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‘SOLUTIONING’: A MODEL OF STUDENTS’ PROBLEM-SOLVING PROCESSES

by

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CONTENTS

ABBREVIATIONS

CPS (Complex problem solving)

CR (Cognitive reassurance)

IPS (Information processing systems)

MC (Mathematical conviction)

LIST OF FIGURES AND TABLES

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DECLARATION

I declare that the material within this thesis has not been presented in any other thesis.

Chapter 3 refers to material that was reported in the following papers:

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Chapters 4 contains material that formed the basis of the following papers:

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ABSTRACT

The aim of this study was to generate a model (or theory) that explains students' concerns as they tackle non-routine mathematical problems. This was achieved by using the grounded theory approach as suggested by Glaser and Strauss (1967) and further developed by Glaser (1978; 1992; 1998; 2001; 2003). The study took place in the context of a problem-solving course offered at the undergraduate level. As methods of data collection, the study made use of the problem-solving rubrics (or scripts) that students generated during the course. Other sources of data included interviews with the students and observations in class.

The model generated as a result of this study suggests that problem solving can be seen as a four-stage process. The process was labelled 'solutioning' and is characterised by students trying to resolve the following concerns:

Generating knowledge;
Generating solutions;
Validating the results, and
Improving the results.

The model also makes reference to pseudo-solutioning as an alternative approach to solutioning. During pseudo-solutioning, instead of tying to resolve the concerns listed above, students focus on trying to satisfy the academic requirement to submit an acceptable piece of work. Thus, pseudo-solutioning can be seen an important variation to solutioning.

After presenting the model of 'solutioning', the study provides an illustration of how it can be used to describe students' processes. This is done in a set of case studies in which three problem-solving processes are considered. The case studies provide a view of how the model developed fits the data and serves to highlight relevant patterns of behaviour observable as students solve problems. The case studies illustrate how the concepts suggested by the model can be used for explaining success and failure in the processes considered.

This study contributes to the study of problem solving in mathematics education by providing a conceptualisation of what students do as they try to solve problems. The concepts that the model suggests are relevant for explaining how students resolve their main concerns as they tackle problems during the course. However, some of these concepts (e.g., 'reducing complexity', 'blinding activities', 'transferring') may also be of relevance to problem solving in other areas.

1. INTRODUCTION

Cognitive scientists have been concerned about investigating problem solving with the aim of learning about human reasoning in tasks that range from well-defined, laboratory-like situations to considerably more complex problems. In doing this, they have investigated the skills and knowledge that influence problem solving and that may lead to success in dealing with a range of situations (Greeno, 1978).

Furthermore, the importance of problem solving has been highlighted by the attention it has received from teachers and researchers in mathematics and science education. In the case of the mathematics education, arguments about the relevance of problem solving are often based on reports such as *Mathematics Counts* (Cockcroft, 1982), in England; and *An Agenda for Action* (NCTM, 1980) and *Curriculum and Evaluation Standards for School Mathematics*, in the United States (NCTM, 1989). These reports stress the importance of teaching higher order skills to students and suggest that problem solving is a viable way of achieving this aim.

It may be said that the aims of cognitive scientists and education practitioners and researchers converge in an interest (be it implicit or explicit) to improve people's ability to solve problems.

The present study recognises the importance of problem solving in mathematics education and supports the idea that efforts should be made to help students become better solvers. It may be argued that, in order to do this, two types of research are required: research that focuses on understanding human problem solving, and research that focuses on developing and/or evaluating ways of teaching effective problem-solving behaviours. While these two types of research can be seen as two separate strands, it may be said that the latter can benefit from the knowledge generated by studies that aim to explain how students solve problems. The present study seeks to generate understanding on mathematical problem solving and the next section takes a closer look at the position adopted to achieve this.

1.1. RATIONALE AND AIM OF THE STUDY

This study took place in the context of the mathematics problem-solving course offered at the University of Warwick. The idea of looking at the course emerged from an interest in investigating mathematical thinking in non-routine, problem-solving situations. In this sense, the course offered a valuable opportunity for observing the way in which students deal with mathematical problems that require them to make decisions such as which knowledge to apply and how to apply it. In a previous study, Mohammad-Yusof (1995; 1998) had looked at students' changes in attitudes and beliefs as a result of following a related course. However, her study focused on affective issues rather than *how* students solved problems.

Consequently, the aim of the present study was defined as that of generating a model of students' problem-solving processes, using the course as the context of the study. It was also established that the model should only make reference to issues of relevance to the students involved and to their problem-solving processes. Thus, it was decided against using a predefined theoretical framework or introducing categories from other studies. The methodology chosen for this study was the grounded theory approach as suggested by Glaser and Strauss (1967) and further developed by Glaser (1978; 1992; 1998; 2001; 2003). In terms of this methodology, the aim was defined as follows:

> To provide a *substantive* theory (or model) that explains students' main concerns as they tackle non-routine mathematical problems.

By a 'substantive' theory it is meant a theory developed for a particular area of enquiry (as opposed to a 'formal' theory that is more conceptual and is not restricted to a particular domain; see Glaser and Strauss, 1967).[1] Furthermore, theories generated from a grounded theory approach aim at conceptualising the main 'concern' or problem that the participants are constantly trying to resolve. In Glaser's words:

> [T]his [main] concern is not a professional interest, nor a general public interest, and not an exceptional interest of the researcher. For better or worse in the researcher's view, it is what the participants are trying to do, and act as such. It is always there and will emerge out of collection, coding and analysing much data, if not squelched by preconceptions.
>
> (Glaser, 2001, pp. 103-104)

Thus, the present study adopted a focus on "what the participants are trying to

[1] See also Section 3.1.2.2.

do" and not a preconceived stance. Consequently, the model that emerged focuses on the main concerns that students try to resolve as they solve problems. As will be seen, the term 'students' is an intentional generalisation that highlights the fact that it is patterns of behaviour that are of interest and not particular cases.

As a result of the study, it was observed that students' main concerns are: *generating knowledge* (i.e., information and understanding), *generating solutions*, *validating the results*, and *improving* them. These concerns can be summarised in a main concern called 'solutioning'. Solutioning has an important variation known as 'pseudo-solutioning' which suggests that, instead of trying to resolve the concerns mentioned above, students focus on generating an academically acceptable piece of work. The model is presented in Chapter 4 and it explains how students resolve these concerns as they tackle non-routine mathematical problems.

The terms 'model' and 'theory' are used interchangeably throughout the thesis. It is assumed that a theory in the social sciences is a model that explains social or psychological behaviour based on the systematic use of evidence (Hoover, 1992; Lewins, 1992). In this sense, the model presented represents also a (substantive) *theory* of problem solving.

Furthermore, the purpose of the study can be classified as 'pure' (Schoenfeld, 2000), in the sense that its aims is to generate understanding about students' processes and not to provide direct answers to questions like 'how to improve the course' or 'what teaching method works best'[2]. In spite of this, there is an explicit interest in generating relevant information about the way in which students solve problems. Information that is relevant may be useful for helping students become better problem solvers and to assist course tutors and course developers. Thus, although the purpose of the study is not 'applied', an indirect aim is that the results should be useful to those interested in improving mathematics instruction.

1.2. OVERVIEW OF THE STUDY

The study consists of five chapters besides the current one. Chapter 2 provides a discussion of the literature in relation to problem solving. The aim of this chapter is to suggest how the present study fits with other work that has been done in the area and the contribution that it intends to make. Thus, the chapter starts by considering early studies in problem solving as conducted in the cognitive sciences. It continues by suggesting how the study of problem solving evolved from using mainly well-defined problems to focusing on problem solving in less laboratory-like contexts. After considering the study of problem solving from a general perspective, the chapter discusses the study of problem solving in mathematics education. This discussion considers salient studies on problem solving in this particular domain and assesses their influence and contribution. A final consideration is made in relation to frameworks for teaching problem

[2] Schoenfeld described the purposes of mathematics education research as follows: 'Pure' (basic science): to understand the nature of mathematical teaching, thinking and learning. 'Applied' (engineering): to use such understanding to improve mathematics instruction. (Schoenfeld, 2000, p. 641)

solving and their impact on practice as well as on research. This chapter highlights the contribution that this study aims to make by studying problem solving using a grounded theory approach.

Chapter 3 provides an account of the methodology used for this study. The chapter is divided in three parts that suggest a 'chronology' of how the views of the methods of data collection and analysis evolved. Part I discusses the context of the study, the methodology chosen, and the first decisions that were made in relation to the methods of data collection. As for Part II, it discusses the 'exploratory' and 'pilot' studies. These preliminary studies served to test and further develop the methods of data collection and analysis established in Part I. Finally, Part III presents the methods of data collection and analysis that were employed in what can be called the 'main' study. In all, this chapter considers important methodological issues in relation to this study and explains how it was conducted.

Chapter 4 presents 'Solutioning' as the model of problem solving that was generated as a result of this study. The model was generated by following the grounded-theory approach and thus provides a conceptualisation of how students solve problems. The model explains the main concerns of students as they tackle non-routine mathematical problems and how these are resolved. Although some aspects of the model could be considered as descriptive, it succeeds in raising important concepts that represent general patterns of behaviour observed in the way students solve problems. It may be argued that some of these concepts may also be relevant to contexts other than the one in which they were generated.

Chapter 5 provides an illustration of how the model can be used to analyse students' processes. This is done in a series of case studies in which three students' rubrics are described using the model as theoretical framework. The descriptions presented in the case studies are intended to be read in conjunction with the rubrics (provided in Appendix 3). The descriptions of the rubrics illustrate how the concepts suggested by the model account for important aspects of students' problem-solving processes. The discussion at the end of the chapter also serves to highlight how the model can be useful for explaining why some students may be more successful than others during mathematical problem solving. This section illustrates how the model fits the criteria for evaluating grounded theories, namely, 'fit', 'relevance' and 'work'.

Finally, Chapter 6 starts by looking at the nature of the study and at the contribution that it makes to the study of problem solving. The second section of this chapter takes a closer look at the model that was generated and discusses some of the most relevant concepts that it proposes. Some indications about the usefulness of these concepts, as well as suggestions for future research, are provided. The study concludes with some final comments and considers the possibility of a 'formal' (as opposed to 'substantive') theory of problem solving.

2. LITERATURE REVIEW

2.1. INTRODUCTION

Problem solving has been of interest to researchers in areas such as philosophy, psychology, artificial intelligence, neuroscience, linguistics, anthropology and education. In relation to mathematical problem solving, it has not been developed in isolation but has been influenced by research into human problem solving. It may be said that findings in the cognitive sciences have had a considerable influence in the research questions and methods employed in the study of mathematical problem solving (Schoenfeld, 1987a).

The aim of this chapter is to discuss the approaches that have been taken towards the study of problem solving and to locate the present study within work that has been previously done in the area. This is first done from the perspective of the cognitive sciences, particularly from the perspective of those psychological studies conducted during the past century and that continue to influence the study of problem solving nowadays. After this, the review takes a closer look at studies of problem solving in mathematics education.

The chapter is structured in five sections. Section 2.2 explores the early studies on problem solving conducted by Gestaltist psychologists. Gestaltists were particularly interested in what they called 'productive' thinking. Although they were able to generate interesting contributions to the study of problem solving, these did not consolidate as theories. As for Section 2.3, it looks at the information-processing approach to problem solving. This approach emerged in part as a reaction to the methodological limitations of the Gestaltist's approach. However, it may be said that other factors such as the development of computers since the 1960's have played an important role its development. One of the criticisms made to this approach is the fact that its theories were, in most cases, based on well-defined, laboratory-like problems. This, critics argue, made generalisations to more sophisticated problems difficult or in some cases unfeasible. Section 2.4 discusses two different reactions to this criticism: the study of problem solving in domain-specific situations and the study of problem solving in complex (usually computerised) environments.

After a discussion on problem solving in the cognitive sciences, Section 2.5 and Section 2.6 deal with the study of problem solving in mathematics education. The former section discusses how the issue of problem solving has been explored in mathematics education and how some of the main theories have emerged. Section 2.6 discusses 'teaching' frameworks that have been developed in mathematics education. Among these are Polya (1957) and Mason, Burton and Stacey's (1982) problem-solving frameworks. Although these were not developed in a 'scientific' way, they have influenced mathematics education practices and have provided a basis for further research, including the present study.

2.2. PROBLEM SOLVING AS A DIRECTED AND PRODUCTIVE ACTIVITY

The Gestalt approach to thinking (and problem solving) emerged as a reaction to the stimulus-response theories put forward by associationists (e.g., Thorndike,

1911) and behaviourists (e.g., Skinner, 1958). Gestaltists argued that problem solving is the result of mentally arranging complex situations into organised wholes. They suggested that problem solving is possible when the observed situation is restructured in such a way that a solution can be 'seen'. Being able to 'see' this solution is the outcome of flashes of insight that allow the solver to reorganise situations into coherent structures (Kohler, 1927). As Dellarosa (1988) put it, Gestaltists "believed perception to be an active, constructive process, not a passive, 'reflexive' one" (p. 9). Thus, for Gestalt psychologists, problem solving was seen as an activity that consisted of giving meaning to a problematic situation rather than experimenting with it in an undirected way. In other words, Gestaltist psychologists maintained that thinking can be productive and that it can lead to generating new ideas and understanding (Wertheimer, 1959).

Making a distinction between productive and reproductive thinking is an important contribution of the Gestalt approach to problem solving (Moates and Schumacher, 1980). Productive thinking involves creating a solution by combining different ideas about the situation in a novel way. Reproductive thinking, on the other hand, involves solving problems "by recall, by mechanical repetition of what has been drilled, by sheer chance discovery in a succession of blind trails" (Wertheimer, 1959, p. 11). In other words, reproductive thinking involves applying learnt habits without trying to examine the situation in a meaningful way.

Gestaltist psychologists' interest in productive thinking manifested in the type of problems that they investigated. These were largely 'insight' problems in which solvers "must discover a crucial element and once this element is discovered all the other elements fall into place and the problem is solved" (Dunbar, 1998, p. 291). An example of such a problem is the 'Six-Stick' problem that requires the solver to arrange six sticks in a way that four equilateral triangles, with each side one stick long, are formed. The 'Six-Stick' problem can be considered an 'insight' problem that requires solvers to discover that such arrangement can only be possible in three dimensions.

Besides 'restructuring' and 'productive thinking', Gestalt psychologists introduced other ideas to the study of thinking and problem solving. Some of these ideas are discussed in Sections 2.2.1. Section 2.2.2 looks at the legacy of the study of problem solving from a Gestaltist approach.

2.2.1. Stages of problem solving and problem-solving set

Gestaltist psychologists suggested that problem solving occurs in stages. In *The Art of Thought*, Wallas (1926), explained problem solving processes as a series of four discrete phases, namely, 'preparation', 'incubation' 'illumination' and 'verification'. Presenting problem solving as a series of steps towards a solution incorporates some of the most important ideas proposed by Gestaltists. Among these ideas, 'illumination' seems to be the main focus of the approach. As can be observed below, it is during this phase that a solution is achieved and where a mental restructuring of the situation may occur. The four phases proposed by Wallas are discussed below.

- *Preparation.* The solver recognises the problem as one where a certain goal must be achieved, starting from a given state and under certain conditions. The solver makes some attempts at understanding and solving the problem.
- *Incubation.* If after some attempts the problem cannot be solved, the solver may stop working and thinking about the problem for a while. Gestaltists maintained that although incubation does not involve any conscious effort at solving the problem, the solver continues to work on the problem in an unconscious way.
- *Illumination.* It is at this stage that sudden flashes of insight occur. Insights are presented as the result of both preparation and incubation, although Gestaltists seem to attribute it mainly to the latter. Insights are those moments when it becomes clear to a solver how a solution can be achieved and they may precede or announce a mental reorganisation of the situation.
- *Verification.* This stage consists of making sure that the situation can be solved in the way 'revealed' by the insight. As Best (1995) suggested, from a Gestaltist perspective "this stage is the least complicated and is usually no more than a simple checking to make sure that the insight worked" (p. 414).

Some of the ideas proposed by Wallas continue to be of relevance nowadays. For instance, Mason, Burton and Stacey (1982) suggested that putting a question aside and mulling on it for a while may lead to moments of insight "in which the whole question or a significant part of it seems to fall into place" (p. 127). These ideas closely relate to Wallas's 'incubation' and 'illumination' stages defined above. Other studies such as Garofalo and Lester's (1985) and the present study consider issues that can be related to preparation and verification. However, contrary to the Gestaltist approach, the present study considers these issues to be at least as important as issues related to the moment when a solution is generated.

Another important idea introduced by Gestaltists was the idea of 'problem-solving set'. This concept can be broadly defined as approaching a new situation in a particular way that is based on prior experience. Although problem-solving set can have positive effects on problem solving (Dominowski and Bourne, 1994) the Gestaltist approach seems to focus on its negative effect. This fact can be related to the idea of problem solving as a process that consists of suitably reorganising information about the situation. From this perspective, an inability to modify or restructure the way one sees the situation is seen as detrimental to the solving process. For instance, Gardner and Runquist (1958) conducted a study to investigate the idea of problem-solving set and its effect on students' problem-solving processes. Their study consisted of presenting students with a series of problems that could be successfully solved by applying a particular method. After this, students were presented with a second series of problems that required the use of simpler (but different) methods. The authors observed that students found it difficult to solve the 'simpler' problems and argued that the familiar method acted as a problem-solving set that prevented students from successfully dealing with the new series of problems.

2.2.2. The legacy of the Gestaltist approach

An important aspect of the Gestalt approach was its aim to direct psychologists' attention to more 'meaningful' thinking processes than the ones behaviourists were dealing with. In other words, the aim of Gestalt psychologists was characterised by an attempt to change psychologists' attention from the study of simple stimulus-response sequences to observing more holistic and complex thinking processes (Dellarosa, 1988). In this way, Gestaltist psychologists proposed a number of ideas that would influence future studies on problem solving and that continue to be of interest nowadays.

According to their critics, however, some of the ideas that Gestaltist psychologists proposed were too vague to be studied in an objective way. This is particularly evident in the fact that ideas such as 'incubation' and 'illumination' are not easy to define or to interpret concisely (Mayer, 1992). For instance, the idea that 'incubation' is due to unconscious mental work is difficult to accept as a testable hypothesis. In addition to this (and possibly as a consequence), many of the studies in which Gestaltist ideas were tested were based on scientifically unacceptable methods such as introspection or anecdotal evidence (Ericsson and Hastie, 1994) and thus could not be accounted for as theories. As Greeno (1978) put it:

> While the studies conducted by Gestaltist psychologists provided numerous interesting examples of thinking processes that were analysed insightfully, few general principles emerged that could lead to the development of a solid body of theory.
>
> (Greeno, 1978, p. 240)

In short, it may be said Gestaltist psychologists' intentions of investigating productive thinking lacked the support of suitable methods for dealing with complex thinking processes (Dunbar, 1998).

In spite of the difficulties, some researchers have been able to investigate some of the ideas proposed by the Gestalt approach in more rigorous ways. In a series of studies, Metcalfe (1986a; 1986b) and Metcalfe and Weibe (1987) provided empirical evidence that supported the idea of insight as a sudden and unexpected restructuring of the problem situation. In the latter study, for example, the authors compared 'warmth ratings' (or students' perceived nearness to a solution) in insight and non-insight problems. In the case of insight problems, they perceived a sudden increase in the students' warmth rating whereas for non-insight problems the change was incremental. Thus, the study provided empirical evidence to "indicate, in a straightforward manner that insight problems are, at least subjectively, solved by a sudden flash of illumination" (Metcalfe and Wiebe, 1987, p. 243).

But even if the issues raised by Gestaltists are scientifically investigated, it may be said that the study of problem solving as a sudden reorganisation of information leads to another important limitation. Although it may be accepted that issues such as illumination and insight play an important role during problem solving, there are other aspects of the process that need to be studied as well. For example, issues such as how the problem is made sense of, or how a new solution is verified or improved once it is achieved, are also worth considering. The present study agrees with other studies (e.g., Siegler and Jenkins, 1989) in that there are important limitations in an approach that focuses mainly on the discovery aspect of problem solving.

2.3. INFORMATION PROCESSING THEORIES

It has been suggested that information-processing theories of problem solving emerged as a reaction to the limitations observed in the Gestalt approach, particularly in relation to its lack of 'scientific' methods for investigating problem solving (Moates and Schumacher, 1980). Furthermore, besides failing to produce 'acceptable' theories (Dellarosa, 1988), Gestaltist explanations were considered as incomplete since they failed to provide "a prescription for problem solving" (Hunt, 1994, p. 217). Consequently, researchers searched for alternative approaches and investigated different avenues. However, attributing the emergence and evolution of the information-processing to the rejection of Gestaltism creates a distorted view of this approach. The information-processing approach to human cognition can be better seen as the result of multiple events in the scientific context (including a mounting criticism to Gestaltism and behaviourism) that allowed this approach to emerge.

The information-processing approach places emphasis on problem solving as a process and focuses on the events that make human cognition possible (Sternberg and Salter, 1982). However, in contrast to the Gestalt psychologists, information-processing theorists do not try to study thinking processes as a whole but proceed by dissecting "complex behaviours into simpler component stages" (Massaro and Cowan, 1993, p. 388). In other words, researchers interested in how information is processed focus on detailed accounts of thinking processes and aim at creating accurate descriptions of these accounts. Dellarosa (1988) suggested that most contemporary models of problem solving have, in some way or another, been influenced by the information-processing approach.

An important idea behind information-processing theories is that humans process information "almost entirely serially, one process at the time, rather than in parallel fashion" (Simon, 1978, p. 273). Short- and long-term memory also play important roles in explaining human information-processing activity. Short-term memory consists of the elements or ideas that are processed at each stage and there is a limit to the amount of information it can handle at a given time (Miller, 1956). As for long term memory, it allows the storage of considerably more information than short-term memory and in a relatively permanent way. Long-term memory may provide information that can be retrieved for solving the problem. Furthermore, it is in long-term memory that relevant new ideas will be stored for retrieval at a later time. Information-processing theory assumes that an ability to manipulate information in the short-tem memory is essential for efficient problem solving (Guenther, 1998).

Information-processing theorists were interested in some of the issues proposed by Gestaltists (such as the stages in which problem solving occurs). However, they were more concerned with rigour and making more objective hypotheses and analyses. For this purpose, they employed think aloud methods and constructed protocols to register solvers' thoughts as they tackled a problem. The use of verbal protocols for recording solvers' thinking processes can be considered as an important methodological contribution to the study of problem solving (Ericsson and Oliver, 1988). In their search for objectivity, psychologists following the information-processing approach were able to create precise descriptions of human problem-solving processes, but as we shall see in Section 2.3.2, such descriptions failed to explain important aspects of this activity.

The development of computers in the 1960's and the following decades provided the necessary resources for the evolution of the information-processing approach. With the availability of computers, researchers started to develop models of problem solving that could be programmed and run in a computer (Ibid). This influenced not only the way in which these models were developed but also the way in which they were validated. Problem-solving models were considered as valid if, once translated into a computer simulation program, they served to emulate human problem solving behaviour. In other words, the possibility of being able to model a theory of problem solving became not only a property of information-processing theories but also a good indicator of the accuracy of the theory (Mayer, 1992). This is not to say that all information-processing models of problem solving were expressed in a computerised form. However, being able to provide accurate descriptions of a solving process (as is required for computer simulations) is a general characteristic of the information-processing approach

(Medin, Ross and Markman, 2001) and this is typified by Newell and Simon's (1972) theory.

2.3.1. Newell and Simon's theory on human problem solving

Newell and Simon (1972) proposed an information-processing theory that models general problem-solving behaviour, i.e., problem solving that is generally observed rather than solver- or situation-specific. Developed over the decade preceding its publication, and noted as one of the most salient in the field, the theory was based on four propositions, namely:

1. Few, and only a few characteristics of IPS [Information Processing Systems] are invariant over task and problem solver.
2. These characteristics are sufficient to determine that a task environment is represented (in IPS) as a problem space, and that problem solving takes place in a problem space.
3. The structure of the task environment determines the possible structures of the problem space.
4. The structure of the problem space determines the possible programs that can be used for problem solving.

(Newell and Simon, 1972, pp. 788-789)

Since these propositions have influenced other information-processing studies of problem solving, it seems important to discuss them further.

The first proposition translates into the idea that there are a few generalisable strategies that solvers use for solving a problem. Newell and Simon maintained that the strategies that they identified were independent of domain-specific knowledge or experience. According to information-processing theories, differences between successful and unsuccessful processes can be explained by the strategy chosen and by the efficiency (or inefficiency) of the solver's short-term memory (Guenther, 1988; Hunt, 1994). Working forwards towards a goal and working backwards from goals to givens are two of the strategies proposed by Newell and Simon. Further studies (see Simon, 1978) have suggested other strategies that in some cases are variations, combinations or extensions of the ones generated by Newell and Simon.

The second proposition suggests that once a problem is understood, solvers generate a series of alternatives that allow them to move from an initial stage to a solution. This series of alternatives is what Newell and Simon called the 'problem space'. A problem space is a 'map' or a description of all the possible routes that can be taken in order to advance towards the goal defined by the problem. Once a problem space is generated, the solver's task is to choose among the alternatives. In Newell and Simon's words:

> Initially, when a problem is first presented, it must be recognized and understood. Then a problem space must be constructed or, if one already exists in long-term memory, it must be evoked. Problem spaces can be changed and modified during the course of solving.
>
> (Newell and Simon, 1972, p. 809)

The third proposition suggests that, for a given problem, it is possible to determine the problem space based on the characteristics of the task. Newell and Simon acknowledged that it was not possible to generalise this statement to any type of problem. However, they maintained that, at least in the type of problems they considered, it was possible to determine all the possible routes directly from the task.

The fourth proposition is particularly related to the third in the sense that, if it is possible to determine all the possible routes that solvers might take to solve a problem, then it is possible to create a general model of human problem-solving behaviour. In Newell and Simon's case, this was done through a computer program (the General Problem Solver) that was able to imitate human problem-solving behaviour in chess, cryptarithmetic and puzzle-like tasks.

It may be said that these four propositions established the assumptions that underlie not only Newell and Simon's but most information-processing theories. Moreover, the influence of these propositions seems to be pervasive in most contemporary studies on problem solving: As Ericsson and Hastie (1994) put it:

> These [fundamental] assumptions [of the information-processing approach] lead to the image of the thought process as movement from location to location, tracing a unique path through a 'problem space' of potential knowledge states. These same assumptions underlie the ubiquitous tendency by investigators to summarise complex thinking strategies in terms of a model series of substages in flow charts.
>
> (Ericsson and Hastie, 1994, p. 48)

Cobb (1987) warned, from a constructivist perspective, against taking this computer-influenced approach to problem solving as the only possibility for analysing problem-solving behaviour. He suggested that information-processing theories of problem solving tend to focus on those activities that can be expressed in formal (programmable) language and leave out other important issues. It may be difficult to disagree with Cobb in the idea that information-processing theories are in some way restricted by their methodological approach. However, it is this well-defined methodological approach that allows information-processing researchers to generate 'scientific' accounts of problem solving, something Gestaltist psychologists failed to achieve (see Dellarosa, 1998). It is to this and other criticisms of the information-processing approach that we now turn.

2.3.2. Criticism made to information-processing theories

An important criticism to information-processing theories is that they describe problem solving but fail to provide explanations that give a sense of better understanding of how a solution is generated. Information processing simulations of how problems are solved do not deal with issues such as the creation of new ideas or assigning meaning to a new situation (see Cobb, 1987). Instead, problem solving is treated as a process where information is transformed by means of operations that can be modelled in a formal way, i.e., by manipulating syntactic rules. Computer programs of problem solving simulate human thinking and aim at reproducing it. However, the actions that these programs conduct cannot be 'meaningful' and they do not generate the information and understanding that eventually allow humans to solve a problem.

Computer simulations of problem-solving processes allow researchers to analyse strategies and resources (such as memory and knowledge) and how they are used for processing information during problem solving. Nonetheless, these simulations are limited in the sense that they do not make evident how strategies are generated or how information is discriminated or brought forward. Dellarosa (1988) explains this situation alluding to the Gestaltist perspective.

> Ironically, Gestalt psychologists are among the strongest critics [to the information-processing approach]. They hold that the most important aspects of human reasoning have not been explained or even exhibited by computer simulation models.
>
> (Dellarosa, 1988, p. 16)

Information-processing psychologists reacted against the Gestaltists' subjective approach to problem solving. In doing so, they chose to focus on generating detailed descriptions of what solvers do as they tackle a problem. However, it has been suggested that an accurate description of problem solving does not constitute a theory but a method for representing a situation (Mayer, 1992). Furthermore, information-processing theories suggest that the solver does not add anything to what is known about the situation. The idea of a problem space suggests that all the necessary information is already there even before the solver starts tackling the problem. This view of problem solving can be considered as 'sterile' (in Poincare's (1982) terms) in the sense that it purports that information is manipulated and implicitly negates the creation of new knowledge or ideas.

Although Gestaltists psychologists were criticised for focusing on difficult-to-measure variables such as 'insight' and 'restructuring', the fact that they were not able to study such issues in a satisfactory way does not mean that they are not relevant. According to constructivists, information-processing theories ignore issues that are related to the meaning humans assign to their mental actions. Cobb (1990) argued that information-processing theories are not adequate enough to deal with the issue of meaning in mathematics. What Poincare (1982) calls fruitful reasoning in mathematics, for instance, is an aspect of problem solving that information-processing theories do not seem to deal with:

> What is the nature of mathematical reasoning? Is it really deductive as is commonly supposed? A deeper analysis shows that it is not, that it partakes in a certain measure of the nature of inductive reasoning, and just because of this is it so fruitful.
>
> (Poincaré, 1982, p. 29)

Anderson, Reder and Simon (2000) argued against the criticism raised by constructivists against the information-processing approach. They suggested that knowledge and understanding can be represented symbolically in such a way that reasoning can be computer simulated. In their argument, Anderson et al. seemed to suggest that this is possible for 'decomposed' and 'decontextualised' mental activities but not for complex problem-solving situations.

From the perspective of the present study, the information-processing approach can be criticised for investigating the researchers' concerns rather than those of the solvers. Information-processing theorists investigate problem solving as an activity that can be simulated. In doing so, they inevitably focus on those activities that are programmable and neglect others that, since they reflect what students are actually trying to do, are of utmost relevance.

Another important criticism that was made to the information-processing approach to problem solving is related to its generalisability. Information-processing theories study problem solving from the perspective of well-defined problems. A well-defined problem is a problem for which the initial state, the goal and the conditions are clearly defined, and for which there is known to be at least one way of reaching the goal (Sternberg and Ben-Zeev, 2001). It may be said that the decomposed and decontextualised mental activities that Anderson et al. allude to belong to this category. Critics of the information-processing approach argued that the theories derived from studying well-defined problems do not apply to the way humans tackle 'real-life' problems and that therefore they cannot be generalised (Mayer, 1992). Ericsson and Hastie (1994) explained this situation in the following way:

> Doubts about the representativeness of highly controlled, limited pre-experimental knowledge laboratory tasks are the basis for the most common criticisms of the information-processing approach. Is this approach relevant to cognition in everyday life? Much of this criticism is not specific to the theoretical assumptions of the information-processing approach, but implicates the methodological tactic of beginning research by studying subject's performance in unnatural, controlled laboratory tasks.
>
> (Ericsson and Hastie, 1994, p. 52)

Information-processing theorists anticipated this type of criticism and suggested that their theories contained elements that were common to a range of problem-solving situations. Newell and Simon (1972) argued that their theory was based on specific problem-solving situations but suggested that further generalisations were feasible. Later, Simon (1973; 1978) suggested that solvers treat ill-defined problems as well-defined problems once a goal has been specified. Nonetheless, this generalisability was not taken for granted and researchers started to look for ways of investigating problem solving in more 'real-life' situations.

It may be said that the study of problem solving from an information-processing approach has led to important contributions. These contributions can be summarised as the identification of important components of human thinking and an interest in the study of the interaction of these components "with their entire context" (Anderson et al., 2000, p. 2). Nevertheless, an interest in studying this interaction in complex situations has raised questions that may require a different approach. The next section discusses the study of problem solving from two different perspectives that emerged from the observed limitations in information-processing theories.

2.4. GOING BEYOND WELL-DEFINED PROBLEMS

The study of well-defined problems by the Gestaltists and information-processing theorists allowed researchers to gain knowledge and understanding of problem solving. However, researchers started questioning whether what was learnt through the study of well-defined problems could be extended to the problems that people experience in their everyday lives or other complex situations. As suggested above, a good part of the criticism made to information-processing theories did not challenge the methods used or the evidence underlying their claims. What seemed to worry critics was whether the results from information-processing theories could inform about cognition outside the context in which they were created.

In other words, one of the main questions that were raised in relation to information-processing theories had to do with the type of problems that they considered. Researchers started to question whether or not studying problem solving in these situations served to inform about thinking and problem solving in more complex situations. They started wondering whether or not what was learnt by looking at well-defined problems could be generalised to solving what can be generally called 'ill-defined' problems. Thus, before considering their reactions towards this criticism, it is important to discuss what is meant by ill-defined problems.

As indicated above (Section 2.3.2), well-defined problems are problems for which the initial state, as well as the goal and the conditions, are clearly specified and for which there is at least one path to the solution. Well-defined problems are associated with simple laboratory problems that can be solved in a relatively short period of time. A problem that is not well-defined can be labelled as 'ill-defined'. Best (1995) suggests that a problem is ill-defined

> if the start state is vague or unspecifiable, if the goal state is unclear, or if the operations required to change the start state into the goal state are unclear.
>
> (Best, 1995, p. 425)

Whereas the definition of well-defined problems was relatively unproblematic, the definition of ill-defined problems was so general that it could hardly be considered a category of problems. Although researchers disagreed about the importance of this lack of specification (Quesada, Kintsch and Gomez, 2003), what seemed to matter most to researchers was moving away from laboratory-like problem-solving situations to situations that resembled the types of problems that people deal with in their everyday lives.

According to Funke and Frensch (1995a), North American and European researchers responded differently to the need of a new approach towards the study of problem solving. On the one side, researchers in North America started to investigate problem solving in areas where content knowledge plays an important role such as physics, writing and solving managerial problems (see Chi, Feltovich and Glaser, 1981; Bryson, Bereiter, Scardamalia and Joram, 1991; Wagner, 1991). Funke and Frensch suggested that, in adopting this approach, North American researchers were abandoning the idea of generating a global theory of problem solving and focused more on a domain specific perspective. In contrast, researchers in Europe concentrated on the study of 'complex problem solving', or CPS. CPS researchers maintained the aim of generating a global theory of problem solving. Their approach was to investigate 'complex' solving situations, usually through computer simulated environments known as microworlds.

2.4.1. The study of problem solving from a domain-specific perspective

The 'domain-specific' perspective to problem solving, or what Funke and Frensch (1995a) called the North American perspective, refers to research conducted mainly in the United States during the 1970's and following decades. Problem-solving studies from this perspective have been more widely published than the studies on 'complex problem solving' (Quesada et al., 2003). It is this perspective, which can be seen as the direct evolution of the information-processing approach, that most studies make reference to and from which most researchers derive their frameworks (Ibid). The evolution from information-processing to domain-specific meant changing the types of problems that were studied but preserving many of the views suggested by earlier by information-processing theories.

The main difference between information-processing and domain-specific theories is that the former suggest the possibility of defining general problem-solving strategies that apply to all areas of knowledge (Simon, 1978). Domain-specific theories, in contrast, maintain that problem solving depends on domain specific knowledge which is particular to each domain and is only acquired after years of experience (Mayer, 1992). This view led researchers to study expertise and how the transition from being a novice to becoming an expert takes place. Other studies focused on the role of previous content-knowledge and on whether this knowledge can be transferred from one situation to another (Guenther, 1998). Thus, *expertise* and *transfer* can be considered two important concerns of the domain-specific perspective of the information-processing approach. These concerns are briefly discussed below.

Expertise

Domain-specific studies suggest that experts' knowledge base makes them better solvers than novices in their area of expertise. The advantage of experts over novices can be explained by the greater amount of information that experts store in their long-term memory and that can be retrieved for dealing with particular situations (Lesgold, 1988). Moreover, studies have suggested that experts have the ability to represent situations in ways that enable them to solve problems in more efficient ways (e.g., Larkin, McDermott, Simon, and Simon, 1980). It seems that not only the amount of knowledge is important for expert problem solvers but also the quality of this knowledge in terms of how clearly it is understood and represented. This seems to be the case in solving physics problems (see Lesgold, 1988) as well as in developing mathematical proofs (Weber, 2001).

Transfer

Another question that has been addressed in relation to the domain-specific view of problem solving is whether knowledge that is generated in one domain can be useful in other domains. A number of studies have pointed towards the idea that knowledge in one domain does not transfer easily to other domains and that solvers find it difficult to generalise from one situation to another (e.g., Hayes and Simon 1977). Gick and Holyoak (1980), for instance, conducted an experiment that suggested that knowledge that has been acquired in one situation is not necessarily applied spontaneously in other domains. They suggested that although the solver may have the capacity to use information from one domain into another, actually using it is not something that solvers usually do.

Hunt (1991) suggested that, in the light of the evidence that studies on problem solving provide, there is little hope for a global theory on problem solving. According to the author, what can be concluded from studies on problem solving is that being 'concrete' (i.e., procedural), rather than conducting pure reasoning at all times, seems to be a common strategy when it comes to dealing with a new situation:

> The picture that emerges is one of a solver who has memorized a huge number of tricks of the trade, rather than one who has derived a powerful strategy for reasoning. Local optimality acquired at the expense of global consistency. This bothers people (particularly academics) who like to think of themselves as Homo sapiens. Shouldn't we value pure reasoning?
>
> (Hunt, 1991, p. 393)

Hunt suggests that seeing problem solving as an activity in which the solver's strategies are determined by the domain leaves little room for a general theory on problem solving. Considering the results of studies on problem solving, he argues that it would be better to focus on how expertise is acquired rather than on describing how problem solving is conducted in different domains. However, the present study can be used as an example to argue against Hunt's view. Although this study can be considered as domain-specific in the sense that it looks at the particular domain of problem solving, it maintains that it is possible to provide a 'conceptual' view of how students solve problems.[3] Other researchers interested in 'complex problem solving' argue that a global theory of problem solving can be developed by studying how solvers perform in what they called 'complex' situations. This approach will be discussed next.

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[3] This conceptual account can be achieved by using an orthodox grounded theory approach. This approach places emphasis on conceptualisation rather than on description of the situation being observed (Glaser, 2001). For instance, instead of describing how students tackle problems, the present study focuses on global issues such as 'reducing complexity'. The present study suggests that students 'reduce complexity' (Section 4.2.1) to deal with situations that appear difficult to handle. This general behaviour has been identified in other studies (e.g., Hazzan, 1991; 2001) and it seems feasible that this strategy is also employed in problem solving in other contexts. Thus, the stance adopted here does not support the idea that domain-specific studies cannot contribute to global theories of problem solving.

2.4.2. Complex problem solving: The aim for a general theory of problem solving

Complex problem solving, or CPS, differs from the domain-specific approach in that its aim is to generate theories of problem solving that are applicable across different contexts. To do this, CPS researchers explore problem solving through activities designed to make the solver's previous knowledge as irrelevant as possible for dealing the problem. These activities usually involve a considerable number of variables that change over time and which effects are not easy to measure. Furthermore, these activities are ill-defined in the sense that they do not have clearly defined goals, initial states or conditions (Funke and Frensch, 1995a). According to CPS supporters, this allows them to explore mental processes that are not involved when 'simple' tasks are presented. Also, this type of tasks allows them to explore problem solving behaviour as a general activity and not as a cognitive process that is described in terms of the specific domain in which it takes place.

Brehmer and Dorner (1993) suggested that the dilemma associated with investigating complex problem-solving situations in simple, laboratory tasks and in complex, real-life situations can be resolved by using computers as a tool for research. They argued that, by generating computer tasks that can simulate complex situations, and by having solvers trying to 'resolve' them, it is possible to study complex problem-solving behaviour in the laboratory without being restricted to simple tasks.[4]

Thus, a common characteristic of CPS studies is that problem-solving is usually explored through microworlds (Quesada et al., 2003). Microworlds are computer environments that can simulate complex fictitious situations involving a large number of interdependent and constantly-changing variables. An example of these environments is the "Firechief" microworld (Omodei and Wearing, 1995), a computerised problem-solving program that requires solvers to extinguish a forest fire. The software provides the solver with resources and with a changing situation that places different demands during the solving process.

In spite of CPS researcher's efforts to generate a unified approach to problem solving, critics suggest that this approach has not been concretised (Quesda et al., 2004). Funke and Frensch's (1995b) compilation of CPS studies can be seen as an attempt to provide a unifying picture of problem solving from the European perspective. However, it seems that the boundary between CPS and other types of problem solving remains largely undefined, as is a global theory of problem solving.

In sum, it may be said that since the Gestaltist psychologists proposed that

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[4] It may be pointed out that the present study shares with CPS the aim of looking at complex problem-solving situations. However, whereas the CPS approach places emphasis on making content-knowledge as irrelevant as possible, this study seeks generate a substantive theory of problem solving. For this reason, the present study can be better classified as a domain-specific study (but with an aim that is not related to the information-processing approach).

problem solving needed to be studied as a *productive* rather than a reproductive activity different approaches to the study of this activity have emerged. The information-processing approach aimed at making the study of problem solving more 'scientific' by placing emphasis on the research methods employed. It seems that using different methods affected both the questions asked and the answers provided to these questions. The information-processing approach was criticised for, among other things, focusing on studying problem solving in the context of well-defined problems. This, critics argued, led to theories that were non-generalisable, and consequently to the emergence of other approaches. The study of problem solving from a domain-specific perspective and from the perspective of complex, context-free situations are the most salient of such alternative views.

Since the aim of this study is to provide a substantive theory that explains students' main concerns as they tackle non-routine mathematical problems, it is important to discuss problem solving from a mathematics education perspective. Mathematics education has been influenced by research in psychology and the cognitive sciences. However, as it will be discussed, mathematics education researchers have also developed their own frameworks that guide present and further research.

2.5. PROBLEM SOLVING IN MATHEMATICS EDUCATION

To be sure, research in mathematical problem solving has not evolved as a subcategory of the cognitive sciences. However, advances in psychology and artificial intelligence have provided mathematical problem solving with frameworks for shaping its questions and methods of research (see Anderson, Reder and Simon, 2000). For instance, the influence of the behaviourist perspective to the study of thinking can be observed in the approach taken by studies conducted in the 1950's, 60's and 70's. In these studies, mathematical problem solving was usually approached from a stimulus-response perspective. The hypotheses that underscored these studies focused, in general, on the relationship between the types of problems and the rate of correct/incorrect responses provided by the solvers (Bell, 1979).

Educational research during the 1950's and through the 1970's focused particularly on quantitative analyses of the product of the mathematical activity (Schoenfeld, 1987a). However, starting in the mid seventies, researchers became aware of the need for approaches that could shed some light into the processes of dealing with mathematics and not only on their products (Ibid). This led researchers to look into the cognitive sciences for theoretical approaches that could be used for studying mathematical performance.

Schoenfeld's (1985a, 1985b) work can be considered as pioneering in adopting a systematic approach to the study of problem-solving processes. In *Mathematical Problem Solving* (1985b) he proposed a framework that can be used for analysing problem-solving behaviour. This framework consists of four areas defined as:

> *Cognitive resources*, the body of facts and procedures at one's disposal; *heuristics*, 'rules of thumb' for making progress in difficult situations; *control*, having to do with the efficiency with which individuals utilise knowledge at their disposal; *belief systems*, one's perspective regarding the nature of the discipline and how one goes about working in it.
>
> (Schoenfeld, 1985b, p. xii)

Widely used as a basis for other studies on problem solving (see, for example, Defranco, 1996; Lerch, 2004), Schoenfeld's four-element framework can be seen as a series of hypotheses derived from two sources. On the one hand, the framework was based on previous studies on problem solving conducted by psychologists and other cognitive scientists. The influence of information-processing theories is evident in the framework, particularly the work of Newell and Simon (1972). On the other hand, the framework incorporated the results of a number of studies that Schoenfeld had conducted in the decade prior to the publication of his 1985 book.

The introduction of the use of verbal protocols to mathematics education can also be, at least in part, attributed to the influence of his work (see Schoenfeld, 1985a). Due to the importance of Schoenfeld's work in contemporary studies on mathematical problem solving, the elements of his framework are discussed in detail.

2.5.1. Schoenfeld's framework for looking at mathematical problem solving

As previously indicated, Schoenfeld's framework consists of four areas that are considered to be important in mathematical problem solving. The first area that he discussed refers to the cognitive resources that students have at their disposal for solving a problem. Schoenfeld suggested that knowing what facts, procedures and skills students can use as they solve a problem is necessary to generate the 'problem space', i.e., the model of the possible routes that can be followed to solve the problem (see Section 2.3.1). He explained that:

> an inventory of what individual problem solvers know and the ways in which they access that knowledge is essential if we are to understand what takes place in a problem-solving session.
>
> (Schoenfeld, 1985b, p. 46).

The second area that Schoenfeld discusses is 'heuristics'. Heuristics are 'rules of thumb' or methods that solvers employ as general strategies for solving problems. Schoenfeld suggested that there is a degree of overlap in the heuristics employed by expert problem solvers, even if they approach novel situations in different ways. He argued for the identification and characterisation of these strategies and suggested they be defined in a level of detail that could make them useful for students. He suggested that previous failures to help students become better solvers through teaching strategies such as those suggested by Polya (1957; see Section 2.6 below) were, in part, due to the generality in which strategies were expressed. Schoenfeld proposed that this situation could be improved by increasing the degree of detail in which these strategies were described (see also Schoenfeld, 1987a):

> Despite the fact that heuristics have received extensive attention in the mathematics education literature, heuristic strategies have not been characterised in nearly adequate detail... In most studies, the characterization of heuristic strategies was not sufficiently prescriptive. Not nearly enough detail was provided for the characterisation to serve as guides to the problem-solving process.
>
> (Schoenfeld, 1985b, p. 73-74)

Control, defined as the way solvers use the information at their disposal, is the third element of Schoenfeld's model. Control refers to metacognitive or managerial decisions that solvers make as they tackle problems. These decisions refer to what students consider needs to be done to achieve a solution. Defining goals and subgoals, choosing a strategy, evaluating and assessing results, and making plans are all examples of metacognitive decisions. Schoenfeld suggests that having a large number of detailed heuristics at one's disposal is only effective if they are efficiently managed.

The fourth element of Schoenfeld's model refers to *belief systems*. The author suggests that:

> The performance of most intellectual tasks takes place within the context established by one's perspective regarding the nature of those tasks.
>
> (Schoenfeld, 1985b, p. 35)

In other words, he suggested that the way problem solving is conducted is partly determined by the beliefs a person holds, particularly in relation to mathematics and to what it means to 'do' mathematics. However, how this happens is not made clear.

Schoenfeld's framework represents one of the most systematic efforts at modelling problem-solving behaviour in mathematics (Lester, 1994) and has been considerably used in other studies (see Section 2.5.2 below). However, the framework does not establish how each of its four elements interrelate during problem solving and thus does not explain problem solving 'as a whole'. The relationship between the elements of the framework is implicitly suggested by Schoenfeld but not clearly addressed. It may be said that the studies described in *Mathematical Problem Solving* (1985b) were not aimed at gathering information about interrelation but were aimed at testing particular aspects of the framework. For instance, in one study, Schoenfeld investigated the use of heuristics on students that had taken a problem-solving course. In another study, the author made a comparison between expert and novice solvers. Through these studies, Schoenfeld was able to gather evidence supporting the importance of the elements of his framework but not how the elements interrelate.

Another criticism that can be made to Schoenfeld's framework is that the four areas that it considers do not necessarily fit what students are trying to do during problem solving. These areas were generated taking the results of previous studies as a basis but they do not always relate to what students actually do (see Section 2.5.2 below). To avoid this pitfall, the present study adopts an approach that focuses on students' concerns as they solve problems and in explaining how they are resolved (see Chapter 3).

2.5.2. Studies related to Schoenfeld's framework

The studies that Schoenfeld presents in *Mathematical Problem Solving* (1985b) are partly based on the methodological framework that he introduced in the first part of this book (although the studies helped to shape the framework as well). These empirical studies are, in many cases, based on the type of studies that were being conducted in the cognitive sciences at that time. For instance, Schoenfeld conducted an adapted version of a study in which Chi, Feltovich and Glaser (1981) compared the way experts and novices categorised physics problems. In his study, Schoenfeld compared how 'experts' (faculty members) and 'novices' (undergraduate students) sorted a set of 32 mathematical problems "accessible to students with a high-school background in mathematics" (Schoenfeld, 1985b, p. 248). Like Chi, Feltovich and Glaser, Schoenfeld found that experts seemed to classify problems by their 'deep structures' whereas novices based their categories on the 'surface structures' of the problems. Furthermore, the way in which Schoenfeld designed the experiment allowed him to determine whether the students' status of 'novice' improved after a course on problem solving. In relation to this, Schoenfeld concluded that:

> This comparison of surface and deep structure proportions provides an indirect indication that the experimental's group problem perceptions became more 'expert-like' as they became better problem solvers.
>
> (Schoenfeld, 1985, p. 258-259)

In another set of studies, Schoenfeld compared the verbal protocols of experts and novices as they solved problems. In these studies, the author noticed important differences in the executive or control skills that they displayed. He argued that experts and novices used their time differently during problem solving. In the case of novices, the time was mostly spent on reading the problem, choosing an approach and working on it for most of the solving session. As for experts, they not only took more time to analyse possible approaches, but also showed evidence of monitoring the activities that were conducted. In all, the author suggested that: "For the most part, students are unaware of, or fail to use the executive skills demonstrated by the expert." (Schoenfeld, 1992, p. 356).

