (p6)

In recommending ways forward for educational research, Sfard stresses that,

> *"Educational research can only do its job properly if it makes room for both Acquisition and Participation Metaphors" (p7);*

that,

> *"The relative advantages of each of the two metaphors make it difficult to give up either of them: Each has something to offer that the other cannot provide. The basic tension between seemingly conflicting metaphors is our protection against theoretical excesses, and is a source of power." (p10).*

She warns of dire consequences for the acceptance of one metaphor rather than the other in guiding research in education:

> *"When a theory is translated into an instructional prescription, exclusivity becomes the worst enemy of success. Educational practices have an overpowering propensity for extreme, one-for-all practical recipes…. Because no two students have the same needs and no two teachers arrived at their best performance in the same way, theoretical exclusivity and didactic single-mindedness can be trusted to make even the best of educational ideas fail." (p11).*

This is what diSessa & Cobb (2004) and Burkhardt & Schoenfield (2003) were referring to when they highlighted the potential for harm caused by the over-extension of grand theory into instructional practice.

So, learning is the development of concepts, whether by acquisition or participation or both. Perhaps there is a connection between these ideas in that one or other is more appropriate to either "grand" or micro-level theoretical frameworks?

I am aware of, and have experienced, the lack of explanatory power of grand theory in the classroom, and understand that "managing the gap" between grand theory and classroom practice (diSessa & Cobb 2004) is a challenge for contemporary theory development. Having established that it is necessary to consider learning in order to be able to understand transfer, that they are both elements of conceptual growth and development, it is vital to acknowledge that learning is, in itself a problematic concept. It is pertinent, at this point, to consider how some of the key theories in this field have attempted to describe learning and the mechanisms by which it is achieved.

2.2 Historical changes in focus

In the first part of the twentieth century, mathematics education was influenced by theories about learning based on notions of conditioning and behaviourism. Thorndike & Woodworth (1901) considered a range of what they called “functions”; these might be thought of as abilities or competencies (e.g. estimating magnitude such as estimating the length of a line on a page). Thorndike and Woodworth studied the effects of training aimed at improving performance in those functions. They were particularly interested in the range of situations in which trained functions appeared to be applied.

In the 1930s, Skinner (1938) found that the efficacy of reinforcement of behaviours was dependant on factors such as its frequency and the length of time between the behaviour and the reinforcement.

Clearly, the impact of behaviourism meant that it was the teacher/trainer, or, more correctly, the programme of rewards and punishments that was considered to be the key to learning; no real consideration was paid to the learner. However, such stimulus-response theories left too many aspects of learning unaccounted for; for example, Steffe 1983 (citing Brownell 1935) points out that,

> *“Only 40% of the responses given by 32 third grade children to 16 addition combinations were taken as memorized associations in spite of the fact that the children had been taught the 100 addition*

> *combinations on the basis of drill theory. The rest were taken as either counting or indirect responses (37%) or guesses (24%).”*

Stimulus-Response theories were, evidently, limited in their scope for describing, and helping us to understand, learning since so much of what is known could not be attributed to what has been taught.

Dewey (1938/1998) changed the focus: he acknowledged that the learner, and the learner’s experience, shaped his/her response to stimuli, to tasks and to learning. Dewey was interested in motivation and the notion that stimuli come from the learner, not from external sources.

From the middle of the twentieth century, more learner-focused research became the catalyst for change in education. The work of Piaget in the 1950s has evolved into a particularly well-known tradition in education. It is interesting to note that, implicit in Piaget’s focus on staged development (Piaget & Inhelder 1969), is a belief that there is a limit to the learning that can be achieved, according to the development stage of the learner and that, while it might be possible for learning to be accelerated within a development stage, it is not possible to accelerate development. The concept of cognitive structures is key to Piaget’s theory. He believed that there are 4 principle structures, or development stages, and that these change through 3 processes of adaptation. The first of these processes is assimilation, in which new experiences “fit in” with existing knowledge. Accommodation occurs when existing mental structures or concepts have to change or expand in order to make sense of the new knowledge. Where new experience conflicts with existing concepts, equilibration must occur for cognitive harmony to be achieved. Piaget’s approach may be described as cognitive constructivism.

Other constructivist theories of learning that emerged from the middle of the twentieth century were put forward by Bruner (1966) and Vygotsky (1962; 1978). These were more socially than cognitively focused. Bruner, more than Piaget, was concerned with the processes of learning and explored the place of language and the role of the teacher. So Bruner accepted the notion, introduced by Dewey and Piaget, that the learner was a key influence in their own learning, but also embraced the significance of the teacher and of communication and attempted to explore and describe how they fitted into the whole learning process. One point on which Piaget and Bruner disagreed was regarding the potential effect of teaching on learning. Bruner (2006) wrote, *" .. any subject can be taught effectively in some intellectually honest form to any child at any stage of development…. It is a bold hypothesis and an essential one in thinking about the nature of a curriculum."* Piaget, on the other hand, maintained that potential learning was limited by the developmental stage attained by the learner. Bruner holds that there are *"three systems of processing information by which human beings construct models in their world, through action, through imagery and through language"* (2006; p68)

Bruner's view is that the mode of presentation of new knowledge must be such that it can be grasped readily by the learner. He describes 3 modes: the enactive mode; the iconic mode and the symbolic mode. Bruner believes that the development of these three modes of representation, "for how we move, perceive and think" (p68), *"is in that order, each depending upon the previous one for its development, yet all of them remaining more or less intact throughout life."* (p69).

Vygotsky's work came to the attention of the western world in "Thought and Language (1962), though it had been conducted in Russia in the 1920s. He fundamentally believed that learning is an interactive process between the learner and his/her environment, including people, which of course, in some contexts, includes teachers. Vygotsky framed his ideas about learning on the notion of 2 levels of performance – assisted and unassisted. The unassisted level may be equated with Piaget's developmental level. The assisted level is that which is achievable with good teaching, sometimes referred to as the learning level. Quite simply, the gap between these 2 levels is what is perhaps the best-known Vygotskian construct; the Zone of Proximal Development (ZPD).

In Vygotsky's view, learning might begin to occur when assistance is offered at appropriate points in a child's learning – i.e. when the child has reached the limits of his/her unassisted performance. Vygotsky stresses that learning is therefore dependent on meaningful social interaction. Clearly, a skilful teacher may have a powerful influence on a child's learning.

A more advanced stage in learning is when the child becomes his/her "own assistant" (Vygotsky 1962). In this, control of learning is being taken up by the child him/herself. This suggests, to me, that the child might be aware of his/her own learning; that meta-cognition is suggested, even for young learners.

This brief overview of some of the major contributors to grand theory in education reveals some themes that are recognisable in education policy and practice in the UK today, such as: the idea that development occurs through stages in some way; that learners' experience has an effect on their learning; that social interaction and language are important elements of learning, including high quality teaching; ultimately, independence in continuing learning is attainable and desirable.

As well as such theories that illuminate processes and features of the development of knowledge and understanding in the broadest sense, there are other theorists who have concerned themselves more with certain aspects of learning and cognitive growth (e.g. van Hiele 1986; Sfard 1991). These often propose a hierarchical framework of concepts whereby knowledge of particular concepts at a particular level of sophistication is pre-requisite for beginning to develop similar or related concepts at a higher level of sophistication.

Pegg & Tall (2002) identify what they call *"a fundamental cycle of growth in the learning of specific concepts"* – a sequence of changing cognitive structures that evolve and develop to become necessary tools or requisites for further development of more sophisticated structures. They show how such cycles are evident within the processes of "conceptual growth" (p41) described by those whose interest is the science of learning and the mechanisms and processes by which learning occurs. Pegg & Tall present comparisons between a range of theories related to broader (i.e. not necessarily concept-specific) long term growth of the individual, including some of those previously mentioned, and suggest that,

> *"What stands out from such theories is the gradual biological development of the individual, growing from dependence on sensory perception through physical interaction and on, through the use of language and symbols, to increasingly sophisticated modes of thought" (p42).*

Mason & Spence (1999) recognise that a belief that knowledge is built on previous knowledge in hierarchies is evident in the current framework for teaching primary mathematics in the UK, the National Numeracy Strategy (DfEE 1999). Current educational policy and practice in UK would appear to have been strongly influenced by research-based beliefs about how children learn; for example, our system acknowledges, and is dominated by, a notion of hierarchical knowledge.

To recap, research throughout 20th century has revealed that there are many influences on learning and on the quality of learning outcomes. Although, in the first half of the century, the focus was on external (to the learner) factors, more recent research found that the impact of the learner him/herself (and of his/her experience) were far more significant influences, on the way new experiences are perceived and understood, than had previously been realised or acknowledged. Moreover, I believe that to focus exclusively on the child's part in the process would be to limit the potential for him/her to learn; I believe that the teacher also influences the child's learning outcomes, through the design of programmes of learning, tasks and experiences and through skilful provision of appropriate levels of support for learning. The teacher's role in designing and facilitating their own interactions with the child, as well as those amongst groups of children, is also, I believe, a significant influence on children's learning; on their conceptual growth and development.

2.3 How is mathematics learned? A focus on abstraction

Some researchers in the domain of mathematics education, and specifically in the area of conceptual growth in mathematics have developed theoretical frameworks describing how concepts are developed to a level at which they are likely to be re-used in new situations, through a process of abstraction. This might be seen as a logical focus since mathematics, particularly that which might be described as “advanced” is not directly accessible through our senses but exists only in some kind of abstract form.

Workers in this area include Dubinsky (1991), Sfard (1991) and Davis (1986). They present mathematics learning as a hierarchical process, progressing through a series of stages. They believe that children begin with physical, enactive experiences through which they are introduced to a process which (it is intended) will come to be understood and used as a condensed form of the original sequence of actions. The next stage is reached when condensed procedures come to be understood as tools and objects themselves which can be called upon in order to “tame” (Dienes, 1960) other processes. The researchers who uphold these theories accept that there are certain similarities and parallels cutting across their theoretical frameworks. Pegg & Tall (2002) and Barnard & Tall (2001) explicitly set out to identify and understand the similarities and commonalities across different theories (Davis 1984, Dubinsky 1991, Gray & Tall 1994 & 2001, Biggs & Collis 1982 - all cited in Pegg & Tall 2002) by contrasting themes and artefacts within them and mapping structurally similar elements across theories. They highlight similarities in stages in development of conceptual growth within the different theories in terms of actions, object, procedures, processes, schemas and entities, implying that although the terminology might be different they are simply different labels for the same (or similar) things.

Pegg & Tall (2002) and Tall & Barnard (2001) attempt to map the way that abstraction is seen to occur in each theory, highlighting the notion that when new concepts are constructed, they may or may not be qualitatively different to earlier concepts; if they are, they move the learner towards a more abstract understanding of that concept. If the new concept is no more sophisticated or abstract than the existing one, it simply enriches the concept already held, acting as potential to trigger a more abstract version in its next construction. Pegg & Tall (2002) also state that there are “different kinds of learning” (p45), depending on whether the focus of the learner’s attention is on the “base objects” or on the actions. There are similarities, here with the distinction between “figurative” and “operative” schemes drawn by Steffe 1983.

However, I would suggest that there are two major shortcomings of these theories, for those concerned with improving the potential for every child to learn in school: firstly, the theories do not describe what it is that makes a new concept qualitatively better, more sophisticated than the previous one – what is it that leads to abstraction; and secondly, there is a lack of clarity about the definition of “abstraction” – specifically, whether the term is being used as noun or verb – outcome or process.

2.4 What is involved in the learning of mathematics?

I now turn to research that is more specifically focussed on **students’** and **children’s** learning of mathematics to discover whether they can offer any further insight.

Pegg & Tall (2002) suggest that learning that develops from a focus on the “base objects”, itself focuses on “the nature and properties of those objects”. Similarly, where the learner has focused on the actions on the objects, the concept that then develops is one that is concerned with the nature and properties of the actions, the use of symbols to represent the actions, and the sequencing of actions to form procedures and, ultimately, processes.

Perhaps children’s difficulties with “abstracting” mathematics from teaching activities arise because children develop an object-based understanding of the concept, rather than an actions-based knowledge. If this is so, it begs the questions of whether there is something inherent in the nature of common teaching activities that enables the development of objects-based knowledge more readily than of actions-based knowledge.

Whilst it might be interesting to consider that there are different types of learning processes, leading to different types of knowledge, it is also interesting to explore the idea that there are different types of learner. In particular, I question whether there is potentially a bifurcation in the process for children who are more or less successful in their mathematics: a point (or multiple points) along the journey at which learners may continue along the same path or may begin to move along another path. Certainly, it seems to me logical to compare the knowledge of high and low attaining children in mathematics in order to illuminate the differences in the nature and quality of their knowledge.

If we can uncover what it is that successful learners are able to do, cognitively, that their less successful peers cannot, we might infer aspects of cognitive development that might lead to the development of understanding and superior performance. A particular aspect of that cognitive development is where there is a change from instrumental to relational understanding (Skemp 1976). If research can illuminate the nature of the gap or difference between the learning that more and less able children are able to achieve, then logically it might be possible for teachers to assist the less able more effectively and enable transfer and the development of knowledge to occur for more learners, more often, more readily.

Gray, Pitta & Tall (2000) focused on the role of imagery in children's development of understanding about number. They concluded that qualitative differences arise between children who concentrate on different aspects of images: that high-achievers tend to describe visual prompts at an impersonal level ("It is …") as well as by relating them to personal, specific and action-based episodes (e.g. "I have five fingers"). Gray et al labelled these 2 types of images "episodic" and "semantic",

> *"to draw a distinction between images arising from memory associated with the recollection of personal happenings and*

events, and images associated with organised knowledge, having meaning and relationships." (p408).

They found that low-achievers did not offer "semantic" responses – that their images were only, or mainly, episodic, action-based, concretised. Gray et al believe that this is evidence that low achievers' thinking is tied up with access to previous experience and that high achievers have detached (at least in some part) that experience from their thinking.

Gray et al also report that,

> *"Though they initially focused on core concepts, the high-achievers could traverse, at will, a hierarchical network of knowledge from which they abstracted these notions or representational features." (p407).*

Steffe (1983) also looked at qualitatively different modes for working mathematically. He was interested in the *"observed difficulties of primary and secondary school students in developing the "proper" mathematical methods" (p109).* He was keen to discover how children make the transition from using operative schemes (e.g. counting) to using figurative schemes (e.g. written algorithm for addition), a transition which, he stresses, occurs through their own construction.

Pegg & Tall (2002) state that knowledge is objects-based or actions-based, but I would suggest that it is also interesting to consider that possession of both types of knowledge might confer an advantage for learners. Gray 1991 (focusing on differences between high and low ability children) considered the alternative strategies used to obtain solutions to basic arithmetical problems. He notes that,

> *"… divergence between the strategies available to the less able and the more able children is revealed. The alternative strategies used are based either on counting, procedural strategies or on the use of selected known knowledge – deductive strategies. Above average children have both available as alternatives; evidence of deduction is rare amongst below average children. The more able*

child appears to build up a growing body of known facts from which new known facts are deduced. Less able children – relying mainly on procedural strategies – do not appear to have this feedback loop available to them." (p551).

In other words, Gray feels that there are some things that low achievers do not or cannot do – i.e. that high achievers have a much wider range of knowledge and strategies at their disposal.

Askew, Bibby & Brown (2001) believe that,

> *"traditional models of remedial programmes in numeracy (that) tend to concentrate on the inculcation of arithmetical "facts" (are) inadequate" (p3).*

This is a logical conclusion if we consider that remedial programmes that are designed this way require low achieving children to work in ways that they have already shown they cannot do. Askew et al feel that it is the children who are able to make links between known and derived facts who are able to expand not only their knowledge of number facts but also their range of strategies for deriving new facts. They found that it is not necessary to "wait for children to be "ready" to be taught new strategies"; that, "through carefully targeted teaching, pupils who have not developed these strategies for themselves can indeed learn them" (p9). This might be seen to be at odds with Piagetian notions of developmental stages being determinant of readiness; however, Piagetian principles would predict that, as long as learners are in a particular developmental stage, their learning can be accelerated within that stage, though not beyond it. Also, this supports the findings of Gray 1991 and supports my own contention that teachers are vital influences on the quality of learning that can be facilitated.

Furthermore, I would point out that it is not only remedial programmes that would appear to be expecting learners to employ strategies for learning which they do not possess. After all, it is not only groups identified as "low achievers" in school who fail to achieve even basic understanding in mathematics: Askew et al (2001) note that there are many children at the end of primary school who rely on procedures such as counting to carry out calculations, rather than use known or derived number facts. This is in line with Brownell's (1935; cited in Steffe 1983) findings more than 60 years earlier in that children who "failed" to use "standard computational methods" used "either counting or indirect strategies … " i.e. they would seem to be "stuck" in an operative mode of thinking, have not moved on to a figurative knowledge (Steffe 1983). We see then that researchers and theorists have been identifying the same issues for a long time; however, this does not mean that we are any closer to understanding those issues well enough to enable us to "teach" mathematics better, so that more learners can develop better quality knowledge and understanding. We know that children often fail to re-use knowledge about concepts that they have learned (or, at least, that teachers think they have taught,) in primary school; that children revert to, or fail to develop from, earlier, more naïve strategies. I want to find out how to help children to learn; how to move on from that naïve knowledge and to avoid the development of an inadequate cognitive toolkit – i.e. one that contains predominantly procedural/instrumental approaches.

A new imperative for education research about learning and transfer

Grand theory does not, in my view, furnish educators, who work at the chalkface, with knowledge that enables them to design or engineer more effective teaching and learning. In other words, as previously stated, grand theory lacks explanatory or engineering power. If mathematics-focused learning theories (e.g. Sfard 1991; Gray and Tall 1994; 2001) are considered, it is clear that a common theme evident within this work is a focus on abstraction as key to the development of higher levels of thinking. These workers see abstraction of concepts as the key to children's ability to transfer or re-use knowledge; that without abstraction transfer cannot occur. These workers appear to use "abstraction" to refer to a process that leads to knowledge which is abstract in its nature – i.e. abstraction that produces abstractions; there is a subtle difference between this and "abstracting" in a less "grand" fashion, as part of the process of concept formation and development. Sfard (1991) sets out her understanding that mathematical concepts are developed through 3 stages: interiorisation, condensation and reification. She explores in some depth the reification phase of the process and points out that there are degrees of abstraction and of integration. She states that abstract notions can be conceived in *"two fundamentally different ways: structurally – as objects, and operationally – as processes." (p1)* (There are similarities here with Pegg & Tall's (2002) comments about learning being objects-based or actions based.) Sfard points out,

> *"the crucial, qualitative, difference between the two modes of thinking lies in the basic, usually implicit, beliefs about the nature of mathematical entities. In other words, there is a deep ontological gap between operational and structural conceptions." (p4)*

It is worth noting, at this point, that Sfard uses the terms "object" and "structural" in a particular way: when she refers to objects, she means "mathematical objects" – i.e. abstract objects that are often, themselves, the result of some process. Indeed, Sfard believes that concepts are hierarchical, moving from operational to structural.

Sfard notes that there have been many accounts of knowledge as dichotomies, such as relational or instrumental (Skemp 1976), conceptual or procedural (e.g. Lesh & Landau 1983; Hiebert & Lefevre 1986). However, she stresses that the 2 types of knowledge that she discusses, structural (focused on mathematical objects) and operational (focused on actions and processes) do not represent a dichotomy but, rather, a duality. There are clear parallels here with the ideas of Pegg &Tall (2002); Steffe(1983). Sfard notes that,

> *" … These two approaches, although ostensibly incompatible, are in fact complementary. It will be shown that the processes of learning and of problem-solving consist in an intricate interplay between operational and structural conceptions of the same notions." (p4)*

Sfard goes on to state,

> *" ….. what is conceived purely operationally at one level should be conceived structurally at a higher level. Such hierarchy emerges in a long series of reifications, each one of them starting where the former ends, each one of them adding a new layer to the complex system of abstract notions." (p16)*

It seems that Sfard's description of mathematics knowledge as a hierarchical system is rather simplistic and, therefore, difficult to reconcile with her suggestion of "intricate interplay". I also feel that her explanation is, from a micro-perspective, at least, incomplete and therefore unsatisfactory. I feel that Sfard, like other theorists who concern themselves with a broad view, does not make clear exactly what it is that happens in reification that marks the point at which it occurs. How can the teachers or the learners know when it has occurred; when they have "understood"? Even if a learner is confident that it has happened, it is often not until that new knowledge is tested that it becomes apparent that it is not secure after all.

Gray & Tall (1994) do not believe that abstraction is always brought about in the same rigid sequence. They focus on the differential mathematical success experienced by individuals and propose that the underlying mental structures that enable some children to think flexibly and solve problems more efficiently are bound up with development of a "proceptual" system. That is to say, those who are most successful in their mathematics are those who are able to deal with different understandings of a mathematical idea (i.e. as process and as object) and to switch effortlessly between them. This is similar to the interplay ("between operational and structural conceptions") mentioned by Sfard (1991, p1), that she says leads to learning and problem solving.

Do these ideas apply to learning in all areas of mathematics?

Gray & Tall (in Boero et al 2002) draw on a range of studies to show that when learners endeavour to use and develop their proceptual knowledge in different situations, they are likely to encounter difficulties that are specific to particular areas of the mathematics. Gray & Tall believe that,

> *" .. it is a laudable aim to have a general theory of construction, but we observe that specifics often overwhelm the broad sweep of such a theory. The acquisition of mathematical knowledge from early years to undergraduate level involves a variety of reconstructions. Each new reconstruction refines that which was established earlier …" (p119)*

It would seem that knowledge and knowing and understanding are not consistent for any individual across different settings. There are important implications for this for educational research and for learning about learning – that is, that though particular knowledge might not be evident in one situation it cannot be inferred that the learner does not possess that particular knowledge; only that, if it does exist, it was not activated in that situation and that it might become evident in a different situation.

Tall, Thomas, Davis, Gray & Simpson (1999) set out *"the transition between process and object"* as presented by different researchers (Piaget, Dienes, Greeno, Dubinsky, Sfard and Gray & Tall). All of these theories are based on the idea that the outcome of the process is an object and that mathematics is about working with these objects, to subsequently create more objects. Moreover the objects are considered as abstract, context-free, formal. Gray & Tall (2007) reflect on their development of ideas and knowledge about abstraction which, they say, occurs through a process of compression. They identify 3 types of concepts that are abstracted in mathematics learning, noting that,

> *"Compression involves taking complicated phenomena, focusing on essential aspect of interest to conceive of them as whole to make them available as an entity to think about." (p25)*

How are concepts and pieces of knowledge (compressed or otherwise) recognised as potentially relevant in any situation? Mason & Spence (1999) distinguish between “knowing-about” and “knowing-to”. “Knowing-to” can be thought of as recognition of a relationship between some elements of existing knowledge and some aspect of a new situation. How does the brain “know-to”? Gray & Tall (2007) do set out their understanding of how the brain is able to link concepts, in physiological terms involving “long-term potentiation” (p26); however, this does not resolve the matter of how similarity is recognised at the level of the individual in a classroom setting.

It is my contention that, if the aim of a system for education is for learners to learn (i.e. to develop their knowledge and understanding), it is necessary to take account of a wide range of factors that are known to impact on learners’ capacity to learn.

However, I have shown that grand theory does not provide us with sufficiently wide-ranging knowledge to achieve this; nor, should we expect it to. The very nature of such broad, generic theories might be helpful for predicting patterns of behaviour or achievement on a grand scale but, since they do not focus on individuals, should not be used to predict outcomes for individuals.

2.5 An alternative focus

The idea that conceptual knowledge changes as it develops is not problematic for me – that much feels obvious and natural. However, I think that within the research there is ambiguous use of the term abstraction and that this confounds central issues about what comprises learning.

The use of the word abstract to describe the end product of the process leads us to conceptualise the process that leads to it as abstraction, which is defined by the Oxford English Dictionary as,

> *“The act or process of separating in thought, of considering a thing independently of its associations; or a substance independently of its attributes; or an attribute or quality independently of the substance to which it belongs.” (Oxford English Dictionary;2008)*

In my opinion, and for the sake of clarity in my presentation of ideas and arguments, I prefer not to use the term “abstraction” as a noun that refers to a piece of abstract knowledge; rather, I choose to refer to those as abstract notions. I am comfortable with the use of the term “abstraction” to denote the process that generates abstract notions. However, I would add another term, from the same root meaning, to describe the process of coming to recognise common features across knowledge resources; this recognition constructs connections between concepts, – “abstracting”. Abstracting does not, therefore, lead directly to abstract notions but to a degree of detachment from contextual references, through a shift in focus to the connections between those references. By distinguishing between abstraction and abstracting and abstract notions, within my own conceptualisation of these terms, I am able to analyse and articulate my own and others’ ideas about these issues more effectively.

I have already noted that Gray, Pitta & Tall (2000) discovered that high achieving children have action-based, episodic concrete mental images as well as semantic images, in their thinking. This implies that, for these children, context has not been lost and that their knowledge remains richly connected to contextual information. This leads me to pose the question of whether high level thinking needs both sorts of images or just one - are the episodic images required for high-level thinking or is possession of semantic images sufficient? That is, do semantic images actually replace episodic images or supplement them?

I believe that, if we are to understand how contextual information is used or lost when concepts are formed and developed, it is necessary to find ways of observing conceptual growth taking place. Interestingly, it is to children’s activity in situations which are designed to facilitate learning that we might turn to be able to observe (or more correctly, infer) their cognitive activity. Children's activity in the classroom is, paradoxically, both a means by which their conceptual growth occurs and the window through which we, as observers, might see it happening.

Schwarz, Hershkowitz & Dreyfus (in Boero 2002) set out to work directly with children to seek and observe "epistemic actions" which they claim are "constituent of abstraction". They describe a model, the "dynamically nested RBC model of abstraction", that they used as an analytical tool with which they were able to observe various processes that occur as part of the broader process of abstraction. Their theory arises very explicitly from a view of abstraction as activity; and from the premise that abstraction can be observed and provoked in a classroom setting, during students' activities. (The authors' intention in their use of the term "abstraction" is unclear.)

Evolving research questions

Issues arising from my review of the literature around theoretical views of learning (generally and within mathematics) enable an expansion of my initial research questions that now includes questions relating to abstraction and transfer in more depth:

- How is knowledge re-used?
- Is it possible to observe different types of transfer?
- How do old and new concepts relate to, and affect, each other?
- How do children recognise situations in which old knowledge is relevant?
- Is it possible to observe abstraction and/or abstracting in mathematics in the primary school?
- What is the relationship between conceptual growth and abstraction?
- What is the relationship between conceptual growth and abstracting?

- What is the relationship between conceptual growth and transfer?
- What are the relationships between abstraction and transfer, and abstracting and transfer?
- What happens to the contextual references in children's knowledge as their knowledge develops? Is it possible that they are preserved?

These are questions, somewhat naively formed at this stage, that I expect to elaborate through continued interrogation of the literature.

2.6 "Micro-level" approach

Many workers are interested in what individual children do while learning. This type of research is at a different grain size to research that culminates in the development of grand theory and other macro-level studies. "Micro-level" findings have the potential to illuminate the gap between grand theory and children in the real world.

In line with the Realistic Mathematics Education (RME) stance that formal mathematics develops out of children's mathematics activity, Gravemeijer & Doorman (1999) believe that informal solution procedures might act as "foothold inventions … that become catalysts for curtailment, formalisation or generalisation" (p117). This seems to be in line with observations and theories about similar phenomena noted in the work of others, such as: Noss & Hoyles (1996); Pratt & Noss (2002); Dubinsky (1991); diSessa (1993); Wagner (2006). These workers take the view that concepts are modified, rather than replaced by "more advanced" concepts. For example, Noss, Healy & Hoyles (1997) believe that,

> *" … abstracting – considered as a process – can be seen as a way of layering meanings on each other, rather than as a way of replacing one kind of meaning (concrete, referential) with another (abstract, de-contextualised). The emphasis is on connections between ways of knowing and seeing, rather than on the replacement of one by another." (p226)*

Resources for learning

Learning can be considered as an interaction between the individual and the world. A range of resources might be utilised in such interactions. Resources can be:-

> External – (acting from the world upon the individual) e.g. environment, materials, teacher, tasks;
>
> Internal – (acting from the individual upon the world) e.g. knowledge, experience, memory, attitudes, skills.

The ways that learners utilise these resources has been a focus for some researchers. (e.g. Lave 1988; Noss & Hoyles 1996)

Noss et al (1997) emphasise that abstraction is not what determines mathematical activity but that it is a resource for activity. Noss & Hoyles (1996) focus on meaning as the key to understanding how mathematics is learned. They do not accept the focus on ascension towards abstraction held by Gray & Tall, Dubinsky, Sfard and others, believing that meaning cannot be found in a de-contextualised world. Noss & Hoyles (1996) state that,

> *"abstraction is a process of connection rather than ascension". (p48)*

Here we see that different authors use the word "abstraction" to mean very different things; something that I believe causes confusion within the literature, leading to a focus on the use/misuse of the word, rather than on the real learning issues. Noss et al (1997) use abstraction to describe a process of abstracting – i.e. at a low level, in a mundane way, as part of the early learning of a concept. Gray & Tall, Dubinsky and Sfard, on the other hand, refer to abstraction at a higher "grand"-er level, as the end product of a series of cognitive transitions as well as a way of describing the process through which those abstractions are achieved.

Noss & Hoyles (1996) feel that to consider the connections that the individual learner makes with their own previous learning is to focus too narrowly and they go on to consider a range of other resources that may be involved in the process of mathematical abstraction. Noss & Hoyles put forward two main ideas. The first of these is webbing. This describes the way that learners construct and repeatedly modify a network of connections within and across concepts. Webbing is ongoing and iterative and has the effect of extending and enriching the links between knowledge resources. Noss & Hoyles go on to propose the notion of "situated abstractions" to describe how learners emerge from learning situations having abstracted some aspects of concepts experienced in that learning situation. They present evidence that abstraction of a new concept is situated, in that it is linked cognitively to the situation in which it was developed and that it is triggered by situations that are perceived as similar. Situated abstractions, themselves, become resources for sense-making where similarities are perceived. We see, then, that resources have potential to facilitate the formation and recognition of connections and that resources are therefore linked to the potential for transfer or re-use. What is not clear is how newly abstracted knowledge (the result of recognition of commonalities across problems or contexts or experiences) is used in unrelated or dissimilar situations. The distinction between abstracting, at the micro-level, and abstraction that creates abstract notions that are conceived as mathematical objects, at the macro-level, is key to describing and defining learning.

The human aspect

We find, then, that many issues are relevant when attempting to understand learning and transfer. Moreover, I believe that re-use of knowledge and the development of concepts is about even more than the classroom, practitioner, technical and practical attributes of an experience. In considering the whole range of available resources, we must include intra-personal resources such as cognition, memory and motivation.

Memory

Some researchers have focused on the human aspects of learning, for example motivation and memory, in their pursuit of understanding about learning and the growth of knowledge.

Clancey (1989) explains that human memory is better characterized as a *capacity* than as a *repository*. A knowledge engineer may represent what someone knows in terms of formal linguistic descriptions, such as the rules in an expert system, but these rules are not literally stored in the expert's head. In fact, knowledge bases often contain models of the world that go beyond what anyone has said before.

Other workers in this area (Suchman, 1987; Agre, 1988; also Clancey, 1992) present similar conclusions, believing that human memory is not a place where things (e.g. schemas, categories, rules, procedures, scripts) are stored: such representations—when they are not stored in the environment—are always constructed each time they are used.

Resources, both internal and external, and their potential to activate existing knowledge, are therefore crucial for enabling learning and the development of knowledge through conceptual development and growth.

Situated cognition

Another group of researchers have focused on the role of setting, and the “situated-ness” of knowledge. This is the notion that knowledge is embedded in the context within which it was generated and only has meaning in those settings or in settings that are perceived as related.

Lave (1988) set out to explore the relationship between education, cognitive theory and everyday practice. She observed the arithmetic strategies used by “just plain folks” in various everyday activities such as shopping and dieting. These were compared with performance on tests in a more controlled setting. She found that there was only very limited use of school mathematics in real world settings, that,

"…… when we investigate learning transfer directly across situations, the results are consistently negative .." *(p68).*

Lave develops a critique of learning transfer research and goes on to assert that knowledge does not develop out of learning transferred from one situation to another. Of course, the logical development of this view is that learning acquired in schools is unlikely to be used in the real, material world, without significant and problematic transformation being necessary.

She analyses the relationships between activity, practice, mind, person and knowledge and goes on to assert that knowledge is constructed by a learner, as a result of the dialectical relationship that exists between the learner, the setting and the activity.

Nunes et al (1993) investigated what they call "street mathematics". They found that fishermen were able to refine and reformulate the calculating strategies they normally used at work and use them in other real-world "street" domains. This is evidence of flexibility and generalisability – that people who engaged in street mathematics can generalise the schemas they develop in a particular "street" setting to other street settings. This shows us that there is some link between everyday maths practice and knowledge that can be activated in other contexts.

It would seem, therefore, that though Lave may have found that transfer did not occur between formal and informal settings, Nunes et al (1993) did find evidence of transfer between informal settings.

I believe that the work by Nunes et al might serve to lead us out of the situationist "cul-de-sac" (Noss & Hoyles, 1996, p33) – to illuminate a way forward to understanding more about how we learn. The situationist view is that knowledge, and knowing, and coming-to-know are highly situated and not transferable to new situations. Since the very nature of mathematics, particularly that of advanced mathematical thinking, depends on working with abstract ideas and pattern, it would seem that (theoretically at least) there can be no mathematical activity if the situated cognitionists' view is correct. This prompts the questions as to whether mathematics is truly formal and abstract: I would question whether, perhaps, it is only the most advanced mathematics that is truly formal; that perhaps the mathematics that young learners need to engage with, in order to develop more advanced understanding, is not. It is possible that even the most advanced mathematics depends on situated roots. There may be no evidence of this in the symbolic notations through which the mathematics is presented but perhaps advanced mathematicians, in making sense of the symbols, rely on concepts and intuitions that are rooted in situated knowledge; knowledge that has extended the range of contexts in which it is considered relevant. Clearly, whilst we might accept that the role of setting in learning is important, we cannot accept the extreme view that learning is completely situated and cannot be abstracted (or "uncoupled" (Lave 1988)) from the setting in which it was created. We need to learn more about the relationship between setting and learning. Furthermore, we need to understand the role of the resources that are available in different settings. This might help us to understand why contextual aspects of mathematics experience are so important.

It is children's response to and utilisation of resources connected to a task that research might usefully consider. A focus on their informal strategies in their response to a situation or problem might help illuminate the learning process, including development towards more formal knowledge.

Recognition of similarity

The Realistic Mathematics Education movement in the Netherlands is committed to the design of "learning trajectories" that provide sequences of contextualised tasks from which, it is intended that, for learners, models will emerge and more formal mathematical knowledge will be achieved. Context problems are therefore one of the keystones of the RME philosophy. Gravemeijer & Doorman (1999) define context problems as "problems of which the problem situation is experientially real to the student." (p111). If we accept that context problems might provide a valid window through which we might attempt to observe and evaluate the learning process, it is important to remind ourselves that, in order to design context problems with any confidence, it is necessary to have confidence in our knowledge of what children will perceive as relevant.

Although it could be argued (from an extreme situationist view) that transfer or re-use of prior learning in new situations does not occur, it is clear to me that in order to function as human beings, it must be possible to apply knowledge or behaviour learned in one context in other situations. I would argue that the question is not of **whether** learning can be transferred or re-used but of **how** it happens. (It is entirely feasible, in the light of my review of research in this field and reflecting on my own developing knowledge and understanding of these issues, that it will be shown that knowledge does not actually transfer, if we accept that the meaning of the verb "to transfer" means that something is moved or conveyed from one place or person to another. I already see that this is not an appropriate description of what happens; that actually, knowledge is caused to change or develop or transform but that it does not re-locate. Transfer is not, therefore, an appropriate term for what I prefer to call re-use of knowledge contributing to conceptual growth. However, for the sake of conciseness and consistency with the prevailing jargon, I shall refer to "transfer" where others do so.)

To accept that, for children to be able to make sense of new mathematics they must be able to connect it to existing knowledge, and that this cannot happen in a contextual vacuum, is to acknowledge that setting plays an important part in teaching and learning. There is a learning paradox which educators need to resolve which I call the "paradox of situations": that is, that the most valuable learning is that which can be useful in new situations – and yet, new learning is only meaningful in the situation in which was acquired. All too often, children appear to learn (even master) something in one lesson that they seem unable to remember in another. Clearly, new learning does sometimes become locked in to specific situations (at least temporarily) and children are often not able to apply it in new situations.

It follows that, for relevance of some previous experience or knowledge to be recognised, some aspect of a new problem must be perceived as similar. This perception of relevance is a vital piece in the jigsaw which is the “gap” between informal knowledge and more formal abstract mathematical understanding which is, I feel often overlooked.

I conjecture that it must be the activation of mental connections and pathways to prior learning through recognition of similarities in new contexts that is key. For prior learning to be activated, any similarities must be recognised by the child. Therefore, one way in which we can facilitate learning is by providing feedback to children about the relevance of previous experience and explicitly pointing out similarities. Of course, good teachers do this all the time (Askew, Brown, Rhodes, Wiliam & Johnson (1997) reported on the links between “connectionist” teaching and high levels of effectiveness).

Some researchers have focused not on learning generally but particularly on the re-use of prior knowledge itself; and most particularly on strategies and mechanisms that facilitate the creation of meaning across concept boundaries – i.e. the perception of similarity. This crossing of boundaries constitutes a type of transfer that I shall show is only one type: there are other processes involved in conceptual change and growth that are also types of transfer. Incidentally, it is interesting to consider that repeated crossing of boundaries, in itself, leads to blurring and dissolution of those boundaries.

diSessa and his colleagues, and Mason and his colleagues, have made a significant contribution to the literature in this area, which I shall now consider.

diSessa & Wagner (2005) challenge research in the field of transfer to address with clarity the nature and role of knowledge, *"We question the assumption that high-level abstractness is the principal quality of knowledge that provides for its applicability across diverse situations." (p121)*

They claim that research often misdirects its focus and studies performance without validating its link with knowledge. diSessa & Wagner go on to say that,

"… theories purporting to explain transfer must be held accountable for describing and determining knowledge – not merely successful or unsuccessful performance." (p121)

diSessa & Wagner (2005) describe Co-ordination Class Theory, which identifies different elements of the "complex knowledge system" which, they belief, characterises knowledge itself,

"We, (and others ….) view knowledge as a complex system of many kinds and instances of knowledge elements and structures. Learning, say, a concept entails co-ordinating a large number of elements in many ways. Furthermore, many of these elements (following a constructivist orientation) come from the prior conceptual competence of the learner." (p125)

They go on to point out that any model of a complex knowledge system will inevitably be complex itself, (as Co-ordination Class Theory is).

diSessa & Wagner (2005) believe that there are different kinds of knowledge and that *they "have different properties in transfer"*. Their focus is "specific conceptual knowledge".

They point out that, previously, interest has been in abstraction - "the problem of how knowledge is generalized so as to become applicable across a wide range of situations". Co-ordination Class Theory, on the other hand, emphasises the earlier phases of the actual construction of knowledge which occurs incrementally and over time, including what I call abstracting.

diSessa & Wagner stress that,

> *"Co-ordination Class Theory is not a theory of transfer. Nonetheless, transfer can be found within it, notably in the relation of a co-ordination class to the multiple contexts in which it can operate. Co-ordination class theory shows how a concept can become robust enough so that it is applicable fluently across a wide range of situations (p139).*

They distinguish between different types of transfer:

- Class A Transfer– *"where an adequately prepared set of ideas is used unproblematically in new situations"* (p148); *"the knowledge whose transfer is at issue is assumed to be, or can be demonstrated to be, well prepared and does not, in principle, require further learning to apply"* (p124). diSessa & Wagner note that this is important for schools who *"want students to be able to solve problems other than the ones used in teaching them concepts" (p125);*
- Class B transfer – knowledge constructed that is *"presuming subjects' persistent effort… sufficiently prepared so that transfer can be reliably accomplished in acceptable periods of time (e.g. in a few hours or days…)" (p125)*
- Class C transfer– How do *"relatively unprepared subjects (students) use prior knowledge in early work in a domain?"* (p125); *"where bits and pieces of "old" knowledge are invoked, productively or unproductively, typically in early stages of learning"* (p148). Class C transfer might be considered as the processes that lead to transferable knowledge. (p125)

It is interesting to consider that the Class A transfer identified by diSessa & Wagner is what is often labelled as "abstraction"; what I have called high level abstraction to produce abstract notions. Class B and Class C transfer might be considered as "abstracting" at a lower level, or "micro-transfer".

Co-ordination Class Theory (unlike abstraction theories that dominated the literature in the 1980s and 1990s) asserts that contextual information in and from learning experiences is not stripped away: on the contrary; constructs of Co-ordination Class Theory (CCT) explicitly include attention to situation-specific knowledge. For example, “readout strategies” (cognitive tools with which (according to CCT) we glean information and understanding about a situation) are thought to be highly context-dependent and particular; and wide “span” (the range of new situations to which knowledge is considered by the learner to be at all relevant) is accomplished,

> *“by accumulating a lot of situation-specific knowledge, rather than by deleting reference to particular features in the abstract and thus generalized knowledge.”(diSessa & Wagner 2005; p140)*

Co-ordination Class Theory would seem to provide an alternative, more realistic characterisation of what occurs, within learners’ cognition, that enables transfer, as opposed to abstraction theories which, I have argued, cannot lead to transfer because they fail to describe the links between existing and new knowledge. One of the underlying principles of CCT is that the similarity between situations that must be perceived is not similarity of structure but similarity of some item or aspect of the situations.

Royer, Mestre & Dufresne (2005) also dismiss the idea that recognition of structural similarity is key to transfer. They point out that there are 2 broad categories of theories about learning and transfer: “environmental”; and “cognitive”. Royer et al stress that,

> *“Rather than transfer being dependent on stimulus similarity, cognitive theory proposed that transfer was dependent on conceptual similarity.” (pxvi)*

This represents a significant shift in the factors that research needs to consider since it stipulates that transfer occurs only where there is similarity between the way that new and previous experiences or problems are conceptualised or understood. That is to say, any similarity that must be recognised lies within the learner, not in the setting itself.

diSessa & Wagner (2005) consider that,

> *"Felt relevance ….. We believe its role has been much underplayed in transfer research." (p147)*

I would add that another aspect of transfer that is not sufficiently attended to or explored (and is not therefore understood) is that of the level of preparedness of prior knowledge that is necessary for Class A transfer to occur. Schwartz, Bransford & Sears (2005) consider a type of transfer which they call "preparation for future learning" ("PFL"). They, like diSessa &Wagner, maintain that transfer is not necessarily the direct application of existing knowledge; that it is also the process by which existing knowledge provides a framework, a resource, that enables learners to know what sort of information is useful in working towards a solution to a new problem so that they are not completely lacking focus or direction. We see, then, that much contemporary research is, therefore, focusing on the early stages of knowledge building.

Wagner (2006) carried out a micro-level analysis of the work of undergraduates learning about "the law of large numbers". He went on to develop further (what he calls a "micro-genetic analysis") the work of one student, Maria.

Wagner (2006), like diSessa, believes that abstraction is **not** the primary source of the generalizeability of knowledge; and that transfer results,

> *"not from the acquisition of a single, sufficiently abstract understanding of a concept, but from the construction of a collection of knowledge resources." (p56).*

Wagner's approach is aligned with diSessa's "knowledge-in-pieces" (diSessa & Sherin 1998) epistemology in that he expected the elements of the co-ordination system, including knowledge resources, to be sensitive to context; that the patterns of re-use of knowledge would be variable. According to Co-ordination Class Theory, these knowledge resources facilitate,

" … conceptual recognition, interpretation and reasoning according to the contextual circumstances under which particular collections of (distributed) elements best function." (Wagner 2006; p56)

In Wagner's study, he showed that,

"Maria was not abstracting structure from the problem situation, but actively structuring it by the most active knowledge frame available to her". (p57)

I believe that the knowledge that she used must have been triggered by contextual features and that resources that are cued are therefore sensitive to contextual variation. However, I do not feel that this means that knowledge is situated, as we usually understand it.

Wagner (2006) describes,

"incremental growth, systematization and organisation of knowledge resources that only gradually extends the span of situations in which a concept is perceived as applicable". (p10)

This reveals a clear relationship between learning and transfer; a link that is developing reciprocally – i.e. learning leads to transfer and as transfer occurs, knowledge (including that of its own span of relevance and efficacy) develops further. Wagner describes this process as one,

"by which ideas once cued only in particular contexts can be actively and flexibly developed, combined and co-ordinated such that they are more likely to be used in an increasingly wider span of situations". (p6)

Obversely,

"pieces of knowledge previously unassociated" are more "likely to be cued together, and contexts that once cued more limited frames were now more likely to cue this larger, enriched frame". (p54)

Wagner (2003) mentions the relationship between knowledge, understanding and transfer and explains that deeper understanding can be understood as extending co-ordination. This was evident in that,

> *"Maria's "rule" was incrementally transferred to a wider span of problem aspects as new co-ordination knowledge took hold". (p63)*

There are several key ideas which I have drawn from the work of diSessa (1993), diSessa & Sherin (1998), diSessa & Wagner (2005) and Wagner (2006), relating to the perception of similarity:-

- Perception of similarity is arguably the main key to transfer. Where no similarity is perceived, relevant knowledge resources will not be "brought to mind". There are many ways in which a situation or problem might be similar (or dissimilar) to another situation or problem, including problem type, aspect or context. A learner's ability to perceive similarities is therefore dependent on him/her having some knowledge of situations where elements of problem type, aspect or context are similar to those of the new situation.
- However, though necessary, it is not sufficient for relevant knowledge resources to exist in the mind of the learner: if they are to be rendered available in new situations similarity (in some way) must be perceived and recognised by the learner. Recognition of similarities is both facilitated and limited by the learner's span of experience and existing knowledge resources of many kinds.
- The perception of relevance emerges in the relationship between the situation and the individual's interpretive knowledge that frames the situation.
- Interpretive knowledge is that which learners develop out of their experience of concepts, skills and connections and associations in all aspects of their learning. It is this that they use to make sense of new situations. One dimension of that knowledge is the development of strategies for extracting information from a

situation (readout) and for making sense/meaning through co-ordinating this with other knowledge and experience.

- Reorganisation and systematization of interpretive knowledge involves integration of descriptive and explanatory knowledge.
- Structure within a problem situation is neither rigid nor objective; it is constructed as part of the problem solving activity itself by the problem solver. This knowledge can only develop through experience with situations which have some similarity, though they do not need to "match" structurally.

At this point in my review of the literature, the importance of internal and external resources – their nature and learners' use of them – as well as a focus on the need to understand the role of learners' perception of similarity has become very clear. It is appropriate, therefore, to further expand my list of research questions to include:

- How do children use all types of available resources, both external and internal?
- Where do similarities (that might be recognised) reside? – in the problem context? – in its structure? – or somewhere else?

2.7 Relationships between knowledge, learning and transfer.

Mutual bootstrapping of conceptual knowledge

Wagner (2006) resonates with my own intuitive beliefs about learning, knowledge and transfer. In the development of my own conceptual knowledge, I am aware of a sense of a hand-over-hand process whereby the growth of span of relevance and of the applicability of situations to knowledge resources (and vice versa) which constitutes conceptual development, is characterised by a small change in one dimension facilitating a small change in another dimension, after which the first dimension is enabled to grow or change a little more. Wagner describes how his main subject revealed her incremental development of a particular concept; bootstrapping different aspects of the concept:

> *"Within the active knowledge framework, Maria expected representativeness and accuracy to be relevant explanatory ideas, but the role they served in the particular problem, and the role they served in co-ordination with other ideas in the framework, had yet to be clarified. These incremental moves are a hallmark of a knowledge-in-pieces perspective on the growth of conceptual understanding, as well as the transfer-in-pieces perspective on how knowledge of a principle develops to span more and more situations." (p62)*

I find the work of diSessa (1988), diSessa & Wagner (2005) and Wagner (2006) very appealing. It provides a framework for identifying the different aspects of the learning process and offers a well drawn rationale for characterising the relationships between knowledge and learning and transfer, taking account of what I call "micro-transfer". Though other theories have offered models (often hierarchical) for moving understanding and knowledge from that of the particular to that of something more generic that might be inserted into a new, structurally similar situation, they have not succeeded, for me, in filling in the gaps – the major shortcoming of abstractionist, or encapsulation theories is in their failure to meaningfully (for me) address the question of perception of similarity and, therefore, relevance. It has been clear to me that application of knowledge in a new situation requires some understanding of the relationship between the existing knowledge and the demands of the new situation as well as some notion of correspondence or mapping between contextual attributes old and new. Wagner (2006) highlights,

> *"the in-separability of the perception of structure in a problem from the knowledge of the principles needed to solve it." (p61)*

We see no more than what our knowledge leads us to expect to see:

> *"seeing complex structures within a given situation depends on our having complex expectations – not necessarily ready-made structures previously stored that are retrieved in their entirety, but complex associations of descriptive and explanatory knowledge resources that have proven to be mutually supportive in other circumstances. These associations are learned." (Wagner 2006; p63)*

Mason (2002) also acknowledges co-evolution of an individual's capacity to recognise connections between resources with the development of some experience with those resources,

> *"Perception and preparedness to be able to perceive emerge together as the result of perceiving and preparing to perceive." (p229)*

To recap, transfer may be understood as follows:-

- Transfer <u>requires</u> an incremental growth of span of relevance;
- Transfer <u>enables</u> an incremental growth of span of relevance;
- These processes together constitute a "hand-over-hand" growth of interpretive knowledge (includes descriptive, explanatory, readout, co-ordination);
- Transfer is dependent on perception of similarity, which is dependent on development of span of relevance;
- Transfer does not require general (abstract) expression; to illustrate this point, Wagner (2006) states that *"We expect no single understanding of the law of large numbers to be applicable anywhere and everywhere, rather different combinations of distributed elements may support its recognition and use in different circumstances" (i.e. is sensitive to context and takes account of it p56)*. This is a key idea, vital for shaping understanding about knowledge and transfer.

How does abstraction relate to transfer?

In considering what transfer is, what it looks like and what it entails, it is clear that abstraction, as opposed to abstracting, is not necessary for transfer (of Class B or Class C) to occur. Interestingly, Maria did develop and express,

> *"an increasingly invariant understanding of a statistical principle as she incrementally constructed an interpretive knowledge frame that widened the span of phenomena to which she understood the law of large numbers to apply. …. Maria's attempts at stating a generalised principle took place only in the aftermath of the development of interpretive knowledge frames at a much finer level of detail."*

Therefore,

> *" .. abstraction was a consequence of transfer and the growth of understanding, not the cause of it." (Wagner 2003 p72)*

It was argued previously that the perception of relevance emerges in the relationship between the situation and the individual's interpretive knowledge that frames the situation. Since the "individual's interpretive knowledge that frames the situation" might be more simply referred to as the learner's understanding of the situation, it is clear that understanding is related to both transfer and abstraction. A learner with deep understanding is likely to perceive similarities with the greatest span of relevant knowledge resources. Therefore, we see that understanding supports transfer and transfer contributes to understanding. Eventually, when sufficient examples have been experienced and understood at some level (facilitated through recognition of similarities), generalisation across situations will become possible – i.e. when multiple instances of transfer have occurred, abstraction is enabled; this in turn will deepen understanding. This model locates abstraction (rather than abstracting) as an outcome of (Class C) transfer, rather than a prerequisite for it.

Why is evidence of transfer so elusive to researchers?

Co-ordination classes are, according to diSessa & Wagner (2005), those concepts which are complex and comprise multiple layers and dimensions of meaning and relationships. It is this complexity that means that co-ordination classes are difficult concepts to learn.

Not all concepts are co-ordination classes but many of the more problematic concepts (in maths and science learning) are (diSessa 1993).

An alternative contribution to the field comes from Mason and Spence (1999) and it is interesting to contrast their ideas with those of diSessa and Wagner and others. Mason & Spence describe a framework that, like the contribution of diSessa and Wagner focuses on the earlier “precursor” stages of knowledge-building in order to illuminate the cognitive mechanisms for “transfer” of that knowledge into new situations. They explain their view that

> *“Knowing-about, that is, knowing-that, -how, and –why forms the heart of institutionalised education: students can learn and be tested on it. But success in examinations gives little indication of whether that knowledge can be used or called upon when required, which is the essence of “knowing-to”. Although knowing-to does of course depend on training in behaviour, it is based, as we shall see, in awareness. It has to do with the structure of attention.” (p138)*

I feel that this latter point is an interesting one; that knowing-to is more than a behaviour that can be trained – it is about awareness and what is attended to (or is not). I agree with Mason & Spence (1999) that knowing-to is clearly not the same as reacting – i.e. it cannot be achieved simply by training.

Once knowing-to has occurred, the other aspects of knowing-about are enabled: without the trigger of knowing-to, all that is known-about is not accessible . After all,

> *"No-one can act if they are unaware of a possibility to act; no-one can act unless they have an act to perform." (Mason & Spence 1999, p135)*

It is interesting to note, at this point, that Mason & Spence seem to support the view that transfer ("knowing-to") precedes abstraction ("knowing-about"). It is also clear that, in order to "know-to", it is necessary to know something; this, I would argue, entails "knowing-about". Papert (1996) introduced what he called the "Power Principle". He described how children working with LOGO were able to learn about angle by working with angle to construct shapes: they were learning by using; Papert posed the question, "What comes first, using it or getting it?" (p4).

Anecdotes of learners failing to re-use knowledge that they are thought to have learned are rife in schools the world over. This might be described using Mason & Spence's (1999) parlance: that learners who have demonstrated that they know-about in some way (know-that, know-how and/or know-why) do not apply any of that knowledge in a situation in which it would facilitate them to access the problem, possibly solving it. Presumably, we might infer, this might be because they do not realise the relevance of the knowledge-about to the new situation and so do not use it – i.e. they do not know-to. In terms of diSessa and Wagner's "CCT" lexicon, their span is not sufficiently developed.

Broudy 1977 (cited in Schwarz et al 2005) believes that there are 3 kinds of knowledge: "replicative", "applicative" and "interpretive". This provides a valuable framework with which to analyse the elements and demands of tasks and through which we might understand the reasons for an apparent lack of transfer. Transfer research is often based on measurements of retention of skills and knowledge after a learning event. Replicative and applicative knowledgeare relatively straightforward to assess and these form the basis of a great deal of what are used as assessments of knowledge. However, replicating and applying old knowledge in the same ways and in similar problem situations is achieved by performance rather than through understanding. Tests and assessments and research tasks that facilitate such "performance" after a learning episode may actually find transfer, if transfer is understood as "the degree to which a behaviour will be repeated in a new situation" (Detterman & Sternberg 1993, cited Schwarz et al 2005). This is the classic definition of transfer. If, however, we hold that transfer actually involves modification and adaptation of old knowledge to new situations, then the third kind of knowledge identified by Broudy - i.e. interpretive - is required and would need to be evident in research tasks. Interpretive knowledge is that which enables learners to interpret the content and context of new situations in order to select appropriate knowledge skills and understanding for addressing a new problem. Schwarz et al point out that research into transfer has not, traditionally, designed methodologies that enable interpretive knowledge to be used and/or observed.

Mason & Spence (1999) believe that,

> *"The state of sensitivity-awareness of the individual, combined with elements of the situation which metonymically trigger or metaphorically resonate with experience, are what produce the sudden knowing-to act in the moment." (p146)*

This reference to "elements of the situation" suggests that this approach to understanding the mental processes involved in transfer of knowledge, allows for attention to contextual information, rather than focusing on structural similarity. I would suggest that, if we accept this account of the processes involved in transfer, it would be helpful to gain some understanding of ways in which educators might effectively sensitise the "triggers" to which Mason & Spence refer. Or, to use Co-ordination Class Theory terminology, strategies for extending span, including testing alignment need to be explored and developed.

Mason & Spence conclude that,

> *"Knowing is not a simple matter of accumulation. It is rather a state of awareness, of preparedness to see in the moment." (p151)*

They go on to consider whether knowing-to can be prepared for (a more appropriate term than taught or trained). They believe that, even if metaphors are deliberately provided, children's uptake and use of them is highly variable; that attempts to "implant" the metaphors and images are unsuccessful and that children need to actively and personally take on board metaphors that might be suggested from within or outside of themselves.

Some of the ideas of Schwarz, Bransford & Sears (2005) resonate with the work of Mason & Spence. Schwarz et al describe how much transfer research has been based on a "sequestered problem solving" (SPS) approach to assessing transfer of knowledge and learning and they posit that this only facilitates the measurement of certain types of knowledge since it looks for "direct application" of old knowledge in a new situation. They propose that this type of research neglects and is blind to a range of modifications and adaptations to old knowledge that might facilitate problem-solving **in the future** rather than in the test situation. This "preparation for future learning" (PFL) is a more helpful view of what transfer actually entails since it extends the range of situations where it might be evident.

Wagner (2003;2006) can be seen to have implanted metaphors (or prepared Maria for learning). He designed a sequence of learning activities for Maria that exposed her to certain ideas. It would seem then that attempts to "design-in" exposure to relevant and potentially helpful metaphors might help knowledge to develop, although Mason & Spence (1999) thought it not worthwhile. This tension in the findings from different studies is not disconcerting: I believe that work in this field, with a sharp focus on the minutiae of conceptual change and growth, is only beginning to develop and that findings from one worker do not necessarily predict outcomes in other settings, even where they appear to be similar.

I think transfer might be different for different knowledge domains. diSessa &Wagner (2005) assume this might be the case:

> *"We, ….., do not presume transfer is homogenous with respect to kind of knowledge or with respect to other such dimensions. Nonetheless, it is useful to examine transfer in a particular, reasonably well-elaborated case."* (p139)

This is what Wagner chose to do when he focused on one undergraduate student for his (2003;2006) study.

Research at this micro-level, focusing on an individual in one setting in order to discover at least how that individual thinks and learns will, I believe, help us to develop experience and understanding of those particular subjects in those particular settings. If a model of learning for transfer is built on the idea that understanding of multiple examples of particular situations is what leads ultimately to generalisation and deeper understanding then it is appropriate that micro-level research is the way forward. That is to say, an accumulation of knowledge about how individuals are able to effect conceptual growth might be the most appropriate way forward if we are to begin to understand learning and transfer in a way that might subsequently contribute to the development of more macro-level theories.

Research into the development of knowledge and understanding presents significant challenges in the 21st century. The theories which dominated the field for most of the latter part of the 20th century have been rigorously challenged. We have now been shown that an abstract understanding in/of mathematics, stripped of any context-based references, is not necessarily an appropriate goal for teachers and learners (or researchers) in primary schools. It is no longer abstraction which is the key objective for mathematics education; there is now a bigger challenge. Research in this area must develop strategies for changing its focus, perhaps for zooming-in on individual instantiations of learning and transfer. I am happy to adopt some of the terminology introduced by diSessa and Wagner (2005) as I summarise some of their ideas which, I believe, are thorough and meaningful; I believe, also, that they reflect, develop and represent realistically and accurately what learners actually have and do:

- in cases where a co-ordination class is well prepared, almost by definition, subjects will be able to "transfer" that knowledge to any related problem within a sensible range;
- mismatch of contextual characteristics will prevent prior knowledge being used but so will underdeveloped readout strategies and naïve or flawed co-ordination knowledge of a concept;
- transfer research has found failure because it is looking for Class A transfer where it is unlikely to be found – i.e. knowledge is unlikely to be sufficiently prepared;
- Class C is a frequent, "blind" (understandably) process involved in the extended preparation that is required for Class A transfer to be enabled;
- investigating Class C *"depends strongly on our ability to see particular knowledge in action even if it does not show up as context-transcending success."* (diSessa & Wagner 2005; p 148).

Some of these points will be discussed in later chapters in relation to findings from my study.

2.8 Partial states of knowledge construction:

Schemas, models – where do they fit in?

Within the field of cognitive science, the terms "schema" and "model" abound. However, it is not always clear exactly what it is that these terms are being used to describe. Generally, these terms are intended to describe the way we represent (internally and externally) what is in our heads. "Schema" is often used to label a kind of mental map; "model" is sometimes equated with some sort of analogue or metaphorical representation.

Fischbein (1987) set out 3 requirements for an effective model: "*comprehensiveness*", *"obviousness"* and *"correctness"*.

Chinnappan (1998) focused on schemas and mental models and considered the nature of,

> *"possible interactions between the state of organisation of available geometric knowledge and the accessing of that knowledge during problem solving". (p214)*

He judged that,

> *"attention to the qualitative aspects of knowledge development and utilisation has the potential to improve current levels of understandings about why some mathematics students experience difficulties in applying previously learnt knowledge." (p214);*

a view I have previously expressed. He goes on to consider the relevance and appropriateness of certain models and representations in different areas of mathematics.

There are some aspects of Chinnapan's view that might be seen to parallel the findings of diSessa and Wagner (2005), as I shall now describe.

Chinnapan (1998) was interested in the relationship between the quality of children's knowledge base and their ability to access and make effective use of that knowledge. He considers whether more able children are more likely to use a greater number of different schemas and more often than lower ability children. This, of course, implies that more able children might have a more sophisticated relational knowledge base than their less able peers. Relational knowledge is that which is rich in connections which, is, of course, the way that diSessa and Wagner (and others) characterise knowledge and conceptual growth. Clearly, Chinnappan also views the extent to which knowledge is relational as key to "ability". He sees mental models as images or representations of what exists/occurs in the mind of the learner. Chinnappan believes that there are *"key concepts that anchor other concepts"* (p203) and that,

> *"Two characteristics are important to understanding geometric schemas; organisation and spread. Organisation refers to the establishment of connections between ideas, whereas spread refers to the extent of those connections." (p203)*

There are similarities, even parallels, here with diSessa & Wagner (2005); for example transfer and alignment correspond at some level with the idea of "organisation", including scope for complexity and sophistication.

The term schema suggests mapping of elements from one case to another; this implies a focus on structure which, I have argued, is not appropriate for describing learning. However, perhaps schemas **are** an appropriate description of what learners are able to construct through repeated experience? Schwarz et al (2005) think so.

Schemas and models contribute to what Schwarz et al describe as "efficiency" – i.e. that through repeated opportunities to work with similar tools to solve similar problems, developing "replicative" and "applicative" knowledge, schemas and models are developed and readily utilised. They go on to stress, however, that the development of interpretive knowledge that equips learners to learn, also increases efficiency,

Schwarz et al find that,

> *" … enhanced learning does indeed occur when people have an opportunity to develop the interpretive knowledge that prepares them to learn." (p11)*

They also stress that learners need to interact and to access additional resources, obtaining feedback. This, clearly, is in stark contrast to much of the transfer research that has been based on "Sequestered Problem Solving" (SPS) i.e. in which,

> *"Tests of the 'direct application' view [that] typically place people in sequestered environments, where they have no access to*

> *'contaminating' information sources other than what they have learned previously, and where they receive no chances to learn by trying out an idea and revising as necessary." (p5)*

There is clearly a conflict and tension in the design and management of research into learning wherein the methodology is based on SPS: if learners are denied opportunities for feedback and revision, they will not be able to show learning.

