IN THE NAME OF ALLAH

THE DOUBLE POINT SURFACES OF IMMERSIONS IN COMPLEX PROJECTIVE SPACES

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Contents

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We review the general theory of cobordism of codimension k immersions of compact manifolds M into a given compact manifold N. Applying the Pontrjagin-Thom construction for immersions, any cobordism class $[F : M^n \looparrowright N^{n+k}]$ corresponds to a unique homotopy class $f \in [N_+^{n+k}, QMO(k)]$. According to Eccles [E96] the cobordism class of the immersion F, as well as the r-fold intersection points of F, can be determined using the Hurewicz image of the mapping f.

We shall apply these techniques to the problem of studying double point manifolds of immersions $M^{k+2} \looparrowright \mathbb{C}P^{k+1}$. The double point manifold of such an immersion is a surface, and the cobordism group of surfaces is completely known.

We shall prove that in the case that k is odd there exists always an immersion $M^{k+2} \looparrowright \mathbb{C}P^{k+1}$ whose double point manifold is cobordant to the projective plane.

For even k, specifically $k = 2$, we show that there exists an immersion $M^4 \looparrowright \mathbb{C}P^6$ whose double point manifold is cobordant to the projective plane.

In the other cases, $M^{k+2} \looparrowright \mathbb{C}P^{k+1}$ with $k > 2$ and $k \equiv 2 \pmod 4$, we determine a homological condition for the double point manifold to be cobordant to the projective plane.

For $k = 4$, we show that the double point of any immersion $M^6 \looparrowright \mathbb{C}P^5$ is a boundary.

In the case $k > 4$ with $k \equiv 0 \pmod 4$ we do not have a complete result and this is an ongoing project.

Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Copyright Statement

Dedicated to Prophet Mohammed

Dedicated to My Parents, Saad Al-Shehry and Thanwa with gratitude for all their support and for encouragements they have made..

and,

Dedicated to my brothers and sisters who sent me their encouragements and endless love all the time.

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Introduction

The idea of cobordism is a purely geometric idea, which aims to distinguish between different manifolds. In an ideal world, one would like to determine whether or not two given smooth n-dimensional manifolds M_0 and M_1 are diffeomorphic, which is of course an open problem as well as it is not an easy problem. It is immediate that if two manifolds M_0 and M_1 are diffeomorphic then the disjoint union $M_0 \sqcup M_1 = \partial W$ where $W = M_0 \times [0, 1]$ is an $(n + 1)$-dimensional manifold. However, the converse is not true, i.e. $M_0 \sqcup M_1 = \partial W$ does not imply that M_0 and M_1 are diffeomorphic. Two manifolds are said to be cobordant if $M_0 \sqcup M_1$ is the boundary of an $(n+1)$-dimensional manifold W. Hence, cobordism provides a way of classifying n-dimensional manifolds where $n > 0$ is arbitrary. This idea was initially considered by Thom.

The problem then reduces to distinguishing between two different manifolds in the same cobordism class. These ideas can be extended to study cobordisms with given structures, and various versions of such cobordism theories do exist. In this thesis, we consider the theory of cobordism of manifolds equipped with an immersion in an ambient space.

The essential idea in this thesis is to use a basic property of immersions. Given an immersion $F : M^n \looparrowright N^{n+k}$ the image of F may have points whose preimage has more than one point. Such points are known as 'self-intersection' points of F. We then observe that if the immersion F is self-transverse then the set of those points whose preimage has r-distinct points known as the r-fold manifold of F is itself a submanifold. We shall only consider the case of double point manifolds of immersions, in the special case of $M^{k+2} \looparrowright \mathbb{C}P^{k+1}$ where the double point manifolds are surfaces.

Previously, Asadi and Eccles [AEa00] have considered the problem of determining the cobordism class of double point manifolds of immersions $M^{k+2} \looparrowright \mathbb{R}^{2k+2}$. Any surface is either a boundary or is cobordant to the real projective plane. For all k, there exists an immersion $M^{k+2} \looparrowright \mathbb{R}^{2k+2}$ with double point manifold which is a boundary. For example take the standard embedding $S^{k+2} \hookrightarrow \mathbb{R}^{2k+2}$ when the double point manifold is empty. They show that there exists an immersion F with double point manifold cobordant to the projective plane if and only if $k \equiv 1 \pmod 4$, or $k \equiv 3 \pmod 4$ and $k+1$ is a power of 2.

This thesis considers the same problem for immersions $M^{k+2} \looparrowright \mathbb{C}P^{k+1}$. Notice that any immersion $M^{k+2} \looparrowright \mathbb{R}^{2k+2}$ gives rise to an immersion $M^{k+2} \looparrowright \mathbb{C}P^{k+1}$ with the same double point manifold. We show that, for odd k, there is always an immersion with double point manifold cobordant to the projective plane. For even k, we only have partial result. For $k = 2$, there exists an immersion with double point manifold cobordant to the projective plane. For $k = 4$, there does not exist an immersion with double point manifold cobordant to the projective plane. For other values of k with $k \equiv 2 \pmod 4$, we give a condition for the existence of an immersion with double point coborant to the projective plane.

The method of our calculation is to translate problems from geometry into homotopy theory through the Pontrjagin-Thom construction. The outline of this thesis is as follows.

Chapter 1 includes background material from differential topology, and homotopy theory. We review the issue of transversality in Chapter 2. In Chapter 3 we recall the basics of the Pontrjagin-Thom construction for embeddings, which are essential for the introduction of the Pontrjagin-Thom construction for immersions introduced in Chapter 4. In these two chapters, we also set up the homological machinery that we are going to use during our calculations. In Chapter 5 we recall some facts about the Steenrod operations, and the Kudo-Araki operations. We shall describe in Chapter 6, how we can determine the cobordant class of the double point manifold of an immersion. Chapters 7 and 8 then illustrate the techniques introduced in previous chapters, and contain the proofs of our results.

Chapter 1

Background

Through this chapter we review some of the geometric background material, and fix our notation.

The notion of homotopy is of fundamental importance for us. The main essence of our thesis is to translate geometric problems into equivalent problems in homotopy theory, and use the methods of algebraic topology to solve these problems. For this reason, we start by recalling some basic facts from homotopy theory.

1.1 Sets of homotopy classes of maps

We consider topological spaces and continuous maps between these spaces.

Definition 1.1.1. *We say two continuous maps $f_0, f_1 : X \to Y$ are* homotopic *if there is a family of continuous maps $f_t : X \to Y$ for every $t \in I = [0,1]$ i.e if there exists a continuous map $F : X \times I \to Y$ such that $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$. We say F is a homotopy between f_0 and f_1, and we write $f_0 \simeq f_1$ (or $F : f_0 \simeq f_1$) to indicate that f_0 is homotopic to f_1.*

It is easy to check that the homotopy relation is an equivalence relation on the set of all continuous maps $X \to Y$, denoted by $\mathrm{Map}(X,Y)$. Let $[X,Y]$ denote the set of homotopy classes of maps from X to Y, i.e.

$$[X,Y] = \mathrm{Map}(X,Y)/\simeq,$$

where $\simeq$ is the homotopy relation.

Notice that if Y is path-connected then the set $[X, Y]$ contains a distinguished class of maps, namely all the constant maps. We will use this as a base point for $[X, Y]$ if one is needed.

If X has a base point x_0 and Y has a base point y_0, let $[(X, x_0), (Y, y_0)]$ denote *the homotopy classes of based maps*, where *a based map* is a map $f : X \longrightarrow Y$, such that $f(x_0) = y_0$. We may write $f : (X, x_0) \to (Y, y_0)$ for such a pointed map. Then $[(X, x_0), (Y, y_0)]$ has distinguished class, namely the class of the constant mapping sending everything to y_0.

Given a map $f : X \longrightarrow Y$, let $[f]$ denote its homotopy class in $[X, Y]$. Notice that it will be clear from the context whether the spaces are based or not . So if there is no confusion we may write $[X, Y]$ for based maps as well.

Next, we introduce a couple of constructions that are central in homotopy theory.

Definition 1.1.2. *For a space X, the* suspension SX *is the quotient of $X \times I$ obtained by collapsing $X \times 0$ to one point and $X \times 1$ to another point.*
If X has a base point $x_0 \in X$ the reduced suspension of X, denoted by ΣX, is obtained from the suspension SX by collapsing the line segment $x_0 \times I$ to a point; equivalently

$$\Sigma X \cong X \wedge S^1 \cong X \times S^1 / X \vee S^1.$$

The point that we collapse $X \vee S^1$ to is the base point of this space.

The space of all paths in a space Y is defined to be the function space $Y^I = PY =$ $\mathrm{Map}(I, Y)$ *[G75]. This space is given the compact-open topology, that is generated by all pairs $\langle K, V\rangle$ of all $f \in \mathrm{Maps}(X, Y)$ with $f(K) \subseteq V$ for $K \subseteq X$ and $V \subseteq Y$ such that K is a compact subset of X and V is an open subset of Y. If Y has a base point $y_0 \in Y$ the space of loops in Y based at y_0 is defined by*

$$\Omega(Y, y_0) = \mathrm{Map}\left((I, \{0, 1\}), (Y, y_0)\right)$$

which is the set of all maps $\alpha : I \to Y$ such that $\alpha(0) = \alpha(1) = y_0$. This set is homeomorphic to $\mathrm{Map}\left((S^1, s_0), (Y, y_0)\right)$ *with the compact-open topology where $s_0 =$* $(1, 0)$ *is the base point of S^1 (when sitting in $\mathbb{R}^2$). The space of loops at Y is a subspace* *of Y^I and is topologized as a subspace of the space Y^I. Recall that $\Sigma S^n \cong S^{n+1}$ (i.e. they are homeomorphic).*

Notice that the loop and suspension give rise to functors. More precisely, for a given map $f : X \to Y$ we have $\Sigma f : \Sigma X \to \Sigma Y$ where $\Sigma f([x,t]) = [f(x), t]$ where we write $[x,t]$ for the class of $(x,t) \in X \times I$ under the identification in the above definition. Similarly, we have $\Omega f : \Omega X \to \Omega Y$ where $\Omega f(\alpha) = f \circ \alpha$.

Our next observation, is a relation between suspension and loop functors.

Theorem 1.1.3. ***Adjointness theorem*** . *Let X, Y be pointed spaces. Then there is a natural bijection $[\Sigma X, Y] \cong [X, \Omega Y]$.*

Proof. Let X be a space with base point x_0. The maps

$$f : X \times I \to Y$$

are in one to one correspondence with maps

$$g : X \to Y^I$$

defined by $f(x,t) = g(x)(t)$, where $(x \in X, t \in I)$. If we take account of the base points, we find the maps $f : \Sigma X \to Y$ are in (1-1)-correspondence with maps

$$g : X \to \Omega Y.$$

So, passing to homotopy class we see that a natural (1-1)-correspondence

$$[\Sigma X, Y] \leftrightarrow [X, \Omega Y],$$

where ΩY is the loop space of Y at its chosen basepoint and the constant loop is taken as the basepoint of ΩY. □

Now we would like to have a group structure on $[X, Y]$. This can be obtained using the basic adjoint relation as following.

Lemma 1.1.4. *Let X and Y be based spaces and all maps and homotopies preserve base points. Then*

(1) $[X, \Omega Y] \cong [\Sigma X, Y]$ *is a group;*

(2) $[X, \Omega(\Omega Y)] \cong [\Sigma X, \Omega Y] \cong [\Sigma^2 X, Y]$ *is an abelian group.*

Proof. See [DK, Lemma 6.41]. □

Definition 1.1.5. *Suppose that X is a pointed space with base point $x_0 \in X$, and $n > 0$. The* n-th homotopy group *of X at x_0, denoted by $\pi_n(X, x_0)$ is defined by*

$$\pi_n(X, x_0) = [(S^n, s_0), (X, x_0)],$$

where s_0 is the base point of $S^n \cong \Sigma S^{n-1}$. When $n = 0$, we have the set of path-connected components of X which is not a group in general and is given by $\pi_0 X = [(S^0, \{0\}), (X, x_0)]$ where $S^0 = \{0, 1\}$.

Notice that $\pi_1(X, x_0)$ is a group, but non-commutative in general. $\pi_n(X, x_0)$ is an abelian group for $n \geq 2$.

This definition depends on the chosen base point. However, there are spaces such that the definition will be independent of this choice. Recall that a space X is called path connected if for any pair of points $x_0, x_1 \in X$ there exists a continuous function $\alpha : I \to X$ such that $\alpha(0) = x_0$ and $\alpha(1) = x_1$. Moreover, for a given such path we have the reverse path associated with α given by $r_\alpha : I \to X$ such that $r_\alpha(t) = \alpha(1-t)$ which satisfies $r_\alpha(0) = x_1$ and $r_\alpha(1) = x_0$.
We also recall the definition juxtaposition of two paths. If we have $\alpha, \beta \in PX$ such that $\alpha(1) = \beta(0)$, we then may define another path $\alpha \star \beta \in PX$ by

$$(\alpha \star \beta)(t) = \begin{cases} \alpha(2t) & 0 \leqslant t \leqslant 1/2, \\ \beta(2t-1) & 1/2 \leqslant t \leqslant 1. \end{cases}$$

Now assume X is path connected and let $x_0, x_1 \in X$ be two distinct points. Choose a path $\alpha \in PX$ such that $\alpha(0) = x_0$ and $\alpha(1) = x_1$. We then define

$$\phi_\alpha : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$$

by

$$\phi_\alpha[f] = [\alpha f r_\alpha].$$

This is easy to check that this map is an isomorphism of groups.

Lemma 1.1.6. *The mapping $\phi_\alpha : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$ is an isomorphism.*

This lemma has a generalization to higher homotopy groups.

Theorem 1.1.7. *Let $n > 1$. For each path $\alpha : I \to X$ there exists an isomorphism*

$$\pi_n(X, \alpha(0)) \longrightarrow \pi_n(X, \alpha(1)).$$

Proof. See [B, Theorem 7.2, Chapter VII]. □

According to above lemma, and the theorem after it, when X is path connected we may relax our notation and just write $\pi_n X$. Moreover, when X has more than one component, we choose to work with the component which has the base point, and hence using this notation for $\pi_n X$ makes sense.

Finally, according to the Adjointness theorem observe that for nonnegative integers n, k

$$\pi_{n+k}X = [S^{n+k}, X] = [\Sigma^k S^n, X] \cong [S^n, \Omega^k X] = \pi_n \Omega^k X.$$

1.2 Stable homotopy groups

Recall from the previous section that suspending a based map $f : X \to Y$ gives rise to a based map $\Sigma f : \Sigma X \to \Sigma Y$. This then allows us to have the following definition.

Definition 1.2.1. *The suspension homeomorphism*

$$\Sigma : \pi_n(X) \to \pi_{n+1}(\Sigma X), n \geq 0,$$

is defined by $\Sigma[f] = [\Sigma f]$, where $f : S^n \longrightarrow X$ and $\Sigma f : \Sigma S^n \cong S^{n+1} \longrightarrow \Sigma X$ is the suspension of f. Clearly it is a natural transformation from the functor π_n to the functor $\pi_{n+1} \circ \Sigma$.

It is useful to know if there are cases when the suspension homeomorphism is an isomorphism. The following theorem identifies one of these cases. Recall that a topological space X is said to be $(k-1)$-connected, if $\pi_i X = 0$ for all $i \leq k-1$.

Theorem 1.2.2. ***Freudenthal suspension theorem*** . *Suppose that X is a $(k-1)$-connected CW complex. Then the suspension map $\pi_i(X) \longrightarrow \pi_{i+1}(\Sigma X)$ is an isomorphism for $i < 2k-1$ and a surjection for $i = 2k-1$.*

Proof. See [H02, Corollary 4.24.] □

Notice that in particular, the n-sphere S^n is $(n-1)$-connected. The following then is an application of the above theorem.

Corollary 1.2.3. *For every* $n \geq 1$

$$\Sigma : \pi_n(S^n) \longrightarrow \pi_{n+1}(S^{n+1})$$

is an isomorphism, and hence $\pi_n(S^n) \cong \mathbb{Z}, n \geq 1$.

Proof. See [S75, Theorem 6.28]. □

From Theorem 1.2.2 for a $(k-1)$-connected CW complex X, the suspension map $\pi_i(X) \to \pi_{i+1}(\Sigma X)$ is an isomorphism for $i < 2k-1$. We note that when we suspend a space, we increase the connectivity of that space by 1. Therefore, for every space X and $i > 0$, there exists l, sufficiently large, such that $(\Sigma^l X)$ is $(k+l-1)$-connected and $i + l < 2(k+l) - 1$, i.e. in

$$\pi_{i+l}\Sigma^l X \to \pi_{i+l+1}\Sigma^{l+1} X \to \pi_{i+l+2}(\Sigma^{l+2} X) \to \cdots \to \pi_{i+l+m}\Sigma^{l+m} X$$

all mappings are isomorphisms where $m > 0$ is arbitrary. This means that for any space X, and $i > 0$, after finitely many suspensions the resulting homotopy group is independent of suspension, and there is a unique group, which we call it the *i-th stable homotopy group* of X, denoted by $\pi_i^S X$. More formally, we may define

$$\pi_i^S X = \text{direct limit } \pi_{i+l}\Sigma^l X.$$

By the Adjointness theorem $\pi_{i+l}\Sigma^l X$ is isomorphic to $\pi_i \Omega^l \Sigma^l X$ for all l and i, where Ω^l denotes the *l-th loop space functor*. There is a natural inclusion of $\Omega^l \Sigma^l X$ in $\Omega^{l+1}\Sigma^{l+1} X$. Let QX denote the direct limit $\lim \Omega^l \Sigma^l X$. According to [G75, Chapter 15 (direct limits)] one can take direct limit of a directed system of spaces and then take the homotopy group, or can take the homotopy group and then take the direct limit of the result directed system of groups. So,

$$\pi_i^S X \cong \lim\, \pi_{i+l}\Sigma^l X \cong \lim\, \pi_i \Omega^l \Sigma^l X \cong \pi_i \lim\, \Omega^l \Sigma^l X \cong \pi_i QX,$$

that is, any stable homotopy group can viewed as an unstable homotopy group as well.

1.3 Immersions

Through this thesis we will work with compact, connected, smooth manifolds of finite dimensions, and smooth maps between such manifolds; M^n will denote a manifold of dimension n.

Definition 1.3.1. *A map $F : M \longrightarrow N$ is called an* immersion *if the Jacobian $dF_x : T_xM \longrightarrow T_{F(x)}N$ is a monomorphism (injective) for every $x \in M$. We denote an immersion F by $F : M \looparrowright N$.*

Definition 1.3.2. *A map $F : M \longrightarrow N$ is called* a submersion *if the Jacobian $dF_x : T_xM \longrightarrow T_{F(x)}N$ is surjective for every $x \in M$.*

Definition 1.3.3. *If $F : M \looparrowright N$ is an immersion and $F : M \to F(M)$ maps M homeomorphically onto its image, then F is called an* embedding *denoted by $F : M \hookrightarrow N$.*

We shall provide examples of immersions in later chapters. Next, we introduce equivalence relations between immersions and embeddings.

Definition 1.3.4. *An* isotopy *between embeddings of manifolds $F_0, F_1 : M \hookrightarrow N$ is a homotopy*

$$F : M \times I \longrightarrow N \ \ ; (x,t) \mapsto F_t(x)$$

such that for each $t \in I$ the map $F_t : M \hookrightarrow N$ is an embedding. That is, it is a homotopy through embeddings.

Definition 1.3.5. *A* regular homotopy *of immersions $F_0, F_1 : M \looparrowright N$ is a homotopy*

$$F : M \times I \longrightarrow N \ \ ; (x,t) \mapsto F_t(x)$$

such that for each $t \in I$ the map $F_t : M \looparrowright N$ is an immersion.

Remark 1.3.6. *In particular, an embedding is an immersion, and isotopic embeddings are regular homotopic.*

This notion will be useful, as in the next subsection we show that the normal bundle of an immersion only depends on the regular homotopy class of the given immersion. This is technically important as we want to calculate the algebraic invariants of a given immersions.

The set of r-fold points of a given immersion is defined as below.

Definition 1.3.7. *Let $F : M^n \looparrowright N^{n+k}$ be an immersion where $k > 0$. For integers $r \geq 1$, we may define the r-fold self intersection sets of F in N as follows*

$$I_r(F) \;=\; \{y = F(x_1) = \cdots = F(x_r) \in N \mid |F^{-1}(y)| \geq r\} \subseteq N.$$

A point $y \in N^n$ is called an r-fold intersection point of F if $y \in I_r(F)$.

Example 1.3.8. *For $n = 1$ and $k = 1$ then the figure eight immersion of the circle $F : S^1 \looparrowright \mathbb{R}^2$ has a single double point, $I_2(F) \subseteq \mathbb{R}^2$.*

In the next chapters, we will set up the framework that we are going to use in order to study the self-intersection points of a given immersion. In our examples we will take N to be Euclidean spaces, and projective spaces. We fix our notation for these spaces.

Definition 1.3.9. *The real projective space $\mathbb{R}P^n$ is the set of lines through the origin in $\mathbb{R}^{n+1}$, i.e. $\mathbb{R}P^n$ is the set of all one-dimensional subspaces of $\mathbb{R}^{n+1}$.*

We may obtain $\mathbb{R}P^n$ by identifying antipodal points in S^n, i.e. $\mathbb{R}P^n = S^n/\{x, -x\}$.

Example 1.3.10. *$\mathbb{R}P^1 \cong S^1$ is called the real projective line. $\mathbb{R}P^2$ is called the real projective plane. We will use this surface in many places. $\mathbb{R}P^\infty = \lim_n \mathbb{R}P^n$ is called infinite real projective space.*

By analogy, we may define complex projective space $\mathbb{C}P^n$.

Definition 1.3.11. *The complex projective space $\mathbb{C}P^n$, of complex dimension n (real dimension $2n$), is the set of complex 1-dimensional subspaces of $\mathbb{C}^{n+1}$.*

We can identify this space with a quotient of the unit $2n+1$ sphere in $\mathbb{C}^{n+1}$ under the action of $U(1) = S^1$, i.e. $\mathbb{C}P^n \cong S^{2n+1}/S^1$. This is because every complex line in $\mathbb{C}^{n+1}$ intersects the unit sphere in a circle. This action is given by

$$z(z_1, z_2, \ldots, z_{n+1}) = (zz_1, zz_2, \ldots, zz_{n+1}),$$

for $z \in S^1$ and $(z_1, z_2, \ldots, z_{n+1}) \in S^{2n+1}$.

Example 1.3.12. *$\mathbb{C}P^1 \cong S^3/S^1 \cong S^2$ is called the complex projective line. $\mathbb{C}P^2$ is called the complex projective plane. $\mathbb{C}P^\infty$ is called infinite complex projective space.*

Notation 1.3.13. *We will write $\mathbb{R}P^n_k$ for the truncated real projective space $\mathbb{R}P^n/\mathbb{R}P^{k-1}$, and $\mathbb{R}P^\infty_k$ for the truncated real projective space $\mathbb{R}P^\infty/\mathbb{R}P^{k-1}$. Similarly, we will write $\mathbb{C}P^n_k$ for the truncated complex projective space $\mathbb{C}P^n/\mathbb{C}P^{k-1}$, also $\mathbb{C}P^\infty_k$ for the truncated complex projective space $\mathbb{C}P^\infty/\mathbb{C}P^{k-1}$.*

1.4 Vector bundles

In order to study the self intersection points of a given immersion, we will use specific invariants of vector bundles. For this reason, we include a brief review of vector bundle theory.

Definition 1.4.1. *A real vector bundle ξ over B is a triple (E, π, B) such that:*
(1) The topological space $E = E(\xi)$ is called the total space.
(2) The continuous map $\pi : E \to B$ called the projection map.
(3) Each $b \in B$ has a fiber $\pi^{-1}(b)$ and each fiber has the structure of a vector space over $\mathbb{R}$, we will write $F_b(\xi)$ for this fibre. Moreover, for each $b \in B$ there exists a neighborhood $U \in B$ of b, and a homeomorphism $\phi : U \times \mathbb{R}^n \to \pi^{-1}(U)$, such that for each $x \in U$ the restriction $\phi : \{x\} \times \mathbb{R}^n \to \pi^{-1}(x)$ is an isomorphism of vector spaces. The pair (U, ϕ) will be called a local coordinate system for ξ about b. We say ξ is n-dimensional if $\pi^{-1}(b) \simeq \mathbb{R}^n$ for all $b \in B$.

Next, we define maps between n-dimensional vector bundles over the same base space.

Definition 1.4.2. *Let $\xi = (E, \pi, B)$ and $\eta = (E', \pi', B')$ be two vector bundles. A bundle map $\eta \to \xi$ is pair of continuous functions $g : E' \to E$ and $\overline{g} : B' \to B$ such that the following diagram commutes*

$$\begin{array}{ccc} E' & \xrightarrow{g} & E \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ B' & \xrightarrow[\overline{g}]{} & B \end{array}$$

So $g(F_{b'}(\eta)) \subseteq F_{\overline{g}(b')}(\xi)$ for all $b' \in B'$. If ξ and η are over the same base space, i.e. $B = B'$ and $\overline{g}(b) = b$, we will say ξ is isomorphic to η if g is a homeomorphism, and it maps $(F_{b'}(\eta))$ isomorphically onto $F_{\overline{g}(b')}(\xi)$. We write $Vect^n(B)$ for the set of n-dimensional bundles over B.

We now provide the reader with a set of examples of vector bundles that we are going to use throughout this thesis.

Example 1.4.3. *The trivial bundle over B is given by $\varepsilon^n_B = (B \times \mathbb{R}^n, \pi, B)$ where $\pi : B \times \mathbb{R}^n \to B$ is the natural projection. More generally, we will say an n-dimensional vector bundle ξ over B is trivial if it is isomorphic to ε^n_B. Such an isomorphism is a* trivialization *of ξ.*

Example 1.4.4. *Let $M \subseteq \mathbb{R}^{n+k}$ be an n-dimensional manifold. The* tangent bundle *of M, denoted by τ_M, has $E(\tau_M) = \bigcup_{x \in M} T_xM$ with*

$$T_xM = \{(x, v) \mid v \in \mathbb{R}^{n+k} \text{ such that } v \text{ is tangent to } M \text{ at } x\}.$$

The space $E(\tau_M) \subseteq M \times \mathbb{R}^{n+k}$ has the subspace topology. The projection $\pi : E(\tau_M) \to M$ map is given by $\pi(x, v) = x$. Notice that $\pi^{-1}(x) = T_xM$. The vector space structure on each fibre T_xM is determined by

$$t_1(x, v_1) + t_2(x, v_2) = (x, t_1v_1 + t_2v_2)$$

where $t_1, t_2 \in \mathbb{R}$. The local triviality condition is satisfied.

Although we used an embedding to define the tangent bundle, it is possible to define the tangent bundle of a differentiable n-dimensional manifold M only using the charts of the differentiable structure. For a given point $x \in M$ with a local chart $\phi : U \xrightarrow{\cong} \mathbb{R}^n$ one identifies the tangent space $T_x M$ with the space spanned by differential operators

$$\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2} \cdots, \frac{\partial}{\partial x^n}.$$

In this way, it is possible to see that τ_M is an n-dimensional vector bundle and is independent of the embedding (we refer the reader to [B75, Chpater 2, Section 4] for more details). The space $E(\tau_M)$ also can be given the structure of a $2n$-dimensional differential manifold [MS74].

However, our definition is of a more geometrical nature. We note that it is always possible to define the tangent bundle of a given manifold in this way as according to Whitney's embedding theorem we always can embed a given manifold in Euclidean space as a submanifold.

Theorem 1.4.5. ***Whitney's embedding theorem*** *. If M^n is a compact n-manifold then there exists a smooth embedding $F : M^n \to \mathbb{R}^{2n+1}$.*

Proof. See [B, Theorem 10.7]. □

In the next proposition we study the pullback of a given vector bundle.

Proposition 1.4.6. *Given a map $f : B^{'} \longrightarrow B$ and a vector bundle ξ given by $\pi : E \longrightarrow B$ there exists a vector bundle $f^*\xi = (E(f^*\xi), \pi', B')$ and a map $\widehat{f} : f^*E \longrightarrow E$ taking the fiber $F_{b'}(f^*\xi)$ isomorphically onto $F_{f(b')}(\xi)$.*
Moreover, if $\eta = (E', p', B')$ is another vector bundle with a bundle map $\eta \to \xi$ then $\eta \cong f^*\xi$ as vector bundles over B', i.e. pullback of a vector bundle along f is unique up to isomorphism of vector bundles over B'.

Proof. We define $E(f^*\xi) = \{(b^{'}, e) \mid b^{'} \in B^{'}, e \in E, f(b^{'}) = \pi(e)\} \subseteq B' \times E$ with the subspace topology. We let $\pi^{'} : E(f^*\xi) \longrightarrow B^{'}$ be defined by $\pi'(b^{'}, e) = b^{'}$. There is a continuous map $\widehat{f} : E(f^*\xi) \longrightarrow E$ given by $\widehat{f}(b^{'}, e) = e$. Then we can check that the following diagram commutes.

$$\begin{array}{ccc} E(f^*\xi) & \xrightarrow{\widehat{f}} & E \\ {\scriptstyle \pi^{'}} \downarrow & & \downarrow {\scriptstyle \pi} \\ B^{'} & \xrightarrow[f]{} & B \end{array}$$

One can be more explicit about local trivializations in the constructed bundle $f^*\xi$. If E is trivial over a subspace $U \subseteq B$ then $f^*\xi$ is trivial over $f^{-1}(U)$ and so is a vector bundle. In particular, if ξ is a trivial bundle then so is $f^*\xi$. This can also be seen directly from the definition, which in the case $E = B \times \mathbb{R}^n$ just says that $f^*\xi$ consists of the triples $(b^{'}, b, e)$ in $B^{'} \times B \times \mathbb{R}^n$ with $b = f(b^{'})$, so we have just the product $B^{'} \times \mathbb{R}^n$.

To show uniqueness, let $\eta = (E', p', B')$ be another vector bundle satisfying the proposition, i.e. there exists a mapping $\widehat{f_1} : E' \to E$ such that

$$\begin{array}{ccc} E' & \xrightarrow{\widehat{f_1}} & E \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle \pi} \\ B' & \xrightarrow{f} & B \end{array}$$

and $\widehat{f_1}$ maps each fibre $F_{b'}(\eta)$ isomorphically onto $F_{f(b')}(\xi)$.

Define $h : E^{'} \to E(f^*\xi)$ by $h(e^{'}) = (p'(e^{'}), \widehat{f_1}(e^{'})) = (b^{'}, e)$, where $e^{'} \in E^{'}$. Since h is continuous and maps each fiber $F_{b'}(\eta)$ isomorphically onto the corresponding fiber $F_{b'}(f^*\xi)$ then h is an isomorphism of vector bundles. □

Remark 1.4.7. *From the uniqueness statement it follows that the isomorphism type of $f^*\xi$ depends only on the isomorphism type of the bundle ξ since we can compose the map $\widehat{f}$ with an isomorphism of E with another vector bundle over B. Thus we have a function $f^* : Vect^n(B) \longrightarrow Vect^n(B^{'})$ taking the isomorphism class of $E(\xi)$ to the isomorphism class of $E^{'}(\eta)$. Often the vector bundle $E^{'}(\eta)$ is written as $f^*\xi$ and called the* bundle induced *by f, or* the pullback *of ξ by f. A map $f : B^{'} \longrightarrow B$ gives rise to a function $f^* : Vect^n(B) \longrightarrow Vect^n(B^{'})$, in the reverse direction.*

Notice that by construction, if ξ is an n-dimensional vector bundle then $f^*\xi$ is also an n-dimensional vector bundle. We provide some easy examples.

Example 1.4.8. *The restriction of a vector bundle ξ over a subspace $A \subseteq B$ can be viewed as a pullback with respect to the inclusion map $A \hookrightarrow B$ since the inclusion $\pi^{-1}(A) \hookrightarrow E$ is certainly an isomorphism on each fiber.*

Example 1.4.9. *If $f : B^{'} \longrightarrow B$ is a constant map, having image a single point $b \in B$, then $f^*\xi = E(B^{'} \times \pi^{-1}(b))$ is a trivial bundle.*

Definition 1.4.10. *If ξ is a bundle over B and η is a bundle over B'. We define a bundle $\xi \times \eta$ over $B \times B'$ by*

$$E(\xi \times \eta) = E(\xi) \times E(\eta) \xrightarrow{\pi\times\pi'} B \times B' .$$

Then the Whitney sum *$\xi \oplus \eta$ of two bundles over B is defined to be $\Delta^*(\xi \times \eta)$ and is called Whitney sum of ξ and η where $\Delta : B \to B \times B$ is the diagonal map $\Delta(b) = (b, b)$. Note that each fiber $F_b(\xi\oplus\eta)$ is canonically isomorphic to $F_b(\xi)\oplus F_b(\eta)$.*

The notion of a pullback bundle has a nice homotopy property.

Theorem 1.4.11. *Let $f, g : B' \to B$ be two homotopic maps, with B' a para-compact space. Let ξ be a vector bundle over B. Then $f^*\xi$ and $g^*\xi$ are isomorphic as vector bundles over B'.*

Proof. See [D66, Theorem 4.7, Chapter 1, Section 4]. □

Definition 1.4.12. *Let ξ, η, and ζ be three vector bundles over a fixed base space B. A short exact sequence*

$$0 \longrightarrow \xi \xrightarrow{g} \eta \xrightarrow{h} \zeta \longrightarrow 0$$

of vector bundles over a fixed base space is given by bundle maps g and h where over each point $b \in B$ we obtain short exact sequences of vector spaces

$$0 \longrightarrow F_b(\xi) \xrightarrow{g} F_b(\eta) \xrightarrow{h} F_b(\zeta) \longrightarrow 0.$$

Now we can define the normal bundle of an immersion. Let $F : M^n \looparrowright N^{n+k}$ be an immersion. We then have two vector bundles over M^n, namely, τ_M and $F^*\tau_N$ where for each $x \in M$

$$\begin{aligned} E(\tau_M) &= \{(x, v) \mid v \in T_xM\}, \\ E(F^*\tau_N) &= \{(x, w) \mid w \in T_{F(x)}N\}. \end{aligned}$$

We define a bundle map from $\tau_M \to F^*\tau_N$ which on the level of total spaces is covered by $g : E(\tau_M) \to E(F^*\tau_N)$ with $g(x, v) = (x, dF_xv)$. Notice that F is an immersion, and therefore the restriction of g to each fibre $F_x(\tau_M) \to F_x(F^*\tau_N)$ is a monomorphism, since g is defined by dF. We then let the normal bundle of F, denoted by ν_F, be defined by the following exact sequence of vector bundles over M

$$0 \longrightarrow \tau_M \xrightarrow{g} F^*\tau_N \xrightarrow{h} \nu_F \longrightarrow 0.$$

More precisely, we have $\nu_F = (E(\nu_F), \pi, M)$ defined by

$$E(\nu_F) = \{(x, v) \mid x \in M, v \in F_x(F^*\tau_N)/F_x(\tau_M)\}$$

and the projection map is given by $\pi(x, v) = x$. Notice that over each point $x \in M$ the above short exact sequence is given by a short exact sequence of vector spaces over $\mathbb{R}$

$$0 \longrightarrow F_x(\tau_M) \xrightarrow{g} F_x(F^*\tau_N) \xrightarrow{h} F_x(\nu_F) \longrightarrow 0$$

which is split, i.e. over each point $x \in M$ we have $F_x(F^*\tau_N) \cong F_x(\tau_M) \oplus F_x(\nu_F)$. This implies that $E(\nu_F) \subseteq E(F^*\tau_N)$. We then give the subspace topology to $E(\nu_F)$.

If we assume that $N \subseteq \mathbb{R}^l$ for some large l, where $\mathbb{R}^l$ has the Euclidean inner product, then, τ_N has a natural Euclidean structure. Therefore, we may think of ν_F as given by

$$\{(x, v) \mid x \in M, v \in F^*\tau_N \text{ and } v \perp T_xM\}$$

over each point $x \in M$. In this case, each fibre ν_F at a point $x \in M$ is given by the orthogonal complement of T_xM, that is

$$F_x(\nu_F) = T_xM^\perp = \{(x, v) \mid v \in F^*\tau_N \text{ and } v \perp T_xM\}.$$

We then may view ν_F as the orthogonal complement of τ_M in $F^*\tau_N$.

Notice that each fibre $F_x(\nu_F)$ is defined as the quotient of two vector space, namely $F_x(\tau_M)$ and $F_x(F^*\tau_N)$. We then may say that ν_F is the quotient of $F^*\tau_N$ by τ_M. In fact for given a map of vector bundles $\xi \to \eta$ over the same base space with $\dim \xi \leqslant \dim \eta$ we can define a general construction as the quotient bundle η/ξ. We refer the reader to [D66] for more details. The following proposition collects these observations.

Definition 1.4.13. *If $M^n \looparrowright N^{n+k}$ is an immersion. The quotient bundle $\nu_F = F^*\tau_N/\tau_M$ is a k-dimensional bundle over M called the normal bundle of the immersion F.*

Proposition 1.4.14. *For any immersion $F : M \looparrowright N$, with N Riemannian, that is, its tangent bundle τ_N has a Euclidian structure, there is a* Whitney sum decomposition

$$F^*\tau_N \cong \tau_M \oplus \nu_F.$$

Proof. Since F is an immersion then $dF_x : T_xM \to T_{F(x)}N$ is a monomorphism. In addition, T_xM is isomorphic to $dF_x(T_xM)$. This implies that T_xM is a subvector space of $T_{F(x)}N$. However,

$$\begin{aligned} F_x(F^*\tau_N) &\cong x \times \{(y, v) \in \tau_N : y = F(x)\}, \\ &\cong \{(F(x), v) \mid v \in T_{F(x)}N\} \\ &\cong T_{F(x)}N. \end{aligned}$$

Hence, T_xM is a subvector space of $T_{F(x)}N$, Therefore τ_M is a subbundle of $F^*\tau_N$. Then we have $F^*\tau_N \cong \tau_M \oplus \nu_F$. □

In particular, when $F : M^n \to \mathbb{R}^{n+k}$ we have

$$F^*\tau_{(\mathbb{R}^{n+k})} = \tau_M \oplus \nu_F.$$

Since $\tau_{(\mathbb{R}^{n+k})}$ is trivial, then $F^*\tau_{(\mathbb{R}^{n+k})}$ is trivial, i.e. $\tau_M \oplus \nu_F = \varepsilon_M^{n+k}$.

