Control of Oscillatory Convolution Operators via Maximal Functions in Weighted L^2 Inequalities.

by

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A thesis submitted to
The University of Birmingham
for the degree of
Doctor of Philosophy

School of Mathematics
The University of Birmingham
APRIL 2010

Abstract

This thesis is concerned with the weighted $L^2(\mathbb{R})$ boundedness of the family of convolution operators corresponding to the kernels $K_t = e^{i\Phi/t}\psi$, where ψ is a smooth cutoff, Φ is a function on $\mathbb{R}$ that looks locally like $(\cdot)^\ell$ in the support of ψ for some integer $\ell \geq 3$ and $0 < t < 1$. Using the techniques of Bennett et al inequalities of the form

$$\int_{\mathbb{R}} |K_t * f|^2 d\mu \lesssim C_t \int_{\mathbb{R}} |f|^2 \mathcal{M}_{t,\ell}(\mu)$$

are proven, where μ is an arbitrary Borel measure on $\mathbb{R}$, and $\mathcal{M}_{t,\ell}$ is a maximal function depending on t and ℓ. The weighted $L^2(\mathbb{R})$ estimates that are derived are shown to be sharp in the sense that the $L^p(\mathbb{R})$ boundedness of $\mathcal{M}_{t,\ell}$ can be used to recover the sharp exponent in t for the $L^p(\mathbb{R}) \to L^p(\mathbb{R})$ constant for convolution with K_t when $\ell' \leq p \leq \ell$.

Acknowledgements

I would not have been able to do this without the help and support of my supervisor, Jon Bennett, to whom I would like to offer my most heartfelt thanks for his enviable wisdom, and his admirable patience. I would like to thank all my fellow postgrads for their continued friendship, but especially the good people of room 315 and Postgrad Mathletic. I would like to thank my family, my friends and Lisa for reasons that I could not possibly summarise in a few sentences. Finally, I would like to thank EPSRC for their generous financial support.

Introduction

The provenance of this thesis can be traced back to 1978 when Stein, at a conference in Williamstown (see [27]), suggested that it would be worthwhile to study two-weighted norm estimates for the disc multiplier operator. In the years leading up to this, the L^p boundedness of the disc multiplier had provoked a great deal of interest and speculation, and one of the fundamental problems of twentieth century harmonic analysis was laid to rest when Charles Fefferman proved that the disc multiplier is bounded only on L^2. Studying weighted L^2 inequalities for the disc multiplier could then help one gain further knowledge of its L^2 behaviour.

It is necessary at this point to clarify some terminology. By a *two-weighted* norm estimate for an operator T, we mean an inequality of the form

$$\int |Tf|^2 w \lesssim \int |f|^2 W$$

for all $f \in L^2(W)$, where w and W are a pair of a Borel measures, or suitable weight functions. In particular, we are interested in the case where w and W are in a correspondence determined by an operator (typically some kind of maximal function) M_T, for which $M_T(w) = W$. This also includes the case where $M_T(w)$ happens to be constant for each w. In this case, we have a functional $w \mapsto C(w)$

such that

$$\int |Tf|^2 w \lesssim C(w) \int |f|^2.$$

We will refer to such an inequality as a *one-weighted* L^2 estimate.

A pivotal point in the history of harmonic analysis was the birth of the study of A_p weights. When investigating the functions w for which the inequality

$$\int_{\mathbb{R}^n} |Tf|^p w \leq C_{p,n} \int_{\mathbb{R}^n} |f|^p w$$

holds for a suitable maximal function or singular integral operator T, it emerged that necessary and sufficient geometric conditions could be placed on w in order for the above to hold. Functions satisfying such conditions are said to belong to the class A_p. One of the canonical texts on the subject is the book by García-Cuerva and Rubio de Francia ([19]) which was among the first to dedicate itself to the treatment of A_p weights. The success of the theory of A_p weights has a led to a good understanding of two-weighted norm estimates for singular integral operators, maximal functions and square functions.

This way of looking at the L^p boundedness of operators proved to be very influential to the point that weighted norm estimates are now a familiar sight to the harmonic analyst, and are central to the study of certain important operators that don't fall under the remit of the A_p theory. In particular, we are interested in integral operators with kernels that display oscillation. What emerges when one examines how estimates for these operators have been studied in the past is that their usages tend to be somewhat disparate - the question of for which oscillatory integral operators T we can expect to have estimates of the form

$$\int |Tf|^2 w \lesssim \int |f|^2 M_T(w)$$

remains largely unaddressed. The aim of this thesis is to begin to explore this question, beginning with a family of oscillatory convolution kernels on $\mathbb{R}$. This family of kernels (which are introduced in Chapter 4) is a natural place to start and they give one an opportunity to see the role that oscillation plays in such estimates.

We begin by reviewing some preliminary material, namely some useful results from Littlewood-Paley theory, and some techniques for analysing certain oscillatory integrals. In the chapter that follows, we review some particular examples of weighted L^2 inequalities for the disc multiplier operator, and for extension operators. Since the material in Chapter 4 makes heavy use of techniques from [5], we pay special attention to the results in this paper. We then go on to prove two-weighted norm inequalities for the aforementioned family of convolution operators, and from this we obtain information about the $L^p(\mathbb{R})$ to $L^p(\mathbb{R})$ boundedness for this family. Finally, we prove a one-weighted estimate for convolution operators with radial oscillatory kernels on $\mathbb{R}^n$.

Contents

Chapter 1

Weighted Littlewood-Paley Theory

Littlewood-Paley theory is a broad term relating to a collection of results concerning the L^p-boundedness of operators with some sort of quadratic nature. The L^p-boundedness of classical Littlewood-Paley operators (or 'square functions' as they are also known) is well understood, and the study of such operators and their applications forms one of the canons of harmonic analysis. A good introduction to the subject is given by Duoandikoetxia in [15]. One of the classical applications of Littlewood-Paley theory is the famous Marcinkiewicz multiplier theorem, which gives sufficient conditions for a Fourier multiplier to define a bounded operator on L^p. Among the various types of square function that are of use in harmonic analysis are ones that involve decomposing the Fourier support of a function f in certain ways. For example, if we let $\{\Delta_j\}$ denote a collection of (unions of) dyadic intervals in $\mathbb{R}$, $\Delta_j = (-2^{j+1}, -2^j] \cup [2^j, 2^{j+1})$, then we may define a family of operators by

$$\widehat{S_j f}(\xi) = \chi_{\Delta_j}(\xi)\hat{f}(\xi)$$

for $j \in \mathbb{Z}$, and a corresponding operator S by

$$Sf(x) = \left(\sum_{j \in \mathbb{Z}} |S_j f(x)|^2 \right)^{\frac{1}{2}}.$$

By Plancherel's Theorem, it is easy to show that $\|Sf\|_2 = \|f\|_2$, and a classical result of Littlewood and Paley also states that $\|f\|_p \sim \|Sf\|_p$ for $1 < p < \infty$.

One may construct a 'smoothed-out' version of the operator S as follows: let $\psi \in \mathcal{S}(\mathbb{R})$ be non-negative, supported in $\{\xi : 1/2 \leq |\xi| \leq 4\}$, and equal to 1 when $1 \leq |\xi| \leq 2$. Let $\psi_j(\xi) = \psi(2^{-j}\xi)$ and define a family of operators $\{\widetilde{S}_j\}$ by

$$\widehat{\widetilde{S}_j f}(\xi) = \psi_j(\xi)\hat{f}(\xi).$$

We may then define a corresponding Littlewood-Paley operator by

$$\widetilde{S}f(x) = \left(\sum_{j} |\widetilde{S}_j f(x)|^2 \right)^{\frac{1}{2}}.$$

1.1 Littlewood-Paley Theory with Weights on $\mathbb{R}$

In order to carry to carry out (and subsequently 'undo') the frequency decomposition that is central to the proof of Theorem 4.2.2 we require several results concerning the weighted L^2 boundedness of the types of square functions described above.

We include here two weighted Littlewood-Paley type lemmas that we will use later on, both of which are adapted from results by Bennett et al in [5]. Our first lemma concerns an 'equally-spaced' frequency decomposition and has its basis in a result from [19] . This is followed by a weighted estimate for a square function arising from a dyadic decomposition.

Lemma 1.1.1. *For $L > 0$, let W_L be a function on $\mathbb{R}$ with* $\text{supp}\widehat{W_L} \subset \{x \in \mathbb{R} : |x| \leq 2L\}$, *such that*

$$\sum_{k\in\mathbb{Z}} \widehat{W}_L(x + kL) = 1$$

for all $x \in \mathbb{R}$, and satisfying the estimate

$$|W_L(x)| \lesssim \frac{L}{(1 + L|x|)^N}$$

for any $x \in \mathbb{R}$ and any $N \in \mathbb{N}$.

For a function f on $\mathbb{R}$, let $f_k(x) = f * (e^{2\pi i k L(\cdot)} W_L)(x)$. Then for any non-negative w on $\mathbb{R}$,

$$\int_{\mathbb{R}} \sum_k |f_k|^2 w \lesssim \int_{\mathbb{R}} |f|^2 |W_L| * w.$$

Proof. Observe that

$$f_k(x) = e^{2\pi i k L x} (f(\cdot) W_L(x - \cdot))\widehat{\ }(kL),$$

and so

$$\sum_k |f_k(x)|^2 = \sum_k |(f(\cdot) W_L(x - \cdot))\widehat{\ }(kL)|^2.$$

Fix $x \in \mathbb{R}$, and let $g_x(y) = f(y) W_L(x - y)$. By the Poisson Summation Formula,

$$\sum_k \hat{g}_x(kL) e^{2\pi i k L y} = \frac{1}{L} \sum_k g_x(y + k/L),$$

and by Plancherel's Theorem,

$$\sum_k |\hat{g}_x(kL)|^2 = L \int_0^{1/L} \left| \frac{1}{L} \sum_k g_x(y + k/L) \right|^2 dy.$$

This may be written as

$$\begin{aligned}
\sum_k |f_k(x)|^2 &= \frac{1}{L} \int_0^{1/L} \left| \sum_k f(y + k/L) W_L(x - y - k/L) \right|^2 dy \\
&\leq \frac{1}{L} \int_0^{1/L} \sum_k |f(y + k/L)|^2 |W_L(x - y - k/L)| \sum_l |W_L(x - y - l/L)| dy \\
&= \frac{1}{L} \sum_k \int_{k/L}^{(k+1)/L} \left(\sum_l |W_L(x - z - (k - l)/L)| \right) |f(z)|^2 |W_L(x - z)| dz \\
&\lesssim \int_{\mathbb{R}} |f(z)|^2 |W_L(x - z)| dz \\
&= |f|^2 * |W_L|(x)
\end{aligned}$$

using the fact that

$$\frac{1}{L} \sum_l |W_L(x - z - (k - l)/L)|$$

is uniformly bounded by our assumptions on W_L. It then follows that

$$\int_{\mathbb{R}} \sum_k |f_k|^2 w \lesssim \int_{\mathbb{R}} |f|^2 |W_L| * w,$$

as claimed. □

Remark. It is proved in [5] that Lemma 1.1.1 holds when f, W_L, and w are similarly defined on $\mathbb{S}^1$, the proof of which forms the basis for the proof of Lemma 1.1.1.

The following lemma may also be found in [5], but we present here a corrected proof that addresses an error in the original found by the author. The major changes to the proof are summarised in the associated corrigendum [6].

Lemma 1.1.2. *For each $k \in \mathbb{N}$ let $Q_k \in C^1(\mathbb{R})$ be such that* $\operatorname{supp} \widehat{Q}_k \subset \{x \in \mathbb{R} : |x| \sim 2^k\}$*, and suppose further that for each $N \in \mathbb{N}$ there is a constant $C_N \geq 0$ such that*

$$|Q_k(x)| + 2^{-k}|Q_k'(x)| \leq C_N \frac{2^k}{(1+2^k|x|)^N}$$

for all k. Then, with M denoting the Hardy-Littlewood maximal function,

$$\int_{\mathbb{R}} \sum_k |f * Q_k|^2 w \lesssim \int_{\mathbb{R}} |f|^2 Mw,$$

Proof. For each $j, k \in \mathbb{Z}$ we define collections of sets E_j and $A_{j,k}$ by

$$\begin{aligned} E_j &= \{x \in \mathbb{R} : Mw(x) > 2^j\} \\ A_{j,k} &= \{x \in \mathbb{R} : B(x, 2^{-k}) \subset E_j\}. \end{aligned}$$

We now choose another similar bump function $\widetilde{Q}_k$ at scale 2^{-k} and an odd function R_k such that $\widetilde{Q}_k * R_k = Q_k$ for every k. Now by Jensen's inequality,

$$\begin{aligned} \int_{\mathbb{R}} \sum_k |Q_k * f|^2 w &\lesssim \int_{\mathbb{R}} \sum_k |R_k * f|^2 |\widetilde{Q}_k| * w \\ &= \sum_j \sum_k \int_{A_{j,k} \setminus A_{j+1,k}} |R_k * f|^2 |\widetilde{Q}_k| * w, \end{aligned} \tag{1.1}$$

and since $|\widetilde{Q}_k * w(x)| \lesssim 2^j$ whenever $x \in A_{j,k} \setminus A_{j+1,k}$, (1.1) is bounded by

$$\sum_j 2^j \sum_k \int_{A_{j,k}} |R_k * f|^2.$$

Now let $\{P_n\}_{n\in\mathbb{N}}$ be a smooth partition of unity on $\mathbb{R}$ with each P_n even, and such that supp $P_n \subset \{x \in \mathbb{R} : |x| \sim 2^{-n}\}$. For uniformity purposes let us suppose that $\{P_n\}$ is constructed in the standard way by scaling a certain fixed smooth function and taking differences. For $k \in \mathbb{N}$ and integers ℓ with $0 \leq \ell \leq \infty$ we now define

$$P_{k,\ell} = \begin{cases} P_{k-\ell} & \text{if} \quad 0 < \ell < \infty \\ \sum_{n \geq k} P_n & \text{if} \quad \ell = 0. \end{cases}$$

Thus for each k, $\{P_{k,\ell}\}_{0\leq\ell\leq\infty}$ forms a partition of unity on $\mathbb{R}$.

We claim that

$$(R_k P_{k,\ell}) * f(x) = (R_k P_{k,\ell}) * (\chi_{E_{j-\ell}} f)(x)$$

for all $x \in A_{j,k}$. To see that the above assertion holds, we write

$$(R_k P_{k,\ell}) * f(x) = \int_{\mathbb{R}} R_k P_{k,\ell}(y) f(x-y) dy.$$

Notice that if y is in the range of integration and $x \in A_{j,k}$ then $|y| \lesssim 2^{-(k-\ell)}$ and $M(w)(x') > 2^j$ for $x' \in B(x, 2^{-k})$. As a consequence,

$$Mw(x-y) > 2^{-\ell} Mw(x) > 2^{j-\ell}.$$

In other words, $x - y \in E_{j-\ell}$ and so

$$(R_k P_{k,\ell}) * f(x) = \int_{\mathbb{R}} R_k P_{k,\ell}(y) \chi_{E_{j-\ell}}(x-y) f(x-y) dy = (R_k P_{k,\ell}) * (\chi_{E_{j-\ell}} f)(x)$$

as claimed.

By Plancherel's Theorem,

$$
\begin{aligned}
&\left(\sum_j 2^j \sum_k \int_{A_{j,k}} |R_k * f|^2\right)^{\frac{1}{2}} \\
= &\left(\sum_j 2^j \sum_k \int_{A_{j,k}} \left|\sum_\ell (R_k P_{k,\ell}) * (f\chi_{E_{j-\ell}})\right|^2\right)^{\frac{1}{2}} \\
\leq &\left(\sum_j 2^j \sum_k \int_{\mathbb{R}} \left|\sum_\ell \widehat{R_k P_{k,\ell}}(\xi)\widehat{f\chi_{E_{j-\ell}}}(\xi)\right|^2 d\xi\right)^{\frac{1}{2}} \\
\leq &\sum_\ell \left(\sum_j 2^j \sum_k \int_{\mathbb{R}} |\widehat{R_k P_{k,\ell}}(\xi)\widehat{f\chi_{E_{j-\ell}}}(\xi)|^2 d\xi\right)^{\frac{1}{2}} \\
= &\sum_\ell \left(\sum_j 2^j \int_{\mathbb{R}} \left(\sum_k |\widehat{R_k P_{k,l}}(\xi)|^2\right) |\widehat{f\chi_{E_{j-\ell}}}(\xi)|^2 d\xi\right)^{\frac{1}{2}}. \qquad (1.2)
\end{aligned}
$$

Fix $\xi \in \mathbb{R}$. If $|\xi|2^{\ell-k} \leq 1$, then

$$
|\widehat{R_k P_{k,\ell}}(\xi)| = \left|\int_{\mathbb{R}} R_k P_{k,\ell}(x)[e^{-ix\xi} - 1]dx\right| \leq 2^{\ell-k}|\xi|2^{-\ell(N-1)},
$$

since $\int R_k P_{k,\ell} = 0$, for any $N \in \mathbb{N}$, and so

$$
\sum_{k:|\xi|2^{\ell-k}\leq 1} |\widehat{R_k P_{k,\ell}}(\xi)|^2 \lesssim 2^{-2\ell(N-1)}. \qquad (1.3)
$$

On the other hand, if $|\xi|2^{\ell-k} > 1$ we integrate by parts once to obtain

$$
|\widehat{R_k P_{k,\ell}}(\xi)| \leq \frac{1}{|\xi|}\int_{\mathbb{R}} |(R_k P_{k,\ell})'(\eta)|d\eta \leq \frac{1}{|\xi|}2^k 2^{-\ell(N-1)} = \frac{2^{-\ell(N-1)}}{|\xi|2^{\ell-k}}. \qquad (1.4)
$$

Hence by (1.3) and (1.4),

$$
\sum_k |\widehat{R_k P_{k,\ell}}(\xi)|^2 \lesssim 2^{-2\ell(N-2)}
$$

for each $N \in \mathbb{N}$, and so by Plancherel's Theorem, (1.2) is bounded by a constant multiple of

$$\begin{aligned} &\sum_{\ell} 2^{-\ell(N-2)} \left(\sum_{j} 2^{j} \int_{\mathbb{R}} |\widehat{f\chi_{E_{j-\ell}}}(\xi)|^2 d\xi \right)^{\frac{1}{2}} \\ = &\sum_{\ell} 2^{-\ell(N-5/2)} \left(\sum_{j} 2^{j-\ell} \int_{\mathbb{R}} |f(x)|^2 \chi_{E_{j-\ell}}(x) dx \right)^{\frac{1}{2}} \\ \lesssim &\left(\int |f|^2 Mw \right)^{\frac{1}{2}}. \end{aligned}$$

□

The use of a dyadic frequency decomposition in Chapter 4 necessitates the use of an inequality to act as a 'reverse' of that in Lemma 1.1.2. The author was unable to find a suitable result in the existing literature and so we will derive the following lemma, which is sufficient for our purposes.

For non-negative integers k let Q_k be a smooth function on $\mathbb{R}$ with supp $\widehat{Q_k} \subset \{|\xi| \sim 2^k\}$ when $k > 0$ and supp $\widehat{Q}_0 \subset \{|\xi| \lesssim 1\}$. Suppose further that

$$\sum_{k \geq 0} \widehat{Q_k} = 1.$$

Let us define a family of operators Δ_k for integers $k \geq 0$ by $\Delta_k f = f * Q_k$, so that

$$f = \sum_{k \in \mathbb{Z}} \Delta_k f.$$

For our purposes, we may choose the Q_k such that $\Delta_j \Delta_k = 0$ if $|j - k| > 1$.

Now for integers $k \geq 1$ let P_k be a collection of odd functions such that $\widehat{P}_k(\xi) = \widehat{Q}_k(\xi)$ for $\xi > 0$ and $\widehat{P}_k(\xi) = -\widehat{Q}_k(\xi)$ for $\xi < 0$. For ease of notation we let $P_0 = Q_0$

and add P_0 to our collection of P_k.

Lemma 1.1.3. *For all weights w,*

$$\int_{\mathbb{R}} |f(x)|^2 w(x)dx \lesssim \int_{\mathbb{R}} \sum_{k\geq 0} |P_k * f(x)|^2 M^3(w)(x)dx \tag{1.5}$$

Before proceeding with the proof, we state a weighted estimate for singular integral operators due to Pérez (from [24]) that will be of use:

Theorem 1.1.4. *Let $Tf = K * f$ where the kernel K on $\mathbb{R}^n$ is C^1 away from the origin, has mean value zero on the unit sphere and satisfies*

$$|K(y)| \leq C/|y|^n \quad \textit{and} \quad |\nabla K(y)| \leq C/|y|^{n+1}$$

for $y \neq 0$. Then for each weight w on $\mathbb{R}^n$,

$$\int_{\mathbb{R}^n} |Tf|^p w \lesssim \int_{\mathbb{R}^n} |f|^p M^{[p]+1}(w)$$

where $[p]$ denotes the integer part of p.

Moreover, this result is sharp in the sense that it fails if $[p]+1$ is replaced with $[p]$.

Proof of Lemma 1.1.3. We begin by splitting f into its 'even' and 'odd' parts. Let

$$T^e = \sum_{k>0} \Delta_{2k}, \quad \text{and} \quad T^o = \sum_{k\geq 0} \Delta_{2k+1},$$

so that $f = \Delta_0 f + T^e f + T^o f$, and define two related operators as follows: Let $\{\epsilon_k\}$

be an arbitrarily chosen sequence with $\epsilon_k \in \{-1, 1\}$ for integers $k \geq 0$, and define

$$\widetilde{T}^e f = \sum_{k>0} \epsilon_{2k} P_{2k} * f, \quad \text{and} \quad \widetilde{T}^o = \sum_{k \geq 0} \epsilon_{2k+1} P_{2k+1} * f.$$

Now, we aim to find operators S^e and S^o such that $S^e \widetilde{T}^e = T^e$, and $S^o \widetilde{T}^o = T^o$. To construct such an S^e, let R_k be an odd function on $\mathbb{R}$ such that $\widehat{R}_k(\xi) = 1$ on $\operatorname{supp}\widehat{Q}_k \cap [0, \infty)$, $\widehat{R}_k(\xi) = -1$ on $\operatorname{supp}\widehat{Q}_k \cap (-\infty, 0]$ and vanishing outside of a slightly larger set, for each $k \in \mathbb{Z}$. With the R_k defined in this way we have $R_k * P_k = Q_k$.

Define $S^e f = \sum_{k>0} \epsilon_{2k} R_{2k} * f$. If we choose the P_k and R_k such that $\widehat{R}_{2j} \widehat{P}_{2k} = 0$ for $j \neq k$, as we may, then $S^e \widetilde{T}^e = T^e$. If in addition we choose the R_k such that they satisfy the decay estimates

$$|R_k^{(j)}(x)| \lesssim \frac{(2^k)^{j+1}}{(1 + 2^k|x|)^N} \quad j = 0, 1$$

for any $N \in \mathbb{N}$ ($N = 2$ is enough here) then S^e is a convolution operator with kernel $\sum_{k \geq 0} \epsilon_{2k} P_{2k}$ that satisfies Theorem 1.1.4 and so we have the inequality

$$\int |S^e f|^2 w \lesssim \int |f|^2 M^3(w)$$

which holds uniformly in the choice of $\{\epsilon_k\}$. We may define S^o in a similar way.

By the triangle inequality,

$$\begin{aligned}
\|f\|_{L^2(w)} &\leq \|\Delta_0 f\|_{L^2(w)} + \|T^e f\|_{L^2(w)} + \|T^o f\|_{L^2(w)} \\
&= \|\Delta_0 f\|_{L^2(w)} + \|S^e \widetilde{T}^e f\|_{L^2(w)} + \|S^o \widetilde{T}^o f\|_{L^2(w)} \\
&\lesssim \|\Delta_0 f\|_{L^2(w)} + \|\widetilde{T}^e f\|_{L^2(M^3 w)} + \|\widetilde{T}^o f\|_{L^2(M^3 w)}
\end{aligned}$$

Since this holds uniformly in our choice of sequence $\{\epsilon_k\}$, we may take $\epsilon_k = r_k(t)$, where r_k is the kth Rademacher function and $t \in [0,1]$. With $\widetilde{T}^e f$ and $\widetilde{T}^o f$ now implicitly functions of $t \in [0,1]$,

$$\begin{aligned}
\|f\|_{L^2(w)} &= \left(\int_0^1 \|f\|^2_{L^2(w)} dt\right)^{1/2} \\
&\lesssim \left(\int_0^1 (\|\Delta_0 f\|_{L^2(w)} + \|\widetilde{T}^e f\|_{L^2(M^3 w)} + \|\widetilde{T^o} f\|_{L^2(M^3 w)})^2 dt\right)^{1/2} \\
&\leq \left(\int_0^1 \|\Delta_0 f\|^2_{L^2(w)} dt\right)^{1/2} + \left(\int_0^1 \|\widetilde{T}^e f\|^2_{L^2(M^3 w)} dt\right)^{1/2} + \left(\int_0^1 \|\widetilde{T}^o f\|^2_{L^2(M^3 w)} dt\right)^{1/2}. \\
&= \|\Delta_0 f\|_{L^2(w)} + \left(\int_0^1 \|\widetilde{T}^e f\|^2_{L^2(M^3 w)} dt\right)^{1/2} + \left(\int_0^1 \|\widetilde{T}^o f\|^2_{L^2(M^3 w)}\right)^{1/2}.
\end{aligned}$$

Observe that $\|\Delta_0 f\|_{L^2(w)}$ is bounded by the square root of the right hand side of (1.5). Now,

$$\begin{aligned}
\left(\int_0^1 \|\widetilde{T}^e f\|^2_{L^2(M^3 w)} dt\right)^{1/2} &= \left(\int_0^1 \left(\int |\sum_{k>0} r_{2k}(t) P_{2k} * f|^2 M^3 w\right) dt\right)^{1/2} \\
&= \left(\int \left(\int_0^1 |\sum_{k>0} r_{2k}(t) P_{2k} * f|^2 dt\right) M^3 w\right)^{1/2} \\
&\lesssim \left(\int \sum_{k>0} |P_{2k} * f|^2 M^3 w\right)^{1/2} \quad \text{by Khinchine's inequality} \\
&\leq \left(\int \sum_{k\geq 0} |P_k * f|^2 M^3 w\right)^{1/2}.
\end{aligned}$$

Similarly, one may show that

$$\left(\int_0^1 \|\widetilde{T}^o f\|^2_{L^2(M^3 w)} dt\right)^{1/2} \lesssim \left(\int \sum_{k\geq 0} |P_k * f|^2 M^3 w\right)^{1/2},$$

which completes the proof. □

1.2 Some Littlewood-Paley Theory on $\mathbb{R}^n$

The weighted L^p-boundedness of certain square functions on $\mathbb{R}^n$ is studied extensively by Wilson in [34], from which one may obtain as a corollary an n-dimensional version of Lemma 1.1.2. More specifically, let $\{Q_k\}_{k\in\mathbb{Z}}$ be a family of smooth radial functions on $\mathbb{R}^n$ such that supp $\widehat{Q}_k \subset \{\xi \in \mathbb{R}^n : |\xi| \sim 2^k\}$. We may suppose further that, if R_k is a function on $\mathbb{R}$ such that $R_k(|x|) = Q_k(x)$,

$$|R_k^{(j)}(x)| \leq C_n \frac{(2^k)^{n+j}}{(1+2^k|x|)^{n+1}} \quad j = 0, 1.$$

As above, one may define a corresponding square function $\widetilde{S}$ by

$$\widetilde{S}(f) = \left(\sum_k |Q_k * f|^2 \right)^{1/2}.$$

Lemma 1.2.1. *For all weights w,*

$$\int_{\mathbb{R}^n} |\widetilde{S}(f)|^2 w \lesssim \int_{\mathbb{R}^n} |f|^2 M(w).$$

Proof. This inequality can be deduced as a corollary from two powerful results proved by Wilson in [34] which combine to show the weighted L^2 boundedness of a square function that pointwise dominates $\widetilde{S}$.