Partial states; grey areas

Wilensky (1993) believes, like Piaget, that interaction and familiarity with a concept lead the learner to make more and more connections between other experiences and the new concept. However, in contrast to Piaget (who was, after all, considering processes and outcomes on a grand scale), Wilensky is concerned more about learning at the level of the individual and believes that concreteness is,

> *".. not a property of an object but rather a property of a person's relationship to an object" (p198)*

and that, as the relationship becomes stronger, it becomes more concrete. Concreteness is, therefore, something to which a learner aspires, rather than from which he/she develops. Abstraction, using Wilensky's terminology, occurs through concretisation.

Wilensky also sees conceptual growth as augmentation of connections and that this may facilitate abstraction. Wilensky's notion of concreteness also suggests a continuum – i.e. partial states, rather than a have/have not model.

diSessa & Wagner (2005) also feel that much educational research in this area is guilty of over simplifying the process of learning. They point out that,

> *" … we should expect no sharp line between "having" and "not having" a concept." (p6)*

They go on to state that,

> *.... "states of partial construction are much more important to describe"   (p6)*

and emphasise that with a “complex knowledge system” perspective, it is necessary to acknowledge all the grey areas, the intermediate states. diSessa & Wagner believe that there is a need to characterize partial constructions, particularly the early phases in the construction of a true co-ordination class.

Schwarz et al (2005), in exploring issues of efficiency and innovation and the balance of these 2 aspects of knowledge, emphasise the need for efficiency in that,

> *“ … if people confronted with a new complex problem, have solved aspects of it before, this helps make these sub-problems routine and easy to solve. This frees attentional bandwidth and enables people to concentrate on other aspects of the new situation that may require non-routine adaptation.” (p30)*

They also explain that efficiency is insufficient for innovation and that both are required if learners are to continue to learn and solve new problems. Schwarz et al note that,

> *“ … innovation is often preceded by a sense of disequilibrium that signals that certain processes or ways of thinking (e.g. previously learned routines) are not quite working properly …” (p32)*

Pratt & Noss (2002) acknowledge the importance of the grey areas when they explore the notion of situated abstractions. They found that recently constructed situated abstractions might be called upon in new situations in which similarities are recognised, but that children will, initially, attempt to use other long-established internal resources. (This resonates with the “disequilibrium” noted by Schwarz et al (2005)). This is because resources are “brought to mind” according to a priority order that is established and modified over time, according to feedback regarding the “success” of resources utilised. diSessa (1993) offers the related notions of cueing priority and reliability priority: cueing priority refers to the likelihood that a resource will be activated as potentially useful in a situation; reliability priority is established according to feedback regarding the usefulness of the resource on previous occasions, taking account of other resources also activated. Thereby, high cueing priority and high reliability priority (diSessa referred to these together as “structured priorities”) take time to develop, and new resources can only have low reliability priority (and will not, therefore, be utilised in novel situations) until they are tried and tested.

Pratt & Noss (2002) put forward a theoretical model in which meanings constructed in one setting might also be valuable in another setting. They observed and analysed the way children made sense of the effects of using a variety of computational devices that simulate everyday situations familiar to the children, but offering enhanced functionality in the virtual world. Pratt & Noss believe that there is a distinction between abstraction and de-contextualisation which is generally overlooked in the literature. They point out that,

> *“A central issue is the extent to which mathematical abstraction depends on decontextualization … “ (p454)*

They acknowledge the differences between macro- and micro-level research and sought to illuminate the ways that the findings of research both macro- and micro- might be related and therefore reconciled. They attempt to achieve this by elaborating the relationship between mathematical abstraction and de-contextualisation. Pratt & Noss maintain that,

> *“situated abstraction is observable as more or less tacitly articulated invariance of relations, framed within the situation itself”. (p457)*

They explain that,

"Situated abstractions emerge during activity as internal resources that serve as relatively general devices for making sense of situations that arise within a setting." (p456)

Pratt & Noss go on to say that situated abstractions are,

" … types of knowledge that enable learners to reflect on the structures within a setting a make sense of phenomena that hold true across it." (p456)

They show that situated abstractions *"are expressed in a language […] that remains embedded in the situation in which it was constructed".* (p456)

This analysis (Pratt & Noss, 2002) suggested that children:

- will, when making sense of devices, articulate situated abstractions of the way they work;
- will, when encountering superficially new situations, initially attempt to use long-established internal resources for making sense of such situations, rather than situated abstractions recently constructed;
- will, subsequently, employ recently constructed mental resources as long as:
  a) feedback from the system emphasises the lack of explanatory power of the long-established resources (increases cueing priority), and
  b) there is sufficient similarity between the old and new contexts.

Pratt & Noss's model contributes to my view that learning is not about detachment from contextual features but development of increasingly rich and intricate networks of attachments comprising aspects of experience of, and within, those features. Therefore, if abstraction depends on decontextualization, it follows that learning cannot be dependant on abstraction. This point is becoming increasingly clear to me: that abstraction (of abstract notions) might be an outcome of learning that, in itself, might enable advanced functioning at high levels within the domain, but it is not the cause of earlier stages of concept development. Therefore, I do not believe that abstraction (understood as decontextualization) is necessary for learning.

Salomon & Perkins (1989) offer a view which acknowledges different models of transfer:

> *" .. we argue that transfer is not at all a unitary phenomenon. Rather, transfer can occur by different routes dependant on different mechanisms and combinations of mechanisms." (p115)*

They propose 2 types of transfer – "high-road" and "low-road", whereby the former,

> *" … occurs by intentional mindful abstraction of something from one context and application in a new context."*

And the latter,

> *" .. depends on extensive, varied practice and occurs by the automatic triggering of well-learned behaviour in a new context." (p113)*

This acknowledgement of different kinds of transfer is most helpful: it accommodates and validates the range of behaviours and outcomes that research has observed. Moreover, it might provide a way forward in that it might provoke future work in this field to clarify its aims and match these to appropriate methodologies and theoretical frameworks.

2.9 Summary of Chapter 2: Reflections on the literature

It appears to me that much of the modern research recently conducted in the area of the development of understanding and conceptual development has converged on the need to explore and develop knowledge about the intermediate states of concept development. I welcomed this approach as I rejected simplistic views that knowledge becomes abstract and reduced to its structural essence so that it can be unproblematically plugged into new situations as necessary. Also, I found that I could not subscribe to the preoccupation with abstraction and the assumption of a have/have not dichotomy that has, at times, dominated this field. The theoretical frameworks that appeal to me are those that acknowledge, and seek to illuminate, the complexity of the processes involved in retaining cognitive links with contextual resources. Our understanding about learners' development and refinement of their knowledge - that may eventually lead to the effective application of that knowledge in new situations - necessitates exploration of the nature of conceptual knowledge at all stages of its development.

Evident within a significant portion of the literature (e.g. diSessa & Sherin 1998; Wagner 2003; 2006; Mason & Spence 1999;) is a persistent intuition on the part of conceptual researchers that concepts are always bootstrapped within relational structures for other concepts. This coincides with my own intuitive beliefs.

diSessa & Wagner's work with Maria (2005) also inspired me. They found that*, "span was particularly hard-won in her (Maria's) learning" (p133)*. This, coupled with my own ultimate professional goal – to improve the quality of primary mathematics teaching, compels me to discover whether younger children expand and develop their knowledge in similar ways to Maria (who is a more mature and sophisticated learner).

I have not, so far, found much of the published research in this field particularly valuable in showing me how to help children to "learn maths better". I had, however, found the reporting of the work of several researchers – their rationales and outcomes - to be very interesting and, as John Mason might say – metaphorically/metonymically resonant.

I have come to believe that the reason why I could not transform the excitement I often felt on reading such reports into positive transformation of my own (and recommendations for others') teaching practice is because the key differences between "abstractionists", "situationists" and any other "…ists" is not in what they found but in what they were looking at. I now believe that, as in every other aspect of our work, the approach we take in our attempts to understand something which we previously failed to understand will be shaped by our deeply held views about what is important. Consequently, some researchers who feel it is important to learn about trends in large populations will inevitably reach conclusions about those populations and might be able to describe patterns in overall behaviour or general outcomes. Conversely, researchers who seek to understand issues at the level of the individual will set about their research in a very different way and will look for and try to understand quite different findings. I now believe that grand theory and other theories that describe vast populations cannot be useful for understanding individual or small groups.

So, my focus is on how children in primary schools use and re-use knowledge and other resources within and across their experiences. It is not specifically transfer that I wish to observe, though it might be one aspect of what I find. I am more interested in all stages of concept development. I wish to design a research methodology that optimises the opportunity for me to infer the way that children use knowledge resources. In order to design an effective methodology I must therefore understand these hypothetical issues and theories about what children's knowledge resources might look like or manifest themselves in order to have any opportunity of observing them. I must also take heed of the theoretical tensions embodied by certain methodologies in the past and ensure that research design is coherent and well-founded and well-matched to my aims.

My research questions thus far stated are:

- How is knowledge re-used?
- Is it possible to observe different types of transfer?
- How do old and new concepts relate to, and affect, each other?
- How do children recognise situations in which old knowledge is relevant?

- Is it possible to observe abstraction and/or abstracting in mathematics in the primary school?
- What is the relationship between conceptual growth and abstraction?
- What is the relationship between conceptual growth and abstracting?
- What is the relationship between conceptual growth and transfer?
- What are the relationships between abstraction and transfer, and abstracting and transfer?
- What happens to the contextual references in children's knowledge as their knowledge develops? Is it possible that they are preserved?

I must now add to this questions relating to the micro-aspects of the process, using some of the vocabulary developed by workers in this field where appropriate:

- How do children respond to or use metaphors in their construction of new knowledge?
- Can children's development of a concept be tracked to observe the states of partial construction? This might illuminate the relationship between old and new concepts and enable identification of increasing span and alignment?

The above list has been compiled in the light of my review of the literature about conceptual change, learning and transfer. It is now necessary to consider an appropriate medium within which I might observe and analyse children learning – I must choose an area of mathematics in which I am likely to be able to evaluate how children are developing knowledge and understanding about a new concept. For reasons which shall be elaborated, I have decided to work in the domain of negative numbers. It is therefore appropriate, at this point, to review the mathematics education literature relating to teaching and learning about negative numbers. This is presented in the next chapter, where I shall also problematise the teaching and learning about negative numbers in relation to what I have learned in the literature about conceptual change and growth. The review and analysis of research about negative numbers in the next chapter will enable the development of more precise, more rigorous research questions about conceptual change and growth, learning and transfer.

Chapter 3: Aims of this study: Negative numbers as a window on transfer

3.1 Core themes

The core themes, identified and explored in the previous chapter, are:

- the microevolution of conceptual knowledge;
- micro transfer within the processes of change and growth of concepts;
- transfer of knowledge within and across the contexts in which learning occurs.

3.2 Negative numbers as a window for observing micro-evolution of knowledge

I need to observe children in situations in which their thinking is provoked to change as they are introduced to a new concept. For reasons which are explained in Chapter 4: Methodology, my research about my core themes is situated in the domain of negative numbers as it is taught in UK primary schools. It is therefore appropriate, at this point, to turn to the literature about negative numbers to learn about the successes and problems associated with teaching and learning in this domain.

3.2.1 Review of the literature relating to teaching and learning about negative numbers

Before entering into a painstaking design process for the creation of teaching and learning tasks and resources to facilitate them, it is necessary to understand about different approaches to teaching in this domain, possible reasons for children's difficulties, and the successes and failures of a variety of strategies and models.

My analysis reveals several emerging themes:

- relevance of negative numbers in primary mathematics teaching;
- descriptions of children's difficulties and the underlying cognitive difficulties;
- dimensions in which negative number situations are cognitively experienced;
- use of metaphors and the development of mental models.

Each of the above is now considered.

3.2.1.1 Relevance of negative numbers in primary mathematics teaching

Some writers, in analysing and describing the cognitive and pedagogical challenge presented by negative numbers, conclude that it might not be appropriate or relevant in the primary curriculum.

Fischbein (1987) claims that negative numbers only exist mathematically – that they cannot be represented concretely or modelled effectively so should not be taught until pupils can cope with *"intra-mathematical justifications" (p.281)*. This would require understanding and facility with algebraic representations of negative numbers which would not normally be expected in the primary school.

However, Ryan & Williams (2007) emphasise the development of pre-algebraic thinking that is enabled through learning about negative numbers, through the facilitation of generalising from knowledge of natural numbers to integers – gaining experience and familiarity with the integer as process.

3.2.1.2 Descriptions of children's difficulties and the underlying cognitive difficulties

Gallardo **(**2002) evaluated 12/13 year-old students' understanding of negative numbers by constructing measures of understanding related to that evident in ancient texts from a range of historical periods and cultures. He found that students of this age were not able to demonstrate formal understanding of a negative number as part of the family of integers. However, they were able to demonstrate some acceptance of: relative or directed numbers; opposite quantities; a negative number as an isolated number or as the result of an operation.

This is in line with the findings of Lytle (1994). When asked to give a meaning for a given negative number, students responded with ideas relating to position, "something missing", the result of subtraction, or "opposite". In a preliminary test Lytle found that, amongst a group of students in 7th Grade classes, *" … most were successful in operating **on** a negative number, but not **with** one."* (p.195). This would mean that, for example, students could multiply $^{-}2$ by 3 but they cannot multiply 3 by $^{-}2$.

Hayes (1996) also presents findings from his secondary school study, showing that many students fail to develop understanding in the topic. He also describes a workshop for 4th year trainee mathematics teachers in which, *"not one could prove or give an explanation for either "0 - $^{-}6 = 6$" or "$^{-}2 \times {}^{-}3 = 6$".* Hayes (1996) goes on to claim that,

> *"A large proportion of students emerge from secondary school with a seriously flawed and incomplete understanding of the real number system. Any area or application of mathematics requiring the use of negative numbers and related concepts is likely to produce difficulties."* *(no page numbers in online text)*

Some workers suggest reasons for such widespread difficulty. For example, Linchevski & Williams (1999) state that,

> *"Traditionally, negative numbers introduce a new aspect into the study of mathematics: for the first time reasoning in an algebraic frame of reference seems to be required." (p.134)*

Tang (2003) believes that the concept of negative numbers is outside the scope of an innate number sense that helps us learn about natural numbers and basic operations on and with them. Findings of a study of 7 and 9 year-olds working with negative numbers (Bristow &

Desforges,1995) support this view since children involved revealed a persistent conception of zero as unpassable, “impenetrable”. Tang believes that conceptual metaphors play a major role in embodying or bringing to life mathematical ideas. Linchevski and Williams (1999) also believe that, by exploiting metaphors, it is possible to extend children’s innate arithmetic.

Peled (1991) presents a useful framework for describing and assessing levels of understanding about negative numbers.

The basic level is when the child is aware of the existence of negative numbers and has some vague understanding of them as opposite to the non-negative integers already known. Another aspect of this basic level is a willingness to move through zero when faced with a simple, and otherwise uncomplicated, problem.

At the next level, children are able to operate on and with both positive and negative numbers, as long as both numbers in the problem have the same sign. When children achieve the most advanced level, the numbers or quantities do not need to be of the same type. (This is in tension with Lytle’s finding mentioned previously in that it seems to suggest that children will find $^{-}2 \times {}^{-}4$ easier than $^{-}2 \times 4$. In my professional experience, this was not generally the case.) Peled believes that children hold multiple images of negative numbers simultaneously and that they call upon different images, depending on the nature of the number problem to be solved.

3.2.1.3 “Dimensions” in which negative numbers are cognitively experienced

Peled (1991), in her framework setting out levels of knowledge about signed numbers, describes knowledge in 2 dimensions: what she calls the “number line dimension”; and the “quantity dimension”. My analysis reveals that these would appear to map onto 2 main types of models for supporting the construction of mental images: firstly, a number line model, in which numbers are used as both points and vectors; and, secondly, a “neutralisation” or “cancellation” model which deals with numbers as quantities.

Many of the teaching experiments which have been conducted in this domain were set up to compare the effectiveness of these 2 models (e.g. Hayes, 1996; Liebeck, 1990). Aspects of some of this work are described in more detail later.

Bruno & Martinon (1996;1999) recognise Peled’s (1991) 2 dimensions, though they would extend the framework to include 3 dimensions by replacing the quantity dimension with distinct “abstract” and “contextual” dimensions. They explain that the contextual dimension includes situations that are described by numbers – i.e. states, combinations, variations and comparisons.

Bruno & Martinon (1996) set out to discover what they call the transferences between dimensions. They found that, for example:-

- When asked to transfer information presented in an abstract format into a presentation on a number line: the least able children found it extremely difficult; the more able found this easier than when changing information from an abstract to a contextual representation;
- When taking information presented in a number line format, and re-presenting it abstractly (symbolically): all groups generally found this easier than when they tried to do it the other way around; all groups were significantly less likely to succeed than when moving from a contextual format to the abstract;
- Less able children consistently performed better when transferring between contextual and number line dimensions (in either direction) than when transfer to or from the abstract dimension was required;
- All students found it more difficult to pass from the abstract to the contextual than in the opposite direction.

In their conclusion, Bruno & Martinon (1996) note,

> *" … we can see that transference from the abstract to the contextual dimension implies greater difficulty than transference from the contextual to the abstract. It is easier to arrive at representations on the number line from the contextual than from the abstract."   (p.168)*

(It is important to note that Bruno & Martinon refer to "abstract" in the sense of a formal symbolic use of the notation for negative numbers.)

3.2.1.4 Use of metaphors and the development of mental models

As previously outlined, the model for the development of mathematical knowledge propounded by Sfard (1991) highlights a stage which Sfard calls reification; wherein a major cognitive shift is required that effectively transforms understanding of mathematical processes into an assimilation of those processes as objects in themselves. The mechanisms by which reification might occur are not well understood. Linchevski & Williams (1999) suggest that, where such intuitive gaps exist, it is appropriate to use extra-mathematical knowledge to support children's learning.

Linchevski & Williams (1999) believe that,

> *"Situations and models must describe a reality that is meaningful to the student in which the extended world of negative numbers already exists and the student's activities allow them to discover it." (p.134)*

However, they also raise the issue that, even where contexts are selected carefully, to reflect children's culture and experience, and activities may be seen as highly "authentic", this "authenticity" seldom *"survives the transfer to the classroom situation" (p.132).* They suggest that this occurs because, although many characteristics of the real-life context can be reproduced, the goals cannot.

Tang (2003) cites Lakoff & Nunez (2000) who propose 4 grounding metaphors for arithmetic. These are fundamental cognitive mechanisms that stimulate and develop arithmetical ideas and understanding. They are:

- “arithmetic as object collection;
- arithmetic as object construction;
- measuring-stick metaphor;
- arithmetic as motion along a path” (p.236)

All of these metaphors are recognisable in contemporary mathematics teaching in the UK. Tang holds, what he calls, an “embodied realistic” view of mathematics and believes that abstract concepts are understood metaphorically. He believes that the use of myth, stories and fantasy is supportive of children’s learning in mathematics. Tang, therefore, does not use “abstract” in the same way as other authors mentioned in this section in that an “embodied realistic” view is, necessarily, connected through metaphors to contextual references.

As previously mentioned, there are 2 main models that have been used for teaching and learning about negative numbers: number line; and neutralisation. Workers in the field do not agree about the effectiveness of the two models. Neither is there consensus regarding the purpose of developing models at all. Sometimes they are seen as tools for learning which are abandoned when concepts become developed in a more formal way. However, models may also be retained in the mind of the learner so that, at times of subsequent uncertainty, the model can be recalled and the concept refreshed or rediscovered. The perception of the purpose of models is, I believe, connected with the researcher’s understanding of the process of abstracting or learning in mathematics.

Janvier (1985) points out that subtraction is difficult to model, whatever type of model is selected, without performing “acrobatics”. He explains that,

> *“ … in many models, subtraction has no contextual meaning but is represented as the addition of the element opposite in nature.” (p.136)*

This is especially true in “neutralisation” scenarios whereby, when the quantity to be subtracted is greater than the starting quantity, it is necessary to first increase the starting quantity by a number (at least) as great as the difference. It is far from intuitive and leads to children learning rules rather than developing understanding. Difficulties with subtraction using a neutralisation model were noted by several workers (e.g. Lytle, 1994; Linchevski & Williams with their “Disco Game”, 1999; Liebeck, with her “Scores and Forfeits”, 1990).

The number line model also has its critics. In developing her rationale for researching a model based on the neutralisation principle, Lytle (1994) cites studies and surveys that found children were less successful when using number lines for arithmetic tasks than using other methods (e.g. Ernest, 1985; Rathmell, 1980; Kuchemann, 1981 – all cited Lytle 1994). The main criticism is that, though children are able to locate points on a number line and to carry out simple addition and subtraction by moving to the right or left, they are not able to represent more sophisticated addition and subtraction calculations as situations and “stories” on the number line. This suggests that children do not intuitively connect the number line with the operation beyond the simplest of structures.

But Bruno & Martinon (1999) say the number line **is** an appropriate model where it represents a way of working which is applicable to **all** integers, not only non-negative numbers. This, of course, precludes the use of numbers to express cardinality. Bruno & Martinon (1999) suggest that numbers that represent the measurement of scalar magnitude would be appropriate since it is relevant for negative as well as non-negative numbers. They report that the number line became an indispensable tool for students solving problems in their experiment; that,

> *“they exhibited more confidence in the results obtained on the number line than through calculations “ (p.808)*

Research by Fischer (2003) and Fischer & Rottmann (2005) suggests that negative numbers might be cognitively represented in the left space, on a “mental number line”. I would point out that, if this is shown to be true, that learners associate negative numbers with “left”, then the case for focusing on a number line model for teaching and learning about negative numbers is strengthened.

Peled, Mukhopadhyay & Resnick (1994) found that children who had not received any instruction about negative numbers had, nonetheless, formed a number line representation of an extended number system. Moreover, Peled et al suggest,

> *“ … children seem able to develop pre-instructional intuitions about purely mathematical entities (the negative numbers) by elaborating previously developed ideas about number (additive composition and partitioning) that were originally rooted in physical experience but have, through practice, become so familiar as to become intuitions in their own right.” (p.109)*

3.2.1.5 Metaphors:

There are several contextual situations that are reported in the literature as being exploited in teaching and learning about negative numbers as metaphors:

- credit/debt (have/owe);
- creation/annihilation;
- temperature;
- journey left/right or forward/back;
- chronology (before/after an event);
- Yin/Yang.

Some are more appropriately modelled by one method than the other; some may be represented using either number line or neutralisation models. In the prevailing curriculum guidance in the UK (currently the Primary National Strategy, Framework for Literacy and Mathematics (DfES 2008)) where a context is suggested for work on negative numbers, it is the context of temperature (e.g. in Year 4 Counting, partitioning and calculating, Unit 1).

3.2.2 Implications for my study

Teaching and learning about negative numbers is, historically, problematic. It is, therefore, a worthwhile focus for my study in that my research will make an epistemological contribution. It is possible, also, that it might ultimately contribute to the development of philosophy, tools, strategies, materials or curricula that improve education in this domain.

Both existing models for teaching about negative numbers have been criticised in some respects so neither would be a more nor less appropriate model than the other, upon which to base my own teaching intervention, as long as weaknesses inherent in the model are acknowledged and addressed,

The issue of "authenticity" should be considered; in particular, I wish to consider ways to replicate the goals of an activity (for example, Linchevski & Williams (1999) reported that the notions of teams and points-scoring was authentic in their study).

Borba & Nunes (2001) found that children were significantly better at solving negative number problems when they were allowed to use writing or manipulatives to represent the problems and the solutions than when they were required to do it orally, without writing or using any apparatus. Though this was not the focus of their study, it does provide evidence that explicit, external support is likely to improve children's potential for learning about negative numbers.

Bruno & Martinon (1999) consider that children are facilitated to extend their number system in different ways, including the negative numbers extension. They point out that, when children learn about different extensions in isolation, they do not understand that they are all part of the same system. Bruno & Martinon go on to analyse sequences of learning about extensions leading to the concept of the real number system. They feel that, where teaching about negative numbers is *backwards from* the real number system (i.e. the real number system is the starting point, as it is taught in Spain), it is more difficult to learn than if it were taught from a starting point of *forwards towards* the real number system (as it is taught in UK). This suggests that the overall structure and aims for extending the number system for children in UK primary classrooms is valid and should be retained.

3.3 How do issues arising from negative numbers research relate to what I have learned about broader learning issues considered in Chapter 2: Literature Review?

It is interesting to note that work in the domain of negative numbers has raised questions and issues about conceptual difficulties that I can relate to what I have learned from my review of broader issues about conceptual learning. For example, Fischbein (1987) and Linchevski & Williams (1999) point out that children in primary schools cannot be expected to have developed sufficient algebraic knowledge to be able to manage the concept of negative numbers effectively. Ryan and Williams (2007), however, claim that work with negative numbers is a good way to develop algebraic thinking. Perhaps, then, Papert's Power Principle (1996) is evident here – that learners learn about new concepts by using them?

I suggest that Gallardo's (2002) observation, that pupils aged 12/13 years could not demonstrate formal understanding of negative numbers but clearly had other related knowledge of negative numbers, is evidence that their concept of negative numbers was partially constructed; that the span of a new concept does not extend rapidly and suddenly to new contexts, but by increments. Lytle's (1994) finding, that students could operate on but not with negative numbers, is evidence that negative numbers were understood as point before they were understood as vector; this also supports a hypothesis that knowledge in this domain is constructed gradually.

Bruno & Martinon's (1995) observation, that children found zero to be "impenetrable" might suggest that zero lies at the boundary of a concept – and that crossing of this boundary is problematic, at least until the concept boundary is moved when span is extended; when the contextual neighbourhood is broadened.

I am able to see that Bruno & Martinon's work on transferences across dimensions in which children encounter negative numbers would be predicted by research in the broader field of conceptual learning and transfer. They found that, in general, students were unable to "translate" information presented in the abstract dimension into a story or problem in a contextual dimension or on a number line. The same students were generally able to transfer between contextual dimensions or from contextual to abstract dimensions. If we consider that abstract knowledge develops from context-based learning it should come as no surprise that students, when presented with information which for them has no contextual anchors or references (since it was not constructed by them from their experience and learning) were not able to invent those references and add context to something which, for them, had not emerged as a development from such references.

3.4 Research questions revisited

It is pertinent, at this stage in my thesis, to refine previously emergent research questions in the light of my developing knowledge about teaching and learning about negative numbers. It should then be possible to articulate more clearly defined research questions that might be rigorously explored in order to illuminate the micro-processes that are inherent in conceptual development and transfer of knowledge.

The principle themes that emerge from my research questions are:

1. What resources shape the nature of transfer and the growth of knowledge about negative numbers?

2. What is the role of the interplay of resources in the micro-transfer of knowledge about negative numbers?
3. What is the relationship between abstracting and transferring knowledge about negative numbers?

3.5 Key constructs for analysis of core themes

In order to elaborate these research questions, it is necessary to be clear about a range of key constructs that emerge as critical from my reviews of the literature on conceptual change and from that on learning about negative numbers.

Table 1 below sets out key constructs that I find valuable for describing and analysing the processes related to my core themes of microevolution of knowledge, and the transfer and micro-transfer that occur as part of conceptual change and growth. These are taken or adapted from theoretical frameworks considered in Chapter 2, either directly (in these cases, authors are acknowledged) or indirectly where I offer my own interpretation of constructs that I feel are unclear in the literature.

Abstract notion	Expression or description of a pattern or relationship using only general terms
Abstracting	Process of coming to recognise common features across knowledge resources; this recognition creates associations between concepts.
Abstraction	The process of generalising at a high level.
Alignment	"Determining the same information reliably across different contexts" … "The information determined in different situations, possibly using different knowledge, must be the same information" (diSessa & Wagner 2005).
Association	Link or connection between ideas or pieces of knowledge; created when commonalities are perceived.

Concept	An aggregation of ideas and pieces of knowledge with at least one common association; continually changing in response to experience.
Conceptual resource/knowledge resource	Any (internal or external) piece of knowledge, or experience, or way of thinking or acting that might be utilised to make meaning in any situation.
Contextual neighbourhood	The full range of concepts associated (through span of resources) with any given situation or resource (Pratt & Noss 2002).
Co-ordination class	"a particular kind of concept whose structure exhibits a complex system of many elements and kinds of knowledge" (diSessa & Wagner 2005; p121)
Cueing Priority	Likelihood that a resource or sense-making mechanism will be activated as potentially useful in any situation (diSessa 1993; p112).
Micro-transfer	Abstracting; construction and perception and utilisation of associations between concepts that strengthens links between concepts; might lead to abstraction.
Readout strategies	"the ways in which people focus their attention and read out information relevant to, but possibly not the same as, the defining information." (diSessa & Wagner 2005; p131) e.g. knowing that the numbers on a timetable refer to time; knowing that journey duration can be inferred from a timetable, even if the knower is not able to calculate the duration.
Reliability Priority	Established according to feedback regarding the usefulness of the resource on previous occasions, taking account of other resources also activated. (diSessa 1993; p112).
Resource in memory	Resource constructed out of own direct experience or indirectly through exposure to experience and knowledge of others.
Sense-Making Mechanism	Internal knowledge resource; cognitive device that facilitates learner to infer meaning – includes logical/deductive processes, situated abstractions.

Situated abstraction	" …Emerge during activity as internal resources that serve as relatively general devices for making sense of situations that arise within a setting" … " (p458)…expressed in a language that remains embedded in the situation in which it was constructed, potentially constraining its validity in new contexts, with different tools and affordances" (from Pratt & Noss 2002).
Span	i. Existence of one or more associations between resources; ii. Evocation or construction of association between resources. (diSessa & Wagner 2005).
Transfer	Application of knowledge constructed in one setting in a different setting, including processes that lead to transferable knowledge
Tuning towards expertise	Change towards more normalised ("expert") forms of knowledge (from diSessa 1993; p114).
Webbing	Modification of span within and across concepts through construction of connections and associations between them (based on Noss & Hoyles 1996).

Table 1. Catalogue of key constructs

A view of the relationships between some of these constructs, and their role in conceptual change and growth and the construction of knowledge, is described in Figure 3 overleaf.

It is very difficult, using a schematic representation of a process, to convey the sense of flux and instability in any (or all) of the elements of my model. Figure 3 suggests a flow of inputs, through interpretive and analytical mediators to outputs that feedback with the effect of modifying those mediating processes. It is important that I should emphasise that a static interpretation of this model is not entirely appropriate; that the state of knowledge, for any individual, is constantly changing.

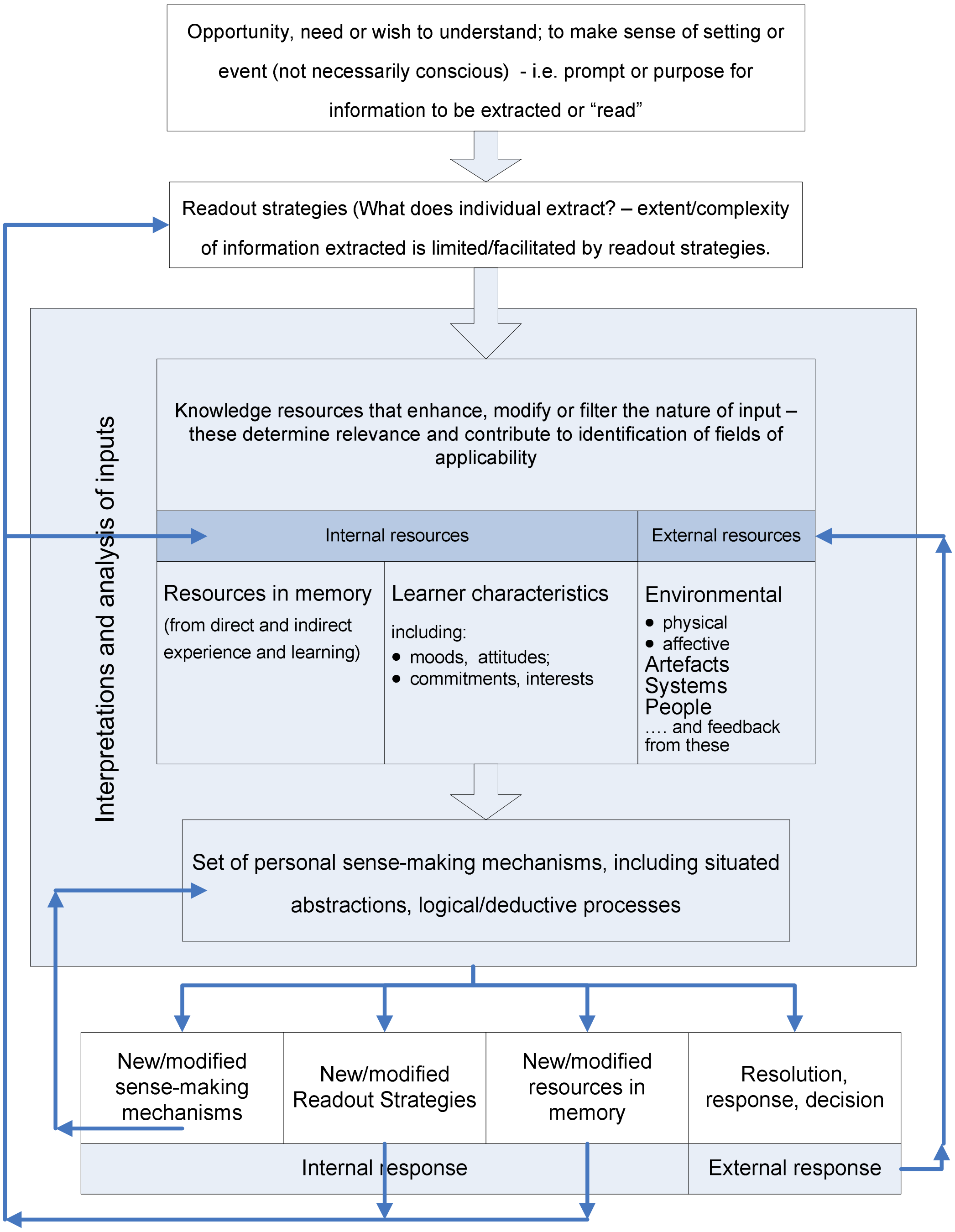

Figure 3: A model of the micro-evolution of knowledge suggested by the literature

Chapter 4: Methodology

4.1.1 Introduction to methodology

I have argued that "Grand", macro-theories are not helpful; they do not tell me what I want to know because they do not describe what I want to understand. Because, as a teacher, I want to be able to optimise my pupils' opportunities to learn, I need to understand how they learn; how and why their concepts change; how they tune towards expertise. These processes are not simple nor directly observable, neither are they similar for all learners. We know, from the failure of "Grand" and macro-level theories to predict learning pathways and outcomes for individuals, that research methods that probe the thinking and behaviour of individual learners are required.

It was to qualitative research methods that I turned in order to discover fine-grain learning processes used by individual children. I recognised that I must acknowledge the drawbacks as well as embrace the benefits associated with qualitative research methods. These considerations therefore form part of my presentation of my methodology.

Denscombe (1998) points out that it is often processes, rather than outcomes, that research seeks to discover. He believes that case studies are appropriate for research about processes,

> *" The real value of a case study is that it offers the opportunity to explain <u>why</u> certain outcomes might happen – more than just find out what those outcomes are." (p31)*

It might be argued that the value of case study research is that it is often possible to generalise from knowledge of the particular. Denscombe stresses this potential, though noting conditions,

> *"Although each case is in some respects unique, it is also a single example of a broader class of things." "The extent to which findings from the case study can be generalised to other examples*

in the class depends on how far the case study example is similar to others of its type." (p35)

This was not my primary goal. I wanted to work with a small number of children and generate a small number of case studies in order to build theory by testing and developing a model of learning which I had developed (Figure 3).

Paradoxically, it would seem that the only way we know how to see thinking is through activity (Schwarz, Hershkowitz & Dreyfus 2002), and yet the thinking that we see in an activity is, at least to some extent, determined by that activity. Therefore, if a type of thinking or behaviour is absent it must not be inferred that children cannot or do not do it – if it is absent it could simply be because the activity doesn't allow, facilitate or encourage it.

Schwarz et al suggest that "theoretical spectacles" are needed – so that behaviours that might be interpreted as evidence of abstraction can be recognised. They believe that "recognising", "building with" and "constructing" are 3 epistemic actions which are constituent of abstraction – and that they are observable. Schwarz et al advocate their "RBC" approach for researching thinking and abstraction.

4.1.2 Research questions

An obvious starting point in designing my study was to pose the question "What do I want to find out?" In the preceding chapters, I have developed 3 key research questions:

- What resources shape the nature of transfer and the growth of knowledge about negative numbers?
- What is the role of the interplay of resources in the micro-transfer of knowledge about negative numbers?
- What is the relationship between abstracting and transferring knowledge about negative numbers?

This list emerged out of my own personal and professional knowledge and experience as well as my review of relevant literature.

4.2 Case study research

> *"Case study is study of a singularity conducted in depth in natural settings". (Bassey 1999; p47)*

A basic tenet of my approach in my research is to accept that, in setting children to work, it is not possible to predict outcomes. At the outset I, the researcher, in this case, could not know what would happen and what the activity might reveal – though of course I hoped to recognise that thinking and capture it. (This is a compelling reason why research in this area has to be, in some way, iterative, with each study providing information that the next can use, so that it can be more effective than the earlier studies at being able to observe and capture that which it seeks to illuminate.)

However, this is also a major criticism levelled at qualitative research, particularly case study; that the notion of emergent design suggests something very loose, undisciplined and lacking in direction and rigour. I would argue that, although I do not claim to have "known" what would emerge from my research, I did have knowledge that equipped me to predict, at least tentatively, a range of possible findings. As a researcher, I cannot avoid conjecture and I concede that my knowledge led me to have some expectations about what I might find. However, I acknowledge this as a strength, rather than a weakness, since it prevents too much "looseness". Notwithstanding this advantage, it was important that I ensured that my research methodology retained an open-ness and readiness to see that which was not expected.

Robson (1993) defends case study research, saying,

> *"Case study need not be of this loose, emergent type"... "In principle, it can be as pre-structured or 'emergent'… as is appropriate for the purposes of your case study". (p148)*

He explains that the aim of any case study might be exploratory or confirmatory or that it might be a combination of these. He warns, however, that,

> *"The looser the original design, the less selective you can afford to be in data selection. Anything might be important." (p149)*

The approach I took in my study of children's changing concepts in a particular mathematics domain, was case study.

It is important to acknowledge that case study is a strategy - not a method (Denscombe 1998; p32) and that there are not, therefore, rigid "rules" about how it should be approached and carried out.

Denscombe also believes,

> *"The aim is to illuminate the general by looking at the particular" (p30).*

My aim was to set out a fine-grained analysis in order to gain insights into how children's ideas change and how they re-use ideas. In developing my analysis, I use it to test conjectures based on my own experience and the literature. In this way, the children's work became a test-bed from which I am better able to evaluate current theory and propose new aspects of that theory.

4.3 Iterative design

Robson (1993) stresses that, when working with qualitative data, interim analysis and iteration are vital. The first iteration of my research took place as part of the "Webkit" project[1], described in Stringer et al (2005). The aims and outcomes of the Webkit project are worth summarising here:

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[1] This is a European project "Webkit: Intuitive physical interfaces to the WWW" (IST-2001-341 171).

4.3.1 Webkit phase

The Webkit project explored the potential for tangible user interfaces (TUIs) to contribute to effective learning of mathematics in a primary school. At that time, I designed equipment and facilitated and led sessions with children to evaluate the way that they re-used knowledge that had been "learned" in a TUI environment, in a different environment – i.e. a major focus of this "Webkit" phase of my study was to discover whether knowledge recently constructed would be considered relevant and be used in a new context – in other words, whether it would transfer to a new environment.

In the lifespan of the project it was not, in the end, possible to evaluate re-usability but it was possible to observe considerable conceptual change in some difficult concept areas (using a TUI). Indeed, it was notable, during analysis of data from Webkit "trials" that some of the most interesting findings related to children's re-use of existing internal resources, as well as to their use of technology and other external resources provided.

What was most interesting, perhaps, was the insight that the research trials provided into the ways children were developing understanding and new knowledge by linking it with experiences in their past, as well as with other new knowledge. Trials "brought to light" interesting insights into the ways children used a wide range of resources available to them.

(My intention here is to summarise how the project impacted on my approach and not to present data; the experience of Webkit was to orientate my methodological perspective, rather than to create data which forms part of the findings of this thesis.) One type of internal resource that children, who were observed and taught as part of Webkit, used on many occasions, was existing knowledge, which included all of the following:

1. secure knowledge – that which had been learned/told/experienced and that children believed and understood;
2. misconceptions;

3. naïve or fragile knowledge (not secure);
4. a vague sense of …, something to do with … (tentative link)
5. “tip of the tongue” (not readily accessible);
6. Knowledge unexpectedly triggered (relevance only unconsciously perceived).

These are all evidence of links with internal resources – i.e. knowledge that had previously been processed in some way.

There was also evidence of links being made “in-action”; “in –the-moment”:

- intrapersonally – i.e. in a child’s own mind; interacting with his/her own processing of his/her own experience both existing and new or recent;
- interpersonally – i.e. with another child’s report or presentation of knowledge that is seen as relevant.

So, from a personal standpoint, as initial trials drew to a close and analysis was completed, my focus had sharpened. My interest had shifted away from a pre-occupation with the use of technology, and particularly TUIs, to a more fundamental desire to understand more about the processes that contribute to conceptual change – how do children construct new concepts, or expand or modify existing concepts?

I look on the Webkit phase of my work as a preliminary iteration from which I obtained a more focused understanding of what I needed to consider in order to engage more effectively with research about transfer. I had realised that, though new technologies might have something to offer that might improve the likelihood that children will transfer new knowledge, we do not actually adequately understand transfer itself. I felt that a focus on the nature of the context or environment, and particularly by narrowly considering this only in terms of the balance of real/virtual, might be irrelevant.

From the end of the Webkit phase, I believed that it was more important to understand better what it is that facilitates or inhibits the re-use of knowledge in new situations. In order to be able to learn about the way children use and re-use resources (to develop knowledge) and knowledge (as a resource) it was necessary to create the conditions where this can be observed – i.e. conditions in which children can have opportunities to recognise, build and construct (Schwarz et al 2002).

The second iteration (the focus of this study) was made up of 4 parts and took place in a different school; only more conventional technologies (i.e. PC and the internet) were used.

4.4 The research setting

4.4.1 The researcher

Denscombe (1998) states that,

> *"Qualitative data, whether words or images, are the product of a process of interpretation" .. " .. the researcher's self plays a significant role in the production and interpretation of qualitative data. … The researcher's self is inevitably an integral part of the analysis, and should be acknowledged as such." (p208)*

I have, in a preceding chapter, described the motivation for my research – i.e. that though no longer a practising primary teacher, in my current role as a teacher of teachers, it is still important to me that I learn more about the ways in which children learn mathematics most effectively. My professional experience in primary classrooms has provided me with innumerable experiences of children apparently failing to transfer knowledge. It has, however, also provided me with even more extensive experience of children successfully learning mathematics by developing knowledge and understanding in very disparate ways. I believe my experience in the classroom also has provided me with appreciation of a vast range of socio-cultural and affective, rather than only cognitive, factors that influence children's learning.

As a teacher and researcher I am part of the learning setting in which my research subjects work. I am, myself, therefore, an external resource and I agree with Denscombe when he goes on to say that the,

> *"researcher's self should not be regarded as a limitation to the research but a crucial resource". (p209)*

So, having described, in very broad terms, what I want to achieve, it is now appropriate to consider the research setting, and to offer some rationale for decisions that were made.

4.4.2 The school

In acknowledging that I would be asking to work with children many times and that I would need co-operation from teachers and parents, I chose to conduct my research in a school in which I have previously worked. It is several years since I worked there so I was not known to any of the children nor most of the staff. The head teacher and some of the teachers, however, did know me and I knew that I could rely on their support.

4.4.3 Class

Because I had already analysed the curriculum and had selected my domain focus, it was appropriate for me to work with children in Year 4. In this year group, children are introduced, for the first time, to negative numbers. It was important, for my research, to explore children's re-use of existing (and particularly recently constructed) knowledge and I felt that in order to be sure whether any existing knowledge in evidence was "recently constructed", it had to relate to something that I could be confident they had only recently been taught. (See my comments in "Chapter 6: Discussion of Findings" regarding "informal" knowledge about negative numbers.) At my research school, there are two Year 4 classes. Both classes are timetabled together for mathematics lessons and are "set" in 3 ability groups across the year group. Children in both classes were therefore “eligible” for inclusion in my research and parental consent was sought for all children in the year group. (See Appendix 1 “Consent Form”)

4.4.4 Children

Children were selected from all of those whose parents consented to their inclusion.

I had learned from the Webkit phase of my research that the groups I had worked with then were too large – that in those groups (of 6-7) some individuals did not engage or contribute, allowing the more confident and vociferous to dominate the group. I chose, for this phase, to work with smaller groups. I did not want to work with individual children as that would constitute more of an interview. Although Wagner (2006) had reported on work with individual students, I considered it important for children to have peers with whom they could share their ideas and thoughts as I believe that they would be more relaxed and would use each other to scaffold their learning. Also, I was concerned that one-to-one interviews might be intimidating for such young children.

Pairing of children might have worked well for discussion and argument within the pairs – however, from a pragmatic viewpoint, this would cause problems if a child was absent on days when I was due to visit. I therefore chose to work with groups of 3 so that an absence would not preclude a session and so that group members would be unlikely to fail to participate.

The class teacher was asked to provide a list of groups of 3 children (for whom consent had been given) who she thought would co-operate and would be supportive of each other. She was asked to exclude any children who would “find it extremely difficult to talk about their ideas and their thinking”.

I did not ask the teacher to consider ability when grouping children. She provided me with a list of 9 groups of 3. At that time she explained that she had considered that children should work with others with whom they are used to “doing maths”. This meant that all 3 children within each group were from the same maths set – that is, each group of 3 comprised children who had been assessed by teachers as being of similar ability. Once I realised this, although initially disappointed because I had expected to work with mixed ability groups, I could see that there would be advantages to this approach:

- that the pace of each activity would match more closely the needs of every child in the (similar ability) group, rather than a “best-fit” match that would be necessary for a mixed ability group;
- that this approach would afford me the opportunity to consider “ability” differences when constructing and developing new concepts, more reliably than I would have been able to do (or had previously intended to do) with mixed ability groupings.

I therefore took the decision to embrace this unforeseen differentiation of my sample groups, rather than re-group them. I selected 3 groups of 3 children – one from each of the 3 ability sets. Gray et al (2000) had focused on ability differences and reported some interesting findings. I considered that their research might support me in analysing and comparing the progress of different groups. It is important to note, at this point, that ability is not here defined but is some construct in the mind of the teacher who made the decision when populating the groups. There was no methodological intention to relate findings to ability since my interest is in the changing thinking of individuals.

At this stage I did not choose the particular children who would become the focus for individual case studies. This decision was made much later, after all sessions with children had taken place and after preliminary analysis of the data.

4.5 Researcher as facilitating observer

Gravemeijer & Doorman (1999) elaborate on the Realistic Mathematics Education principle that cognitive growth requires reinvention of the mathematics by the learner. They point out that reinvention can, and should, be supported by tasks and other interventions that enable it, so-called “guided reinvention”. This would seem to be in line with the Vygotskian construct of the “More Knowledgeable Other”; that, another person, more knowledgeable than the learner, can support conceptual change that would not have been possible independently.

There were many reasons why I elected to be the person who led my planned sessions with the 3 groups of children. The first of these is that I felt that, since I wanted to observe children’s conceptual change and to optimise their potential for this, I should make myself available as their “More Knowledgeable Other” and to guide their reinvention where appropriate.

I did not consider it appropriate for the teacher to lead the research sessions. I did not believe it would be possible for her to fully understand what I was aiming to achieve and feared that the outcomes of the sessions might be adversely affected by something that she might do or say (of fail to do or say). Because I was known to the school, the staff and the head teacher (and parents) were very happy for me to work with groups of children without any other adult present.

My role was not of observer – this implies a passivity that I believed was not in the interests of the children, or of the research. To have been only an observer would be to prevent me from responding to children or redirecting them if I felt the activity was moving in an unwanted direction. Also, I felt that the children would be very accepting of me if I behaved something like a teacher – that they were used to sometimes working with other adults who they think of as teachers and they would therefore not regard our sessions as unusual or abnormal.

At the same time, however, I wanted to avoid any kind of didactic “teaching”. The aim of the research was to provoke children’s conversations and activities that reveal their thinking processes, as they construct new knowledge and when they engage with a new mathematical domain. The guidance I wanted to provide was mainly through the tasks that I would design for them. Although I was happy to respond to questions in a way that encouraged them to think for themselves, I determined to avoid any kind of direct instruction.

I therefore intended to be a resource but to minimise any direct teaching input. My role would be “facilitating observer”.

4.6 Ethics

Although my role was not of “participant observer” per se, it was still appropriate to pay regard to the ethics of being a researcher working closely within my research context, with the research subjects. The children, after all, though informed about the nature and aims of our sessions, might well have forgotten that I am not one of their teachers and consider me as a member of staff. I was conscious that I might become aware of confidential material and I discussed this with the teacher. We agreed that I would pass on any concerns that might arise from my interaction with the children to her. Other general concerns about anonymity and confidentiality are addressed by my checklist below.

Another disadvantage of participant observation that I felt was relevant for me as “facilitating observer” is that it can be difficult to separate my (it feels natural) wish to help children to learn, from my interest as a researcher in observing what children can achieve without anything but the minimum of direct intervention by a teacher.

Denscombe (1998) states that,

> *“The success of participant observation depends of being able to walk a tightrope between the involvement and passion associated with full participation and the cool detachment associated with research observation.” (p154)*

Bassey (1999) noted that respect for the persons involved in, and affected by, case study research is shown by consideration of 4 points:

- Permission to conduct research must be obtained: *in the context of my study, I obtained permission from the head teacher, and any class teachers involved. I then went on to obtain written consent from parents and ensured that children understood the purpose of our sessions together;*
- Agreement for transfer of the ownership of the record of utterances and actions to the researcher: *in the context of my research, this was agreed as part of the initial consent agreement with parents;*
- Decision to identify or conceal the identity of individuals and setting*: I achieved this by using pseudonyms throughout my reporting of my research;*
- Permission to publish the report: *this was obtained at the outset from the head teacher.*

It was important for me to consider whether my decision to work in a school where I had previously worked was ethically sound. If I could not show this it would be necessary to arrange to conduct my research in a different school. Siedman (1991) believes that,

> *" … the easier the access (to interviewees), the more complicated the interview." (p31)*

On the other hand, Fraser (1997) feels that, provided due consideration of ethical issues is carried out at all stages, research conducted by an "insider" can be,

> *"… the most appropriate, most effective and least threatening strategy….." (p169)*

I considered the moral and ethical implications of my research methodology and, as advocated by Somekh (1995), compiled my own set of "ground rules" which were:-

- All parties are informed of the purpose of the research;

- Informed consent of the school, teachers, parents and children was sought and obtained;
- Children would be given frequent opportunities to withdraw from the research;
- The identity of children's teachers will not be recorded;
- The research will not disrupt children's and teachers' timetables and learning;
- Any disruption that does become necessary will be negotiated with teachers and kept to a minimum;
- The research will not impinge on children's play-time;
- Respondents' real names will not be used;
- The school will not be named;
- Interviews will take place in a quiet, private place in school which does not interfere with the normal routines of the school;
- If respondents show signs of distress at any point, the interview will be accelerated or terminated.

I believe that adherence to these "ground rules" ensured that the potential for inequalities of power and status and for role conflict were acknowledged, respected and assuaged. Respect was shown for the school and its routines, as well as for the integrity of the research.

4.7 Task design

4.7.1 Curriculum analysis

In the UK, the statutory curriculum for mathematics is set out in The National Curriculum for England, Key Stages 1-4 (DfEE/QCA 1999). The Programme of Study for Number (Ma2) includes requirements that, by the end of Year 6,

"Pupils should be taught to,

… ; recognise and continue number sequences formed by counting on or back in steps of constant size from any integer, extending to negative integers when counting back;. ; order a set of negative integers, explaining methods and reasoning;… ” (pp21-22)

In the National Numeracy Strategy Framework for Teaching Mathematics from Reception to Year 6 (DfEE 1999), the end of Key Stage objectives set out in the National Curriculum are broken down into Yearly Teaching Programmes (non-statutory). The concept of negative numbers receives no mention until Year 4 and is developed in Year 5:

Year 4 “Recognise negative numbers in context (e.g. on a number line, on a temperature scale).” (Section 3, p18)

Year 5 “Order a given set of negative and positive integers (e.g. on a number line, on a temperature scale); Calculate a temperature rise or fall across 0ºC.” (Section 3, p22)

Therefore, teachers are not advised to teach about negative numbers before Year 4 and children are not expected to calculate with negative numbers until Year 5. I concluded from this consideration of the statutory requirements and non-statutory guidance that children in Year 4 classes, at or near the beginning of the academic year, would be unlikely to have received teaching about negative numbers in school. This was confirmed by their class teacher (Appendix 2 shows the schedule for my interview with the class teacher).

It is appropriate, at this point, to set out my view of an appropriate progression in preparing to learn about negative numbers. This is:

- Secure knowledge of whole numbers greater than zero;
- Knowledge of how to compare whole numbers greater than zero;
- Knowledge of how to order whole numbers greater than zero;
- Knowledge of how to count to find difference between positive numbers;

- Knowledge of strategies for calculating difference between positive numbers.

I must assume some existing knowledge. At age 8-9 years, children should have, available to them, internal knowledge resources about positive integers, and should be able to compare them and order them. They will be able to count and to use counting knowledge to evaluate or calculate the difference between positive integers. They will have experience of using number lines to count or calculate these differences.

What children must learn is that the number system extends through zero and beyond. Children of this age understand that positive integers increase or “go up” or “get higher” as they move further away from zero and that the numbers near zero are “low” numbers. Anything on the other side of zero is therefore likely to be conceived as lower than, or below, zero. Indeed, children may have encountered the expressions “below zero” or “sub-zero” in the everyday world.

Once aware of such an extension to the number system, pupils must learn that the “ $^{-}$ “ sign denotes numbers below zero. This is likely to cause some difficulty since children will have a great deal of experience with the “-“ sign, used as the symbol for subtraction. Also, the fact that negative numbers are often referred to as “minus numbers” is likely to contribute to conceptual difficulties relating to the meaning of the sign or the “minus” label. Yet another possible source of difficulty in this area is that the positive integers with which pupils are so familiar are unlikely to have been signed nor referred to as positive in their experience so far. It is not therefore a matter of correspondence between “old” and “new” numbers that needs to be learned or that might be used to support development of the knowledge about the “new” numbers; it is more demanding than that.

Once children learn that the number system is more extensive than they had previously known, and how to recognise and refer to the numbers below zero, they must learn about the symmetry of the order of numbers about zero. In the positive domain, pupils will have learned that “high”, “up”, “further from zero” or “big”, even “right” (direction of movement towards higher numbers on a number line) all have some equivalence in their knowledge. On the other side of zero, however, these directional, positional concepts and any relationships between them will be challenged. Reconciliation, or alignment, of old and new knowledge must be achieved if children are to be able to move on and function mathematically within their new extended number system.

Pupils must develop the ability to traverse the extended number system, in small steps and in both directions. As this ability begins to develop, they should then respond to questions and problems that require bridging through zero and to those involving larger values both positive and negative.

4.7.2 The nature and range of the tasks

4.7.2.1 Variety

From the Webkit phase of my research I had learned that children used resources that were provided in different ways – some children enjoyed and exploited resources that others did not seem to find at all interesting or helpful (Pratt & Simpson 2004a & b). Their use of internal resources was also varied – for example, one child displayed advanced knowledge about maps and globes; another could remember the temperature in Greece when on holiday there.

It was clear, therefore, from the outset, that, if children were to be facilitated to display their thinking processes, a variety of tasks, using several different internal, as well as physical and virtual resources, would be required in order to provoke such behaviours and processes in the children. This is as my model of learning (Figure 3) would predict – i.e. that children will construct different types of internal resources from experience with all kinds of activities and (formal and informal) learning episodes.

My model would predict that evidence of knowledge relating to any particular concept that is apparent in a single context is not evidence of rich, robust conceptual knowledge – that this can only be inferred from multiple demonstrations in different contexts.

A range of tasks, in different contexts and involving different resources, was also vital in order to provide the scope for children to engage with the different dimensions for negative numbers – i.e. quantity (abstract or contextual) and number line dimensions (Peled 1991; Bruno & Martinon 1996).

Bruno & Martinon evaluated transferences between dimensions. Since conceptual growth and change includes such transferences, I gave children the chance to demonstrate this by including opportunities to work in/with different dimensions through my provision of a variety of tasks.

If we accept that it is possible to use existing knowledge in new settings then there are two possible explanations when research fails to discover evidence of that re-utilisation:

- that it did not happen in the conditions created;
- that the methodology was not able to "see" it – it was not visible through the methodological lens that was available within that study.

It was therefore crucial that I designed tasks that optimise the possibility that children's thinking processes are made "visible", either through their actions and utterances or through my "theoretical spectacles".

4.7.2.2 Images and symbolism:

From my review of the literature it was clear that children use images, metaphors and symbolism in different ways and to different extents, depending on the child, the context and available resources. Gray, Pitta & Tall (2000) found,

> *"The objects of thought of the low achievers were analogues of perceptual items that seemed to force them to carry out*

procedures in the mind, as if they were carrying out the procedures with perceptual items on the desk in front of them. Their images were essential to the action; they maintained the focus of attention. For these children, mathematics involved action and to carry out the action they used "real" things."… "Symbolic images played considerably less part in processing for low achievers that they did for high achievers." (p409)

Therefore, it was important that I should be alert to such differences and design tasks that would facilitate children of all "abilities" to reveal their use of images.

It was also important to consider different types of transfer, as described by diSessa & Wagner (2005):

- Class A Transfer– *"where an adequately prepared set of ideas is used unproblematically in new situations"* (p148); *"the knowledge whose transfer is at issue is assumed to be, or can be demonstrated to be, well prepared and does not, in principle, require further learning to apply"* (p124). diSessa & Wagner note that this is important for schools who *"want students to be able to solve problems other than the ones used in teaching them concepts" (p125);*
- Class B transfer – knowledge constructed that is *"presuming subjects' persistent effort… sufficiently prepared so that transfer can be reliably accomplished in acceptable periods of time (e.g. in a few hours or days…)" (p125);*
- Class C transfer– How do *"relatively unprepared subjects (students) use prior knowledge in early work in a domain?"* (p125); *"where bits and pieces of "old" knowledge are invoked, productively or unproductively, typically in early stages of learning"* (p148). Class C transfer might be considered as the processes that lead to transferable knowledge. (p125)

In "Chapter 2: Literature Review", I argued that it would not be reasonable to expect to find Class A transfer of knowledge recently constructed.

Therefore, I did not set out to look for children exhibiting Class A transfer of negative numbers knowledge from our first session to later sessions (though it is, of course, possible that knowledge they had previously learned outside school is "well-prepared" and might transfer in a "Class A" fashion). I hoped to be able to observe mainly Class B and Class C transfer of knowledge constructed in our early work together, in subsequent sessions: my interest was in children's "thinking-in-change" (Noss & Hoyles, 1996).

4.7.3 The tasks

Task 1: "Journey"

Gravemeijer & Doorman (1999) state,

> *"Context problems can function as anchoring points for the reinvention of mathematics by the students". (p111)*

It was my aim, in designing "Journey", to instil a sense of purpose. Ainley, Pratt & Hansen (2006) emphasise the importance of a sense of purpose in mathematics tasks; that children need to believe that their efforts make a difference in a way that they care about. By using a mythical character with whom all of the children can be expected to have some affinity, I hoped that they would engage with the aim of getting Father Christmas back to the North Pole so that he can deliver presents on Christmas Eve and they would do their best to make it happen.

In the Webkit phase of my study, one of the tasks developed and used was "Journey". In that iteration, children worked with a large map that was electronically linked to a computer. The map, in that phase, **was** the Tangible User Interface.

In this iteration, the map was used again, though without any connection to a computer. It was simply a large map upon which children could move a model of Father Christmas .

Figure 4.7.3.1a: Photograph of the map and Father Christmas figure being used in Webkit.

Children were told that Father Christmas had gone on holiday just before Christmas, to have a break before his busiest night of the year. They were encouraged to think about where he might have gone, that it would have been somewhere really hot. I told the children that on Christmas Eve he had to travel back to the North Pole and we discussed the idea that, as he travelled, he would find that it got colder and colder. I explained that they would be planning his journey and pointed out that it would be a real nuisance if he had to keep putting on extra clothes and taking them off as he travelled – that it would be better if, once he had put extra clothes on he didn't need to take them off again. I demonstrated the database to the children and checked that they were all able to use it to look up the temperature in the countries shown on the map. I showed 2-3 different countries on the database so that the children could see the type of information shown for each country and could see how Father Christmas's clothes varied. Figures 4.7.3.1c (overleaf) show the various states of undress in which Father Christmas appeared.

Information for each country on the database was displayed as a page/slide that showed Father Christmas in appropriate clothing, the temperature in that country (average daytime temperature for that country's capital in December) and the country's national flag. Figure 4.7.3.1b shows some examples.

Figure 4.7.3.1b Examples of information pages in database.

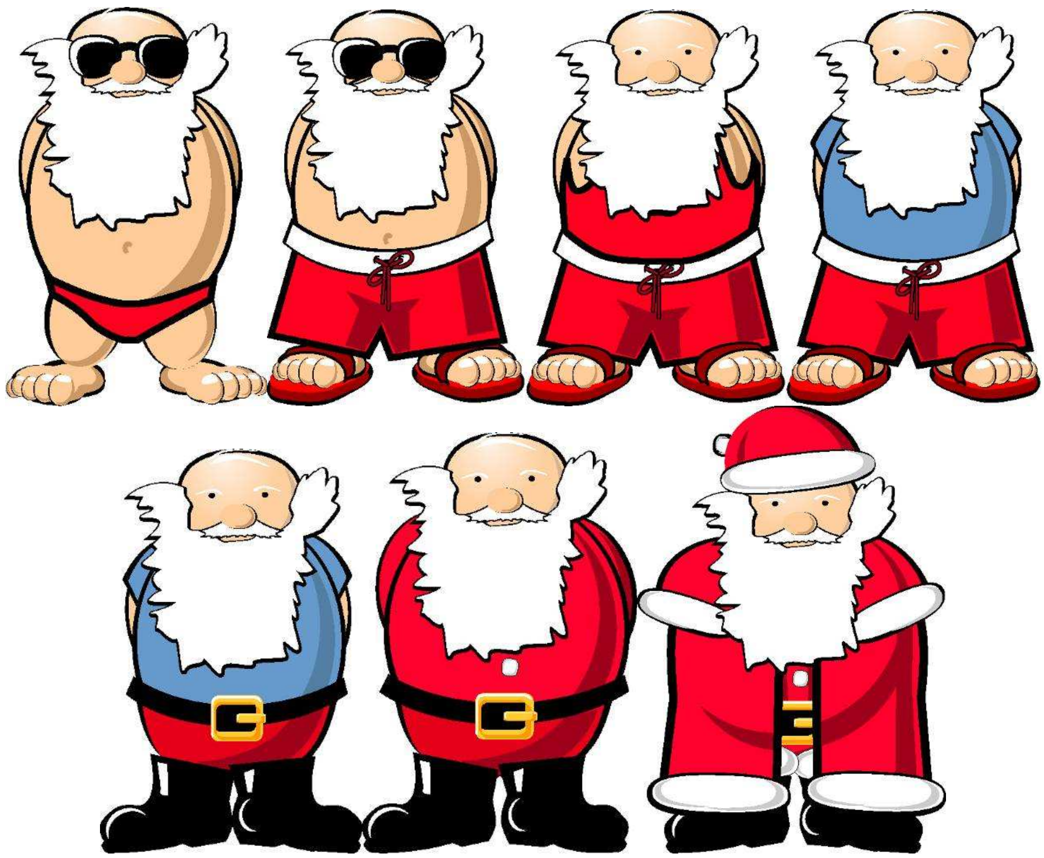


Figure 4.7.3.1c: Father Christmas in various states of undress.