We note that the notion of the 'orthogonal complement' of 'tangent bundle' can be generalized as follows. Suppose $\eta = (E, \pi, B)$ is a vector bundle which possesses a Riemannian metric, as explained in Section 1.5. Let ξ be a sub-bundle of η, i.e. $\xi = (E', \pi', B)$ where $E' \subseteq E$ and π' is just given by the restriction of π on E'. In particular, we have $F_b(\xi) \subseteq F_b(\eta)$. We also have the orthogonal complement of $F_b(\xi)$ inside $F_b(\eta)$, defined by

$$F_b(\xi)^\perp = \{(b, v) \in F_b(\eta) \mid \langle v, w\rangle_b = 0 \text{ for all } w \in F_b(\eta)\}$$

where $\langle -, -\rangle_b$ denotes the inner product on $F_b(\eta)$ coming from the Riemannian structure on η. We then define the vector bundle $\xi^\perp$ as a sub-bundle of η, to have

$$E(\xi^\perp) = \bigcup_{b \in B} F_b(\xi)^\perp$$

and the projection map $E(\xi^{\perp}) \longrightarrow B$ is given by $(b, v) \longmapsto b$. The local triviality of this bundle and the vector space structure on each fibre are inherited from η. We call $\xi^{\perp}$ the orthogonal complement of ξ in η.

Theorem 1.4.15. *$E(\xi^{\perp})$ is the total space of a sub-bundle $\xi^{\perp} \subseteq \eta$. Furthermore η is isomorphic to the Whitney sum $\xi \oplus \xi^{\perp}$.*

Proof. See [MS74, Thm 2.10]. □

Remark 1.4.16. *From Proposition 1.4.14, since τ_M is a sub-bundle of $F^*\tau_N$. Then by Theorem 1.4.15 we have $F^*\tau_N \cong \tau_M \oplus \tau_M^{\perp}$. Hence the normal bundle of F, $\nu_F \cong \tau_M^{\perp}$.*

Corollary 1.4.17. *A cross section of τ_M is called* a (tangent) vector field *on M, and a cross section of ν_M is called* a (normal) vector field *on M.*

Finally, we describe the 'universal bundles' and show that any vector bundle over a para-compact space is the pullback of a universal bundle. First we recall definition of the Stiefel and Grassmann manifolds.

Before defining the Stiefel space let us mention that an n-frame in $\mathbb{R}^{n+k}$ is an n-tuple of linearly independent vectors of $\mathbb{R}^{n+k}$.

Definition 1.4.18. (1) *The Stiefel manifold $V_n(\mathbb{R}^{n+k})$ is the set of all orthonormal n-frames in $\mathbb{R}^{n+k}$ i.e.*

$$V_n(\mathbb{R}^{n+k}) = \{(\upsilon_1, \ldots, \upsilon_n) \ \mid \upsilon_i \in S^{n+k-1}, \upsilon_i \cdot \upsilon_j = \delta_{ij}\},$$

where δ_{ij} is Kronecker's delta function.

(2) *The Grassmann space of n-dimensional subspaces of $\mathbb{R}^{n+k}$, denoted by $G_n(\mathbb{R}^{n+k})$, is the set of all n-dimensional vector subspace of $\mathbb{R}^{n+k}$, that is n-dimensional planes in $\mathbb{R}^{n+k}$ passing through the origin.*

Notice that because $V_n(\mathbb{R}^{n+k})$ is a closed subset of the compact space $(S^{n+k-1})^n$, it is a compact space when it is topologized with the subspace topology. Moreover, according to the above definition, there is a natural surjection $V_n(\mathbb{R}^{n+k}) \longrightarrow G_n(\mathbb{R}^{n+k})$ sending an n-frame to the subspace it spans, and $G_n(\mathbb{R}^{n+k})$ is topologized by giving it the quotient topology with respect to this surjection. So $G_n(\mathbb{R}^{n+k})$ is compact as well. We also record the following fact.

Lemma 1.4.19. *The Grassmann manifold $G_n(\mathbb{R}^{n+k})$ is a compact topological manifold of dimension nk. The correspondence $V \longrightarrow V^{\perp}$, which assigns to each n-plane its orthogonal n-plane, defines a homeomorphism between $G_n(\mathbb{R}^{n+k})$ and $G_k(\mathbb{R}^{n+k})$.*

Proof. See [MS74, Lemma 5.1]. □

Next, we introduce the notion of infinite dimensional Grassmann spaces. Notice that there are natural inclusions $\mathbb{R}^m \to \mathbb{R}^{m+1}$ which allows to think of a k-plane in $\mathbb{R}^m$ as a k-plane in $\mathbb{R}^{m+1}$. Using these inclusions we obtain inclusion maps $G_n(\mathbb{R}^{n+k}) \to G_n(\mathbb{R}^{n+k+1})$. We now can formulate the definition of infinite dimensional Grassmannians.

Definition 1.4.20. *The infinite dimensional Grassmann manifold of n-planes in $\mathbb{R}^\infty$, is the set of all n-dimensional linear subspace of $\mathbb{R}^\infty$*

$$BO(n) = G_n(\mathbb{R}^\infty) = \lim_{k\to\infty} G_n(\mathbb{R}^{n+k+1}) = \bigcup_{k=0}^{\infty} G_n(\mathbb{R}^{n+k})$$

and is topologized by the weak topology, i.e. a set in $G_n(\mathbb{R}^\infty)$ is open (or closed) if and only if it intersects each $G_n(\mathbb{R}^{n+k})$ in an open (or closed) set. Here the direct limit is taken over the natural inclusions $G_n(\mathbb{R}^{n+k}) \to G_n(\mathbb{R}^{n+k+1})$ as $k \to \infty$.

The space $BO(n)$ is a limit of compact spaces, and in particular, it is paracompact [MS74, Corollary p. 66].

Remark 1.4.21. *We note that by analogy one can define the complex Grassmannian manifold $G_n(\mathbb{C}^{n+k})$ to be the space of all complex n-dimensional subspaces of $\mathbb{C}^{n+k}$. Similarly, we have the Grassmannian manifold of complex n-dimensional subspaces in $\mathbb{C}^\infty$ which we denote by $BU(n)$.*

Note that when $n = 1$ we have $G_1(\mathbb{R}^{1+k}) = \mathbb{R}P^k$, and $V_1(\mathbb{R}^{1+k}) = S^k$. In particular, we have

$$BO(1) = G_1(\mathbb{R}^\infty) = \mathbb{R}P^\infty, \quad BU(1) = G_1(\mathbb{C}^\infty) = \mathbb{C}P^\infty.$$

Let A_n denote the set of all $n \times n$ real matrices. The set of all non-singular matrices is the general linear group $GL(n,\mathbb{R})$, which is an open subset of A_n. We have the group $O(n)$ of orthogonal real matrices which is a subgroup of $GL(n, \mathbb{R})$ and then the group $O(n-1)$ is a subgroup of $O(n)$. From these groups we can construct the Stiefel space as

$$V_n(\mathbb{R}^{n+k}) \cong O(n+k)/O(k).$$

This description is useful, when we consider the generalized J-homomorphism in Chapter 7. We also have

$$G_n(\mathbb{R}^{n+k}) \cong V_n(\mathbb{R}^{n+k})/O(n).$$

Now we are ready to introduce the 'universal bundles'. First, we look at the canonical bundles over finite dimensional Grassmann manifolds.

Example 1.4.22. *The canonical n-vector bundle $\gamma_k^n(\mathbb{R}^{n+k})$ over $G_n(\mathbb{R}^{n+k})$ has the total space*

$$E(\gamma_k^n(\mathbb{R}^{n+k})) = E(\gamma_k^n) = \{(X, x) \mid X \in G_n(\mathbb{R}^{n+k}), x \in X\}.$$

This is to be topologized as a subset of $G_n(\mathbb{R}^{n+k}) \times \mathbb{R}^{n+k}$. The projection map $\pi : E(\gamma_k^n) \longrightarrow G_n(\mathbb{R}^{n+k})$ is defined by $\pi(X, x) = X$.

Lemma 1.4.23. *$\gamma_k^n(\mathbb{R}^{n+k})$ is n-dimensional vector bundle.*

Proof. See [H03, Lemma 1.15]. □

In the special case of $n = 1$, γ_k^1 is called the canonical line bundle.

Next, we define the Gauss map which tells us how to classify a given bundle over a para-compact space.

Given a smooth n-manifold $M \subseteq \mathbb{R}^{n+k}$ the generalized Gauss map

$$\overline{g} : M \longrightarrow G_n(\mathbb{R}^{n+k})$$

can be defined as the function which carries each $x \in M$ to its tangent space $T_xM \in G_n(\mathbb{R}^{n+k})$. In the following diagram

$$\begin{array}{ccc} E(\tau_M) & \xrightarrow{g} & E(\gamma_k^n) \\ \downarrow & & \downarrow \\ M & \xrightarrow[\overline{g}]{} & G_n(\mathbb{R}^{n+k}) \end{array}$$

$\overline{g}$ is covered by a bundle map

$$g : E(\tau_M) \longrightarrow E(\gamma_k^n),$$

where $g(x, v) = (T_xM, v)$. We will use the abbreviated notation

$$g : \tau_M \longrightarrow \gamma_k^n$$

for this bundle map. From the uniqueness of pullbacks in Proposition 1.4.6, we deduce that that $\tau_M \cong \overline{g}^*\gamma_k^n$. Not only tangent bundles, but all other n-vector bundles over a para-compact space can be mapped into the bundle γ_k^n providing that k is sufficiently large. For this reason γ^n over $BO(n)$ is called the "*universal bundle*".

The universal bundles are analogous to the canonical bundles, but they are defined over the infinite dimensional Grassmann spaces $BO(n)$.

Example 1.4.24. *The universal n-plane bundle γ^n over $BO(n)$ has the total space*

$$EO(n) = E(\gamma^n) = \{(X, x) \mid X \in BO(n), x \in X\}.$$

This is to be topologized as a subset of $BO(n) \times \mathbb{R}^\infty$. The projection map $\pi : EO(n) \longrightarrow BO(n)$ is defined by $\pi(X, x) = X$.

Lemma 1.4.25. *The bundle γ^n satisfies the local triviality condition.*

Proof. See [MS74, Lemma 5.4]. □

In this case the Gauss map

$$\overline{g} : M \longrightarrow G_n(\mathbb{R}^\infty) = BO(n)$$

will be defined as the map which carries each $x \in M$ to its tangent space in $BO(n)$. This is covered by a bundle map $g : E(\tau_M) \longrightarrow EO(n)$, where $g(x, v) = (T_xM, v)$. The covering means that $(\overline{g})^*\gamma^n$, the induced bundle by $\overline{g}$ on M is isomorphic to τ_M.

Now we turn back to the general case.

Theorem 1.4.26. *Any n-vector bundle ξ over a para-compact base space admits a bundle map $\xi \longrightarrow \gamma^n$.*

Proof. See [MS74, Theorem 5.6]. □

Two bundle maps, $f, g : \xi \longrightarrow \gamma^n$ are called bundle-homotopic if there exists a one-parameter family of bundle maps

$$h_t : \xi \to \gamma^n, 0 \leq t \leq 1,$$

with $h_0 = f, h_1 = g$, such that h_t is continuous as a function in both variables. In other words the associated function

$$h : E(\xi) \times [0,1] \to E(\gamma^n)$$

must be continuous.

Theorem 1.4.27. *Any two bundle maps from an n-vector bundle to γ^n are bundle-homotopic.*

Proof. See [MS74, Theorem 5.7]. □

Corollary 1.4.28. *Any n-vector bundle ξ over a para-compact space B determines a unique homotopy class of maps*

$$f : B \longrightarrow BO(n).$$

Proof. Let $F : \xi \longrightarrow \gamma^n$ be any bundle map, and let f be the induced map of base spaces. □

Notice that the above theorems together with Proposition 1.4.6 imply that given any n-vector bundle $\xi = (E, \pi, B)$ there exists a map $f : B \to BO(n)$, unique up to homotopy, such that $\xi = f^*\gamma^n$.

Definition 1.4.29. *The mapping $f : B \to BO(n)$ is called the classifying map for the vector bundle ξ.*

Recall the notation $[B, G_n(\mathbb{R}^{n+k})]$ for the set of homotopy classes of maps

$$f : B \longrightarrow G_n(\mathbb{R}^{n+k}).$$

Theorem 1.4.30. *For para-compact B, the map $[B, BO(n)] \longrightarrow Vect^n(B)$, $[f] \longmapsto f^*(\gamma^n)$, is a bijection.*

Proof. Given $f : B \longrightarrow BO(n)$, this corresponds to the pullback $f^*\gamma^n$, thus vector bundles over a fixed base space are classified by homotopy classes of maps into $BO(n)$.

From last four theorems we can say that if ξ is an n-dimensional bundle then there is a continuous map $f : B \longrightarrow BO(n)$ such that $\xi \cong f^*\gamma^n$. Furthermore, $f_0^*\gamma^n \cong f_1^*\gamma^n \Leftrightarrow f_0 \simeq f_1$. This means that

$$Vect^n(B) \longleftrightarrow [B, BO(n)].$$

□

Now, we want to use the notion of the classifying map for vector bundles together with the uniqueness of pullback bundles to show that the isomorphism class of the normal bundle only depends on the regular homotopy class of the given immersion.

Theorem 1.4.31. *Suppose $F_0, F_1 : M^n \looparrowright N^{n+k}$ are two regularly homotopic immersions in N^{n+k}. Then*

$$\nu_{F_0} \cong \nu_{F_1}.$$

Proof. Let F_0 and $F_1 : M \looparrowright N$ be regular homotopic immersions with a regular homotopy $F : M \times I \to N$. Let $G : M \times I \to N \times I$ be the map $G(x,t) = (F(x,t), t)$. Then G is an immersion, the normal bundle of G restricted to $M \times 0$ gives the normal bundle of F_0 and the normal bundle of G restricted to $M \times 1$ gives the normal bundle of F_1.

Let $g : M \times I \to BO(k)$ be the classifying map for the normal bundle of G. Then, if $i_0 : M \to M \times I$ is the map $i_0(x) = (x, 0)$ and $i_1 : M \to M \times I$ is the map $i_1(x) = (x, 1)$, then $g \circ i_0$ is the classifying map for the normal bundle of F_0 and $g \circ i_1$ is the classifying map for the normal bundle of F_1.

However, i_0 is homotopic to i_1, the homotopy is the identity map $M \times I \to M \times I$ and so $g \circ i_0$ is homotopic to $g \circ i_1$. Hence the normal bundle of F_0 is isomorphic to the normal bundle of F_1 (since homotopic maps to $BO(k)$ correspond to isomorphic bundles). □

We also have the following observation which is a comparison between the isomorphism classes of vector bundles over two different base spaces.

Theorem 1.4.32. *A homotopy equivalence $f : B' \longrightarrow B$ of para-compact spaces induces a bijection*

$$f^* : Vect^n(B) \longrightarrow Vect^n(B').$$

In particular, every vector bundle over a contractible para-compact base space is trivial.

Proof. If g^* is a homotopy inverse of f^* then we have

$$f^* g^* \cong \mathbf{1}^* \cong \mathbf{1}$$

and

$$g^* f^* \cong \mathbf{1}^* \cong \mathbf{1}.$$

□

Remark 1.4.33. *If we have an immersion $M^n \looparrowright N^{n+k}$ then τ_M is an n-dimensional vector bundle, i.e. ν_F is a k-dimensional vector bundle over M. Hence, since our manifolds are compact, then we have a unique map $\nu(F) : M \longrightarrow BO(k)$ which classifies the normal bundle ν_F. This is called* normal map *of the immersion.*

1.5 Suspension and Thom spaces

Throughout this section we consider those n-vector bundles $\xi = (E, \pi, B)$ which possess a Riemannian metric. This means that on each fibre $F_b(\xi) \cong \mathbb{R}^n$ there is a positive definite Riemannian product, that is an inner product

$$\langle -, - \rangle_b : F_b(\xi) \times F_b(\xi) \longrightarrow \mathbb{R}$$

for each $b \in B$ where $\mathbb{R}$ is the set of all nonnegative real numbers. In this case, we can make sense of the length of a vector $x \in F_b(\xi)$ denoted by $|x|$ where we define

$$|x|^2 = \langle x, x \rangle_b.$$

We note that in our calculations in this thesis we are dealing with compact manifolds which always possess a Riemannian metric on their tangent bundle.

Definition 1.5.1. *For a vector bundle ξ, having the base space B and total space E,* the disc bundle *of ξ is defined by $D(\xi) = \{x \in E(\xi) \mid |x| \leq 1\}$, that consists of all vectors in $E(\xi)$ of length ≤ 1.*

Definition 1.5.2. *For a vector bundle ξ,* the sphere bundle *of ξ is defined by $S(\xi) = \{x \in E(\xi) \mid |x| = 1\}$, that consists of all vectors in $E(\xi)$ of length $= 1$.*

Definition 1.5.3. The Thom space *of a real vector bundle ξ, denoted by $T(\xi)$, is the quotient space $D(\xi)/S(\xi)$.*

Remark 1.5.4. *Let $\pi : E \to B$ be n-dimensional real vector bundle over the compact space B. Then for each point b in B, the fiber F_b is n-dimensional real vector space. We can form an associated sphere bundle $Sph(\xi)$ by taking the one-point compactification of each fiber separately. Finally, from the total space $E(\xi)$ we may obtain the* Thom space *$T(\xi)$ by identifying all the new points to a single point ∞, which we take as the basepoint of $T(\xi)$.*

Proposition 1.5.5. *If ξ is a real vector bundle with a compact base space, $T(\xi)$ is homeomorphic to the one-point compactification of $E(\xi)$, i.e. $T(\xi) \equiv E(\xi)_+$, where $E(\xi)_+$ is the one-point compactification of $E(\xi)$.*

Proof. Observe that $D(\xi) - S(\xi)$ and $E(\xi)$ are homeomorphic. The one-point compactification of $D(\xi) - S(\xi)$, and the one-point compactification of $E(\xi)$ are homeomorphic, [G75]. $E(\xi)_+ \cong (D(\xi) - S(\xi))_+ \cong D(\xi)/S(\xi) \cong T(\xi)$. □

Proposition 1.5.6. *Let ξ and η be two real vector bundles over compact spaces. then the Thom space $T(\xi \times \eta)$ and the space $T(\xi) \wedge T(\eta)$ are homeomorphic.*

Proof. By using Proposition 1.5.5, the Thom space $T(\xi \times \eta)$ is the one-point compactification of $E(\xi \times \eta)$. However $E(\xi \times \eta) = E(\xi) \times E(\eta)$, and so $E(\xi \times \eta)_+ \cong E(\xi)_+ \wedge E(\eta)_+$. Hence $T(\xi \times \eta) = T(\xi) \wedge T(\eta)$. □

Theorem 1.5.7. *The Thom space $T(\xi \oplus \varepsilon^n)$ is homeomorphic to the suspension $\Sigma^n(T(\xi))$.*

Proof. Notice that $\xi \oplus \varepsilon^n$ isomorphic to $\xi \times \mathbb{R}^n$, where $\mathbb{R}^n$ is the n-dimensional real vector bundle over a point. Then by Proposition 1.5.5 we have

$$T(\xi \oplus \varepsilon^n) \cong T(\xi \times \mathbb{R}^n) \cong T(\xi) \wedge T(\mathbb{R}^n).$$

Hence $T(\xi) \wedge \mathbb{R}^n_+ = T(\xi) \wedge S^n \cong \Sigma^n T(\xi)$. □

Example 1.5.8. *Suppose that ε^n_X is the n-dimensional trivial vector bundle over X. Then $E(\varepsilon^n_X) = X \times \mathbb{R}^n$, where $E(\varepsilon^n_X)$ is the total space of ε^n_X. Then we have $T(\varepsilon^n_X) \cong \Sigma^n(X_+)$. In particular, if ε^n_* is the trivial n-dimensional vector bundle over a point $*$, then $T(\varepsilon^n_*) \cong (\{*\} \times \mathbb{R}^n)_+ \cong S^0 \wedge S^n \cong S^n$.*

Finally, we record one of the important properties of Thom complexes.

Theorem 1.5.9. ***Thom Isomorphism**. Let ξ be a k-dimensional vector bundle over B and $T(\xi)$ the related Thom space then $H^n(B) \cong \widetilde{H}^{n+k}(T(\xi))$ and $H_n(B) \cong \widetilde{H}_{n+k}(T(\xi))$ where the homology and cohomology groups have $\mathbb{Z}/2$-coefficients.*

Proof. See [D66, Theorem 16.10.3]. □

Remark 1.5.10. *The Thom isomorphism is natural in the sense that if $\xi \to \xi'$ is a map of Euclidean bundles then $(T(\overline{F}))^* \circ \phi' = \phi \circ (\overline{F})^*$ where $T(\overline{F}) : T(\xi) \to T(\xi')$ is the induced map of Thom complexes, $\overline{F} : B \to B'$ is the map of base spaces and ϕ, ϕ' denote the appropriate Thom isomorphisms.*

1.6 Stiefel-Whitney classes

In this section we introduce some algebraic invariants associated with vector bundles. These will be useful when we describe a systematic way to determine cobordism classes of given immersions.

First of all we will introduce four axioms which characterize the Stiefel-Whitney cohomology classes of a vector bundle. The coefficient group will be $\mathbb{Z}/2$, the group of integers modulo 2, and we write H^*X for $H^*(X;\mathbb{Z}/2)$ and H_*X for $H_*(X;\mathbb{Z}/2)$.

Theorem 1.6.1. ***Axiom*** 1. *To each vector bundle ξ there corresponds a sequence of cohomology classes,*

$$w_i(\xi) \in H^i B(\xi), \text{ for } i = 0, 1, 2, \ldots,$$

called the Stiefel-Whitney classes *of ξ. The class $w_0(\xi) = 1 \in H^0 B(\xi)$ and $w_i(\xi) = 0$ for $i > n$ if ξ is an n-plane bundle.*

Axiom 2. Naturality. *If $f : B(\xi) \to B(\eta)$ is covered by a bundle map from ξ to η, then*

$$w_i(\xi) = f^* w_i(\eta).$$

Axiom 3. The Whitney Product Theorem. *If ξ and η are vector bundles over the same base space, then*

$$w_k(\xi \oplus \eta) = \sum_{i=0}^{k} w_i(\xi) \cup w_{k-i}(\eta)$$

where $\cup$ (which we will omit when it is clear) denotes the cup product.

Axiom 4. *For the line bundle γ_1^1 over the circle $\mathbb{R}P^1$, the Stiefel-Whitney class $w_1(\gamma_1^1) \in H^1 \mathbb{R}P^1 \cong \mathbb{Z}/2$, is non-zero.*

In the case when $\xi = \varepsilon_B^n$ is the trivial bundle over B, we have $w_i(\xi) = 0$ for all $i > 0$ [MS74, Proposition 2, Chapter 4]. For the special case of the universal n-plane bundle we have the following.

Theorem 1.6.2. *$H^* BO(k) \cong \mathbb{Z}/2[w_1, w_2, \ldots w_k]$ a polynomial ring with coeffcients in $\mathbb{Z}/2$, where $w_i \in H^i(BO(k))$ is the i-th universal Stiefel whitney class, has dimension $|w_i| = i$.*

Proof. See [MS74, Theorem 7.1]. □

Remark 1.6.3. *In the special case of the universal bundle γ^k the Thom isomorphism $H^n BO(k) \cong \widetilde{H}^{n+k} MO(k)$ is given by $w^I \leftrightarrow w^I w_k$, since $w_k \in H^k BO(k) \cong \mathbb{Z}/2$ is the Thom class. (See [D66, Theorem 16.10.3]).*

Remark 1.6.4. *Given any n-plane bundle ξ, over a para-compact base space, it is classified by a map $f : B(\xi) \longrightarrow BO(n)$. By Axiom 2, we have*

$$w_i(\xi) = f^*(w_i).$$

The total Stiefel-Whitney class of an n-dimensional vector bundle ξ over B is defined to be the element

$$w(\xi) = 1 + w_1(\xi) + \cdots + w_n(\xi)$$

of the ring H^*B. In particular, $w(\varepsilon_B^n) = 1$ for all $n > 0$. The Whitney product theorem can now be expressed by the simple formula

$$w(\xi \oplus \eta) = w(\xi)w(\eta).$$

Let ξ be an n-dimensional vector bundle over a compact space B. Suppose that there is an m-dimensional vector bundle η over B such that $\xi \oplus \eta$ is a trivial bundle ε_B^{n+m}. Since $w_k(\varepsilon_B^{n+m}) = 0$ for $k > 0$, then by the Whitney Product Theorem for $k > 0$

$$\sum_{i=0}^{k} w_i(\xi)w_{k-i}(\eta) = 0.$$

Therefore if the Stiefel-Whitney classes of the bundle ξ are known then we can compute the Stiefel-Whitney classes of bundle η from the above formula. We have $w(\xi)w(\eta) = 1$ and so $w(\eta)$ is the multiplicative inverse of $w(\xi)$ which is often written $\overline{w}(\xi)$.

Suppose a differentiable manifold M^n has an immersion in Euclidean space $\mathbb{R}^{n+k}$. Then the immersion has a k-dimensional normal bundle ν such that $\tau_M \oplus \nu \cong \varepsilon^{n+k}$ by Proposition 1.4.14. Since $w(\varepsilon^{n+k}) = 1$, then $w(\tau_M)w(\nu) = 1$, so $w(\nu)$ is the formal inverse of $w(\tau_M)$. We often write $w(M^n) = w(\tau_M)$, and call $w(M^n)$ the total Stiefel-Whitney class of M^n; then $w(\nu)$ is called *the dual Stiefel-Whitney class* and is denoted by $\overline{w}(M^n)$. Notic that $\overline{w}_r(M^n) = 0$ for $r > k$. This give us the following result.

Lemma 1.6.5. ***Whitney duality theorem*** . *If τ_M is the tangent bundle of a manifold in Euclidean space and ν is the normal bundle then $w(\tau_M)w(\nu) = 1$ or (equivalently) $w_i(\nu) = \overline{w_i}(\tau_M)$.*

We provide some examples which illustrate how the Steifel-Whitney classes can be calculated.

Lemma 1.6.6. *For the tangent bundle τ_{S^n} of the sphere S^n, the class $w(\tau_{S^n}) = w(\nu_{S^n})$ is equal to 1.*

Proof. For the standard embedding $S^n \subseteq \mathbb{R}^{n+1}$,the normal bundle ν is trivial. Hence $w_i(\nu_{S^n}) = 0$ for $i > 0$, and $w_0(\nu_{S^n}) = 1$, then $w(\nu_{S^n}) = 1$. Therefore from Theorem 1.6.5, we deduce that $w(\tau_{S^n}) = 1$. □

Lemma 1.6.7. *The group $H^i\mathbb{R}P^n$ is cyclic of order 2 for $0 \leq i \leq n$ and is zero for higher values of i. Furthermore, if a denotes the non-zero element of $H^1\mathbb{R}P^n$ then each $H^i\mathbb{R}P^n$ is generated by the i-fold cup product a^i.*

Proof. See [D66, Lemma 4.3]. □

Thus $H^*\mathbb{R}P^n$ can be described as the algebra with unit over $\mathbb{Z}/2$ having one generator a and one relation $a^{n+1} = 0$.

Remark 1.6.8. *[MS74, Remark p.42]. For the canonical map $f : S^n \to \mathbb{R}P^n$, this lemma can be used to compute the homomorphism*

$$f^* : H^n\mathbb{R}P^n \to H^nS^n$$

providing that $n > 1$. In fact

$$f^*(a^n) = (f^*a)^n = 0$$

since $f^*a \in H^1S^n = 0$.

Example 1.6.9. *The total Stiefel-Whitney class of the canonical line bundle γ_n^1 over $\mathbb{R}P^n$ is given by*

$$w(\gamma_n^1) = 1 + a.$$

Proof. The standard inclusion $j : \mathbb{R}P^1 \to \mathbb{R}P^n$ is clearly covered by a bundle map from γ_1^1 to γ_n^1. Therefore

$$j^*w_1(\gamma_n^1) = w_1(\gamma_1^1) \neq 0.$$

This shows that $w_1(\gamma_n^1)$ cannot be zero, hence must be equal to a. The remaining Stiefel-Whitney classes of γ_n^1 are determined by Axiom 2. □

Theorem 1.6.10. *The Whitney sum* $\tau_{\mathbb{R}P^n} \oplus \varepsilon^1$ *is isomorphic to the* $(n+1)$*-fold Whitney sum* $\gamma_n^1 \oplus \gamma_n^1 \oplus \ldots \oplus \gamma_n^1$*. Hence the total Stiefel-Whitney class of* $\mathbb{R}P^n$ *is given by*

$$w(\mathbb{R}P^n) = (1+a)^{n+1} = 1 + \binom{n+1}{1} a + \binom{n+1}{2} a^2 + \cdots + \binom{n+1}{n} a^n,$$

where a *is the generator of* $H^1\mathbb{R}P^n$*.*

Proof. See [D66, Theorem 4.5]. □

Remark 1.6.11. *The total Stiefel-Whitney class of the normal bundle of* $\mathbb{R}P^n$ *is given by*

$$w(\nu_{\mathbb{R}P^n}) = \overline{w}(\tau_{\mathbb{R}P^n}) = \overline{(1+a)^{n+1}} = (1+a)^{-n-1}.$$

Theorem 1.6.12. *Let* M^n *be a manifold. If* M^n *can be immersed in* $\mathbb{R}^{n+k}$*, then* $\overline{w}_i(M) = 0$ *for* $i > k$*. If* M^n *can be embedded in* $\mathbb{R}^{n+k}$*, then* $\overline{w}_i(M) = 0$ *for* $i \geq k$*.*

Proof. See [E81, Theorem 17.10.2]. □

Example 1.6.13. *If the* 9*-dimensional manifold* $\mathbb{R}P^9$ *can be immersed in* $\mathbb{R}^{9+k}$*, then* $\tau_{\mathbb{R}P^9} \oplus \nu_{\mathbb{R}P^9} = \varepsilon^{9+k}$*. Using Theorem 1.6.10 we find that*

$$w(\mathbb{R}P^9) = (1+a)^{10} = 1 + a^2 + a^8 + a^{10} = 1 + a^2 + a^8,$$

because $a^{10} \in H^{10}(\mathbb{R}P^9) = 0$ *and then*

$$w(\tau_{\mathbb{R}P^9}) \cdot w(\varepsilon) = w(\tau_{\mathbb{R}P^9}) \cdot 1 = w(\gamma_9^1)^{10}.$$

Hence

$$w(\nu^k_{\mathbb{R}P^9}) = \overline{(1 + a^2 + a^8)} = 1 + a^2 + a^4 + a^6.$$

In particular, $w_6(\nu_{\mathbb{R}P^9}) \neq 0$*. Hence* $\dim(\nu_{\mathbb{R}P^9}) \geq 6$ *and so* $k \geq 6$*.*

Proposition 1.6.14. $\mathbb{R}P^n$ *can be embedded in* $\mathbb{R}^{n+1}$ *only if* $n = 2^r - 1$ *for some* r *and can be immersed in* $\mathbb{R}^{n+1}$ *only if* $n = 2^r - 1$ *or* $n = 2^r - 2$*. If* $n = 2^r$ *then there is no immersion of* $\mathbb{R}P^n$ *in* $\mathbb{R}^{2n-2}$ *and no embedding in* $\mathbb{R}^{2n-1}$*.*

Proof. We know that if $\mathbb{R}P^n$ can be immersed in $\mathbb{R}^{n+1}$ then $\overline{w}(\mathbb{R}P^n) = 1$ or $\overline{w}(\mathbb{R}P^n) = 1 + a$. In the first case since $w(\mathbb{R}P^n) = (1+a)^{n+1}$, by Theorem 1.6.10 we would have $(1+a)^{n+1} = 1$, which implies $n + 1 = 2^r$, for some r.

In the latter case $(1 + a)^{n+2} = 1$, which implies $n + 2 = 2^r$. The former case must hold if $\mathbb{R}P^n$ can be embedded. If $n = 2^r$, then

$$\overline{w}(\mathbb{R}P^n) = (1+a)^{-(n+1)} = (1+a)^{-n}(1+a)^{-1} = (1+a^n)\cdot(1+a+\cdots+a^n) = 1+a+\cdots+a^{n-1},$$

and so the final statement follows from Prop.1.6.14. □

Remark 1.6.15. *Whitney showed that any differentiable n-manifold can be immersed in $\mathbb{R}^{2n-1}$ and embedded in $\mathbb{R}^{2n}$.[M58, 1.32]*

1.7 Chern classes

In this section we define the Chern classes of complex vector bundles, cohomology groups have coefficients $\mathbb{Z}$.

Theorem 1.7.1. *[D66]. For each complex vector bundle ξ over a space B there are classes $c_i(\xi) \in H^{2i}(B;\mathbb{Z})$ with the following properties:*

(1) $c_0(\xi) = 1 \in H^0B$ and $c_i(\xi) = 0$ for $i > \dim \xi$;

(2) if ξ and η are isomorphic, it follows that $c(\xi) = c(\eta)$, and if $f : B_1 \longrightarrow B$ is a map, then we have $f^*(c(\xi)) = c(f^*(\xi))$;

(3) for vector bundles ξ and η over B, the relation $c(\xi \oplus \eta) = c(\xi)c(\eta)$ (cup multiplication) holds;

(4) for the canonical line bundle λ_1^1 over $S^2 = \mathbb{C}P^1$, the element $c_1(\lambda_1^1)$ is a generator of $H^2(S^2,\mathbb{Z})$;

(5) for the canonical line bundle λ^1 over $\mathbb{C}P^\infty$, the element $c_1(\lambda^1)$ is a generator b of the polynomial ring $H^*(\mathbb{C}P^\infty;\mathbb{Z})$.

Definition 1.7.2. *Let ξ be a complex n-vector bundle over B. For $i \leq n$, we define the total Chern class of ξ denoted $c(\xi) \in H^*(B(\xi);\mathbb{Z})$ as follows,*

$$c(\xi) = 1 + c_1(\xi) + \ldots + c_n(\xi).$$

Proposition 1.7.3. *For the tangent bundle $\tau_{\mathbb{C}P^n}$ there is the relation $c(\mathbb{C}P^n) = (1+b)^{n+1}$, where b is a generator of $H^2(\mathbb{C}P^n;\mathbb{Z})$ [D66, Prop. 4.5].*

We have the following relation between w_2 and c_1.

Lemma 1.7.4. *Given a complex vector bundle ξ (over a para-compact space B) then $w_2(\xi) \in H^2B$ is the mod 2 restriction of $c_1(\xi) \in H^2(B;\mathbb{Z})$.*

Proof. See [D66, Cor.11.5] □

Chapter 2

Self-transversality and multiple point of immersions

Let $F : M^m \to N^n$ be a smooth map of manifolds, in order to have $F^{-1}(y)$ as a submanifold of M, we need y to be a regular value of F. Firstly, we will explain briefly how smooth maps pull regular values back to submanifolds.

2.1 Regular values and Sard's theorem

Definition 2.1.1. *Let $F : M^m \to N^n$ be a smooth map of manifolds.*

(1) $p \in M$ is a critical point *of F if $dF_p : T_p(M) \to T_{F(p)}(N)$ has rank $< n$.*

(2) $p \in M$ is a regular point *of F if $dF_p : T_p(M) \to T_{F(p)}(N)$ has rank $= n$.*

(3) $q \in N$ a critical value *of F if $q = F(p)$ is the image of critical point p.*

(4) $q \in N$ is a regular value *of F when $F(p) = q$, this implies that p is regular point. So if $q \notin F(M)$, then q is a regular value.*

Definition 2.1.2. *Let M be a manifold of dimension m and $Z \subseteq M$ a subspace such that for each point $p \in Z$ we can find a smooth chart (V, ϕ) in the maximal atlas of M around p in M with $Z \cap V = \phi^{-1}(\mathbb{R}^k \subseteq \mathbb{R}^m)$. Then Z is called* a submanifold *of dimension k (or codimension $m - k$) in M.*

From the next proposition there is, a close relationship between submanifolds and embeddings.

Proposition 2.1.3. *If M is a submanifold of N, then the inclusion map, $i : M \to N$, is an embedding. Conversely, if $F : M \to N$ is an embedding, then $F(M)$ with the subspace topology is a submanifold of N and F is a diffeomorphism between M and $F(M)$.*

Proof. See [O83, Chapter 1]. □

We will mention the most useful property of regular values.

Theorem 2.1.4. *[B04]. Let $F : M^m \to N^n$ be a smooth map between smooth manifolds. If $q \in N$ is a regular value of F, Then $F^{-1}(q)$ is either a smooth submanifold of M of dimension $m - n$ or the empty set.*

Proposition 2.1.5. *Let Z be the preimage of a regular value $q \in N$ under the smooth map $F : M^m \to N^n$. Then the kernel of the derivative $dF_p : T_p(M) \to T_q(N)$ at any point $p \in Z$ is precisely the tangent space to Z, $T_p(Z)$.*

Proof. Since F is constant on Z, dF_p is zero on $T_p(Z)$. But $dF_p : T_p(M) \to T_q(N)$ is surjective, so the dimension of the *kernel* of dF_p must be

$$\dim T_p(M) - \dim T_q(N) = \dim M - \dim N = \dim Z.$$

Thus $T_p(Z)$ is a subspace of the *kernel* that has the same dimension as the complete *kernel*; hence $T_p(Z)$ must be the *kernel.* □

Definition 2.1.6. (1) *Let S be a subset of $\mathbb{R}^m$. Then S has* measure zero *if for every $\epsilon > 0$, there exists a cover of S by a countable number of open cubes $C_1, C_2, \ldots$ in $\mathbb{R}^m$ such that $\sum_{i=1}^{\infty} vol[C_i] < \epsilon$.*
(2) *Let M be a smooth manifold and S a subset of M. Then S is of* measure zero *if there exists a countable open cover $U_1, U_2, \ldots$ of S and charts $\phi_i : U_i \to \mathbb{R}^m$ such that $\phi_i(U_i \cap S)$ has measure zero in $\mathbb{R}^m$.*

The following is the main result concerning regular values.