Defranco (1996) further studied the notion of expert mathematical problem solver suggested by Schoenfeld (1985b; 1992). In a first experiment, Defranco replicated Schoenfeld's work on problem-solving expertise by collecting information in the form of verbal protocols from six PhD mathematicians. This cohort was considered as a group of experts and Defranco expected that, based on Schoenfeld's work, their displayed problem-solving behaviour would include a positive expression of the following:

> Domain knowledge (the tools a problem solver brings to bear on a problem), problem-solving strategies (Polya-like heuristics), metacognitive skills (issues of control – selecting strategies and solution paths to explore or abandon, appropriate allocation of one's resources, etc), and a certain set of beliefs (a particular world view of mathematics).
>
> (Defranco, 1996, p. 196)

The resulting data indicated that the PhD mathematicians did not display the expected attributes of experts. In the light of this, Defranco proceeded to study the problem-solving behaviour of two groups defined in the following way:

> Group A – eight men who earned a doctorate or its equivalent in mathematics and have achieved national or international recognition in the mathematics community, and
>
> Group B – eight men who earned a doctorate in mathematics but have not achieved such recognition.
>
> (Defranco, 1996, p. 196)

By comparing the problem-solving behaviour and comments of these two groups, Defranco concluded that group A mathematicians displayed the problem-solving attributes that Schoenfeld assigned to experts. Group B mathematicians, on the other side, behaved more as novices. This led Defranco to suggest that being a problem-solving expert goes beyond possessing the content knowledge necessary for tackling mathematical problems. His study supports the idea put forward by Schoenfeld that "other attributes such as problem-solving skills, metacognitive skills, and a certain set of mathematical beliefs" (Defranco, 1996, p. 209) are also necessary.

In another study conducted by Lerch (2004) the author investigated the problem-solving processes of a group of undergraduate students. Although she stated that the purpose of her study was to build understanding of students' processes "from the students' perspective" (Lerch, 2004, p. 22), what she meant by this statement is not clear. Instead of analysing how students 'do' mathematics (her stated aim), the analysis was based on Schoenfeld's framework and on discussing how issues in relation to 'control and 'beliefs' were manifested in students' processes.

In relation to the issue of control, Lerch noted that: "None of the students exhibited the types of control decisions Schoenfeld (1985b) identified as important for successful problem solving."(Lerch, 2004, p. 28). In relation to students' belief systems, the discussion focused on the choices that students made during their processes and on relating those choices to students' interpretation of the problem or to the resources that they had at their disposal. In both cases, it may be said that Lerch's analysis seemed to be limited rather than enhanced by the theoretical framework used. Her analysis leaves the reader wanting to read more about what students actually did as they solved the problems. By discussing students' problem-solving process in terms of what they do not do suggests that interesting aspects in relation to what students do were actually left out. This is particularly true considering her aim to understand how students 'do' mathematics 'from their own perspective'.

In sum, Schoenfeld's framework has been used to investigate problem solving in a number of studies. Schoenfeld's own studies and the two studies presented above are examples of such investigations. Though the framework has helped to advance what is known about mathematical problem solving, it is apparent in some instances that researchers seem to employ the framework without considering its limitations or its appropriateness for their studies. It was suggested that one of the limitations of Schoenfeld's framework is that, although it proposes four constructs that can be used for analysing problem-solving behaviour, these constructs are not sufficiently interrelated. It is unlikely that merely providing more detailed descriptions of these elements will increase what we know about the interrelations among them. What may be needed is a more holistic perspective in which problem solving is not seen as a collection of elements but as interrelated patterns of behaviour. As for the present study, it may be said that adopting a grounded-theory approach provided the research methods necessary for investigating patterns of behaviour manifest as students solve problems. This resulted in the development of a model that highlights students' concerns and explains how these are constantly resolved. As a result, the model explains problem solving in a holistic way.

2.5.3. Other studies on problem solving

This section discusses other contributions to the study of problem solving. One such study is Garofalo and Lester's (1985) report on self-monitoring in mathematical performance. This study is related to Schoenfeld's (1985b; 1987a) work in that it suggests going beyond merely studying cognitive aspects of problem solving and proposes that metacognitive aspects should be carefully studied as well. Garofalo et al. proposed a framework for studying problem-solving performance that includes four main activities: orientation, organisation, execution and verification. As the authors explained, these activities are based on Polya's (1957) four phases but are more broadly defined in the sense that they make reference to specific metacognitive behaviours associated with each activity.

Garofalo et al.'s framework has been used in other studies of problem solving. For example, Pugalee (2004) used it for describing the problem-solving processes of ninth-grade algebra students and Stillman and Galbraith (1998) used it both in the design of the methods of data collection and in the analysis of the data for their study. In spite of its proven applicability, it can be suggested that the application of the framework may limit the analysis of important problem-solving behaviours. For instance, in these two studies, the use of the framework led to restricting the analysis of verification strategies to those conducted at the end of the process. Using the framework, both Pugalee and Stillman et al. concluded that students do not usually verify their solutions at the end of the process. However, they suggested that there was some verification being conducted *during* the process rather than at the end of it but, restricting to the framework, they decided to focus only on the latter. In relation to the present study, it may be said that this sort of limitations was avoided by adopting an approach that allows the researcher to focus on those issues that emerge as important during the course of the study (see Section 3.1.2).

Taking a different approach to Garofalo et al., Nunokawa (1994, 2001), investigated problem solving from the perspective of the "problem-solver's structures for representing a problem situation" (Nunokawa, 1994, pp. 275-276). He believed that investigating problem solving from the perspective of how solvers understand or make sense of the problematic situations can shed insight into the general process that allows them to solve problems. In his earlier paper, he suggested that the solver's structures are continuously being modified during problem solving. This process takes place until the problem is represented in such a way that the solver can apply his or her knowledge to the situation and thus generate a solution. Nunokawa (2001) also explored the relationship between the solver's structures and the goals and subgoals established during the process. He concluded that setting subgoals can help the solver to construct better structures but that it can also delimit their scope.

Nunokawa's work can be seen as an effort to study problem solving in a holistic rather than in an atomistic way; or, in other words, as an effort to explain problem solving from the perspective of one global activity instead of looking at its components in isolation. In this way, his aim to explain problem solving in terms of an activity that makes the process possible bears resemblance to the approach adopted by Gestaltist psychologists. Although Nunokawa proposed an interesting view of problem solving, the model was based upon close analyses of individuals' problem-solving processes and no claims were made about the generalisability of the ideas suggested.

In contrast to the approach adopted by Nunokawa, some studies focus on particular aspects or isolated activities of the problem-solving process. Stylianou

(2002), for instance, looked particularly at the visual representations used by an experienced problem solver (a mathematician) and at the process of generating them. As the author pointed out, representations were used by the solver as a way of gathering the resources to be able to solve the problem. Thus, representing the situation in a visual way is not an end but a means for achieving a particular aim. As Stylianou put it:

> For the mathematician who participated in our study, the drawing of a diagram was not a goal in itself but a means to aid them in gaining more information of the problem situation.
>
> (Stylianou, 2002, p. 310)

Moreover, Stylianou used problem-solving problems specially chosen for observing visual representations. She chose problems "such that their solution invites, or can be facilitated by the use of a visual representation" (p. 307). This serves her purpose of studying a particular aspect of problem solving in a somewhat isolated way. However, this approach also raises some interesting questions. It has been suggested that studies that look at isolated problem-solving activities may lead to incomplete analyses of problem solving. Hunt (1991) suggested that studying different aspects of problem solving with the aim of putting the obtained knowledge together may not serve to increase our understanding of how problems are tackled. First, the aim of looking at isolated activities is usually unclear, unless it is to merely describe these activities. Second, looking at them separately does not add a lot to what we know of problem solving unless they are put back together and explained as part of a more general process.

Thus it may be said that although a number of studies have contributed to the study of problem solving in the past, few have aimed to provide a view of problem solving as a whole. Moreover, it may be said that no studies have been identified as aiming to provide a theory that explains students' main concerns, as the present study does.

2.6. FRAMEWORKS FOR TEACHING PROBLEM SOLVING

An important section of the literature in mathematical problem solving comprises frameworks that were developed with didactical rather than theoretical or research purposes. Polya's book *How to Solve It* (1957) can be considered as one of the most influential frameworks for teachers and students (Schoenfeld,1987b; Lester, 1994). The aim of this book was to provide advice and a list of strategies, summarised in four stages, that teachers could use to help students become better problem solvers:

> First, we have to *understand* the problem; we have to see clearly what is required. Second, we have to see how the various items are connected, how the unknown is linked to the data, in order to obtain the idea of the solution, to make a *plan*. Third, we *carry out* our plan. Fourth, we *look back* at the completed solutions, we review and discuss it.
>
> (Polya, 1957, pp. 5-6; emphasis in the original)

However, the strategies that Polya provides in *How to Solve It* go beyond this simplistic process. The book also provides a comprehensive 'dictionary of heuristics' that includes more detailed strategies and further practical advice.

In spite of the enthusiasm with which Polya's framework was adopted, it has been suggested that providing strategies that describe how expert mathematicians solve problems is of little use for helping students become better solvers. Schoenfeld (1985b; 1987a, 1987b) suggested that the problems with Polya's strategies was that they were too general for students to apply them directly and that they needed to be detailed to a prescriptive level of detail if they were to be of use. In relation to this, Stewart (1989) argued that *How to Solve It* should not be read as a prescription for problem solving but as a book that provides general advice as to how to solve problems. Advice which, as he said, might be useful for the more receptive students but of little use to students who expect to become better solvers by following some rules by rote.

Another important problem-solving framework that has influenced mathematical problem solving is the one developed by Mason, Burton and Stacey (1982). The present study was conducted in the context of a problem-solving course based on this framework. The details of the framework will be discussed elsewhere (Section 3.1.1.2). The important thing to mention here is the fact that Mason et al., as well and Polya's framework, have influenced mathematics teaching more than frameworks developed in a more 'scientific' way (in the sense that they provide generalisations and that these are arrived at by using a systematic research methodology; see Lewins, 1992). This may be in part due to the fact that theoretical frameworks and didactical frameworks are generated with different aims in mind. While the aim of didactical frameworks is to help students become better solvers, the aim of theoretical frameworks is to develop understanding of how students solve problems. Ideally, the latter aim should also provide information useful for helping students become better solvers. However, it seems that this does not necessarily occur in a direct way.

An important characteristic of the effect of frameworks for teaching problem solving is that their application has been the source of inspiration not only for teaching but for empirical research as well. The aim behind the latter can be to evaluate or further develop the framework being used. In the case of Polya's framework, researchers have found that it is not sufficient to teach strategies in order for students to become better solvers (see Schoenfeld, 1987b). In relation to Mason et al.'s framework, it has been suggested that teaching it may benefit students. Mohammad-Yusof, (1995, 1998) found that a course based on this framework had a positive effect on students' beliefs and attitudes towards doing mathematics (although these did not endure the influence of other more 'traditional' courses and students seemed to change their attitudes and beliefs back to 'normal' some time after taking the course). As for the present study, it took place in the context of problem-solving course based on Mason et al.'s framework. Although the aim is not to evaluate the effect of the course on students but to gain understanding on students' problem-solving processes, the use of the framework is an important aspect of the context of the study.

Glaser (1976) suggested that the theoretical ideas generated in laboratory-like situations cannot be directly translated into didactic situations. One reason for this is that the situations from which they were derived are simplified versions of the reality in which learning and instruction take place. However, it seems that the situations created by the application of non-'scientific' problem-solving frameworks provide an opportunity for studying problem solving in a more 'natural' context and not merely through simplified tasks. For example, like in the present study, problem-solving situations can be designed in such a way that they are not limited to twenty- or thirty-minutes tasks. Studies based on situations like this may allow the generation of models that are scientific and yet tied to the practice since the outset. This can be achieved through methodologies such as grounded theory that allow the researcher to model patterns of behaviour and to generate explanations of complex situations (see Chapter 3).[5]

2.7. SUMMARY AND THE PRESENT STUDY

This chapter provided a view of the study on problem solving, first from the perspective of the cognitive sciences and then from the perspective of mathematics education. As discussed in Section 2.2, the Gestaltist approach to the study of problem solving emerged as a reaction to a view of problem solving as a stimulus-response activity. Gestaltist psychologists defined problem solving as a sudden reorganisation of ideas that allow the solver to 'see' the solution. For them, problem solving was a productive activity that could be explained as a

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[5] This position is not against Anderson, Reder and Simon's (2000) idea that the study of mathematical learning can benefit from researching isolated cognitive tasks. What this study argues is that the study of complex problem solving behaviour requires different methodological and theoretical approaches. Some have been developed through what can be called the post-information-processing approaches discussed above (Section 2.4). This study proposes the use of yet another approach.

four-stage process: preparation, incubation, illumination and verification. Gestaltists paid special attention to the illumination phase and elaborated on issues such as insight and intuition. In doing this, it may be said that they neglected looking more deeply into other aspects of problem solving such as how information is generated or how solutions are verified. As for the present study, it may be said that the methodological approach adopted is such that it allows relevant factors to emerge and provides methods for establishing relationships among them. In doing this, it remedies the suggested limitation of focusing on the discovery aspect of problem solving.

The information-processing approach to problem solving emerged in part as a reaction to the non-rigorous research methods employed by Gestaltist psychologists. However, as discussed in Section 2.3, other factors such as the advances in computer technology had an important effect. The information-processing approach focused on problem solving as a cognitive process which activities can be decontextualised and studied in isolation. An important characteristic of the information-processing approach is that some of its major theories represented computer simulations of human problem-solving processes. Such is the case of Newell and Simon's (1972) influential theory. Information-processing theories were criticised for providing accurate descriptions of problem-solving situation but failing to explain how the process took place. However, possibly the most important criticism made to these theories was that they were based on studies that made use of well-defined (or laboratory-like) problems. This raised questions regarding the generalisability of their results and led to different approaches.

Section 2.4 discussed two approaches that emerged as a result of criticisms made to the information processing approach, particularly in relation to criticism made to the types of problems employed. This criticism led to two different perspectives: the domain-specific approach to problem solving and the study of 'complex problems solving', or CPS. The domain-specific approach seemed to abandon the aim of generating a global theory of problem solving and focused on looking at problem solving in situations where content-knowledge is important. As for the CPS approach, it maintained the aim of generating a content-free theory, relevant to problem solving in general situations. It may be said that the present study can be related to the study of problem solving from a domain-specific perspective. Nonetheless, this study does not adhere to the information-processing approach but adopts a methodology that allows the conceptualisation (rather than the description) of patterns of behaviour observed in students' problem-solving processes.

After discussing problem solving in general, Section 2.5 looked at problem solving studies in mathematics education. Although research into mathematical problem solving has not evolved as a subcategory of the cognitive sciences, mathematic education has had a considerable influence from the latter. This influence can be observed in studies such as Schoenfeld's (1985a, 1985b, 1987a). Schoenfeld's work represents one of most serious efforts to study mathematical problem solving in a rigorous way and it has had a strong influence in other studies. Other important studies on problem solving include Garofalo and Lester's (1985) metacognitive framework and Nunokawa's (1994, 2001) explanations of problem-solving from the perspective of the 'structures' that students use for solving problems. These studies have certainly served to advance what is known about problem solving in mathematics education. However, it seems that a holistic and generalised explanation of problem solving has not been proposed yet. The present study aims to contribute to the study of mathematics education by proposing a model of problem solving that focuses on students' concerns as they solve problems and on explaining how these are resolved. In doing this, the model not only raises concepts (in the form of conceptualised patterns of behaviour) but it also suggests how they interrelate.

Finally, Section 2.6 looked into the role that frameworks for teaching problem solving play in mathematics education. It was suggested that these frameworks provide models that can be used to help students become better solvers. Moreover, these frameworks can provide valuable opportunities for investigating mathematical problem solving in organised, yet non-laboratory situations. The present study took place in the context of a problem-solving course that made use of Mason et al.'s (1982) framework for problem solving. In order to make sense of complex problem-solving situations without needing to oversimplify them, the study was conducted using the grounded theory approach.

3. METHODOLOGY

The aim of the study is to provide a *substantive* theory that explains students' main concerns as they tackle non-routine mathematical problems. This was done in the context of a problem-solving course offered at the university (undergraduate) level. The previous chapter provided a picture of how other studies have served to increase what we know about problem solving and suggested how the present study aims at making a contribution in this area.

This third chapter discusses the methodology that was used to achieve this aim and that guided the present study. The chapter consists of three parts, each of which considers methodological issues that can be considered as 'chronologically' different:

- Part I (Section 3.1) – Discusses the context of the study and the first decisions that were made in relation to the methodology and the methods of data collection.
- Part II (Section 3.2) – Explains how an 'exploratory' and a 'pilot' study were conducted to further define the methods of data collection and explore the methods of data analysis. Its aim is to describe how the evolution of this study's methodology took place.
- Part III (Section 3.3) – Presents the methods of data collection and analysis as a result of what was discussed in Parts I and II. These methods were employed for collecting and analysing the data during the 'main' study.

Sections 3.1, 3.2 and 3.3 are written in such a way that they can be read almost independently. In fact, it is possible to read only Part I or Part III of this chapter and get an idea of the methodology. Part II is intended for those interested on how the methods of data collection were developed and how grounded theory was adopted. Thus, the reader will note that some issues that are raised in Part I are considered again in Parts II or III. Every effort was made to ensure clarity of the ideas discussed while at the same time keeping repetition to a minimum.

3.1. PART I: METHODOLOGY AND METHODS OF DATA COLLECTION

This first part of the methodology chapter considers the context in which the study took place and the methodological approach that was taken. As Section 3.1.1 discusses in detail, the study took place in the context of a mathematics problem-solving course at undergraduate level. Section 3.1.2 discusses the rationale behind choosing grounded theory as the methodology for this study and provides a general view of the methods involved.[6] The third section of this part, 3.1.3, looks at the type of data that was considered suitable for the study and explains why qualitative methods of data collection were chosen. This short section leads directly into Part II of the methodology chapter, where the methods of data collection and analysis are further explored.

3.1.1. The context of the study

The aim of this section is to provide a picture of the context in which the study took place. It provides details concerning the course, including a general profile of the students. Other details include a description of the problem-solving framework upon which the course is based and an explanation of the technique of rubric writing. The type of problems that students dealt with during the course are also considered.

[6] Further details of the methodology are provided in Parts II and III of this chapter (Sections 3.2 and 3.3, respectively).

3.1.1.1. *The problem-solving course at Warwick*

The problem-solving course started to be offered at the University of Warwick shortly after the publication of Mason, Burton and Stacey's book *Thinking Mathematically*, in 1982. The course was developed by David Tall, then senior lecturer in Science Education at Warwick, as an option open to mathematics and computer-sciences students (David Tall, personal communication). It was structured as a ten-week course aimed at providing students with the opportunity to reflect on their own problem-solving processes and to develop personal problem-solving strategies.

Originally, the problem-solving course was offered to second- and third-year undergraduate students doing mathematics, physics and computer science degrees. In the academic year 2002/2003, however, it was decided that the course would also be offered to first-year BA(QTS) (i.e., teacher-training) students. During this academic year, the proportion of mathematics, computer-sciences and BA(QTS) students was roughly the same.

The course was organised in the following way. During the weekly sessions, students were introduced to some aspect of Mason et al.'s (1982) framework. Then, to practice the ideas to which they were introduced, students were also required to tackle one or more problems and asked to keep a record of their work in the form of a 'rubric'. The latter problem-solving activities were an important part of the course and thus students were also asked to tackle problems at home. Discussing solutions in the light of the framework was another important activity that usually took place either at the beginning or at the end of each session.

The assessment of the course was made through a final assignment and a written examination. Both methods of assessments comprised a problem-solving section and a section where students were required to critically discuss their problem solving processes and to comment on the effectiveness of their strategies.

The course continues to be offered at the University of Warwick. Over the years, different lecturers have been in charge of it and this has resulted in variations in specific aspects of the course. However, the course continues to be the same in essence and students' active participation in the course continues to be especially important. Further specifications of the course will be provided throughout this chapter (including Parts II and III), particularly in relation to those years in which the present study was conducted.

Due to its relevance in understanding the course, Mason et al.'s framework for solving problems is discussed in detail in the next section. Other important aspects of the course such as rubric writing (the technique for keeping a written record of one's problem-solving process) and the types of problems that students tackled during the course will be considered in the subsequent subsections (3.1.1.3 and 3.1.1.4, respectively).

3.1.1.2. Mason, Burton and Stacey's problem-solving framework

One of the strengths of Mason et al.'s framework is that it provides students with a view of mathematics as a creative process in which they too can take part. The framework provides students with some practical advice as to what to do when feeling 'stuck' and unable to make any progress. It also provides some useful suggestions as to what to do when ideas seem to be emerging. The aim of this section is to provide a summarised picture the framework to which students were exposed. For a detailed picture of the framework, the reader may refer to *Thinking Mathematically*.[7]

In general, the framework that Mason et al. present in *Thinking Mathematically* consists of three processes: 'specialising', 'generalising' and 'conjecturing', and three phases: 'entry', 'attack' and 'review' (see Figure 1). The processes represent actions that can be help students generate information and move between phases. Rubric writing can be considered as part of the framework since it is the media in which ideas are communicated and stored for later review. Due to its importance in terms of data collection for this study, rubric writing will be considered in detail in a Section 3.1.1.3.

[7] In terms of the problem-solving course, the framework provided students with ideas as to what to do in order to solve problems. However, the aim of this study is to *explain* how students solve problems and not to explore the influence of the framework in a direct way. It is expected that the model that will be developed will show evidence of Mason et al.'s work. However, the framework did not guide the development of the model.

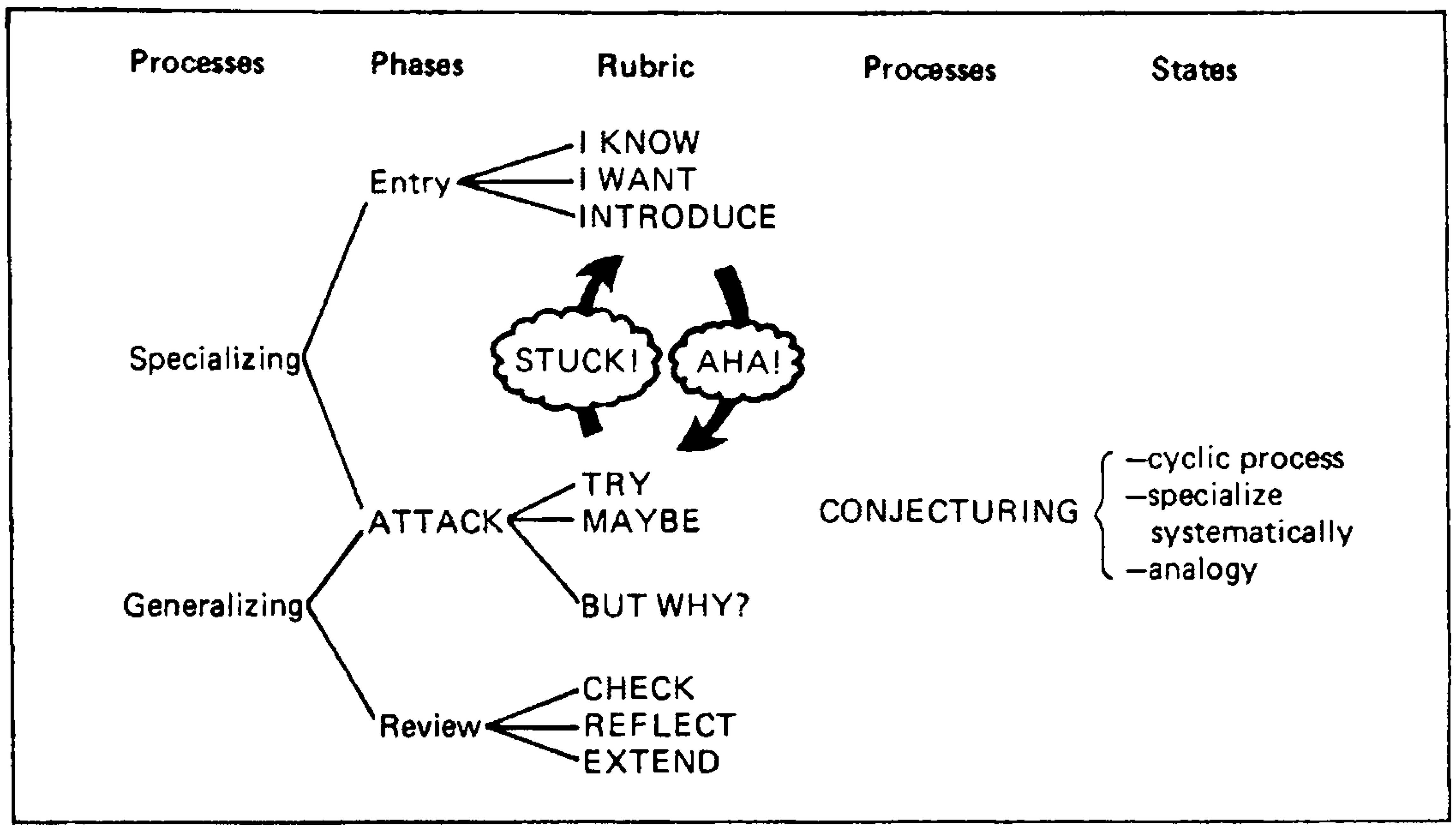

Figure 1: Mason et al.'s problem-solving framework

Specialising and generalising

Specialising means "turning to examples to learn about the question" (Mason et al., 1982, p. 3). Generalising, on the other hand, is about noticing underlying patterns in the data, even if they cannot as yet be articulated. Specialising can be done with the intention of getting a feel for the question. However, it may also be used for gathering information for specific purposes such as generalising or for testing a generalisation. The patterns noticed through generalising may lead to:

> WHAT seems likely to be true (a conjecture);
>
> WHY it is likely to be true (a justification);
>
> WHERE it is likely to be true, that is, a more general setting of the question (another question!).

(From Mason et al., 1982, p. 24)

Conjecturing

When a pattern is articulated, it usually becomes a conjecture. As Mason et al. suggest, conjecturing can be considered as "the recognising of a burgeoning generalisation" (Mason et al., 1982, p. 82). Conjectures are statements that are not proven or justified but that seem reasonable. During the course of a solving process, some conjectures are more relevant than others and most may need to be modified in one way or another. In Mason et al.'s words:

> Conjectures… form the backbone of mathematical thinking. Some property is thought to be true. A conjecture about it often begins as a vague feeling lurking in the darkness at the back of the mind. Gradually it is dragged forward by attempting to state it as clearly as possible, so it can be exposed to the strong light of investigation. If it is found to be false it is either modified or abandoned. If it can be convincingly justified, then it takes its place in the series of conjectures and justifications that will eventually make up the resolution.
>
> (Mason et al., 1982, p. 64)

The Entry phase

Although it is not uncommon for students to read the question and then jump straight into trying to find a solution, Mason et al. suggest that achieving a solution in this way is rarely possible. Thus, the authors suggest that an entry phase can and should exist in order to prepare for better attacking the problem. The entry phase is described as follows:

> Entry begins when I am faced with a question. Usually the question has been written down, so most of the Entry phase can be summarised as **really read it**! In other cases, the question presents itself, perhaps from work on another question or from a situation outside mathematics. Then the Entry phase work is largely in formulating the question precisely and in deciding exactly what I want to do...
>
> The other activity that usually takes place during entry is to make some technical preparations for the main attack, such as deciding on a notation, or a means of recording the results of specialising.
>
> (Mason et al., 1982, pp. 28-29; emphasis in the original)

In general, the entry phase consists of finding out what the problem is asking (or how the question will be defined) and gathering information for the next phase. Asking questions such as "What do I KNOW?", "What do I WANT?" and "What can I INTRODUCE?" can be helpful for guiding the entry phase.

The Attack phase

The entry phase, together with specialising and generalising, may swiftly lead to articulating conjectures that seem to be true. The attack phase is the next phase of the process in which the aim is to distil the 'underlying structure' (see Mason et al., 1982, pp. 85-94) and to provide a justification aimed at convincing not only yourself but also a 'friend' and if possible an 'enemy'. This phase consists mainly of investigating why a conjecture must be true and trying to provide a convincing justification.

The Review phase

The review phase may take place as a solving session is about to be concluded, regardless of whether a satisfactory solution has been obtained or not. Before leaving the process, solvers are advised to review their work and to look back at it. The intention of the review phase is to learn from what was done and to suggest possible ways of improving the results when reconsidering the problem (if and when this is possible). Mason et al. summarise the review phase in three activities:

> CHECK the resolution
>
> REFLECT on the key ideas and key moments
>
> EXTEND to a wider context
>
> (Mason et al., 1982, p. 39)

The review phase may be particularly useful when students take breaks from their processes. Reviewing their work and highlighting what has been achieved so far can make it easier for students to come back to the problem and to continue working at it. This aspect of the framework is made possible through rubric writing, a technique for keeping a written record of one's problem-solving process.

3.1.1.3. Rubric writing

Rubric writing is an important aspect of Mason's framework. Students were introduced to this technique for recording mathematical experiences during the course and required to practice it. Rubric writing was of particular importance to the study since the product of this activity (i.e., students' 'rubrics') became one of the main sources of data.

During rubric writing students are required to write down the ideas that occur to them during the process together with their thoughts and feelings about it. Rubric writing is not a description of how a problem was tackled but a written account of a mental process as it occurs. According to Mason et al., recorded experiences promote reflection and make available material that can be studied and analysed at a later time, thus providing an opportunity for developing mathematical thinking. During rubric writing, the aim is to record three things:

- All the significant ideas that occur to the student as he searches for a resolution to a question

Together with

- What the student is trying to do
- What the student is feeling about the situation.

(From Mason et al., 1982, p. 11)

In order to facilitate the process of rubric writing, students were introduced to terms such as 'Stuck' and 'Aha!' (see Figure 1 on Section 3.1.1.2). It was suggested to students that they use these terms to make the process of recording ideas easier and to facilitate the retrieval of key ideas once the rubric was complete. Furthermore, students were encouraged to develop their own vocabulary for monitoring and documenting their work. In this way, rubric writing allowed students to keep a record of their process that could be studied easily at a later time.

During the course, rubrics were also used for communicating ideas to others as well as for keeping a personal record of one's process. Rubric writing was not the only way in which students communicated their processes in the course. However, whenever this was the required method of communication, rubrics were comprehensive accounts of how a problem was tackled and of the results obtained. This study tapped into this fact and as a result rubrics were used as the main method of data collection.[8]

3.1.1.4. The problems

A comprehensive list of the problems that students had to tackle during the course are given in Appendix 1. These problems are 'non-routine' in the sense that they do not require students to deal with any particular mathematical topic or procedure. Also, these problems can be considered as 'ill-defined' in the sense that the goal, the initial state or the conditions are not necessarily clearly specified (see Section 2.4). The following is an example of such a problem:

[8] Before making this decision, the use of rubric writing as a method for collection was carefully investigated. For further details on this preliminary study see Section 3.2.1.2.

Cartesian Chase:

A game of two players is played on a rectangular grid with a fixed number of rows and columns. Play begins in the bottom left hand square when the first player puts a counter. On his turn, a player may move the counter one square up, one square right or one square diagonally (up and right). The winner is the player who gets the counter to the top right square.

Other problems may, at first sight, appear as well-defined. However, these problems are usually simple and students soon realise that there is a need to extend them and define more challenging situations. Thus, what initially may seem like a well-defined problem is in fact an ill-defined problem where the goal and the conditions need to be further specified. The following problem exemplifies this situation.

Ins and Outs:

Take a strip of paper and fold it in half (always placing the right hand edge on top of the left hand edge). Unfold it several times and observe the sequence of 'in' and 'out' creases. For example three folds produces:

in in out in in out out

What sequence would arise from 10 folds?

Answering the question posed by this problem is hardly challenging for any undergraduate student. However, students were aware of the need to regard this problems as one where a general answer was required and where predicting the sequence for 10 folds would not suffice. In general, students were expected to regard all problems as ill-defined even if, superficially, they appeared to be well-defined.

This section looked at the problem-solving course in which this study took place. As said, the course provided students with the opportunity to reflect upon their own problem-solving processes and had the aim of helping them become better solvers. Rubric writing was particularly important for the purposes of this study since it provided access to students' problem-solving process. Once the context of the study was defined, it became necessary to choose an appropriate methodology for studying how students solve problems during the course. Section 3.1.2 looks at the methodology that was chosen for the purposes of this study.

3.1.2. Grounded theory: First considerations

Grounded theory is an inductive approach through which data from complex social and psychological situations is analysed with the purpose of identifying the main concerns of those involved and how these are resolved (Glaser, 2001). An important aspect of the grounded-theory methodology is that its methods are designed for generating abstract conceptualisations of observed patterns of behaviour rather than detailed descriptions of the situation. Furthermore, the grounded theory approach offers a fully-fledged process for generating theory from the data.[9]

In the case of the present study, the aim was to provide a model of students' problem-solving processes. It was also established that the model should make reference to issues that appeared relevant to the participants and not just to the researchers as external observers. Since problem solving is a complex process in which a considerable number of different activities take place, an approach aimed at generating detailed descriptions (such as the information-processing approach) would not have been a feasible alternative. However, as Glaser's put it, grounded theories aim

> to generate a conceptual theory that accounts for a pattern of behaviour which is relevant and problematic for those involved. The goal is not voluminous description, nor clever verification.
>
> (Glaser, 1978, p. 93)

Thus, the grounded theory methodology appeared as a suitable option. The main concerns of students' as they solve problems had to be issues of relevance to them and this would assure a study of interest for others involved such as course tutors and researchers.

Grounded theory was first presented by sociologists Barney Glaser and Anselm Strauss in 1967. The methodology emerged as a reaction to a perceived over

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[9] Further details of the grounded theory methods are discussed in Part III (Section 3.3) of this chapter.

reliance on quantitative methods in sociological studies (Charmaz, 2000) and responded to a "preoccupation with the quantitative testing of propositions derived from a few highly abstract, 'grand theories'" (Pidgeon, 1996, p. 76). Grounded theory thus emerged not as a 'verificational' method but as an approach to data that would allow theory to be directly derived from it. In general, grounded theory is designed to allow relevant variables to emerge from the data and the result is a theory that fits the situation and serves to make grounded predictions about it (Glaser, 1978; Glaser and Strauss, 1967; Strauss and Corbin, 1990; 1998).

As the literature on grounded theory suggests, this methodology had two important effects on research in the social sciences and psychology. Firstly, as Pidgeon put it "it [grounded theory] became used (and cited) as a manifesto by researchers who wished to break out of the confines of existing types of theory" (Pidgeon, 1996, p. 79). As a result of this, grounded theory is sometimes taken as an umbrella term for a number of methodologies based on approaching data without a predefined theory or set of categories in mind (Lee, 2000). Secondly, grounded theory was also recognised as comprehensive methodology involving the systematic use of specific research techniques. This 'orthodox' view of grounded theory is particularly related to Glaser's (1978; 1992; 1998; 2001; 2003) work. In relation to the preliminary research conducted in this study, it may be said that grounded theory was taken as a general approach that made emphasis on allowing ideas to emerge from the data. As the study evolved, however, Glaser's more 'orthodox' approach was adopted.[10]

3.1.2.1. Brief outline of the methodology

The grounded theory methodology is based on two main activities: constant comparative analysis and memo writing (Glaser, 1992). The method of constant comparisons (as the former is also known) refers to the process of constantly comparing the data with the aim of generating hypotheses and abstracting patterns of behaviour. Memo writing is the process through which emergent hypotheses, as well as any other ideas about the data, are recorded for further comparisons. Comparing the data and memo writing are activities that continue to be in use throughout the study as other activities take place.

Grounded-theory studies are guided by a sequence of activities that incorporate constant comparative analyses and memo writing. The process through which grounded theories are generated can be summarised as follows:

- Grounded theory starts with 'open coding', which is a line-by-line analysis of the data designed to allow main categories to emerge. During open coding the researcher tries to find out what the participants are trying to do or, in other words, their main concern.

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[10] For a contrast between a general and a more 'orthodox' approach to grounded theory see Parts II and III of this chapter (Sections 3.2 and 3.3, respectively).

- Once the main concern of the participants is identified, the researcher concentrates on those categories that serve to explain it. Coding the data in this way is known as selective coding. During selective coding, the researcher 'theoretically samples' the data and focuses on those incidents that refer to the main concern.
- Coding and theoretical sampling result in an extensive memo fund that will need to be resorted. Sorting the memo fund leads to an outline that expresses the theory and contains its main categories. Once the memos are sorted, the theory has already been generated and then next step is 'writing-up' the theory.
- Writing-up is the last process of the grounded theory methodology, although it may take a few months before the theory is ready to be presented.

3.1.2.2. The outcome of studies using the grounded theory methodology

The grounded theory approach may be used to develop substantive or formal theories. Glaser and Strauss (1967) explained the difference between substantive and formal theories in the following way:

> By substantive theory, we mean that developed for a substantive area, or empirical, area of sociological inquiry, such as patient care, race relations, professional education, delinquency, or research organizations. By formal theory, we mean that developed for a formal, or conceptual area of sociological enquiry, such as stigma, deviant behaviour, formal organization, socialization, status congruency, authority and power, reward systems, or social mobility.
>
> (Glaser and Strauss, 1967, p. 32)

In this way, a theory of problem solving in a context such as the one considered in this study can be seen as substantive, as it belongs to a particular area of inquiry. On the other hand, a study that focuses on issues that may be relevant across a number of areas (such as 'blinding behaviours', or behaviours that disturb sense making in complex tasks) could be considered as formal.

Although Glaser and Strauss presented grounded theory as a methodology to be used in sociological inquiry, further accounts of the theory have suggested its suitability in other areas such as psychology and education (Haig, 1995; Glaser, 1992). In relation to the present study, it may be said that the grounded theory methodology will be used to generate a substantive theory of mathematical problem solving.

Once it was decided that the grounded theory methodology would be used in this study, the next question concerned the type of data that needed to be gathered. The next section looks into the methods of data collection considered for the purposes of this study.

3.1.3. The methods of data collection

The grounded theory approach to data collection can be summarised in the idea that 'all is data' (Glaser, 1978; 1998; 2001). In other words, grounded theory is based on the idea that if the aim is to gain understanding about a particular situation, any information that can help to increase understanding of it can be considered as data. The methodology is designed in such a way that it allows the researcher to incorporate any kind of data into the analysis process, if this data informs about the area or topic of study. Thus, the information that is gained through planned interviews or focus groups can be as useful as the information gathered in more opportunistic ways. This is also the case for the literature which, although it should not be initially considered, should be reviewed at later stages and should be treated as "more data for constant comparison" (Glaser, 1998, p. 67).

To be sure, grounded theory does not use statistical methods to arrive at generalised conclusions about the data. However, and consistent with the idea that 'all is data' mentioned above, grounded theory studies can make use of data arrived at quantitatively (Glaser and Strauss, 1967). As Glaser's (1992) put it:

> Qualitative analysis means any kind of analysis that produces findings or concepts and hypotheses, as in grounded theory, that are not arrived at by statistical methods. To repeat, qualitative analysis may be done with data arrived at quantitatively or qualitatively or in some combination. As we grounded theorists say when doing a book or paper and theoretically sampling for more data, 'It's all data for the analysis. Whether soft or hard it is just grist for the mill of constant comparison and analysing.'
>
> (Glaser, 1992, p. 11)

In the case of this study, however, collecting quantitative data was not considered as a viable option. As indicated before, the aim of the present study is to provide a model that explains students' main concerns as they solve non-routine mathematical problems. Collecting quantitative data and putting it in a suitable form for it to be analysed from the grounded theory perspective was seen as an unfeasible task. Furthermore, suitable data of this kind was not available from previous studies either. As for qualitative data, its potential for providing access to students' concerns as they solved problems made it more suitable for the purpose of this study. It was therefore decided that the best option was to make use of methods of qualitative data collection.

Once it was decided that study would make use of qualitative methods of data collection, the next step involved finding ways of accessing students' problem solving processes and generating suitable data. Problem solving is a mental activity that may take place without the solver communicating or interacting with others. Finding ways of tapping into students' thinking as they try to solve a problem has been as important in problem solving as the way in which the information obtained is analysed (see Ericsson and Simon, 1980; Ginsburg, 1981; Schoenfeld, 1985a, 1985b).

In the light of this situation, an exploratory study was conducted to select the methods of data collection that were to be used. The aim of this exploratory study was to examine possible methods for accessing students' processes as they solve non-routine mathematical problems. Two methods were carefully considered, namely, verbal and textual reports. As a result of the exploratory study, textual reports in the form of rubrics were chosen as a method of data collection. Part II of this chapter describes how verbal and textual reports were evaluated during the exploratory study. It also looks at the way a pilot study was conducted to further specify the methods of data collection and analysis that were used in what came to be considered the 'main' study.

3.2. PART II: PRELIMINARY STUDIES

Part I of the methodology chapter (Section 3.1) discussed the context of the study and introduced the methodology to be used and the methods of data collection. The aim of this second part of the methodology chapter (Section 3.2) is to discuss, in detail, how the methods of data collection and analysis evolved during two preliminary studies: an exploratory study and a pilot study. During the exploratory study (Section 3.2.1), two methods of data collection were considered, namely, verbal and textual reports. As a result of the exploratory study it was decided that textual reports were to be used as the main source of data. As for the pilot study (Section 3.2.2), it represented the first attempt at collecting and analysing data and helped define the methods of data analysis (i.e., the approach that would be taken towards grounded theory) that were to be used in the main study. Section 3.2.3 evaluates the pilot study and considers its implications.

3.2.1. An exploratory study

The aim of the exploratory study was to select a method of data collection that could serve the purposes of this study by providing insight into students' problem-solving processes. Verbal reports in the form of think-aloud protocols constituted the first method considered (Section 3.2.1.1). This method was suggested by a review of the literature and then adapted for the purposes of this study. However, some of its limitations led the researcher to consider other possibilities such as the use of textual reports (Section 3.2.1.2), particularly in the form of the rubrics that students generated during the course. What follows is a discussion of the advantages and disadvantages of both methods. A careful evaluation of both suggests why textual reports were chosen as the most suitable alternative in terms of the purposes of this study.

3.2.1.1. Verbal reports

Verbal reports were developed by information-processing theorists as a means of gathering data that could be useful for generating accurate descriptions of mental processes (Ericsson and Simon, 1980). To be sure, the aim of this study is not to develop a theory of problem solving from an information-processing perspective. However, verbal reports have been found useful for gaining insight into subjects' thinking processes regardless of whether information-processing methods of analysis are employed (Yang, 2003). In the exploratory study, verbal reports in the form of think-aloud protocols were considered only as a method of data collection. This subsection explores the pros and cons of using them for the purposes of this study. It also explains why think-aloud protocols were not considered as suitable and how this led to considering textual reports.

The type of verbal report explored in this study was based on the think-aloud protocol suggested by Schoenfeld (1985a; 1985b). In general terms, problem-solving sessions based on this method consist of a student or group of students tackling a problematic situation. While doing so, students are prompted to verbalise their thinking; in other words, they are asked to comment on what they are doing or trying to do or on whatever occupies their mind as they tackle the problem. Students are not required to recall or discuss specific actions other than the ones that they choose to focus on as they solve the problem. Furthermore, with the intention of being as unintrusive as possible, the interviewer limits his or her interventions to encouraging students to verbalise their thoughts. Problem sessions conducted in this way may be taped and video recorded in such a way that they can be transcribed for future analysis.

Considering verbal reports raises the question of what type of information this technique makes available to the researcher. According to Ericsson and Simon (1980), if the solver is asked to verbalise only the information that s/he is attending to, and not "information that would not be heeded in the normal course of processing" (p. 229), then what is reported is information that is stored in the students' short-term memory. In other words, verbal reports provide data about the ideas that students attend to as they tackle a problem, so long as the interviewer limits her interventions to prompting students to think aloud and to share their thoughts. Ericsson and Simon suggested that data gathered in this way may provide information about the cognitive processes that take place as the solver tackles a problem.

It is because they give access to the solvers' cognitive processes that think-aloud methodologies are recognised as a useful source of data for learning about human problem solving activities (Yang, 2003). However, as Ericsson and Simon pointed out, it cannot be ascertained that this data covers the whole range of psychological activities conducted by the student as a problem is solved. For instance, some information may not be available in the short-term memory (e.g., the reason why a given activity is being chosen) or the solver may simply fail to report all that is considered during the solving process. To remedy this situation, Ericsson and Simon suggested that further information about the solver's thinking process can be obtained by making use of 'retrospective reports'.

> The most general retrospective verbalizing instruction asks the subject to report everything he or she can remember about the cognitive process studied. If the subject is asked immediately after performing the process, the model predicts that some previously heeded information will still be in STM [short-term memory], permitting direct reporting... and facilitating retrieval of additional information stored in LTM [long-term memory]
>
> (Ericsson and Simon, 1980, p. 226)

In an attempt to harness the advantages of verbal protocols and retrospective reports, the first method of data collection tested during the exploratory study was *think-aloud problem-solving sessions followed by a review of the video.* More precisely, this method was composed of the following two parts:

1. A one-to-one problem solving session in which the student would be asked to think-aloud as s/he solved a non-routine mathematical problem. The session was to be audio and video-recorded.
2. Immediately after this solving phase, the researcher and the student would watch the video and look closely at what the student said and did as s/he tried to solve the problem. During this second part, the student would be asked to try to expand on what s/he had done as the situation was dealt with, particularly if there was something about a stream of thought that was not pointed out before.

Once the method was clearly delineated, the next step was to test it. For the purposes of the exploratory study, the problem solving sessions were organised and conducted in the following way:

- Warwick University students that had taken the Problem Solving course during the autumn term 2000 were contacted and asked to participate. Six students agreed to take part in the study and attended individual meetings. The meeting took place in a classroom with which the participants were already familiar, but arranged for the purposes of the study.
- Each meeting consisted of a twenty-minute session during which the student was asked to solve a non-routine mathematical problem and to think aloud. (Here the interviewer was not allowed to intervene, unless it was to prompt the student to share his or her thoughts.) The student was audio and videotaped as a solution was attempted.
- Immediately after the solving session, the video was played and watched by the student and the interviewer. The aim of this phase was to expose students to their own performance and to allow them to expand on the way the information was processed while trying to solve the problem.

Once the problem-solving sessions were conducted, the researcher had a considerable amount of data at her disposal. This data consisted of:

- Videotapes containing a recording of students' problem-solving processes;
- Notes made by the students during the session;

- An audio-recording of the discussion that took part during the retrospective reports; and
- Further notes on the students' comments and processes.

The next step was to decide whether this information could be useful for the purposes of the study and, if so, how.

As can be seen, the 'think-aloud problem-solving sessions followed by a review of the video' method was designed in such a way that it would combine the advantages of using the think-aloud protocol with those of the retrospective reports. In practice, it may be said that this method proved to be useful since it captured details of the processes through which students solved problems. Furthermore, the comments offered during the second phase (review of the video) provided extra information such as the reasons for choosing a given course of action (e.g., having used a similar technique in the past, knowing a better alternative but discarding it because of lack of proficiency).

However, although verbal reports in the form described above seemed to be useful for providing insight into students' problem-solving processes, a closer analysis revealed a number of limitations. The purpose of this study was to generate a model that explains students' main concerns as they solve problems. In order to build such model, it was necessary to gather information about what students were trying to do as they solved problems. In these terms, verbal protocols seemed to impose a number of restrictions which may have prevented students from displaying 'natural' problem-solving behaviour. First, as

Schoenfeld (1985a; 1985b) suggested, solving problems for someone else at the same time that one is being video- and tape-recorded is stressful and far from a natural. Some students may even find it difficult to communicate their thoughts in front of a camera, a tape-recorder or an interviewer.

Second, there was the question of the relationship between the activities carried out during the sessions and those that students conducted as they solved problems during the course. It was suggested that during the 'think-aloud...' sessions, students were unable to carry out activities that were 'natural' to them as solvers. For instance, when students were genuinely interested in solving a problem it was not uncommon that they explained their thoughts to a friend or a classmate who, if nothing else, listened to their ideas. The following is a brief list of some of the activities that students were unable to conduct during the 'think aloud...' problem-solving sessions.

- Communicating with others;
- Using other sources of information that may be either directly useful or serve to trigger new ideas; and
- Taking time to mull about the problem or just to "take a break" from it.

Third, as suggested in the last point of the list above, the 'think aloud...' method placed considerable restrictions in the way students managed their time for dealing with the problem. The fact that students had twenty minutes to attempt to solve a problem (whereas during the course they had, in some cases, a whole week or more) seemed to make a considerable difference. During a twenty-minute session, students might approach problems in a different way to when they are allowed to manage their time and are provided with resources to genuinely try to develop – rather than quickly discover – a solution.

In all, these characteristics of the 'think aloud…' method suggested that the information obtained did not reflect students' processes during the course. It may be argued that the information provided in such problem solving sessions leaves out some activities that are relevant when students are in class or at home trying to deal with a problematic situation. In this way, it was decided that a different alternative needed to be explored. This alternative was the use of textual reports.

3.2.1.2. Textual reports

The limitations of the 'think-aloud problem-solving sessions followed by a review of the video' method led to the search for other methods of data collection. The next option considered was the use of the textual reports produced by students during the problem-solving course; i.e., their rubrics. As mentioned before, rubrics are students' own written accounts of how a problem is solved. In them, students report their ideas, what they are trying to do and their feelings as they tackle a problem. Furthermore, since rubrics may be written for communicating a solving process, they are usually comprehensive accounts that give a clear picture of how a solution was obtained. This subsection discusses how, after examining this option and assessing its limitations, it was decided that students' rubrics provided a suitable method of data collection.

The rubrics that were analysed for the purposes of the exploratory study belonged to the group of students that had taken the problem-solving course during the autumn term 2000. (These were the same students who were asked to participate in the one-to-one problem solving sessions described above.) As part of the course, students were introduced to rubric writing and were required to practice this technique as they solved non-routine mathematical problems. It was observed that, in general, rubrics contained detailed descriptions of how the problem was tackled. Some rubrics were even created over a number of days and this was indicated by reports of having put the problem aside and coming back to it at a later time. In their rubrics, students wrote – among other things – about

- Their interpretation of the problem;
- How they came across interesting or possibly useful facts and the decisions that these findings would lead to;
- The dead ends and difficulties they were facing together with what they were trying to do about them;
- Their thoughts and evaluations of different aspects of their process; and
- Their feeling about the problem-solving situation.