I was conscious, in designing this task, that children would need to draw upon concepts that they might have previously developed to some extent:

- Knowledge that movement towards the north pole and away from the equator leads, in general, to colder temperatures;
- Knowledge that hotter temperatures are represented by higher numbers;
- experience with number lines;
- experience with maps.

Task 2: "Cards"

Table 2 (overleaf) shows the information that was used to create cards for Task 2. The list of countries included in this task is not the same as in "Journey". This time, the focus is on Europe so there are often multiple countries with the same temperature. There are also smaller differences

Albania	¯6°
Belarus	¯3°
Belgium	3°
Bosnia & Herzegovina	¯1°
Bulgaria	¯1°
Croatia	0°
Cyprus	9°
Czech Republic	0°
Denmark	0°
Estonia	¯2°
Finland	¯4°
France	5°
Germany	1°
Gibraltar	13°
Greece	12°
Hungary	1°
Iceland	0°
Italy	8°
Latvia	¯2°
Lithuania	¯3°
Luxembourg	1°
Macedonia	0°
Malta	12°
Moldova	¯2°
Monaco	8°
Netherlands	4°
Norway	¯3°
Poland	0°
Portugal	12°
Romania	1°
Russia	¯6°
Slovakia	1°
Slovenia	¯1°
Spain	7°
Sweden	¯2°
Switzerland	0°
Turkey	¯2°
Ukraine	¯3°

Table 2. Information used to create Task 2 “Cards”

between the temperatures for the countries in the list than there would have been if I had included a broader geographical area, as in “Journey”. In this task, children were asked to place the cards on the desk in order, with “the highest” at the top edge of the desk. I deliberately chose the word “highest” rather than “hottest” because I wanted to gently encourage expansion of the focus that had, in “Journey” been on hotter/colder rather than higher/lower. Also, once zero had been reached, children would be forced to consider the negative numbers on the higher/lower continuum which I expected to cause some difficulty.

Task 3: “Quiz”

Lytle (1994) found that children are unsuccessful with problems using negative numbers except for the most basic tasks involving no more than “simple location” and addition or subtraction. In “Quiz” it was intended that children should be given the opportunity to show that they can engage with slightly more demanding tasks than Lytle seems to consider them capable of. I created an interactive quiz in which children selected the questions they wanted to answer, in any of the 5 (colour coded) question types available. The countries included were the same as those in “Cards” so children were to use the cards as their main resource for temperature information. Figures 4.7.3.3 a-b show the home page of the quiz and an example of each question type.

For the first time in our work together, children were required to operate **with** and **on** the values they encountered.

The map and thermometer icons presented on every “Quiz” page were hyperlinks to a (non-interactive) map of Europe and to software called “Thermometer” produced by the National Numeracy Strategy (Figure 4.7.3.3c). It is freely available and downloadable for all teachers and others using the “Standards Site” website: (http://www.standards.dfes.gov.uk/primary/teachingresources/mathematics/nns_itps/thermometer/)

These icons were provided for children and were pointed out to them at the beginning of the “Quiz” activity.

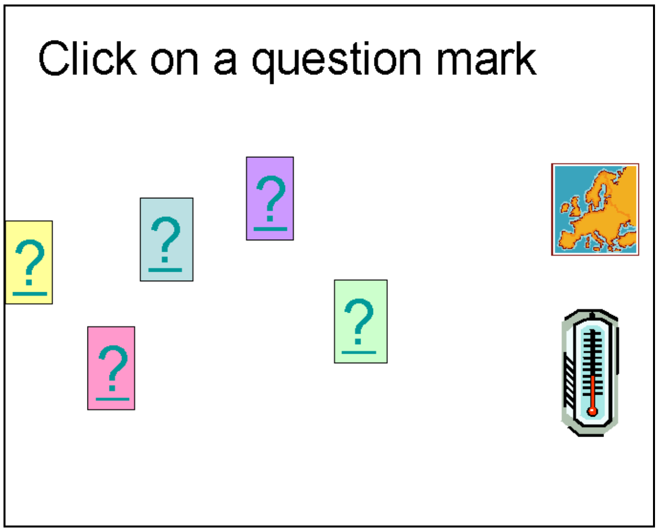


Figure 4.7.3.3a Quiz home page

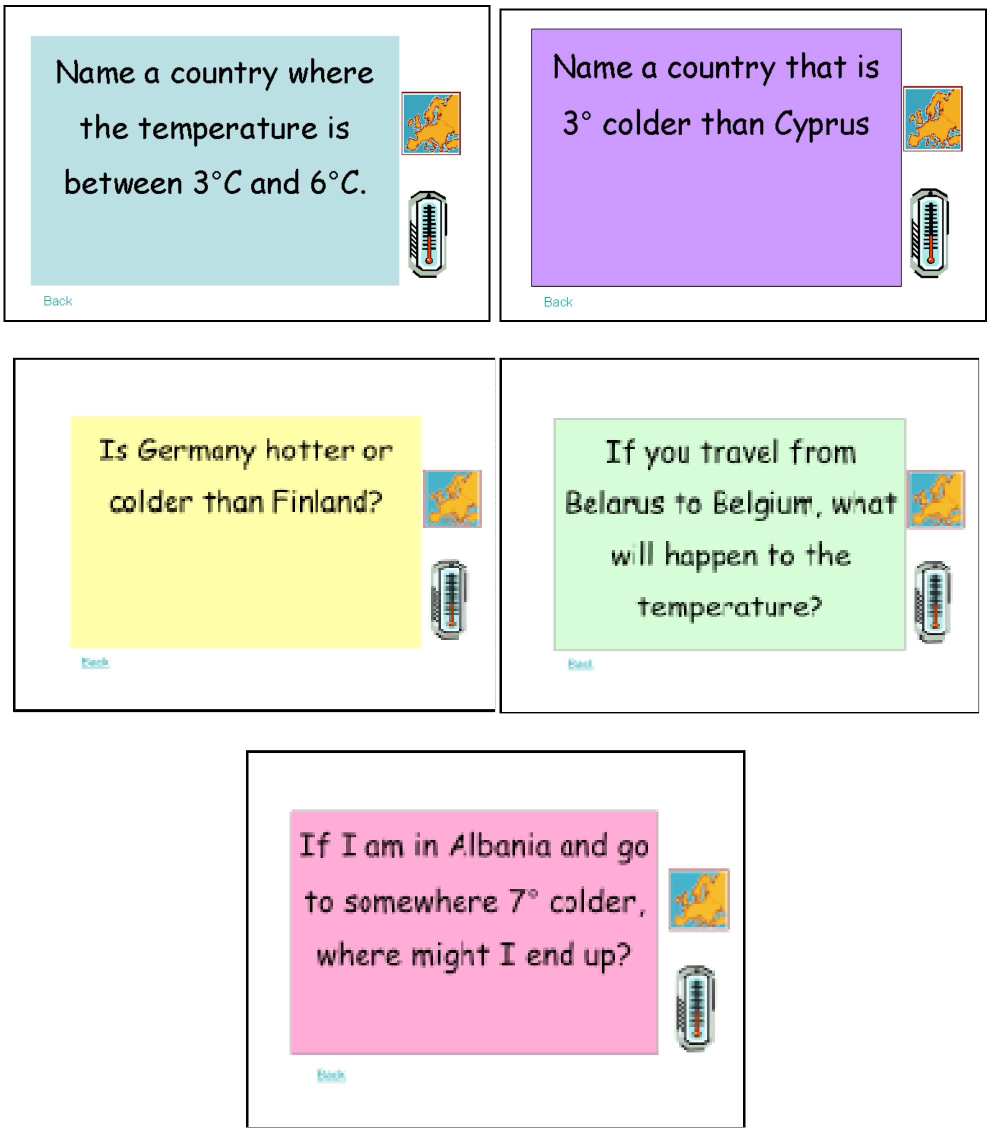


Figure 4.7.3.3b Samples of different types of questions in "Quiz"

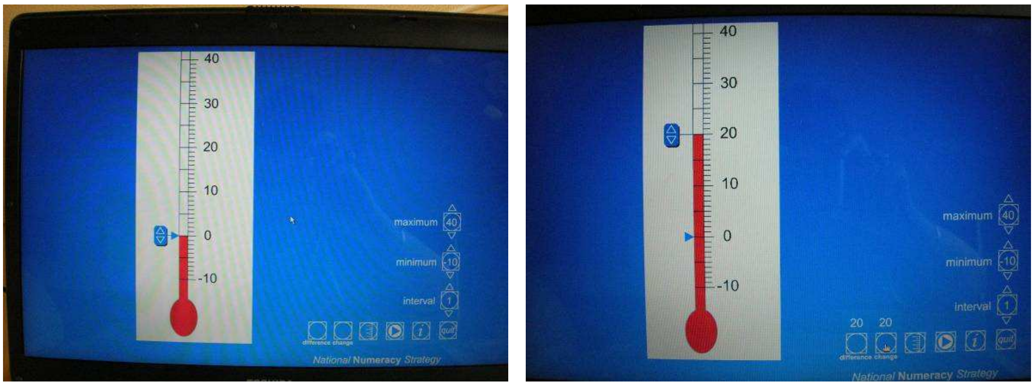


Figure 4.7.3.3c: Photographs of Thermometer in use

Task 4: "Balloons"

For the final task I wanted there to be no mention of Father Christmas or of temperatures or of travel. This is so that the only obvious (to me at least) theme or concept that previous tasks had in common with this one is negative numbers. I sought a PC based activity that the children would experience as a game with some element of competition – this might be against each other or against the clock. I purchased a suite of games for use in primary schools called "Primary Games 4" (Primary Games 2005). This includes many "games" including 2 relating to negative numbers

Figure 4.7.3.4a : Photograph of "Primary Games 4" contents screen

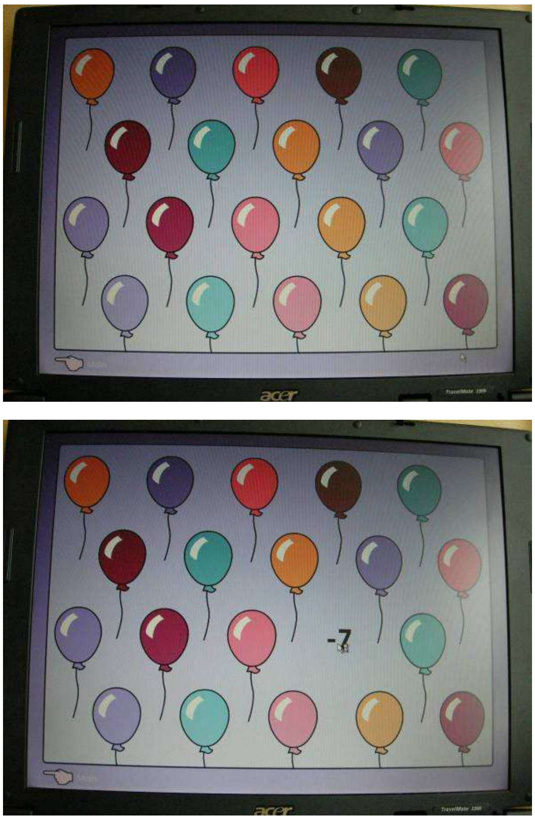


Figure 4.7.3.4 b: Numbers revealed as balloons burst

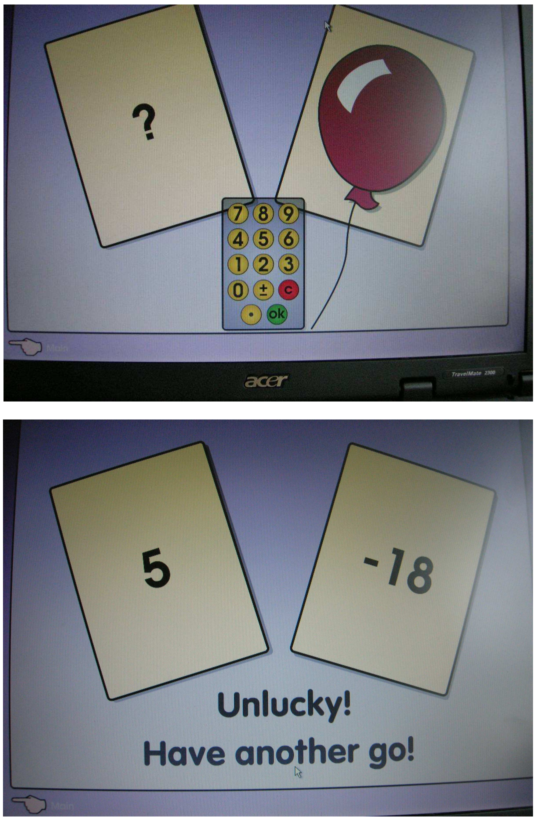


Figure 4.7.3.4c: Checking the answer

Figures 4.7.3.4 b-c show photographs of the screen during various stages of “Balloon Burst”. The game required children to watch a screenful of balloons as some of them burst and disappear in turn, revealing a value in the position of the burst balloons before the value disappears too. Children needed to add together the values revealed and type in their answer. They therefore needed to add a string of positive and negative numbers together. Difficulty levels could be adjusted and were set by me before each game starts, sometimes through negotiation with the group.

It is helpful to summarise the tasks here, as a reminder of the opportunities for conceptual development that were intentionally facilitated or afforded. The tasks begin with numbers embedded in a temperatures context, so the language used includes “warmer”, “increase”; and the sense of travel and change is strong. The first task, “Journey” also uses a map and children’s knowledge relating north to cold and decreasing temperatures (in the northern hemisphere) is also involved. In “Cards” and “Quiz” the activities are still linked to temperatures and countries, and the virtual thermometer is introduced. A number line model has thereby been implied through the thermometer and the notion of travel north/south. The children are also encouraged to record their thinking in any way that is helpful for them, thus affording them the opportunity to de-contextualise their activities and their thinking and to consider them symbolically. The final task “Balloons” does not include any references to resources, models or images used in previous sessions; it focuses on the addition and subtraction of positive and negative numbers. The children are forced to confront their understanding of the minus sign, whether “$^{-}$“ or “-“ and to begin to address the tension between the meanings of “minus” – i.e. as preposition, adjective or noun.

4.8 Data Collection

My selection of strategies and instruments for gathering data has taken account of the need to capture evidence of children’s thinking-in-change. This included use of metaphors, images, number lines – as well as external physical resources provided including map, Father Christmas model, cards. Decisions made took into consideration lessons learned from the Webkit phase of my research.

Bassey (1999) feels that,

> *"There are three major methods of collecting research data: asking questions (and listening intently to the answers), observing events (and noting carefully what happens) and reading documents." (p81)*

My approach was mainly a combination of the first two of these, though the third was also included in the form of my curriculum analysis. My evidence was therefore obtained from multiple sources, as Robson (1993) advocates:

> *"Case study is a strategy for doing research which involves an empirical investigation of a particular contemporary phenomenon within its real life context using multiple sources of evidence". (p5)*

The collection of evidence from multiple sources was intended to facilitate description and evaluation of children's thinking through observation and inference of their actions and utterances.

In devising my strategy for data collection it was important to acknowledge that the phenomenon under scrutiny is not well defined – the purpose of the study is to discover hidden processes that are not well understood. It was not possible to know, with any certainty at this stage, what is interesting or valuable. Therefore, it was vital that a range of data sets should be gathered.

My data included:

- National Curriculum and National Numeracy Strategy Framework for Teaching Mathematics;
- Recording of pre-session discussion with teacher (see Appendix 2);

- Recordings of sessions with each group of children (3 groups, 3 or 4 sessions with each group, each session approximately one hour);
- Field notes;
- Researcher's write-up of each session (see Appendices 3-5);
- Researcher's account of each session for 2 focus children ("Chapter 5: Analysis of Findings");
- Children's jottings.

The accounts of each session for each focus child were the main data for analysis. Accounts were compiled using recordings and field notes and write-ups. It is therefore important to clarify issues relating to recordings and write-ups and accounts:

Recordings

Experience from Webkit had shown that use of video to record sessions with children was problematic. Though some difficulties had been anticipated and measures introduced to minimise loss of data, there remained some issues that had not been resolved:

- with one camera it was not possible to see all children's faces **and** hands **and** the map **and** computer;
- technical support was needed from the school;
- the camera operator was not always clear about aims for data capture;
- risk of loss (or failure to capture) data due to technical breakdown.

A major problem with video data had been in transcribing group activities: it was often impossible to know who was speaking – and often impossible to know the words that were being spoken because children did not speak "one at a time" when engrossed in the Webkit tasks.

I therefore decided not to use video recording for this study. The type of recording that I conducted was through use of a program called "Camtasia" (TechSmith 2005). This creates a type of video recording of the screen activity on a PC together with a synchronous audio recording. Using this would provide a record of what children said and what they were looking at or doing on the screen at the same time. I chose to control children's contributions so that they spoke only one at a time, explaining to them that this was necessary for me to be able to listen later.

Although I realised that I would have no permanent record of children's movements, gestures and expressions, I felt that field notes and my own experience of the sessions, when considered alongside Camtasia recordings (audio and PC screen) would enable the construction of an accurate write-up of each session. I also knew, from Webkit experience, that even with video recording, I would not have been able to construct a "complete" record without significant input of equipment, personnel, technical support and training.

Write-ups

Transcription of whole sessions was not to be attempted. Recordings were reviewed in one-minute segments and a brief description of each minute was written, consisting of approximately 3-5 lines of text. As well as recordings, I relied on my own field notes and my memory to write up reviews of every session. It was therefore vital that these were written up as soon as possible after each session. Denscombe (1998) points out,

> *"Field notes are urgent business." (p151)*

I acknowledge that any summary or description that I created cannot be scientific, objective or value-free. It is inevitable that my descriptions are based, to some degree, on my interpretation of events and involve selection, on my part, of what to include. As Mason (2002) notes,

> *"Description is a cornerstone of all research. Any description is based on making distinctions and drawing attention to relationships, through the process of stressing some features and consequently ignoring or down-playing others. In some research, description provides data for analysis, the description becoming a substitute for "the real thing." (p227)*

The write-up for each session was a (summary) time-stamped log of all children's activities and progress through each task. This facilitated focused transcription of segments that were considered of interest in subsequent analysis, should it be required.

Accounts

Write-ups would provide the data to enable the decision as to which children were suitable for in-depth study – who would become my chosen case studies. Once case study children were selected, detailed accounts of their activity and performance in each session were compiled. As well as providing sufficient summary information to facilitate selection of cases, the write-ups provided a valuable indicator of sections of the recordings that should be "re-viewed" (and possibly transcribed at this stage) in order to enable the writing of an account for each "case".

Mason distinguishes between accounts-of and accounts-for,

> *"An account-of describes as objectively as possible by minimising emotive terms, evaluation, judgements and explanation.… By contrast, an account-for introduces explanation, theorising and perhaps judgement and evaluation." (p40)*

He goes on,

> *"To account-for something is to offer interpretation, explanation, value-judgement, justification, or criticism. To give an account-of is to describe or define something in terms that others who were present (or might have been present) can recognise". (p41)*

The accounts that I created - after the case studies have been identified, and using the same data from which write-ups were created, though in a more focused and “zoomed-in” way - are “accounts-of”. Later, I created “accounts-for” in which I attempted to interpret and analyse children’s actions and utterances in relation to learning processes which I believe might be indicated. It is appropriate for me to acknowledge, at this point, the potential for some bias to creep into my “accounts-of”, since these are compiled from my own (inevitably, to some extent, subjective) experience of the sessions and my own professional interpretation of events. It is important that I should be aware of this and to try to be as objective as possible when reviewing sessions to compile write-ups. However, as Mason (2002) points out, if my interpretations are the same as those that others with my knowledge, and in the same situation, would generate, it is appropriate to accept them as accounts-of.

At all stages I recorded “memos to self” (Denscombe 1998, p211). These included any observations, questions and remarks relating to discrepancies and consistencies in events or data. They were intended to serve as reminders and prompts to reflect that would support subsequent analysis and conclusions.

Mason (2002 reminds us,

> *“Fidelity to some “actual event” is a highly contentious and problematic issue, since for most events, all that remains afterwards are stories told by participants, which are bound to be selective.”… “... all accounts are fictions, and the degree of fictionality is not the issue. Rather, in common with literature, the criterion is whether readers recognise something in their own experience, and whether this leads to informing future practice.” (p234)*

It is pertinent, at this point, to consider issues of honesty and integrity of research generally, and to relate this to my study.

4.9 Trustworthiness of the data and of the research

Robson believes that,

> *"The concepts of "internal validity", "external validity" (or generalizability) "reliability" and "objectivity" […] represent the criteria which have been developed in response to these questions within the experimental and survey traditions." (p403)*

He cites Lincoln & Guba (1985, cited by Robson 1993) who agree that these "conventional criteria" are not appropriate and propose: credibility, transferability, dependability and confirmability as more meaningful criteria for qualitative research.

Transferability

In case study research, the case(s) that are selected are cases of a class. They are not, however, necessarily representative of that class and should not be assumed as such. Therefore, I do not claim transferability in a straightforward way. However, my research aims to build theory and contribute to knowledge about learning; it can achieve this only through using any individuals that I select for study as cases (rather than exemplars) of a broader class of learners.

I aim to provide sufficient information to enable others to reach their own judgements about the applicability of my findings to other cases or situations.

Confirmability

Robson (1993) notes that confirmability is similar to objectivity in other types of research but that the emphasis is shifted from objectivity as an attribute of the enquirer to confirmability as an attribute of the case study itself.

Credibility

Robson suggests that, for their research to be credible, researchers must,

> *" … demonstrate that the enquiry was carried out in a way which ensures that the subject of the enquiry was accurately identified and described". (p403)*

Bassey (1999) also supports Lincoln & Guba's (1985, cited in Bassey 1999) concept of "trustworthiness". Bassey suggests that strategies that contribute to credibility are:

- prolonged involvement with data sources;
- persistent observation of emerging issues;
- triangulation;
- sharing data and interpretations with a "critical friend". (p76)

If I consider these as conditions for credibility (though I understand that those working in the field of qualitative research methods such as Lincoln & Guba, Bassey, Robson would not be so prescriptive) I see that my methodology fulfilled the first of these most certainly. With respect to the second, my involvement was for 3-4 hours for each small group; whether this is "prolonged" is not clear but seemed sufficient for my purpose and demanding in terms of the resource I could bring to the task as a lone researcher. Sharing my thoughts with professional colleagues happened frequently. Such discussions arose informally but provided valuable collaboration through which I was able to develop and test my ideas.

Triangulation contributes to dependability as well as credibility of research

The notion of triangulation may be interpreted in different ways. I might argue that, from a post-modern perspective, the essential purpose of triangulation is problematic, since I believe that there is no single truth that can be located by seeking some intersection of research findings discovered by different methods. However, if I accept that, by using multiple methods and sources to collect data, my research methodology is necessarily triangular (Cohen, Manion & Morrison 2007), then I must clarify the ways in which the variety of data-collection methods enhances the trustworthiness of my data and subsequent interpretation.

The variety of tasks with which the children engage, the group setting in which they work and the flexibility with which I can support and respond to them, constitute different methods for acquiring research data; in creating opportunities for the children to utilise a wide range of external and internal resources in unpredictable ways (which I hoped to capture), I also employed a variety of methods for obtaining evidence of their thinking-in-change. For example, children's response to the map and the data acquired through observing and recording this might have complemented or contradicted the data acquired through observing and recording discussions they had amongst themselves. The actions and utterances provoked by questions from me would generate another set of data that might also complement or contradict data generated in other ways. Robson comments,

> *"Both correspondences and discrepancies are of value. If two sources give the same messages then, to some extent, they cross-validate each other. If there is a discrepancy, its investigation may help in explaining the phenomenon of interest". (p383)*

Different methods for generating data were:

- provision of external physical resources (map, Father Christmas model, cards, standard stationery items);
- provision of "virtual" and electronic external resources (temperatures database, interactive "Quiz", "Thermometer" Interactive Teaching Program, "Balloons" and other PC "games");
- interactions within each group;
- interactions with me, including instructions, questions, support, encouragement to reflect and explain.

In summary, it would appear that, in considering the trustworthiness of case study research,

> *"The case study relies on the trustworthiness of the human instrument (the researcher) rather than on the data collection techniques per se". (Robson, 1993; p160)*

4.10 Data analysis

> *"When methods generating qualitative data form the only, or a substantial, aspect of the study, then serious and detailed attention needs to be given to the principles of their analysis". (Robson 1993, p371)*

4.10.1 Primary analysis

Analysis of data was iterative and reflexive, beginning with the preliminary analysis that took place in constructing write-ups. This was **Stage1** of my data analysis.

Stage 2 of my data analysis was the identification of one individual in each group who would be a suitable candidate for case study. Since write-ups, in effect, "tagged" events throughout all sessions, they enabled me to identify the extent to which individuals contributed to sessions. I could then evaluate those whose actions and utterances seemed to imply conceptual change and growth that I could attempt to describe and analyse.

Having identified one child in each group – i.e. 3 candidates for case study, **Stage 3** of my data analysis was the creation of accounts for each of these 3 candidates.

My raw data, particularly Camtasia recordings, were considered again, at this stage, with a focus on one particular child in each group, in order to construct "accounts-of" each session. (I maintain that it is not necessary or helpful to transcribe whole sessions – that transcription of selected sections was far more effective, if transcription was necessary. This is supported in the research methods literature (e.g. Hutchinson 1988; Robson 1993; Bassey 1999). At this stage, some data, that which did not appear to illuminate the work and progress of the focus child in each group, was omitted, though held in storage for later retrieval should new insights require its re-analysis at a later stage.

Mason (2002) points out that, in creating “accounts-of”, a degree of professional interpretation of incidents is acceptable as long as others sharing that professional culture would be likely to reach the same interpretation,

> *“Whenever there is any uncertainty as to whether a slide is taking place from account-of to accounting-for, ask yourself whether what is being described is behaviour, whether it is negotiably visible or audible to others who share a similar culture to your own, for the focus of accounts-of is negotiable recognition by participants and by experienced colleagues of some phenomenon, prior to accounting-for it”. (p42)*

“Accounts-of” were presented as a chronological list of events, actions and utterances for each session.

The next stage, **Stage 4**, of the analysis was the creation of “accounts-for”. These were written by considering each item of the “account-of” in turn and inferring and making judgments about the child’s cognitive processes at that point in the task. Each item in the list was presented as a row in a table showing the “account-of” and the corresponding “account-for” each item.

My decision to present results of my data analysis in this tabular format was a deliberate, considered choice. Mason (2002) noted,

> *“Finding a way to retain complexity while still saying something useful is extremely difficult”. (p237)*

It might also contribute to methodological trustworthiness:

> *"Maintaining complexity is usually more valuable than achieving simplicity when human interactions are involved." (Mason 2002, p46)*

Tabular presentation of my findings retained the complexity which is, I believe, vital if it is to represent issues of learning, cognition and transfer which, I have shown, are inherently complex. This tabular format also provoked and enabled further analysis and dissemination.

It was previously noted that, in writing "accounts-for", the writer selects what is considered and what is ignored. It was very important, in order to be able ultimately to respond to my research questions, to include comments, inferences and questions that would illuminate the construction and re-use of knowledge. Children's use of all kinds of resources must be described, based on my own understanding derived from my own learning and experience. In particular, constructs that relate to a model of learning developed by myself (Figure 3).

My study aims mainly to build theory rather than to test it, though my development of a model of learning (achieved following a thorough review of the literature as well as my own professional experience) provides me with a preliminary vocabulary and architecture for "describing" and "accounting-for" what children do and how their thinking changes. Of course, it is possible that children and events might not have been consistent with the theoretical positions I took in the light of my earlier review. In this respect, I am testing theory in order to be able to build further theory.

It was important that my "accounts-for" should, therefore, consider more than what children do and say; I also needed to record what I thought they were thinking.

Stage 5 entailed selection of 2 of the 3 accounts who I would develop into deeper case studies which I could analyse and discuss more fully at the level of micro-processes. This reduction was necessary to enable sufficient depth within the resource allocation (in terms of the maximum number of words) that is available to me. At Stage 5, my rationale for selection of cases was to consider each case's potential for elaborating my research questions. It was clear that "G" and "C" stood out in this respect, and that these two cases provided rich and contrasting accounts. "N"'s account was the least rich: interestingly, "N" was from the group that the teacher had identified as "low ability".

4.10.2 Specific analyses

There are points in my discussion in Chapter 6 at which I carried out further analyses of the accounts as data in their own right (secondary level, derived data):

- **Stage 6a** of my data analysis entailed analysis of types, or classes, of transfer according to a taxonomy put forward by diSessa & Wagner (2005) which I presented in Chapter 2: Literature Review. In order to analyse each transfer event (recorded as a row in the accounts for both children created at Stage 4 of my data analysis), it was necessary to clarify, within my own understanding, features of knowledge that are associated with different classes of transfer.
    - Firstly, I concluded that re-use of any knowledge – old or new, effective or not – constitutes transfer of knowledge. This would mean that, where re-use of knowledge was not evident in any row in the accounts, it was judged that transfer had not occurred; and that transfer had occurred in all other rows. Such rows therefore represented transfer events.
    - Next, I considered that a key distinction between transfer classes is the notion of preparedness of knowledge. My interpretation of preparedness of knowledge, as suggested by diSessa & Wagner, and its effect on transfer is that:-
        - **Confidence** or lack of it is not an indicator of any particular class of transfer: confidence may or may not be evident in any class of transfer;

- **Long-lived "-ness"** does seem to influence the ways in which knowledge is likely to be transferred, in that knowledge that was only very recently constructed is unlikely to be sufficiently aligned with other knowledge to be considered "well-prepared". Class A transfer is effective and reliable re-use of knowledge – such reliability is achieved through multiple experiences with the knowledge. Therefore, Class A transfer is unlikely where very new knowledge is re-used; Class C transfer might be of new or old knowledge.
- diSessa & Wagner describe Class A transfer as effective and unproblematic use of well-prepared knowledge. Therefore **effectiveness** is a necessary, though not sufficient, condition of Class A transfer. On the other hand, Class C transfer is not determined by effectiveness, since it is not necessarily productive nor unproductive. Invocation of some prior knowledge, old or new, is in itself Class C transfer; it could be argued that its effectiveness is in its potential to be productive. Identification of Class C transfer by an observer presumes the learner's perception of relevance of a piece of knowledge, at least in its potential to be productive, even where it is found not to be so.

So, factors that determine preparedness and therefore transfer class are effectiveness and reliability of knowledge. Having excluded learner confidence and long-lived-ness as determinants of transfer class, it was possible to devise a key that would help in assessment of class of transfer where transfer was determined in accounts generated. This is shown in Figure 4.10. Findings from this analysis of transfer types are discussed in Chapter 6.

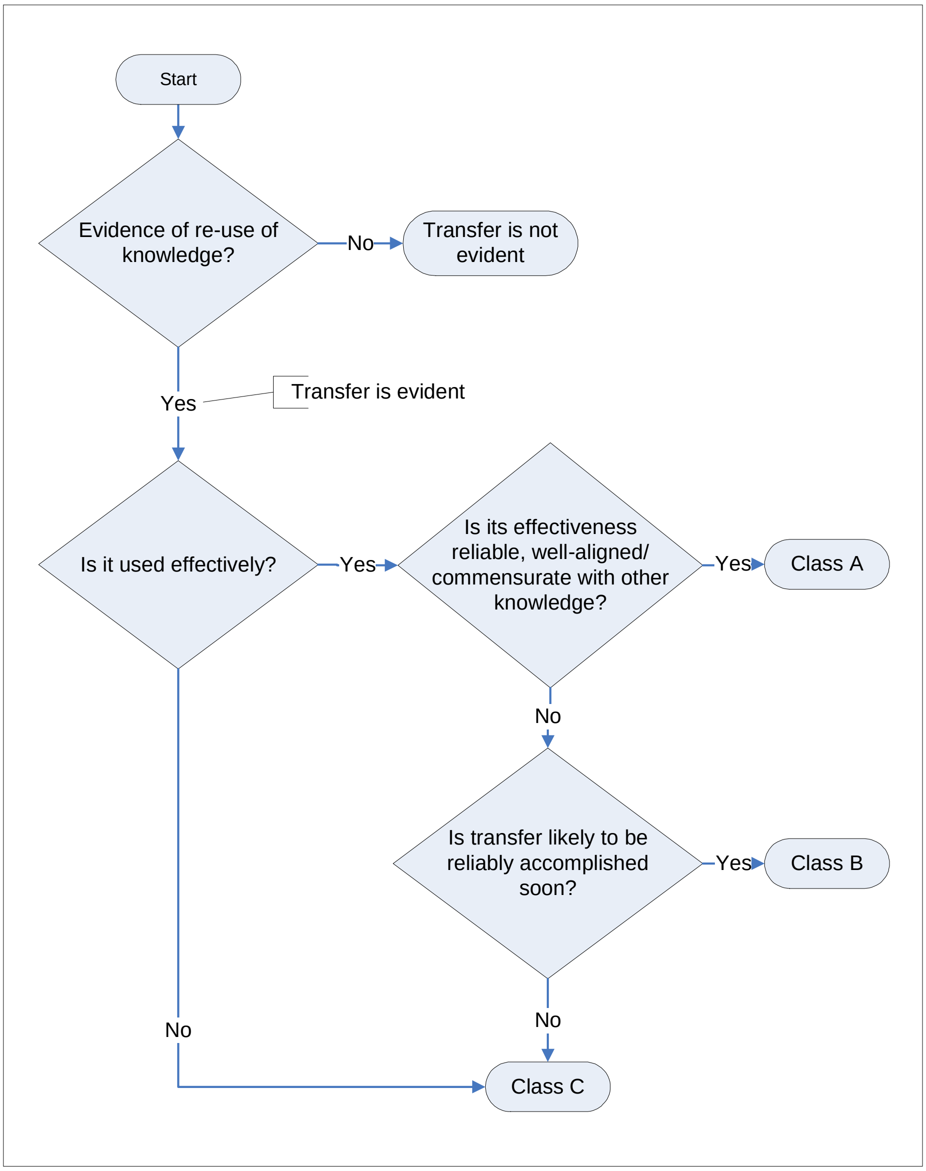


Figure 4.10 Key to determine Transfer Class

- **Stage 6b** of my data analysis aimed to discover whether C's and G's ability to transfer was sensitive to different elements of the problem or task, (Wagner, 2006). Each row of the analysis grids was examined and where C or G demonstrated success or failure, I considered which facets of a problem or task were apparent at that point in our work. Wagner defined 3 facets of problems:
    - o problem type: the problem can be "distinguished by legitimate mathematics descriptors";
    - o problem aspect; "any detail of a problem or problem situation that can be a focus of attention";

- problem context: "the cover story in which the problem is embedded". (p13)

Results of my analysis of problem type/aspect/context in relation to the boys' success and failure in the tasks are shown in Chapter 6.

- **Stage 6c** of my data analysis was carried out to consider my two cases in relation to Bruno & Martinon's (1996) work relating to transferences between dimensions when working with negative numbers. Each row, in each boy's analysis grid, was analysed to consider whether there was evidence of transference between "Quantity: abstract", "Quantity: contextual" and "Number line" dimensions. The number of occurrences in either direction was recorded to provide a representation of the extent of each boy's ability to make connections between dimensions, evidence of richly associated conceptual resources and flexible re-use of those resources. Findings are shown in Chapter 6.

Retrospective note

Initially, accounts-for included explicit tracking of the growth of specific concepts – i.e. the analysis grids originally had 3 columns. However, the third column was abandoned when it became clear that it was not possible to be at all confident about its content. This type of adjustment is what Robson (1993) calls "playing with the data",

> *" … case study design is flexible, with the final version evolving through interaction with the case … "playing with the data" at this intermediate stage may well assist in identifying themes which can form the basis for a workable descriptive framework. Even with a theoretical frame, initial exploration of this kind may give an early warning of its inadequacy, and perhaps lead to a beneficial recasting." (p378)*

Robson commented on the use of matrices to present research findings,

> *"It should not be thought that this is an automatic process for getting at "the truth" about the case. It is an attempt to provide an integrated summary of what I know about it, but is necessarily more suggestive than definitive." (p399)*

I feel this is true of my analysis grids.

4.11 Concluding thoughts

Denscombe (1998) writes,

> *"The logic behind concentrating efforts on one case rather than many is that there may be insights to be gained from looking at the individual case that can have wider implications and, importantly, that would not have come to light through the use of a research strategy that tried to cover a large number of instances". (p30)*

I firmly believe that the phenomenon that is my focus of interest – i.e. children's conceptual change when working in a new domain – is concerned with processes that are difficult to observe. Therefore, knowledge about those processes can only be built by looking and listening very hard, and with minimal distraction and "noise", to the way that one child achieves it. Case study is a highly appropriate strategy for uncovering these processes.

Robson (1993) explains,

> *"Support for the theory may be qualified or partial in any particular case, leading to revision and further development of theory …. ." (p162)*

There is every reason to believe that the small number of case studies that emerged from my research will support some aspects of theory, thereby strengthening that theory and enabling further development.

Chapter 5: Analysis of Findings

5.1 Introduction to analysis

In this chapter I present my account and analysis of individual children's experience and performance in a series of group sessions that took place in their school over a period of a few weeks. Constructs developed and described by researchers into conceptual change provide a basis for a vocabulary and architecture to describe what I observed and inferred about children's knowledge as they progressed through the tasks. For each of 2 focus children, I have compiled an account of their changing skills, knowledge and understanding relating to a mathematics concept of which they have no schooled experience. Table 1 sets out an elaboration of the constructs that I have adopted, adapted or created. Many of these constructs are embedded within the model for the micro-evolution of conceptual resources shown in Figure 3.

I do not believe it is possible to introduce and prepare conceptual knowledge to a highly developed and robust state in a short treatment, of the kind that it is possible to administer as a guest in the children's school. However, by focusing on understanding the nature of the children's knowledge and thinking; by making considered and informed inferences about what they do and say, I was able to describe a learning journey which includes micro-developmental processes involved in the growth of conceptual knowledge. This, itself, implicitly includes what is known as "transfer" or "application". (I maintain that transfer is not, actually, an appropriate word for what occurs: I show that what happens is an extension of the span of situations to which knowledge is appropriately and effectively applied – i.e. nothing is moved from one location to another; it is simply that what is, at first, sensed as applicable in one situation, comes to be understood as relevant in other situations.)

Three groups of 3 children aged 8/9 years worked together on a series of tasks related to negative numbers. This is a domain with which they had no previous experience in school, though of course they may have begun to construct a concept of negative numbers from their experiences outside school. The tasks, which were described in detail in Chapter 4, took place over a period of 5-6 weeks, each group spending 3 or 4 sessions with me. The 2 children I have selected for in-depth analysis were in different groups; two boys who I refer to as "G" and "C".

Analysis and discussion of the findings of my study are presented in this chapter and the next. To begin, the record of the boys' contributions to their respective group's activity, alongside my analysis of those contributions, is presented and discussed, focussing on the boys' conceptual resources and conceptual changes at different stages of our work together. Those records and accompanying analysis are presented in tabular form for the sake of economy of words; in order both to avoid repetition and to furnish the reader with as comprehensive an account of as many aspects of the task, and of the children's conceptual change, as possible. The two columns, "Account of" and "Account for" (Mason 2002) form the analysis grids for each of "C" and "G".

Having used write-ups as data to inform selection of cases for deep study, analysis grids are the outputs of analysis of raw data (recordings and field notes). Once created, analysis grids, since they now contain case-specific analysis derived from primary data, become the main data source for subsequent discussion. For discussion of any event (row in the grid), both my description and analysis of it, should be considered together. For each boy, I also provide a summary of that child's conceptual development during the tasks.

For reasons set out in Chapter 4: Methodology, dialogue was transcribed only where points of particular interest had been identified through consideration of write-ups of all group sessions. C's and G's accounts comprise details of all of their contributions to their respective groups' work with me in my role of "researcher as facilitating observer". Where appropriate and helpful, their actual words are noted; at other times, events and utterances are summarised and paraphrased.

For each of the 2 boys selected for individual analysis, every contribution by C to his group's activity and discussion, and by G to his group's activity and discussion, is described and presented as a row in each grid, in the "Account of" column. (Where there is no contribution by the focus child, no record is noted; therefore, the activity described in the grids' rows are not necessarily continuous, though they are chronological.) The "Account for" column of each table contains analysis of conceptual change for each entry (row) in the grid.

My own knowledge and understanding, hypothesising and further consideration of conceptual mechanisms, processes and changes are ongoing and iterative throughout this chapter and the next, "Chapter 6: Discussion of Findings".

Having considered both children in a very task-focused way in this current chapter, I shall, in the next chapter, broaden my consideration (of the boys' changing concepts) to incorporate the usefulness of particular models of learning in describing and facilitating our understanding of the cognitive processes involved. I shall go on to evaluate my own emerging model for characterising the micro-development of knowledge.

5.2 Case studies

(Descriptions were derived from primary data (minute-by-minute reflections on recordings and field notes). They are therefore recorded in the present tense and analysis and associated comments are consistent with this.)

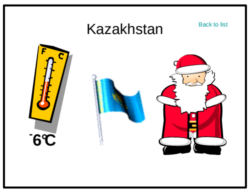
Case study 1: "C"

C has conceptual resources relating to "Temperatures in different countries and parts of the world" ("TW"). Also, he shows that he takes an interest in, and is able to remember facts about Ancient Egypt:

“C”: Events, actions and utterances (selected from researcher’s write-up, Camtasia recordings and field notes)	
“Account of” : Description of C’s contributions to the discussion.	**“Account for” : Conceptual changes (inferred by researcher)**
1. Even before I introduce the task, C is leaning over the map saying “Spain, Spain, where’s Spain, Spain? Where’s Spain?”	C expects to find Spain on the map – he knows that countries are represented on maps. He has been on holiday to Spain and has constructed resources relating to Spain.
2. His interest in Spain is sustained throughout much of the first session. Having agreed that Father Christmas (“FC”) would prefer a warm destination for his holiday, C suggests Spain.	C’s concept of Spain leads him to expect it to be warm there. This resource within his “Spain” concept may span to other similar resources which form parts of C’s concepts about other countries or parts of the world. If there are associations between such resources, a further concept “Temperature in different parts of the world” (“TW”) has been formed, though it might comprise only a few elements, and associations between them may be weak or unformed.
3. When I say that we should send FC somewhere as hot as we can, C, straight away, says “That’s Africa.”	His “TW” concept contains sufficient alignment between its “Spain” and “Africa” components to enable C to compare them and to judge that Africa is hotter than Spain. He doesn’t have conceptual resources about anywhere being hotter than Africa – at least none that span to this situation.
4. He sees Egypt on the map and wants to go there because “It’s really hot there. There used to be people like this .. mummies, pyramids, .. musca	C has some knowledge about Egypt – not only resources about the temperature but also others about ancient Egyptian civilisations. We know that he has learned at least some of this information from a book.

…muscats, what are they called, muscats, it's called suffinks" *(he means sphynx)* He says that he read about Egypt in a book.	

We see that C does have some "sense" of values that he associates with "hot".

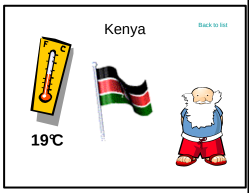
5. While S talks about her trip to Pakistan, C  becomes bored and distracted and is looking at the map. The display for Kazakhstan shows $^{-}6^{o}$. C says "That's not a lot."	C has conceptual resources that are connected to form concepts of "Temperature in different countries/parts of the world" ("TW") and "Knowledge about numbers used to represent temperatures and the hot/cold ness they represent" ("NT"). There is some span across these 2 concepts and, it would seem, alignment - at least to some extent as C is able to judge whether a number, as a representation of a temperature, is or is not "a lot".
6. He spots Germany and calls out "Germany, Germany. Why don't we send him to Germany – that's a hot place."	C thinks Germany is hot. It is not possible to begin to infer where this idea comes from.

7. He also shows, when he says in minute 10, “That’s not a lot!”, that the value for temperature in Germany (1 degree) is not one he associates with warmth; he is surprised. When no-one responds to his “That’s not a lot!” comment he perseveres and tries to resolve his uncertainty and asks N “Is that a lot?” When he gets the response “No”, C is satisfied and doesn’t pursue the question any further.	C expects Germany to be hot, based on some inference or association that is not clear. The value that is displayed is not one that C associates with high temperatures and, though there is some tension/conflict between the 2 concepts (TW and NT), C initially trusts his knowledge about numbers used to represent temperature values (NT). However, it would seem that he is not completely confident and in the absence of reassurance from others, C checks with his friend N, whose judgement he trusts. N confirms what C had thought.
8. When I then go on to talk about what we see in Germany and state the temperature, C says “Oh my God, that’s not a lot.” This is interesting because we know that C didn’t know what to think about a temperature of 1 degree, but because N has given C some information that he trusts, C seems happy to have a reason to contribute to the	Now confident that what he says is correct, C repeats his comment more emphatically to the group. Alignment of the 2 concepts involved (“TW” and “NT”) is reinforced. C is also attending to FC’s clothes and it appears that he is aligning his readout strategies that enable him to attend to, and interpret relevant features of FC’s clothes with his now more secure knowledge about “TW” and “NT”

discussion. He also points out that "He was wearing a lot, there, too." (It would seem that this was post hoc reasoning on C's part.)	

C's confidence in his knowledge has grown, due to approval of C's ideas by his friend. With new-found confidence, C was able to go on to develop these concepts further, seeking and testing alignment with other conceptual resources and extending span within and across concepts.

Next, C has to deal with conflicts arising within his conceptual resources:

9. When the display shows Kenya 19º, C laughs at the image of  FC in his t-shirt shorts and flip flops. C says "Flip flops, it must be hot!" C also remarks that, "If it gets hotter, he'll take off his top or his shorts."	C is reinforcing for himself his understanding that hotter means less clothing (co-development of readout strategies and concepts relating to temperature ("TW") and the number system as it represents temperature ("NT")).
10. S asks whether Madagascar is hot. N says he doesn't think so and C agrees that he doesn't think so either. This is certainly a feature of C's contributions to the	We have already seen that C trusts N's opinion. C shows repeatedly that he feels secure in agreeing with what N says. N's opinion is a resource that C frequently uses. Although I suspect that C would not have expressed a view if N had not, it is not actually clear that this is true.

group – he waits until N says something and copies what N says	
11. When they click on Madagascar and see that it is 21º they are surprised. C says "Oh my God! This must be playing tricks .. because that's *(Madagascar*) hotter than that" *(Kenya*).	2 conflicts are evident within C's conceptual resources: He has said that he thinks Madagascar is not hot – and yet he finds that Madagascar is 21 and 21 is a number that he associates with "hot". Within his "NT" concept C has yet to develop a system of graduation between hot and cold, as well as alignment between this and the numbers themselves. For C, Madagascar is hotter (higher number) than Kenya, but also, Madagascar is further from Equator than Kenya and further away from equator means less hot. These conflicts are not resolved.

Although the conflicts are evident, C does not resolve them, nor does he appear to make any attempt to do so. Perhaps C does not have adequate internal resources to facilitate any efforts to analyse and address the contradictions that he does, at least, seem to recognise? These are conflicts between: C's confidence in N's knowledge and C's own "NT" concept (recently reinforced); and between a situated abstraction, "nearest equator is hottest" connected with his "TW" concept and his "NT" concept.

C continues to enjoy making contributions to the group's activity; at the same time he also further aligns conceptual resources relating to numbers and temperatures:

12. C wants to go to Spain - he says he wants to see what it is because he's been to Spain and it's "real hot". When it's C's turn he goes to Spain. The display shows 7º. C	He is shocked when he finds that it is 7º (though he pretends not to be). His shock might be evidence of a conflict between his "TW" and "NT" concepts – i.e. he "reads" 7 as not hot but his resources relating to Spain have led him to expect a "real hot" number. It is not clear whether he chooses, at this point, to ignore his uncertainty, or whether (without vocalising it) he resolves it by drawing upon another conceptual resource that enables him to reason that 7º is hotter than other countries they have visited so perhaps, in comparison, 7º is

	seems shocked and hesitates. I ask him if this is what he expected Spain to be. He blustered, “Yes, yes. I knew Spain was hot.”	“hot” after all.
13.	When the group inadvertently click on Jordan (9º) C thinks “It’s less hot than Spain”. He claims this, however, without being able to remember what Spain was.	It is not clear whether C has remembered the Spain temperature incorrectly and does actually think that Spain was more than 9º or whether he has just become very confused.
14.	N suggests that they go to Iceland; he knows it’s really cold there. When I ask how he knows, C joins in with “Because it’s white!”. He then makes a joke. When he says “Ice, ice .. get it?!”	It is not clear whether C has any knowledge or expectation about Iceland. His joke might have been used to mask a lack of knowledge or it may not.
15.	After S tells us that she knows something about the Arctic, C is quick to join in, telling us that it’s really, really cold there. When S starts to tell us about a TV programme called Serious Arctic, C talks over her, saying that	It would appear that C does have a resource that leads him to think it is “really, really cold” in the Arctic. It is not clear whether this resource has spanned to his other “TW” and “NT” concepts.

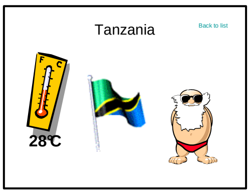
he's seen that too	
16. C thinks that Sweden is a bit hot and a bit cold, and that Poland is "medium". When I ask C how he knows this he says that he doesn't know how he knows. I say "It's funny how sometimes we know things but we don't know how we know." C adds, "And sometimes you know and you forget." Maybe, C doesn't actually know anything about these countries but he is (successfully) including himself without committing himself?	C's non-committal comments suggest that he does not have conceptual resources relating to Sweden or Poland or that, if he does, any resources that he has constructed for either Sweden or Poland do not span to other concepts.
17. I explain the task and check their understanding by asking whether the places he visits will get hotter or colder. N answers quickly, and C repeats what N says "Colder."	It is not clear whether C himself correctly interpreted the task or whether he "tailgates" N's reply because he trusts N to give me the correct answer.

At some points, C's comments suggest that he has not remembered information that was determined previously in the task:

18. C is keen to be the person who chooses where FC will start and places him at the North Pole. I point out that that is where he will finish. C hadn't been listening very carefully or had not understood instructions.	It would seem that C had either not heard or not assimilated information about the task previously because he gives the wrong answer now.
19. He quickly says that he wants him to start in Madagascar "because that's hot – not hot as in sexy of course". When I ask why he wants it to be there, and not anywhere else, he says it's because he knows its really hot there. He doesn't seem to remember any of the information about other (hotter) countries from a few minutes ago.	It is interesting that he had discovered previously that Niger was 25 º but chooses still to use Madagascar as the start of the journey. The information (that may or may not have become a resource in memory) that Niger is 25 º did not span effectively to C's "TW" and "NT" concepts. Niger was not triggered in response to the "hottest" prompt. Furthermore, C's Madagascar = 21 memory resource does not appear to have been triggered.

At other times, it is clear that he has remembered:

20. As they start to plan FCs journey, C says something that confirms that he has remembered that Kenya was cooler	C remembers that Kenya was cooler than Madagascar. He refers to it as "cold."

than Madagascar when he lands on it saying "That was cold, wasn't it?" N queries this but C insists that Kenya was colder than Madagascar.	
21. S is not completely clear on whether they are looking for hotter or colder countries. C tells her "We've got to beat, got to lose 21. Look at what he's wearing."	C now shows that resources relating to the rules of the task are now associated with each other and with other resources, enabling him to engage with the task more meaningfully. He is interested in FC's clothes as an indicator of how hot/cold it his. This shows that C might use FC's clothes as a resource from which he will infer hot/coldness and compare different countries.
22. N decides to visit Tanzania to see if they should go there next. When the display shows FC in his swimming trunks and sunglasses, C giggles and tells N that they should start there. 	C is developing and modifying readout strategies that enable him to infer whether the temperature in the country being visited satisfies their needs of the task at any particular point in the journey. C appears to have understood the aim of the task. This may be seen as his construction of a situated abstraction "more clothes = a move in the correct direction when aiming for a colder country".

However, C's memory resources are not always so accessible, nor is his apparent confidence and success necessarily due to robust conceptual resources relating to numbers:

23. C thinks Kenya should be the next stop because it should be easy to "beat" 28º.	C knows that 28º is very high compared to other countries they have visited during the whole session. His comments suggest that he does not actually remember the temperature in Kenya (19º) and simply thinks that it is likely that it is not 28º or more because almost all the countries they have tried have not been that high.
24. Kenya is 19º and when I point out that FC has got "a lot more clothes" here, C very confidently says "Yes, that's good, that's good."	C is focused on the rules of the task and is satisfied that this move is in line with aims for FC's journey and with his "NT" concept. He does not attend to my hint about "a lot more clothes", my (too subtle for C) attempt to suggest a too-big temperature difference. This reinforces the situated abstraction he has constructed, "more clothes = a move in the correct direction when aiming for a colder country".
25. C groans when the display for Sudan shows 25º – he seems to understand immediately that 25 is not lower than 19 so Sudan cannot be the next stop.	Both of these concepts (NT and "aim of the task) are sufficiently aligned for C to participate and make effective judgments in the task.
26. Until now C seems to have understood the objectives very well but when they visit Ethiopia (16º) he shows the first sign of confusion. His first reaction is "Yes! 16!" But then he says worriedly, "But it's less hot. Last time he was having a coat on, wasn't he?" He revisits Kenya to check. "He's got no coat on. Blue t-shirt	The temperature in the last country was 19 (Kenya) and C is initially confident that a move to a country with a temperature of 16 is valid. However, he questions his judgement when he (mis-) remembers that FC had a coat on at the last stop so he "goes back" to check. He is reassured when he realises that FC has even more clothes on now than before. It appears that C's attention to FC's clothes is a readout strategy upon which he is quite dependant for giving him confidence in his decisions about appropriate journey moves.

and shorts. Let's see, Ethio .. It's the same – blue t-shirt and .. trousers, and boots – he has got more clothes on now so that's alright."	
27. When they see that Chad is 24º, C says "We lost that. We gotta go to a different one. He asks the others where they should go. They want to go to Niger. When they see that it is 25º, C says "Oh, that's rubbish now."	C had decided that Ethiopia 16º had been the previous stop so understands that they should not go to a country where the temperature is 24. I think "We lost that" is a reference to winning or losing at each step – i.e. whether they click on country that fits the requirements of the task at each juncture – in this case they needed one that was less than 16 and got one that is more, so they "lost" and need to find another one. When the other boys go to one that is even hotter C is frustrated. He had quite clearly understood that he was aiming to find a country cooler than 16 and the others are clicking on countries that are increasingly removed from his aim. His own reasoning strategies are functioning effectively, though he is not able to find what he seeks.

We see that C uses other resources (i.e. Father Christmas's clothes) to give him confidence in his reasoning about numbers. C has already shown that he enjoys aspects of competition and challenge within the task and in Row 27 we see that he gets frustrated when he feels that others are lagging behind in their understanding of the aims of the game.

In Row 28, we see, again, that C has greater trust in N's knowledge than he has in his own, especially at a point when C's excitement about his changing conceptual resources had been subdued by others' lack of response to him.

28. C mumbles something about, “Do we want it warmer?” No-one responds. When the display shows 6º C says “Aahh…” and it is only when N starts to say “Nnnno” that C joins in with “No!”	C is beginning to lose confidence in his own ability to make sense of and resolve problems within the task because the others do not appear to share his disappointment with the hot countries they have been visiting. He questions whether he has perhaps understood it the wrong way round when he asks “Do we want it warmer?”. They do not reply and C, at first, feels relieved when a country that is 6º appears on the computer screen. He feels pleased “Aahh ..” that they have found a lower temperature than 16. However, N says that this is not a valid move, C quickly agrees with him, even though this is in conflict with C’s own sense-making. He trusts N’s knowledge more than his own.
29. Iran is 6º. C echoes N saying “That’s going to be hard to beat.”	N has decided that they did need a stop that was lower, not higher, than 16 so they have agreed to go to Iran (6º) which N says “will be hard to beat”. C agrees with him. There are 2 reasons why C is happy to agree with N: firstly, because he has renewed confidence in his own knowledge about the aims of the task – N’s decision to go to Iran will have reinforced C’s understanding that they were looking for numbers lower, not higher, than 16; and secondly, C has faith in N’s judgements, generally.

With N’s support, C is able to achieve increasing span and alignment of relevant conceptual resources:

30. C says that “Portugal’s gonna be too hot though”. Portugal is 12º. N says, and C echoes, “But he’s got exactly the same clothes on though”.	C’s conceptual resources about Portugal lead him to expect the temperature to be more than 6º – i.e. too hot to be a valid next move in the task. The display shows that Portugal is 12º and C “reads” very efficiently that he was correct : 12º is hotter than 6º. The span and alignment of C’s conceptual resources that he has perceived as relevant in this situation is reinforced. N’s “But” suggests a tension in his understanding of the situation – a misalignment of relevant conceptual resources in that he is not sure whether he should make decisions based on clothes or numbers when the 2 readouts are not in alignment (at least in as much as his thinking leads him to believe). C agrees with him – again this could be because his conceptual environment is similar to N’s and he

	perceives the same tension: alternatively, it might be simply because N thinks so and C has confidence in anything N says, regardless of what C's sense-making mechanisms lead him to think.
31. He clicks on Turkmenistan … The display shows 5º. C says "Yes. We beat it. We beat it, N***. Yes!" He is very excited.	Every time C sees that his concerted resources – i.e. his interpretation of the rules (based on associations between resources within and across concepts) and of how to "read" the numbers and the clothes – leads to success, these resources are more robustly aligned.

Next, we see how easily C becomes confused and distracted. He appears to find it hard to maintain his focus on particular aspects of the task and on his thinking:

32. C states that they will go to UK next, followed by Poland. He says that Poland is cold. UK is 6º. C quickly says "Now we go to Poland". C is confused at this point. He looks at the flag moving within the display about UK and says "Wait a minute, something's wrong. We're meant to be going to U .. Merica and that's British." He has got mixed up with United States.	When the UK temperature is first displayed, C does not seem to realise that, if the previous country was 5º, it is not appropriate to go next to UK, 6º. He simply accepts the inclusion of UK as a way-point on the journey and thinks they should now go to Poland (which he believes is cold) as he had intended. Very soon, however, he queries whether the display they see is actually the correct one for UK. This could be because a situated abstraction that he has constructed, that "movement north = lower temperatures" is conflicting with his concepts of "TW" and UK. This may have led him to think that UK should be less than Turkmenistan. C gets confused between USA and UK, itself evidence of an association between them, perhaps because the beginning of the 2 names is the same.

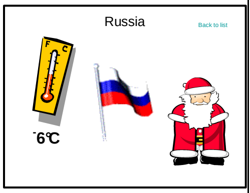
In Row 33 we see another example of C's lack of recall about, even recent, experience and about learning that might have resulted from that experience.

33. Whilst looking for Poland on the list of countries, the children suggest other countries that they might visit. C gets excited at the prospect of visiting Germany but then remembers that they have been there. He can't however remember the temperature in Germany.	Near the beginning of the journey C wanted to visit Germany because he thought it was hot. He found out, at that time, that it is 1º . The excitement he expresses about the prospect of visiting Germany is likely to have the same basis as it did previously. The information he learned earlier has not been re-used by C – i.e. the span of his resources relating to Germany did not extend into his changing concept about temperatures in different countries.

C soon gets excited about zero and we see how he deals with temperatures represented by negative numbers:

34. When they visit Czech Republic C is very excited about the temperature being zero, squealing with pleasure. He confidently tells me that this is less than one..	C has a resource that zero is less than one and is particularly excited about visiting a country with a temperature of 0º. C seems to attach some special importance to zero – perhaps it is simply that he is especially confident about his conceptual resources relating to zero and it is this confidence that excites him. On the other hand, C's excitement might not relate to confidence at all.

35. N asks "Is that minus?" (They are looking at Russia). C replies "Yes, minus 6!" I ask if that is colder than zero and C tells me most emphatically "Yes." He starts to say "We done it! That's it! You can't do any more." S says "But you might get better. You might get minus 8." I ask "Is that colder than minus 6?" C replies quickly "Yes." C tries to explain "Minus 6 means you take 6 away from 6. 6, 5, 4, 3, 2, 1, zero." He tells me "You could have zero, zero, zero, zero – it all means the same."	C believes that, although he agrees that $^{-}6$ is colder than zero, they have achieved the objective of the task and that it is not worth continuing because now that they are visiting countries with temperatures of below zero and "You can't do any more". S tries to make him see that there are differences between "minus" numbers and that the task is not yet completed. C's readout strategies do not facilitate him working with numbers below zero – he doesn't perceive any distinction between them. He explains that "Minus 6 means you take 6 away from 6". His concept of the number system does not effectively extend below zero, although he accepts that minus numbers do exist and seems happy to agree with S that some minus numbers are "better" than others.
36. I say "This says minus 6 – how does that compare to zero?" C tells me "It's colder – look he's wearing more clothes now anyway. He's wearing that big coat, cloak thing.	C's interpretive resources for "reading" FC's clothes support him in his "It's colder" response. He seems to be persuading himself and trying to persuade me that his answer is correct by referring to the clothes that FC is wearing. This suggests that C's understanding about FC's clothes spans effectively to his concept about the number system. His confidence in his sense-making mechanisms that enable him to evaluate the significance of FC clothes, and alignment of any judgements with his number system concept (repeatedly tested and reinforced throughout the task) enables C to make inferences about extending his number system below zero.

C showed that he was focusing on Father Christmas's clothes to enable him to compare temperatures. His lack of resources relating to negative numbers meant that he did not possess adequate readout strategies to be able to extract meaning from the numbers themselves.

Next, it is possible to infer the beginning of a significant change in C's contextual neighbourhood. Interestingly, C seems to be invoking a number line model to "read" values in relation to their proximity to zero. Between rows 37 and 39, C's sense-making mechanisms change considerably.