Let $0 \leq \alpha \leq 1$, and let $\mathcal{C}_\alpha$ denote the family of functions $\phi : \mathbb{R}^n \to \mathbb{R}$ such that ϕ has support in $\{x : |x| \leq 1\}$, $\int \phi = 0$, and

$$|\phi(x) - \phi(x')| \leq |x - x'|^\alpha$$

for all x, x' in $\mathbb{R}^n$. For $(t,y) \in \mathbb{R}^{n+1}_+$, let

$$A_\alpha(f)(t,y) = \sup_{\phi \in \mathcal{C}_\alpha} |f * \phi_y(t)|,$$

where ϕ_y denotes the dilation $y^{-n}\phi(y^{-1}\cdot)$, and define a corresponding square function by

$$G_\alpha(f)(x) = \left(\int_{\Gamma(x)} (A_\alpha(f)(t,y))^2 \frac{dtdy}{y^{n+1}} \right)^{1/2}$$

where $\Gamma(x) = \{(t,y) : |x-t| < y\}$, the cone of aperture one. One may also define a similar-looking square function, the underlying convolution kernels of which are not required to have compact support. For $0 < \alpha \leq 1$ and $\epsilon > 0$, let $\mathcal{C}_{\alpha,\epsilon}$ be the set of functions $\phi : \mathbb{R}^n \to \mathbb{R}$ such that

- $\int \phi = 0$
- $|\phi(x)| \leq (1+|x|)^{-(n+\epsilon)}$
- for all x and x' in $\mathbb{R}^n$,

$$|\phi(x) - \phi(x')| \leq |x-x'|^\alpha((1+|x|)^{-(n+\epsilon)} + (1+|x'|)^{-(n+\epsilon)}).$$

Notice that the functions in $\mathcal{C}_{\alpha,\epsilon}$ are not required to have compact support. Then as before we define

$$\widetilde{A}_{\alpha,\epsilon}(f)(t,y) = \sup_{\phi \in \mathcal{C}_{\alpha,\epsilon}} |f * \phi_y(t)|,$$

with the corresponding square function

$$\widetilde{G}_{\alpha,\epsilon}(f)(x) = \left(\int_{\Gamma(x)} (\widetilde{A}_{\alpha,\epsilon}(f)(t,y))^2 \frac{dtdy}{y^{n+1}}\right)^{1/2}.$$

A third square function that is also of relevance here is the discretised form of $\widetilde{G}$, which is defined by

$$\tilde{\sigma}_{\alpha,\epsilon}(f)(x) = \left(\sum_{k\in\mathbb{Z}} (\widetilde{A}_{\alpha,\epsilon}(f)(x,2^k))^2\right)^{1/2}.$$

It is shown by Wilson in [34] that

$$\widetilde{G}_{\alpha,\epsilon}(f) \sim \tilde{\sigma}_{\alpha,\epsilon}(f) \tag{1.6}$$

with implicit constant depending on α, ϵ and n. The two aforementioned theorems from [34] are the following:

Theorem 1.2.2. *For $0 < \alpha \leq 1$ and $1 < p \leq 2$,*

$$\int_{\mathbb{R}^n} |G_\alpha(f)|^p w \leq C(p,n,\alpha) \int_{\mathbb{R}^n} |f|^p M(w).$$

Theorem 1.2.3. *Let $0 < \alpha' \leq \alpha \leq 1$ and $\alpha' < \epsilon$, then for all $x \in \mathbb{R}^n$,*

$$\widetilde{G}_{\alpha,\epsilon}(f)(x) \leq C(\alpha,\alpha',\epsilon,n) G_{\alpha'}(f)(x).$$

An immediate corollary of these two theorems and (1.6) is that

$$\int_{\mathbb{R}^n} |\tilde{\sigma}_{\alpha,\epsilon}(f)|^2 w \leq C(\alpha,\epsilon) \int_{\mathbb{R}^n} |f|^2 M(w),$$

so in order to complete the proof of the lemma it will suffice to show that

$$|Q_k * f| \lesssim \widetilde{A}_{1,1}(f)(\cdot, 2^{-k}), \tag{1.7}$$

since this implies that $\widetilde{S}(f) \lesssim \tilde{\sigma}_{1,1}(f)$. Recall that

$$A_{1,1}(f)(\cdot, 2^{-k}) = \sup_{\phi \in \mathcal{C}_{1,1}} |f * \phi_{2^{-k}}(\cdot)|,$$

and so (1.7) will follow from the observation that $(2^{-k})^n Q_k(2^{-k}\cdot) \in \mathcal{C}_{1,1}$.

For ease of notation let $\widetilde{Q}_k = (2^{-k})^n Q_k(2^{-k}\cdot)$. By our assumptions on Q_k, the first two criteria for inclusion in $\mathcal{C}_{1,1}$ are immediate for $\widetilde{Q}_k$. It only remains to be seen that

$$|\widetilde{Q}_k(x) - \widetilde{Q}_k(x')| \lesssim |x - x'|((1 + |x|)^{-(n+1)} + (1 + |x'|)^{-(n+1)}) \tag{1.8}$$

for all pairs of points $x, x' \in \mathbb{R}^n$. By the decay assumption on Q_k, (1.8) clearly holds when $|x - x'| \geq 1$, so we only need to consider the case when $|x - x'| < 1$. Suppose that $|x - x'|$ is fixed, then since Q_k is radial the left hand side of (1.8) is largest when x and x' are collinear with the origin, so we may assume that both x and x' lie on, say, the x_1-axis. By the Mean Value Theorem, there exists a point $c \in [x, x']$ such that

$$\begin{aligned} |\widetilde{Q}_k(x) - \widetilde{Q}_k(x')| &= |x - x'| \cdot |\partial_{x_1} \widetilde{Q}_k(c)| \\ &\lesssim |x - x'|((1 + |x|)^{-(n+1)} + (1 + |x'|)^{-(n+1)}). \end{aligned}$$

□

Chapter 2

Oscillatory Integral Techniques

It will often be the case that we wish to understand the behaviour of integrals with an oscillatory factor in the integrand, generally integrals of the form

$$\int e^{i\lambda\phi(x)}\psi(x)dx,$$

where ϕ and ψ are smooth, and real and complex valued respectively. We are interested in results which give us estimates on the decay of such integrals as $\lambda \to \infty$. The results in this section can be found in [28].

2.1 Important results

Our first situation deals with the case where ψ has compact support, and ϕ has no stationary points in the support of ψ.

Lemma 2.1.1. *Let ϕ and ψ be smooth real-valued functions such that ψ has compact support in (a,b), and $\phi'(x) \neq 0$ for all $x \in [a,b]$. Then*

$$\left|\int_a^b e^{i\lambda\phi(x)}\psi(x)dx\right| = O(\lambda^{-N}) \tag{2.1}$$

as $\lambda \to \infty$ for all $N \geq 0$.

Remark. We would expect to see this sort of behaviour, since as λ increases there will be a large amount of oscillation in the integrand, and therefore a great deal of cancellation.

Proof. First, we notice that

$$e^{i\lambda\phi(x)} = \frac{1}{i\lambda\phi'(x)}\frac{d}{dx}(e^{i\lambda\phi(x)}).$$

If we define a differential operator D by

$$Df(x) = \frac{1}{i\lambda\phi'(x)}\frac{d}{dx}f(x)$$

then $D(e^{i\lambda\phi(x)}) = e^{i\lambda\phi(x)}$, and so $D^N(e^{i\lambda\phi(x)}) = e^{i\lambda\phi(x)}$ for all $N \in \mathbb{N}$. If we define another differential operator D^* by

$$D^*f(x) = -\frac{d}{dx}\left(\frac{f(x)}{i\phi'(x)}\right),$$

then

$$\begin{aligned}\int_a^b e^{i\lambda\phi(x)}\psi(x)dx &= \int_a^b D^N(e^{i\lambda\phi(x)})\psi(x)dx \\ &= (-\lambda)^{-N}\int_a^b e^{i\lambda\phi(x)}(D^*)^N\psi(x)dx,\end{aligned}$$

a Lebesgue integral, since $\phi' \neq 0$ on $[a, b]$. If we integrate by parts then we see that this holds for $N = 1$, and then inductively for all $N \in \mathbb{N}$. Therefore

$$\begin{aligned}\left|\int_a^b e^{i\lambda\phi(x)}\psi(x)dx\right| &\leq \lambda^{-N}\int_a^b |(D^*)^N\psi(x)|dx \\ &= C_N\lambda^{-N}\end{aligned}$$

for some $C_N < \infty$. $\square$

Our next result concerns the dependence of the decay rate in λ on the order of vanishing of the stationary points of ϕ.

Lemma 2.1.2 (Van der Corput)**.** *Suppose that ϕ is real-valued and smooth in (a, b), and that $|\phi^{(k)}(x)| \geq 1$ for all $x \in (a, b)$. Then*

$$\left| \int_a^b e^{i\lambda\phi(x)} dx \right| \leq c_k \lambda^{-1/k} \tag{2.2}$$

holds whenever

1. *$k \geq 2$, or*

2. *$k = 1$ and $\phi'(x)$ is monotonic.*

Furthermore, the constant c_k is independent of ϕ and λ.

Proof. Suppose that $k = 1$ and $\phi'(x)$ is monotonic. Define $I_\lambda = \int_a^b e^{i\lambda\phi(x)} dx$. Then

$$\begin{aligned} I_\lambda &= \int_a^b \frac{1}{i\lambda\phi'(x)} \frac{d}{dx} e^{i\lambda\phi(x)} dx \\ &= \left[\frac{e^{i\lambda\phi(x)}}{i\lambda\phi'(x)} \right]_a^b - \frac{1}{i\lambda} \int_a^b \frac{d}{dx} \left(\frac{1}{\phi'(x)} \right) e^{i\lambda\phi(x)} dx \end{aligned}$$

which implies that

$$|I_\lambda| \leq \frac{1}{\lambda|\phi'(b)|} + \frac{1}{\lambda|\phi'(a)|} + \frac{1}{\lambda} \int_a^b \left| \frac{d}{dx} \left(\frac{1}{\phi'(x)} \right) \right| dx.$$

Using the fact that $\phi'(x)$ is monotone, and the Fundamental Theorem of Calculus,

$$\begin{aligned} |I_\lambda| &\leq \frac{2}{\lambda} + \frac{1}{\lambda}\left|\int_a^b \frac{d}{dx}\frac{1}{\phi'(x)}dx\right| \\ &\leq \frac{4}{\lambda} \end{aligned}$$

proving Case 1.

We now proceed by induction on k. Assume that the result holds for some $k \geq 1$, and suppose that $|\phi^{(k+1)}(x)| \geq 1$ on $[a, b]$. Since ϕ is smooth we can assume, without loss of generality, that $\phi^{(k+1)}(x) \geq 1$. Let c denote the point at which the minimum value of $|\phi^{(k)}(x)|$ is attained on $[a, b]$. Suppose that for some δ to be determined later, we have $|x - c| \geq \delta$. If $\phi^{(k)}(c) \geq 0$ then by the Mean Value Theorem, for some $\theta \in (c, x)$,

$$|\phi^{(k)}(x)| \geq |\phi^{(k)}(x) - \phi^{(k)}(c)| = |(x - c)\phi^{(k+1)}(\theta)| \geq \delta|\phi^{(k+1)}(\theta)| \geq \delta$$

If $\phi^{(k)}(c) < 0$, then for some $\theta \in [c, x]$,

$$\begin{aligned} |\phi^{(k)}(x)| &= |-\phi^{(k)}(x)| \\ &\geq |-\phi^{(k)}(x) - (-\phi^{(k)}(c))| \\ &= |-(x - c)\phi^{(k+1)}(\theta)| \\ &\geq \delta|\phi^{(k+1)}(\theta)| \\ &\geq \delta. \end{aligned}$$

So $|\phi^{(k)}(x)| \geq \delta$ whenever $|x - c| \geq \delta$. Now, we rewrite I_λ as

$$I_\lambda = \left(\int_a^{c-\delta} + \int_{c-\delta}^{c+\delta} + \int_{c+\delta}^{b}\right) e^{i\lambda\phi(x)}dx,$$

unless $c-\delta \leq a$, in which case the first integral doesn't appear. Now by our inductive hypothesis,

$$\begin{aligned}\left|\int_a^{c-\delta} e^{i\lambda\phi(x)}dx\right| &= \left|\int_a^{c-\delta} e^{i\lambda\delta(\delta^{-1}\phi(x))}dx\right| \\ &\leq c_k(\lambda\delta)^{-1/k}.\end{aligned}$$

Similarly,

$$\left|\int_{c+\delta}^{b} e^{i\lambda\phi(x)}dx\right| \leq c_k(\lambda\delta)^{-1/k}.$$

In addition, we have the elementary estimate

$$\left|\int_{c-\delta}^{c+\delta} e^{i\lambda\phi(x)}dx\right| \leq 2\delta,$$

which implies that

$$|I_\lambda| \leq 2(c_k(\lambda\delta)^{-1/k} + \delta).$$

The case $k = 1$, and hence the result, follows by taking $\delta = {c_k}^{k/(k+1)}\lambda^{-1/(k+1)}$, so that $c_{k+1} = 4c_k$. □

Van der Corput's Lemma has the following useful corollary:

Corollary 2.1.3. *Suppose that ψ is differentiable, then under the assumptions on ϕ in van der Corput's Lemma we have*

$$\left|\int_a^b e^{i\lambda\phi(x)}\psi(x)dx\right| \leq c_k\lambda^{-1/k}\left(|\psi(b)| + \int_a^b |\psi'(x)|dx\right)$$

Proof. We write

$$\int_a^b e^{i\lambda\phi(x)}\psi(x)dx = \int_a^b F'(x)\psi(x)dx,$$

with

$$F(x) = \int_a^x e^{i\lambda\phi(t)}dt.$$

Integrating by parts and using the estimates obtained from van der Corput's Lemma gives the desired result. □

2.2 An Application

In particular, we will be interested in decay estimates for integrals of the following form:

Claim 2.2.1. *Let $\ell \geq 2$ be an integer, and let $\xi \in \mathbb{R}$. Then*

$$\left|\int_{\mathbb{R}} e^{i(x^\ell - \xi x)}dx\right| \lesssim |\xi|^{-\frac{\ell-2}{2(\ell-1)}}.$$

Proof. We will apply van der Corput's Lemma. Let $\phi(x) = x^\ell - \xi x$, then on the interval $I := [-\frac{1}{2}(\xi/3)^{\frac{1}{\ell-1}}, \frac{1}{2}(\xi/3)^{\frac{1}{\ell-1}}]$ we have $|\phi'(x)| \gtrsim |\xi|$, giving the estimate

$$\left|\int_I e^{i\phi(x)}dx\right| \lesssim |\xi|^{-1}$$

by van der Corput's Lemma. However, for x outside of I, $|\phi''(x)| \gtrsim |\xi|^{-\frac{\ell-2}{\ell-1}}$ and so

$$\left|\int_{\mathbb{R}\setminus I} e^{i\phi(x)}dx\right| \lesssim |\xi|^{-\frac{\ell-2}{2(\ell-1)}}.$$

Overall, this gives us an estimate of

$$\left|\int_{\mathbb{R}} e^{i\phi(x)} dx\right| \lesssim |\xi|^{-\frac{\ell-2}{2(\ell-1)}}.$$

$\square$

Chapter 3

Some Weighted L^2 Theory

We a present here a selection of examples of how weighted L^2 estimates have been used in the analysis of various fundamental objects in modern harmonic analysis. This chapter is not intended to be an exhaustive review but rather to demonstrate how the use of weighted norm estimates is ingrained in modern harmonic analysis, and to show how their application to certain important operators provides the motivation for the results in Chapter 4.

3.1 The Disc Multiplier

The disc multiplier operator T on $\mathbb{R}^n$ is defined by

$$\widehat{(Tf)}(\xi) = \chi_D(\xi)\hat{f}(\xi)$$

where D is the set $\{x \in \mathbb{R}^n : |x| \leq 1\}$. While this operator is clearly bounded on L^2, determining the L^p boundedness of T for $p \neq 2$ proved to be a difficult problem. For $n = 1$, T can be written as a linear combination of Hilbert transforms and so boundedness of T may be deduced from this observation. For $n > 1$, it was originally conjectured that T is bounded on L^p if and only if

$$\frac{2n}{n+1} \leq p \leq \frac{2n}{n-1}.$$

However, a clever counterexample shows that T is bounded only on L^2, an "unfortunate fact" according to the author of the counterexample, Charles Fefferman (see [18]).

It was proposed in 1978 by Stein that weighted inequalities for the disc multiplier should be studied in order to better understand its L^2 behaviour. The question of which maximal functions control T in weighted L^2 is open in general. It is conjectured that

$$\int_{\mathbb{R}^n} |Tf(x)|^2 w(x) dx \lesssim C_s \int_{\mathbb{R}^n} |f(x)|^2 \mathcal{M}_s(w)(x) dx, \tag{3.1}$$

for any $s > 1$, where $\mathcal{M}$ is the universal maximal function on $\mathbb{R}^n$, defined by taking maximal averages over arbitrary rectangles in $\mathbb{R}^n$, and $\mathcal{M}_s(w)$ is $(\mathcal{M}(w^s))^{\frac{1}{s}}$.

Inequality (3.1) can be proven to hold for radial weights due to an elegant argument by Carbery et al, which we will now sketch, whereby the problem can be reduced to certain weighted estimates for the Hilbert transform. The full proof can be found in [10].

For a suitable test function f on $\mathbb{R}^n$, we consider its spherical harmonic expansion

$$f(x) = \sum_{k,j} f_{k,j}(|x|) \mathcal{Y}_j^{(k)} \left(\frac{x}{|x|} \right).$$

Basic properties of spherical harmonics (see [29]) allow us to expand the Fourier transform of f as

$$\hat{f}(\xi) = \sum_{k,j} \frac{i^{-k}}{|\xi|^{(n-2)/2}} H_{k+(n-2)/2}(f_{k,j}(s)s^{(n-2)/2})(|\xi|)\mathcal{Y}_j^{(k)}\left(\frac{\xi}{|\xi|}\right),$$

where H_ℓ is the Fourier-Hankel transform of order ℓ defined by

$$H_\ell g(r) = \int_0^\infty g(s)J_\ell(rs)sds,$$

with J_ℓ denoting the Bessel function of order ℓ. Since we have that $(\widehat{Tf})(\xi) = \hat{f}(\xi)\chi_{|x|\leq 1}(\xi)$, we may write

$$Tf(x) = \sum_{k,j} \frac{1}{|x|^{(n-2)/2}} T_{k+(n-2)/2}(f_{k,j}(s)s^{(n-2)/2})(|x|)\mathcal{Y}_j^{(k)}\left(\frac{x}{|x|}\right),$$

where

$$T_\ell g(r) = \int_0^\infty g(s)\left[(rs)^{1/2}\int_0^1 J_\ell(st)J_\ell(rt)tdt\right]ds.$$

If we write our desired weighted norm inequality

$$\|Tf\|^2_{L^2(w)} \leq C_\alpha \|f\|^2_{L^2(\mathcal{M}_\alpha(w))}$$

in polar coordinates, then it becomes clear that proving (3.1) is equivalent to proving that

$$\int_0^\infty |T_\ell g(r)|^2 w_0(r)dr \leq C_\alpha \int_0^\infty |g(r)|^2 \mathcal{M}_\alpha w_0(r)dr,$$

uniformly in ℓ, where $w_0(r)$ and $\mathcal{M}w_0(r)$ are given by $w(r,0,\ldots,0)$ and $\mathcal{M}_\alpha w(r,0,\ldots,0)$

respectively. We may simplify the kernel of T_ℓ using the identity

$$\begin{aligned} & 2(rs)^{1/2}\int_0^1 J_\ell(rt)J_\ell(st)tdt \\ = \; & \frac{\mu_\ell(s)\sigma_\ell(r)}{r-s} - \frac{\sigma_\ell(s)\mu_\ell(r)}{r-s} + \frac{\mu_\ell(s)\sigma_\ell(r)}{r+s} + \frac{\sigma_\ell(s)\mu_\ell(r)}{r+s} \\ = \; & \sum_{i=1}^4 K_\ell^i(r,s) \end{aligned}$$

where

$$\mu_\ell(r) = r^{1/2}J_\ell(r), \quad \text{and} \quad \sigma_\ell(r) = r^{1/2}J_\ell'(r)$$

for $r > 0$. As a consequence of this, (3.2) can be obtained by proving the four inequalities

$$\int_0^\infty \left| \int_0^\infty K_\ell^i(r,s)g(s)ds \right|^2 w_0(r)dr \leq C_\alpha \int_0^\infty |g(r)|^2 \mathcal{M}_\alpha w_0(r)dr,$$

for $i = 1, 2, 3, 4$. The inequalities corresponding to $i = 1, 3$ are equivalent to, with H denoting the Hilbert transform,

$$\int_{\mathbb{R}} |H(g\mu_\ell)|^2 \sigma_\ell^2 w_0 \leq C_\alpha \int_{\mathbb{R}} |g|^2 \mathcal{M}_\alpha w_0, \tag{3.2}$$

and the inequalities corresponding to $i = 2, 4$ are equivalent to

$$\int_{\mathbb{R}} |H(g\sigma_\ell)|^2 \mu_\ell^2 w_0 \leq C_\alpha \int_{\mathbb{R}} |g|^2 \mathcal{M}_\alpha w_0, \tag{3.3}$$

where g is a function defined on $\mathbb{R}$, and the functions $w_0, \sigma_\ell, \mu_\ell, \mathcal{M}_\alpha w_0$ have been extended from functions on $[0, \infty)$ to even functions on $\mathbb{R}$.

In order to prove inequality (3.2), the authors then go on to show, using estimates on J_ℓ and J'_ℓ, and properties of A_p weights, that

$$\int_{\mathbb{R}} |H(g\mu_\ell)\sigma_\ell|^p w_0 \leq C_{p,\alpha} \int_{\mathbb{R}} |g|^p M_\alpha w_0,$$

provided that $p > 4/3$ and $M_\alpha w_0$ is finite almost everywhere, where $M_\alpha(w) = M(w^\alpha)^{1/\alpha}$ for $\alpha > 1$. Notice that this is stronger than required to prove (3.2) since, in particular, the maximal function M_α is small than $\mathcal{M}_\alpha$.

Now the weighted L^2 properties of the Hilbert transform are well understood, and in particular we have that

$$\int_{\mathbb{R}} |H(g\sigma_\ell)|^2 \mu_\ell^2 w_0 \leq C_\gamma \int_{\mathbb{R}} |g|^2 \sigma_\ell^2 M_\gamma(w_0 \mu_\ell^2)$$

for every $\gamma > 1$. The proof of inequality (3.1) is concluded by showing that for every $\alpha > 1$, there exist $\gamma > 1$ and $C = C_{\alpha,\gamma}$ such that

$$\sigma_\ell^2(s) M_\gamma(w_0\mu_\ell^2)(s) \leq C\mathcal{M}_\alpha(w_0)(s), \qquad s > 0,$$

thus establishing inequality (3.3) and proving the main result.

It is a useful feature of inequalities of the form

$$\int |Tf|^2 w \lesssim \int |f|^2 \mathcal{M}_T(w),$$

where T is some operator and $\mathcal{M}_T$ a maximal function, that L^p bounds on the operator T can be obtained from the L^p boundedness of the maximal function $\mathcal{M}_T$, and this will be a recurring theme of the material that follows. If T denotes the disc multiplier operator then we can derive the following mixed-norm inequality: for

$2n/(n+1) < p < 2n/(n-1)$ there exists $C_{p,n}$ such that

$$\int_0^\infty \left(\int_{\mathbb{S}^{n-1}} |Tf(r,\omega)|^2 d\omega \right)^{p/2} r^{n-1} dr \leq C_{p,n} \int_0^\infty \left(\int_{\mathbb{S}^{n-1}} |f(r,\omega)|^2 d\omega \right)^{p/2} r^{n-1} dr \quad (3.4)$$

where $d\omega$ denotes surface measure on $\mathbb{S}^{n-1}$. This inequality is established as follows: let us write the above mixed norm as

$$\|g\|^p_{L^p_r L^2_\omega} = \int_0^\infty \left(\int_{\mathbb{S}^{n-1}} |g(r,\omega)|^2 d\omega \right)^{p/2} r^{n-1} dr,$$

then our first observation is that if $2 < p < 2n/(n-1)$,

$$\|g\|^2_{L^p_r L^2_\omega} = \sup \int_{\mathbb{R}^n} |g(x)|^2 w_0(|x|) dx, \quad (3.5)$$

where the supremum is taken over all functions $w_0 \in L^{(p/2)'}(r^{n-1}dr)$ of unit norm. We can apply this followed by inequality (3.1) to obtain

$$\begin{aligned} \|Tf\|^2_{L^p_r L^2_\omega} &= \sup \int_{\mathbb{R}^n} |Tf(x)|^2 w_0(|x|) dx \\ &\leq \sup C_\alpha \int_{\mathbb{R}^n} |f(x)|^2 \mathcal{M}_\alpha(w_0)(x) dx, \end{aligned}$$

where the supremum is taken as above. It is at this stage that the we use the boundedness of $\mathcal{M}_\alpha$: as the authors demonstrate, $\mathcal{M}_\alpha$ is bounded on $L^{(p/2)'}_{rad}(\mathbb{R}^n)$, the set of radial functions in $L^{(p/2)'}$, for each $\alpha > 1$ provided that $\alpha n < (p/2)'$. Since $L^{(p/2)'}_{rad}(\mathbb{R}^n)$ can be identified with $L^{(p/2)'}(r^{n-1}dr)$, it follows from the boundedness of $\mathcal{M}_\alpha$ as described above that

$$\begin{aligned} \sup_{w_0} \int_{\mathbb{R}^n} |f(x)|^2 \mathcal{M}_\alpha(w_0)(x) &\leq C_{\alpha,p} \sup_{v_0} \int_{\mathbb{R}^n} |f(x)|^2 v_0(x) dx \\ &= C_{\alpha,p} \|f\|^2_{L^p_r L^2_\omega}, \end{aligned}$$

where w_0 and v_0 are taken to be in $L^{(p/2)'}(r^{n-1}dr)$, and so inequality (3.4) is established for $2 < p < 2n/(n-1)$. For $2n/(n+1) < p < 2$ we appeal to duality and the self-adjointness of T on $L^2(\mathbb{R}^n)$.

3.2 The Extension Operator and the Bochner-Riesz Means

The extension operator $\mathcal{E}$ on the unit sphere in $\mathbb{R}^n$ is defined by

$$\mathcal{E}f(\xi) = \widehat{fd\sigma}(\xi) = \int_{\mathbb{S}^{n-1}} f(x)e^{-2\pi i x\cdot\xi}d\sigma(x),$$

with $d\sigma$ denoting the Lebesgue measure on $\mathbb{S}^{n-1}$. Like the disc multiplier, the extension operator is central to modern harmonic analysis, and in dimensions greater than 2, the L^p boundedness of $\mathcal{E}$ remains one of harmonic analysis' most fundamental unsolved problems. The extension operator is the adjoint of the restriction operator $\mathcal{R}$ defined by the map

$$\mathcal{R} : f \longrightarrow \hat{f}|_{\mathbb{S}^{n-1}},$$

and so by duality, L^p bounds on $\mathcal{E}$ are equivalent to certain L^p bounds on $\mathcal{R}$.

The so-called restriction conjecture on $\mathbb{R}^n$ (formulated in terms of $\mathcal{E}$) states that $\mathcal{E}$ is bounded from $L^p(\mathbb{S}^{n-1})$ to $L^q(\mathbb{R}^n)$ if and only if

$$\frac{1}{q} \leq \frac{n-1}{n+1}\cdot\frac{1}{p'} \quad \text{and} \quad \frac{1}{q} < \frac{n-1}{2n}.$$

It is easily shown by testing the boundedness of $\mathcal{E}$ on certain functions that the conjectured range of p and q is the best possible. The restriction conjecture is known to be true in $\mathbb{R}^2$, but is open in all higher dimensions although partial progress on the conjecture has reduced matters somewhat. The most celebrated partial result is the Stein-Tomas Theorem, which establishes boundedness of $\mathcal{E}$ from $L^2(\mathbb{S}^{n-1})$ to $L^q(\mathbb{R}^n)$ when

$$\frac{1}{q} \leq \frac{n-1}{n+1} \cdot \frac{1}{2}.$$

These results, along with further information regarding the extension operator, can be found in [28] and [36].

As with the disc multiplier, we may study weighted inequalities for the extension operator to better understand its L^p boundedness. One may consider global and local weighted L^2 inequalities for the extension operator of the form

$$\int_{\mathbb{R}^n} |\widehat{fd\sigma}(\xi)|^2 d\mu(\xi) \lesssim \int_{\mathbb{S}^{n-1}} |f|^2 \mathcal{M}(\mu) d\sigma,$$

and

$$\int_{B(0,1)} |\widehat{fd\sigma}(R\xi)|^2 d\mu(\xi) \lesssim \int_{\mathbb{S}^{n-1}} |f|^2 \mathcal{M}_R(\mu) d\sigma, \qquad R \geq 1,$$

respectively. As was the case with the disc multiplier, information about the L^p boundedness of the maximal function $\mathcal{M}$ (or $\mathcal{M}_R$) could then be used to extract corresponding bounds for the extension operator.

It is conjectured that

$$\int_{\mathbb{R}^n} |\widehat{fd\sigma}(x)|^2 d\mu(x) \lesssim \int_{\mathbb{S}^{n-1}} |f(\omega)|^2 \mathcal{M}(\mu)(\omega) d\sigma(\omega), \tag{3.6}$$

or equivalently, for any $R > 1$ and measure μ supported in $B(0,1)$

$$\int_{B(0,1)} |\widehat{fd\sigma}(Rx)|^2 d\mu(x) \lesssim \frac{1}{R^{n-1}} \int_{\mathbb{S}^{n-1}} |f(\omega)|^2 \mathcal{M}(\mu)(\omega) d\sigma(\omega), \tag{3.7}$$

where the maximal function $\mathcal{M}$ is defined by

$$\mathcal{M}(\mu)(\omega) = \sup_{T \| \omega} \frac{\mu(T)}{w(T)^{n-1}},$$

with the supremum taken over all infinite rectangles T in $\mathbb{R}^n$ with $n-1$ short sides of length $w(T)$, and remaining side doubly infinite in the direction of ω.

The conjectured inequality (3.7) is known to be true when μ is radial, in which case $\mathcal{M}_R(\mu)$ is constant and equal to

$$\|\mathcal{M}(\mu)\|_\infty = \sup_T \frac{\mu(T)}{w(T)^{n-1}},$$

with the supremum taken over tubes with dimensions as described above. A proof of this may be found in [4] where, for suitable functions f and radial weight functions V it is shown that

$$\int |\widehat{fd\sigma}(R\xi)|^2 V(\xi) d\xi \lesssim \frac{C(V)}{R^{n-1}} \int |f(\omega)|^2 d\sigma(\omega)$$

with $C(V)$ equal to the supremum of the X-ray transform of V. If V is radial with support in the unit ball then $C(V)$ and $\|\mathcal{M}(V)\|_\infty$ coincide.