The content of these written documents can be related to Leron and Hazzan's (1997) 'virtual monologues'. In these monologues, the researchers used external representations (mainly students' notes and comments) in order to re-create students' thought processes during problem solving. Their method was to write a monologue in the first person through which the researcher would try to describe students' thinking, including thoughts about how they felt about the situation and about their progress. The idea was that "conventional analyses of students' sometimes concentrate too much on cognitive issues and neglect social and affective factors and thus "fall short of describing the student's mind in all its richness and complexity" (Leron and Hazzan, 1997, p. 266). By using 'virtual monologues' the authors aimed to analyse students' problem solving processes from the students' own perspectives. As for the rubrics used in this study, it may be said that they constituted a variation of the 'virtual monologue', but with the advantage that it is the student who wrote them and not the researcher.

In terms of the purposes of this study, other advantages of using students' rubrics for learning about their problem solving processes were the following:

- Students were trained during the course to use rubrics to report their thinking process as they occurred.
- Students were able to document metacognitive activities like reviewing or explaining their reasoning if, and when, they occurred.
- The problem solving process was not affected by the presence of an interviewer or any recording equipment (camera, tape-recorder).
- Students were free to communicate with others about their process but were required to document this activity.
- Students were free to access other sources of information that could give them clues as to how to go about in their process (again, students were asked to acknowledge this).
- Students did not have to restrict their problem-solving process to a 20- or 30-minute session.

- It was left to the students to decide how best to manage their solving time. However, they were advised of the benefits of taking 'breaks' from the problem or mulling about it for a while (see Mason, 1982).

These advantages suggested that this method of data collection was a viable alternative to the use of verbal reports. Textual reports share with verbal protocols the advantage of giving access to students' problem solving processes as they occur. Unlike verbal reports, however, rubric writing and its use as a method of data collection can be considered an 'unobtrusive' method (see Lee, 2000). This is to say that, as the solving process takes place, neither the researcher nor the environment is an intruding element with the potential to affect directly what the student does (or otherwise) as he tackles the problem. Moreover, since rubric writing is a necessary part of the course and was not introduced as part of the researcher's data collection method, this also added to its quality as an unobtrusive method.

A suggested limitation of using rubrics was that they present 'polished' versions of a problem-solving process. After all, rubrics were not created to provide information for the study but as part of an academic course. Thus, it was suggested that students' rubrics do not confer all that occurs as they solve a problem. Instead, students may be selective and present only those aspects of the problem that, in their view, suggest a coherent process. However, this was not considered a genuine drawback to using rubrics for the purposes of this study. Whereas it is true that rubrics may not contain all that crosses students' minds as they solve problems, what they contain is inevitably informative about what they are trying to do as they solve problems during the course. If it had emerged that that the students' main concern as they solve problems is not to generate a solution but to satisfy the tutor, then that is what the theory would have been about.[11]

Furthermore, following the idea that 'all is data' it may be argued that rubrics need not be the only source of information used. The grounded theory methodology makes use of the method of constant comparison. This method requires that the data and the hypotheses that are generated be constantly compared against further data. What this data will consist of is something that will be determined by the questions raised in the emerging hypotheses. In the case of the present study, these data consisted of information from other rubrics as well as information gathered through informal interviews or observations.

As a result of considering the advantages and disadvantages of verbal and textual reports, it was considered that the latter was a suitable method for learning about students' problem-solving processes. Also, the use of other sources such as informal interviews and observations were considered as useful methods of data collection. In this way, the exploratory study provided the researcher with methods of data collection that could be used for learning about students'

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[11] As is discussed in Chapter 4, this was actually an observed pattern of behaviour albeit not the main one identified. It may be said that students' main concerns as they tackle non-routine mathematical problems is to 'solution' (a process composed of other concerns such *generating knowledge*, *generating solutions*, and *validating* and *improving* them. However, it also emerged that students may 'pseudo-solution'. This behaviour suggests that students sometimes may opt for trying to satisfy an academic requirement rather than to genuinely solve the problem.

concerns as they solved problems. The next step after deciding on the methods of data collection was to make decisions as to how the data was to be analysed. It was decided that the best way to do this was by actually collecting and analysing data in order to evaluate the alternatives in a more objective way. This process led to what was called the 'pilot' study.

3.2.2. A pilot study

Once the methodology and methods of data collection were defined through an exploratory study, a first attempt at collecting and analysing the data was conducted. This attempt can be seen as a pilot study that allowed the researcher to further define the methods of data collection and to gain knowledge in relation to using the grounded theory methodology. A number of recommendations emerged as a result of this study. Although some of these recommendations made reference to practicalities related to data collection, the most important contribution of the pilot study had to do with the analysis of the data. Important lessons about data analysis under the grounded theory approach were learnt during this phase.

This section presents the pilot study. A first short subsection (Section 3.2.2.1) describes the context in which the pilot study took place. A second subsection (Section 3.2.2.2) deals with how data was collected and concludes with a list of recommendations for the main study. Section 3.2.2.3 briefly discusses the way the data was analysed during the pilot study and reflects on some of the problems that emerged. This discussion leads to the next section (3.2.3), where a more thorough discussion of the analysis of the data takes place.

3.2.2.1. Context of the pilot study

The pilot study was conducted during the autumn term 2001. During the course that year, students attended a two-hour 'lecture' and a one-hour seminar each week, for ten weeks. The course consisted of two main aspects:

1. During 'lectures', students were introduced to aspects of Mason et al.'s (1982) problem-solving framework. This provided the students with vocabulary and techniques that could be useful for solving problems and for reflecting on their processes.
2. Students were prompted to solve one or two problems in class (during 'lectures') and one or two more problems at home. The latter were to be discussed in the subsequent seminar, in which students were expected to share their solutions and solution processes.

Furthermore, during the fifth week of the course (i.e., half way through the course), students were given the instructions for a final assignment. This assignment was to be submitted five weeks after the end of the course and consisted of three parts, as described in the following excerpt from the instructions sheet:

The assignment is in three parts.

1. Work on one of the problems below ['Liouville' and 'Jogger's Dog'; see Appendix 1] , writing a rubric for your solution process. You should submit your original work without spending too long tidying it up – but it must be LEGIBLE!
2. Write a commentary on your solution, analysing the use of problem-solving techniques, and evaluating their effectiveness.
3. Critically discuss the relevance of problem-solving techniques in other areas of your University course.

Marks will be given particularly for clarity of reflection on and discussion of your work as well as for the elegance of the problem solution.

(From the instruction sheet for final assignment, autumn 2001)

3.2.2.2. *Data collection during the pilot study*

Taking into account the organisation of the course, three potential sources of information were identified. It was hoped that, from these sources, three different types of data would be made available by the end of the term:

- Students' rubrics for the problems tackled in class;
- Students' rubrics for the problems tackled at home, and to be discussed in the seminars; and
- Students' rubrics for the final assignment.

The process of collecting these data was conducted as follows:

- During the first session, the researcher was introduced to the students. She provided an explanation of the study and of the type of information that she wanted to collect. Students were told that their participation was optional and that they could withdraw their participation at any stage of the study. They were also ensured that no real names would be used in any of the reports.
- Students' written work was collected every week in order to archive a photocopy of their work. This was the case in both 'lectures' and seminars (i.e., work done in class and at home).
- At the end of the course, students' final assignments were photocopied. These assignments consisted of (a) the rubric corresponding to a problem solved by students during the vacation period, (b) students' written comments on their solutions, and (c) a comment on the use of problem solving skills in areas outside the course (see the description of the requirements for the final assignment above).

In practice, although data was successfully collected from these three sources, the quality of some of the data was not as expected. It had been assumed that all rubrics would be of similar informative quality regardless of whether they were tackled for lecture or seminar purposes or as part of the final assignment. However, this was not the case. In general, the weekly rubrics were so poorly documented that they were not considered useful for analysis purposes. The reasons for this became evident even before the data collection phase had been completed.[12] These reasons for this can be put as follows:

- During class ('lecture' or seminar), students were working mainly in teams and much of the communication was verbal. This did not encourage (nor effectively allowed) students to record their thoughts as they occurred.
- The fact that it was not a course requisite to submit their weekly rubrics failed to stimulate students to document their work in a clear and comprehensive way.
- During class, students had a maximum of two hours to tackle the problem. Progress was inconsistent among students but it can be said that many of them found their processes interrupted by this time restriction

The situation was very different in the case of the final assignment. In this case, students focused on communicating their ideas, thoughts and emotions with the reader. Furthermore, the fact that the assignment was set for evaluation purposes seemed to encourage students to share their work in a clear manner. As mentioned in the assignment instruction sheet, marks were given "for clarity of reflection on and discussion of your work as well as for the elegance of the problem solution". The result was that rubrics represented relatively integral problem-solving processes, as they occurred and from the stand-point of the

[12] The grounded theory approach provides the researcher with the flexibility to modify the methods of data collection in order to avoid situations like this one, in which data that was evidently useless had to continue being gathered. However, the fact that the data collection process was attached to an academic course meant that changes could not possibly be made. Also, and as will be discussed in the next section, this situation can be partially attributed to an incomplete understanding of the methodology. The latter situation changed throughout the study as the researcher became better acquainted with the methods of grounded theory.

solver. This made the final assignment a useful source of information for the purposes of this study.

Some important recommendations emerged as a result of the data collection process associated with the pilot study. These were taken into account for the main study and can be summarised as follows:

- It is unnecessary to collect the rubrics that students' produce in class; field observations are more suitable in this case.
- Solving one problem at home per week (instead of two) may help increase the quality of the rubrics that students submit each week.
- Collecting students' rubrics for assessment purposes (i.e., assigning a mark to their submission) could help increase their quality.
- The rubrics that students submitted for their final assignments represent a useful source of information; final-assignment rubrics should continue to be collected in the same way.

As said, the aim of the pilot study was to collect and to analyse data in order to explore the feasibility of methods that were chosen. Although the data-collection phase presented some difficulties, it may be said that useful lessons were learnt. Moreover, the data-collection phase provided useful data in the form of the rubrics that students created for the final assignments. These rubrics were analysed for the purposes of the pilot study and the next subsection looks at how this was done.

3.2.2.3. *Data analysis during the pilot study*

This section provides a critical description of how some of the data gathered from the pilot study was analysed. The first attempt at analysing the data made use of the methods of data analysis proposed by Strauss and Corbin (1990; 1998). The authors presented these methods as techniques for generating grounded theories that can be adapted for the purposes of particular studies. In general, the analysis of the data during the pilot study consisted of two stages that can be related to the 'open coding' and the 'axial coding' techniques as suggested by Strauss and Corbin.

Open coding and axial coding are important techniques in the Strauss and Corbin methods of data analysis. Open coding can be described as an exhaustive analysis with the intention of 'opening up' the data and generating categories. Axial coding can be interpreted as the process of elaborating these categories further by investigating their properties and dimensions (i.e., the possible variations of these properties) and their relation to other categories. During the pilot study, open and axial coding were conducted in the following way:

Open coding

During open coding rubrics were analysed line-by-line with the aim of coding the instances observed and looking for categories. At this point, however, the researcher had observed that students seemed to approach mathematical problem solving in two different ways, namely, by trying to develop a solution and by trying to *discover* it (de-Hoyos, Gray and Simpson, 2002). Although confirming this hypothesis was not the intention of open coding, the first attempt at analysing the data was influenced by this previous observation. In this way, open coding became a verification exercise rather than an exercise to 'open up' the data (which was the original aim). It may be said the aim of open coding was unintentionally modified from detecting relevant categories to verifying the categories already observed.

Axial coding

Besides open coding, the other technique that was used was the 'axial coding' technique (Strauss and Corbin, 1990; 1998). According to the authors, in axial coding the

> focus is on specifying a category (*phenomenon*) in terms of the conditions that give rise to it; the *context* (its specific set of properties) in which it is embedded; the... *strategies* by which it is handled, managed, carried out; and the *consequences* of those strategies.
>
> (Strauss and Corbin, 1990, p. 97; emphasis in the original)

Strauss and Corbin's description of this technique was found to be unclear and difficult to put in practice.[13] This led to a general interpretation of axial coding as the process of further specifying observed categories. In practice, the application of this technique resulted in continuing to code rubrics with the aim of developing the observed (or predefined) categories further. As more information was gathered, the categories evolved into two fully-fledged categories that were

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[13] As the study progressed, it became clear why this was the case and what the drawbacks of axial coding were. See Section 3.2.3.2.

named 'panning for gold' and 'building a solution'. Panning for gold and building a solution were presented as two approaches to problem solving (de-Hoyos, 2002). However, although these categories constituted interesting results in themselves, they did not represent, and were not leading to, a holistic model of problem solving.

3.2.3. Evaluation of the pilot study and implications

This section provides an evaluation of the activities that were conducted for the analysis of the data collected during the pilot study. It also discusses a second attempt at 'open coding' the data and how this exercise led to noticing a discrepancy between the methods of grounded theory suggested by Strauss and Corbin (1990; 1998) and those proposed by Glaser (1978; 1998; see also Glaser, 1992). As it will be seen, considering the differences between both methods led to choosing the latter author's approach.

Although the way in which the data was analysed during the pilot study was not altogether ineffective, it may be said that it failed to produce the expected results. That is, the conclusions that were drawn from the pilot study, however interesting, were in no sense leading towards a theory that would account for students' problem solving process as a whole. These results prompted a closer evaluation of the methods employed.

3.2.3.1. The effect of preconceptions in analysing the data

As Glaser and Strauss (1967) suggest, an important aspect of grounded theory as a methodology is the need to approach the data without any preconceptions about what its categories (core or otherwise) will be. The reason behind this idea is that categories must be allowed to emerge from the data. However, this is not a statement that can be easily interpreted. On the one hand, the researcher does not approach the study as a 'tabula rasa' but with a set of views about its elements and salient problems. As Strauss and Corbin (1990) put it, these views can be the result of familiarity with the relevant literature in the field, of personal or professional experience, or even of interaction with the data. In the authors' opinion, all these experiences may be a source of 'theoretical sensitivity', i.e., a sort of background knowledge that makes the researcher sensitive to what the data 'tells' about the area of inquiry. On the other hand, the researcher must not make any assumptions about the relevance of any preconceived ideas in relation to the study (Glaser and Strauss, 1967). Whilst it is possible that some of these might become relevant categories during the analysis, it is also possible that they be considered only tangential or even irrelevant to the emergent theory. This relevance must be achieved and is something that the data should either confirm or refute.

However feasible the above considerations may be, the initial phases of the analysis of the data from the pilot study presented some difficulties. Although the idea of allowing categories to emerge can hardly be considered unacceptable, *how* they will be allowed to emerge is a different problem. In this study, this problem was partially overcome by using preconceived categories to guide an initial attempt at 'open coding'. Moreover, instead of allowing the data to inform about the relevance of these categories, the researcher set out to identify those instances that would provide further information about them.[14] This obviously led to developing these particular categories further, albeit in a somewhat artificial way. Even though this was not in accordance with the grounded theory methodology, it was the best reading of the related literature at that time.

3.2.3.2. The use of axial coding

As mentioned in Section 3.2.2.3, the way axial coding was interpreted during the pilot study did not seem to be leading to a holistic model of students' problem-solving processes. This suggested a review of the way this method was used and a search of possible ways of improving the quality of the analysis and the results.

Strauss and Corbin (1990; 1998) suggest that categories should be specified further by axial coding. This involves considering categories and trying to relate them by asking specific questions such as '*How are category* x *and category* y *related?*' '*Under what circumstances does this category occur?*' or '*What causes or consequences are related to it?*' However, as it was later realised in the present study, asking questions like these raises the need of generating logico-deduced answers and prevents the emergence of grounded relationships among categories. For instance, taking two categories and inquiring about their relationship implies that they are related and increases the possibility of the researcher imposing his or her preconceptions on the data. In other words, rather than concentrating on the data and on trying to 'see' what is actually happening, axial coding leads to trying to answer questions that may be 'suitable' to the

[14] Although this practice is not in itself methodologically unacceptable, it does contradict a basic grounded theory principle, namely, allowing the theory emerge from the data (see Glaser, 1992).

researcher but not necessarily relevant to the situation being studied.

According to Glaser (1992), axial coding is an unnecessary step in grounded theory if categories and their relationships are allowed to emerge. Contrary to Strauss and Corbin (1990), he suggested that the researcher should avoid trying to put categories together in artificial ways but that relations should also emerge from the data in the form of theoretical codes. In Glaser's words:

> Needless to say, theoretical coding families emerge as connections between categories and their properties. If one category is a condition of a property, then this will emerge as such.
>
> (Glaser, 1992, p. 62)

Axial coding was thus put aside and the aim was to follow Glaser's (1978; 1992; 1998) suggestion that constantly comparing the data and memo writing suffice for generating theoretical codes. Theoretical codes can be seen as the relationships among categories that Strauss and Corbin might have referred to, but that, at least in the present study, were not achieved through their method of axial coding.

3.2.3.3. Open coding revisited

According to Glaser (1978), open coding is the initial procedure of the grounded theory methodology. Its purpose is to 'open up' the data through a line-by-line analysis that consists of looking for relevant categories. In the case of the pilot study, open coding was first conducted as a process of isolating problem solving instances and evaluating them in terms of previously conceived ideas. This eventually led to the situation and results described above. The limitations in these results led to a new attempt at making use of the grounded theory approach, this time considering more rigorously what the methodology requires in each phase.

In a second attempt at open coding, an emphasis was put on the need to approach the data without any assumptions on the relevance of previous observations. This was not difficult to achieve but this time a different problem emerged. Again, incidents were isolated and each was given a suitable name. Each of these names was considered a code and it was expected that a later grouping of codes would lead to relevant categories. However, this process did not yield the expected results as categories did not seem to be emerging from the data. As a result, there seem to be no other option than to force them by turning to previously conceived ones. A further analysis of the literature suggested that this process was closer to the labelling procedure recommended by Strauss and Corbin (1990; 1998) than to the constant comparative method originally suggested by Glaser and Strauss (1967; see also Glaser, 1978; 1992).

As suggested by the literature on grounded theory, there seems to be a discrepancy regarding how open coding is to be conducted as a technique for data analysis. On the one hand, according to Glaser (1978), open coding consists of comparing incident with incident and later with emergent codes and categories. In other words, his method is mainly based on the constant comparative method (see Part I of this chapter). During this phase, the analyst takes each incident in turn and asks questions like:

> "What is this data a study of?"
>
> "What category does this incident indicate?"
>
> "What is actually happening in the data?"
>
> (From Glaser, 1978, p. 57)

Focusing on these questions, categories should soon begin to emerge, as well as their properties and dimensions.

On the other hand, this is a different stance to the procedure of 'labelling phenomena' suggested by Strauss and Corbin (1990; 1998). For them, open coding consists of 'conceptualising' data. The authors define this technique as follows:

> **Therefore, conceptualising our data becomes the first step in analysis**. By breaking down and conceptualising we mean taking apart an observation, a sentence, a paragraph, and giving each discrete incident, idea, or event, a name, something that stands for or represents a phenomenon.
>
> (Strauss and Corbin, 1990; emphasis in the original)

Glaser (1992) challenged this interpretation of open coding by stating that:

> By breaking down and conceptualising the data we do not mean taking apart a single observation, sentence or paragraph, and giving each discrete incident, idea or event a conceptual name, which indicates something that stands for or represents a phenomenon...
>
> We do mean comparing incident to incident and/or to concepts as the analyst goes through his data. We look for patterns so that a pattern of many similar incidents can be given a conceptual name as a category.
>
> (Glaser, 1992, p. 40; emphasis as in the original)

It can be seen, therefore, that there are two different descriptions of open coding. With respect to this study, Strauss and Corbin's techniques did not seem to promote the emergence of general categories but required their creation once a number of incidents had been labelled. After experimenting with the constant comparative method, it was observed that categories (and they relationships) began to emerge in a natural way.

3.2.3.4. Implications of the pilot study

As suggested above, Glaser and Strauss seemed to have developed the grounded theory methodology in different and apparently conflicting directions (Charmaz, 2000). On the one hand, Strauss's depiction of grounded theory seems to put the method of constant comparison out of the centre of the methodology and includes a number of new techniques (see Strauss and Corbin, 1990, 1998). Paradoxically, instead of adding rigour to the methodology, these techniques propose a more open approach to 'doing' grounded theory. They may be defined as a set of tools that are available for researchers who may decide which to use and, to an extent, how to apply them. In this sense, Strauss and Corbin made grounded theory a flexible approach that the researcher may use together with other available techniques for data collection and analysis. Glaser, on the other hand, insists that constant comparison and memo writing are sufficient tools for generating the type of theory that fits and accounts for the situation under study (Glaser, 1978; 1992). Moreover, he has made grounded theory a comprehensive methodology that provides fully defined guidelines to take the researcher from data collection to the generation of a theory.

In sum, the pilot study had implications on the way data was going to be collected and analysed during the main study. In terms of data collection, it provided practical recommendations to assure that the information generated was useful for informing about students' problem-solving processes. As for the analysis of the data, it allowed the researcher to become familiar with the selected methodology. This included not only learning about its practicalities but also learning about the different options available. Through the pilot study, it was possible to investigate these options and make informed choices regarding the methodology that was going to be used for the main study.

It may be said that the analysis conducted during the pilot study adopted a version of grounded theory that was closer to Strauss and Corbin's (1990; 1998) more flexible approach. However, a number of difficulties emerged while working in this way and this led to attempts to resolve them by resorting to Glaser's more rigorous methodology. As a result, it was decided that Glaser's grounded theory methods were better suited for the purpose of this study, and thus they were fully adopted. This led to the results that will be presented in Chapter 4. The next part of this chapter discusses the use of the grounded theory methodology during the main study.

3.3. PART III: METHODS OF DATA COLLECTION AND ANALYSIS DURING THE MAIN STUDY

Part II of the methodology chapter (Section 3.2) discussed how the pilot study was the first attempt at collecting and analysing data. This first attempt provided useful experience that served to define the methods of data collection and analysis to be used during the 'main' study. The aim of this third and last part of the methodology chapter (Section 3.3) is to discuss these methods. Section 3.3.1 discusses how data was collected whilst the second section, 3.3.2, explains how the data was analysed following the grounded theory methodology proposed by Glaser (1978; 1992; 1998). The final section, 3.3.3, discusses the criteria for evaluating the product of studies conducted by following this methodology.

3.3.1. Data collection during the main study

Data collection for the main study took place, mainly, during the autumn term of the academic year 2002/2003, although, after this period, further data continued to be gathered in various forms.[15] The academic term, as is usual, consisted of a 10-week period that lasted from October to December 2002. Students attended weekly, three-hour sessions in which three main activities usually took place. The first of these activities consisted of introducing students to some aspect of Mason et al.'s (1982) framework for solving problems. This activity was led by the lecturer and it was usually followed by a problem-solving period in which students were given the opportunity to tackle one or two problems. The third part

[15] Also, it must be stressed that although the information that was collected at this stage consisted of a considerable amount of data, the information collected during the exploratory and pilot studies was equally important for the analysis.

of the session consisted of a whole class discussion in which students considered their processes in the light of the ideas that were presented at the start of the session.

Besides attending lectures, each week, starting from week two, students were required to tackle a problem at home and to submit a rubric of their work. These rubrics were going to be marked and taken into account for assessment purposes. The satisfactory completion of the weekly assignments accounted for 10% of the final assessment for the course. The fact that the rubrics were going to be marked ensured good quality work from most students during the term. In this way, the weekly rubrics constituted an important source of information for the study.[16]

Other sources of information were the final assignment that students had to submit at the end of the course and the observations made by the researcher in every session. Furthermore, once the course was completed, the researcher conducted a number of informal interviews with some of the students that participated in the course. The objective of these interviews was to investigate emergent issues further, particularly when the information was not available in the data that had already been gathered. Further information regarding the

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[16] Assigning a percentage of the final mark to the submission of the assignments was one of the differences between the way course was conducted during the academic year 2002/2003 and previous years. The other difference had to do with the students taking the course. As mentioned in Part I of this chapter (Section 3.1.1.1), the course during the academic year 2002/2003 also included first year students doing BA(QTS) degrees. This was in contrast to previous years when only second- and third-year mathematics, computer sciences and physics students took the course.

process of data collection is provided below:

Weekly Rubrics

From week two to week ten, the researcher collected students' rubrics for solving a non-routine mathematical problem and made a photocopy of each of them. The mathematical problem that students had to tackle was given by the lecturer and students were expected to tackle it outside of class. They were expected to work individually but discussing their ideas with others or looking for information in other sources was not discouraged as long as students documented this in their rubric and their process was clear to follow. Rubrics were to be considered for assessment purposes and students were required to submit a rubric for at least 7 out of the 9 given problems. The problems given during the course are listed in the table below and Appendix 1 provides a full description of the problems.

Week 2: Steps
Week 3: Ins and Outs
Week 4: Cartesian Chase
Week 5: Diagonals of a Rectangle
Week 6: Hat Numbers
Week 7: As Easy as ABC?
Week 8: Arithmagons
Week 9: Visible Points
Week 10: Square Takeaway

Table 1: List of problems considered during the academic year 2002/2003

Final Assignment

Students' final assignments were also photocopied for the purposes of the study.

The submission deadline for these assignments was the first week of the spring term (i.e., at least five weeks after the course had ended). However, the guidelines for completing this piece of work were available to the students from the fifth week of the course. The assignment consisted of two parts:

- A choice of two non-routine mathematical problems: 'Faulty Rectangles' and 'Sums of Diagonals' (see Appendix 1). Students were supposed to choose one and to tackle it, producing a rubric that was going to be read and marked by the lecturer.
- A written commentary on their solution "analysing the use of problem-solving techniques, and evaluating their effectiveness".

Observations

The researcher was present in every session and had the opportunity to get to know the students and observe them in class. This was useful in terms of making her sensitive to the situation, i.e., in terms of fostering theoretical sensitivity (see 3.2.3.1; Glaser, 1978; 1992; Strauss and Corbin, 1990).

In relation to the above methods of data collection, permission was obtained from students to use their work. They were also notified of the fact that they could withdraw their participation from the study at any stage and that all names would remain anonymous.

Informal interviews

At least 7 informal interviews were conducted once the course had ended. The aim of these interviews was to gain information and understanding on issues that were raised by the analysis of the data but for which no further information was available. Thus, the interviewees were selected on the basis of the information that was sought. For instance, if the researcher wanted to gather further information about an emerging category such as guessing an answer versus deducing it logically, she would contact students whose rubrics gave evidence of these behaviours. Not all students agreed to participate but those who did provided the researcher with useful information. During these interviews students were asked general questions regarding a particular aspect of their process. For example, students could be asked to explain "what were you trying to do" in relation to their rubrics. Students were allowed to talk freely once the question was posed. Following the grounded theory methodology interviews were not tape-recorded. Instead, field notes were written immediately after each interview was conducted (Glaser, 1998).

Once the theory was complete, further data was gathered by presenting it to the research supervisors (one of who was the lecturer in charge of the course) and to one of the participants of the course. In these cases, the readers' comments were treated "as more data" (see Glaser, 1998) to be compared against the theory. Each time the theory was presented, the readers' comments, questions and impressions were carefully considered. In some cases, such comments resulted in the researcher going back to the data to clarify aspects of the theory.

3.3.2. Data analysis during the main study

Although in grounded theory data collection and analysis are both ideally conducted in tandem from the beginning (Glaser and Strauss, 1967), this was not possible in this study. The first reading of the collected data was made during the course, as rubrics were being submitted. However, due to schedule restrictions, the analysis had to be postponed until after the course had ended. This did not have negative consequences for further stages; a large amount of useful data was being made available and it was also possible to contact the students once the course had finished. Moreover, the fact that the researcher had become familiar with the rubrics' mathematical content allowed her to start the data analysis right after the data collection phase.

The analysis of the data was done in accordance to the grounded theory methods described in Glaser and Strauss (1967) and Glaser (1978; 1992; 1998). The way the data was analysed and how this eventually led to the theory being presented in this study will be described in the next sub-sections. The main stages of the process that will be described are the following:

- Open coding;
- Selective coding;
- Sorting, and
- Writing-up.

As Glaser (1998) put it, grounded theory's processes are sequential, subsequent and simultaneous. This is to say that although the process of generating a theory has several stages and they are conducted in a particular order, previous stages may continue to be in operation. It is important to keep this in mind as the following sections are considered.

Although every effort will be made to provide an as clear picture as possible of how the data was analysed, it is simply impossible to express the analysis process in full. Among other things, the analysis consisted of constantly comparing incidents from more than 300 scripts. Thus, the analysis process that will be described consists of a general view of the process. To partially remedy this situation, Appendix 2 provides a full account of the analysis of two rubrics. However, the reader must consider that the concepts that comprise the theory presented in this study were derived from comparing a large number of rubrics as well as a considerable amount of supporting data. In other words, these concepts represent patterns of behaviour that may or may not be evident in a single rubric.

3.3.2.1. Initial analysis of the data: open coding

Open coding, as the first analysis activity conducted in grounded theory studies, involves "running the data open" (Glaser, 1978, p. 56), i.e., starting to learn about salient issues of the situation through the data that has been made available. Open coding was introduced in Part I (Section 3.1.2.1) and further discussed in Part II (Sections 3.2.2.3 and 3.2.3.3) of this chapter.

As suggested by Glaser, the ultimate aim of open coding is to find a "core category", i.e., a category that seems to integrate other categories and that can serve to explain the situation as a whole. This aim guides the analysis and once a core category is identified, the researcher proceeds with selective coding (as will be discussed in the next sub-section). Thus, during open coding:

> [The researcher] constantly looks for the 'main theme', for what – in his view – is the *main concern or problem for the people in the setting*, for what sums up a pattern of behaviour, the substance of what is going in the data…
>
> (Glaser, 1978, p. 94; emphasis added)

During the main study, open coding consisted of taking one rubric at a time and analysing it line-by-line but without paying – at least initially – too much attention to the detail. As the researcher analysed students' rubrics, questions like the following were kept in mind:

> What is the student's main concern here? What is s/he trying to do? What does this tell me about? How does this relate to something I noticed before?
>
> (See Glaser, 1978).

The following table provides two examples that illustrate how open coding was conducted. Each example corresponds to a different rubric. However, they are related in that both were coded as 'looking for information and understanding'. As the researcher coded the rubric, the following two quotes caught her attention. In them, students seemed to be trying to learn more about the situation. In other words, they seemed to be looking for information and understanding in order to be able to solve the problem.

Sample passage [all names are pseudonyms]:	Coded as:
"So 23 ways of doing this. So far we have 24 possibilities, but loads more – beginning to look like combinations and permutations. Can't really remember them so want to look at what happens with '2's and even '3's and find a pattern." (Camille, Steps, p. 1)	*Looking for information and understanding.*
"Arg, I don't see where this is leading me. This is clearly a recursive/repeating pattern but I don't know if it helps? I've set up a spreadsheet but nothing clear. Maybe factorising the formula will make things clear." (Julian, As Easy as ABC?, p. 4)	*Looking for information and understanding.*

Table 2: Example of open coding

During the analysis of students' rubrics, the researcher followed a number of informal and often implicit guidelines. These guidelines were practical ideas as to how to make the most of the analysis. Some of these guidelines were borrowed from the literature on grounded theory (particularly Glaser, 1978; 1992; 1998) and some emerged as a result of the experience gained by attending grounded theory seminars and as the analysis progressed. The guidelines are summarised below:

- The analysis should be done line-by-line and not through skim-reading. This is important in order to avoid making false assumptions as to what the student is trying to do at each stage of the rubric.
- Since the grounded theory methods of analysis are based on constant comparisons, it is not necessary to 'linger' for too long on particular incidents. If one incident does not suggest a code or an observation, it is better to continue reading the data and making comparisons rather than to make forced assumptions.

- In relation to the previous point, Glaser (grounded theory seminar, London, 2003) suggested that the researcher should not get too attached to any single idea, no matter how good it sounds. Further data will verify its relevance (or otherwise, in which case the idea should be left behind).

To provide a better picture of how the data was analysed, Appendix 2 presents the process of analysis of two complete rubrics. As it can be seen in these rubrics, the method of constant comparison was used opportunistically. This is to say that, as the researcher analysed each incident, she was vigilant about other incidents with which it would be worth-while comparing.

It is likely that the codes generated in the examples provided above, as well as in the rubrics provided in Appendix 2, would have been compared to find more about ideas such as 'looking for information and understanding'. Moreover, it is possible that this comparison would have prompted ideas about how information was sought, for what purpose, etc. These ideas would have been noted down in the form of a memo and stored for future comparisons. As will be discussed next, 'constant comparisons' and 'writing memos' are important activities during open coding. These activities start with open coding but continue to be in use throughout the process. These activities have been mentioned previously (Sections 3.1.2.1 and 3.2.3.3) and they are worth considering in more detail.

Constant comparative analysis

The method of constant comparative analysis starts by comparing incidents to other incidents within the data (Glaser, 1978). This comparison generates codes (or concepts) that are later compared to more incidents and eventually to other codes. This method quickly begins to generate ideas about the data and to answer the question that grounded theorists are meant to pose (see Section 3.2.3.3). However, although these ideas emerge from the data, they ideas are not fixed; instead, they are subject to further comparisons and as a result of this are constantly modified to make them more suitable representatives of the situation.

Memo writing

Writing memos is another important aspect of grounded theory as it represents the media in which relationships among categories and its elements are not only stored but also developed. According to Glaser:

> Memos are the theorising write-up of ideas about codes and their relationships as they strike the analyst while coding.
>
> (Glaser, 1978, p. 83).

Thus, an important characteristic of memos is that they should be written down as soon as an idea comes to the researcher's mind. (And, as the researcher analyses the data by making constant comparisons, these ideas are constantly emerging.) Recording ideas as soon as they emerge is so important to grounded theory that writing memos – even if only brief ones – is given temporal priority over coding or any other activity. In practical terms, this means that the researcher must stop any activity to write any emerging idea about the situation.

The following table provides examples of memos written as the researcher analysed the data. Within each memo, it can be seen how an effort was made to make them more 'conceptual' and less anecdotal. This was done by avoiding references to the particulars of the incident being considered and concentrating more on those aspects that could lead to conceptualising general patterns of behaviour (Glaser, 1998). In general, memo writing meant that ideas about what was happening in the data were put on paper. This provided a fund of ideas to work with and modify as necessary. As the analysis progressed, memos started to reflect an increased understanding of the situation.

Memo: Variability in the degree of generating information: Some students learn a lot about the situation before they try to provide an answer. Other students, on the other hand, seem to be giving tentative solutions without understanding. Ideas that seem misinformed or underinformed... Investigate the consequences of these approaches
Memo: There seems to be a 'minimum information principle'. Reaching a level of sufficient understanding of the situation is necessary in order for students to start generating a solution. It seems that students do not try to gain more information than what is necessary. If they can generate a solution with what they know, they are more likely to choose to do this than to try to learn more about the situation. Thus, if students can generate a solution with a very surface understanding, then they will not try to gather deeper understanding. They will only do the latter if they have to.
Memo Reducing complexity is a strategy to start dealing with a complex situation. A 'strategy' since it is a consciously chosen route and not something that they do as a result of the circumstances. Students may use this strategy to learn about the situation or to generate a solution. It seems to be a recurrent activity. Reducing complexity helps make the situation simpler and easier to be dealt with.
Memo When students try to generate a solution, students try deduction first (although they do not necessarily succeed!).

Table 3: Examples of memos written during the analysis

3.3.2.2. Selective coding

While open coding allows the researcher to find as many categories as possible to explain the problem that those involved are trying to resolve, during selective coding the researcher "becomes selective and focuses on a particular problem" (Glaser, 1978, p. 56). This 'problem' is the core category which, as mentioned above (Section 3.3.2.1), integrates or sums up what is going on in the data. Finding a core category and focusing on it does not mean that other categories become irrelevant or that they will be ignored. What this means is that the researcher will integrate these categories around a main category that will serve to unify and relate emerging ideas.

In some cases, core categories are best represented by 'basic social processes', which constitute a type of core category (Glaser, 1978). Basic social (or psychological) processes suggest that the situation under study can be explained by a set of identifiable changes that participants go through as they deal with their respective problematic situations. In the case of this study, it became evident that students' problem-solving processes could be best conceptualised as a four-stage basic psychological process which was called 'solutioning' (see Chapter 4).

Once solutioning had emerged as a core category, the researcher moved on to selective coding. Selective coding was similar to open coding in that constant comparative analysis and memo writing continued to be the main activities. However, the difference was that, whereas open coding considered all codes as possibly relevant, selective coding demanded that all the attention be put on those codes that referred to solutioning.

Selective coding continued for several months. Through this process, the researcher accumulated a vast memo fund. Within these memos, some contained brief or incipient ideas about the data. Others – particularly the later ones – contained more sophisticated conceptualisations that seemed to start 'organising' the situation. However, this organisation was not possible merely through memo writing and this prompted the researcher to stop the analysis (at least temporarily) and to start trying to sort the memos into an outline. This implied moving to the next stage of the grounded theory methodology.

3.3.2.3. Sorting the memo fund

Sorting is the process through which the researcher organises memos into an outline that is not preconceived but that, again, should be allowed to emerge. As suggested above, the transition from selective coding to memo sorting comes naturally after a period of analysis and when the researcher notices that memos start being repetitive, restating what was already said and not adding much to the analysis. According to Glaser (1998), this transition is experienced by the researcher as a feeling of exhaustion in relation to analysing data and a willingness to start doing 'something' with the memos available.

In the present study, the sorting process was conducted following Glaser's (1978; 1998) guidelines. The following passage is about the process of sorting memos piles and succinctly details how the researcher should proceed:

> At the start, the researcher faces virtually one large pile of memos. He should enter the pile anywhere, no matter, and pick a memo. Place the memo somewhere on a table; it does not matter where. He should usually choose a large table, like a dining table. It is important to have lots of space. Then pick another memo and see by comparing how it is related to the first one picked. Upon comparison they will relate empirically in some fashion like the substantive area is integrated.
>
> Thus, the researcher just keeps picking a memo off the original pile, constantly comparing and the memo will relate theoretically and substantively to other memos. As an integration emerges, resorting the memos occurs as they fit somewhat differently. Just keep sorting, comparing and resorting and the integration of the theory emerges.
>
> (Glaser, 1998, pp. 189–190)

In this way, memos sorting allowed a general outline to emerge. The emergence of the outline exposed a number of concepts that were not fully saturated in the sense that their variation was not yet fully accounted for. However, this did not pose an unsolvable problem. It simply meant that the researcher had to go back to the data and code selectively in relation to these concepts. This process generated more memos and these were sorted in the same way as with the original memo fund.

3.3.2.4. Writing up the theory

Writing-up the theory consisted of two main stages. The first one was a relatively straightforward task during which the sorted memos were 'poured' into a 'Word' document that resulted in the very first draft of the theory. Once the first draft was complete, the second stage was to 'rework' the theory (Glaser, 1998). Reworking was a longer process than transferring the memos to 'paper'. It required clarifying ideas and making sure that they conveyed what was meant to be said (Glaser, 1978). In some cases, reworking consisted mainly of polishing paragraphs or changing the order in which ideas were presented to ensure clarity. In other cases, reworking the theory evidenced areas that were not sufficiently developed. This raised the need of going back to the data, generating new memos and integrating them.

3.3.3. Criteria for evaluating the theory

There are several important points that need to be taken into account to evaluate a grounded theory. First, grounded theories are not intended as accurate descriptions but as 'modifiable conceptualisations' (Glaser, 2003). As a result, grounded theories cannot be evaluated in terms of how accurately they describe, but in terms of how well they explain the main concerns of those involved in the situation being investigated. Being modifiable means that, as relevant data becomes available, the theory does not need to be discarded but it can be easily modified to incorporate new variations (Ibid). This is possible due to the fact that grounded theories do not aim to provide a detailed account of the particulars. Instead, their aim is to provide an abstract interpretation of how the participants are constantly trying to resolve their main concerns (Glaser and Strauss, 1967).

Having a different objective to other methodologies, it is expected that grounded theories be evaluated using different criteria. After all, as Merrik (1999) suggests, the 'quality' of a proposed theory must be evaluated against what it purports and not necessarily by using the canons established for verifying the validity of fundamentally different studies. Since grounded theories are meant to be theories that explain, they must be evaluated by criteria that considers the suitability (as opposed to the accuracy) of their explanations (Glaser, 1978).

The three criteria that are used to evaluate grounded theories are: 'fit', 'relevance', and 'work' (Galser and Strauss, 1967; see also Glaser, 1978; 1998; 2003). These criteria can be seen as substitutes for 'validity' and 'reliability' as commonly established by other 'qualitative-data analysis' (or QDA) methodologies (see Glaser, 1978; 2003). Fit, relevance and work are discussed next.

Fit

According to Glaser (1998) 'fit' refers to whether each of the concepts being put forward by the theory represents "the pattern of data it purports to denote" (p. 236). For instance, in relation to this study, looking at concepts such as 'control' mechanisms (Schoenfeld, 1985b) would have caused a lack of fit in the theory since there was no pattern in the data to suggest that 'control' was an issue of concern for students as they tried to solve problems. Instead – and to provide two examples from the results of this study – it was observed that students constantly try to 'generate knowledge' and that 'looking for patterns' can, in some cases, help them to achieve this (how this occurs is explained in Chapter 4).

Grounded theories are theories that have fit because their concepts are derived from the data and not imported from other theories or derived by speculation or logical deduction. Concepts emerge as a result of constantly comparing incidents from the data and finding suitable names to describe the patterns of behaviour being observed. Thus, this process generates concepts (or codes) that are later compared against other incidents, and then against other concepts (Glaser, 1978). By making further comparisons, these concepts are organised into a flexible theory that is modifiable. This means that the concepts may be modified according to new information in order to make them better representatives of the pattern observed, i.e., in order to assure fit.

Relevance

During a grounded-theory study, the researcher is interested in finding out what is the main concern of those involved; i.e., what is the issue that they are trying to resolve and how. This leads to generating hypotheses about what the main concern is and that explain how it is processed. By doing this, "emergent concepts will relate to the true issues of the participants in the substantive area" (Glaser, 1998, p. 236). This implies that the theory will be of 'relevance' to those involved and to those interested in affecting the situation.

Work

The idea that a grounded theory should 'work' is closely related to 'fit' and 'relevance'. If a theory fits the data and serves to explain how those involved resolve (or fail to resolve) issues that are relevant to them, then the theory should serve to make predictions and to introduce desired changes. For instance, a theory that explains that students may fail to find key ideas if they focus on trying to discover them (see Chapter 4) can be helpful for predicting possible outcomes and for suggesting what actions need to be encouraged or discouraged.

The case studies provided in Chapter 5 will illustrate how the model developed in this study satisfies the criteria of 'fit', 'work' and 'relevance'.

3.3.4. Brief summary and final considerations

This final part of the methodology chapter looked at the methods of data collection and analysis for the main study. It may be said that the former emerged mainly from the information obtained during the exploratory study. As for the methods of data analysis, they were the result of a continuous process of learning about the grounded theory methods. Although this learning curve started to take place since the outset of the study, new aspects of it continued to become evident at later stages. This part (Part III) described the process through which the theory of problem solving that is presented Chapter 4 emerged.

The theory presented next is composed of interrelated concepts. The concepts (as well as their relationships) were created as a plausible hypothesis based on the data. A process of constant-comparisons served to constantly modify these hypotheses for best fit, relevance and work. Furthermore, the theory is general enough to be modified and improved as new data becomes available. Finally, it must be stressed that the theory that will be presented should be evaluated in terms of how well it fits the data, how relevant it is to students taking the problem-solving course, and how it can be useful for helping students understand their own solving processes.

Conscientiously following a particular methodology ensures that the product of a research study will be characterised by certain qualities. However, this also implies that there are other qualities that the product will not have, either because they are not contemplated or are not regarded as most relevant by the adopted methodology. In the light of this consideration, it is important to stress that grounded theories are not verified theories but plausible explanations of how participants resolve, in this case, the problem of tackling non-routine mathematical problems. In this way, the theory that is presented next explains general patterns of behaviour observed as students try to generate knowledge, as they try to generate and validate solutions, and as they try to improve their results. These patterns, however, are not presented as verified hypotheses. Instead, they are integrated into a grounded theory that is modifiable as new data becomes available.

4. THE MODEL: 'SOLUTIONING'

4.1. INTRODUCTION

This section presents the model of problem solving that was developed as a result of this study. The model represents a *substantive* theory that explains students' main concerns as they tackle problems. The aim of the model is not to provide an accurate description of how problem solving took place in this particular situation. Instead, following the grounded theory methodology, the model aims to conceptualise students' main concerns as they try to solve problems and to explain how these concerns are resolved (see Glaser, 1992; 1998). In doing this, it is hoped that the model will raise issues that are of relevance to the participants and to those interested in problem solving. It is also expected that the model will raise issues of interest for researchers in problem solving in general.

The *basic psychological process* observed as students solve problems was called 'solutioning'. Solutioning can be seen as a process where students usually start by generating knowledge (Section 4.2). Once this is achieved, they proceed by generating solutions and validating them (Sections 4.3 and 4.4, respectively). A further stage of solutioning consists of improving the results (Section 4.5). Solutioning may be a cyclic process and the activities that comprise it can be recurrent, meaning that may be conducted time and again during the process.

Besides the stages mentioned above, the model also discusses 'pseudo-solutioning' as an important variation to solutioning. This can be considered as an alternative approach to problem solving that students may take when they fail to solution. Pseudo-solutioning is discussed in Section 4.6. The chapter then concludes with Section 4.7, where final comments are made in relation to the theory generated.

Each stage of solutioning represents an area of concern for students as they tackle problems. The 'selection' and organisation of the sections were not based on a particular pre-existing theoretical framework or on previous experience. They were the result of following a rigorous methodology that focuses on modelling patterns of behaviour observed in situations that can be considered problematic for those involved. Thus, by following this methodology, the result is a substantive theory of problem solving that focuses on those issues that students are constantly trying to resolve as they tackle complex problems (see Glaser, 1978).

Finally, a note on the presentation of the theory should be made. In order to make the theory more accessible to the reader, a number of examples are given. The reader must be warned that these examples should be read as illustrations of the concepts being discussed.[17] They are not presented as proofs and thus not all concepts are necessarily accompanied by an example. Moreover, in some cases, examples are presented in the form of discussions within the text rather than in the form of verbatim quotes. This does not only coincide with the grounded-theory methodology but it also allows the researcher to present complex

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[17] This is also the case for some explanations that may sound descriptive to the reader. As Glaser (1978) points out, "Indicators for the concepts that are descriptive statements are used only for illustration and imagery. They support the concept, they are not the story itself." (p. 134)

situations that cannot be conveyed in a brief quote. The following quote from Glaser (1978) clarifies the issue of presenting the final product of a grounded theory in a more succinct way:

> The *credibility* of a theory should be won by its integration, relevance and workability, not by illustration used as if it were proof. The assumption of the reader, he should be advised, is that all concepts are grounded and that this massive grounding effort could not be shown in a writing. *Also that as grounded they are not proven; they are only suggested.* The theory is an integrated set of hypotheses, not of findings. Proofs are not the point. Illustrations are only to establish imagery and understanding as vividly as possible when needed. It is not incumbent upon the analyst to provide the reader with description or information as to how each hypothesis was reached.
>
> (Glaser, 1978, p. 134; emphasis in the original)

4.2. GENERATING KNOWLEDGE[18]

An important activity in students' problem-solving process is to generate knowledge about the situation or, in other words, to generate relevant information and to gain understanding. This is usually conducted at the start of the process, particularly if students know little or nothing about the situation. For this reason, generating knowledge seems a good place to start the discussion on students' problem solving processes. However, it must be made clear that the

[18] The term 'knowledge' is used as representing both information and understanding.

need to generate knowledge will continue to emerge throughout the process.

4.2.1. Strategies for, and ways of, generating knowledge

A common strategy to generate information and understanding is to reduce the complexity of the situation being dealt with. When reducing complexity, students "start at the beginning" and focus on intentionally simplified or even trivial versions of the situation. The aim behind reducing complexity seems to be to start gathering the information and understanding that will allow students to eventually move on to more sophisticated cases. Reducing complexity may help students gain access to complex situations by reducing them to simpler, more manageable ones.[19] The following example illustrates this idea. In it Hillary started 'simple' with the intention of working her way up to the desired solution.

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[19] Other studies make reference to similar strategies in different contexts. Hazzan (1999; 2001), for instance, reported that students reduce abstraction when dealing with abstract algebra concepts. Students may do this, for example, by focusing on concrete aspects of the situation when it cannot be conceptualised as a whole. Reducing complexity and Hazzan's concept of reducing abstraction are related in the sense that, in both cases, students simplify a complex situation in order to be able to deal with it. Nonetheless, reducing complexity and reducing abstraction differ in that the former is usually is a consciously chosen activity. As for reducing abstraction, Hazzan suggests that this seems to be an unconscious rather than a conscious 'choice'.

Right, let's think about this. Start simple and work my way up to step 25. Ok, what's the probability of me landing on step 0? Has to be 0 as I start on step 1. So how about the probability of landing on step number 1? Well, I start there, so it is a certainty – probability 1. Right, now it starts to get interesting: what's the probability of landing on step number 2? Let's think about this...

(Hillary, Steps, p. 1)

Another important strategy that helps students generate knowledge is data organising. By developing ways of representing information or arranging the available information in convenient ways students make the process of generating knowledge more manageable. Introducing a notation, a table of values or a diagram are common examples of data organising. The following examples illustrate the latter cases.

I will use a table to search for some patterns:

P	1	2	3	4	5	6	7
$D_p(1,1)$	1	4	10	20	35	56	58

(Keith, Sums of Diagonals, p. 2)

If I try to draw a diagram of the possible outcomes this may help give me a better idea of what is happening and may lead to further development.

(Lila, Steps, p. 1)

Besides making use of some strategies, students generate information and gain understanding about the situation in many ways. The following is a brief list of the types of activities that students conduct for this purpose. The list is not exhaustive and other activities may be included from further research:

Specialising

A common way in which students generate information and understanding is by 'specialising', i.e., by dealing with particular aspects of the situation. When students specialise, they focus on isolated aspects of the situation and thus on simplified versions of the problem. For this reason, it may be said that specialising is intrinsically about reducing complexity. Most students specialise at one point or another in their processes and the choice seems to be made in a 'natural' way (“*My instinct to this problem is to start from the easiest case.*”) However, during the course, students were specifically introduced to Mason et al.'s (1982) idea of specialising. This fact may account for the students' tendency to specialise and to label their activity in that way.