37. Finland is -4. C doesn't react straight away. Then he says "Oh, it's the same – he's got the same clothes on." I ask how minus 4 fits in with zero and minus 6. C doesn't know. I ask if minus 4 is colder than -6. C says "Yes, probably", though is clearly uncertain.	C is still relying on his resources and readout strategies relating to FC clothes. Span of his number system concept has not yet extended effectively and will need further testing, adjustment and reinforcement.
38. N wants to go to Norway. They see that it is -3 and C says "That's even less than	C thinks $^{-}3$ is less than $^{-}4$. He has not referred to FC clothes this time. He is utilising his number system concept, extending its span to reason that the value containing the digit 3 is less than the one containing "4". He hasn't yet learned that this is not the correct way

that” (Finland -4°).	to extend the number system. His use of the word “even” suggests that his number line resources have modified and he realises that cold does not mean coldest – that, some minus temperatures are “less” than others. His conceptual resources, including readout strategies and sense-making mechanisms did not include this previously.
39. To finish off, I ask the group a few quick questions to check their understanding. I ask for a temperature that is warmer than -6 and C tells me “minus 2” I ask “Is minus one even warmer?” C says “Minus zero, minus zero, that’s warmer.” I ask “Which is warmest, minus one or minus four?” C and then N tell me “One.” When I ask why, C tells me “Because its closer to that, minus zero and that’s warmer.”	C’s sense making mechanism for comparing numbers has modified. Now, he thinks $^{-}2$ is warmer than $^{-}6$ and that $^{-}1$ is even warmer. He goes on to tell me that minus zero is warmer than that and that $^{-}1$ is warmer than $^{-}4$. Although he calls zero “minus zero” he does seem to have learned that the digits increase as they become further away from zero and that they get smaller towards zero, and that movement towards zero from a minus temperature is towards “warmer”. Alignment is being established between relevant concepts and every question that I give C is an opportunity to test this alignment, reinforcing his understanding and his confidence in this new expansion of his number system concept, a part of this concept is now about “minus numbers” in the context of this task.
Session 2 of 4	
40. C picks up the Russia card and says “That’s high. Minus 6. That was one of the highest ones, wasn’t it.?”	C seems to be attending to the digit 6. There were many examples in the previous task that were higher than 6 so it is unlikely that C thinks that $^{-}6$ was the warmest temperature that they included previously. However, $^{-}6$ was the lowest of the negative numbers previously considered – it is likely that C is focusing on the digit 6 and remembers that they didn’t encounter any minus numbers with a “higher” digit than 6. It is also possible that he is thinking about the position (on the map)of the countries that FC visited towards the end

	of his journey that were those furthest North. Perhaps "most northerly" or "at the end/upper stages of the journey" are what C is considering when he says the -6 was "one of the highest ones."

Next, C is working with a set of cards showing each country's name and temperature. At first, he is not able to "access" any information determined in the previous session, which had taken place a few days earlier.

41. C has put the -6 card amongst unsigned numbers on the table. N tells C "Russia is not meant to be in the hot section. Minus six". C says "Don't you remember the last time we came? Minus was hot."	Now that signed (minus) and unsigned numbers are juxtaposed (not like in previous task) C shows that his readout strategies have not modified to take proper account of the minus sign. C's comment "Minus was hot" does not align with anything he has said or done previously.
42. C goes on, "N***, don't you see the pattern? 3, 12, 9, 7, 6, 5 .. " I interrupt and point out, "It's not 3, is it, it's 13." C starts again "13, 12, 9, 7, 6, 5 .." He looks for a 4. N is still unhappy that the cards are not correct. I ask him to tell C why he's not happy.	C is not "reading" the minus sign. He does not assign it any importance at all. Although in the previous task he appeared to have learned how to order negative numbers, it is clear that his readout strategies have not yet evolved for him to "read" the minus sign appropriately here – the juxtaposition with unsigned numbers might be leading him to overlook the signs completely. When N tries to explain, C argues with him (unlike in previous session), insisting that minus does not mean cold. C will not acknowledge that the presence of the sign affects the number order.

He says "Because minus means cold." C replies "No, it doesn't." They both agree that it is -6 but C doesn't agree that this affects the order of the numbers	
43. I ask C to read out the numbers from the ordered cards. He reads 13 to zero in order but doesn't mention any signs.	Readout strategies do not facilitate perception or interpretation of the minus signs.
44. C thinks that zero is the bottom of the list, that "Of course zero is the lowest number."	Although he had demonstrated, at the end of the previous session, that his conceptual resources included readout strategies and resources relating to an extension to the number system beyond zero into "minus numbers", C now reverts to something he believed before that change (to his contextual neighbourhood) had occurred. It appears that the extended span of his concept of the number system and its alignment with other relevant conceptual resources had not been tested and reinforced sufficiently for those adjustments and developments to persist into this new setting.
45. N tries to explain to him that minus numbers are below zero – are colder than zero, that there are things below zero. C suddenly says "We need some more room then. I was thinking minus was hot but it's cold isn't it?"	It seems that N's explanation has stimulated the resources constructed by C in the previous session and has created associations between them that C had not previously perceived. C is now able to use new associations to see that his earlier thinking had been incorrect. It appears that C had associated "high" minus numbers with increasing temperatures in the same way that high unsigned temperatures are associated with increasing temperatures. This suggests that C had applied a situated abstraction, that "higher number = higher temperature" to negative numbers as well as unsigned numbers. Now that his resources are connected more effectively, he realises that he must create more space on the table because there are numbers where he didn't previously realise any might be (i.e. his concept of the number system has extended and

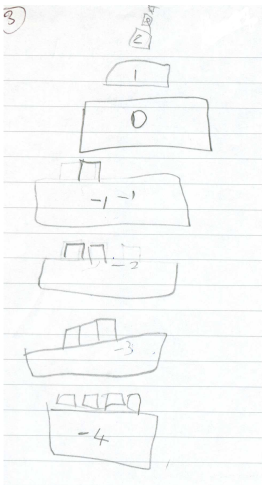
	now includes memory resources and sense-making mechanisms that facilitate continuation beyond zero where numbers are designated using a – sign and that these numbers represent temperatures that are increasingly cold.)
46. C reads the new list where -6 is now below 0 but still doesn't say the minus word.	Readout strategies are still not sufficiently evolved for him to know that he needs to say "minus" as part of the number name.

Next, we see the moment when (initiated by himself) C imposes a kind of symmetry around zero that reveals co-ordination of his resources:

47. C says "That's 6 under zero. "Minus 6," he continues "minus 4, minus 2, minus one, minus one" He doesn't see anything wrong with what he says. I point to the -4 card and ask C what this one is if that one is 6 under zero. He tells me, "4 under zero" I ask him if there is anything wrong with the list and he tells me "These are upside down". S reads the list	C has put all the numbers with minus signs below zero but has positioned the highest digits at the top, closest to zero and the others in order of increasing digits. At the end of the previous session, C had ordered a list of negative numbers like this list, in the context of the "Journey". The span of this new memory resource appears to have extended (though not effectively) to this problem. He is using the "higher number = higher temperature" situated abstraction that was evident previously. It seems that, when (Row 45) C appeared to have adjusted this situated abstraction, the abstraction changed to "higher number = higher temperature and minus temperatures are below zero so even the "highest" minus numbers are below zero". This modification to C's resources is not illogical but does not, of course, include any indication that "high" minus numbers (i.e. digits representing high value" are "lower" or "further down" than minus numbers with "low" digits. His exchange with me helps him because I lead him to articulate and consider the signed numbers in relation to zero – "6 below zero" and "4 below zero". This seems to help C to align his knowledge about

properly. I ask what made them change their mind. C says that it was when he said 6 under zero because then they realised that one under zero is hotter than 6 under zero. …..	numbers with his knowledge about decreasing temperatures.

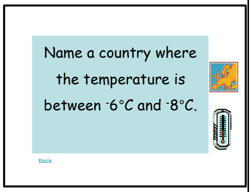
We also hear about his “snow” metaphor:

48. ……. C talks about snow – that higher snow shows that it is colder so -6 is higher snow than -1.	C appears to use an image of a pile of snow to help him compare minus temperatures (scan of his diagram in Figure 5.2).  (Figure 5.2 C’s “piles of snow”) Each “minus” degree is represented by an amount of snow that is increased for every additional “minus” degree. Therefore, more minus degrees = more snow = colder. C talks about colder, rather than lower suggesting that he uses his conceptual resources relating to ”Journey” as the main focus for appropriate alignment of relevant conceptual resources.

Next, we see that C's concepts are aligning and changing.

49. C says Norway is 3 (i.e. doesn't mention minus)	C's readout strategies still do not lead him to perceive the need to say "minus" as part of the number name.
50. I ask C to describe why Netherlands (4^{o}) is where he's put it. At first C asks "Who agrees with me to put it there?" He goes on to say "Because it's under the 5. It's 4."I ask him why he didn't put it "down there" (towards the bottom of the table) because that would be under the 5 wouldn't it? But he knows it shouldn't go in the "minus section". .. "Because that would be under zero."	C is right. When I challenge his use of "under the 5" by suggesting that anywhere on the table that is not at the level of 5 or above would be a correct answer to this question, he confidently (and correctly) tells me that to put it below zero would put it in "the minus section" and that would mean that it was "under zero". This shows that C realises that $^{-}5$ is quite different to 5 – something he did not believe at the beginning of the session. From this I understand that C's readout strategies and his conceptual resources relating to minus numbers, the number system and temperatures have modified.
51. They start the quiz. C says "I got it right. Look! There's Norway and there's Russia" C thinks Norway is hotter than Russia. C explains "Because it  says it on the cards." I argue, pointing out	Span and alignment of C's "NT" concept, resources relating to Father Christmas and his concept of the number system are tested; C rises to the challenge, justifying his judgements effectively.

that it does not say "Norway is hotter than Russia" on the cards. C replies, "It says 3 minus, that means 3 behind zero and that's 6 behind zero… " S says "So that's hotter than Russia." C "Yes, that's what said". I ask "How do you know?" C says "because 6 under zero is real cold but 3 under is only a little bit cold"	
52. Next question: "If I travel from Denmark to Estonia, what will happen to the temperature? C jokes "It'll go higher". I ask "How are you going to find out?" N uses the cards, He says it changes by 2. C agrees. I ask "Does it go up or down?" N says down. I ask "hotter or colder?" C says colder.	(Denmark is 0º; Estonia is $^{-}$2º.) C's first remark "It'll go higher" is not correct. When he pauses and listens again to the question and to N, he agrees with N and adds that if the temperature goes down, it gets colder. It might be that C's resources and sense-making mechanisms (SMMs) relating to up/down and hotter/colder are in some tension with each other. When he focuses on the direction of change of temperature, rather than the direction of change of number, he is able to make sense of the challenge and responds confidently.
53. New question – "Name a country where the temperature is between 3º and 6º" C	C responds well to this question, answering it correctly and explaining well.

says "I know what it is. It could be 4" S says 5. C says "Neverland because that's 4".	
54. "Name a country between 6 Celsius and 8 Celsius." He has misread it – it  actually says -6 and -8. N says "Seven" and C laughs. N says "It could be, though, couldn't it?" C agrees (though hesitantly) "It could be."	C's readout strategies are still not reliable regarding noticing the minus sign and responding to it appropriately. When N suggests a correct answer to the question as it was read, C's laughter might suggest that (a) he thinks N is making a joke and that this is a silly answer or (b) he did perceive and interpret the minus signs in the question, he just didn't say them and so he reasons that N's answer is wrong (as it is to the correct question.) When N persists, C sees that N thinks he is right and, hesitatingly, agrees with N, even though his own conceptual resources lead him to believe that N has not got it right.
55. C says Spain is 7. S says no because "there isn't a minus – it's gotta go down here" C argues because "minus means under zero." C insists that "It can't be something under zero because we need between 6 and 8."	It appears now that C did not "see" the minus signs in the question (lack of effective RS) He does know that a minus number would not go between 6 and 8 so his concept about negative numbers is evolving and developing.
56. I say that the lowest one on our list is -6. C still argues that we shouldn't be looking for minus anything.	C is still adamant that they should not be considering a minus answer because he did not see the minus signs in the question.

Even when N tries to tell C that Spain won't do because "We need minus", C argues that "It doesn't say minus anywhere – it says Celsius". Eventually he does see the minus signs in the question.	

Next, C's evolving concept of an extended number system is effective, though a difficulty with the notion of "between" is suggested:

57. Next question "…. country between 0 and -2?" S says, there's only one and it's got to be minus. They look for countries. N says "It could be any of these 3 – I'm choosing .." C says "I don't get it – because -1 is going to be under zero but what about -2. I ask "Why can't it be 1 rather than -1?! C says "because -2 isn't on top."	He now recognises that the answer will be below zero. SMMs and other resources that would enable him to interpret "between" correctly have not been constructed (and/or are not effectively associated) and he does not realise that $^{-}1$ is the only possible answer. He does understand that the minus sign is important and that a number that doesn't have a minus sign can't be the answer because the lack of the sign means that it is "on top" i.e. above zero and that will not do. C is therefore demonstrating that he is beginning to be able to interpret and co-ordinate knowledge from different conceptual backgrounds in order to address this problem.

In Row 58 C doesn't remember that the map had not been as helpful as he had wanted previously:

58. "If I'm in Albania and go to somewhere that is 7° colder, where  might I be?" C suggests going to the map. N and S say it won't help. C agrees "No, that wouldn't do anything would it?".	C agrees when reminded that the on-screen map was not helpful in a previous question even though his conceptual resources about maps generally had led him to expect that they might find the information he thinks they need on a map.

There is an indication in Row 59 that C's counting strategies are not appropriately evolved:

59. I repeat " the question says 7 degrees colder" The boys count down, C counts 7 cards.	C counts one card for each degree. He does not question whether this correspondence should be assumed. This might suggest that C's RS lead him to "see" each card as an interval on an invisible number line.

Session 3 of 4	
60. I ask what we did with the cards last time. C says we went to countries .	The previous session included ordering cards containing the names of countries but C's "We went to countries" might be referring to the session before, when Father Christmas made his journey using a map.
61. S clicks on the question "Name a country 12 degrees lower than Portugal". They've had this question before but	C has remembered a fact from the previous session. This is evidence of formation of a resource in memory. For some reason, Russia $^{-}6$ is memorable for him – i.e. the span of resources associated with this question includes Russia $^{-}6$.

can't remember the answer. C remembers that Russia was -6. The children find the Russia card and confirm that C is right – it is -6. But does this answer the question? I ask what else we need to know to answer this question.	
62. Eventually C suggests that they look at the temperature in Portugal and "go 7 down, -6, 12" He is very uncertain, confused. C looks for the Portugal card. It shows 12º. C counts back on his fingers.	It is interesting that C counted down 7º for his last question in the previous session. Perhaps "counting down 7" is a resource that is triggered when he gets a question to which he thinks the response requires counting down. It would appear that "counting down" is not a well prepared concept for C - i.e. span to other resources is limited, where it exists at all. This would suggest that, for C, concepts relating to numbers, counting and temperatures are not yet aligned fluently. He seems to reject this strategy when, having started to count down from 12 … (see next row)

C reveals some of his internal resources relating to "the number system", "zero", "information on maps" that do not appear to be adequately associated with other resources to enable C to function effectively, even in areas where we know he has some experience:

63. He finishes on zero. I ask "So, are you looking for zero?" C says that Russia could be the answer because it's under zero. I challenge this. C insists that Russia would do	….he continues all the way to zero (i.e. he correctly counts back 12 and reaches the answer zero). However, he doesn't know he has got to the answer and gets confused again, thinking that perhaps he should find an answer that is below zero. It is possible that he thinks that zero cannot be an answer, perhaps because zero has not been the answer to any question yet. It is possible that he does not include zero within his concept of numerals – that zero is not like the other numbers. Indeed, in Row 34, C did show that he feels zero is special.

64. S gets all the minus cards together and puts them in one pile. She makes another pile using all the other cards. She says that she is going to sort out which is the lowest. C corrects her – "the highest, you mean, out of the coldest?"	C is focusing on the digits when he considers the highest signed digits as being "high" whether they are preceded by a minus sign or not. He has previously shown that his conceptual resources do include a SMM that leads him to believe that these (minus) numbers represent cold temperatures.
65. I say that this is going to take ages – what else can we do? C suggests looking at the map.	C has forgotten that the map didn't actually help the last time the group tried to use it for this reason. This is the second time that C has failed to remember this. It would seem that, for C, the association within his conceptual resource that leads him to expect maps to provide temperature information is somehow more powerful that the memory resource constructed from his recent experience with this map within this task. He has not "learned from his mistake".

Again, C seeks alternatives to offering zero as an answer; the alternatives that he offers show that his concept of "minus numbers" (that had appeared to be forming in our previous work) is not secure.

66. I return children's attention to the question. C still thinks the answer is Russia. I open the Thermometer program and recap that we know that Portugal is 12º and Russia is -6º. C says "But Russia is better, it's lower."	The question has asked for a country with a temperature 12º lower than Portugal. C believes that if he finds a country with a temperature a lot lower than Portugal he will succeed with this question. This might be because he has no confidence in his own ability to work out what is 12º lower and he thinks that he has a good chance of getting the right answer if he can find a country that is much lower than Portugal. It is possible that this indicates that C's concept of "12" is not effectively connected with other resources- e.g. for counting, numbers, calculations . It might also suggest that C does not distinguish between zero and any minus number – that they are all equally correct as answers to this question. We saw in Row 47 that C's concept of minus numbers used to represent cold temperatures

	appeared to be beginning to align with his resources and SMMs relating to "Journey" in that higher digits preceded by minus signs equated to increasingly cold temperatures. However, this concept has not developed sufficiently robustly to be used effectively here.

Within the task, at this stage, children have been introduced to a piece of software that presents an image of a thermometer which the children can adjust to display a variety of values on the scale and within the thermometer bulb and tube (see Fig. 4.7.3.3c). The quiz questions provoke children to increase and decrease the temperature shown. A "Change Box" on the display shows the magnitude and direction of any difference in the temperature currently displayed and the temperature displayed previously by displaying a positive (unsigned) value or a negative (signed) value. C shows that his ability to perceive, interpret and utilise the minus sign is very limited and he becomes confused; not surprising when we remember that he is only just beginning to perceive the minus sign when it precedes a numeral.

67. I point out that the change box shows -12 and ask what this is all about. N says that it shows that we counted down 12. I ask again why the change box shows minus 12. C says "It's because it's below zero". N says "because of counting down minuses."	C's concept of minus numbers includes a resource that makes him think they are below zero. This is the only (or the first) response he believes is applicable here. His readout strategies have evolved so that he sees the signed number and "reads" it as a temperature below zero.

C does begin to interpret the minus sign as an indicator of change as we see in the first part of Row 68. Subsequently, C's conceptual resources relating to the minus sign are associated with other resources:

68. I encourage N to use the mouse and then to "be the teacher" and explain to C and S. C feeds back that if you "go down" 2 it's minus 2 and if you go down 12 it's minus 12. He clicks on "Name a country 1 degree warmer than Finland". N says that the first step is to get Finland. The children find the card for Finland which shows -4º. C wants to use Czech Republic (which is 0º) as the answer. I ask "Is it 1 degree warmer?" C agrees that it is not	C does suggest a country that is warmer than $^{-}4$. This type of response is in line with his answer in Row 66. We have seen that he is able to describe conceptual resources that enable him to visualise the comparative coldness of $^{-}4$ and $^{-}6$ (piles of snow) but, at the same time, he does not seem able to differentiate between negative numbers or between zero and any negative number when quantifying coldness. This suggests that the resource that enables him to perceive and describe the direction of the difference between given "minus numbers" does not also enable him to quantify those differences. C is not influenced by N's explanation about "go(ing) down" meaning minus. It is possible that he hasn't heard N. It is also possible that he has heard but that what N says does not influence C - because C's conceptual resources do not span to include the change box display relating to minus numbers as a process or event , as N is trying to explain.
69. The children say that Norway is the answer to the question. I ask S how that can be right – since one has got 3 on the card and one has got 4 – "How can the one with 3 be warmer"? C says because "that's only 3 below and that's 4 below zero. He repeats "That's 3 under zero colder and	C's explanation is in line with his earlier description of piles of snow. The language he uses shows that this concept is still evolving because he combines 2 forms of speech incorrectly "3 under zero colder". His RS have not so far been able to support him in reading the minus sign as anything other than a reference to a position on the number line below zero. His experiences with recent questions are likely to have provoked his consideration that the minus sign is sometimes also an indicator of movement, process or change. His use of the expression "3 under zero colder" suggests that C is testing a new way of combining the relevant words to discover whether it helps him work more successfully within the quiz.

that's 4 under zero colder. -4 is colder than -3."	
70. Question: "Name a country that is 3º colder than Luxembourg". They look for the Luxembourg card. C interrupts "You've got to take away". I ask, "Why? It doesn't tell you to take away". C replies "But its like taking away, isn't it?" He counts back on his fingers. N does too 1, 0, -1, -2.	C's RS and inferential reasoning have developed so that he now knows that to move to a colder country is to "take away" from the starting temperature.

But C's growing concept is far from being robust and secure:

71. C talks through the problem correctly. Then he questions himself "Oh I don't know. I'm lost"	He does not have sufficient confidence in his knowledge to stand up to his own testing of it.

His confidence in his ability to succeed with these tasks appears to swell and then be immediately dashed:

72. They click on the question "Name a country 4º warmer	C now knows that it is important, when referring to a minus number, to say something as well as the name of the digit. He says the word Celsius, though he should say the word minus. N reminds C that he is

	than Norway". I ask S what we need to do first. S finds the Norway card. C says "3 minus". Then he says excitedly "It's 1! It's 1!" C plays the part of the teacher with the thermometer. He talks about 3 Celsius but puts thermometer on -2. N notices it's not 3 and tells C. C moves it to -3 and still talks about 3 Celsius. The display is showing -3 and C still reads it as 3. Eventually he corrects himself and says minus 3. He moves the thermometer to 3. I ask why. C is confused - he is not listening. He calls out 7. N agrees, 7.	not "saying" the number properly and C eventually starts to refer to the numbers in the way that N wants him to. C's concept is not robust enough to stand up to challenge from N so C assumes that his understanding had been wrong, though it had actually only been his language that was not correct. C changes his strategies for operating with this task to be opposite to what he had thought because he thinks that he had understood things the wrong way round. Now he is trying to work through the problem where his sense-making strategies are at odds with his other resources. This results in confusion for C.
73.	C reads the question out again. N says "Oh warmer, 3 Celsius add .." C questions why he's doing minus. N says that he's right because it starts with 3 minus … N says -2, -1, 0 C	C now focuses on the minus sign only as an instruction to "go down", take away. At first he overlooks the association that he had displayed previously when he used the minus sign as an indicator of position on the number line. His readout strategies are now challenged and he sees only first minus sign that he (appropriately) reads as position indicator (i.e. below zero) but also believes he must take away to solve this problem so is anxious that N is "going up". C's resources are not sufficiently evolved to equip him to see that he must "go up" because he needs a country that is warmer. Although the resources

says "You're not going down, you're going up. N agrees, he is going up	and associations between them may have been constructed, C seems overwhelmed by the need to co-ordinate a wide range of resources, including readout strategies, memory resources and SMMs from different concepts.
74. C put thermometer on -3. He counts up 4 and gets 1. C is confused because "it didn't say minus on the question, only on the card". He still doesn't see/say the minus sign.	This is further evidence of C's preoccupation with the minus sign. He reluctantly accepts that the start value should be $^{-}3$ but seeks justification for this by looking for a reference to "minus" in the question.

C's insecurity has led to him becoming confused and using inappropriate strategies to attack this part of the task.

Soon after this, we see a demonstration of a more effective approach by C. It might be that he is able to re-use knowledge about number bonds and "bridging through ten" (that he had already learned in some other setting) in this setting:

75. "Name a country 10º warmer than Sweden". They find the Sweden card. C says 6 minus. The temperature is -2. S writes 2-10. She changes this to -2 + 10. C says that they've got to add 10 because it says warmer. C says the	C is correct. He knows that in order to get warmer they must add something to the start temperature. When I ask how he knows the answer he tells me very confidently that 2 and 8 is 10. C appears to have an association within his conceptual resources that links 2, 8 and 10. This might suggest that he is using basic "knowledge" of number bonds to 10 (which might be memory resources relating to the relationships between 2, 8 and 10) to help him bridge through zero when adding on the number line. However, it might suggest something far simpler – just that the mention of 2 and 10 (or, in this case, $^{-}2$ and 10) cues or triggers 8 as a response, simply because of this association between 2,8 and 10.

answer is 8. I ask him how he knows and he tells me "2 and 8 is 10".	
76. "If we start at Belarus which is -3 (I put the thermometer at -3) and want to know what the temperature is if it's 20º warmer than this…? N moves the thermometer up and counts up 20. I have to help with the count and controlling the thermometer red bar accurately. N sees the answer is 17. I ask how we could write this down. C writes -3 + 20 = 17. I ask him to read it out to me and check that he is happy with this. He does so and tells me that he is.	The question I pose is similar to the previous (Row 75) question and C, again, shows confidence in using conceptual resources relating to number bonds to derive his (correct) answer. It is still not possible to infer whether his response is a simple reaction to a trigger or is evidence of a more sophisticated calculating strategy (i.e. bridging through 10, or in this case 0). His ability to represent the situation symbolically might suggest that his responses are based on something more that a "knee-jerk response" and that he might be bridging.
77. We re-open the thermometer and set it at -2. N needs to add 30. He goes to 30 (i.e. adds 32). When he corrects himself, C thinks he has made a mistake. I recap and confirm and ask how we	This is another opportunity for C to reinforce the resources he has successfully employed with this sort of problem in Rows 75 and 76. N's error causes C to question his own knowledge but he perseveres and regains confidence in his (correct) answer. It is clear here that C is "bridging" effectively through zero, using conceptual resources that relate numbers to each other (i.e. "knowledge" of number bonds").

would write this one down. C writes -2 + 30 = 28. I ask N to perform the change on the thermometer again. As he moves it through zero, C says "There, that's 2 warmer so it's going to be another 28."	

It is interesting to observe whether C was able to move **downwards** through zero with similar ease:

78. New one – start at 6º and get 8º colder. N slides and counts down 8. C thinks the answer should be -3 – he is counting down on his fingers – 6, 5, 4, 3, 2, 1, -1, -2. (i.e. faulty counting strategy).	Now that he needs to evaluate a change in the opposite direction, C elects to count down using his fingers. His bridging strategy is compromised here through his reversion to his faulty counting-on-his-fingers strategy.

But then C shows, again, that his resource relating to zero is not as well established as resources for numerals on either side of it:

79. C writes numbers as a vertical number line. Now he answers correctly. But his first count is his start number so he should get the wrong answer. I ask N if he	Even when using a number line on paper, C omits to take zero into account as a position on the line. This could be symptomatic of a wider problem relating to zero for C. He needs better understanding about zero (more effective conceptual resources and linkage between them) if he is to effectively develop concepts relating to numbers and operating with numbers. His counting strategies are also faulty and will benefit from more effective resources that relate zero to his concept of the number system.

can explain how C gets the right answer when the method is wrong. (It is because he is not including zero as one of his series of numbers.)	

This separation of zero from the other numerals within C's "number system" concept will inevitably lead to errors with calculating strategies that involve counting procedures.

In our last session together, C failed to recognise the relevance of information he had determined, and strategies and sense-making mechanisms he had used, in the earlier sessions:

Session 4 of 4	
80. Balloons are -3, 2, 9. C reads the numbers but doesn't say word minus. N is quick to correct him. C says "It's 11 – 'cause you add 9 and 2 together … S says "But then you minus 3 away" C says "It's gonna be … 11, 10, 9 "	C's RS are still unreliable in that he fails to acknowledge neither (i) the existence of the minus sign nor (ii) the need to vocalise the minus sign. C adds the 2 unsigned numbers together correctly – i.e. his resources relating to addition of 2 single digit integers is sound and spans to this setting. However, he does not attempt to include the $^-3$ in his calculation until S tells him what to do. Even then, his faulty counting strategies (seen in other tasks) lead him to reach an incorrect solution.
81. Balloons are 1, -7, 1. C giggles and says "That's zero!" S says "1 add 1 minus 7. C repeats, "Zero." He enters 0 as the answer	C does seem to "see" the minus sign this time because we see that his answer does not correspond with addition of only the positive numbers. He is not able to subtract or count back 7 from 2, however. His conceptual resources relating to addition, though apparently adequate with single digit unsigned integers, does not equip him to operate with minus numbers effectively. He has had some experience with adding and subtracting negative numbers (including bridging through multiples of 10) in our previous session when he

	used a number line model to support his working. Memory resources and other resources constructed in that session, themselves associated with a number line-based SMM, have not spanned to this new task. Here, he knows he must add 3 numbers together. There is nothing in this setting that he "reads" (and recognises) as a prompt to consider a number line approach.
82. C reads out next set of balloons: -4, 8, -1 (at first he omits to say the minus sign but corrects himself.) C says 8 take away 4 take away 1 … 3	C's readout strategies are still unreliable in "reading" the minus sign. He appears to have knowledge of (what I know as) commutativity when he reorders the numbers so that he starts with the unsigned 8 and then "reads" the minus sign as an instruction to "take away" for the other 2 minus values.

Modification of C's conceptual resources relating to the minus sign, that had been achieved in previous sessions, is only partially evident in this final session:

83. Balloons are 1, -5, -9 (But C doesn't say minus again) S says "Zero, No. It's got to be minus something". C says "add 5 and then you'll know what minus it is … 15, -14!" C keeps saying "Minus 14, minus 14" I ask him how he got to this answer. He tells me "I added 9 and 5 that makes 15 but then I've got one that makes it 14." I ask why minus? C	RS are still not reliable for reading minus sign – it seems that he does "see" it but doesn't see the need to say it. He reads the minus sign as an instruction to subtract. He gives the correct solution and his explanation of how he arrives at it suggests that he could be using either a number line model or cancellation model. He tells me that he adds 9 and 5 and then adjusts for the 1 by decreasing his answer by 1. It is possible that his methods are quite arbitrary and that he has no preferred model.

says "Because it's got minuses there" I ask "Why don't you just add all 3 numbers and call it minus? "C says "No, because one of them isn't minus" I get C to think it through again, pointing out to him that he has made a careless error. He sees that 9 and 5 make 14, not 15 so adjusts his answer to -13.	
84. 5, 5, 10, 3, -1 S says " 5 and 5 is 10, 20, then that's 23. C says "take away 1" S says "22. I ask "Why are you taking away a number? You're supposed to be adding all the numbers together. N says "Because it's a minus". C agrees, "'Cause there's a minus 1 there. It's telling you to go below. " I ask "Why does that mean take away?"	C very explicitly tells me that he reads the minus sign as an instruction to "go below".

Interestingly, C is able to explicitly relate ideas arising in this session to earlier tasks:

85. I persevere "You haven't explained it very well yet. Why, when you're adding a minus number, does it mean you take it away?" C says ""Cos it's like temperature – 'cos you could have 19 degrees and if you take minus one off it'll be …?" He isn't able to reach an answer. I ask "So, it's like temperature?" N says "Sure, minus, below" C agrees "Yeh, cos we did it last time and I remember. Santa Claus taking his clothes off - stripping"	It is C who mentions that there is some connection with temperature and it is clear that he has a concept about minus numbers in the context of temperature. An image of Father Christmas taking his clothes off is revived for C, showing that his concepts about temperature, minus numbers and the "Journey" task all span each other. Also apparent, however, is a lack of alignment across these concepts which would enable C to work more effectively when trying to operate with minus numbers.

Other conceptual connections are implied by C's actions and words, even where he doesn't articulate any acknowledgement of any recognition of relevance:

86. New balloons 9, -6, 6, 5, -8. S says "They make 11, then add 9 ….. 20." C says "Then take away 6". But he can't count back, gets confused. He	C's faulty counting strategies lead him to become confused and to provide the wrong answer. However, there is again evidence of sound resources relating to some number bonds (as well as other resources relating to strategies for calculating, perhaps - though not necessarily - including explicit knowledge of the commutative law) when he suggests taking 8 away first.. Perhaps it is his sound number bonds resources that cause him to be uncomfortable with the answers he reaches by counting back? His confusion when he tries to count back

suggests, "Why don't we just take away 8 first?" "12 .." He counts back falteringly from 12 ending up with an uncertain "7?"	,and his uncertainty with the answers he reaches using this method, together suggest some conflict within his own contextual neighbourhood.
87. N asks "Shall we try to work it out again?" I ask "Does it help to take the 8 away first?" C says "Yes, because you add the highest number first …. " N and C get very confused when trying to count back 8 and then 6.	C appears to have internalised a situated abstraction (we do not know whether this was given or constructed) that it is appropriate to start with the highest numbers when calculating. The span of this abstraction extends to this type of question and his RS is to actively seek out "highest numbers".
88. 7 balloons 1, 1, -2, 0, -4, 4, -4 C checks that we've got the right number of balloons. He says "4 add 1 add 1 is 6. Take away -4 …" Boys are distracted.	C starts appropriately and doesn't make errors in the early stages of this solution.
89. C's turn: -2, -3, 0, -1, -3, 3, -3. C recaps the numbers. C says "There's a lot of minuses." S suggests "Which ones are minuses? Take out the ones that aren't." C looks at his list again,	Again, C employs his "highest first" strategy. It appears that its use in earlier questions has reinforced its cueing and reliability priorities for these questions and C's RS are now tuned to focus on this aspect.

"Which is the highest minus? Oh no. I don't .."	
90. …. C says "3,6,9, (he's trying to add together the 3 minus 3s) 10,11 (adding the -2) add 3 (the only +3), that's 11 take away 3. 8" . I ask "Why take 3 away from 11?" C can't explain. He thinks it's because "It must be minus because there's lots of minuses."	As in row 83, C's comment " … add 3, that's 11 take away 3" suggests that his conceptual resources include a sense of negating what has been achieved so far, when adding a value with a sign opposite to those accumulated, and we know he groups together like signs and adds them together as the first stage in his calculation. This leads him to consider the addition of an unsigned number to the total of the signed (minus/negative) numbers, as "taking away". This might suggest the use of some conception of operations with signed numbers as movement along a number line that C is not conscious of and does not articulate.
91. C thinks I should "put all the adds together and all the minuses together."	C thinks it might be correct to group together numbers of the same type "adds" and "minuses". This might simply be a vocalisation of the strategy he has been using. It might also mark the point where he first realises that this is what he is doing.
92. I ask "What if I say 4 take away -1?" and I write 4 - -1? C is shocked . "What??!!" N says "5 because when it says add you take away so it might be …" C is incredulous, "You really have gone mad!"	Until now, all problems have required C to add values together and this one clearly sets out subtraction of a minus number. With the previous addition problems, C effectively re-ordered elements so that he could move the minus sign to appear after a value. He doesn't have RS that enable him to "read" this question that might prompt strategies that he has and could invoke in order to attempt to solve the problem. N's remark about taking away when the question says add is an articulation of what C was doing in Row 90 and yet he doesn't recognise it. That particular aspect of his concept of operating with signed numbers did not "stick" – did not become a part of the concept that is called up now – perhaps because he did not have the opportunity to test and reinforce it at that time.

C's story

It is clear that C brought to the tasks a range of qualities and knowledge. He is naturally inquisitive and wants to please and he participated with enthusiasm.

He had some conceptual resources about temperatures in different countries and about numbers. At first, C needed to align these 2 concepts so that he achieved a sense of the range of numbers that typically represent hot and cold temperatures. He learned how to use a set of external resources comprising representations of Father Christmas wearing different clothes, provided within the task, to help him align 2 key concepts: "Numbers as they are used to represent temperatures" ("NT"); and "Temperatures in different countries and parts of the world" ("TW"). This was achieved through C's own active creation of associations across the concepts that effectively connected them at particular points, structuring the relationship between the broader concepts.

At times, C encountered incongruity and conflict. When this occurred he seemed resigned to tolerating the conflict rather than actively pursuing lines of thought that might have helped him resolve it. C lacks confidence in his knowledge and in his ability to solve problems independently – we see this in his (sometimes undeserved) respect for his friend N's opinion. This lack of confidence is likely to be part of the reason that C did not attempt to resolve conflicts that he experienced.

C is very sociable and was eager to please me, the "teacher". These traits, as well as supporting his learning, sometimes compromised learning in that, sometimes, C was so keen to interact and make gratuitous contributions that he did not listen. Consequently, information that may have been determined failed to become internalised and was not, therefore, available to him subsequently. There were several incidents where C failed to recall some resource that he had appeared to have learned in a previous task or dialogue.

It emerged that C's contextual neighbourhood relating to numbers was very fragmented and there were many resources that were not adequately connected to others – these include the notions of “between” and “zero”. Although he appeared to have been exposed to representations of negative numbers, he was not able to relate this experience to anything else within his contextual neighbourhood and was unable to derive meaning from them. He used a range of task-based resources to help him construct internal resources about negative numbers: firstly to realise that the (positive) number system with which he was familiar does extend through zero and beyond to negative numbers; next, to recognise the symmetry of integers on either side of zero, and then to be able to order negative numbers, with the support of task resources. C also went on to increase and decrease values by moving in both directions along a number line, beginning with both positive and negative values and moving through zero; he was able to do this within a “Journey” model and also in purely symbolic mode.

It is interesting to observe that C had particular difficulty in developing readout strategies that reliably perceived and processed the existence of the minus sign/symbol when it precedes a numeral to indicate a negative value; it took a long time for C to begin to articulate the “minus” when referring to a negative number. He did, however, achieve this: he was able to modify his resources effectively so that he could extract meaning from the minus sign that enabled him to begin to construct his concept of negative numbers. C’s other poorly connected resources relating to “zero” and “between” and his flawed strategies for counting with his fingers also interfered with his ability to learn from the tasks, in that the necessary co-ordination is not achievable where resources do not span and/or are not aligned.

C did use existing conceptual resources to support his work in a new task; we saw this when he successfully used number bonds to bridge through multiples of 10 (if that is what he was doing). He also used strategies (conceptual resources) that cannot have been constructed through activities with me, such as grouping like terms together and working from the highest value first. Presumably these strategies were previously reinforced and consolidated thoroughly and repeatedly for them to have become connected within C's knowledge and for high structural priorities to be established; since we saw that any new conceptual resources and associations between them that C was able to build for himself within the new task (thinking-in-change) were fragile and that associations, with little or no reinforcement, often faded and did not span to new situations.

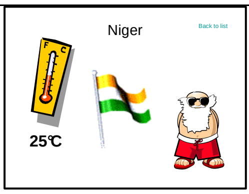
In the final task, C needed to aggregate a number of integer values, both positive and negative. He had to confront the notion that the minus sign is both an indicator of position relative to zero and also an indicator of the direction of change to a value. C seemed able to operate these two facets of this concept where they did not need to be used together. His strategy for working with mathematical operations with minus numbers was to invoke his knowledge of commutativity (or at least of strategies based on this) to justify re-ordering the values so that the negative number becomes the subtrahend in the expression and he can use the minus sign as an instruction to subtract, an idea that he is comfortable with. However, when there were too many minus signs for this strategy to be helpful to him, C could not see his way through the problem.

C's learning journey was a bumpy one. He was able to take on some new knowledge and to align it with existing knowledge. However, he needed repeated opportunities to reinforce those new connections in order to sustain any increased span. Without those opportunities to consolidate, new connections faded and, though not necessarily lost altogether, they were often not recalled in new settings even though relevance had previously been discovered. In other words, the cueing priority and reliability priority of newly constructed resources remained low until or unless associations with other resources are constructed, tested and reinforced.

Case study 2: "G"

G already has some conceptual resources relating to temperatures in different parts of the world ("TW") and he has a secure understanding of the number system as it is used to represent temperature ("NT"). G and his group are very quick to understand the aims and rules of the task. (However, some of G's conceptual resources are not so rational):

"G": Events, actions and utterances (selected from researcher's write-up, Camtasia recordings and field notes)	
"Account of" : Description of G's contributions to the discussion.	**"Account for" : Conceptual changes (inferred by researcher)**
1. Having introduced the group to the countries list and clicked on Madagascar as an example to show the boys what each page looks like, G is the one who very quickly says "Well, that's wrong, 21. It's really hot in the film", when he sees the temperature displayed.	G Reads 21º as "twenty one degrees" and knows that this is a representation of "how hot/cold" His understanding of 21º conflicts with his understanding of Madagascar as he thinks Madagascar is **very** hot and 21 is not **very** hot. The 2 concepts.", "NT" ("Numbers as they are used to represent temperature") and "Madagascar" do span to each other and are already aligned. That alignment is challenged here.
2. M starts to answer "Well, these are quite .." but G interrupts with "hot, especially near the equator".	G's conceptual resources include readout strategies (RS), resources in memory and sense-making mechanisms (SMMs) that enable him to understand that all of the countries indicated are hot; those that are closest to the Equator are very hot. G is able to judge whether a country is "hot" by considering its position on the globe in relation to the equator. This is a sense-making mechanism (SMM) that he uses to organise information about temperatures in different countries.

 3. When they click on Niger (25º) G quickly notices that “He’s taken clothes off - we can’t go there.”	G attends to the clothes worn by FC on the screen. He knows that less clothing = higher temperature (another SMM, perhaps a “situated abstraction”).
4. (The boys) understood the basic rules straight away and using their sound understanding of number order and its relationship with relative temperature (i.e. hotter, colder). When trying to decide whether to go to Libya, someone suggested that Algeria might be better but G disagrees, thinking that Algeria is “really hot”. When I ask him why he thinks that, he tells me that it’s because “It’s got sand”.	G’s “NT” concept has already begun to form –i.e. he has a collection of associated resources that he uses to extract or impose meaning about “Numbers as representations of temperature.” He has a “higher numbers = hotter, lower numbers = colder” SMM. It would appear that G has formed a situated abstraction, “sandy countries are hot”. Interestingly, there is nothing in the external resources provided that shows sand so we cannot know why G immediately associated Algeria with sand. It seems that he has an association between resources for Algeria and sand.

G has one French parent and has visited France often. He therefore has constructed conceptual resources, associated with his “TW” concept, that relate to France.

5. G and the other boys are surprised that Spain is not hotter and M wants to go to France because he thinks it is hot there. G says “It isn’t”. When the France temperature is displayed (5º) G says “I told you.”	He does not consider 5º to be warm/hot. G’s resources relating to “Spain” and the resources within his “Numbers as they are used to represent temperature” (“NT”) concept that relate to “5º” do not span to each other – they cannot begin to be aligned until one spans to the other. The span of his “France” concept does extend to his “Spain” concept, in that at least one resource element in his “Spain” concept is associated with at least one in his “France” concept, and this connection will contribute towards achieving alignment within his “NT” concept.

G shows that he can control the pace of the task:

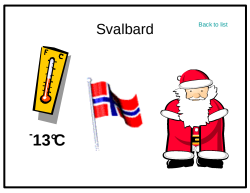
6. Throughout the task, G prefers to visit countries that are close, rather than those that are “far away”. Given a choice between 2 possible destinations, he always chooses to go to the closest one. (I have checked that he understands that the rules of the game allow him to “jump over” countries.) When I ask him why, he smiles and says that he’s enjoying himself and doesn’t want it to	It would seem that G believes that closer will mean less difference in temperature. He thinks that, by keeping the temperature differences small, more countries will be included in the journey and so the game will take longer to complete.

be over too quickly.	

G and the other boys in his group quickly manoeuvre Father Christmas to increasingly cold countries. Now we see how G deals with temperatures below zero:

7.	… but G points out, “But that’s bad because now we have to find somewhere that’s zero.”	G’s initial reaction shows that he may not recognise zero as a representation of a temperature – i.e. that his “NT” concept and any resources that he has relating to zero do not span to each other. It is possible that he does associate zero with temperature, but that he doesn’t know that places with a zero temperature are not any harder to find than others. His “TW” concept does not appear to include a resource relating to zero. This is not to say that he does not have, somewhere in his conceptual resources, a resource for zero but that, if it has been constructed, it does not span to this situation. Now that he is learning that zero is a number used to represent a temperature, the 2 concepts are associated or connected (span is established between them).
8.	(I ask) “As you’ve been to Poland, a country that is zero degrees, what are you looking for now?” M says, “minus” and G agrees. I ask “Minus what?” G and M say together, “Temperatures”.	G had previously seen “zero” as somehow more problematic than other numbers. His readout strategies (RS) appear to be evolving in parallel with the extension of the span of relevant concepts. He now seems to accept zero and negative numbers as an extension of the number system he knows. The span of G’s conceptual resources about directed numbers is expanded to include temperatures as a relevant context for cueing these resources.
9.	M and L take the lead in making decisions to go to Sweden ($^{-}2$) and then Norway ($^{-}3$). At this time, G makes contributions to the conversation though it	G’s trust in the other boys’ confidence with this concept may be seen as a way for him to “read” or infer information about this situation. G also infers from my reaction to the other boys’ ideas that the sequence of numbers they have just generated is an exemplar of the concept he is beginning to develop.

is clear from these that he doesn't understand much about negative numbers: he comments on FC's clothes; (he expresses uncertainty as to whether Norway could be colder than $^{-}2$; after Norway he asks "Now what?"; he makes a joke that suggests he thinks there is a possibility that Svalbard could be "hotter" than Sweden and Norway.) Finally they go to Svalbard ($^{-}$13). G hesitates briefly and looks at me. M announces "We made it!" Then G repeats, still looking at me, "We made it."	G extends the span and tests and improves the alignment of his conceptual resources relating to negative numbers by monitoring the other boys and my response to their ideas about the final stages of the game.



G does not focus on Father Christmas's clothes for long:

10. It is interesting to note that, although the boys were guided (and amused) by FC clothes for the first 2 or 3 country visits, their focus then switched to the numbers themselves. They didn't make any further reference to the clothes.	FC's clothes had been used by G as a resource from which he was able to "read" information about the temperatures, including comparing temperatures. At this stage in the activity, G uses his knowledge about the numbers themselves, rather than any visual resources to support his decision making. He is able to "read" and interpret the numbers efficiently in relation to the task. G makes judgements about effective game moves, based on his evolving conceptual resources relating to the numbers. G has begun to develop his ability to co-ordinate information that he infers using newly extended and aligned conceptual resources. G is able to enjoy success within the activity using his ability to co-ordinate conceptual resources within and across concepts – i.e. those relating to the number system, and to a general temperature gradient from Equator to North Pole.

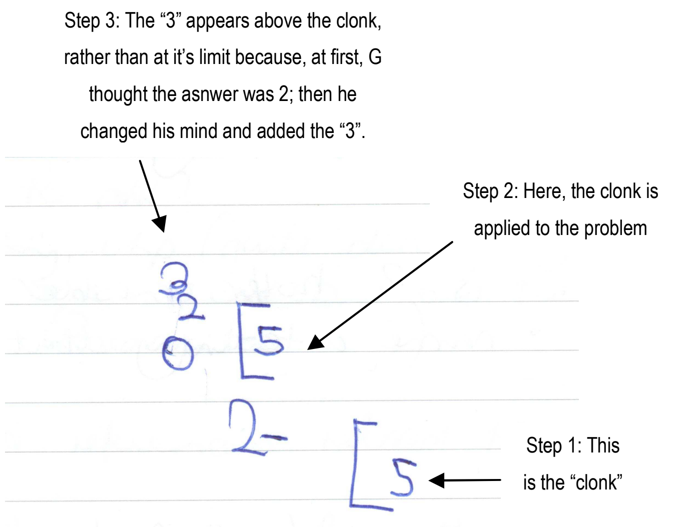
Towards the end of the first session with the group, G is asked to compare negative numbers:

11. I ask the boys how much the temperature changes if we go from Svalbard ($^{-}$13º) to Norway ($^{-}$3º). G says 10º. When I ask whether it is decreasing or increasing, he says it is decreasing. I ask again "Is it getting higher or lower? Increasing or decreasing?" G, looks	G's concept of the number system includes a sense-making mechanism (SMM) that enables him to compare values represented as unsigned digits. He is able to compare and make judgements about the relative values of unsigned numbers and uses the same SMM here, at first. He has a conceptual resource (in this case it might be a resource in memory or a readout strategy) that a "minus number" is an indication of a cold temperature. He also has resources relating to the Equator being hot and the North Pole being cold. He infers that a change from a value represented by the digit 13 (ignoring the sign) to a value represented by the digit 3 is a decrease in value. The fact that I questioned this, in itself, may have suggested to G that he has made an error. His evolving "TW" concept also leads him to infer that the temperature

	at the map and changes his mind - he tells me that it is increasing, not decreasing, because it is getting warmer.	should increase if the change is southward. These inferences are in conflict with each other. G resolves the conflict by judging that the question is about the change in temperature and that the concept of temperature is founded on measures of how hot/how cold. He chooses to focus on whether the move from Svalbard to Norway is an increase in temperature (getting hotter) or a decrease (getting colder). He judges that such a move would result in an increase in temperature. It would seem that he is willing to accept that, although the digits themselves are decreasing, the presence of the minus sign changes the "rules" that he thought he knew. This constitutes a modification to a SMM that had previously been effective. Although G is confident about the magnitude of the change, his "decreasing" response is an indication that his conceptual resources relating to directed numbers are not securely connected to other conceptual resources – i.e. effective span has not been established - through associations between resources. The span of G's "TW" and "NT" concepts has already extended to include each other but some components within these concepts are much better established than others and they are not aligned with each other. He aligns these conflicting inferences about "increase/decrease" by focusing on the context of the problem and reasoning that "getting warmer" equates to an increase in temperature, regardless of whether the digit values are increasing or decreasing. G's engagement with this particular question provides evidence of alignment being tested and evolution of all relevant conceptual resources. This is evidence that span of G's conceptual resources relating to increase/decrease are being extended to be effective in determining information in the contexts of temperature and/or directed numbers.
12.	In the next (similar) example I ask what happens to the temperature if you travel from Norway ($^{-}3^{o}$) to Sweden ($^{-}2^{o}$). G says, "It gets	G focuses on the digits. He has constructed a SMM that enables him to reason that "smaller" digits represent increasing temperatures when the values are negative. Span of his "increase/decrease" concept has extended effectively into the contexts of temperature and/or directed numbers. He is co-ordinating different concepts effectively and shows this with his final remark which suggests that he is able to call up different SMMs, depending on the context – i.e. "increase" or "higher"

smaller, so it increases by one." Then he adds, "But it does kind of decrease" and smiles.	or "bigger" are judgments that might have different meaning when referring temperature than when referring to the digit value of a number. He has achieved this co-ordination very quickly, through only limited experience with these problems.

Before completing the first session, I tested G's changing concepts about numbers and temperatures. G's response provides fascinating insight to his thinking:

13. I ask the group what would happen when you start at $^{-}2$ and add 5. M says 3 and G says 2. G wants to explain how he would do it. He makes a mark on the paper and says "This is $^{-}2$ and this is a 5 clonk". He explains that he thinks of the "amount" to be added as a brick-like object that he lays over what I understand as a section of a number line on his diagram. He then changes his mind about the answer and tells me that "It's 3, not 2 – I was getting mixed up before." I ask him where he learned to do the "clonk" thing.	G's "clonk" strategy:  Figure 5.3 G's "clonk" G appears to choose to use a number line model to help him with addition tasks – i.e. the use of the word "add" in my question is heard ("read") as a cue to employ strategies that he has developed and has found effective for addition – G's use of a number line is a sense-making mechanism.. He does not appear to be at all anxious about the inclusion of negative numbers in the problem to be solved – his RS have developed to include negative numbers. G infers that, as with unsigned numbers, he can use a number line model to solve addition problems with negative numbers i.e. G recognises that a strategy that he finds helpful when adding in other contexts might be helpful in this context. This suggests that there are

He says nowhere – he just thinks of it like that. I ask if someone has taught him to think of it like that and he says no – it's his own thing.	existing associations between his number system concept and his negative numbers concept that prompt consideration of his number line SMM for addressing the problem. His eventual success with this question using this strategy will reinforce this expansion of the span of the 2 concepts. The relevance of this strategy has also tested and confirmed the alignment of conceptual resources about addition with his number system concept and his negative numbers concept.

G's "clonk" is a conceptual tool that he has developed previously that is available to him as a resource. That it was perceived by G (not necessarily consciously) as potentially relevant in this new situation shows that the span of the clonk extends to aspects of this new situation.

At the beginning of the next session, G demonstrates that conceptual resources that he constructed in the previous session are not only cued in this session but are more secure; evident through G's new-found confidence:

SESSION 2 OF 3	
14. When countries with temperatures below zero appear from the pack, the boys take them in their stride, except to notice that Turkey's temperature is not what they expected. G reminds them that "This is around Christmas, though".	Negative numbers are read efficiently. G's conceptual resources include a SMM that enables him to reason that Christmas temperatures are lower than might normally be associated with countries that we visit for holidays in the summer. Whereas during the previous activity, G was uncertain about ordering negative numbers, today he was more confident – his RS, resources in memory and SMMs have modified and he now works effectively with negative numbers and is able to order them. He did not make mistakes. Span and alignment of resources has increased, linking resources that G has relating to numbers.

Conceptual resources relating to maps are cued but are not entirely helpful:

<table>
<tr>
<td>15. G thinks they should click on the map because then it might tell them temperature information. When the map opens, he says – “Yes, look, 15 degrees, 30 degrees ” (He has seen the numbers with a º symbol marked on the map. He doesn’t realise that they are lines of latitude and longitude.)


</td>
<td>G is confident that he can interpret information about temperature from maps. Since he expects to see temperature information on the map, when he sees numbers with the º symbol, he assumes these are temperature labels. This shows that he has an internal resource for the º symbol that is associated with temperature – i.e. G’s conceptual resources about maps lead him to have expectations relating to temperature information. If he has resources relating to lines of latitude and longitude, they are not associated with resources he is using here. It is possible, even likely, that he does not have any resources relating to latitude and longitude. G’s concept of maps will now be modified as knowledge in the form of memory resources are added. Associations with his “TW” concept might also be constructed, extending span of these concepts.</td>
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Next, we see further evidence of G’s ability to co-ordinate related concepts effectively, using new as well as more established resources. He appears to have resolved conflicts that interfered with this co-ordination only a few days previously.

17. G says that he thinks Iceland is cold. I ask why he thinks that. M replies, saying “Because it’s further up”. G agrees that	G appears to have constructed a situated abstraction relating temperature to proximity to the Equator. His conceptual resources equip him to infer that the fact that the equator is further away southwards is some kind of equivalent measure of a country’s coldness in its inversion to its north- or “up-

"The equator is further down."	ness". Demonstration of knowledge of equivalence of different but related measures shows that different conceptual resources are being co-ordinated effectively.
18. In response to a quiz question, the boys need to find a country with a temperature lower than ⁻6. G suggests that they "go to" Slovakia because he remembers that when they used "the big map" last time there was one that was ⁻ 32 and he thinks it was Slovakia.	G's interpretation of the question leads him to seek a country with a low temperature and he remembers that there was a country with a very low temp of ⁻32 and that this would solve his problem, if he could remember which country that was. He is aware that he has a resource in memory that is the name of that country and he tries very hard to recall what the name of the country with the lowest temperature was. He remembers some of the letters in the name of the country he is trying to recall: S L and V. On recognising some of the letters in Slovakia, he reasons that this might be the country with the very low temperature in the previous activity. G is co-ordinating resources from different contexts to help him respond to this new problem.
19. The next question asks whether Croatia is hotter or colder than Estonia. G is quick to find the card for Croatia and tell the others what they need to find . "Look for a E" he suggests.	The boys have already ordered the cards so that each card represents a country with a higher temperature than the one shown on the card immediately below it on the desk. G's SMMs enable him to reason that cards that are positioned towards the top of the desk have higher temperatures than those below and he encourages the others to locate the second card so that he can compare the positions of the 2 cards.

G is able to filter out redundant contextual information and focus on the numbers and the mathematics required to solve the problem in hand. As G's concepts of the number system, "TW" and "NT" evolve, he finds that the inadequate span of relevant conceptual resources sometimes "trips him up". This is only a temporary setback which effectively establishes that span for the future:

20. Question – "If you travel from Russia to Sweden, what will happen to the temperature? G finds the Russia card on the table ($^-6$) and counts up to the Sweden temperature ($^-2$) and says "It will go 4 degrees higher. 4 degrees higher, or 3, I don't know which." G is confused about whether he should count Sweden itself in the count.	G is able to "read" negative numbers. His uncertainty is uncharacteristic of the way he works with difference problems. When working with positive (unsigned) numbers he works them out efficiently and effectively. G's conceptual resources include a set of associated resources that he employs when counting comparing and calculating with numbers. (These might be seen as a concept "Counting, comparing and calculating ("CCC")" within a broader concept about numbers.) These "CCC" resources do not extend to resources relating to negative numbers, though he has already shown that the span of his number system concept and associated RS does include some associations with negative numbers. These 2 concepts, "CCC" and "Negative numbers" are not aligned and this is the reason that G finds himself forced to question something that he is surprised to find he is not confident about after all. Previously he has coped well with tasks involving ordering and comparing greater/smaller, higher/lower. So this marks a point of departure for G in that he appears to be working just beyond the scope of resources with which he is confident and secure.
21. Once he has decided that he should have "counted" Sweden, he also begins to think that he should have counted Russia at the beginning of the count so is confused again. (Something he was confident with previously has now been called into question).	There is something in G's conceptual resources that makes him think that rules should be consistently applied . This forces him to question a strategy with which he had previously been confident and that had been a successful part of well established concepts about numbers and counting – i.e. he begins to think that he should count the start number as his first count when counting to another number. G does not resolve this uncertainty at this point and is content to move on, letting someone else take the lead for a short time. G's in-the-moment reasoning might have led to construction of a new SMM, (that, since the "end" number is counted, the "start" number should be counted too). However, this was not successfully aligned with other resources. It is interesting to note that this question does not present G with a similar dilemma every time he is confronted by (what I recognise as) similar challenges later in the task. It appears that G's more established resources, particularly his counting strategies have high priority and are therefore readily cued in (at least most) subsequent,

	similar challenges. To achieve high priority, the new resource would need to provoke feedback that shows that the resource has explanatory value in this situation. This feedback is not generated here.
22. Throughout, he is very secure in the fact that he must include an invisible count at the $^{-}5$ position even though there is no card for $^{-}5$. (He taps the table when a card is not present for any value).	G's conceptual resources relating to the number system appear to include a well-established number line model. Span is being tested and extended as he finds that what works with positive numbers also seems to work with negative numbers.

With further challenges, G shows his increasing ability and confidence:

 23. G argues, saying that the question says from Denmark to Estonia so that means its going lower. He explains that, "If the question had said from Estonia to Denmark it would have been going up but it doesn't so its going down. That way it's getting hotter, that way it's getting colder. It's not the same." (Denmark is 0°; Estonia is $^{-}2$°)	G interprets the question appropriately and he effectively co-ordinates a range of conceptual resources, including resources in memory and sense-making mechanisms – e.g. • Counting, comparing and calculating with numbers; • Knowledge about negative numbers as used to represent temperatures; • Direction of change is important in some situations and not others.

24. A later question is "Find a country where the temperature is between 1 and $^{-}1$" The boys agree immediately that there are a lot to choose from on the table. M wants to answer Romania. G argues, explaining that he can't have 1, it has to be less than one.	G is very confident in his ability to interpret the precise meaning of the question and effectively co-ordinates relevant conceptual resources including those relating to: • Negative numbers • "NT" • Counting, comparing and calculating with numbers • Knowledge about negative numbers as used to represent temperatures • Between 2 values
25. Next question "If I travel from Belarus ($^{-}3$) to Belgium (3) what happens to temperature?" - they all agree an answer but it is wrong because they have counted Belarus as their first count. When I question this they are happy to stick to their judgment.	This type of error is uncharacteristic for this group and for G. We saw (in 20-21 above) that G's previously secure counting strategies (particularly relating to the inclusion of the "start" number) were challenged. At that time, he did not find the new idea, suggested by his in-the-moment reasoning, to be successful. The new idea – that, for consistency, because the end point is counted, the start point should be too – did not seem to have changed his contextual neighbourhood. However, here, G's incorrect response might suggest that his conceptual resources include the new idea – that a new resource was constructed, and that it is associated with the current challenge – perhaps it forms a part of a "How to apply rules" concept?

The issue of whether to include the start number when counting arises again in the next extract and G is able to become more confident in his own ability to be successful through repeated testing of the alignment of his concepts:

26. I decide to use the thermometer ITP to model a number line method for working on the Belarus to Belgium question. G is able to position the starter pointer on $^{-}3$ without any difficulty. He can also help M to move it to 3. When I show that the change has, therefore, been 6 not 7 as they had answered previously, G quickly says "That's because we counted that one, Belarus". He is quick to accept their mistake and to see why it arose. (Maybe he had doubted their answer in the first place?)	G is quick to learn how to move the interactive display on the screen thermometer and is confident to help his friend. The highest reliability priority for his counting (include "end" number but not "start" number) strategy is re-established when the whole range of his conceptual resources relating to it are tested and their alignment reinforced. Associations between these resources and those relating to negative numbers are also tested and reinforced. G's "NT" concept is strongly connected to his evolving negative numbers concept.
27. L uses cards and gets muddled counting up from one to the other (Slovakia and Albania). M thinks the difference is 8 and G says its 7. M argues and G argues back, reminding him that last time they found out that they shouldn't count the first place.	G is now more confident again. He is aware that his confidence has come from the previous example in which he was able to see for himself that the start number should not be included in a count procedure. Co-ordination of relevant conceptual resources is secure. G's RS and SMMs relating to the thermometer scale are secure.

G occasionally refers to his impression or experience of the development of his own conceptual resources and his ability to co-ordinate them – i.e. metacognitive references:

28. M clicks the change box which shows 7 and G says that he thinks the difference will also show 7. He's right. I ask him why he thought that would happen. He tells me that he's noticed the 2 boxes show the same thing.	G has been observant. Well developed RS mean that G has noticed a relationship between the displays in the 2 boxes and uses this make a prediction.
29. They do a few questions that they make up themselves in which they predict what the change and difference boxes will say. They are correct but only choose examples in which the temperature rises.	The group, including G, spontaneously decide to give themselves "practice" questions. This strategy is evidence of metacognition – i.e. the boys are consciously aware of processes that help them to learn and can see the potential benefit of employing one of those processes in this situation. They are proactive in their own learning. They make up questions for themselves to give themselves opportunities to hone their RS and reinforce their emerging concepts This is reinforcing associations across one set of RS, SMMs and other resources but is not extending span any further.
30. I pose a new type of question "If I am in Moldova and go somewhere that is 9º warmer, what will the new temperature be?" G has control of the thermometer and confidently counts up one degree at a time from the Moldova temp (-2), using the	G is able to solve the new problem (that has a slightly different structure, as I perceive it) without any difficulty. Relevant conceptual resources have sufficient span to be triggered and are well aligned, each incorporating an appropriate range of conceptual resources which act as anchors, enabling G to work unproblematically. Concepts used are thereby becoming increasingly secure; strongly connected.

	thermometer scale as a vertical number line.	
31.	I set G a similar question that will have a negative change. At first the boys don't notice that the change box shows a negative change. When they do, it is only M at first who seems to remember or understand what this means. Then G sees it "Oh, yes, because minuses mean that you're going down." Very soon, however, before moving on to the next question, he says, "I'm confused again now. Not sure now."	G has formed a situated abstraction "minuses mean that you're going down" which is cued when he sees a decrease in temperature and a minus sign in the change box. This situated abstraction, as a resource for making sense, is however weak and somewhat elusive – this is evident when G finds it hard to maintain his grasp on it and tells me so. This is evidence of his own awareness of the changing state of his contextual neighbourhood.
32.	In trying to explain to G, I reminded him that I had asked him to go from Netherlands (4º) to somewhere that was 5º colder. He remembered "And we had minus 5, because	With the association between "minus" and "down" very recently cued; G now also acknowledges a connection between "colder" and "down", and makes the link between "colder", "minus" and "down". These 3 resources are now connected, each spans to the others. They form parts of broader concepts about "Number System" and "Counting" which are, themselves, part of larger and increasingly complex concepts and systems.

it went down".	