From here onwards we will consider local weighted estimates for the extension operator at scale R, where we take $R \geq 1$ to be a fixed large parameter. Let B denote a δ-neighborhood of a point on the surface of $\mathbb{S}^{n-1}$, for small δ, and define a function $g(x) = e^{ia \cdot x} \chi_B(x)$ for $a \in \mathbb{R}^n$. It is well known that $|\widehat{gd\sigma}(Rx)|$ is large for x belonging to a certain rectangle in $\mathbb{R}^n$. More precisely, $|\widehat{gd\sigma}(x)| \gtrsim \delta^{n-1}\chi_T(x)$ where T is a rectangle in $\mathbb{R}^n$ with $n-1$ short sides of length δ^{-1} and one long side of length δ^{-2}. Applying this information to the inequality

$$\int_{B(0,1)} |\widehat{gd\sigma}(Rx)|^2 d\mu(x) \lesssim C(\mu, R) \int_{\mathbb{S}^{n-1}} |g(\omega)|^2 d\sigma(\omega)$$

suggests that it may be the case that

$$\int_{B(0,1)} |\widehat{gd\sigma}(Rx)|^2 d\mu(x) \leq \frac{C}{R^{n-1}} \sup_{R^{-1}\leq\alpha\leq R^{-1/2}} \left\{ \frac{\mu(T(\alpha, \alpha^2 R))}{\alpha^{n-1}} \right\} \|g\|^2_{L^2(\mathbb{S}^{n-1})}, \quad (3.8)$$

for all $g \in L^2(\mathbb{S}^{n-1})$, where $T(\alpha, \alpha^2 R)$ denotes a rectangle in $\mathbb{R}^n$ with arbitrary position and orientation, and having $n-1$ short sides of length α, and one long side of length $\alpha^2 R$, for $R^{-1} \leq \alpha \leq R^{-1/2}$. If inequality (3.8) were true it would imply inequality (3.7) for radial measures, since $\mu(T(\alpha, \alpha^2 R))/\alpha^{n-1} \leq \|\mathcal{M}(\mu)\|_\infty$ uniformly in α and R. However, it is proven in [1] that (3.8) fails, and for radial weights it fails by a factor of $\log\log R$. Furthermore, this factor of $\log\log R$ is sharp in the following sense:

Theorem 3.2.1. *Let μ be a non-negative radial Borel measure supported on $B(0,1)$. There exists a constant $0 < C < \infty$, depending on at most n, such that*

$$\int_{B(0,1)} |\widehat{gd\sigma}(R\xi)|^2 d\mu \lesssim C\frac{\log\log R}{R^{n-1}} \sup_{R^{-1}\leq\alpha\leq R^{-1/2}} \left\{ \frac{\mu(T(\alpha, \alpha^2 R))}{\alpha^{n-1}} \right\} \|g\|^2_{L^2(\mathbb{S}^{n-1})} \quad (3.9)$$

for all $g \in L^2(\mathbb{S}^{n-1})$ and $R \geq 1$. Conversely, there exists a constant $0 < c < \infty$, depending on at most n, such that for each $R \geq 1$ there is a non-negative radial Borel measure μ supported on $B(0,1)$ for which

$$\int_{B(0,1)} |\widehat{d\sigma}(Rx)|^2 d\mu(x) \geq c\frac{\log\log R}{R^{n-1}} \sup_{R^{-1}\leq\alpha\leq R^{-1/2}} \left\{ \frac{\mu(T(\alpha, \alpha^2 R))}{\alpha^{n-1}} \right\}.$$

For a function $f \in L^2(\mathbb{S}^{n-1})$, we may express f in terms of its spherical harmonics as

$$f = \sum_{\ell} \sum_{m=1}^{a_\ell} c_{\ell,m} \mathcal{Y}_m^{(\ell)}.$$

The action of the extension operator on the basis elements of the spherical harmonics allows us to write

$$\widehat{f d\sigma}(\xi) = \sum_{\ell} \sum_{m=1}^{a_\ell} c_{\ell,m} |\xi|^{-\frac{(n-2)}{2}} J_{\ell + \frac{n-2}{2}}(\xi) \mathcal{Y}_m^{(\ell)} \left(\frac{\xi}{|\xi|} \right),$$

where J_ν denotes the Bessel function of order ν. If we write the left hand side of inequality (3.9) in polar coordinates and use the standard orthogonality properties of the spherical harmonics, then proving Theorem 3.2.1 is reduced to demonstrating that

$$\int_{|x| \leq 1} \frac{|J_\nu(R|x|)|^2}{|Rx|^{n-2}} d\mu(x) \lesssim \frac{\log\log R}{R^{n-1}} \sup_{R^{-1} \leq \alpha \leq R^{-1/2}} \left\{ \frac{\mu(T(\alpha, \alpha^2 R))}{\alpha^{n-1}} \right\},$$

which is achieved through some subtle analysis of the left hand side of the above inequality.

The second part of Theorem 3.2.1, which shows that inequality (3.9) is optimal, is proved by the authors using the following example: let μ be a radially non-increasing Borel measure supported in $B(0,1)$. The supremum on the right hand side of (3.9) is attained by a rectangle centred at the origin with long side parallel to the x_1-axis. For such a rectangle $T = T(\alpha, \alpha^2 R)$ we have that

$$\frac{1}{\alpha^{n-1}} \mu(T(\alpha, \alpha^2 R)) \lesssim \frac{1}{\alpha} \int_0^\alpha t d\mu(t) + \int_\alpha^{\alpha^2 R} d\mu(t). \tag{3.10}$$

We now construct our measure μ. First, we define a collection of positive real numbers $\{\alpha_j\}_{j=0}^k$ by setting $\alpha_0 = 0$, $\alpha_1 = 2/R$ and

$$\alpha_{j+1} = R\alpha_j^2 \quad \text{for } 2 \leq j < k$$

where k is such that $\alpha_k \sim R^{-1/2}$. We now define our measure μ by

$$d\mu(t) = \sum_{j=0}^{k} \frac{1}{\alpha_{j+1}} \chi_{(\alpha_j, \alpha_{j+1}]}(t)dt.$$

Observe that since $\alpha_j = \frac{2^{2^j}}{R}$, we have that $k \sim \log\log R$. A well-known asymptotic formula states that

$$\widehat{d\sigma}(x) = c|x|^{-\frac{n-1}{2}} \cos(|x| - \pi/4) + O(|x|^{-\frac{n+1}{2}}) \quad \text{as } |x| \to \infty,$$

which one may use to obtain

$$\int |\widehat{d\sigma}(Rx)|^2 d\mu(x) \gtrsim \frac{1}{R^{n-1}} \int_{\frac{1}{R} \leq |x| \leq 1} \frac{1}{|x|^{n-1}} d\mu(x) \gtrsim \frac{\log\log R}{R^{n-1}}.$$

This gives the desired inequality, since by (3.10) we have

$$\sup_{R^{-1} \leq \alpha \leq R^{-1/2}} \left\{ \frac{\mu(T(\alpha, \alpha^2 R))}{\alpha^{n-1}} \right\} \lesssim 1.$$

The Bochner-Riesz means are a family of operators that have a deep connection to both the disc multiplier and the extension operator and arise from partial Fourier inversion. The partial Fourier integrals on $\mathbb{R}^n$ are defined as

$$S_R(f)(x) = \int_{|\xi| \leq R} e^{2\pi i x \cdot \xi} \hat{f}(\xi) d\xi.$$

They are, in other words, operators whose multiplier corresponds to the characteristic function of the disc $\{|\xi| \leq R\}$, and related questions of convergence concerning Fourier inversion can be formulated in terms of these operators and the related maximal function, $S_*(f)(x) = \sup_{R>0} |S_R(f)(x)|$. As one would expect, almost-everywhere convergence of $S_R(f)$ to f is controlled by the L^p-boundedness of the maximal function S_*, and the celebrated Carleson Theorem shows that S_* is a bounded operator on $L^p(\mathbb{R})$ for $1 < p < \infty$ (see, for example, the text of Grafakos [20]).

As we know, operators corresponding to disc multipliers fail to be bounded on any $L^p(\mathbb{R}^n)$ when $n \geq 2$ other than the trivial case $p = 2$. The Bochner-Riesz means present an alternative way of summing an inverse Fourier transform that is similar to a disc multiplier, but with control over the 'roughness' of the cutoff. For positive R and non-negative δ define an operator S_R^δ by

$$S_R^\delta(f)(x) = \int_{|\xi| \leq R} \hat{f}(\xi) \left(1 - \frac{|\xi|^2}{R^2}\right)^\delta e^{2\pi i x \cdot \xi} d\xi.$$

Notice that when $\delta = 0$ the above operator corresponds to the disc multiplier $\{|\xi| \leq R\}$.

It is natural to ask whether $S_R^\delta(f)$ converges to f in L^p norm, and this reduces to the L^p boundedness of the operator $S^\delta := S_1^\delta$.

The operator S^δ is expressible as a convolution operator with kernel K^δ, where

$$K^\delta(x) = \frac{\Gamma(1+\delta)}{\pi^\delta |x|^{n/2+\delta}} J_{n/2+\delta}(2\pi x), \tag{3.11}$$

with J_μ denoting the Bessel function of order μ. It follows that K^δ is an L^1 function when $\delta > (n-1)/2$, and so for this range of δ the operator S^δ is bounded on all L^p $(1 \leq p \leq \infty)$. The application of Besssel function asymptotics to the formula (3.11) shows that

$$\frac{2n}{n+1+2\delta} < p < \frac{2n}{n-1-2\delta} \tag{3.12}$$

is a necessary condition for S^δ to be bounded on $L^p(\mathbb{R}^n)$. Equivalently put, if S^δ is bounded on $L^p(\mathbb{R}^n)$ then we must have $\delta > \delta(p)$ where

$$\delta(p) = n\left|\frac{1}{p} - \frac{1}{2}\right| - 1/2$$

when $\delta < (n-1)/2$. In fact, it is conjectured that S^δ is bounded on $L^p(\mathbb{R}^n)$ whenever p satisfies (3.12), This conjecture was proven to be true when $n = 2$ by Carleson and Sjölin (see [13]). Standard arguments may then be used to deduce the norm convergence of S_R^δ to f as $R \to \infty$ when $f \in L^p(\mathbb{R}^2)$. Further information about the $L^p(\mathbb{R}^n)$ boundedness of S^δ can be found in [28] and [31].

It is not surprising that almost-everywhere convergence of $S_R^\delta(f)$ to f is controlled by the maximal function

$$S_*^\delta(f)(x) = \sup_{R>0} |S_R^\delta(f)(x)|$$

and it is known from [8] that S_*^δ is bounded on $L^p(\mathbb{R}^2)$ when $\delta > 0$ and $2 \le p \le 4$. There is, however, a more recent weighted $L^2(\mathbb{R}^2)$ estimate for S_*^δ that is strong enough to recover the known $L^p(\mathbb{R}^2)$ boundedness. The following theorem is proved by Carbery and Seeger in [11]:

Theorem 3.2.2. *Given $\delta > 0$ there exists $\epsilon_\delta > 0$ and an operator W_δ bounded on $L^q(\mathbb{R}^2)$ for $2 - \epsilon_\delta \le q \le \infty$ such that*

$$\int_{\mathbb{R}^2} |S_*^\delta(f)(x)|^2 w(x)dx \le C_{\delta,s} \int_{\mathbb{R}^2} |f(x)|^2 M_s W_\delta(w)(x)dx, \tag{3.13}$$

where $M_s(f) = M(|f|^s)^{1/s}$ *for* $s > 1$.

This supercedes an earlier result by Carbery ([9]) where it is shown that for each $q \geq 2$ there exists a maximal function $W_{\delta,q}$, bounded on $L^q(\mathbb{R}^2)$, for which (3.13) holds with $W_{\delta,q}$ in place of $M_s W_\delta$. Further to this, $W_{\delta,q}(w) = W_{\delta,2}(w^{q/2})^{2/q}$ and $W_{\delta,q}$ is bounded on $L^p(\mathbb{R}^2)$ for $2 \leq p \leq 4$.

The proof of Theorem 3.2.2 gives a direct construction of W_δ and relates it to the Kakeya maximal function: Let R_N denote the family of rectangles centered at the origin with the property such that the ratio of the larger to the smaller sidelength is equal to 2^N. Define

$$\mathfrak{M}_N(f)(x) = \sup_{x \in R \in R_N} \frac{1}{|R|} \int_R |f(x+y)| dy.$$

Then W_δ satisfies the estimate

$$W_\delta(w) \leq C_\epsilon \sum_{j \geq 1} 2^{-j\epsilon} (\mathfrak{M}_{j/2}(w^2)(x))^{1/2}$$

for $\epsilon < 2\delta$. It is conjectured that a weighted L^2 estimate of the form (3.13) holds for S_*^δ with the maximal function given on the right by $\sum_{j\geq 1} 2^{-j\epsilon} \mathfrak{M}_{j/2}$ for $\epsilon < 2\delta$, which is perhaps not surprising given the acknowledged connection between Kakeya-type maximal functions and the Bochner-Riesz operators. In particular, there is a sense in which the Bochner-Riesz operators are controlled by maximal functions of Kakeya type, which is explored in [20] and [14].

Such weighted estimates for Bochner-Riesz operators are often proved via weighted estimates for a related square function: Let ϕ be a smooth real-valued bump function with support in $[-1, 1]$ and let $\phi^\alpha(x) = \phi((|x| - 1)/\alpha)$. Let $\hat{\psi}^\alpha = \phi^\alpha$ and we define, with $\psi_t^\alpha = t^{-2}\psi^\alpha(t^{-1}\cdot)$, the square function

$$T^\alpha(f)(x) = \left(\int_0^\infty |\psi_t^\alpha * f(x)|^2 \frac{dt}{t}\right)^{1/2}.$$

In [9], for example, it is shown that if T^α satisfies

$$\int_{\mathbb{R}^2} |T^\alpha(f)(x)|^2 w(x)dx \lesssim \int_{\mathbb{R}^2} |f(x)|^2 Q_\alpha(w)(x)dx,$$

for a maximal function Q_δ then S_*^δ is controlled in a weighted L^2 inequality by an operator of the form $\sum_{k>0} C_k Q_{2^{-k}}$.

It is known that there is a relationship between norm estimates for the Bochner-Riesz means and norm estimates for the restriction operator. For $n \geq 2$, it is shown by Tao in [31] that if $S^{\delta(p)+\alpha}$ is bounded on $L^p(\mathbb{R}^n)$ for some p then the localised restriction estimate

$$\|\mathcal{R}(f)\|_{L^p(\mathbb{S}^{n-1})} \lesssim R^{2\alpha}\|f\|_{L^p(B(0,R))}$$

holds. Furthermore, certain weighted estimates for Bochner-Riesz means and the extension operator are shown to be equivalent in [12]. The functions $\phi^{2^{-k}}$ for $k > 0$ are a convenient way to decompose the multiplier $(1-|\xi|^2)^\delta$ and it is often useful to consider them as multipliers in their own right. Let Φ be a non-negative bump function of one variable, and let T_δ be the operator with multiplier given by $\Phi(|\xi| - \delta^{-1})$ (similar to a rescaled ϕ^δ). Then the estimate

$$\int_{|x|\leq 1} |T_\delta(f)(x)|^2 w(x)dx \leq A\int_{\mathbb{R}^n} |f(x)|^2 dx$$

for all $f \in L^2(\mathbb{R}^n)$ is equivalent to

$$\int_{|x|\leq 1} |\widehat{gd\sigma}(Rx)|^2 w(x)dx \leq \frac{B}{R^{n-1}} \int_{\mathbb{S}^{n-1}} |g(\omega)|^2 d\sigma(\omega)$$

for all $g \in L^2(\mathbb{S}^{n-1})$ with $R = 1/\delta$, where the constants A and B are equivalent. It has already been mentioned that there is a connection between the Bochner-Riesz operators and Kakeya-type maximal functions, and it is conjectured that

$$\int |T_\delta(f)|^2 w \lesssim \int |f|^2 \mathcal{M}^\delta(w) \tag{3.14}$$

where $\mathcal{M}^\delta$ is the maximal average of w taken over rectangles with eccentricity δ^{-1}. If, then, (3.14) did hold, by the above equivalence we would have

$$\int_{|x|\leq 1} |\widehat{gd\sigma}(Rx)|^2 w(x)dx \lesssim \frac{1}{R^{n-1}} \frac{\sup w(T)}{|T|} \int_{\mathbb{S}^{n-1}} |g(\omega)|^2 d\sigma(\omega)$$

where the supremum is taken over all rectangles in the unit ball of eccentricity less than R. This might lead one to conjecture that

$$\int_{|x|\leq 1} |\widehat{gd\sigma}(Rx)|^2 w(x)dx \lesssim \frac{1}{R^{n-1}} \int_{\mathbb{S}^{n-1}} |g(\omega)|^2 \sup \frac{w(T)}{|T|} d\sigma(\omega)$$

where the supremum is taken over rectangles T in the unit ball of direction ω with eccentricity less than R. The above considerations are noted in [5] in justification that weighted estimates for the extension operator of the above kind could rightly be considered as being of Stein-type.

3.3 The Extension Operator on a Hypersurface

For functions f on $\mathbb{R}^{n-1}$ consider the map $f \mapsto e^{it\Delta} f$ where

$$e^{it\Delta} f(x) = \int_{\mathbb{R}^{n-1}} e^{-i\pi t|\xi|^2 + 2\pi i x\cdot\xi} \hat{f}(\xi) d\xi.$$

It is well known (and verifiable using a straightforward application of the Fourier transform) that $u(x,t) = e^{it\Delta} f(x)$ is the solution to the initial value problem for the free Schrödinger equation,

$$\begin{aligned} &i\partial_t u + \Delta_x u = 0 \quad (x,t) \in \mathbb{R}^{n-1} \times \mathbb{R} \quad n \geq 2 \\ &u(x,0) = f(x). \end{aligned}$$

Strichartz inequalities can naturally lead one to consider one-weighted estimates for the Schrödinger operator $e^{it\Delta}$. For example, it is known that

$$\|e^{it\Delta} f\|_{L^r_{x,t}(\mathbb{R}^n \times \mathbb{R})} \lesssim \|f\|_{\dot{H}^s(\mathbb{R}^n)},$$

for $0 \leq s < n/2$ and $r = \frac{2(n+2)}{n-2s}$ where $\dot{H}^s(\mathbb{R}^n)$ is the homogeneous Sobolev space. This is equivalent by duality and Hölder's inequality to the weighted estimate

$$\|e^{it\Delta} f\|^2_{L^2_{x,t}(V)} \lesssim \|V\|_{L^{(r/2)'}_{x,t}(\mathbb{R}^n \times \mathbb{R})} \|f\|^2_{\dot{H}^s(\mathbb{R}^n)}$$

for all V in $L^{(r/2)'}_{x,t}(\mathbb{R}^n \times \mathbb{R})$. This raises the possibility of other functionals $V \mapsto C(V)$ such that

$$\|e^{it\Delta} f\|^2_{L^2_{x,t}(V)} \lesssim C(V) \|f\|^2_{\dot{H}^s(\mathbb{R}^n)}.$$

Such functionals are investigated in [3], where it is shown that $e^{it\Delta}$ is controlled in the above one-weighted sense by Morrey-Campanato norms.

For $\alpha > 0$ and $1 \leq p \leq \frac{n+2}{\alpha}$, let

$$\mathcal{L}^{\alpha,p}_{par} = \{F \in L^p_{loc}(\mathbb{R}^n \times \mathbb{R}) : \|F\|_{\mathcal{L}^{\alpha,p}_{par}} < \infty\}$$

where

$$\|F\|_{\mathcal{L}^{\alpha,p}_{par}} = \sup_{(x,t)\in\mathbb{R}^n\times\mathbb{R}, r>0} r^\alpha \left(r^{-(n+2)} \int_{C(x,t,r)} |F(y,s)|^p dyds \right)^{1/p}$$

with $C(x,t,r)$ denoting the 'parabolic box' $B(x,r) \times (t-r^2, t+r^2)$. The main result from [3] is that if $n/4 \leq s \leq n/2$ and $1 < p \leq \frac{n+2}{2s+2}$ then

$$\|e^{it\Delta} f\|^2_{L^2_{x,t}(V)} \lesssim \|V\|_{\mathcal{L}^{2s+2,p}_{par}(\mathbb{R}^n\times\mathbb{R})} \|f\|^2_{\dot{H}^s(\mathbb{R}^n)}.$$

This Morrey-Campanato norm permits weights V that do not belong to any L^p space, such as $V(x,t) = |x|^{-a}|t|^{-b}$ where $ap < n$, $bp < 1$ and $a + 2b = 2s + 2 + \frac{n+2}{p}$ with p and s as above.

The Schrödinger operator is also of interest from the point of view of extension operators arising from hypersurfaces. Let $n \geq 2$, and let S be a bounded hypersurface in $\mathbb{R}^n$ with everywhere non-vanishing Gaussian curvature. For any such S, we may define a corresponding extension operator - let σ denote the induced Lebesgue measure on S, and for $f \in L^1(S)$ consider

$$\widehat{fd\sigma}(\xi) = \int f(x) e^{-2\pi i x\cdot\xi} d\sigma(x).$$

The extension operator for the surface S is thus defined by the mapping $f \mapsto \widehat{fd\sigma}$.

It is interesting to note that if S is taken to be the base of the paraboloid, then $\widehat{fd\sigma}$ coincides with the solution to the Schrödinger equation in the following way:

If we take S to denote the section of the paraboloid

$$\{\xi = (\xi', \xi_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : \xi_n = |\xi'|^2/2, \quad 0 \leq \xi_1, \ldots, \xi_{n-1} \lesssim 1\} \tag{3.15}$$

and let $d\sigma$ be the induced Lebesgue measure on S, then

$$\widehat{gd\sigma}(x) = \int_{|\xi'|\leq 1} e^{-2\pi i x'\cdot\xi' + \pi i |\xi'|^2 x_n} \hat{f}(\xi') d\xi' = e^{ix_n\Delta} f(x'), \tag{3.16}$$

where $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$, and $\hat{f}(\xi') = g(\xi', |\xi'|^2)(1 + |\xi'|^2)^{1/2}$.

For a general bounded hypersurface S in $\mathbb{R}^n$, there has been much recent activity on weighted estimates of the form

$$\int_{B(0,1)} |\widehat{gd\sigma}(Rx)|^2 d\mu(x) \leq \frac{C(\mu)}{R^\gamma} \|g\|^2_{L^2(S)}.$$

It is of particular interest to consider the pairs of exponents $\gamma > 0$ and $0 \leq \eta \leq n$ for which the inequality

$$\int_{B(0,1)} |\widehat{gd\sigma}(Rx)|^2 d\mu(x) \lesssim R^{-\gamma} \sup_{x\in\mathbb{R}^n, r>0} \frac{\mu(B(x,r))}{r^\eta} \|g\|^2_{L^2(S)} \tag{3.17}$$

holds for all $g \in L^2(S)$, $R \geq 1$, and Borel measures μ supported in $B(0,1)$. Such inequalities have a deep connection to certain areas of geometric measure theory, and particularly Falconer's work on distance sets:

For a compact set E in $\mathbb{R}^n$, the distance set $\Delta(E)$ of E is defined by

$$\Delta(E) = \{|x - y| : x, y \in E\}.$$

There are significant open questions concerning the relationship between the Hausdorff dimension of E and the Lebesgue measure of $\Delta(E)$ (see [23], or [36] for a discussion of the theory). What is known (due to Falconer - [17]) is that if $E \subset \mathbb{R}^n$ has Hausdorff dimension greater than $(n+1)/2$ then $\Delta(E)$ has positive Lebesgue measure. It was shown by Mattila (see [23]) that $|\Delta(E)| > 0$ if there exists a Borel measure μ supported on E such that

$$\int_1^\infty \left(\int_{\mathbb{S}^{n-1}} |\hat{\mu}(t\omega)|^2 d\sigma(\omega) \right)^2 t^{n-1} dt < \infty. \tag{3.18}$$

This naturally leads one to consider the measures μ for which we can expect to have a good rate of decay in t for the inner integral of the expression above, ie. for which measures μ and exponents $\beta > 0$ do we have

$$\int_{\mathbb{S}^{n-1}} |\hat{\mu}(t\omega)|^2 d\sigma(\omega) \leq C_\beta t^{-\beta}.$$

A natural class of measures to consider here are those that have finite *α-dimensional energy.* If μ is a non-negative compactly supported measure on $\mathbb{R}^n$, and $\alpha \in (0, n)$, then the α-energy of μ is given by

$$I_\alpha(\mu) = \int \int \frac{d\mu(x) d\mu(y)}{|x-y|^\alpha},$$

which, by Plancherel's Theorem, is equal to $c_{\alpha,n} \int |\hat{\mu}(\xi)|^2 |\xi|^{\alpha-n} d\xi$. Mattila's result (3.18) is exploited by Bourgain in [7] to obtain an improvement on Falconer's result for the dimension on E when $n = 2$.

There is a close connection between measures μ for which $I_\alpha(\mu) < \infty$, the Hausdorff dimension of the supports of such measures, and measures μ which satisfy $\mu(B(x,r)) \lesssim r^\alpha$ for all $x \in \mathbb{R}^n$ and $r > 0$. As is detailed in [23], for a set $A \subset \mathbb{R}^n$, the Hausdorff dimension of A is equal to

$$\sup\{\alpha : \exists \mu \quad \text{with} \quad I_\alpha(\mu) < \infty\},$$

and

$$\sup\{\beta : \exists \mu \quad \text{with} \quad \mu(B(x,r)) \leq r^\beta \quad \text{for} \quad x \in \mathbb{R}^n, \quad r > 0\}$$

where in each case the μ are assumed to be finite Radon measures on $\mathbb{R}^n$ with compact support in A.

Inequality (3.17) has an elegant connection to the concepts from geometric measure theory described above. Fix $\eta \in [0,n]$ and let $\gamma(\eta)$ be the supremum of the numbers for which (3.17) holds. It is noted in [2] that, in two dimensions, we have

$$\begin{aligned} & \eta/2, \quad 1 \leq \eta \leq 2 && (3.19) \\ \gamma(\eta) \quad = \quad & 1/2, \quad 1/2 \leq \eta \leq 1 && (3.20) \\ & \eta, \quad 0 \leq \eta \leq 1/2. && (3.21) \end{aligned}$$

The first of the these results, (3.19) is due to Wolff [35], and (3.20) and (3.21) to Mattila [22], in which it is shown that if μ is a finite compactly supported Radon measure on $\mathbb{R}^n$, and $0 \leq \alpha \leq \frac{1}{2}(n-1)$,

$$\int |\hat{\mu}(R\omega)|^2 d\sigma(\omega) \leq cR^{-\alpha} I_\alpha(\mu).$$

This tells us that if $n = 2$, $0 \leq \alpha \leq 1/2$ and $0 \leq \beta \leq \alpha$,

$$\int |\widehat{gd\mu}(R\omega)|^2 d\sigma(\omega) \lesssim R^{-\beta} I_\alpha(\mu) \|g\|_{L^2(d\sigma)},$$

when $g \equiv 1$ on S. Now when $n = 2$, one can also show by decomposing the integral $I_\alpha(\mu)$ into dyadic annuli that

$$I_\alpha(\mu) \lesssim \sup_{x \in \mathbb{R}^2, r>0} \frac{\mu(B(x,r))}{r^\alpha},$$

which gives

$$\int |\widehat{gd\mu}(Rx)|^2 d\sigma(x) \lesssim R^{-\beta} \sup_{x \in \mathbb{R}^n, r>0} \frac{\mu(B(x,r))}{r^\alpha} \|g\|^2_{L^2(d\mu)}, \tag{3.22}$$

when $n = 2$, $0 \leq \alpha \leq 1/2$ and $0 \leq \beta \leq \alpha$. One can then show that if inequality (3.22) holds for $g \equiv 1$, then it must also hold for general g in $L^2(d\mu)$. (For example, it is not hard to see that if (3.22) holds for $g \equiv 1$ then it holds for g equal to 1 on subsets of S, and subsequently for g essentially constant on subsets of S.) Finally, (3.22) may be dualised to show that, for $n = 2$, inequality (3.17) holds when $0 \leq \eta \leq 1/2$ and $0 \leq \gamma \leq \eta$. In other words, (3.21) holds. Since $\gamma(\eta)$ is non-decreasing in η, and $\gamma(1/2) = \gamma(1) = 1/2$, this forces (3.20) to be true.

The situation is less straightforward in higher dimensions. As described above, it is known that $\gamma(\eta) = \eta$ for $0 \leq \eta \leq (n-1)/2$, and $\gamma(n) = n - 1$. However, the upper and lower bounds for $\gamma(\eta)$ with η in the region $(\frac{n-1}{2}, n)$ do not coincide. Arguments that lead to lower bounds arise due to Sjölin in [26] and the more recent [16] by Erdoğan. An example that leads to a new upper bound for $\gamma(\eta)$ can be found in [2], where it is shown that for all bounded hypersurfaces S, if (3.17) holds for all $g \in L^2(S)$, all $R \geq 1$, and all Borel measures μ supported in $B(0,1)$, then

$$\gamma \leq (\eta + 1)\left(\frac{n-1}{n+1}\right). \tag{3.23}$$

if $(n-1)/2 < \eta < n$.