> I will start by specialising and using squares, since they seem more straightforward, and then progress to rectangles.
>
> (Hannah, Cartesian Chase, p. 2)

Importing

In order to start making sense of the situation, students sometimes 'import' ideas or information from sources other than the problem and the situation that it presents. These ideas may be relevant to the problem and in that sense that they may help students to better understand the situation and deal with it. Recalling past knowledge or experience are common ways of importing information.

> I know a similar problem. Diagonals of a Rectangle, which seems to be related and I think I can use my solution.
>
> (Emilio, Visible Points, p. 1)

> Fault line – brings to mind brick walls. In a brick wall you couldn't have such a line because the wall would be too weak. Conjecture that brick laying pattern may prove the answer. I will carry on specialising and will come back to this conjecture later.
>
> (Kirk, Faulty Rectangles, pp. 1–2)

Students may also import information from other sources such as their notes (or any bibliographical reference, from that matter). Sharing ideas with fellow classmates may also be a way of gaining information and/or understanding. Importing requires that borrowed ideas are evaluated in terms of their relevance and applicability to the present situation. Importing can provide useful information but also presents the risk of considering irrelevant ideas that may have to be abandoned at a later time. This was the case for the idea that considering how brick walls are constructed could provide useful information for tackling the 'Faulty Rectangles' problems (see example above). Several students considered this idea and either forgot about it or consciously discarded it at a later time.

Hands on investigations

Another way of generating knowledge is by taking a 'hands-on' approach and carrying out the basic operations that are relevant to the situation. For instance, in 'Faulty Rectangles' students physically constructed rectangles with dominoes and observed what combinations could lead to fault-free rectangles. Another example can be given in relation to 'Ins and Outs', where students conducted hands-on investigations by folding pieces of paper and observing the sequences of folds that were generated. Hands-on investigations provide students with first hand experience of the situation and may lead to gaining important knowledge and understanding.

> Shall I try playing it? Use a chessboard and a pawn.
>
> (Jules, Cartesian Chase, p. 1)

Close analysis of information

An important way of finding more about the situation is by carefully analysing the information that is available or that has been made available. In some cases, information and understanding may emerge easily by looking at the data. In other cases, however, students have to make conscientious efforts in order to generate knowledge. By considering (or reconsidering) available information and trying to understand it, it may be possible to derive further information and understanding from it. This may involve reviewing the data and making deliberate efforts at drawing out observations and ideas.

Ok, let's look at our previous example.

N=4

Stage 1: 1, 2, 4 [Divisors of N]

Stage 2: (1), (1, 2), (1,2,4) [Divisors of divisors of N]

Is there any significance about the numbers at stage 1?

(Jared, Liouville, p. 6)

It is not uncommon for students to combine these knowledge-generating activities by either conducting them at the same time or by sharing information from one activity to another. For instance, when specialising, students may take a hands-on approach. Another example is when students conduct a close analysis of information that has been generated after a period of specialising. After all, there is no imposed limit to what students can do in order to generate information and understanding.

The need to generate knowledge will continue to emerge throughout the process. New information and understanding may be required at any stage, from situations in which students are looking for new ideas to situations where they are trying to take an idea further. Consequently, students may be involved in the activities discussed above at any time during their process.

Moreover, students make reference to the information they observe in the form of written or verbal observations. Trying to gain knowledge about the situation leads students not only to *noticing* but also to *making a note* of those new pieces of information that may be relevant in terms of generating a solution. The next subsection looks at the observations that students make as a result of dealing with the data.

4.2.2. Making observations

The information and understanding that students generate may become manifest in the form of observations. In fact, making observations can be seen as a consequence of trying to generate knowledge. Observations are facts or ideas about the situation that students may find interesting or relevant, and that they point out in a written or verbal way. In some cases (like in the example below), these observations may lead directly to an initial solution.

> AHA! The pattern behind the centre is just a pattern of the previous one, while those behind is just the opposite way around [...]
>
> Therefore, if we repeat this, we would be able to generate a sequence after 10 folds.
>
> (Karina, Ins and Outs, pp. 1–2)

Although observations may lead to generating a solution (see Section 4.2.3), in some cases they may involve information that may or may not be used at a later time. This is to say that not all observations will be useful in the same way. Observations may inform students about ways to generate a solution (like in the example above) but they may also provide less central (though not necessarily unimportant) information. Furthermore, important observations may be easily identified as such. Alternatively, it may take the student time and effort to be able to tell whether a certain piece of information is relevant or not. The first example below illustrates a situation in which the student realised, almost immediately, the importance of an observation. In the second example, the student noted an observation but was unsure of its relevance in relation to solving the problem.

> Slope=(4-1)/(4-1)=1.
>
> AHA! The gradient of slope 1 is 1. I can use the same method and apply it to slope 2.
>
> Slope=(9-1)/(5-1)=2.
>
> Aha! I got it!
>
> (Patrick, Sums of Diagonals, p. 2)

> Obviously, I can only pull out the numbers 1 and 2 and the difference between these is 1.
>
> Hmm... could this always be the case (wild guess)? Or is it too early to tell.
>
> (Aminta, Hat Numbers, p. 1)

When students come across an observation, they may adopt a 'pragmatic' approach. Adopting a pragmatic approach involves focusing not only on the observation itself but also on how it can be used for generating a solution. When students adopt a pragmatic approach towards making observations they ask themselves questions like "How can this [idea, fact, etc.] be used?" The example below (as well as Patrick's example above) illustrates a case where the student considered observations in a pragmatic way.

I realised that I'd go through a point if the two lengths of the sides had a factor in common and so could be split into a number of identical smaller rectangles one after the other.

How can I use this?

(Rafael, Visible Points, p. 3)

A pragmatic approach may help to discriminate unimportant ideas and thus may help in making the process more efficient. Thinking in terms of how ideas can be used seems to lead to starting to generate a solution sooner than if observations are made without considering their usefulness or applicability. As the examples below suggest, a pragmatic approach can help students decide more efficiently whether an idea is useful and, if so, how.

Points (i, j), where i, j are positive.

Defined to be BELOW (m, n) where m, n are positive when $i \leq m$ and $j \leq n$.

$\therefore$ (i, j) is below itself – not particularly important.

(Dylan, Visible Points, p. 1)

Already know why this works for primes so maybe can use prime factorisation of numbers to prove the conjecture. See how multiplying a number by a prime alters the number of divisor of divisors…

(Julia, Liouville, p. 3)

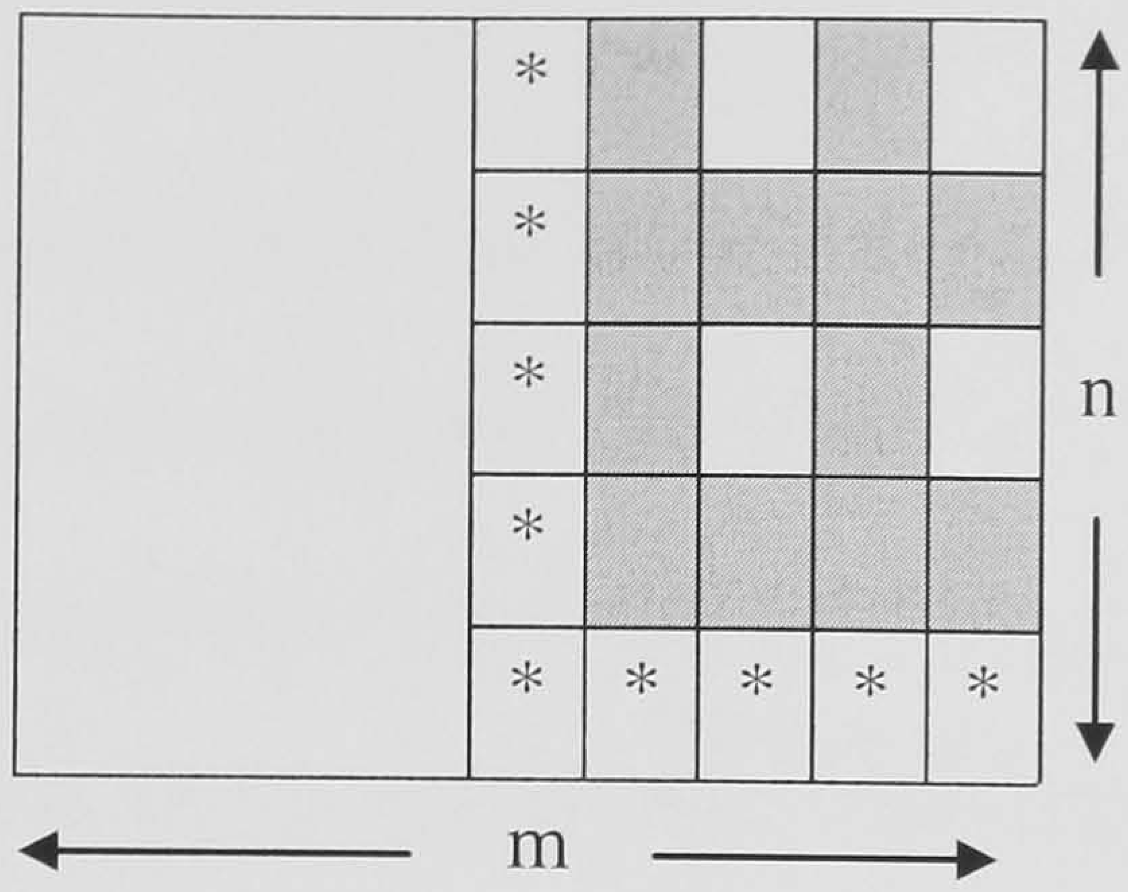
4.2.3. The role of key ideas

Having discussed how students generate knowledge about the situation and how this knowledge becomes manifest, this subsection will look at 'key ideas' as knowledge that is crucial to solving the problem and that students employ directly to generate a solution. This subsection considers how 'looking for patterns' may lead to gaining useful information and understanding but may also become a 'blinding' activity. Also, 'key searching' is discussed as an activity in which students try to discover special features about the situation.

As said in the previous section, some of the observations that students make during problem solving lead directly to generating a solution. Since these observations usually refer to crucial aspects of the situation they can be called key ideas. Solutions are usually based on a key plan or idea that provides hints as to how a solution can be obtained. In order to deal with 'Diagonals of a Rectangle', for instance, some students used the fact that there was a relationship between the highest common factor of the rectangle's dimensions and the number of rectangles crossed. This fact was the key idea on which most (if not all) students who provided a solution for this problem based their processes.

Key ideas sometimes emerge as sudden realisations of important aspects of the situation. These ideas may appear as important breakthroughs (as the student below suggests) and give students the feeling of having discovered how to generate a solution.

AHA! This is a huge breakthrough! This means that for any board with m, n≥4 there is a definite route to victory.



In order to make sure that it is you (and not your opponent) that takes this route, you must be careful. Anything that happens before the row marked (*) is not important. As long as we can guarantee that our opponent moves to (*), we have won, since we can then move to a definite win position.

(Leonard, Cartesian Chase, p. 5)

In other cases, key ideas emerge as less of a surprise. In these cases, key ideas may come gradually as knowledge and understanding increase. In the case of the Arithmagons problem, for instance, the key idea was usually related to realising that the number of sides plays an important role in whether or not the required numbers can be uniquely determined. However, for most students this realisation did not come as surprise but after working with a variety of cases producing a range of results; i.e., after a process of knowledge generation.

In either case, it seems that being able to arrive at a key idea requires a good deal of understanding of the situation. Being able to see a key idea means also being able to see its significance, its importance in relation to the situation and how it can be of use.[20]

Thus, key ideas can be seen as the product of gathering sufficient and relevant information and understanding to be able to start generating a solution. Even if they arrive as sudden realisations they seem to be the result of a continuous process of gaining knowledge about the situation. The following sub-sections look at ways in which students generate, and search for, key ideas.

Looking for patterns

Looking for patterns is a way of learning about the situation that can lead to finding key ideas. Looking for patterns may involve looking for salient or relevant information about the situation and can result in starting to generate a solution. It may be said that students look for patterns hoping that, when they find one, they will be able to transform it into a formula or to make a general statement about the situation.

Looking for patterns can be a useful activity that generates relevant information. For instance, noticing a pattern in the way the creases were formed in the 'Ins

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[20] In relation to this, Raman (2003) observed that the key ideas that more experienced solvers use to provide a mathematical proof "give a sense of understanding and conviction" and show "why a particular claim is true" (p. 5). In more general terms, Barnes (2000) suggested that when students and more experienced mathematicians are able to see a key idea the following takes place:

> There is a claim to a sudden realisation of new knowledge or understanding. Usually this knowledge is 'seen' with great clarity, or experienced with a high degree of confidence or certainty.
>
> (Barnes, 2000, p. 34)

and Outs' problem allowed most students who attempted this problem to tell what the creases for the 10th fold would look like. Furthermore, since looking for patterns involves knowledge-generating activities such as data organising and analysing the data, it usually leads to gaining understanding and learning about the situation. The following quote exemplifies a situation in which looking for patterns led to gaining knowledge:

> We notice a pattern here for any two consecutive terms, adding the numerators gives the denominator of the 2nd, and the denominator are all powers of 2.
>
> So,
>
> 5 steps: P(25)=21/32=21/2^5
>
> 6 steps: P(25)=43/64=43/2^6.
>
> I can see the general formula would be P(25)=?/2n.
>
> (Leonard, Steps, pp. 7-8)

Thus, in many cases, it may be said that looking for patterns is a fruitful activity. However, looking for patterns can also become a 'blinding' activity that hinders the attainment of information and understanding. Focusing mainly on looking for patterns and neglecting trying to see other aspects of the situation may decrease the possibility of gaining useful knowledge. In the example mentioned above, most students were able to see how creases were formed and thus were able to tell how the 10th fold would look like. However, very few students were able to provide a general (non-recursive) formula for this sequence. Students that were able to provide such a general formula did so not by continuing to look for patterns but by gaining a deeper understanding of how the sequence of 'ins' and 'outs' was generated.

Looking for patterns can be useful way of generating knowledge and may even lead to finding key ideas. However, to be able to eventually generate a satisfactory solution, other information and understanding need to be sought as well. Focusing exclusively on trying to find a pattern can lead to a dead end if it prevents students from genuinely learning about the situation.

Key searching

As mentioned above, key ideas allow students to start generating a solution. Finding a key idea is certainly related to successful problem solving and students seem to be aware of this. For this reason, students may look for key ideas by looking for patterns (as was mentioned above). Another way of looking for key ideas is by 'key searching'. Key searching means looking for key ideas in a direct way by trying to discover special features about the problem or by trying to find "what is so special" about the situation.

> I'm looking to see if the number left in the hat has some special quality... Still stuck! Maybe I should go back and try the odd numbers. After all, as this may be the missing clue to the solution...
>
> (Aminta, Hat Numbers, pp. 2–4)

As students try to gain knowledge and understanding of the situation, it is likely that they will eventually come across key ideas. Paradoxically, though, key ideas seem less likely to emerge if students focus on actively seeking them. The reason for this may be that consciously searching for key ideas may divert students' attention from trying to learn about the situation. During key searching, students seem to be so concerned about trying to find some "special" clue or quality that they may neglect other important information. In the case of the Liouville problem, for instance, at least two students spent the majority of their process trying to figure out what was so special about sequences of numbers that if added and then squared give the same value as when they are cubed and then added. In these extreme cases, students were neither able to make any significant progress, nor capable of identifying any of the key ideas that allowed other students to generate a satisfactory solution. The following is an excerpt from one of these students' work.

> I'm looking to find out why the calculations in the problem are equal, so that I can hopefully prove my conjecture. My proof has got to contain something in relation to the fact that the numbers used are the number of divisors of the divisors of the original number (!!) What is so special about these numbers?
>
> (Emma, Liouville, p. 4)

Thus, it may be said that when students search for key ideas, they may ignore important information that, if not a solution in itself, can be used towards that end.

In general, not all students engage in key searching and those who do may eventually abandon this activity and try to generate information and understanding in other ways. However, the implications of key searching make this activity an important one to consider. There is no evidence to suggest that key-searching is related to mathematical background. Students who key-searched belong to all three-groups in the class, i.e., mathematics, computer sciences and BA(QTS) students. What can be suggested is that key-searching may be related to the features of the problems involved. This hypothesis is supported by the fact that more students key-searched in the 'Liouville' problem than in any other. There is not sufficient evidence to take this hypothesis further. This issue can only be suggested for further research.

4.2.4. Generating understanding and situational reasoning

This subsection deals more closely with the issue of generating understanding. This issue plays an important role in being able to generate a solution and students will seek to generate understanding about the situation at one point or another during their processes. Thus, a good place to start a discussion on the characteristics of generating understanding during problem solving is by considering the following quote from Thurston:

> On a more everyday level, it is common for people first starting to grapple with computers to make large-scale computations of things they might have done on a smaller scale by hand. They might print out a table of the first 10,000 primes, only to find that their printout isn't something they really wanted after all. They discover by this kind of experience that what they really want is usually not some collection of answers – what they want is *understanding*.
>
> (Thurston, 1995, p. 29; emphasis in the original)

Although Thurston's assertion was made in reference to professional mathematicians, it may be said that it applies to students as well.

Gaining understanding is an important aspect of the problem solving process. Most students try to gain understanding of the situation to be able to start generating a solution. As a student put it, it is easier to generate a solution by "understanding the underlying principles" of the situation. In general, and as it is illustrated in the next example, it seems that having a better understanding of the situation empowers students and allows them to generate a solution and to look for ways of taking it further.

> Ok. We can see how this game will operate. If you are given your turn when the piece is in coordinates (0, 1), (1, 1) or (1, 0) then you win – you can simply move the piece to the top right and hey presto.
>
> Can we give any more information than this about how to win?
>
> (Jan, Cartesian Chase, p. 1)

Situational reasoning

An important way of gaining understanding is by reasoning in terms of how the data is created, or how it stems from the situation. This form of reasoning has been called 'situational reasoning'.[21] Although not all students try to gain understanding in this way, and those who do may not do so all the time, it may be said that thinking in terms of situational reasoning is a common practice. In the following quote, for instance, the student came to realise the 'underlying principle' behind the entries in a given table. This provided her with empowering understanding of the situation. In other words, this provided her with sufficient understanding as to know how to 'work out' any entry for the table.

> I can't believe how I missed how every entry in the grid is the product of its coordinates...
>
> This means that given any coordinates we can work out what the entry is.
>
> (Nadia, Sums of Diagonals, p. 4b)

Trying to think in terms of situational reasoning, may lead to a kind of understanding that allows students to make informed decisions as to what to do next. In other words, it leads to what Skemp (1976) called 'relational understanding'. This type of understanding allows students to know "both what to do and why" (p. 20) and for this reason it is usually an important asset during

[21] Simon (1996) observed a behaviour that is similar to thinking in terms of situational reasoning. He observed that in some problem-solving situations students tried to make sense of "how the system in question works" (p. 198). He used the term 'transformational reasoning' to refer to this way of thinking about the situation.

problem solving. The understanding achieved by the student in the following example is relational in the sense that it provides information that can be useful for understanding the situation and deciding what to do next and why. Furthermore, the understanding seems to have been generated by reasoning in terms of situational reasoning:

> Let's try to think logically about specifically when a diagonal would pass through a corner.
>
> AHA! I think the diagonal will pass through a corner when n and m have a common factor greater than 1. This makes a lot of sense because it implies that the rectangle can be split up into smaller rectangles with the same diagonal, and therefore the diagonal would pass through the corners.
>
> (Hannah, Diagonals of a Rectangle, pp. 3–4)

Thus, it may be said that situational reasoning may help students gain valuable understanding about the situation. However, students do not always reason in terms of situational reasoning and, instead, may decide to consider other aspects of the situation such as observable patterns. Furthermore, they also may generate understanding by trying to generate knowledge in the ways discussed above.

Having discussed important aspects of generating knowledge, we now to turn to the issue of generating a solution once information and understanding have been achieved.

4.3. GENERATING SOLUTIONS

The previous sections looked at how students generate knowledge about the situation. However, while it is important, it may be said that generating knowledge is not the final aim of problem solving but a means for making necessary resources available. The aim of problem solving is to generate a solution and students will start attempting to do this as soon as sufficient knowledge has been gathered. Two ways in which students may try to generate a solution is by reasoning 'deductively' and 'inductively'. Reasoning in terms of situational reasoning can also play an important role in generating a solution. Furthermore, students may also rely on 'guessing' to generate a solution and this may lead to ungrounded ideas.

In order to start trying to generate a solution, students need to reach a *point of sufficient understanding* in which what they know about the situation allows them to start generating a solution. In other words, a point of sufficient understanding is the minimum amount of understanding (about the situation) that students have to gain in order to be able to solution. Trying – or being forced – to generate a solution below that point (i.e., with insufficient understanding) is usually a difficult task, likely to turn into a frustrating experience. Moreover, students will usually start generating solution as soon as they reach this point.[22]

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[22] Simon (1978) seems to have noticed the importance of reaching a point of sufficient understanding before a solution can be attempted. This idea is implicit in the following quote:

4.3.1. Generating solutions through inductive and deductive reasoning

Once students are ready to generate a solution, they may rely on deductive reasoning in the sense that they may proceed by following logical implications from one idea to another until a conclusion is reached. Reasoning deductively seems to be held in high regard by most students since, whenever possible, they will try to arrive at a solution in this way. In the Liouville problem, for instance, most students' first attempt at generating a solution involved providing some version of the following deductive argument.

> A prime number n has divisors 1 and n only, by definition.
>
> 1 has one divisor (1)
>
> n has two divisors (1, n)
>
> The sum of the number of divisors of divisors is therefore 1+2=3 and squared this is 9.
>
> The sum of cubes of the number of divisors of divisors is $1^3+2^3=9$.
>
> So the two numbers are equal for prime numbers.
>
> (Julia, Liouville, p. 2)

Also, as one student put it:

> The solving process appears to exercise overall control in the sense that it begins to run as soon as enough information has been generated about the problem space to permit it to do anything. When it runs out of things to do it calls the understanding process back to generate more specifications for the problem space.
>
> (Simon, 1978, p. 285)

I generally try to use deduction. Deduction is 'more valid' in mathematics although I often use inductive arguments.

(Leonard, interview)

When students reason deductively, they sometimes base their arguments on a relevant piece of mathematical knowledge. This piece of knowledge may consist of a mathematical concept or a procedure. In other words, students may build a deductive argument by making use of a concept or a definition to devise a logical chain of reasoning that leads to a solution. In the example above, for instance, the student based her deduction on the mathematical definition of 'prime number'. The way she made use of this definition allowed her to generate a logical chain of reasoning and to achieve an initial solution. Alternatively, students may apply a previously known procedure that guides them through an already validated process and thus allows them to create a logical chain of reasoning. The Arithmagons problem provides a good example of this situation. In most solutions to this problem, students based their arguments on procedures for solving systems of linear equations. Although making use of known procedures may be more straightforward than deciding how to apply a concept, in the sense of constructing logical chains of reasoning, the former can also be considered a deductive argument.

Whenever there is the possibility of generating a deductive argument from the knowledge available, students will usually follow this route. However, when this is not the case, one option is to continue trying to generate information and understanding until it is possible to generate a deductive argument. Alternatively, students might start trying to generate a solution in different ways.

As an alternative to reasoning deductively, students may reason inductively by making tentative conjectures or generalisations out of the information that is available. Reasoning in this way may involve *plausible* – as opposed to *logical* – reasoning. As said, making deductions involves deriving ideas that are a logical consequence of the information available. However, when students reason inductively, ideas may also be derived by the solvers' subjective perception of the situation. The following example illustrates inductive reasoning.

> All the results are in a range 48–63...
>
> Notice that the last two results are equal.
>
> Conjecture 1: the percentage of visible points converges to a number.
>
> (Aminta, Visible Points, p. 4)

Generating ideas inductively may lead to inaccuracies or even to incorrect solutions. This is not to say that deductive reasoning is foolproof. What this suggests is that, due to the nature of inductive reasoning, students sometimes have to accept, and deal with, the fact that they are working with imperfect results. However, this is usually not a serious problem since ideas can be re-examined and modifications can be made. Moreover, checking whether a tentative solution is correct and makes sense allows students to improve their solution and increases their knowledge and understanding of the situation (see Sections 4.4 and 4.5). This, together with the fact that an initial solution – i.e., a starting point – is already available, seems to outweigh the possible drawbacks of generating a solution in an inductive way. The example below illustrates the learning process that takes place as students generate solutions in an inductive way.

> Ok, I think I see a pattern forming. I'm not sure quite how to express it, so first I'll just say it. The touching squares go in steps. Obviously, there are always the same number of vertical steps as there are m rows. Now if both n and m are even, then the diagonal will pass through a corner which will touch all four squares that have this corner. Therefore, any two rows one above the other will overlap by two squares. If either n or m are odd, then the diagonal does not pass through any corners and so there is only an overlap of the square.
>
> Oops – I've just found a counterexample to my theory of the diagonal only passing through corners when n and m are even. I'll have to rethink that.
>
> (Hannah, Diagonals of a Rectangle, pp. 2–3)

As said, most students will try to work deductively if at all possible and if not they may choose to work inductively. However, inductive and deductive reasoning are not mutually exclusive as this generalisation may suggest. In fact, it may be said that students combine both approaches and that they complement each other. For instance, after reasoning inductively and generating some feasible conjectures, students may turn to the deductive approach.

Besides reasoning inductively and deductively, students may generate a solution as a result of reasoning in terms of situational reasoning. The previous section discussed how thinking in these terms may provide students with information as to what to do next and why. Since this information is easily translated into a solution, and since this is sometimes done in a direct and even immediate way, situational reasoning can be considered as another way of generating a solution. It seems that solutions achieved in this way are more 'transparent' than solutions arrived at by deduction or induction. When students reason in terms of how data is created, or in terms of how it stems from the situation, it may become evident what a solution should look like and why.

4.3.2. Guessing and ungrounded ideas

It was mentioned before that tentative solutions that are generated inductively or in any other way are usually a good place to start generating a more comprehensive solution. However, there does seem to be an exception to this case. In some cases, students' reasoning can be better explained as 'guessing'. When students guess a solution, their reasoning is unclear and it is usually difficult to tell where ideas come from. Yet, from the comments that students make, it usually becomes evident that they may be testing their luck and proposing ideas without going through conscientious reasoning about the situation.

> Try completely new approach. Convert sequence into a straight number using binary (might get lucky).
>
> (Sebastian, Ins and Outs, p. 5)

> We can see by looking at the diagram that there are three points that would not be visible. Could I work this out algebraically so that it applies to any size grid square?
>
> Maybe it could be (i–j)/j, that would be (9–3)/3=6/3=2. That doesn't work!
>
> Maybe (i–j)/i would be better: (9-3)/2=3. Would this work for other (i, j)?
>
> [...]
>
> (Gina, Visible Points, pp. 3–4)

Ideas that are arrived at by guessing are usually ungrounded, i.e., they are more the product of inventiveness and trial-and-error than of carefully analysing the data. Although the relation between guessing and ungrounded ideas is somewhat evident (i.e., a guessed idea is ungrounded), guessing a solution is not the only way in which students may generate ungrounded ideas. For instance, trying to invent a situation to explain how data is created may also lead to generating ungrounded ideas, particularly when used without considering sufficient empirical data. In other words, in an attempt to provide an account of how data is created students may fall into 'making up' an explanation that is more the product of their creativity than of what they know about the situation.

Solutions based on ungrounded ideas can cause problems and frustration by leading to inconsistencies. Such was the case of a student that provided an interesting explanation (reproduced below) as to why it is *not* possible to build a fault-free rectangle (see the 'Faulty Rectangles' problem). Since fault-free rectangles *can* be built, she found it hard to elaborate her idea further. In general, although ungrounded ideas can be problematic, a positive aspect of them could be that the frustration that they cause may become, in some cases, a good place for starting to learn about the situation.

> What I want to find is that every rectangle made up from dominoes can be split up into two rectangles that are made up from dominoes. Where they join is the fault line. Ideally, then, each rectangle is split up into rectangles which also have fault lines on and on until all that is left is single dominoes.
>
> (Hannah, Faulty Rectangles, p. 8)

Summarising, students may generate an initial solution by reasoning deductively, inductively or situationally. Although students may have a 'predilection' for deductive reasoning, it seems that this predilection is based more on their beliefs about mathematics (deductive reasoning being 'more valid') than on the results that they obtain from reasoning in this way. Inductive reasoning may allow students to generate initial solutions that can later be improved. Moreover, thinking in terms of situational reasoning is another way of generating 'transparent' solutions. Although the last two types of reasoning may not be the students' first choices, they can be efficient ways of generating results.

Once a solution is generated, it may be validated and improved. The next two sections look at 'validating' (Section 4.4) and 'improving' results (Section 4.5).

4.4. VALIDATING RESULTS

During their problem solving processes, students look for ways of validating the ideas that they are generating. To do this, they seek for 'mathematical conviction' and 'cognitive reassurance'. In other words, students try to verify that their results are mathematically or numerically consistent and seek to explain why this is the case. When students validate their results in these ways their main concern is being on the 'right track' and having a clear understanding of the situation. Thus, the arguments that they produce can be considered as personal 'proofs' aimed at convincing themselves that their results are acceptable.

Once students have achieved a satisfactory solution, they may seek to provide a formal mathematical proof of their work. However as the quote below suggests, providing a formal argument seems to have a different purpose than making sure that a solution is correct and makes sense.

> This certainly seems to hold for all m, n [where m and n are natural numbers], but whether or not I can prove it is a different matter.
>
> (Leonard, Diagonals of a Rectangle, p. 19)

It seems that trying to provide a formal mathematical argument that proves that a solution is true is more a way of improving a solution than of making it convincing to themselves. For this reason, providing a formal proof will be discussed in the next section below (Section 4.5).

4.4.1. Seeking 'mathematical conviction' and 'cognitive reassurance'

Mathematical conviction (hereafter MC) refers to those situations in which students seek to verify that their results are mathematically correct. 'Cognitive reassurance' (hereafter CR), on the other hand, refers to situations in which students try to reassure themselves that the ideas that they are generating make sense.

Students seek MC by looking for errors and inconsistencies in their work. To do this, they may review their process to make sure that the chosen procedures were properly conducted. Besides verifying their procedures, students may check to see whether general results work in particular cases. If the results obtained from particular cases are as expected or match with previous data, then they can be accepted.

> I will now see if it works for the numbers I have so far.
>
> (Jasmine, Sums of Diagonals, p.6)

> Check: Does this match the examples I have tried so far?
>
> (Julia, Liouville, p. 10)

Students achieve CR by explaining why ideas are true or by providing justifications as to why they should be accepted. Seeking CR may vary in its degree of complexity. In its simplest form it can involve making mental approximations of the solution and comparing the actual solution to this expected value. However, it is not uncommon for students to provide sophisticated explanations to justify the reasonableness of a solution.

> So P(landing on step 25)=13/18446240= Very small number!
> This can't possibly be right. It can't be that small a chance that you land on step 25! Something has gone wrong but I can't see for the life of me what!
>
> (Kylie, Steps, p. 3)

> Why does it work? Aha! Looking at any diagonal, moving down one adds 1 to the first element, 2 to the second, etc. And then finally one more element equal to the new 'x'.
>
> (Marcus, Sums of Diagonals, p. 3)

It may be said that some activities lead students directly to achieving CR. It was observed that students sometimes gain understanding and generate results by reasoning in terms of situational reasoning. For instance, thinking about the process that gives rise to a pattern may lead to developing a formula that models the situation. Reasoning in this way usually provides the student with a clear understanding of why the result must be true and therefore with CR. In other words, when students generate results by reasoning in terms of situational reasoning, CR may be automatically achieved. In cases like this, students find that there is no need to further validate their results as they stem as a logical consequence of the information available.

What I did was change the bottom line from [two horizontal] to [two vertical] using the two dominoes to block the possible fault line on the newly formed internal line. Then I filled in spaces until I arrived at a new rectangle (continued this process downwards another step since it didn't fit). I will try this a few more times.

I would say that this constitutes a proof for me that 5x2M rectangles can be formed, with M an integer greater than or equal to 3 and that the rectangle is fault-free.

(Marvin, Faulty Rectangles, p. 5)

MC and CR can be sought in tandem, one after another. Once generating an idea or a result, students may check to see that it is mathematically correct. Then they may proceed by verifying whether it makes sense. The example below illustrates this situation.

This looks like the number of creases is 2^a-1.

Check for a=6.

From previous formula creases = 31+32=63=2^6-1.

I can see this would be true because each time I am doing n+(n+1) to get the next term which is equal to 2n+1, so each time I am doubling the previous number (which is less than 2^n as 1 is one less than2^1=2) which would give me 2^n=2 and then adding one so I get 2^n-1.

(Jasmine, Ins and Outs, p. 4)

This is not to say, however, that after seeking MC students will always proceed by seeking CR. In some cases, students may not be interested in explaining why ideas are true so long as they seem correct. In other cases, students may be able to verify that their results are correct but may find it difficult to provide an explanation as to why this is the case.

> It does seem to be the case that the Liouville results are always identical, regardless of the chosen starting number. *Sadly, I have no theories as to why this occurs.*
>
> (Conrad, Liouville, p. 5; emphasis added)

Seeking MC and CR are recurrent activities that students usually conduct throughout the process. Constantly seeking MC and CR carries the advantage that the ideas that are being generated can be modified if they are not correct or do not make sense. Continuously trying to verify that ideas are correct and make sense ensures that inconsistencies are brought to the fore and provides an opportunity to amend them.

4.4.2. Validating results as a desirable condition

Achieving MC and CR is a desirable condition in the sense that students prefer to work in a context in which the results that they are obtaining and on which they are building their solution seem to be correct and make sense. Building on incorrect ideas or results can lead to introducing errors that may later jeopardise the process. By constantly validating their ideas, students ensure that further results are built on safe bases.

> Now I want to check it again that my result is right before I go any further from here. Therefore I count the number of grid squares that are touched by the diagonal again from the grid squares that I have already drawn. And it's correct!
>
> (Anibal, Sums of Diagonals, p. 4)

Successfully validating results by achieving MC and CR provides students with the feeling of being on the 'right track' and ensures that the ideas and results that are being generated are dependable. This in turn allows students to 'move on', i.e., to continue advancing the process. It may be said that achieving MC and CR is a desirable condition for students to move on. Failure to achieve MC or CR may require students to reconsider aspects of their process and to try to correct any inconsistencies.

Although achieving MC and CR is usually a desirable condition, in some cases it can be seen as almost indispensable in order to move on to the next stage. This may be the case, for instance, when students are trying to extend an idea or a solution further, either to other relevant cases or to another domain. Before engaging in extending a solution, students usually require that this solution or idea is validated in the sense that it is correct and makes sense.

However, in spite of its relevance, validating results by seeking MC and CR is not indispensable and students may sometimes neglect or avoid this activity. If the idea that they are working with is not crucial or is uncontroversial, students may choose to trust their work and move on without seeking to verify whether it is correct and makes sense. In this case, validating a result or idea can be postponed. As one student explained, checking results can be omitted if the idea being considered is not so relevant and seems correct. In his opinion, "there is no time to stop and check every single result" (Patrick, interview).

Furthermore, the need to achieve a solution, together with time or knowledge limitations, may force students to avoid seeking MC and CR. In other words, students may avoid seeking MC and CR when moving on is more pressing than making sure results are correct and make sense. By not questioning their results, students can prolong a course of action even it is flawed or inaccurate. Alternatively, students may seek MC and CR in superficial or even spurious ways. In the latter case, students may accept arguments that are not necessarily rigorous but that provide sufficient conviction or reassurance as to allow the student to move on.

Finally, seeking MC and CR does not imply that ideas or results will always be satisfactorily validated. Students' 'checks' and explanations may not always be accurate. In some cases, students may be aware of this inaccuracy and decide to move on regardless (or decide to go back and take corrective measures). For instance, students may not be able to fully explain why an idea is true but, in spite of this, may decide to move on. In other cases, students may be unaware that their results or explanations are flawed and thus they move on without hesitation. Moving on under these circumstances is not necessarily problematic since inaccuracies can be corrected as further information and understanding are obtained.

When MC and CR cannot be fully achieved, students may simply move on with whatever conviction they have at their disposal. This seems a better option than coming to a full stop and putting the problem aside. After all, conviction and reassurance may be possible at a later time, when new information and understanding are made available. The following quote illustrates this situation.

> I can't seem to get the correct algebraic answer but it will equal the equivalent to 4, which can be written as 2^2. [Accepts formula and moves on.]
>
> (Carolyn, Sums of Diagonals, p. 5)

4.5. IMPROVING THE RESULTS

This section looks at what can be considered as the last stage of the solutioning process. Once a solution is achieved and has been validated, the next step is usually to acknowledge the need to improve the results. This is particularly true when students consider that the answer is correct but not ready to be presented as it is. If time and resources allow, they may try to improve the presentation of a result or try to extend it to other domains.

> OK – I'm happy that's worked out in that case. I'm definite there is a more elegant explanation which might be worth looking for. Argument sounds a little awkward to me at the moment – could do with being more persuasive.
>
> Right. Review here – there's a few different ways to go…
>
> Have shown for odd x even, if I could show for even x even I'd be done!
>
> (Rafael, Faulty Rectangles, p. 12)

Improving a solution can be a straightforward task that involves making simple modifications or additions. However, this is not always the case and the modifications needed to improve a solution can vary from being straightforward to very laborious and time-consuming. Improving a solution usually involves dealing with situations that are more complex than the initial solution. Thus, having to deal with progressively more complex situations can make it difficult – or even impossible – for some students to improve their solutions further. The probability of this being the case seems to be higher when students lack the necessary mathematical background to deal with more sophisticated mathematical ideas. Lack of time or energy can also prevent students from improving their solutions. Under these circumstances, some students will decide to stop their process and will present their solution as it is.

> Reached a dead end at the moment so I am unable to progress any further. If I had been able to solve this problem properly I could have also extended it to look at the rest of the items on my brainstorm.
>
> (Lydia, Cartesian Chase, p.13)

Students who are able to improve their solutions do so by improving its presentation or by extending it. Section 4.5.1 looks at the former case while section 4.5.2 looks at how students improve their solutions by extending them.

4.5.1. Improving the presentation of the results

Improving the presentation of a result or a solution may involve presenting it in a more compact (or 'mathematical' way). The following quote provides an example of a student that decided to improve her results in this way.

> I wonder if I could improve this further by rewriting my formula as a closed expression, i.e., an equation in x and n with no summation signs.
>
> (Hillary, Sums of Diagonals, 15)

Another way in which students may seek to improve the presentation of their results is by attempting to produce a formal mathematical proof of their work. Once a satisfactory solution or initial solution is generated, it may be improved by providing a more rigorous argument. Providing a formal mathematical argument is a way of putting an already satisfactory solution in such a way that it can be presented as a *final product* to others. In other words, providing a formal mathematical proof involves elaborating a deductive argument that not only satisfies the student's understanding but also satisfies certain mathematical requisites.

Producing a formal mathematical proof is something that some students attempted in their rubrics during the course. For instance, in 'Sums of Diagonals' various students proved their general formulas by mathematical induction. However, in general, it may be said that providing a rigorous mathematical proof is usually considered a secondary or dispensable aim. It may be suggested that such was the opinion of the student in the example below.

> My formulas are very general and because of the way they were obtained they don't really need any formal proof or justification, as these are evident in the method.
>
> (Nadia, Sums of Diagonals, p. 7b)

In general (and as illustrated in the following quote), students seem more concerned about producing arguments that are convincing, both for themselves and for a sceptical reader, than of providing formal mathematical proofs.[23] Moreover, when it comes to improving their solution, they seem to be more concerned about extending them.

> I believe I have the correct answer, although I have no concrete proof. I believe that, as a possible extension, it would be possible to get an answer involving trigonometry... Other extensions [could be]...
>
> (Roberto, Diagonals of a Rectangle, p. 5)

[23] A 'proof' in the sense of a rigorous mathematical argument that follows the rules and requirements of formal mathematics.

4.5.2. Extending the results

Once students generate a solution, it is not uncommon for them to try to improve it by extending it. Extending a solution is, in a way, a process of generating a new solution. However, in this case, the process may be characterised by previous results and how these were obtained. Solutions may be extended by showing that they account for all possible cases or by making their results valid for a wider domain. For instance, as in the example below, students may consider cases that were previously ignored and will make sure that their solution accounts for those cases as well. Another example of extending is when students successfully generate a solution in the two-dimensional plane and then move on to considering a similar situation in three dimensions.

> Obviously, this solution is only valid for $m,n>4$.
>
> For $m=4$, $n\neq4$ or $m\neq4$, $m=4$, we must once again ensure that we land in one of our guaranteed winning positions.
>
> For $m,n=4$, we must make sure that we can start from a definite win.
>
> In fact, for $m=4$, $n=i$ $(i=1,2,3,4)$ or $n=4$, $m=i$, $(i=1,2,3,4)$ we must make sure that we start from a definite win.
>
> For m, $n=3$ we must make our opponent start to guarantee success.
>
> For cases where $m,n=i$ $(i=1,2)$ the game is trivial and no fun to play.
>
> (Leonard, Cartesian Chase, p. 6)

Transferring

When students generate a solution, they sometimes notice that the ideas or the methods that they used can be applied to other situations as well. In other words, they notice that some of their ideas can be *transferred* and thus be made useful for solving, or dealing with, other cases – i.e., for extending.

> Aha! If I can do this for a number with two divisors that are prime, I could probably do it for a number with exactly 3, 4, … or more non-trivial divisors, all which are prime.
>
> (Jason, Liouville, p. 3)

> Can I use the same process as earlier to generate more even x even fault free rectangles?
>
> (Camille, Faulty Rectangles, pp. 2b–3)

Although transferring means that previously developed ideas will be used in other situations, this is not necessarily a simple task. Transferring may require students to make some changes to the ideas or procedures to be transferred to make them suitable for the new situation. These changes can be relatively simple, such as when students decide to introduce a new, more efficient notation:

> The largest secret number 'a' was found by adding the two largest side numbers and subtracting the remaining side numbers…I think [this] rule is most likely to work with arithmagons with >3 sides.
>
> As I am seeing a general rule for arithmagons with n sides, I will need to alter my notation for improved clarity. Instead of x, y and z for the side numbers I will use s_1, s_2, s_3, …, s_n…
>
> (Jules, Arithmagons, p.5)

Although adapting previously used methods or ideas can be a simple task, in other cases it can be a complicated or even impractical activity.

> My proof that there was a path came from visualising, again, what the path should be, since anything other than the circle seemed unlikely, and bearing in mind the complete symmetry of the circle. Unfortunately, this reliance on the symmetry of the circle meant I couldn't extend the theory to irregular circles very easily.
>
> (Albert, Jogger's Dog, Commentary)

Thus, in some cases, *adapting* can be a considerably complicated activity. In situations like this, students will find that looking for new ways of generating a solution may be a better option. In a way, finding new ways of solutioning may suggest a need to start the solving process all over again. However, this is not necessarily the case. The knowledge and understanding that may have been gained about the situation are likely to make this 'new' process a more efficient one. Of course, this will be the case only if there is persistence in extending the solutions; they may well decide to stop at this stage.

4.6. FAILING TO SOLUTION AND PSEUDO-SOLUTIONING

This section discusses situations in which students may fail to generate satisfactory solutions and some of its possible consequences. Among these consequences, 'pseudo-solutioning' emerged as an interesting approach to problem solving. Pseudo-solutioning can be seen as a digression from trying to provide a solution. When students pseudo-solution, they seem more concerned about providing an academically acceptable rubric than about generating a solution. Needless to say, not all students engaged in this behaviour. However, pseudo-solutioning does explain an important variation observed in relation to solutioning.[24]

This section is organised in the following way: After looking briefly into why students fail to generate satisfactory results (Section 4.6.1), Section 4.6.2 discusses two ways in which students may pseudo-solution. A final section (Section 4.6.3) considers some of the implications of pseudo-solutioning.

4.6.1. Why students fail to generate satisfactory results

Before considering pseudo-solutioning, it is important to discuss why students fail to generate satisfactory solutions or results. A *lack of mathematical knowledge* and failing to reach a *point of sufficient understanding* were observed as the main reasons.

It seems that, in order to generate satisfactory results, students need to have the necessary knowledge or, in other words, to gain sufficient information and understanding about the situation. A *lack of mathematical knowledge* can be observed when students cannot access the information necessary to deal with a mathematical situation. They may be able to realise that there is a procedure that can be used but are unable to make use of it; or they may not be aware of this

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[24] Vinner (1997) observed a similar behaviour in students and labelled it pseudo-conceptual or pseudo-analytical behaviour. Although he recognised the limitations of this type of behaviour, he recognised the fact that it may help students achieve certain goals (such as gaining some sort of 'credit' or providing an initial solution).

(and consequently) will not know how to proceed. The following example illustrates a lack of mathematical knowledge as experienced by a student. In it, the student noticed a pattern and seemed aware of the fact that there was a mathematical procedure for dealing with it. Nonetheless, as she acknowledged it, her lack of mathematical knowledge prevented her to generate a solution.

> AHA! I have detected a pattern in the above sequence. In the '2nd difference' the numbers are consecutive. I have also noticed that the '1st difference' the numbers are triangle numbers. All I have to do now is to find a formula for n. This is a bit tricky because the differences have gone to three rows. I think that this means the formula will have n^3 in it. I know there is more to the formula than just n^3, but I don't have a clue as to how to find it.
>
> I'm stuck! I don't know how to find the formula for the above sequence.
>
> (Annette, Sums of Diagonals, p. 5)

Another reason why students fail to generate satisfactory solutions is because of not being able to reach a *point of sufficient understanding* (Section 4.3.1). This is usually the case when students do not devote enough time to generate knowledge. As suggested in Section 4.3.1, trying to generate a solution before reaching a point of sufficient understanding does not usually lead to satisfactory results.

As a result of failing to generate satisfactory results, students may decide to persist in their attempt or may to give up and abandon the process. However, there seems to be a third alternative, namely, to pseudo-solution.

4.6.2. Pseudo-solutioning by focusing on manageable ideas and making inaccurate deductions

Pseudo-solutioning is characterised by a concern with trying to generate an acceptable rubric rather than a concern for generating a solution. Pseudo-solutioning may be due to a combination of students' difficulties in generating a solution plus the (academic) need to submit an acceptable account of their work. After all, some results and a genuine attempt at solving the given problem provide a higher probability of achieving a mark than not presenting any results. Two ways in which students may pseudo-solution are 'focusing on manageable ideas' and 'making inaccurate deductions'.

Focusing on manageable ideas

'Focusing on manageable ideas' involves focusing on ideas that seem manageable but are not necessarily relevant. Manageable ideas are ideas that seem easy to handle for the student or that are related to a familiar context. Focusing on manageable ideas suggests a situation in which students concentrate on less problematic issues and neglect the ideas that were being considered in the first place. It seems that students are more likely to be distracted by irrelevant, manageable ideas when they are failing to generate satisfactory results. When results are difficult to obtain, dealing with manageable ideas may be an alternative for achieving some results, albeit these may not be necessarily significant. Pseudo-solutioning in this way may provide results but not necessarily a solution.

The following is an example of focusing on manageable ideas. It shows Hillary's attempt at trying to find a formula to model an observed pattern. Having identified some familiar patterns – i.e., 'triangle' and 'square' numbers – she then decided to focus on how they could be extended – i.e., on what 'pentagon' numbers would look like. This instance was considered as a digression from her aim. However, this digression seemed more promising option (it was the "logical progression", after all) than considering how to derive a formula for the sequences she was dealing with.

> AHA! Sums of slopes with gradient 1 have differences which are consecutive triangle numbers and sums of slopes with gradient 2 have differences which are consecutive square numbers.
> Sums of slopes with gradient 3 will have differences which are consecutive pentagon numbers.
> This seems to be the logical progression, but what is a pentagon number? Ok, well, triangle and square numbers can be formed by arranging dots into those shapes, so using that method I will try to form some 'pentagon numbers'...
>
> (Hillary, Sums of Diagonals, p. 5)

Making inaccurate deductions

Students make 'inaccurate deductions' by making flawed arguments for generating or validating a solution. These arguments may be confusing in the sense that it is not clear or evident how they lead to the alleged result.

Alternatively, they may violate what Goldin (2003) called "the integrity of mathematical knowledge". In other words, students may ignore or fail to recognise "elementary concepts underlying mathematical knowledge" (Idem, p. 179). For instance, they may ignore the fact that an example does not prove the rule or that a formula being true in a particular case does not mean it is true for all cases.

The following quote provides an example of making inaccurate deductions. In it, the student suggested that having found a fifth example was enough evidence to generalise that "every multiple of 5 dominoes... will make a fault-free rectangle". Whether she was convinced by her deduction or whether she accepted it because of the pressure of having to present some results is – as suggested above – a question that needs to be further investigated.

> YES! Finally found another fault-free rectangle, and it's a multiple of 5. That means I've found five rectangles now which are fault free and are a multiple of 5. I think that's quite enough evidence for me to believe that every multiple of 5 dominoes from 15 onwards will make a fault-free rectangle without me having to find a rectangle with 100 dominoes in.
>
> (Rita, Faulty Rectangles, p. 17)

Related to making inaccurate deductions is the fact that, during pseudo-solutioning, students may avoid validating their results or may do it in less rigorous ways. By not seeking to validate their results students avoid bringing inconsistencies to the fore and jeopardising their solutions. Evaluating results in terms of whether they are correct and/or make sense makes logical flaws or inaccuracies evident and forces students to take measures (such as having to discard an emerging solution). If validation is not sought, or is sought in a limited way, the solver can avoid the risk of having to cope with an inconsistent or flawed solution.