During this phase of our work together, G seems to be increasingly conscious of the way the changes in his contextual neighbourhood make him feel:

33. I ask a supplementary question, "If I am in Germany (1 degree) and I go to somewhere that shows a change on here of 2, where might I be?" G hesitates but then says "Is it 3?" He goes on to explain "Because it went up because it wasn't a minus. I get it now."	G's "I get it now" remark suggests that he is aware that resources of different kinds are connecting together within his contextual neighbourhood. Other related resources within the evolving conceptual system are tested, requiring RS and conceptual resources to trigger extension of span to include "up", "warmer", "not minus".
34. I encourage L to have a go at one of these questions and G says "I'm still a little bit confused." I go through another example for L. G is getting excited now "I get it, I get it, because if you go up." He says he needs to do some more straight away. He answers the next question correctly …… but then he goes on to say that	G is aware of the fragility of his evolving concepts and knows that he needs to repeatedly reinforce span and alignment of new resources, particularly while they are beginning to evolve and connections are being constructed. His error, even after getting correct answers, is evidence of that fragility. Span and alignment need to be tested and reinforced repeatedly in order to establish high cueing and reliability priorities.

the difference will be -4 (this is incorrect)	

G's construction of associations between resources are often quite transparent:

35. As soon as I say to G, by way of explanation, that the difference between 5º and 1º is (pointing to thermometer) "those 4 degrees", he says "It's how big the gap is? So you can't have a minus gap - you can't ever get to a minus." I work through a few more examples with the boys and G is able to predict what will appear in the change and difference boxes consistently correctly, moving in both directions. G is dissatisfied that there are no "add" numbers in the difference box because "We said it would add when you go up".	G is very quick to take up the metaphor of a gap and seems to immediately "see" that it is not possible to have negative gap. He notices that the values are unsigned and questions why there are not "add" signs. This is likely to be because he associates "add" and "up" together. (Later, he reasons that a sign, in this situation, is irrelevant.)

36. G asks "Are we going to discuss about last week, going to zero?"	G appears to think there might be some connection between the work we are doing in both sessions. He has identified a particular aspect of our previous discussion that has not arisen yet during this session. His question might imply that he thinks that the previous discussion about "going to zero" is relevant here. G has formed resources about "going to zero" that he thinks are relevant in today's discussion – i.e. there is at least one resource that is common in both contexts and the span of the "going to zero" extends to aspects of today's work.

G rises to the challenge of representing their activity in written form without any real difficulty (and with some enjoyment):

37. I ask G to "write down what we are doing with the thermometer" and tell the group that we are going to start on 3 and go up 10 degrees. G thinks we should write 3 + 10. For a similar question with starting point of $^{-}3$ the boys agree that we should write $^{-}3$ + 10 and that the answer will be 7. They model it using the thermometer and see that they are right. G is excited "I get it. I get it. If you go down it puts it as a minus. It's as if you're doing the sum."	G spontaneously extracts the mathematics from the situation. He easily uses the 2 numbers involved in the question, relating them to each other in terms of starting with one temperature value and "going up" by a number of degrees, using the + symbol to show that the first quantity/number is increased by second quantity/number. G "sees" the similarity between the screen thermometer display and his own tentative attempts to express the temperature changes symbolically: these 2 situations have at least one resource in common. He is excited about this. The span of G's established conceptual resources relating to working with numbers and increasing quantities has extended to be perceived as applicable to the temperature context. G is able to further reinforce his emerging hypotheses about the mathematics within the temperature problems.

38. I ask G if he can tell me a sum that will give us a minus number in the change box. Straight away he models 13 minus 1,2,3,4,5. I prompt him to finish, saying ".. equals … ?" He hesitates, seems a little surprised but offers "equals 8". G then volunteers, "See, it's as if, look I started on 13 then you add the minus 5. I found it quite … funny." (He means peculiar, rather than comical.)	G articulates, quite clearly, his experience of taking on board new experiences and his growing ability to make sense of what happens. He notices the point at which he realised that there really is a relationship between changing temperatures as events or journeys and symbolic mathematical expressions. He is able to sense in a perceptual way that his ability to co-ordinate a changing range of conceptual resources is evolving – he is metacognisant. His reference to feeling "funny" might refer to the fact that he is scrutinising his own thinking and that this is an unfamiliar experience.

Next, we see that the "add/minus minus" problem is not yet resolved and G laughs about being confused:

39. I ask "so you're adding a minus number?" G replies "Yes. Now you're confusing yourself" He laughs and goes on to muse "But <u>minus</u> minus 5 doesn't make any sense?"	G is making a joke at his own expense – about the fact that someone other than him might be confused. However, when he stops to really think about my question he realises that he has been doing something that seemed to make perfectly good sense but that actually conflicts with what he thought he knew. G is likely to hold a concept that includes resources relating to "add" and "minus" as operations and that, logically therefore the element in the expression following "add" or "minus" should be an item that is the thing to be operated with or upon. Therefore, in G's experience, the word to describe one operation cannot be followed immediately by another word describing an operation without something in between. This SMM needs to be modified before he is happy to use the

	expression "minus minus". It is the ambiguity of the word "minus" which is likely to be the reason for his confusion. His conceptual understanding of "minus" has, until now, only involved minus as an operation and he interprets it as an instruction to "take away". Now that he has developed a conceptual system incorporating negative (or, as he calls them, minus) numbers as points on a number line he must modify his "minus sign is an instruction to subtract" rule.
40. He offers to explain again. This time he doesn't actually say the "add minus 5" part and M challenges him over it, telling G that he was wrong all the time and now he "must know it because he didn't say add minus this time". G denies having said it at all and says he's confused again.	G avoids saying "add minus" because his more established conceptual resources lead him to consider this as nonsense. G even finds it hard to believe that he had said "add minus" before because he knows that it conflicts with his existing conceptual framework. Well established knowledge elements have high cueing priority and the conceptual systems which they inhabit are themselves structured by them. G needs to acknowledge that there are new, different ways of interpreting "minus" – that it is not only an operation or an instruction to perform it. This would amount to the construction of new RS. Bootstrapping of changes to RS and conceptual resources needs to occur for G to be able to move forward in his understanding – to be able to co-ordinate resources from different concepts effectively.

G's determination to make sense of his situation is evident when he perseveres with his attempts to resolve the conflict and tension that he perceives:

41. G now refers to that step simply as "minus 5" but seems very uncertain and remembers that with a positive change everyone was happy that the change was "added" ….. L says	G continues to try to make sense of conflicting messages and interpretations – his own and the group's – about the question of whether a change is "added". G again voices his confusion, of which he is uncomfortably aware. He really wants to find meaning, make sense, of these mathematical situations and events.

	"It's 13 take away 5, basically". G is just getting more confused and says so.	
42.	L has written 13 - $^{-}$5 = 8. G says that's not right - it's not the same at all because its got 2 minuses. I ask G to do 13 take away minus 5 on the thermometer. He says "How do you do that? How do you take away a minus? That's why I don't think it's right." He recaps starting on 13, doing a minus 5 to get to 8.	G seems to be the only member of the group for whom the 2 symbols written adjacently carry a bothersome significance which he finds hard to accept. Using the thermometer he focuses on the minus symbol preceding the 5 as an instruction to subtract. This is the only thing he can think of to do – the only resource within his contextual neighbourhood that spans to this situation is his "minus 5" strategy. He thinks he should be trying to "take away a minus" but doesn't even know whether that is possible, and certainly doesn't know how to do it – he has no resources relating to "take away minus", or "minus minus".
43.	I agree that this is what happens when you "Do" a minus 5. I go on to ask "What about when you take away a $^{-}$5 instead of doing one?" G asks "Is it add?" I show him 13 on the thermometer and ask "If this is where I am after a minus 5 was done, where was I before that $^{-}$5 was done? G hesitates briefly, then says "18". I ask again "So, if I am	G seems to interpret from the "instead" in my question that he is looking for something other than (even opposite to?) the previously approved idea. He offers an "educated guess" that if he took away before, perhaps he should trying adding this time. I didn't think this would be helpful to pursue so used the thermometer to make my questions more concrete. He succeeds with the first question. It seems that G hadn't been entirely convinced the first time we had talked through this question, even though he had come up with right answer – that, although the experience introduced new memory resources into his conceptual resources, associations with other resources had not formed. He is excited that he thinks he really understands now – is able to perceive associations between resources now.. But his conceptual resources also include some which lead him to believe that, when he learns new things, he doesn't "understand" very securely at the beginning and that it is only with further reinforcement that he feels

	at 13 because someone had done a minus 5 and I want to take away that ⁻5 …?" G says "You go up! So I was right?!" I ask, "So 13 take away minus 5 is .. ?" G responds "18, I get it." But immediately he adds "I get it a little bit, but not much"	his evolving resources become robust and reliable.
44.	We work through another example (10 - ⁻2 = 12) and G says "So take away a minus is not like add a minus – they're different"	He is pleased to be able to restate his earlier ideas, with more confidence this time.
45.	M starts to talk me through how he bridges downwards through zero. But G interrupts, "Oh no! I need to do some more. I'm losing it."	G doesn't want to be distracted from reinforcing his new ideas. He is sufficiently metacognisant to know that, unless he reinforces the span and alignment of evolving conceptual resources, he will not be able to use them to make sense of situations and problems in future - that they will not span effectively to be called up in situations like this one and he will be back to " square one". The efficacy of span across concepts relies on the establishment of high cueing priority between relevant RS, SMMs and other resources and G actively seeks to establish high cueing priority and reliability priority through reinforcement.
46.	G suggests using a ruler but can't remember how it will help and can't do anything useful with it. He chooses to draw thermometers to help	G has resources in memory that lead him to believe that models and images often help him understand or work things out. He tries to recall those he has used in previous work so that he can evaluate whether they might help here. He wants to be able to describe to others (and, at the same time to himself) what his new conceptual resources are and how they work in concert with other

him explain his thinking	(perhaps more established) resources. G is (though not consciously) hoping to find conceptual resources within his contextual neighbourhood that have sufficient span to connect with to his new thoughts and will help him to anchor his new knowledge.

Session 3 of 3	
47. They want a new game and I suggest "Balloon Burst". I explain that 5 balloons will pop, one after another. Each time a balloon pops it will reveal a number that will stay on the screen for 2 seconds. The boys' task is to add the 5 numbers together. G asks if they can have a piece of paper. When I ask why he wants it he tells me that, if he needs to go over it again, he might not remember the numbers.	From this I infer that G intends to try to add the numbers as they appear but realises that he might want to check or review his addition of the 5 numbers. This would become necessary for him to be able to confirm or correct an answer. It is likely that G has experience of performing mental calculations on strings of numbers. From this he will have developed a collection of associated resources relating to "mental calculations". Although I have not told the boys to add mentally, I infer from G's comments that he intends to do it this way, perhaps with the support of jottings.
48. The numbers are 9, 21, $^{-}21$, 8, $^{-}12$. G says quickly 17 minus 12, that's 5, 5. He seems to have immediately realised that the 21 and $^{-}21$	We have already seen that G's "readout" of the minus sign as an instruction to subtract has high cueing priority. If this has led to his reading of the 21 and $^{-}21$ as "21 minus 21", this might explain why he was able to reach a point where he was able to disregard these 2 numbers from the list so quickly that he didn't even mention them. It also seems that he is able to operate with these numbers very quickly and efficiently. His knowledge of commutativity is implicit, and forms

cancel each other out and that 9 and 8 are 17. He is very quick to do this, offers no explanation and the other boys don't question him.	part of his contextual neighbourhood, as a resource in memory and/or a SMM.
49. For the second question, L calls out the numbers (correctly $^{-}11$, $^{-}24$, 9, 14, 5). G goes to use the on-screen calculator to add and subtract the numbers as they appeared. I tell him that it won't do that for him – that it's not actually a calculator, only a way of entering their answer when they work it out. After a few seconds, G reaches to enter his answer.	G's RS recognise the calculator on screen and he does not hesitate to attempt to try to use it as his "Calculators" resources are cued. He quickly adapts his "Calculators" resources so that he does not automatically assume that everything that looks like a calculator can be used as one; associated RS must also be modified. G now has a new resource relating to other possible uses for such "pseudo calculators". This currently forms part of his collection of resources relating to playing these games but is not yet associated with resources in other concepts.

In the final session, G quickly reveals his mental addition strategies when he plays the computer game "Balloon Burst":

50. I stop him and ask him to explain how he worked it out (balloons: $^{-}11$, $^{-}24$, 9, 14, 5). He says 9 and 14 and 5. I ask why	G has previously shown that he seems to have conceptual resources that give him strategies for adding lists of numbers. Here we see evidence that G might have some experience of grouping similar elements together when calculating with lists of values. He has decided to defer consideration of the numbers which are preceded by a minus sign. This may or may not be because he is not very

	those 3 and he tells me because “they’re the only ones which are the adds.	confident about what to do with them. He refers to the numbers preceded by the + sign as “the adds”. This might be because (as we soon find) he refers to the others as the “minuses” and he applies the same “rule” to naming both kinds of numbers. This is evidence that G’s RS are naive when working with directed (signed) numbers.
51.	Then I added the other 2 minuses together – I got 35 and then it’s 35 minus 28 and that equals 7. He is surprised to see that the answer is $^{-}7$.	He adds the numbers together as he had for the + numbers and for a reason that is not clear, he subtracts the + total from the – total. From this it is possible to tentatively infer a great deal about G’s conceptual resources relating to mathematics. It is possible that he is simply subtracting the smaller value (28) from the larger value (35), ignoring the signs. This would be predicted if we consider that G’s RS are likely to have developed from working with addition and subtraction with only positive numbers where the problems he will have experienced within his maths teaching will have been designed to avoid moving into the negative domain. His conceptual resources relating to “Addition” and “Subtraction” have not yet spanned to include operations with negative numbers. His recent experience with negative numbers in the context of the different tasks has not been recognised as relevant – i.e. the span of the conceptual resources relating to the different tasks has not extended to G’s concepts of addition and subtraction. G’s 2 sets of concepts about operations on numbers and what he calls minus numbers are not yet aligned so the span of each does not, at this time, extend to include the other. Furthermore, G’s RS pertaining to symbols for number operations are so well established that it seems he cannot see the minus sign as anything other than an instruction to subtract, though he is inconsistent in what he subtracts from what. This is evidence of the highest level of cueing priority for this interpretation of the minus sign; an aspect of G’s RS that needs to be modified if he is to develop more effective conceptual resources for working with negative numbers. His surprise is evidence that he recognises the inadequacy of his conceptual resources. It betrays some anxiety – understandable when we consider that he is now forced to question something in which he had felt secure.

The novelty of working with negative numbers is suggested in G's lack of expectation that values might be negative:

52. G reads out the numbers for the next question: 9, $^{-}$8, $^{-}$48, 48, $^{-}$34, 33. It is interesting to note that, with each negative number, he starts to say the number without saying "minus" – each time, before he finishes saying the number, he stops himself and says it again, with the "minus" this time.	This suggests that G's RS are changing to be able to take account of signs – i.e. that where the sign is shown, it should be "spoken". This is not something he is used to as his experience of "saying" numbers does not include any reference to signs. However, his "Minus Numbers" concept has evolved considerably and the span of both of these related concepts would appear to include the other as he decides that it is appropriate to speak the sign as well as the number as he did in earlier activities. He often fails to do it at first but corrects himself.

Next, we see how a comment by M leads G to confront, and begin to resolve, a conflict in his conceptual resources. This provokes a significant change in G's contextual neighbourhood:

53. M says "9 add $^{-}$8, which is 1". G says "You can't do that". No-one follows up on his remark until after they have entered the wrong answer to the question and start to think it through again. At this point G asks "Can you add minus numbers and plus numbers?" It is interesting that he did not realise that this is what they were doing with earlier questions. Maybe there is something about this question that emphasises to him that this is what is happening. I think, though, that it was M's articulating "9 add $^{-}$8.." that has triggered something in G's understanding. My suspicion is confirmed when he repeats to	M talks aloud as he writes down what they need to do. As he speaks the numbers, including the "minus" sign, he includes the word "add" as he recognises that the task is to add the numbers together. G finds that this conflicts with the interpretation that his own conceptual resources facilitate – he believes M has made a mistake. No-one else acknowledges his remark. When the boys go over the problem again G questions whether it is appropriate to add "minus numbers" and "plus numbers". He has not referred to the unsigned numbers as "plus numbers" before (though he did refer to them as "adds"). So perhaps, we are seeing his concept of "minus numbers" expanding to include "the other numbers" to become a more inclusive concept about signed or directed numbers? If G's knowledge is extending in this way, I think it is almost coincidental as his conscious focus is the concatenation of 2 words, both of which he previously understood as instructions to carry

himself and looks at M, "9 ADD MINUS ..??" M explains that he had written the numbers down and then went back and put all the add signs in because they need to add all the numbers every time.	out an operation on a number: + (add or plus) and – (minus). It is his mention of "add minus" and his discomfort when M said the same thing previously that leads me to believe this is the cause of G's anxiety at this point. It would seem that G's evolving "Minus numbers" or "Signed numbers" concept is undergoing expansion to include resources that conflict with elements from other collections of conceptual resources that he was beginning to align. In his previous experience an instruction (a sign) has always been followed by a number, not another sign. G's discomfort, his difficulty in aligning related conceptual resources, is evident here.

G recognises the relevance of number lines:

54. The boys all look at the list of numbers. M says "If we cross out those we're on zero still" He is referring to $^{-}48$ and 48. M tries to explain why the 1 that is left when 9 and $^{-}8$ are added can be put with $^{-}34$ and 33 to make zero. He doesn't explain it very well, though uses gesticulations, sweeping both his hands to the left and to the right. G quickly takes over and explains it very clearly, using a number line model to describe the movements through zero to 33 and back to zero. He is able to talk about moving to the right as adding and moving to the left as taking away or minus. He uses the signs of the numbers as instructions to add or minus (used as a verb), assuming a + sign where	M's arm movements appear to have triggered a resource within G's contextual neighbourhood. G now realises that he can think of these operations as movement along a number line. The image of M's movements cued resources relating to number lines. These resources are now associated with this activity, though that association was not evident previously. It would appear that alignment has not yet been achieved but association (span) between conceptual resources has begun to be constructed. Feedback, and perhaps repetition, will be necessary if cueing and reliability priorities of this new resource, compared to that of existing resources, are to change.

there is no minus sign.	

Even though G appears to have resolved his difficulty with this idea, he continues to grapple with this modification of his contextual neighbourhood:

55. When M tries to explain the way he works through the list, he says 9 add minus 8 and G interrupts "You can't say that. You can't add a minus". He is still resistant to this notion within his own understanding of what he is doing. G believes that the add sign is superfluous as he says "9 minus 8 is already there". He doesn't see any need for the add sign and thinks it confuses the question.	G is still very uncomfortable with add and minus being spoken in tandem. He feels that this doesn't make any sense. He appears to "read" the minus prefix as the indicator that he needs that tells him which way to move on the number line. He clings onto his belief that the "add" word cannot occur immediately preceding the "minus" word (cueing priority is still very high). His developing concepts ("Minus numbers" and "Calculations") contain conflicting resources, they are not yet aligned in this respect, even though span of each does extend to the other.

G actively seeks ways to work with the task that make sense to him, according to the conceptual resources that he possesses.

56. The boys do not come to agreement over this and I ask whether $9 - {}^{-}8$ is the same as $9 + {}^{-}8$. G ignores my question. He has noticed that by rearranging the order of the list of numbers he can make sense of it – he can avoid doing $9 + {}^{-}8$ by switching the order and thinking of it as ${}^{-}8 + 9$.	My question challenges G as his conceptual resources do not accommodate the idea that 2 operation words can be adjacent within an expression. Before he could answer my question G's concepts must be modified. G's SMM relating to the commutative law enables him to restate the problem in a way that makes it possible for him to solve. This way he can solve the problem using his existing conceptual resources without any imperative to change or expand them. This is preferable to G.

G continues to use a number line model to play the “Balloon Burst” game:

57. When I ask G to focus on the $^{-}$34 + 33 section of the list, he talks about being “in the minus section” (minus is a place/location). He is quite comfortable with starting “in the minus section” and moving up by adding a number which is not minus. G can explain that if he starts at $^{-}$34 and takes 33 away from 34 he knows it will be minus one, “still in the minuses”. He isn’t able to tell me why he is “taking away” 33 from 34 when he is adding the 2 numbers together	It is not surprising to me that since G’s “number line” concept is well established, and now that he has learned that this is useful in solving the problems presented by this game, he “reads” minus as a section of the number line, i.e. a pseudo-concrete representation; a location. It is possible that G is able to recognise, implicitly, that in order to move up 33 (and I think he equates moving up, or to the right, with adding) and if the starting point is a minus value greater than 33, he can calculate where he will end up by subtracting 33 from the start (negative) number.
58. I ask him to do one with easier numbers, $^{-}$8 + 6. Immediately he tells me “that’s $^{-}$2”. ”He says “6 is less than 8 so you’re still in the minuses.”	I believe that “still in the minuses” implies that G’s interpretation of this operation, is that, from a starting position of $^{-}$8, by adding 6, it is necessary to move towards zero but that zero will not be reached. It is likely that G has a sense making mechanism, that has probably emerged from a number line model, that enables him to deduce that, if the number added is less than the start value, the fact that there is a difference means that there is a “gap” between the answer and zero – i.e. upon completion of the operation, “You’re still in the minuses”.
59. G goes on to tell me that “if you take away from $^{-}$8 you get bigger digits because you go	He seems to interpret “taking away” as the move is in the opposite direction to adding. G’s thinking aloud is acting as a window onto the modification of his “Minus numbers” concept and to the alignment of this with other

that way and you're on the minus side". (His gesticulations suggest that he represents the calculation/situation to himself as movement in one direction or the other along a number line).	related conceptual resources. We see that he is able to co-ordinate the notions that taking away leads to smaller digits on one side of zero and to larger digits on the other side of zero.

G attempts to resolve the difficulty of "minus minus"; he once more shows his ability to perceive some potential relevance of resources that he has previously constructed:

60. The other boys explain their strategies which include swapping signs around for what appear to be arbitrary reasons. G joins in and explains that another way to do it is to swap the numbers around "because you can do that" and that because the 8 is a minus "I'd take away that minus sign and change this to a minus to get 6-8 which is minus 2."	G is using his SMM relating to the commutative law again here. Although he talks about changing the minus sign, he is actually changing the order of the number terms so that a positive number (i.e. unsigned) appears first and the minus sign of the negative number can be treated as an instruction to subtract.

61.	I redirect G and the other boys to my earlier question : 9 - - 8. They keep saying it aloud "Nine minus minus 8". M says "minus, minus" several times. G says "I know, it's hard – minus minus." They are not at all confident about this one.	G doesn't attempt to change the order of the numbers . It is possible that this strategy is cued but that it doesn't help with this problem. G doesn't appear to remember how he had succeeded with "minus minus" previously – i.e. by using an "undoing" strategy. Perhaps cueing of the commutative strategy is blocking cueing of the undoing strategy because it's recent effectiveness has earned it high reliability priority (for now, at least)?
62.	L suggests adding words to make it simpler "because people who aren't very good at maths might be good at English". He and M exchange banter and G interrupts "Stop – you're confusing me right now." He seems to be trying to process something in his head, trying to make sense of something.	G appears to be actively trying to organise and secure his conceptual resources and connections between them. Things that the other boys are saying are not in line with his own contextual neighbourhood and he finds it confusing to hear them whilst trying to review his own thoughts. This shows that, for G at least, making sense of his own conceptual resources and experience, involving modification of old concepts through alignment with new resources, requires effort and concentration for which he needs to be free of distractions.

Next, G recognises the usefulness of other strategies that form part of his conceptual resources:

63.	I ask G if he can explain. He says that "-5 add 4, if that was a 5 it would be zero but it's not, it's 4 so	G describes a compensation strategy for calculation. This is something that is likely to have been taught in school and G has realised (not necessarily consciously) that it might be relevant here – there is at least one resource that is common to existing resources that relate to "compensation" and to the current task. We have already

the zero changes to -1. I did this with that game with the countries. I said that when we went to warmer and colder places it was like a sum." This is the first time that he's made any reference to our previous sessions, even though the calculations have been very similar.	seen evidence of the co-ordination of "Calculations" and "Minus numbers" concepts; this now suggests that the span of another collection of conceptual resources (relating to "Compensation") has been extended. This will improve alignment of all these concepts. It is interesting that G also recognises that some resources relating to these concepts also form part of his concepts relating to "Journey" and "Quiz" tasks.

Next, we see whether the boys are able to create appropriate questions using the numbers given.

64. I ask them if they could turn these questions into ones about countries and temperatures. They are quite excited about this and think that they can. L starts by saying tentatively "It was -5 at Antarctica…" The others interrupt and G points out that "It would be way less than that!"	To be successful, the boys will need to co-ordinate several concepts that have not previously been used in concert. Even if we see that span of these concepts is sufficient for relevance to be perceived in working with the challenge that I have set, I would expect to see that alignment has not yet been established. It is interesting to see that G is not comfortable with the Antarctica temperature – there is some dissonance within his contextual neighbourhood. I believe that this shows that concepts that he employs in the context of this problem are "TW" and "NT" and/or "Antarctica". The idea that the temperature in Antarctica might be -5º does not align with his existing conceptual resources. In his attempt to solve the problems posed by the challenge, G feels it necessary to invent contextual details that are as realistic (according to the current state of his concepts) as he can. This may, of course, simply show that he thinks that is what I want him to do.

65.	L ignores their suggestions for alternative starting countries and continues " .. And then he needs to go to …. England where it was 4 degrees." G says "And you'd find out how much there was between them, how much warmer that was." I ask warmer or colder? G repeats "Warmer. And it's 9"	We see that G's concepts relating to difference, temperature, "Journey" and "Quiz" are all being used appropriately. G is using the 2 given numerals, together with the preceding sign, as the temperatures of the 2 countries and then working out the difference between them. He is quite comfortable with executing procedures like these as he was consistently successful with them in previous work.
66.	I ask G to do the same with $^{-}2$ take away $^{-}4$. He says "If Father Christmas starts in Norway, minus 4 and goes to Russia which is minus 2, how much warmer is it?" Straight away he sees that this can't be right but can't think of an alternative question. He says "I know this isn't right. It's not like those – there's something missing. I don't know what, though."	G uses the same strategy but this time he thinks he has done something wrong – he recognises that his response is not in line with his evolving concept relating to "take away minus". His initial response follows the format of questions from "Journey". This suggests that the resources formed about the "Journey" activity, though only recently constructed, are quite well established – they span to this new task. However, G's RS have not yet developed in ways that would enable him to "read" the signs preceding numbers as both instructions to operate and as indicators of location on a number line. G has shown that he possesses some knowledge about this (e.g. he has talked about "in the minuses" and he does read the minus sign as an instruction to move to the left on a number line). Although G's "take away minus" resource has begun to be connected with other resources in some ways, his RS that must evolve to enable him to fully utilise his new concept of negative numbers, lag behind.

So, we, like G, are able to see the point at which his conceptual resources about operating with signed and unsigned numbers begin to fail him. In my analysis, I describe how readout strategies and other types of conceptual knowledge co-develop and suggest that such development of each is mutually dependent on that of the other.

I prompt the boys to use some sort of number line to help them with the hardest problems but they do not respond as I had expected:

67. I ask the boys to tell me what pictures or memories they have in their heads at the moment. They say countries, Father Christmas, temperatures, cards, lines on the map. I ask whether they might be able to talk me through the way they work these problems out if we drew some of these things. I suggest we see whether a line might help us now and draw a vertical line on the page. G thinks this is a good idea “Yes, and here is minus (indicating the area to the left of the line) and here is the positive (indicating the right side).	It is interesting that G interpreted my line as dividing a space into minus and positive. I had expected the boys to see the line as a vertical number line, like the scale on the thermometer they had been using. G did not interpret it this way, even though he had been using vertical number lines and using the interactive screen thermometer in previous sessions. Although those experiences will have formed part of G’s conceptual resources, those new resources were not cued in this situation.

68. L then said “But I was going to do this – he drew onto the line several markers crossing it at intervals from top to bottom. G writes on a 0 next to one of the markers in the middle and goes on to label all the other markers (intervals of 1).	As soon as L marks intervals on the vertical line G seems to recognise it as a vertical number line and adds the number labels appropriately. It seems that his “Number line” concept did not span to the previous unmarked line but it does extend to the one with L’s marks. The span of his “Number line” resources has now extended to include blank vertical lines. This means that, in future, his contextual neighbourhood has changed so that G might “read” a blank vertical line as a potential number line.

G goes on to recognise further connections within and across his concepts and I encourage him to integrate other resources and expand his contextual neighbourhood even further:

69. I ask him to show me how this helps with 4 + $^{-}5$. He is excited and says, “This is like the cards and the thermometer!”	This suggests to me that G had not, until now, connected vertical number lines and “Journey” – i.e. his “Journey” resources did not actually contain a resource about vertical number lines and/or did not span to his “Number line” concept (despite the fact that he appeared to have perceived a connection whilst using the thermometer as part of “Journey”.) G’s conceptual resources relating to the “Journey” and “Temperatures” tasks and number lines and signed numbers are now more connected (span has extended) and he has now recognises that the image of a vertical number line has some of the same properties as the thermometer in the Temperatures activity and can be used in the same way.
70. He shows me how he works with the 4 + $^{-}5$ problem: he moves from 4 to $^{-}1$, counting 4 (to zero), 5 as he goes. I ask him to do it the other way around, to swap the numbers	G is able to use the vertical number line effectively and his RS are changing, enabling him to read the “minus” sign appropriately.

	round and start with the $^{-}5$. He does it confidently, counting 1,2,3,4 up to $^{-}1$.	
71.	G tries to resolve the “minus minus 8” problem again but is frustrated “How do you minus minus 8? – I don’t understand!” I re-read the problem, emphasising the “take away” aspect of the problem. I then ask the boys “What would I do if the question said $^{-}9$ + $^{-}8$?” L shows on the (vertical) number line a movement from $^{-}9$ to $^{-}17$. L thinks this confirms what he thought the answer was but G points out “But that’s the answer to add. I get it. If it’s take away do I go the other way?!”	Despite previous experience with this type of problem (and eventual success with it) (see Rows 43-44, Session 2) G is not able to access any conceptual resource that might help him with the “minus minus” problem. (This might be due to inadequate RS, insufficient associations between resources, low cueing priority or lack of appropriate SMMs.) Input from me helps G to focus on one particular aspect of the statement that he is able to relate to his broad mathematics contextual neighbourhood, including concepts of addition and subtraction, as well as a SMM that equips him to infer that , as subtract is the opposite of add, perhaps he should move in the opposite direction on the number line. Until now, G’s RS have not modified to attend to what until now seemed (to G) to be superfluous words. RS will now change in this way.

72. I re-iterate $^{-}9$ add $^{-}8$ means we have to start at $^{-}9$ and DO a $^{-}8$ … G picks up and continues my explanation "So if it's take away $^{-}8$ you take away one that was done before. $^{-}9$ - $^{-}8$. That's just like undoing this into an add." The "this" he refers to is the minus sign preceding the 8 in the expression. After a pause, he adds "We did this with that thermometer thingy".	I give G the opportunity to reflect upon and reinforce what he has just done. He goes further to link the word undoing with reference to a process that appears to have been reversed, producing the effect of adding. He remembers that he did something like this in the previous session. In doing so, we see that he is already beginning to further extend the span of his naïve "Minus numbers" concept to other existing conceptual resources and is testing alignment.

At the end of our session, when trying to solve difficult problems, G describes how he visualises a number line:

73. I give the boys several more examples for them to work out on the number line and they quickly abandon the number line and are able to do them mentally. G says he doesn't need to draw a number line because he can "see	G and the others only draw number lines until they feel confident to work mentally. G does not abandon use of his "Number line" resource, however; he simply uses a mental representation rather than a concrete diagrammatic one.

one in my head".	

As our sessions came to an end, G demonstrates that he has modified his contextual neighbourhood:

74. I ask them to make up a "minus minus" question using words. G is able to give me an appropriate Father Christmas scenario.	G is able to work with all relevant concepts simultaneously and effectively. Span and alignment of all relevant conceptual resources is adequate for this task. RS have already evolved to enable him to extract appropriate information in a meaningful way.
75. When I ask him to explain using some sort of picture or diagram, he draws a vertical number line and makes it look like a thermometer. The pointer is Father Christmas's hat.	G shows very explicitly that he can draw upon different conceptual resources effectively and appropriately. He has formed a system for integrating and co-ordinating a range of concepts from different aspects of his experience.

G's story

From the beginning, G demonstrates that he understands each task and is able to make good progress towards the goal of the task. He is able to co-ordinate a varied range of conceptual resources, including readout strategies, resources in memory, conceptual associations and sense-making mechanisms (about direction of movement, number order and Father Christmas's clothing), effectively and without difficulty.

Although G has had some informal experience with "minus temperatures", he had not previously interpreted the signed numbers used to represent sub-zero temperatures as relative values that could be compared, ordered and increased or decreased by mathematical operations. Within the sessions we had together, he was able to build connections between resources he had about numbers and a variety of other concepts. Those concepts became more organised in relation to each other and developed into a conceptual system that G was able to use in other contexts.

The analysis and commentary above also reveal examples of other connections that G had previously created that did not become so securely or widely connected. Such resources, including generalisations developed by G from previous experience, are not necessarily very rational, though without knowing their source this is difficult to judge. These are likely to fail to become established through lack of reinforcement of span and alignment, so structural priorities will not become high.

Other pieces of knowledge that G uses as tools for learning include his "clonk" (Figure 5.3). This is a tool that he recognises might be relevant in the context of this task and he finds that it does help him to solve the problems that arise.

G is aware of his own thinking and learning and often refers to the way he is feeling at several points as he works through the tasks. He describes confusion, anxiety, optimism and joy; clearly emotional responses to his own activity and development.

G is very articulate and is able to talk about his thinking and his approach to the problems within the tasks. It is clear that conceptual resources that he has already constructed are used at various points to help him make progress with the challenges he faces; for example, we see resources about calculation strategies and number lines and commutativity. Also clear is G's active search for meaning – he really wants to be able to make sense of and solve the problems he encounters.

5.4 Concluding remarks for Chapter 5: Analysis of findings

Analysis of the two boys' conceptual change whilst engaged with the series of tasks has been most illuminating. Both boys were enthusiastic and well-motivated and they eagerly interacted with the tasks and with their peers in pursuit of the goals of the different activities. It has been possible to infer much about changes to their respective contextual neighbourhoods, using constructs outlined in Table 1, and the model for conceptual change that they describe (Figure 3), as tools for analysis and elaboration.

It is clear that, when beginning to work in a new domain, there are multiple cognitive processes involved in making sense of all types of inputs that must be accommodated harmoniously within the extensive pool of conceptual resources already available within each boy's "knowledge". The evidence presented also suggests that other forces exert an influence too – socio-cultural factors (such as roles, relationships and rules) appear to have affected both boys' contributions and performance.

Both C and G drew upon the external resources available, though not always in the same ways or to the same extent. However, it was clear that they were only able to make progress, that is to be able to make sense of negative numbers in the context of different tasks, by making connections between a variety of different kinds of internal conceptual resources. The construction of those connections was not straightforward or predictable nor did it occur as an even and gradual increase in terms of the numbers of resources in memory, or associations, or sense-making mechanisms. The learning trajectory for each boy was uneven in its content and pace – indeed, sometimes even appearing to go backwards.

It would seem that both boys' conceptual change is predicted by the model shown in Figure 3; in some, broad and general respects, both boys' progress in their learning in a new domain, their tuning towards expertise, can be described by the model. However, there were many differences in the ways that resources and associations were constructed, both **within** each boy's learning journey as well as differences **between** the 2 boys. In the next chapter, the conceptual changes that occurred for both C and G will be considered further.

Chapter 6: Discussion of Findings

In the previous chapter I presented my analysis of each boy's experiences of a sequence of tasks. In each of the two case studies I was able to identify or infer conceptual changes and to make inferences about internal and external resources that were being used, including existing knowledge.

In compiling my analysis for "Chapter 5: Analysis of Findings", it was apparent to me that several themes were emerging that connected both boys' case studies. In this chapter, I shall identify these themes and, in discussing how the boys' experiences and outcomes relate to those themes, will be able to further illuminate any conceptual change that occurred. Reconsidering the case studies thematically will also facilitate some comparison between the case studies, from which tentative questions about ability differences might be posed.

6.1 Re-use of existing knowledge

My knowledge of the National Curriculum (DfEE/QCA 2000) and National Numeracy Strategy (DfEE 1999), both current at the point in the design and data collection phases of my study, led me to believe that children in Year 4 in a UK primary school would not yet have received any teaching about negative numbers. It was confirmed by the class teacher that the children had not. Therefore it was appropriate to infer that any knowledge that children had about negative numbers had been acquired or developed informally, outside of their lessons in school.

Any existing knowledge about negative numbers was not, however, without interest for me and, in acknowledging that some might exist, I was interested to discover how children would use existing knowledge resources in their work with me.

One aspect of the re-use of existing knowledge that I am keen to consider is, therefore, the re-use of long-standing, well-established knowledge.

6.1.1 Re-use of well-established knowledge

Throughout all sessions children said and did things that revealed knowledge resources that they had previously begun to develop. Sometimes, for me to probe their knowledge further might have disturbed the flow of the group's work and on those occasions I chose not to follow up on their remarks and am dependant therefore on my own inferences about their knowledge. At other times, I did probe further and was able to obtain a more direct account of the child's existing knowledge.

Both boys revealed knowledge about temperatures in different parts of the world, a concept I called "TW". It was evident that this knowledge had originated in a wide range of sources. For example, C knew something about Egypt: that it is hot; that there is a "suffinks" (sphynx) there. He also knew about extremely cold environments. C had no direct experience or formal teaching about either of these subjects and had developed these elements of his "TW" concept largely from TV and from books. For C, this knowledge meant that he had some understanding (of what it is like in a very hot or very cold country) that he brought to the task in Session 1, and it contributed to further development of "TW" as the task progressed. G brought to the task his experience of travel to Europe, particularly France. He had some knowledge of how temperatures in France compared with UK and Spain.

G showed that he was able to access and utilise internal resources about the significance of the Equator on a map or globe. He showed, from the beginning of the "Journey" task in Session 1, that he knew that countries on or close to the equator are hot. He was able to judge whether a country is hot by considering its position on the map in relation to the equator.

Whilst working on the "Journey" task, It became clear that C's exposure to, or experience with, negative numbers in the context of values that he might have seen and heard in everyday life did not help him; he did not appear to have any knowledge of temperatures below zero; (at least, he was not able to access and co-ordinate resources effectively). G, on the other hand, in Row (G) 9 shows that he did have some knowledge that there are “minus” temperatures

9.	(I ask) “As you’ve been to Poland, a country that is zero degrees, what are you looking for now?” M says, “minus” and G agrees. I ask “Minus what?” G and M say together, “Temperatures”.	G had previously seen “zero” as somehow more problematic than other numbers. His readout strategies (RS) appear to be evolving in parallel with the extension of the span of relevant concepts. He now seems to accept zero and negative numbers as an extension of the number system he knows. The span of G’s conceptual resources about directed numbers is expanded to include temperatures as a relevant context for cueing these resources.

As the sessions progressed, and the boys revealed their existing mathematical knowledge, they both used calculation strategies that they had not been taught by me but that have been taught and reinforced by their Mathematics teacher over an extended period of time. For example: C uses his fingers to count up and down (though often unsuccessfully, as I shall discuss later); C also habitually wants to start with the highest value when adding; he also demonstrates sound knowledge of many number bonds – something he will have started to learn in school 3 years ago. G also uses mathematical knowledge within our sessions that he must have learned previously: for example, in Row (G) 22, he shows that his counting strategies using a number line are sound:

22.	Throughout, he is very secure in the fact that he must include an invisible count at the $^{-}5$ position even though there is no card for $^{-}5$. (He taps the table when a card is not present for any value).	G’s conceptual resources relating to the number system appear to include a well-established number line model. Span is being tested and extended as he finds that what works with positive numbers also seems to work with negative numbers.

So, it is clear that both boys are able to remember and spontaneously use long established knowledge in the context of our work together – i.e. a new setting or context; different to the one(s) in which they previously developed those concepts and strategies.

It is interesting to consider whether this re-use of long-standing, well established knowledge constitutes “transfer”. At times the existing knowledge does seem to help G solve a new problem. For example, in Row (G) 56, he recognises the relevance of his knowledge of the commutative law and finds that it helps him to solve the problem at hand:

56. The boys do not come to agreement over this and I ask whether 9 - ⁻8 is the same as 9 + ⁻8. G ignores my question. He has noticed that by rearranging the order of the list of numbers he can make sense of it – he can avoid doing 9 + ⁻8 by switching the order and thinking of it as ⁻8 + 9.	My question challenges G as his conceptual resources do not accommodate the idea that 2 operation words can be adjacent within an expression. Before he could answer my question G's concepts must be modified. G's SMM relating to the commutative law enables him to restate the problem in a way that makes it possible for him to solve. This way he can solve the problem using his existing conceptual resources without any imperative to change or expand them. This is preferable to G.

At other times, for both boys, it does not appear to have any direct bearing on the new problem or task objective but does help comprise a "backdrop" knowledge that enables the child to achieve a richer understanding of the new setting. An example of this is reported in Row (C) 4:

4. He sees Egypt on the map and wants to go there because "It's really hot there. There used to be people like this .. mummies, pyramids, .. musca …muscats, what are they called, muscats, it's called suffinks" *(he means sphynx)* He says that he read about Egypt in a book.	C has some knowledge about Egypt – not only resources about the temperature but also others about ancient Egyptian civilisations. We know that he has learned at least some of this information from a book.

As I argued in Chapter 4: Methodology, any re-use of knowledge, whether used directly or indirectly, and whether it proves to be effective in solving a new problem or not, **is** transfer.

6.1.2 Re-use of more recent learning

Another aspect of the matter of re-use of knowledge concerns knowledge that is less well-established – that which has been learned more recently, even **very** recently. In the data and in my analysis it is apparent that G re-uses newer knowledge on many occasions. For example, in Row (G) 9, (shown above) his sense-making mechanisms facilitate incorporation of zero into his number system concept, even though only minutes beforehand he had perceived zero to be more difficult than the positive integers he was very familiar with. Also, between Session 1 and Session 2, G remembered how to order negative numbers – something he had only learned to do at the end of the first session.

C, on the other hand, is less able to re-use knowledge recently learned. He does not remember or think to apply related knowledge that he seemed to have recently learned. For example, in Row (C) 18 he has forgotten the aim of the task:

18. C is keen to be the person who chooses where FC will start and places him at the North Pole. I point out that that is where he will finish. C hadn't been listening very carefully or had not understood instructions.	It would seem that C had either not heard or not assimilated information about the task previously because he gives the wrong answer now.

And, in Row (C) 13, he has forgotten the (recently discovered) temperature for Spain:

13. When the group inadvertently click on Jordan (9º) C thinks "It's less hot than Spain". He claims this, however, without being able to remember what Spain was.	It is not clear whether C has remembered the Spain temperature incorrectly and does actually think that Spain was more than 9º or whether he has just become very confused.

I should re-iterate, at this point, that my understanding and use of "transfer" includes re-use of resources in memory. Therefore, failure to evoke a resource in memory (given that such a resource has revealed itself on another occasion) is an example of failure to transfer.

It is actually quite difficult, when searching through the data, to find examples of C successfully re-using recently learned knowledge. This would be consistent with a "knowledge-in-pieces" model for learning (diSessa 1988) in that C is, at this stage, unable to co-ordinate the range of resources available to him. C's behaviour is, perhaps, evidence that higher priority meanings are triggered rather than recently constructed memory resources which have only low priority at this time.

There are, however, a few examples, such as Row (C) 20 where C is very confident that he has remembered something correctly:

20. As they start to plan FCs journey, C says something that confirms that he has remembered that Kenya was cooler than Madagascar when he lands on it saying "That was cold, wasn't it?" N queries this but C insists that Kenya was colder than Madagascar.	C remembers that Kenya was cooler than Madagascar. He refers to it as "cold."

6.1.3 Different types of transfer

As I set out in Chapter 2: Literature Review, diSessa & Wagner (2005) identified 3 types of transfer: Class A, B and C. To recap:

- Class A Transfer– "where an adequately prepared set of ideas is used unproblematically in new situations" (p148); "the knowledge whose transfer is at issue is assumed to be, or can be demonstrated to be, well prepared and does not, in principle, require further learning to apply" (p124). diSessa & Wagner note that this is important for schools who *"want students to be able to solve problems other than the ones used in teaching them concepts" (p125);*
- Class B transfer – knowledge constructed that is *"presuming subjects' persistent effort… sufficiently prepared so that transfer can be reliably accomplished in acceptable periods of time (e.g. in a few hours or days…)" (p125);*
- Class C transfer– How do *"relatively unprepared subjects (students) use prior knowledge in early work in a domain?"* (p125); *"where bits and pieces of "old" knowledge are invoked, productively or unproductively, typically in early stages of learning"* (p148). Class C transfer might be considered as the processes that lead to transferable knowledge. (p125)

Adopting this typology enables me to distinguish between different kinds of transfer evident in the boy's work. Consideration of the examples above and the analysis in Chapter 5: Analysis of Findings, shows that both boys demonstrated Class A transfer when they were able to re-use long established knowledge to solve problems they encountered in our sessions together. Knowledge about numbers, temperatures and calculation strategies are examples of what diSessa & Wagner referred to as "well-prepared knowledge".

The table below summarises my analysis of the transfer types that were evident in each row of both boys' analysis grids (where transfer was indicated at all). For each event, transfer type was evaluated according to the key presented in Chapter 4: Methodology (Figure 4.10)

Transfer type	**Child "C" (row numbers)**	**Number of instances**	**Child "G" (row numbers)**	**Number of instances**
Class A	76, 77	**2**	11*, 13*†, 15*, 22*, 23*, 48*, 49*, 73*, 74*, 75*, 2, 17, 24, 27, 4†,	**15**
Class B	30, 31, 78, 79, 24†, 25†, 26†, 27†, 29†, 52†, 53†, 85†, 86†, 88†	**14**	12, 20, 26, 30, 33, 34, 37, 38, 39, 60,	**10**
Class C	2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 15, 19, 20, 21, 22, 28, 32, 34, 35, 36, 38, 39, 40, 41, 45, 47, 48, 50, 51, 54, 55, 57, 58, 59, 61, 62, 64, 65, 66, 67, 69, 70, 72, 73, 74, 75, 80, 82, 83, 84, 87, 90, 91	**53**	1, 3, 5, 6, 9, 14, 16, 18, 21, 25, 28, 29, 31, 32, 35, 36, 40, 43, 44, 46, 47, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72	**41**
Total number of rows recording transfer		**69**		**66**
* = includes use of knowledge resources developed during our sessions (i.e. "new" knowledge) † = borderline with class below (e.g. B/C is recorded in Class B with †				

Table 3: Occurrences of different types of transfer for both boys in all tasks.

Although such an analysis is based only on my interpretation of events, it has been carried out on 3 occasions, over a period of a year, and has been shown to be reliable. (Of 167 row entries, only 8 were evaluated differently at the first re-analysis and 3 at the second re-analysis. Overall trends and patterns were not affected by such a low number of discrepancies). Table 3 shows the final evaluations.

Figures 6.1a and 6.1b show each Transfer Class as a proportion of all transfer events for each boy (each “transfer event” recorded as a row in analysis grids in Chapter 5):

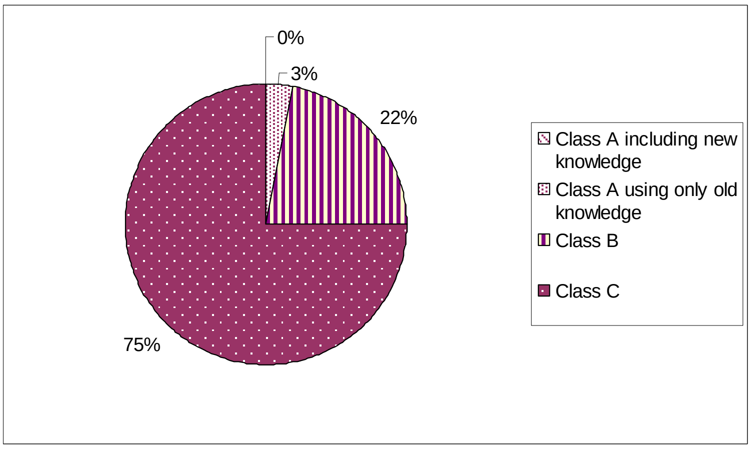


Figure 6.1a Classes of transfer evident in C’s transfer events

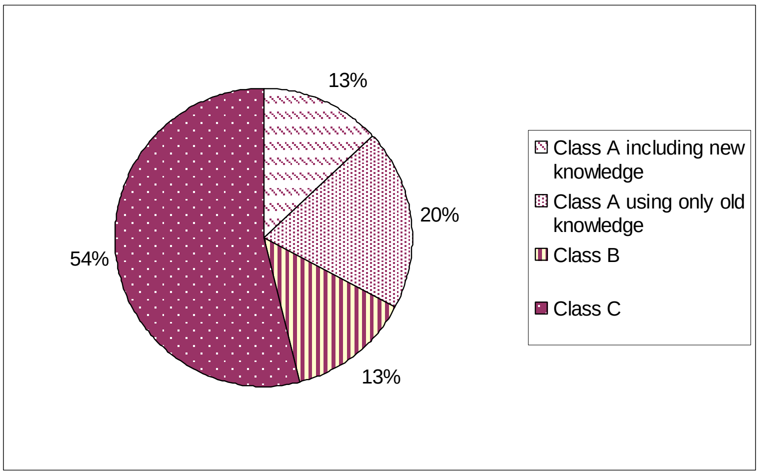


Figure 6.1b Classes of transfer evident in G’s transfer events

It is clear that the pattern of transfer types is different for the 2 boys in some respects. For instance, Class A was rare for C and much more common for G; almost 33% of G’s instances of transfer were Class A. It is important to point out that, although, many of G’s Class A transfer occurrences were based wholly on “old” knowledge, some (shown * in Table 3) incorporated “new” resources that were used in conjunction with “old” ones. C’s Class A accounted for less than 3% of all his transfer events. Class C transfer, although evident in a high number of instances for the 2 boys, accounted for a much higher proportion of any transfer achieved by C than for G. These differences will be considered later with other disparities between the boys' learning. It is useful to point out here, however, that although both boys exhibited all 3 classes of transfer, the sophistication of the knowledge being transferred and the confidence with which it was applied were very dissimilar.

Some patterns did emerge from my analysis of transfer types:

- Where C was able to demonstrate Class A, it was only to use, at one point in our work together, a "bridging through 10" strategy, using knowledge of number bonds;
- G's Class A happened across all phases of our work. At first it was to show and use knowledge about "TW", maps, equator etc. Then we saw that he had learned to increase and decrease temperatures in the quiz and to use the thermometer. Later with "Balloons" he used knowledge of the commutative law to help him tackle and solve problems, incorporating his new understanding of negative numbers. G's re-use of knowledge as he progressed through the "Journey" and "Quiz" tasks are often Class A or Class B transfer. It appears that, in these phases, his knowledge had become either "well-prepared" (diSessa & Wagner 2005) or "sufficiently prepared" such that transfer could be "reliably accomplished soon". Therefore, it is clear that his knowledge had developed beyond that exploratory, undiscriminating use of resources that characterises Class C transfer;

As well as identifying patterns of transfer types for the 2 boys, analysing the data in this way reinforced for me the fact that what is most interesting in reflecting on my work with these children is not, actually, the instantiations of transfer, of whatever class. The real interest and value is in acknowledging, and trying to understand, all the other developments and contributions and obstacles and deviations that affect the direction, extent and pace of children's learning in a new domain. (Whilst I acknowledge that there is sense in which analysis of transfer classes is open to interpretation, I would point out that I have a systematic way of working that does promote reliability and validity.) It appears to me that, to categorise and classify only according to class of transfer is to reduce such experiences (data) to a point where too much data has been lost. This just serves to remind me that transfer is only a part of a broad and highly complex process of learning and that it is important to attend to all aspects of conceptual change involved in learning and the growth of knowledge and understanding.

My analysis of transfer types does, however, illustrate very well the point that Class A transfer only accounts for a proportion of the transfer that occurs in a learning episode; that a great deal of learning occurs without much Class A transfer in evidence. Indeed for some children, like C, Class A transfer is quite rare. This is, as previously noted, in line with diSessa & Wagner (2005) who describe a theory of co-ordination of knowledge resources. Also, it supports the argument within the research literature (e.g. diSessa & Wagner, 2005; Schwartz et al, 2005) that blames underpinning methodologies and principles for any apparent lack of transfer that they are reported to have revealed. These workers believe that the early stages in the construction of knowledge - in which learners link together pieces of knowledge from different experiences and sources, though without understanding the nature of the connections, and without being able to co-ordinate the different pieces in anything but a fairly haphazard or experimental way –are necessary for conceptual growth and transfer. Schwartz et al (2005) called this “Preparation for Future Learning”, (PFL). Indeed, the most common type of transfer observed in my sessions with C and G was what diSessa & Wagner (2005) would call Class C and what Schwartz et al (2005) would call PFL, as these workers would predict.

6.1.4 Failure to re-use existing knowledge

There are points during our sessions at which the boys might have been re-using some aspects of their knowledge but also seemed to fail to use other knowledge that they have already shown were available to them – i.e. failure to transfer. These instances are noted below:

Occasions when C failed to transfer:

- Row 13: He thinks Jordan (9°) is cooler than Spain, even though he found out that Spain was 7° only a few minutes previously;
- Row 18: He thinks that Father Christmas could start at the North Pole, even though he was been told that Father Christmas must finish at the North Pole;
- Row 19: Although he does re-use knowledge about Madagascar, he doesn't remember that there were some countries that he has "visited" in the game that had hotter temperatures than Madagascar;
- Row 33: He has previously discovered the temperature in Germany but can't remember it – even forgets that he has "been there";
- Row 40/41: He thinks $^{-}6^{\circ}$ is a "high" temperature, despite having achieved some success with ordering negative numbers in previous session; he is re-using only a very small part of knowledge he appeared to have constructed in the previous session;
- Row 42: He argues with N, insisting that minus does not mean cold, even though he was happy in previous session that minus values represented very cold temperatures;
- Row 44: He claims that "Of course zero is the lowest number" – the list contains negative numbers that, in the previous session, he had ordered correctly relative to zero;
- Row 81: It does not occur to him to use a number line to help him count up or down through zero, even though he had achieved this in the previous session.

For C, there are also many occasions when he omits to read or mis-reads the minus sign.

Occasions when G fails to transfer:

- Row 67: He did not recognise a vertical line on the page as a prompt to use it as a vertical number line, despite working with a vertical thermometer both on screen and on paper, including thermometers and vertical number lines that he himself had drawn;

When C failed to transfer knowledge, it was often knowledge that was very new and perhaps was not sufficiently associated with other existing knowledge resources that he did not re-use. It appears that new pieces of knowledge did not span (through associations with existing resources) to new situations and were not, therefore, called up in a subsequent situation where they would have been relevant.

There was a point on which both boys failed to transfer knowledge that they had shown was available to them; this was their failure to spontaneously use a number line model where it would have been appropriate in the later tasks. Both C and G had, at previous points in the tasks, successfully used number lines, including those they had thought to draw for themselves, having recognised their usefulness for the questions they were working on at the time. It is interesting that they both, when faced with other situations in which a number line would have supported them, did not recognise that relevance. Although, in G's case, an alternative interpretation was provided in the analysis grid, even this would support the inference that cueing priority and/or reliability priority of number lines in that situation was low.

I shall, later, consider children's use of number lines in more depth.

6.1.5 Perception of similarity

It is clear from many of the boys' comments that their ability or propensity to re-use knowledge is linked to some aspect of the task that reminds them of something; that "brings to mind" something they know or have experience of. Previously, in my review of the literature, I showed that perception of similarity is key to learning since, without it, connections between old and new knowledge cannot develop. I argued that conceptual development occurs through recognising and building and organising connections within and across concepts.

In Chapter 3, I set out a model of learning which I had created in the light of my own experience and reading (see Table 1 and Figure 3). My analysis of the data, presented in Chapter 5, enables me to explore instantiations of associations between concepts and contexts being utilised and developed; of the extension of span, and of the alignment that must also be established between existing and new resources, in order for effective co-ordination of concepts to be enabled.

6.2 Improving span and alignment

What follows is a reflective summary of some of the events, actions and utterances during the tasks in which the extension and testing of span and alignment is evident. This is not an exhaustive list but does serve to illustrate the wide range of concepts that were developing for both boys in different ways in all our work together.

6.2.1 C's conceptual resources: development of span and alignment

Rows (C)3-4:

3. When I say that we should send FC somewhere as hot as we can, C, straight away, says "That's Africa."	His "TW" concept contains sufficient alignment between it's "Spain" and "Africa" components to enable C to compare them and to judge that Africa is hotter than Spain. He doesn't have conceptual resources about anywhere being hotter than Africa – at least none that span to this situation.
4. He sees Egypt on the map and wants to go there because "It's really hot there. There used to be people like this .. mummies, pyramids, .. musca …muscats, what are they called, muscats, it's called suffinks" *(he means sphynx)* He says that he read about Egypt in a book.	C has some knowledge about Egypt – not only resources about the temperature but also others about ancient Egyptian civilisations. We know that he has learned at least some of this information from a book.

C knows that Spain is hot. He knows that Africa is very hot – "as hot as we can get". He also knows that Egypt is hot. All of these knowledge pieces come from different sources (holidays, TV, books respectively) and C thinks of them all in response to seeing a map and to questions and discussions about temperatures in different parts of the world. As noted in the analysis grid, “His TW concept contains sufficient alignment between its Spain and Africa elements to enable C to compare them and to judge that Africa is hotter than Spain.” As the task develops it is evident that C uses these pieces of “old” knowledge to structure his understanding of the task, as a framework for considering temperatures in other countries, on different sections of the map – i.e. he is able to construct and use a sense-making mechanism.

The following extracts show that C is able to compare temperature values:

Row (C) 25:

25. C groans when the display for Sudan shows 25° – he seems to understand immediately that 25 is not lower than 19 so Sudan cannot be the next stop.	Both of these concepts are sufficiently aligned for C to participate and make effective judgments in the task.

And Row (C) 27:

27. When they see that Chad is 24°, C says “We lost that. We gotta go to a different one. He asks the others where they should go. They want to go to Niger. When they see that it is 25°, C says “Oh, that’s rubbish now.”	C had decided that Ethiopia 16° had been the previous stop so understands that they should not go to a country where the temperature is 24. I think “We lost that” is a reference to winning or losing at each step – i.e. whether they click on country that fits the requirements of the task at each juncture – in this case they needed one that was less than 16 and got one that is more, so they “lost” and need to find another one. When the other boys go to one that is even hotter C is frustrated. He had quite clearly understood that he was aiming to find a country cooler than 16 and the others are clicking on countries that are increasingly removed from his aim. His own reasoning strategies are functioning effectively, though he is not able to find what he seeks.

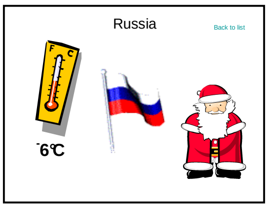
Alignment has been established that enables C to know that a country with a temperature higher than their current location is <u>not</u> a valid next step in the task. This alignment has been achieved within the task: C did not understand previously how to use his knowledge about the Journey task itself in concert with his NT concept.

Row (C) 31:

31. He clicks on Turkmenistan … The display shows 5º. C says "Yes. We beat it. We beat it, N***. Yes!" He is very excited.	Every time C sees that his concerted resources – i.e. his interpretation of the rules (based on associations between resources within and across concepts) and of how to "read" the numbers and the clothes – leads to success, these resources are more robustly aligned.

Here, alignment of all related conceptual resources are further reinforced.

Row (C) 36:

36. I say "This says minus 6 – how does that compare to zero?" C tells me "It's colder – look he's wearing more clothes now anyway. He's wearing that big coat, cloak thing. 	C's interpretive resources for "reading" FC's clothes support him in his "It's colder" response. He seems to be persuading himself and trying to persuade me that his answer is correct by referring to the clothes that FC is wearing. This suggests that C's understanding about FC's clothes spans effectively to his concept about the number system. His confidence in his sense-making mechanisms that enable him to evaluate the significance of FC clothes, and alignment of any judgements with his number system concept (repeatedly tested and reinforced throughout the task) enables C to make inferences about extending his number system below zero.

Here, C shows that he is able to co-ordinate different resources in order to actively create new knowledge which adds to his concept of the number system.

Row (C) 39:

39. To finish off, I ask the group a few quick questions to check their understanding. I ask for a temperature that is warmer than $^{-}6$ and C tells me "minus 2" I ask "Is minus one even warmer?" C says "Minus zero, minus zero, that's warmer." I ask "Which is warmest, one or minus four?" C and then N tell me "One." When I ask why, C tells me "Because its closer to that, minus zero and that's warmer."	C's sense making mechanism for comparing numbers has modified. Now, he thinks $^{-}2$ is warmer than $^{-}6$ and that $^{-}1$ is even warmer. He goes on to tell me that minus zero is warmer than that and that $^{-}1$ is warmer than $^{-}4$. Although he calls zero "minus zero" he does seem to have learned that the digits increase as they become further away from zero and that they get smaller towards zero, and that movement towards zero from a minus temperature is towards "warmer". Alignment is being established between relevant concepts and every question that I give C is an opportunity to test this alignment, reinforcing his understanding and his confidence in this new expansion of his number system concept, a part of this concept is now about "minus numbers" in the context of this task.

For the first time, C seems to understand how "minus numbers" are ordered in relation to zero.

Row (C) 51:

51. They start the quiz. C says "I got it right. Look! There's Norway and there's Russia" C thinks Norway is hotter than Russia. C explains "Because it says it on the card." I argue, pointing out that it does not say "Norway is hotter than Russia" on the card. C replies, "It says 3 minus, that means 3 behind zero and that's 6 behind zero… " S says "So that's hotter than Russia." C "Yes, that's what said". I ask "How do you know?" C says "because 6 under zero is real cold but 3 under is only a little bit cold" 	Span and alignment of C's "NT" concept, resources relating to Father Christmas and his concept of the number system are tested; C rises to the challenge, justifying his judgements effectively.

Span and alignment of resources relating to zero, below zero and "minus numbers" are tested and reinforced.