3.4 The Extension Operator - Weighted Inequalities on $\mathbb{S}^1$

As we have seen, weighted L^2 inequalities for the extension operator on the circle are well understood if the weight in question is radial. Here we present a result from [5] that establishes for the extension operator L^2 inequalities on $\mathbb{R}^2$ that are weighted with very different measures - ones that are supported on $\mathbb{S}^1$:

Theorem 3.4.1. *For all $R > 1$ and measures μ supported on $\mathbb{S}^1$,*

$$\int_{\mathbb{S}^1} |\widehat{gd\sigma}(Rx)|^2 d\mu(x) \lesssim \frac{\log R}{R} \int_{\mathbb{S}^1} |g(\omega)|^2 M\mathcal{M}^R(\mu)(\omega) d\sigma(\omega), \tag{3.24}$$

and

$$\int_{\mathbb{S}^1} |\widehat{gd\sigma}(Rx)|^2 d\mu(x) \lesssim \frac{1}{R} \int_{\mathbb{S}^1} |g(\omega)|^2 M\mathcal{M}^R M^2(\mu)(\omega) d\sigma(\omega), \tag{3.25}$$

where

$$\mathcal{M}^R(\mu)(\omega) = \sup_{\substack{T \| \omega \\ R^{-1} \leq \alpha \leq R^{-2/3}}} \frac{\mu(T(\alpha, \alpha^2 R))}{\alpha},$$

and M is the Hardy-Littlewood maximal function.

As before, the form of the maximal function $\mathcal{M}^R$ is suggested by the example $g(\omega) = e^{ia\cdot\omega}\chi_C(\omega)$ where C is a δ-cap on $\mathbb{S}^1$ and $a \in \mathbb{R}^2$.

We will now give a sketch of the proof. The proof of Theorem 3.4.1 has extra significance here in that, philosophically and technically, it is very similar to the proof of Theorem 4.2.2 and any areas that we may skip over here will get a full exposition in Chapter 4.

Let g be a function on $\mathbb{S}^1$. We begin by defining a collection of sets that partition $\mathbb{Z}$, and treating separately the cases where g has Fourier frequencies supported in each of these sets. By subsequently decomposing the frequencies of g further, we reach a stage where the geometric action of the operator $g \mapsto \widehat{gd\sigma}(R\cdot)|_{\mathbb{S}^1}$ can be more easily identified. This enables one to derive a collection of weighted inequalities, each of which is valid for g on $\mathbb{S}^1$ with Fourier frequencies supported in one of our original collection of sets, from which (3.24) and (3.25) can be deduced using Littlewood-Paley theory.

The initial frequency decomposition is based on the following collection of sets: fix p such that $1 \leq 2^p \leq R^{2/3}$ and define

$$\begin{aligned} \mathcal{A}_p &= \begin{cases} \{j \in \mathbb{Z} : R - j \sim 2^{-p}R\} & \text{if } 1 < 2^p < R^{2/3}, \\ \{j \in \mathbb{Z} : 0 \leq R - j \leq R^{1/3}\} & \text{if } 2^p = R^{2/3}, \end{cases} \\ \mathcal{B}_p &= \begin{cases} \{j \in \mathbb{Z} : j - R \sim 2^{-p}R\} & \text{if } 1 < 2^p < R^{2/3}, \\ \{j \in \mathbb{Z} : 0 \leq j - R \leq R^{1/3}\} & \text{if } 2^p = R^{2/3}, \end{cases} \end{aligned}$$

along with

$$\begin{aligned} \mathcal{C}_0 &= \{j \in \mathbb{Z} : |j| \leq R/2\} \\ \mathcal{C}_\infty &= \{j \in \mathbb{Z} : |j| > 3R/2\}. \end{aligned}$$

The sets described above, along with $-\mathcal{A}_p$ and $-\mathcal{B}_p$, form an approximate partition of $\mathbb{Z}$. It is important to note that the operator

$$g \mapsto \widehat{gd\sigma}(R\cdot)|_{\mathbb{S}^1}$$

coincides with convolution on $\mathbb{S}^1$ with $e^{iR\cos\cdot}$. If g has Fourier frequencies supported in $\mathcal{A}_p$ (for example) then we may write

$$g(\theta) = \sum_{j\in\mathcal{A}_p} \alpha_j e^{ij\theta}$$

and so for $|x| = 1$,

$$\widehat{gd\sigma}(Rx) = \sum_{j\in\mathcal{A}_p} \alpha_j J_j(R) e^{ij\arg(x)}. \tag{3.26}$$

Since we have the estimate

$$|J_k(s)| \le cs^{-1/2} \min\left\{k^{1/6}, \left|\frac{|s|+|k|}{|s|-|k|}\right|^{1/4}\right\},$$

we have control over $J_j(R)$ when $j \in \mathcal{A}_p$, and similarly for $-\mathcal{A}_p$, $\pm\mathcal{B}_p$. How this manifests itself in the maximal function will become clear later on. Theorem 3.4.1 is established by way of the following result:

Proposition 3.4.2. *Let μ be a measure supported on $\mathbb{S}^1$.*

1. *If g has Fourier frequencies supported in either $\mathcal{A}_p$, $-\mathcal{A}_p$, $\mathcal{B}_p$, or $-\mathcal{B}_p$, then*

$$\int_{\mathbb{S}^1} |\widehat{gd\sigma}(Rx)|^2 d\mu(x) \lesssim \frac{1}{R}\int_{\mathcal{S}^1} |g(\omega)|^2 \mathcal{M}_p(\mu)(\omega) d\sigma(\omega).$$

2. *If g has frequencies supported in $\mathcal{C}_0$ or $\mathcal{C}_\infty$, then*

$$\int_{\mathbb{S}^1} |\widehat{gd\sigma}(Rx)|^2 d\mu(x) \lesssim \frac{1}{R}\int_{\mathbb{S}^1} |g(\omega)|^2 \mathcal{M}_0(\mu)(\omega) d\sigma(\omega),$$

where, for p such that $1 \leq 2^p \leq R^{2/3}$,

$$\mathcal{M}_p(\mu)(\omega) = \sup_{T \| \omega} \frac{\mu(T(2^{p/2}/R, 2^p/R))}{2^{p/2}/R}.$$

Let us start with part 1 of the proposition, and suppose that g has Fourier frequencies supported in $\mathcal{A}_p$. Using the formula (3.26) one may argue that

$$\int_{\mathbb{S}^1} |\widehat{gd\sigma}(Rx)|^2 d\mu(x) = \int_{\mathbb{S}^1} |\widehat{gd\sigma}(Rx)|^2 P_p * \mu(x) d\sigma(x)$$

where P_p is any function satisfying $\hat{P}_p(j) = 1$ when $|j| \leq 4 \cdot 2^{-p}R$. We may choose P_p to be the kernel of an approximation to the identity on $\mathbb{S}^1$ at scale $2^p/R$ such that, for any $N \in \mathbb{N}$,

$$|P_p(x)| \lesssim \frac{2^{-p}R}{(1 + 2^{-p}R|x|)^N}$$

for all $x \in [-\pi, \pi]$. In other words, restricting the frequencies of g allows us to smooth-out the measure μ, in this case at scale $2^p/R$.

Now let ϕ_p be a bump function at scale $2^p/R$ such that $\hat{\phi}_p(j) = 1$ when $j \in \mathcal{A}_p$. Then $\hat{g} = \hat{g}\hat{\phi}_p$, implying that $g = \phi_p * g$, and so (bearing in mind that the extension operator coincides on the circle with a convolution operator) we have $\widehat{gd\sigma}(x) = \widehat{\phi_p d\sigma} * g(x)$ for $x \in \mathbb{S}^1$. One might then be tempted to reason that the map $g \mapsto \widehat{gd\sigma}(R\cdot)|_{\mathbb{S}^1}$ can be understood by its action on such functions as ϕ_p. However, a bump function at this scale is not smooth enough for any such action to be established. For this reason, a further frequency decomposition is to be carried out, but first it is necessary to dominate the weight $P_p * \mu$ by a function with an increased level of smoothness in order to recover some partial orthogonality from the forthcoming second frequency decomposition: first, we dominate $P_p * \mu$ pointwise by

$$\psi_1(\theta) = \sup_{|\eta-\theta|\leq 2^{-p/2}} |P_p * \mu(\eta)|.$$

Now let Θ_p be a non-negative function on $\mathbb{S}^1$ with non-negative Fourier coefficients supported in $\{j \in \mathbb{Z} : |j| \leq 2^{p/2}\}$ We may choose Θ_p such that for each $N \in \mathbb{N}$,

$$\Theta_p(\theta) \lesssim \frac{2^{p/2}}{(1 + 2^{p/2}|\theta|)^N}$$

and such that there is an absolute constant $c > 0$ (independent of p) for which $\Theta_p(\theta) \gtrsim 2^{p/2}$ whenever $|\theta| \leq c2^{-p/2}$. Let $\psi_2 = \Theta_p * \psi_1$. The scale of the local supremum above and the conditions on Θ_p allow one to argue that $\psi_1 \lesssim \psi_2$ as follows:

Lemma 3.4.3. $\psi_1 \lesssim \psi_2$

Proof. By the properties of Θ_p,

$$\Theta_p * \psi_1(\theta) \gtrsim 2^{p/2} \int_{|\phi|\lesssim c2^{-p/2}} \psi_1(\theta - \phi)d\phi.$$

By elementary considerations, either

$$\psi_1(\theta') \geq \psi_1(\theta) \quad \text{for all} \quad \theta - 2^{-p/2} \leq \theta' \leq \theta,$$

or

$$\psi_1(\theta') \geq \psi_1(\theta) \quad \text{for all} \quad \theta \leq \theta' \leq \theta + 2^{-p/2},$$

and so $\Theta_p * \psi_1(\theta) \gtrsim \psi_1(\theta)$ uniformly in θ. ☐

Therefore it suffices to control

$$\int_{\mathbb{S}^1} |\widehat{gd\sigma}(Rx)|^2 \psi_2(x)dx.$$

We now carry out our second frequency decomposition as follows: for $L \in \mathbb{N}$, let W_L be a function on $\mathbb{S}^1$ with frequencies supported in $\{j \in \mathbb{Z} : |j| \leq 2L\}$ such that

$$\sum_k \widehat{W}_L(j + kL) = 1$$

for all j in $\mathbb{Z}$. We also choose W_L (as we may) such that for each $N \in \mathbb{N}$,

$$|W_L(\theta)| \lesssim \frac{L}{(1 + L|\theta|)^N}$$

for all $\theta \in [-\pi, \pi]$. If for each q with $0 \leq q \leq 2^{-3p/2}$ we write

$$g_q(\theta) = \int_{-\pi}^{\pi} g(\phi) e^{-i(q2^{p/2} + R(1-2^{-p+1}))\phi} W_{2^{p/2}}(\theta - \phi) d\phi,$$

then

$$g(\theta) = e^{iR(1-2^{-p+1})\theta} \sum_{q=0}^{2^{-3p/2}R} e^{iq2^{p/2}\theta} g_q(\theta).$$

Now let Φ_p be a function on $\mathbb{S}^1$ satisfying

$$\widehat{\Phi}_p(j) = \begin{cases} 1 & \text{if} \quad |j| \leq 2^{p/2+1} \\ 0 & \text{if} \quad |j| \geq 2^{p/2+2}, \end{cases}$$

so that $g_q = \Phi_p * g_q$ for each q. We may choose Φ_p such that for each $N \in \mathbb{N}$, the derivatives of Φ_p satisfy

$$|\Phi_p^{(k)}(\theta)| \lesssim \frac{2^{(k+1)p/2}}{(1+2^{p/2}|\theta|)^N} \tag{3.27}$$

for all $\theta \in [-\pi, \pi]$. With this notation, we have the formula

$$\widehat{gd\sigma}(Re^{i\phi}) = e^{iR(1-2^{-p+1})\phi} \sum_{q=0}^{2^{-3p/2}R} e^{iq2^{p/2}} g_q * \Psi_{p,q}(\phi),$$

where

$$\Psi_{p,q}(\phi) = \int_{-\pi}^{\pi} e^{iR[(1-2^{-p+1}+q2^{p/2}/R)\theta + \cos\theta]} \Phi_p(\phi-\theta)d\theta,$$

and as a result,

$$\begin{aligned}
&\int_{-\pi}^{\pi} |\widehat{gd\sigma}(Re^{i\phi})|^2 \psi_2(\phi)d\theta \\
&= \sum_{q,q'} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} g_q(u)\overline{g_{q'}(v)} \\
&\times \left(\int_{-\pi}^{\pi} \Psi_{p,q}(\phi-u)\overline{\Psi_{p,q'}(\phi-v)}\psi_2(\phi)e^{i(q-q')2^{p/2}\phi}d\phi\right) dudv.
\end{aligned}$$

Since $\widehat{\Psi}_{p,q}(k)$, $\widehat{\Psi}_{p,q'}(k)$, $\hat{\psi}_2(k) = 0$ when $|k| \geq 4 \cdot 2^{p/2}$, one can argue that

$$\int_{-\pi}^{\pi} \Psi_{p,q}(\phi-u)\overline{\Psi_{p,q'}(\phi-v)}\psi_2(\phi)e^{-(q-q')2^{p/2}\phi}d\phi = 0$$

whenever $|q-q'| > 12$. Since $|g_q(y)\overline{g_{q'}(z)}| \leq \frac{1}{2}(|g_q(y)|^2 + |g_{q'}(z)|^2)$, by symmetry it suffices to control

$$
\begin{aligned}
&\sum_{|q-q'|\le 12}\int_{\mathbb{R}}|g_q(y)|^2\left(\int_{\mathbb{R}}\int_{\mathbb{R}}|\Psi_{p,q}(\phi-u)||\Psi_{p,q'}(\phi-v)|\psi_2(\phi)d\phi dv\right)du\\
&=\sum_{|q-q'|\le 12}\int_{\mathbb{R}}|g_q(y)|^2\left(\int_{\mathbb{R}}|\Psi_{p,q}(\phi-u)|\psi_2(\phi)d\phi\right)\left(\int_{\mathbb{R}}|\Psi_{p,q'}(v)|dv\right)du.
\end{aligned}
\tag{3.28}
$$

Since we have the decay estimate (3.27), one may argue in the manner of Lemma 2.1.1 to obtain the following result, which can be thought of as identifying the geometric action of convolution with $e^{iR\cos\cdot}$.

Lemma 3.4.4.

$$|\Psi_{p,q}(\phi)|\lesssim\frac{2^{p/4}}{R^{1/2}}H_p(\phi-\pi/2)$$

uniformly in q, where H_p satisfies

$$H_p(\phi)\lesssim\frac{2^{p/2}}{(1+2^{p/2}|\phi|)^N}$$

for each $N\in\mathbb{N}$.

As a consequence, we have that

$$\int_{-\pi}^{\pi}|\Psi_{p,q'}(v)|dv\lesssim\frac{2^{p/4}}{R^{1/2}},$$

and so in inequality (3.28) this gives

$$\int_{-\pi}^{\pi}|\widehat{gd\sigma}(Re^{i\phi})|^2\psi_2(\phi)d\phi\lesssim\frac{2^{p/2}}{R}\int_{-\pi}^{\pi}\sum_q|g_q(u)|^2\psi_3(u-\pi/2)du.$$

Now Littlewood-Paley theory (see the remark following the proof of Lemma 1.1.1) allows us to deduce that

$$\int_{-\pi}^{\pi} |\widehat{gd\sigma}(Re^{i\theta})|^2 \psi_2(\theta) d\theta \lesssim \frac{1}{R} \int_{-\pi}^{\pi} |g(\theta)|^2 2^{p/2} \psi_4(\theta - \pi/2) d\theta$$

where $\psi_4 = |W_L| * \psi_3$. In order to establish part 1 of Proposition 3.4.2 it remains to be proved that $2^{p/2}\psi_4(\cdot - \pi/2)$ is dominated pointwise by $\mathcal{M}_p(\mu)$. The can be seen as follows: let C_λ denote the arc on $\mathbb{S}^1$ centred at $-\pi/2$ of length $2\lambda \cdot 2^p/R$, where $1 \leq 2^p \leq R^{2/3}$ and $1 \leq \lambda R/2^p$. Let $T = T(2^{p/2}/R, 2^p/R)$ have long side parallel to the x-axis. By arguing that C_λ is contained in at most $4\lambda^2$ of such rectangles T, it follows that $\mu(C_\lambda) \leq 4\lambda^2 \sup_T \mu(T)$. We now apply this fact to show that

$$2^{p/2} P_p * \mu(\theta - \pi/2) \lesssim \mathcal{M}_p(\mu)(\theta) \tag{3.29}$$

for all $\theta \in [-\pi, \pi]$: By rotational symmetry we may assume that $\theta = 0$. The kernel P_p is dominated by

$$\sum_{1 \leq 2^k \leq R/2^p} \frac{1}{2^{kN}} \frac{\chi_{\{|\cdot| \lesssim 2^k 2^p/R\}}}{2^k 2^p/R},$$

for any $N \in \mathbb{N}$, and so

$$\begin{aligned} P_p * \mu(-\pi/2) &\lesssim \frac{R}{2^p} \sum_{1 \leq 2^k \leq R/2^p} \frac{\mu(C_{2^k})}{2^{k(N+1)}} \\ &\lesssim \frac{R}{2^p} \left(\sum_{1 \leq 2^k \leq R/2^p} \frac{2^{2k}}{2^{k(N+1)}} \right) \sup_T \mu(T), \end{aligned}$$

where the supremum is taken over all rectangles $T = T(2^{p/2}/R, 2^p/R)$ with long side parallel to the x-axis. Hence

$$2^{p/2}P_p * \mu(-\pi/2) \lesssim \sup_{T\|(1,0)} \frac{\mu(T(2^{p/2}/R, 2^p/R))}{2^{p/2}/R} = \mathcal{M}_p(\mu)(0).$$

Geometric considerations allow one to argue that for any measure ν on the unit ball in $\mathbb{R}^2$,

$$\mathcal{M}_p(\nu)(\theta - \phi) \lesssim (1 + 2^{p/2}|\sin\phi|)\mathcal{M}_p(\nu)(\theta). \tag{3.30}$$

from (3.29) and (3.30) it follows that

$$2^{p/2}\psi_1(\cdot - \pi/2) \lesssim \mathcal{M}_p(\mu).$$

Next, let χ_p be a bump function at scale $2^{-p/2}$ satisfying

$$|\chi_p(\phi)| \lesssim \frac{2^{p/2}}{(1 + 2^{p/2}|\phi|)^{\ell}}$$

for some $\ell > 2$. Then by (3.30),

$$\begin{aligned} \chi_p * \mathcal{M}_p(\mu)(\theta) &\lesssim \mathcal{M}_p(\mu)(\theta) \int (1 + 2^{p/2}|\phi|)|\chi_p(\phi)| d\phi \\ &\lesssim \mathcal{M}_p(\mu)(\theta) \int \frac{2^{p/2}}{(1 + 2^{p/2}|\phi|)^{\ell-1}} d\phi \\ &\lesssim \mathcal{M}_p(\mu)(\theta). \end{aligned}$$

Since ψ_2, ψ_3 and ψ_4 are obtained by successive convolutions with such bump functions at scale $2^{-p/2}$, it follows that $2^{p/2}\psi_4(\cdot - \pi/2) \lesssim \mathcal{M}_p(\mu)$ and the proof of part 1 of Proposition 3.4.2 is complete.

Part 2 of Proposition 3.4.2 is considerably easier to prove. If g has frequencies supported in $\mathcal{C}_0$, one can replicate the argument with $p = 1$. If g has frequencies supported in $\mathcal{C}_\infty$, one can argue in the following way. Let Φ be a Schwartz function on $\mathbb{R}^2$ such that $\widehat{\Phi}$ is equal to 1 on the ball of radius 2 in $\mathbb{R}^2$. If we write $\Phi_{1/R} = R^2\Phi(R\cdot)$ then $|\widehat{gd\sigma}(R\cdot)|^2 = |\widehat{gd\sigma}|^2 * \Phi_{1/R}$ and so

$$\int_{\mathbb{S}^1} |\widehat{gd\sigma}(R\omega)|^2 d\mu(\omega) = \int_{\mathbb{R}^2} |\widehat{gd\sigma}(Rx)|^2 \Phi_{1/R} * \mu(x)dx$$

where $*$ now denotes convolution on $\mathbb{R}^2$. Using polar coordinates, and the rapid decay of Φ, it suffices to assume that μ is essentially constant at scale $1/R$. In which case,

$$\begin{aligned}\int_{\mathbb{S}^1} |\widehat{gd\sigma}(Rx)|^2 d\mu(x) &\lesssim \|\mu\|_\infty \|\widehat{gd\sigma}(R\cdot)\|_2^2 \\ &\lesssim \|\mu\|_\infty \sup_{|j|\geq 3R/2} |J_j(R)|^2 \|g\|_2^2 \\ &\lesssim \frac{1}{R}\int_{\mathbb{S}^1} |g(\omega)|^2 \mathcal{M}_0(\mu)(\omega)d\sigma(\omega),\end{aligned}$$

using the fact that $\mathcal{M}_0(\mu)$ is constant and of the order of $\|\mu\|_\infty$, which completes the proof of Proposition 3.4.2. We are now in a position to deduce Theorem 3.4.1 from Proposition 3.4.2 using Littlewood-Paley theory.

Let A_p^+, A_p^-, B_p^+, B_p^-, C_0 and C_∞ be appropriate smoothed out Littlewood-Paley convolution operators associated to the intervals $\mathcal{A}_p^+$, $\mathcal{A}_p^-$, $\mathcal{B}_p^+$, $\mathcal{B}_p^-$, $\mathcal{C}_0$ and $\mathcal{C}_\infty$ respectively, so that

$$\begin{aligned} g &= \sum_p A_p^+ g \sum_p A_p^- g + \sum_p B_p^+ g \sum_p B_p^- g + C_0 g + C_\infty g \\ &= g_{A^+} + g_{A^-} + g_{B^+} + g_{B^-} + g_0 + g_\infty. \end{aligned}$$

Now $\widehat{g_A^+ d\sigma} = \sum_p \widehat{A^+ g d\sigma}$, and since for any fixed R there are $\sim \log R$ intervals $\mathcal{A}_p$, we have

$$|\widehat{g_{A^+}d\sigma}|^2 \lesssim \log R \sum_p \left|\widehat{A^+ g d\sigma}\right|^2.$$

As a consequence of this fact and Proposition 3.4.2,

$$\int_{\mathbb{S}^1} |\widehat{g_{A^+}d\sigma}(Rx)|^2 d\mu(x) \lesssim \frac{\log R}{R} \int_{\mathbb{S}^1} \sum_p |A^+ g(\omega)|^2 \mathcal{M}_p(\mu)(\omega) d\sigma(\omega),$$

and so by Lemma 1.1.2,

$$\int_{\mathbb{S}^1} |\widehat{g_{A^+}d\sigma}(Rx)|^2 d\mu(x) \lesssim \frac{\log R}{R} \int_{\mathbb{S}^1} |g(\omega)|^2 M\mathcal{M}^R(\mu)(\omega) d\sigma(\omega).$$

The terms $g_{A^-}, \ldots, g_\infty$ may all be treated in the same way, thus proving the first claimed inequality of Theorem 3.4.1. As for the second inequality, the fact that the map $g \mapsto \widehat{gd\sigma}(R\cdot)$ coincides with convolution on $\mathbb{S}^1$ allows one to apply classical Littlewood-Paley theory (see [32] and [33]) and Proposition 3.4.2 to obtain

$$\begin{aligned} \int_{\mathbb{S}^1} |\widehat{g_{A^+}d\sigma}(Rx)|^2 d\mu(x) &\lesssim \int_{\mathbb{S}^1} \sum_p |\widehat{A_p^+ g d\sigma}(Rx)|^2 M^2(\mu) d\sigma(x) \\ &\lesssim \frac{1}{R} \int_{\mathbb{S}^1} \sum_p |A_p^+ g(\omega)|^2 \mathcal{M}_p M^2(\mu)(\omega) d\sigma(\omega), \end{aligned}$$

to which one applies Lemma 1.1.2 which yields

$$\int_{\mathbb{S}^1} |\widehat{g_{A^+}d\sigma}(Rx)|^2 d\mu(x) \lesssim \frac{1}{R} \int_{\mathbb{S}^1} |g(\omega)|^2 M\mathcal{M}^R M^2(\mu)(\omega).$$

Again, a similar treatment of $g_{A^-}, \ldots, g_\infty$ completes the proof.

As a corollary of Theorem 3.4.1 one is able to deduce the following inequality:

Corollary 3.4.5. *For all $g \in L^3(\mathbb{S}^1)$,*

$$\|\widehat{gd\sigma}(R\cdot)\|_{L^3(\mathbb{S}^1)} \lesssim R^{-1/3}\|g\|_{L^3(\mathbb{S}^1)}. \tag{3.31}$$

Note that inequality (3.31) can be viewed as a consequence of a result of Greenleaf and Seeger in [21]. Using a standard duality argument (see the proof of Corollary 4.3.2 in the following chapter), to prove Corollary 3.4.5 it is sufficient to show that

$$\|\mathcal{M}^R(\psi d\sigma)\|_{L^3(\mathbb{S}^1)} \lesssim R^{1/3}\|\psi\|_{L^3(\mathbb{S}^1)}$$

for all $g \in L^3(\mathbb{S}^1)$ and $R \geq 1$. This is achieved as follows: it is convenient here to work with functions on $\mathbb{R}$ here rather than on $\mathbb{S}^1$. let Φ be a compactly supported bump function at scale 1, $\Phi_y = y^{-1}\Phi(y^{-1}\cdot)$, and $\beta \geq 0$. For $j \in \mathbb{N}$ with $1 \leq 2^j \leq R^{1/3}$, and for $\psi \in L^3(\mathbb{S}^1)$ define

$$\psi^*_{\beta,j}(t) = \sup_{(x,y)\in\Gamma^j_R(t)} y^\beta|\Phi_y * \psi(x)|$$

where

$$\Gamma^j_R(t) = \{(x,y) \in \mathbb{R}\times\mathbb{R}^+ : 0 < y < 2^{-j}R^{-1/3}, |x-t|^2 y < 2^j R^{-1}\}.$$

Then

$$\mathcal{M}^R(\psi d\sigma)(\theta) \lesssim \sup_j R^{1/2}2^{-j/2}\left(\psi^*_{\frac{1}{2},j}(\theta+\pi/2) + \psi^*_{\frac{1}{2},j}(\theta-\pi/2)\right)$$

and so it suffices to show that

$$\|\psi^*_{\frac{1}{2},j}\|_3 \lesssim 2^{j/6}R^{-1/6}\|\psi\|_3. \tag{3.32}$$

By Stein's method of analytic interpolation (see [28]), inequality (3.32) can be obtained from the estimates

$$\|\psi^*_{\frac{3}{2},j}\|_{L^1} \lesssim 2^{j/2} R^{-1/2} \|\psi\|_{H^1} \tag{3.33}$$

where H^1 denotes the Hardy space, and

$$\|\psi^*_{0,j}\|_\infty \lesssim \|\psi\|_\infty.$$

The second of these follows directly from the definition of $\psi^*_{0,j}$, and the first can be proved by testing on H^1-atoms. For an H^1-atom a with corresponding support interval I, one can use the pointwise bound

$$|\Phi_y * a(x)| \lesssim \begin{cases} 1/|I|, & \text{if} \quad y \lesssim |I| \quad \text{and} \quad |x| \lesssim |I| \\ |I|/y^2, & \text{if} \quad y \gtrsim |I| \quad \text{and} \quad |x| \lesssim y \\ 0, & \text{otherwise.} \end{cases}$$

to show that

$$\|a^*_{\frac{3}{2},j}\|_{L^1} \lesssim 2^{j/2} R^{-1/2}$$

from which the estimate (3.33) follows.

Chapter 4

Weighted L^2 Estimates for a Family of Oscillatory Convolution Kernels on $\mathbb{R}$

4.1 Bessel Potentials

In order to provide a more transparent example of how frequency decompositions can be used to prove two-weighted L^2 inequalities, we consider the following example, which can be considered a simpler version of Theorem 4.2.2. Define an operator T_s, for $0 < s < 1$, by

$$\widehat{T_s f}(\xi) = (1 + |\xi|^2)^{-s/2} \hat{f}(\xi).$$

The operators T_s are known as the Bessel potentials. While this operator may appear different to the operators we will encounter in the next section, the multipliers exhibit a similar type of decay in both cases for which the dyadic frequency decomposition is particularly appropriate.

We will use a frequency decomposition based on the following collection of sets, which form an approximate partition of $\mathbb{R}$:

$$\begin{aligned}\mathcal{A}_p &= [2^p, 2^{p+1}] \quad \text{for integers } p \geq 0 \\ \mathcal{A}_0 &= [-1, 1].\end{aligned}$$

This is a natural frequency decomposition to use since the multiplier $(1 + |\xi|^2)^{-s/2}$ is effectively "constant" (to the order of 2^{-ps}) on the sets $\mathcal{A}_p$. The aim is to find maximal functions $\mathcal{M}_{p,s}$ such that

$$\int_{\mathbb{R}} |T_s f(x)|^2 d\mu(x) \lesssim \int_{\mathbb{R}} |f(x)|^2 \mathcal{M}_{p,s}(\mu) dx$$

if $\text{supp}(\hat{f}) \subset \mathcal{A}_p$, for p a non-negative integer, and then use Littlewood-Paley theory to derive a weighted L^2 inequality for f with unrestricted Fourier support.