4.6.3. Implications of pseudo-solutioning

It is unlikely that pseudo-solutioning, in itself, will contribute to students generating a fully-fledged solution. Nonetheless, pseudo-solutioning presents the advantage that it allows students to *itemise*, i.e., to generate a number of results that, if nothing else, can be presented as evidence of having 'seriously' tried to solve the problem. In the academic context in which students were working, producing a proof of attempt was not something to be neglected. In fact, producing evidence of having attempted to solve the problem was an important element of the 'didactical contract' (Brousseau and Warfield, 1999) that students and the tutor assumed since the start of the course. In this context, it seems that failing to generate a solution caused some students to focus more on producing results than on trying to achieve a more ambitious solution. In the end, for these students, producing some results may have looked like a better option than not producing any.

Pseudo-solutioning seems to be an important consequence of failing to generate satisfactory solutions or results during problem solving. On one level, that students pseudo-solution suggests that successful problem solving requires an understanding of the situation to be tackled as well as mathematical knowledge.

On another level, pseudo-solutioning points to some sort of didactical contract, accepted by the tutor and the student. This means that the context in which problem solving situations take place affects the way in which this activity is conducted. In the case of this study, the academic requirement of having to produce a rubric seemed to have led students to a position of thinking in terms of 'an attempt to generate a solution is better than no solution at all'.

Finally, whether students are aware of their own pseudo-solutioning is a question that needs further investigation. In some instances, students seemed unaware of their unacceptable procedures. In others, they seemed to be trying to make it look as if they were genuinely attempting to achieve a mathematical solution while ignoring important flaws in their work. It seems clear that students with a poor mathematical background are more likely to pseudo-solution than students with a sound mathematical background. In other words, at least in the problem-solving context being considered, it was more likely to see a BA(QTS) student engaging in pseudo-solutioning. This can be explained by the fact that students with a sound mathematical background may be more critical of the procedures that they choose to follow, even if they experience difficulties in generating a solution.

4.7. FINAL COMMENTS

The theory has been presented as a series of sections in which different aspects of solutioning are discussed. Since the theory consists of concepts and explanations as to how these concepts are related, a final summary of the theory was not considered appropriate (see Glaser, 1978). For instance, the theory raises issues such as seeking 'mathematical conviction' (MC) and 'cognitive reassurance'

(CR). These concepts could be explained in a few words; nonetheless, this would not make justice to the theory. The theory goes beyond merely suggesting names for observed patterns of behaviour. It establishes relationships between concepts and explains how these are related for resolving the main concerns of the participants. In this way, seeking MC and CR – to continue using the same example – are relevant to validating results and help to explain how students resolve this concern. Moreover, the discussion of these concepts within the theory highlights some of the conditions in which seeking MC and CR take place and explains its possible consequences. Thus, a summary is not provided; instead, a list of the main sections of the theory are listed in the following table:

GENERATING KNOWLEDGE • Strategies for, and ways of, generating knowledge • Making observations • The role of key ideas • Generating understanding and situational reasoning
GENERATING SOLUTIONS • Generating solutions through inductive and deductive reasoning • Guessing and ungrounded ideas
VALIDATING RESULTS • Seeking MC and CR • Achieving MC and CR as a desirable condition
IMPROVING THE RESULTS • Improving the presentation of a result • Extending results
FAILING TO SOLUTION AND PSEUDO-SOLUTIONING • Why students fail to generate satisfactory results • Pseudo-solutioning by focusing on manageable ideas and making inaccurate deductions • Implications of pseudo-solutioning

Table 4: Main sections of 'Solutioning'

Since it is a process, solutioning tends to follow a general order in terms of the activities that are conducted. It is easy to see why solutioning usually starts with generating knowledge and ends once a solution has been sufficiently improved. Also, being a process, solutioning implies that resolving an aspect of the process leads, in turn, to the need of resolving other aspects. For instance, generating a solution may lead to the need to validate it and then to improve it. However, this does not mean that the different stages are not recurrent. For example, the need to generate knowledge is continuously remerging at almost every stage; and validating may be done with any partial solution or idea that is generated. Also, the need to conduct some activities may continue to re-emerge. This is, for instance, the case for activities such as reducing complexity, thinking in terms of situational reasoning, and transferring previously used ideas. The usefulness of these activities results in students applying them again and again, whenever it is pertinent and viable.

5. USING THE MODEL: THREE CASE STUDIES

5.1. INTRODUCTION

Chapter 4 presented the model of problem solving that was generated as a result of this study. The model was developed using the grounded theory methodology to investigate how students solve problems in the context of a particular undergraduate course. The model does not make any reference to particular students; instead, it suggests patterns of behaviour observed across participants. Following a grounded theory approach, these patterns emerged as a result of a process of constant comparison and memo writing that served to ensure that observations were grounded on the data. The model includes some examples but these are merely aimed to illustrate particular patterns. In other words, these illustrations "are only to establish imagery and understanding as vividly as possible when needed" (Glaser, 1978, p. 134).

As for the present chapter, it moves from investigating general patterns of behaviour to looking at particular cases, i.e., at particular students solving a particular problem. The aim is to allow the reader to appreciate students' problem-solving processes from a concrete rather than a generalised perspective. This is done through a set of three case studies that describe students' processes using the model as theoretical framework. In them, the generic term 'students' is substituted by a particular name. Equally, reference will be made to a particular problem rather than to a category of problems. Like any detailed view of a situation, this will provide a view of the details but may put some aspects of the general picture, at least momentarily, out of focus.

It is important to stress that while the model explains students' problem-solving processes, the reverse is not necessarily true. This is to say that the processes presented next can be described by the model but they do not, in themselves, define the model. The reason for this is that the model was generated by a systematic process of comparing incidents from more than 300 rubrics. Thus, single rubrics do not make or falsify the pattern. This is an important issue to take into account when considering the following case studies. Nonetheless, it may be said that the rubrics were selected to represent each group of students (mathematics, computer-sciences and teacher-training students) and serve to illustrate the model in detail.

5.1.1. Fit, relevance and work in relation to the model developed

As mentioned in Section 3.3.3, 'fit', 'relevance' and 'work' are the criteria for judging grounded theories (Glaser, 1978). The case studies presented next illustrate how the model *fits* the data, is *relevant* and *works*. *Fit* can be seen in the way the concepts suggested suitably describe observable patterns of behaviour (e.g., 'pseudo-solutioning' suitably designates a situation in which the student's concern is to generate an academically acceptable piece of work rather than a solution). Furthermore, the case studies illustrate how concepts such as 'situational reasoning', 'transferring ideas' and 'focusing on manageable ideas' – among others – serve to discuss activities that are *relevant* to students' problem-solving process. This relevance is evident in the way these concepts contribute to explaining how students resolve their main concerns (e.g., 'transferring ideas' indicates an important strategy that students use to resolve their concern for improving results).

The discussion presented in the last section of this chapter (Section 5.5) also shows how the model *works* for explaining success and failure in students' processes. In other words, by comparing students' processes using the concepts provided by the theory, it is possible to suggest why some students were better able to provide a satisfactory answer than others (e.g., students who 'guess' are less expected to succeed than those who rely on 'deductive' or 'inductive' reasoning).

Evaluating students' problem-solving processes using the theory is not done in terms of personal attributes but in terms of concrete actions taken or avoided. In this sense, the model also works for helping students become better solvers by providing them with information as to what actions are more effective and which can lead to difficulties during the process.

5.1.2. The use of case studies: A descriptive view

The use of case studies was chosen as a way of discussing, in a more concrete way, the patterns of behaviour suggested by the theory. They also provide a means for following a number of problem-solving processes from beginning to end. The case studies provide a detailed view of the work of three students working, independently, on one problem. The problem that will be considered is the 'Sums of Diagonals' problem (see Appendix 1; further comments on this problem are also given in Appendix 3) and the students are Leonard, Patrick and Carolyn (all pseudonyms).

The 'Sums of Diagonals' problem was chosen for two reasons. First, since students who tackled this problem did so for their final assignment, they invested more time and effort in it. This meant longer and richer processes, suitable for illustrating a wider range of patterns of behaviour. The second reason for choosing the 'Sums of Diagonal' problem was that this problem can be tackled successfully in at least two ways, one more deductive than the other. In terms of the rubrics, this meant that even students without a strong mathematical background were able to display a fully-fledged mathematical process. In other words, the problem did not present insurmountable obstacles for non-mathematics students.

The students chosen for the case studies belonged to each group represented in the course. Leonard was a third-year, mathematics student; Patrick a second-year, computer-science student, and Carolyn was starting a four-year, teaching degree (a BA(QTS) degree). By including a student from each of these three groups it was hoped that variability would be maximised and more patterns of behaviour would be illustrated.

To make evident the use of the theory, numbers in square brackets are used to indicate the section that discusses the elements of the theory that the analysis makes use of. Furthermore, the reader is encouraged to follow the process by examining the students' rubrics, which can be found in Appendix 3 (the page numbers indicated in brackets refer to these rubrics).

5.2. CASE 1: LEONARD

Like the rest of the students considered for the case studies, Leonard participated in the course in the academic year 2002/2003. He participated enthusiastically in the course and showed willingness in relation to sharing his ideas through rubric writing. He also agreed to participate in interviews to talk about his work in the course. In Leonard's process, it is possible to see activities aimed at generating knowledge, generating solutions and validating and improving them. His mathematical background allowed him to deal with complex situations and to extend and prove his solutions.

Leonard's first activities were aimed at generating knowledge [4.2]. He started by looking at the problem statement and by analysing the diagonals of slope 1. He complemented this analysis by specialising [4.2.1] and then he proceeded to briefly analyse diagonals of slope 2 (p. 1).

After writing down his observations, Leonard continued trying to find more about the problem. Before doing this, he established what exactly he wanted to find out:

> I would like a formula that would allow me to find the sum of a particular diagonal when we input the slope of the line and the number from the top row in that line. (p. 2)

After this, he proceeded by organising data [4.2.1] and "putting together a table of diagonals of slope 1" (p. 2). As a result of analysing the table looking for patterns, he was able to make an important observation [4.2.2], (p. 3). It may be suggested that, since the observation made provided a hint as to how to generate a solution, it constituted a key idea [4.2.3].

As can be seen in pages 3 and 4 of Leonard's rubric, his first solution was generated deductively by making use of familiar mathematical procedures [4.3.1]. These procedures involved algebraic manipulations to find the formula of an equation for which the third differences are constant.

Once generating an initial solution (p. 4), Leonard proceeded to validate it by seeking mathematical conviction [4.4.1], (p. 5). At this point he did not seem very concerned about whether it also made sense and thus it may be said that he did not seek cognitive reassurance. Instead, after deciding that "The formula clearly works!" (p. 5) he decided to continue by improving his solution [4.5] and, in particular, by extending it [4.5.2]. In his own words:

> However, this is only for the case when the diagonal has slope 1. I will now try to apply the same method for diagonals of slope 2. (p. 5)

Furthermore, as the quote above suggests, Leonard intended to extend his solution by trying to transfer ideas that had proven to be useful in a previous case [4.5.2]. To be able to do this, he first tried to generate further information and understanding. Thus, he organised data in a similar way as before, i.e., by putting together a table of diagonals, this time of slope 2 [4.2.1], (p. 5).

Noticing that "Once again the 3rd differences are equal..." (p. 6) made him confident about the transferability of his previous results. This also allowed him to improve his initial solution as intended and then to validate his results by achieving mathematical conviction [4.4.1], (p. 7).

Still with the intention of further extending his solution, Leonard decided that he needed to generate further information and understanding [4.2], (p. 7). Before extending his solution to slopes other than 1 and 2, he decided to look at a couple of particular cases (slopes 3 and 4). He mentioned that:

> I'm looking for a link between S_r for slope 1 and S_r for slope 2. Before I can make any assumptions, I should consider at least two more slopes. (p. 7)

Leonard proceeded by looking at the sums of diagonals of slopes 3 and 4, (pp. 8-11). In each case, he first investigated the situation and the proceeded by transferring and adapting the ideas that he had used in the cases for slopes 1 and 2 [4.5.2]. This allowed him to generate a formula in a deductive way [4.3.1] and then to validate it by achieving mathematically conviction [4.4.1]. He worked in this way with diagonals with slope 3 and then with diagonals with slope 4.

He continued by organising the data that he had accumulated into a short list and reflecting on what he had done so far [4.2.1], (p. 12). He then created a more detailed table that included the results for diagonals with gradients 1, 2, 3, 4 [Ibid], (p. 12).

By analysing the data, he observed another key idea (or crucial pattern) [4.2.3], (p. 13). As he put it:

> It seems that a_g, b_g, c_g are sequences that are arithmetic progressions of the form Ag+B. (p. 13)

Leonard made use of this key idea to work deductively [4.3.1] towards a general formula for diagonals of a general slope 'g' (pp. 13-14). Like in previous cases, he proceeded to validate his results by seeking mathematical conviction [4.4.1], (p. 15). Following this check, he concluded that "The general formula works for g=5" (p. 15).

It may be said that Leonard validated his results as soon as they emerged. However, these validations were not formal proofs but verifications to see whether the results were mathematically consistent [4.4]. Up to this point, he did not verify whether his results made sense but only whether they worked in particular cases. In other words, he had been seeking mathematical conviction but not cognitive reassurance [4.4.1].

Once he had generated a general solution and validated it, Leonard did not stop there but continued trying to improve his solution further. At this point, he decided to continue by providing a more convincing argument for why his results were true. As he put it:

I have a formula which fulfils my initial requirements. However, I do not have any reasons why the formula is what it is, nor have I proved it works! I suspect that proving it could help to see why the formula is what it is, and hence answer both questions. So that will be my next step.

Extension

I will now extend the problem by looking deeper into my solution and finding out what is really going on. (p. 16)

It may be said that in trying to explain why his results were true, Leonard was not trying to validate his results but to improve his solution[25]. To do this, he decided to generate knowledge by investigating "where sums come from" (p. 16). In other words, he decided to think in terms of situational reasoning [4.2.4].

To start gaining further understanding of the situation, Leonard recurred to specialising and focused on a particular case [4.2.1], (p. 17). This made his analysis more succinct and easier to handle. As a result of the knowledge gained, he was able to generate a general formula for diagonals of slope 1 – something he had already done through a different route (p.17).

He repeated this process by transferring the method used for diagonals of slope 1 to diagonals of slope 2 (p. 18). By analysing the results and suggesting a

[25] The model explains that students may try to explain why results are true in order to validate them. However, in Leonard's case, seeking to explain why results were true was done with the aim of improving his solution. This may be used to suggest that seeking to explain why results are true can *also* be done with the purpose of improving the results. This observation does not make the model invalid but improves its capacity for explaining how students resolve their main concerns during problem solving (see Glaser, 1978; 1998).

plausible pattern (i.e., by a process of inductive reasoning [4.3.1]) he generated a formula for calculating the sums of diagonals of any slope 'g'. After producing this general formula, he then generated a table showing the formulas for diagonals of slopes 1, 2, 3, 4 and 'g' (p. 19). He also introduced a further column showing that the results he had obtained this time were the same as the results he had obtained before starting to think in terms of situational reasoning.

When his result seemed complete, Leonard noticed an area of opportunity for further improving his results. In his own words:

> I have proved that for all g ∈ □ the sums work, since they came directly from the sequences a_n. However, the formulae results have only been proved for specific values of g and r. But if I can show that ∀ g, each of these sums ⇔ each of these formulae, then my proof is complete. (p. 19)

Leonard proceeded to generate a proof by mathematical induction, which can be considered a deductive way of improving a solution [4.5.1], (pp. 20-23). In this way, he concluded with a mathematical proof and a formula that accounted for the sum of any diagonal with slope 'g' (g ∈ □) and starting on a specific row. Leonard closed his process with a final remark on his results and a suggestion for possible future extensions.

5.3. CASE 2: PATRICK

Patrick is a second-year computer science student. Like other students from this course, he seemed confident about his mathematical abilities but was not necessarily aware of certain issues in relation to advanced mathematics. In his rubric, he provided several 'proofs' for his results or observations. Although these proofs are not rigorous mathematical proofs, they do suggest a concern for providing arguments as to why results must be true.

Patrick started by generating knowledge [4.2], (p. 1). First he tried to understand what the problem was about and to understand the different elements in it. To interpret the term 'slope', he decided to make use of what he had "learnt from mathematics" (p. 1) about slopes. In other words, he decided to import information [4.2.1].

After establishing that what he knew about slopes could be applied to this problem, he decided that he also needed some representation to make data handling more efficient (p. 2). Thus, he engaged in data organising [4.2.1] and introduced a suitable notation. Once this was accomplished, he engaged in data organising once more, this time by creating a list in which he could "see some sums" (p. 3).

He proceeded by looking for patterns in the list of values that he had constructed [4.2.3], (p. 3). Not being able to spot any pattern or make any observation, he decided to organise the data [4.2.1] in another way, this time in a tree-diagram (p. 3).

Analysing the tree diagram allowed him to notice a pattern, namely, that the second differences in the list of numbers he was observing were consecutive numbers (p. 3). He continued to work with this idea and deriving further observations from it [4.2.2].

After validating his observations in terms of mathematical conviction [4.4.1], he decided that, in spite of the results he had obtained by using a tree-diagram, he needed a more efficient way of organising the data [4.2.1], (pp. 3-4).

It seems that seeking a better a way of organising the data led Patrick to reasoning in terms of situational reasoning [4.2.4], (p. 4). As he put it:

> REFLECT: Drawing tree diagrams is one method to find the sum but it is too clumsy. Because I need to draw the tree from the beginning to n if I want to find 1.n.
>
> Let me don't look at the sums now, *but the way how it's built up*. (p. 4; emphasis added)

After analysing the data, Patrick was able to generate an initial solution by reasoning inductively [4.3.1]. (p. 4). After this, he immediately proceeded to improve this result by rewriting it in a clearer way.

After reviewing his work (pp. 4-5). Patrick continued by trying to improve his results by extending them to diagonals with slope 2 [4.5], (p. 5). To extend his results in this way, he first tried to generate further knowledge by analysing the relevant data. He then decided to explore whether what he had done in the previous case could be transferred to this new case [4.5.2], (p. 5). In his own words:

> I've done the formula for slope 1, can I use the same idea to develop a formula for slope 2? (p. 5)

After generating knowledge by specialising and data organising [4.2.1], (p. 5), Patrick was able to generate a formula for the sums of diagonals of slope 2 (p. 6). This formula was generated inductively [4.3.1] and expressed as a conjecture to be validated. Before starting to validate his result, he mentioned that the formula emerged as a "guess from the previous calculation" (p. 6). However, in terms of the model, his formula was not a 'guess' (see [4.3.2]) but the result of reasoning inductively about the situation.

Patrick proceeded to validate his result by seeking cognitive reassurance [4.4.1], (p. 6). He found himself unable to fully explain his formula and, after making an observation [4.2.2], (p. 6), decided to analyse information that he had previously generated [4.2.1]. (p. 7). This allowed him generate sufficient understanding as to be able to fully explain his "conjecture" (pp. 7-8).

After reflecting on what he had achieved and how, Patrick made another observation (p. 8). He stated it as a "conjecture", suggesting his intention of validating it next. To do this, he first explained how he was able to generate this observation by reasoning inductively [4.3.1]. Then he provided an explanation as to why the observation must be true and that thus validated his results in terms of cognitive reassurance [4.4.1], (p. 8).

It seems that the knowledge generated, together with the results already obtained, led Patrick to deciding to further improve his results [4.5] by writing "a more general equation that covers all slopes" (p. 9). To do this, he first took some time to review how he had generated the solutions for diagonals of slopes 1 and 2. The aim of this analysis seemed to be to find a way of adapting his previous work to his new aim of generating a more general formula [4.5.2], (pp. 9-10).

After comparing the two formulas he had previously generated (and how he had generated them), Patrick succeeded in providing a formula for any slope 'm' (p. 10). To do this he worked inductively [4.3.1] by making use of his previous results to make plausible generalisations (p. 10).

After reflecting and summarising his work, Patrick noticed that by working only with integer slopes he had made a "hidden assumption" (p. 11). This, for him, seemed to suggest another area of opportunity for improving his results as he then decided to extend his solution to cases where the slope is not an integer [4.5.2].

To extend his result to diagonals with non-integer slopes, Patrick decided to test his luck and conjectured that his previous results were valid for non-integer as well as for integer slopes (p. 11). This 'guess' [4.3.2] showed him that his formula was not as 'general' as he thought:

> Reflect: general formula is no longer 'general', it doesn't cover the case when slope is not integer. (p. 11)

This failed exercise led him to investigate whether his results could be adapted to the new situation and, if so, how [4.2], (pp. 11-12). Soon this analysis started to suggest that previous methods could be transferred and what adaptations needed to be conducted [4.5.2], (p. 12). This process involved both inductive and deductive reasoning: inductive, by making plausible generalisations; and deductive, by using this information to generate logical chains of reasoning, [4.3.1], (pp. 12-13).

After providing a formula for sums of diagonals with non-integer slopes, Patrick reflected on and reviewed his process (p. 13). In this way, he found a way of further improving his results by providing a more compact formula using summations [4.5.1], (p. 14).

To provide this general formula Patrick looked at previous cases and, by a combination of inductive and deductive reasoning [4.3.1], eventually arrived to "the general formula for non-integral slope" (p. 14). He validated his results by seeking mainly mathematical conviction [4.4.1], (p. 15) and then reflected on his work and summarised his results (p. 16).

Before closing his process, Patrick started to work on another extension to the problem. However, this extension implied a different interpretation of the problem given. After generating some knowledge about the situation, Patrick decided to quit this route and finally closed his process.

5.4. CASE 3: CAROLYN

Carolyn is a first-year student doing a four-year BA(QTS) teaching degree . Like most students doing this degree and that participated in the course, she doubted her ability to provide satisfactory solutions. Her confidence seemed to increase during the course, at least to the point of providing longer and more complete processes towards the end of it. It may be said that Carolyn's process is characterised by a focus on looking for patterns that bordered on key searching.

Carolyn started off by looking for patterns [4.2.3], (p. 1). Instead of trying to understand what the problem was exactly about, she began by looking for salient features about the grid of numbers provided. This allowed her to notice "a line of symmetry running through the grid" (p. 1) and to make some speculations about the meaning of this observation.

She then decided to investigate the situation and to do this she recurred to reducing complexity [4.2.1], (p. 1). As she put it, she decided to work by "starting from the smallest possible diagonal" (p. 1). She also decided to generate some data and organise it in a table of values "so it will be possible to identify a pattern" (p. 1), [4.2.1].

Not being able to spot an obvious pattern, Carolyn decided to re-organise the data, this time in a tree-diagram that allowed her look at the first, second and third differences between a list of numbers [4.2.1], (p. 2). At this point, she came across a pattern that, for other students dealing with this problem, was a key idea [4.2.3] that led them to start generating a solution. Although she seemed to realise that this was an important piece of information, she declared herself unable to find the formula. In her own words:

> AHA! There is an obvious pattern as there is a set of consecutive steps between them. From knowing this I would be able to find the sum of the diagonal from 8 to 8 but at the moment, I am unable to find the formula. (p. 2)

Two comments can be made in relation to this incident. On the one hand, it can be suggested that a lack of mathematical knowledge [4.6.1] may have prevented Carolyn form generating a solution. On the other hand, it can be suggested that a more pragmatic approach [4.2.2] to the observation made could have led to an initial solution. However, Carolyn did not seem to be interested in finding out how she could use the idea of consecutive steps to generate a solution. Without even trying to do this (or at least not reporting so), she put this observation aside and continued looking for patterns [4.2.3], (p. 2).

Carolyn continued by further analysing the data she had previously organised [4.2.1] and in this way she was able to generate some observations [4.2.2], (p. 2). In this way, she made a note on the difference between starting on an even or an odd number (pp. 2-3) and created another table of values to explore this idea further [4.2.1], (p. 3).

After analysing the data she claimed to have found what can be called a key idea [4.2.3]:

> AHA! I've just seen it, I can't believe I didn't see it before. With the odd numbers, the total is the starting number squared, plus any odd steps before you.
>
> E.g., 5) => (5x5)+(3x3)+(1x1)=35
>
> 7) => (7x7)+35=84. (p. 3)

Carolyn's quest for a pattern seemed to have succeeded when she pointed out the pattern mentioned above. After this, she tried to generate a solution by translating this pattern into a formula (p. 3). Although this formula seemed to fit her observations, it seemed more like a 'guess' made from one or two particular cases than the result of careful observation or conscientious reasoning about the situation [4.3.2]. In fact, it may be said that up to now she had not conducted a very meaningful search for knowledge. Looking for patterns seems to have been distracting her from trying to gain understanding about the situation, turning this into a somehow 'blinding' activity [4.2.3].

Once having achieved an initial solution, Carolyn acknowledged that she could not explain why the formula should be true (p. 3). Thus she decided to validate her solution by trying to "explain the reason for the formula" (p. 4), i.e., by seeking cognitive reassurance [4.4.1].

To do this, she resorted to trying to generate knowledge about the situation. She analysed the grid by focusing on a particular case ('3x3' diagonals), [4.2.1], (pp. 3-4). After not being able to make any progress, she suggested an intention of going back to the start (p. 4). However, before doing this, she made another observation.

> AHA! I don't know if this will help me but I may have found a way to explain the reason for the formula.
> For 6 by 6, the numbers are 6, 10, 12, 12, 10, 6. These can also be written as 6x1, 5x2, 4x3, 3x4, 2x5 and 1x6. As each number decreases by 1, the number it is timesed by, increases by 1. (p. 4)

What Carolyn learnt by this form of situational reasoning [4.4.4] allowed her to validate her formula for the sums of diagonals of slope 1 [4.4.1], (p. 4). Because of her previous results, she first verified this formula for cases where the starting number was even (pp. 4-5). It seems that although she was not able to achieve mathematical conviction, this did not prevent her from achieving cognitive reassurance about her results (p. 5). After validating her results for even numbers, she then briefly tried to validate her result in cases where the starting point was an odd number (p. 5-6).

Carolyn's used an algebraic process to 'validate' her results. This process is confusing and difficult to follow. It may be said that the validation was done in less than rigorous way and that her argument was inaccurate. A reason for this may be that Carolyn did not have the mathematical knowledge to verify her results [see 4.6.1]. Another influencing factor may be that the need to move on and complete the solving process was an important aim at that time. A combination of these two factors may also be a plausible explanation [see 4.6.2].

After looking at what she had achieved so far, Carolyn continued by trying to improve her solution by extending it [4.5.2], (p. 6). She had been investigating diagonals of slope 1 and now she had decided to investigate "diagonals where for every move across, you move up 2" (p. 6).

She looked at a few particular cases [4.2.1] and was able to make some observations in relation to them [4.2.2], (p. 6-7). She then decided to test whether the method she had used before applied to these cases as well [4.5.2], (p. 7). Without investigating its suitability, Carolyn tried to generate a solution by transferring a previously used method. Again, it may be said that she proceeded by trying to 'guess' a solution rather than trying to generate it based on what she knew about the situation [4.3.2].

This led to problems that reflected in her not being able to obtain the expected results (p. 7). She then decided that she had made an algebraic error and, after correcting the mistake, she continued with the process (pp. 7-8). She eventually accepted the answer and even provided a (somewhat confusing) validation. The process is unclear and the argument suggests that she incurred in making inaccurate deductions [4.6.2], (p. 8).

To extend her solution even further, Carolyn decided to see if the formula applied to "diagonals with the ratio x=y" (p. 8). In this case she transferred the previously used method and then tried to adapt it so that it would fit the new situation [4.5.2]. After manipulating data, she was able to come to some results that "worked" [see 4.6.2], (pp. 8-10).

Carolyn continued trying to extend her solution in a similar way for some time. Her process of transferring a previous result without analysing the conditions in which the transference was to take place seemed to be causing her problems. She seemed to be assuming (guessing) that the transference was viable and then fixing the formulas by manipulating them. Not surprisingly, this led to a few dubious results as well as to some confusion and frustration [4.3.2], (p. 10).

After acknowledging that she had not generated the expected results, she then decided to organise the results she had generated so far [4.2.1]. As she did this, she was also aiming to improve the presentation of the results by expressing them in a more succinct way [4.5.1], (p. 10). In Carolyn's words:

> STUCK! This didn't work. I will have to work through it algebraically and then it will give me the formula and it may help me understand further. Before I do this I will just write any formulas out so far so I may see a connection and so they will be easy to find if I need to refer to them later.
>
> There may be an easier way to writing $2y^3-xy^2$+(previous diagonal). Aha! I can use the signal Σ which will add up all numbers between a given section. (p. 10)

Carolyn expressed her formulas using summations and made a list of her results [4.2.1], (p. 11). She then started looking for patterns that could give her a clue to a more general formula [4.5.2], (p. 11). After further organising some data in a table of values and analysing it [4.2.1], (p. 11), she thought she had found a pattern (p. 12). However, although she was able to make some predictions from this pattern, she was not able to provide a more general result [see 4.6.1], (pp. 11-12).

Although she was able to make some observations, Carolyn did not succeed in providing a more general solution. Overall, she seemed to be aware that, although she had itemised [4.6.3], she had failed to provide a satisfactory solution.

> STUCK! I feel that at this moment in time, I can no longer go on with this part of the question. *I have discovered a lot so far but there is still lots to find out.* I am going to move further on but there is still lots more to find out. I am going to move further on and then might come back to this part of the question. (p. 13; emphasis added)

Under these conditions Carolyn decided to move on by looking at the problem in a different way. This time she decided to investigate multiplication rather than addition within the diagonals (p. 14). As before, she relied on looking for patterns more than on trying to generate knowledge about the situation. She generated some results by a process closer to 'guessing' than to induction or deduction. More specifically, she proceeded by looking at particular examples and then trying to generate formulas out of them, hoping that these could be generalisable. This led to a few results that were superficially validated (pp. 16-18).

Before closing her process Carolyn provided a comment on her work and summarised her results. Among other things, she recognised that she did not investigate the situation sufficiently and that she would try to do so if she were to start again:

> I think that I did not conduct my investigation to the problem very well. I feel that as soon as I got a new idea I rushed to do it rather than continuing with what I was doing and making a note of it so I could move onto it later.
>
> If I had the chance to do this again or come back to the problem, I would return to each type of diagonal and spend some time working on it... (p. 20)

5.5.DISCUSSION OF THE CASE STUDIES

Leonard, Patrick and Carolyn's processes present a number of similarities. For instance, all of them looked for patterns at one point or another during their processes. This helped them to generate some results, particularly at early stages. Other similarities suggest that that improving their results, mostly by extending their solutions, was important for them. However, their processes also suggest a number of differences that suggest why some were more successful than others and thus are worth considering more in detail.

There are important differences in the way these students generated knowledge. Leonard and Patrick started by generating knowledge and, in particular, by analysing the data. In doing this, they were trying to understand what the problem was about and making decisions on what they wanted to achieve. Carolyn, on the other hand, started by looking for patterns in the grid of numbers provided. Leonard turned to looking for patterns only after briefly analysing diagonals of slopes 1 and 2. As for Patrick, he also looked for patterns but soon decided to think in terms of situational reasoning [4.2.4]. Looking for patterns seemed to be a 'blinding' activity [4.2.3] for Carolyn in the sense that it may have distracted her from learning about important aspects of the situation.

Looking for patterns allowed Leonard to generate an initial solution early in his process. As for Patrick, looking for patterns allowed him to generate some observations; however, it was when he turned to situational reasoning that he was able to generate his first results. Carolyn was also able to spot some patterns but seemed to be having difficulties in translating her observations into formulas. Eventually she succeeded and noted a pattern that she was able to translate into a formula.

It may be said that Leonard's reasoning for generating an initial solution can be described as mainly deductive. After noticing a pattern or making an observation, he usually proceeded by employing previously-known mathematical procedures to generate a solution. Patrick's reasoning seemed to be inductive rather than deductive. By looking at how several sums of diagonals were created he was able to generate a tentative initial solution. Carolyn's processes differed from Leonard and Patrick's in that her solution did not seem to follow from her knowledge about the situation. Instead, she seemed to be guessing [4.3.2] by following a process of trial and error, testing her luck until she was able to find a formula that matched the observed pattern.

In general, Leonard relied strongly on deductive reasoning, although inductive reasoning was not entirely absent. Patrick relied more on inductive reasoning but he seemed to combine both in some cases. As for Carolyn, her process seemed to be characterised by relying more on guessing than on using her knowledge of the situation to reason deductively or inductively [4.3]. It may be suggested that Carolyn's reliance on guessing contributed to her not being able to produce satisfactory results.

As for validating results, during his process, Leonard seemed more concerned about seeking MC than CR [4.4.1]. It was not until he decided to improve his solution that he started trying to provide an argument to explain why his results were true. Patrick seemed constantly concerned about whether his results were correct but also about whether they made sense. Thus, he provided a number of explanations which he labelled as 'proofs'. In spite of the use of this term, these explanations were not rigorous mathematical proofs but validations generated with the aim of achieving cognitive reassurance. As for Carolyn, it may be said that her validations are confusing and unconvincing. It is difficult to tell to what extent she was convinced by her arguments. However, the fact that she chose to carry on regardless of inconsistent results suggests that she might have been more concerned about completing her rubric than on providing a convincing result. In this sense, she may have engaged into pseudo-solutioning [4.6].

A common feature of the students' processes is that they all tried to improve their solutions. In general, they tried to extend their results. To do this they sought to transfer previously used ideas (or 'methods') to new situations. In doing this, Leonard and Patrick usually engaged in generating further knowledge about the situation before deciding whether the method could be transferred. Also, in cases where ideas proved transferable, they conducted the necessary adaptations before generating the new (extended) solution. In one occasion, Patrick proceeded by assuming that his solution could be adapted without conducting any prior investigation. Since he was testing his luck rather than carefully reasoning about the situation, it can be said that he proceeded by guessing. After noticing that his method was not directly transferable, he proceeded to generate information and understanding that would allow him to proceed with his aim. As for Carolyn, her transferences were commonly characterised by guessing [4.3.2]. Instead of trying to find out whether or not ideas were transferable, she guessed that they were. This led to difficulties and some confusion that eventually caused her to give up.

Other ways of improving results included Leonard's successful attempt at generating a mathematical proof of his results. Moreover, Patrick further improved his results of presenting them in a more compact way by using summations. As for Carolyn, she briefly tried to improve the presentation of her solution but then she decided to look at a different interpretation of the problem. In her case, the latter can also be considered as an instance of 'focusing on

[more] manageable ideas' [4.6.2].

As this section suggests, the model presented in Chapter 4 can be used for describing students' processes and for highlighting important patterns of behaviour. Moreover, the discussion presented in this last section compared the three processes and highlighted interesting variations in issues such as generating knowledge and results, and validating and improving the solutions. In other words, these case studies serve to illustrate how the model 'fits' the data and is useful for highlighting 'relevant' problem-solving behaviours. In this sense, it may be suggested that the case studies also illustrate how the model 'works' for evaluating students processes and making predictions about the possible consequences of engaging in certain activities.[26] The next section provides an overall discussion on these and other aspects of the theory and of the study through which it was generated.

[26] It may be stressed that the case studies were not designed to validate the model. As said in Section 3.3.3, grounded theories are evaluated by their 'fit', 'relevance' and 'work' (Glaser, 1978). These characteristics are ensured *during* the analysis and not attained after the theory is generated. The case studies are meant to *illustrate* how the theory satisfies these criteria.

6. DISCUSSION

6.1. THE NATURE OF THE STUDY

This study was conducted with the aim of providing a *substantive* theory (or model) that explains students' main concerns as they tackle non-routine mathematical problems. Chapter 2 discussed previous research on problem-solving and suggested how the present study aims to make a contribution to this area. After discussing the methodology chosen and how the methods of data collection and analysis evolved for this study, Chapter 3 concluded with a description of how the study was conducted following the grounded theory methods of research. This led to Chapter 4, which presented the model developed. The model discussed students' main concerns as they solve problems and explained how these are resolved during the problem-solving process. Whereas Chapter 4 provided a conceptualisation of students' main concerns, Chapter 5 adopted a more concrete approach and described a set of three students' processes using the model as a theoretical framework. These case-studies serve to illustrate how the theory satisfies the criteria of 'fit', 'relevance' and 'work' against which grounded theories should be evaluated.

As discussed in Chapter 3, the development of the model that is provided as a result of this study was based on a systematic use of evidence. In relation to this, Hunt (1991) suggested that the study of problem solving can be done from a scientific or an engineering perspective. A scientific perspective allows generalisations and is based on a systematic process of data collection and analysis. As for engineering investigations, they

> are supposed to provide instructions to solve a particular problem. Engineering explanations are often derived from scientific principles, but there is no necessary reason that this be so, even in the physical sciences. Engineering explanations are valid if they work and if it's clear when they should be used...
>
> (Hunt, 1991, p. 384)

In these terms, it may be said that the model presented in this study is 'scientific'. The use of the grounded theory approach provided a systematic methodology for generating this model. Moreover, the model suggests patterns of behaviour observed in students' processes; these patterns are generalisations of what students do as they solve problems. As for the problem-solving framework on which the course is based (i.e., Mason et al.'s, 1982), it can be seen as an 'engineering' explanation that is useful for teaching students how to approach non-routine mathematical problems and for guiding them through the process of doing mathematics (see Section 2.6).

It is not uncommon that studies on problem solving look at this activity from the perspective of what seems relevant from the researcher's point of view. For instance, Gestaltist psychologists looked at problem solving from a *Gestaltist* perspective, i.e., as a process of cognitive reorganisation (Section 2.2). Moreover, information-processing theorists analysed problem-solving in detail and provided accurate descriptions of such processes (Section 2.3). In mathematics education, researchers have relied on predefined issues to study problem solving behaviour. For instance, Schoenfeld (1983; 1987a) focused on issues such as heuristics, control and beliefs; and Garofalo and Lester (1985) focused on metacognition (Section 2.5). The present study – following a grounded theory approach – proposed looking at students' processes from the perspective of *'what the students are trying to do'* (see Section 1.1). Also, the study focuses on conceptualising patterns of behaviour rather than on providing accurate descriptions of how students solve problems. Furthermore, it does not consider predefined categories such as heuristics, beliefs or metacognition since they did not emerge as representative of students' concerns. Instead, the study focuses on the issues such as 'generating knowledge', 'generating solutions', and 'validating' and 'improving' them. These issues emerged as the students' main concerns as they solve problems and the model presented in Chapter 4 explains how students try to resolve them.

Since the present study aims to generate a substantive theory of problem solving, it can be classified as 'domain-specific' (see Section 2.4.1). However, as said before, the aim of this study is not to provide a description of the domain-specific actions that students conduct for solving problems. Instead, the aim is to conceptualise patterns of behaviour observed in students as they try to solve problems. These patterns of behaviour are present in the model in the form of concepts such as 'reducing complexity', 'seeking cognitive reassurance' and 'pseudo-solutioning', to name a few.[27] These concepts, although generated in a

[27] Although it may be evident, it is important to stress that the theory also explains how these concepts are related. For instance, the theory explains how 'making observations' is a *consequence* of trying to generate knowledge (Section 4.2.2). Furthermore, making observations is related to key ideas since key ideas *are* relevant observations (Section 4.2.3). The reader may

domain-specific situation, are independent of any particular student. In other words, they were generated by comparing the work of numerous students working on different problems. From the analysis, these concepts emerged as *patterns of behaviour* and not as descriptions of any particular student's work. Although they cannot be generalised to other situations without a systematic process of further comparisons, it seems likely that they will serve to highlight important problem-solving issues in other areas.

Sfard (2004) suggested that the results from studies on mathematics education are difficult to generalise due to the vast national, economical and cultural diversity in which they are embedded. It may be said that this is true not only on a macro- but also on a micro-level since the personal or individual differences in groups of students can also be vast. Sfard's comment can be used to suggest that, while it may important to consider the differences, it may also be necessary to look at general patterns of behaviour. These patterns may be conceptualised and presented as information for helping practitioners improve their practices and, in a more general context, they may aid in decision making and in policy development. As for the model presented in this study, it suggests patterns of behaviour observable in the way students deal with non-routine mathematical problems. These patterns were generated in a particular context. However, due to their abstract nature, they can be discussed in a broader context as well. The next section considers some of these patterns (or concepts) and discusses their relevance in both the context in which they were developed and in a more general domain.

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identify other relationships by referring to the theory. See also Section 6.2.

6.2. EXPLAINING HOW STUDENTS DEAL WITH THEIR MAIN CONCERNS DURING PROBLEM SOLVING

6.2.1. A closer look at the nature of the model

The model or theory that was presented in Chapter 4 explains the main concerns of students as they tackle non-routine mathematical problems. Before discussing some of the main concepts that constitute the model, this section takes a closer look at the nature of the model that was generated. The model was called 'solutioning' and can be seen as a process in which students try to resolve four main concerns:

Generating knowledge;

Generating solutions;

Validating the results, and

Improving the results.

Explaining how students resolve these concerns is done in terms of concepts that represent conceptualised patterns of behaviour. In the theory, concepts are related in ways that were not prescribed or logically deduced but that were (like the concepts themselves) observed in the data. For instance, in analysing rubric after rubric for at least eight months, it was observed that, students may generate understanding by reasoning 'situationally'. However, they may also use this form of reasoning to generate solutions. Furthermore, when students generate results by recurring to situational reasoning, 'cognitive reassurance' may be automatically achieved. This is just one example of a complex relationship among concepts and other examples can be found in the theory.

Besides establishing relationships between concepts, the theory accounts for variation in the way students solve problems. 'Pseudo-solutioning' is a particularly important example in this respect. This concept accounts for an alternative approach to solutioning in which, instead of aiming to solve the problem, students seem more concerned about satisfying an academic requirement. Another example of variation is the following: Most students try to generate solutions at some point or another during their processes. However, there are important differences in the way they do this. Whereas some students may show a predilection for generating solutions in a 'deductive' way, other students may prefer to do this 'inductively' or even by reasoning in terms of 'situational' reasoning. Moreover, some students may resort to 'guessing'. These different ways of generating results may lead to different consequences. In this sense, the theory may also be used for predicting what these consequences may be (e.g., it is possible to predict that guessing is likely to lead to inconsistencies and frustration).

The next subsections discuss some of the main concepts that emerged by looking at how students try to resolve their main concerns during problem solving. Not all the concepts mentioned in the theory are discussed; the ones included here were considered as the most salient but the reader may find others more significant.

6.2.2. Generating knowledge

The general concept of 'generating knowledge' (Section 4.2) emerged as an important aspect of problem solving during the course. However, it is not difficult to see that this concept can be of importance in other areas where new information has to be generated for decisions to be made. For instance, big companies or even nations recognise the importance of generating reliable knowledge for achieving their goals. Generating knowledge is a process that might be worth studying more in detail to find ways of making it more efficient and to correct any actions that may prevent its full realisation.

This study raises some important issues in relation to generating knowledge. It was observed that students generate knowledge by making use of strategies such as 'reducing complexity' and 'data organising' (Section 4.2.1). Although these strategies do not in themselves lead to generating knowledge, they may be used to make the process more efficient. Reducing complexity allows students to start dealing with situations that are too complex to be dealt with as they are presented. As for data organising, it allows data to be managed in such a way that facilitates the generation of further information.

Reducing complexity is a concept which importance has been previously recognised in doing mathematics. For instance, reducing complexity is related to Mason et al.'s (1982) idea of 'specialising' and to Hazzan's (1999; 2001) idea of 'reducing abstraction' (see Section 4.2.1). In this sense, the present study does not propose an altogether new idea. However, this study's contribution is to establish a relationship between reducing complexity and other aspects of problem solving. For instance, the model suggests that students may consciously choose to reduce complexity to start generating knowledge. Moreover, it is suggested that reducing complexity is an idea that goes beyond mathematical problem solving. It is relevant to other areas of doing mathematics and it may also be relevant to explain certain aspects of generating knowledge in other areas such as generating specialists' reports or academic writing.

Another important concept discussed in relation to generating knowledge is that of a 'blinding activity' (Section 4.2.3). A blinding activity is an activity that involves focusing so strongly on achieving a goal in a particular way that relevant information may be overlooked. In this study, it was observed that looking for patterns sometimes becomes a blinding activity that prevents students from generating the necessary knowledge. But blinding activities are not necessarily exclusive of the context of mathematical problem solving. For instance, in conducting research in the social sciences, it is not uncommon that a data-collection method turns into a blinding activity that, instead of allowing the researcher to learn about the situation, prevents her from looking into issues that seem of importance. Learning about blinding activities may help to increase awareness of situations that might hinder knowledge generation and may allow solvers to deal better with them.

Generating knowledge during the course was characterised by some patterns of behaviour being more effective than others. For instance, while 'situational reasoning' (Section 4.2.4) is a concept that, in general, suggests an effective behaviour, 'key searching' (Section 4.2.3) can be classified as an ineffective way of trying to generate knowledge. In relation to the latter, it may be said that assuming the problem has a special quality that once discovered will lead to solving it automatically is not necessarily an erroneous approach. Key searching may not be altogether unsuitable for dealing with 'insight' problems like the ones used by some psychologists in their experiments (see Section 2.2). Moreover, the problems that Polya (1957) used were problems that can be solved once a particular idea is realised and in which the solver can benefit from receiving a 'hint' from someone who 'knows' or who has already solved the problem successfully. Thus, 'key searching' as a way of generating knowledge (or even a solution) may have its roots in earlier or alternative views of problem solving. However, as Stewart (1989) points out, this approach is not applicable to the types of problems that mathematicians deal with or, for that matter, to any situation that represents a genuine problem for which the solver is interested in generating a solution. In general, an awareness of what behaviours are more efficient for generating knowledge is something that students can definitely benefit from.

As for 'situational reasoning', it may be said that this concept highlights an important aspect of mathematical thinking. Thinking in terms of how the data stems from the situation may provide information that can be useful not only for generating knowledge but also for generating solutions and validating them (see Sections 4.3.1 and 4.4.1, respectively). After all, situational reasoning brings to mind Lakatos' (1976) idea that mathematicians deal not only with deductive reasoning but with other forms of reasoning as well. To what extent situational reasoning as suggested in this study resembles other forms of reasoning used by mathematicians is something that needs to be further investigated. Nonetheless, this study contributes by highlighting the versatility of this way of reasoning in students and points out its importance in mathematical problem solving.

6.2.3. Generating solutions

Most students' intention as they generate knowledge is to achieve sufficient information and understanding to start, eventually, generating a solution. As the model suggests, students will start generating a solution as soon as they reach a 'point of sufficient understanding' (Section 4.3). This concept serves to explain the transition from generating knowledge to generating a solution in terms of students' knowledge growth. The idea of a point of sufficient understanding can be useful for teachers since it empowers them to start dealing with an issue that they may have noticed before but were not able to point out directly. Furthermore, this concept plays a role in predicting situations in which students may engage in pseudo-solutioning (see Section 4.6). It seems that the idea of reaching a point of sufficient understanding is important in the school and university problem-solving environment, where students are expected to generate solutions and to answer questions in a limited amount of time. However, this concept may also be relevant to other areas where performance in new situations is important.

This study suggested that students generate solutions by reasoning 'deductively', 'inductively' or by 'guessing', and sometimes even by reasoning 'situationally' (Section 4.3). The terms deductive and inductive reasoning were used in a free way in the sense that they express patterns of behaviour observed in students' processes and do not necessarily satisfy the definitions given in other studies. This approach allowed the researcher to provide an explanation of how students generate solutions that is grounded on the data. As for the idea of 'guessing' as a way of generating solutions (Section 4.3.2), it may be said that this is an important concept that points to an activity that might be more common than we would wish to acknowledge. It is therefore important to understand the nature of this way of 'reasoning' and the circumstances in which students choose to engage in it. This study's contribution is to suggest this pattern of behaviour and to delineate its relationship to other concepts (e.g., to ungrounded ideas). However, further research is necessary not only in relation to how students generate solutions by guessing but also in relation to reasoning deductively and inductively. Doing this will help to improve the theory since, as Glaser (1978) suggested, grounded theories are not reified but modifiable and thus can always be improved as new data becomes available.

6.2.4. Validating results

Validating results has a tradition as an important aspect of mathematics education, particularly in relation to problem solving. Students are encouraged to 'look back' (Polya, 1957) and ensure that their solutions are convincing to themselves and to others (Mason et al., 1982). Encouraging students to validate their results seems to have a dual purpose. On the one hand, it helps students to make sure that their results are correct. On the other, it encourages reflection on the activities that were conducted and provides an opportunity for future learning.

In spite of students being recommended to validate their results, some studies have reported that checking solutions before submitting them or closing the process is not a common habit (Schoenfeld, 1985b; Stillman and Galbraith, 1998). However, the results of this study point in a different direction. They suggest that students do validate their results, although they may do it *during* the processes and not necessarily once it is concluded (Section 4.4). In other words, students do not wait until final results are ready to start seeking validation. Instead, they may be constantly seeking to validate the emerging results in different ways. The study goes further by explaining students' validations in terms of the concepts of seeking 'mathematical conviction' and 'cognitive reassurance' (Section 4.4.1). It may be said that these terms serve to explain how students resolve an important concern such as a making sure that their results are consistent and that they make sense.

Validating can be seen as a way of monitoring one's results as problems are being solved and thus it may resemble Schoenfeld's (1985b) idea of 'control'. However, a closer look reveals considerable differences. On the one hand, Schoenfeld focuses on describing activities that students do not conduct or conduct with less frequency than 'expert' problem solvers. As Stacey (1988) suggested in her review of *Thinking Mathematically*, Schoenfeld's ideas in relation to control can be summarised as follows:

> They [students] tend to make decisions hastily without considering other possibilities, fail to monitor progress, and often do not question whether a chosen approach will yield helpful information.
>
> (Stacey, 1988, p. 356)

In contrast, the present study discusses validating as something that students *are* concerned about and the model explains how they go about achieving it. For instance, the model suggests that validating results is a desirable condition and that students try to achieve it by seeking mathematical conviction and cognitive reassurance. However, it also suggests that, to a certain extent, failing to successfully validate the results should not prevent students from moving on with their process and continuing to generate knowledge about the situation. In this way, by explaining what students are trying to do, the model provides information that can help students understand their own processes and to control them in a more efficient way.

Validation and proof

An important issue in relation to validating results is the distinction between this activity and trying to generate a formal mathematical proof. As mentioned in the model presented in Chapter 4, validating was done with a different purpose than trying to formalise the results obtained. Students who aimed to prove[28] their results did so with the aim of improving an already accepted result by formalising it rather than with the aim of convincing themselves or others of its validity.