Row (C) 81:

81. Balloons are 1, $^{-}7$, 1. C giggles and says "That's zero!" S says "1 add 1 minus 7. C repeats, "Zero." He enters 0 as the answer	C does seem to "see" the minus sign this time because we see that his answer does not correspond with addition of only the positive numbers. He is not able to subtract or count back 7 from 2, however. His concept of addition, though apparently adequate with single digit unsigned integers, does not equip him to operate with minus numbers effectively. He has had some experience with adding and subtracting negative numbers (including bridging through multiples of 10) in our previous session when he used a number line model to support his working. Memory resources and other resources constructed in that session, themselves associated with a number line-based SMM, have not spanned to this new task. Here, he knows he must add 3 numbers together. There is nothing in this setting that he "reads" (and recognises) as a prompt to consider a number line approach.

Span of C's number line concept, sometimes effective within the "Journey" and "Quiz" tasks does not extend to the "Balloons" task. He does not recognise the relevance of number lines in the new setting.

6.2.2 G's conceptual resources: development of span and alignment

At the very beginning, G shows that there is some tension within his concept of "Madagascar":

Row (G) 1:

1. Having introduced the group to the countries list and clicked on Madagascar as an example to show the boys what each page looks like, G is the one who very quickly says "Well, that's wrong, 21. It's really hot in the film." when he sees the temperature displayed.	G Reads 21º as "twenty one degrees" and knows that this is a representation of "how hot/cold" His understanding of 21º conflicts with his understanding of Madagascar as he thinks Madagascar is **very** hot and 21 is not **very** hot. The 2 concepts.", "NT" ("Numbers as they are used to represent temperature") and "Madagascar" do span to each other and are already aligned. That alignment is challenged here.

This suggests alignment between G's concepts about numbers used to represent temperatures ("NT") and Madagascar. However, this alignment is challenged and will need to be reinforced.

Row (G) 12:

As G attempts to resolve problems with negative numbers in the context of the "Journey" task, many changes in span and alignment can be inferred:

12. I ask the boys how much the temperature changes if we go from Svalbard ($^-13°$) to Norway ($^-3°$). G says 10°. When I ask whether it is decreasing or increasing, he says it is decreasing. I ask again "Is it getting higher or lower? Increasing or decreasing?" G, looks at the map and changes his mind - he tells me that it is increasing, not decreasing, because it is getting warmer.	G's concept of the number system includes a sense-making mechanism (SMM) that enables him to compare values represented as unsigned digits. He is able to compare and make judgements about the relative values of unsigned numbers and uses the same SMM here, at first. He has a conceptual resource (in this case it might be a resource in memory or a readout strategy) that a "minus number" is an indication of a cold temperature. He also has resources relating to the Equator being hot and the North Pole being cold. He infers that a change from a value represented by the digit 13 (ignoring the sign) to a value represented by the digit 3 is a decrease in value. The fact that I questioned this, in itself, may have suggested to G that he has made an error. His evolving "TW" concept also leads him to infer that the temperature should increase if the change is southward. These inferences are in conflict with each other. G resolves the conflict by judging that the question is about the change in temperature and that the concept of temperature is founded on measures of how hot/how cold. He chooses to focus on whether the move from Svalbard to Norway is an increase in temperature (getting hotter) or a decrease (getting colder). He judges that such a move would result in an increase in temperature. It would seem that he is willing to accept that, although the digits themselves are decreasing, the presence of the minus sign changes the "rules" that he thought he knew. This constitutes a modification to a SMM that had previously been effective. Although G is confident about the magnitude of the change, his "decreasing" response is an indication that his conceptual resources relating to directed numbers are not securely connected to other concepts – i.e. effective span has not been established - through associations between resources. The span of G's "TW" and "NT" concepts has already extended to include each other but some components within these concepts are much better established than others and they are not aligned with each other. He aligns these conflicting inferences about "increase/decrease" by focusing on the context of the problem and reasoning that "getting warmer" equates to an increase in temperature, regardless of whether the digit values are increasing or decreasing. G's engagement with this particular question provides evidence of alignment being tested and evolution of all relevant conceptual resources. This is evidence that span of G's concept of increase/decrease is being extended to be effective in determining information in the contexts of temperature and/or directed numbers

Row (G) 15:

15. When countries with temperatures below zero appear from the pack, the boys take them in their stride, except to notice that Turkey's temperature is not what they expected. G reminds them that "This is around Christmas, though".	Negative numbers are read efficiently. G's conceptual resources include a SMM that enables him to reason that Christmas temperatures are lower than might normally be associated with countries that we visit for holidays in the summer. Whereas during the previous activity, G was uncertain about ordering negative numbers, today he was more confident – his RS, resources in memory and SMMs have modified and he now works effectively with negative numbers and is able to order them. He did not make mistakes. Span and alignment of resources has increased, linking resources that G has relating to numbers.

At the beginning of the second session, G is able to "read" and order negative numbers effectively. He had not been able to do this at the beginning of the previous session. Learning about negative numbers has taken place in the first session that G is able to use here. Moreover, he is able to use it efficiently and without prompting – the result of previous reinforcement of span and alignment. This is tested and extended further in Row 22 in which G's actions suggest that he is using a (well-established) number line model to support the development of his knowledge in this new domain.

Row (G) 22

22. Throughout, he is very secure in the fact that he must include an invisible count at the $^{-}5$ position even though there is no card for $^{-}5$. (He taps the table when a card is not present for any value).	G's conceptual resources relating to the number system appear to include a well-established number line model. Span is being tested and extended as he finds that what works with positive numbers also seems to work with negative numbers.

In Row (G) 30, G is able to re-use knowledge to address a different type of problem – evidence of effective span and alignment, recently modified:

30. I pose a new type of question "If I am in Moldova and go somewhere that is 9° warmer, what will the new temperature be?" G has control of the thermometer and confidently counts up one degree at a time from the Moldova temp ($^{-}2$), using the thermometer scale as a vertical number line	G is able to solve the new problem (that has a slightly different structure, as I perceive it) without any difficulty. Relevant conceptual resources have sufficient span to be triggered and are well aligned, each incorporating an appropriate range of conceptual resource which act as anchors, enabling G to work unproblematically. Concepts used are thereby becoming increasingly secure; strongly connected.

In Row (G) 37, G shows that he is able to perceive similarity across settings, including between narrative and iconic and symbolic contexts.

37. I ask G to "write down what we are doing with the thermometer" and tell the group that we are going to start on 3 and go up 10 degrees. G thinks we should write 3 + 10. For a similar question with starting point of $^{-}3$ the boys agree that we should write $^{-}3$ + 10 and that the answer will be 7. They model it using the thermometer and see that they are right. G is excited "I get it. I get it. If you go down it puts it as a minus. It's as if you're doing the sum."	G spontaneously extracts the mathematics from the situation. He easily uses the 2 numbers involved in the question, relating them to each other in terms of starting with one temperature value and "going up" by a number of degrees, using the + symbol to show that the first quantity/number is increased by second quantity/number. G "sees" the similarity between the screen thermometer display and his own tentative attempts to express the temperature changes symbolically: these 2 situations have at least one resource in common. He is excited about this. The span of G's established conceptual resources relating to working with numbers and increasing quantities has extended to be perceived as applicable to the temperature context. G is able to further reinforce his emerging hypotheses about the mathematics within the temperature problems.

In Row (G) 63, G has recognised the relevance of a number line model and has been using it effectively.

63. I ask G if he can explain. He says that "$^{-}5$ add 4, if that was a 5 it would be zero but it's not, it's 4 so the zero changes to $^{-}1$. I did this with that game with the countries. I said that when we went to warmer and colder places it was like a sum." This is the first time that he's made any reference to our previous sessions, even though the calculations have been very similar.	G describes a compensation strategy for calculation. This is something that is likely to have been taught in school and G has realised that it might be relevant here – there is at least one resourcenode that is common to existing resources that relate to "compensation" and to the current task. We have already seen evidence of the integration of "Calculations" and "Minus numbers" concepts; this now suggests that the span of another collection of conceptual resources (relating to "Compensation") has been extended. This will improve alignment of all these concepts. It is interesting that G also recognises that some resources relating to these concepts also form part of his concepts relating to "Journey" and "Quiz" tasks.

Here, he recognises that earlier tasks in which he increased and decreased temperature values required similar knowledge and strategies to those he has used now. So, span across the tasks or settings is established through a new association. It is through the establishment of multiple associations that alignment emerges.

I have outlined a few examples in which extension and testing or reinforcement of span and alignment can be inferred. There are many other such examples within the complete analysis grid in Chapter 5: Analysis of Findings. Span and alignment are constructs used by diSessa & Wagner (2005) to describe: how connections between concepts begin to become constructed; and how those connections, at first tentative and uncertain, can become robust such that they anchor the concepts together, providing reciprocal reference points across concepts. Connections might be across small pieces of knowledge or across more complex knowledge systems or concepts. In all cases, the connection itself is an association of common attributes within the learner's experience and knowledge. Some commonality or similarity across settings is crucial for growth of concepts. Moreover, learners' perception of that commonality or similarity is also crucial.

Both C and G showed that, where they perceived some kind of similarity – i.e. where existing knowledge resources spanned sufficiently to aspects of another task or setting to be triggered in that situation - the boys were able to begin to develop understanding of concepts on both sides of the associative “bridge”. As conceptual knowledge grew, the boys learned more about the nature and the extent of the associations between different aspects and elements of related concepts, sometimes leading to conflation of ideas at first thought to be separate.

However, in cases where the boys failed to recognise any similarity between elements of their knowledge and experience in a new task or setting and existing knowledge (previously demonstrated), conceptual modification did not occur and the boys failed to make efficient progress with the task.

6.3 What is similar?

I have stated that it is not sufficient for similarities between facets of one task and another to be perceptible to an observer; that it is essential for similarity to be perceived by the learner if conceptual change is to occur. Moreover, I have been able to provide evidence from my findings that, on numerous occasions, the boys did not appear to recognise structural similarities between tasks, even where they had previously demonstrated some proficiency with a concept in a previous task.

A view of mathematics learning set out in “Chapter 2: Literature Review”, holds that, for transfer or application of knowledge to occur, some similarity in the structure of a problem is recognised and matched to knowledge of problems with similar structural characteristics. I argued that this view does not explain why children who have been exposed to particular structures fail to recognise them in new situations.

I find an alternative stance more credible; that recognition of similarity does not rely on a focus on elements that are external to the learner such as structural characteristics – that perception of similarity is necessarily subjective and for associations to be developed there must be connections with elements of experience and knowledge that are internal to the learner and that are recognised by the learner, though not necessarily consciously.

Wagner (2006) found that his subject's re-use of knowledge was sensitive to different elements of the problem or task. These were:

- Problem type: the problem can be "distinguished by legitimate mathematics descriptors";
- Problem aspect; "any detail of a problem or problem situation that can be a focus of attention";
- Problem context: "the cover story in which the problem is embedded". (p13)

He found that his subjects' use of resources varied across a range of problem types, problem aspects and problem contexts. He illuminates the development of his subject's understanding of concepts through descriptions of his subject's interpretations of, and effectiveness with, a growing range of problem types, aspects and contexts.

Previous analyses, presented in "Chapter 5; Analysis of Findings" set out my inferences about the connections and associations that C and G perceived and developed. It is interesting at this point in my discussion to consider, from the analysis grids, which facets of a problem or task were associated with success or failure for both boys – i.e. whether it was problem type, aspect or context (as defined by Wagner 2006).

6.3.1 Effective re-use of knowledge

- C Row 3-4: C's ability to make judgments about "how hot" depend on his knowledge about measures of temperature – i.e. sensitivity to problem type/problem aspect;
- C Rows 25 & 27 & 31: C had successfully aligned 2 concepts, using problem aspect (temperature change and direction of that change) to support this alignment;
- C Row 36: C co-ordinates a wider range of conceptual resources effectively, focusing on a different problem aspect – i.e. understanding that colder temperatures are associated with more clothes;

- C Rows 39 and 51: C's concept of negative numbers is developing; he can now order negative numbers in relation to zero – i.e. problem type;
- G Row 12: G attempts to co-ordinate concepts that contain conflicting ideas. He uses the problem context to support him in making decisions about how to resolve the conflict;
- G Row 15: The setting of the "Journey" task (i.e. problem context) appears to be at the front of G's mind as he confidently solves problems in the next ("Cards") task. There is a similarity between an aspect of the 2 tasks in that they both involved the names of countries linked with numbers;
- G Row 22: G's decision to use a number line model to solve problems with the ordered cards is likely to have been triggered by the problem type;
- G Row 30: Within the same context, a change in the structure of the problem does not cause G any difficulty – he recognises similarity in problem type;
- G Row 63: In a new context, G recognises similarities in problem type.

6.3.2 Knowledge not re-used effectively

- C Row 81: When working on the "Balloons" task, C fails to recognise relevance of number lines that he had used proficiently in earlier work together – i.e. problem context;
- G Row 67: G failed to recognise a vertical line as a prompt to draw a number line when the problem context changed significantly.

In summary, I do not find these reflections on the boys' effective and ineffective re-use of knowledge in relation to problem type, aspect or context to be very enlightening. However, there is one pattern that is perhaps worthy of note – i.e. that, where type is similar and context is the same, C was able to form associations between resources but that, where context is different across tasks (even where type is the same) C's resources did not span effectively. Therefore, C's ability to transfer across contexts was very limited.

Also, it is important to note that there were several instances (described previously) where C's failure to re-use knowledge does not seem to relate to problem type, aspect or context.

6.4 Beginning to learn about negative numbers

6.4.1 What do children do/say that suggests or illuminates the trajectory of growth of their specific conceptual knowledge?

The main focus for my study is the growth of knowledge about a new concept, specifically about negative numbers. As described in "Chapter 4: Methodology" the tasks were designed to facilitate the growth of knowledge about negative numbers, along a learning pathway that my own knowledge led me to believe is logical and progressive.

For each of the boys I have re-visited the analysis grid and have constructed Tables 4a and 4b shown below. These tables set out the learning trajectories for each boy.

Session / task	Row	What does C do or say?/What can be inferred about C's thinking?
Session 1: "Journey"	35	All minus numbers are same as zero
	36	Compares $^{-}6$ with zero using FC clothes
	37	Compares $^{-}6$ with $^{-}4$ using FC clothes
	38	Thinks $^{-}3$ less than $^{-}4$
	39	Thinks closer to zero means warmer
Session 2: "Cards" and "Quiz"	44	Thinks "zero is lowest number". Does not discriminate between signed and unsigned numbers
	47	Begins to see digit as indicator of distance from zero – able to order "minus numbers"
	50	Separates signed and unsigned numbers
	53	C does know what is between 3 and 5
	54	C does not know what is between $^{-}6$ and $^{-}8$
	57	Realises $^{-}6$ and $^{-}8$ are "minus numbers". Still doesn't know what is between the 2 values
	59	Faulty counting strategy
Session 3: "Quiz" including use of thermometer	61 & 66	Thinks $^{-}6$ might be the answer to 12 lower than 12
	67	Knows that "go down" same as "minus"
	68	Thinks zero might be the answer to 1° warmer than $^{-}4^{o}$
	69	Knows $^{-}3$ is warmer than $^{-}4$ because it's closer to zero
	70	Knows "colder" same as "take away"
	71-74	Very confused about minus sign and "Celsius"
	75	Knows "warmer" same as "add to"
	76-77	Records temperature changes symbolically (correctly)
Session 4: "Balloons"	79	Spontaneously uses number line though uses faulty counting strategy
	80	Fails to "read" minus sign
	81	Thinks zero is lowest number
	84	Knows "minus" same as "go below, take away"
	85	Recognises similarity with temperature tasks
	86-91	Consistently effective within the game, using commutative law and elementary calculation strategies

Table 4a – Brief summary of selected events in C's learning trajectory (full account in Chapter 5)

Table 4a sets out how C's knowledge of an extended number system, including negative numbers, grew steadily during "Journey". In the next session, as he started work with "Cards" and then "Quiz", C's knowledge about the extended number system faltered at first but soon recovered, slow to recognise the relevance of learning from Session 1. Also, a poor understanding about "between" hampered his effectiveness, as did faulty counting strategies. Upon returning to “Quiz” in Session 3, C’s extended number system knowledge was, again, very weak, never recovering to the level he had demonstrated in Session 1. In Session 3 he was able to show growing understanding of relationships and connections between some basic key concepts, though he was working at a generally low level of difficulty. He also showed that he was effective in a symbolic environment.

In the “Balloons” task in Session 4, C again started at a very low level of understanding. At one point he experienced a moment of recognition that adding the numbers on the balloons was similar in some way to earlier work with temperatures and Father Christmas. After this, Table 4a shows a marked increase in his knowledge in this area.

C’s trajectory was, therefore, rather erratic, showing a series of developments and relapses in his knowledge. Although the links across the tasks, in terms of concepts involved, might be clear or obvious to an onlooker, they were not clear to C. When he did perceive such connections, as he did towards the end of “Balloons”, he was able to quickly improve his effectiveness in the task, displaying increasingly secure knowledge

G’s development and re-use of knowledge is described in Table 4b. G’s knowledge and effectiveness in the same conceptual areas, far outstripped C’s. It is interesting to note that there is evidence that G had, available to him, a range of relevant knowledge resources, and that those resources became enhanced and connected, from the earliest stages of our work together; C’s fledgling knowledge about an extended number system was not discernible until a much later stage. C’s resources did not achieve effectiveness beyond some basic tasks and problems. G, on the other hand, was able to move his learning forward, not only developing understanding of an extended number system but also showing that he could function and operate and reason mathematically within it. There were occasions where G was forced to confront some tension or misconception in his knowledge but, where these occurred, he quickly regained any lost ground and moved to a new level.

Session/ task	Row	What does G do or say?/What can be inferred about G’s thinking?
Session 1: “Journey”	9	Already aware of “minus numbers”
	10	G learns largely by observing the other boys about ordering negative numbers
	12	Sense making, co-ordination of related concepts (higher/lower, increase/decrease, warmer/colder)
	13	Further evidence of effective co-ordination
	14	“Clonk” (number line) strategy to bridge zero
Session 2: “Cards” and “Quiz”, including use of thermometer	15	Negative numbers “read” efficiently
	20	Able to describe direction of change and difference between 2 negative numbers
	21	Loses confidence in counting strategies temporarily. Conflict not resolved
	25	Counting strategy challenged again – still no resolution
	26	Sees what he was doing wrong
	30	Successful on question with slightly different structure
	32	Sees connection between “Minus” and “down”
	33	This connection reinforced. Knows temperature rise makes positive (unsigned) change
	34	Asks for more examples (reinforcement) to give him more confidence
	35	Adopts “gap” explanation of difference.
	37	Can record correctly using symbols
	39	Acknowledges that he is adding a minus number, though cannot accept that minus minus is possible
	40	Loses confidence in his understanding of “add minus” just learned
	43	Able to keep up with my “minus minus” explanation (“undoing”)
	45	Wants more practice
Session 3: “Balloons”	48	Effective when adding 5 numbers
	49-51	Largely effective but retains focus on digits rather than sign, leading to wrong answer
	53	Confused again about whether “minus” and “plus” numbers can be added together
	55	Reverts to thinking “You can’t add a minus”
	59	Knows that subtracting from a minus number gives solution “ with bigger digits and you’re on the minus side”
	63	Describes a compensation strategy for $^{-}5 + 4$
	65	Co-ordinates multiple concepts: difference, temperature, “Journey” task, calculations/operations
	71-2	Achieves understanding (again) that $9 - {}^{-}8$ means “undoing” a minus 8 previously executed

Table 4b – Brief summary of selected events in G’s learning trajectory (full account in Chapter 5)

In the “Journey” task, G worked very hard to make sense of the resources that must be employed simultaneously in order to meet the challenges of the task. In this session, we saw that, following a significant development in G’s knowledge, he seemed to experience something of a relapse. It was sometimes only after further opportunities to “learn” a particular extension to his knowledge that it became more reliable. This was evident in all sessions, with different concepts.

In “Quiz”, G extended his ability to co-ordinate relevant key concepts and went on to operate in different ways using all parts of his extended number system. This ability to operate mathematically using new knowledge resources was also evident when the task changed. Any temporary regression during “Balloons” was due to a demand for a more advanced knowledge than had been developed or tested in “Quiz”, as well as to the requirement for de-contextualisation.

I should point out that the reason that an additional session was conducted for C’s group was because their rate of progress through the tasks was much slower than for G’s group.

6.4.2 What did the boys find difficult?

C:

In the early stages of our work together, C had some difficulty with remembering or understanding the objective of the “Journey” task. He also focused on trying to judge whether any given number was “a lot”, wanting to understand “how hot” each number is. He did not realise for a while that he needed to focus on comparing and ordering numbers to play the game because he was distracted by the context.

C showed that he often fails to re-use new information. There were several occasions where he appeared to have forgotten something which he had known a short time previously. (This is likely to be due to low cueing priority of new resources.)

A major difficulty for C was in his apparent lack of perception of the minus sign used to denote negative numbers. He failed to “see” the sign on numerous occasions throughout the sessions.

When his group first encounters zero in “Journey”, C was very excited. From his subsequent responses it appears that this excitement was because he thought they had achieved the objective of the task – that zero was the lowest number they would find. It is clear that C had no concept that included a world beyond zero. This belief – that “zero is the lowest number” – and C’s apparent “blindness” to the minus sign persisted into Session 2, even after he had acknowledged and worked with negative numbers in Session 1.

Another difficulty for C was with the idea of “between”: he did not understand that the numbers quoted are excluded (i.e. that neither 3 nor 6 can be “between 3 and 6”). He also made many errors because of a faulty counting process.

(It is possible that these misconceptions are linked in that C might perceive some connection: he might think that the “rules” about counting-on and counting-back - that 7 is 3 more than 4 because the start number is not counted but the end number **is** included – apply to “between” as well. If this were true, it is understandable that C thinks the values between 4 and 7 are 5,6 and 7.)

In C’s 3rd session (of 4) he showed that he had learned to compare 2 numbers and evaluate which is higher or warmer but that he was not able to count or calculate the difference between them. He did begin to experience some success with this before the end of Session 3.

C was comfortable and secure with his knowledge of “-“ as an instruction to subtract or take away, though his newly extended number system concept was more fragile.

In the main, C did not appear to conceive of negative numbers in a quantity dimension (Peled 1991). He dids have some knowledge of negative numbers in a number line dimension, though this wasnaïve and incomplete. His experience with subtraction of positive integers enabled him to conceive of negative vectors but his experience with negative points was very recent and any development of this concept was very limited.

C's concept of zero was another source of difficulty for him. He did not include it when counting in either direction, compounding his counting errors.

In general, C is very "rules-bound". He did remember some rules about numbers, counting and calculations and often used them effectively. He did, sometimes, apply such rules (i.e. "old" knowledge) in the new task setting, with some success but without the understanding that would help him to know why the rules "work" (or do not). He applied old knowledge quite blindly, as Wagner (2006) describes as characteristic of Class C transfer.

C found it difficult to add strings of numbers, getting confused and making mistakes. At the end of Session 4, it occured to C that he could "put all the adds together and all the minuses together". This was the first indication that he might perceive numbers as quantities, as well as points and vectors, and that he was intuitively exploring a neutralisation model, rather than a number line model, for operating with sets of mixed positive and negative numbers.

G:

G did not encounter any real difficulty until, when working on the "Quiz" task, he questioned his own knowledge about counting; he founds that he was unsure of whether the destination number should be included in the count. He did not resolve this question at this point; rather, he seemed to extrapolate that, if the destination number is to be included when counting on or back, then there is no logical reason why that start number should not also be included (Row (G) 21). Having voiced his uncertainty at that point in the task, G did not obviously accept or reject the idea that destination numbers, and possibly start numbers, should be included when counting on or back to find difference. In subsequent counting-on or counting-back activities, G reverted to his established knowledge (that the destination number is included but the start number is not) without remark. Again, this supports the notion of reliability priority and its role in conceptual change, transfer and knowledge.

There are other occasions during the tasks when G showed that he was aware of uncertainty and inconsistencies in his knowledge – i.e. when he was actively, even consciously, extending span and aligning concepts. An example of this occured when the boys were learning how to use the virtual thermometer and how it might help them: G has, within his resources, a sense-making mechanism “minus means you’re going down” and was, at first, comfortable in using that SMM in the new context. However, he quickly began to doubt its relevance and I speculate that this is because he found little in the new context that was similar to previous experience where “minus means you’re going down”. He did not reject the impulse to apply this knowledge to the new setting; rather, he actively sought to build his confidence in its relevance by talking about his thinking and asking for more examples to work with.

In a similar way, G grappled with the notion of “add a minus” – at first, he questioned the relevance of his existing knowledge to the new setting, perceiving some kind of similarity but seeking the reassurance of success with the new knowledge in the new setting before he had sufficient confidence in its relevance to accept and include it in his conceptual knowledge.

Soon afterwards, he questioned again whether it was appropriate to consider adding a minus, though he very quickly accepted it this time (the level of reliability priority has increased) and soon moved on to consider how to “minus minus”.

G made very good progress through the tasks, his concept of negative numbers developing very quickly. He showed that he intuitively adopted a number line model for counting and calculating differences between temperatures and, later, between numbers. He also showed, when working on “Balloons”, that his knowledge included “neutralisation” strategies, though when he did this, it was still linked to a number line image (Rows (G) 57-8). He was able to understand numbers as points and vectors (as on a number line) and as quantities that might be traded and balanced against each other.

Did the boys find the same things difficult?

The nature of the difficulties experienced by both boys was quite disparate. C was unlikely to re-use new knowledge: there were several occasions when he failed to mention or re-use knowledge that he had demonstrated he previously had available to him. On these occasions he appears to have “forgotten” what he had recently appeared to have learned; however, it is actually more likely that new resources simply were not triggered – not that they had been lost. Within his more established knowledge, he relied heavily on rules and sometimes had misconceptions and very fragile knowledge of basic mathematics - e.g. zero, between, counting.

C’s flawed existing knowledge, itself largely instrumental rather than relational, coupled with the low cueing priority of knowledge about new information and recent experience, meant that C’s learning was slow. New concepts (i.e. extension of the number system, comparing, ordering and operating with negative numbers in different contexts) developed in a very fragmentary and uneven fashion. Overall, the difficulties that C had in extending span of his resources meant that relevance was not recognised and conceptual growth was, therefore, limited and marked by “relapses” in his conceptual development.

G, on the other hand, only demonstrated difficulty where he was trying to align new knowledge with old. His own awareness of the development of new concepts contributed directly to his confidence in his knowledge as it expanded and adjusted its scope and span. It was this fluctuating confidence that was the main reason for dips in G’s effectiveness throughout the tasks.

6.4.3 Does the literature predict these trajectories for conceptual growth about negative numbers?

Peled (1991) set out a hierarchy of knowledge about negative numbers. She maintained that children's experience and representation of negative numbers is in two dimensions: a number line dimension and a quantity dimension. Peled set out 4 levels of knowledge through which children develop their knowledge in each dimension.

Number line dimension:

In the number line dimension, the growth of knowledge starts with knowing that negative numbers exist *"to the left of zero on the number line "* (Peled 1991; p 146) and understanding that their order reflects that of positive numbers. At the next level, children learn that the direction of movement along the number line when numbers are added or subtracted is the same on the extension to the number line – i.e. that addition (of an unsigned number) is movement to the right, whether operating from a positive or a negative number. At this level, children will "agree" to move to the right, through zero, if the start number is negative and the number to be added is larger. At Level 3, children learn how to deal with operations on pairs of numbers with the same sign; that just as in the positive world where adding a positive number to a positive number gives a value that is even more positive, the same happens in the negative world. In this way, children understand that adding a negative number to a negative number increase the negativity of the answer (${}^{-}3 + {}^{-}2 = {}^{-}5$) and that subtracting a negative from a negative number, the answer will be less negative than the start number (i.e. ${}^{-}3 - {}^{-}2 = {}^{-}1$). At Level 4, for the first time, children are able to perform operations with pairs and strings of numbers which have different signs, and starting from both positive or negative start numbers.

Quantity dimension:

In this dimension, numbers are *"amounts of things"* (Peled, 1991 p 148), though some things carry unfavourable, or negative, connotations; things such as debt. At Level 1 in the quantity dimension the way that numbers are ordered (that is, the interpretation of whether numbers are large or small) is different to that in the number line dimension because a large negative amount is a smaller value than a smaller negative amount which, in turn, is of smaller value than a small positive amount.

In the context of “Journey” (and therefore temperatures), the conflict between a decrease/increase comparison and the higher/lower comparison exemplifies the difference between the number line dimension and the quantity dimension. In Row (G)12 G compares $^{-}13$ and $^{-}3$. At first he believes that $^{-}3$ is lower than $^{-}13$; saying that the change is a decrease, indicating that he is thinking in the number line dimension (Level 3). However, he changes his mind, perhaps because he takes into consideration the temperatures context, and says that a change from $^{-}13$ to $^{-}3$ is an increase – this is Level 1 in the quantity dimension.

At Level 2, children are able to subtract a large number from a smaller positive number by bridging through zero and the answer is designated a deficiency by adding a ”–“ sign.

At Level 3 children can, from a negative start number, add and subtract negative quantities. Amounts need to be of the same type until Level 4 when they can be of different types, as in the number line dimension where *”the effect of the operation is determined by the operation and the sign of the second number”* (Peled 1991, p 149), regardless of whether the first number is positive or negative, or whether the sign is the same or different to the first number.

It is interesting to consider G’s and C’s development in their knowledge about negative numbers in relation to Peled’s framework. G achieved a secure Level 3 in the number line dimension and was beginning to demonstrate Level 4 knowledge. In the quantity dimension, he confidently worked with Level 2 representations, though any performance beyond this level was only evident when linked with a number line as well.

C did achieve Level 1 in the number line dimension, though this was hard-won. He does not appear to have developed his knowledge beyond this level, His knowledge did not seem to be even at Level 1 in the quantity dimension; however, in Row (C) 64, he referred to what a friend called the lowest temperature as “the highest … of the coldest” – this suggests that he does have some naïve knowledge about negative numbers in the quantity dimension. Also, at the end of the final session his comment strongly indicated a readiness for development in this dimension when he suggested putting “all the adds together and all the minuses together” (Row (C) 91).

There appears to be a significant disparity between the extent to which the boys developed their concept of negative numbers during our work together.

Bruno & Martinon (1996; 1999) develop their own framework for analysing knowledge of negative numbers. They expanded Peled’s (1991) framework to include the number line dimension as she had but they divided the quantity dimension into distinct “abstract” and “contextual” dimensions. Bruno & Martinon (1996) summarise:

> *“Pure and symbolic mathematical knowledge is found in the abstract dimension. Use of numerical knowledge in concrete situations is found in the contextual dimension. Finally, identification of numbers with points on the number line is found in the number line dimension.” (p 161)*

Considering this classification, both C and G may be seen to have held more knowledge about negative numbers in the quantity dimension than my interpretation of Peled’s framework had revealed.

My analysis of the boys’ achievements, related to Bruno & Martinon’s framework are shown in Figures 6.4(C) & 6.4(G).

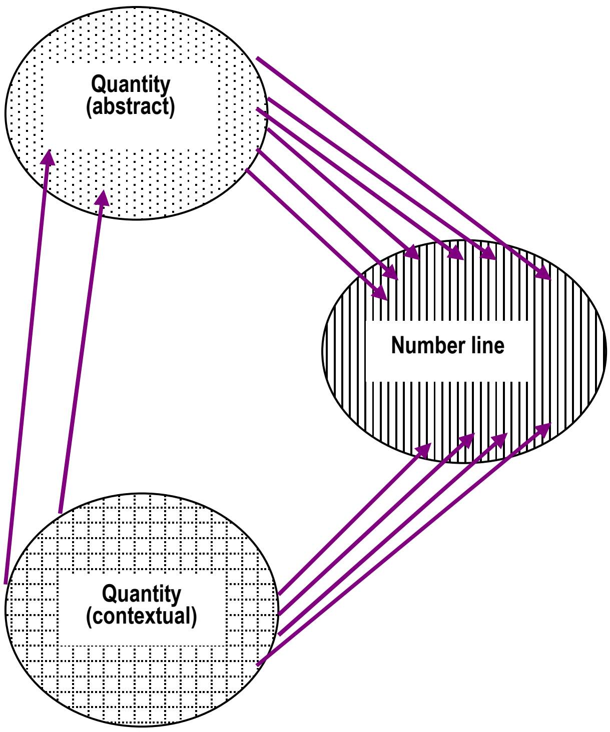


Figure 6.4(C) C's successful transferences between dimensions for negative numbers. (see Bruno & Martinon 1996)

(Each arrow represents an occurrence of transference between 2 dimensions)

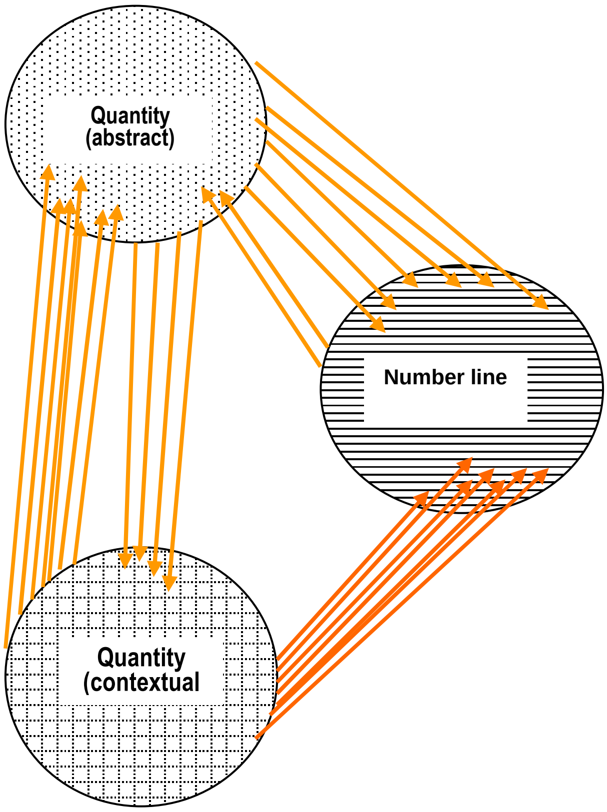


Figure 6.4(G) G's successful transferences between dimensions for negative numbers. (see Bruno & Martinon 1996)

(Each arrow represents an occurrence of transference between 2 dimensions)

Bruno & Martinon considered children's ability to translate a situation or operation presented in one dimension into the same situation or operation described in another dimension. A particular interest for them was to contrast the work of groups of more and less able children.

Bruno & Martinon found that, for all groups, transferences from abstract to number line were more difficult than from contextual to number line. This was not true for C nor G who both achieved abstract to number line transferences quite reliably at the end of our final session.

Bruno & Martinon also found that children find it difficult to transfer between abstract and contextual. Although C did not get the opportunity to translate from the abstract to the contextual, he did show that he could do the reverse; something which Bruno & Martino only observed in their most able subjects. G was successful in both directions between abstract and contextual.

Bruno & Martinon found that, amongst the less able children, translation between number line and contextual dimensions, in either direction, was much more likely to occur than any transference involving the abstract dimension. As already noted, C did achieve translation from abstract to number line. We see, then, that C's development of a concept of negative numbers was not as Bruno & Martinon's framework would have predicted since he achieved a level of transference between dimensions that only Bruno & Martinon's most able subjects achieved.

G also achieved more than Bruno & Martinon's framework would have predicted in that he did experience some success in translation from abstract to contextual, something that Bruno & Martinon's subjects found extremely difficult.

Generally, my analysis shows that Bruno & Martinon's framework under-estimates what the boys were actually able to do.

6.5 Resources

6.5.1 What internal and external resources do children use to support their work in this area? How are they used?

A range of resources were available in each session. Resources may be considered as internal or external or a combination of these. External resources include those that were intentionally provided as part of the task as well as some others used by the children, though their use had not been anticipated. External resources were:-

- Large map
- Father Christmas model
- Database of temperatures in many countries
- Animated page linked to database, showing, for each country represented, a thermometer, the country's flag, the temperature represented as a number, a picture of Father Christmas wearing clothing appropriate to that temperature
- A set of cards, each showing the name of a country and the associated temperature
- Thermometer (Interactive Teaching Program)
- Quiz questions displayed on the PC, interactive to the extent that children selected their own questions
- (non-interactive) map linked to "Quiz"
- Paper and pencils
- CD-ROM based suite of games, including "Balloons"
- Ruler
- Globe
- Researcher as facilitating observer

Internal resources include children's memories, attitudes, skills and knowledge. They include the wide range of conceptual resources described in Figure 3 and Table 1 ("Model of learning" and "Catalogue of constructs".)

It was notable that the 2 boys did not always use resources in the same way.

Both boys used knowledge gained from previous experience relating to geography and travel. At times, this resource was the basis of knowledge and ideas about the temperature in different countries, and about variations in temperature relating to a country's position on the map, in relation to the north pole, or the equator, or to another country.

C uses knowledge about Spain and about Egypt (which he has learned from books and TV). He appears to have some knowledge about Germany and about the Arctic. These internal resources all facilitate C's engagement with the "Journey" task. His "NT" concept is not well developed at the outset, though he does have some knowledge of temperature comparisons – for example, he thinks Spain is hot and that Madagascar is hotter than Kenya.

G has knowledge about France that he uses as a resource for making predictions about comparisons with Spain and UK.

C pays a lot of attention to the image of Father Christmas wearing appropriate clothes in each country. It was noted previously that C uses this resource to enable him to compare the temperatures in countries that the boys visit in the "Journey" task, showing that he is not able to compare the numbers directly and uses Father Christmas's clothes as an interpretive tool. C is amused by the images of Father Christmas, especially when he has only few clothes on. G, on the other hand, shows little interest in Father Christmas's clothes after the first 2 or 3 countries.

G's use of the cards shows that he does not pay attention, at first, to the name of the country on the card; he focuses on the number when solving the numerical aspect of the problems presented and is not distracted or interested by the country name until (or unless) he needs to look at it in order to furnish the answer to the quiz question.

An interesting resource, drawn upon heavily by C, is his respect for his friend N's opinion. There are many instances in the plain account of the work of C's group that show that C trusts what N thinks and says, more than he trusts his own knowledge.

The thermometer ITP was adopted by G as a helpful resource. He was able to use it to support him when counting or calculating temperature increases and decreases. C, however, was not able to use it effectively.

At one point, G picked up a ruler when he is struggling to understand a number line problem, as if he associates a ruler with a number line. However, he was unable to remember how to use it as a tool for working with number line problems.

An important resource for both boys was their existing knowledge about numbers: how to order them and calculate difference between them; how to add and subtract; any previous exposure to, or experience with, negative numbers.

6.5.2 Abstractions as resources

A further type of internal resource, used by both boys, is the set of any previously formed situated abstractions that they re-use in the tasks. There are many examples of evidence of these, including:-

C

- “Minus 6 means you take 6 away from 6” etc etc (Row 35);
- $^{-}3$ is less than $^{-}4$ (Row 38);
- (Last time we came) “minus was hot” (Row 41);
- “zero is the lowest number” (Row 44);
- Snow metaphor (Row 48);
- Learns that “minus means under zero” (Row 55);
- “Go down” is same as “minus” (Row 68);
- “Minus means take away” (minus as sign or operator is associated with take away by C) (Row 82 and subsequently);
- 20 - 6 - 8 is same as 20 - 8 – 6 (Row 86);
- When adding, add the highest number first (Rows 87 & 89).

G

- 21 does not match with expectation for Madagascar (Row 1);
- Countries near the equator are very hot (Row 2);
- Countries that have sand are “really hot” (Row 5);
- France is not hotter than Spain (Row 6);
- Travelling to countries close by will mean temperature difference is small and game will last longer (Row 7);
- Clonk (Row 14);
- Further away from equator on the map is colder (Row 17);
- 9 + $^{-}8$ is same as $^{-}8$ + 9 (Row 56).

In most cases, these previously formed abstractions were helpful to the boys in addressing the challenges they faced in our work together. For example, G’s previously formed abstractions generally support his development of new conceptual knowledge related to our tasks. This was not always true, however: for instance, C’s abstraction that “minus 6 means you take 6 away from 6” and that “minus 5 is 5 away from 5” led him to deduce that all “minus numbers” had the same value as each other and the same value as zero and that “zero is the lowest number”.

Another of C’s abstractions that hindered, rather than helped, him was the link he had constructed between “go down” and “minus”. He did not discriminate between the minus sign as “-“ or as “$^{-}$“. This led him to over-simplify the “rule” that he thought he had recognised, leading to difficulties such as in Row 73 in which C had seen the minus sign attached to the start number as an indication that he should “go down”, paying no attention to the operator or to the question context. (Spooner, (2002) pointed out that children’s over-application of a generalisation in this way is often the root of a misconception.)

Some of those listed above are abstractions that may have been generated within the context of our tasks. For example, it is not clear whether C’s “snow” metaphor was actually something he brought with him to the task (as he wanted me to think) or whether it was something he noticed while we were working.

There were several instances where both boys formed abstractions within our work together. As outlined in “Chapter 2: Literature Review”, Pratt & Noss (2002) believe that abstractions formed in a setting will be expressed and understood by children in language and images linked to the settings in which they are developed, and that these “situated abstractions” are internal resources for sense-making within the setting in which they are constructed.

Pratt & Noss also felt that, in new settings – even where similarities with a previous setting are perceived - children will only attempt to use situated abstractions recently generated in that previous setting after they have explored the effectiveness of longer-established knowledge and found it lacking. This is indicative of the effect of a system of priority for triggering or cueing of internal resources. Pratt & Noss believe that high cueing priority takes time to develop since effectiveness of resources can only be established through repeated reinforcement. My interpretation is that effective co-ordination of different resources is dependant on the cueing of the most appropriate resources, including (not excluding) those recently formed. Therefore, it is to be expected that the capacity to co-ordinate relevant resources in new settings develops iteratively and slowly since it is dependant on newly constructed resources achieving high cueing priority. Several examples have been noted in which recently formed resources and associations were not readily cued because other (longer-held) resources had higher cueing priority.

6.5.3 Other resources

Use of metaphors

The first task, ”Journey” introduced a metaphor to help the boys understand the task. They were already familiar with the notion that Father Christmas needs to be at the North Pole on Christmas Eve and that he travels all over the world with magical ease. They were already familiar with the type of clothes that he is normally depicted wearing – i.e. red and white suit with boots, hat and perhaps cloak. The key metaphors that were inherent within the task were: Father Christmas homeward bound on Christmas Eve, travelling through progressively colder temperatures, necessitating the addition of more clothing. Williams & Linchevski (1997) and Linchevski & Williams (1999) believe that the use of metaphors is valuable to children learning about negative numbers but stress that it is difficult to devise authentic goals and that, where authenticity is lacking the metaphor's potential to support learning is limited. The metaphors inherent in "Journey", including the goals of the activity, appear to have been sufficiently authentic, and therefore helpful, for both boys. Both of them spontaneously referred to and re-used these metaphors in the subsequent "Quiz" task, even though no images of Father Christmas, nor any suggestion of number lines or movement in a north/south orientation were present in "Quiz".

Both of the boys also showed that they had constructed their own idiosyncratic metaphors that spanned to the new situation and were therefore used in the context of our tasks. C told me about the "piles of snow" that he thought about when comparing sub-zero temperatures. I was not completely convinced that this metaphor existed previously for C and suspected that he constructed it while we were working on the task. It was, nonetheless, a valuable metaphor that helped him make sense of the world of temperatures below zero that he seemed to be discovering for the first time.

G's "clonk" (which has previously been described in Chapter 5: Analysis of Findings) was very helpful to him. More of an image than a metaphor, it supported G in moving effectively along a number line in both directions.

6.6 Affective factors

6.6.1 Mindfulness

G, on several occasions, showed that he was aware of his own thinking and sometimes voiced his confidence (or lack thereof) in particular aspects of his knowledge. He showed a mature understanding of some of the processes that help him to learn, including the need to reinforce new knowledge. He told us that he sometimes had a sense of new knowledge slipping away; that it had started to hold some meaning for him but that he was not able to re-use it without “some more practice”. This might be seen as evidence of an intuitive drive to increase cueing priority.

In Chapter 2: Literature Review, it was noted that Salomon & Perkins (1989) described 2 types of transfer – “low-road” and “high-road”. One of the key differences between these 2 types is the level of mindfulness (or conversely, automaticity) that is involved. The authors argue that low-road transfer is achieved through varied practice and that high-road transfer is achieved through mindful abstraction that “depends on conscious control and analytic awareness” (p128). It would appear that G is demonstrating mindfulness and that he will achieve high-road transfer of those concepts or resources. It could be argued that G is so analytical in his thinking that he is able to analyse processes and behaviours after they have become automatic (automaticity is achieved through low-road transfer) and achieve high-road transfer post-hoc, something that Salomon & Perkins accept is possible for some learners. However, I would argue that G’s comments about his own thinking and his learning needs are very much “in-action” and therefore not retrospective. He is, therefore, on these occasions, working towards high-road transfer. One exciting outcome for G’s learning about learning, and knowledge of himself as a learner is pointed out by Salomon & Perkins,

> *“The payoff of such activities, of course, is not just particular transfers made, but the establishment of an expectation for transfer”. (p136)*

G is likely, therefore, to continue to seek, and be confident that he will be able to find, meaning in new situations.

C did not show much awareness of his own understanding, though he did sometimes bemoan his lack of understanding, demonstrating some awareness of that. We saw that C did become able to use recently learned knowledge, though only after repeated opportunities for reinforcement. Salomon & Perkins (1989) point out, when considering low-road transfer,

> *"Transfer occurs to the extent that a new circumstance calls on a complex of procedures overlapping a complex that was previously well-exercised. Varied practice would yield more transfer by exercising a wider variety of related complexes …" .(p120)*

The need to "exercise" complexes or concepts or other internal resources was something both boys were conscious of, though G's needs revolved around his wish to analyse and generalise, as well as to be able to remember his new knowledge in the future. This conscious focus on future use of knowledge currently under development indicates a high level of mindfulness and is something that C did not demonstrate.

6.6.2 Confidence, confusion and conflict

Both boys display at least some confidence throughout our work together, with occasional lapses. There are many occasions where C shows his lack of confidence and his awareness of this. Sometimes, he is able to purposefully choose for himself external resources that he thinks will support him. In Row (C) 50 he chooses to use others in the group:

50. I ask C to describe why Netherlands (4º) is where he's put it. At first C says "Who agrees with me to put it there?" He goes on to say "Because it's under the 5. It's 4." I ask him why he didn't put it "down there" (towards the bottom of the table) because that would be under the 5 wouldn't it? But he knows it shouldn't go in the "minus section". .. "Because that would be under zero."	C is right. When I challenge his use of "under the 5" by suggesting that anywhere on the table that is not at the level of 5 or above would be a correct answer to this question, he confidently (and correctly) tells me that to put it below zero would put it in "the minus section" and that would mean that it was "under zero". This shows that C realises that $^{-}5$ is quite different to 5 – something he did not believe at the beginning of the session. From this I understand that C's readout strategies and his concepts about minus numbers, the number system and temperatures have modified.

On another occasion he thinks that the image of FC and the clothes he is wearing will help him:

26. Until now C seems to have understood the objectives very well but when they visit Ethiopia (16º) he shows the first sign of confusion. His first reaction is "Yes! 16!" But then he says worriedly, "But it's less hot. Last time he was having a coat on, wasn't he?" He revisits Kenya to check. "He's got no coat on. Blue t-shirt and shorts. Let's see, Ethio .. It's the same – blue t-shirt and .. trousers, and boots – he has got more clothes on now so that's alright."	The temperature in the last country was 19 (Kenya) and C is initially confident that a move to a country with a temperature of 16 is valid. However, he questions his judgement when he (mis-) remembers that FC had a coat on at the last stop so he "goes back" to check. He is reassured when he realises that FC has even more clothes on now than before. It appears that C's attention to FC's clothes is a readout strategy upon which he is quite dependant for giving him confidence in his decisions about appropriate journey moves.

C sometimes showed confidence where (I would suggest) it was feigned rather than real – e.g.

12. C wants to go to Spain because he says he wants to see what it is because he's been to Spain and it's "real hot". When it's C's turn he goes to Spain. The display shows 7º. C seems shocked and hesitates. I ask him if this is what he expected Spain to be. He blustered, "Yes, yes. I knew Spain was hot."	He is shocked when he finds that it is 7º (though he pretends not to be). His shock might be evidence of a conflict between his "TW" and "NT" concepts – i.e. he "reads" 7 as not hot but his resources relating to Spain have led him to expect a "real hot" number. It is not clear whether he chooses, at this point, to ignore his uncertainty, or whether (without vocalising it) he resolves it by drawing upon another conceptual resource that enables him to reason that 7º is hotter than other countries they have visited so perhaps, in comparison, 7º is "hot" after all.

However, in general, where C demonstrated or voiced confidence it was genuinely felt – e.g.

24. Kenya is 19º and when I point out that FC has got "a lot more clothes" here, C very confidently says "Yes, that's good, that's good."	C is focused on the rules of the task and is satisfied that this move is in line with aims for FC's journey and with his "NT" concept. He does not attend to my hint about "a lot more clothes", my (too subtle for C) attempt to suggest a too-big temperature difference. This reinforces the situated abstraction he has constructed, "more clothes = a move in the correct direction when aiming for a colder country".

… and ….

34. When they visit Czech Republic C is very excited about the temperature being zero, squealing with pleasure. He confidently tells me that this is less than one..	C has a resource that zero is less than one and is particularly excited about visiting a country with a temperature of 0°. C seems to attach some special importance to zero – perhaps it is simply that he is especially confident about his conceptual resources relating to zero and it is this confidence that excites him.

Where C shows well-founded confidence, he also clearly shows that he takes pleasure in his confidence.

Sometimes, C's confidence wavers (e.g. Row (C) 77). When this happens, he seems to want to recover some level of confidence as quickly as possible:

77. We re-open the thermometer and set it at $^{-}2$. N needs to add 30. He goes to 30 (i.e. adds 32). When he corrects himself, C thinks he has made a mistake. I recap and confirm and ask how we would write this one down.. C writes $^{-}2 + 30 = 28$. I ask N to perform the change on the thermometer again. As he moves it through zero, C says "There, that's 2 warmer so it's going to be another 28."	This is another opportunity for C to reinforce the resources he has successfully employed with this sort of problem in Rows 75 and 76. N's error causes C to question his own knowledge but he perseveres and regains confidence in his (correct) answer. It is clear here that C is "bridging" effectively through zero, using conceptual resources that relate numbers to each other (i.e. "knowledge" of number bonds").

He seeks any explanation for what he "sees" and quickly adopts new ideas without any attempt to explore and understand links with other existing resources. C appears to want to find meaning in what he does but, in his unquestioning and superficial acceptance of new ideas, he fails to construct associations and therefore span across resources.

G, on the other hand, is not so easily convinced by new ideas; he needs and demands to be able to reinforce and consolidate any connections he discovers between new and existing knowledge.

His confidence is evident in the way he responds to challenges and in the way he helps his peers such as in Row (G) 24 :

24. A later question is "Find a country where the temperature is between 1 and ⁻1" The boys agree immediately that there are a lot to choose from on the table. M wants to answer Romania. G argues, explaining that he can't have 1, it has to be less than one.	G is very confident in his ability to interpret the precise meaning of the question and effectively co-ordinates relevant conceptual resources including those relating to: Negative numbers "NT" Counting, comparing and calculating with numbers Knowledge about negative numbers as used to represent temperatures Between 2 values

Manifestations of G's confidence, or the lack of it, are linked with his capacity for mindfulness and his desire to understand.

27. L uses cards and gets muddled counting up from one to the other (Slovakia and Albania). M thinks the difference is 8 and G says its 7. M argues and G argues back, reminding him that last time they found out that they shouldn't count the first place.	G is now more confident again. He is aware that his confidence has come from the previous example in which he was able to see for himself that the start number should not be included in a count procedure. Co-ordination of relevant conceptual resources is secure. G's RS and SMMs relating to the thermometer scale are secure.

Also, in Row (G) 61, G's behaviour might be interpreted as suggesting that he feels that he does not expect his available resources to help with the problem in hand – that he acknowledges the limits of his resources:

61. I redirect G and the other boys to my earlier question : 9 - ⁻8. They keep saying it aloud "Nine minus minus 8". M says "minus, minus" several times.G says "I know, it's hard – minus minus." They are not at all confident about this one.	G doesn't attempt to change the order of the numbers . It is possible that this strategy is cued but that it doesn't help with this problem. G doesn't appear to remember how he had succeeded with "minus minus" previously – i.e. by using an "undoing" strategy. Perhaps cueing of the commutative strategy is blocking cueing of the undoing strategy because it's recent effectiveness has earned it high reliability priority (for now, at least)?

In Row (G) 73, he shows further awareness of his thinking when he tells us that he visualises a number line:

73. I give the boys several more examples for them to work out on the number line and they quickly abandon the number line and are able to do them mentally. G says he doesn't need to draw a number line because he can "see one in my head".	G and the others only draw number lines until they feel confident to work mentally. G does not abandon use of his "Number line" resource, however; he simply uses a mental representation rather than a concrete diagrammatic one.

G's confidence often appears to be linked with changing cueing priorities. For example, in Rows (G) 15 and (G) 26:

15. When countries with temperatures below zero appear from the pack, the boys take them in their stride, except to notice that Turkey's temperature is not what they expected. G reminds them that "This is around Christmas, though".	Negative numbers are read efficiently. G's conceptual resources include a SMM that enables him to reason that Christmas temperatures are lower than might normally be associated with countries that we visit for holidays in the summer. Whereas during the previous activity, G was uncertain about ordering negative numbers, today he was more confident – his RS, resources in memory and SMMs have modified and he now works effectively with negative numbers and is able to order them. He did not make mistakes. Span and alignment of resources has increased, linking resources that G has relating to numbers.

26. I decide to use the thermometer ITP to model a number line method for working on the Belarus to Belgium question. G is able to position the starter pointer on $^{-}3$ without any difficulty. He can also help M to move it to 3. When I show that the change has, therefore, been 6 not 7 as they had answered previously, G quickly says "That's because we counted that one, Belarus". He is quick to accept their mistake and to see why it arose. (Maybe he had doubted their answer in the first place?)	G is quick to learn how to move the interactive display on the screen thermometer and is confident to help his friend. The highest reliability priority for his counting (include "end" number but not "start" number) strategy is re-established when the whole range of his conceptual resources relating to it are tested and their alignment reinforced. Associations between these resources and those relating to negative numbers are also tested and reinforced. G's "NT" concept is strongly connected to his evolving negative numbers concept.

The basis for G's confidence is his already robust conceptual knowledge of relevant areas. When G encounters ideas and challenges that require him to extend and deepen his knowledge, his confidence level drops. G ultimately shows that his conceptual resources are sufficient for him to construct new connections (therefore extending span) and to develop understanding of the nature of those connections (improve alignment). However, it is not until he has achieved this that he voices any confidence in his knowledge. Even then, he actively seeks opportunities to reinforce new knowledge before he expresses his own confidence in it.

In Rows (G) 20-21 we see an example of G's lack of confidence as it coincides with the point at which he recognises misalignment or non-alignment of resources:

20. Question – "If you travel from Russia to Sweden, what will happen to the temperature? G finds the Russia card on the table (-6) and counts up to the Sweden temperature (-2) and says "It will go 4 degrees higher. 4 degrees higher, or 3, I don't know which." G is confused about whether he should count Sweden itself in the count.	G is able to "read" negative numbers. His uncertainty is uncharacteristic of the way he works with difference problems. When working with positive (unsigned) numbers he works them out efficiently and effectively. G's conceptual resources include a set of associated resources that he employs when counting comparing and calculating with numbers. (These might be seen as a concept "Counting, comparing and calculating ("CCC")" within a broader concept about numbers.) These "CCC" resources do not extend to resources relating to negative numbers, though he has already shown that the span of his number system concept and associated RS does include some associations with negative numbers. These 2 concepts, "CCC" and "Negative numbers" are not aligned and this is the reason that G finds himself forced to question something that he is surprised to find he is not confident about after all. Previously he has coped well with tasks involving ordering and comparing greater/smaller, higher/ lower. So this marks a point of departure for G in that he appears to be working just beyond the scope of resources with which he is confident and secure.
21. Once he has decided that he should have "counted" Sweden, he also begins to think that he should have counted Russia at the beginning of the count so is confused again. (Something he was confident with previously has now been called into question).	There is something in G's conceptual resources that makes him think that rules should be consistently applied . This forces him to question a strategy with which he had previously been confident that had been a successful part of well established concepts about numbers and counting – i.e. he begins to think that he should count the start number as his first count when counting to another number. G does not resolve this uncertainty at this point and is content to move on, letting someone else take the lead for a short time. G's in-the-moment reasoning might have led to construction of a new SMM, (that, since the "end" number is counted, the "start" number should be counted too). However, this was not succesfully aligned with other resources. It is interesting to note that this question does not present G with a similar dilemma every time he is confronted by (what I recognise as) similar challenges later in the task. It appears that G's more established resources, particularly his counting strategies have high priority and are therefore readily cued in (at least most) subsequent, similar challenges. To achieve high priority, the new resource would need to provoke feedback that shows that the resource has explanatory value in this situation. This feedback is not generated here.

The ways in which G and C deal with such crises in their learning trajectories are generally quite different. C's lack of confidence (perhaps itself linked to poorly developed cueing priority for new resources) often leads him to doubt his own ideas and seek the reassurance or advice of others, especially N, as in Rows (C) 7 and (C) 28:

7. He also shows, when he says in minute 10, "That's not a lot!", that the value for temperature in Germany (1 degree) is not one he associates with warmth; he is surprised. When no-one responds to his "That's not a lot!" comment he perseveres and tries to resolve his uncertainty and asks N "Is that a lot?" When he gets the response "No", C is satisfied and doesn't pursue the question any further.	C expects Germany to be hot, based on some inference or association that is not clear. The value that is displayed is not one that C associates with high temperatures and, though there is some tension/conflict between the 2 concepts, C initially trusts his knowledge about numbers used to represent temperature values. However, it would seem that he is not completely confident and in the absence of reassurance from others, C checks with his friend N, whose judgement he trusts. N confirms what C had thought.

28. C mumbles something about, "Do we want it warmer?" No-one hears or replies. When the display shows 6° C says "Ahh…" and it is only when N starts to say "No" that C joins in with "No!"	C is beginning to lose confidence in his own ability to make sense of and resolve problems within the task because the others do not appear to share his disappointment with the hot countries they have been visiting. He questions whether he has perhaps understood it the wrong way round when he asks "Do we want it warmer?". They do not reply and C, at first, feels relieved when a country that is 6° appears on the computer screen. He feels pleased "Aahh .." that they have found a lower temperature than 16. However, N says that this is not a valid move, C quickly agrees with him, even though this is in conflict with C's own sense-making. He trust N's knowledge more than his own.

Where external resources, including his friends, do not provide explanations and solutions, C does not persevere in seeking to resolve the problem using only his own internal resources:

11. When they click on Madagascar and see that it is 21° they are surprised. C says "Oh my God! This must be playing tricks .. because that's (Madagascar) hotter than that (Kenya).	2 conflicts are evident within C's conceptual resources: He has said that he thinks Madagascar is not hot – and yet he finds that Madagascar is 21 and 21 is is number that he associates with "hot". Within his "NT" concept C has yet to develop a system of graduation between hot and cold, as well as alighnment between this and the numbers themselves. For C, Madagascar is hotter (higher number) than Kenya, but also, Madagascar is further from Equator than Kenya and further away from equator means less hot. These conflicts are not resolved.

In Row (C) 32, it could be interpreted that the UK / Umerica issue effectively rescued C since it provided a distraction from the conflict within his resources that had become evident to him (though not necessarily consciously):

32. C states that they will go to UK next, followed by Poland. He says that Poland is cold. UK is 6º. C quickly says "Now we go to Poland". C is confused at this point. He looks at the flag moving within the display about UK and says "Wait a minute, something's wrong. We're meant to be going to U .. Merica and that's British." He has got mixed up with United States.	When the UK temperature is first displayed, C does not seem to realise that, if the previous country was 5º, it is not appropriate to go next to UK, 6º. He simply accepts the inclusion of UK as a way-point on the journey and thinks they should now go to Poland (which he believes is cold) as he had intended. Very soon, however, he queries whether the display they see is actually the correct one for UK. This could be because a situated abstraction that he has constructed, that "movement north = lower temperatures" is conflicting with his concepts of "TW" and UK. This may have led him to think that UK should be less than Turkmenistan. C gets confused between USA and UK, itself evidence of an association between them, perhaps because the beginning of the 2 names is the same.

G, on the other hand, usually confronts conflict where he finds it. This was a significant feature of G's learning trajectory described in Table 4b. Rows (G) 53 & (G) 55 exemplify G's determination to find meaning in a situation where, at first, there was incongruity and tension:

53. M says "9 add $^{-}8$, which is 1". G says "You can't do that". No-one follows up on his remark until after they have entered the wrong answer to the question and start to think it through again. At this point G asks "Can you add minus numbers and plus numbers?" Interesting that he did not realise that this is what they were doing with earlier questions. Maybe there is something about this question that emphasises to him that this is what is happening. I think, though, that it was M's articulating "9 add $^{-}8$.." that has triggered something in G's understanding. My suspicion is confirmed when he repeats to himself and looks at M, "9 ADD MINUS ..??" M explains that he had written the numbers down and then went back and put all the add signs in because they need to add all the numbers every time.	M talks aloud as he writes down what they need to do. As he speaks the numbers, including the "minus" sign, he includes the word "add" as he recognises that the task is to add the numbers together. G finds that this conflicts with the interpretation that his own conceptual resources facilitate – he believes M has made a mistake. No-one else acknowledges his remark. When the boys go over the problem again G questions whether it is appropriate to add "minus numbers" and "plus numbers". He has not referred to the unsigned numbers as "plus numbers" before (though he did refer to them as "adds"). So perhaps, we are seeing his concept of "minus numbers" expanding to include "the other numbers" to become a more inclusive concept about signed or directed numbers? If G's knowledge is extending in this way, I think it is almost coincidental as his conscious focus is the concatenation of 2 words, both of which he previously understood as instructions to carry out an operation on a number: + (add or plus) and – (minus). It is his mention of "add minus" and his discomfort when M said the same thing previously that leads me to believe this is the cause of G's anxiety at this point. It would seem that G's evolving "Minus numbers" or "Signed numbers" concept is undergoing expansion to include resources that conflict with elements from other collections of conceptual resources that he was beginning to align. In his previous experience an instruction (a sign) has always been followed by a number, not another sign. G's discomfort, his difficulty in aligning related conceptual resources, is evident here.

55. When M tries to explain the way he works through the list, he says 9 add minus 8 and G interrupts "You can't say that. You can't add a minus". He is still resistant to this notion within his own understanding of what he is doing. G believes that the add sign is superfluous as he says "9 minus 8 is already there". He doesn't see any need for the add sign and thinks it confuses the question.	G is still very uncomfortable with add and minus being spoken in tandem. He feels that this doesn't make any sense. He appears to "read" the minus prefix as the indicator that he needs that tells him which way to move on the number line. He clings onto his belief that the "add" word cannot occur immediately preceding the "minus" word (cueing priority is still very high). His developing concepts ("Minus numbers" and "Calculations") contain conflicting resources, they are not yet aligned in this respect, even though span of each does extend to the other.