Suppose first that we have $\text{supp}(\hat{f}) \subset \mathcal{A}_0$, and let ϕ_0 be a smooth function such that $\hat{\phi}_0(\xi) = 1$ when $|\xi| \leq 1$ and $\hat{\phi}_0 = 0$ when $|\xi| \geq 2$. Then $f = f * \phi_0$ and so, with μ a Borel measure on $\mathbb{R}$ we have

$$\begin{aligned}&\int_{\mathbb{R}} |T_s f(x)|^2 d\mu(x) \\ &= \int_{\mathbb{R}} |T_s(\phi_0 * f)(x)|^2 d\mu(x) \\ &= \int_{\mathbb{R}} |T_s(\phi_0) * f(x)|^2 d\mu(x) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} f(y)\overline{f(z)} T_s(\phi_0)(x-y) \overline{T_s(\phi_0)(x-z)} d\mu(x) dy dz.\end{aligned}$$

Since $|f(y)\overline{f(z)}| \leq \frac{1}{2}(|f(y)|^2 + |f(z)|^2)$, it suffices by symmetry to bound

$$\begin{aligned}&\int_{\mathbb{R}} \int_{\mathbb{R}} |f(y)|^2 \int_{\mathbb{R}} |T_s(\phi_0)(x-y)||T_s(\phi_0)(x-z)| d\mu(x) dy dz \\ &= \int_{\mathbb{R}} |f(y)|^2 \left(\int_{\mathbb{R}} |T_s(\phi_0)(x-y)| d\mu(x) \right) \left(\int_{\mathbb{R}} |T_s(\phi_0)(z)| dz \right) dy.\end{aligned}$$

We now wish to estimate $T_s(\phi_0)$. Integrating by parts yields, for any $N \in \mathbb{N}$,

$$\begin{aligned} T_s(\phi_0)(x) &= \int_{\mathbb{R}} e^{2\pi i x\xi}\hat{\phi}_0(\xi)(1+|\xi|^2)^{-s/2}d\xi \\ &= (-1)^N(2\pi i x)^{-N}\int_{\mathbb{R}} e^{2\pi i x\xi}\frac{d^N}{d\xi^N}[\hat{\phi}_0(\xi)(1+|\xi|^2)^{-s/2}]d\xi, \end{aligned}$$

and so

$$\begin{aligned} |T_s\phi_0(x)| &\lesssim |x|^{-N}\int_{-2}^{2}|\frac{d^N}{d\xi^N}[\hat{\phi}_0(\xi)(1+|\xi|^2)^{-s/2}]|d\xi \\ &\leq C_N|x|^{-N} \end{aligned}$$

since

$$|\frac{d^N}{d\xi^N}[\hat{\phi}_0(\xi)(1+|\xi|^2)^{-s/2}]| \lesssim C_N$$

for all ξ. Since $T_s(\phi_0)(x)$ is clearly a bounded function, we have that $|T_s(\phi_0)(x)| \lesssim H_0(x)$ where $H_0(x) \lesssim (1+|x|)^{-N}$ for all $N \in \mathbb{N}$.

We have thus proved that

$$\int_{\mathbb{R}} |T_s f(x)|^2 d\mu(x) \lesssim \int_{\mathbb{R}} |f(x)|^2 H_0 * \mu(x)dx$$

when $\text{supp}(\hat{f}) \subset [-1,1]$. We now seek to prove similar weighted L^2 inequalities for functions f with $\text{supp}(\hat{f}) \subset \mathcal{A}_p$ for $p > 0$. To do this, we proceed as before. Let ϕ_p denote a smooth function with $\hat{\phi}_p(\xi) = 1$ when $\xi \in \mathcal{A}_p$ and $\hat{\phi}_p(\xi) = 0$ when ξ lies outside of a slightly larger interval containing $\mathcal{A}_p$, so that $f = \phi_p * f$. For uniformity purposes, we take ϕ_p to be a dilation of a smooth function ϕ such that $\hat{\phi}(\xi) = 1$ if $1/2 \leq \xi \leq 2$, and $\hat{\phi}(\xi) = 0$ if ξ lies outside of the interval $[1/4, 2]$. In particular,

$|\hat{\phi}_p^{(k)}(x)| \lesssim 2^{-kp}$. We then have

$$\begin{aligned} & \int_{\mathbb{R}} |T_s f(x)|^2 d\mu(x) \\ = & \int_{\mathbb{R}} |T_s(\phi_p) * f(x)|^2 d\mu(x) \\ = & \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} f(y)\overline{f(z)} T_s(\phi_p)(x-y)\overline{T_s(\phi_p)(x-z)} d\mu(x) dy dz, \end{aligned}$$

and so, as before, it suffices to bound

$$\int_{\mathbb{R}} |f(y)|^2 \left(\int_{\mathbb{R}} |T_s(\phi_p)(x-y)| d\mu(x) \right) \left(\int_{\mathbb{R}} |T_s(\phi_p)(z)| dz \right) dy.$$

Since $\widehat{T_s(\phi_p)}(\xi) = (1+|\xi|^2)^{-s/2}\hat{\phi}_p(\xi) \sim 2^{-ps}\hat{\phi}_p(\xi)$, one might expect $T_s(\phi_p)$ to look like $2^{-ps}\phi_p$. This is, in fact, true in the following sense: we may choose ϕ_p such that for any $N \in \mathbb{N}$,

$$|\phi_p(x)| \lesssim \frac{2^p}{(1+2^p|x|)^N},$$

and for such a ϕ_p we will show that

$$|T_s(\phi_p)(x)| \lesssim 2^{-ps} H_p(x)$$

where H_p satisfies

$$H_p(x) \lesssim \frac{2^p}{(1+2^p|x|)^N}$$

for any $N \in \mathbb{N}$. We integrate by parts N times to obtain

$$\begin{aligned} |T_s(\phi_p)(x)| &= \left|\int_{\mathbb{R}} e^{ix\xi}(1+|\xi|^2)^{-s/2}\hat{\phi}_p(\xi)d\xi\right| \\ &\leq |x|^{-N}\int_{\mathbb{R}} |\frac{d^N}{d\xi^N}[(1+|\xi|^2)^{-s/2}\hat{\phi}_p(\xi)]|d\xi. \end{aligned}$$

Now for ξ in the support of $\hat{\phi}_p$,

$$|\frac{d^N}{d\xi^N}[(1+|\xi|^2)^{-s/2}\hat{\phi}_p(\xi)]| \lesssim 2^{-p(N+s)},$$

and so

$$|T_s(\phi_p)(x)| \lesssim |x|^{-N} \cdot 2^p \cdot 2^{-p(N+s)} = 2^{-ps}\frac{2^p}{(2^p|x|)^N}.$$

Again, since $|T_s(\phi_p)| \lesssim 2^p$ we may conclude that there is a function H_p satisfying

$$H_p(x) \lesssim \frac{2^p}{(1+2^p|x|)^N}$$

for any natural number N such that

$$|T_s(\phi_p)(x)| \lesssim 2^{-ps}H_p(x),$$

as claimed. We have therefore proven that if $\text{supp}\hat{f} \subset \mathcal{A}_p$,

$$\int_{\mathbb{R}} |T_s(f)(x)|^2 d\mu(x) \lesssim \int_{\mathbb{R}} |f(x)|^2 2^{-2ps} H_p * \mu(x)dx.$$

Notice how the behaviour of the multiplier $(1+|\xi|^2)^{-s/2}$ on $\mathcal{A}_p$ has been "encoded" in the weight in the form of a factor of 2^{-2ps}. We also showed that if supp$\hat{f} \subset \mathcal{A}_0$,

$$\int_{\mathbb{R}} |T_s(f)(x)|^2 d\mu(x) \lesssim \int_{\mathbb{R}} |f(x)|^2 H_0 * \mu(x) dx.$$

Our overall aim is to find a maximal function $\mathcal{M}$ such that $\int |T_s(f)|^2 d\mu \lesssim \int |f|^2 \mathcal{M}(\mu)$ for all f, and so if we can find $\mathcal{M}$ such that $\mathcal{M}(\mu)$ dominates $H_0 * \mu$ and $2^{-2ps} H_p * \mu$ pointwise for all $p \in \mathbb{N}$ this would clearly be a strong candidate for our requirements. We have several choices here: firstly, by using the change of variable $r = 2^{-p}$ we could take $\mathcal{M}$ to be defined by

$$\mathcal{M}_s(\mu)(x) = \sup_{0<r<1} r^{2s} |\Phi_r * \mu(x)|$$

where Φ_r is an appropriate kernel of an approximation to the identity at scale r, eg. a standard bump function or the Poisson kernel, for example. We could also take $\mathcal{M}$ to be the fractional maximal function M_{2s}, defined by

$$M_{2s}(\phi)(x) = \sup_{r>0} \frac{1}{2r^{1-2s}} \int_{-r}^{r} |\phi(x-y)| dy.$$

We may now move beyond the case where $\hat{f}$ is supported in one of the $\mathcal{A}_p$'s. Let $\{A_p\}_{p\in\mathbb{N}}$ denote an enumeration of the smoothed-out Littlewood-Paley convolution operators associated to the intervals $\mathcal{A}_0$, $\mathcal{A}_p$, and $-\mathcal{A}_p$ for $p \in \mathbb{N}$. Lemma 1.1.3 may be applied here to obtain

$$\begin{aligned}\int_{\mathbb{R}} |T_s f|^2 d\mu &= \int_{\mathbb{R}} |\sum_p A_p T_s f|^2 d\mu \\ &\lesssim \sum_p \int_{\mathbb{R}} |A_p T_s f|^2 M^3(\mu) \\ &= \sum_p \int_{\mathbb{R}} |T_s A_p f|^2 M^3(\mu) \\ &\lesssim \sum_p \int_{\mathbb{R}} |A_p f|^2 \mathcal{M} M^3(\mu)\end{aligned}$$

whichever our choice of $\mathcal{M}$. By Lemma 1.1.2,

$$\int_{\mathbb{R}} |T_s f|^2 d\mu \lesssim \int_{\mathbb{R}} |f|^2 M \mathcal{M} M^3(\mu).$$

In particular we have that T_s is controlled (in the weighted L^2 sense) by the fractional maximal function M_{2s}. This is also the case for the fractional integral operators. The fractional integral operator of order α on $\mathbb{R}^n$ is given by

$$I_\alpha(f) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy$$

for $0 < \alpha < n$, which can also be realised as a multiplier operator with multiplier equal to $C_\alpha |\cdot|^{-\alpha}$. It is shown by Pérez in [25] that the operators I_α satisfy the weighted norm estimate

$$\int_{\mathbb{R}^n} |I_\alpha(f)(x)|^p w(x) dx \lesssim \int_{\mathbb{R}^n} |f(x)|^p M_{p\alpha} M^{[p]+1}(w)(x) dx \qquad (4.1)$$

where $[p]$ denotes the integer part of p for $1 < p < \infty$. Furthermore this is sharp in the sense that (4.1) fails if $[p]+1$ is replaced by $[p]$.

4.2 A Family of Oscillating Kernels on $\mathbb{R}$

Define a new kernel on $\mathbb{R}$ by

$$K_t(x) = e^{i\Phi(x)/t}\psi(x)$$

for small t, where the functions Φ and ψ are as-yet unspecified.

It is desirable for the phase function Φ to locally resemble the model cases $(\cdot)^\ell$ for integers $\ell \geq 3$, and so we suppose that Φ is a C^∞ function satisfying

$$\Phi^{(k)}(x_0) = 0 \quad \text{for} \quad 0 \leq k \leq \ell - 1, \quad \text{and} \quad \Phi^{(\ell)}(x_0) > \epsilon > 0, \tag{4.2}$$

for some ϵ.

As one might expect from the main result in [5], trigonometric phases such as $\Phi(x) = x - \sin x$ satisfy the above conditions as do the aforementioned model cases $\Phi(x) = x^\ell$ for integers $\ell \geq 3$.

Let $0 \leq k \leq \ell - 1$, then by Taylor's theorem, for each fixed x we have

$$\begin{aligned}
\Phi^{(k)}(x) &= \Phi^{(k)}(0) + x\Phi^{(k+1)}(0) + \cdots + x^{\ell-k}\Phi^{(\ell)}(y_{x,k}) \\
&= x^{\ell-k}\Phi^{(\ell)}(y_{x,k})
\end{aligned}$$

for some $y_{x,k} \in (0, x)$. As a result we have functions Φ_k for $0 \leq k \leq \ell - 1$ such that

$$\Phi^{(k)}(x) = x^{\ell-k}\Phi_k(x).$$

Since $|\Phi^{(\ell)}|$ is bounded below in a neighborhood of the origin, so is each of the Φ_k, and so we choose the smooth cutoff function ψ so that $|\Phi^{(\ell)}|$ is bounded below on $\operatorname{supp}\psi$.

For uniformity purposes, we wish to make the conditions on Φ and ψ more quantitative. In addition to (4.2), Let $\{A_j\}$ be a collection of positive constants for integers $j \geq 0$, and suppose that

$$\|\Phi^{(j)}\|_\infty \leq A_j.$$

By the Mean Value Theorem, there exists a neighbourhood V of x_0, depending only on ϵ and $A_{\ell+1}$ such that $\Phi^\ell(x) \geq \epsilon/2$ for $x \in V$. Finally, let ψ be a smooth function with support in V such that $\int |\psi'| \leq B$ for some positive constant B.

Notice that if ϕ is a local diffeomorphism on $\mathbb{R}$ with $\phi(y_0) = x_0$ for some y_0, then the new phase function $\Phi \circ \phi$ satisfies the hypotheses (4.2) at the point y_0 with a different value of ϵ. Due to this diffeomorphism invariance, we may suppose that $x_0 = 0$.

As in the case of the Bessel potentials, we will proceed using a frequency decomposition motivated by the following estimates for $\widehat{K}_t$.

Proposition 4.2.1.

$$|\widehat{K}_t(\xi)| \lesssim \begin{cases} t^{1/\ell}, & |\xi| \leq t^{-1/\ell} \\ t^{\frac{1}{2(\ell-1)}}|\xi|^{-\frac{\ell-2}{2(\ell-1)}}, & t^{-1/\ell} \leq |\xi| \leq 2A_1 t^{-1} \\ |\xi|^{-N} \quad \textit{for any } N \in \mathbb{N}, & |\xi| \geq 2A_1 t^{-1}, \end{cases}$$

with implicit constant depending on ℓ, the A_j, B and ϵ.

Proof. By corollary 2.1.3, the first two claimed estimates follow from corresponding estimates on the integral

$$\int_I e^{i(\Phi(x)/t - x\xi)}dx$$

that are uniform in I, where I is an interval contained in supp ψ. If we write the phase of the above integral as $p(x)/t$ where $p(x) = \Phi(x) - tx\xi$, then $p^{(\ell)}(x)$ is bounded below on I, and so by van der Corput's Lemma,

$$\left| \int_I e^{i(\Phi(x)/t - x\xi)} dx \right| \lesssim t^{1/\ell}.$$

For the second estimate, let $I_1 = \{x \in I : |x| \lesssim |t\xi|^{\frac{1}{\ell-1}}\}$, with suitably small implicit constant, and $I_2 = \{x \in I : |x| \gtrsim |t\xi|^{\frac{1}{\ell-1}}\}$. If we write the phase of the integral in question as $\xi p_1(x)$, where $p_1(x) = \Phi(x)/t\xi - x$, then $|p_1'(x)| \gtrsim 1$ for $x \in I_1$, and so

$$\left| \int_{I_1} e^{i(\Phi(x)/t - x\xi)} dx \right| \lesssim |\xi|^{-1}$$

by van der Corput's Lemma. If one writes the phase as $t^{-\frac{1}{\ell-1}} \xi^{\frac{\ell-2}{\ell-1}} p_2(x)$ where $p_2(x) = \Phi(x)(t\xi)^{-\frac{\ell-2}{\ell-1}} - x\xi^{\frac{1}{\ell-1}} t^{\frac{1}{\ell-1}}$ then $|p_2''(x)| \gtrsim 1$ for $x \in I_2$. This gives an estimate of

$$\left| \int_{I_2} e^{i(\Phi(x)/t - x\xi)} dx \right| \lesssim t^{\frac{1}{2(\ell-1)}} |\xi|^{-\frac{\ell-2}{2(\ell-1)}}.$$

Overall, the estimate becomes

$$\left| \int_I e^{i(\Phi(x)/t - x\xi)} dx \right| \lesssim \max\{|\xi|^{-1}, t^{\frac{1}{2(\ell-1)}} |\xi|^{-\frac{\ell-2}{2(\ell-1)}}\},$$

but for $t^{-1/\ell} \leq |\xi|$ we have that $|\xi|^{-1} \leq t^{\frac{1}{2(\ell-1)}} |\xi|^{-\frac{\ell-2}{2(\ell-1)}}$, and so the second estimate is complete.

For the third estimate, suppose that $|\xi| \geq 2A_1 t^{-1}$ and write $\widehat{K}_t(\xi) = \int e^{i\xi h(x)} \psi(x) dx$, where $h(x) = \Phi(x)/t\xi - x$. Then for all x in the support of the integrand, $|h'(x)| \geq 1/2$ and $|h^{(j)}(x)| \leq A_j$ for integers $j \geq 2$. Proceed in the spirit of Lemma 2.1.1 and define a differential operator D by

$$Df(x) = \frac{1}{ih(x)} f'(y),$$

then integrating by parts N times yields

$$|\widehat{K}_t(\xi)| \lesssim |\xi|^{-N} \int |({}^tD)^N \psi(x)| dx$$

where

$${}^tDf(x) = \frac{d}{dx}\left(\frac{f(x)}{h'(x)}\right).$$

By our assumptions on ψ, and our estimates on the derivatives of h,

$$\int |({}^tD)^N \psi(x)| dx \le C_N$$

with C_N depending on the A_j's for each $N \in \mathbb{N}$, and so $|\widehat{K}_t(\xi)| \le C_N |\xi|^{-N}$, as claimed. □

This motivates a frequency decomposition using the following collection of sets:

$$\begin{aligned}
\mathcal{A}_0 &= \{\xi \in \mathbb{R} : |\xi| \lesssim t^{-1/\ell}\} \\
\mathcal{A}_p &= \{\xi \in \mathbb{R} : \xi \sim 2^{-p}/t\} \quad \text{for } p \text{ such that } 1 \lesssim 2^p \lesssim t^{-(\ell-1)/\ell} \\
\mathcal{A}_\infty &= \{\xi \in \mathbb{R} : |\xi| \gtrsim t^{-1}\},
\end{aligned}$$

along with the sets $-\mathcal{A}_p$, with implicit constants depending only on A_1, which will be used to prove the following:

Theorem 4.2.2. *For all Borel measures μ there exist constants C_1 and C_2 depending* *on ℓ, the A_j, B and ϵ such that*

1.

$$\int_{\mathbb{R}} |K_t * f(x)|^2 d\mu(x) \leq C_1 t^{\frac{1}{\ell-1}} \int_{\mathbb{R}} |f(x)|^2 M^2 \mathcal{M}_{t,\ell} M^3(\mu)(x) dx \tag{4.3}$$

2.

$$\int_{\mathbb{R}} |K_t * f(x)|^2 d\mu(x) \leq C_2 t^{\frac{1}{\ell-1}} \log(t^{-1}) \int_{\mathbb{R}} |f(x)|^2 M^2 \mathcal{M}_{t,\ell}(\mu)(x) dx \tag{4.4}$$

where M^k denotes the k-fold composition of the Hardy-Littlewood maximal function M, and $\mathcal{M}_{t,\ell}$ is given by

$$\mathcal{M}_{t,\ell}(\phi)(x) = \sup_{(y,r)\in\Gamma_{t,\ell}(x)} r^{\frac{\ell-2}{\ell-1}} |P_r * \phi(y)|$$

where P_r is the kernel of a suitable approximation to the identity at scale r, and $\Gamma_{t,\ell}(x)$ is the region

$$\{(y,r) : 0 < r \leq t^{1/\ell}, \quad \text{and} \quad |y - x| \leq t^{\frac{1}{\ell-1}} r^{-\frac{1}{\ell-1}}\}.$$

Proof. We will suppose first that supp$\hat{f} \subset \mathcal{A}_0$. Although the following argument is less technical than when $\hat{f}$ is supported in $\mathcal{A}_p$, it gives us a clear philosophical and theoretical framework for that case.

Let ϕ_0 and P_0 be functions on $\mathbb{R}$ such that $\hat{\phi}_0(x), \widehat{P}_0(x) = 1$ if $|x| \lesssim t^{-1/\ell}$ and satisfying the estimates

$$|\phi^{(k)}(x)| \lesssim \frac{(t^{-1/\ell})^{k+1}}{(1 + t^{-1/\ell}|x|)^N},$$

and

$$|P_0(x)| \lesssim \frac{t^{-1/\ell}}{(1+t^{-1/\ell}|x|)^M}$$

for every $N, M \in \mathbb{N}$. Then

$$\begin{aligned}
&\int_{\mathbb{R}} |K_t * f(x)|^2 d\mu(x) \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \left[\int_{\mathbb{R}} e^{2\pi i (y-z)x} d\mu(x) \right] \widehat{K_t}(y) \overline{\widehat{K_t}(z)} \hat{f}(y) \overline{\hat{f}(z)} dy dz \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{\mu}(z-y) \widehat{K_t}(y) \overline{\widehat{K_t}(z)} \hat{f}(y) \overline{\hat{f}(z)} dy dz \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{P}_0(z-y) \hat{\mu}(z-y) \widehat{K_t}(y) \overline{\widehat{K_t}(z)} \hat{f}(y) \overline{\hat{f}(z)} dy dz \\
&= \int_{\mathbb{R}} |K_t * f(x)|^2 P_0 * \mu(x) dx \\
&= \int_{\mathbb{R}} |(K_t * \phi_0) * f(x)|^2 P_0 * \mu(x) dx \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} f(y) \overline{f(z)} K_t * \phi_0(x-y) \overline{K_t * \phi_0(x-z)} P_0 * \mu(x) dx dy dz.
\end{aligned}$$

Since $|f(y)\overline{f(z)}| \leq \frac{1}{2}(|f(y)|^2 + |f(z)|^2)$, it suffices by symmetry to bound

$$\begin{aligned}
&\int_{\mathbb{R}} \int_{\mathbb{R}} |f(y)|^2 \int_{\mathbb{R}} |K_t * \phi_0(x-y)| |K_t * \phi_0(x-z)| |P_0 * \mu(x)| dx dy dz \\
&= \int_{\mathbb{R}} |f(y)|^2 \left(\int_{\mathbb{R}} |K_t * \phi_0(x-y)| |P_0 * \mu(x)| dx \right) \left(\int_{\mathbb{R}} |K_t * \phi_0(z)| dz \right) dy.
\end{aligned}$$

In the following lemma we observe that bump functions at an appropriate scale are left looking similar when convolved with K_t. Identifying this action of K_t is a crucial stage in the proof, and allows to move from working with oscillatory integrals to working with bump functions.

Lemma 4.2.3.

$$|K_t * \phi_0(x)| \lesssim t^{1/\ell} H_0(x),$$

where H_0 *satisfies*

$$H_0(x) \lesssim \frac{t^{-1/\ell}}{(1 + t^{-1/\ell}|x|)^N}$$

for integers $0 < N < (\ell - 1)^2$.

Proof. Let $\Psi(x) = \int_{\mathbb{R}} e^{i\Phi(y)/t}\psi(y)\phi_0(x-y)dy$, and let $\{\nu_n\}_{n\in\mathbb{Z}}$ be a smooth partition of unity on $\mathbb{R}$ with $\text{supp}\nu_n \subset \{|x| \sim 2^n\}$. It is important to note here that, for uniformity purposes, $\{\nu_n\}$ and any other partitions of unity that are used in this proof are constructed in the standard way from a fixed smooth function and taking differences. Define

$$\eta_j = \begin{cases} \nu_j(t^{-1/\ell}\cdot) & \text{if} \quad j > 0 \\ \sum_{n\leq 0} \nu_n(t^{-1/\ell}\cdot) & \text{if} \quad j = 0. \end{cases}$$

Now $\{\eta_j\}_{j\geq 0}$ defines a partition of unity on $\mathbb{R}$. Write

$$I_j(x) = \int_{\mathbb{R}} e^{i\Phi(y)/t}\psi(y)\phi_0(x-y)\eta_j(x-y)dy$$

so that

$$|\Psi(x)| \leq \sum_{2^j t^{1/\ell} \gtrsim |x|} |I_j(x)| + \sum_{2^j t^{1/\ell} \lesssim |x|} |I_j(x)|. \tag{4.5}$$

To deal with the first sum in inequality (4.5) above, it will be sufficient to prove the uniform estimate

$$|I_j(x)| \lesssim 2^{-jN}$$

for any $N \in \mathbb{N}$. Integrating by parts, we may deduce that

$$\begin{aligned} I_j(x) &= \int_{\mathbb{R}} \frac{d}{dy}\left(\int_{-1}^{y} e^{i\Phi(z)/t}\psi(z)dz\right)\phi_0(x-y)\eta_j(x-y)dy \\ &= -\int_{\mathbb{R}}\left(\int_{-1}^{y} e^{i\Phi(z)/t}\psi(z)dz\right)\frac{d}{dy}(\phi_0(x-y)\eta_j(x-y))dy. \end{aligned}$$

Now as indicated previously,

$$\left|\int_{-1}^{y} e^{i\Phi(z)/t}\psi(z)dz\right| \lesssim t^{1/\ell}$$

uniformly in y, and so, for each $N \in \mathbb{N}$,

$$\begin{aligned} |I_j(x)| &\lesssim t^{1/\ell} \cdot t^{1/\ell} 2^j \frac{(t^{-1/\ell})^2}{(1 + t^{-1/\ell} \cdot t^{1/\ell} 2^j)^N} \\ &\leq 2^{-j(N-1)}, \end{aligned}$$

as required.

Now suppose that $ct^{1/\ell}2^j \leq |x|$, for some constant c. If we take c to be suitably large, then $|y| \sim |x|$ for all y in the support of $\eta_j(x - \cdot)$.

If we define a differential operator D (in the spirit of Lemma 2.1.1) by

$$Df(y) = \frac{t}{i\Phi'(y)} f'(y),$$

then

$$|I_j(x)| \lesssim t^N \int_{\mathbb{R}} |({}^tD)^N(\psi(y)\phi_0(x-y)\eta_j(x-y))|dy,$$

where

$${}^tDf(y) = -\frac{d}{dy}\left(\frac{f(y)}{\Phi'(y)}\right).$$

For $0 \leq N \leq \ell - 1$ we may write $({}^tD)^N f(y)$ as a sum of terms of the form

$$f^{(p)}(y)(\Phi'(y))^{-n}\Pi_i(\Phi^{(q_i)}(y))^{m_i}$$

where $p + \sum_i m_i q_i = n$, $n - \sum_i m_i = N$, and $0 \leq p \leq N$. By our observations on the derivatives of Φ,

$$|f^{(p)}(y)(\Phi'(y))^{-n}\Pi_i(\Phi^{(q_i)}(y))^{m_i}| \sim |f^{(p)}(y)||y|^{\sum_i m_i(\ell-q_i)}|y^{\ell-1}|^{-n} = |f^{(p)}(y)||y|^{-(\ell N-p)}.$$

It follows that $|({}^tD)^N f(y)|$ is controlled by a sum of terms of the form $|y|^{-k}|f^{(\ell N-k)}(y)|$ for $(\ell-1)N \leq k \leq \ell N$, the number of which depends only on N.

In order to bound $|I_j(x)|$, it therefore suffices to control

$$t^N \int_{\mathbb{R}} |y|^{-k}|(\psi(y)\phi_0(x-y)\eta_j(x-y))^{(\ell N-k)}|dy$$

where $(\ell-1)N \leq k \leq \ell N$. Since $|y| \sim |x|$ for all y in the support of $\eta_j(x-\cdot)$, for each $k, N \in \mathbb{N}$ we have

$$\begin{aligned} t^N \int_{\mathbb{R}} |y|^{-k} |(\psi(y)\phi_0(x-y)\eta_j(x-y))^{(\ell N-k)}| dy &\lesssim t^N |x|^{-k} t^{1/\ell} 2^j \cdot \frac{(t^{-1/\ell})^{\ell N-k+1}}{(1+t^{-1/\ell}\cdot t^{1/\ell}2^j)^N} \\ &\leq \frac{2^{-j(N-1)}}{(t^{-1/\ell}|x|)^k} \\ &= 2^{-j(N-1)} t^{1/\ell} \frac{t^{-1/\ell}}{(t^{-1/\ell}|x|)^k}, \end{aligned}$$

from which the desired bound for $|K_t * \phi_0|$ follows.