Theorem 2.1.7. ***Sard's theorem***. *If $F : M \to N$ is any smooth map of manifolds, then almost every point in N is a regular value of F.*

The assertion in Sard's theorem that (almost every point) of N is regular value for M means that the points that are not regular values constitute a set of measure zero. Since the complement of the regular values are the critical values, Sard's theorem may be restated as follows:

Theorem 2.1.8. ***Sard's theorem (restated)***. *If $F : M \to N$ be a smooth map of manifolds, then the set of critical values of F has measure zero in N.*

Proof. See [GP, Theorem p.40]. □

2.2 Transversality

Transversality can be viewed as a generalization of the notion of regular value. We define the basic notions of transversality and show briefly that transverse maps pull submanifolds back to submanifolds.

We can neatly define the transversality of manifolds using tangent spaces.

Definition 2.2.1. *Let M be a smooth manifold, W and Z submanifolds of M. Then W and Z are* transverse *at $p \in M$, denoted $W \pitchfork_p Z$, if either:*
(1) $p \notin W \cap Z$ *or*
(2) $p \in W \cap Z$ *and* $T_pM = T_pW + T_pZ$.

We say that W and Z are transverse, which we denoted by $W \pitchfork Z$, when $W \pitchfork_p Z$ for every $p \in X$.

Theorem 2.2.2. *If $W \pitchfork Z$ in M^m then $W \cap Z$ is a submanifold of M^m of dimension*

$$\dim(W \cap Z) = \dim(W) + \dim(Z) - \dim(M).$$

Proof. See [B, Theorem 7.7.] □

Proposition 2.2.3. *Two submanifolds W and Z are transverse at $p \in M$, $W \pitchfork_p Z$, if either:*
1- $p \notin W \cap Z$ *or*
2- $\nu_pW \cap \nu_pZ = \{0\}$, *where* $\nu_pW = (T_pW)^\perp$ *and* $\nu_pZ = (T_pZ)^\perp$, *(Remark 1.4.16).*

Proof. Suppose U and $V \subseteq \mathbb{R}^n$, then $U + V = \mathbb{R}^n$ if and only if $U^\perp \cap V^\perp = \{0\}$. □

Proposition 2.2.4. *[GP, Ex.4]. Let W and Z be transverse submanfolds of M. If $p \in W \cap Z$, then $T_p(W \cap Z) = T_p(W) \cap T_pZ$.*

The following is a generalization of the definition of transversality 2.2.1, where we define what it means for a smooth map to be transverse to a submanifold.

Definition 2.2.5. *Let $F : M^m \to N^n$ be a smooth map of smooth manifolds, let W be a submanifold of N of dimension k. Then F is* transverse *to W at $p \in M$, if either:*

(1) *$F(p) \notin W$ or*

(2) *$F(p) \in W$ and $T_{F(p)}N = (dF)_p(T_pM) + T_{F(p)}W$.*

By applying Theorem 2.2.2 we get

$$(dF)_p(T_pM) \cap T_{F(p)}W = m + k - n.$$

Lemma 2.2.6. *Let $F : M \to N$ be a smooth map of manifolds, W a submanifold of N. Then F is* transverse *to W at p if and only if*

$$\nu_pF \cap \nu_{F(p)}W = \{0\}.$$

Proof. Similar to the proof of Proposition 2.2.3. □

Proposition 2.2.7. *Let $F : M \to N$ be a map of smooth manifolds, W a submanifold of N. If $\dim M + \dim W < \dim N$, then $F \pitchfork W$ if and only if the image of F is disjoint from W, that is, $F(M) \cap W = \emptyset$.*

Proof. Suppose there $p \in M$ with $F(p) \in W$. Then

$$\begin{aligned} \dim[(dF)_p(T_pM) + T_{F(p)}W] &\leq \dim(dF)_p(T_pM) + \dim T_{F(p)}W \\ &\leq \dim M + \dim W \\ &< \dim N \\ &= \dim T_{F(p)}N \end{aligned}$$

Therefore, $(dF)_p(T_pM) + T_{F(p)}W \neq T_{F(p)}N$. And so F is not transverse to W at p. So $F(M) \cap W = \emptyset$. □

Remark 2.2.8. (1) *Suppose that* $\dim M \geq \dim N$ *and that* W *consists of a single point* $q \in N$. *Then* $F \pitchfork W$ *if and only if* q *is a regular value of* F. *For* F *is transversal to* q *if* $(dF)_p(T_pM) = T_{F(p)}N$ *for all* $p \in F^{-1}(q)$, *which is to say that* q *is a regular value of F. So transversality includes the notation of regularity as a special case.*
(2) *It also follows immediately that submersions are transverse to every submanifold.*
(3) *If* F *is an embedding then* $F \pitchfork W$ *if and only if* $F(M) \pitchfork W$ *in the sense of Definition 2.2.1.*

Theorem 2.1.4 provides a useful tool for generating manifolds. We can generalize it by the next theorem.

Theorem 2.2.9. *If* $F : M^m \to N^n$ *is transverse to a submanifold* W *in* N, *then* $F^{-1}(W)$ *is a submanifold of* M *with* $codim F^{-1}(W) = codim W$, *that is,*

$$\dim M - \dim F^{-1}(W) = \dim N - \dim W.$$

Proof. See [S75, Theorem 12.17.] □

Corollary 2.2.10. *Let* M *be a smooth manifold,* W *and* Z *submanifolds. If* $W \pitchfork Z$ *then* $W \cap Z$ *is a submanifold of* M *with*

$$codim(W \cap Z) = codim W + codim Z.$$

Proof. Let $i : Z \to M$ be the inclusion map. Then $(di)_p(T_pZ) = T_pM$, so $W \pitchfork Z$ implies $i \pitchfork Z$. $W \cap Z = i^{-1}(Z)$ so we are done by Theorem 2.2.9. □

We are going to state the next proposition which is a generalization of Proposition 2.1.5.

Proposition 2.2.11. *[GP, Ex.5]. Let* $F : M \to N$ *be a map transverse to a submanifold* W *in* N *so that* $Z = F^{-1}(W)$ *is a submanifold of* M. *Then* $T_p(Z)$ *is the preimage of* $T_{F(p)}(W)$ *under the linear map* $dF_p : T_p(M) \to T_{F(p)}(N)$.

Proof. The proof is a generalization of the proof of Proposition 2.1.5. □

Now we are considering the case of a map $F : M^n \looparrowright N^{n+k}$, and the definition of a self-transverse immersion as follows.

Definition 2.2.12. *An immersion $F : M^n \looparrowright N^{n+k}$ is self-transverse at the set of distinct points $\{x_1, x_2, \dots x_r\}$ such that $F(x_1) = F(x_2) = \cdots F(x_r)$ when*

$$\dim(\nu_{x_1} F + \nu_{x_2} F + \cdots + \nu_{x_r} F) = rk.$$

An immersion F is self-transverse if it is self-transverse at all sets of distinct points $\{x_1, x_2, \dots x_r\}$ such that $F(x_1) = F(x_2) = \cdots = F(x_r)$.

Example 2.2.13. *Imagine a smooth curve in $\mathbb{R}^2$ intersecting itself transversely at a point. With only two dimension in which to move, it impossible to remove this intersection through an arbitrarily small deformation. However, if we now embed the curve in $\mathbb{R}^3$, we can remove the intersection.*

Given an immersion $F : S^1 \looparrowright \mathbb{R}^3$ then $\nu_{x_1} F$ and $\nu_{x_2} F$ are 2-dimensional. Then given distinct points $\{x_1, x_2\}$ such that $F(x_1) = F(x_2)$ we cannot have $\dim(\nu_{x_1} F + \nu_{x_2} F) = 4$ since $\dim \mathbb{R}^3 = 3$. So it cannot be self-transverse at $\{x_1, x_2\}$. Therefore, a self-transverse curve in $\mathbb{R}^3$ can have no double points and so will be an embedding.

2.3 The double points of immersions in Euclidean spaces

Let $F : M^n \looparrowright \mathbb{R}^{n+k}$ be an immersion of a compact closed smooth n-dimensional manifold in Euclidean space. A point $y \in (\mathbb{R}^{n+k})$ is a double point of F when it is the image of two distinct points $x_1, x_2 \in M$.

We write $M^{(2)}$ for the Cartesian product of two copies of M with itself and $F^{(2)} : M^{(2)} \to \mathbb{R}^{2n+2k}$ for the map induced by F between the products.

Let $\Delta_2(\mathbb{R}^{n+k}) = \{(u, u) \mid u \in \mathbb{R}^{n+k}\}$ be the diagonal of $\mathbb{R}^{2n+2k}$. Since we have the inclusion map $i : \mathbb{R}^{n+k} \to \Delta_2(\mathbb{R}^{n+k}) \subseteq \mathbb{R}^{2n+2k}$, then for $y \in \mathbb{R}^{n+k}$,

$$di_y : T_y\mathbb{R}^{n+k} = \mathbb{R}^{n+k} \to T_{(y,y)}\Delta_2(\mathbb{R}^{n+k}) \subseteq T_{(y,y)}(\mathbb{R}^{2n+2k}) = \mathbb{R}^{2n+2k},$$

where $T_{(y,y)}\Delta_2(\mathbb{R}^{n+k}) = \Delta_2(\mathbb{R}^{n+k})$ and $di_y(\mathbb{R}^{n+k}) = \Delta_2(\mathbb{R}^{n+k})$. Hence

$$\nu_{(y,y)}\Delta_2(\mathbb{R}^{n+k}) = \Delta_2(\mathbb{R}^{n+k})^{\perp} = \{(u_1, u_2) \in \mathbb{R}^{2n+2k} \mid u_1 + u_2 = 0\}$$

is the orthogonal complement of $\Delta_2(\mathbb{R}^{n+k})$.

Proposition 2.3.1. *Given an immersion* $F : M^n \looparrowright \mathbb{R}^{n+k}$*, suppose that* $F(x_1) = F(x_2) = y, x_1 \neq x_2 \in M$*. Then* $F^{(2)}$ *is transverse to* $\Delta_2(\mathbb{R}^{n+k})$ *at* $(x_1, x_2) \in M^{(2)}$ *if and only if* F *is self-transverse at* $\{x_1, x_2\}$*.*

Proof. First of all, observe that F is self-transverse at $\{x_1, x_2\}$ if and only if $\dim(\nu_{x_1}F + \nu_{x_2}F) = 2k$ or, equivalently, $\nu_{x_1}F \cap \nu_{x_2}F = \{0\}$. Suppose that $F^{(2)}$ is transverse to $\Delta_2(\mathbb{R}^{n+k})$ at (x_1, x_2). Then by Lemma 2.2.6,

$$\nu_{(x_1,x_2)}F^{(2)} \cap \nu_{(y,y)}\Delta_2(\mathbb{R}^{n+k}) = \{(0,0)\}.$$

Suppose $u \in \nu_{x_1}F \cap \nu_{x_2}F$. Then $u \in \nu_{x_1}F$ and $u \in \nu_{x_2}F$ so that $-u \in \nu_{x_2}F$. Since $\nu_{x_1}F \times \nu_{x_2}F = \nu_{(x_1,x_2)}F^{(2)}$, then $(u, -u) \in \nu_{(x_1,x_2)}F^{(2)}$.
Since also $u \in \nu_{x_1}F$ and $-u \in \nu_{x_2}F$ and $u + (-u) = 0$, then $(u, -u) \in \nu_{(y,y)}\Delta_2(\mathbb{R}^{n+k})$. Thus $(u, -u) \in \nu_{(x_1,x_2)}F^{(2)} \cap \nu_{(y,y)}\Delta_2(\mathbb{R}^{n+k}) = \{(0,0)\}$, i.e $(u, -u) = (0, 0)$. So $u = 0$, Hence $\nu_{x_1}F \cap \nu_{x_2}F = \{0\}$ and therefore F is self-transverse at $\{x_1, x_2\}$.

Conversely, suppose that F is self-transverse at $\{x_1, x_2\}$ so that $\nu_{x_1}F \cap \nu_{x_2}F = \{0\}$. Let $(u, v) \in \nu_{(x_1,x_2)}F^{(2)} \cap \nu_{(y,y)}\Delta_2(\mathbb{R}^{n+k})$, then $(u, v) \in \nu_{(x_1,x_2)}F^{(2)}$ and $(u, v) \in \nu_{(y,y)}\Delta_2(\mathbb{R}^{n+k})$. Hence $u \in \nu_{x_1}F, v \in \nu_{x_2}F$ and $u + v = 0$. Hence $u = -v \in \nu_{x_2}F$. Then $u \in \nu_{x_1}F \cap \nu_{x_2}F = \{0\}$ and so $u = 0$. So $(u, v) = (0, 0)$ and so
$\nu_{(x_1,x_2)}F^{(2)} \cap \nu_{(y,y)}\Delta_2(\mathbb{R}^{n+k}) = \{(0,0)\}$. Therefore $F^{(2)}$ is transverse to $\Delta_2(\mathbb{R}^{n+k})$ at (x_1, x_2). □

2.4 Triple points of immersions in Euclidean spaces

Given an immersion $F : M^n \looparrowright \mathbb{R}^{n+k}$ of a connected n-dimensional compact closed smooth manifold in Euclidean space, a point $y \in \mathbb{R}^{n+k}$ is a triple intersection point of F when it is the image of three distinct points x_1, x_2, x_3 of M.

Let $\Delta_3(\mathbb{R}^{n+k}) = \{(u, u, u) \mid u \in \mathbb{R}^{n+k}\} \subseteq \mathbb{R}^{3n+3k}$ be the diagonal of $\mathbb{R}^{3n+3k}$. Then for $y \in \mathbb{R}^{n+k}$, we have $T_{(y,y,y)}\Delta_3(\mathbb{R}^{n+k}) = \Delta_3(\mathbb{R}^{n+k})$. And so $\nu_{(y,y,y)}\Delta_3(\mathbb{R}^{n+k})$ is the orthogonal complement of $\Delta_3(\mathbb{R}^{n+k})$. Also for$(u_1, u_2, u_3) \in \mathbb{R}^{3n+3k}$ and for $u \in \mathbb{R}^{n+k}$

we have

$$\begin{aligned}(u_1, u_2, u_3) \cdot (u, u, u) &= u_1 \cdot u + u_2 \cdot u + u_3 \cdot u \\ &= (u_1 + u_2 + u_3) \cdot u = 0 \\ &\Leftrightarrow u_1 + u_2 + u_3 = 0.\end{aligned}$$

Hence $\nu_{(y,y,y)}\Delta_3(\mathbb{R}^{n+k}) = \{(u_1, u_2, u_3) \in \mathbb{R}^{3n+3k} \mid u_1 + u_2 + u_3 = 0\}$.

Remark 2.4.1. *The immersion $F : M^n \looparrowright \mathbb{R}^{n+k}$ is self-transverse at the set of distinct points $\{x_1, x_2, x_3\}$ if and only if*

$$\dim(\nu_{x_1}(F) + \nu_{x_2}(F) + \nu_{x_3}(F)) = 3k.$$

Lemma 2.4.2. *For linear subspaces U_1, U_2, U_3 of dimension k,*

$$\dim(U_1 + U_2 + U_3) = 3k \Leftrightarrow U_1 \cap U_2 = \{0\} \text{ and } (U_1 + U_2) \cap U_3 = \{0\}.$$

Proof. Suppose that $\dim(U_1 + U_2 + U_3) = 3k$, then $\dim(U_1 + U_2) = 2k$ and $(U_1 + U_2) \cap U_3 = \{0\}$. Since $\dim U_1 = k, \dim U_2 = k$, then $\dim U_1 + \dim U_2 = 2k$. Since also $\dim(U_1 + U_2) = \dim U_1 + \dim U_2$, $\dim(U_1 \cap U_2) = 0$. Hence $U_1 \cap U_2 = \{0\}$ and $(U_1 + U_2) \cap U_3 = \{0\}$.

Conversely, suppose that $U_1 \cap U_2 = \{0\}$ and $(U_1 + U_2) \cap U_3 = \{0\}$. Then $\dim(U_1 + U_2) = 2k$ and $(U_1 + U_2) \cap U_3 = \{0\}$. Hence

$$\dim(U_1 + U_2 + U_3) = 3k.$$

□

Proposition 2.4.3. *Given an immersion $F : M^n \looparrowright \mathbb{R}^{n+k}$, suppose that $F(x_1) = F(x_2) = F(x_3) = y$ where x_1, x_2, x_3 are distinct points of M. Then $F^{(3)}$ is transverse to $\Delta_3(\mathbb{R}^{n+k})$ at (x_1, x_2, x_3) if and only if F is self-transverse at $\{x_1, x_2, x_3\}$.*

Proof. Suppose that $F^{(3)}$ is transverse to $\Delta_3(\mathbb{R}^{n+k})$ at (x_1, x_2, x_3). By Lemma 2.2.6, we have $\nu_{(x_1,x_2,x_3)}F^{(3)} \cap \nu_{(y,y,y)}\Delta_3(\mathbb{R}^{n+k}) = \{(0, 0, 0)\}$.
Suppose that $u \in (\nu_{x_1}(F) + \nu_{x_2}(F)) \cap \nu_{x_3}(F)$. Then $u \in (\nu_{x_1}(F) + \nu_{x_2}(F))$ and $u \in \nu_{x_3}(F)$, so that $-u \in \nu_{x_3}(F)$. Since $u \in (\nu_{x_1}(F)+\nu_{x_2}(F))$ we can write $u = u_1+u_2$,

where $u_1 \in \nu_{x_1}(F)$, $u_2 \in \nu_{x_2}(F)$.

Since $u = u_1 + u_2, u_1 + u_2 + (-u) = 0$. Then $(u_1, u_2, -u) \in \nu_{(y,y,y)}\Delta_3(\mathbb{R}^{n+k})$. Also $(u_1, u_2, -u) \in \nu_{(x_1,x_2,x_3)}F^{(3)} = \nu_{x_1}F \times \nu_{x_2}F \times \nu_{x_3}F$. So $(u_1, u_2, -u) = (0, 0, 0)$. Therefore, $u_1 = u_2 = -u = 0$.

Hence $u = 0$ and so $(\nu_{x_1}(F) + \nu_{x_2}(F)) \cap \nu_{x_3}(F) = \{0\}$. Also from Proposition 2.3.1 we can prove that $\nu_{x_1}(F) \cap \nu_{x_2}(F) = \{0\}$. Hence, by Lemma 2.4.2,

$$\dim(\nu_{x_1}F + \nu_{x_2}F + \nu_{x_3}F) = 3k.$$

And so F is self-transverse immersion at $\{x_1, x_2, x_3\}$.

Conversely suppose that F is self-transverse at $\{x_1, x_2, x_3\}$. Then by Lemma 2.4.2, $\nu_{x_1}(F) \cap \nu_{x_2}(F) = \{0\}$ and $(\nu_{x_1}(F) + \nu_{x_2}(F)) \cap \nu_{x_3}(F) = \{0\}$.

Let $(u, v, w) \in \nu_{(x_1,x_2,x_3)}F^{(3)} \cap \nu_{(y,y,y)}\Delta_3(\mathbb{R}^{n+k})$. Then $(u, v, w) \in \nu_{(x_1,x_2,x_3)}F^{(3)}$ and $(u, v, w) \in \nu_{(y,y,y)}\Delta(\mathbb{R}^{n+K})$. Hence $u \in \nu_{x_1}(F), v \in \nu_{x_2}(F), w \in \nu_{x_3}(F)$ and $u + v + w = 0$ which implies $u + v = -w$.

Since $u + v \in (\nu_{x_1}(F) + \nu_{x_2}(F))$ then $-w \in (\nu_{x_1}(F) + \nu_{x_2}(F))$ and so $w \in (\nu_{x_1}(F) + \nu_{x_2}(F))$. Since $w \in \nu_{x_3}(F)$, $w = 0$ by our second hypothesis, and so $u + v = 0$. Since $u + v = 0$, $u = -v \in \nu_{x_2}(F)$. Hence $u \in \nu_{x_1}(F) \cap \nu_{x_2}(F) = \{0\}$ and so $u = 0$ and $v = -u = 0$. So $(u, v, w) = (0, 0, 0)$. Hence $\nu_{(x_1,x_2,x_3)}F^{(3)} \cap \nu_{(y,y,y)}\Delta_3(\mathbb{R}^{n+k}) = \{(0, 0, 0)\}$ and so $F^{(3)}$ is transverse to $\Delta_3(\mathbb{R}^{n+k})$ at (x_1, x_2, x_3) □

Now we are going to consider the general case of r-fold points of an immersion in a manifold N.

2.5 r-Fold points of immersions in manifold N

Given a self-transverse immersion $F : M^n \to N^{n+k}$ of a compact n-dimensional manifold of M in N. Let $i : N \to \Delta_r(N) \subseteq N^{(r)}$ be the inclusion map, where $\Delta_r(N) = \{(u, u, \ldots, u) \mid u \in N\} \subseteq N^{(r)}$, the diagonal of $N^{(r)}$. Then for $y \in N$, we have $di_y(T_yN) = T_{(y,\ldots,y)}\Delta_r(N) = \{(u, \ldots, u) \mid u \in T_yN\} = \Delta_r(T_yN) \subseteq (T_yN \times \ldots \times T_yN) = T_{(y,\ldots,y)}(N \times \ldots \times N)$ is the tangent space of the diagonal. Hence, the orthogonal complement of tangent space of the diagonal defined by

$\nu_{(y,\ldots,y)}\Delta_r(N) = \Delta_r(T_{(y,\ldots,y)}N)^\perp = \{(u_1,\ldots,u_r)|u_i \in T_yN, \Sigma_{i=1}^r u_i = 0\}$.

By a generalization of Lemma 2.4.2 we obtain the following lemma.

Lemma 2.5.1. *An immersion $F : M^n \looparrowright N^{n+k}$ is self-transverse at $(x_1,\ldots,x_r)$ if and only if*

$$\nu_{x_1}(F) \cap \nu_{x_2}(F) = \{0\},$$

$$(\nu_{x_1}(F) + \nu_{x_2}(F)) \cap \nu_{x_3}(F) = \{0\},$$

$$\vdots$$

$$(\nu_{x_1}F + \cdots + \nu_{x_{r-1}}F) \cap \nu_{x_r}F = \{0\}.$$

Lemma 2.5.2. *Given an immersion $F : M^n \looparrowright N^{n+k}$, suppose that $F(x_1) = \cdots = F(x_r) = y$ where $x_1,\ldots,x_r$ are the distinct points of M. Then $F^{(r)} : M^{(r)} \to N^{(r)}$ is transverse to $\Delta_r(N)$at $(x_1,\ldots,x_r)$ if and only if F is self-transverse at $\{x_1,\ldots,x_r\}$.*

Proof. Suppose that $F^{(r)}$ is transverse to $\Delta_r(N)$ at $(x_1,\ldots,x_r)$. Then by Lemma 2.2.6, we have $\nu_{(x_1,\ldots,x_r)}F^{(r)} \cap \nu_{(y,\ldots,y)}\Delta_r(N) = \{(0,\ldots,0)\}$.

Using Lemma 2.5.1, suppose $u \in (\nu_{x_1}F + \cdots + \nu_{x_{i-1}}F) \cap \nu_{x_i}F$, for $2 \le i \le r$. Then $u \in (\nu_{x_1}F + \cdots + \nu_{x_{i-1}}F)$ and $u \in \nu_{x_i}F$, so that $-u \in \nu_{x_i}F$.

Since $u \in (\nu_{x_1}F + \cdots + \nu_{x_{i-1}}F)$, we write $u = u_1 + \cdots + u_{i-1}$ where $u_1 \in \nu_{x_1}F,\ldots,u_{i-1} \in \nu_{x_{i-1}}F$. Since $u = u_1 + \cdots + u_{i-1}$, then $u_1 + \cdots + u_{i-1} + (-u) = 0$, $(u_1,\ldots,u_{i-1},-u) \in \nu_{(y,\ldots,y)}\Delta_r(N)$, and $(u_1,\ldots,u_{i-1},-u,) \in \nu_{(x_1,\ldots,x_i)}F^{(r)}$. So
$(u_1,\ldots,u_{i-1},-u) \in \nu_{(x_1,\ldots,x_r)}F^{(r)} \cap \nu_{(y,\ldots,y)}\Delta_r(N) = \{(0,\ldots,0)\}$. Hence
$(u_1,\ldots,u_{i-1},-u) = (0,\ldots,0) \Rightarrow u_1 = \cdots = u_{i-1} = -u = 0$.
Therefore, $u = 0$ and so $(\nu_{x_1}F + \cdots + \nu_{x_{i-1}}F) \cap \nu_{x_i}F = \{0\}$ for $2 \le i \le r$. By Lemma 2.5.1, F is self-transverse at $\{x_1,\ldots,x_r\}$.

Conversely, suppose that F is self-transverse at $\{x_1,\ldots,x_r\}$ so that
$(\nu_{x_1}F + \cdots + \nu_{x_{i-1}}F) \cap \nu_{x_i}F = \{0\}$ for all $i,\ 2 \le i \le r$.
Let $(w_1,\ldots,w_r) \in \nu_{(x_1,\ldots,x_r)}F^{(r)} \cap \nu_{(y,\ldots,y)}\Delta_r(N)$. Then $(w_1,\ldots,w_r) \in \nu_{(x_1,\ldots,x_r)}F^{(r)}$ and $(w_1,\ldots,w_r) \in \nu_{(y,\ldots,y)}\Delta_r(N)$. So $w_1 \in \nu_{x_1}F,\ldots,w_r \in \nu_{x_r}F$ and $w_1 + \cdots + w_r = 0$.

Hence $w_1 + \cdots + w_{r-1} = -w_r$. Since $w_1 + \cdots + w_{r-1} \in (\nu_{x_1}F + \cdots + \nu_{x_{r-1}}F)$, $-w_r \in (\nu_{x_1}F + \cdots + \nu_{x_{r-1}}F)$ and so $w_r \in (\nu_{x_1}F + \cdots + \nu_{x_{r-1}}F)$. As $w_r \in \nu_{x_r}F$ then

$w_r \in (\nu_{x_1}F + \ldots + \nu_{x_{r-1}}F) \cap \nu_{x_r}F = \{0\}$ by Lemma 2.5.1. Therefore

$$w_r = 0.$$

Similarly, by the induction we can prove that $w_1 = \cdots, w_{r-1} = 0$. So

$$(w_1, \ldots, w_r) = (0, \ldots, 0).$$

Hence $\nu_{(x_1,\ldots,x_r)}F^{(r)} \cap \nu_{(y,\ldots,y)}\Delta_r(N) = \{(0,\ldots,0)\}$, and so by Lemma 2.2.6 $F^{(r)}$ is transverse to $\Delta_r(N^{n+k})$ at $(x_1, \ldots, x_r)$. □

Theorem 2.5.3. *Suppose $F : M^n \looparrowright N^{n+k}$ is a self-transverse immersion of a compact closed smooth manifold M in smooth manifold N. Then the r-fold self intersection sets $I_r(F)$ is itself the image of an immersion*

$$\theta_r(F) : \Delta_r(F) \looparrowright N^{n+k}.$$

Proof. See Eccles-Grant [G06]. Suppose F is a self-transverse immersion of manifold M in N, then for each $r \geq 1$, $F^{(r)} : (M)^{(r)} \looparrowright (N)^{(r)}$ is transverse to the diagonal $\Delta_r(N)$.

Let $\overline{\Delta}_r(F) = \{(x_1, \ldots, x_r) \in M^{(r)} | F(x_1) = \ldots = F(x_r), i \neq j \Rightarrow x_i \neq x_j\}$, where $M^{(r)}$ is the r-fold cartesian product of M with itself.
$F^{'}(M, r) = \{(x_1, \ldots, x_r) \in M^{(r)} : i \neq j \Rightarrow x_i \neq x_j\} \subseteq M^{(r)}$ is an open submanifold of $M^{(r)}$ and so has dimension rn. Hence by Lemma 2.5.2 and Proposition 2.2.11,

$$\overline{\Delta}_r(F) = (F^{(r)})^{-1}(\Delta_r(N)),$$

is a submanifold of $F^{'}(M, r) \subseteq M^{(r)}$, of codimension $(n+k)(r-1)$. Thus $\overline{\Delta}_r(F)$ has dimension $rn - (n+k)(r-1) = n - k(r-1)$.

The symmetric group Σ_r acts freely on $\overline{\Delta}_r(F)$ by permuting the coordinates. Factoring out by this action give a compact manifold of dimension $n - (r-1)k$

$$\Delta_r(F) = \overline{\Delta}_r(F)/\Sigma_r.$$

This is the $r$fold point manifold of F in N. We may define a map

$$\theta_r(F) : \Delta_r(F) \looparrowright N$$

by $\theta_r(F)[x_1,\ldots,x_r] = F(x_1)$. We show that this map is an immersion as follows.

Since $F : M^n \looparrowright N^{n+k}$ is an immersion, then $F^{(r)} : F^{'}(M,r) \subset M^{(r)} \longrightarrow N^{(r)}$ is an immersion. By restricting to $\overline{\Delta}_r(F)$ in $F^{'}(M,r)$ and to $\Delta_r(N)$ in $N^{(r)}$, we deduce that the restriction $g : \overline{\Delta}_r(F) \to \Delta_r(N)$ is an immersion too. This follows from the following diagram.

$$\begin{array}{ccccc} F^{'}(M,r) & \xrightarrow{F^{(r)}} & N^{(r)} & \xrightarrow{p} & N \\ \uparrow & & \uparrow & \nearrow_{s} & \\ \overline{\Delta}_r(F) & \xrightarrow[g]{} & \Delta_r(N) & & \end{array}$$

The diagonal $s : \Delta_r(N) \subset N^{(r)} \to N$ is a diffeomorphism. Then by composing g and s, we get the map $sg = \alpha : \overline{\Delta}_r(F) \to N$ which is an immersion. Consider the following diagram.

$$\begin{array}{ccc} \Delta_r(F) & \xrightarrow{\theta_r(F)} & N \\ {\scriptstyle q}\uparrow & \nearrow_{\alpha} & \\ \overline{\Delta}_r(F) & & \end{array}$$

Since the space $\Delta_r(F)$ is obtained by factoring out $\overline{\Delta_r}(F)$ by the symmetric group action, and since α is an immersion, then $\theta_r(F) : \Delta_r(F) \looparrowright N$ is immersion. $\square$

Chapter 3

Pontrjagin-Thom theory for embeddings

The classical Pontrjagin-Thom theory establishes a relation between the cobordism classes of embeddings $M^n \hookrightarrow N^{n+k}$ and the set of homotopy classes of maps $N_+^{n+k} \to MO(k)$. If N^{n+k} is compact, then N_+^{n+k} is given by N^{n+k} together with a disjoint base point. For $N^{n+k} = \mathbb{R}^{n+k}$ then $N_+^{n+k} \cong S^{n+k}$. Moreover, $MO(k)$ is the Thom complex of the universal n-plane bundle γ^n. We start with the cobordism theory of embeddings.

3.1 Cobordism group of embeddings

Suppose n and k are fixed non-negative integers, M is a compact smooth n-dimensional manifold without boundary, and (M, F) be a pair where $F : M^n \hookrightarrow N^{n+k}$ is an embedding. Roughly speaking, we say that two manifolds M_1 and M_2 are said to be cobordant if their disjoint union, denoted by $\sqcup$ is the boundary of some other manifold. More formally, we have the following definition.

Definition 3.1.1. *Given two embeddings $F : M_1^n \hookrightarrow N^{n+k}$ and $G : M_2^n \hookrightarrow N^{n+k}$, we say (M_1, F) is cobordant to (M_2, G), written $(M_1, F) \sim (M_2, G)$ if the following conditions hold,*

(1) *There exists $(n+1)$-dimensional manifold W such that $\partial W = M_1 \times 0 \sqcup M_2 \times 1$.*

(2) *There exists an embedding* $H : W \longrightarrow N^{n+k} \times [0,1]$ *such that,*
$H|_{M_1 \times 0} = F \times 0$ *and* $H|_{M_2 \times 1} = G \times 1$.
(3) $x \in \partial W$ *if and only if* $H(x) \in N^{n+k} \times \{0,1\}$.
(4) *There exists* $\epsilon > 0$ *such that* $(N^{n+k} \times [0,\epsilon)) \cap F(W) = F(M_1) \times [0,\epsilon)$, *and*

$$(N^{n+k} \times (1-\epsilon, 1]) \cap H(W) = G(M_2) \times (1-\epsilon, 1].$$

We say that (W, H) is a cobordism between (M_1, F) and (M_2, G). The cobordism relation $\sim$ is an equivalence relation on the set of all pairs (M, F). Let $Emb_k(N^{n+k})$ be the set of equivalence classes of such pairs. We shall denote the class of (M, F) by $[(M, F)]$.

Before introducing the Pontrjagin-Thom construction, we need to recall the tubular neighborhood theorem.

Theorem 3.1.2. ***Tubular neighborhood theorem***. *Let* $F : M^n \hookrightarrow N^{n+k}$ *be an embedding. Then there exists an open neighborhood of* M^n *in* N^{n+k} *which is diffeomorphic to* $E(\nu_F)$ *under a diffeomorphism which maps each point of* $x \in M$ *to the zero normal vector at* x.

Proof. See [MS74, Theorem 11.1]. □

Our next theorem introduces the Thom-Pontrjagin construction.

Theorem 3.1.3. ***Thom***. *The Pontrjagin-Thom construction induces a function*

$$\tau : Emb_k(N^{n+k}) \longrightarrow [N^{n+k}_{+}, MO(k)].$$

Proof. Let $\alpha \in Emb_k(N^{n+k})$ and let (M, F) be a representative of α. Given an embedding $F : M^n \hookrightarrow N^{n+k}$ where M is a closed connected smooth n-dimensional manifold, we have the following commutative diagram.

$$\begin{array}{ccc} E(\nu_F) & \xrightarrow{\widehat{F}} & E(\gamma^k) \\ \downarrow & & \downarrow \\ M & \xrightarrow{\overline{F}} & BO(k) \end{array}$$

in which $\overline{F} : M \longrightarrow BO(k)$ is the normal map of F which classifies the normal bundle

ν_F over M. Since M^n is compact, $\overline{F}$ is closed [G75], and hence

$$\widehat{F} : E(\nu_F) \longrightarrow E(\gamma^k)$$

is closed. This consequently induces a map

$$T(\overline{F}) : T(\nu_F) \longrightarrow MO(k)$$

which is induced by ν_F on the associated Thom space.

Since $M \cong F(M)$ is a submanifold of N^{n+k}, by Theorem 3.1.2, M has a tubular neighborhood U in N^{n+k} such that U is diffeomorphic to the total space $E(\nu)$ of the normal bundle over M. Notice that M is compact, so $T(\nu_F) \cong E(\nu_F)_+$. Thus we obtain a map

$$L : U \cong E(\nu_F) \longrightarrow E(\nu_F)_+ \cong T(\nu_F).$$

Now we have a map which is defined on $U \subset N_+^{n+k}$ but not all of N_+^{n+k}. We extend this map to N_+^{n+k} by sending $N_+^{n+k} - U$ to t_0, the base point of the Thom space of normal bundle ν_F. Let r denotes this continuous extension,

$$r : N_+^{n+k} \longrightarrow T(\nu_F).$$

If we compose the map $T(\overline{F})$ with r, the composition provides the required map

$$f : N_+^{n+k} \longrightarrow MO(k).$$

A similar construction starting from a cobordism $H : W^{n+1} \longrightarrow N^{n+k} \times [0,1]$ leads to a homotopy $f' : (N^{n+k} \times [0,1])_+ \to MO(k)$. Hence the homotopy class of f only depends on the cobordism class of (M, F).

The map f is called *Pontrjagin-Thom Construction* associated with the embedding F. Now we define the Thom map

$$\tau : Emb_k(N^{n+k}) \longrightarrow [N_+^{n+k}, MO(k)]$$

by $\tau(\alpha) = [f]$, where $[f]$ denotes the homotopy class of f. □

Now we want to construct the inverse to the function τ in the above theorem.

Theorem 3.1.4. *There is one-to-one map*

$$t : [N_+^{n+k}, MO(k)] \to Emb_k(N^{n+k}).$$

Proof. Suppose that $\alpha \in [N_+^{n+k}, MO(k)]$ is represented by a map

$$f : N_+^{n+k} \to MO(k).$$

Choose a representative f which is transverse to $BO(k) \subset MO(k)$. Then by the methods of chapter 2 and using Theorem 2.2.9, $f^{-1}(BO(k)) = M^n$ is a submanifold of N of $\dim n$. This gives an embedding map $M^n \hookrightarrow N_+^{n+k}$.

$$\begin{array}{ccc} N_+^{n+k} & \xrightarrow{f} & MO(k) \\ {\scriptstyle i}\uparrow & & \uparrow \\ M^n & \xrightarrow{f_1} & BO(k) \end{array}$$

According to the above diagram, we get a map of vector bundle $\nu_i \to \gamma^k$ which is isomorphism in each fiber, where ν_i is the normal bundle of the embedding i and γ^k is the normal bundle of the map $BO(k) \to MO(k)$. This show that $f_1^*(\gamma^k) = \nu_i$. Hence we have an embedding $M^n \hookrightarrow N^{n+k}$.