[28] In the sense of a formal mathematical proof, as the term was used in the model.

Although the differentiation made in the model between validating and proof can be seen as clear-cut, this is not necessarily the case in mathematics or in mathematics education. Whether an argument made with the purpose of 'validating' can be considered a proof is a debatable issue (see Davis and Hersh, 1981). In relation to this, Tall (1995) suggested that there are different forms of proof

> which might be appropriate at various stages of cognitive development, dependent on different representations of knowledge that may be available.
>
> (Tall, 1995, p. 27).

Bell (1976) adopted a similar position to Tall and suggested a classification of the different functions of proof that included verification, illumination and systematisation. According to this classification, 'systematisation' relates to what is usually accepted as a formal mathematical proof. However, the other two categories suggest a more flexible definition. Moreover, Balacheff (1986) also seemed to support the idea of a relationship between proof and validation. As he put it:

> By proof we not only mean a strict mathematical proof (what we call in French: la *demonstration*) but rather the various means of control (such as semantic control, logical control...) within the problem solving process. We call these *procedures of validation.*
>
> (Balacheff, 1986, p. 10; emphasis in the original)

Thus, if the definition of proof is broadened as to include Balacheff calls 'procedures of validation', then it may be said that some of the validations that students generated during the course can be considered as proofs. After all, some of these resemble the 'alternative' forms of proof proposed by Tall (i.e., visual, enactive, manipulative). However, as Tall (Ibid) warns, these proofs are no substitute for formal proofs. Non-formal proofs may fail to convey all the information needed to make the conjecture a fully-fledged theorem. Thus, although there may be good reasons for extending the definition of proof to other forms of argumentation, it is still necessary to differentiate between validation as a way of monitoring the process and proof as a way of establishing a formalised mathematical concept.

It may be said that the students that participated in this study had different conceptions of proof and thus made reference to it in different ways. For instance, Leonard (Section 5.2) seemed to be aware of the difference between validating and generating a formal mathematical proof. Patrick (Section 5.3), on the other hand, seemed unaware of this difference, or at least chose not to use the definition of proof in a strict, formal sense. It may be said that Patrick's use of the term 'proof' did not alter his validations or made them more or less efficient.

However, regardless of the term students used to describe their actions, there was a clear distinction between what they were trying to achieve. By using the term 'proof' as referring to formal mathematical arguments, the model is able to highlight the distinction between validating and improving the results by formalising them.

6.2.5. Improving the results

Hunt (1991) suggested that although we like to think about ourselves as Homo sapiens, we tend to look for situations in which we can 'economise' our reasoning. For instance, after going through the hurdle of creating a strategy or method that works, it is not uncommon that it will be re-applied in related situations to lessen the amount of reasoning that needs to be done in the future. However, this 'economy of thought' does not necessarily mean that reasoning is suspended or that simpler forms of thinking take place. For instance, as Gray and Tall (1994) suggested, when some aspects of thinking become less demanding, others can be attended to, making reasoning a more efficient process. Thus, what Hunt suggests should bother scientists is possibly that which should amaze them most, i.e. a capacity to create mental tools to deal with complex situations without having to reason them from scratch.

The concept of 'transferring' (Section 4.5.2) is a good example of the situation described above. Once having generated an acceptable result, it was not uncommon for students to re-utilise the method that gave rise to the solution to generate further results. This allowed students to improve their results by extending them. However, these extensions were not necessarily 'more of the same' but usually implied dealing with more complex situations. It may be suggested that transferring helped students generate increasingly more sophisticated results by allowing them to 'free' some of their attention and to concentrate on other aspects of the situation.

Although it may be said that in the case of this study transferring was in most cases successful, other studies on problem solving have suggested that knowledge in one domain does not transfer from one situation to another (see Section 2.4.1). In relation to this, Best (1995) mentioned that, in terms of problem solving "there is little generalisation from one domain of knowledge to another" (p. 444). However, this study suggests that transferences may take place and can be successful when students are able to relate the source and the target domains; or, in other words, when it is up to them to decide what will be transferred and how. It may be the case that transferring is a delicate issue that can be easily inhibited if certain conditions are not satisfied. Investigating transference situations may provide further insight into these conditions and may serve to inform about this activity in other domains.

6.2.6. Pseudo-solutioning

'Pseudo-solutioning' (Section 4.6) is a concept that points to an interesting pattern of behaviour. It may be said that pseudo-solutioning and solutioning indicate dealing with different concerns. While the latter implies trying to generate a solution, the former implies trying to satisfy an academic requirement. Pseudo-solutioning can be seen as a flawed way of dealing with a situation that is accepted when no better alternatives are available. Situations like this are important to study in mathematics education and in other areas as well. Moreover, a generalised version of this concept may be used to explain situations in which people take pseudo-measures that address part of a problem but that do not resolve what is generally the main concern.

Pseudo-solutioning suggests that some students may have needs that are not being addressed by the course. After all, this behaviour seems related to an inability to deal successfully with the problems given and, ultimately, to not being able to generate a solution. The aim of the course is to help students become better solvers. However, questions may be raised in relation to the benefits of the course for students who pseudo-solution, particularly for those who turned to this approach on a regular basis. These students may need extra help in issues such as acquiring certain mathematical knowledge, or in developing efficient strategies for generating information and eventually reaching a point of sufficient understanding.

Although the present study hopes to make a useful job in pointing out relevant issues in relation to pseudo-solutioning, some of these require further consideration. For example, further investigations can be made in relation to 'making inaccurate deductions' and the circumstances in which this activity takes place. Another issue that needs further investigation is the role of academic requirements in mathematical performance (see also Vinner, 1997). In the case of this study, the academic requirement of having to submit a rubric was an important intervening variable that, combined with failing to generate a solution, seemed to have led some students to pseudo-solutioning. However, the existence of academic requirements is not particular to the context of this study or to problem solving. As Pichat and Ricco (2001) suggested, problem solving is a *complex system of interfering constraints* where students have to consider other aspects (academic, social) besides the mathematical content itself.

6.3. FINAL COMMENTS

The previous section looked at some of the concepts that constitute the model (or theory) of problem solving developed in this study. Different things were discussed in relation to various concepts but in general, the following issues were considered:

- The relevance of the concepts in both the context in which they were developed and beyond, and
- Exemplifications of how the model provides understanding of problem solving and of how it can be of use.

Other issues could be considered and more examples could be given. However, the aim is not to be exhaustive but to suggest how the model, in explaining students' main concerns as they tackle problems, satisfies an implicit aim of generating understanding on mathematical problem solving (see Chapter 1).

Schoenfeld (1985b; 1987a) suggested that the reason why problem-solving frameworks such as Polya's (1957) were not useful for helping students become better solvers was because they did not provide sufficiently detailed descriptions of problem-solving heuristics (see Section 2.5.1). However, as Schoenfeld himself later argued (1987b), presenting students with clearly delineated prescriptions as to how to solve problems does not make them better solvers. The present study claims that it is not by providing more accurate descriptions of how problems can be solved successfully that students can be helped in this respect. What may be needed is a conceptualisation that allows students to abstract those aspects of problem solving that are genuinely relevant to them as solvers and that, if addressed, can make a difference on the types of results obtained. This is not a simple task but, as is well known, increasing a person's ability to solve problems successfully is not an easy task.

6.3.1. Grounded theory revisited

The reader may be reminded that grounded theories are not evaluated in terms of validity and reliability in the same way as in other 'qualitative data analysis' (or QDA) methodologies (Glaser, 2003). Instead, theories generated following a grounded theory approach are evaluated in terms of 'fit', 'relevance' and 'work'. As said before (Section 3.3.3), these qualities are achieved by following a rigorous methodology in which data is constantly compared and memos are incessantly being written. Other aspects of the methodology help to achieve this aim as well and they are discussed in Chapter 3. Chapter 5 provides an illustration of how the theory presented in this study satisfies these criteria. In other words, Chapter 5 illustrates how the concepts suggested by the theory *fit* the data, how they deal with patterns of behaviour that are *relevant* for explaining how students resolve their main concerns, and how the theory *works* for explaining why some students are better able to provide satisfactory results and why others fail to do this.

As a result of using the grounded theory methodology, the model suggests patterns of behaviour that are grounded on the data but that are not quantitatively or otherwise validated. These patterns of behaviour can be seen as hypothesis that, if necessary, can be validated by further research. Other approaches place more emphasis on validating the results. For instance, in the information-processing approach, the models generated can be validated by computer simulations that are expected to emulate human problem solving. The extent to which these simulations resemble the observed problem solving behaviour determines the adequacy of the model (Mayer, 1992). However, validations of this sort do not feature within the grounded theory methods.

In spite of this disadvantage, the grounded theory approach was considered as a suitable methodology. Grounded theory allows the researcher to conceptualise the participants' main concerns and thus its results are of relevance to those involved. Another advantage of using grounded theory is that it is a methodology that guides the researcher from data collection to generating a theory. Other approaches (e.g., phenomenography, information-processing) present well-developed data-collection methods, but do not provide much guidance as to how the data analysis is to be conducted (Akerlind, 2002; Yang, 2003). In relation to this, Glaser (1998) mentioned that the methods of data collection and analysis that he presented were developed by a process of conducting research on how grounded theory studies are conducted. It may be suggested that a similar process may help to make other methodologies more accessible to those interested in using them.

6.3.2. From a substantive to a formal theory of problem solving

Before concluding this study, it seems relevant to raise the issue of whether it is possible to generate a formal theory of solutioning that applies to areas other than mathematical problem solving. Problem solving is an activity that requires the solver to generate knowledge in order to later generate a solution. Also, once a solution is achieved, it may be validated and then improved. To start with, it seems feasible that solutioning could be extended to other processes such as the development of new concepts in the social sciences. After all, the methods that are known as scientific involve some version of the stages mentioned Chapter 4. Moreover, behaviours such as 'reducing complexity', 'validating' and the idea that there is a 'point of sufficient understanding' below which it is difficult to generate new knowledge are not necessarily alien to this domain.

To be sure, a formal theory of solutioning would not be an exact copy of the one presented here. It would be necessary to make further comparisons of relevant data and to memo extensively before being able to provide a grounded version of solutioning as a formal rather than as a substantive theory (see Strauss and Glaser, 1967; Glaser, 1978). However, the important thing is not what aspects of solutioning remain but what we eventually learn about problem solving in more general domains. After all, as discussed in Chapter 2, the aim of generating a general theory of problem solving is far from being fulfilled. For this reason, trying to generate a formal theory in this way seems a worthwhile aim.

Finally, the theory suggested in this study contributes to the study of problem-solving by providing a grounded theory that explains students' concerns as they tackle non-routine mathematical problems. Some of the concepts suggested by the theory have been previously highlighted in other studies (e.g., Hazzan, 1999; 2001; Vinner, 1997). However, many of the concepts suggested have not been identified before. But possibly more important than the novelty of the concepts is the fact that they are presented as part of a holistic model that explains how students solve problems. In this sense, even the concepts that have been previously discussed are presented in a way that sheds light to novel relationships.

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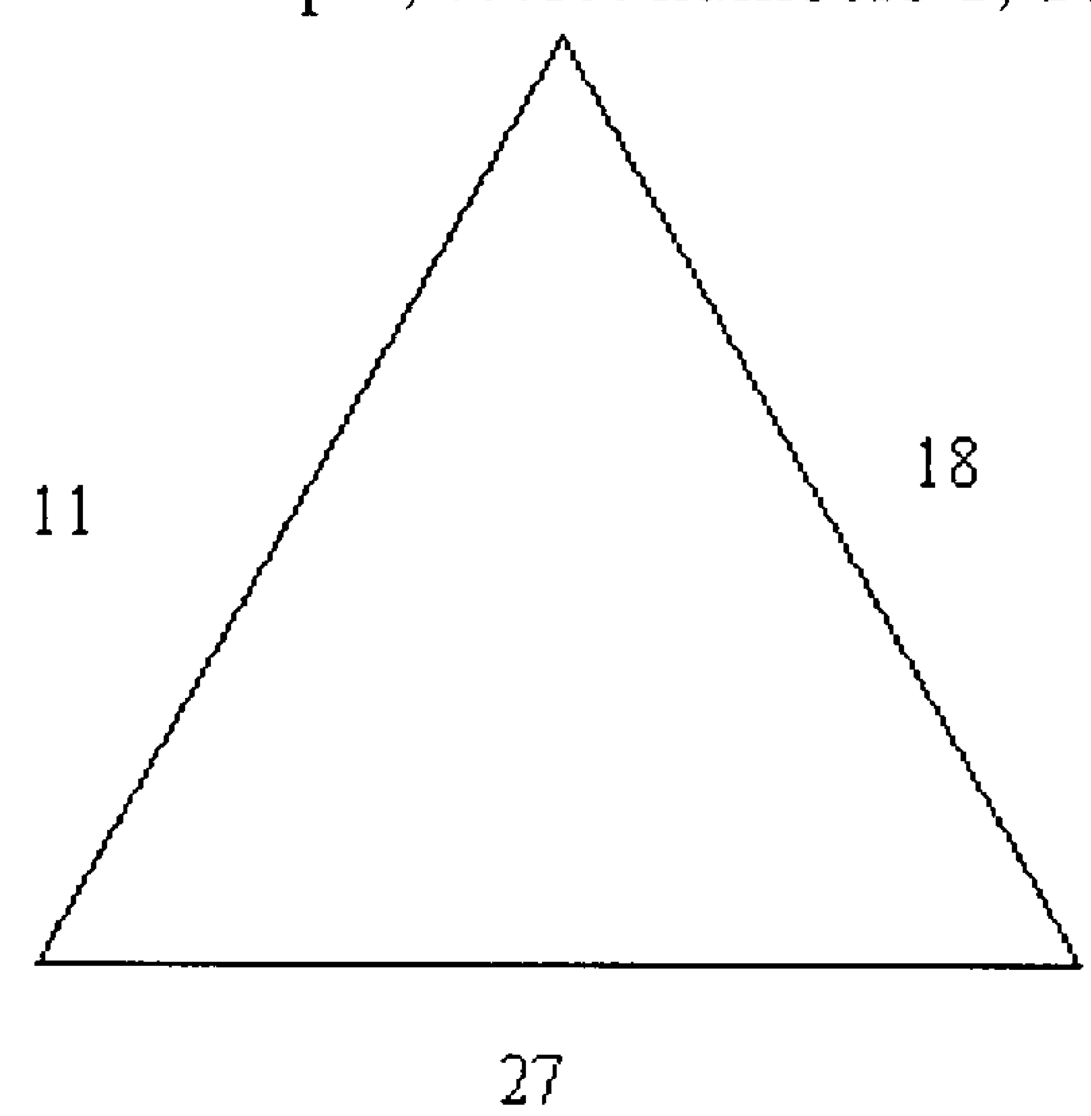
APPENDICES

CONTENTS:

APPENDIX 1: THE PROBLEMS

(In alphabetical order)

Arithmagons:
A secret number is assigned to each vertex of a triangle. On each side of the triangle is written the sum of the secret numbers at its ends. Find a simple rule for revealing the numbers.
For example, secret numbers 1, 10, 17 produce:



Generalise to other polygons.

Comment: *This problem can be easily solved by, e.g., using systems of linear equations. Thus, many students generated an initial solution almost immediately and this allowed them to consider secret numbers in other polygons or even in three-dimensional shapes. However, some students were misled by the initial simplicity of the problem. Although they had generated a solution by an algebraic procedure, they spent a considerable amount of time trying to justify that their results were true for any triangle.*

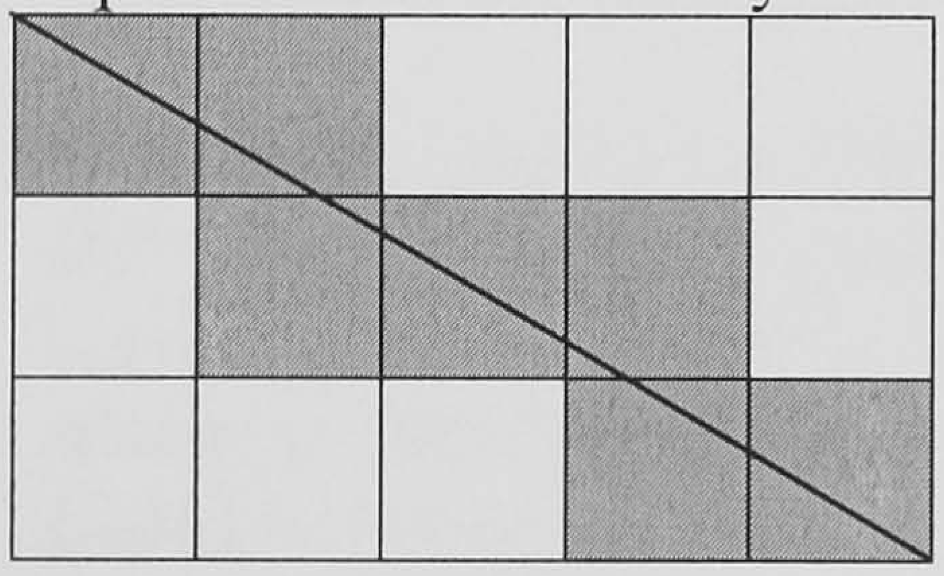
As Easy as ABC?:
Let a,b,c be positive numbers with the property that abc=1. What can you say about (a–1+1/b)(b–1+1/c)(c–1+1/a)?

Comment: *This was a more difficult problem than the one above. Some students were able to conclude that the expression would always be less than 1, but were unable to provide a justification for this. Other students did not go past trying to understand and learn about the problem. Many students tried an algebraic approach to deal with this problem but were unable to produce any results in this way. There is no available solution to this problem.*

Cartesian Chase:
A game of two players is played on a rectangular grid with a fixed number of rows and columns. Play begins in the bottom left hand square when the first player puts a counter. On his turn, a player may move the counter one square up, one square right or one square diagonally (up and right). The winner is the player who gets the counter to the top right square.

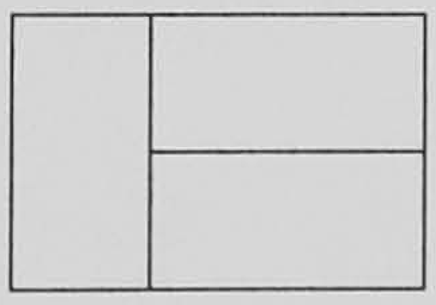
Comment: *An interesting solution to this problem involves a list of rules for winning the game. These rules indicate that only a small portion of the grid is relevant for determining the winner and that anything that happens before a few squares close to the top-right corner is irrelevant. An important characteristic of the problem is the vagueness of its instructions for the solver. Many students spent a considerable amount of time trying to define what they wanted to achieve with this problem and some did not get past this point.*

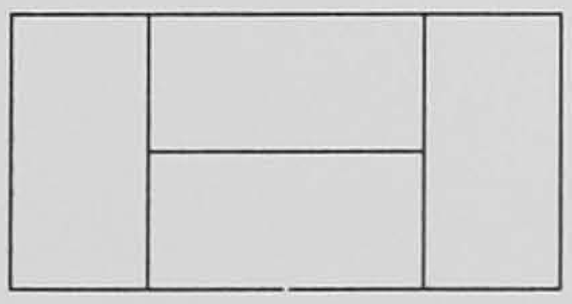
Diagonals of a Rectangle:
On squared paper, draw a rectangle and draw in a diagonal. How many grid squares are touched by the diagonal. E.g.



Comment: *Providing a formula for some particular cases of the situation described in this problem was not difficult. Many students were able to notice a pattern in n by n rectangles but not all were able to provide a formula valid for all cases. Those who did based their general formula on a pattern related to the highest-common factor between the dimensions of the rectangle.*

Faulty Rectangles:
These rectangles are made from 'dominoes' (2 by 1 rectangles). Each of these large rectangles has a 'fault line' (a straight line joining opposite sides).





What fault free rectangles can be made?

Comment: *In general, fault-free rectangles can be made if at least one of their sides is even and if their dimensions are 5 by 6 or greater. But there is an exception to this rule: fault-free rectangles with dimensions 6 by 6 cannot be made. Most students concluded that fault-free rectangles are possible but not all provided an expression that defined their dimensions. Why 6 by 6 rectangles cannot be made is something many students were able to notice but none provided a satisfactory solution as to why this is the case.*

Hat Numbers:

A hat contains 1992 pieces of paper numbered 1 through 1992. A person draws two pieces of paper at random from the hat. The smaller of the two numbers drawn is subtracted from the larger. That difference is written on a new piece of paper which is placed in the hat. The process is repeated until one piece of paper remains. What can you tell about the last piece of paper left?

Comment: *Satisfactory solutions for this problem provided a way of predicting whether the last number to be taken out of the hat would be odd or even. Very few students arrived at this solution. Many found it difficult to explore this problem in a useful way.*

Ins and Outs:

Take a strip of paper and fold it in half (always placing the right hand edge on top of the left hand edge). Fold it several times and observe the sequence of 'in' and 'out' creases. For example three folds produces:

in in out in in out out

What sequence would arise from 10 folds?

Comment: *The first difficulty that this problem presented to the students was that it is not possible to fold pieces of paper ten times. Therefore they devised a way of generating the necessary information by looking at how the creases emerged. Most students were able to predict the pattern of 'ins' and 'outs' recursively but very few students provided a more general formula.*

Jogger's Dog:

A jogger runs, at a constant speed, around a circular track. The jogger's dog runs, always toward the jogger, at constant speed. What sort of paths does the dog describe?

Comment: *This was an optional problem in one of the final assignments and it was tackled by no more than five students. The problem required the students to make a number of assumptions and thus the available responses vary considerably. Among the responses given, students suggested that the dog would follow a circular path around the jogger.*

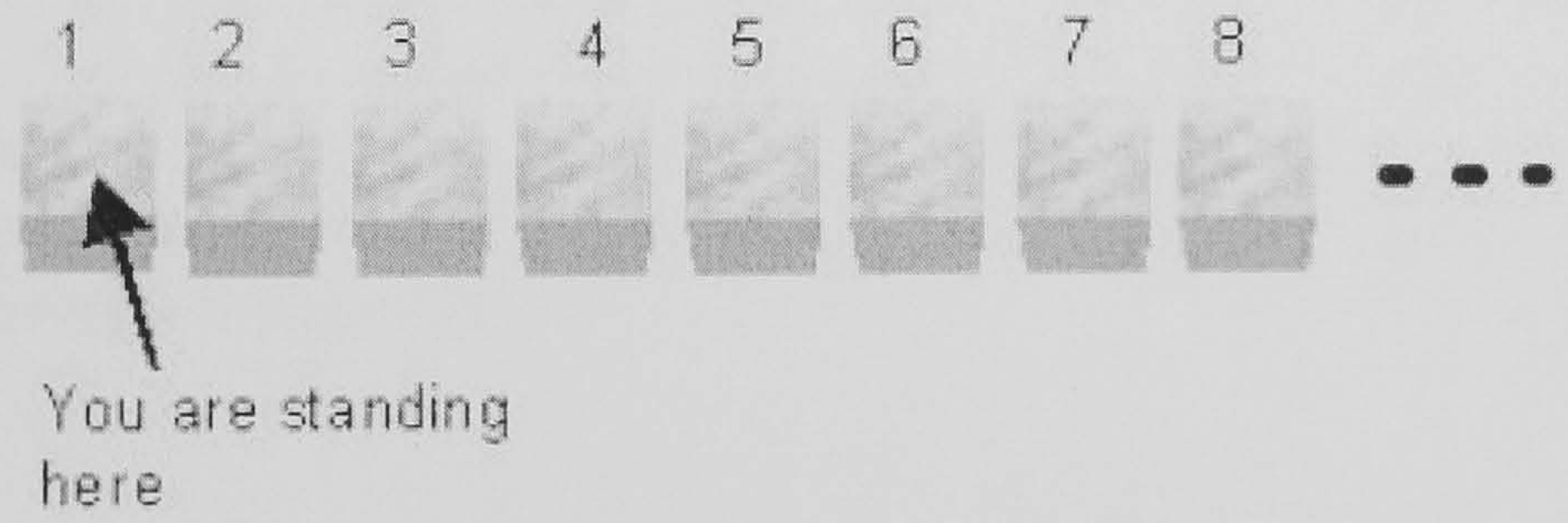
Liouville:
Take any number and find all of its positive divisors. Find the number of divisors of each of these divisors. Add the resulting numbers and square the answer. Compare it with the sum of the cubes of the numbers of divisors of the original divisors

Comment: *Students noticed that the two quantities to be compared in the Liouville problem are always equal. Successful explanations for this observation made artful use of the fact that all natural numbers can be expressed as the product of its primes. Providing a fully-fledged argument to justify why both expressions are equal was a complex task. This problem was given as an option for a final assignment and this may have contributed to a considerable number of students persevering on it.*

Square Take-away:
Take a rectangular piece of paper and remove from it the largest possible square. Repeat the process with the left-over rectangle. What different things can happen? Can you predict when they will happen?

Comment: *Students investigated this problem by physically removing squares from rectangular pieces of paper. They easily arrived at the conclusion that this process leads to a squared pieces of paper and thus to removing the whole area. Although students noticed some patterns, none of them provided a satisfactory way of predicting how long would it take for the whole rectangle to be removed.*

Steps:
You are standing at the beginning of an infinitely long path, as shown below:



You throw a fair coin which has the number "1" written on one side, and the number "2" on the other. You walk forward the number of steps shown on the side of the coin that lands face up. For example, if you throw the coin and it comes up "2" you take 2 steps forward to land on the 3rd step of the path - 2 steps from where you were on step number 1.
You now repeat the exercise - throw the coin again and walk forward the number of steps that comes up on the coin. If you throw the coin 24 times you are certain to have landed on, or past, spot number 25. What is the probability that you will land on step number 25?

Comment: *Successful responses to this problem suggested that the probability of landing on step 25 is 2/3. Students explored this problem through different routes such as tree-diagrams and from the perspective of combinations and permutations. Successful solutions seemed to rely also on a careful analysis of the 'accumulation' of probabilities up to step 25.*

Sums of Diagonals:

Investigate the sums of diagonals of different slopes in the grid below.

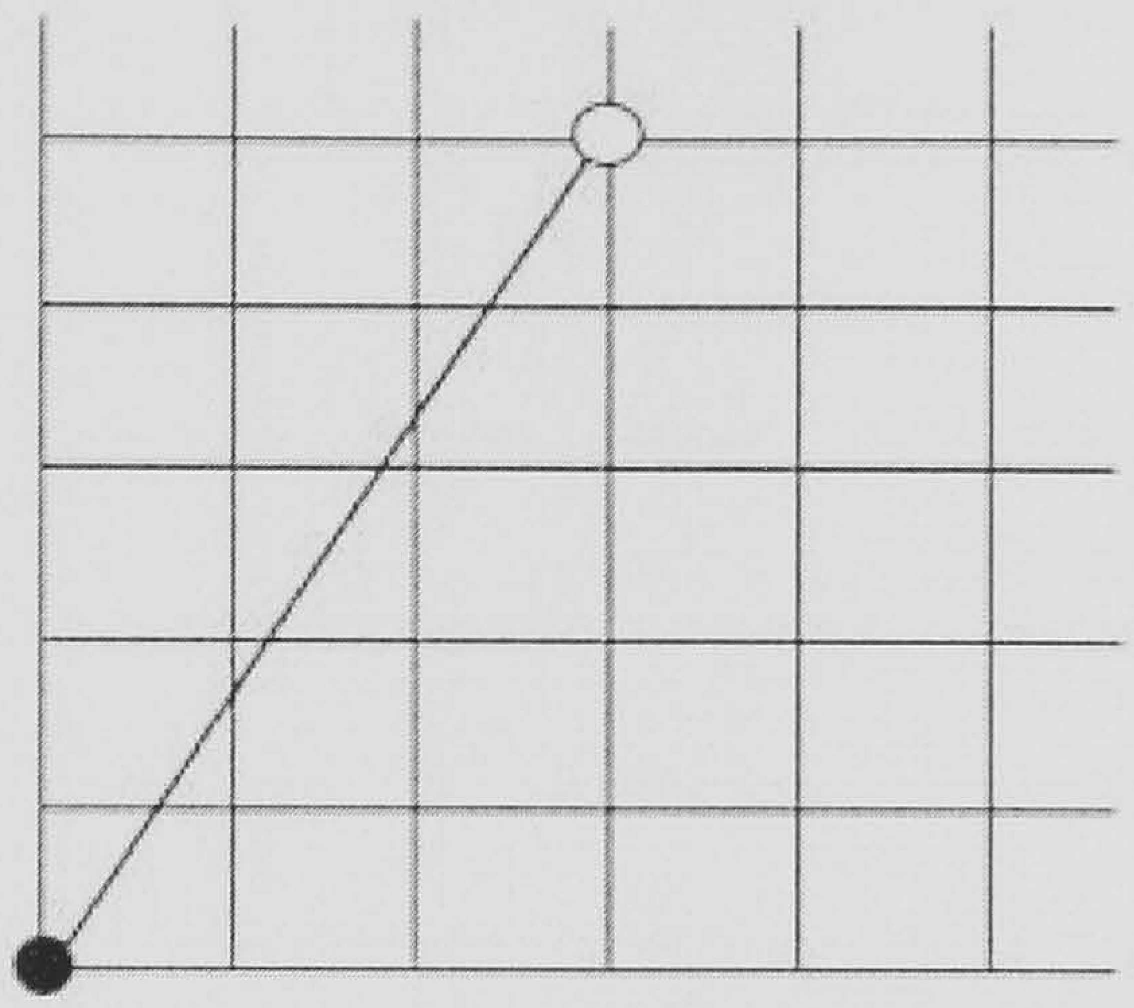
1	2	3	4	5	6	7	8	9	...
2	4	6	8	10	12	14	16	18	...
3	6	9	12	15	18	21	24	27	...
4	8	12	16	20	24	28	32	36	...
5	10	15	20	25	30	35	40	45	...
6	12	18	24	30	36	42	48	54	...
7	14	21	28	35	42	49	56	63	...
8	16	24	32	40	48	56	64	72	...
9	18	27	36	45	54	63	72	81	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

Comment: *This problem can be solved by noticing that the third differences between the sums of diagonals of particular slopes are constant. Some students were able to deduce a formula from this observation. Other students generated a formula for the sums of diagonals of different slopes by observing patterns and generalising them. Students were able to generate important information about the problem by investigating how each element in the sums is generated.* (Note: Since rubrics for this problem were considered for the case studies presented in Chapter 5, further information about the solution to 'Sums of Diagonals' it is provided in Appendix 3.)

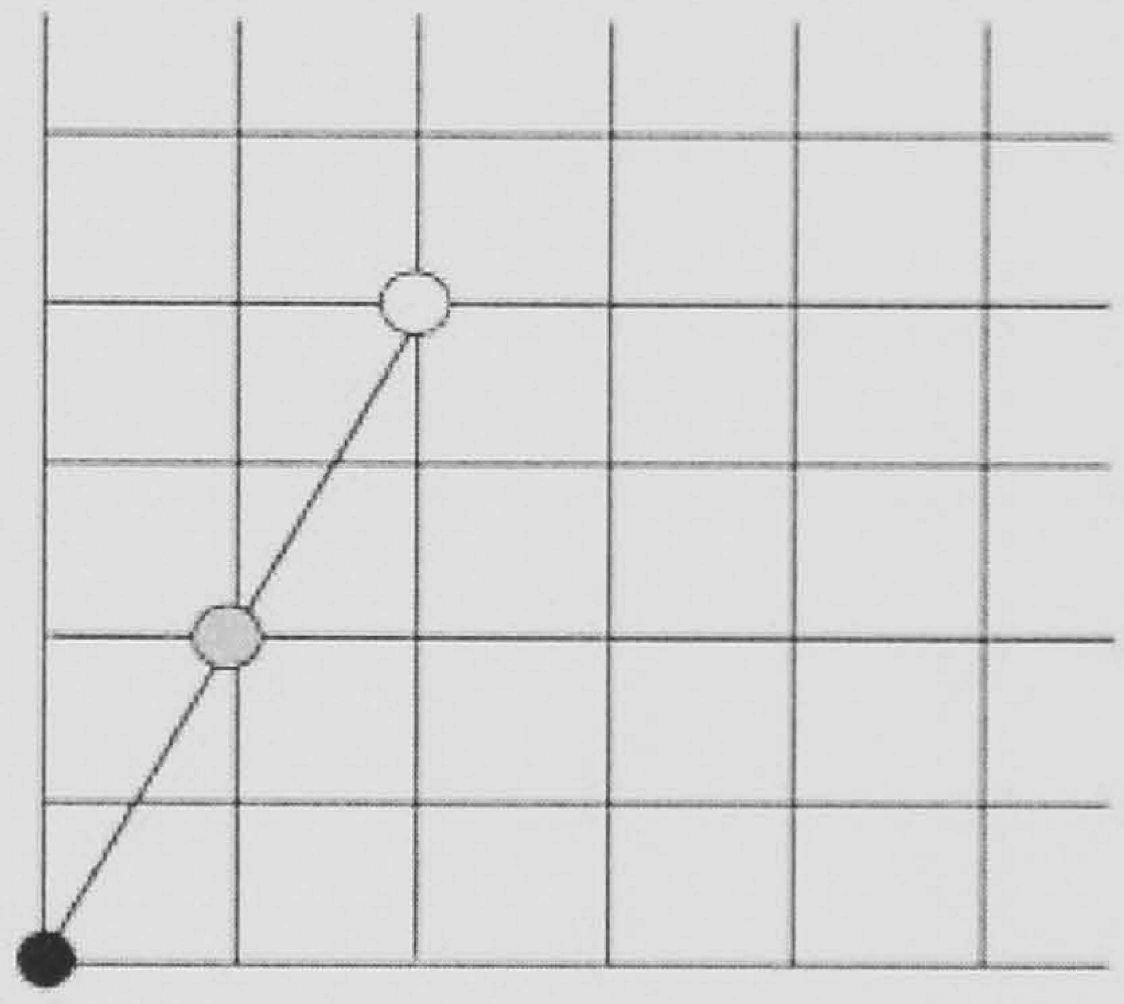
Visible Points:

A point (i, j) in the plane, with non-negative integers coordinates i and j, is **below** a point (m, n) with non-negative integer coordinates when and $i \leq m$ and $j \leq n$.

A point (m, n) in the plane, with m and n non-negative integers, is visible from (0, 0) if the straight line joining (0, 0) to (m, n) passes through no other points below (m, n).



The point (3, 5) is visible from (0, 0)



The point (2, 4) is **not** visible from (0, 0): the straight line from (0, 0) to (2, 4) passes through the point (1, 2) which is below (2, 4)

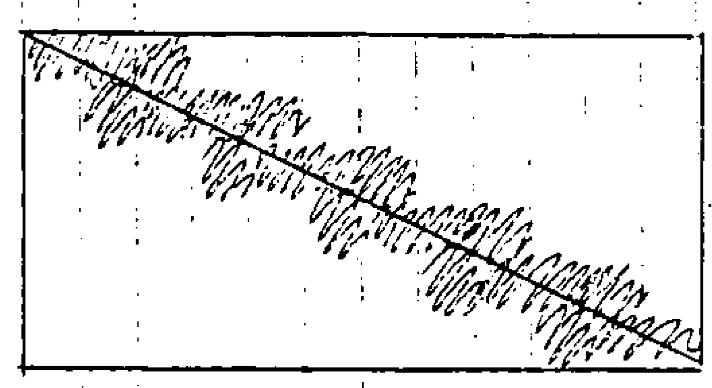
As m and n increase, what percentage of points is visible from (0, 0)?

Comment: *Students solved this problem a few weeks after having solved the 'Diagonals of a Rectangle' problem. It was not uncommon for them to find a relationship between these problems, particularly with respect of making use of highest-common factors to investigate the situation. A number of students were able to suggest that, for points with non-zero coordinates, a point would be visible if the highest common-factor of its coordinates was 1. No student provided a general formula that represented the percentage of visible points from (0, 0).*

APPENDIX 2: SAMPLE ANALYSIS OF TWO RUBRICS

DIAGONALS OF A RECTANGLE

Hannah
(Maths)



OK, well I've started by drawing any rectangle and a diagonal through it, just as they've asked me.

Now, first things first, let's look at the problem. Well, all it says is to draw a rectangle, a diagonal through it, and see how many squares it touches. I've come across some ambiguity already in the rectangle I've drawn above, in that the diagonal passes through a corner. I've just taken this as touching all the squares that have this corner.

Wants to learn about the problem - what's it about

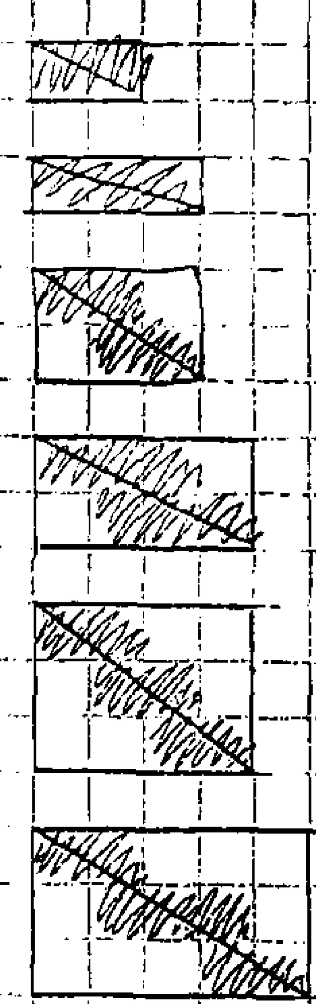
Now what do I want to know? Well, the number of grid squares touched by the diagonal. But this will change depending on the number of columns (n) and the number of rows (m). So, we want a relationship between the number of squares touched, and n and m.

Right, at the moment, ~~I have absolutely~~ no idea! So I'll start by investigating and drawing some rectangles: call the number of squares touched r.

Highlights aspects of situat^n as a result of investigating it. Interesting things start to emerge

To start getting an idea of the situation, "works with it", so to speak.

To learn about situation: tries out some examples, orders the info.



$n=2,\ m=1,\ r=2$

$n=3,\ m=1,\ r=3$

$n=3,\ m=2,\ r=4$

$n=4,\ m=2,\ r=6$

$n=4,\ m=3,\ r=6$

$n=5,\ m=3,\ r=7$

AHA!

OK, I think I see a pattern forming. I'm not sure quite how to express it, so first I'll just say it. The touching squares go in steps. Obviously there are always the same number of vertical steps as there are m rows. Now if both n and m are even, then the diagonal will pass through a corner, which will touch all four squares that have this corner. Therefore, any two rows one above the other will overlap by two squares.

If either n or m are odd, then the diagonal does not pass through any corners, and so there is only an overlap of one square.

STUCK though, because I'm not sure how to work out the length of each row of touched squares.

~~I'm see~~ Perhaps this is ~~not the best way~~ to unstick myself, but I'm going to investigate a bit further, an

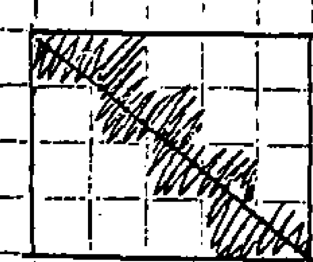
Keeps eyes open for patterns – pattern searching?

Points out what she found (how does this relate to looking for patterns?)

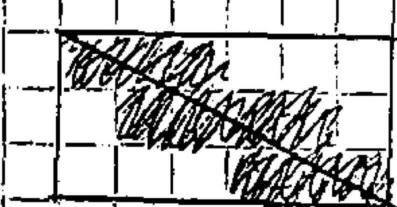
Seeking more info. about the situation. Wants to gain understanding.

see if I can get any more of a pattern.

Looks for patterns



$n=5$, $m=4$, $r=8$

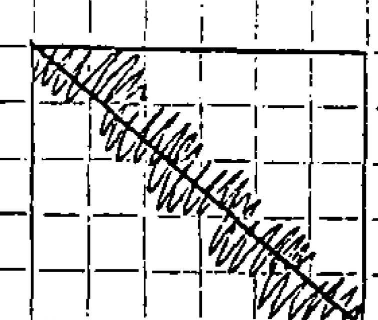


$n=6$, $m=3$, $r=10$

Ooops - I've just found a counterexample to my theory of the diagonal only passing through corners when n and m are even. I'll have to rethink that!

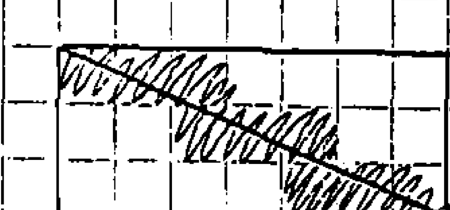
Still needs to gather more info...

$n=6$, $m=4$, $r=10$



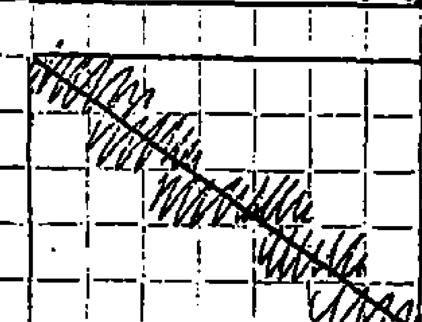
$n=6$, $m=5$, $r=10$

as before, to do this tries out examples + orders info. conveniently



$n=7$, $m=3$, $r=9$

$n=7$, $m=4$, $r=10$



$n=7$, $m=5$, $r=11$

Well, I think that's enough examples for the moment. My theory about the corners and n & m being even is definitely not right - I've had quite a few counterexamples now! Let's try to think logically about specifically when a

Needs to find out more about the situation.

diagonal would pass through a corner

Highlights important observation

AHA!! I think the diagonal will pass through a corner when n and m have a common factor greater than 1. This makes a lot of sense because it implies that the rectangle can be split up into smaller rectangles with the same diagonal, and therefore the diagonal would pass through corners.

Considers 'veracity' of observation – explains it

Ok, so I should have to find different formulas for n and m coprime, and n and m not coprime.

Let's start with n and m coprime.

The trend seems to be that the number of squares touched $r = n+m-1$

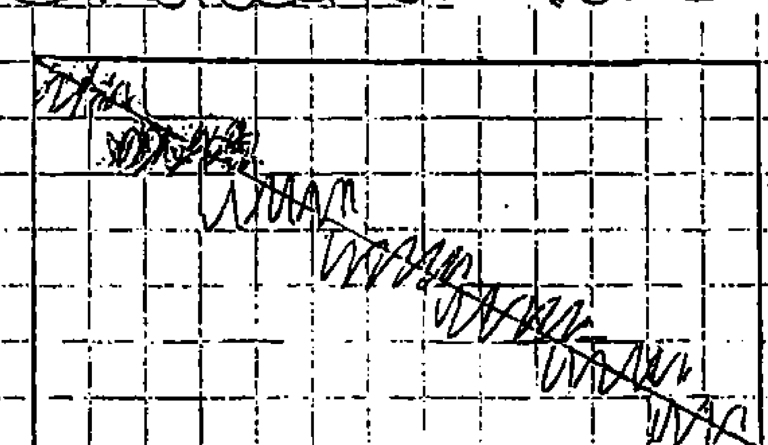
Decides to try to generate solution (but how?!)

If we consider a small rectangle, say, $n=3$, $m=1$ then this is obvious, because the diagonal passes through all the squares, which is $n+m-1$

But why is this necessarily true for a larger rectangle?

Deduction?

Consider a random rectangle with n, m coprime



In this case $n=13$, $m=7$, $r=19$

There are always going to be m 'steps' of squares that are touched.

Induction?

AHA! I think I understand the pattern.

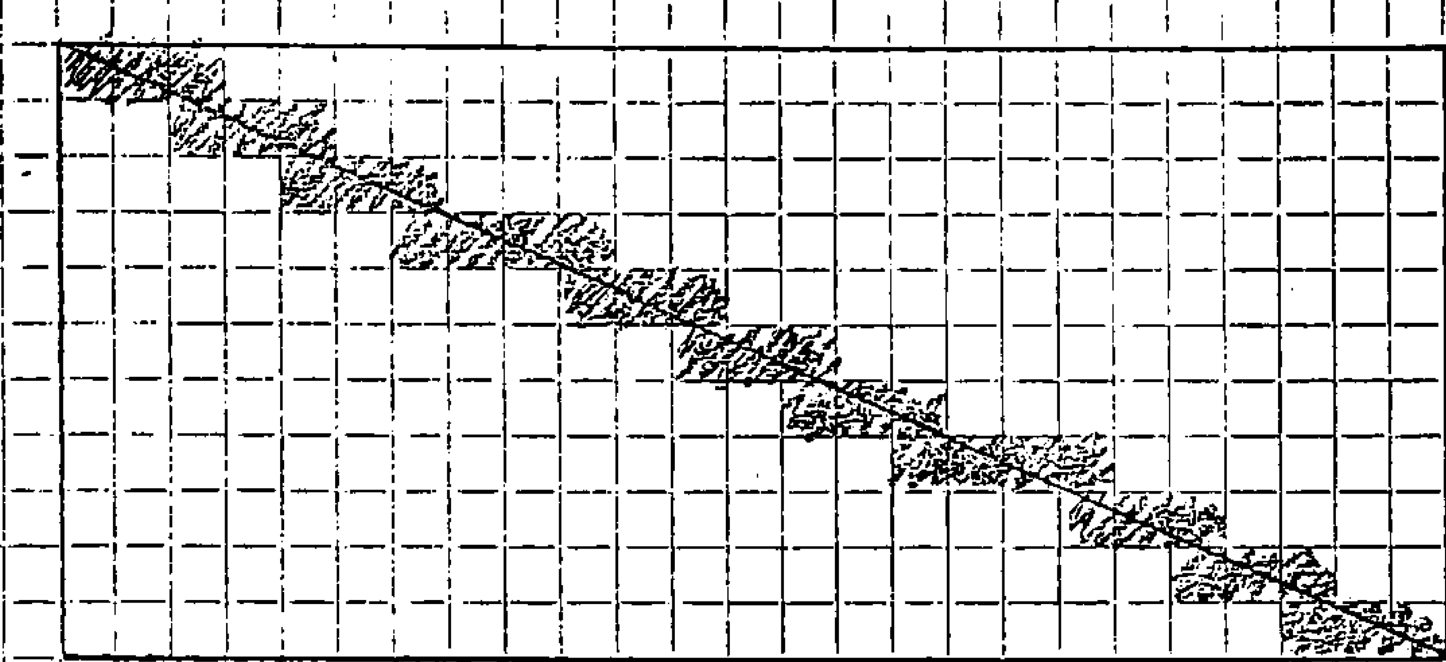
There are always n squares touched, because the diagonal travels from one side of the rectangle to another. Also, there are m steps. Each step is placed underneath the last square of the step above it. Because there are m rows, there are m-1 transitions from one step to the next, and each of these transitions adds an extra square, since the steps overlap as I have described.

Satisfied with result once explained: achieved reassurance throug verification

Therefore, there are n+m-1 squares touched by the diagonal.

I will now specialise artfully with another random rectangle with n,m coprime to check that my formula still holds.

Further checks to see whether it's actually consistent



n=25
m=11
r=35
For this rectangle my formula is valid

Right - yay!

Let's move on to when n and m are not coprime as I think this might be more difficult.

Seeks to improve result, wants to take it further

Now, whenever the diagonal touches a corner, instead of the one-square overlap that is usual between steps, there is a two-square overlap. Apart from that, the pattern is the same.

observes...

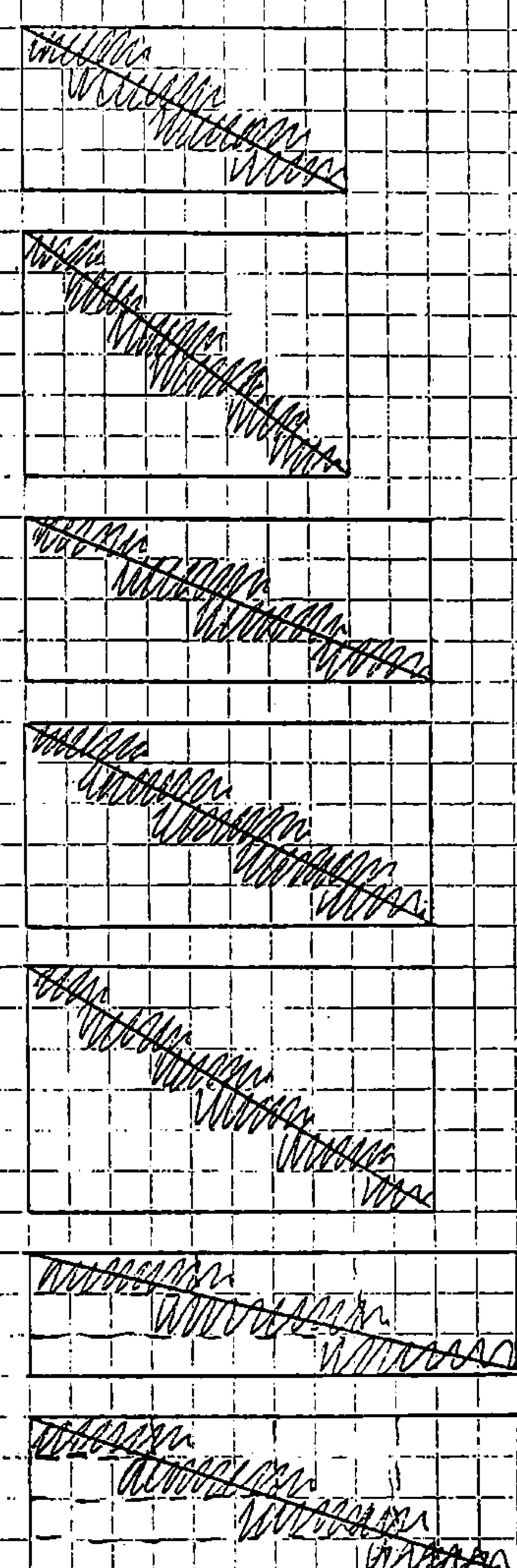
So, I need to find a way of working out how many corners the diagonal will touch, call this c.