Having considered mindfulness, confidence, confusion and conflict, it is clear to me that these are all valuable resources for learning. G showed that his progress was, in no small part, due to affective resources. His readout strategies, resources in memory and other resources, including sense-making mechanisms, co-evolved and we saw that G possessed an "expectation for transfer" (Salomon & Perkins 1989, p 136). This dialectical relationship between engagement with confusion, confrontation of conflict, expectation for transfer and confidence to persevere was apparent in G's achievements during our work together. C did not demonstrate these features and his learning did not progress in the same way. Perhaps, for C, until high cueing priority for new resources is established through repeated reinforcement of new associations, it is unreasonable to expect that sense-making mechanisms and readout strategies can evolve. Perhaps learning opportunities for C must take into account his need to develop affective and attitudinal behaviours so that he has confidence that he is capable of finding meaning where he does not at first see it. It is, after all, only through success that C can change his attitudes and expectations of himself and his confidence in his ability to learn and his regard of himself.

Chapter 7: Conclusion

7.1 Overview – the big picture

Research questions developed from my review of the literature and my professional experience were:

- What resources shape the nature of transfer and the growth of knowledge about negative numbers?
- What is the role of the interplay of resources in the micro-transfer of knowledge about negative numbers?
- What is the relationship between abstracting and transferring knowledge about negative numbers?

These questions are quite broad and it was not possible to analyse the "big picture" without breaking it down into more focused elements. I shall now reflect on what I have learned about different aspects of learning and hope, ultimately, to draw together my understanding of the "parts" to form a better understanding of broader issues that make up the big picture.

7.2 Re-use of knowledge

7.2.1 Abstraction and transfer

My observations and analysis of the 2 boys' achievements during our work together revealed that there were many different types of knowledge that were re-used as they started to work in the new domain of negative numbers. Both boys used a range of knowledge resources that were well-established (i.e. "old" knowledge) including direct primary experience and learning as well as learning from secondary sources such as television, friends and family. "Old" knowledge sometimes provided background understanding that helped to make sense of aspects of the new domain, such as C's knowledge about Egypt and Spain as it helped him to begin to construct a referential framework to solve problems in "Journey". At other times, "old" knowledge was more directly useful – for example, G used his knowledge of (what I know as) the commutative law to help him solve problems with negative numbers in "Balloons". (Perhaps it is important to remind ourselves, here, that G's perspective on what I know as the commutative law might be somewhat situated.)

Different types of transfer were evident for both boys in all tasks. Using diSessa & Wagner's (2005) taxonomy, it was possible to identify some Class A transfer (i.e. of well-prepared knowledge in a new setting), even though I had not set out to look for it. Knowledge about numbers, temperatures and calculation strategies are examples of well-prepared knowledge that was effectively re-used in new settings, including in the new domain of negative numbers. There were also many examples of Class B (developing towards A) and Class C (tentative application, sometimes blindly, of ideas and resources that are considered potentially relevant) transfer for both boys.

However, the 2 boys demonstrated disparate patterns of types of transfer. C's Class A was extremely limited, whereas G successfully implemented well-prepared conceptual resources, some of them quite recently constructed, in new problem settings.

Recently constructed knowledge resources were, however, only rarely transferred in a Class A manner. In "Chapter 6: Discussion of Findings" I considered the notion of cueing priority and argued that Class A transfer of new knowledge is not possible until new resources have been reinforced through reiteration of their relevance in appropriate situations. It is only when new resources have been successfully re-used in a range of settings that span and alignment of those resources can extend and improve. G was able to reinforce his new knowledge and expand the contextual neighbourhood of those resources, enabling re-use of them, even within the few sessions that we had together. C, on the other hand, was not able to do this and demonstrated a high rate of failure to transfer, compared to G. I argued that this was because cueing priority for new knowledge was not reinforced for (or by) C.

In order for the boys to recognise situations in which old knowledge is relevant, they must have available to them resources in the new situation and in the old knowledge (the concept in its current state of construction) that are associated with each other. Such associations are constructed by the learner in response to experience. We saw that, in general, G was able to perceive such associations but C was not. It is unhelpful to glibly attribute such differential achievements in terms of re-use of knowledge to “ability differences”. It is, I believe, imperative that we understand more about what it is that G did, in other aspects of his learning, that C did not, that might account for such a disparity.

It was evident, then, that the triggering or cueing of conceptual resources to facilitate sense making and problem solving in new settings is dependant on the recognition of relevance of those resources; this recognition is prompted by the learner’s perception of something in the problem or setting that resonates with available resources. This is my summary of the processes involved in “knowing-to” (Mason 2002).

7.2.2 G’s development of his concept of negative numbers

It is clear to me that learning, for G and C, required changes and developments within each boy’s contextual neighbourhood. I attempt to describe and represent changes in G’s contextual neighbourhood (Figures 7 a-c) that are suggested by his actions and utterances during the tasks. (I chose not to present C’s conceptual changes in this way as G’s experience and development is most likely to provide a sufficiently rich, “thick” description to illuminate relevant issues.)

It is important to note that there are limitations of presenting, diagrammatically, processes that are, after all, extremely complex:

- firstly, the concept labels (dark grey boxes) are something that I impose – i.e. the learner is not aware of “a concept of ..” at the level of their own sense-making. As shown previously, in Table 1, I consider a concept as an aggregation of associated resources; I believe that it continues to change and grow in response to an individual's experience. I do not conceive of a concept as something that is a static collection – it is not fixed in time nor location. Dark grey boxes in Figures 7a-c are representations of, what I perceive as, themes that are very richly connected through a dense web of associations;

- secondly, the representation of conceptual resources as boxes that are shaded the same shade of light grey does not imply that all conceptual resources represented are at the same level of abstractness/situatedness – i.e. the same grain size. For different grain sizes, we might imagine considering a range of conceptual resources through a magnifying lens:
    - at a high magnification, a number of resources that are highly context-bound and with few or no associations with other resources would be visible, though only small sections of the entire conceptual web would be visible in any one view;
    - as some of these resources become increasingly connected, those more densely-connected parts of the conceptual web, would appear as patches (when viewed at a lower magnification); at this lower level of magnification, and as connectedness between associations increases, becoming intense in some areas, some level of abstraction of commonalities occurs; relationships between these patches (notions at an interim level of abstractness) are only visible as I, the viewer, "zoom-out";
- thirdly, of course, there are many concepts, associations and other conceptual resources that are not represented on any of the diagrams since it would be impossible to acknowledge, exhaustively, all associations that might be involved. It is important

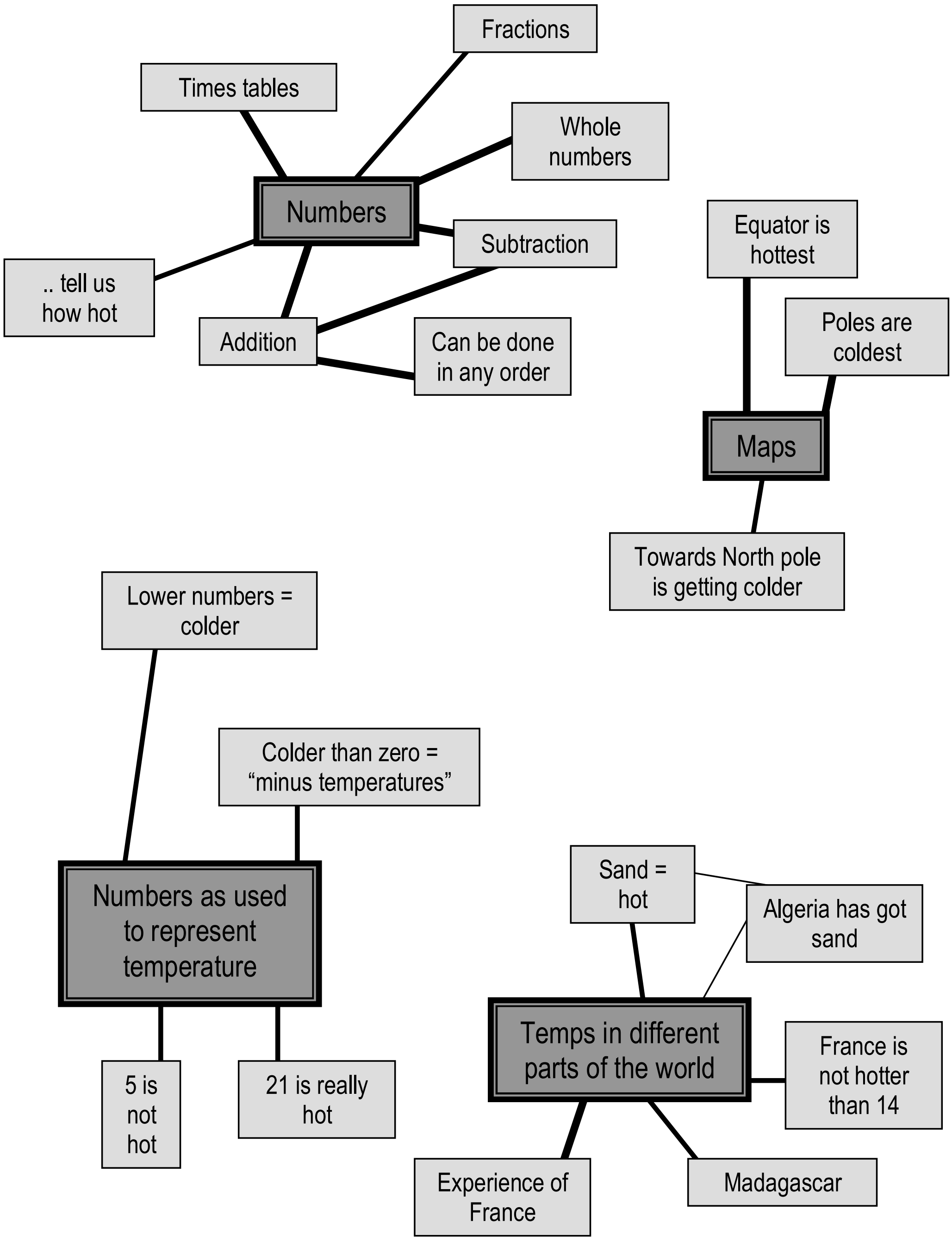
to realise that there are many unseen resources that might play a part in G's development of a concept of negative numbers.

To accurately and comprehensively represent G's conceptual resources and the existence and strength of associations between them, at any one level of magnification, would not be possible. However, I strongly believe that some diagrammatic representation of the micro-evolution of knowledge, for the sake of my thesis, is necessary. I have therefore combined elements from different levels of magnification in my representation of conceptual resources (light grey boxes) in Figures 7a-c.

Figures 7a-c are now described;

- In Figure 7a, (overleaf) we see that, as we began our sessions together, G had already formed collections of associated resources relating to "Numbers", "Maps", "Numbers used to represent temperatures" and "Temperatures in different parts of the world". Some associations between resources were stronger than between others;
- Figure 7b shows the process of webbing - new resources are being added and associations are formed and strengthened; the result of experience of working on the tasks. In the main, new and old resources seem to complement each other – sometimes, however, new resources introduce tension or dissonance;
- In Figure 7c, after several new resources relating to negative numbers are added to G's contextual neighbourhood, and associations between them and with other resources are constructed, it seems appropriate to consider that he has now formed a concept of negative numbers that should continue to develop according to his experience.

Figure 7d reproduces Figures 7a-c on a single page, so that general changes throughout the process are rendered more visible, though details within each box are less visible.



Key: association weak; association more established; strong association; conflict

Figure 7 a
G’s contextual neighbourhood. A representation of concepts that he uses as he begins his work with “Journey”.

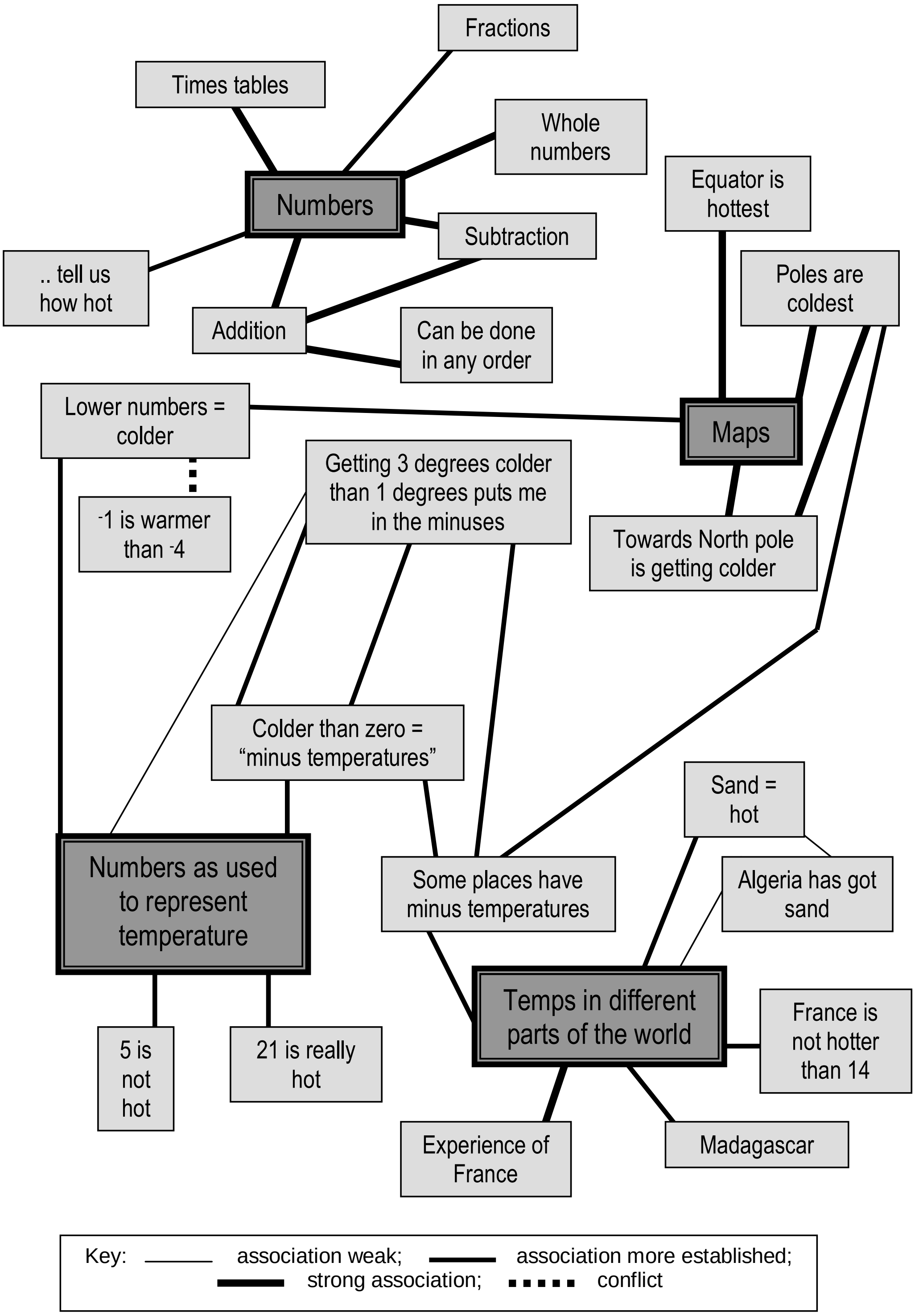


Figure 7 b
G's contextual neighbourhood. A representation of concepts that he uses in his work with "Journey". New associations have been constructed between resources. Sometimes, new associations introduce tension (see dotted line).

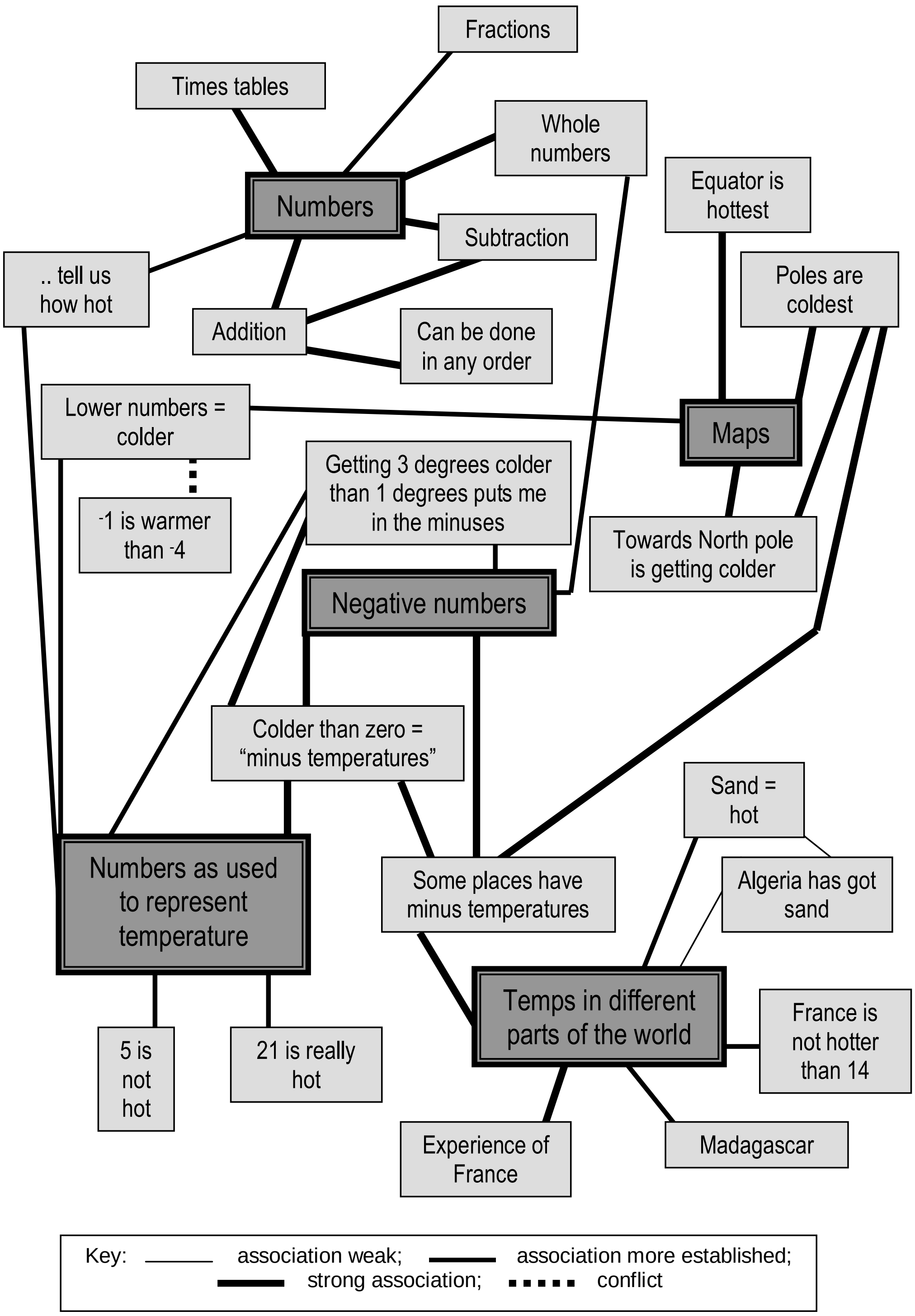


Figure 7 c
G’s contextual neighbourhood. A representation of concepts that he uses in his work with “Journey”. As there are now several associations between resources relating to negative numbers, G might now be seen to have started to construct a concept of negative numbers.

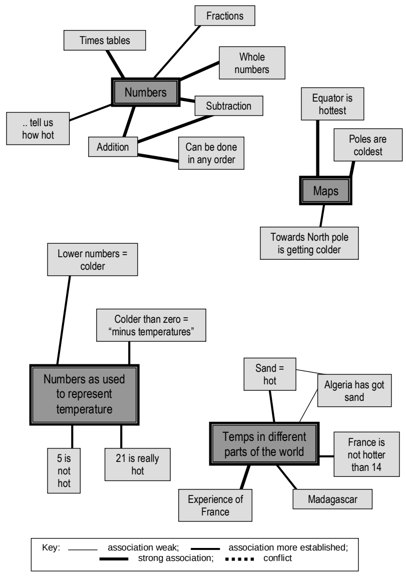


Figure 7 a
G's contextual neighbourhood. A representation of concepts that he uses as he begins his work with "Journey".

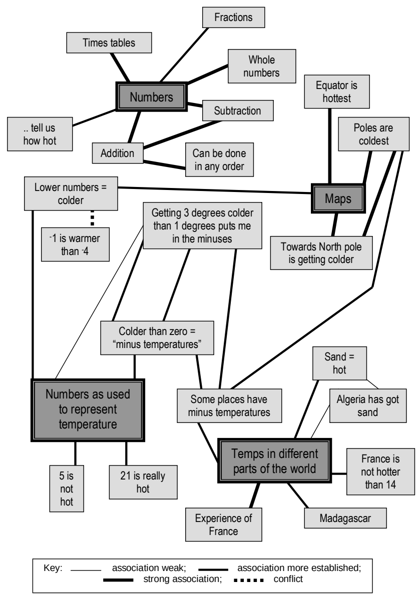


Figure 7 b
G's contextual neighbourhood. A representation of concepts that he uses in his work with "Journey". New associations have been constructed between resources. Sometimes, new associations introduce tension (see dotted line).

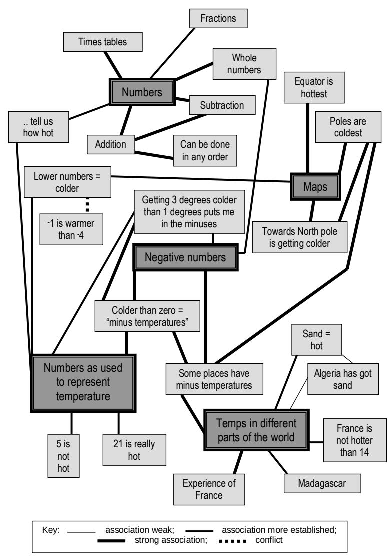


Figure 7 c
G's contextual neighbourhood. A representation of concepts that he uses in his work with "Journey". As there are now several associations between resources relating to negative numbers, G might now be seen to have started to construct a concept of negative numbers.

Figure 7d:

Overview of Figs 7a-c illustrating formation and strengthening of associations and emergence of a new concept – "Negative numbers"

Figures 7 a-c represent my impression of the micro-evolution of G's thinking-in-change about negative numbers. It is clear to me that abstraction has not been a prominent feature of this process. Therefore, the development of mathematical conceptual knowledge does not require abstraction.

7.2.3 Relationships between conceptual growth, abstraction and transfer

Having stated that, on the basis of my analysis of G's and C's achievements, I find that abstraction is not necessary for conceptual change, I should consider the nature of the relationship between abstraction and conceptual change, if one exists. I return to Figures 7 a-c and note that, in compiling these representations, I deliberately chose to record some resources using G's own words – e.g. "Getting 3 degrees colder than 1 degrees puts me in the minuses". Others were recorded in more abstract terms. I believe that abstraction, or lack of it, is implicit in the language used to articulate conceptual resources – that is, new resources that have not been abstracted are available with contextual bells and whistles still firmly attached. As these resources are reinforced and modified through other relevant experiences, the label for the concept becomes increasingly abstract. For example, G's "Getting 3 degrees colder than 1 degrees puts me in the minuses" resource became something he referred to later as "I **can** take a big number away from a smaller number – I just go minus". (This is an example of a situated abstraction (Pratt & Noss 2002)). This shows that, although it wasn't necessary for G to abstract new concepts in order to learn mathematics, abstraction may, and does, occur. Wagner (2006) argued

> *" .. abstraction was a consequence of transfer and the growth of understanding, not the cause of it." (p86)*

My findings would support this view. A complementary conclusion is that conceptual growth, as represented in Figures 7 a-c, cannot occur without transfer: transfer is substantiated in the construction of associations between resources, old and new. Furthermore, I would re-state that:

- to construct a web of associations is to extend span of applicability and relevance of resources;
- tuning of the appropriate application of those resources leads to proper alignment;
- reinforcement of the span and alignment of associated resources improves cueing and reliability priorities which then becomes self sustaining and self regulating.

All of these are constituent of conceptual development.

It is logical to argue that the disparity in the patterns of incidence of different transfer types between the 2 boys is connected to their capacity for transfer – i.e. their propensity for extension of span and improvement of alignment. This might also be related to the notion of webbing – that the extent and density of a learner’s web will predict their capacity to transfer – and therefore, their prospect of success in their early work in a new domain. I found that most of C’s transfer events were Class C and that he often needed to re-start his webbing process in this new domain of negative numbers, making slow progress. G, on the other hand, showed that he could construct associations between knowledge resources old and new, at each stage extending span and improving alignment. This enabled rapid and effective progress through the tasks.

7.3 Resources

My model of learning presented in “Chapter 5: Analysis of Findings” depicts conceptual development through the interpretation of inputs leading to modification of conceptual resources, including sense-making mechanisms.

I have already mentioned that G retained references to contextual details in his construction of conceptual resources. There are many examples in the analysis grids of both boys doing this. Even when some sense-making mechanisms and other resources had become abstract, the contextual details in which they had initially been embedded were still accessible. An example of this was when C often referred to Father Christmas's clothes to justify, to others, his judgements about temperature comparisons.

We saw that C was sometimes able to use pieces of old knowledge (i.e. internal resources) to provide a framework for evaluating temperatures. He actively constructed sense-making mechanisms during the tasks that helped him interpret what he found – i.e. he learned to co-ordinate different resources in order to make decisions and to have confidence in them. In this way, C used this kind of internal resource to develop interpretive knowledge. However, although there was evidence of the extension and improvement of span and alignment within "Journey" and "Quiz", C did not "know-to" use a number line in "Balloons". G, as I have already noted, made more effective use of internal resources, partly because he was able to perceive relevance more readily than C could demonstrate.

The boys' use of metaphors (C's piles of snow and G's clonk) provided other interesting examples of their use of internal resources.

C appeared to be more dependent on the external resources provided – for example, he relied on Father Christmas's clothes and other images more than G, who was more focused on the numbers themselves.

Other resources that appeared to be linked with success were affective, rather than cognitive. Mindfulness was a strong feature of G's work, including his drive to practise and reinforce new internal resources (what Salomon and Perkins (1989) would refer to as "exercising" new "complexes of procedures"). G showed that he needed to be able to convince himself of the validity of new pieces of knowledge and actively sought to find or construct associations within his contextual neighbourhood. For G, confidence and mindfulness are mutually supportive and are powerful resources for his learning. Conflict and tension within his contextual neighbourhood are also valuable resources in that they provoke G to analyse and resolve such dissonance within his knowledge resources.

C did not display these characteristics in the same way and often seemed to accept, without question or challenge, new ideas that arose in the tasks, even where they did not align with existing resources. It seemed that, where span did not exist between resources and settings, misalignment was not therefore perceptible nor addressed.

7.4 Perception of similarity

Similarity between problems and/or their contexts is only perceptible through associations that exist or are constructed. Indeed, a common aspect of the structure of a problem or setting **<u>might</u>** be a similarity that is recognised by learners in mathematics, prompting the use of particular strategies and mechanisms. However, the application of appropriate knowledge in a new setting does not rely on the recognition of structural similarity; rather, it relies on the perception (conscious or unconscious) by the learner of some association between the new situation and some existing resource. This association might concern structure but, as we saw with G and C, might relate to any mathematical or non-mathematical feature of the task or setting.

Perception of similarity is, therefore, necessarily subjective. We saw C fail to perceive similarity on many occasions, even where tasks and aspects of it might appear similar to an observer.

The capacity for perception of similarity or the recognition of applicability of a resource occurs through a system of priorities that determine the likelihood that a resource will be triggered. This likelihood is a measure of the connectedness and consistency of the resource with aspects or elements of the new situation. Increased priority can only be realised through reinforcement of relevance. G showed that he was able to see similarity across narrative, iconic and symbolic settings for negative numbers, evidence that he constructed and aligned resources that spanned the different settings.

7.5 Beginning to learn about negative numbers

In the domain of negative numbers, the boys used the images and other resources provided in different ways. C was quite dependant on images of Father Christmas in different clothes to provide the basis for his sense-making mechanism for interpreting temperature values. In the earlier phases of our sessions together, G did not appear to use the imagery provided but preferred to focus on the numbers themselves, successfully comparing and ordering them. Both boys relied quite heavily on a number line model (often drawing one) to support them in making sense of changes in temperature in “Quiz” and in adding signed and unsigned numbers in “Balloons”. It was their use of a number line that supported the boys in extending their knowledge about numbers to include a world on the other side of zero. G was also able to use a number line to help him make sense of taking away or “undoing” the prior addition of a negative number.

Learning trajectories compiled for both boys show points at which I inferred losses or slips in knowledge. For G, these corresponded with lapses in confidence which, I found, were associated with his perception of conflict or tension within his contextual neighbourhood. These were addressed and resolved and G’s trajectory recovered and his conceptual resources continued to expand and connect with each other. C’s conceptual knowledge from Session 1 faltered but recovered in Session 2. In Session 3, his knowledge seemed to lapse again before it began to recover. Such lapses were characteristic of C’s learning trajectory. C found it difficult to “read” or interpret negative numbers whether embedded in a context such as “Journey” or presented only symbolically within “Balloons” – he consistently failed to perceive the minus sign until the latter stages of our work together.

Within this domain, Peled (1991) and Bruno & Martinon (1996; 1999) set out a hierarchy of knowledge based on “number line” and “quantity” dimensions for conceptualising negative numbers. I found that both G and C demonstrated higher attainment in the number line dimension than in the quantity dimension, though G’s facility in both dimensions was higher than C’s. However, when Bruno & Martinon’s distinction between abstract quantity and contextual quantity was taken into account, the discrepancy between achievements in the number line and contextual quantity dimensions was much less than when considering Peled’s broader quantity dimension. Therefore, I believe that both boys developed their conceptual knowledge about negative numbers in a quantity dimension almost as well as they did in a number line dimension. This is interesting when I consider that the “Journey”, “Cards” and “Quiz” tasks were all based on a number line model of an extended number system.

Linchevski and Williams (1999) and Williams & Linchevski (1997) hold that neutralisation is not an effective model for teaching subtraction of negative numbers. I could not evaluate this as I did not explicitly introduce neutralisation in the teaching tasks. However, in “Balloons”, G may have been using neutralisation strategies since it is possible that his effectiveness with compensation strategies might be masking his use of a neutralisation model. Moreover, I can see that, if we consider that mental calculation strategies based on compensation - for a portion (of the total to be added or subtracted) that had been, for expediency, previously added or subtracted - themselves incorporate a view of “number as quantity” inherent within them. So, perhaps application of compensation strategies for calculation is itself indicative of a neutralisation model for working with all numbers, including negative numbers.

7.6 Reflections

It is clear that conceptual resources are continually developed in the light of experience and learning in all settings. Concepts are constructed and modified through associations with other knowledge resources; previously existing knowledge is <u>one</u> of many types of knowledge resources that are involved in conceptual change. Other types of resources, evident in my work with C and G are as my model of learning predicted. They include external and internal resources, including memory resources and sense-making mechanisms which themselves include abstractions generated by the learner which are, to various degrees, situated in the context in which they were first created. Cueing and reliability priorities might be seen as a resource. Other internal resources with significance, that appear to be underplayed in the transfer literature, are affective; mindfulness, together with confidence, and the capacity to resolve dissonance within the contextual neighbourhood.

There is also another type of resource that I have not observed nor analysed rigorously and yet feel deserves consideration, if not by me in this study, then by myself or by others in further research. This resource is social, rather than cognitive or personal: the use by learners of their peers. There were many examples of G's pleasure in his badinage with others in his group creating opportunities to articulate (and to challenge) his knowledge. C always had something to contribute, though in his case, this often amounted to clownish remarks that did little to facilitate the construction and reinforcement of new conceptual resources. C was very dependent on his friend N, for support of his ideas and he actively sought his approval in order to feel any confidence.

Since I have not analysed N's experience and contributions, it is not appropriate to offer authoritative judgements regarding his influence on C's learning. I did find the interaction between C and N very interesting, however. Vygotsky (1978) presented the notion of "More Knowledgeable Other" ("MKO"), referring to someone who has more experience, facility or knowledge of a concept or process and who facilitates a less knowledgeable learner to construct and develop those concepts or processes. N, had he been effective as C's MKO, would have scaffolded C to a more sophisticated level of development in the mathematical domains in which we worked. It would appear that N was not an effective MKO, even though C sought his advice and approval on many occasions. C did, therefore, actively try to involve N as an agent for C's learning, though this was largely ineffective.

Transfer was very difficult for C. The cueing priority of newly constructed resources, including sense making mechanisms, did not get reinforced. In the absence of reinforcement, the level of cueing priority for any new resource did not improve and new resources were not triggered for C. We saw several examples of C using only “old” resources to try to address problems for which he needed to apply new resources, indicating that, for C, it was very difficult for new knowledge to become sufficiently prepared for transfer of that knowledge to occur. The intrapersonal and social affective dimensions of G’s experience and contributions – mindfulness, confidence and resilience – were not evident in C’s work.

Although conceptual change for C was hard won and therefore limited, it was possible to examine instances where C’s contextual neighbourhood changed and to offer possible reasons for the apparent fragility of his knowledge. It has also been possible to present a micro-evolution of G’s struggle to accommodate new meanings for negative numbers. Indicators that his contextual neighbourhood underwent significant changes were observable in his actions and utterances. Span of his conceptual resources was constantly changing – sometimes expanding and sometimes contracting through conflation of previously unassociated resources. Generally, however, the span of G’s resources, perceived as relevant for particular tasks, changed towards a normalised view.

Although I did not set out to explore what learners of different abilities might achieve (not least because I feel that “ability” is a problematic concept), I find that the serendipitous selection of groups by the children’s class teacher has revealed considerable differences between the conceptual changes observable (or at least inferrable) in 2 boys identified by their teacher as belonging to different “ability groups”. G is in the “high ability” group and is described by his teacher as “very bright”. C is in the “middle ability” group and his teacher points out that he is “at the lower end of that group”. Whilst I have tried to avoid reference to the different ability “status” of the boys in my analysis and discussion, there are clearly appreciable differences in the ways that the 2 boys were able to develop their conceptual resources. Gray, Pitta & Tall (2000) found that high achieving children tended to hold “semantic” images as part of their conceptual resources, compared to more “episodic” images held by low achieving children. Analysis of C’s and G’s work during our sessions revealed that G, at first articulating episodic descriptions of conceptual resources did, later, provide semantic references to those resources. Gray (1991) emphasised that there are two general types of resources available to learners and he explains that less able children have only one type available to them, whereas more able children have both types available – i.e. that more able children can use procedural strategies and can also build on their knowledge by deduction to create new resources that also become available to them. Gray believes that

> *“More able children appear to be doing a qualitatively different sort of mathematics than the less able.” (p551)*

It would seem that G certainly had a wide range of resources available to him including: efficacy with basic counting and other procedural resources; deductive sense-making mechanisms, including situated abstractions, helping him to develop episodic-style resources into more abstract concepts based on generalisations; as well as intrapersonal resources such as mindfulness. C’s resources did not develop in the same way as G’s, seeming to remain largely procedural and dependant on external agents (e.g. task-based images, his friend “N”) to trigger their application. In this way, perhaps it is true to say that G and C were “doing a qualitatively different sort of mathematics”: there clearly were notable differences in the ways that one “less able” boy and one “more able” boy were able to develop and change their contextual neighbourhoods relating to the number system and an extension to this. However, as hinted previously, I am uncomfortable with the notion of “ability” as a label for children and other learners since I feel it implies something fixed and I do not believe that capacity for conceptual change is fixed. The notion of “knowing-to” (Mason 2002) captures, meaningfully, what is necessary for new resources to be triggered and reinforced and their cueing priority thereby increased. “Knowing-to”, for me, implies potential and is necessarily fluid and transient since “knowing-to” refers to an “in-the-moment” phenomenon.

I stated at the commencement of the thesis that my motivation for undertaking research into learning in mathematics was to discover how to help children to learn. I believe that my data suggests that a research focus on individual children, which I advocated in the light of my literature review, is justified; the data reveals that the two boys interacted with the tasks in very different ways and that there was great disparity in the conceptual change that they each achieved. One of the differences that the boys brought to our work together was the level of achievement already attained and it is interesting to consider how children of different “abilities” are able to construct and modify conceptual knowledge. It is important to point out, however, that C and G were not studied as examples of classes of learners – i.e. low ability and high ability. Rather, they were studied as two individual cases of 8-9 year olds extending their knowledge about the number system into the domain of negative numbers. Notwithstanding any reservations I have about the notion of “ability”, it is nonetheless helpful to acknowledge evidence of cognitive, personal and social processes and attributes that are evident for G and not for C and to recognise their role in achievement of mathematical knowledge. In so doing, I believe that it is possible to improve knowledge about a pedagogy for mathematics in the primary school.

7.7 Towards a theoretical framework

Having argued, in my review of the literature, against a focus on abstraction as the key to transfer, it is not my intention to engage further with that argument here. I prefer to suggest that my data is evidence that abstraction is a consequence of transfer, and that, although abstraction is the process that enables generalisation and pattern identification that **is** the development of mathematical thinking, it is transfer (rather than abstraction) that should be a key aim for mathematics teaching in the primary school since without transfer, abstraction will not occur. In order for transfer to be facilitated it is necessary to appreciate the nature of cognitive processes involved as well as the interplay between all kinds of internal and external resources. I have shown that children respond to stimuli, metaphors and other external resources that might be available to them in different ways; moreover, their interpretation - both of the problem itself and of what they believe is expected of them – will, in part, determine their effectiveness with the problem. I found that 2 boys, presented with the same problem in the same context and provided with similar external resources, interpreted and responded to the challenges of the tasks in dissimilar ways. G was very good at interpreting the evolving demands of the tasks and the resources available to him, both internal and external. He used his old and new knowledge in a sophisticated interplay which itself created new associations and modified the span of his conceptual resources. New resources that were constructed included associations and sense-making mechanisms that facilitated effective interpretation of inputs and interim understandings, as well as enabling transformation of naïve knowledge resources into "well prepared" knowledge from which patterns and generalities might be abstracted.

G was able to demonstrate not only an ability to traverse a web of concepts related through associated resources - interplay between resources that facilitated transfer - but he also displayed a transformational interplay (between "operational and structural conceptions" (Sfard; 1991) or "episodic and semantic images" (Gray 1991)) that enabled him to abstract conceptual knowledge.

Wagner adopted a framework based on Co-ordination Class Theory (CCT) to explore the development of his students' mathematical knowledge. Wagner's (2006) assertion that,

> *"Maria was not abstracting structure from the problem situation, but actively structuring it by the most active knowledge frame available to her", p(57)*

supports my notion of transformational interplay between conceptual resources. Transformational interplay facilitates interpretation of problems and settings and facilitates transfer. Such development of resources might also lead to abstraction.

It is appropriate, at this point, to reflect upon and review my model for learning. At the end of "Chapter 3: Aims", I presented a model for learning that I had compiled in the light of my research and experience (see Figure 3). This had evolved partly in the light of CCT and adapted and incorporated some of its constructs. It also, however, included constructs that explicitly identify social and personal influences on conceptual growth and change.

It is now clear that readout strategies are interpretive resources used by learners at the first encounter with an opportunity, or prompt, to understand. The beginning of the interpretation and analysis of inputs is, therefore, when readout strategies come into play, not afterwards as Figure 3 shows. Other factors involved with interpretations and analysis of inputs were identified in Figure 3 as internal knowledge resources in memory and sense-making mechanisms; I am satisfied that this categorisation is appropriate. Of the internal resources shown in Figure 3, both boys demonstrated that memory resources and personal (learner) characteristics did contribute to their perception of relevance, as did the range of external resources that were available.

I now believe that the model I had constructed is too simplistic and does not adequately portray the interplay **between** resources that occurs. From the boys' responses and contributions it was clear that there was an abundance of links across resources, and loops of interpretive responses that traversed between all kinds of resources, that sometimes incorporated sense-making mechanisms in the early stages of analysis. Figure 3 implies a straightforward flow - from inputs through readout strategies and then through internal resources before sense-making mechanisms are invoked. This, I now find unrepresentative of the processes that I observed and inferred. Interpretive knowledge is therefore extremely complex and would be better represented by a network of links and loops and arrows showing that all elements are able to feedback to others. Sense-making mechanisms should be represented as agents incorporated into the processing of resources, mediating within the process rather than appearing as a separate and final step in the interpretation and analysis of inputs.

I am satisfied that the internal and external responses to the input that provoke the process are appropriately represented. However, the influence of learner characteristics (including moods, attitudes, propensities and interests) is understated in Figure 3. Figure 7e shows an evolution of the model previously presented, emphasising the role of learner characteristics.

7.8 Limitations

7.8.1. Resources provided

Most of the resources provided were suitable and children were able to utilise them as intended, as well as in unanticipated ways. The "Thermometer Interactive Teaching Program" was too difficult to understand and manipulate for most of the children, as was evident in the analysis of data (Chapter 5). One reason for my incorporation of this resource was to implant an image of a vertical number line (further to our earlier map work) and movement along it; however, a simpler, more user-friendly version might have achieved this more efficiently and effectively.

7.8.2. Age of children

I believe I was justified in my decision to work with 8 – 9 year olds as they began to construct knowledge about negative numbers. I see now that it would be even more illuminating to conduct a longitudinal study to follow the children's development of their knowledge in this domain. To study their conceptual change over a period of 4-5 years, during which time they would be expected (curriculum expectations) to attain greater levels of facility with negative numbers would reveal much more about the ways that individuals' conceptual knowledge in a particular domain modifies (both appropriately and inappropriately) over an extended period.

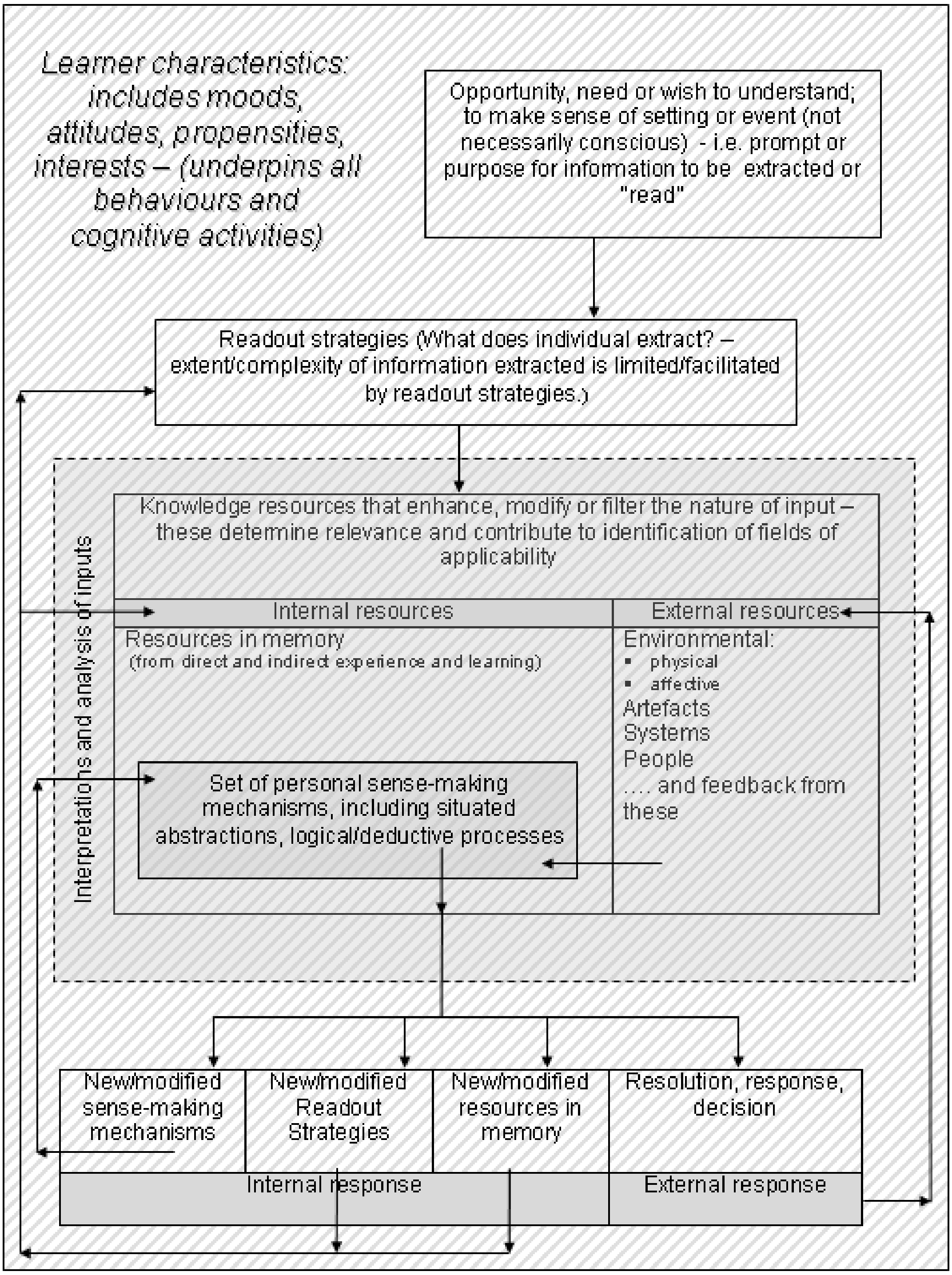


Figure 7e: Model of the micro-evolution of knowledge, amended in response to analysis and discussion of my findings

It was not possible to evaluate the effectiveness of teaching a neutralisation model for knowledge about negative numbers (and I had not intended to do so). Perhaps this would have been possible had I worked with a greater number of groups, some of whom might have engaged in tasks based on neutralisation.

7.8.3 Complexity of processes

As previously described, my representation of the micro processes involved in conceptual growth and change had been overly simplistic; my study shows that those processes and the interplay between them is extremely complex. It has been possible to infer the elements, micro-processes and links between them with varying degrees of confidence since the complexity of the relationships masks, at least to some extent, their visibility. In subsequent studies, greater confidence in analysis might be achieved as the micro-processes become better understood and research is able to "zoom in" to study at increasingly smaller grain sizes.

7.8.4 Attention to social influences

It was not my desire, nor within the scope of my expertise, to analyse the influence of social aspects of learning. However, I do believe, in the light of my analysis and discussion, that these factors cannot be excluded from consideration of how children construct and refine knowledge. I feel that intrapersonal and interpersonal factors were powerful mediators for learning.

7.9 Implications for the future

7.9.1 For future research

I have shown that contextual attributes are not "filtered-out" in the development of concepts but that references and links to situation-specific knowledge resources form the basis of the growth of knowledge and understanding. Therefore, for future research to have value in contributing to knowledge about children's development of concepts, it must not focus on the decontextualisation of abstract knowledge.

What individual children said and did was not simply a product of their knowledge but was clearly related to an aggregation of social and personal factors – i.e. individuals' experience of our sessions was certainly shaped by cognitive factors, processes and outcomes, but also by personal and social behaviours, expectations, perceptions, attitudes and relationships.

Intra-personal factors appear to significantly affect the experience and learning of individuals and I would suggest that collaboration between workers in the fields of psychology and education are vital in order to reach a greater understanding about learning (at the level of micro-processes).

At the earliest stages of the design of my methodology for my research, I decided not to work with individual children. The reasons for this are set out in "Chapter 4: Methodology". However, having made this decision, I was conscious of a range of issues that are relevant to working with groups of children – i.e. pertaining to children's own experience of being part of the group; but also relating to the performance of a group as an entity rather than only considering individuals. It is not within the scope of this thesis to consider, in any depth, issues relating to collaborative working; suffice it to say that I recognise that there is an extensive and constantly developing literature in this field that might also illuminate the issues which I seek to understand. A related, interesting story to be told would be to consider and contrast the experiences of individuals within the same group. Consider that: there are differential cognitive contributions and developments; disparate intra- and inter-personal interactions and relationships. It would seem logical to expect that, in telling the individuals' stories, they may seem to be describing different episodes when, in fact, they are recounting the same episode through different lenses.

7.9.2 For teaching and learning

It is a fundamental aim of teaching that learners will be enabled to construct knowledge and be able to implement that knowledge appropriately in future. Transfer is therefore implicit in this fundamental aim. I have shown that transfer and the growth and change of conceptual knowledge are intertwined in a positive feedback loop – i.e. transfer cannot occur unless similarity is recognised and when transfer does occur, associations are constructed, span of concepts is modified and further transfer is enabled. Therefore, failure to transfer will both contribute to poor conceptual development and will be a result of it.

There are many factors that enhance or restrict conceptual change and transfer including: the effectiveness of readout strategies; mindfulness; and opportunities to reinforce new resources and associations between them. It would therefore enhance learning if teachers were to provide these opportunities and to provide explicit reminders and prompts that are likely to activate new knowledge resources which have not yet achieved high cueing priority.

Some children are able to learn more quickly, due at least in part to qualities and behaviours that are well developed within them – such as mindfulness and confidence and an expectation to transfer and to understand, including engagement with, rather than avoidance of, conflict. Teachers who attempt to facilitate the development of these qualities and behaviours will be helping their pupils to develop resources for learning.

With regard to the teaching of negative numbers, there was great disparity in the extent to which C and G were able to engage with, and make progress with this new concept. Those differences have previously been thoroughly discussed and analysed; to focus on the differences between the two case studies is not appropriate here. Rather, in the concluding paragraphs of my thesis, it is most helpful to consider more general findings about learning about negative numbers.

Both boys were able to work effectively with negative number problems and contexts that used number lines and neutralisation. Even though I had intended to focus on only one of these models for teaching about negative numbers – i.e. a number line model, they both demonstrated an ability to utilise notions and images of neutralisation as well as of number lines. I would therefore encourage teachers to embrace both models and to work with them simultaneously and in concert together.

Predictably, there were sometimes problems with interpreting the minus sign in its usage to denote a negative number – readout strategies that confer some significance to this particular symbol (arguably any symbol) were slow to develop for one of the boys and contributed to poor progress. Explicit checking and reminders about the minus sign when negative numbers are first introduced are therefore likely to benefit those children who don't "see" it.

In one of the case studies it was interesting to note that the same child, having engaged with some success with the questions and tasks that related to journey or temperatures contexts, reverted to application of rules and procedures when the context was removed. For both boys the context had facilitated the construction of sense-making mechanisms about the new numbers; moreover, further work within the context triggered these SMMs and reinforced all associated resources. The context therefore supported conceptual change and growth. For one of the boys, the support provided by the initial context, in its capacity to cue these SMMs, was vital and there were significant consequences when this support was withdrawn. When addressing new problems in a different context he was unable to perceive any similarity with the previous task. The resources that were cued in the new situation were based on rules and procedures and were not well connected with the new problem. The new context was not sufficiently similar, and/or new associations were not sufficiently reinforced, for transfer and further conceptual change to be enabled.

The importance of context and the analysis of potential similarities is an important consideration for teachers when designing and evaluating learning tasks. Teachers must consider whether children are likely to perceive relevance; if not, teachers should be committed to either changing the task, introducing more overt associations and/or explicitly articulating those links. The construction and reinforcement of associations between resources should be a focus for teachers because it is only through increasing networks of these associations that transfer and conceptual growth can occur.

7.10 Moving forward

My research was conducted using an amalgamation of ideas from a variety of theories old and new. In order to be able to observe (or at least infer) micro-processes related to conceptual learning, it was necessary to exploit aspects of learning processes that had been identified by others. Through rigorous and purposeful design and analysis of children's engagement and achievements with learning tasks I have been able to make my own contribution to theory relating to children's construction of mathematical knowledge (summarised in 7.11). I find that, even though it has been possible to infer trajectories for learning about negative numbers for C and G, this knowledge is not sufficient to be able to predict learning about negative numbers for other children, or other learning pathways for the same two boys. This is because their conceptual growth appears to be influenced by a range of factors, of which cognition is only one aspect. Analysis of my data suggests that cognitive and affective aspects of achievement and performance are deeply connected.

I found, when conducting my review of the literature in the field of educational research about learning and transfer, that workers have focused on either the cognitive or the social and cultural dimensions of learning; I chose, when planning my research, to concentrate on the area of cognition. I am now convinced that, for theory about conceptual development to, itself, develop, it must take consideration of affective aspects of learning as well as cognitive aspects, since my data shows that these two are deeply connected. Upon my most recent perusal of the literature, I attempted to discover whether others have described learning with respect to both cognition and affect, acting in combination, as I believe they do, to impact on conceptual development in complex ways. The notion of "transformational interplay" that I have introduced is something that is alluded to by workers in the field of educational psychology: Snow (1989) claims that conation is intertwined with cognition, emotion and behaviour. He suggests that it is because these processes are so difficult to separate that conation is rarely studied in its own right. Conation, Huitt (1999) explains,

> *"refers to the connection of knowledge and affect to behaviour … is closely associated with the concept of volition".*

Corno & Kanfer (1993) also urge that research should seek to understand better the interaction between factors:

> *"The emphasis given to the dynamic interplay between volition and other psychological determinants of action (i.e. cognition and affection) represents a third distinction between the present aptitude approach and current self-regulation research perspectives. Although most researchers agree that learning and performance are joint functions of these factors, little is understood about the way these factors interact." (p307-8).*

I would point out that "volition", "self-regulation", "conation" are just a few of the constructs that appear to be related to C's and G's progress and achievements when working with a new domain. As I suggested previously, researchers in education, mathematics education, psychology and educational psychology must now collaborate in order to construct knowledge about ways in which children construct knowledge. I have emphasised that my research found that cognition, affective factors and mindfulness are deeply connected and that knowledge about one of these factors does not necessarily help us to know about conceptual learning. I suggest that they are sufficiently intertwined that research should concede that it is not helpful to explore them separately. Research should, therefore, find ways of exploring all three components in combination: cognition, affect and conation, since my research informs me that it is their combined effect that constitutes the achievement of effective conceptual learning.

7.11 Summary of this final chapter

I am now able to respond, succinctly, to my three research questions first set out in Chapter 3:

What resources shape the nature of transfer and the growth of knowledge about negative numbers?

> All kinds of internal and external resources are involved. One of the principle findings of my research is that intrapersonal resources – particularly mindfulness and an expectation for transfer – were strongly associated with pupils' success in constructing knowledge about negative numbers that they could use effectively.

What is the role of the interplay of resources in the micro-transfer of knowledge about negative numbers?

> Micro-transfer requires the perception of some similarity across any aspects of different problems and settings; where no similarity is perceived, micro-transfer is not initiated. It occurs within a web of all kinds of knowledge resources including interpersonal and intrapersonal skills, knowledge, commitments and beliefs. Interplay occurs across resources, forging links and shaping priorities. It also occurs across resources at different levels of abstraction, transforming knowledge and understanding through perception of generalities. The exploitation of all kinds of resources relating to a concept generates feedback to all other resources relating to that concept and, through association, to other concepts.

What is the relationship between abstracting and transferring knowledge about negative numbers?

Abstracting was the process that facilitated recognition of the potential relevance of existing resources in new situations. Abstracting therefore facilitated micro-transfer and knowledge resources were re-used in new situations, albeit sometimes in haphazard or unproductive ways. It was often possible to build on this low-level transfer and to be able to extend the span of perceived relevance of existing knowledge so that it became sufficiently prepared for higher levels of transfer. However, it was clear that abstraction, as it is normally understood, was not necessary for conceptual change, though it might be a consequence of it.

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Appendices

Appendix 1: Letter to parents

(Name of school omitted for confidentiality)

Dear Parent,

Some of you may remember Mrs Simpson, who taught at ** ******* from 1995 – 2001. While she was here, she developed an interest and conducted research into the way children learn Mathematics. She has continued to research in this field and will be working with ** ******* over the coming year, particularly with children in Year 4. She will be teaching and observing children while they work with her in pairs and small groups and will need to ask children questions about they way they think about and understand Mathematics.

If you are happy for your child to take part in Mrs Simpson's research, would you please return the reply slip below before Friday 23 September. Please be assured that any information gathered will be anonymous and confidential.

Yours faithfully,

R S******

Head Teacher

………………………………………………………………………………………
……..

I/We give consent for ………………………………. (Child's name) in Class ……..

to take part in Mrs Simpson's research, which, I/we understand, might be published in the context of her academic work.

Signed ……………………………………. (Parent/Guardian)

Date: ……………………

Appendix 2: Schedule for interview with class teacher prior to commencing research sessions with children.

Interview to be:

- Fairly unstructured in format;
- Key questions 1-7 noted below;
- Freedom to exploit relevant issues that may arise

1. Confirm year group/age of children in class?
2. Introductory script: “I need to be able to observe and work with children while they learn something new in Maths. I have selected negative numbers because it appears to be introduced for the first time in Year 4. Is this correct - will the children begin to learn about negative numbers while they are in Year 4? Have they begun any work in this area yet? I need to establish whether negative numbers is a new concept for children in your class.”
3. “What do the children in your class already know about the number system? e.g. integers, fractions, decimals”?
4. “What experience do the class have with number lines?”
5. “What have they learned in school about maps?”
6. “Are they used to talking aloud about their thinking in Maths?”
7. “In the light of your experience with this age group and your knowledge of your class, what would you expect children’s difficulties in this area to be?”

Appendix 3a: Write-up for Session 1 H, R, W

Minute number:

1. children arriving and sitting down
2. I tell children that they won't need pens or pencils as they won't need to write anything today
3. I start to set the scene for the work to be done
4. We talk about temperatures at Christmas. I ask children if they recognise any of the countries on the map
5. They know Madagascar – have seen the film. They know Jordan is the name of someone in the same class. Rome is familiar for one whose brother went there recently.. Children talk about holidays and where there are family members living – H's Dad works in N.Ireland during the week. R doesn't know where her Dad is – could be in Africa for all she knows.
6. Children want me to read out all country names that end in … istan. They mention hearing about recent earthquake in Pakistan. We locate Pakistan on map.
7. I indicate Africa on map and explain that all these countries are part of a continent called Africa. I ask them to read out the names of some of the African countries.
8. I indicate Europe on the map and ask children to read out some of the names of countries in Europe. H's Mum has been to Iceland – the supermarket – H is joking – doesn't really think her Mum has been to the country. I pick up the globe and ask what they know about the globe.
9. R points to the equator and knows that it is hot on the equator. They expect the black line on the map to be the equator because it is in the middle of the map. I explain that our map shows only part of the globe and that most of our map is north of the equator
10. someone knows that the sea is hot in hot countries. I explain the "Father Christmas on holiday" scenario. H says she doesn't believe in Father Christmas
11. H tells me she could see Scotland from Northern Ireland when she was there. R has a relative in Scotland
12. I explain why we need to consider temperatures for FC's journey
13. H think Kenya will be a good starting place – as she says it's boiling there
14. I check that children can use touchpad on laptop. They look up Kenya – is 19 degrees. They like the flag.
15. H hasn't been to Kenya. I encourage children to try a couple of countries each

16. W asks does that say Spain? He looks up temp = 7 degrees. R says that's not hot. They talk about what they would wear for 7 degrees.
17. go to Tanzania. Children laugh. W describes how the flag would have to be different to the Jamaica flag.
18. W's turn. I remind him to look at the map first to choose a country that he wants to check. W chooses Egypt but clicks on UK (6 degrees)
19. Go to Egypt (some trouble with touchpad control) 19 degrees. Go to Jordan (by choosing from list, not map)
20. I ask which other countries will have the same temperature, at the same level on the map. They say Iran. They lookup = 6 degrees. R's turn – she wants to look at Madagasgar (so far has been H who wanted to go there). I ask R whether she thinks it's going to be hot. She says a little bit hot.
21. Difficulties with touchpad (it keeps reading clicks where children's touch is too heavy – leads to them being taken to places they don't want to go to. Children plan where they're going when it's their turn – H wants Czech Republic, W wants Tanzania.
22. They look up Madagasgar = 21 degrees. I ask if they know what UK temperature is today. H says 12 degrees. Cherice clicks on Ghana by mistake – she wanted Iceland (H says it's "cold, man"). Ghana = 28 degrees. I ask where it is on map. W says dunno.
23. Someone has got FC head off model. Children click on Iceland = 0 degrees (Oh my God, zero!")
24. I remind children they should be using the map to plan their journey, not randomly visiting pages from the list
25. They are confident that they can work and discuss together, use the map and computer information properly. They decide to check Kenya's temperature as it looks like a good starting point.
26. They have trouble using touchpad. Distracted, interested in …
27. Finland, even though they know it's cold. H suggest they start from the top of the list and work down. I remind them that they should start from the map, not the list.
28. Click on Zimbabwe = 22 degrees – agree to start there. Click on UAE = 21 degrees. They think they've got to stay with same clothes so UAE is no good.
29. They revisit Zimbabwe page to check clothes.
30. They click on Yemen but didn't mean to – want to look at Zambia and Democratic Republic of Congo. They click on Latvia (2 degrees) "He looks weird – he's wearing a coat". H says "2 minus!". R asks "What does that mean?" H explains "that means it's freezing, freezing"

31. Someone is suggesting a journey by bouncing the model around on the map. Dem Rep Congo = 26 degrees, H notices that FC has taken his top off and says they "need the same clothes"
32. They look at Angola (– they are trying to decide using the map at last). They have changed the start country so that FC doesn't start with too many clothes because they want to include some stops where he would have less clothes than Kenya so best to start somewhere else – this is my interpretation – they couldn't explain reasons for the change)
33. R has a turn using the PC – she looks at Gabon but it's hotter and FC has "pants on"
34. They try Central African Republic = 25 degrees and FC has shorts. They like this one.
35. Children are excited now that they've got started.
36. They want to try Egypt. It is 16 degrees and FC has blue T-shirt. H thinks they can't use it because he's put more clothes on. I remind her of rules. H want Libya next. W is suggesting Jordan. R say the flag is just green.
37. H and W plan routes over land. I point out that FC can travel over sea, can fly. Looking for Greece. H asks if I'm going to listen to what they are saying.
38. H doesn't want to go to Greece but to Morocco. W has got Morocco football kit. His uncle has been there. They are struggling with the touchpad. They laugh at their difficulties with touchpad.
39. Russia comes up by mistake. H says it's 6 minus. I help them get to Morocco (14 degrees) Bell goes. Children ask if they can carry on. They want to complete the task.
40. W takes over the PC and wants to go to Spain. H comes back from asking teacher a question and says "Do Portugal". W starts looking for Portugal. H calls out to friend "We're playing on this – it's better than footie
41. H describes the "game" to her friend. Portugal = 12 degrees. I ask W whether he can remember what temperature and clothes were in Morocco. He can
42. They want Spain next
43. (eventually) Spain = 7 degrees. I remind them that they don't have to stop at every country on the route.
44. Someone suggests France but they quickly change their mind and go to UK then Norway. They remind each other that it's OK for him to put clothes on.
45. Norway = -3 degrees. I ask what's happening to the numbers. H says its getting colder. They consider Russia. I encourage them to go to Svalbaard (-13 degrees). They're pleased – "We won!" I ask what's happened to this number – "It's getting higher and it's gone

colder" W says he has seen numbers like this before in America, then changes his mind.

46. I ask So what is this sign in front of it?. They say take away. I give each child a job to do as I review their journey.