Similarly a homotopy $(N^{n+k} \times [0,1])_+ \to MO(k)$ gives rise to a cobordism $W^{n+1} \longrightarrow N^{n+k} \times [0,1]$. Hence we have defined a function

$$t : [N_+^{n+k}, MO(k)] \to Emb_k(N^{n+k}).$$

□

Theorem 3.1.5. *The map τ is bijection,*

$$Emb_k(N^{n+k}) \cong [N_+^{n+k}, MO(k)].$$

Proof. The map t is inverse to τ. [S68, Theorem p. 18]. □

If $N^{n+k} = \mathbb{R}^{n+k}$ then $N_+^{n+k} = S^{n+k}$ and we obtain a one to one correspondence

$$\tau : Emb_k(\mathbb{R}^{n+k}) \longrightarrow [S^{n+k}, MO(k)] = \pi_{n+k}MO(k)$$

where the right hand side is a group. On the other hand $Emb_k(\mathbb{R}^{n+k})$ is a group under the disjoint union operation, and it is Thom's result that the mapping τ defined above is a group isomorphism. The method of proof is similar to the proof of Theorem 3.1.3.

Given an embedding $F : M^n \hookrightarrow \mathbb{R}^{n+k}$, it corresponds to an element $\alpha \in \pi_{n+k}MO(k)$. We can construct a map $f : S^{n+k} \longrightarrow MO(k)$ be the Pontrjagin-Thom construction and $\tau(\alpha) = [f]$ where $[f]$ is the homotopy class of f.

We are going to show that the cobordism class of M determines and is determined up to cobordism by certain Hurewicz images and so it natural to ask how this information can be retrieved.

We will define the Hurewicz homomorphism which illuminates the close relation between homology and homotopy.

Definition 3.1.6. ***The Hurewicz homomorphism***

$$h : [N_+^{n+k}, MO(k)] \to H_{n+k}MO(k)$$

is defined by setting $h(\alpha) = h([f]) = f_*[N]$, *where* $[N] \in H_{n+k}N^{n+k}$ *is the fundamental homology class of* N, *and* $f_*[N] \in H_{n+k}MO(k)$, *for*

$$f_* : H_{n+k}N^{n+k} \longrightarrow H_{n+k}MO(k)$$

the map induced by f.

If $N^{n+k} = \mathbb{R}^{n+k}$ then *The Hurewicz homomorphism*

$$h : \pi_{n+k}MO(k) \to H_{n+k}MO(k)$$

is defined by setting $h(\alpha) = h([f]) = f_*(g_{n+k})$ for $f : S^{n+k} \longrightarrow MO(k)$, such that g_{n+k} is a generator of $H_{n+k}S^{n+k}$ and $f_* : H_{n+k}S^{n+k} \longrightarrow H_{n+k}MO(k)$ is induced by f.

Next we need to explain characteristic numbers of manifolds.

3.2 Stiefel-Whitney numbers and cobordism

Let M be a closed connected smooth n-dimensional manifold with an embedding. Recall that our manifolds are compact and connected. Hence, according to [MS74, Theorem A.8] there exists a unique non-zero fundamental homology class $[M] \in H_n(M)$, and for any cohomology class $v \in H^n M$, the Kronecker product $\langle v, [M] \rangle \in \mathbb{Z}/2$ is defined. We will use the abbreviated notation $v[M]$ for this Kronecker product.

Let $F : M^n \hookrightarrow N^{n+k}$ be an embedding. For $I = (i_1, \ldots, i_r)$ a sequence of non-negative integers with degree $i_1 + 2i_2 + 3i_3 + \cdots + ri_r = n$ we can form the monomial

$$w^I(\nu_F) = w_1^{i_1}(\nu_F) w_2^{i_2}(\nu_F) \ldots w_r^{i_r}(\nu_F) \in H^n M.$$

Definition 3.2.1. *The Normal Stiefel-Whitney number of an embedding F corresponding to a monomial $w_1^{i_1} w_2^{i_2} \ldots w_r^{i_r} = w^I$ of degree n is the number*

$$w^I(\nu_F)[M] = \langle w^I(\nu_F), [M] \rangle \in \mathbb{Z}/2.$$

Notation 3.2.2. *In the case of an embedding $F : M^n \hookrightarrow \mathbb{R}^{n+k}$, we have $\nu_F \oplus \tau_M \cong \varepsilon^{n+k}$, then $w(\nu_F) = w(\tau_M)^{-1}$, and so the normal Stiefel-Whitney numbers do not depend on F and so can be written $\overline{w}^I[M]$. In the case of an embedding $F : M^n \hookrightarrow N^{n+k}$ we call $w^I(\nu_F)[M]$ the normal Stiefel-Whitney number of F corresponding to monomial $w_i^{i_1} \ldots w_r^{i_r}$ and denoted it $\overline{w}^I[F]$, since it may depend on the choice of embedding (see Example 3.3.10).*

Remark 3.2.3. *It is also possible to define tangent Stiefel-Whitney number of a manifold M using the tangent bundle. This is more usual.*

The example below illustrates how to calculate the normal Stiefel-Whitney numbers .

Example 3.2.4. *Given an embedding $\mathbb{R}P^2 \hookrightarrow \mathbb{R}^l$, for sufficiently large l. Then the normal Stiefel-Whitney numbers of $\mathbb{R}P^2$ are described as follows.*
According to Theorem 1.6.10, $\tau_{\mathbb{R}P^2} \oplus \varepsilon^1 = \gamma_2^1 \oplus \gamma_2^1 \oplus \gamma_2^1$, where γ_2^1 is Hopf line bundle.
Then $\overline{w}(\mathbb{R}P^2) = w(\nu_{\mathbb{R}P^2}) = w(\tau_{\mathbb{R}P^2})^{-1} = (1+a)^{-3} = (1+a)$.
Then $\overline{w}_1(\mathbb{R}P^2) = a$, $\overline{w}_2(\mathbb{R}P^2) = 0$. Hence

$$\overline{w}_1^2[\mathbb{R}P^2] = 1, \ \overline{w}_2[\mathbb{R}P^2] = 0.$$

We are interested in the Stiefel-Whitney numbers as they distinguish between given manifolds belonging to different cobordism classes. This is the outcome of the following results.

Theorem 3.2.5. *[Pontrjagin]. If B is a smooth compact $(n+1)$-dimensional manifold with boundary equal to M, then the Stiefel-Whitney numbers of M are all zero.*

Proof. See [D66, Theorem 4.9]. □

We may say that Thom's theorem is an inverse for Pontrjagin's theorem.

Theorem 3.2.6. *[Thom]. If all of the Stiefel-Whitney numbers of M are zero, then M can be realized as boundary of some smooth compact manifold.*

Proof. See [BG88] □

Combining Theorem 3.2.5 and Theorem 3.2.6 we have the following.

Theorem 3.2.7. *Two smooth closed n-manifolds belong to same cobordism class if and only if their corresponding normal Stiefel-Whitney numbers are equal for all I.*

$$\overline{w}^I[M] = \overline{w}^I[N].$$

Proof. See [G75, Proposition 30.21]. □

Thus is a similar result for the case of tangent Stiefel-Whitney number (see[D66, Theorem 17.9.7]).

Now we will calculate the normal Stiefel-Whitney numbers of two different manifolds of the same dimension in order to demonstrate that they are not cobordant.

Example 3.2.8. Given $\mathbb{R}P^2 \times \mathbb{R}P^2 \hookrightarrow \mathbb{R}^l$ and $\mathbb{R}P^4 \hookrightarrow \mathbb{R}^l$, their normal Stiefel-Whitney numbers are calculated a below

$\overline{w}(\mathbb{R}P^2 \times \mathbb{R}P^2) = w(\nu_{\mathbb{R}P^2 \times \mathbb{R}P^2}) = w(\tau_{\mathbb{R}P^2 \times \mathbb{R}P^2})^{-1} = (1+a)^{-3}(1+b)^{-3}$, where

$$H^*(\mathbb{R}P^2 \times \mathbb{R}P^2) \cong H^*\mathbb{R}P^2 \otimes H^*\mathbb{R}P^2 \cong \mathbb{Z}/2[a]/(a^3) \otimes \mathbb{Z}/2[b]/(b^3) = \mathbb{Z}/2[a,b]/(a^3, b^3).$$

Since $(1+a)^4 = 1 + a^4 = 1$, then $(1+a)^3(1+a) = 1$, and so $(1+a)^{-3} = 1+a$. Hence $\overline{w}(\mathbb{R}P^2 \times \mathbb{R}P^2) = (1+a)(1+b) = 1 + (a+b) + ab$.

So $\overline{w}_1(\mathbb{R}P^2\times\mathbb{R}P^2) = a+b, \overline{w}_2(\mathbb{R}P^2\times\mathbb{R}P^2) = ab$. Then $\overline{w}_1^4(\mathbb{R}P^2\times\mathbb{R}P^2) = (a+b)^4 = a^4+b^4 = 0$, $\overline{w}_1^2\overline{w}_2(\mathbb{R}P^2\times\mathbb{R}P^2) = (a+b)^2(ab) = (a^2+b^2)ab = 0$, $\overline{w}_1\overline{w}_3(\mathbb{R}P^2\times\mathbb{R}P^2) = (a+b)a^3b^3 = 0$, $\overline{w}_2^2(\mathbb{R}P^2\times\mathbb{R}P^2) = a^2b^2, \overline{w}_4(\mathbb{R}P^2\times\mathbb{R}P^2) = 0$. Hence

$$\overline{w}_2^2[\mathbb{R}P^2\times\mathbb{R}P^2] = 1,$$

and the other characteristic numbers are zero.

Since we have $\overline{w}(\mathbb{R}P^4) = (1+a)^{-5} = (1+a)^3 = 1+a+a^2+a^3$, then $\overline{w}_1(\mathbb{R}P^4) = a, \overline{w}_2(\mathbb{R}P^4) = a^2, \overline{w}_3(\mathbb{R}P^4) = a^3, \overline{w}_4(\mathbb{R}P^4) = 0$, and then $\overline{w}_1^4 = \overline{w}_1^2\overline{w}_2 = \overline{w}_1\overline{w}_3 = \overline{w}_2^2 = a^4, \overline{w}_4 = 0$. Hence

$$\overline{w}_1^4[\mathbb{R}P^4] = \overline{w}_1^2\overline{w}_2[\mathbb{R}P^4] = \overline{w}_1\overline{w}_3[\mathbb{R}P^4] = \overline{w}_2^2[\mathbb{R}P^4] = 1,$$

$$\overline{w}_4[\mathbb{R}P^4] = 0.$$

We deduce that the above two manifolds have different normal Stiefel-Whitney numbers.

Our calculations in Example 3.2.8 show that the manifolds $\mathbb{R}P^4$ and $\mathbb{R}P^2\times\mathbb{R}P^2$ are not cobordant because they do not have the same normal Stiefel-Whitney numbers.

3.3 Reading off the Stiefel-Whitney numbers

Now, we describe a systematic way how to read off the normal Stiefel-Whitney numbers of a given embedding $F: M^n \hookrightarrow N^{n+k}$. This answers the question that we posted at the end of Section 3.1.

We start by describing the cohomology of $MO(k)$. To determine the cohomology group of $MO(k)$, remember that $MO(k) = D(\gamma^k)/S(\gamma^k)$ where

$$\begin{aligned} E(\gamma^k) &= \{(X,x) \in BO(k)\times\mathbb{R}^\infty : x \in X\}, \\ S(\gamma^k) &= \{(X,x) \in E(\gamma^k) : |x| = 1\}, \\ D(\gamma^k) &= \{(X,x) \in E(\gamma^k) : |x| \leq 1\}. \end{aligned}$$

We then have the following observations.

Lemma 3.3.1. *$D(\gamma^k)$ is homotopy equivalent to $BO(k)$.*

Lemma 3.3.2. *$S(\gamma^k)$ is homotopy equivalent to $BO(k-1)$.*

Lemma 3.3.3. *$MO(k)$ is homotopy equivalent to $BO(k)/BO(k-1)$.*

Proof. This lemma follows by the following diagram :

$$\begin{array}{ccccc} S(\gamma^k) & \xrightarrow{i} & D(\gamma^k) & \xrightarrow{q} & MO(k) \\ \downarrow f & & \downarrow g & & \downarrow h \\ BO(k-1) & \xrightarrow{i} & BO(k) & \xrightarrow{q} & BO(k)/BO(k-1) \end{array}$$

Here f and g are homotopy equivalences by Lemma 3.3.1 and Lemma 3.3.2, respectively. Note that the induced map h makes the diagram commutative. □

Lemma 3.3.3 and Theorem 1.6.2 enable us to calculate the cohomology of $MO(k)$. Consider the cofibration sequence

$$BO(k-1) \xrightarrow{i} BO(k) \xrightarrow{q} BO(k)/BO(k-1) .$$

In the long exact cohomology sequence the induced homeomorphism
$i^* : \widetilde{H}^*BO(k) \longrightarrow \widetilde{H}^*BO(k-1)$ is given by $i^*(w_i) = w_i$ for $i \leq k-1$ and $i^*(w_i) = 0$ by Theorem 1.6.1. Hence i^* is an epimorphism. This means that the long exact sequence breaks up into the following short exact sequence

$$0 \longrightarrow \widetilde{H}^*BO(k)/BO(k-1) \xrightarrow{q^*} \widetilde{H}^*BO(k) \xrightarrow{i^*} \widetilde{H}^*BO(k-1) \longrightarrow 0 .$$

Since i^* is ring homomorphism, then $ker(i^*) = w_k\mathbb{Z}/2[w_1, w_2, \dots w_k]$.
But $\widetilde{H}^*BO(k)/BO(k-1) \cong ker(i^*)$. By Lemma 3.3.3 $BO(k)/BO(k-1)$ is homotopy equivalent to $MO(k)$ so we have the following corollary.

Corollary 3.3.4. $\widetilde{H}^*MO(k) \cong \mathrm{Z}/2[w_1, w_2, \dots w_k]/\mathbb{Z}/2[w_1, w_2, \dots w_{k-1}] \cong w_k\mathbb{Z}/2[w_1, w_2, \dots w_k]$.

A basis for $H^*MO(k)$ leads to a dual basis for $H_*MO(k)$. To be more explicit we will describe the homology of $MO(k)$ briefly in term of the basis $e_{i_1}, e_{i_2}, \dots, e_{i_k}$.

let $e_i \in H_iBO(1) = H_i\mathbb{R}P^\infty \cong \mathbb{Z}/2$ be the non-zero element for all $i \geq 0$. Let

$$\mu_k : BO(1)^{(k)} = BO(1) \times BO(1) \times \ldots \times BO(1) \rightarrow BO(k)$$

be the map which classifies the product of k copies of the universal line bundle. Then for the sequence

$I = (i_1, i_2, \ldots, i_k)$ of non-negative integers We define

$$e_I = e_{i_1} e_{i_2} \ldots e_{i_k} = (\mu_k)_*(e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_k}) \in H_*BO(k). \qquad (3.3.5)$$

By the homotopy commutativity of the product,

$$\begin{aligned} e_{i_{\sigma(1)}} e_{i_{\sigma(2)}} \ldots e_{i_{\sigma(k)}} &= (\mu_k)_*(e_{i_{\sigma(1)}} \otimes \ldots \otimes e_{i_{\sigma(k)}}) \\ &= (\mu_k)_*(e_{i_1} \otimes \ldots \otimes e_{i_k}) \\ &= e_{i_1} e_{i_2} \ldots e_{i_k} \end{aligned}$$

for each $\sigma \in \Sigma_k$, where Σ_k is the permutation group on k elements. Thus each such element can be written as $e_{i_1} e_{i_2} \ldots e_{i_k}$ where $0 \le i_1 \le i_2 \le \ldots \le i_k$ and it follows by a counting argument that

$$\{e_{i_1} e_{i_2} \ldots e_{i_k} \mid 0 \le i_1 \le i_2 \le \ldots \le i_k\}$$

is a basis for $H_*BO(k)$.

The inclusion map $i : BO(k-1) \to BO(k)$ induces a map in homology $i_* : H_*(BO(k-1)) \to H_*BO(k)$ given by $i_*(e_{i_1} \ldots e_{i_{k-1}}) = e_0 e_{i_1} \ldots e_{i_{k-1}}$.

Hence using Lemma 3.3.3, it follows that

$$\{e_{i_1} e_{i_2} \ldots e_{i_k} \mid 1 \le i_1 \le i_2 \le \ldots \le i_k\}$$

is a basis for $\widetilde{H}_*MO(k)$. Next, we record one of the important properties of Thom complexes.

Theorem 3.3.6. *Thom isomorphism*. *Let γ^k be universal bundle over $BO(k)$ and $MO(k)$ the related Thom space. Then the Thom isomorphism $T : H_*BO(k) \to \widetilde{H}_*MO(k)$ is given by*

$$T(e_{i_1} e_{i_2} \ldots e_{i_k}) = e_{i_1+1} e_{i_2+1} \ldots e_{i_k+1}.$$

Proof. Recall that $\mu_k : BO(1)^{(k)} \to BO(k)$, then $(\mu_k)_* : H_*BO(1)^{(k)} \to H_*BO(k)$ and

$$(\mu_k)_*(e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_k}) = e_{i_1} e_{i_2} \ldots e_{i_k} \in H_*BO(k).$$

By naturality, we have the following diagram

$$\begin{array}{ccc} H_*BO(1)^{(k)} & \xrightarrow{(\mu_k)_*} & H_*BO(k) \\ \downarrow T & & \downarrow T \\ \widetilde{H}_*MO(k)^{(k)} & \xrightarrow{(\mu_k)_*} & \widetilde{H}_*MO(k) \end{array}$$

where T denotes the Thom isomorphism. The Thom isomorphism of $T : H_iBO(1) \to \widetilde{H}_{i+1}MO(1) \cong \mathbb{Z}/2$ is given by $T(e_i) = e_{i+1}$. Therefore, by naturality the Thom isomorphism $T : H_*BO(k) \to \widetilde{H}_*MO(k)$ is given by

$$\begin{aligned} T(e_{i_1}e_{i_2}\dots e_{i_k}) &= T(\mu_k)_*(e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_k}) \\ &= (\mu_k)_*(Te_{i_1} \otimes Te_{i_2} \otimes \dots \otimes Te_{i_k}) \\ &= T(e_{i_1}e_{i_2}\dots e_{i_k}) \\ &= e_{i_1+1}e_{i_2+1}\dots e_{i_k+1}. \end{aligned}$$

□

Recall the map $\mu_k : BO(1)^k \to BO(k)$. Write $a_i \in H^1BO(1)^k$ for the generator $1 \otimes \dots \otimes w_1 \otimes \dots \otimes 1$, with w_1 in the i-th place. Then $H^*BO(1)^k$ is the polynomial ring $\mathbb{Z}/2[a_1, \dots, a_k]$. Evaluation of the total Stiefel-Whitney class of the product of the line bundles gives

$$\mu_k^*(1 + w_1 + \dots + w_k) = (1 + a_1)(1 + a_2)\dots(1 + a_k)$$

from which it follows that $\mu_k^* w_i = \sigma_i(a_1, a_2, \dots, a_k) = \sigma_i$, the i-th elementary symmetric polynomial $\Sigma a_1a_2\dots a_i = \Sigma a_{j_1}a_{j_2}\dots a_{j_i}$ for $1 \leq j_1 < \dots j_i \leq k$. More generally

$$\mu_k^* w^J = \mu^* w_1^{j_1}\dots w_k^{j_k} = \sigma_1^{j_1}\dots\sigma_k^{j_k} = \sigma^J. \tag{3.3.7}$$

This implies to the following result.

Proposition 3.3.8. *For $w^J \in H^*BO(k)$ and $e_I \in H_*BO(k)$ the Kronecker product $\langle w^J, e_I\rangle$ is given by the coefficient of a^I when σ^J is written as a polynomial in $a_1, \dots a_k$.*

Proof. See [AEb00, Proposition. 3.4]. □

We now state our main theorem which explains the relation between the normal Stiefel-Whitney numbers and the Hurewicz homomorphisms. This provides the main computational tool for us that we are going to use in our calculations in next chapters.

Theorem 3.3.9. *Suppose $F : M^n \hookrightarrow N^{n+k}$ is an embedding which corresponds to $\alpha \in [N_+^{n+k}, MO(k)]$ under the Pontrjagin-Thom construction. The normal Stiefel-Whitney numbers of F are given by the Kronecker product*

$$\overline{w}^I[F] = w^I(\nu_F)[M] = \langle w^I w_k, h(\alpha)\rangle.$$

Proof. Let $T(\nu_F)$ be the Thom space of normal bundle of F. Let ν_F be the classifying map of the normal bundle of M and h denote the Hurewicz homomorphism. Let $\alpha \in Emb_k(N^{n+k})$ represent the embedding $M^n \hookrightarrow N^{n+k}$ under the Pontrjagin-Thom construction. Therefore, α is the homotopy class of a composition of the following form.

$$f : N_+^{n+k} \xrightarrow{r} T(\nu_F) \xrightarrow{T(\overline{F})} MO(k) .$$

By Definition 3.1.6

$$h(\alpha) = f_*([N]) = (T(\overline{F})r)_*([N]) \in H_{n+k}MO(k).$$

Now consider the following commutative diagram, where θ_* and ϕ_* denote the Thom isomorphisms in homology.

$$\begin{array}{ccc} H_{n+k}MO(k) & \xrightarrow{\theta_*} & H_nBO(k) \\ \uparrow{\scriptstyle T(\overline{F})_*} & & \uparrow{\scriptstyle (\overline{F})_*} \\ H_{n+k}T(\nu_F) & \xrightarrow{\phi_*} & H_nM \end{array}$$

Therefore, $\theta_*h(\alpha) = \theta_*T(\overline{F})_*r_*([N]) = (\overline{F})_*\phi_*r_*([N])$. However, $r_*([N])$ is non-zero, where

$$r_* : H_{n+k}N_+^{n+k} \longrightarrow H_{n+k}T(\nu_F),$$

and ϕ_* is an isomorphism, therefore $\phi_*r_*([N]) = [M] \in H_n(M) \cong \mathbb{Z}/2$,

$$(\overline{F})_*[M] = (\overline{F})_*\phi_*r_*([N]) = \theta_*T(\overline{F})r_*([N]) = \theta_*h(\alpha).$$

As a result,

$$\begin{aligned}
\overline{w}^I(\nu_F)[M] &= \langle (\overline{F})^*(w^I), [M] \rangle \\
&= \langle w^I, (\overline{F})_*([M]) \rangle \\
&= \langle w^I, \theta_* h(\alpha) \rangle \\
&= \langle \theta^* w^I, h(\alpha)) \rangle \\
&= \langle w^I w_k, h(\alpha) \rangle
\end{aligned}$$

where $\theta^* : H^n(BO(k)) \to H^{n+k}(MO(k))$ denotes the Thom isomorphism in cohomology and is given by $w^I \leftrightarrow w^I w_k$. Hence $h(\alpha)$ determines characteristic numbers of embedding $F : M^n \hookrightarrow N^{n+k}$. □

Let us give an example of how this theorem works out.

Example 3.3.10. *The normal Stiefel-Whitney numbers of*

$$F : \mathbb{R}P^2 \hookrightarrow \mathbb{C}P^n.$$

Let $G : \mathbb{C}P^n \to \mathbb{R}^l$ be an embedding for l large.

In Example 3.2.4 we found that $w(\nu_{G\circ F}) = 1 + a$. Since $\nu_{G\circ F} = \nu_F \oplus F^*\nu_G$ then

$$1 + a = w(\nu_{G\circ F}) = w(\nu_F)F^*w(\nu_G).$$

Moreover, $c(\tau_{\mathbb{C}P^n}) = (1 + b)^{n+1}$, where $b \in H^2(\mathbb{C}P^n)$ and c is the total Chern class by Proposition 1.7.3 and Definition 1.7.2. So $c(\nu_{\mathbb{C}P^n}) = (1 + b)^{-n-1}$. Hence

$$c_1(\nu_{\mathbb{C}P^n}) = -(n + 1)b.$$

Since the class $w_2(\nu_{\mathbb{C}P^n})$ is the mod 2 restriction of $c_1(\nu_{\mathbb{C}P^n})$ by Lemma 1.7.4 , then

$$w_2(\nu_{\mathbb{C}P^n}) = \begin{cases} b & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

In the case of n odd, then $F^*w(\nu_G) = F^*w(\nu_{\mathbb{C}P^n}) = 1$ and so $w(\nu_F) = w(\nu_{G\circ F}) = 1 + a$.

On the other hand when n is even, then $F^*w(\nu_G) = 1 + F^*(b) = 1 + \lambda a^2$, for some $\lambda \in \mathbb{Z}/2$. Then $1 + a = (1 + \lambda a^2)w(\nu_F)$. Hence

$$w(\nu_F) = (1 + a)(1 + \lambda a^2)^{-1} = (1 + a)(1 + \lambda a^2) = 1 + a + \lambda a^2.$$

Now we have two possibilities for λ. If $\lambda = 1$ then $w(\nu_F) = 1+a+a^2$. So $w_1(\nu_F) = a$, $w_2(\nu_F) = a^2$. Hence by Theorem 3.3.9 the normal Stiefel-Whitney numbers are

$$w_1^2(\nu_F)[\mathbb{R}P^2] = \langle w_1^2 w_2, h(\alpha)\rangle = 1,$$

$$w_2(\nu_F)[\mathbb{R}P^2] = \langle w_2^2, h(\alpha)\rangle = 1.$$

From (3.3.8) since we have $\mu_k^*(w_1^2) = (\Sigma a_1)^2 = \Sigma a_1^2$, $\mu_k^*(w_2) = \Sigma a_1 a_2$ and $\mu_k^*(w_k) = \Sigma a_1 \dots a_k = a_1 \dots a_k$. Then

$$\begin{aligned} \mu_k^*(w_1^2 w_k) &= \Sigma a_1^2 a_1, \dots a_k \\ &= \Sigma a_1^3 a_2 \dots a_k, \end{aligned}$$

$$\begin{aligned} \mu_k^*(w_2 w_k) &= (\Sigma a_1 a_2) a_1, \dots a_k \\ &= \Sigma a_1^2 a_2^2 a_3 \dots a_k. \end{aligned}$$

Hence, by Proposition 3.3.8,

$$\begin{aligned} \langle w_1^2 w_k, e_1^{k-1} e_3\rangle &= 1, \\ \langle w_2 w_k, e_1^{k-1} e_3\rangle &= 0, \\ \langle w_1^2 w_k, e_1^{k-2} e_2^2\rangle &= 0, \\ \langle w_2 w_k, e_1^{k-2} e_2^2\rangle &= 1. \end{aligned}$$

Hence

$$h(\alpha) = e_1^{k-1} e_3 + e_1^{k-2} e_2^2.$$

When $\lambda = 0$ then $w_1(\nu_F) = a \in H^1(\mathbb{R}P^2)$, $w_2(\nu_F) = 0$ so

$$w_1^2(\nu_F)[\mathbb{R}P^2] = 1, \quad w_2(\nu_F)[\mathbb{R}P^2] = 0.$$

Hence, as above

$$h(\alpha) = e_1^{k-1} e_3.$$

Proposition 3.3.11. *Given an embedding $L^2 \hookrightarrow \mathbb{C}P^n$, for n is even corresponding to $\alpha \in [\mathbb{C}P^n_+, MO(2n-2)]$, then $h(\alpha) \in H_{2n}MO(2n-2)$ determines L^2 up to cobordism. $L^2 \sim \mathbb{R}P^2$ if and only if*

$$h(\alpha) = e_1^{2n-1}e_3 + e_1^{2n-2}e_2^2,$$

or

$$h(\alpha) = e_1^{2n-1}e_3.$$

L^2 is a boundray if and only if

$$h(\alpha) = e_1^{2n-2}e_2^2,$$

or

$$h(\alpha) = 0.$$

Proof. The cobordism class of the embedding $L^2 \hookrightarrow \mathbb{C}P^n$ corresponds to

$$\alpha \in Emb_{2n-2}(\mathbb{C}P^n) \cong [\mathbb{C}P^n_+, MO(2n-2)] \cong [\mathbb{C}P^n_{n-1}, MO(2n-2)].$$

Then the cofiber sequence

$$S^{2n-2} \to \mathbb{C}P^n_{n-1} \to S^{2n}$$

induces a short exact sequence

$$0 \to \pi_{2n}MO(2n-2) \to [\mathbb{C}P^n_{n-1}, MO(2n-2)] \to \pi_{2n-2}MO(2n-2) \to 0,$$

where $\pi_{2n}MO(2n-2) \cong \mathbb{Z}/2$ and $\pi_{2n-2}MO(2n-2) \cong \mathbb{Z}/2$. Therefore, $[\mathbb{C}P^n_{n-1}, MO(2n-2)]$ has order 4. Now we shall show that the Hurewicz image

$$h\colon [\mathbb{C}P^n_{n-1}, MO(2n-2)] \to H_{2nMO(2n-2)}$$

is a monomorphism.

The embedding $F\colon \mathbb{R}P^2 \hookrightarrow \mathbb{R}^{2n} \subset \mathbb{C}P^n$ gives $\theta \in [\mathbb{C}P^n_+, MO(2n-2)]$. From Example 3.3.10 since $\mathbb{R}P^2 \not\hookrightarrow \mathbb{R}^3$, $w_2(\nu_F) = 0$, so $w(\nu_F) = 1 + a^2$. Hence

$$h(\theta) = e_1^{2n-1}e_3 \in H_{2n}MO(2n-2).$$

The embedding $G\colon S^2 = \mathbb{C}P^1 \hookrightarrow \mathbb{C}P^n$ gives $\phi \in [\mathbb{C}P^n_+, MO(2n-2)]$. Since

$$\begin{aligned} w(\tau_{\mathbb{C}P^1})w(\nu_G) &= w(G^*\tau_{\mathbb{C}P^n}) \\ (1+b)^2 w(\nu_G) &= (1+b)^{n+1}, \end{aligned}$$

so (since n is even), then

$$\begin{aligned} w_2(\nu_G) &= (n+1)b \\ &= b. \end{aligned}$$

This gives

$$h(\phi) = e_1^{2n-2}e_2^2 \in H_{2n}MO(2n-2)$$

as in Example 3.3.10. Thus $h([\mathbb{C}P^n_+, MO(2n-2)]) \subseteq H_{2n}MO(2n-2)$ has order 4 and so h is an isomorphism. Hence $h(\alpha)$ determines $\alpha \in Emb_{2n-2}(\mathbb{C}P^n)$ and therefore determines the embedded manifold (up to cobordism).

The above examples give the result in the theorem once we observe that if $h(\alpha) = e_1^{2n-1}e_3 + e_1^{2n-2}e_2^2$, then the surface is cobordant to $\mathbb{R}P^2 \sqcup S^2 \sim \mathbb{R}P^2$. □

In the final two chapters we will study the double point manifold of an immersion $M^{n+1} \looparrowright \mathbb{C}P^n$ and this will be a surface embedded in $\mathbb{C}P^n$. In chapter 7 we will study the values of $h(\alpha)$. The above example shows that we need to take account of the ambient space of the embedding in order to determine the Stiefel-Whitney numbers of the surface via the Hurewicz image.

If $N = \mathbb{R}^{n+k}$ we can restate Thom's theorem in the following form.

Theorem 3.3.12. ***Thom***. *Suppose that $M^n \hookrightarrow \mathbb{R}^{n+k}$ represents $\alpha \in \pi_{n+k}MO(k)$. Then the Hurewicz image $h(\alpha) \in H_{n+k}MO(k)$ determines the cobordism class of M.*

Proof. Theorem 3.3.9 shows that the Hurewicz image $h(\alpha) \in H_{n+k}MO(k)$ determines the characteristic numbers of M. Then from Theorem 3.2.7 this determines the cobordism class of M. □

For $F : M^n \hookrightarrow N^{n+k}$, by Example 3.3.10 we have shown that the normal bundle depend on the embedding F. However if $N = \mathbb{R}^{n+k}$ we have the next Proposition.

Proposition 3.3.13. *Given an embedding $F : M^n \hookrightarrow \mathbb{R}^{n+k}$, then the stable normal bundle is independent of the embedding F. Hence the normal Stiefel-Whitney numbers are given by Kronecker product*

$$\overline{w}^I[M] = \langle w^I w_k, h(\alpha)\rangle.$$

We will provide a simple example to show the Hurewicz image arising from the real projective plane in Euclidean spaces. Then illustrate the relation between homology classes and normal Stiefel-Whitney numbers.

Example 3.3.14. For an embedding $F : \mathbb{R}P^2 \hookrightarrow \mathbb{R}^4$ the homology class $h(\alpha) \in H_4MO(2)$ is shown in the following table.

$\mathbb{R}P^2 \hookrightarrow \mathbb{R}^4$	$(\mathbb{R}P^2)^{(2)} \hookrightarrow (\mathbb{R}^4)^{(2)}$	$(\mathbb{R}P^2)^{(3)} \hookrightarrow (\mathbb{R}^4)^{(3)}$
$k = 2$	$k = 4$	$k = 6$
$\alpha \in \pi_4 MO(2)$	$\alpha \in \pi_8 MO(4)$	$\alpha \in \pi_{12} MO(6)$
$h(\alpha) = e_1 e_3$	$h(\alpha) = e_1^2 e_3^2$	$h(\alpha) = e_1^3 e_3^3$

Here the embedding $F^{(2)} : (\mathbb{R}P^2)^{(2)} \hookrightarrow (\mathbb{R}^4)^{(2)}$ is the Cartesian product of two copies of $\mathbb{R}P^2$ and so on. By Example 3.2.4 the normal Stiefel-Whitney numbers of $\mathbb{R}P^2$ are given by

$$\overline{w}_1^2[\mathbb{R}P^2] = 1 \text{ and } \overline{w}_2[\mathbb{R}P^2] = 0.$$

Then by Proposition 3.3.13

$$\langle w_1^2 w_2, h(\alpha)\rangle = 1 \text{ and } \langle w_2^2, h(\alpha)\rangle = 0$$

However,

$$\langle w_1^2 w_2, e_1 e_3\rangle = 1 \text{ and } \langle w_2^2, e_1 e_3\rangle = 0$$

also

$$\langle w_1^2 w_2, e_2^2\rangle = 0 \text{ and } \langle w_2^2, e_2^2\rangle = 1$$

and so

$$h(\alpha) = e_1 e_3.$$

The other examples are calculated similarly.

In the second case of Example 3.3.10, when $\lambda = 0$ we get similar result for the normal Stiefel-Whitney numbers as in example 3.2.4 and same result of homology class $h(\alpha)$ as in above example. However, we get different normal Stiefel-Whitney numbers for embedding manifold in complex projective case. Hence we deduce that the normal bundle depends on the embedding in general.

3.4 Pontrjagin-Thom theory for embeddings with ξ-structures

The Pontrjagin-Thom theory for embeddings has a well developed generalisation where one puts a specific structure on the normal bundle.

Definition 3.4.1. *Suppose ξ and ν are arbitrary $\mathbb{R}^k$-bundles, not necessarily over the same base space. We say ν has a ξ-structure if there is a map of bundles $\nu \to \xi$ which induces isomorphism on fibres, i.e. if the bundle map is covered by $\widehat{F} : E(\nu) \to E(\xi)$ and $\overline{F} : B(\nu) \to B(\xi)$ then the mapping $\widehat{F}$ maps $F_b(\nu)$ isomorphically onto $F_{\overline{F}(b)}(\xi)$. Given an embedding $F : M^n \hookrightarrow N^{n+k}$ we say that it has a ξ-structure if the normal bundle ν_F has a ξ-structure.*

Notice that if $F : M^n \hookrightarrow N^{n+k}$ has a ξ-structure, we then obtain a mapping $T(\nu_F) \to T(\xi)$ by *Thomification*. Similar to the previous theory, there are notions such as regular homotopy between two embeddings $F_0, F_1 : M^n \hookrightarrow N^{n+k}$ with ξ-structures and cobordism of embeddings with ξ-structure. Notice that in this way, given any embedding $F : M^n \hookrightarrow N^{n+k}$ with a ξ-structure, the Tubular Neighborhood Theorem provides a mapping $N_+ \to T(\nu_F)$ where composition with $T(\nu_F) \to T(\xi)$ yields a mapping $N_+ \to T\xi$. This then defines a mapping

$$\tau : Emb_k^{\xi}(N^{n+k}) \longrightarrow [N_+^{n+k}, T\xi]$$

where $Emb_k^{\xi}(N^{n+k})$ is the set of all coboridsm classes for embeddings $F : M^n \hookrightarrow N^{n+k}$ with a ξ-structure. We refer the reader to [S68, chapter 2] for details on this. We mention the main theorem which is analogous to the classical one.

Theorem 3.4.2. *The Pontrjagin-Thom construction*

$$\tau : Emb_k^{\xi}(N^{n+k}) \longrightarrow [N_+^{n+k}, T\xi]$$

is a one to one correspondence.

Proof. See [S68, Theorem on page 18]. □

This theorem will be useful when we introduce the Pontrjagin-Thom construction for immersions.

Chapter 4

Pontrjagin-Thom theory for immersions

The Pontrjagin-Thom construction, which originally was developed for embeddings, has been extended to the case of immersions. In this version of the Pontrjagin-Thom theory, the cobordism classes of immersions $M^n \looparrowright N^{n+k}$ are related to stable homotopy classes of maps $N_+^{n+k} \to MO(k)$.

In this chapter, and based on The compression theorems of Rourke and Sanderson [RS01], we provide a new proof of this extended Pontrjagin-Thom theory for the case of immersions $M^n \looparrowright N^{n+k}$.

We start with the cobordism theory of immersions.

Definition 4.0.1. *Let $F : M_0^n \looparrowright N^{n+k}$ and $G : M_1^n \looparrowright N^{n+k}$ be two immersions of closed n-manifolds in N^{n+k}. Then F and G are said to be* cobordant*, denoted by $F \sim G$, if the following conditions hold,*

(1) *There exists $(n+1)$-manifold W^{n+1} such that $\partial W^{n+1} = M_0 \times 0 \sqcup M_1 \times 1$,*

(2) *There exists an immersion $H : W^{n+1} \looparrowright N^{n+k} \times I$, where $I = [0,1]$, and a projection map $\pi : N^{n+k} \times I \to I$ such that $H|_{M_0 \times 0} = F \times 0$ and $H|_{M_1 \times 1} = G \times 1$.*

(3) *For $H(W) \subset N^{n+k} \times I \subset N^{n+k} \times \mathbb{R}$,*

$$\pi H(x) \in \{0,1\} \Leftrightarrow x \in \partial W^{n+1},$$

$$\pi H(x) = 0 \Leftrightarrow x \in M_0 \times 0 \;\; and \;\; \pi H(x) = 1 \Leftrightarrow x \in M_1 \times 1.$$

This relation is an equivalence relation on the set of immersions of n dimensional manifolds M^n into N^{n+k}. We write $[M^n, F]$ for the equivalence class of such an immersion. We write $Imm_k(N^{n+k})$ for the set of all cobordism classes of immersions $M^n \looparrowright N^{n+k}$. Next, we put the structure of a group on $Imm_k(N^{n+k})$. First, we record the following fact.