Since in all other ways the pattern is the same, the number of squares touched will then be n+m-1+c

deduction

since each corner touched adds an extra square I'm going to specialise again, systematically, and investigate the number of corners the diagonal touches with n and m not coprime

Seeks to generate further info (compare to previous cases)



$n=8, m=4, c=3, r=14$
$r=n+m+c-1$ ✓

$n=8, m=6, c=1, r=14$
$r=n+m+c-1$ ✓

$n=10, m=4, c=1, r=14$
$r=n+m+c-1$ ✓

$n=10, m=5, c=4, r=18$
$r=n+m+c-1$ ✓

$n=10, m=6, c=1, r=16$
$r=n+m+c-1$ ✓

$n=12, m=3, c=2, r=16$
$r=n+m+c-1$ ✓

$n=12, m=4, c=3, r=18$
$r=n+m+c-1$ ✓

Right, well, if m is a factor of n, then there always seems to be m-1 corners touched. This is fairly obvious, because if m is a factor of n, then the rectangle can be split up into m rectangles all with the same

Points out observation
Explains it...
(Does looking for info cause pointing or noticing interesting observations? See previous cases.)

diagonal, as can be seen in green on the last two rectangles drawn. Therefore, the diagonal will touch $m-1$ corners.

re-uses previous idea

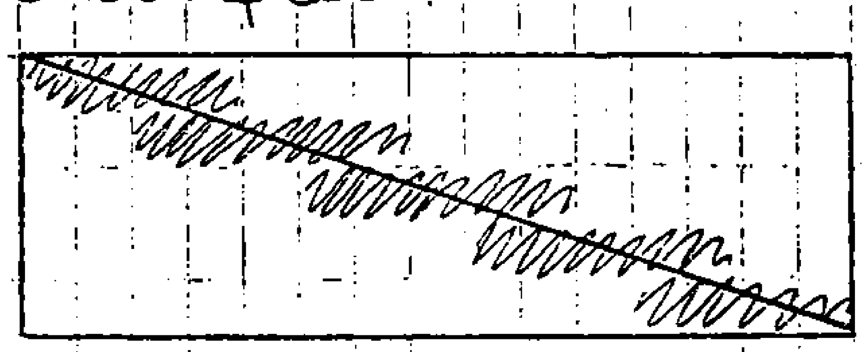
Also, if m is not a factor of n, then consider the highest common factor of m and n. With the reasoning the same as it is above, the rectangle can be split up into $hcf(m,n)$ rectangles all with the same diagonal, and so the diagonal will pass through $hcf(m,n)-1$ corners.

generates solution deductively

Phew! My brain hurts now!, but I think this all makes sense, and will be true for all m, n not coprime.

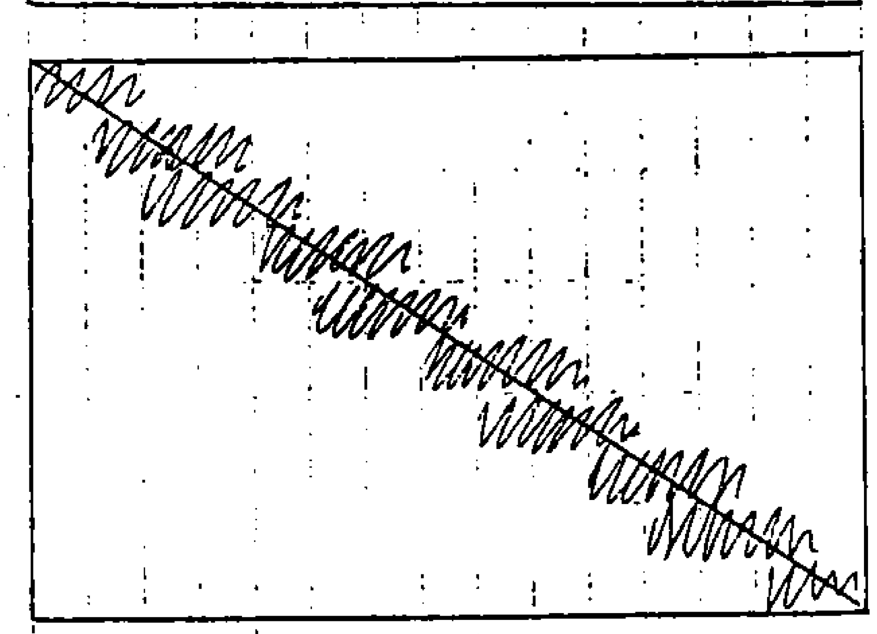
I'll specialise artfully again, and pick two random examples

checks result to see if it is consistent (already trusts that it makes sense)



$n=15, m=5, c=4, r=23$

$r = n+m+c-1$ ✓



$n=15, m=10, c=4, r=28$

$r = n+m+c-1$ ✓

$hcf(m,n)=5$ $c = hcf(m,n)-1$ ✓

Well, I think I've finished now, so I'll write out my solutions.

Going over my notation:

n = number of columns

m = number of rows
c = number of corners touched
r = number of squares touched

So, for n, m coprime $r = n + m - 1$
For n, m not coprime
$r = n + m + c - 1$
where $c = hcf(m, n) - 1$
so $r = n + m + hcf(m, n) - 2$

Right, now what was my initial aim? I wanted to find a relationship between n and m and the number of squares touched, which I have done for all n, m. So I've answered the question.

My reasoning seems to make sense, and I think probably my key moments in solving the problem were realising that there would have to be different formulae for n, m coprime, and for n, m not coprime. Also that the formula for n, m not coprime would depend on hcf(m, n).

Was this the key idea that allowed her to generate a solution? Was she looking for it? Some students (eg, Perry) actively look for key ideas. She didn't. But found one...

I got stuck earlier because I was not able to ~~d~~ discover a way of knowing the length of each row of touched squares. I abandoned this because I got sidetracked down another path, but if I had more time, I could try to see if that had any bearing on the number of touched squares, and if the length of those rows was related to n and m.

Lila
(BA (arts))

Diagonals of a ~~Triangle~~ Rectangle!

what do I know?

Investigates problem – to see what it's about

Okay the question directs us as to what needs to be done, explaining how to draw a diagonal line on a rectangle to see how many grid squares touch the diagonal? The diagram given illustrates this idea.

STUCK!

Okay there doesn't actually seem to be a direct question leaving plenty of room for developing and expanding the problem. ?

Realises she needs more info...

The problem does not say what size triangles to use, what to find as such, neither does it tell us fully what is means by grid squares touching the diagonal line, how do we differentiate what one's should be ~~incl~~ included ~~the~~ when the line goes straight through a point.

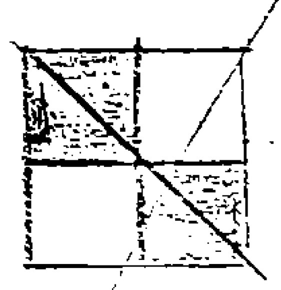
AHA.

Start somewhere...

I shall begin with the example on the page and record how many grid squares are touched by the diagonal line. I shall then perhaps specialise by using an systematic approach to see if there are any similarities between the rectangles I draw and whether there is a relationship between them that will later help me to generalise.

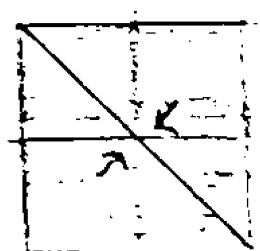
Plans to analyse info carefully to find out – to generate more info.

First I think I need to make clear what is meant by the diagonal line touching the square grids. For example in the case would the amount of squares touching the diagonal be:



would you class this a being 2 grid squares

OR



would class it as having 4 grid squares touching because the line touches the tip on g'

This is another way to generate info: by making assumptions/decisions

For this investigation I shall take it as being the first idea that in a case where the line goes straight through a point you do not need to colour in cover? corresponding sides.

Observes, makes a note

NOTE: I can already see that their may be a problem when drawing these diagrams about the degree of accuracy used. For example if the line is too thick it may cross squares where it might not have done when drawn with a fine line.

To help me I have printed off squares onto a piece of paper to use to draw on my rectangles and try and work out a pattern.

Looks at particular cases to investigate.

I have drawn out rectangles off different sizes starting at 1×1, 1×2, 1×3 .....
2×2, 2×3, 2×4 ....
3×3, 3×4, 3×5 ....

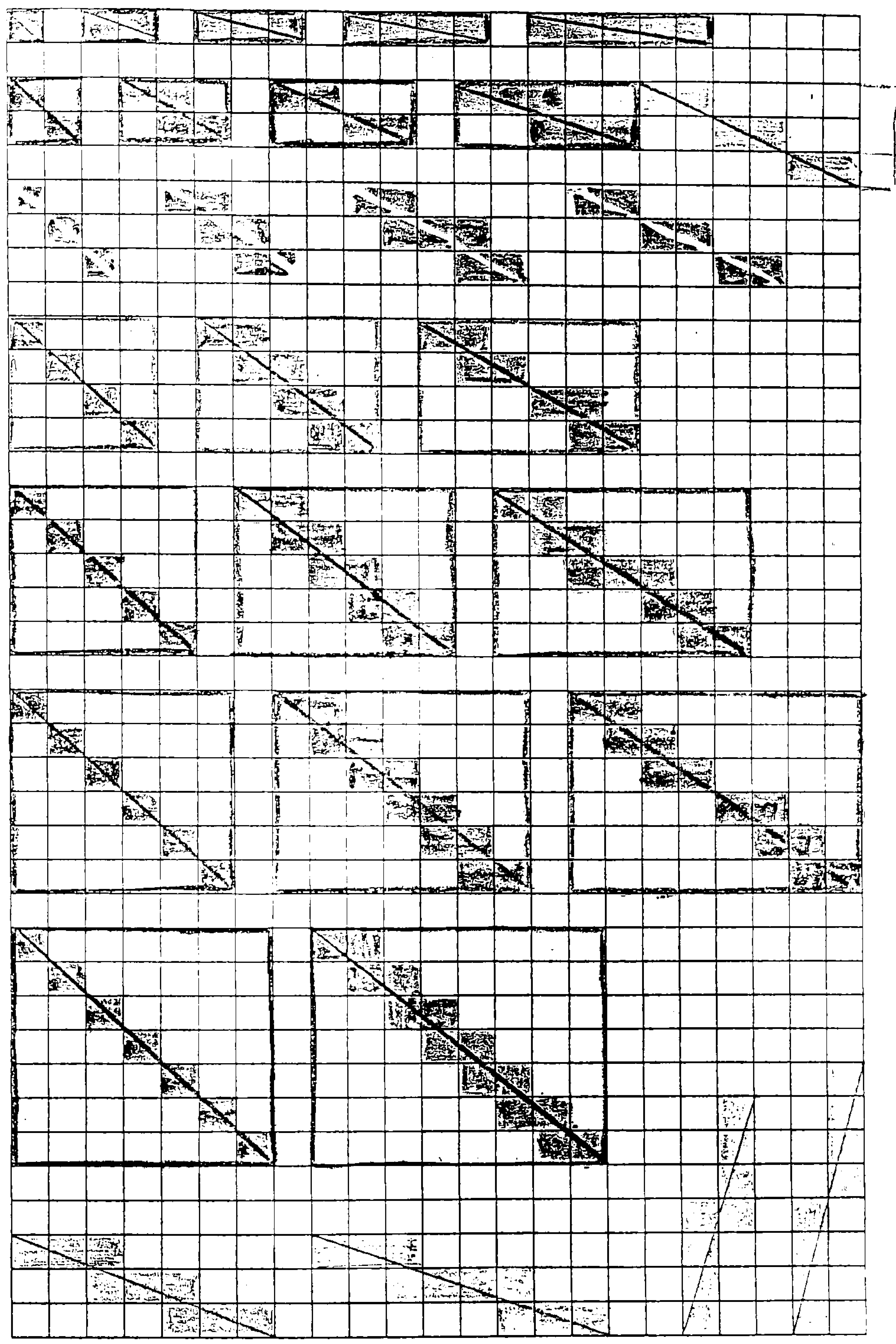
To record my results I shall use a cayley table.

Organises the data

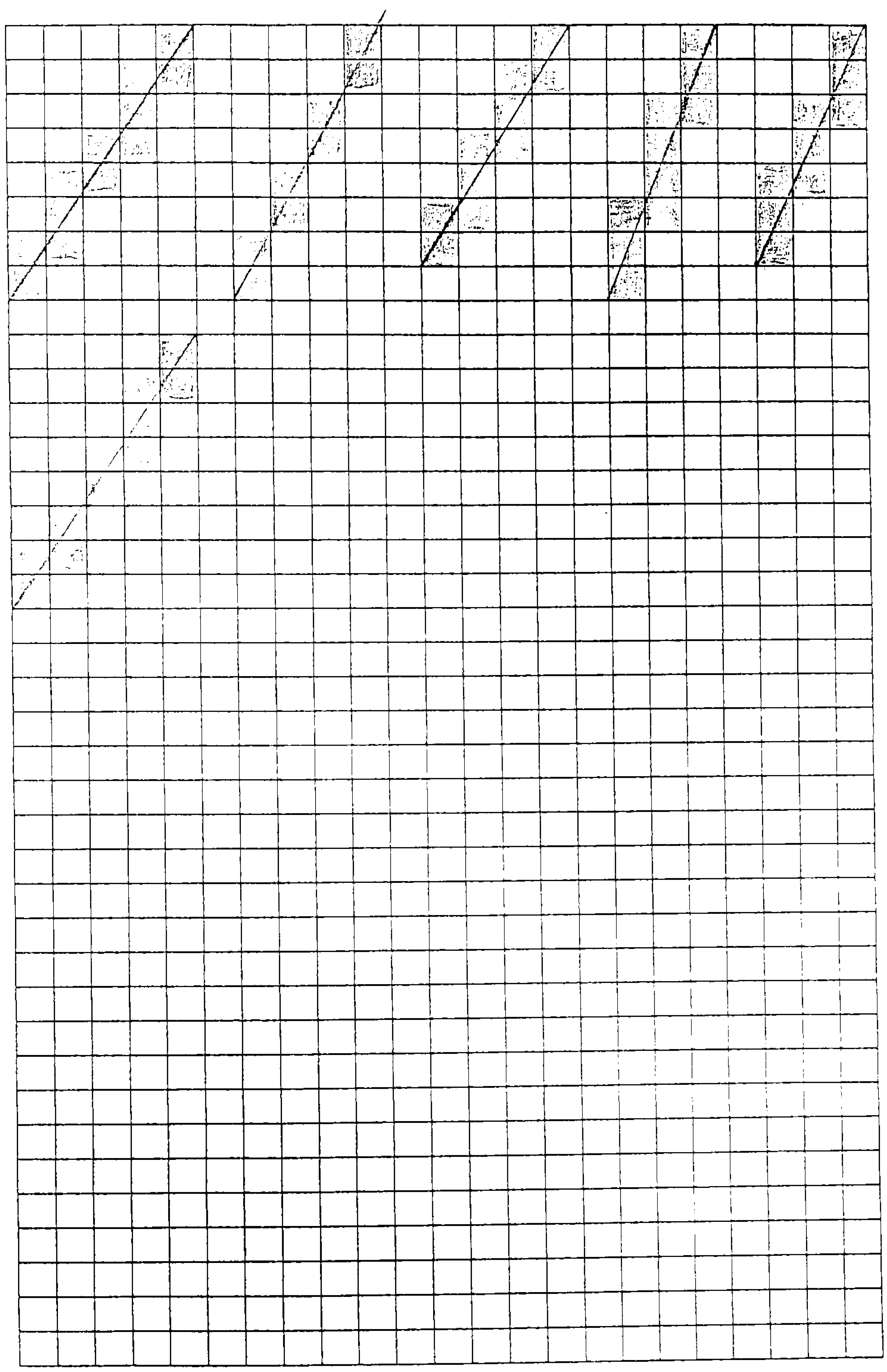
by	1	2	3	4	5	6	7	8
1	~~1~~	2	3	4	5	6	7	8
2	2	~~2~~	4	~~4~~	6	6	~~8~~ 8	~~8~~ 8
3	3	4	3	6	7	6	9	10
4	4	~~4~~	6	4	8	8	10	8
5	5	6	7	8	5	10	11	12
6	6	6	6	8	10	6	12	12
7	7	8	9	10	11	12	7	14

Both representations are a bit of a blur at the moment I cannot seem to take any of the information in, both from the table or the diagrams which I have drawn.

v



P4



Looks for a pattern but unable to find one

It is very difficult to generalise and come up with a pattern because it varies from rectangle to rectangle to generalise in this situation is proving to be difficult.

I can see that when both sides are equal the number of squares touching the diagonal has ~~and~~ the same number as the length of one side.

observation (as a result of looking for patterns?)

STUCK This is extremely hard, every direction that I think about will work for one or two cases and then not for the others, which leads me no further towards the problem. I have stared these results for ages getting no where fast.

(where does this idea come from?)

AHA what if there is a relationship between the amount of squares on the grid and the proportion of those squares touching the diagonal line.

Comparing this process to Hannah's, she seems to be trying possible solutions almost at random. When Hannah noted a possible pattern, she was able to explain why she thought so. This is not the case here. There may be an important difference in the ways in which ss arrive to tentative solutions...

For example on a 3 by 2 rectangle you would have 3x2 squares which is equal to 6. The are 4 squares touching the diagonal line, therefore $\frac{4}{6}\frac{2}{3}$ $\rightarrow$ $\frac{2}{3}$ rds of the square are touched by the line. Perhaps this might help me to generalise.

DAMN!! ~~but~~ I have just noticed that you could have a 6 by 1 rectangle which has the same area as the 3 x 2 rectangle however with this rectangle there will be 6 squares touching the diagonal line. which cancels out the idea of the area being important.

STUCK I thought I was finally getting somewhere with this problem, but I'm back to square one.
I am beginning to get frustrated because I can not see a solution to the problem and don't seem to be going anywhere fast.

Providing ungrounded tentative solutions (guessing) leads to frustration

Review

Investigate further!

I have drawn all many rectangles and obtained my results (specialising) however this hasn't led me to any significant answers or solutions.
I can not see any patterns developing.
I have ruled out the idea of the area being related to the (problem) number of squares touched by the diagonal.
I am really STUCK I can not see where to go from here. I feel I have reached a dead end, not knowing where to go from here.

APPENDIX 3: RUBRICS CONSIDERED FOR THE CASE STUDIES

The following rubrics belong to Leonard, Patrick and Carolyn. These rubrics were analysed by taking the model presented in Chapter 4 as theoretical framework. The analysis is presented in Chapter 5 as a series of three case studies.

The problem that students solve in the following rubrics is the 'Sums of Diagonals' problem (see Appendix 1). This problem requires students to investigate, on a multiplication table, sums of diagonals of different slopes. By listing the sums of diagonals of a given slope, it becomes evident that the third differences of these numbers are constant. For instance, the sums of diagonals of slope 1 are 1, 4, 10, 20, 35, ... (where all the third differences are equal to 1). Thus, it is possible to generate third order equations for diagonals of slopes 1, 2, 3, etc. These equations have *r* (the top-row number for the diagonal) as independent variable. Equations for slopes 1, 2 and 3 are listed below:

Sum of diagonals of slope 1:	$S_r = \frac{1}{6}r^3 + \frac{1}{2}r^2 + \frac{1}{3}r$
Sum of diagonals of slope 2:	$S_r = \frac{1}{3}r^3 + \frac{1}{2}r^2 + \frac{1}{6}r$
Sum of diagonals of slope 3:	$S_r = \frac{1}{2}r^3 + \frac{1}{2}r^2$

Equations for the sums of diagonals of other slopes can be found in a similar way. Furthermore, these equations can be generalised into an equation for any slope g (where g is a natural number).

Sum of diagonals of slope g:	$S_r = \frac{g}{6}r^3 + \frac{1}{2}r^2 + (\frac{1}{2} - \frac{g}{6})r$

As said, it is possible to reach these formulas once noticing that the third differences of the sums of diagonals of a given slope constant. However, it is also possible to reach a general solution by analysing how each number in the sums of diagonals is generated (i.e., by reasoning situationally).

Leonard's, Patrick and Carolyn's rubrics are presented next.

LEONARD'S RUBRIC

Sums of Diagonals

This is what we know:

Consider a times-table grid as shown.

1	2	3	4	5	6	.....
2	4	6	8	10	12	....
3	6	9	12	15	18	....
4	8	12	16	20	24	....
5	10	15	20	25	30	....
6	12	18	24	30	36	....
⋮	⋮	⋮	⋮	⋮	⋮	⋱

A diagonal of slope 1 (gradient 1) consists of all numbers which are 1 up and 1 right from the previous number.

eg. In this grid we have the following diagonals of slope 1.

1
2, 2
3, 4, 3
4, 6, 6, 4
5, 8, 9, 8, 5
⋮

Diagonals of slope 2 numbers which are 2 up and 1 right from the previous number.

What we want

I would like to find a formula that would allow me to find the sum of a particular diagonal when we input the slope of the line and the number from the top row in that line.

I will start by putting together a table of diagonals of slope 1. Let r be the number in the diagonal which is in the top row. let a_n be the sequence of numbers in the diagonal. Let S_r be the sum of a_n, Let D_r be $S_r - S_{r-1}$.

r	a_n	S_r	D_r
1	1	1	
2	2, 2	4	3
3	3, 4, 3	10	6
4	4, 6, 6, 4	20	10
5	5, 8, 9, 8, 5	35	15
6	6, 10, 12, 12, 10, 6	56	21
7	7, 12, 15, 16, 15, 12, 7	84	28
8	8, 14, 18, 20, 20, 18, 14, 8	120	36
9	9, 16, 21, 24, 25, 24, 21, 16, 9	165	45
10	10, 18, 24, 28, 30, 30, 28, 24, 18, 10	220	55

AHA! – Examining D_r, there is a clear pattern. The 3rd differences of S_r are equal which means that the general formula for S_r can be represented by

$$S_r = ar^3 + br^2 + cr + d.$$

$r = 2$, $4 = 8a + 4b + 2c + d$ (1)

$r = 3$, $10 = 27a + 9b + 3c + d$ (2)

$r = 4$, $20 = 64a + 16b + 4c + d$ (3)

$r = 5$, $35 = 125a + 25b + 5c + d$ (4)

Solving (1) to (4) should yield a, b, c, d.

(2) – (1)

$6 = 19a + 5b + c$ (*)

(4) – (3)

$15 = 61a + 9b + c$ (**)

(3) – (2)

$10 = 37a + 7b + c$ (***)

(**) – (*)

$9 = 42a + 4b$ (†)

(**) – (***)

$5 = 24a + 2b$ (††)

$\dagger - 2 \dagger\dagger$

$-1 = -6a$

$\Leftrightarrow$ $a = 1/6$

In $\dagger$ gives

$9 = 7 + 4b$

$\Leftrightarrow 4b = 2$

$\Leftrightarrow$ $b = 1/2$

In $*$ gives

$6 = \frac{19}{6} + \frac{5}{2} + c$

$\Leftrightarrow 36 = 19 + 15 + 6c$

$\Leftrightarrow 6c = 2$

$\Leftrightarrow$ $c = 1/3$

In 1 gives

$4 = \frac{8}{6} + 2 + \frac{2}{3} + d$

$\Leftrightarrow$ $24 = 8 + 12 + 4 + 6d$

$\Leftrightarrow$ $6d = 0$

$\Leftrightarrow$ $d = 0$

$\therefore S_r = \frac{1}{6}r^3 + \frac{1}{2}r^2 + \frac{1}{3}r$

4

Check

$S_2 = 4$ $\quad$ $S_6 = 56$

$S_3 = 10$ $\quad$ $S_7 = 84$

$S_4 = 20$ $\quad$ $S_8 = 120$ $\quad$ and $S_1 = 1$.

$S_5 = 35$ $\quad$ $S_9 = 165$

The formula cleary works!

However this is only for the case when the diagonal has slope 1. I will now try to apply the same ~~[illegible]~~ method for diagonals of slope 2. This time I will let $D_r = S_r - S_{r-1}$, but also $D_{2(r)} = D_r - D_{r-1}$, and $D_{3(r)} = D_{2(r)} - D_{2(r-1)}$, etc.

r	a_n	S_r	D_r	$D_{2(r)}$	$D_{3(r)}$
1	1	1	/	/	/
2	2, 3	5	4	/	/
3	3, 6, 5	14	9	5	/
4	4, 9, 10, 7	30	16	7	2
5	5, 12, 15, 14, 9	55	25	9	2
6	6, 15, 20, 21, 18, 11	91	36	11	2
7	7, 18, 25, 28, 27, 22, 13	140	49	13	2
8	8, 21, 30, 35, 36, 33, 26, 15	204	64	15	2
9	9, 24, 35, 42, 45, 44, 39, 30, 17	285	81	17	2

Once again the 3rd differences are equal, which means $S_r = ar^3 + br^2 + cr + d$ for some $a, b, c, d \in \mathbb{R}$.

[I note at this point that so far the slope of the diagonal has been equal to the 3rd difference. This could be an avenue to check out later.]

$r=1 \iff 1 = a+b+c+d$ ①

$r=2 \iff 5 = 8a+4b+2c+d$ ②

$r=3 \iff 14 = 27a+9b+3c+d$ ③

$r=4 \iff 30 = 64a+16b+4c+d$ ④

② − ①

$4 = 7a+3b+c$ (*)

③ − ②

$9 = 19a+5b+c$ (**)

④ − ③

$16 = 37a+7b+c$ (***)

(**) − (*)

$5 = 12a+2b$ (†)

(***) − (**)

$7 = 18a+2b$ (††)

†† − †

$2 = 6a$

$\iff \underline{\underline{a = 1/3}}$

⑥

In (T) gives

$$5 = 4 + 2b$$

$$\Leftrightarrow 1 = 2b$$

$$\Leftrightarrow b = 1/2$$

In (✱) gives

$$4 = \frac{7}{3} + \frac{3}{2} + c$$

$$\Leftrightarrow 24 = 14 + 9 + 6c$$

$$\Leftrightarrow 1 = 6c$$

$$\Leftrightarrow c = 1/6$$

In (1) gives

$$1 = \frac{1}{3} + \frac{1}{2} + \frac{1}{6} + d$$

$$\Leftrightarrow d = 0$$

$$\therefore \quad S_r = \frac{1}{3} r^3 + \frac{1}{2} r^2 + \frac{1}{6} r$$

check

$S_1 = 1$	$S_5 = 55$
$S_2 = 5$	$S_6 = 91$
$S_3 = 14$	$S_7 = 140$
$S_4 = 30$	$S_8 = 204$

STUCK!

I'm looking for a link between S_r for slope 1 and S_r for slope 2. Before I can make any assumptions, I should consider at least two more slopes. For ease of calculation, I will now consider r up to 5 instead of 10.

Slope 3

r	a_n	S_r	D_r	$D_{2(r)}$	$D_{3(r)}$
1	1	1	/	/	/
2	2, 4	6	5	/	/
3	3, 8, 7	18	12	7	/
4	4, 12, 14, 10	40	22	10	3
5	5, 16, 21, 20, 13	75	35	13	3

AHA!

As I suspected the 3rd difference is 3.

So,

$$S_r = ar^3 + br^2 + cr + d$$

$r=1$, $\quad 1 = a + b + c + d \quad$ (1)

$r=2$, $\quad 6 = 8a + 4b + 2c + d \quad$ (2)

$r=3$, $\quad 18 = 27a + 9b + 3c + d \quad$ (3)

I also suspect that $d=0$ again. I will assume this and see if it works afterwards.

So with $d=0$,

(2) $- 2\times$(1) gives

$$4 = 6a + 2b \iff 2 = 3a + b \quad (*)$$

$2\times$(3) $- 3\times$(2)

$$18 = 30a + 6b \iff 3 = 5a + b \quad (**)$$

$(**) - (*)$

$1 = 2a$

$a = 1/2$

In $(*)$ gives

$2 = \frac{3}{2} + b$

$\Leftrightarrow b = 1/2$

In ① gives

$1 = \frac{1}{2} + \frac{1}{2} + c$

$\Leftrightarrow c = 0$

$\therefore \quad S_r = \frac{1}{2} r^3 + \frac{1}{2} r^2$

Check: $S_1 = 1$
$S_2 = 6$
$S_3 = 18$
$S_4 = 40$
$S_5 = 75$.

Slope 4

r	a_n	S_r	D_r	$D_{2(r)}$	$D_{3(r)}$
1	1	1	/	/	/
2	2, 5	7	6	/	/
3	3, 10, 9	22	15	9	/
4	4, 15, 18, 13	50	28	13	4
5	5, 20, 27, 26, 17	95	45	17	4

So our solution is in the form

$$S_r = ar^3 + br^2 + cr.$$

$r=1 \iff 1 = a+b+c$ (1)

$r=2 \iff 7 = 8a+4b+2c$ (2)

$r=3 \iff 22 = 27a+9b+3c$ (3)

(2) $- 2\times$(1)

$5 = 6a+2b$ (*)

$2\times$(3) $- 3\times$(2)

$23 = 30a+6b$ (**)

(**) $- 3\times$(*)

$8 = 12a$

$\iff \underline{\underline{a = 2/3}}$

10

In (*) gives

$$5 = 4 + 2b$$

$$\Leftrightarrow 2b = 1$$

$$\Leftrightarrow b = 1/2$$

In (1) gives

$$1 = \frac{2}{3} + \frac{1}{2} + c$$

$$\Leftrightarrow c = -1/6$$

$$\therefore S_r = \frac{2}{3} r^3 + \frac{1}{2} r^2 - \frac{1}{6} r$$

Check: $S_1 = 1$
$S_2 = 7$
$S_3 = 22$
$S_4 = 50$
$S_5 = 95$

lets just stop for a moment to try and see what we have done so far, and make sure I'm not getting side-tracked.

- I have worked formulae for slopes 1 to 4.
- I am looking to work out a formulae for a slope of gradient g.
- I need to analyse the results I have so far and make a hypothesis.

So far I have the following values of a, b, c in

$$S_r = ar^3 + br^2 + cr$$

for slopes 1 to 4.

Gradient of Slope (g)	a	b	c
1	$1/6$	$1/2$	$1/3$
2	$1/3$	$1/2$	$1/6$
3	$1/2$	$1/2$	0
4	$2/3$	$1/2$	$-1/6$

It seems that a_g, b_g, c_g are sequences that are arithmetic progressions of the form $Ag+B$

For $a_g = \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3},$

$g=1 \Leftrightarrow \frac{1}{6} = A+B$ ①

$g=2 \Leftrightarrow \frac{1}{3} = 2A+B$ ②

② − ① $\frac{1}{6} = A$

In ① gives

$\frac{1}{6} - \frac{1}{6} = B$

$\Leftrightarrow B=0$

$\therefore a_g = \frac{g}{6}$

For $b_g = \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2},$

$A=0,\ B=\frac{1}{2}$

$\therefore b_g = \frac{1}{2}$

For $c_g = \frac{1}{3}, \frac{1}{6}, 0, -\frac{1}{6},$

we have

$g=1 \iff \frac{1}{3} = A+B$ ①

$g=2 \iff \frac{1}{6} = 2A+B$ ②

② − ①

$-\frac{1}{6} = A$

In ① gives

$B = \frac{1}{3} + \frac{1}{6}$

$\iff B = \frac{1}{2}$

$\therefore c_g = \frac{1}{2} - \frac{g}{6}$

Hypothesis

This means that for a slope of gradient g

$$\boxed{S_r = \left(\frac{g}{6}\right)r^3 + \left(\frac{1}{2}\right)r^2 + \left(\frac{1}{2} - \frac{g}{6}\right)r}$$

We should now check our result works for $g=5$.

$\underline{\underline{g=5}}$

r	a_n	S_r	D_r	$D_{2(r)}$	$D_{3(r)}$
1	1	1	/	/	/
2	2, 6	8	7	/	/
3	3, 12, 11	26	18	11	5
4	4, 18, 22, 16	60	34	16	5
5	5, 24, 33, 32, 21	115	55	21	5

We require $\underline{S_r = \frac{5}{6}r^3 + \frac{1}{2}r^2 - \frac{1}{3}r}$

Check: $S_1 = 1$ $\quad S_4 = 60$

$S_2 = 8$ $\quad S_5 = 115$

$S_3 = 26$

$\therefore$ The general formula works for $g=5$.

Reflection

I have a formula which fulfils my initial requirements. However, I do not have any reason why the formula is what it is, nor have I ~~guessed~~ proved it works! I suspect that proving it could help to see why the formula is what ~~it~~ it is, and hence answer both questions. So that will be my next step.

Extension

I will extend the problem by looking deeper into my solution and finding out what is really going on.

I think examining the sequences a_n is the best way to go since this is where the Sums come from. Maybe the fact the third differences are always equal to the gradient will help me here. The problem stinks of ~~a~~ an induction proof, but I really don't know where to start.

When I look at the actual sequences themselves, there are many patterns, but one pattern seems to stand out more than any other.

Consider $g = 1$,

then we have

$a_n =$ 1
2, 2
3, 4, 3
4, 6, 6, 4
5, 8, 9, 8, 5
⋮

This can be written as

1×1
$2\times1, 1\times2$
$3\times1, 2\times2, 1\times3$
$4\times1, 3\times2, 2\times3, 1\times4$
⋮
$n\times1, (n-1)\times2, (n-2)\times3, \ldots, 2\times(n-1), 1\times n$
⋮

This means that we can express S_r as a sum.

$$S_r = \sum_{n=1}^{r} (r-n+1)n$$

Likewise, for $g=2$, we have

1×1
$2\times1,\ 1\times3$
$3\times1,\ 2\times3,\ 1\times5$
$4\times1,\ 3\times3,\ 2\times5,\ 1\times7$

The only difference here being that each new column starts with every other number (i.e. 1, 3, 5, 7, ...) instead of all numbers.

$$\Rightarrow \quad S_r = \sum_{n=1}^{r}(r-n+1)(2n-1)$$

In fact, these patterns exist for all different values of g.

I will just take a few moments to tabulate my results and see how my proof is coming along.

I can see that in general for slope $=g$.

$$S_r = \sum_{n=1}^{r}(r-n+1)\left[gn-(g-1)\right]$$

$$\Leftrightarrow S_r = \sum_{n=1}^{r}(r-n+1)\left(g(n-1)+1\right)$$

g	S_r (sum)	S_r (formula)
1	$\sum_{n=1}^{r}(r-n+1)\,n$	$\frac{1}{6}r^3+\frac{1}{2}r^2+\frac{1}{3}r$
2	$\sum_{n=1}^{r}(r-n+1)(2n-1)$	$\frac{1}{3}r^3+\frac{1}{2}r^2+\frac{1}{6}r$
3	$\sum_{n=1}^{r}(r-n+1)(3n-2)$	$\frac{1}{2}r^3+\frac{1}{2}r^2$
4	$\sum_{n=1}^{r}(r-n+1)(4n-3)$	$\frac{2}{3}r^3+\frac{1}{2}r^2-\frac{1}{6}r$
$\vdots$	$\vdots$	$\vdots$
g	$\sum_{n=1}^{r}(r-n+1)(g(n-1)+1)$	$\left(\frac{g}{6}\right)r^3+\left(\frac{1}{2}\right)r^2+\left(\frac{1}{2}-\frac{g}{6}\right)r$

STUCK! I have proved that for all $g \in \mathbb{N}$, the sums work, since they came directly from the sequences a_n. However, ~~the~~ formulae results have only been proved for specific values of g and r. But if I can show that $\forall g$, each of these sums $\Leftrightarrow$ each of these formulae, then my proof is complete.

AHA!

I can see a possible proof which would require 2 smaller inductive proofs.

I must first prove that for $g=1$,

Step 1 $\quad S_r = \sum_{n=1}^{r} (r-n+1)n \iff S_r = \frac{1}{6}r^3 + \frac{1}{2}r^2 + \frac{1}{3}r$

$\forall r \in \mathbb{N}$

and then use this to prove that

Step 2 $\quad S_r = \sum_{n=1}^{r} (r-n+1)(g(n-1)+1) \iff S_r = \left(\frac{g}{6}\right)r^3 + \left(\frac{1}{2}\right)r^2 + \left(\frac{1}{2} - \frac{g}{6}\right)r$

$\forall g \in \mathbb{N}$.

Step 1

For $g=1$, $S_1 = \sum_{n=1}^{1} (1-n+1)n$

$S_1 = 1 \times 1$

$S_1 = 1$

for $r=1$, $\frac{1}{6}r^3 + \frac{1}{2}r^2 + \frac{1}{3}r = 1$

$\therefore$ The result is true for $r=1$.

Suppose the result is true for $r=k$

i.e. $S_k = \sum_{n=1}^{k}(k-n+1)n \Leftrightarrow S_k = \frac{1}{6}k^3 + \frac{1}{2}k^2 + \frac{1}{3}k$

then

$$S_{k+1} = \sum_{n=1}^{k+1}\big((k+1)-n+1\big)n$$

$$\Leftrightarrow \quad = \sum_{n=1}^{k}\big((k+1)-n+1\big)n + \left[\cancel{(k+1)} - \cancel{(k+1)} + 1\right]\left[k+1\right]$$

$$\Leftrightarrow \quad = \sum_{n=1}^{k}(k-n+1)n + \sum_{n=1}^{k} n + \left[k+1\right]$$

$$\Leftrightarrow \quad = \left[\frac{1}{6}k^3 + \frac{1}{2}k^2 + \frac{1}{3}k\right] + \left[\frac{k(k+1)}{2}\right] + \left[k+1\right]$$

$$\Leftrightarrow \quad = \frac{1}{6}k^3 + k^2 + \frac{11k}{6} + 1$$

$$\Leftrightarrow \quad = \frac{1}{6}k^3 + \frac{1}{2}k^2 + \frac{1}{2}k^2 + \frac{1}{2}k + \frac{1}{3}k + k + \frac{1}{2} + \frac{1}{3} + \frac{1}{6}$$

$$\Leftrightarrow \quad = \frac{1}{6}\left[k^3 + 3k^2 + 3k + 1\right] + \frac{1}{2}\left[k^2 + 2k + 1\right] + \frac{1}{3}\left[k+1\right]$$

$$\Leftrightarrow \quad = \frac{1}{6}(k+1)^3 + \frac{1}{2}(k+1)^2 + \frac{1}{3}(k+1)$$

$\therefore$ If the result is true for $r=k$, then it is true for $r=k+1$
But the result is true for $r=1$, hence true for $r=2$, $r=3$, etc
And hence by the principle of mathematical induction, the result is true $\forall r \in \mathbb{N}$.

Step 2

New consider

$$S_r = \sum_{n=1}^{r}(r-n+1)\left(g(n-1)+1\right) \Leftrightarrow S_r = \left(\frac{g}{6}\right)r^3 + \left(\frac{1}{2}\right)r^2 + \left(\frac{1}{2}-\frac{g}{6}\right)r$$

The result is clearly true for $g=1$. (This is what I proved in step 1).

Suppose this result is true for $g=p$.

Then,

$$S_{r(p+1)} = \sum_{n=1}^{r}(r-n+1)\left((p+1)(n-1)+1\right)$$

$$\Leftrightarrow \quad = \sum_{n=1}^{r}(r-n+1)\left(p(n-1)+1+(n-1)\right)$$

$$\Leftrightarrow \quad = \sum_{n=1}^{r}(r-n+1)\left(p(n-1)+1\right) + \sum_{n=1}^{r}(r-n+1)(n-1)$$

$$\Leftrightarrow \quad = S_{r(p)} + \sum_{n=1}^{r}\left(rn-n^2+2n-r-1\right)$$

$$\Leftrightarrow \quad = S_{r(p)} + (2+r)\sum_{n=1}^{r}n - \sum_{n=1}^{r}n^2 - r^2 - r$$

$$\Leftrightarrow \quad = S_{r(p)} + \frac{r(r+1)(r+2)}{2} - \frac{r(r+1)(2r+1)}{6} - r^2 - r$$

$$\Leftrightarrow \quad S_{r(p+1)} = \left(\frac{p}{6}\right)r^3 + \left(\frac{1}{2}\right)r^2 + \left(\frac{1}{2}-\frac{p}{6}\right)r + \frac{r(r+1)(r+2)}{2} - \frac{r(r+1)(2r+1)}{6} - r^2 - r$$

$$\Leftrightarrow S_{r(p+1)} = \left(\frac{p}{6} + \frac{1}{2} - \frac{1}{3}\right)r^3 + \left(\frac{1}{2} \cancel{+\frac{3}{2}} \cancel{-\frac{1}{2}} \cancel{-1}\right)r^2 + \left(\frac{1}{2} - \frac{p}{6} \cancel{+1} - \frac{1}{6} \cancel{-1}\right)r$$

$$\Leftrightarrow S_{r(p+1)} = \left(\frac{p+1}{6}\right)r^3 + \left(\frac{1}{2}\right)r^2 + \left(\frac{1}{2} - \frac{(p+1)}{6}\right)r$$

$\therefore$ If the result is true for $g=p$, then it is true for $g=p+1$.

But the result is true for $g=1$, hence true for $g=2$, $g=3$, etc.

And Hence, by the principle of mathematical induction, the result is true $\forall g \in \mathbb{N}$. ∎

Result! I have now proved the following result

When given a diagonal with slope, $g \in \mathbb{N}$, and number, r, in the first row (ie 1st term of the sequence), the sum, $S_{r(g)}$, of that diagonal is defined by

$$\boxed{\begin{aligned} S_{r(g)} &= \sum_{n=1}^{r} (r-n+1)\big(g(n-1)+1\big) \\ &= \left(\frac{g}{6}\right)r^3 + \left(\frac{1}{2}\right)r^2 + \left(\frac{1}{2} - \frac{g}{6}\right)r \end{aligned}}$$

∎

Checking the result works here is not necessary, since this result has already been checked earlier. This proof just checks the result works for all positive values of g and r.

The reasons I say all positive values of g and r are as follows.

1) $r = 0$ does not appear in the grid.
2) $r = -\alpha$, $\alpha \in \mathbb{N}$ does not appear in the grid $\forall \alpha$
3) If $g = 0$ or $g = -\alpha, \alpha \in \mathbb{N}$, then the sequence, a_n, would be infinite, with each term being larger than the previous term. Hence $S_{r(g)} = \infty$ $\forall g \leq 0$.

Reflection

The investigation certainly seems to be complete, and all of the loose ends (e.g. limits of r and g) appear to have been tied up.

Extension

One might ask what happens when you "sum your sums" of gradient 1, or gradient 2, etc. Obviously, if we sum all of slopes of gradient 1, we will be summing to infinity. But for a finite number of diagonals, can we predict the outcome?

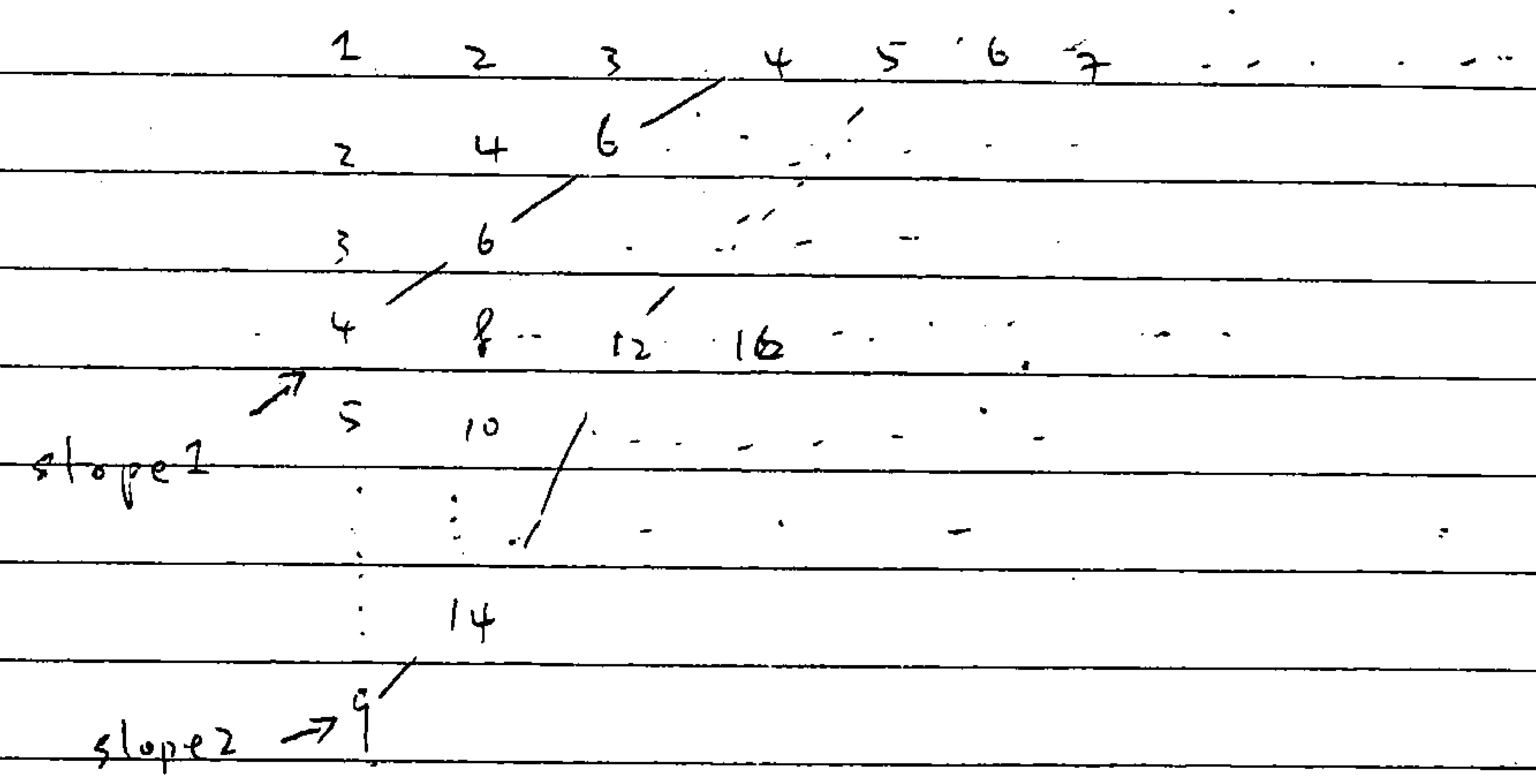
Others might ask what happens with different grids e.g. Complex numbers, or alternate numbers which are positive and negative.

PATRICK'S RUBRIC

Problem Solving - Part 1

Sums of diagonals

Q : Investigate the sums of diagonals of different slopes in the grid.



A resolution:

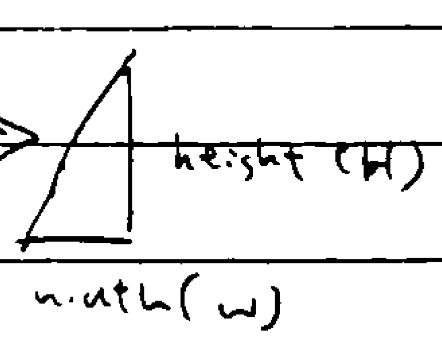
First of all, I want to know more about the question, especially the word "slope". It seems to be one of the crucial words in the question.

STUCK : What's mean by slope 1 and slope 2? Are the just ~~the~~ names of ~~the~~ slopes or actually mean the slope of "slope n" is n?

H. 1 and 2 in "slopes 1" and "slopes 2" are only indices. I have to think a systematic way to ~~define~~ create slope 3, 4, 5 ... and so on. But what if ~~it~~ they are actually the values of the slope?

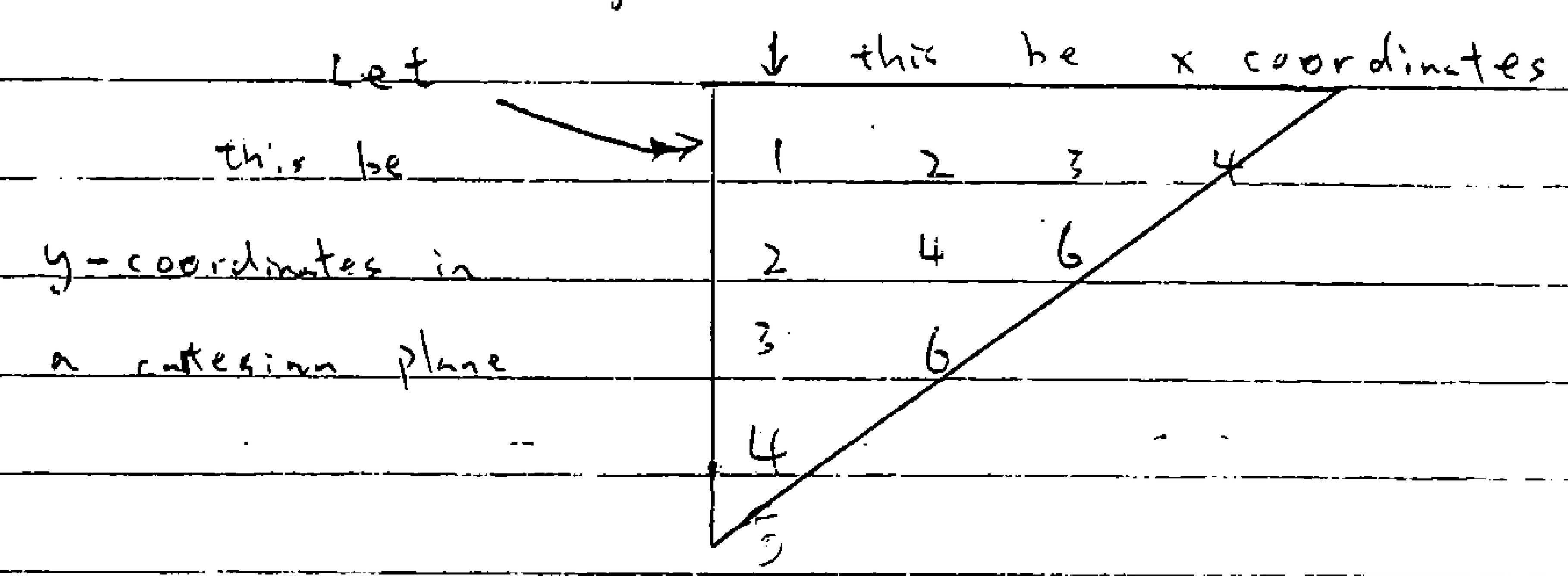
I've learnt the word "slope" from mathematics. In short, it's just the height divided by the width in a triangle

$$\text{Slope} = \frac{H}{W} \longrightarrow$$



What if I apply this to the question?