□

The previous lemma yields, for $\mathrm{supp}(\hat{f}) \subset \mathcal{A}_0$, the inequality

$$\begin{aligned} \int_{\mathbb{R}} |K_t * f(x)|^2 d\mu(x) &\lesssim t^{2/\ell} \int_{\mathbb{R}} |f(x)|^2 H_0 * |P_0 * \mu|(x) dx \\ &= t^{\frac{1}{\ell-1}} \int |f(x)|^2 H_0 * |t^{\frac{\ell-2}{\ell(\ell-1)}} P_0 * \mu(x)| dx. \end{aligned}$$

Since P_0 is a kernel of an approximation to the identity at scale $t^{1/\ell}$, $|t^{\frac{\ell-2}{\ell(\ell-1)}} P_0 * \mu| \lesssim \mathcal{M}_{t,\ell}\mu(x)$, and so $H_0 * |t^{\frac{\ell-2}{\ell(\ell-1)}} P_0 * \mu(x)| \lesssim M\mathcal{M}_{t,\ell}\mu(x)$. As a result,

$$\int_{\mathbb{R}} |K_t * f(x)|^2 d\mu(x) \lesssim t^{\frac{1}{\ell-1}} \int_{\mathbb{R}} |f(x)|^2 M\mathcal{M}_{t,\ell}\mu(x) dx \tag{4.6}$$

whenever $\mathrm{supp}(\hat{f}) \subset \mathcal{A}_0$.

We now consider the case where f has Fourier frequencies supported in $\mathcal{A}_p$, and proceed as before. Now,

$$\begin{aligned}
&\int_{\mathbb{R}} |K_t * f(x)|^2 d\mu(x) \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \left[\int_{\mathbb{R}} e^{2\pi i (y-z)x} d\mu(x) \right] \widehat{K_t}(y) \overline{\widehat{K_t}(z)} \hat{f}(y) \overline{\hat{f}(z)} dy dz \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{\mu}(z-y) \widehat{K_t}(y) \overline{\widehat{K_t}(z)} \hat{f}(y) \overline{\hat{f}(z)} dy dz \qquad (4.7) \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \widehat{P}_p(z-y) \hat{\mu}(z-y) \widehat{K_t}(y) \overline{\widehat{K_t}(z)} \hat{f}(y) \overline{\hat{f}(z)} dy dz \\
&= \int_{\mathbb{R}} |K_t * f(x)|^2 P_p * \mu(x) dx,
\end{aligned}$$

for any function P_p satisfying $\widehat{P}_p(x) = 1$ whenever $|x| \lesssim 2^{-p}/t$. We may choose P_p to satisfy

$$|P_p(x)| \lesssim \frac{2^{-p} t^{-1}}{(1 + 2^{-p} t^{-1} |x|)^N}$$

for every $N \in \mathbb{N}$. The proof in this case now diverges from the proof in the previous case for the following reason: before, we were able to identify the action of K_t on a bump function at a reciprocal scale to the support of $\hat{f}$. However, a bump function at scale $2^p t$ with Fourier support in $\{|x| \sim 2^{-p}/t\}$ is not smooth enough for this action to be satisfactorily identified, and so it is necessary to carry out a further 'equally spaced' frequency decomposition as follows: let W_L be a function on $\mathbb{R}$ with $\operatorname{supp} \widehat{W_L} \subset \{x \in \mathbb{R} : |x| \leq 2L\}$ such that

$$\sum_{k \in \mathbb{Z}} \widehat{W}_L(x + kL) = 1$$

for all $x \in \mathbb{R}$. The value of L is to be determined later, and will depend on p. We may also specify that W_L satisfies

$$|W_L(x)| \lesssim \frac{L}{(1+L|x|)^N}$$

for all $x \in \mathbb{R}$ and for any $N \in \mathbb{N}$. Let

$$f_k(x) = \int_{\mathbb{R}} f(y) W_L(x-y) e^{-2\pi i(2^{-p}/t+kL)y} dy,$$

for integers k with $0 \leq k \leq (2^p tL)^{-1}$ so that

$$f(x) = e^{2\pi i 2^{-p} t^{-1} x} \sum_{k=0}^{(2^p tL)^{-1}} e^{2\pi i k L x} f_k(x).$$

Let ϕ_p be a function on $\mathbb{R}$ satisfying

$$\widehat{\phi}_p(\xi) = \begin{cases} 1 & \text{if} \quad |\xi| \leq 2L \\ 0 & \text{if} \quad |\xi| \geq 4L \end{cases}$$

so that $f_k = \phi_p * f_k$ for each k, since $\mathrm{supp}\hat{f}_k \subset \{|x| \leq 2L\}$. We may also choose ϕ_p such that for each $N \in \mathbb{N}$,

$$|\phi_p^{(c)}(x)| \lesssim \frac{L^{c+1}}{(1+L|x|)^N}$$

for every $x \in \mathbb{R}$. Then $K_t * f(x)$ may be written as

$$\begin{aligned} &\sum_{k=0}^{(2^p tL)^{-1}} \int_{\mathbb{R}} e^{i\Phi(x-y)/t + 2\pi i(2^{-p}/t+kL)y} \psi(x-y) \int_{\mathbb{R}} f_k(z) \phi_p(y-z) dz dy \\ &= e^{2\pi i 2^{-p} t^{-1} x} \sum_{k=0}^{(2^p tL)^{-1}} e^{2\pi i k L x} f_k * \Psi_{p,k}(x), \end{aligned} \tag{4.8}$$

where

$$\Psi_{p,k}(x) = \int_{\mathbb{R}} e^{i\Phi(w)/t - 2\pi i(2^{-p}/t + kL)w} \psi(w)\phi_p(x-w)dw.$$

For our purposes it will be necessary to fix $L = 2^{p/(\ell-1)}$. Referring back to (4.7), we have replaced the arbitrary Borel measure μ with the weight $P_p * \mu$. However, it is necessary to dominate $P_p * \mu$ by a function with an increased level of smoothness in order to recover some partial orthogonality from the frequency decomposition we have just carried out. For our new weight to have Fourier support in $\{x : |x| \lesssim L\}$ is desirable, and so we construct such a weight as follows: firstly, we bound $P_p * \mu$ pointwise by

$$\psi_1(x) = \sup_{|y-x| \leq L^{-1}} |P_p * \mu(y)|,$$

and secondly let $\psi_2(x) = \Theta_p * \psi_1(x)$ where Θ_p is a non-negative function on $\mathbb{R}$ such that $\widehat{\Theta}_p$ is non-negative and supported in $\{\xi \in \mathbb{R} : |\xi| \lesssim L\}$. We may also choose Θ_p such that

1. for each $N \in \mathbb{N}$,

$$\Theta_p(x) \lesssim \frac{L}{(1+L|x|)^N}, \quad \text{and}$$

2. There exist constants $C, c > 0$ independent of p such that $\Theta_p(x) \geq CL$ whenever $|x| \leq cL^{-1}$.

Using the argument from Lemma 3.4.3 allows us to argue that $\psi_2 \geq C\psi_1$ and so it suffices to control $\int_{\mathbb{R}} |K_t * f(x)|^2 \psi_2(x)dx$. Now from (4.8) we may write

$$\begin{aligned}
&\int_{\mathbb{R}} |K_t * f(x)|^2 \psi_2(x)dx \\
&= \int_{\mathbb{R}} \sum_{k,k'} e^{-2\pi i L(k-k')x} f_k * \Psi_{p,k}(x) \overline{f_{k'} * \Psi_{p,k'}(x)} \psi_2(x)dx \\
&= \sum_{k,k'} \int_{\mathbb{R}} \int_{\mathbb{R}} f_k(y)\overline{f_{k'}(z)} \\
&\times \left(\int_{\mathbb{R}} \Psi_{p,k}(x-y)\overline{\Psi_{p,k'}(x-z)}\psi_2(x)e^{-2\pi i(k-k')Lx}dx \right) dydz.
\end{aligned}$$

Now $\widehat{\Psi}_{p,k}(\xi) = \widehat{\Psi}_{p,k'}(\xi) = \hat{\psi}_2(\xi) = 0$ when $|\xi| \geq 4 \cdot L$ and so

$$\int_{\mathbb{R}} \Psi_{p,k}(x-y)\overline{\Psi_{p,k'}(x-z)}\psi_2(x)e^{-2\pi i(k-k')Lx}dx = 0$$

whenever $|k-k'| > 12$. To see this, we write the above integral as

$$\begin{aligned}
&\int\int\int\int e^{2\pi i[(x-y)\xi-(x-z)\eta-x\nu-(k-k')Lx]}\widehat{\Psi}_{p,k}(\xi)\overline{\widehat{\Psi}}_{p,k'}(\eta)\widehat{\psi}_2(\nu)dxd\xi d\nu d\eta \\
&= \int\int\int e^{2\pi i[-y\xi+z\eta]}\widehat{\Psi}_{p,k}(\xi)\overline{\widehat{\Psi}}_{p,k'}(\eta)\widehat{\psi}_2(\nu) \int e^{2\pi ix[\xi-\eta+\nu-(k-k')L]}dxd\xi d\eta d\nu \\
&= \int\int\int e^{2\pi i[-y\xi+z\eta]}\widehat{\Psi}_{p,k}(\xi)\overline{\widehat{\Psi}}_{p,k'}(\eta)\widehat{\psi}_2(\nu)\delta(\xi-\eta+\nu-(k-k')L)d\xi d\nu d\eta.
\end{aligned}$$

Since $\widehat{\Psi}_{p,k}(\xi) = \widehat{\Psi}_{p,k'}(\xi) = \hat{\psi}_2(\xi) = 0$ when $|\xi| \geq 4 \cdot L$, the integrand above is zero when $|k-k'| > 12$.

Since $|f_k(y)\overline{f_{k'}(z)}| \leq \frac{1}{2}(|f_k(y)|^2 + |f_{k'}(z)|^2)$, by symmetry it suffices to control

$$\begin{aligned}
&\sum_{|k-k'|\leq 12} \int_{\mathbb{R}} |f_k(y)|^2 \left(\int_{\mathbb{R}} \int_{\mathbb{R}} |\Psi_{p,k}(x-y)||\Psi_{p,k'}(x-z)|\psi_2(x)dxdz \right) dy \\
&= \sum_{|k-k'|\leq 12} \int_{\mathbb{R}} |f_k(y)|^2 \left(\int_{\mathbb{R}} |\Psi_{p,k}(x-y)|\psi_2(x)dx \right) \left(\int_{\mathbb{R}} |\Psi_{p,k'}(z)|dz \right) dy.
\end{aligned}$$

Our next lemma identifies the action of K_t on bump functions at scale L^{-1}.

Lemma 4.2.4. *With our choice of L,*

$$|\Psi_{p,k}(x)| \lesssim t^{1/2} 2^{\frac{p(\ell-2)}{2(\ell-1)}} H_p(x)$$

where H_p satisfies

$$H_p(x) \lesssim \frac{2^{\frac{p}{\ell-1}}}{(1+2^{\frac{p}{\ell-1}}|x|)^N}$$

for $0 \leq N \leq (\ell-1)^2$.

Proof. Let $\{\eta_n\}$ be a partition of unity on $\mathbb{R}$ with $\operatorname{supp} \eta_n \subset \{x \in \mathbb{R} : |x| \sim 2^n\}$. Define

$$\eta_{p,j} = \begin{cases} \eta_j(2^{\frac{p}{(\ell-1)}} \cdot) & \text{if} \quad j > 0 \\ \sum_{n \leq 0} \eta_n(2^{\frac{p}{(\ell-1)}} \cdot) & \text{if} \quad j = 0. \end{cases}$$

Then $\operatorname{supp} \eta_{p,0} \subset \{|x| \lesssim 2^{\frac{-p}{(\ell-1)}}\}$, $\operatorname{supp} \eta_{p,j} \subset \{|x| \sim 2^{\frac{-p}{(\ell-1)}+j}\}$, and $\{\eta_{p,j}\}_{j\geq 0}$ is a partition of unity on $\mathbb{R}$ for every p. Recall that

$$\Psi_{p,k}(x) = \int_{\mathbb{R}} e^{i\Phi(y)/t - 2\pi i(2^{-p}/t + kL)y} \psi(y) \phi_p(x-y)dy.$$

With this in mind, we write $c_k = 2\pi(2^{-p}/t + kL)$ and let

$$I_{p,k,j}(x) = \int_{\mathbb{R}} e^{i(\Phi(y)/t - c_k y)} \psi(y) \phi_p(x-y) \eta_{p,j}(x-y)dy.$$

Now the phase of the integral in question has a stationary point when $\Phi'(y) = c_k t$, which occurs for $y \sim (c_k t)^{1/(\ell-1)} =: y_k$.

Since we expect the main contribution of this integral to occur around stationary points of the phase, it makes sense to decompose the integral relative to $|x - y_k|$.

Note that $y_k \sim 2^{-\frac{p}{\ell-1}}$. We write

$$|\Psi_{p,k}(x)| \leq \sum_{2^j \gtrsim 2^{\frac{p}{(\ell-1)}}|x-y_k|} |I_{p,k,j}(x)| + \sum_{2^j \lesssim 2^{\frac{p}{(\ell-1)}}|x-y_k|} |I_{p,k,j}(x)|$$

and consider each of the sums separately. Since $|y_k| \leq 2^{-p/(\ell-1)}$, it suffices to show that

$$|\Psi_{p,k}(x)| \lesssim t^{1/2} 2^{\frac{p(\ell-2)}{2(\ell-1)}} \cdot \frac{2^{\frac{p}{\ell-1}}}{(1 + 2^{\frac{p}{\ell-1}}|x - y_k|)^N}$$

for every $N \in \mathbb{N}$, and $0 \leq k \leq (2^p tL)^{-1}$. Fix x and suppose that $2^j \gtrsim 2^{\frac{p}{(\ell-1)}}|x - y_k|$, then integrating by parts we have

$$\begin{aligned} I_{p,k,j}(x) &= \int_{\mathbb{R}} \frac{d}{dy} \left(\int_{-1}^{y} e^{i(\Phi(z)/t - c_k z)} \psi(z) dz \right) \phi_p(x - y) \eta_{p,j}(x - y) dy \\ &= -\int_{\mathbb{R}} \left(\int_{-1}^{y} e^{i(\Phi(z)/t - c_k z)} \psi(z) dz \right) \frac{d}{dy} (\phi_p(x - y) \eta_{p,j}(x - y)) dy. \end{aligned}$$

By Proposition 4.2.1, we have the estimate

$$\left| \int_{-1}^{y} e^{i(\Phi(z)/t - c_k z)} \psi(z) dz \right| \lesssim t^{\frac{1}{2(\ell-1)}} |2^{-p}/t|^{-\frac{\ell-2}{2(\ell-1)}} = t^{1/2} 2^{\frac{p(\ell-2)}{2(\ell-1)}},$$

uniformly in y, and so

$$\begin{aligned} |I_{p,k,j}(x)| &\lesssim t^{1/2} 2^{\frac{p(\ell-2)}{2(\ell-1)}} \cdot 2^{\frac{-p}{\ell-1}+j} \cdot (2^{\frac{p}{\ell-1}})^2 \cdot 2^{-Nj} \\ &= t^{1/2} 2^{\frac{p(\ell-2)}{2(\ell-1)}} \cdot 2^{\frac{p}{\ell-1}} \cdot 2^{-(N-1)j} \end{aligned}$$

for any $N \in \mathbb{N}$, which is sufficient.

We now suppose that $2^j \lesssim 2^{\frac{p}{\ell-1}}|x - y_k|$. Then

$$|I_{p,k,j}(x)| \leq t^N \int_{\mathbb{R}} |({}^tD)^N \psi(y)\phi_p(x-y)\eta_{p,j}(x-y)|dy \tag{4.9}$$

for any $N \in \mathbb{N}$ where

$$({}^tD)g(y) = -\frac{d}{dy}\left(\frac{g(y)}{\Phi'(y) - y_k^{\ell-1}}\right),$$

with the operator ${}^tD$ corresponding to a differential operator D in the spirit of Lemma 2.1.1. For $0 \leq N \leq \ell - 1$ we may write $({}^tD)^N g(y)$ as a sum of terms of the form

$$g^{(p)}(y)(\Phi'(y) - y_k^{\ell-1})^{-n}\Pi_i(\Phi^{(q_i)}(y))^{m_i}$$

where $p + \sum_i m_i q_i = n$, $n - \sum_i m_i = N$, and $0 \leq p \leq N$. By our observations on the derivatives of Φ,

$$|g^{(p)}(y)(\Phi'(y))^{-n}\Pi_i(\Phi^{(q_i)}(y))^{m_i}| \sim |g^{(p)}(y)||y|^{\sum_i m_i(\ell - q_i)}|y^{\ell-1}\Phi_1(y) - y_k^{\ell-1}|^{-n}.$$

By the above relations, $(\ell-1)n - \sum_i m_i(\ell - q_i) + p = \ell N$, and so we may bound $|({}^tD)g(y)|$ by a sum of terms of the form

$$|y|^\alpha |y^{\ell-1}\Phi_1(y) - y_k^{\ell-1}|^{-\beta}|\frac{d^\gamma}{dy^\gamma}g(y)|,$$

where $(\ell-1)\beta - \alpha + \gamma = \ell N$, and $0 \leq \gamma \leq N$.

Thus, $|I_{p,k,j}(x)|$ may be bounded by a sum of terms (the number of which depending only on N) of the form

$$t^N \int_{\mathbb{R}} |y|^\alpha |y^{\ell-1}\Phi_1(y) - y_k^{\ell-1}|^{-\beta} |\frac{d^\gamma}{dy^\gamma}(\psi(y)\phi_p(x-y)\eta_{p,j}(x-y)|dy.$$

Now if $2^j c \le 2^{\frac{p}{\ell-1}}|x-y_k|$ with c sufficiently large then, for all y in the range of integration we have $|x-y_k| \sim |y-y_k|$ and $|y| \lesssim |x-y_k|$. We also have that $|y\Phi_1(y)^{1/(\ell-1)} - y_k| \sim |y-y_k|$, and so

$$\begin{aligned}\frac{|y|^\alpha}{|y^{\ell-1}\Phi_1(y) - y_k^{\ell-1}|^\beta} &\lesssim \frac{|y|^\alpha}{|y\Phi_1(y)^{1/(\ell-1)} - y_k|^{(\ell-1)\beta}} \\ &\sim \frac{|y|^\alpha}{|y-y_k|^{(\ell-1)\beta}} \\ &\sim |x-y_k|^{\alpha-(\ell-1)\beta}\end{aligned}$$

and so we may bound (4.9) by a constant multiple of

$$\begin{aligned}&t^N|x-y_k|^{\alpha-(\ell-1)\beta}(2^{\frac{-p}{\ell-1}+j})\frac{(2^{\frac{p}{\ell-1}})^{\gamma+1}}{(1+2^{\frac{p}{\ell-1}}\cdot 2^{\frac{-p}{\ell-1}+j})^N} \\ &\le t^N|x-y_k|^{\alpha-(\ell-1)\beta}(2^{\frac{p}{\ell-1}})^\gamma 2^{-j(N-1)} \\ &= 2^{-j(N-1)}\frac{t^N(2^{\frac{p}{\ell-1}})^{\ell N}}{(2^{\frac{p}{\ell-1}}|x-y_k|)^{\ell N-\gamma}} \\ &= 2^{-j(N-1)}\cdot(t2^{\frac{\ell p}{\ell-1}})^{N-1/2}\cdot t^{1/2}2^{\frac{p(\ell-2)}{2(\ell-1)}}\frac{2^{\frac{p}{\ell-1}}}{(2^{\frac{p}{\ell-1}}|x-y_k|)^{\ell N-\gamma}}\end{aligned}$$

from which the desired bound follows, since $t2^{\frac{\ell p}{\ell-1}} \lesssim 1$.

□

If we let $\psi_3 = H_p * \psi_2$, then we have that

$$\int_{\mathbb{R}} |K_t * f(x)|^2 d\mu(x) \lesssim \int_{\mathbb{R}} \sum_k |f_k(x)|^2[t2^{\frac{p(\ell-2)}{\ell-1}}\psi_3(x)]dx.$$

Now on applying Lemma 1.1.1, our weighted estimate for convolution with K_t becomes

$$\int_{\mathbb{R}} |K_t * f(x)|^2 d\mu(x) \lesssim \int_{\mathbb{R}} |f(x)|^2 [t2^{\frac{p(\ell-2)}{\ell-1}} \psi_4(x)] dx, \tag{4.10}$$

where $\psi_4 = |W_L| * \psi_3$.

Claim 4.2.5.

$$t2^{\frac{p(\ell-2)}{\ell-1}} \psi_4(x) \lesssim t^{\frac{1}{\ell-1}} M\mathcal{M}_{t,\ell}(\mu)(x).$$

Proof. Writing ψ_4 out in full we have that

$$t2^{\frac{p(\ell-2)}{\ell-1}} \psi_4(x) = |W_L| * H_p * \Theta_p * t2^{\frac{p(\ell-2)}{\ell-1}} \psi_1(x)$$

where

$$t2^{\frac{p(\ell-2)}{\ell-1}} \psi_1(x) = t2^{\frac{p(\ell-2)}{\ell-1}} \sup_{|y-x| \leq L^{-1}} |P_p * \mu(y)|.$$

Now if we write $r = 2^p t$, then bearing in mind that $L = 2^{\frac{p}{\ell-1}}$ the expression becomes

$$t2^{\frac{p(\ell-2)}{\ell-1}} \psi_1(x) = t^{\frac{1}{\ell-1}} \sup_{|y-x| \leq t^{\frac{1}{\ell-1}} r^{\frac{-1}{\ell-1}}} r^{\frac{\ell-2}{\ell-1}} |P_r * \mu(y)|$$

where P_r is a smooth bump function satisfying $|P_r(x)| \lesssim r^{-1}/(1 + r^{-1}|x|)^N$ for any $N \in \mathbb{N}$. By assumption, p is a non-negative integer with $1 \lesssim 2^p \lesssim t^{-\frac{\ell-1}{\ell}}$, and so

$t \lesssim r \lesssim t^{1/\ell}$. Taking the supremum over $r \in (0, t^{1/\ell}]$ yields that

$$\begin{aligned} t2^{\frac{p(\ell-2)}{\ell-1}}\psi_1(x) &\leq t^{\frac{1}{\ell-1}} \sup_{0<r\leq t^{1/\ell}} \sup_{|y-x|\leq t^{\frac{1}{\ell-1}}r^{-\frac{1}{\ell-1}}} r^{\frac{\ell-2}{\ell-1}}|P_r * \mu(y)| \\ &= t^{\frac{1}{\ell-1}} \sup_{(y,r)\in\Gamma_{t,\ell}(x)} r^{\frac{\ell-2}{\ell-1}}|P_r * \mu(y)| \\ &= t^{\frac{1}{\ell-1}}\mathcal{M}_{t,\ell}(\mu)(x), \end{aligned}$$

where $\Gamma_{t,\ell}(x)$ is defined to be the region

$$\{(y,r) : 0 < r \leq t^{1/\ell} \quad \text{and} \quad |y-x| \leq t^{\frac{1}{\ell-1}}r^{\frac{-1}{\ell-1}}\}.$$

Now since each of W_L, H_p and Θ_p is a bump function at the same scale, we have

$$t2^{\frac{p(\ell-2)}{\ell-1}}\psi_4(x) \lesssim M\mathcal{M}_{t,\ell}(\mu)(x),$$

as required. □

It then follows that if $\text{supp}\hat{f} \subset \mathcal{A}_p$, we have

$$\int_{\mathbb{R}} |K_t * f|^2 d\mu \lesssim t^{\frac{1}{\ell-1}} \int_{\mathbb{R}} |f|^2 M\mathcal{M}_{t,\ell}(\mu) \tag{4.11}$$

The final case to consider is when $\hat{f}$ is supported in $\mathcal{A}_\infty$. Consider a bump function Ψ such that $\widehat{\Psi}(\xi) = 1$ for $x \in [1,2]$, and $\widehat{\Psi}(\xi) = 0$ for x outside a slightly larger interval. For p such that $2^p \geq t^{-1}$, define a new function Ψ_p by $\widehat{\Psi}_p = \widehat{\Psi}(2^{-p}\cdot)$. Then $\widehat{\Psi}_p(\xi) = 1$ on $\{\xi \sim 2^p\}$, and $|\widehat{\Psi}_p^{(k)}(\xi)| \lesssim 2^{-pk}$. We can estimate the action of convolution with K_t on Ψ_p as follows. Integrating by parts N times (in the manner of Lemma 2.1.1) on the expression

$$K_t * \Psi_p(x) = \int e^{ix\xi}\widehat{\Psi}_p(\xi)\widehat{K}_t(\xi)d\xi$$

yields

$$|K_t * \Psi_p(x)| \lesssim |x|^{-N} \int |\frac{d^N}{d\xi^N}(\widehat{\Psi}_p(\xi)\widehat{K}_t(\xi))|d\xi.$$

Now one may show by using the integration by parts argument from the final part of the proof of Proposition 4.2.1 that whenever $|\xi| \gtrsim t^{-1}$ we have

$$|\frac{d^N}{d\xi^N}\widehat{K}_t(\xi)| \lesssim |\xi|^{-M}$$

for all natural numbers N and M, with implicit constant depending on N, M and the A_j. Using this and the above estimate on the derivatives of $\widehat{\Psi}_p$, along with the assumption that $\text{supp}\widehat{\Psi}_p \subset \{\xi \sim 2^p\}$, it follows that

$$|K_t * \Psi_p(x)| \lesssim 2^{-pM}|x|^{-N}$$

for all N, $M \in \mathbb{N}$. We may also use the rapid decay of $\widehat{K}_t$ to obtain the trivial estimate

$$|K_t * \Psi_p(x)| \leq \int |\widehat{\Psi}_p(\xi)\widehat{K}_t(\xi)|d\xi \lesssim 2^{-pM}$$

for all $M \in \mathbb{N}$. It therefore follows that

$$|K_t * \Psi_p(x)| \lesssim 2^{-pM}H_p(x), \tag{4.12}$$

for all $M \in \mathbb{N}$, with implicit constant depending on M and the A_j, where H_p is a non-negative bump function satisfying

$$H_p(x) \leq C_N \frac{2^p}{(1+2^p|x|)^N}$$

for all $N \in \mathbb{N}$.

Now let $\{\Psi_p\}_{p\in\mathbb{N}}$ be a smooth partition of unity on $\mathbb{R}$, constructed in the standard way, with $\operatorname{supp}\widehat{\Psi}_p \subset \{\xi : |\xi| \sim 2^p\}$. If $\hat{f}$ is supported in $\mathcal{A}_\infty$ then

$$f = \sum_{2^p \gtrsim t^{-1}} f * \Psi_p$$

and so

$$\int |K_t * f|^2 d\mu = \int |\sum_{2^p \gtrsim t^{-1}} (K_t * \Psi_p) * f|^2 d\mu.$$

By multiplying out the integrand and using Fubini's Theorem, $\int |K_t * f|^2 d\mu$ is controlled by

$$\sum_{2^p, 2^q \gtrsim t^{-1}} \int |f(y)|^2 \left(\int |K_t * \Psi_p(x-y)| d\mu(y) \right) \left(\int |K_t * \Psi_q(z)| dz \right) dy.$$

By estimate (4.12), for any $M \in \mathbb{N}$ this is dominated by a constant multiple of

$$\sum_{2^p \gtrsim t^{-1}} \int |f(y)|^2 2^{-pM} H_p * \mu(y) dy \lesssim \int |f(y)|^2 \left[\sup_{0<r<t} r^M H_r * \mu(y) \right] dy,$$

where H_r is the kernel of a suitable approximation to the identity at scale r. We may clearly choose M large enough that the bound

$$\sup_{0<r<t} r^M H_r * \mu \leq t^{\frac{1}{\ell-1}} \mathcal{M}_{t,\ell}(\mu)$$

holds uniformly in μ and t, and so for f with $\mathrm{supp}\hat{f} \subset \mathcal{A}_\infty$,

$$\int |K_t * f|^2 d\mu \lesssim t^{\frac{1}{\ell-1}} \int |f|^2 \mathcal{M}_{t,\ell}(\mu). \tag{4.13}$$

We may now suppose that f has unrestricted Fourier support, and apply Lemma 1.1.2. Let $\{A_k\}$ denote an enumeration of the smoothed-out Littlewood-Paley convolution operators associated with the intervals $\mathcal{A}_0$, $\mathcal{A}_p$ and $-\mathcal{A}_p$ for $p \in \mathbb{N}$ with $1 \lesssim 2^p \lesssim t^{-\frac{\ell-1}{\ell}}$, and $\mathcal{A}_\infty$. Then

$$f = \sum A_k f. \tag{4.14}$$

Since for any fixed t there are $\sim \log(t^{-1})$ intervals $\mathcal{A}_p$, we have that

$$|\sum K_t * A_k f|^2 \lesssim \log(t^{-1}) \sum_k |A_k(K_t * f)|^2.$$

Hence, we may apply inequalities (4.6), (4.11) and (4.13), and Lemma 1.1.2 to obtain

$$\begin{aligned}
\int |K_t * f|^2 d\mu &= \int |\sum_k K_t * A_k f|^2 d\mu \\
&\lesssim \log(t^{-1}) \sum_k \int |K_t * A_k f|^2 d\mu \\
&\lesssim t^{\frac{1}{\ell-1}} \log(t^{-1}) \int \sum_k |A_k f|^2 M \mathcal{M}_{t,\ell}(\mu) \\
&\lesssim t^{\frac{1}{\ell-1}} \log(t^{-1}) \int |f|^2 M^2 \mathcal{M}_{t,\ell}(\mu),
\end{aligned}$$

which establishes (4.4).