Lemma 4.0.2. *Let $F : M^n \looparrowright N^{n+k}$ be an immersion. Then there exists large l such that the composition*

$$i \circ F : M^n \looparrowright N^{n+k} \longrightarrow N^{n+k} \times \mathbb{R}^l$$

is regular homotopic to an embedding $F_1 : M^n \hookrightarrow N^{n+k} \times \mathbb{R}^l$, where i is the standard inclusion $i : N^{n+k} \longrightarrow N^{n+k} \times \mathbb{R}^l$

Proof. Applying Whitney's embedding theorem, Theorem 1.4.5, we can find $l > 0$ and an embedding $G : M^n \hookrightarrow \mathbb{R}^l$. Define

$$H : M^n \times I \to N^{n+k} \times \mathbb{R}^l,$$

by

$$(x, t) \to (F(x), tG(x)),$$

for $(x, t) \in M \times I$. Notice that $H(x, 0) = i \circ F(x)$.
The mapping F is immersion and hence $i \circ F$ is an immersion. Moreover, G is an embedding which in particular means it is an immersion. This implies that $H_t : M^n \times I \to N^{n+k} \times \mathbb{R}^l$ defined by $H_t(x) = H(x, t)$ is an immersion for all $t \in [0, 1]$. Hence, H is a regular homotopy.

On the other hand, since G is one to one, hence H_1 given by

$$H_1(x) = (F(x), G(x))$$

is a one to one immersion, i.e. H_1 is an embedding as M is compact. We can choose $F_1 = H_1$. This completes the proof. $\square$

We now introduce the group operation on $Imm_k(N^{n+k})$.

Definition 4.0.3. *Let* $F : M_1^n \looparrowright N^{n+k}$ *and* $G : M_2^n \looparrowright N^{n+k}$ *be two immersions, then the sum*

$$F \sqcup G : M_1^n \sqcup M_2^n \looparrowright N^{n+k}$$

is an immersion, where the symbol $\sqcup$ *be the disjoint union.*

Theorem 4.0.4. *Let* $F : M_1 \looparrowright N$ *and* $G : M_2 \looparrowright N$ *be two immersions. If* $i \circ F$ *and* $i \circ G$ *are regular homotopic to embeddings*

$$F_1 : M_1 \hookrightarrow N \times \mathbb{R}^l \ \text{ and } G_1 : M_2 \hookrightarrow N \times \mathbb{R}^l.$$

Then there exist an embedding $F \sqcup G \colon M_1 \sqcup M_2 \hookrightarrow N^{n+k} \times \mathbb{R}^l$ *which is regular homotopic to* $i \circ (F \sqcup G)$.

Proof. Suppose that $i \circ F : M_1 \looparrowright N^{n+k} \to N^{n+k} \times \mathbb{R}^l$, $i \circ G : M_2 \looparrowright N^{n+k} \to N^{n+k} \times \mathbb{R}^l$. Applying Lemma 4.0.2 we find that the composition $i \circ F$ is regular homotopic to an embedding $F_1 : M_1 \hookrightarrow N^{n+k} \times \mathbb{R}^l$. Also $i \circ G$ is regular homotopic to an embedding $G_1 : M_2 \hookrightarrow N^{n+k} \times \mathbb{R}^l$.

Now we notice that $\mathbb{R}^l \cong \mathbb{R}^{l-1} \times (0, \infty)$ as well as $\mathbb{R}^{l-1} \times (-\infty, 0)$, using $\mathbb{R} \cong (-\infty, 0) \cong (0, \infty)$. Then let

$$F_2 : M_1 \hookrightarrow N^{n+k} \times \mathbb{R}^l \cong N^{n+k} \times \mathbb{R}^{l-1} \times (-\infty, 0) \subset N^{n+k} \times \mathbb{R}^{l-1} \times \mathbb{R}.$$

$$G_2 : M_2 \hookrightarrow N^{n+k} \times \mathbb{R}^l \cong N^{n+k} \times \mathbb{R}^{l-1} \times (0, \infty) \subset N^{n+k} \times \mathbb{R}^{l-1} \times \mathbb{R}.$$

F_2 is clearly regular homotopic to F_1 and so to $i \circ F$, also G_2 is similarly regular homotopic to G_1 and so to $i \circ G$. Next we have the map

$$F_2 \sqcup G_2 : M_1 \sqcup M_2 \to N^{n+k} \times \mathbb{R}^l$$

is an embedding and is regular homotopic to $i \circ (F \sqcup G)$. □

The Pontrjagin-Thom theorem for immersions can deduced from the Pontrjagin-Thom theorem for embeddings. In one direction, starting with a given immersion, we record the following fact.

Lemma 4.0.2 allows us to define a group homomorphism

$$Imm_k(N^{n+k}) \longrightarrow [N_+^{n+k}, QMO(k)].$$

Lemma 4.0.5. *There is a group homomorphism*

$$\tau : Imm_k(N^{n+k}) \longrightarrow [N_+^{n+k}, QMO(k)].$$

Proof. Let $F : M^n \looparrowright N^{n+k}$ be an immersion. Then according to Lemma 4.0.2 there exists $l > 0$ such that the composition

$$M^n \looparrowright N^{n+k} \longrightarrow N^{n+k} \times \mathbb{R}^l$$

is regular homotopic to an embedding. The normal bundle of this embedding is given by $\nu_F \oplus \varepsilon^l$ and its Thom space is $T(\nu_F \oplus \varepsilon^l) \cong \Sigma^l T(\nu_F)$ by Theorem 1.5.7. Applying the Pontrjagin-Thom construction to this embedding we obtain a mapping

$$(N^{n+k} \times \mathbb{R}^l)_+ \cong \Sigma^l N_+^{n+k} \longrightarrow \Sigma^l T(\nu_F).$$

Moreover, notice that ν_F is a k-vector bundle, hence we obtain a mapping

$$T(\nu_F) \to T(\gamma^k) = MO(k).$$

Hence, composition gives

$$\widetilde{f} : \Sigma^l N_+^{n+k} \longrightarrow \Sigma^l MO(k).$$

We then have the adjoint,

$$N_+^{n+k} \longrightarrow \Omega^l \Sigma^l MO(k).$$

Composing this map with the inclusion $\Omega^l \Sigma^l MO(k) \to QMO(k)$ then gives a map,

$$f : N_+^{n+k} \longrightarrow QMO(k).$$

Moreover, let $[M_1, F]$, $[M_2, G]$ be two immersions which are cobordant through the immersion $H : W^{n+1} \looparrowright N^{n+k} \times I$. Applying the same technique as above yields a homotopy, say $\tau([W^{n+1}, H]) : (N^{n+k} \times I)_+ \to QMO(k)$ between F and G. Hence, we have obtained a mapping

$$\tau : Imm_k(N^{n+k}) \longrightarrow [N_+^{n+k}, QMO(k)].$$

Now for an immersions $F, G : M^n \looparrowright N^{n+k}$, the last suspension coordinate will be $\mathbb{R}$. Let $Y = F_2 \sqcup G_2 : M \hookrightarrow N^{n+k} \times \mathbb{R}^l$ be the embedding of Theorem 4.0.4 which represent $[F]+[G]$. So by taking the sum of last suspension coordinates and applying Theorem 4.0.4 we find that $\widetilde{y} \simeq \widetilde{f} + \widetilde{g}$ where $y = \tau([Y])$ and $\widetilde{y} \in [\Sigma^l N_+^{n+k}, \Sigma^l MO(k)]$ correspond to Y. Hence

$$\tau([F \sqcup G]) = \tau([F]) + \tau([G])$$

so that τ is a homomorphism. Notice that $QMO(k)$ is a loop space. Hence the set $[N_+^{n+k}, QMO(k)]$ is a group. □

4.1 The compression theorem

In order to describe an inverse mapping

$$[N_+^{n+k}, QMO(k)] \longrightarrow Imm_k(N^{n+k})$$

we need to start with a given map $N_+^{n+k} \to QMO(k)$ and show that it represents a unique cobordism class of immersions $M^n \looparrowright N^{n+k}$. This can be done using the Pontrjagin-Thom theory for embeddings with ξ-structure together with the Compression theorem of Rourke and Sanderson. Before proceeding we recall some definitions.

Definition 4.1.1. *Let $F : M^n \to N^{n+k}$ be an immersion. A normal vector field on M is a mapping $s : M^n \to E(\nu_F)$ such that $s(x) \in F_x(\nu_F)$ for each $x \in M^n$. More briefly, $\pi \circ s = 1_M$ where $\pi : E(\nu_F) \to M$ is the projection and 1_M is the identity function on M.*

Notice that ν_F is a k-vector bundle, i.e. $F_x(\nu_F) \cong \mathbb{R}^k$ for each $x \in M^n$. Suppose $s_1, \ldots, s_l$ are normal vector fields on M^n. We then have the following definition.

Definition 4.1.2. *We say the vector fields $s_1, \ldots, s_l : M \to E(\nu_F)$ are linearly independent if and only if the $\{s_1(x), \ldots, s_l(x)\}$ is a linearly independent subset in $F_x(\nu_F)$ for each $x \in M^n$.*

Definition 4.1.3. *Let $F : M^n \hookrightarrow N^{n+k} \times \mathbb{R}$ be an embedding, then the normal vector field is called straightened if it is parallel to the given $\mathbb{R}$ direction, namely, $s(x) = (0, e_1) \in F_x(\nu_F) \subseteq T_{F(x)}N \times \mathbb{R}$ for all $x \in M$ where $e_1 = 1 \in \mathbb{R}$.*

Now we want to construct the inverse to the homomorphism of Lemma 4.0.5. To do this we need to use The compression theorem which will now be explained.

Theorem 4.1.4. ***The compression theorem*** *. Let M^n be a compact manifold embedded in $N^{n+k} \times \mathbb{R}$ and equipped with a non-trivial normal vector field. Assume $k \geq 1$. Then the vector field can be straightened by an isotopy of M.*

Proof. See [RS99, section 2.] □

Definition 4.1.5. *An embedding $M \hookrightarrow N^{n+k} \times \mathbb{R}$ is called compressible if it projects by vertical projection to an immersion in N^{n+k}.*

We think of $\mathbb{R}$ as vertical and the positive $\mathbb{R}$ direction as upwards. Theorem 4.1.4 moves M to a position where it is *compressible*. We could say that the vector field always points vertically up.

More generally, we can straighten a sequence of vector fields. More precisely, suppose that M is embedded in $N^{n+k} \times \mathbb{R}^l$ with l independent normal vector fields, then M is isotopic to an embedding in which each vector field is parallel to the corresponding copy of $\mathbb{R}$.

Definition 4.1.6. *let M^n be embedded in $N^{n+k} \times \mathbb{R}^l$ and suppose that M is equipped with l linearly independent normal vector fields. Then we say that the embedding is* parallel *if the l vector fields are parallel to the l coordinate directions in $\mathbb{R}^l$,i.e. $s_i(x) = (0, e_i)$ for $1 \leq i \leq l, x \in M$.*

Theorem 4.1.7. ***Multi-compression theorem*** *. Suppose that M^n is embedded in $N^{n+k} \times \mathbb{R}^l$ with l independent normal vector fields and $k \geq 1$. Then the l vector field can be straightened by an isotopy of M to a parallel embedding .*

Proof. See [RS01, Theorem 4.5.] □

The following Theorem is an application of the compression theorem.

Theorem 4.1.8. *Suppose $F : M^n \hookrightarrow N^{n+k} \times \mathbb{R}^l$ is a parallel embedding with l linearly independent normal vector fields. Then*

$$F_1 = p_1 \circ F : M^n \looparrowright N^{n+k}$$

is an immersion, i.e F is a compressible embedding, where $p_1 : N^{n+k} \times \mathbb{R}^l \to N^{n+k}$ is the projection map.

Proof. Suppose that $F : M^n \hookrightarrow N^{n+k} \times \mathbb{R}^l$ is a parallel embedding and consider $F_1 = p_1 \circ F : M^n \to N^{n+k}$ as shown in next figure.

$$\begin{array}{ccc} M^n & \xrightarrow{F} & N^{n+k} \times \mathbb{R}^l \\ & \searrow^{F_1} & \downarrow p_1 \\ & & N^{n+k} \end{array}$$

Let $F(x) = (F_1(x), F_2(x))$, where $F_1(x) \in N^{n+k}$ and $F_2(x) \in \mathbb{R}^l$. Then

$dF_x : T_xM^n \to T_{F(x)}(N^{n+k} \times \mathbb{R}^l) = T_{F_1(x)}N^{n+k} \times T_{F_2(x)}\mathbb{R}^l = T_{F_1(x)}N^{n+k} \times \mathbb{R}^l$.

For $0 \leq i \leq l$, $s_i(x) = (0, e_i) \in F_x(\nu_F) \subseteq T_{F_1(x)}N^{n+k} \times \mathbb{R}^l$. Therefore, $\{0\} \times \mathbb{R}^l \subseteq F_x(\nu_F)$. Then $(dF_x(T_xM)) \cap (\{0\} \times \mathbb{R}^l) = \{0\}$. Hence $\{0\} \times \mathbb{R}^l$ will not lie in

$$dF_x(T_xM) \subseteq T_{F(x)}(N^{n+k} \times \mathbb{R}^l).$$

Now by taking the projection map

$$dp_1 : T_{F(x)}(N \times \mathbb{R}^l) = T_{F_1(x)}N^{n+k} \times \mathbb{R}^l \to T_{F_1(x)}N$$

the kernel $dp_1 = \{0\} \times \mathbb{R}^l$.

$$\begin{array}{ccccc} T_xM^n & \xrightarrow{dF_x} & T_{F(x)}(N^{n+k} \times \mathbb{R}^l) & \xrightarrow{=} & T_{F_1(x)}N^{n+k} \times \mathbb{R}^l \\ \downarrow (dF_1)_x & & \downarrow dp_1 & & \downarrow dp_1 \\ T_{F_1(x)}(N^{n+k}) & \xrightarrow{=} & T_{F_1(x)}(N^{n+k}) & \xrightarrow{=} & T_{F_1(x)}N^{n+k} \end{array}$$

Then

$$(dF_1)_x : T_xM^n \longrightarrow T_{F_1(x)}(N^{n+k})$$

is a monomorphism if and only if

$$dF_x(T_xM) \cap \text{ kernel } dp_1 = dF_x(T_xM^n) \cap (\{0\} \times \mathbb{R}^l) = \{0\}$$

which we have proved. Hence

$$F_1 : M^n \to N^{n+k}$$

is an immersion which completes the proof. □

Theorem 4.1.9. *There is a function*

$$t : [N_+^{n+k}, QMO(k)] \longrightarrow Imm_k(N^{n+k}),$$

inverse to the function τ *of Theorem 4.0.5.*

The proof of this theorem is similar to that of the case of embeddings, replacing embeddings by immersions in cobordism theory corresponds to replacing homotopy groups of Thom complexes by stable homotopy groups.

Proof. Let $\alpha \in [N_+^{n+k}, QMO(k)]$ be represented by a map

$$f : N_+^{n+k} \longrightarrow QMO(k)$$

where $QMO(k) = \lim\, \Omega^l \Sigma^l MO(k)$. Thus, there exists $l \geqslant 0$ such that we may realize f as a mapping $f : N_+^{n+k} \to \Omega^l \Sigma^l MO(k)$. We then consider the adjoint of f as a mapping

$$g : \Sigma^l N_+^{n+k} \longrightarrow \Sigma^l MO(k).$$

Notice that $\Sigma^l N_+^{n+k} \cong (N^{n+k} \times \mathbb{R}^l)_+$ and $\Sigma^l MO(k) \cong T(\gamma^k \oplus \varepsilon^l)$ where ε^l is the trivial l-dimensional bundle over $BO(k)$. Hence, we may consider g as a mapping

$$(N^{n+k} \times \mathbb{R}^l)_+ \longrightarrow T(\gamma^k \oplus \varepsilon^l).$$

By the generalised Pontrjagin-Thom construction for embeddings this mapping represents an embedding

$$M^n \hookrightarrow N^{n+k} \times \mathbb{R}^l$$

where the normal bundle of this embedding admits a splitting $\nu \oplus \varepsilon^l$. This then satisfies the conditions of Theorem 4.1.8. Hence, by the Multi compression theorem, by composing with the projection $N^{n+k} \times \mathbb{R}^l \to N^{n+k}$ we obtain a mapping

$$M^n \hookrightarrow N^{n+k} \times \mathbb{R}^l \longrightarrow N^{n+k}$$

which is isotopic to an immersion $F : M^n \looparrowright N^{n+k}$ with $\nu_F \cong \nu$.

Moreover, one may start with a homotopy and end up with a cobordism class by applying the analogous construction. Hence, we have defined a function

$$t : [N_+^{n+k}, QMO(k)] \longrightarrow Imm_k(N^{n+k}).$$

It is straightforward to check that t and τ are inverse functions. □

Corollary 4.1.10. *The homomorphism*

$$\tau : Imm_k(N^{n+k}) \cong [N_+^{n+k}, QMO(k)]$$

is an isomorphism.

Notice that in this particular case, the Pontrjagin-Thom theory provides

$$Imm_k(\mathbb{R}^{n+k}) \to [S^{n+k}, QMO(k)] \cong \pi^S_{n+k}MO(k),$$

which relate the $(n+k)$-th stable homotopy group of $MO(k)$ to the cobordism of codimension k immersed in $\mathbb{R}^{n+k}$.

4.2 Stiefel-Whitney numbers of immersions

The cobordism class of manifolds are determined by their normal Stiefel-Whitney numbers. This is similar to the case of embedded manifolds and we may apply a similar construction to determine the Stiefel-Whitney numbers of an immersion $M^n \looparrowright N^{n+k}$. Let $f : N_+^{n+k} \to QMO(k)$ represent an element α in $[N_+^{n+k}, QMO(k)]$. Then there exists $l \geqslant 0$ such that $f : N_+^{n+k} \to \Omega^l\Sigma^l MO(k)$. This gives the adjoint mapping $\Sigma^l N_+^{n+k} \to \Sigma^l MO(k)$.

Definition 4.2.1. *The stable Hurewicz homomorphism*

$$h^S : [N_+^{n+k}, QMO(k)] \longrightarrow H_{n+k}MO(k)$$

is defined by the composition

$$[N_+^{n+k}, \Omega^l\Sigma^l MO(k)] \cong [\Sigma^l N^{n+k}, \Sigma^l MO(k)] \xrightarrow{h} H_{n+k+l}\Sigma^l MO(k) \cong H_{n+k}MO(k)$$

where h is the Hurewicz homomorphism defined in previous section.

The following theorem is a generalisation of the result of Asadi and Eccles [AEb00, Lemma 2.2].

Theorem 4.2.2. *Suppose an immersion $F : M^n \looparrowright N^{n+k}$ corresponds to an element $\alpha \in [N_+^{n+k}, QMO(k)]$. Then the normal Stiefel-Whitney numbers of the immersion are determined by*

$$\langle w^I(\nu_F), [M]\rangle = \langle w^I w_k, h^S(\alpha)\rangle.$$

The proof of this theorem is based on Theorem 3.3.9, Lemma 4.0.2, and the Thom-Pontrjagin theory for immersions.

Proof. Suppose $F : M^n \looparrowright N^{n+k}$ is an immersion, corresponding to an element $\alpha \in [N_+^{n+k}, QMO(k)]$. Under The Thom-Pontrjagin construction we represent α by the map

$$f : N_+^{n+k} \to QMO(k) = \text{direct lim } \Omega^l \Sigma^l MO(k).$$

On the other hand, according to Lemma 4.0.2, there exists $l > 0$ such that F is regular homotopic to an embedding $F_1 : M^n \hookrightarrow N^{n+k} \times \mathbb{R}^l$ with $\nu_{F_1} \cong \nu_F \oplus \varepsilon^l$, i.e. F_1 corresponds to an element $\widetilde{\alpha} \in [\Sigma^l N_+^{n+k}, \Sigma^l MO(k)]$ which is represented by

$$\widetilde{f} : \Sigma^l N_+^{n+k} \to \Sigma^l MO(k).$$

Observe that, according to the proof of the Thom-Pontrjagin theory for immersions, we may think of $\widetilde{\alpha}$ as the stable adjoint of α. In particular, this implies that

$$h^S(\alpha) = h(\widetilde{\alpha}).$$

Applying Whitney's product theorem we see that $w_i(\nu_{F_1}) = w_i(\nu_F \oplus \varepsilon^l) = w_i(\nu_F)$ for all $i \leqslant k$ and $w_i(\nu_{F_1}) = 0$ for $i > k$. This means that ν_F and ν_{F_1} have the same Stiefel-Whitney classes, and hence the same Stiefel-Whitney numbers. This implies that

$$\langle w^I(\nu_F), [M]\rangle = \langle w^I(\nu_{F_1}), [M]\rangle = \langle w^I w_k, h(\widetilde{\alpha})\rangle = \langle w^I w_k, h^S(\alpha)\rangle.$$

This completes the proof. □

Corollary 4.2.3. *For an immersion $F : M^n \looparrowright \mathbb{R}^{n+k}$ corresponding to $\alpha \in \pi_{n+k} QMO(k)$, the normal Stiefel-Whitney numbers of the manifolds are determined by*

$$\overline{w}^I[M] = \langle w^I w_k, h^S(\alpha)\rangle.$$

Chapter 5

Steenrod operations and Kudo-Araki operations

We have observed that determining the cobordism class of a manifold M^n depends on determining the Steifel-Whitney numbers of the stable normal bundle of M. According to Theorem 4.2.2 determining the normal Steifel-Whitney numbers of F can be done by calculating the image of the fundamental class $[N_+^{n+k}]$ under the stable Hurewicz homomorphism

$$h^S : [N_+^{n+k}, MO(k)]^S \to H_{n+k}MO(k).$$

In this chapter, we will describe the homology of $QMO(k)$ and introduce some algebraic tools that help us to do calculations in the homology ring $H_*QMO(k)$ namely the Steenrod operations, and the Kudo-Araki operations.

5.1 Steenrod operations

Roughly speaking, a cohomology operation θ is homomorphism from the additive group H^*X to itself, which assigns a class $\theta(x) \in H^*X$ to every given $x \in H^*X$. The class $\theta(x)$ does not need to have the same dimension as x. We refer the reader to [MT68] for a general theory of (co-)homology operations.

A Steenrod operation, is a cohomology operation satisfying some specific properties. The following theorem introduces Steenrod squares.

Theorem 5.1.1. *There are unique cohomology operations Sq^i, $i = 0, 1, 2, \ldots$, called the i-th* "Steenrod squares" *which, are homomorphisms*

$$Sq^i \colon H^n X \to H^{n+i} X$$

defined for all $n \geq 0$, satisfying the following properties:

(1) *$Sq^0(x) = x$, (the identity homomorphism);*

(2) *$Sq^i(x) = 0$ for all $x \in H^n X$, $i > n$;*

(3) *$Sq^i(x) = x^2$ if $x \in H^i X$;*

(4) *Sq^1 is the Bockstein (connecting) homomorphism associated with the coefficient sequence $0 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathbb{Z}/4 \longrightarrow \mathbb{Z}/2 \longrightarrow 0$;*

(5) *(Stability) if $\sigma^* : H^n X \longrightarrow H^{n+1} \Sigma X$ is the suspension isomorphism, then*

$$Sq^i \sigma^* = \sigma^* Sq^i;$$

(6) *(Naturality) for any map $f : X \to Y$, then*

$$Sq^i f^* = f^* Sq^i;$$

(7) *(Cartan formula): $Sq^i(x \cup y) = \sum_{j+k=i} (Sq^j x) \cup (Sq^k y)$, where $x \cup y$ denotes the cup product;*

(8) *(Adem relations). If $0 < a < 2b$, then*

$$Sq^a Sq^b = \sum \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^c,$$

where the binomial coefficient is taken mod 2.

Proof. See [MT68, Chapter 2]. □

Because of condition 5 in the above theorem we say that the Steenrod operations are "stable operations".

Notice that Sq^i is a function, so it makes sense to take about composition of two given squares, such as $Sq^i Sq^j$, and so on. In general, a Steenrod operation θ will be a linear combination of compositions of Steenrod squares, for example

$$\theta = Sq^4 Sq^2 + Sq^1 Sq^5$$

is a Steenrod operation.

Definition 5.1.2. Let $I = (i_1, \ldots, i_s)$ be a sequence of nonnegative integers which denoted by $Sq^I = Sq^{i_1} \cdots Sq^{i_s}$. Then Sq^I is called *admissible* if $i_j \geq 2i_{j+1}$ for every $j < s$. The *excess* of I is given by $ex(I) = i_1 - i_2 - \ldots - i_r$. We also define $\dim(I) = i_1 + i_2 + \ldots + i_s$, and the length of I by $l(I) = s$.

Proposition 5.1.3. *[MT68, Cor.1]. Every Steenrod square can be written in terms of*

$$Sq^{2^i},\ i \geq 0.$$

For example $Sq^2Sq^2 = Sq^3Sq^1 = Sq^1Sq^2Sq^1$ and $Sq^1Sq^1 = 0$.

Next, we describe the action of the Steenrod squares on the cohomology of some well-understood spaces, namely real and complex projective space, that we are going to deal with during our calculations in the next chapters. First, we recall the following description.

Proposition 5.1.4. *The following isomorphisms describe the cohomology rings of projective spaces,*

$$\begin{aligned} H^*\mathbb{R}P^\infty &\cong \mathbb{Z}/2[a], \\ H^*\mathbb{R}P^n &\cong \mathbb{Z}/2[a]/(a^{n+1}), \\ H^*\mathbb{C}P^\infty &\cong \mathbb{Z}/2[b], \\ H^*\mathbb{C}P^n &\cong \mathbb{Z}/2[b]/(b^{n+1}), \end{aligned}$$

where $\dim a = 1$ *and* $\dim b = 2$.

Proof. See [G75, Corollary 26.35]. □

We then have the following description.

Proposition 5.1.5. *Let $a \in H^1\mathbb{R}P^\infty$, $b \in H^2\mathbb{C}P^\infty$ be non-zero elements. Then*

(1) $Sq^i a^n = \binom{n}{i} a^{n+i}$,

(2) $Sq^{2i} b^n = \binom{n}{i} b^{n+i}$,

(3) $Sq^{2i+1} b^n = 0$.

Proof. See [G75, Proposition 27.20]. □

We notice that, by definition, the Steenrod operations $Sq^i \colon H^nX \to H^{n+i}X$ are linear functions between $\mathbb{Z}/2$-vector spaces. By vector space duality, we obtain $\mathbb{Z}/2$-homology operations

$$Sq^i_* : H_{n+i}X \longrightarrow H_nX.$$

These operations have properties, similar to cohomology operations. For example, if $f : X \to Y$ is given, then the operations Sq^i_* are natural, i.e.

$$f_*Sq^i_*x = Sq^i_*f_*x$$

where $x \in H_*X$. We calculate the action of these operation, using the actions of the cohomology operations Sq^i and the Kronecker pairing

$$\langle -, - \rangle : H^*X \times H_*X \to \mathbb{Z}/2.$$

Example 5.1.6. *Consider $H_*\mathbb{R}P^\infty$. For each k, we have a generator $e_k \in H_k\mathbb{R}P^\infty$ such that $\langle a^j, e_k \rangle = \delta_{j,k}$ where $\delta_{j,k}$ is the Kronecker delta function. In order to calculate $Sq^i_*e_k$ we notice that*

$$\langle a^j, Sq^i_*e_k \rangle = \langle Sq^ia^j, e_k \rangle = \langle \binom{j}{i} a^{i+j}, e_k \rangle = \binom{j}{i} \delta_{i+j,k}.$$

This implies that

$$Sq^i_*e_k = \binom{k-i}{i} e_{k-i}.$$

In $H_*\mathbb{C}P^\infty$ we have generators $a_{2k} \in H_{2k}\mathbb{C}P^\infty$ such that $\langle b^j, a_{2k} \rangle = \delta_{j,2k}$.

A similar calculation as above shows that

$$Sq^{2i}_*a_{2k} = \binom{k-i}{i} a_{2k-2i},$$

and $Sq^{2i+1}_*a_{2k} = 0$ for all $i \geq 0$.

5.2 The homology of $QMO(k)$ and Kudo-Araki operations

In order to describe the homology of $QMO(k)$ we need to describe the $\mathbb{Z}/2$-homology operations known as Kudo-Araki operations. First, we note that the space $QMO(k)$

is a loop space. If we have two loops $f, g \in QMO(k)$ we then can consider the loop sum $f \star g \in QMO(k)$, as introduced in Section 1.2. This means that we have a mapping

$$QMO(k) \times QMO(k) \longrightarrow QMO(k).$$

In homology, this mapping induces a product

$$H_*QMO(k) \otimes H_*QMO(k) \longrightarrow H_*QMO(k).$$

This gives the structure of a ring to $H_*QMO(k)$ which is known as the Pontrjagin ring, and the ring product is known as the Pontrjagin product [W78, Chap.7].

Theorem 5.2.1. *Let X be an infinite loop space. Then for each $j = 0, 1, 2, \ldots,$ there exist operations*

$$Q^j \colon H_n X \longrightarrow H_{n+j} X,$$

which satisfying the following prorerties:

(1) *Q^j raises dimensions by j, where $j = 0, 1, 2, \ldots$.*

(2) *$Q^j x = 0$ if $j < n$ for any $x \in H_n X$;*

(3) *$Q^j x = x^2$ if $j = n$, where the square is the Pontrjagin product;*

(4) *$Q^j 1 = 0$ if $j > 0$, where $1 \in H_0 X$ is the identity element of the Pontrjagin ring;*

(5) *the Cartan formula holds:*

$$Q^j(xy) = \sum_{i+k=j} (Q^i x)(Q^k y);$$

(6) *Q^j commutes with homology suspension i.e.*

$$\sigma_* Q^j = Q^j \sigma_*,$$

where $\sigma_* : \widetilde{H}_* \Omega X \to \widetilde{H}_* X$ is the homology suspension;

(7) *the Adem relation. If $a > 2b$, then*

$$Q^a Q^b x = \sum_r \binom{r-b-1}{2r-a} Q^{a+b-r} Q^r x;$$

(8) *the Nishida relations:*

$$Sq^i_* Q^a x = \sum_r \binom{a-i}{i-2r} Q^{a-i+r} Sq^r_* x.$$

Proof. See [CLM76, Theorem 1.1.]. □

These operations satisfy other variations of Cartan formula [CLM76]. As with the Steenrod squares we may consider iterations of these operations.

Definition 5.2.2. *Let $J = (j_1, \dots, j_r)$ be a sequence of nonnegative integers. Then the term $Q^J = Q^{j_1} \cdots Q^{j_r}$ is called admissible if $j_i \leqslant 2j_{i+1}$ for $1 \leqslant i \leqslant r-1$. The excess of J is given by $ex(J) = j_1 - j_2 - \dots - j_r$. We also define* $\dim(J) = j_1 + j_2 + \dots + j_r$*, and length of J by $l(J) = r$.*

The significance of these definition is that $Q^J x$ is a Pontrjagin square if $ex(J) = \dim(x)$ and vanishes if $ex(J) < \dim(x)$. Also if J is a non-admissible sequence, then Q^J can be written in terms of admissible sequences using Adem relations in the same way that this is done for Steenrod squares.

The homology ring H_*QX, when X is a path connected space, can be described as follows. Let $\{x_\mu\}$ be a homogeneous basis for $\tilde{H}_*X \subseteq H_*QX$, the reduced $\mathbb{Z}/2$-homology of X. We then have [CLM76]

$$H_*QX \cong \mathbb{Z}/2[Q^J x_\mu | J \text{ admissible}, \mathrm{ex}(J) > \dim x_\mu].$$

Thus a basis for H_*QX is provided by the monomials in the polynomial generators.

Now, for $X = MO(k)$ we have the following. Recall from Chapter 3 that $\tilde{H}_*MO(k)$ has a homogeneous basis

$$\{e_I : I = (i_1, \dots, i_k) \text{ such that } 1 \leq i_1 \leq i_2 \cdots \leq i_k\}.$$

This then implies that

$$H_*QMO(k) = \mathbb{Z}/2[Q^J e_I \mid J \text{ admissible }, \mathrm{ex}(J) > \dim e_I].$$

The action of the Steenrod operations Sq^r_* on the classes e_I is calculated using naturality. More precisely, note that

$$e_I = (\mu_k)_*(e_{i_1} \otimes \dots \otimes e_{i_k}).$$

This implies that

$$Sq^r_* e_I = Sq^r_*(\mu_k)_*(e_{i_1} \otimes \dots \otimes e_{i_k}) = (\mu_k)_* Sq^r_*(e_{i_1} \otimes \dots \otimes e_{i_k}). \tag{5.2.3}$$

On the other hand, similar to the diagonal Cartan formula in cohomology in Theorem 5.1.1, we have diagonal Cartan formula in homology; if $x \in H_*X$ and $y \in H_*Y$ then

$$Sq^r_*(x \otimes y) = \sum_j (Sq^{r-j}_* x) \otimes (Sq^j_* y).$$

We shall apply this formula to (5.2.3) to determine $Sq^r_* e_I$. This of course, can be tedious when $l(I)$ is too big, but we will work with cases when $l(I)$ is reasonably small, where $l(I) = k$ is the *length* of a sequence I.

Definition 5.2.4. *We say that a homology class $x \in H_nX$ is $\mathcal{A}$-annihilated if and only if $Sq^i_* x = 0$ for all $i > 0$.*

From Proposition 5.1.3 we deduce the following result

Corollary 5.2.5. *Let $x \in H_nX$. Then x is $\mathcal{A}$-annihilated if and only if $Sq^{2^i}_* x = 0$ for all $i \geq 0$, $2^{i+1} \leq \dim x$.*

Example 5.2.6. *A basis for $H_4QMO(2)$ is given by the following set*

$$\{e_1e_3,\ e_2^2,\ e_1^2 \cdot e_1^2\}$$

where $\cdot$ denotes the Pontrjagin product in $H_*QMO(2)$ coming from the loop space structure on $QMO(2)$. The following table shows the Steenrod squares of these elements.

	Sq^1_*	Sq^2_*
e_1e_3	0	0
e_2^2	0	e_1^2
$e_1^2 \cdot e_1^2$	0	0

For instance, in the case of calculating $Sq^r_* e_2^2$ we have the following.

$Sq^1_*(e_2^2) = (Sq^1_* e_2)(Sq^0_* e_2) + (Sq^0_* e_2)(Sq^1_* e_2) = e_1e_2 + e_1e_2 = 0.$

$Sq^2_*(e_2^2) = (Sq^2_* e_2)(Sq^0_* e_2) + (Sq^1_* e_2)(Sq^1_* e_2) + (Sq^0_* e_2)(Sq^2_* e_2) = e_1e_1 = e_1^2$. *And for $r > 2$, $Sq^r_* e_2^2 = 0$.*

In the case of the element $e_1^2 \cdot e_1^2$ we also need to use Cartan formula as follow

$$Sq^1_*(e_1^2 \cdot e_1^2) = Sq^1_*(e_1^2) \cdot e_1^2 + e_1^2 \cdot Sq^1_*(e_1^2) = 0.$$

By using the diagonal Cartan formula in homology and the Nishida relations we can find the $\mathcal{A}$-annihilated elements of $H_*QMO(k)$.
In the above example, we find clearly that the elements e_1e_3 and $e_1^2 \cdot e_1^2$ are $\mathcal{A}$-annihilated. However the element e_2^2 is not.

5.3 The cup-coproduct in $H_*QMO(k)$

We describe the primitive classes in $H_*QMO(k)$. This is useful when we wish to determine the Hurewicz image $f_*[N_+^{n+k}]$ for a given map $f : N_+^{n+k} \to QMO(k)$. The cup-coproduct or briefly coproduct is denoted by the symbol ψ. The map $\psi : H_*X \longrightarrow H_*X \otimes H_*X$ is induced by the diagonal map $X \to X \times X$. We notice that this map is the vector space dual of the cup-product

$$H^*X \otimes H^*X \longrightarrow H^*X.$$

Definition 5.3.1. *The homology class $u \in H_nX$ is called primitive if*

$$\psi(u) = u \otimes 1 + 1 \otimes u,$$

where,

$$\psi : H_nX \to H_n(X \times X) \cong \sum_j H_jX \otimes H_{n-j}X.$$

Let a denote the generator of $H^1BO(1)$. Then a^i is the generator of $H^iBO(1)$ which is dual to $e_i \in H_iBO(1)$. Since $a^j \cup a^{i-j} = a^i$, then by the definition of ψ we have

$$\psi(e_i) = \sum_{j=0}^{i} e_j \otimes e_{i-j}.$$

The coproduct ψ is the mapping which in homology is induced by the diagonal mapping $X \to X \times X$. This implies that ψ is natural with respect to mappings of spaces, that is for $g : X \to Y$ we have

$$\psi g_*(u) = (g_* \otimes g_*)\psi(u)$$

where $(g_* \otimes g_*)(a \otimes b) = g_*(a) \otimes g_*(b)$ and $u \in H_*X$. Moreover, we have the following.