CHECK! For slope 1, I can pick out a triangle from the grid like this:



$$\text{slope} = \frac{4-1}{4-1} = 1$$

AHA: the gradient of slope 1 is 1

I use the same method and apply it to slope 2,

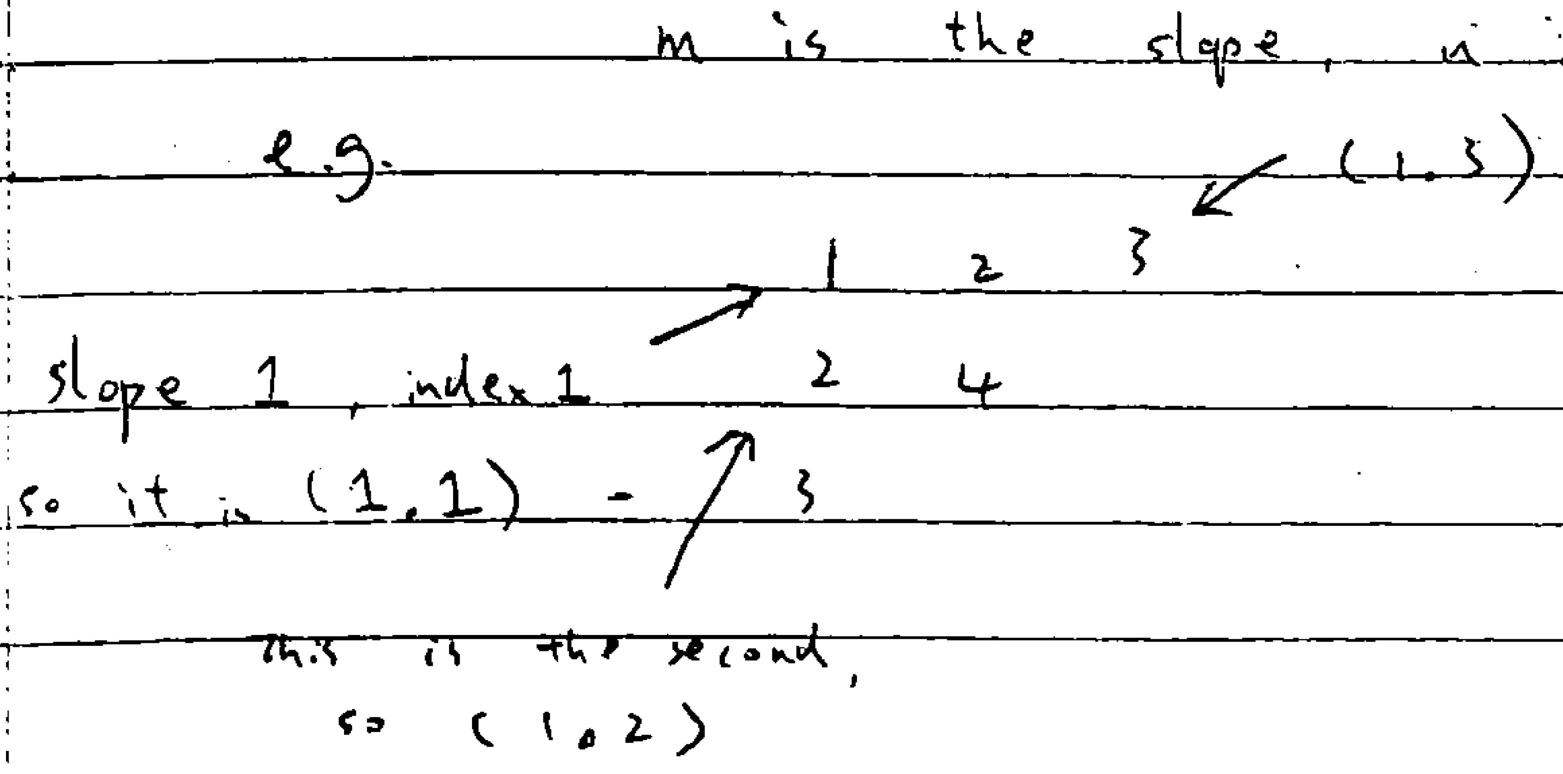
$$\cancel{\text{slope}} = \frac{9-1}{5-1} = 2$$ AHA! I got it!

So I'm going to stick to this method and develop slope 2, 3, 4 ...
I look at slope 1 first

It's impossible for me to draw the grid here everytime I refer to it, so I'm going to use some representation:

Introduce: Reference to ~~the~~ diagonals m, n
m is the slope, n is the index

e.g.



Let me focus on slope 1 and see some sums.

1.1 : $1 = 1$

1.2 : $2 + 2 = 4$

1.3 : $3 + 4 + 3 = 10$

1.4 : $4 + 6 + 6 + 4 = 20$

1.5 : $5 + 8 + 9 + 8 + 5 = 35$

STUCK! What do I want?

Any pattern here? General eq^{n}?

Anything related to triangle numbers, square numbers?

NO!

How about the difference between sums?

1 4 10 20 35

3 6 10 15

→ 3 4 5

AHA! they are consecutive numbers! what does it mean?

CHECK! What if I put 6 at the end?

1 4 10 20 35 56

3 6 10 15 21

3 4 5 6

Is $1.6 = 56$?

CHECK! $1.6 = 6 + 10 + 12 + 12 + 10 + 6 = 56$

Yes, it follows the rule. But what if I want to find 1.n? It's impossible for me to keep drawing tree diagram!

REFLECT: Drawing tree diagram is one method to find the sum but it is too clumsy. Because I need to draw the tree from the beginning to n if I want to find 1, n

Let me don't look at the sums now, but the way how it's built up.

I'm now looking at 4, 6 → 6, 4, if I say 4 is n then the next number I should add to n is $(n-1) \times 2$, then $(n-2) \times 3$ and finally $(n-3) \times 4$.

$$\begin{array}{cccc} n-3 & n-2 & n-1 & n \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 & 4 \\ 2 & 4 & 6 \leftarrow (n-1)\times 2 & \\ 3 & 6 \leftarrow (n-2)\times 3 & & \\ (n-3)\times 4 \rightarrow 4 & & & \end{array}$$

Can I find a general formula for it?

For 1, 4 sum is $4 + 2(4-1) + 3(4-2) + 4(4-3)$

For 1, n, sum is $n + 2(n-1) + 3(n-2) + 4(n-3) \dots + n$

and the last term should be n for slope 1, n.

let me rewrite the formula to include the last few terms, otherwise people don't know when to stop.

For 1, n, the sum is $n + 2(n-1) + 3(n-2) + \dots (n-2)[n-(n-3)] + (n-1)[n-(n-2)] + n[n-(n-1)]$

REFLECT: It's quite hard to see the pattern of sums. It's more easy to find out a general formula by looking at how the sum is built up.

P

REVIEW : I've done slope 1. n the general formula is

$$n + 2(n-1) + 3(n-2) + \cdots (n-2)[n-(n-3)] + (n-1)[n-(n-2)] + n[n-(n-1)]$$

For slope 2, let me just write out some numbers which belong to slope 2,

	1	2	3	4	5
	2	4	6	8	10
2.1 →	3	6	9	12	15
	4	8	12	16	20
2.2 →	5	10	15	20	25
	6	12	18	24	
2.3 →	7	14	21		
	8	16			
2.4 →	9				

According to the representation that I introduced, 2.1 should include number 3 and 2 as shown because

$$\frac{3-1}{2-1} = 2$$

As I'm now only doing integral numbers not every number is included in slope 2.

I've done the formula for slope 1; can I use the same idea to develop a formula for slope 2?

CHECK !

For 2.1, the sum is $2 + 3(2-1) = 5$

2.2, " " $3 + 3(3-1) + 5(3-2) = 14$

2.3, " " $4 + 3(4-1) + 5(4-2) + 7(4-3) = 30$

2.4, $5 + 3(5-1) + 5(5-2) + 7(5-3) + 9(5-4) = 55$

Conjecture 1: Formula for 2.n is

$$(n+1) + 3(n) + 5(n-1) + \cdots (2n+1)(1)$$

The above formula is just a guess from the previous calculation of 2.1, 2.2, 2.3 and 2.4

Since the number "1" is belonged to slope 1, so slope 2 starts with "2".
For 2.1, the first term is 2 which is 1+1
For 2.2, the first term is 3 which is 2+1
So for 2.n, first term should be n+1

The next term is a neighbour of previous number. so it's (n+1-1). But this is not the actual number I want. A coefficent should be added to find the actual term. In this case, 3 should be added

For more clear, I divide a term into 2 parts
For example $5\,|\,(n-1)$

coefficient variable

The variable part can be easily obtain by subtracting the previous number by 1 as they are neighbour. But how about the coefficient part?
We can see that it's obviously 1, 3, 5, 7, 9... pattern, but why?

STUCK! Why coefficents are in 1, 3, 5, 7, 9... pattern

The only think I can capture is the difference of those number, they are all differed by 2, is this related? Another STUCK!

Looking back to the slope rule that I defined,

$$\text{slope} = \frac{\text{height (vertical difference)}}{\text{width (horizontal difference)}}$$

slope is 2 in this case

For slope 2, height and width are in fixed ratio which is 2 : 1. So 1 movement (from 1 number to the one just next to it) in horizontal means 2 movements in vertical so that the ratio ~~of~~ 2 can be kept.

AHA! I think I got it!

For example,

4 5

8 1

2

12 [illegible]

$\frac{2}{1} = 2$!

From 5 to 4 (there's 1 horizontal movement, so 2 vertical movement is required to keep slope 2, the next one should be $(5-1) \times 3$ because 2 ~~move~~ vertical movements ~~leads to~~ triple the original value ($4 \to 12$)

Similarly, from $5 \to 3$ (horizontal), 4 ~~move~~ vertical movements are required

3 4 5

6 8

4 9 12

12

15

~~so~~ So the third term should be $(5-2)\,|\,(4+1)$

coefficient

include ~~the~~ original number

P

Now I know why the coeffient pattern is 1, 3, 5, 7, 9 .....

So the formula for 2.n is proved.

Reflect: Once again, it's more easy to build up ~~from~~ the formula by understanding the underlying principle. By knowing how the slope works and how numbers are "jumping" around, I think it's ~~so~~ easier to further develop a more general formula.

Review: slope 1.n:

$n + 2(n-1) + 3(n-2) + \dots (n-1)[n-(n-2)] + n[n-(n-1)]$

slope 2.n: (not include number "1")

$(n+1) + 3(n) + 5(n-1) + \dots (2n+1)(1)$

Coefficient for slope 1: 1, 2, 3, 4, 5 ...

" " slope 2: 1, 3, 5, 7, 9 ..

Conjecture 2: The difference between consecutive coefficient in slope m.n is m.

I make this conjecture just by looking at the pattern of slope 1 and 2, because the difference b/w consecutive coef. in slope 1 is 1, slope 2 is 2.

For keeping slope m, one movement in horizontal means m movements in vertical because the slope rule is: $\frac{\text{Vertical displacement}}{\text{horizontal displacement}}$

So conjecture 2 is proved.

By using conjecture 2, I know the pattern of coefficient of slope 3 should be 1, 4, 7, 10...

I did formulae for slope 1 and 2, but I realise that I can't write out all formulae for every slope. What is the formula for slope m? I think I need a more general equation that cover all the slopes.

For slope 2, I use "n+1" as the first term, this is because I want to match the index of slope. But what if people don't know about index of slope. I'm thinking an easier way.

My formulae are actually built up from the rightmost number in the diagonal. For example,

3

4 <- rightmost number

3

3 is the rightmost number in the above case. If I let n to be the rightmost number in my formula for slope 2, the formula will be come

$$n + 3(n-1) + 5(n-2) + \dots (2n+1)(n-(n+1))$$

which is much more similar to the formula for slope 1. Only the coefficients are different.

This change doesn't affect the formula for slope 1. Then I re-state my formula:

slope 1: $n + 2(n-1) + 3(n-2) + \dots n[n-(n-1)]$

slope 2: $n + 3(n-1) + 5(n-2) + \dots (2n+1)[n-(n-1)]$

where n is the rightmost number in the diagonal

The next thing I want to sort out is the coefficient of the formula because I want one general formula.

Let me look at the coefficient again.

slope 1 : 1, 2, 3, 4, 5 ...

2 : 1, 3, 5, 7, 9 ...

3 : 1, 4, 7, 10 ...

I know the difference b/w consecutive coef. is th slope of the diagonal, so for slope m, it should be,

$1, \ 1+m, \ 1+m+m, \ 1+m+m+m \ldots$

$\Rightarrow 1, \ 1+m, \ 1+2m, \ 1+3m, \ldots$

This can be used in the general formula.

So the general formula for slope m is

$n + (1+m)(n-1) + (1+2m)(n-2) + \ldots + [1+(n-1)m][n-(n-1)]$

where n is the rightmost number in the diagonal

REFLECT: After comparing the formulae of slope 1 and 2 step by step, the difference between two can be easily noticed and thus a general formula was produced.

REVIEW: def^n of "n" is changed from index of slope to the rightmost number. (3 4 3 righmost)

General formula for slope m

$n + (1+m)(n-1) + (1+2m)(n-2) + \ldots + [1+(n-1)m][n-(n-1)]$

The slopes I've tested or worked ~~we~~ were all integers. It seems that there is a hidden ~~assumptiy~~ ~~is~~ assumption in my rubric which assuming that slopes are integer.

Hidden Assumption: Slopes are integer.

But what if slope is not integer? Is my general formula cover this issue?

Conjecture 3: General equation works if slope is not integer

CHECK! let's consider a case which slope is $\frac{1}{2}$ (random)

$$\begin{array}{c} \\ 2\left\{\begin{array}{c} \\ \\ \\ \end{array}\right. \end{array} \begin{array}{ccccc} \multicolumn{5}{c}{\overbrace{\hspace{8em}}^{4}} \\ 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 6 & 8 & \\ 3 & 6 & 9 & & \end{array} \qquad \text{slope} = \frac{2}{4} = \frac{1}{2}$$

Put, $m = \frac{1}{2}$ and $n = 5$ in my general formula,

$$5 + (1+\tfrac{1}{2})(5-1) + (1+\tfrac{2}{2})(5-2) + (1+\tfrac{3}{2})(5-3) + (1+\tfrac{4}{2})(5-4)$$

$$\cancel{+(1+\tfrac{3}{2})(5-1)}$$

$$= 5 + 6 + 6 + 5 + 3 > 5 + 6 + 3$$

obviously WRONG!

Conjecture 3 disproved!

Reflect: general formula is no longer "general", it doesn't cover the case when slope is not integer

STUCK! How to cover those cases which slopes are not integer?

let me look back to the example used in checking just now

$$2\left\{\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 6 & 8 & \\ 3 & 6 & 9 & & \end{array}\right.$$

Slope $= \frac{1}{2}$

I recall my memory on how to build equation when slope is integer.

AHA! I think I can use the same method here!

For the above case, I can rewrite $5+6+3$ to $5 + (5-2)\times 2 + (5-4)\times 3$

slope $\frac{1}{2}$

Actually, I can view the grid in this way

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 6 & 8 & \\ 3 & 6 & & & \end{array}$$

(horizontal 2, vertical 1)

So In $5 + (5-2)\times 2 + (5-4)\times 3$,

this is width and this is displacement

Since I can't put just one variable like what "m" did in the general eq^{n}, I think I need to introduce 2 other variables which are the height and width.

~~Variable~~

Introduce: n is the rightmost number

h is the vertical displacement from one number to next number

w is the horizontal displacement from

one number to next one

Let me rewrite the formula in more general way,

$n + (n-w) \times (h+1) + (n-2w) \times (2h+1) + (n-3w)(3h+1) \ldots$

AHA! I found something interesting!

Conjecture 4 : coefficient of h and w in the equation above are always the same

The coefficient of w and h is actually leading the pointer to the next number in the diagonal. So it is increased by one each time in order to go to the ~~no~~ next number. They should be the same so that the ratio of h and w can be kept as well as the slope.

Conjecture 4 is proved.

REFLECT : From the previous experience on building up the general formula for integral slope, recalling the process facilitated in building up the equation for non-integral slope.

let i be the coefficien of w and h,

rewriting the eq$^{\underline{n}}$

$n + (n-w) \times (h+1) + (n-iw) \times (ih-i) + (n-$?

Stuck! What should I write here? I shouldn't use I again!

REVIEW: - eq$^{\underline{n}}$ for non integral slope

$n + (n-w) \times (h+1) + (n-2w) \times (2h+1) - \ldots$

- coefficient of h and w are the same

- Stuck because of putting i into the eq$^{\underline{n}}$

Hey! everyterm looks similar to each other and
the coefficient increased by 1 each time,
why don't use $\sum$?

STUCK! General term? Something like $\sum_{i=0}^{?} (\ )(\ )$?
eq$^{\underline{n}}$ $n + (n-w)(h+1) + (n-2w)(2h+1) \ldots$

$(n-iw)(ih+1)$

when $i=0$, $(n-0)(0+1) \Rightarrow n$

AHA! it should be $\sum_{i=0} (n-iw)(ih+1)$

REFLECT: Compare terms and look at the position
which is increasing (i position)
Then check if n can be obtained by
putting $i=0$.

STUCK! What is the superscipt? When to end?
$\sum_{i=0}^{?} (n-iw)(ih+1)$

AHA! (as i starts from 0)
The superscript is actully the number of terms-1!
So it can be obtained by $\underline{\dfrac{\text{rightmost number } (n)-1}{w}}$

Introduce: $t = \dfrac{n-1}{w}$

So the general for non-integral slop is
$$\sum_{i=0}^{t} (n-iw)(ih+1)$$

REFLECT: look at the role of each ~~part~~ variable
(e.g. superscript) can easily define and obtain
the variable.

Conjecture 5: The equation $\sum_{i=0}^{t} (n-iw)(ih+1)$ is not only for non-integral slopes, but also intergral slopes.

Each integer can be represented by fraction,

e.g. $2 = \frac{8}{4} = \frac{4}{2}$

$3 = \frac{6}{2}$

The $eq^{\underline{n}}$ should work if we just put the height and width into ~~the~~ it.

CHECK!

1 2 3 4

2 4 6

3 6

4

slope = 1

$t = \frac{4-1}{1} = 3$, $n = 4$, $h = 1$, $w = 1$

$\sum_{i=0} (4 - i)(i+1)$

$= 4 + (4-1)(1+1) + (4-2)(2+1) + (4-3)(3+1)$

$= 4 + 6 + 6 + 4$

$= 20$ ✓

Conjecture 5 is proved.

REFLECT: The $eq^{\underline{n}}$ $\sum_{i=0}^{t} (n-iw)(ih+1)$ cover both integral and non-integral slopes

Solution: For finding out the sums of diagonals of different slope, one should know the right most number of the diagonal and the slope, the apply the general $eq^{\underline{n}}$.

General eq$^{\underline{n}}$ $\sum_{i=0}^{t} (n-iw)(ih+1)$

t = number of terms − 1
calculated by $\frac{n-1}{w}$

n = rightmost number of diagonal
w = horizontal displacement from one num
to the next number within the diagonal
h = vertical displacement from one number
to the next number within the diagonal

REFLECTION : - hard to see pattern from the sums
- think representation for slopes
- easy to build formula if we know
where do the numbers come from
- from less general to more general case
(integral slope) (eq$^{\underline{n}}$ for both
eq$^{\underline{n}}$ for integral & slope
non integral)
- recall memory, learn from previous
experience

Extend!

By looking at the numbers ~~table~~ grid and slope 1 and 2 on the gs paper, I find that there is a hidden assumption. The general formula works if the diagonal touches both the first row and the first column.

What if the diagonal doesn't touch the first row or first column, or even neither? Let me check if the general rule can still be applied.

CHECK! If I want to find diagonal A which shown below.

1 2 3 4
2 4 6
3 6
Diagonal A → 4

The sum is $4+6+6=16$

Try apply my general formula, diagonal A has slope 1 and rightmost number is 6, the sum should be:

$$6+(1+1)(6-1)+(1+2)(6-2)+\dots(6)(1)$$

6, 10, 12, 6

Obviously the sum is greater than 16. My rule seems no longer "general" as it can't cover all the case!

STUCK! How to cover the case like diagonal A?

Diagonal min is no longer useful in this case.

STUCK! What information is needed so that I can

(A

refer to a unique diagonal?

~~Can~~ If I know the rightmost number and the bottom number, are they enough?

4 ← rightmost

6

6

bottom (or leftmost) → 4

NO! Because there are two "6" in the above case, if I only have the rightmost & leftmost number, I can refer to 2 different diagonals which are

① 6 ② 6

6

4 4

STUCK! What information is needed to specify a unique diagonal?

AHA! The number at the right angle corner!

right angle → 2 4 6 3 6

3 6 4

4

o Right angle corner o Leftmost o rightmost

Conjecture 6: Right-angle number (I refer it as ~~top~~ top-left number below), leftmost number and rightmost number are the key identifiers of a diagonal. Each combination of these 3 numbers will give an unique diagonal.

I just want to get a random combination

Say Top-left number = 4
Rightmost number = 12
leftmost number = 8

The triangle will look like:

1	2	3	4	5	6
2	4	6	8	10	12
3	6	9	12	15	
4	8				

In my mind, there are 2 tests to check if another diagonal can be created using these set of numbers

Test 1: Fix the top-left number. Move the rightmost number horizontally or leftmost number vertically to see if it meet 12 or 8 in other location

Obviously, numbers are increasing in the direction of right and down. Numbers in each row and column are unique and won't show again. It's impossible to get 2 same no. in a row or column.

4 6 8 10 12 → increasing
6 9 12
8
↓
increasing

No other diagonal will be found from this test.
Test 1 passed

Test 2: As in the test 1, top-left ~~to~~ number was fixed and other 2 numbers were moved. ~~To~~ This time, I look for another top-left number which is 4, then try to find if the rightmost and leftmost can be 12 and 8 respectively.

OH! NO! I found another diagonal which has the same values (4, 12, 8)

4 5 6 7 8 9 10 11 12
8

Result are different! Only 2 numbers involved.

I'm going to stop here.

Reflect: - Not as easy as I thought
- different combinations appeared
- need further define the characteristics of diagonal
- def^n of diagonal

CAROLYN'S RUBRIC

Sum of diagonals

1	2	3	4	5	6	7	8	9	10	11
2	4	6	8	10	12	14	16	18	20	
3	6	9	12	15	18	21	24	27	30	
4	8	12	16	20	24	28	32	36	40	
5	10	15	20	25	30	35	40	45	50	
6	12	18	24	30	36	42	48	54	60	
7	14	21	28	35	42	49	56	63	70	
8	16	24	32	40	48	56	64	72	80	
9	18	27	36	45	54	63	72	81	90	
10	20	30	40	50	60	70	80	90	100	
11										

$4+6+6+4=20.$

$9+14+15+12+5=55.$

I have noticed that there is a line of symmetry running through the grid from the top left through to bottom right. Does this mean that for example 9 to 5 will give the same result as 5 to 9? Will diagonals that go to and from the same number (eg 4 to 4) need a different formula than those that go to a different number (eg 9 to 5)? Once the formula is found will it work on different grids or a section of the multiplication square (eg a square with the corners 20, 40, 36, 72)? Hopefully I will be able to answer all of these questions by the end of this investigation.

I think it would be best to start from the smallest possible diagonals ~~from~~ which go to the same number and record the results so it will be possible to identify a pattern.

start & finish	total numbers passed	total.
1	1	1
2	2+2	4
3	3,4,3	10
4	4~~3~~, 6,6,4	20
5.	5,8,9,8,5	35
6.	6,10,12,12,10,6.	56
7	7,12,15,16,15,12,7	84

* these first three are ~~squa~~ numbers. Wait! there not, im mistaken

I can not see an obvious pattern with the results recorded. I will try and reorganise them to see if that helps me.

1	2	3	4	5	6	7	8
1	4	10	20	35	56	84	120
+3	+6	+10	+15	+21	+28	+36	
+3	+4	+5	+6	+7	+8		

AHA! There is an obvious pattern as there is a set of consecutive steps between them. From knowing this I would be able to find the sum of the diagonal from 8 to 8 but at the moment, I am unable to find the formula.

Using these previous calculations, I think that from 8 to 8, the sum of the diagonals is 120. I will now look on my multiplication square and see if I am correct.

8+14+18+20+20+18+14+8 = 120. My calculations was correct.

I have just noticed that if you go from 4 to 4, you have 4 steps. The same seems to be true for every other set. This means that if you start from even, you hav two lots of each number you step on. If you have

5

an odd set of steps, all your numbers are repeated apart from the middle value. Does this mean that there may be different solutions for odd and even number of steps?

2	2, 2	4	1	1	1
4	4, 6, 6, 4	20	3	3, 4, 3	10
6	6, 10, 12, 12, 10, 6	56	5	5, 8, 9, 8, 5	35
8	8, 14, 18, 20, 20, 18, 14, 8	120	7	7, 12, 15, 16, 15, 12, 7	84

All even steps are even and have an even ~~sum~~ total. all odd steps are in a pattern of odd, even, odd, even.... and if the middle value is even, the answer is even.

AHA! I've just seen it, I cant believe it works I didnt see it before. With the odd numbers, the total is the starting number squared, plus any odd steps before squa

eg 5) ⇒ $(5\times5)+(3\times3)+(1\times1)=35$

7) ⇒ $(7\times7)+35=84$.

It is also true for the even steps. At the moment, I cant see why its true but I should be able to write it into a formula.

sum of diagonal $x \to x = x^2 + (x-2)^2 + (x-4)^2 \ldots\ldots$ until read 0 or 1.

To try and understand this further, I am going to look at the grid to try and see why it happens.

1	2	3	4	5	6
2	4	6	8	10	12
3	6	9	12	15	18
4	8	12	16	20	24
5	10	15	20	25	30
6	12	18	24	30	36

For 3 to 3 the answer just happ to be 10 (9+1) which are its corners. For 5 to 5 the answers are its corner and the middle value.

This hasnt really helped me as it has just showed me in a different way, something I already knew. STUCK!! I cant see how to progress this any further I am going to go back to the start of my work and read it over again so it may give me the chance to see something I may not have seen before.

AHA! I dont know if this will help me but I may have found a way to explain the reason for the formula.

For 6 to 6, the numbers are 6,10,12,12,10,6. These can also be written as 6x1, 5x2, 4x3, 3x4, 2x5 & 1x6. As each number decreases by 1, the number it is timesed by, increases by 1.

In general termes:

$$(x\times 1)+(x-1\times 2)+(x-2\times 3)\ldots\ldots(x-(x-1)+x)$$

For example: 4,6,6,4.

$$(4\times 1)+(3\times 2)+(2\times 3)+(1\times 4)$$

$$(x\times \overset{x-3}{1})+(x-1\times \overset{x-2}{2})+(x-2\times \overset{x-1}{3})+(x-3\times \overset{4}{\cancel{x}})$$

$$(x\times x-3)+(x-1\times x-2)+(x-2\times x-1)+(x-3\times x)$$

$$(x^2-3x)+(x^2-3x+2)+(x^2-3x+2)+(x^2-3x)$$

$$4x^2-12x+4.$$

$$64-48+4=20.$$

wow! It worked! I wonder if the formula $4x^2-12x+4$ will be general in the other cases. I will try with another even set of steps before I move onto odd!

6: 6,10,12,12,10,6. where $x=6$

$$(x\times 1)+(x-1\times 2)+(x-2\times 3)+(x-3\times 4)+(x-4\times 5)+(x-5\times 6)$$

$$(x\times x-5)\ (x-1\times x-4)+(x-2\times x-3)+(x-3\times x-2)+(x-4\times x-1)+(x-5\times x)$$

$$x^2-5x+x^2-5x+4+x^2-5x+6+x^2-5x+6+x^2-5x+4+x^2-5x$$

$$6x^2-30x+20.$$

Both equations are similar but with slight alteration At the begining the 4 and 6 can become another x so it becomes x^3. As for the 12 and 30, $12 = 4 \times 3$ and $30 = 6 \times 5$ so each could be written as: $x \times x-1 = x^2 - x$ Once added to the $-x$ it becomes $-x^3 + x^2$. This is difficult to explain in writting so I am going to write it down algebraically.

where $x=4$

$4x^2 - 12x + 4$

$x(x^2) - (x \times x-1)x + 4$

$x^3 - (x^2 - x)x + 4$

$\cancel{x^3} - \cancel{x^3} + x^2 + 4$

$x^2 + 4$

$x=6$

$6x^2 - 30x + 20$.

$x(x^2) - (x \times x - 1)x + 20$

$x^3 - (x^2 - x)x + 20$

$\cancel{x^3} - x^3 + x^2 + 20$

$x^2 + 20$.

STUCK! from here I cant see what the 4 and 20 have in common. The relationship between 4 & 4 is 1 whereas between 20 & 6 is not.

AHA! when $x=4$ $x^2+4=20$ which has to be added to 6^2 for the next answer.

when $x=2$

$(x \times x-1) + (x-1 \times x)$ — $2 \times 1 + 1 \times 2$. — $(x \times 1) + (1 \times x)$

$x^2 - x + x^2 - x$ — $2+2$ — $x + x$

$x^4 - 2x$ — 4. — $2x = 4$.

I cant seem to get the correct algebraic answer but it will equal the equivilant to 4 which can be written as 2^2.

This brings me back to the fact that the sum of an even diagonal is itself squared plus and smaller even numbers squared.

I wonder if it will be the same general formula for odd numbers? I will try an examble to see.

$x=5$

$(x \times x-4)+(x-1 \times x-3)+(x-2 \times x-2)+(x-3 \times x-1)+(x-4 \times x)$

$x^2-4x+x^2-4x+3+x^2-4x+4+x^2-4x+3+x^2-4x$

$5x^2-20x+10$

$x^3-(x \times x-1)x+10.$

$x^3-x^3+x^2+10$

$x^2+10 \leftarrow$ where 10 is the sum of 3^2 and 1^2!

so what have I found out so far.

- with diagonals to and from the same number, the start number is the number of steps.
- The sum of an odd diagonal is itself squared plus any smaller odd numbers squared.
- The sum of an even diagonal is itself squared plus any smaller even number squared.

So far I have looked at diagonals where for every move you make across, you move up one. I will now investigate diagonals where for every move across, you move up 2. I wonder if this will have simular results as moving across two and up one? Will the formula be linked to the previous one I have found?

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100
11									

$3+2=5$

$5+6+3=14$

$7+10+9+4=30$

$9+14+15+12+5=55$

Will the answers to these diagonals be the 1st/start number times the last number and then adding smaller similar sets of numbers? I have noticed that the numb you start with is odd. This is because if it was even, it wouldnt land on the other edge. I have also noticed that the final number $(x) + (x-1) =$ the start number. I will work through the sums algebraically first to see if I get a similar formula to before.

when $x=7, y=4$. $\qquad x = 2y-1$.

$$(x) + (x-2\times y-2) + (x-4\times y-1) + (x-6\times y)$$

$$(x\times y-3) + (x-2\times y-2) + (x-4\times y-1) + (x-6\times y).$$

$$xy - \cancel{3y}^{3x} + xy - 2x - 2y + 4 + xy - x - 4y + 4 + xy - 6y.$$

$$4xy - 3\cancel{y}^{x} - 15y + 8$$

exchange $x = 2y-1$.

$$4(2y-1)y - 3(2y-1) - 15y + 8$$

$$8y^2 - 4y - 6y + 3 - 15y + 8$$

$$8y^2 - 25y + 11$$

check

$8(4^2) - 25(4) + 11.$

$128 - 100 + 11$

$= 17$

STUCK! I cant simplify this any further and at the moment it is not in a form that I can work from. AHA! I can see where I have got stuck! After exchangi 4 for y, you get 17 rather than 30. This must mean that I have made an algebraic error. I will go throu again and try and find where I have gone wrong. I have found my mistake. ($3y$ instead of $3x$). I will alter the rest of the formula and see if I can continue further.

$$4xy - 6x - 12y + 8$$

$$4(2y-1)y - 6(2y-1) - 12y + 8$$

$$8y^2 - 4y - 12y + 6 - 12y + 8$$

$$8y^2 - 28y + 14$$ ← AHA! I recognise that as the answer

when you start on 5 and finish on 3.

* I can also see that $-28y$ can be converted to $-7 \times 4y$, and when $y=4$, $\Rightarrow 7y^2 \cdot 8y^2$ can also be converted to $2y^3$.

Knowing this I can predict that the diagonal starting at 11 & finishing on 6 with the ratio $2y-1=x$ will have the sum of:

$2y^3 - 11y^2 + 55 = 91$. when $y=6$.

You have to add 55 because this is the answer to the previous diagonal. I will now draw on the diagonal and see if this is correct.

$11+18+21+20+15+6=91$.

- Yes! This worked. So in general:

with a diagonal, of the ratio $2y-1=x$:

$2y^3 - xy^2 +$ (other previous diagonals).

I wonder if this formula will work for the diagonal with the ratio $x=y$. I will see if it works.

when $x \,\&\, y = 5$

$2(5^3) - (5 \times 5^2) + 20 = 145$

two formulas

STUCK! I cant see how these are connected.

AHA! Is the 2 at the start due to the fact that for every one across, you go 2 up!

AHA! $2y-1$

$2y \times y^2 - 1 \times y^2$

$= 2y^3 - y^2$ OH! I thought I sort of had it then!

I also wondered why to find the sum, do you add all previous diagonals whereas for diagonals ($x=y$), you add every other diagonal? Is it due to the fact that every diagonal of $2y-1=x$, starts on an odd number?

I was wondering if you have to add 1, as there is no diagonal that passes it. But, in the previous set of diagonals, 1 was added to the odd set and there was no diagonals through it! I will investigate with $x=3$ & $y=2$. If $2y^3 - xy^2 = 5$, then 1 is not added.

$$2y^3 - xy^2$$
$$2(2^3) - 3(2^2) =$$
$$16 - 12 = \underline{4}$$

This means that 1 must be added.

Now that I know this formula, can it be manipulated for diagonals for steps 2 across and one up and for 1 across and 3 up?

I think that the formula for 2 across and one up will be the same as the previous formula but with the x and y's reversed.

eg $2x^3 - yx^2 +$ (previous diagonals)

1	2	3	4	5	6	7	8	9	10	11
2	4	6	8	10	12	14	16	18	20	
3	6	9	12	15	18	21	24	27	30	
4	8	12	16	20	24	28	32	36	40	
5	10	15	20	25	30	35	40	45	50	
6	12	18	24	30	36	42	48	54	60	
7	14	21	28	35	42	49	56	63	70	
8	16	24	32	40	48	56	64	72	80	
9	18	27	36	45	54	63	72	81	90	
10	20									

$2+3=5$

$3+6+5=14$

$4+9+10+7=30$

Wait! these are the same answers for one across and two up, so I wonder if my new formula will still work. I will try it for $x=5$ and $y=9$.

$$2(5^3) - 9(5^2) + 14 = 39.$$

$5+12+15+14+9=55$

STUCK! my formula doesnot work. I thought I had found how to manipulate one formula to the other!
AHA! I inserted 14 rather than 30.

$2(5^3)-9(5^2)+\cancel{14}\,30=55.$

I will just try the following diagonal in the series to make sure the answers right.

$x=6 \quad y=11$

$2(x^3)-x^2y+$ (previous diagonal).

$2(6^3)-6^2\times 11+55=91.$

$6+15+20+21+18+11=91.$

Yes I have managed to do it. I wonder if it can also work for $3y-1=x$ if I change the 2 at the begining to a 3.

$3y^3 - xy^2+$ (previous)-

$3(2^3)-4(2^2)+1=9$

$4+2=6.$

$3y^3-xy^2+$ (previous)

$3(3^3)-7(3^2)+6=24.$

$7+8\cancel{10}+3=\cancel{20}.\ 18$

STUCK! This didnt work. I will have to work through it algebraically and then it will give me the formula and it may help me to understand further. Before I do this I will just write my formulas out so far so I may see a connection and so they will be easy to find if I need to refer back to them later.

There must be an easier way than writting $2y^3-xy^2+$ (previous diagonal).
AHA! I can use the signal Σ which will add up all number between a given selection.

- when $x=y$ and both x & y are even.

$$\sum_{2n}^{x} x^2$$

- when $x=y$ and x & y are odd

$$\sum_{2n-1}^{x} x^2$$

- when $2y-1=x$

$$\sum_{n=1}^{y} 2y^3 - xy^2$$

- when $2x-1=y$

$$\sum_{n=1}^{x} 2x^3 - x^2y$$

For the diagonal formula, I need to work out the relationship between the start and finish number.

x	1	4	7	10
y	1	2	3	4

$+3$ $+3$ $+3$

relationship.

STUCK! I cant see what the relationship is! I will write it out in the form of a table as I used before and see if I can progress any further.

start & finish.	total no passed	total.
4 & 2.	4, 2	6
7 & 3	7, 8, 3	18
10 & 4	10, 14, 12, 4	40.

5 4 12 17 10 28

AHA! I can see a pattern! The start number times the finish number, −3, is the sum of the diagonal. I should now be able to predict the following diagonal:

Start: 13   finish: 5
sum from counting: 13 + 20 + 21 + 16 + 5 = 75
sum from formula: $(10 \times 5) - 3 = 47$.

stuck! It didnt work! I will just try and do it algebraically.

eg when $x = 10$, $y = 4$.

$$(x \times y - 3) + (x - 3 \times y - 2) + (x - 6 \times y - 1) + (x - 9 \times y)$$

$$xy - 3x + xy - 2x - 3y + 6 + xy - x - 6y + 6 + xy - 9y.$$

$$4xy - 6x - 18y + 12.$$

$$xy^2 - (x - y)x - 18y + 12$$

$$xy^2 - x^2 + yx - 18y + 12.$$

I have ~~been~~ managed to simplify the formula to this. Before I go any further I will just test it with a few examples to make sure I have converted it correctly.

~~$x = 4 \quad y = 2$~~

~~$4(2^2) - (2^2) - (4 \times 2) - 18($~~

$x = 10 \quad y = 4$

$$xy^2 - x^2 + yx - 18y + 12$$

$$(10 \times 4^2) - (10^2) + (10 \times 4) - (18 \times 4) + 12.$$

$$= 160 - 100 + 40 - 72 + 12.$$

$$= 40.$$

Yes It worked! I still think some of the numbers may need to be converted so I will try with another example and see what results I get.

13.

$x = 13 \quad y = 5$

$$\overset{13\times1}{(x \times y-4)} \times \overset{10\times2}{(x-3 \times y-3)} \times \overset{7\times3}{(x-6 \times y-2)} \times \overset{4\times4}{(x-9 \times y-1)} \times \overset{1\times5}{(x-12 \times y)}$$

$xy - 4x + xy - 3x - 3y + 9 + xy - 2x - 6y + 12 + xy - x - 9y + 9 + xy - 12y.$

$5xy - 10x - 30y + 30.$

$xy^2 - 10x - 30y + 30$

$= 325 - 130 - 150 + 30$

$= 75.$

It worked! now I have to find what the two formulas have in common so I can remove some ov these number

$xy^2 - 10x - 30y + 30$ where $x = 13$ & $y = 5$

$xy^2 - 6x - 18y + 12.$ where $x = 10, y = 4.$

STUCK! I cant see any correlation. But while investigatin the formula I realised the relationship:

$3y - 2 = x$

This formula looks similar to the previous one $(2y-1=x)$ but having looking at it, it wont work!

$x = 4, y = 2.$

$3(2^3) - 2(4 \times 2^2)$

$= 24 - 32$

$= -8.$

STUCK! I feel that at this moment in time, I can no longer go on with this part of the questio I have discovered a-lot so far but there is still lots more to find out. I am going to move further on and then I might come back to this part of the question.

of the paper

So far I have looked at the multiplication square. I will now look at the addition square and see if there is any correlation between those diagonals.

1	2	3	4	5	6	7	8	9	10
2	4	5	6	7	8	9	10	11	12
3	5	6	7	8	9	10	11	12	13
4	6	7	8	9	10	11	12	13	14
5	7	8	9	10	11	12	13	14	15
6	8	9	10	11	12	13	14	15	16
7									
8									
9									
10									

Wait! would this be the correct way to display this, as this may be classed as missing out a row and a column (starting 1+2=3, ...)

I have just had an idea! If I go back to the multiplication square, but if I multiply the diagonals rather than adding them, will there be a common rule?

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

$2\times2=4$

$3\times4\times3=36$

$4\times6\times6\times4=576$

$5\times8\times9\times8\times5=14400$

This may be difficult to spot a pattern as we are working with numbers with large ranges betw[een] them due to the multiplication. Im not sure if there will be any particular pattern, but ill work through it algebraically and see.

when $x=2$

$x \times x = 4$

$x^2 = 4$.

when $x=3$

$x \times (x-1 \times x-1) \times x$

$(x \times x-2) \times (x-1 \times x-1) \times (x-2 \times x)$

$(x^2-2x) \times (x^2-2x+1) \times (x^2-2x)$

$(x^4-2x^2+x^2-2x^3+4x^2-2x) \times (x^2-2x)$.

$x^6-2x^4+x^4-2x^5+4x^4-2x^3-2x^5+4x^3-2x^3+4x^4-8x^2+4x$

$x^6 - 4x^5 + 7x^4 + 6x^3 - 8x^2 = 36$.

$729 - 972 + 567 + 162 - 72 = 414$!..

STUCK! rather than having x as $(x \times x-2)$, I might see if it will make a difference with the formula as its not working at the moment.

when $x=3$

$x \times (x-1 \times x-1) \times x$

$x^2 (x^2-2x+1)$

$x^4-2x^3+x^2 = 36$.

when $x=4$.

$x \times (x-1 \times x-2) \times (x-2 \times x-1) \times x$

$x^2 \times (x^2-3x+2)^2$

$x^2(x^4-3x^3+2x^2-3x^3+9x^2-6x+2x^2-6x+4)$

$x^6-3x^5+2x^4-3x^5+9x^4-6x^3+2x^4-6x^3+4x^2$

16

$$x^6 - 6x^5 + 13x^4 + -12x^3 + 4x^2 = 576.$$

WELL! I have managed to convert my multiplications into equations but it hasn't seemed to have helped me much. I am going to fiddle around with them and see what I can find.

x^2

$x^4 - 2x^3 + x^2 \Rightarrow x^2(x^2 - 2x + 1)$

$x^6 - 6x^5 + 13x^4 - 12x^3 + 4x^2 \Rightarrow x^2(x^4 - 6x^3 + 13x^2 - 12x^1 + 4)$

I am just trying to see if there is a similar pattern as before where the diagonal is a equation encoorporating previous answers.

$$\begin{array}{r l}
 & x^2 - 4x + 4 \\
x^2 - 2x + 1 \,\big) & x^4 - 6x^3 + 13x^2 - 12x + 4 \\
- & x^4 - 2x^3 + x^2 \\
 & -4x^3 + 12x^2 - 12x + 4 \\
- & -4x^3 + 8x^2 - 4x \\
 & 4x^2 - 8x + 4 \\
 & 4x^2 - 8x + 4
\end{array}$$

CHECK: $(x^2 - 2x + 1)(x^2 - 4x + 4)$

$x^4 - 4x^3 + 4x^2 - 2x^3 + 8x^2 - 8x + x^2 - 4x + 4$

$x^4 - 6x^3 + 13x^2 - 12x + 4$

AHA! Both x^2 and $(x^2 - 2x + 1)$ can be removed from diagonal 4, leaving $(x^2 - 4x + 4)$, But what does 3 have to do with $(x^2 - 2x + 1)$ and 4 have to do with $(x^2 - 4x + 4)$?

$(x^2 - 2x + 1) \Rightarrow (x - 1)^2$

$(x^2 - 4x + 4) \Rightarrow (x^2 - 2)^2$

Does this mean that for the fifth diagonal, it will be the previous diagonals ~~&~~ multiplied by $(x-3)^2$ or is it just a coincidence that all previous diagonals are multiplied by $(x-(x-2))^2$? I will now find out.

So when $x=5$, will the diagonal be equal to:

$x^2 \cdot (x-1)^2 \, (x-2^2) \, (x-3)^2$.

$5^2 \times 4^2 \times 3^2 \times 2^2 = 14400$.

Wow! It worked. I also have seen a pattern which I never expected. $(5^2 \times 4^2 \times 3^2 \times 2^2)$ I should be able to predict the diagonal ~~for~~ of multiplication for the diagonal starting at 7!

$7^2 \times 6^2 \times 5^2 \times 4^2 \times 3^2 \times 2^2 = 25401600$

$7 \times 12 \times 15 \times 16 \times 15 \times 12 \times 7 = 25401600$

OH! now I see why the formula is as it is! If you take any row, eg 5! When you write out how each number is made and then times them, the numbers can be rearranged to give the sum of all previous square numbers.

$5 \times (4 \times 2) \times (3 \times 3) \times (2 \times 4) \times 5$

$5 \times 4 \times 2 \times 3 \times 3 \times 2 \times 4 \times 5$

$5^2 \times 4^2 \times 3^2 \times 2^2$.

1^2 could be added to the end of that equation as where 5 is placed, it could also be written as 5×1 and 1×5. However it is not needed as $1^2 = 1$ and anything multiplied by 1, remains the same.

I will now try multiplying diagonals for lines of $2y-1=x$.

when $x=3$, $y=2$

$3 \times 1 \times 2 \times 1 = 3 \times 2 \times 1^2 = 6$. ← this didnt fit this pattern but it may fit another as $2y-1=x$

earlier did not fit the origonal pattern

when $x=5$, $y=3$

$5 \times 1 \times 3 \times 2 \times 3 \times 1 = 90 = 5 \times 3^2 \times 2 \times 1^2$

when $x=7$, $y=4$.

$7 \times 1 \times 5 \times 2 \times 3 \times 3 \times 4 \times 1 = 7 \times 5 \times 4 \times 3^2 \times 2 \times 1^2 = 2520$.

when $x=9$, $y=5$

$9 \times 1 \times 7 \times 2 \times 5 \times 3 \times 3 \times 4 \times 5 \times 1 = 9 \times 7 \times 5^2 \times 4 \times 3^2 \times 2 \times 1^2 = 113400$.

STUCK! this one is more ~~hav~~ difficult to work out because of the way the diagonal falls. The diagonal lands on all the y numbers but only the odd x numbers. This means that numbers such as 2, 4, 6 will never be squared. Only the odd y values will be squared in the diagonal.

I feel that this is a good place to stop. I think I have made a good start to a problem that is far from finished.

At the start while organising my thoughts on how to enter the problem, I ~~know~~ noticed the line of symmetry From that point, I knew that there would be some sort of formula that would work for some or all of the diagonals on the grid. It would have to be a general rule, as it was an ongoing grid which didnt stop at 10, ~~so~~ so would have to work for all numbers

I did not manage to prove or disprove all of the points I made at the start of the investigation. I did manage to show that going from 9 to 5 for example gives the same answer as going from 5 to 9, which was due to the symmetry. So far, I have proved that diagonals to and from the same number need a different formula to diagonals to and from different numbers. I still believe though that there has to be a common connection between all sets of diagonals. I didnt hav chance to work on a selection of the grid. I think tha if I did, there may only be a formula in selection on the line of symmetry (eg square with corners 16, 32, 32 an 64). I managed to move onto multiplying the diagonals rather than addition, but there was no connection between the two formulas. This is because I altered the function too much. However I did manage to find a formula for one type of diagonal.

I think I started the investigation well, as I worked through the diagonals in numerical order and recorded my results in a number of ways so I would have a greater chance in finding a pattern. I moved from finding a sequence to looking for the formula. I found most of my formulas by converting my numbers to algebra and then checking it with examples. I think I may have spent too much time checking examples but I wanted to make sure that my formula didnt just work for a few examples. I also tried to look for the reason behind the formula but I didnt get very far.

I later went on to looking at other diagonals in the grid. When I got stuck at a part, I checked through what I had previously written to ensure I had not made a mistake. In one case I had made a mistake so was easily able to become unstuck and move on. I spent quite a time trying to prove that both sorts of diagonals were closely connected I later came to the conclusion that two distinct formulas were needed.

When I moved onto another grid, I was unsure whether or not to use the addition square as I coul not show 3+1 on it. I dont feel that I had discover enough information on the original problem to move onto an extension. I dont think that the way I extended was the correct move to make as I can not see a direct link between the two formulas.

My main key turning points was when I realised that the sum of the diagonals was the start number squared plus alternate numbers squared. I realised why this was the case when I returned to look at my grid. It did take me, however, to put the formula into a complete equation.

S7a

I think that I did not conduct my investigation to the problem very well. I feel that as soon as I got a new idea, I rushed to do it rather than continuing with what I was doing and making a note of it so I could move onto it later.

If I had the chance to do this again or come back to the problem, I would return to each type of diagor and spend some more time working on it. I would also like to investigate the selection of the grid which I mentioned earlier. I think a good way to extend it would bee to investigate the diagonal as earlier but also the squares inside the triangle it makes.

eg:

1	2	3	4	5
2	4	6	8	10
3	6	9	12	15
4	8	12	16	20

for diagonal 3 to 3
sum of triangle would be:
1+2+3+2+4+3 = 15

I would definetly spend more time on the first grid and not move on until I understood the reasons for the formula first.

So, what did I find out from doing this problem!:

- Line of symmetry through the grid.
- with diagonals of $x=y$, x & y are the number of squares the diagonal covers.
- even diagonals of $x=y$, the formula is the number squared plus any smaller even number squared:

$$\sum_{2n}^{x} x^2$$

- odd diagonals of $x=y$, formula is the odd number squared plus any previous odd number squared.

$$\sum_{2n+1}^{x} x^2$$

- ~~whit~~ with a diagonal of $2y-1=x$, $\sum_{n=1}^{y} 2y^3 - xy^2$
- when a diagonal is $2x-1=y$ $\sum_{n=1}^{x} 2x^3 - x^2y$.

- in the multiplication diagonals, when $x=y$, the formula is : $x^2 \times (x-1)^2 \times (x-2)^2 \times \cdots\cdots \times 1^2$