Alternatively, let $\{P_k\}_{k\geq 0}$ be functions such that supp $\widehat{P}_0 \subset \{|\xi| \lesssim t^{-1/\ell}\}$, and for integers k such that $2^k \gtrsim t^{-1/\ell}$, P_k is odd with supp $\widehat{P}_k \subset \{|\xi| \sim 2^k\}$. Then by our Littlewood-Paley Lemma 1.1.3, and inequalities (4.6), (4.11) and (4.13) we have

$$\begin{aligned}
\int_{\mathbb{R}} |K_t * f|^2 d\mu &\lesssim \int_{\mathbb{R}} \sum |P_k * (K_t * f)|^2 M^3(\mu) \\
&= \sum \int_{\mathbb{R}} |K_t * (P_k * f)|^2 M^3(\mu) \\
&\lesssim t^{\frac{1}{\ell-1}} \int_{\mathbb{R}} \sum |P_k * f|^2 M \mathcal{M}_{t,\ell} M^3(\mu).
\end{aligned}$$

By Lemma 1.1.2 we may therefore conclude that

$$\int_{\mathbb{R}} |K_t * f(x)|^2 d\mu(x) \lesssim t^{\frac{1}{\ell-1}} \int_{\mathbb{R}} |f(x)|^2 M^2 \mathcal{M}_{t,\ell} M^3(\mu) dx,$$

completing the proof.

□

4.3 Corollaries of Theorem 4.2.2

A one-weighted L^2 estimate for convolution with K_t may be deduced as a simple consequence of Theorem 4.2.2. However, it may also be proven directly without the need to resort to a second frequency decomposition, using little more than the decay estimates on $\widehat{K}_t$.

Corollary 4.3.1.

$$\int_{\mathbb{R}} |K_t * f(x)|^2 d\mu(x) \lesssim t^{\frac{1}{\ell-1}} \log(t^{-1}) \|\mathcal{M}_{t,\ell}(\mu)\|_\infty \int_{\mathbb{R}} |f(x)|^2 dx, \tag{4.15}$$

with implicit constant depending on ℓ, the A_j, B and ϵ.

Proof. Firstly, suppose that supp$\hat{f} \subset \mathcal{A}_0$. Then for a suitable bump function P_0 at scale $t^{1/\ell}$ we have

$$\begin{aligned}\int |K_t * f|^2 d\mu &= \int |K_t * f|^2 P_0 * \mu \\ &\lesssim \|P_0 * \mu\|_\infty \int |K_t * f|^2 \\ &= \|P_0 * \mu\|_\infty \int |\widehat{K}_t|^2 |\hat{f}|^2 \\ &\lesssim \sup_{\xi \in \mathcal{A}_0} |\widehat{K}_t(\xi)|^2 \|P_0 * \mu\|_\infty \int |\hat{f}|^2 \\ &\lesssim t^{2/\ell} \|P_0 * \mu\|_\infty \int |f|^2,\end{aligned}$$

and $|t^{2/\ell} P_0 * \mu| = |t^{1/(\ell-1)} (t^{1/\ell})^{\frac{\ell-2}{\ell-1}} P_0 * \mu| \lesssim t^{1/\ell-1} \mathcal{M}_{t,\ell}(\mu)$, which implies that

$$t^{2/\ell} \|P_0 * \mu\|_\infty \lesssim t^{\frac{1}{\ell-1}} \|\mathcal{M}_{t,\ell}(\mu)\|_\infty.$$

For f with $\hat{f}$ supported in $\mathcal{A}_p$ we argue in a similar way to obtain, for a suitable bump function P_p at scale $2^p t$,

$$\begin{aligned}\int |K_t * f|^2 d\mu &\lesssim [(2^{-p}/t)^{-\frac{\ell-2}{2(\ell-1)}} t^{\frac{1}{2(\ell-1)}}]^2 \|P_p * \mu\|_\infty \int |f|^2 \\ &= t(2^p)^{\frac{\ell-2}{\ell-1}} \|P_p * \mu\|_\infty \int |f|^2 \\ &\lesssim t^{\frac{1}{\ell-1}} \|\mathcal{M}_{t,\ell}(\mu)\|_\infty \int |f|^2,\end{aligned}$$

since $t(2^p)^{\frac{\ell-2}{\ell-1}} P_p * \mu = t^{1/(\ell-1)} (2^p t)^{\frac{\ell-2}{\ell-1}} P_p * \mu \lesssim t^{1/(\ell-1)} \mathcal{M}_{t,\ell}(\mu)$.

It follows from (4.13) that

$$\int |K_t * f|^2 d\mu \lesssim t^{\frac{1}{\ell-1}} \|\mathcal{M}_{t,\ell}(\mu)\|_\infty \int |f|^2$$

when supp$\hat{f} \subset \mathcal{A}_\infty$, and so the desired one-weighted inequality holds for functions

f that have Fourier support in a piece of our frequency decomposition. Again, we use the fact that there are $\sim \log(t^{-1})$ intervals $\mathcal{A}_p$ in our frequency decomposition, and Littlewood-Paley theory, to conclude that inequality (4.15) holds for f with unrestricted Fourier support. □

As indicated previously, a simple duality argument can be applied to weighted inequalities such as (4.3) which allows us to obtain L^p bounds on our operator via L^p bounds on our controlling maximal function $\mathcal{M}_{t,\ell}$. As one might hope, our maximal function $\mathcal{M}_{t,\ell}$ is sharp in the sense that it allows us to recover the L^p to L^p operator norm of convolution with K_t, for certain p. Like Corollary 3.4.5, the operator norm is already known as a consequence of [21].

Corollary 4.3.2. *The inequality*

$$\|K_t * f\|_p \lesssim t^{1/\ell}\|f\|_p, \tag{4.16}$$

with implicit constant depending on ℓ, the A_j, B and ϵ, holds for all $f \in L^p(\mathbb{R})$ if and only if $\ell' \leq p \leq \ell$.

Proof. To see that the claimed range of p is necessary, consider the following example. Fix a $t \in (0,1)$, and define a function f by $f(x) = e^{-i(-x)^\ell/t}\chi_{[-t,t]}(x)$. Consider the special case when $K_t(x) = e^{ix^\ell/t}\psi(x)$ for an integer $\ell \geq 3$, and a smooth cutoff ψ with support in $(-1,1)$. Then

$$\begin{aligned} |K_t * f(x)| &= \left|\int_{-t}^{t} e^{i(x-y)^\ell/t - i(-y)^\ell/t}\psi(x-y)dy\right| \\ &\gtrsim \left|\int_{-t}^{t} \cos\left(\frac{1}{t}\sum_{j=1}^{\ell-1}(-1)^j\binom{\ell}{j}y^j x^{\ell-j}\right)\psi(x-y)dy\right|. \end{aligned}$$

Let $T = \{x : |x| \lesssim 1\}$. We may take the implicit constant sufficiently small (depending only on ℓ) that $T \subset \operatorname{supp}(K_t * f)$, and for all $x \in T$ we have

$$\left| \frac{1}{t} \sum_{j=1}^{\ell-1} (-1)^j \binom{\ell}{j} y^j x^{\ell-j} \right| \lesssim 1/8,$$

and so

$$|K_t * f(x)| \gtrsim t\chi_T(x).$$

As a consequence, $\|K_t * f\|_p \gtrsim t$. Now $\|f\|_p = t^{1/p}$, and so if we assume inequality (4.16) to be true for some $1 \leq p < \infty$, we must have $t \lesssim t^{1/\ell+1/p}$. Since all implicit constants are independent of t, we must have $1/\ell + 1/p - 1 \leq 0$, which rearranges to $\ell' \leq p$. By duality we must also have $p \leq \ell$.

We now proceed using a standard duality argument. For $f \in L^\ell(\mathbb{R})$,

$$\begin{aligned}
\|K_t * f\|_\ell^2 &= \|(K_t * f)^2\|_{\frac{\ell}{2}} \\
&= \sup_{\|g\|_{(\frac{\ell}{2})'}=1} \left| \int |K_t * f|^2 g \right| \\
&\lesssim t^{\frac{1}{\ell-1}} \sup_{\|g\|_{(\frac{\ell}{2})'}=1} \left| \int |f|^2 M^2 \mathcal{M}_{t,\ell}(M^3 g) \right| \quad \text{by Theorem 4.2.2} \\
&\leq t^{\frac{1}{\ell-1}} \sup_{\|g\|_{(\frac{\ell}{2})'}=1} \|(f)^2\|_{\frac{\ell}{2}} \|M^2 \mathcal{M}_{t,\ell}(M^3 g)\|_{(\frac{\ell}{2})'} \quad \text{by Hölder's inequality} \\
&\leq t^{\frac{1}{\ell-1}} \|M^2 \mathcal{M}_{t,\ell} M^3\|_{(\frac{\ell}{2})' \to (\frac{\ell}{2})'} \|f\|_\ell^2.
\end{aligned}$$

In order to show that

$$\|K_t * f\|_\ell \lesssim t^{1/\ell} \|f\|_\ell$$

for all $f \in L^{\ell}(\mathbb{R})$, it therefore suffices to show that

$$\|\mathcal{M}_{t,\ell}\|_{(\frac{\ell}{2})' \to (\frac{\ell}{2})'} \lesssim t^{\frac{\ell-2}{\ell(\ell-1)}}, \tag{4.17}$$

since M is bounded on L^p for all $p > 1$. Write $\mathcal{M}_{1,\ell} = \mathcal{M}_\ell$, then by scaling in t, (4.17) is equivalent to the estimate

$$\|\mathcal{M}_\ell\|_{(\frac{\ell}{2})' \to (\frac{\ell}{2})'} \lesssim 1, \tag{4.18}$$

which is established as follows:

For integers $k \geq 0$, let $P_{k,r}$ be compactly supported bump functions at scale $2^k r$ respectively such that

$$P_r(x) \lesssim \sum_{k \geq 0} 2^{-kN} P_{k,r}(x)$$

for $N \in \mathbb{N}$, and define a new maximal function $\widetilde{\mathcal{M}}_{\ell,k}$ by

$$\widetilde{\mathcal{M}}_{\ell,k}(\phi)(x) = \sup_{(y,r) \in \Gamma_\ell(x)} r^{\frac{\ell-2}{\ell-1}} |P_{k,r} * \phi(y)|.$$

where $\Gamma_\ell(x)$ denotes $\Gamma_{1,\ell}(x)$. Since we have

$$\mathcal{M}_\ell(\phi)(x) \lesssim \sum_{k \geq 0} 2^{-kN} \widetilde{\mathcal{M}}_{\ell,k}(\phi)(x)$$

for functions ϕ, it will suffice to show that

$$\|\widetilde{\mathcal{M}}_{\ell,k}\|_{(\frac{\ell}{2})' \to (\frac{\ell}{2})'} \lesssim 1, \tag{4.19}$$

uniformly in k.

A simple scaling argument shows that

$$\widetilde{\mathcal{M}}_{\ell,k}(\phi)(2^k x) \lesssim 2^{-k}\widetilde{M}_{\ell,0}(\phi_{2^k})(x), \tag{4.20}$$

where ϕ_{2^k} denotes the dilation $2^k\phi(2^k\cdot)$. Using (4.20) and a change of variables, we have

$$\begin{aligned}\int |\widetilde{\mathcal{M}}_{\ell,k}(\phi)(x)|^{(\frac{\ell}{2})'}dx &= 2^k\int |\widetilde{\mathcal{M}}_{\ell,k}(\phi)(2^k x)|^{(\frac{\ell}{2})'}dx \\ &\lesssim (2^k)^{1-(\frac{\ell}{2})'}\int |\widetilde{\mathcal{M}}_{\ell,0}(\phi_{2^k})(x)|^{(\frac{\ell}{2})'}dx.\end{aligned}$$

As we will go on to show below,

$$\|\widetilde{\mathcal{M}}_{\ell,0}\|_{(\frac{\ell}{2})'\to(\frac{\ell}{2})'} \lesssim 1, \tag{4.21}$$

and so

$$\begin{aligned}\int |\widetilde{\mathcal{M}}_{\ell,k}(\phi)(x)|^{(\frac{\ell}{2})'}dx &\lesssim (2^k)^{1-(\frac{\ell}{2})'}\int |\phi_{2^k}(x)|^{(\frac{\ell}{2})'}dx \\ &= \int |\phi(x)|^{(\frac{\ell}{2})'}dx,\end{aligned}$$

which establishes (4.19).

All that remains now is to prove (4.21). Define

$$\mathcal{M}_\ell^\beta(\phi)(x) = \sup_{(y,r)\in\Gamma_\ell(x)} r^{\frac{\ell\beta}{\ell-1}}|P_r * \phi(y)|$$

where P_r is a compactly supported bump function at scale r. By Stein's method of analytic interpolation (again, see [28]), inequality (4.21) can be obtained from the estimates

$$\|\mathcal{M}_\ell^0(\phi)\|_\infty \lesssim \|\phi\|_\infty$$

and

$$\|\mathcal{M}_\ell^1(\phi)\|_{L^1} \lesssim \|\phi\|_{H^1}.$$

The first estimate is elementary, and the second may be verified by testing on atoms. Let a be an H^1-atom with support interval I (by translation invariance we may suppose that I is centered at the origin). For an atom a as described above, we have the pointwise bound

$$r^{\frac{\ell}{\ell-1}}|P_r * a(x)| \lesssim \begin{cases} r^{\frac{\ell}{\ell-1}}/|I|, & \text{if} \quad r \lesssim |I| \quad \text{and} \quad |x| \lesssim |I| \\ |I|/r^{2-\frac{\ell}{\ell-1}}, & \text{if} \quad r \gtrsim |I| \quad \text{and} \quad |x| \lesssim r \\ 0, & \text{otherwise.} \end{cases}$$

First, suppose that $|I| \geq 1$, so that our pointwise estimate becomes

$$r^{\frac{\ell}{\ell-1}}|P_r * a(x)| \lesssim \begin{cases} r^{\frac{\ell}{\ell-1}}/|I|, & \text{if} \quad |x| \lesssim |I| \\ 0, & \text{otherwise.} \end{cases} \tag{4.22}$$

If $|x| \leq 4|I|$, then $\mathcal{M}_\ell^1(a)(x) \lesssim 1/|I|$ (since $r^{\frac{\ell}{\ell-1}}|P_r * a(x)| \lesssim r^{\frac{\ell}{\ell-1}}/|I|$) which contributes ≈ 1 to $\|\mathcal{M}_\ell^1(a)\|_{L^1}$. On the other hand, suppose that $x \geq 4|I|$. Then the right hand side of (4.22) is maximised for $(y,r) \in \Gamma_\ell(x)$ when $r \approx (x-|I|)^{-(\ell-1)}$, and so

$$\mathcal{M}_\ell^1(a)(x) \approx |I|^{-1}(x-|I|)^{-\ell}.$$

This contributes

$$|I|^{-1}\int_{x\geq 4|I|}(x-|I|)^{-\ell}\approx |I|^{-\ell}\leq 1$$

to $\|\mathcal{M}_\ell^1\|_{L^1}$. The case when $x\leq -4|I|$ may be treated similarly, and so combining these estimates we have that $\|\mathcal{M}_\ell^1(a)\|_{L^1}\lesssim 1$ when $|I|\geq 1$.

Now suppose that $|I|<1$. For any $(y,r)\in\Gamma_\ell(x)$, $r^{\frac{\ell}{\ell-1}}|P_r*a(y)|\lesssim |I|^{\frac{1}{\ell-1}}$ so for $|x|\leq 4|I|^{-\frac{1}{\ell-1}}$ we have the estimate $\mathcal{M}_\ell^1(a)(x)\lesssim |I|^{\frac{1}{\ell-1}}$ which contributes ≈ 1 to $\|\mathcal{M}_\ell^1(a)\|_{L^1}$. If $x\geq 4|I|^{-\frac{1}{\ell-1}}$, then again we have

$$\mathcal{M}_\ell^1(a)(x)\lesssim |I|^{-1}(x-|I|)^{-\ell}$$

which contributes

$$|I|^{-1}\int_{x\geq 4|I|^{-\frac{1}{\ell-1}}}(x-|I|)^{-\ell}\approx |I|^{-1}(|I|^{-\frac{1}{\ell-1}})^{-(\ell-1)}=1$$

to $\|\mathcal{M}_\ell^1(a)\|_{L^1}$, and so we may conclude that

$$\|\mathcal{M}_\ell^1\|_{H^1\to L^1}\lesssim 1$$

as required.

This is establishes that $\|K_t*f\|_\ell\lesssim t^{1/\ell}\|f\|_\ell$, which allows one to deduce using duality that $\|K_t*f\|_{\ell'}\lesssim t^{1/\ell}\|f\|_{\ell'}$ for all $f\in L^{\ell'}(\mathbb{R})$. One may now interpolate to show that $\|K_t*f\|_p\lesssim t^{1/\ell}\|f\|_p$, for all $f\in L^p(\mathbb{R})$, whenever $\ell'\leq p\leq \ell$.

$\square$

Chapter 5

Higher Dimensions

An obvious extension of the results in chapter 5 to consider is that of analogues of Theorem 4.2.2 in higher dimensions. For example, define a convolution kernel by

$$K_\lambda(x) = e^{\lambda i \Phi(x)}\psi(x),$$

where Φ is a suitable smooth function on $\mathbb{R}^n$, $\lambda \in [1, \infty)$, and ψ is an appropriate cut-off supported in a neighborhood of the origin. It is natural to address the matter of determining the functions Φ for which we find a maximal function $\mathcal{M}_\Phi$ or a correspondence $w \mapsto C_w$ such that

$$\int_{\mathbb{R}^n} |f * K_\lambda|^2 w \lesssim \int_{\mathbb{R}^n} |f|^2 \mathcal{M}_\Phi w,$$

or

$$\int_{\mathbb{R}^n} |f * K_\lambda|^2 w \lesssim C_w \int_{\mathbb{R}^n} |f|^2$$

for all weights w on $\mathbb{R}^n$. As before, the maximal function $\mathcal{M}_\Phi$ should be suitably geometric in nature. We present here some partial answers to the above problems.

5.1 Corollaries of Theorem 4.2.2 Continued

Theorem 4.2.2 may be used to prove two-weighted estimates for certain convolution kernels on $\mathbb{R}^n$. For example, for $1 \le j \le n$ let Φ_j and ψ_j be functions on $\mathbb{R}$ satisfying the hypotheses of Theorem 4.2.2, and define

$$K_j(x) = e^{i\Phi_j(x)/t}\psi_j(x)$$

for $x \in \mathbb{R}$. We may then define a kernel K on $\mathbb{R}^n$ by

$$K(x) = \prod_{1 \le j \le n} K_j(x_j)$$

for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

Corollary 5.1.1. *Let $\widetilde{\mathcal{M}}_{t,\ell,j}$ denote $M^2\mathcal{M}_{t,\ell}$ acting in the jth variable. Then with K as defined above,*

$$\int_{\mathbb{R}^n} |K * f|^2 \mu \lesssim (t^{\frac{1}{\ell-1}} \log(t^{-1}))^n \int_{\mathbb{R}^n} |f|^2 \widetilde{\mathcal{M}}_{t,\ell,n} \widetilde{\mathcal{M}}_{t,\ell,n-1} \cdots \widetilde{\mathcal{M}}_{t,\ell,1}(\mu). \tag{5.1}$$

Proof. For functions $f : \mathbb{R}^n \to \mathbb{C}$ and $g : \mathbb{R} \to \mathbb{C}$, let $g *_j f$ denote convolution in the j variable of g with f. More precisely, for $x = (x_1, \ldots, x_n)$,

$$g *_j f(x) = \int_{\mathbb{R}} f(x_1, \ldots, x_j - y, \ldots, x_n) g(y) dy.$$

Then as a consequence of Theorem 4.2.2,

$$\int_{\mathbb{R}^n} |K_j *_j f(x)|^2 d\mu(x) \lesssim t^{\frac{1}{\ell-1}} \log(t^{-1}) \int_{\mathbb{R}^n} |f(x)|^2 \widetilde{\mathcal{M}}_{t,\ell,j}(\mu)(x) dx. \tag{5.2}$$

Observe that

$$K * f(x) = K_n *_n K_{n-1} *_{n-1} \cdots *_2 K_1 *_1 f(x),$$

and so by repeated iterations of (5.2) we obtain (5.1).

□

Recall the uniformity considerations on the phase and associated cutoff of the kernel that preceded Theorem 4.2.2. As a consequence of these, a two-weighted estimate on $\mathbb{R}^2$ can be obtained if the phase function $\Phi : \mathbb{R}^2 \to \mathbb{R}$ has a suitably 'weak' dependance on one of the variables. This notion of 'weakness' is made clear below.

Suppose that $\Phi : \mathbb{R}^2 \to \mathbb{R}$ satisfies the hypotheses of Theorem 4.2.2 in the first variable, uniformly in the second variable. More precisely, suppose that

$$\partial_1^k \Phi(0, \cdot) = 0$$

for $1 \leq k \leq \ell - 1$, and there exists $\epsilon > 0$ such that

$$\partial_1^\ell \Phi(0, x_2) > \epsilon$$

for all x_2. Let $\psi : \mathbb{R}^2 \to \mathbb{R}$ be a suitable smooth cutoff around zero with compact support in $[-1, 1]^2$, and define a kernel $K(x_1, x_2) = e^{i\Phi(x_1,x_2)/t}\psi(x_1, x_2)$. If g is a function from $\mathbb{R}^2$ to $\mathbb{C}$ then we will sometimes write $g(x, y) = g_y(x)$ for notational convenience.

Notice that a function such as $\Phi(x_1, x_2) = \Phi_1(x_1)\Phi_2(x_2)$ where Φ_1 satisfies the hypotheses from Theorem 4.2.2 and Φ_2 is bounded below gives an example of a phase satisfying the above conditions.

In what follows we let $\star$ denote convolution on $\mathbb{R}^2$ and $*$ denote convolution on $\mathbb{R}$ to avoid ambiguity. We denote by $\|\cdot\|_{L_2^p(L_1^q)}$ the mixed norm corresponding to an L^q norm in the first variable followed by L^p in the second, i.e.

$$\|f\|_{L_2^p(L_1^q)} = \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x,y)|^q dx\right)^{p/q} dy\right)^{1/p}.$$

Corollary 5.1.2. *Let w be a weight function on $\mathbb{R}^2$, then with K as defined above,*

$$\int_{\mathbb{R}^2} |K \star f(x_1,x_2)|^2 w(x_1,x_2)dx_1dx_2 \lesssim t^{\frac{1}{\ell-1}} \log(t^{-1}) \int_{\mathbb{R}^2} |f(x_1,x_2)|^2 \mathfrak{M}(w)(x_1,x_2)dx_1dx_2,$$

where

$$\begin{aligned} \mathfrak{M}(w)(x_1,x_2) &= \int_{\mathbb{R}} \widetilde{\mathcal{M}}_{t,\ell,1}(w)(x_1,y)\chi(y-x_2)dy \\ &= \chi *_2 \widetilde{\mathcal{M}}_{t,\ell,1}(w)(x_1,x_2) \end{aligned}$$

for any non-negative compactly-supported bump function χ on $\mathbb{R}$ at scale 1 with $\chi(x) = 1$ when $x \in [-1,1]$. If $2 \le p < \infty$ then we have the mixed norm estimate

$$\|K \star f\|_{L_2^p(L_1^\ell)} \lesssim t^{1/\ell} \|f\|_{L_2^p(L_1^\ell)}.$$

Proof. We begin by using the support of K to write

$$\begin{aligned} K \star f(x_1,x_2) &= \int_{\mathbb{R}}\int_{\mathbb{R}} K(x_1-y_1,x_2-y_2)f(y_1,y_2)dy_1dy_2 \\ &= \int_{|x_2-y_2|<1} K_{x_2-y_2} * f_{y_2}(x_1)dy_2 \\ &= \int_{|x_2-y_2|<1} K_{x_2-y_2} * f_{y_2}(x_1)\chi(x_2-y_2)dy_2 \end{aligned}$$

for a suitable non-negative bump function χ at scale 1. By the Cauchy-Schwarz inequality we obtain

$$|K \star f(x_1, x_2)| \lesssim \|K_{x_2 - \cdot} * f_{\cdot}(x_1)\chi(x_2 - \cdot)\|_{L^2(\mathbb{R})}.$$

As a consequence,

$$\begin{aligned}
& \int_{\mathbb{R}^2} |K \star f(x_1, x_2)|^2 w(x_1, x_2) dx_1 dx_2 \\
\leq\ & \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}} |K_{x_2 - y} * f_y(x_1)|^2 \chi^2(x_2 - y) dy \right) w(x_1, x_2) dx_1 dx_2 \\
=\ & \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |K_{x_2 - y} * f_y(x_1)|^2 w(x_1, x_2) dx_1 \right) \chi^2(x_2 - y) dx_2 dy \\
\lesssim\ & t^{\frac{1}{\ell - 1}} \log(t^{-1}) \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} |f(x_1, y)|^2 \widetilde{\mathcal{M}}_{t,\ell,1}(w)(x_1, x_2) dx_1 \chi^2(x_2 - y) dx_2 dy \\
=\ & t^{\frac{\ell}{\ell - 1}} \log(t^{-1}) \int_{\mathbb{R}^2} |f(x_1, y)|^2 \left(\int_{\mathbb{R}} \widetilde{\mathcal{M}}_{t,\ell,1}(w)(x_1, x_2) \chi^2(x_2 - y) dx_2 \right) dx_1 dy.
\end{aligned}$$

Relabeling the variables gives the desired result.

To prove the mixed norm inequality we write

$$\begin{aligned}
\|K \star f\|^2_{L_2^p(L_1^\ell)} &= \|(K \star f)^2\|_{L_2^{p/2}(L_1^{\ell/2})} \\
&= \sup \int_{\mathbb{R}^2} |(K \star f(x, y))^2| w(x, y) dx dy
\end{aligned}$$

where the supremum is taken over w on $\mathbb{R}^2$ with $\|w\|_{L_2^{(p/2)'}(L_1^{(\ell/2)'})} = 1$. With w described thus we have

$$\begin{aligned}
& \int_{\mathbb{R}^2} |K \star f(x, y)|^2 w(x, y) dx dy \\
\lesssim\ & t^{\frac{1}{\ell - 1}} \int_{\mathbb{R}^2} |f(x, y)|^2 \mathfrak{M}(w)(x, y) dx dy \\
\leq\ & t^{\frac{1}{\ell - 1}} \|f\|^2_{L_2^p(L_1^\ell)} \|\mathfrak{M}(w)\|_{L_2^{(p/2)'}(L_1^{(\ell/2)'})} \quad \text{by applying Hölder's Inequality twice.}
\end{aligned}$$

It therefore suffices to show that

$$\|\mathfrak{M}(w)\|_{L_2^{(p/2)'}(L_1^{(\ell/2)'})} \lesssim t^{\frac{\ell-2}{\ell(\ell-1)}} \|w\|_{L_2^{(p/2)'}(L_1^{(\ell/2)'})}.$$

We use the fact that $\mathfrak{M}(w)$ is $\widetilde{\mathcal{M}}_{t,\ell,1}(w)$ convolved with a non-negative bump function χ on $\mathbb{R}$ in the second variable to obtain

$$\begin{aligned}
\|\mathfrak{M}(w)(\cdot, y)\|_{(\ell/2)'} &= \| \int_{\mathbb{R}} \widetilde{\mathcal{M}}_{t,\ell,1}(w)(\cdot, z)\chi(z-y)dz\|_p \\
&\leq \int_{\mathbb{R}} \|\widetilde{\mathcal{M}}_{t,\ell,1}(w)(\cdot, z)\|_p \chi(z-y)dz
\end{aligned}$$

by Minkowski's Inequality. It follows that

$$\begin{aligned}
\|\mathfrak{M}(w)\|_{L_2^{(p/2)'}(L_1^{(\ell/2)'})} &\lesssim \|\widetilde{\mathcal{M}}_{t,\ell,1}(w)\|_{L_2^{(p/2)'}(L_1^{(\ell/2)'})} \\
&\lesssim t^{\frac{\ell-2}{\ell(\ell-1)}} \|w\|_{L_2^{(p/2)'}(L_1^{(\ell/2)'})}.
\end{aligned}$$

As a consequence,

$$\int_{\mathbb{R}^2} |K \star f(x,y)|^2 w(x,y)dxdy \lesssim t^{2/\ell} \|f\|^2_{L_2^p(L_1^\ell)} \|w\|_{L_2^{(p/2)'}(L_1^{(\ell/2)'})}$$

and so taking the supremum over w of unit mixed-norm yields the desired inequality.

□

5.2 A One-Weighted estimate on $\mathbb{R}^n$

As in the one-dimensional case, a one-weighted estimate for a family of convolution kernels of the above type is readily obtainable using estimates on the Fourier transforms of those kernels. As before, the frequency decomposition we use will be motivated by the behaviour of the Fourier transforms of the kernels in different regions of $\mathbb{R}^n$.

Let Φ be an even function on $\mathbb{R}$ such that

- $\Phi \in C^\infty$ in a neighborhood of the origin.
- $\Phi^{(k)}(0) = 0$ for $0 \leq k \leq \ell - 1$, and $\Phi^{(\ell)}(0) \neq 0$

for an integer ℓ greater than 2, and ψ is a smooth, even function with compact support containing the origin chosen such that $\Phi^{(\ell)}$ is bounded below in its support. We let $K_\lambda(x) = e^{i\lambda\Phi(|x|)}\psi(|x|)$ for $\lambda \geq 1$ and $x \in \mathbb{R}^n$.