Proposition 5.3.2. *The coproduct on a homology class $u \otimes v \in H_*(X \times Y) \cong H_*X \otimes H_*Y$ is calculated by*

$$\psi(u \otimes v) = \psi(u) \otimes \psi(v).$$

Proof. Assume that the map $\Delta : X \to X \times X$ is the diagonal map, then we have a commutative diagram

$$\begin{array}{ccc} X \times Y & \xrightarrow{1} & X \times Y \\ \downarrow{\scriptstyle \Delta\times\Delta} & & \\ X \times X \times Y \times Y & & \downarrow{\scriptstyle \Delta} \\ \downarrow{\scriptstyle 1\times\tau\times 1} & & \\ X \times Y \times X \times Y & \xrightarrow{1} & X \times Y \times X \times Y \end{array} \tag{5.3.3}$$

where $\tau : X \times Y \to Y \times X$ is the map which switches components, i.e. $\tau(x, y) = (y, x)$. Then given

$$\psi(u) = \sum u' \otimes u'' \text{ and } \psi(v) = \sum v' \otimes v'',$$

we have

$$\begin{aligned} \psi(u \otimes v) &= \textstyle\sum (u' \otimes v') \otimes (u'' \otimes v'') \\ &= \psi(u) \otimes \psi(v) \end{aligned}$$

where $u \in H_*X$ and $v \in H_*Y$. □

Remark 5.3.4. *Recall that for $I = (i_1, \ldots, i_k)$ with $0 \le i_1 \le i_2 \le \ldots \le i_k$ we defined $e_I = (\mu_k)_*(e_{i_1} \otimes \cdots \otimes e_{i_k}) \in H_*BO(k)$. Hence, we may calculate $\psi(e_I)$ by naturality. We may express $\psi(e_I)$ in the following form:*

$$\psi(e_I) = e_I \otimes 1 + \sum e_{I-M} \otimes e_M + 1 \otimes e_I + A_I$$

where the sum $\sum e_{I-M} \otimes e_M$ runs over terms where both M and $I - M$ have only nonzero entries. The class A_I is a sum of terms of the form

$$e_{j_1} e_{j_2} \cdots e_{j_r} \otimes e_{k_1} \cdots e_{k_r}$$

where $J = (j_1, \ldots, j_r)$ and $K = (k_1, \ldots, k_r)$ are increasing sequences, and at least one of them has an entry equal to 0, and at least one of the entries in both of J or

K is nonzero, with the convention that $e_0 = 1$. For example, gives $e_2e_3 \in H_*BO(2)$, we may calculate that

$$\begin{aligned}\psi(e_2)\psi(e_3) &= (e_2 \otimes 1 + e_1 \otimes e_1 + 1 \otimes e_2)(e_3 \otimes 1 + e_2 \otimes e_1 + e_1 \otimes e_2 + 1 \otimes e_3) \\ &= e_2e_3 \otimes 1 + e_2^2 \otimes e_1 + e_1e_2 \otimes e_2 + e_2 \otimes e_3 + \\ &\quad e_1e_3 \otimes e_1 + e_1e_2 \otimes e_1^2 + e_1^2 \otimes e_1e_2 + e_1 \otimes e_1e_3 + \\ &\quad e_3 \otimes e_2 + e_2 \otimes e_1e_2 + e_1 \otimes e_2^2 + 1 \otimes e_2e_3.\end{aligned}$$

Here the sum $\sum e_{I-M} \otimes e_M$ is given by

$$e_1e_2 \otimes e_1^2 + e_1^2 \otimes e_1e_2$$

whereas $A_{(2,3)}$ is given by

$$\begin{aligned}&e_2^2 \otimes e_1 + e_1e_2 \otimes e_2 + e_2 \otimes e_3 + e_1e_3 \otimes e_1 + \\ &e_1 \otimes e_1e_3 + e_3 \otimes e_2 + e_2 \otimes e_1e_2 + e_1 \otimes e_2^2.\end{aligned}$$

Notice that the class e_2e_3 maps to a nontrivial class in $H_*MO(2)$. On the other hand we know that classes such as e_2 belong to the kernel of the projection map $H_*BO(2) \to H_*MO(2)$. Notice that the class $e_0^2 = 1$ maps to $1 \in H_0MO(2)$. This implies that in $H_*MO(2) \otimes H_*MO(2)$ we have

$$\psi(e_2e_3) = e_2e_3 \otimes 1 + e_1e_2 \otimes e_1^2 + e_1^2 \otimes e_1e_2 + 1 \otimes e_2e_3.$$

In general, we observe that A_I, in the expression for $\psi(e_I)$ belongs to the kernel of the projection map $H_*BO(k) \otimes H_*BO(k) \to H_*MO(k) \otimes H_*MO(k)$, i.e. in $H_*MO(k) \otimes H_*MO(k)$ we have

$$\psi(e_I) = e_I \otimes 1 + \sum e_{I-M} \otimes e_M + 1 \otimes e_I$$

where the sum $\sum e_{I-M} \otimes e_M$ runs over terms where both M and $I - M$ have only nonzero entries.

Definition 5.3.5. *The reduced coproduct $\widetilde{\psi} : H_*X \to H_*X \otimes H_*X$ is defined by*

$$\widetilde{\psi}(u) = \psi(u) - 1 \otimes u - u \otimes 1.$$

Then, according to the above example, in $H_*MO(2)$ we have

$$\widetilde{\psi}(e_2e_3) = e_1e_2 \otimes e_1^2 + e_1^2 \otimes e_1e_2 = \widetilde{\psi}(e_2)\widetilde{\psi}(e_3).$$

This is an example of a general result.

Lemma 5.3.6. *Let $I = (i_1, \dots, i_k)$ be an increasing sequence with $i_1 \geq 1$. Then for $e_I \in H_*BO(k)$*

$$\widetilde{\psi}(e_I) = \sum e_{I-M} \otimes e_M + A_I,$$

using the notation of Remark 5.3.4.

Consequently, in $H_*MO(k) \otimes H_*MO(k)$ we have an expression of the form

$$\widetilde{\psi}(e_I) = \sum e_{I-M} \otimes e_M$$

where both M and $I - M$ have only nonzero entries.

Proof. This is straightforward from Remark 5.3.4, and Definition 5.3.5. □

Theorem 5.3.7. *For $e_I = e_{i_1} \cdots e_{i_r} \in H_*MO(r)$ and $e_J = e_{j_1} \cdots e_{j_s} \in H_*MO(s)$, the reduced coproduct satisfies*

$$\widetilde{\psi}(e_Ie_J) = \widetilde{\psi}(e_I)\widetilde{\psi}(e_J)$$

where $e_Ie_J = e_{i_1} \cdots e_{i_r}e_{j_1} \cdots e_{j_s}$.

Proof. From Lemma 5.3.6, we calculate that in $H_*BO(r)$

$$\psi(e_I) = e_I \otimes 1 + 1 \otimes e_I + \sum_M e_{I-M} \otimes e_M + A_I.$$

Moreover, in $H_*BO(s)$ we have

$$\psi(e_J) = e_J \otimes 1 + 1 \otimes e_J + \sum_N e_{J-N} \otimes e_N + A_J.$$

Hence $\psi(e_Ie_J)$ in $H_*BO(r+s)$ is given by

$$\begin{aligned} \psi(e_Ie_J) &= e_Ie_J \otimes 1 + 1 \otimes e_Ie_J + (\textstyle\sum_M e_{I-M} \otimes e_M)(\textstyle\sum_N e_{J-N} \otimes e_N) + A_{(I,J)} \\ &= e_Ie_J \otimes 1 + 1 \otimes e_Ie_J + \textstyle\sum_{M,N} e_{I-M}e_{J-N} \otimes e_Me_N + A_{(I,J)} \end{aligned}$$

where $(I, J) = (i_1, \ldots, i_r, j_1, \ldots, j_s)$. Moreover,

$$\begin{aligned} A_{(I,J)} &= \sum (e_I e_{J-N}) \otimes e_N + e_I \otimes e_J + \\ &\quad \sum (e_{I-M} e_J) \otimes e_M + \sum e_{I-M} \otimes (e_M e_J) + \\ &\quad e_J \otimes e_I + \sum e_{J-N} \otimes (e_I e_N) + \\ &\quad \text{other terms coming from products with } A_I \text{ or } A_J \,. \end{aligned}$$

This implies that $\psi(e_I e_J)$ in $H_*MO(r+s)$ is given by

$$\psi(e_I e_J) = e_I e_J \otimes 1 + 1 \otimes e_I e_J + \sum_{M,N} e_{I-M} e_{J-N} \otimes e_M e_N$$

or equivalently

$$\widetilde{\psi}(e_I e_J) = \sum_{M,N} e_{I-M} e_{J-N} \otimes e_M e_N = \widetilde{\psi}(e_I)\widetilde{\psi}(e_J).$$

This completes the proof. □

Example 5.3.8.

$$\begin{aligned} (\psi e_2^4)^3 &= (e_2^4 \otimes 1 + e_1^4 \otimes e_1^4 + 1 \otimes e_2^4)^3 \\ &= e_2^{12} \otimes 1 + e_1^{12} \otimes e_1^{12} + 1 \otimes e_2^{12} \end{aligned}$$

Hence

$$\widetilde{\psi}(e_2^4)^3 = e_1^{12} \otimes e_1^{12}.$$

The following result determines those classes e_I that are primitive in $H_*MO(k)$.

Lemma 5.3.9. *For $I = (i_1, i_2, \ldots, i_k)$, such that $i_1 \leq i_2 \leq \ldots \leq i_k$. The element e_I is primitive in $H_*MO(k)$ if and only if $i_1 = 1$.*

Proof. Notice that $\widetilde{\psi}(e_i) = 0$ if and only if $i = 1$. Suppose that $e_I \in H_*QX$ is primitive, then

$$\widetilde{\psi}(e_I) = \widetilde{\psi}(e_{i_1} \ldots e_{i_k}) = \widetilde{\psi}(e_{i_1}) \ldots \widetilde{\psi}(e_{i_k}) = 0.$$

Hence $\widetilde{\psi}(e_{i_j}) = 0$ for some i_j. Then $i_j = 1$ and then $i_1 = 1$ since $1 \leq i_1 \leq i_j$.

Conversely, Suppose that $i_1 = 1$, then $\widetilde{\psi}(e_{i_1}) = 0$ and so $\widetilde{\psi}(e_I) = 0$. Hence e_I is primitive. □

In H_*QX the coproduct is calculated as follows. Let $\psi(u) = \sum u^{'} \otimes u^{''}$, then the diagonal Cartan formula [CLM76, Part I, Theorem 1.1] is given by

$$\psi(Q^j(u)) = \sum_{i+k=j} Q^i u^{'} \otimes Q^k u^{''}.$$

Using this we have the following.

Proposition 5.3.10. *If $u \in H_*MO(k)$ is primitive, then $Q^n u$ is primitive in $H_*QMO(k)$.*

Proof. Suppose that u is primitive, then $\psi(u) = u \otimes 1 + 1 \otimes u$. Then

$$\begin{aligned} \psi(Q^n u) &= \Sigma_i Q^i u \otimes Q^{n-i} 1 + \Sigma_i Q^i 1 \otimes Q^{n-i} u \\ &= Q^n u \otimes 1 + 1 \otimes Q^n u, \end{aligned}$$

since $Q^i 1 = 0$ for $i > 0$. Then $\widetilde{\psi}(Q^n u) = 0$. Hence $Q^n u$ is primitive □

In Example 5.2.6, by Lemma 5.3.9 and Proposition 5.3.10, it is obvious that the elements $e_1 e_3,\ e_1^2 \cdot e_1^2 \in H_4QMO(2)$ are primitive, but not the element e_2^2. Hence we deduce that the elements $e_1 e_3$ and $Q^2 e_1^2$ are $\mathcal{A}$-annihilated and primitive.

We note that the Pontrjagin product

$$H_*QMO(k) \otimes H_*QMO(k) \to H_*QMO(k)$$

is induced by the addition of loops $QMO(k) \times QMO(k) \to QMO(k)$. The fact that the coproduct is natural then implies that for $u, v \in H_*QMO(k)$ we have

$$\psi(u \cdot v) = \psi(u) \cdot \psi(v).$$

Let us mention a useful Lemma which gives the primitive elements required in the study of manifolds immersed in Euclidean spaces [AEa00].

Lemma 5.3.11. *Let $k > 2$. Then a basis for the cup coproduct primitive classes in $H_{2k+2}QMO(k)$ is given by the following set of elements*

$$\{e_{i_1} e_{i_2} \dots e_{i_k} | 1 = i_1 \leq i_2 \leq \cdots \leq i_k\} \cup \{e_2^{k-2} e_3^2 + e_1^k \cdot e_1^{k-2} e_2^2,\ e_1^{k-1} e_2 \cdot e_1^{k-1} e_2,\ Q^{k+2} e_1^k\}.$$

For $k = 1$, a basis for the primitives in $H_4QMO(1)$ is given by

$$\{Q^3 e_1,\ \ e_1 \cdot e_1 \cdot e_1 \cdot e_1\}.$$

For $k = 2$, a basis for the primitives in $H_6QMO(2)$ is given by

$$\{e_1e_5, \;\; e_3^2 + e_1^2 \cdot e_1^2 + e_1^2 \cdot e_1^2 \cdot e_1^2, \;\; e_1e_2 \cdot e_1e_2, \;\; Q^4e_1^2\}.$$

Proof. See [AEa00, Lemma 2.6]. ∎

Chapter 6

Determing the double point manifolds of $F : M^n \looparrowright N^{n+k}$

In this chapter, we turn to our core problem in this thesis, determining the cobordism class of double point manifolds of a given immersion. We describe the general machinery here, and leave the detailed calculations to the next chapters.

Given an immersion $F : M^n \looparrowright N^{n+k}$, by Definition 1.3.7, $I_r(F)$ will be the set of r-fold self-intersection points of F, i.e. points of N which are the image under F of at least r distinct points of the manifold under F. We always can choose F to be a self-transverse immersion up to regular homotopy [B].

Moreover, a cobordism between self-transverse immersions can be taken to be self-transverse [AEb00]; it is obvious that such a cobordism will induce a cobordism of the immersions of the r-fold self-intersection sets. By Theorem 2.5.3 the self-transversality of F implies that $I_r(F) \subseteq N^{n+k}$ is itself the image of $\theta_r(F) : \Delta_r(F) \looparrowright N^{n+k}$, the r-fold self-intersection manifold which is of dimension $n - k(r-1)$.

Let the immersion F correspond to a map $f : N_+^{n+k} \longrightarrow QMO(k)$ under the Pontrjagin-Thom construction. We will show that the Stiefel-Whitney numbers of the r-fold self-intersection manifold can be determined from $f_*[N_+^{n+k}] \in H_{n+k}QMO(k)$.

When $N_+^{n+k} = S^{n+k}$ then $f_*[N_+^{n+k}]$ has to be $\mathcal{A}$-annihilated and primitive and then this case was considered by Asadi-Eccles [AEa00]. We first consider the general case, when N^{n+k} is an arbitrary manifold. The co-coproduct structure of $H_*N_+^{n+k}$

and the action of the Steenrod algebra place restrictions on the possible values of $f_*[N_+^{n+k}]$ but in general it need not be $\mathcal{A}$-annihilated and primitive.

We then focus on the case $r = 2$ and we refer to $\Delta_2(F)$ as the double point manifold of F. Our goal is then to describe a machinery that determines the cobordism class of $\Delta_2(F)$. This will be built upon the tools that we have described in section 4.2. We will use homotopy theory to carry out this task. In order to do this, we need to introduce another set of tools from homotopy theory, namely the stable James-Hopf invariants. The applications of the James-Hopf invariant to the problem of our study was first observed by A. Szücs [S76I],[S76II], P. Vogel [V74] and Koschorke and Sanderson [KS78].

6.1 Stable James-Hopf invariants and $\Delta_2(F)$

We start by recalling some facts about QX. Suppose X is a path connected space. According to [BE74, Theorem B] the space QX admits a splitting

$$QX \simeq \prod_{r=1}^{\infty} QD_rX \tag{6.1.1}$$

where D_rX is defined by

$$D_rX = \frac{X^{\wedge r} \times_{\Sigma_r} W\Sigma_r}{\{*\} \times_{\Sigma_r} W\Sigma_r}.$$

Here $X^{\wedge r}$ denotes the r-fold smash product $X \wedge X \wedge \cdots \wedge X$, Σ_r denotes the permutation group on r elements, and $W\Sigma_r$ a contractible space with a free Σ_r-action. The group Σ_r acts on $X^{\wedge r}$ by permuting the factors. The space D_rX is known as the r-adic construction on X. In particular, $D_1X = X$. Projection onto the r-th factor gives natural maps

$$h^r : QX \longrightarrow QD_rX$$

known as the r-th stable James-Hopf maps. These have stable adjoint

$$\Sigma^\infty QX \longrightarrow \Sigma^\infty D_rX$$

which can be used to construct a stable splitting [BE74, Theorem C]

$$QX \simeq_{\text{stable}} \bigvee_{r=1}^{\infty} D_rX. \tag{6.1.2}$$

We are interested in the case $r = 2$ where $W\Sigma_2$ can be taken as S^∞ with Σ_2 acting by the antipodal action. The space D_2X is called *the quadratic construction* on X. The projection onto the second factor in the first splitting yields a map

$$h^2 : QX \longrightarrow QD_2X$$

known as the second stable James-Hopf map [AEa00] which induces the 2nd *stable James-Hopf invariant*

$$h^2_* : [N^{n+k}_+, QX] \longrightarrow [N^{n+k}_+, QD_2X].$$

The r-adic construction can be done also on vector bundles. In particular, we have a 'universal bundle' $D_2(\gamma^k)$ given by

$$(EO(k) \times EO(k)) \times_{\Sigma_2} S^\infty \longrightarrow (BO(k) \times BO(k)) \times_{\Sigma_2} S^\infty.$$

Next, let $F : M^n \looparrowright N^{n+k}$ be an immersion. Then the double point manifold $\Delta_2(F)$ has dimension $n - k$, i.e. considering the double point manifolds of a given immersion defines a mapping

$$\theta_2 : \mathrm{Imm}_k(N^{n+k}) \longrightarrow \mathrm{Imm}_{2k}(N^{n+k}).$$

The normal bundle of the immersion $\theta_2(F) : \Delta_2(F) \looparrowright N^{n+k}$ has a $D_2(\gamma^k)$-structure [AEa00]. Hence, applying the generalised Pontrjagin-Thom construction for immersions to $\theta_2(F)$ yields a mapping

$$f_2 : N^{n+k}_+ \to QT(D_2(\gamma^k)).$$

Note that $T(D_2(\gamma^k)) = D_2T(\gamma^k) = D_2MO(k)$. Hence,

$$f_2 : N^{n+k}_+ \to QD_2MO(k).$$

On the other hand the immersion F corresponds to a mapping, unique up to homotopy, $f : N^{n+k}_+ \to QMO(k)$ under the Pontrjagin-Thom construction. According to Koshchorke and Sanderson [KS78], see also [AEb00], f_2 is given by the composition

$$N^{n+k}_+ \xrightarrow{f} QMO(k) \xrightarrow{h^2} QD_2MO(k).$$

Notice that $D_2(\gamma^k)$ is a $2k$-bundle. The classifying map for this bundle is a map

$$(BO(k) \times BO(k)) \times_{\Sigma_2} W\Sigma_2 \longrightarrow BO(2k).$$

This then induces a map of Thom spaces

$$\xi : D_2MO(k) \longrightarrow MO(2k)$$

which corresponds to forgetting the additional structure on the normal bundle of the double point manifold and viewing it only as a vector bundle of dimension $2k$. This then induces a mapping

$$\xi_* : [N_+^{n+k}, D_2MO(k)] \to [N_+^{n+k}, MO(2k)].$$

The main result of [KS78], see also Szücs [S76I], is the following theorem.

Theorem 6.1.3. *The following diagram is commutative.*

$$\begin{array}{ccccc} \mathrm{Imm}_k(N^{n+k}) & & \xrightarrow{\theta_2} & & \mathrm{Imm}_{2k}(N^{n+k}) \\ \cong \downarrow \tau & & & & \cong \downarrow \tau \\ [N_+^{n+k}, QMO(k)] & \xrightarrow{h_*^2} & [N_+^{n+k}, QD_2MO(k)] & \xrightarrow{(Q\xi)_*} & [N_+^{n+k}, QMO(2k)] \end{array} \tag{6.1.4}$$

This commutative diagram provides the main tool in our calculations when we pass on to the homology of the spaces involved here.

6.2 The homology of James-Hopf maps

We start by describing H_*D_rX. The splitting (6.1.2) gives rise to a decomposition of homology as

$$\widetilde{H}_*QX \cong \bigoplus_{r=1}^{\infty} \widetilde{H}_*D_rX.$$

Define the height function ht on the monomial generators of $\widetilde{H}_*QX$ by $ht(x_\mu) = 1$, $ht(Q^ix) = 2ht(x)$ and $ht(x \cdot y) = ht(x) + ht(y)$, where $x \cdot y$ represents the Pontrjagin product and $x_\mu \in H_*X$. It is known that H_*D_rX is generated by the monomial generators of H_*QX which have height r [G73].

Notice that projecting onto the r factor in the first splitting (6.1.1) provides a map $QX \to QD_rX$ which in homology induces

$$h_*^r : \widetilde{H}_*QX \longrightarrow \widetilde{H}_*QD_rX$$

which maps the elements of height r nontrivially, and all of the elements of other heights belong to its kernel.

If a basis for $H_*QMO(k)$ is written in terms of the standard monomial generators, then the terms of height 1 in $f_*[N^{n+k}]$ determine the characteristic numbers of F as in Theorem 4.2.2. We shall show that the height 2 terms determine the characteristic numbers of double point manifold of F.

Let us recall that given any space X, the homology suspension $\sigma_* : H_n\Omega X \to H_{n+1}X$ is induced by the evaluation map [W78]. More precisely, the identity mapping $1 : \Omega X \to \Omega X$ has the adjoint $e : \Sigma\Omega X \to X$ usually known as the evaluation map. In homology, we then obtain

$$\sigma_* = e_* : H_n\Omega X \cong H_{n+1}\Sigma\Omega X \longrightarrow H_{n+1}X.$$

In particular, we may consider the identity map $1 : \Omega^l\Sigma^l X \to \Omega^l\Sigma^l X$ and its iterated adjoint $\Sigma^l\Omega^l\Sigma^l X \to \Sigma^l X$. This then induces the iterated homology suspension

$$H_n\Omega^l\Sigma^l X \cong H_{n+l}\Sigma^l\Omega^l\Sigma^l X \longrightarrow H_{n+l}\Sigma^l X \cong H_nX.$$

By analogy, in the case of $QX =$ direct limit $\Omega^l\Sigma^l X$ we consider the identity mapping $1 : QX \to QX$ and its stable adjoint $\Sigma^\infty QX \to \Sigma^\infty X$. The stable mapping $\Sigma^\infty QX \to \Sigma^\infty X$ agrees with the projection on to the first factor $D_1X = X$ in the splitting (6.1.2). This induces the homology suspension

$$p_1 : H_nQX \longrightarrow H_nX.$$

Next, notice that the stable James-Hopf map

$$h^2 : QX \longrightarrow QD_2X$$

has stable adjoint

$$\Sigma^\infty QX \longrightarrow \Sigma^\infty D_2X.$$

This induces a mapping in homology with agrees with the mapping $p_2 : H_*QX \to H_*D_2X$ defined by the composition

$$p_2 = p_1 \circ h_*^2 : H_*QX \longrightarrow H_*QD_2X \longrightarrow H_*D_2X.$$

Notice that, in general, $H_*\Omega X$ is a ring. The homology suspension has the property that it kills products [S75]. Hence, in this case, the mapping $p_1 : H_*QD_2X \longrightarrow H_*D_2X$ kills all of the product elements. Finally, notice that elements of height 1 in H_*QD_2X correspond to elements of height 2 in H_*QX. Hence, using the effect of p_1, we see that the mapping p_2 maps all elements of height 2 isomorphically whereas it kills all of terms of height other than 2.

Now we want to describe ξ_*. Let $X = MO(k)$. Then $H_*D_2MO(k)$ has a basis

$$\{e_{i_1}e_{i_2}\dots e_{i_k}\cdot e_{j_1}e_{j_2}\dots e_{j_k},\ Q^Je_{i_1}e_{i_2}\dots e_{i_k} \mid \mathrm{ex}(J) > \dim I\},$$

where $I = (i_1, \dots, i_k)$ and $\dim I = i_1 + i_2 + \dots + i_k$.

Theorem 6.2.1. *The homomorphism $\xi_* : H_{2k+2}D_2MO(k) \to H_{2k+2}MO(2k)$ is determined by the following values:*

$$\begin{aligned}
\xi_*(e_1^k \cdot e_1^{k-1}e_3) &= e_1^{2k-1}e_3;\\
\xi_*(e_1^k \cdot e_1^{k-2}e_2^2) &= e_1^{2k-2}e_2^2;\\
\xi_*(e_1^{k-1}e_2 \cdot e_1^{k-1}e_2) &= e_1^{2k-2}e_2^2.
\end{aligned}$$

$$\xi_*(Q^{k+2}e_1^k) = \begin{cases} 0 & \text{for } k \equiv 0 \pmod 4;\\ e_1^{2k-1}e_3 & \text{for } k \equiv 1 \pmod 4;\\ e_1^{2k-2}e_2^2 & \text{for } k \equiv 2 \pmod 4;\\ e_1^{2k-1}e_3 + e_1^{2k-2}e_2^2 & \text{for } k \equiv 3 \pmod 4. \end{cases}$$

Proof. See [AEa00, Lemma 2.4]. □

6.3 Normal Steifel-Whitney numbers of $\Delta_2(F)$

According to Pontrjagin-Thom theory the cobordism class of a manifold M with an immersion $F : M^n \looparrowright N^{n+k}$ determines the normal Stiefel-Whitney numbers of F. In general, the normal Stiefel-Whitney numbers of F may not determine M. For example if $F : M^n \to M^n \times \mathbb{R}$ the normal bundle is trivial so all the normal Stiefel-Whitney numbers are zero. So there is no information about M. Then the Stiefel-Whitney numbers of F may not determine the Stiefel-Whitney numbers of M.

According to Theorem 4.2.2 these numbers depend on stable Hurewicz homomorphism of $f : N_+^{n+k} \to QMO(k)$ where f corresponds to F under the Pontrjagin-Thom construction.

The main result in this chapter, which provides the framework for our calculations in the next chapters, is that the normal Stiefel-Whitney numbers of the self intersection immersions $\theta_r(F)$ can be determined by the unstable Hurewicz homomorphism

$$h : [N_+^{n+k}, QMO(k)] \to H_{n+k}QMO(k).$$

We will provide some details below.

Given a map $f : N_+^{n+k} \to \Omega^l\Sigma^l MO(k) \subseteq QMO(k)$ we have the adjoint mapping

$$\widetilde{f} : \Sigma^l N_+^{n+k} \to \Sigma^l MO(k).$$

Recall that we have the *homology suspension* map

$$\sigma_*^l : H_{n+k}\Omega^l\Sigma^l MO(k) \to H_{n+k+l}\Sigma^l MO(k).$$

Proposition 6.3.1. *Suppose $f : N_+^{n+k} \to \Omega^l\Sigma^l MO(k)$ is any map and $\widetilde{f} : \Sigma^l N_+^{n+k} \to \Sigma^l MO(k)$ is its adjoint. Then the diagram below commutes for all $n+k$.*

$$\begin{array}{ccc} H_{n+k}N_+^{n+k} & \xrightarrow{f_*} & H_{n+k}\Omega^l\Sigma^l MO(k) \\ \downarrow{\cong} & & \downarrow{\sigma_*^l} \\ H_{n+k+l}\Sigma^l N_+^{n+k} & \xrightarrow{\widetilde{f}_*} & H_{n+k+l}\Sigma^l MO(k) \end{array}$$

Proof. See [S75, Proposition 15.43.]. □

According to Theorem 6.1.3 and the cobordism of immersions in chapter 4 the normal Stiefel-Whitney numbers (and so the cobordism class) of $\Delta_2(F)$ corresponding to $\alpha \in [N_+^{n+k}, QMO(k)]$ are determined by (and determine) the Hurewicz image

$$h^S(\beta) = \xi_* p_2 h(\alpha) \in H_{n+k}MO(2k)$$

where the element $\beta = (Q\xi)_* \circ h^2_*(\alpha) \in [N^{n+k}_+, QMO(2k)]$ corresponds to the immersion $\theta_2(F) : \Delta_2(F) \looparrowright N^{n+k}$. To determine it we will state and prove the next main theorem.

Theorem 6.3.2. *Suppose that the self-transverse immersion $F : M^n \looparrowright N^{n+k}$ corresponds to a continuous function $f : N^{n+k}_+ \to QMO(k)$. Then the Steifel-Whitney numbers of the normal bundle of $\theta_2(F) : \Delta_2(F) \looparrowright N^{n+k}$ are given by*

$$\overline{w}^I[\theta_2(F)] = \langle w^I(\nu_{\theta_2(F)}), \Delta_2(F)\rangle = \langle w^I w_{2k}, \xi_* p_2 h(\alpha)\rangle.$$

Proof. Given a map $f : N^{n+k}_+ \to QMO(k)$, then

$$f_* : H_{n+k}N^{n+k}_+ \longrightarrow H_{n+k}QMO(k).$$

The composition of ξ_* and p_2 induces the following map

$$H_{n+k}QMO(k) \xrightarrow{p_2} H_{n+k}D_2MO(k) \xrightarrow{\xi_*} H_{n+k}MO(2k)\ .$$

So we have the next diagram.

$$\begin{array}{ccccc} H_{n+k}N^{n+k} & \xrightarrow{f_*} & H_{n+k}QMO(k) & \xrightarrow{p_2} & H_{n+k}D_2MO(k) \\ & \searrow & & \swarrow{\scriptstyle \xi_*} & \\ & & H_{n+k}MO(2k) & & \end{array}$$

Since $h^S = p_1 \circ h$, then we get the following commutative diagram.

$$\begin{array}{ccccc} [N^{n+k}_+, QMO(k)] & \xrightarrow{h^2_*} & [N^{n+k}_+, QD_2MO(k)] & \xrightarrow{(Q\xi)_*} & [N^{n+k}_+, QMO(2k)] \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} \\ H_{n+k}QMO(k) & \xrightarrow{h^2_*} & H_{n+k}QD_2MO(k) & \xrightarrow{(Q\xi)_*} & H_{n+k}QMO(2k) \\ \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_1} \\ H_{n+k}QMO(k) & \xrightarrow{p_2} & H_{n+k}D_2MO(k) & \xrightarrow{\xi_*} & H_{n+k}MO(2k) \end{array} \qquad (6.3.3)$$

So the immersion $\theta_2(F) : \Delta^{n-k}_2(F) \looparrowright N^{n+k}$ corresponds to the element

$$\beta = (Q\xi)_* \circ h^2_*(\alpha) \in [N^{n+k}_+, QMO(2k)]$$

where $\alpha \in [N^{n+k}_+, QMO(k)]$ and $h([f]) = f_*[N^{n+k}_+] = h(\alpha)$.

From Theorem 4.2.2 we can find the Stiefel-Whitney numbers of $\theta_2(F)$ by evaluating the stable Hurewicz image of the element β. However,

$$\begin{aligned} h^S(Q\xi)_* h^2_*(\alpha) &= p_1 h(Q\xi)_* h^2(\alpha) \\ &= \xi_* p_1 h h^2_*(\alpha) \\ &= \xi_* p_1 h^2_* h(\alpha) \\ &= \xi_* p_2 h(\alpha). \end{aligned}$$

Hence, the stable Hurewicz image of $(Q\xi)_* h^2_*(\alpha)$ is $\xi_* p_2 \circ h(\alpha)$. The theorem now follows from Theorem 4.2.2. □

Remark 6.3.4. *In the case of $N^{n+k} = \mathbb{R}^{n+k}$, the Steifel-Whitney numbers in this theorem are the Steifel-Whitney numbers of $\Delta_2(F)$ (see Notation 3.2.2). Hence $h(\alpha)$ determines the cobordism class of $\Delta_2(F)$. This is not true in general.*

In the next two chapters we will investigate the case of $N = \mathbb{C}P^{k+1}$.

Chapter 7

The double point manifolds of $F : M^{k+2} \looparrowright \mathbb{C}P^{k+1}$ when k is odd.

Suppose $F : M^{k+2} \looparrowright \mathbb{C}P^{k+1}$ is an immersion. We shall apply the methods of Chapter 6 to determine the cobordism class of double point manifolds of immersions F. Notice that in this case, the double point manifold will be a surface and the cobordism classes of surfaces are completely known, so it is either a boundary or cobordant to the projective plane.

Previous work has been done in the case of immersions $F : M^{k+2} \looparrowright \mathbb{R}^{2k+2}$ [AEa00]. In this case the double point manifold must be a boundary if $k \equiv 0, 2 \pmod 4$, or $k \equiv 3 \pmod 4$ and $\alpha(k+1) > 1$, where α is the number of digits 1 in the dyadic expression; there exists an immersion for which it is cobordant to the projective plane when either $k \equiv 1 \pmod 4$, or $k \equiv 3 \pmod 4$ and $k+1$ is a power of 2. In this chapter we are going to investigate the double point manifold of immersions $F : M^{k+2} \looparrowright \mathbb{C}P^{k+1}$ when k is odd.

If $k \equiv 1 \pmod 4$, there exists an immersion F in complex projective space with double point manifold cobordant to the projective plane. This result follows from the result for immersions in Euclidean spaces. If $k \equiv 3 \pmod 4$ we show that for all values of k there exists immersion $F : M^{k+2} \looparrowright \mathbb{C}P^{k+1}$ with double point manifold cobordant to projective plane. We are going to state the next theorem for the case $k = 11$ as an illustration of the case $k \equiv 3 \pmod 4$.

7.1 The case $k = 11$.

Theorem 7.1.1. *An immersion $F : M^{13} \looparrowright \mathbb{C}P^{12}$ corresponding to a function $f : \mathbb{C}P^{12}_{+} \longrightarrow QMO(11)$ has double point manifold cobordant to the projective plane if and only if*

$$f_*(a_{24}) = c\, Q^{13}e_1^{11} + c_1 e_1^{10}e_2 \cdot e_1^{10}e_2 + c_2 e_1^{11} \cdot e_1^{9}e_2^{2} + \varphi,$$

where φ has height one.

Proof. According to the techniques explained in Section 6.2, we need to calculate the Hurewicz image of f. Notice that here $h([f]) = f_*([\mathbb{C}P^{12}_{+}]) = f_*(a_{24}) \in H_{24}QMO(11)$. Applying lexicographic ordering, the group $H_{24}QMO(11)$ has a basis as follows

$$\{e_1^{10}e_2 \cdot e_1^{10}e_2,\ e_1^{11} \cdot e_1^{10}e_3,\ e_1^{11} \cdot e_1^{9}e_2^{2},\ Q^{13}e_1^{11}\} \cup \chi,$$

where, χ denotes a basis for the elements of height one.

In order to eliminate the impossible values for $f_*(a_{24})$ we will use the action of Sq^1_*, and the homology coproduct. More precisely, the relations

$$Sq^1_* f_*(a_{24}) = f_* Sq^1_*(a_{24}) = 0$$

and

$$\begin{aligned}
\widetilde{\psi}(f_*(a_{24})) &= (f_* \otimes f_*)(\widetilde{\psi}(a_{24})) \\
&= (f_* \otimes f_*)(a_2 \otimes a_{22} + a_4 \otimes a_{20} + a_6 \otimes a_{18} + a_8 \otimes a_{16} + a_{10} \otimes a_{14} + \\
&\quad a_{12} \otimes a_{12} + a_{14} \otimes a_{10} + a_{16} \otimes a_8 + a_{18} \otimes a_6 + a_{20} \otimes a_4 + a_{22} \otimes a_2). \\
&= f_*(a_2) \otimes f_*(a_{22}) + f_*(a_4) \otimes f_*(a_{20}) + f_*(a_6) \otimes f_*(a_{18}) + \\
&\quad f_*(a_8) \otimes f_*(a_{16}) + f_*(a_{10}) \otimes f_*(a_{14}) + f_*(a_{12}) \otimes f_*(a_{12}) + \\
&\quad f_*(a_{14}) \otimes f_*(a_{10}) + f_*(a_{16}) \otimes f_*(a_8) + f_*(a_{18}) \otimes f_*(a_6) + \\
&\quad f_*(a_{20}) \otimes f_*(a_4) + f_*(a_{22}) \otimes f_*(a_2).
\end{aligned}$$

However, we notice that

$$f_*(a_{2i}) = 0, \text{ for } \quad i \leq 5$$

since $H_{2i}(QMO(11)) = 0$.

The group $H_{12}QMO(11) \cong \mathbb{Z}/2$ is generated by $e_1^{10}e_2$, and so $f_*(a_{12}) = \alpha' e_1^{10}e_2$ for $\alpha' \in \mathbb{Z}/2$. By applying Sq_*^1 to the homology class we have, $Sq_*^1(f_*(a_{12})) = f_*(Sq_*^1(a_{12})) = f_*(0) = 0$ but $Sq_*^1(\alpha' e_1^{10}e_2) = \alpha' e_1^{11}$, and so $\alpha' = 0$. Hence

$$f_*(a_{12}) = 0.$$

This implies that $\widetilde{\psi}(f_*(a_{24})) = 0$, so that $f_*(a_{24})$ is primitive. Calculation gives the following table

	Sq_*^1	$\widetilde{\psi}$
$e_2^9e_3^2$	$e_1e_2^8e_3^2$	A
$e_1^{11} \cdot e_1^9e_2^2$	0	A
$e_1^{10}e_2 \cdot e_1^{10}e_2$	0	0
$Q^{13}e_1^{11}$	0	0
$e_1^{11} \cdot e_1^{10}e_3$	0	B
$e_2^{10}e_4$	$e_2^{10}e_3$	$B+C$
$e_1e_2^8e_3e_4$	$e_1e_2^8e_3^2$	0
$\vdots$	$\vdots$	$\vdots$

where

$$\begin{aligned} A &= e_1^{11} \otimes e_1^9e_2^2 + e_1^9e_2^2 \otimes e_1^{11}, \\ B &= e_1^{11} \otimes e_1^{10}e_3 + e_1^{10}e_3 \otimes e_1^{11}, \\ C &= e_1^{10}e_2 \otimes e_1^{10}e_2. \end{aligned}$$

We have four elements of height two $\{e_1^{11} \cdot e_1^9e_2^2,\ e_1^{10}e_2 \cdot e_1^{10}e_2,\ Q^{13}e_1^{11}, e_1^{11} \cdot e_1^{10}e_3\}$. Then from above table observe that the element $e_1^{11} \cdot e_1^9e_2^2$ is not primitive, in addition there is no linear combination with other elements to give $\mathcal{A}$-annihilated and primitive element. A only appears in the coproduct of which gives an $e_2^9e_3^2$ and $e_1^{11} \cdot e_1^9e_2^2$.