47. Lower numbers, more clothes.

48. R asks what minus numbers mean. I say we'll do it next time.

Appendix 3b: Write-up for Session 2 H, R and W

Minute number:

1. I tell children will be playing a quiz. H reports that Charlie (from the other group) “thought minus was hot but it isn’t”. I shuffle cards (about ½ of them). I ask children to lay down cards so that numbers are in order – highest ones here and lowest there. H asks whether zero is minus.
2. Someone is saying 8, then 7, then it could Russia (this is -6). H argues that it couldn’t – because it’s minus. She explains to R that it means its really cold. R still doesn’t understand.
3. H starts laying -1, -2 cards out. R says “Oh does it mean its even colder than that?” I press H to explain again. H says “If it’s got a - that means it’s minus and minus means (R says take away). H says “No, well it could in Maths but we’re not doing Maths are we, means a country is colder than that one that doesn’t have a line (a minus sign)
4. R understands now. I ask her if she’s noticed the minus sign . R says yes. I ask did she think it was important . R says yes, she thought it was take away 6 or something.
5. Children have moved cards with zero away from end of the table (i.e. have realised that zero isn’t the lowest value). They cooperate well to lay out cards. H is not happy with what they’ve done.
6. I ask R to read down the list of numbers. 0 0 0 1 -1 1 -1 1 -2 -2 -3 -6 7 8 12 13 (or similar). I re-state original task. R not sure that what they’ve done is right. W thinks it is right.
7. I ask H what’s wrong. She doesn’t explain very clearly – says that shouldn’t match that. R seems to understand and says all minuses should be together. R says to H “But those zeros haven’t got minuses either (i.e. shouldn’t we be picking these up too?). H says “No, but they go there of course,”
8. H says 13 12 8 7 1 13’s the highest. Zero is the lowest. The list is in 2 parts. W thinks lowest no is minus 0
9. H says “I get it – pretend that’s colder than that – do you have to put that one before …?” R says “Can we check it out on the computer?” I tell her “No, you need to think it through” H reads 13 12 8 7 1 1 1 2 2 2 2 2 1 2 2 3 3 3 6. R says H has said it in the wrong order. W says yes, because she didn’t say minus.
10. W points to the negative numbers. H says that she didn’t know that she had to say minus (she repeats this). R still thinks that H did it wrong. H still thinks she was right. I remove some of the duplicates to make more space on the table so that all cards fit in one column.

11. H says “that’s the hottest because it’s got 13 and it hasn’t got minus. W reads 13 12 8 7 1 1 0 -1 -2 -3 -6 (omitted to say the first “minus” but immediately self-corrected.
12. I give children rest of cards
13. W has put a zero card at the end of the table on it’s own. I ask him why. H knows where he should put it.
14. I ask W why he’s put 5 next to 7 – he can’t explain
15. more interpolation of cards
16. more interpolation of cards
17. H tells us that she’s got a new coat
18. children complete interpolation of cards. I tell them we’ll do the quiz now.
19. I recap that on the table the card for Gibraltar 13 is at one end and the card for Russia -6 is at the other end. I open the powerpoint quiz and W reads from the PC screen “Click a question mark”
20. I show children the links on the screen to a world map and to a thermometer – they are excited with this. Back to the main screen. W wants a blue question.
21. Name country between 1 degree C and -1 degree C H ignores minus sign when reading number. So does W. R asks what does the C mean? H says I wish England was here. H reports that had told her friend that England is one of the richest countries and he had said no.
22. Slovenia. I ask H “Is this one?” H says it is minus one. I ask W to show a country that’s 1 (he does) and can show a -1 country. I ask the quiz question again – i.e. between 1 and -1. Someone suggests Germany. I point out that Germany is 1, not between.
23. H says zero. W suggests Denmark. H says Czech Republic (correct). R’s turn – country 12 degrees colder than Portugal. H says “We don’t have one” she says it 3 times
24. They locate Portugal on the list. Don’t seem able to attack the question so I ask for countries colder than Portugal. W says “So we can pick any of them?” I stress 12 degrees colder. W think should just go to bottom of the list. R suggests Germany – she says she counted back 12 (cards). I ask so is each card 1 degree?
25. H mentions rain outside. I repeat “So we need to find somewhere 12 degrees colder than that”. Children have lost interest because it was too hard. I get R to click on the thermometer. It shows zero Someone says “Oooh, that’s quite cold.” I explain that we must set the temperature at 12 degrees and then count back to see what’s 12 degrees less than that.
26. H says “I know what 12 is. I know where 12 is” I demonstrate how to reduce temperature on thermometer. I ask “What do you think it’s gonna be?” H says “I know already, 12”

27. I model 12 – 12 using thermometer.
28. H calls zero “minus zero”. I explain that we need to find a country with a temperature of zero degrees.
29. New question: Name a country with a temperature between 0 degrees and … H says 2 minus. H tells me names of counries that are zero. R knows that we need -1 countries.
30. I suggest doing another similar question – between 10 and 15. H goes to cards and inidicates all those under 15
31. Another question – between 3 and 6. W and H say that’s easy, get it right. Another question between -6 and -8. W/H call out Spain (7). I ask why they’ve gone to that (+ve) end of the table. R read question again (correctly). “But we don’t have a country below -6.”
32. I ask “What have we got that might help us find a country to answer the question?” They open thermometer and
33. set temp at 6. H thinks the question was about “six minus and eight. I ask whether this thermometer helps. Children don’t know. I open the map
34. I say “See where Russia is? We need to find a country that’s colder. Children don’t know, fed up, distracted
35. They mention countries all over the map. I ask them to think about map we used last time.
36. “What happened to the temperature when we move that (north) way? W says it got colder. H asks – “So we could go up?” W says “So every holiday, I’m going to go down, south. H says I’m always gonna go up. I recap that we need to find countries colder than Russia so which way should we go? H says we just go up. W and H suggest Iceland, Greenland. W says his Mum goes to Iceland shopping. New question “Is Estonia hotter or colder than Croatia?” H say hotter, changes her mind because
37. “It says there, that’s zero, that’s -2”

Appendix 3c: Write-up for Session 3 H, R and W

Minute number:

1. – 6 Children play "Reflect – a sketch". H is not good at this game. They realise quickly that they need to put corners on the intersections of the lines. R copies rather than reflects.
7. H asks "How come we're not doing the country thing? I liked that." R adds "When he changes his clothes and he's wearing pants, its funny." H asks if I've got a 1/2/3/4 player game
8. H asks for another game that is part of the same games package
9. talk about games they like
10. change to "Swimming Pool" game. Straight away H says "I don't get it. What are you supposed to do? No, I don't get it."
11. Start "Balloon Burst" I explain the game
12. (2 2 -2) They read 2,2, divide 2. 2 and 2 is 4, divide …. R says you add 2 and 2 is 4, then take away 2 is 2. H yeh, but it's divide. R says "It's take away" She checks her answer – correct.
13. (4 5 1) H says 6 take away 4 is 1 … is 2. R says 1 add 5 = 6 then takeaway 4 is 2. (1 5 -2) R says 6 minus 2 is 4 – they check – correct
14. (-4 -3 -4) minus 4, minus 3, minus 4 R say zero because you don't add anything. H says no, it's minus something. W says minus zero? H says 4 add 4 is 8, add …. I know we're not adding, add 3 more, 9 10 11 so it's minus 11. R says no its not because all of them are minuses
15. Check answer = -11. H says "Told you!" R says how can it be minus 11 when there was nothing? (5 -3 4) H says 4 add 5 is 9, take away 3 … R says "I think it's 7" She notices that H doesn't count down with her fingers accurately. R is right.
16. (-4 4 2) 6 6 H says its 2, 4 minus 4.
17. R asks "Is it 2? because minus 4 is take away 4. H shall I tell them it? You add them 2, 4 and 2 is 6, then take away 4 is 2. (-1 -4 -3) R says "Is it-8?" H says yeh. Correct. (3 -3 -4) R asks^? H says no. 3 take away 3, zero, add 4 is 4.
18. R says "But it's minus 4. They check their -4 answer – correct. (1 5 -4) W enters -1 – wrong. Should be 2
19. – 21 5 balloons. I give children paper so that they can record the 5 numbers (3 3 2 -1 -2) Sometimes R forgets to say minus. She writes numbers like A on sheet
22. R says "That's all a zero on it's own (she means that one number cancels out another) (2 0 -4 1 4) – they almost get this one right
23. (-1 5 -2 3 2) No-one paying attention (-4 -3 5 0 0)

24. – 27 Children are asking me whether I am a teacher – I explain that I used to be. Nimh says -4 and -3 is -7, add 5 is … ? H wonder whether starting with the 5 and taking away would help. She is able to tell me how to bridge through zero (5 to get to zero then 2 left because 5 from 7 is 2) She's not sure whether answer is 2 or -2 – ip, dip sky blue …
28. (1 5 -1 -1 1) one add 5 is 6, add another one then take away 2. The answer is 5
29. Can explain why adds positives together first i.e. would be harder the other way. I ask why are you taking things away when I've asked you to add all these numbers together?
30. R says because they've "got a little minus on" which means take away.
31. I show the addition sum written out properly. H says you shouldn't write it like that. She has a go at writing it herself but is unhappy with it and concedes that mine is right. I ask how they have learned to do it this way – who has told them? where have they done it? H says she has never done it with Miss Swain or Miss Marriott.
32. (5 3 -4 -3 -4) H says 4 and 4 is 8, add 3 is 11, take away 8. then she gets lost or distracted. H says she thinks the answer is 3.
33. W thinks so too because H says so
34. (5 -2 4 -1 -3) R asks "Is that normal 5 or minus 5? Is it 3?" H agrees
35. (1 5 -1 1 4) I ask H and R to give W a clue. H suggest adding 5 and 4 first (biggest positive numbers) W can't do this but eventually gets there. R suggests take away 1. H would do that last.

FINISH ALL SESSIONS – H, R and W

Appendix 4a: Write-up for Session 1 C, S and N

Minute number:

1. I set up map while children watch. They are excited.
2. C notes that FC has got no body. List of countries is showing on PC screen. C is looking for Spain on list while I finigh setting up. I start to talk about Christmas Eve being a busy night for FC.
3. I explain that FC likes to go on holiday before Christmas Eve. Children think he'd prefer somewhere hot for his holiday. C and N suggest Spain, not Africa because it's too hot. Or he could go to Turkey, perhaps?
4. I say let's send him to the hottest place we can. The children say Africa, S says Egypt as it's very hot there. C talks about mummies and sphynx. He has read about it in a book. Someone mentions Togo
5. Children notice Black Sea (they ask is it black?) They ask is there one called the Blue Sea? N says there is a sea where there is dirty water and clean water. They see Pakistan. S says its next to India. She says something about 3 months. She's been to Pakistan. S says it's really hot. When she arrived on the plane it was so hot that she couldn't breathe. They agree that FC should go to Africa for his holiday.
6. Children correct me when I mention Egypt. They don't think that Egypt is in Africa. I explain that it is. They notice Niger on the map and laugh (because it is like N's name?). They discuss things that they have seen on the news on TV.
7. S asks where is Iraq? S talks about dirty toilets when she went to India. She thinks she was in Uzbekistan (looking at map when she says this).
8. S still talking about her trip to Pakistan. She says it took a whole day to get somewhere. She saw whales. Boys mention Germany and think it's hot there. I explain about continents.
9. Boys think Russia is hot. I explain the table – that it shows us what temperature it is on Christmas Eve in each of the countries on the list. The ones with the red spot are on the list. N says some places are hot on Christmas Eve. C tells S "Go to Germany". S looks for Germany on the list. She clicks on Kazahkstan.
10. Children talk about what they can see on the screen. I say that's the temperature. C says "That's not a lot". Germany is showing on the screen. S asks "When are we going to see FC?" C points at temperature (1 degree) and asks "Is that a lot?" N says "No". N finishes off my reading of the temp .. "Celsius"
11. Back with the list, children are still asking "So is that hot or not?" I ask "What do you think?" Children say yes, because he wasn't

wearing a lot, Go to Egypt. 16 degrees. S laughs because he's "taken his clothes off!" I start to explain the game

12. I demonstrate 2 pages. C wants to go to Madagasgar. C notices that the colour of the country name changes on the list once it as been "used". I introduce the game
13. FC has been on his holidays in Africa. All discuss the red line – children know this is the hottest place. Somebody goes to classroom to get a globe.
14. Children look at equator on globe. I ask children to look for Europe and Africa on globe. They spot Madagasgar on both globe and map.
15. I show them Kenya, where FC starts his journey. I ask someone to look up the temperature in Kenya. I say that he's got his flip-flops on. One of the boys says that it must be hot.
16. They talk about clothes. I ask what which countries (indicating map) might be hotter. They try Madagasgar. C says this is playing tricks because Madagascar is hotter than Kenya. N says it's nearer the equator. (They are interested in Madagasgar because of the film I think – they don't really know whether its likely to be hotter than Kenya but are interested to know what it is and are surprised when they see that it's not hotter.
17. N tries Nigeria (wanted to do Niger but thinks he can't because he expect Niger to be hotter than Nigeria. They go to Niger (25 degrees) They laugh at FC in shorts and shades.
18. C want to go to Spaiin – he thinks it's hot – has been there. They visit Turkey (1 degree) C still wants to go to Spain but has trouble clicking on it. (7 degrees. C says "Spain's hot"
19. Ukraine is -3 degrees. They don't make any comment, seem to have visited in error. They are looking for Jordan. Is 9 degrees. N says is hotter than Spain. C says "No, it's less hot than Spain". They can't remember what Spain was.
20. They go to Spain again (7). I recap the temperatures of the countries FC has visited so far. N asks "Can we try Iceland?" ask "What will happen there?" N says it will be very very very cold but can't explain why he thinks that. C hints that there could be a clue in the name Iceland.
21. I ask what we think about other countries "up there" (indicating top section of the map). They read some of the country names and that it will be cold. S says it's because they are closer to the Arctic. I ask what do we know about the Arctic? How do we know?
22. Children say that they haven't been there and that it's not that someone has told them what it's like. N says he has seen it on TV. S says she has seen it on TV too – a programmed on in the mornings called "Serious Arctic". She tells us aout the programme – mentions teenagers who have very cold hands.

23. C or N say “Poland and Finland is cold” He can’t tell me how he knows. N notices latitude and longitude lines on the map. I explain that they are different.
24. I tell children they must plan a journey for FC starting with Kenya and ending up in Svalbaard. I point out that he will not have many clothes on when he starts. I ask whether the places that he visits will get hotter or colder. No hesitation – the boys say colder. I explain about clothes being added gradually.
25. I ask children where they want to start.. C says Madagsgar is hot (and adds “but not as in sexy”). C says “That was cold, wasn’t I?” (Kenya). N says no. They check Madagascar temperature (21 degress) C says he’ll go to Kenya and see if it’s colder.
26. Children talk about “beat the number” “go less”. They check Tanzania – 28 degrees, FC wearing trunks – children giggle. C says “We’ll start from there”
27. I check that children understand that the next place can be the same or with more clothes, but not less. They laugh about what “less clothes” might mean
28. They think they could go to Kenya next as it should be easy to beat 28. They notice they have been to Kenya before. N says Yes, FC has more clothes. C says That’s good. N says he wants to try that country “Sunderland”. I correct him “Sudan”
29. Sudan is 25. Oooh unsure what do do. They are disappointed. Ethiopia is 16. S says Yes! She notices that FC has boots on now.
30. I recap: Tanzania 28, Kenya 19. I ask if they want to “Go here next?” C says no because it is “less hot”. I ask “Is that wrong?” C says “last time he had a coat on”. They check – no coat previously.
31. Someone says “Look, trousers in Ethiopia, not Kenya. They are confused, undecided. S is looking for Chad on the list.
32. Chad = 24. S says this is kind of hot. They want to go to Niger (“N’s country”)
33. They know they need to get colder. Niger is 25. N says told you it was hot. C says that’s rubbish – Go to the one under it. They keep talking about Portugal. I show them where it is on the map and say its too far to go. S says try Pakistan. I explain that there is no red dot so Pakistan is not on our temperature table.
34. They look for a country with a red dot . They go to Iran (6). N says it will be “hard ot beat. He’s got everything on”. S says “Not his hat” They go to Portugal (12). C and N notice that FC has got the same clothes on.
35. I remind them “So you’re looking for somewhere colder than 6”. They say Germany’s cold, S says could do Scotland.. It rains all the time and it’s really cold. C doesn’t want to go to UK yet – too far away. He suggests they go there once they are a bit closer.

36. Turkmenistan (5).”Yes, we beat it”. S notices that FC has hat on. C says now he needs his sunglasses. They decide to go to UK next, then Poland.
37. UK = 6. S says they are not allowed to go backwards. C says We’re meant to go to America but that’s British. We’re going to Poland.
38. S would like to go to Germany (1). She says “Yes. I got it right”. C says it’ll be hard to beat. “We’re going to Poland” N asks “Can I go to Russia”. They talk about whether they are allowed to go “back”
39. Czech Republic = 0 “Zero!” Children are very happy about this. S says this is colder than any oof them. N say “But it might be minus in other countries”. C says “can’t go to Russia because it’s too hot for that. Let’s just try”
40. He clicks on Poland by mistake (0). Russia (-6) C laughs. Ns says “Is that minus?” C calls it six minus.
41. C is very excited “Minus 6. That means you’re not allowed any more than 6.” S says it might get better. -8 or -10. I ask if that would be colder. S says yes. I say that I don’t understand “minus numbers”. C explains that it means “for -6, you must take 6 away from 6. I challenge this and pretend I still don’t understand – that that doesn’t make any sense to me. S agrees that it is confusing. C shrugs his shoulders and says “no idea.”
42. Iceland next – 0. They think it’s warmer than -6 and suggest going there first.
43. They want to check Finland = -4. C says “But it’s the same because the clothes are the same. I ask “Is -4 colder than -6?” C doesn’t know. N thinks not.
44. N wants to look at Norway (-3). C says “That one’s even less than that!” They look at Sweden (-2) and are disappointed. They agree with me that Russia is colder than all of these. C suggests leaving Russia out and just going to these. The bell goes. N says he would rather carry on with this than have playtime. They look at Svalbaard. Someone says “Oh no! That was just one less”
45. -13. C says “Russia is hotter than that.” S says no it’s colder, not hotter. I ask which is coldest -6 or -13. They agree that -6 is hotter. I ask for a temperature warmer than -6. C says -2. S says -1. C says -0.
46. I ask which is warmest 1 or -4? N says -4. While he tries to explain he gets confused and changes his mind.

Appendix 4b: Write-up for Session 2 C, S and N

Minute number:

1. I tell children that we'll being doing a quiz and they must do something with these cards (show children a reduced pile of cards – i.e. not all of them)
2. I explain that each card has the name of a country and a number. C or N says "Russia is minus 6" C says "I love Spain" S says "And that one's minus 13. The boys speak the names of some of the countries on the cards. S and/or C call out "minus 1!"
3. I ask for children to put cards in order with the "highest numbers" here and the "lowest numbers" here. Someone says "Russia's high. It's minus 6 – that's high"
4. N wants to do the hottest first. 9's hotter than Spain isn't it? he asks. C talks about 6 6 6 6 then 5 5 5 5. C says "I think that one's colder than that one". N tells C "Russia is not meant to be in the hot section". They both agree that it is -6 but C doesn't understand. (He only "sees" the number, not the sign)
5. C reads 13 as 3
6. N is not happy because he thinks minus means cold and C has put Russia in a postion where the numeral is correct but he ignores the signs. I ask C to read out the numbers from the ordered cards. He reads 13 to zero in order but doesn't mention any signs. N says "He's wrong" S does it the same way as C.
7. As S passes Russia (-6), in the sequence, C says "See, you don't have to say the minus" N read the list and includes all (minus" words. N explains that minus means cold, below zero. S says "minus counts".
8. C and N argue about whether minus goes below zero.
9. C reads the new list where -6 is now below 0 but still doesn't say the minus word. S and N say "all the minus ones are colder and should be below zero."
10. S mentions "lowest cold". N mentions 6 under zero, 4 under zero. S reads the list properly.
11. I ask what made them change their mind. S and C say that it was when they said 6 under zero because then then they realised that one under zero is hotter than 6 under zero
12. C talks about snow – that higher snow shows that it is colder so -6 is higher snow than -1
13. I remove some cards with duplicate temperatures to make room to add in other cards
14. N mentions "my country". Someone says Turkey.

15. C interpolates cards. N says Portugal. I ask children to describe where they are putting things. C does one but says hotter than, rather than colder – he corrects himself immediately.
16. C wants to “go for” Norway, N for Hungary – they laugh at the name Hungary. C says Norway is 3 (i.e doesn’t mention minus)
17. Someone jokes about Neverland/Netherlands – Peter Pan. Someone remarks “Poland’s hot”. I ask C to describe why Netherlands is where he’s put it. C says it’s “under the 5” But he knows it shouldn’t go in the “minus section”.
18. One more each. N remarks “No Niger in here! My country”
19. I tell children to stop rushing. N says “It’s hotter than France but colder than Italy” (good at explaining)
20. I explain the question mark icons on the screen. First question “Is Norway hotter or colder than Russia”
21. C says “Yes it is, yes it is. Give me Norway! Yes. I got it right.” I tell C to calm down.
22. Children are laying out cards neatly on the table.
23. C says “I got it right. Look! There’s Norway and there’s Russia” C thinks Norway is hotter than Russia.
24. C explains “Because it says it on the card. Three minus means 3 behind zero and that’s 6 behind zeo. S says “So that’s hotter than Russia.” C “Yes, that’s what said”. I ask “How do you know?”C says “because 6 under zero is real cold but 3 under I’s only a little bit cold”
25. Next quiz question”Name country 12 degrees colder than Portugal”. They look for Portugal on the cards.
26. N says “It is 12. Zero. Iceland” He understands that any that are zero will work. N says 12 less than 12 is zero
27. S says “I don’t get it.” I tell N to expalin. C say it’s easy – he counts the cards but skips some – he is corresponding one count with one card and shouldn’t
28. I point out the thermometer icon and say it might help.
29. Children have never see the thermometer ITP before. I demonstrate what it does.
30. C tries to put the thermometer on 12. He counts down 12 from the starting point (which was 20 degrees). N says “It’s gone past 10” Cs “wanders” with the virtual thermomenter. N says 12 is 2 more than 10.
31. S thinks its 28 (has she counted down from 40 – max label on therm?)
32. I and N explain to S how things go up from zero.
33. Thermometer shows 15 but S thinks is 25. S correctly reads 19.

34. I show that if it starts at 12, we can use the change display to help find 12 below the starting temperature.
35. They count down and use the change display to check they have counted correctly.
36. Back to the quiz. "If travel from Denmark to Estonia, what will happen to the temperature?C jokes "It'll go higher"
37. I ask "How are you going to find out?" N uses the cards, He says it changes by 2.
38. I ask "Does it go up or down?" N says down. I ask "hotter or colder?" C says colder. S says "higher or lower? lower". New question on quiz – "Name a country where the temperature is between 3 degrees and 6 degrees" S misreads the question and says she doesn't understand the degree symbol
39. C says "I know what it is. It could be 4" Ss says 5. C says Neverland because that's 4. He reads the next question "Name a country between 6 celsius and 8 celsius. He has misread it – it actually says -6 and -8.
40. Chalrie says Spain is 7. S says no because "there isn't a minus – it's gotta go down here" C argues because minus means under zero.
41. I say that the lowest one on our list is -6. C still argues that we shouldn't be looking for minus anything. Eventually he does see the minus signs in the question. The boys know that they need a -7 country and
42. we don't have one. They check the map on screen but it doesn't help. The latitude and longitude lines are confusing them.
43. Children discuss countries they see on the map. They try to click on the map and on the £1000 banner on screen.
44. They chat about travel, airports, driving
45. Back to the quiz. S asks about why time goes backwards in UK. C wants to watch the prize banner until someone wins.
46. Next question "country between 0 and -2?" S says, there's only one and it's got to be minus. They look for countries. N says "It could be any of these 3 – I'm choosiing .." C says "I don't get it – because -1 is going to be under zero but what about -2. I ask "Why can't it be 1 rather than -1?! C says
47. .."because -2 isn't on top."Next question "country between 10 and 15. Children are bored with this question type. They look at a "pink question" "If I'm in Albania and go to somewhere that is 7 degrees colder, where might I be?"
48. C suggests going to the map. N and S say it won't help. S says Albania is 6. She suggest Spain. I repeat " the question says 7 degrees colder" The boys count down, C counts 7 cards.

49. C says “Lets finish it quick. Is it Spain?” I ask what is the temperature in Spain? Is it-7 degrees?” They have looked for a country that is 7, not -7 – i.e have ignored the sign.

50. Finish. Children go out to play.

Appendix 4c: Write-up for Session 3 C, S and N

Minute number:

1. I ask what we did with the cards last time. N says “Put them from highest temperature to lowest”. Cs says we went to counries . N says they also answered question, some were temperature questions.
2. I say “Remember the quiz? We’re going to do some more questions and look at a couple of other things.”
3. S clicks on the question “Name a country 12 degrees lower than Portugal”. They’ve had this question before but can’t remember the answer, C remembers that Russia was -6.
4. I look for the list of temps in my bag – can’t find it. The children find the Russia card and confirm that C is right – it is -6. But does this answer the question? I ask what else we need to know to answer this question.
5. Eventually C suggests that they look at the temperature in Portugal and “go 7 down, -6, 12. C looks for the Portugal card. It shows 12 degrees. C counts back on his fingers.
6. He finishes on zero. I ask “So, are you looking for zero?” C says that Russia could be the answer because it’s under zero. I challenge this. C insists that Russia would do. They remark that the map didn’t help and C mentions the £100 banner.
7. I ask if they can see Portugal on the map. S finds it..
8. I ask whether they still think Russia is the answer. Ss says no. I offer pen and paper and children accept – they think it will help. S suggest writing down Portugal and all the ones that are minus and then they will be able to see which one is the lowest.
9. S repeats “then they will be able to see which one is the lowest.”. C says “But we don’t know how long the minuses go down” He looks through the cards for the “lowest minus”. C says it’s probably Russia and says that they need to write Russia at thebottom of the list.
10. S gets all the minus cards together.
11. S makes 2 piles – one of “minuses” and one for all the others. She says that she is going to sort out which is the lowest.
12. C corrects her – “the highest, you mean, out of the the coldest?” N argues “No, the coldest, I think she means. C replies “Yes, that’s what I said.” C is playing with mouse and map – is not really paying attention to the discussion or is pretending not to.
13. N says “Russia is a big one, isn’t it?” I ask N to explain what he’s doing on the paper. He is writing names of countries as if they were positioned against a vertical number line. (see children’s

annotations). N mentions that he is in the lowest group for spelling and is surprised that he can spell some of the names.

14. C interrupts and takes over saying the numbers in order -1, -2, -3, -4, -5, -6. I say that this is going to take ages – what else can we do? C suggests looking at the map. I point out that they have been looking at the map already but does it help? C says No. N asks why is Asda on the map (advertising banner)
15. I return children's attention to the question. C still thinks the answer is Russia. I open the Thermometer ITP and recap that we know that Portugal is 12 degrees and Russia is -6. C says "But Russia is better, it's lower." I ask "But is it 12 degress lower?" N says "We have to go to 12, C, I've worked it out – it's zero. 12 lower than 12, 12 lower than Portugal is zero."
16. C counted down on his fingers. They find the card with Czech Republic (which is 0 degrees). N puts the thermometer on 12 – he says we chould have counted down to make sure.
17. I get children to count down 12. S doesn't hink that the thermometer actually shows 12 anyway. I move the marker to zero and show children the "difference" box on screen which is showing 12.
18. I point out that the change box shows -12 and ask what this ms all about. N says that is shows that we counted down 12. I ask again why the change box shows minus 12. C says its because its below zero. N says "because of counting down minuses".
19. C asks me about whether I am a teacher and I reply. I ask N to explain and he says the same again, adding "it means take away, sort of."
20. I move the thermometer up and down so that the "change" display keeps switching between negaive and not. N is still confused though he thinks he understands something "and then when the thermometer goes down it's like taking something away so you get the takeaway sign."
21. I encourage N to use the mouse and then to be the teacher and explain to C and S. C feeds back that if you "go down" 2 its minus 2 and if you go down 12 its minus 12.
22. I ask why there is not a minus sometimes. N replies that this happens when "you go higher". C says that he understands but S says she does not.
23. Boys are chatting about the thermometer confidently.
24. I demonstrate again for S. I ask whether they can predict what will be in Change box. N can do it. C is not sure.
25. S is still unsure.
26. S still does not understand.
27. We return to the quiz questions. N's turn.

28. He clicks on “Name a country 1 degree warmer than Finland. N says that the first step is to get Finland. S thinks the answer to the question is Egypt. N says that he thinks Finland “is pretty cold, isn’t it?” I ask S why she thinks Egypt is the answer. She replies, because its hot. The children find the card for Finland which shows -4 degrees. C wants to use Czech Republic (which is 0 degrees) as the answer. I ask “Is it 1 degree warmer?”C agrees that it is not.
29. Bell goes. We stop for playtime.
30. After playtime, we resume. The children say that Norway is the answer to the question.I ask S how that can be right – that one has got 3 on the card and one has got 4 – “How can the one with 3 be warmer”
31. C says because “that’s only 3 below and that’s 4 below zero.
32. C repeats “That’s 3 under zero colder and that’s 4 under zero colder. -4 is colder than -3.” They click on another question “Name a country that is 3 degrees colder than Luxembourg”
33. They look for the Luxembourg card. N says he’s already found one. He seems to be very uncertain and confused. I ask him to explain.
34. C says that “you’ve got to take away”. I ask why.
35. .. that it doesn’t tell you to take away. C says “But its like taking away, isn’t it?” He counts back on his fingers. N does too 1, 0, -1, -2. I suggest writing something down like a teacher would on a whiteboard.
36. N writes 1 – 3 = -2. I ask N to use the thermometer. He does it correctly (He uses “mathematical” language and talks about taking away, rather than the temperature language.)
37. I ask C to demonstrate the same problem. I ask what is 1, why 1? – they all seem to have forgotten
38. C talks through the problem correctly. Then he questions himself – he’s not sure about something but doesn’t know what.
39. They click on the question “Name a country 4 degrees warmer than Norway”. I ask S what we need to do first. S finds the Norway card. C says “3 minus”
40. C says excitely “It’s 1! Its 1!” C plays the part of the teacher with the thermometer.
41. He talks about 3 Celsius but puts thermomeer on -2. N notices its not 3. C moves it to -3 and still talks about 3 celsius. I challenge him.
42. The display is showing -3 and C still reads it as 3. Eventually he corrects himself and says minus 3. He moves the thermometer to 3. I asks why. C is confused is not listening. He calls out 7. N agrees, 7.

43. N reads out what he has written. “3 minus ..” C interrupts “But it’s not takeaways” N says “It is. It starts with 3 minus take away 4”. I ask why take away 4? N says because it said in the question. C read the question out again. N says “Oh warmer, 3 celsius add ..” C questions why he’s doing minus. N says that he’s right because it starts with 3 minus … N says -2, -1, 0 C says “You’re not going down, you’re going up. N agrees, he is going up and he repeats the numbers.

44. S says “If it’s add, you go forward, if it’s take away, you go lower. N says word celsius for minus when he reads values. C put thermometer on -3. He count up 4 and gets 1.

45. C is confused because “it didn’t say minus on the question, only on the card”. He still doesn’t see/say the minus sign.

46. N explains how he “pictures minuses”. “I think of a tube of ice, blocks of ice, big blocks of ice is zero”. He draws this, he says “ 1 celsius, 2 celsius etc and writes down -1, -2 as he speaks

47. C draws his picture too. He also draws ice cubes.

48. He makes is colder and asks “What is infinity?” N says it’s where numbers never end. N adds numbers to C’s diagram. S extends it to -4.

49. They click on new question. “Name a country 3 degrees colder than Cyprus”

50. N looks for the Cyprus card. S finds it. It shows 9 degrees. N wants to write it down 9 degrees – 3 degrees = 6. C says “We’ve got no 6 ones.”

51. N explains that if the question says colder, in this language, it means take away.

52. New question “Name a country 10 degrees warmer than Sweden”. S wants to explain. They find the Sweden card. C says 6 minus.

53. The temperature is -2. S writes 2-10. She changes this to -2 + 10. C says that they’ve got to add 10 because it says warmer. N agrees.

54. C says the answer is 8. I ask S to use the thermometer to work through the problem.

55. S sets the thermometer at -2. She slides the red up and counts up 10. I ask “What temperature are we at now?” eventually they agree that it’s 8

56. S knows that if the question had said colder, they would have done a minus.

57. The group talks about the other thermometer on-screen buttons

58. I ask “If we start at Belarus which is -3 (I put the thermometer at -3) and want to know what the temperature is if its 20 degrees warmer than this…? N moves the thermometer up and counts up 20.

59. I have to help with the count and controlling the thermometer red bar accurately.

60. N sees the answer is 17.

61. I ask how we could write this down. C writes -1 + 20 = 17

62. I say "Lets start at -4 and go

63. 30 degrees warmer. N counts up as thermometer moves. Something goes wrong with the application and I quit the thermometer program.

64. I check that Camtasia is still working.

65. We reopen the thermometer and set it at -2. N needs to add 30. He goes to 30 (i.e. adds 32). When he corrects himself,

66. C thinks he has made a mistake. I recap and confirm and ask how we would write this one down.

67. C writes -2 + 30 = 28. New one – start at 6

68. and get 8 degrees colder. Ns slides and counts down 8. C thinks the answer shoud be -3 – he is counting down on his fingers – 6, 5, 4, 3, 2, 1, -1, -2

69. N can see that the correct answer is -2. C writes

70. numbers as a vertical number line. Now he answers correctly. But his first count is his start number so he should get wrong answer

71. I ask N if he can explain how C gets the right answer when the method is wrong. It is because he is not icluding zero as one of his series of numbers

72. Bell has gone. Children are confused with task and tired, now. I ask C why he didn't draw a number line – he doesn't know.

Appendix 4d: Write-up for Session 4 C, S and N

Minute number:

1. -6 S wants to do Swimming Pool Sid. Ne thinks dimensions 9 by 9 is an area of 18. They find that 10 by 9 is 90 but can't work out that the 2 dimensions are multiplied together.
7. … even when I try to prompt it. S says "If we do 10 and 2 it will be 20" The boys ignore her. Change game to Reflect a Sketch.
8. S is quick to see how to do it and helps the boys.
9. – 12 C understands it.
13. S explains how she wowrks these out, what she looks at.
14. – 17 They change to a horizontal line of reflection
18. N says "We're doing better than the one before."
19. They go to Balloon Burst. While I explain the game, S asks "Can we use paper?" C asks why. S says "Because you have to add"
20. balloons are -3, 2, 9 C reads the numbers but doesn't say word minus. N is quick to correct him. C says "It's 11 – 'cause you add 9 and 2 together … S says "But then you minus 3 away" C says "It's gonna be … 11, 10, 9 " S says "I thought so – you added only 2" (I think she means take away)
21. balloons are 1, -7, 1 S says "1 add 1 minus 7. C says "Zero!" He enters 0 as the answer and sees the answer on screen and corrects to "Minus 5 actually" Niamh asks "What are we doing?" C counts on his fingers 6, 7, 8, 9, 10 … it's 9.
22. C reads out next set of balloons are -4, 8, 1 (last one should be -1 but C doesn't say minus) C says 8 take away 4 take away 1 … 3
23. Ballons are 1, -5, -9 (But C doesn't say minus again) S says "Zero, No. It's got to be minus something" C says "add 5 and then you'll know what minus it is … 15, -14! I added 9 and 5 that makes 15 but then I've got one that makes it 14." I ask why minus? C says "Because it's got minuses there" I ask "Why don't you just add all 3 numbers and call it minus?"C says "No, because one of them isn't minus"
24. I get C to think it through again and he spots his error He types in -13 (correct)
25. I set the game to 5 balloons. S says "So we know it was minus because for this one it was -5 because it went over zero. New balloons are 0, -4, 9, 2, 8 Ss says "You could do the tens and units thingy, She writes the numbers down and asks "Are we adding or taking away?"
26. I say "S has added all those numbers together and she's got 23. What do you do now?" S says "Zero doesn't count because you don't take anything away." Boys want to put 23 in as the answer.

27. They take another look. C rushes into the next set of balloons 0, 4, -2, -2, 7 C says “Zero, which is nothing, then 7,8,9, 10” S says “Then you minus 2” C says “Wait, 7, 8, 9, 10, 11 minus 9” N asks “Are we sure?” C says Yes. (he’s wrong of course – they hadn’t written all the numbers down properly)
28. 5, 5, 10, 3, -1 S says “ 5 and 5 is 10, 20, then that’s 23. C says “take away 1” S says “22. I ask “Why are you taking away a number? You’re supposed to be adding all the numbers together. N says “Because it’s a minus” C agrees “’Cause there’s a munus 1 there” I ask “Why does that mean take away?”
29. I persevere “You haven’t explained it very well yet. Why, when you’re adding a minus number, does it mean you take it away?” C says “’Cause it’s like temperature – ‘cause you could have 19 degrees and if you take minus one off it’ll be …” I ask “So, its like temperature?” N says “Sure, minus, below” C agrees “Yeh, cos we did it last time and I remember. Santa Calus taking his clothes off- stripping”
30. They type in 22 (correct) New ballooons 9, -6, 6, 5, -8 S says “They make 11, then add 9 ….. 20. “
31. C says “Then take away 8” But he can’t count back, gets confused. C “7?”
32. N asks “Shall we try to work it out again?” I say “Does it help to take the 8 away first?
33. C says “Yes, because you add the highest no first …. “
34. I say “Let me show you somethin else You’ve got 6 and -6 “ Niegel says “That’s zero” I say “So you just need to add the others together … Get me a ruler.
35. We are going to use a ruler to help us. We’ve already agreed that +6 and -6 is zero ….
36. N says “add 9 is 14 take away minus 8 .. 9 C says “not 9, 6”(he’s right)
37. 7 balloons 1, 1, -2, 0, -4, 4, -4 Cs check that we’ve got the right number of balloons. He says “4 add 1 add 1 is 6. Take away -4 …” Boys are distracted. Ss says “Minus 4. You showed us the way of adding all these ones (positive) before we did thes ones.” I said “I didn’t show you that. Has someone else shown you that?”
38. S says “Altogether they make 6, take away 4, that’s zero, then take away another 4, that’s minus 4. (correct)
39. -41 -2, -3, 0, -1, -3, 3, -3 C recaps the numbers. Ns says “) and 3, only 2” S separates munuses. C says “3,6,9,10,11 add 3 that’s 11 take away 3. 8” I ask “Why take 3 away from 11? C can’t explain. They get it right.

42. I write -9 9 and ask “Whats the difference?” S says “Well, you add the line and then it tells you to take away it …” I write 2+3+-4+-1+2=
43. S has already latched onto the “pairing to cancel” idea but gets itslightly wrong Ns keeps running total 5, 1, 0, 2
44. 0, 1, 0, 5, 3, -1, -3 N speaks his running total “1, 1, 6, 9, 8, 5, “ I ask “What if I say 4 take away -1? and I write 4 - -1
45. N says “5 because when say add you take away so it might be … “
46. I ask “What is 3 times -2?” N says “You might have to divide it. S says “3 times 2 is …” no answer.

Appendix 5a: Write-up for Session 1 L, M & G

Minute number:

1. I show children the map and explain list of countries and the way the on screen database works, including the different types of information it holds. They laugh at FC clothes
2. Children can tell me there are countries and some continents on the map. They know that it is hot near the equator. They ask where India is and understand why it is not on the map.
3. I explain the journey task
4. I emphasise “We don’t want FC to have to take clothes off .. “They realise “So we’re not allowed to make him get hotter” Boys discuss which hot place to start with.
5. M suggests “Let’s click on Meroon” (similar to his name) 23 degrees. L points to Nigeria “I think that’s hotter”
6. They check Nigeria. They suggest taking turns, G thinks Nigeria is a good one to start with M says No. They argue about whether to go to Niger or Libya
7. M is excited “It’s 25! He’s taken his thing off.” They decide to go to Chad instead of Nigeria. M, “Yes, 24 . Put him on Chad. Lets go to Chad. I ask them to recap. Meroon 21, then Nigeria.
8. M and G argue whether Meroon was 21 or 23 then agree it was 24. I check “So, it’s getting hotter?” They check Nigeria – 21 degrees. They revisit to check Meroon 23. They look at Niger 25. They remember that they had decided not to go there and went to Chad instead 24. I say it’s too hot. G says – FC has same clothes. M insist “But it is the temperature that is most important.
9. They try Benin. I ask “What are you looking for?” G says, something colder. Benin is 25 and the say this is too hot. They think they are stuck unless they “jump over”. I tel them that they can – they had thought they must move to adjacent countries. Boys are all very excited. They find that Burkina Faso is 25. M says “We could jump over to Sunday”
10. They check Sudan 25, Libya is 13. L says Yes, that’s better. I want to slow them down. G wants to go to Algeria. M says No as its realy, really hot (but they haven’t looked it up yet) When I ask why they think that
11. he says because it’s got sand. Algeria is 13. They check Morocco. I recap – Meroon 23, Nigeria 21 What’s next? Libya? .. (is 13). They are excited and don’t want to stop. They check Spain and find that it is 7 degrees.
12. “It doesn’t seem possible – I’ve been there and its really hot. They argue about whether France is hot. I ask for a recap. Spain 7,

France 5 the UK. Someone says it's too big a jump. They find UK = 6. They think this is "too high – that was 5"

13. OK Belgium. Let's see Belgium" They find that it isn't on the list. "You'll have to go to Germany. There, 1 degree" Boys very satisfied with this. "We'll go there. But that's bad because we have to look for somewhere that's zero. Poland = zero. They carry on, talk about Norway and Sweden.
14. I ask "Hold on – if that's zero, what are we looking for?" Chorus, emphatic "minus". I ask "Is that colder, then?" "Yes. Let's check out Sweden." They find it is -2 "Yes, that." They laugh at FC coat. They try Norway , -3 degrees. L starts to recap, I continue Poland was zero, Sweden -2, Norway -3.
15. They check Svalbaard -13. "We made it!" I check route with them Meroon 23, Nigeria 21,
16. Algeria 13, Spain 7, France 5, Germany 1, Poland 0, Sweden -2, Norway -3, Svalbaard -13
17. I tell them I want them to think about the numbers from 23 to -13 – "Is that colder or hotter?! They chorus, confidently "Colder". I ask what has happened with FC clothes. They tell me he has been putting more on. I ask "What's happening to the numbers?" M says they are getting smaller.. when it gets past zero its into minuses which is really cold"
18. I ask why is this (5) hotter than this (-13)? M says because that one (5) is over zeroand its not a minus so its hotter. Minus is colder. I ask "So will any minus be colder than even abig plus number? G says Yes, the big plus numbers are hotter because they're closer to the equator. M agrees – Yes, the equator is like the hottest because it's like the oven. And that's like the freezer up there. I ask "So give me a number, any number , L, that's bigger than -20" M laughs "-19". L say 30
19. I ask "Give me a number that's smaller than -20." L says -29. I ask for one between -20 and -29. L says -26. I ask "If you're somewhere that's -13 and go to Norway, how much is the temperature changing? 10 degrees, they all tell me emphatically. I ask whether it is increasing or decreasing. Lowering, decreasing is the reply. They correct themselves, increasing.
20. I ask "And if I go from Norway to Sweden?" M says it gets smaller – then corrects, saying "It increases by one. But it does kind of decrease as well. M says "But its minus" I ask "So if I said what is minus 2 add 5 ?" M answers 3. G says 2 – they argue G explains because you add on to get to the zero. M says 2 take away 5 is 3 which means you'll get 3. I ask M to draw what he means.
21. As he draws, he say "You're on -2 and you have 5 so you take away … 2 which makes zero which means you have 3 left" G wants to do it. I intervene "But I asked you to add 5, not take

anything away. G mentions “clonk” M understands him – he describes it as a brick and draws a diagram

22. I ask if anyone has taught him to do this L wants to explain his way. It is similar but includes going “up” to zero and then past zero. Number line model) G thought and still thinks answer is 2 – he has come confusion around the zero. Others are both confident with 3.

Appendix 5b: Write-up for Session 2 L, M & G

Minute number:

1. I ask what boys remember from last time. L recaps the FC journey task.
2. Boys ask if their faces are recorded or just their voices. I explain that the screen is recorded too and that they should avoid knocking on the table
3. M tells me that webMs can be used on MSN. I shuffle the country cards. L asks “Are we playing dominos or something?” I explain that there is no map today but that we do have some of the same information on the cards. I ask them to order the cards with the highest number here and the lowest one there
4. They realise immediately that they don’t know where to put first cards because they don’t know what others will be. G says some will go up to 20 and knows that not every no will be represented. Boys start to put cards on the table
5. More placing of cards on table. They realise there are sometimes more than one country with the same number. This doesn’t worry them
6. More cards. They take turns and do each one confidently and correctly.
7. More. The boys are well-motivated and enthusiastic.
8. More
9. More. They need to keep moving the cards to create space for new ones.
10. Someone mentions Monaco Grand Prix
11. The read the list of cards in unison.
12. Coldest is Russia. There are seven countries with zero. I open the quiz slideshow
13. Question asks for country between 0 and -2. M immediately says -1. G says Bulgaria. They read the question correctly and say “Minus”
14. Question asks for country between -6 and -8. They agree to use Russia -6 because it’s the lowest one they’ve got. They understand that they need a -7. I ask whether there is something else we could use to find out an answer to the question
15. M suggests Google. I show them the thermometer ITP. The boys think the map will help and click on the map. They mistake the latitude longitude lines for temperature labels. I point this out and the boys agree they must ignore these.

16. I ask “Is there anything else on the map that could help us?” M says could click for information to see how hot or cold it is. G says maybe it doesn’t work like that. L says “I know a cold country” M says “Egypt’s cold” (he is joking) someone thinks Ireland is cold. Someone else thinks Iceland is colder. I ask why. He says because it’s further up. The equator is further down and it’s the hottest place. M says he’s been to Ireland and it’s not that cold there.
17. L remembers that there had been one that was 13minus on the other map. They think it might have been Slovakia (remembering the S and L sound in Svalbaard?) But then they remember that the question needs between -6 and -8.
18. L clicks on a “high” yellow question. Is Estonia hotter or colder than Croatia. M says Go on the map. G says Croatia is here (looking at cards). He finds that Estonia is -2 and says So it’s colder.
19. M says “between 3 and 6” (Reading from a new question) L says 3 and 6, up there! M says Minus! G says No, it’s not minus. L holds up cards and says France and Netherlands.
20. G reads next question 4 degrees warmer than Norway. M remembers we had Norway last time. They find the Norway card and M counts up 4 (not sure whether he is counting cards or looking at the temps). M says Germany, Germany. L asks if the colours relate to continents because the blue ones were about Europe.
21. I recap the question. I ask M to explain how he got to the answer. He says Norway is 3 so you need to go up to -2, up to -1, up to -0, to 1, not -0, to 0 12 degrees colder than Portugal. They find the Portugal card quickly
22. Thy count down together in ones to 0 and know they need a country with 0. Between 10 and 15. “It’s out of these – Portugal”
23. They are not bothered that there isn’t a card with 15. I ask them how they do it if there is no 15 card. Someone repeats the question. I say “So you’re thinking just about the numbers, not the cards?”
24. Question : In Norway, go 4 degrees hotter, where am I? Someone says Norway’s down there. L/G “Oh its back to the ones. M says Romania, Hungary then jokes about hungry. Someone says that Romania is in Harry Potter – dragon
25. Question: travel from Russia to Sweden, what happens to the temperature? Someone says “Russia’s down here. It’ll go 1, 2, 3, 4 degrees higher or 3 – I don’t know which. Not sure whether to count 2s. I say “So you’re imagining that there’s a -5 a counting that as your first. And Finland as your second, -3 means you have gone up 3, to go to Sweden because Sweden is in 2 – i.e they don’t know whether it should be included in the count or not.

26. Someone asks “Why don’t we start on that on (Russia) and end up at Sweden – they get the answer 5. I suggest they pick another green question because it will be similar and might help them.

27. Question: If travel from Denmark to Estonia, what happens to the temperature? Someone answers, 3 degrees lower. M was doing it in the wrong direction. M says So, its still the same. L insists it does matter – that Denmark to Estonia is not same as Estonia to Denmark. M says it is the same and he counts 1,2,3 in both directions to show it is the same.

28. L says Yeah but you’re going up not down. G tried to explain to M why the direction is important for this question. M still thinks it’s the same. Someone says that way its getting hotter and that way its getting colder. L find both cards and stresses that they have to do DOWN.

29. I ask M to re-read the question. He says the temperature will go by 3. He keeps repeating this. I ask “go what by 3?” M says go down by 3.

30. Question: If travel from Ukraine to Turkey what happens to the temperature? Someone says one degree higher, up by one. BELL for playtime

31. Question: Is Norway hotter or colder than Russia. I ask where would Norway have to be if it was colder than Russia? They tell me “that way” (off the table) i.e. they have the concept of coldest . Question: country between 1 and -1. M reads it as one and one, corrects to minus one to one,

32. then corrects again when challenged by G. Gets the right answer. Question: country one degree cooler than Croatia

33. I ask them what they are “looking at” to help them. They can’t explain but seem to use cards and relative positions of groups of cards with equal values.

34. I take control of the mouse and fire questions at them. They find them easy. Question: start in Monaco and go somewhere 4 degrees hotter

35. They start with Monaco card and argue about how to count cards. When they don’t have cards for every value they have to find other strategies to visualise, mark where they’d be to help them to count up/down

36. They get the right answer. Question: start Czech Republic and go somewhere 3 degrees hotter . They get it right, tapping table to show missing values within line of cards. Question: If travel from Belarus to Belgium what happens to the temperature? They locate both cards and answer “will go up 7 degrees”

37. M counted Belarus as the first count. I ask why. M says because the question to start on that one. I model use of the thermometer ITP for Belarus -3 to Belgium 3.

38. G puts thermometer on -3 correctly. L (eventually) moves it to 3.

39. I explain that this ITP is good for showing what happens when you go from one temperature to another. I point out the “Change” box, displayin 6 (i.e. not 7).

40. I ask what happens if you travel from Slovakia to Albania? They find the 2 cards. They try to count the difference with cards and argue. L reminds them that last time they were wrong to count the start number

41. They touch the zeros and count as one, two etc. I ask them to do it with the thermometer

42. They get the correct answer. The change box shows 7. M asks if he can check the “difference”

43. (it would seem that M has something in his mind regarding change and difference. Question, what happens to temperature if travel from Germany (1) to Sweden (-2). M immediately says its going to go down 3

44. I tell M to tell L what to do with the thermometer. Change is -3 this time. M says this is because it has gone down 3 but in the difference box its just changed by 3.

45. I recap that the size of the difference is the same whichever way round you do it but the change tells us which direction you went in

46. L picks Russia -6 and Gibraltar 13. M has control of the mouse. L works it out by counting cards and reaches answer of 19. L changes his mind about the direction he should count in.

47. M does the thermometer correctly. With the next example, M can predict what is in change and difference boxes.

48. I ask where will I be if start in Moldova and go somewhere 9 degrees warmer. L works out using thermometer what temp will be. M finishes it off, saying Spain is 7.

49. They work out Netherlands to 5 degrees colder. They get correct answer ..

50. … using both cards and thermometer

51. I ask If I wanted to change the -5 showing in the change box to 5, what would I have had to do? M says add 10 (not what I meant). I explain that if the question had been from cold to warmer, we wouldn’t have had a minus answer in the change box. G says “Oh! .. because minuses mean you’re going down!”

52. I encourage them to do something to the thermometer – he isn’t sure what I want. G says “You’ve confused me again now”

53. I explain again by asking about more examples – “If I’m in Germany and I go to somewhere that makes the change box show ? where might I be?” … If I’m in Germany and go somewhere so that the change = 2 not -2, where might I be? G says 3 – says he

gets it now. I ask L (G interrupts that he is still a bit confused again) ..

54. If we are in France (5) and we go somewhere that creates a "change" of -4, what will temperature be? L answers correctly
55. I ask if the change is 4 not -4, what will temperature be? L answers correctly. I slide the thermometer to 9 – G think the difference will be -4 and is surprised when both boxes show 4. He asks why
56. I tell him that the difference is those 4 degrees. G says "So it can't get to a minus" I ask "I wonder if we could get the difference to a minus. G says No.
57. I demonstrate that whether the move is up or down, the difference does not have a minus. G is still unsure. G says to M "But we said it would add when you go up" (has he noticed there's never a + sign?)
58. I demonstrate 11+6. Boys predict change and difference will both be 6. I moves thermometer down – we all count interval of 8. I ask can they predict change and difference boxes. L thinks "One's gonna be -8 and that one 8. M says No, -9 and 9.
59. I give boys paper and say "Use paper, if it helps, use thermomenter if it helps. I suggest we start at 3 and go up/add/ get warmer by 10. I tell boys I don't know what to write – what should I do?
60. M says add 10 to 3, 13. L asks is it -3 or normal 3? 13. I ask "If I start at -3, then what?" L answers 7.
61. I ask Is he right? G and M use thermometer and get it right and can predict change and difference boxes. G says "I get it. When you go down, that one takes off how much you took off because it puts a minus as if you're doing a sum"
62. I ask Can you think of a sum that would show us a minus number in that change box? L says 13 – 5 – it's gonna be -5. I say "That's interesting. So you're actually adding a minus 5. G says "Now you're confusing yourself – because minus minus 5 doesn't make any sense. M says 13 add -5? G tries to explain.
63. G writes as he speaks 13 – 5 = 8. They are unhappy with the concept of adding a minus.
64. G write 13 = -5 … they confuse each other with their explanations
65. G seems to be trying to explain that the + sign is superfluous/redundant so is OK to leave it out. They keep "accusing" G of saying 13 add minus 5 and he keeps trying to explain it, though it's not clear whether he only ever said it because I hd (who said it first?) G says "I've got a number, then it goes down (M interjects "minus") G continues, "Then it goes down, minus 5 and I get 8 so..

66. .. the number was ? M says "Now you're just confusing me (hints of understanding subtraction of minus numbers, doing/undoing) I ask G what he thinks of what M has written 13- -5 G "I think that's wrong. You don't have to put … All you have to put is 13 minus 5

67. I ask "Does it mean the same?" G insists "no because it's got 2 minus. G questions meaning of first minus – "Is it add because its minus …" M thinks he's being ridiculous. I ask G to do 13 takeaway -5 with thermometer. G asks "How do I take away -5?" I say "That's what I want you to think about" G "That's why I don't think it's right"

68. I recap 13 – 5 = 8 on the thermometer and says "that's what happens when you DO a minus 5. But what happens when you takeaway a minus 5?! I start the thermometer at 13. "Imagine that I did a -5 to get there, where must I have been before?? They answer 18.

69. So we're saying that 18 – 5 would be 13? We check with thermometer and it is correct. M says "It's like 18-5=13? I say "So, if I'm at 13 and I want to take away the -5 that somebody did to me?" I repeat it. Someone says (triumphantly) "You have to go up 5!"

70. G says "So I was right! Because 13 takeaway a -5 is … 18! I get it! I get a little bit but not much actually." I ask another (10 - -2 =?) Someone says "You go up 2 to 12.

71. I tell them "So when you see that (--) it actually means something special. Boys say "take away a minus" I ask them "What shall I ask you next?" Someone suggests 3 take away 5. I say "Can you show me how you work that out .. do a little diagram"

72. Someone says "Because you take 3 away and you've 2 left …

73. Boys (L?) draw a thermometer to illustrate. L counts down 2, 1 and this line is 0 and we've only taken away 3 so we have to go into the minus.

74. I ask 11 – 15 Someone asks why we're not using the cards to help. I ask "What could you do with them to help? Or what else could you use to help?" L says "We could use a ruler. That's going to 11, we need another object

75. M says "I've got it 11 – 15 = 4 because 11 is smaller than 5, it equals -4 … (corrects himself) 11 is smaller than 15, there's a 4 in between so you end up with -4. Coz 11 is smaller so will end up with -4"

76. I ask "Can you draw something?" M writes what he has been saying, Eventually is says that when you're taking 14 away from 11, there's a 4 number gap in between so you get to zero and you've still got 4 left

77. L (diagram) draws a number line. He counts down to zero and shows that he has only taken away 11.

78. He says he was keeping count up to 15 “while he moved through minus”
79. I give G new question – 6 add minus 8. G is a bit confused because “it’s easier than the ones they’ve just done”
80. G’s explanation is similar to L’
81. … “have to keep going because 8 is 2 more than 6. I give them “one last really hard one -3 go down by 2” they respond -5 (no problem?) I then ask -3 takeaway -2. They want to draw it “It’s -1!”

Appendix 5c: Write-up for Session 3 L, M & G

Minute number:

1. to 7 Swimming pool Sid. They don’t know about area. G says think you have to “times them two”. Others don’t listen and suggest dimensions of 3 x 2 will give an area of 32. Then they try 4 by 8. G and L are first to understand how it works. They realise they can use the tables square on the wall to find factor pairs very quickly.
8. New game – Balloon Burst. 5 balloons. M asks for paper. L asks if he can write. They plan who’s going to write, call out etc.
9. balloons are 9, 21, -21, 8, -12 L says 17 minus 12 .. 5 that’s 5
10. balloons are -11, -24, 9, 14, 5 G says -11 from zero, that makes - 11 … M says No, add those together then take those minuses away…. That’s easy…..
11. L says 9 + 14 + 5, which is 28 … because they’re the only ones which are the adds. Then …1 add the other 2 minuses together – I got 35 and then it’s 35 minus 28 and that equals 7. (should be -7) L asks G to write the number next time.
12. balloons are 20, 17, -14, 15, 15 M says OK so that’s 30. 17 add 20, that’s 37, that’s 67 .. minus 14 … 3 … 53 L wants to add them up as he goes along.
13. New balloons 17, 13, 17, 17, 10 74 It’s either 74 or 64 but I’m gonna go 74
14. M asks for Space Invaders on main meu. I send them back to balloon burst and set it for 6 balloons 9, -8, -48, 48, -34, 33 M says OK so 9 … is it take away or plus? .,.. no no it’s plus. (he write it down 9 + -8 which is 1… G says You can’t do that. L says I think it’s 68. (answer is 0)
15. I ask boys to slow down and look at this more carefully G asks Can you add plus numbers to minus numbers. I ask if he was surprised when M wrote it down this way. G says he was. I ask M why he wrote them down this way – he doesn’t realise what I’m asking about (does not see significance?) and just tells me why he wrote them down at all – i.e. he couldn’t remember them unless he wrote them down
16. I point out to M that he then went back through the list of numbers and put all these add signs down. M says he needed to see what the numbers are. L says “So it got all muddled up.
17. M cancels -48 and 48 – he explains that they “equal zero” – “that’s minus and that’s normal”. He goes on 9 -8 is 1.. minus 34 is -33 … add 33 is zero.
18. M tries to explain to others how to follow his “string”. G doesn’t understand how you can add a minus. L says you must start with the 8 and put the minus there so it’s 8 minus 9.

19. G says No, "9 minus 8 so there shouldn't be the add sign" G keeps asking what's the point of putting the add sign?

20. I ask is 9 - -8 (written) same as 9 + -8? M says No, that's gonna be still 9. L says Yes that's right because we're in -8 at the moment, then add 9 and hat equals 1. Then -48 plus 48 is still 1. I ask them to focus on -34 +33. L says "in -34, add 33, that's -1" I take away 33 from 34 and that leaves me with -1 and I'm still in the minuses.

21. I ask why take 33 away from 34 if you're adding 33? L says because that's positive and that's negative and you take the positive away from the negative. I ask Why. L replies because its easier than saying …… that that's in minuses and you take away … ?

22. I ask if there is a rule or picture. M tries to explain using partitioning. I ask what is -8 + 6?

23. L says that equals -2. G agrees with -2 and says "But I did the other differently, same as L, took the 6 away from 8. I as why. G says "Because if you take away 8 …. if you take away from -8 then get the higher digits because its on the minus side. L says "Just imagine it's not in the minus 8 – 6 = 2 and then put it back in the minuses, minus 2.

24. M says "I was just gonna do that" I ask why would you do that? M says "I would do this, 8 and instead of having minus 6 just put 8, no just put 6 -8 seeing as the 8 is a minus (ie reordering) G says same as L (though he doesn't realise that what he says)

25. I redirect them to 9 - -8. L says "Is it 9 add 8? G says I know – its 1. M says "No its not, G, we're in -8." They try to re-order and get -8 +9 =1.

26. I explain that these operations are not commutative 4 – 8 is not the same as 8 – 4. M says It's -1 then. L asks "Who agrees that it's 1?"

27. I ask "Is there another way of explain it e g for someone who's not as good at maths as you?" L asks "By adding words into it?

28. e.g. increase?" G says that's even more confusing. I say "Oh, I was thinking of 4 somethings add -5 something else – your idea might be better?"

29. L says "I could make it 4 t-shirts ad -5 t-shirts. They laugh at the idea of -5 t-shirts. M says that -5 is 1/5 of a t-shirt. Someone says "Just do what G does, change the order so -5 +4

30. G says "If that was a 5 it would be zero but it's not a 5, it's a 4 s … zero take away 1 is minus 1. I ask them if this reminds them of last time. G says 50 -13. L says "It makes me think of countries with degrees. I ask if they can turn this into some sort of story about Father Christmas or countries.

31. L says "It was -5 at Antarctica." They laugh and say it would be more .. "And then he needed to go to …

32. England and in England it was 4 degrees … Actually I don't know what I would do … You'd find out how much was between them

33. G says "I know what I'm doing but I don't know how to say it. I ask the boys what else is in their minds at the moment. They say temperature, countries, adding degrees. I draw blank number lines horizontally and veritcally. They talk about the "minus side" and the plus side in the space on either side of the line (rather than at 2 ends or 2 sides of a point called zero)

34. G puts numbers on the line. I ask "how would you do the sum on there?" Gs talks aloud "Start on -5 and move up 4 to add 4" L says "Oh I get it, minus 1 add minus 1 is minus 2!"

35. I say "So can you show me it the other way around, 4 – 5? M says "It'll be the same because it's like times tables – it doesn't matter which way you put it round. G counts back along the line, realising that 0 does count. M doesn't understand "What are you doing?"

36. M says -9 minus, minus 8. L is excited "Ooh, I get this!" M says eleven. G says 9-8 is 1 ..

37. L says "I get it. I think it's minus 17 because you've to add the minuses together. These 2 have got to be bigger numbers.

38. I ask "Why can't it be -17?" L says "It can't be because when you add you get bigger numbers. M/G says "But it's not adding, it's minusing. G says "Minus minus … I don't understand" M says "minus 9 minus minus 8.. 9 minus 8 equals 1… so minus 9 minus minus 8 equals minus 1 – it's -11.

39. You add 9 plus 8, that's 11 and because its going up the minus" L corrects him "It's 17" M realises his mistake. L explains how to bridge through 10.

40. I ask them to use a number line to show it. They say they can't. G asks "How can you minus a minus 8? I don't understand. I explain the "undo" strategy. L sees it straight away. They can do them easily now

41. In their next game, Ghostbusters, the boys very easily respond that to get from -50 to 50 they need to score 100.

42. At the end of the session G says to me "That line thingy made me think of a thermometer" L agreed with him.