It is well known that the Fourier transform of a radial function is itself radial, and can be written in terms of the Hankel transform. Let f be a radial function on $\mathbb{R}^n$, and let f_0 be the function on $[0, \infty)$ such that $f(x) = f_0(|x|)$. Then

$$\widehat{f}(\xi) = C_n H_{\frac{n-2}{2}}(f_0)(|\xi|),$$

where, for $t \in [0, \infty)$,

$$H_\nu(f_0)(t) = t^{-\nu} \int_0^\infty f_0(r) J_\nu(rt) r^{1+\nu} dr,$$

which is the modified Hankel transform of f_0. If f is radial we will sometimes abuse notation and write $H_\nu(f)$ when we mean $H_\nu(f_0)$. In order to estimate $\widehat{K_\lambda}$ it therefore suffices to consider the corresponding Hankel transform.

We will go on to prove a one-weighted L^2 estimate using a similar argument to that of Corollary 4.3.1. As before, we need to estimate the decay of $\widehat{K_\lambda}$ which will then give an indication of the frequency decomposition to proceed with.

Lemma 5.2.1.

$$|\widehat{K}_\lambda(\xi)| \lesssim \begin{cases} \lambda^{-1/\ell}, \quad |\xi| \lesssim \lambda^{1/\ell} \\ \lambda^{\frac{-1}{2(\ell-1)}}|\xi|^{-\frac{\ell-2}{2(\ell-1)}}, \quad \lambda^{1/\ell} \lesssim |\xi| \lesssim \lambda \\ |\xi|^{-N} \quad \textit{for any } N \in \mathbb{N}, \quad |\xi| \gtrsim \lambda. \end{cases}$$

The following lemma gives a description of the action of convolution with K_λ on a bump function which has Fourier support in the region where $\widehat{K}_\lambda$ is rapidly decreasing.

Lemma 5.2.2. *Let Ψ_p be a smooth radial function on $\mathbb{R}^n$ such that* $\operatorname{supp}\widehat{\Psi}_p \subset \{|\xi| \sim 2^p\}$ *for some p with $2^p \gtrsim \lambda$. Let $\tilde{\Psi}_p$ be a function on $\mathbb{R}$ such that $\Psi_p(x) = \tilde{\Psi}_p(|x|)$ and assume further that $\frac{d^k}{dt^k}H_{\frac{n-2}{2}}(\tilde{\Psi}_p)(t)$ is bounded uniformly in t and p. Then*

$$|K_\lambda * \Psi_p(x)| \lesssim 2^{-pN}Q_p(x)$$

where

$$Q_p(x) \lesssim \frac{(2^p)^n}{(1+2^p|x|)^M}$$

for all $N, M \in \mathbb{N}$.

The proofs of these Lemmas are left until after the proof of Theorem 5.2.3.

These estimates on $\widehat{K}_\lambda$ suggest that the following collection of sets is an appropriate frequency decomposition with which to proceed:

$$\begin{aligned} \mathcal{A}_0 &= \{\xi \in \mathbb{R}^n : |\xi| \lesssim \lambda^{1/\ell}\} \\ \mathcal{A}_p &= \{\xi \in \mathbb{R}^n : |\xi| \sim 2^{-p}\lambda\} \quad \text{for } p \text{ such that } 1 \lesssim 2^p \lesssim \lambda^{(\ell-1)/\ell} \\ \mathcal{A}_\infty &= \{\xi \in \mathbb{R} : |\xi| \gtrsim \lambda\}. \end{aligned}$$

Theorem 5.2.3.

$$\int_{\mathbb{R}^n} |K_\lambda * f(x)|^2 d\mu(x) \lesssim \lambda^{-\frac{1}{\ell-1}} \log(\lambda) \|\mathcal{M}_{\lambda,\ell}(\mu)\|_\infty \int_{\mathbb{R}^n} |f(x)|^2 dx \tag{5.3}$$

for all non-negative Borel measures μ on $\mathbb{R}^n$. There exists a function P_r on $\mathbb{R}^n$ such that

$$\mathcal{M}_{\lambda,\ell}(\mu)(x) = \sup_{0\leq r\leq \lambda^{-1/\ell}} r^{\frac{\ell-2}{\ell-1}} |P_r * \mu(x)|,$$

and P_r satisfies

$$P_r(x) \lesssim \frac{(r^{-1})^n}{(1+r^{-1}|x|)^N}$$

for all $N \in \mathbb{N}$.

Proof. We argue in the manner of Corollary 4.3.1, and suppose that supp$\hat{f}$ is restricted to an $\mathcal{A}_p$, $\mathcal{A}_0$ or $\mathcal{A}_\infty$. Firstly, suppose that supp$\hat{f} \subset \mathcal{A}_0$, and let P_0 be a function on $\mathbb{R}^n$ such that $\widehat{P}_0(\xi) = 1$ when $\xi \lesssim \lambda^{1/\ell}$. Then

$$\begin{aligned}
\int |K_\lambda * f|^2 d\mu &= \int |K_\lambda * f|^2 P_0 * \mu \\
&\leq \|P_0 * \mu\|_\infty \int |K_\lambda * f|^2 \\
&= \|P_0 * \mu\|_\infty \int |\widehat{K}_\lambda|^2 |\hat{f}|^2 \\
&\leq \sup_{\xi\in\mathcal{A}_0} |\widehat{K}_\lambda(\xi)|^2 \|P_0 * \mu\|_\infty \int |\hat{f}|^2 \\
&\lesssim \lambda^{-2/\ell} \|P_0 * \mu\|_\infty \int |f|^2.
\end{aligned}$$

For f with $\hat{f}$ supported in $\mathcal{A}_p$, the above argument yields

$$\int |K_\lambda * f|^2 d\mu \lesssim \lambda^{-1}(2^p)^{\frac{\ell-2}{\ell-1}} \|P_p * \mu\|_\infty \int |f|^2$$

where P_p is a function on $\mathbb{R}^n$ with $\widehat{P}_p(\xi) = 1$ for $|\xi| \lesssim 2^{-p}\lambda$. Now we may choose P_p to satisfy

$$|P_p(x)| \lesssim \frac{(2^{-p}\lambda)^n}{(1 + 2^{-p}\lambda|x|)^N}$$

for all $N \in \mathbb{N}$, and since $\lambda^{-1}(2^p)^{\frac{\ell-2}{\ell-1}} P_p * \mu = \lambda^{-\frac{1}{\ell-1}}(2^p/\lambda)^{\frac{\ell-2}{\ell-1}} P_p * \mu$ we have

$$\lambda^{-1}(2^p)^{\frac{\ell-2}{\ell-1}} \|P_p * \mu\|_\infty \lesssim \lambda^{-\frac{1}{\ell-1}} \|\mathcal{M}_{\lambda,\ell}(\mu)\|_\infty.$$

Similarly, one may show that $\lambda^{-2/\ell}\|P_0 * \mu\|_\infty \lesssim \lambda^{\frac{1}{\ell-1}} \|\mathcal{M}_{\lambda,\ell}\|_\infty$.

Finally, we consider the case when $\mathrm{supp}\hat{f} \subset \mathcal{A}_\infty$. Let $\{\Psi_p\}_{p\in\mathbb{N}}$ be a smooth partition of unity on $\mathbb{R}^n$, constructed in the standard way, with $\mathrm{supp}\widehat{\Psi}_p \subset \{\xi : |\xi| \sim 2^p\}$ and each Ψ_p radial. If $\hat{f}$ is supported in $\mathcal{A}_\infty$ then

$$f = \sum_{p:2^p \gtrsim \lambda} f * \Psi_p$$

and so

$$\int |K_\lambda * f|^2 d\mu = \int |\sum_{2^p \gtrsim \lambda} (K_\lambda * \Psi_p) * f|^2 d\mu.$$

By multiplying out the integrand and using Fubini's Theorem, $\int |K_\lambda * f|^2 d\mu$ is controlled by

$$\sum_{2^p, 2^q \gtrsim \lambda} \int |f(y)|^2 \left(\int |K_\lambda * \Psi_p(x-y)| d\mu(y) \right) \left(\int |K_\lambda * \Psi_q(z)| dz \right) dy.$$

By Lemma 5.2.2, for any $M \in \mathbb{N}$ this is dominated by a constant multiple of

$$\sum_{2^p \gtrsim \lambda} \int |f(y)|^2 2^{-pM} Q_p * \mu(y) dy \lesssim \int |f(y)|^2 \left[\sup_{0<r<\lambda^{-1}} r^M Q_r * \mu(y) \right] dy,$$

where Q_r is as in the lemma. We may clearly choose M large enough that the bound

$$\sup_{0<r<\lambda^{-1}} r^M Q_r * \mu \lesssim \lambda^{-\frac{1}{\ell-1}} \mathcal{M}_{\lambda,\ell}(\mu)$$

holds uniformly in μ and λ, and so for f with $\text{supp}\hat{f} \subset \mathcal{A}_\infty$,

$$\int |K_\lambda * f|^2 d\mu \lesssim \lambda^{-\frac{1}{\ell-1}} \int |f|^2 \mathcal{M}_{\lambda,\ell}(\mu).$$

We have therefore established that

$$\int_{\mathbb{R}^n} |K_\lambda * f|^2 d\mu \lesssim \lambda^{-\frac{1}{\ell-1}} \|\mathcal{M}_{\lambda,\ell}(\mu)\|_\infty \int_{\mathbb{R}^n} |f|^2$$

whenever the support of $\hat{f}$ is restricted to $\mathcal{A}_0$, $\mathcal{A}_p$ for some p or $\mathcal{A}_\infty$. Since there are $\sim \log(\lambda)$ sets in our frequency decomposition we can conclude that, for f with unrestricted Fourier support,

$$\int_{\mathbb{R}^n} |K_\lambda * f|^2 d\mu \lesssim \lambda^{-\frac{1}{\ell-1}} \log(\lambda) \|\mathcal{M}_{\lambda,\ell}(\mu)\|_\infty \int_{\mathbb{R}^n} |f|^2$$

as claimed.

□

Proof of Lemma 5.2.1. Let us abuse notation and write

$$H_\nu(K_\lambda)(t) = t^{-\nu} \int_0^\infty e^{i\lambda\Phi(r)} \psi(r) J_\nu(rt) r^{1+\nu} dr$$

for $t > 0$. Using Fubini's Theorem and the formula

$$J_\nu(z) = C_\nu z^\nu \int_{-1}^1 e^{izs}(1-s^2)^{\nu-1/2} ds$$

we have

$$H_\nu(K_\lambda)(t) = C_\nu \int_{-1}^1 (1-s^2)^{\nu-1/2} \left[\int_0^\infty e^{i(\lambda\Phi(r)+rst)} \psi(r) r^{1+\nu} dr \right] ds.$$

From Proposition 4.2.1, we know that

$$\left| \int_0^\infty e^{i(\lambda\Phi(r)+rst)} \psi(r) r^{1+\nu} dr \right| \lesssim \begin{cases} \lambda^{-1/\ell}, & t \lesssim \lambda^{1/\ell} \\ \lambda^{\frac{-1}{2(\ell-1)}} |ts|^{\frac{-(\ell-2)}{2(\ell-1)}}, & \lambda^{1/\ell} \lesssim t \lesssim \lambda, \end{cases}$$

and since $|\cdot|^{\frac{-(\ell-2)}{2(\ell-1)}}$ is integrable on $[-1,1]$ we have

$$|H_\nu(K_\lambda)(t)| \lesssim \begin{cases} \lambda^{-1/\ell}, & t \lesssim \lambda^{1/\ell} \\ \lambda^{\frac{-1}{2(\ell-1)}} |t|^{\frac{-(\ell-2)}{2(\ell-1)}}, & \lambda^{1/\ell} \lesssim t \lesssim \lambda. \end{cases}$$

It remains to be shown that $H_\nu(K_\lambda)(x)$ has rapid decay when $x \gtrsim \lambda$. Let D denote the differential operator

$$Df(t) = \frac{1}{t}\frac{d}{dt} f(t),$$

then it is easily verified by integration by parts and induction (see [30]) that

$$\begin{aligned} H_\nu(K_\lambda)(x) &= (-1)^N H_{\nu+N}(D^N K_\lambda)(x) \\ &= (-1)^N x^{-(\nu+N)} \int_0^\infty D^N K_\lambda(t) J_{\nu+N}(tx) t^{\nu+N+1} dt, \end{aligned} \tag{5.4}$$

where N is an integer that will be treated as fixed.

One may write $D^N K_\lambda$ as

$$D^N K_\lambda(t) = \sum_{j=0}^{N} D^j(e^{i\lambda\Phi(t)}) D^{N-j}\psi(t),$$

and for each $j = 1, \dots, N$ there exists functions $\Phi_{j,k}$ such that

$$D^j(e^{i\lambda\Phi(t)}) = e^{i\lambda\Phi(t)} \sum_{k=1}^{j} \lambda^k \Phi_{j,k}(t).$$

Each $\Phi_{j,k}$ is expressible as a linear combination of terms (the number of which depending on j) of the form

$$\prod_l (D^{p_l}\Phi)^{q_l}$$

for some integers p_l and q_l with $\sum_l p_l q_l = j$. It is shown in [30] that the operator D maps even Schwartz functions to even Schwartz functions, so if we assume, as we may, that Φ extends to an even Schwartz function on $\mathbb{R}$, then we must have that $\Phi_{j,k}$ is also Schwartz for each j and k. Consequently, one may write $D^N K_\lambda(t)$ as a sum of terms (the number of which depending only on N) of the form

$$e^{i\lambda\Phi(t)} \lambda^a \Psi(t) D^b \psi(t)$$

where a and b are integers with $0 \leq a, b \leq N$, and Ψ is a Schwartz function which is equal to $\Phi_{j,a}$ for some $0 \leq j \leq N$. So, $H_\nu(K_\lambda)(x)$ may be controlled by a sum of terms, the number of which depending only on N, of the form

$$\frac{\lambda^a}{x^N} x^{-\nu} \left| \int_0^\infty e^{i\lambda\Phi(t)} D^c\psi(t)\Psi(t) J_{\nu+N}(tx) t^{\nu+N+1} dt \right|. \tag{5.5}$$

Since $D^c\psi$ and Ψ are both even Schwartz functions, so is their product and so for ease of notation we relabel $D^c\psi \cdot \Psi$ as Ψ.

Note that $|\lambda^a/x^N| \lesssim 1$. If μ is an integer, then by the classical asymptotic theory of Bessel functions one may write

$$J_\mu(r) = r^{-1/2} e^{ir} \sum_{j=0}^{\mu} a_j r^{-j} + r^{-1/2} e^{-ir} \sum_{j=0}^{\mu} b_j r^{-j} + e(r), \tag{5.6}$$

for constants a_j and b_j and a function e with $|e(r)| \leq r^{-(\mu+1)}$. If $\mu \in \mathbb{Z} + 1/2$, then there exists constants a_j and b_j for which

$$J_\mu(r) = r^{-1/2} e^{ir} \sum_{j=0}^{\mu-1/2} a_j r^{-j} + r^{-1/2} e^{ir} \sum_{j=0}^{\mu-1/2} a_j r^{-j}.$$

See, for example, Stein [28].

Suppose that ν (and by implication $\nu + N$) is an integer. Consider the integral term in (5.5): By using the asymptotic formula (5.6), it is bounded by the sum of three terms I_1, I_2 and I_3 where the first two correspond to substituting the summations in (5.6) and the third to the error term. Specifically,

$$\begin{aligned} I_1(x) &= \sum_{j=0}^{\nu+N} a_j \left| \int_0^\infty e^{i(\lambda\Phi(t)+tx)} \Psi(t) (tx)^{-(j+1/2)} t^{\nu+N+1} dt \right|, \\ I_2(x) &= \sum_{j=0}^{\nu+N} b_j \left| \int_0^\infty e^{i(\lambda\Phi(t)-tx)} \Psi(t) (tx)^{-(j+1/2)} t^{\nu+N+1} dt \right|, \end{aligned}$$

and

$$I_3(x) = \left| \int_0^\infty e^{i\lambda\Phi(t)} \Psi(t) e(tx) t^{\nu+N+1} dt \right|.$$

Fix a value of j in the sum defining I_1 and call this $I_{1,j}$, i.e.

$$I_{1,j}(x) = a_j |x|^{-(j+1/2)} \left| \int_0^\infty e^{i(\lambda\Phi(t)+tx)} \Psi(t) t^{\nu+N+1/2-j} dt \right|$$

for $0 \leq j \leq \nu + N$.

Write the phase of $I_{1,j}(x)$ as $xh(t)$, where $h(t) = \lambda\Phi(t)/x + t$. Then in the range of integration, $|h'(t)| \gtrsim 1$, and $|h^{(k)}(t)| \lesssim 1$ for $k \geq 2$, and

$$I_{1,j}(x) = a_j |x|^{-(j+1/2)} \left| \int_0^\infty e^{ixh(t)} \Psi(t) t^{\nu+N+1/2-j} dt \right|.$$

Once again, we proceed in the manner of Lemma 2.1.1. Let D_1 be a differential operator defined by

$$D_1 f(t) = \frac{1}{ih'(t)} f'(t),$$

then

$$I_{1,j}(x) = a_j |x|^{-(j+1/2)} \left| \int_0^\infty D_1^N(e^{ixh})(t) \Psi(t) t^{\nu+1/2+N-j} dt \right|.$$

If we integrate this expression by parts $\nu + N - j$ times, then

$$I_{1,j}(x) = a_j |x|^{-(j+1/2)} |x|^{-(\nu+N-j)} \left| \int_0^\infty e^{ixh(t)} (D_1^*)^{\nu+N-j} (\Psi(t) t^{\nu+N+1/2-j}) dt \right|,$$

with

$$D_1^* f(x) = \frac{d}{dt}\left(\frac{f(t)}{ih'(t)}\right),$$

where the vanishing of the boundary terms is due to the compact support of Ψ and the fact that $(D_1^*)^k(\Psi(t)t^{\nu+N+1/2-j})|_{t=0} = 0$ for $1 \leq k \leq \nu + N - j$. Since Ψ is a compactly supported even Schwartz function,

$$(D_1^*)^{\nu+N-j}(\Psi(t)t^{\nu+N+1/2-j})$$

is always bounded on supp Ψ. Therefore

$$I_{1,j}(x) \lesssim |x|^{-(\nu+N-1/2)},$$

and as a consequence, the same bound holds for $I_1(x)$. Since a similar argument shows that $I_2(x) \lesssim |x|^{-(\nu+N-1/2)}$, it only remains to deal with I_3. To these ends, we simply use the bounds on the error term e and the compact support of Ψ to obtain

$$I_3(x) \lesssim |x|^{-(\nu+N)} \int_0^\infty |\Psi(t)| t dt \lesssim |x|^{-(\nu+N)},$$

and our estimates on I_1, I_2 and I_3 are complete.

Since $H_\nu(K_\lambda)$ is controlled by terms of the form (5.5), which we have shown to have decay $|x|^{-N}$ for $x \geq \lambda$ and any $N \in \mathbb{N}$, it then follows that $|H_\nu(K_\lambda)(x)| \lesssim |x|^{-N}$ as well, and the estimate is complete for $\nu \in \mathbb{N}$. If ν is a half-integer then the above argument may be replicated but without the error term.

□

Proof of Lemma 5.2.2. Our first observation is that

$$\left| \frac{d^N}{dt^N} H_{\frac{n-2}{2}}(K_\lambda)(t) \right| \leq C_{N,M} t^{-M} \tag{5.7}$$

for $N, M \in \mathbb{N}$ when $t \gtrsim \lambda$. This holds since for any $N \in \mathbb{N}$ we have

$$\frac{d^N}{dt^N} H_{\frac{n-2}{2}}(K_\lambda)(t) = t^N H_{\frac{n-2}{2}+N}(K_\lambda)(t)$$

which is dominated by $C_{N,M} t^{-M}$ for any $M \in \mathbb{N}$ by Lemma 5.2.1.

In what follows we let H denote $H_{\frac{n-2}{2}}$ and H_N denote $H_{\frac{n-2}{2}+N}$ for notational convenience. Now by Hankel transform inversion and (5.4) for any natural number N we have

$$\begin{aligned} K_\lambda * \Psi_p(x) &= H(H(K_\lambda)H(\Psi_p))(|x|) \\ &= (-1)^N H_N(D^N(H(K_\lambda)H(\Psi_p))(|x|) \\ &= (-1)^N |x|^{-(\frac{n-2}{2}+N)} \\ &\quad \times \int_0^\infty D^N(H(K_\lambda)H(\Psi_p))(t) J_{\frac{n-2}{2}+N}(t|x|) t^{\frac{n-2}{2}+N+1} dt. \end{aligned} \tag{5.8}$$

By (5.7), $|D^N(H(K_\lambda)H(\Psi_p))(t)|t^{\frac{n-2}{2}+N+1} \leq C_{N,M} 2^{-pM}$ for t in the support of $H(\Psi_p)$ for any $M \in \mathbb{N}$, and we also have that $|J_{\frac{n-2}{2}+N}(t|x|)| \leq C_{n,N}$ uniformly in t and x. Applied to (5.8) this yields

$$|K_\lambda * \Psi_p(x)| \leq C_{N,M} |x|^{-(\frac{n-2}{2}+N)} 2^{-pM} \tag{5.9}$$

for any $N, M \in \mathbb{N}$.

Finally, we use the fact that $t^{-\nu}J_\nu(t)$ is bounded uniformly in t to observe that

$$\begin{aligned} |K_\lambda * \Psi_p(x)| &= |H(H(K_\lambda)H(\Psi_p))(|x|)| \\ &= \left| \int_0^\infty H(K_\lambda)(t)H(\Psi_p)(t)\frac{J_{\frac{n-2}{2}}(t|x|)}{(t|x|)^{\frac{n-2}{2}}}t^{n-1}dt \right| \\ &\lesssim \int_0^\infty |H(K_\lambda)(t)H(\Psi_p)(t)|t^{n-1}dt \\ &\lesssim 2^{-pN} \end{aligned} \tag{5.10}$$

for any $N \in \mathbb{N}$. The estimates (5.9) and (5.10) may then be combined to yield

$$|K_\lambda * \Psi_p(x)| \lesssim 2^{-pN}\frac{(2^p)^n}{(1+2^p|x|)^M}$$

for any $M, N \in \mathbb{N}$ as claimed.

□

The existence of the one-weighted estimate (5.3) could reasonably lead one to conjecture that a two-weighted estimate for convolution with K_λ holds on $\mathbb{R}^n$, where the controlling maximal function is some suitable n-dimensional analogue of $\mathcal{M}_{t,\ell}$. A major obstacle towards proving an estimate of this kind using the methods used in Chapter 4 would be the probable necessity of the second (the 'equally spaced') frequency decomposition. If the Fourier support is decomposed initially by dyadic annuli then it is not beyond the realms of possibility that the second may involve some kind of tiling of the annuli with rectangles in $\mathbb{R}^n$. The Littlewood-Paley theory associated with such a decomposition would inevitably involve some Kakeya-type information.

A more modest approach would be to consider weighted norm estimates of the form

$$\int_{\mathbb{R}^n} |K_\lambda * f|^2 d\mu \lesssim C_\lambda \int_{\mathbb{R}^n} |f|^2 \mathcal{M}(\mu)$$

that are valid for radial functions f, radial weights μ and radial kernels K_λ. An approach such as this would naturally lend itself to the exploitation of properties of Bessel functions and associated Hankel transforms. There is, however, still no obvious way of carrying out a "nice" second frequency decomposition to correspond to the "equally spaced" decomposition on $\mathbb{R}$.

List of References

[1] J. A. Barceló, J. M. Bennett, and A. Carbery. A note on localised weighted inequalities for the extension operator. *Journal of the Australian Mathematical Society*, 84(03):289–299, 2008.

[2] J. A. Barceló, J. M. Bennett, A. Carbery, A. Ruiz, and M. C. Vilela. Some special solutions of the Schrödinger equation. *Indiana Univ. Math. J.*, 56(4):1581–1593, 2007.

[3] J. A. Barceló, J. M. Bennett, A. Carbery, A. Ruiz, and M. C. Vilela. Strichartz inequalities with weights in Morrey-Campanato classes. *Collect. Math.*, 61:49–56, 2010.

[4] J. A. Barcelo, A. Ruiz, and L. Vega. Weighted estimates for the Helmholtz equation and some applications. *J. Funct. Anal.*, 150(2):356–382, 1997.

[5] Jonathan Bennett, Anthony Carbery, Fernando Soria, and Ana Vargas. A Stein conjecture for the circle. *Math. Ann.*, 336(3):671–695, 2006.

[6] Jonathan Bennett, Anthony Carbery, Fernando Soria, and Ana Vargas. Corrigendum for "A Stein conjecture for the circle". *Math. Ann.*, 344(1):117, 2009.

[7] Jean Bourgain. Hausdorff dimension and distance sets. *Israel J. Math.*, 87(1-3):193–201, 1994.

[8] Anthony Carbery. The boundedness of the maximal Bochner-Riesz operator on $L^4(\mathbf{R}^2)$. *Duke Math. J.*, 50(2):409–416, 1983.

[9] Anthony Carbery. A weighted inequality for the maximal Bochner-Riesz operator on $\mathbf{R}^2$. *Trans. Amer. Math. Soc.*, 287(2):673–680, 1985.

[10] Anthony Carbery, Elena Romera, and Fernando Soria. Radial weights and mixed norm inequalities for the disc multiplier. *J. Funct. Anal.*, 109(1):52–75, 1992.

[11] Anthony Carbery and Andreas Seeger. Weighted inequalities for Bochner-Riesz means in the plane. *Q. J. Math.*, 51(2):155–167, 2000.

[12] Anthony Carbery, Fernando Soria, and Ana Vargas. Localisation and weighted inequalities for spherical Fourier means. *J. Anal. Math.*, 103:133–156, 2007.

[13] Lennart Carleson and Per Sjölin. Oscillatory integrals and a multiplier problem for the disc. *Studia Math.*, 44:287–299. (errata insert), 1972. Collection of articles honoring the completion by Antoni Zygmund of 50 years of scientific activity, III.

[14] Antonio Cordoba. The Kakeya maximal function and the spherical summation multipliers. *Amer. J. Math.*, 99(1):1–22, 1977.

[15] Javier Duoandikoetxea. *Fourier analysis*, volume 29 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001. Translated and revised from the 1995 Spanish original by David Cruz-Uribe.

[16] M. Burak Erdoğan. A note on the Fourier transform of fractal measures. *Math. Res. Lett.*, 11(2-3):299–313, 2004.

[17] K. J. Falconer. On the Hausdorff dimensions of distance sets. *Mathematika*, 32(2):206–212 (1986), 1985.

[18] Charles Fefferman. The multiplier problem for the ball. *Ann. of Math. (2)*, 94:330–336, 1971.

[19] José García-Cuerva and José L. Rubio de Francia. *Weighted norm inequalities and related topics*, volume 116 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1985. Notas de Matemática [Mathematical Notes], 104.

[20] Loukas Grafakos. *Modern Fourier analysis*, volume 250 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2009.

[21] Allan Greenleaf and Andreas Seeger. On oscillatory integral operators with folding canonical relations. *Studia Math.*, 132(2):125–139, 1999.

[22] Pertti Mattila. Spherical averages of Fourier transforms of measures with finite energy; dimension of intersections and distance sets. *Mathematika*, 34(2):207–228, 1987.

[23] Pertti Mattila. *Geometry of sets and measures in Euclidean spaces*, volume 44 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995. Fractals and rectifiability.

[24] C. Pérez. Weighted norm inequalities for singular integral operators. *J. London Math. Soc. (2)*, 49(2):296–308, 1994.

[25] Carlos Pérez. Sharp L^p-weighted Sobolev inequalities. *Ann. Inst. Fourier (Grenoble)*, 45(3):809–824, 1995.

[26] Per Sjölin. Estimates of spherical averages of Fourier transforms and dimensions of sets. *Mathematika*, 40(2):322–330, 1993.

[27] E. M. Stein. Some problems in harmonic analysis. In *Harmonic analysis in Euclidean spaces (Proc. Sympos. Pure Math., Williams Coll., Williamstown, Mass., 1978), Part 1*, Proc. Sympos. Pure Math., XXXV, Part, pages 3–20. Amer. Math. Soc., Providence, R.I., 1979.

[28] Elias M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, volume 43 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.

[29] Elias M. Stein and Guido Weiss. *Introduction to Fourier analysis on Euclidean spaces*. Princeton University Press, Princeton, N.J., 1971. Princeton Mathematical Series, No. 32.

[30] Krzysztof Stempak. A note on Zemanian spaces. *Extracta Math.*, 12(1):33–40, 1997.

[31] Terence Tao. The Bochner-Riesz conjecture implies the restriction conjecture. *Duke Math. J.*, 96(2):363–375, 1999.

[32] J. Michael Wilson. A sharp inequality for the square function. *Duke Math. J.*, 55(4):879–887, 1987.

[33] J. Michael Wilson. Weighted norm inequalities for the continuous square function. *Trans. Amer. Math. Soc.*, 314(2):661–692, 1989.

[34] Michael Wilson. The intrinsic square function. *Rev. Mat. Iberoam.*, 23(3):771–791, 2007.

[35] Thomas Wolff. Decay of circular means of Fourier transforms of measures. *Internat. Math. Res. Notices*, (10):547–567, 1999.

[36] Thomas H. Wolff. *Lectures on harmonic analysis*, volume 29 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2003. With a foreword by Charles Fefferman and preface by Izabella Łaba, Edited by Łaba and Carol Shubin.