The element $e_1^{11} \cdot e_1^{10}e_3$ is $\mathcal{A}$-annihilated. However, it is not primitive and also there is no linear combination with other elements given a primitive element.

Finally, the elements $e_1^{11} \cdot e_1^9e_2^2 + e_2^9e_3^2 + e_1e_2^8e_3e_4$, $e_1^{10}e_2 \cdot e_1^{10}e_2$ and $Q^{13}e_1^{11}$ are primitive and are $\mathcal{A}$-annihilated by Sq_*^1. Hence

$$f_*(a_{24}) = c\, Q^{13}e_1^{11} + c_1 e_1^{10}e_2 \cdot e_1^{10}e_2 + c_2 e_1^{11} \cdot e_1^9e_2^2 + \varphi.$$

Hence there are eight possibilities for $f_*(a_{24})$.

We are now in a position to determine the double point manifold of an immersion F. By using diagram (6.3.3) and referring to Theorem 6.2.1, we find that

$$\xi_* p_2 h(\alpha) = c(e_1^{21}e_3 + e_1^{20}e_2^2) + c_1 e_1^{20}e_2^2 + c_2 e_1^{20}e_2^2 \in H_{24}QMO(22).$$

This implies that

$$\xi_* p_2 h(\alpha) = c e_1^{21}e_3 + (c + c_1 + c_2)e_1^{20}e_2^2.$$

Applying Proposition 3.3.11 we get that for an embedding $L^2 \hookrightarrow \mathbb{C}P^{12}$ corresponing to $\alpha \in [\mathbb{C}P^{12}_+, MO(22)]$, L^2 is cobordant to $\mathbb{R}P^2$ if and only if

$$h(\alpha) = e_1^{21}e_3 + e_1^{20}e_2^2,$$

or

$$h(\alpha) = e_1^{21}e_3.$$

Hence, the double point manifold of $F : M^{13} \looparrowright \mathbb{C}P^{12}$ is cobordant to the projective plane if and only if $c \neq 0$. □

We are now going to show this case does arise from the double point of an immersion and we can give a general result for all $k \equiv 3 \pmod 4$.

7.2 The case $k \equiv 3 \pmod 4$

The case $k = 11$ is the first case where we obtain a different result from the case of immersed manifolds in $\mathbb{R}^{2k+2}$. In [AEa00] Eccles and Asadi have shown that the double point manifold of any immersion in Euclidean spaces is a boundary because $\alpha(11 + 1) = \alpha(12) = 2 > 1$. However, we will show that there is an immersion F in complex projective space with double point manifold cobordant to the projective plane.

Now we are going to state the main theorem of this study which shows the existence of an immersion $F : M^{k+2} \looparrowright \mathbb{C}P^{k+1}$ with double point manifold cobordant to the projective plane for $k \equiv 3 \pmod 4$.

Theorem 7.2.1. *For $k \equiv 3 \pmod 4$, there exists a map $f : \mathbb{C}P_+^{k+1} \to QMO(k)$, such that*

$$f_*(a_{2k+2}) = Q^{k+2}e_1^k.$$

Suppose that $a_{2k+2} \in H_{2k+2}\mathbb{C}P_+^{k+1}$ is the generator. We want to prove that there exists a map f such that

$$f_*(a_{2k+2}) = Q^{k+2}e_1^k \in H_{2k+2}QMO(k).$$

This is constructed as a map

$$\underbrace{\mathbb{C}P_+^{k+1} \to \Sigma^k \mathbb{R}P_k^{k+2} \to QMO(k)}_{f}.$$

We prove this theorem by breaking the proof into two propositions as follows.

Proposition 7.2.2. *There exists a map $f_1 : \mathbb{C}P_+^{k+1} \longrightarrow \Sigma^k \mathbb{R}P_k^{k+2}$, such that*

$$(f_1)_*(a_{2k+2}) = \sigma^k b_{k+2}.$$

Proof. Let $\mathbb{C}P_k^{k+1}$ be the truncated complex projective space $\mathbb{C}P_k^{k+1} = \mathbb{C}P^{k+1}/\mathbb{C}P^{k-1}$. For k odd we can form $\mathbb{C}P_k^{k+1}$ from S^{2k} by attaching a $(2k+2)$-cell e^{2k+2} via the suspension of the Hopf map $\eta_{2k} : S^{2k+1} \to S^{2k}$ because $Sq^2 : H^{2k}\mathbb{C}P_k^{k+1} \to H^{2k+2}\mathbb{C}P_k^{k+1}$ is non-zero. Then

$$\mathbb{C}P_k^{k+1} = S^{2k} \cup_{\eta_{2k}} e^{2k+2}.$$

Let $\mathbb{R}P_k^{k+2} = \mathbb{R}P^{k+2}/\mathbb{R}P^{k-1}$, then we can form $\mathbb{R}P_k^{k+2}$ from $\mathbb{R}P_k^{k+1}$ by attaching a $(k+2)$- cell e^{k+2} via a map $\phi'_{k+1} : S^{k+1} \longrightarrow \mathbb{R}P_k^{k+1}$ and

$$\mathbb{R}P_k^{k+2} = \mathbb{R}P_k^{k+1} \cup_{\phi'_{k+1}} e^{k+2}.$$

Let $\phi_k : S^k \longrightarrow \mathbb{R}P^k$ be the double cover map. Then

$$\mathbb{R}P^{k+1} = \mathbb{R}P^k \cup_{\phi_k} e^{k+1}.$$

Thus we have a cofibre sequence:

$$S^k \xrightarrow{\phi_k} \mathbb{R}P^k \xrightarrow{i} \mathbb{R}P^{k+1} \xrightarrow{P_{k+1}} S^{k+1}.$$

The composition

$$S^k \xrightarrow{\phi_k} \mathbb{R}P^k \xrightarrow{P_k} \mathbb{R}P^k/\mathbb{R}P^{k-1} \cong S^k$$

is a map of degree

$$1 + (-1)^{k+1} = \begin{cases} 0 & \text{if } k \text{ is even,} \\ 2 & \text{if } k \text{ is odd.} \end{cases}$$

Since k is odd then $P_k \circ \phi_k = 2\iota : S^k \to S^k$, where $\iota \in \pi_k S^k \cong \mathbb{Z}$ generated by identity map $\iota : S^k \longrightarrow S^k$. So we have the following diagram of cofibration sequences.

$$\begin{array}{ccccccc} S^k & \xrightarrow{\phi_k} & \mathbb{R}P^k & \xrightarrow{i} & \mathbb{R}P^{k+1} & \xrightarrow{P_{k+1}} & S^{k+1} \\ \downarrow{=} & & \downarrow{P_k} & & \downarrow{P_k} & & \downarrow{=} \\ S^k & \xrightarrow{2\iota} & S^k & \xrightarrow{i} & \mathbb{R}P_k^{k+1} & \xrightarrow{P_k} & S^{k+1} \end{array} \tag{7.2.3}$$

Since we have $k \geq 3$ we are in the stable range and so the cofibrations give exact sequences of stable homotopy groups. We have the following diagram.

$$\begin{array}{ccccc} S^{k+1} & \xrightarrow{\phi_{k+1}} & \mathbb{R}P^{k+1} & \xrightarrow{i} & \mathbb{R}P^{k+2} \\ \downarrow{=} & & \downarrow{P_k} & & \downarrow{P_k} \\ S^{k+1} & \xrightarrow{\phi'_{k+1}} & \mathbb{R}P_k^{k+1} & \xrightarrow{i} & \mathbb{R}P_k^{k+2} \end{array} \tag{7.2.4}$$

Diagrams (7.2.3) and (7.2.4) give the following diagram.

$$\begin{array}{ccccccccc} & & & & S^{k+1} & & & & \\ & & & & \downarrow{P\circ\phi_{k+1}} & & & & \\ \longrightarrow S^k & \xrightarrow{2\iota} & S^k & \xrightarrow{i} & \mathbb{R}P_k^{k+1} & \longrightarrow & \mathbb{R}P^{k+1}/\mathbb{R}P^k = S^{k+1} & \longrightarrow & \Sigma\mathbb{R}P_k^k \end{array}$$

$$\phi : S^{k+1} \dashrightarrow S^k, \qquad P_{k+1}\circ\phi_{k+1} : S^{k+1} \longrightarrow \mathbb{R}P^{k+1}/\mathbb{R}P^k$$

The map $P_{k+1} \circ \phi_{k+1} : S^{k+1} \longrightarrow S^{k+1}$ is trivial since k is odd. Hence the attaching map $P \circ \phi_{k+1}$ pulls back to $\phi : S^{k+1} \longrightarrow S^k$. This gives the following commutative diagram of cofibre sequences defining the function g_1.

$$\begin{array}{ccc}
S^{k+1} & \xrightarrow{\ 1\ } & S^{k+1} \\
{\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle P\circ\phi_{k+1}} \\
S^{k} & \xrightarrow{\ i\ } & \mathbb{R}P_k^{k+1} \\
\downarrow & & \downarrow{\scriptstyle i} \\
(S^k \cup_\phi e^{k+2}) & \xrightarrow{\ g_1\ } & \mathbb{R}P_k^{k+2} \\
\downarrow & & \downarrow \\
S^{k+2} & \xrightarrow{\ 1\ } & S^{k+2}
\end{array}$$

Suppose that $\phi \in, \pi_{k+1}S^k \cong \mathbb{Z}/2$, it is generated by the map $\phi : S^{k+1} \longrightarrow S^k$. Then we have the mapping cone $C_\phi = S^k \cup_\phi e^{k+2}$.

The induced diagram in homology shows that $(g_1)_* : H_kC_\phi \to H_k\mathbb{R}P_k^{k+2}$ and $(g_1)_* : H_{k+2}C_\phi \to H_{k+2}\mathbb{R}P_k^{k+2}$ are isomorphisms.

Hence by naturality $Sq^2_* : H_{k+2}C_\phi \to H_kC_\phi$ is non-zero. This shows that ϕ is non-trivial and so is homotopic to $\eta_k : S^{k+1} \to S^k$ the suspension of the Hopf map. Hence $C_\phi \simeq S^k \cup_{\eta_k} e^{k+2}$ and so

$$\Sigma^k C_\phi \simeq S^{2k} \cup_{\eta_{2k}} e^{2k+2} = \mathbb{C}P_k^{k+1}$$

Taking the k-th suspension of g_1 leads to

$$\Sigma^k g_1 : S^{2k} \cup_{\eta_{2k}} e^{2k+2} \longrightarrow \Sigma^k\mathbb{R}P_k^{k+2}$$

such that $(f_1)_* : H_{2k+2}\mathbb{C}P_+^{k+1} \to H_{2k+2}\Sigma^k\mathbb{R}P_k^{k+1}$ is an isomorphism. Since $H_{k+2}\mathbb{R}P_k^{k+2} \cong \mathbb{Z}/2$. It is generated by b_{k+2}. Then

$$(\Sigma^k g_1)_*(a_{2k+2}) = \sigma^k b_{k+2}.$$

Finally we compose $\Sigma^k g_1$ with the quotient map $q\colon \mathbb{C}P_+^{k+1} \to \mathbb{C}P_k^{k+1}$ to obtain the required map $f_1 = \Sigma^k g_1 \circ q$ which completes the proof. □

Proposition 7.2.5. There exists a map $f_2 : \Sigma^k\mathbb{R}P_k^{k+2} \longrightarrow QMO(k)$, such that

$$(f_2)_*(\sigma^k b_{k+2}) = Q^{k+2}e_1^k \text{ modulo elements of height one .}$$

To prove this we describe two maps and then take their composition to get the required map as in Eccles [E96].

Definition 7.2.6. We define the J-homomorphism

$$J : V_3(\mathbb{R}^{k+3}) \longrightarrow \Omega^{k+3}\Sigma^3 MO(k).$$

Suppose that $G(k+3)$ is a closed subgroup of the orthogonal group $O(k+3)$ with inclusion map

$$i : G(k+3) \longrightarrow O(k+3).$$

Then if ξ is a $(k+3)$-dimensional vector bundle over S^{k+2} represented by $\xi \in \pi_{k+2}BO(k+3)$, a $G(k+3)$-structure on ξ is a choice of element $\overline{\xi} \in \pi_{k+2}BG(k+3)$ such that $i_*\overline{\xi} = \xi$. We have the following exact sequence:

$$\pi_{k+2}O(k+3)/G(k+3) \to \pi_{k+2}BG(k+3) \to \pi_{k+2}BO(k+3).$$

The normal bundle ν of the standard embedding $S^{k+2} \hookrightarrow \mathbb{R}^{2k+5}$ is trivial, and so each element of $\pi_{k+2}O(k+3)/G(k+3)$ determines a $G(k+3)$-structure on ν. With this structure, the embedded sphere represents an element of $\pi_{2k+5}MG(k+3)$ by the Pontrjagin-Thom construction [E96]. This process defines the generalized J-homomorphism

$$J_* : \pi_{k+2}O(k+3)/G(k+3) \longrightarrow \pi_{2k+5}MG(k+3).$$

The image of this map J_* consists of those elements which may be represented by the standard embedding $S^{k+2} \hookrightarrow \mathbb{R}^{2k+5}$ with some $G(k+3)$-structure.

When $G(k+3)$ is the trivial group, we get the classical J-homomorphism

$$\pi_{k+2}O(k+3) \longrightarrow \pi_{2k+5}S^{k+3}.$$

On the other hand when $G(k+3) = O(k)$, $O(k+3)/O(k)$ will be the real Stiefel manifold $V_3(\mathbb{R}^{k+3})$ as defined after Remark 1.4.21, and $MG(k+3)$ is the suspension $\Sigma^3 MO(k)$. So we get a map from the homotopy group of the real Stiefel manifold to the homotopy of the suspended Thom complex of $O(k)$.

$$J_* : \pi_{k+2}V_3(\mathbb{R}^{k+3}) \longrightarrow \pi_{2k+5}\Sigma^3 MO(k).$$

Assume that for the classifying space of γ^k we take the infinite Grassmannian $G_k(\mathbb{R}^\infty)$ of k-dimensional linear subspaces of $\mathbb{R}^\infty$. Then the total space $EO(k)$ of the universal bundle is given by

$$EO(k) = \{(u, U) | u \in U, U \in G_k(\mathbb{R}^\infty)\}.$$

Next, let $\upsilon = (\upsilon_1, \upsilon_2, \upsilon_3) \in V_3(\mathbb{R}^{k+3})$ be an orthogonal 3-frame in $\mathbb{R}^{k+3}$. We can write $U = \langle \upsilon_1, \upsilon_2, \upsilon_3 \rangle^\perp \subseteq \mathbb{R}^{k+3} \subseteq \mathbb{R}^\infty$ for the subspace of $\mathbb{R}^{k+3}$ orthogonal to υ_1, υ_2 and υ_3. Then a point of $\mathbb{R}^{k+3}$ may be written uniquely as $u + t_1\upsilon_1 + t_2\upsilon_2 + t_3\upsilon_3$ where $t_i \in \mathbb{R}$ for $i = \{1, 2, 3\}$ and $u \in U$. Define a continuous map $J(\upsilon) : \mathbb{R}^{k+3} \longrightarrow EO(k) \times \mathbb{R}^3$ by

$$J(\upsilon)(u + t_1\upsilon_1 + t_2\upsilon_2 + t_3\upsilon_3) = ((u, U), t_1, t_2, t_3).$$

One point compactification induces a map

$$J(\upsilon) : S^{k+3} \longrightarrow MO(k) \wedge S^3,$$

i.e. $J(\upsilon) \in \Omega^{k+3}\Sigma^3 MO(k)$, and then, by [E96, Proposition 2.1.], the continuous map $J : V_3(\mathbb{R}^{k+3}) \longrightarrow \Omega^{k+3}\Sigma^3 MO(k)$ induces the generalized J-homomorphism

$$J_* : \pi_{k+2}V_3(\mathbb{R}^{k+3}) \longrightarrow \pi_{k+2}\Omega^{k+3}\Sigma^3 MO(k) \to \pi_{2k+5}\Sigma^3 MO(k).$$

Definition 7.2.7. We define the hyperplane reflection map

$$\lambda : \mathbb{R}P_k^{k+2} \longrightarrow V_3(\mathbb{R}^{k+3}).$$

A point $a \in \mathbb{R}P^{k+2}$ is a line through the origin of $\mathbb{R}^{k+3}$, and we may use this to define a map

$$\lambda_{k+3} : \mathbb{R}P^{k+2} \longrightarrow O(k+3).$$

The element $\lambda_{k+3}(a)$ is given by the hyperplane reflection map which is given by reflection in the hyperplane orthogonal to the line $a \in \mathbb{R}P^{k+2}$. Given a point $x \in \mathbb{R}^{k+3}$, we can write it uniquely as $x = x_1 + x_2$, such that $x_1 \in \text{span}\langle a \rangle, x_2 \in a^\perp$. Then $\lambda_{k+3}(a)(x) = -x_1 + x_2$. This is an orthogonal map represented by an element $\lambda_{k+3}(a) \in O(k+3)$.

In the following commutative diagram the right hand is a sequence of groups and the left is a cofibre sequence. Since $\mathbb{R}^k = \mathbb{R}^k \times \{0\} \subseteq \mathbb{R}^{k+3}$, we get the following commutative diagram defining the map λ

$$\begin{array}{ccc} \mathbb{R}P^{k-1} & \xrightarrow{\lambda_k} & O(k) \\ \downarrow & = & \downarrow i \\ \mathbb{R}P^{k+2} & \xrightarrow{\lambda_{k+3}} & O(k+3) \\ \downarrow & & \downarrow \\ \mathbb{R}P^{k+2}_k & \xrightarrow{\lambda} & O(k+3)/O(k) \end{array}$$

where the vertical maps are the standard inclusions and $O(k+3)/O(k) = V_3(\mathbb{R}^{k+3})$.

Proof of Proposition 7.2.5. The k-th adjoint of the continuous map J gives a map

$$\widetilde{J} : \Sigma^k V_3(\mathbb{R}^{k+3}) \longrightarrow \Omega^3\Sigma^3 MO(k) \to QMO(k). \tag{7.2.8}$$

Moreover, by taking k-th suspension of the hyperplane reflection map we get a map

$$\widetilde{\lambda} : \Sigma^k\mathbb{R}P^{k+2}_k \longrightarrow \Sigma^k V_3(\mathbb{R}^{k+3}). \tag{7.2.9}$$

Then by taking the composition of the maps (7.2.8) and (7.2.9) we have

$$f_2 = \widetilde{J} \circ \widetilde{\lambda} : \Sigma^k\mathbb{R}P^{k+2}_k \longrightarrow QMO(k).$$

Next we need to describe this map in homology. Consider the following commutative diagram.

$$\begin{array}{ccccc} \Sigma^k\mathbb{R}P^{k+2}_k & \xrightarrow{\widetilde{\lambda}} & \Sigma^k V_3(\mathbb{R}^{k+3}) & \xrightarrow{\widetilde{J}} & QMO(k) \\ \downarrow i & & = & & \downarrow h^2 \\ \Sigma^k\mathbb{R}P^{\infty}_k \cong D_2S^k & \xrightarrow{i} & QD_2S^k & \xrightarrow{(QD_2)i} & QD_2MO(k) \end{array}$$

This follows from the diagram in the proof of Theorem 3.4 in [E96]. Since $\sigma^k b_{k+2} \in H_{2k+2}\Sigma^k\mathbb{R}P^{k+2}_k$ corresponds to $Q^{k+2}g_k \in H_{2k+2}QD_2S^k$ which maps to $Q^{k+2}e_1^k \in H_{2k+2}QD_2MO(k)$,

$$(f_2)_*(\sigma^k b_{k+2}) = Q^{k+2}e_1^k \text{ modulo elements of height one .}$$

This completes the proof of Proposition 7.2.5. □

Proof of Theorem 7.2.1. By the composition of the functions of Proposition 7.2.2 and Proposition 7.2.5 we get the following

$$\mathbb{C}P_+^{k+1} \xrightarrow{f_1} \Sigma^k \mathbb{R}P_k^{k+2} \xrightarrow{f_2} QMO(k) .$$

Hence we may define

$$f = f_2 \circ f_1 : \mathbb{C}P_+^{k+1} \longrightarrow QMO(k).$$

Then

$$f_*(a_{2k+2}) = (f_2)_*(f_1)_*(a_{2k+2}) = (f_2)_*(\sigma^k b_{k+2}) = Q^{k+2}e_1^k,$$

which completes the proof. □

Corollary 7.2.10. *There exist an immersion* $F : M^{k+2} \longrightarrow \mathbb{C}P^{k+1}$ *which has a double point manifold* $\theta_2(F) : \Delta_2(F) \looparrowright \mathbb{C}P^{k+1}$ *cobordant to the projective plane.*

Proof. Suppose that $F : M^{k+2} \looparrowright \mathbb{C}P^{k+1}$ is an immersion corresponding to the map

$$f : \mathbb{C}P_+^{k+1} \longrightarrow QMO(k),$$

which is given above. The self-transverse immersion F represents the element $\alpha \in [\mathbb{C}P_+^{k+1}, MO(k)]^S$, where α is the homotopy class of f. Then by Theorem 6.3.2 the double point manifold

$$\theta_2(F) : \Delta_2(F) \looparrowright \mathbb{C}P^{k+1}$$

corresponds to the element $h_*^2(\alpha) \in [\mathbb{C}P_+^{k+1}, D_2MO(k)]^S$, where

$$h_*^2 : [\mathbb{C}P_+^{k+1}, MO(k)]^S \longrightarrow [\mathbb{C}P_+^{k+1}, D_2MO(k)]^S.$$

is the Hopf invariant.

Suppose that $[g] = \xi_* \circ h_*^2 \circ ([f])$, where $[g] \in [\mathbb{C}P_+^{k+1}, MO(2k)]^S$. This element corresponds to the immersion

$$\theta_2(F) : \Delta_2(F) \looparrowright \mathbb{C}P^{k+1}.$$

By Theorem 6.3.2 the element $g_*(a_{2k+2}) \in H_{2k+2}QMO(2k)$ determines (and is determined by) the characteristic numbers of the immersion $\theta_2(F) : \Delta_2 \looparrowright \mathbb{C}P^{k+1}$.

Since $f_*(a_{2k+2}) = Q^{k+2}e_1^k$, $p_2 f_*(a_{2k+2}) = Q^{k+2}e_1^k$. Hence, by Theorem 6.2.1, we deduce that

$$g_*(a_{2k+2}) = \xi_* p_2(Q^{k+2}e_1^k) = e_1^{2k-1}e_3 + e_1^{2k-2}e_2^2 \in H_{2k+2}QMO(2k).$$

This shows that the immersion $\theta_2(F) : \Delta_2(F) \looparrowright \mathbb{C}P^{k+1}$ is cobordant to an immersion $\mathbb{R}P^2 \looparrowright \mathbb{C}P^{k+1}$ with $b_2 \mapsto a_1$ as we explained in Example 3.3.10. □

7.3 The case $k \equiv 1 \pmod 4$

We show that in this case, it is possible to have an immersion $F : M^{k+2} \looparrowright \mathbb{C}P^{k+1}$ whose double point manifold is not a boundary. This occurs directly from the solution of the problem in the Euclidean case.

Theorem 7.3.1. *There always exists an immersion $F : M^{k+2} \looparrowright \mathbb{C}P^{k+1}$ whose double point manifold is cobordant to the projective plane.*

Proof. According [AEa00, Theorem 4.1] there always exists an immersion $F : M^{k+2} \looparrowright \mathbb{R}^{2k+2}$ whose double point manifold is cobordant to a projective plane.

Let $i : \mathbb{R}^{2k+2} \hookrightarrow \mathbb{C}P^{k+1}$ be an embedding which always exists as $\mathbb{C}P^{k+1}$ is a $(2k+2)$-dimensional manifold. The composition

$$i \circ F : M^{k+2} \looparrowright \mathbb{R}^{2k+2} \hookrightarrow \mathbb{C}P^{k+1}$$

provides an immersion whose double point manifold is not a boundary. □

Chapter 8

The double point manifolds of $F : M^{k+2} \looparrowright \mathbb{C}P^{k+1}$ when k is even

In the previous chapter we deduced that for all odd values of k there exists an immersion F with double point manifold cobordant to the projective plane.

In this chapter we deal with the case when k is even, and divide it into two cases, $k \equiv 2 \pmod 4$ and $k \equiv 0 \pmod 4$.

In the case of $k \equiv 2 \pmod 4$ we start with the specific case of $k = 2$ and show that there exists an immersion with double point manifold cobordant to the projective plane. In general we derive a condition for the double point manifold to be cobordant to the projective plane. However we have not shown that these immersions exist.

For $k \equiv 0 \pmod 4$ we consider the specific example of $k = 4$ but we do not obtain a general result in this case.

8.1 The case $k \equiv 2 \pmod 4$.

Assume $k = 4r + 2$. We start by studying immersions $F : M^{k+2} \looparrowright \mathbb{C}P^{k+1}$ with $k = 2, 6, 10$. According to the Pontrjagin-Thom theory, we have to calculate the possible values of $f_*[\mathbb{C}P^{k+1}_+]$ where $f : \mathbb{C}P^{k+1}_+ \to QMO(k)$ corresponds to F. Of course, these are just potential values and in order to show the existence of the desired immersions we either have to construct the immersion F directly or construct a mapping f which has the given Hurewicz image. We will do this in the case of $k = 2$.

8.1.1 The double point manifold of $F : M^4 \looparrowright \mathbb{C}P^3$

Theorem 8.1.1. *An immersion $F : M^4 \looparrowright \mathbb{C}P^3$ corresponding to a map $f : \mathbb{C}P^3_+ \to QMO(2)$ has double point manifold cobordant to the projective plane if and only if $\widetilde{f}_*(a_4) = e_2^2 + e_1e_3$, where $\widetilde{f} : \mathbb{C}P^3_+ \twoheadrightarrow MO(2)$ is the stable map corresponding to f.*

Proof. Our goal is to prove that for some immersion F the double point manifold is cobordant to projective plane. The Pontrjagin-Thom construction gives a function $f : \mathbb{C}P^3_+ \to QMO(2)$ representing a homotopy class $\alpha \in [\mathbb{C}P^3_+, QMO(2)]$.
We need to calculate $f_*(a_6)$ and in order to eliminate possible values we also need to find out about $f_*(a_2)$ and $f_*(a_4)$.

First of all, since $f_*(a_2) \in H_2QMO(2) \cong \mathbb{Z}/2$ generated by e_1^2, then for $\alpha' \in \mathbb{Z}/2$.

$$f_*(a_2) = \alpha' e_1^2.$$

Hence we have two possibilities for $f_*(a_2)$, which are either 0 or e_1^2.

It remains to determine the possibilities for $f_*(a_4), f_*(a_6)$. A basis of $H_4QMO(2)$ is given by

$$\{e_1e_3,\ e_2^2,\ e_1^2 \cdot e_1^2\}.$$

To eliminate the impossible values of $f_*(a_4)$, we consider the following table

	Sq^1_*	Sq^2_*	$\widetilde{\psi}$
e_1e_3	0	0	0
e_2^2	0	e_1^2	$e_1^2 \otimes e_1^2$
$e_1^2 \cdot e_1^2$	0	0	0

Consider the reduced coproduct of $f_*(a_4)$ as follows

$$\begin{aligned}\widetilde{\psi}(f_*(a_4)) &= f_*(a_2) \otimes f_*(a_2), \\ &= \alpha'(e_1^2 \otimes e_1^2).\end{aligned}$$

Then $f_*(a_4)$ is not primitive if $\alpha' = 1$. Hence

$$f_*(a_4) = \alpha' e_2^2 + \text{primitive terms.}$$

Since $Sq^1_* f_*(a_4) = 0, Sq^2_* f_*(a_4) = f_*(a_2) = \alpha' e_1^2$, then $f_*(a_4)$ is not $\mathcal{A}$-annihilated if $\alpha' = 1$. The $\mathcal{A}$-annihilated and primitive elements in $H_4QMO(2)$ are spanned by $\{e_1e_3,\ e_1^2 \cdot e_1^2\}$. Then for some $\beta, \gamma \in \mathbb{Z}/2$.

$$f_*(a_4) = \alpha' e_2^2 + \beta e_1 e_3 + \gamma e_1^2 \cdot e_1^2,$$

Our task is now to consider the possibilities for $f_*(a_6)$. Let

$$h(\alpha) = f_*(a_6) \in H_6(QMO(2)),$$

where $a_6 = [\mathbb{C}P^3_+] \in H_6(\mathbb{C}P^3_+)$ is the fundamental class. A basis for $H_6QMO(2)$ is given by

$$\{e_1e_5,\ e_2e_4,\ e_3^2,\ e_1^2 \cdot e_1e_3,\ e_1^2 \cdot e_2^2,\ e_1e_2 \cdot e_1e_2,\ Q^4e_1^2,\ e_1^2 \cdot e_1^2 \cdot e_1^2\}.$$

To eliminate the possible values of $f_*(a_6) \in H_6QMO(2)$, we consider the following table:

	Sq^1_*	Sq^2_*	$\widetilde{\psi}$
e_1e_5	0	e_1e_3	0
e_2e_4	$e_1e_4 + e_2e_3$	$e_1e_3 + e_2^2$	$B + C$
e_3^2	0	0	A
$e_1^2 \cdot e_1e_3$	0	0	B
$e_1^2 \cdot e_2^2$	0	$e_1^2 \cdot e_1^2$	$A + D$
$e_1e_2 \cdot e_1e_2$	0	$e_1^2 \cdot e_1^2$	0
$Q^4e_1^2$	$Q^3e_1^2$	$e_1^2 \cdot e_1^2$	0
$e_1^2 \cdot e_1^2 \cdot e_1^2$	0	0	D

where

$$\begin{aligned} A &= e_1^2 \otimes e_2^2 + e_2^2 \otimes e_1^2, \\ B &= e_1^2 \otimes e_1e_3 + e_1e_3 \otimes e_1^2, \\ C &= e_1e_2 \otimes e_1e_2, \\ D &= e_1^2 \otimes e_1^2 \cdot e_1^2 + e_1^2 \cdot e_1^2 \otimes e_1^2. \end{aligned}$$

By applying the reduced coproduct of $f_*(a_6)$ we will obtain

$$\begin{aligned}\widetilde{\psi}(f_*(a_6)) &= f_*(a_2)\otimes f_*(a_4) + f_*(a_4)\otimes f_*(a_2)\\ &= \alpha^{'}(e_1^2\otimes e_2^2) + \alpha^{'}\beta(e_1^2\otimes e_1e_3) + \alpha^{'}\gamma(e_1^2\otimes e_1^2\cdot e_1^2)\\ &+ \alpha^{'}(e_2^2\otimes e_1^2) + \alpha^{'}\beta(e_1e_3\otimes e_1^2) + \alpha^{'}\gamma(e_1^2\cdot e_1^2\otimes e_1^2)\\ &= \alpha^{'}A + \alpha^{'}\beta B + \alpha^{'}\gamma D.\end{aligned}$$

Then $f_*(a_6)$ is not primitive if $\alpha^{'} = 1$. Hence

$$f_*(a_6) = \alpha^{'}\ e_3^2 + \alpha^{'}\beta(e_1^2\cdot e_1e_3) + \alpha^{'}\gamma(e_1^2\cdot e_1^2\cdot e_1^2) + \text{primitive terms}.$$

Applying Sq^1_* and Sq^2_* to the homology class $f_*(a_6)$ implies that $Sq^1_*f_*(a_6) = 0, Sq^2_*f_*(a_6) = 0$, so that $f_*(a_6)$ is $\mathcal{A}$-annihilated. Hence for some $\varphi \in \mathbb{Z}/2$

$$f_*(a_6) = \alpha^{'}\ e_3^2 + \alpha^{'}\beta(e_1^2\cdot e_1e_3) + \alpha^{'}\gamma(e_1^2\cdot e_1^2\cdot e_1^2) + \varphi,$$

where $\phi = e_3^2 + e_1^2\cdot e_2^2 + e_1e_2\cdot e_1e_2 + e_1^2\cdot e_1^2\cdot e_1^2$ is the unique non-zero $\mathcal{A}$-annihilated and primitive element in $H_6QMO(2)$. Hence

$$f_*(a_6) = \alpha^{'}\ e_3^2 + \alpha^{'}\beta(e_1^2\cdot e_1e_3) + \alpha^{'}\gamma(e_1^2\cdot e_1^2\cdot e_1^2) + \delta(e_3^2 + e_1^2\cdot e_2^2 + e_1e_2\cdot e_1e_2 + e_1^2\cdot e_1^2\cdot e_1^2).$$

The following table summarizes the possible value for $f_*(a_{2i})$ for $1 \leq i \leq 3$.

$f_*(a_2)$	$f_*(a_4)$	$f_*(a_6)\pmod{\phi}$
0	0	0
0	e_1e_3	0
0	$e_1^2\cdot e_1^2$	0
0	$e_1e_3 + e_1^2\cdot e_1^2$	0
e_1^2	e_2^2	e_3^2
e_1^2	$e_2^2 + e_1e_3$	$e_3^2 + e_1^2\cdot e_1e_3$
e_1^2	$e_2^2 + e_1^2\cdot e_1^2$	$e_3^2 + e_1^2\cdot e_1^2\cdot e_1^2$
e_1^2	$e_2^2 + e_1e_3 + e_1^2\cdot e_1^2$	$e_3^2 + e_1^2\cdot e_1e_3 + e_1^2\cdot e_1^2\cdot e_1^2$

Applying diagram (6.3.3) we deduce that

$$p_2h(\alpha) = \alpha^{'}\beta e_1^2\cdot e_1e_3 + \delta(e_1^2\cdot e_2^2 + e_1e_2\cdot e_1e_2),$$

and then from Theorem 6.2.1 we have

$$\xi_* p_2 h(\alpha) = \alpha' \beta e_1^3 e_3 + \delta(e_1^2 e_2^2 + e_1^2 e_2^2).$$

Hence

$$\xi_* p_2 h(\alpha) = \alpha' \beta e_1^3 e_3 \in H_6 MO(4).$$

We are now in a position to deduce that the immersion $F : M^4 \looparrowright \mathbb{C}P^3$ has a double point manifold cobordant to the projective plane if and only if $h(\alpha)$ has $\alpha' = \beta = 1$ as observed on the line 6 and 8 of the above table. This mean that $f_*(a_4) = e_2^2 + e_1 e_3 + \gamma e_1^2 \cdot e_1^2$ or equivalently $\widetilde{f}_*(a_4) = e_2^2 + e_1 e_3$. □

Next we are going to show that there exists an immersion with double point manifold cobordant to the projective plane by constructing a stable map $\mathbb{C}P^3_+ \nrightarrow QMO(2)$ whose Hurewicz image in homology gives the right terms to have this property. This map is constructed from two other maps which we now describe.

Proposition 8.1.2. *There exists a map $f_1 : \mathbb{C}P^3_+ \longrightarrow QMO(2)$ such that its stable adjoint $\widetilde{f}_1 : \mathbb{C}P^3_+ \nrightarrow MO(2)$ satisfies*

$$(\widetilde{f}_1)_*(a_4) = e_1 e_3.$$

We shall define $\widetilde{f}_1$ as the composition of two maps which are $X : \mathbb{C}P^3 \longrightarrow S^4$ and $Y : S^4 \nrightarrow MO(2)$. We break the proof into small lemmas.

Lemma 8.1.3. *There is a map $X : \mathbb{C}P^3_+ \longrightarrow S^4$ such that $a_4 \longmapsto g_4$.*

Proof. We have the truncated projective space $\mathbb{C}P^3_2 = \mathbb{C}P^3/\mathbb{C}P^1$, this is given by a cell complex $S^4 \cup_\alpha e^6$ obtained by attaching the 6-cell e^6 via a map $\alpha : S^5 \longrightarrow S^4$. The homotopy group $\pi_5 S^4 \cong \mathbb{Z}/2$ is generated by suspension of the Hopf map. The Hopf map is detected by

$$Sq^2 \neq 0 : H^4 S^4 \cup_\alpha e^6 \longrightarrow H^6 S^4 \cup_\alpha e^6.$$

However, $Sq^2 = 0 : H^4 \mathbb{C}P^3_2 \longrightarrow H^6 \mathbb{C}P^3_2$ and so α is trivial and so $\mathbb{C}P^3_2 \simeq S^4 \vee S^6$.

$$\begin{array}{ccc} \mathbb{C}P^3_+ & \xrightarrow{q} & \mathbb{C}P^3_2 \simeq S^4 \vee S^6 \\ & \overset{X}{\searrow} & \downarrow p \\ & & S^4 \end{array}$$

Now by the composing of the quotient map q with the projection map p as in the above diagram we obtain a map $X : \mathbb{C}P^3_+ \longrightarrow S^4$ such that

$$X_*(a_4) = g_4 \in H_4 S^4.$$

□

Lemma 8.1.4. *There is a stable map $Y : S^4 \nrightarrow MO(2)$ such that $g_4 \mapsto e_1 e_3$.*

Proof. The double cover map

$$\phi_2 : S^2 \longrightarrow \mathbb{R}P^2$$

is the attaching map of the 3-cell in $\mathbb{R}P^3$, i.e. $\mathbb{R}P^2 \cup_{\phi_2} e^3 = \mathbb{R}P^3$. We have a cofiber sequence

$$S^1 \xrightarrow{i} \mathbb{R}P^2 \xrightarrow{p} S^2 .$$

This cofibration gives an exact sequence

$$\pi_2^S S^1 \xrightarrow{i_*} \pi_2^S \mathbb{R}P^2 \xrightarrow{p_*} \pi_2^S S^2 .$$

The composition $p \circ \phi_2 : S^2 \longrightarrow \mathbb{R}P^2/\mathbb{R}P^1 = S^2$ has degree 0 and so is trivial in homotopy [H02]. This means that $p_*[\phi_2] \in \pi_2^S S^2$ is trivial, and hence by exactness there exists $[\phi_1] \in \pi_2^S S^1$, such that

$$[\phi_2] = i_*[\phi_1].$$

This means that there exists a stable map $\phi_1 : S^2 \nrightarrow S^1 = \mathbb{R}P^1$ such that the following diagram is commutative

$$\begin{array}{ccccc} & & S^2 & & \\ & \overset{\phi_1}{\swarrow} & \downarrow{\scriptstyle \phi_2} & \overset{0}{\searrow} & \\ \mathbb{R}P^1 = S^1 & \xrightarrow{i} & \mathbb{R}P^2 & \xrightarrow{p} & S^2 = \mathbb{R}P^2/\mathbb{R}P^1 \\ & & \downarrow{\scriptstyle i} & & \\ & & \mathbb{R}P^3 & & \end{array}$$

where $[\phi_1] \in \pi_2^S S^1 \cong \mathbb{Z}/2$ is generated by the Hopf map $\eta : S^3 \to S^2$ which is detected by Sq^2. But $Sq^2 : H^1\mathbb{R}P^3 \longrightarrow H^3\mathbb{R}P^3$ is trivial. Moreover, by naturality we see that

$$Sq^2 : H^1 S^1 \cup_{\phi_1} e^3 \longrightarrow H^3 S^1 \cup_{\phi_1} e^3$$

is trivial. So $[\phi_1] = 0$, and so $[\phi_2] = 0 \in \pi_2^S \mathbb{R}P^2$. Hence there is a stable equivalence

$$\mathbb{R}P^3 \simeq \mathbb{R}P^2 \vee S^3.$$

Therefore, there is a stable map $\phi_3 : S^3 \nrightarrow \mathbb{R}P^2 \vee S^3 \simeq \mathbb{R}P^3$ such that the following diagram commutes

$$\begin{array}{ccc} S^3 & \xrightarrow{\ \phi_3\ } & \mathbb{R}P^3 \\ & \searrow^{1} & \downarrow \\ & & S^3 = \mathbb{R}P^3/\mathbb{R}P^2 \end{array}$$

and so $(\phi_3)_*(g_3) = e_3$.

We define a stable mapping $Y : S^4 \nrightarrow MO(2)$ by the composition

$$S^4 \xrightarrow{\Sigma\phi_3} \Sigma\mathbb{R}P^3 \xrightarrow{\ i\ } \Sigma\mathbb{R}P^\infty = \Sigma MO(1) \longrightarrow MO(2)$$

where $\mathbb{R}P^\infty = MO(1)$. The mapping $\Sigma MO(1) \to MO(2)$ sends σe_3 to $e_1 e_3$. Hence,

$$Y_*(g_4) = e_1 e_3.$$

This completes the proof. □

Now we are ready to prove Theorem 8.1.2.

Proof of Theorem 8.1.2. From Lemma 8.1.3 and Lemma 8.1.4 we get the following map:

$$\widetilde{f_1} : \mathbb{C}P^3_+ \xrightarrow{\ X\ } S^4 \xrightarrow{\ Y\ } MO(2)\ .$$

By the composition we get the required map

$$(\widetilde{f_1})_*(a_4) = Y_* \circ X_*(a_4) = e_1 e_3.$$

□

Proposition 8.1.5. *There exists a map* $f_2 : \mathbb{C}P^3_+ \longrightarrow QMO(2)$*, such that its stable adjoint* $\widetilde{f_2} : \mathbb{C}P^3_+ \to MO(2)$ *satisfies*

$$(\widetilde{f_2})_*(a_4) = e_2^2.$$

Proof. Let η be the universal complex line bundle over the base space $BU(1) = \mathbb{C}P^{\infty}$, where $U(1)$ is the unitary group of degree 1 which corresponds to the circle group, that is, $U(1) = S^1$. The bundle η is a 2-dimensional real vector bundle and so is classified by a map

$$g : BU(1) \longrightarrow BO(2).$$

This gives a diagram.

$$\begin{array}{ccc} E(\eta) & \longrightarrow & EO(2) \\ \downarrow & & \downarrow \\ BU(1) & \xrightarrow{g} & BO(2) \end{array}$$

We have $EO(2)$ which is the total space of the universal 2-plane bundle over the base space $BO(2)$. This map of bundles induces a map

$$MU(1) \xrightarrow{\overline{g}} MO(2)\ ,$$

where $MU(1)$ is the Thom complex of universal complex line bundle over $BU(1)$. Now by the Thom isomorphism 3.3.6 we have the following diagram

$$\begin{array}{ccc} H_2BU(1) & \xrightarrow{g_*} & H_2BO(2) \\ \cong \downarrow T & & \cong \downarrow T \\ H_4MU(1) & \xrightarrow{\overline{g}_*} & H_4MO(2) \end{array}$$

We have a cohomology map

$$g^* : H^*BO(2) \longrightarrow H^*BU(1).$$

Since $H^1BU(1) = 0$ for dimensional reasons, then $g^*(w_1) = 0$. Since $w_2(\eta)$ is the mod 2 restriction of $c_1 \in H^2BU(1)$ by Lemma 1.7.4, then

$$g^*(w_2) = c_1.$$

By duality since $a_2 \in H_2BU(1)$, then $g_*(a_2) = e_1^2 \in H_2BO(2)$. From this, we deduce that if a_4 is the generator of $H_4MU(1)$ then $(\overline{g})_*(a_4) = e_2^2 \in H_4MO(2)$ since $T(e_1^2) = e_2^2$ by Theorem 3.3.6. We obtain $\widetilde{f_2}$ by restricting $\overline{g}$ to $\mathbb{C}P^3_+$.

$$\begin{array}{ccc} \mathbb{C}P^3_+ & \longrightarrow & \mathbb{C}P^{\infty} = MU(1) \\ & \searrow^{\widetilde{f_2}} & \downarrow \overline{g} \\ & & MO(2) \end{array}$$

and then $(\widetilde{f}_2)_*(a_4) = e_2^2 \in H_4MO(2)$ which gives Lemma 8.1.2. □

Theorem 8.1.6. *There is an immersion $F : M^4 \looparrowright \mathbb{C}P^3$, such that the double point manifold of F is cobordant to the projective plane.*

Proof. The idea is to prove that there exists a map $f : \mathbb{C}P^3_+ \to QMO(k)$ such that

$$f_*(a_4) = e_2^2 + e_1e_3 + \gamma e_1^2 \cdot e_1^2,$$

and use Theorem 8.1.1. This corresponds to a stable map $\widetilde{f} : \mathbb{C}P^3 \nrightarrow MO(2)$ such that

$$\widetilde{f}_*(a_4) = e_2^2 + e_1e_3.$$

Now it is straightforward to prove this result by taking the sum of maps constructed in Proposition 8.1.2 and Proposition 8.1.5. Define

$$\widetilde{f} = \widetilde{f}_1 + \widetilde{f}_2 : \mathbb{C}P^3 \nrightarrow MO(2).$$

Then

$$\widetilde{f}_*(a_4) = (\widetilde{f}_1)_*(a_4) + (\widetilde{f}_2)_*(a_4) = e_2^2 + e_1e_3 \in H_4MO(2)$$

This completes the proof. □

8.1.2 The double point manifold of $F : M^8 \looparrowright \mathbb{C}P^7$

Similarly to the previous example, we will observe that it is enough to consider $f_*(a_8)$ rather than $f_*(a_{14})$.

Theorem 8.1.7. *An immersion $F : M^8 \looparrowright \mathbb{C}P^7$ corresponding to a map $f : \mathbb{C}P^7_+ \to QMO(6)$ has double point manifold cobordant to the projective plane if and only if the map f has the property*

$$f_*(a_8) = e_1^4e_2^2 + e_1^5e_3.$$

Proof. Since $f_*(a_2) \in H_2QMO(6) = 0$, then $f_*(a_2) = 0$. Since $f_*(a_4) \in H_4QMO(6) = 0$, then

$$f_*(a_4) = 0.$$

We have $H_6QMO(6) \cong \mathbb{Z}/2$ generator by e_1^6, and so

$$f_*(a_6) = \alpha' e_1^6.$$

The group $H_8QMO(6)$ is spanned by $\{e_1^5e_3,\ e_1^4e_2^2\}$.

	Sq_*^1	Sq_*^2	Sq_*^4	$\widetilde{\psi}$
$e_1^5e_3$	0	0	0	0
$e_1^4e_2^2$	0	e_1^6	0	0

The reduced coproduct of $f_*(a_8)$ is obtained by the following:

$$\widetilde{\psi}(f_*(a_8)) = f_*(a_2) \otimes f_*(a_6) + f_*(a_4) \otimes f_*(a_4) + f_*(a_6) \otimes f_*(a_2) = 0.$$

Then $f_*(a_8)$ is primitive.

Since $Sq_*^1 f_*(a_8) = 0$, $Sq_*^2 f_*(a_8) = f_*(a_6) = \alpha' e_1^6$, and $Sq_*^4 f_*(a_8) = f_*(a_4) = 0$. Then $f_*(a_8)$ is not $\mathcal{A}$-annihilated if $\alpha' = 1$. For $Sq_*^2 f_*(a_8) = \alpha' e_1^6$, then the coefficient of $e_1^4e_2^2 \in f_*(a_8)$ is α'. Hence

$$f_*(a_8) = \alpha' e_1^4e_2^2 \text{ modulo the other basis elements .}$$

From the above table we deduce that the only $\mathcal{A}$-annihilated and primitive elements in $H_8QMO(6)$ is $e_1^5e_3$. Hence, for some $\beta \in \mathbb{Z}/2$

$$f_*(a_8) = \alpha' e_1^4e_2^2 + \beta e_1^5e_3.$$

A basis for $H_{10}QMO(6)$ is given by the set $\{e_1^5e_5,\ e_1^4e_2e_4,\ e_1^4e_3^2,\ e_1^2e_2^4,\ e_1^3e_2^2e_3\}$. Consider the following table.

	Sq_*^1	Sq_*^2	Sq_*^4	$\widetilde{\psi}$
$e_1^5e_5$	0	$e_1^5e_3$	0	0
$e_1^4e_2e_4$	$e_1^5e_4 + e_1^4e_2e_3$	$e_1^5e_3 + e_1^4e_2^2$	0	0
$e_1^4e_3^2$	0	0	0	0
$e_1^2e_2^4$	0	0	e_1^6	0
$e_1^3e_2^2e_3$	0	$e_1^5e_3$	0	0

Since $\widetilde{\psi}(f_*(a_{10})) = 0$, then $f_*(a_{10})$ is primitive.

We have $Sq^1_*(a_{10}) = 0$, $Sq^2_* f_*(a_{10}) = 0$, and $Sq^4_* f_*(a_{10}) = f_*(a_6) = \alpha' e_1^6$. Hence $f_*(a_{10})$ is not $\mathcal{A}$-annihilated if $\alpha' = 1$. Since $Sq^4_* f_*(a_{10}) = f_*(a_6) = \alpha' e_1^6$, then from the above table $f_*(a_{10}) = \alpha' e_1^2 e_2^4$ modulo the other basis elements . Hence

$$f_*(a_{10}) = \alpha' e_1^2 e_2^4 + \gamma e_1^4 e_3^2 + \delta(e_1^5 e_5 + e_1^3 e_2^2 e_3).$$

We will not need to find $f_*(a_{12})$ because $Sq^2_* f_*(a_{14}) = f_*(Sq^2_* a_{14}) = 0$.

Now a basis of $H_{14}QMO(6)$ is given by

$$\{e_1^5 e_9,\ e_1^4 e_3 e_7,\ e_1^4 e_4 e_6,\ e_1^4 e_5^2,\ e_1 e_2^4 e_5,\ e_2^5 e_4,\ e_2^4 e_3^2,\ e_1^3 e_2^2 e_7,\ e_1^3 e_2 e_3 e_6,\ e_1^3 e_2 e_4 e_5,\ \ldots,$$

$$\ldots, e_1^2 e_2^2 e_3 e_5,\ e_1^6 \cdot e_1^5 e_3,\ e_1^6 \cdot e_1^4 e_2^2,\ e_1^5 e_2 \cdot e_1^5 e_2,\ Q^8 e_1^6\},$$

where $\ldots$ is other elements of height one.

We are almost ready to invoke the $\mathcal{A}$-annihilated and primitive elements of $H_{14}QMO(6)$ as shown in the following table:

	Sq^1_*	Sq^2_*	Sq^4_*	$\widetilde{\psi}$
$e_2^4 e_3^2$	0	0	$e_1^4 e_3^2$	A
$e_1^6 \cdot e_1^4 e_2^2$	0	$e_1^6 \cdot e_1^6$	0	A
$e_1^5 e_2 \cdot e_1^5 e_2$	0	$e_1^6 \cdot e_1^6$	0	0
$Q^8 e_1^6$	$Q^7 e_1^6$	$e_1^6 \cdot e_1^6$	0	0
$e_1^6 \cdot e_1^5 e_3$	0	0	0	B
$e_2^5 e_4$	$e_1 e_2^4 e_4 + e_2^5 e_3$	$e_1 e_2^4 e_3 + e_2^6$	$e_1^4 e_2 e_4$	$B + C$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

where,

$$\begin{aligned} A &= e_1^6 \otimes e_1^4 e_2^2 + e_1^4 e_2^2 \otimes e_1^6, \\ B &= e_1^6 \otimes e_1^5 e_3 + e_1^5 e_2 \otimes e_1^6, \\ C &= e_1^5 e_2 \otimes e_1^5 e_2. \end{aligned}$$

To find the non primitive elements of $f_*(a_{14})$ we are going to apply the reduced coproduct as below:

$$\begin{aligned}\widetilde{\psi}(f_*(a_{14})) &= f_*(a_6)\otimes f_*(a_8)+f_*(a_8)\otimes f_*(a_6)\\ &= \alpha' e_1^6\otimes(\alpha' e_1^4e_2^2+\beta e_1^5e_3)+(\alpha' e_1^4e_2^2+\beta e_1^5e_3)\otimes\alpha' e_1^6\\ &= (\alpha')^2(e_1^6\otimes e_1^4e_2^2)+\alpha'\beta(e_1^6\otimes e_1^5e_3)+(\alpha')^2(e_1^4e_2^2\otimes e_1^6)+\alpha'\beta(e_1^5e_3\otimes e_1^6)\\ &= \alpha' A+\alpha'\beta B.\end{aligned}$$

Then $f_*(a_{14})$ is not primitive if $\alpha'=1$. Hence

$$f_*(a_{14})=\alpha' e_1^6\cdot e_1^4e_2^2+\alpha'\beta e_1^6\cdot e_1^5e_3+\text{primitive terms}.$$

Hence, from the table above the primitive elements of $H_{14}QMO(6)$ are spanned by

$$\{e_2^4e_3^2+e_1^6\cdot e_1^4e_2^2,\ e_1^5e_2\cdot e_1^5e_2,\ Q^8e_1^6\}\cup P_{14}$$

where P_{14} are primitive elements of height one. Therefore,

$$f_*(a_{14})=\alpha\ e_1^6\cdot e_1^4e_2^2+\alpha\beta\ e_1^6\cdot e_1^5e_3+b_{14}\ (e_2^4e_3^2+e_1^6\cdot e_1^4e_2^2)+c_{14}\ e_1^5e_2\cdot e_1^5e_2+$$

$$d_{14}\ Q^8e_1^6+\varphi,$$

where $b_{14},\ c_{14},\ d_{14}$ are $\in\mathbb{Z}/2$ and $\varphi\in P_{14}$.

Since $Sq_*^1(f_*(a_{14}))=0,\ Sq_*^2(f_*(a_{14}))=0$ and $Sq_*^4(f_*(a_{14}))=0$. Then $f_*(a_{14})$ is $\mathcal{A}$-annihilated.

For $Sq_*^1(f_*((a_{14}))=0$, we have $d_{14}=0$ which yields

$$f_*(a_{14})=\alpha'\ e_1^6\cdot e_1^4e_2^2+\alpha'\beta\ e_1^6\cdot e_1^5e_3+b_{14}\ (e_2^4e_3^2+e_1^6\cdot e_1^4e_2^2)+c_{14}\ e_1^5e_2\cdot e_1^5e_2+\varphi',$$

where $Sq_*^1\varphi'=0$.

As $Sq_*^2(f_*((a_{14}))=0$, we obtain

$$\alpha' e_1^6\cdot e_1^6+b_{14}\ e_1^6\cdot e_1^6+c_{14}\ e_1^6\cdot e_1^6=0.$$

In this equation we have:

Coefficient of $e_1^6\cdot e_1^6:\qquad \alpha'+b_{14}+c_{14}=0$. Hence,

$$f_*(a_{14})=\alpha' e_1^6\cdot e_1^4e_2^2+\alpha'\beta\ e_1^6\cdot e_1^5e_3+b_{14}\ (e_2^4e_3^2+e_1^6\cdot e_1^4e_2^2)+c_{14}\ e_1^5e_2\cdot e_1^5e_2+\varphi'',$$

where $\varphi^{''}$ is the set of an $\mathcal{A}$-annihilated and primitive elements of height one in $H_{14}QMO(6)$ and $Sq^2_*\varphi^{''} = 0$.

From diagram (6.3.3) we obtain that

$$p_2h(\alpha) = \alpha^{'}e_1^6 \cdot e_1^4e_2^2 + \alpha^{'}\beta\, e_1^6 \cdot e_1^5e_3 + b_{14}\, e_1^6 \cdot e_1^4e_2^2 + c_{14}\, e_1^5e_2 \cdot e_1^5e_2.$$

Then by Theorem 6.2.1

$$\xi_*(p_2h(\alpha)) = (\alpha^{'} + b_{14} + c_{14})\, e_1^{10}e_2^2 + \alpha^{'}\beta e_1^{11}e_3$$

since $\alpha^{'} + b_{14} + c_{14} = 0$. Then

$$\xi_*(p_2h(\alpha)) = \alpha^{'}\beta\, e_1^{11}e_3.$$

Hence the double point manifold of an immersion F is cobordant to the projective plane if and only if for $\alpha^{'} = \beta = 1$. Notice that $\alpha^{'} = \beta = 1$ if and only if

$$f_*(a_8) = e_1^4e_2^2 + e_1^5e_3.$$

Hence, it is enough only to consider $f_*(a_8)$. This completes the proof. □

8.1.3 The double point manifold of $F : M^{12} \looparrowright \mathbb{C}P^{11}$

This is our final example, and hopefully will make the general pattern clear.

Theorem 8.1.8. *An immersion $F : M^{12} \looparrowright \mathbb{C}P^{11}$ corresponding to a map $f : \mathbb{C}P^{11}_+ \to QMO(10)$, has double point manifold cobordant to the projective plane if and only if the map f has the property*

$$f_*(a_{12}) = e_1^8e_2^2 + e_1^9e_3.$$

Proof. Notice that

$$f_*(a_{2i}) = 0 \; for\, i < 5.$$

We have $H_{10}QMO(10) \cong \mathbb{Z}/2$ generator by e_1^{10}, then for some $\alpha^{'} \in \mathbb{Z}/2$,

$$f_*(a_{10}) = \alpha^{'}e_1^{10}.$$

A basis for $H_{12}QMO(10)$ is given by $\{e_1^9e_3,\ e_1^8e_2^2\}$.

Consider the following table

	Sq_*^1	Sq_*^2	Sq_*^4	$\widetilde{\psi}$
$e_1^9e_3$	0	0	0	0
$e_1^8e_2^2$	0	e_1^{10}	0	0

Since the reduced coproduct of $f_*(a_{12})$ is trivial, then $f_*(a_{12})$ is primitive.

We have $Sq_*^1(a_{12}) = 0$, $Sq_*^2f_*(a_{12}) = f_*(a_{10}) = \alpha' e_1^{10}$, $Sq_*^4f_*(a_{12}) = 0$. Then $f_*(a_{12})$ is not $\mathcal{A}$-annihilated if $\alpha' = 1$.

Because $Sq_*^2f_*(a_{12}) = \alpha' e_1^{10}$, then the coefficient of $e_1^8e_2^2$ is α'. So

$$f_*(a_{12}) = \alpha' e_1^8e_2^2 \text{ modulo the other basis elements .}$$

From the above table it is obvious that the only non-zero $\mathcal{A}$-annihilated and primitive element in $H_{12}QMO(10)$ is $e_1^9e_3$. Hence for some $\beta \in \mathbb{Z}/2$,

$$f_*(a_{12}) = \alpha' e_1^8e_2^2 + \beta e_1^9e_3.$$

A basis of $H_{14}QMO(10)$ is given by $\{e_1^9e_5,\ e_1^8e_2e_4,\ e_1^8e_3^2,\ e_1^7e_2^2e_3,\ e_1^6e_2^4\}$. Consider the following table.

	Sq_*^1	Sq_*^2	$\widetilde{\psi}$
$e_1^9e_5$	0	$e_1^9e_3$	0
$e_1^8e_2e_4$	$e_1^9e_4 + e_1^8e_2e_3$	$e_1^9e_3 + e_1^8e_2^2$	0
$e_1^8e_3^2$	0	0	0
$e_1^7e_2^2e_3$	0	$e_1^9e_3$	0
$e_1^6e_2^4$	0	0	0

Since $\widetilde{\psi}(f_*(a_{14})) = 0$, then $f_*(a_{14})$ is primitive. We have $Sq_*^1(a_{14}) = 0$, $Sq_*^2f_*(a_{14}) = 0$, $Sq_*^4f_*(a_{14}) = 0$ and $Sq_*^8f_*(a_{14}) = 0$. Then $f_*(a_{14})$ is $\mathcal{A}$-annihilated . Hence

$$f_*(a_{14}) = \gamma e_1^8e_3^2 + \delta e_1^6e_2^4 + \epsilon(e_1^9e_5 + e_1^7e_2^2e_3),$$

where $\gamma, \delta, \epsilon \in \mathbb{Z}/2$.

We will not need to find $f_*(a_{16}),\ f_*(a_{18})$ and $f_*(a_{20})$ for similar reasons to Theorem 8.1.7. A basis for $H_{22}QMO(10)$ is given by the set

$$\{e_1^9e_{13},\ e_1^8e_7^2,\ e_1e_2^6e_3^3,\ e_2^8e_3^2,\ \ldots,\ e_1^{10}\cdot e_1^9e_3,\ e_1^{10}\cdot e_1^8e_2^2,\ e_1^9e_2\cdot e_1^9e_2,\ Q^{12}e_1^{10}\},$$

where the symbol ... denote to other elements of height one. We consider the following table.

	Sq_*^1	Sq_*^2	Sq_*^4	Sq_*^8	$\widetilde{\psi}$
$e_2^8e_3^2$	0	0	0	$e_1^8e_3^2$	A
$e_1^{10}\cdot e_1^8e_2^2$	0	$e_1^{10}\cdot e_1^{10}$	0	0	A
$e_1^9e_2\cdot e_1^9e_2$	0	$e_1^{10}\cdot e_1^{10}$	0	0	0
$Q^{12}e_1^{10}$	$Q^{11}e_1^{10}$	$e_1^{10}\cdot e_1^{10}$	0	0	0
$e_1^{10}\cdot e_1^9e_3$	0	0	0	0	B
$e_2^9e_4$	$e_1e_2^8e_4+e_2^9e_3$	$e_1e_2^8e_3+e_2^{10}$	0	$e_1^8e_2e_4$	$B+C$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

where,

$$\begin{aligned} A &= e_1^{10}\otimes e_1^8e_2^2+e_1^8e_2^2\otimes e_1^{10}, \\ B &= e_1^{10}\otimes e_1^9e_3+e_1^9e_3\otimes e_1^{10}, \\ C &= e_1^9e_2\otimes e_1^9e_2. \end{aligned}$$

Next we are going to apply the reduced coproduct on basis elements to find out a non primitive elements as below:

$$\begin{aligned} \widetilde{\psi}(f_*(a_{22})) &= f_*(a_{10})\otimes f_*(a_{12})+f_*(a_{12})\otimes f_*(a_{10}) \\ &= \alpha'(e_1^{10}\otimes e_1^8e_2^2+e_1^8e_2^2+\otimes e_1^{10})+\alpha'\beta(e_1^{10}\otimes e_1^9e_3+e_1^9e_3\otimes e_1^{10}) \\ &= \alpha'A+\alpha'\beta B. \end{aligned}$$

Then $f_*(a_{22})$ is not primitive if $\alpha'=1$. Hence

$$f_*(a_{22})=\alpha' e_1^{10}\cdot e_1^8e_2^2+\alpha'\beta e_1^{10}\cdot e_1^9e_3+\text{primitive terms}.$$

From the table above the primitive elements of $H_{22}QMO(10)$ are spanned by

$$\{e_2^8e_3^2 + e_1^{10} \cdot e_1^8e_2^2,\ e_1^9e_2 \cdot e_1^9e_2,\ Q^{12}e_1^{10}\} \cup P_{22}$$

where P_{22} is a basis of primitive elements of height one. Therefore,

$$f_*(a_{22}) = \alpha^{'}\ e_1^{10} \cdot e_1^8e_2^2 + \alpha^{'}\beta\ e_1^{10} \cdot e_1^9e_3 + b_{22}\ (e_2^8e_3^2 + e_1^{10} \cdot e_1^8e_2^2) + c_{22}\ e_1^9e_2 \cdot e_1^9e_2 +$$

$$d_{22}\ Q^{12}e_1^{10} + \varphi,$$

where $b_{22},\ c_{22},\ d_{22} \in \mathbb{Z}/2$ and $\varphi \in P_{22}$.

Since $Sq_*^1(f_*(a_{22})) = 0,\ Sq_*^2(f_*(a_{22})) = 0,\ Sq_*^4(f_*(a_{22})) = 0$ and $Sq_*^8(f_*(a_{22})) = f_*(a_{14})$. Then $f_*(a_{22})$ is not $\mathcal{A}$-annihilated if ($\gamma \neq 0$ or $\delta \neq 0$).

Since $Sq_*^1(f_*(a_{22})) = 0$, then $d_{22}(Q^{11}e_1^{10}) = 0 \Rightarrow d_{22} = 0$ and then

$$f_*(a_{22}) = \alpha^{'}\ e_1^{10} \cdot e_1^8e_2^2 + \alpha^{'}\beta\ e_1^{10} \cdot e_1^9e_3 + b_{22}\ (e_2^8e_3^2 + e_1^{10} \cdot e_1^8e_2^2) + c_{22}\ e_1^9e_2 \cdot e_1^9e_2 + \varphi^{'},$$

where $Sq_*^1\varphi^{'} = 0$. For $Sq_*^2(f_*((a_{22})) = 0$, then

$$\alpha^{'}e_1^{10} \cdot e_1^{10} + b_{22}\ e_1^{10} \cdot e_1^{10} + c_{22}\ e_1^{10} \cdot e_1^{10} = 0$$

Since the coefficient of $e_1^{10} \cdot e_1^{10}$: $\quad \alpha^{'} + b_{22} + c_{22} = 0$. Hence

$$f_*(a_{22}) = \alpha^{'}e_1^{10} \cdot e_1^8e_2^2 + \alpha^{'}\beta\ e_1^{10} \cdot e_1^9e_3 + b_{22}\ (e_2^8e_3^2 + e_1^{10} \cdot e_1^8e_2^2) + c_{22}\ e_1^9e_2 \cdot e_1^9e_2 + \varphi^{''}$$

where $\varphi^{''}$ is the set of an $\mathcal{A}$-annihilated primitive elements of height one in $H_{22}QMO(10)$ with $Sq_*^2\varphi^{''} = 0$. Using diagram (6.3.3)

$$p_2h(\alpha) = \alpha^{'}e_1^{10} \cdot e_1^8e_2^2 + \alpha^{'}\beta e_1^{10} \cdot e_1^9e_3 + b_{22}\ e_1^{10} \cdot e_1^8e_2^2 + c_{22}\ e_1^9e_2 \cdot e_1^9e_2.$$

Then by Theorem 6.2.1 we have

$$\xi_*(p_2h(\alpha)) = \alpha^{'}\beta e_1^{18}e_3 + (\alpha^{'} + b_{22} + c_{22})e_1^{18}e_2^2.$$

Since $\alpha^{'} + b_{22} + c_{22} = 0$. Then

$$\xi_*(p_2h(\alpha)) = \alpha^{'}\beta e_1^{18}e_3.$$

Hence the double point manifold of immersion F is cobordant to projective plane if and only if $\alpha^{'} = \beta = 1$. Similar to the previous example, we observe that it is enough to consider $f_*(a_{12})$. $\square$

In the next section we are going to use the same method in general to give a condition for the existence of an immersion $F : M^{k+2} \looparrowright \mathbb{C}P^{k+1}$ with double point manifold cobordant to the projective plane for general $k \equiv 2 \pmod 4$.

8.1.4 The double point manifold of $F : M^{k+2} \looparrowright \mathbb{C}P^{k+1}$

We now give our main theorem. We observe that in the case of determining the double point manifolds of a given immersion $F : M^{k+2} \looparrowright \mathbb{C}P^{k+1}$ it is possible to reduce the calculation from calculating $f_*(a_{2k+2})$ to calculating $f_*(a_{k+2})$. We have the following theorem.

Theorem 8.1.9. *An immersion $F : M^{k+2} \looparrowright \mathbb{C}P^{k+1}$ corresponding to a map $f : \mathbb{C}P^{k+1}_+ \to QMO(k)$, for $k \equiv 2 \pmod 4$, i.e. $k = 4r + 2$, and $r > 0$ has double point manifold cobordant to the projective plane if and only if*

$$f_*(a_{4r+4}) = e_1^{4r} e_2^2 + e_1^{4r+1} e_3.$$

Proof. Given a function $f : \mathbb{C}P^{k+1}_+ \to QMO(k)$, since $k = 4r + 2$, then

$$f : \mathbb{C}P^{4r+3}_+ \to QMO(4r + 2).$$

Since $H_{2i}QMO(4r + 2) = 0$, for all $i < 2r + 1$. Then

$$f_*(a_{2i}) = 0.$$

The homology group $H_{4r+2}QMO(4r + 2) \cong \mathbb{Z}/2$ generated by e_1^{4r+2}. Then

$$f_*(a_{4r+2}) = \alpha' e_1^{4r+2},$$

for some $\alpha' \in \mathbb{Z}/2$. Moreover, since $H_{4r+4}QMO(4r+2) \cong (\mathbb{Z}/2)^2$ generated by $e_1^{4r+1}e_3$ and $e_1^{4r}e_2^2$ and $Sq^2_*(a_{4r+4}) = a_{4r+2}$. Then

$$f_*(a_{4r+4}) = \alpha' e_1^{4r} e_2^2 + \beta e_1^{4r+1} e_3.$$

A basis for $H_{8r+6}QMO(4r + 2)$ is given by the set

$$\{e_1^{4r+2} \cdot e_1^{4r+1} e_3,\ e_1^{4r+2} \cdot e_1^{4r} e_2^2,\ e_1^{4r+1} e_2 \cdot e_1^{4r+1} e_2,\ Q^{4r+4} e_1^{4r+2}\} \cup \chi.$$

Here χ denoted to elements of height one. The homology class of $f_*(a_{8r+6})$ has reduced coproduct

$$\begin{aligned}\widetilde{\psi}(f_*(a_{8r+6})) &= f_*(a_{4r+2}) \otimes f_*(a_{4r+4}) + f_*(a_{4r+4}) \otimes f_*(a_{4r+2}) \\ &= \alpha^{'}(e_1^{4r+2} \otimes e_1^{4r}e_2^2 + e_1^{4r}e_2^2 \otimes e_1^{4r+2}) + \\ &\quad \alpha^{'}\beta(e_1^{4r+2} \otimes e_1^{4r+1}e_3 + e_1^{4r+1}e_3 \otimes e_1^{4r+1}e_3) \\ &= \alpha^{'}A + \alpha^{'}\beta B.\end{aligned}$$

where

$$\begin{aligned}A &= e_1^{4r+2} \otimes e_1^{4r}e_2^2 + e_1^{4r}e_2^2 \otimes e_1^{4r+2}, \\ B &= e_1^{4r+2} \otimes e_1^{4r+1}e_3 + e_1^{4r+1}e_3 \otimes e_1^{4r+1}e_3.\end{aligned}$$

Then $f_*(a_{8r+6})$ is not primitive when $\alpha^{'} = \beta = 1$. Hence

$$f_*a_{8r+6} = \alpha^{'}e_1^{4r+2} \cdot e_1^{4r}e_2^2 + \alpha^{'}\beta e_1^{4r+2} \cdot e_1^{4r+1}e_3 + \text{primitive terms.}$$

The set of primitive elements of $H_{8r+6}QMO(4r+2)$ is spanned by

$$\{e_2^{4r}e_3^2 + e_1^{4r+2} \cdot e_1^{4r}e_2^2,\ \ e_1^{4r+1}e_2 \cdot e_1^{4r+1}e_2,\ \ Q^{4r+4}e_1^{4r+2}\} \cup P_{8r+6},$$

where P_{8r+6} is the set of primitive elements of height one in $H_{8r+6}QMO(4r+2)$. Hence

$$f_*(a_{8r+6}) = \alpha^{'}e_1^{4r+2} \cdot e_1^{4r}e_2^2 + \alpha^{'}\beta e_1^{4r+2} \cdot e_1^{4r+1}e_3 + b\,(e_2^{4r}e_3^2 + e_1^{4r+2} \cdot e_1^{4r}e_2^2) +$$

$$c\,e_1^{4r+1}e_2 \cdot e_1^{4r+1}e_2 + d\,Q^{4r+4}e_1^{4r+2} + \varphi,$$

where $\varphi \in P_{8r+6}$.

Now by applying Sq_*^1 on $f_*(a_{8r+6})$ we get that $Sq_*^1 f_*(a_{8r+6}) = 0$ then $d\,Q^{4r+4}e_1^{4r+2} = 0$ and so $d = 0$. Hence

$$f_*(a_{8r+6}) = \alpha^{'}e_1^{4r+2} \cdot e_1^{4r}e_2^2 + \alpha^{'}\beta e_1^{4r+2} \cdot e_1^{4r+1}e_3 + b\,(e_2^{4r}e_3^2 + e_1^{4r+2} \cdot e_1^{4r}e_2^2) + c\,e_1^{4r+1}e_2 \cdot e_1^{4r+1}e_2 + \varphi^{'},$$

where $Sq_*^1\varphi^{'} = 0$. For $Sq_*^2 f_*(a_{8r+6}) = 0$. Then

$$\alpha^{'}e_1^{4r+2} \cdot e_1^{4r+2} + b\,(e_1^2e_2^{4r-2}e_3^2 + e_1^{4r+2} \cdot e_1^{4r+2}) + c\,e_1^{4r+2} \cdot e_1^{4r+2} = 0$$

Coefficient of $e_1^{4r+2} \cdot e_1^{4r+2}$: $\alpha' + b + c = 0$. Hence

$$f_*(a_{8r+6}) = \alpha' e_1^{4r+2} \cdot e_1^{4r} e_2^2 + \alpha' \beta e_1^{4r+2} \cdot e_1^{4r+1} e_3 + b\, (e_2^{4r} e_3^2 + e_1^{4r+2} \cdot e_1^{4r} e_2^2) + c\, e_1^{4r+1} e_2 \cdot e_1^{4r+1} e_2 + \varphi'',$$

where φ'' is set of height one in $H_{8r+6}QMO(4r+2)$ with $Sq_*^2 \varphi'' = 0$.

Next by using diagram (6.3.3) we deduce that

$$p_2 h(\alpha) = \alpha' e_1^{4r+2} \cdot e_1^{4r} e_2^2 + \alpha' \beta e_1^{4r+2} \cdot e_1^{4r+1} e_3 + b\, e_1^{4r+2} \cdot e_1^{4r} e_2^2 + c\, e_1^{4r+1} e_2 \cdot e_1^{4r+1} e_2.$$

Therefore, by Theorem 6.2.1

$$\xi_*(p_2 h(\alpha)) = \alpha' \beta e_1^{8r+3} e_3 + (\alpha' + b + c) e_1^{8r+2} e_2^2.$$

Since $\alpha' + b + c = 0$. Then

$$\xi_*(p_2 h(\alpha)) = \alpha' \beta e_1^{8r+3} e_3$$

Then the double point manifold of an immersion F is cobordant to the projective plane if and only if $\alpha' = \beta = 1$. □

In the case $r = 0$ we have been able to construct a map with required property showing that an immersion exists with double point manifold cobordant to the projective plane. For $r > 0$, a construction would be more difficult and this has not been achieved.

8.2 The case $k = 4$

We show that any immersion $F : M^6 \looparrowright \mathbb{C}P^5$ has a double point manifold which is a boundary.

Theorem 8.2.1. *Given any immersion $F : M^6 \looparrowright \mathbb{C}P^5$, then the double point manifold of F is a boundary.*

Proof. According to Section 6.2, a basis of $H_{10}QMO(4)$ is given by the set

$$\{e_1 e_2 e_3 e_4,\ e_1 e_2^2 e_5,\ e_1 e_3^3,\ e_1^2 e_2 e_6,\ e_1^2 e_3 e_5,\ e_1^2 e_4^2,\ e_1^3 e_7,\ e_2^2 e_3^2,\ e_2^3 e_4,\ e_1^4 \cdot e_1^2 e_2^2,\ e_1^4 \cdot e_1^3 e_3,$$

$$e_1^3 e_2 \cdot e_1^3 e_2,\ Q^6 e_1^4\}.$$

Next we need to eliminate the impossible values for $f_*(a_{10})$ by using the action of the Steenrod algebra, and the homology coproduct as we explained in chapter 5.

We start with $f_*(a_2)$. Since $f_*(a_2) \in H_2QMO(4) = 0$, then

$$f_*(a_2) = 0.$$

Since $H_4QMO(4) \cong \mathbb{Z}/2$ generated by e_1^4, then

$$f_*(a_4) = \alpha' e_1^4.$$

A basis for $H_6QMO(4)$ is given by $\{e_1^2e_2^2,\ e_1^3e_3\}$. The homology class $f_*(a_6) \in H_6QMO(4)$ has the reduced cup coproduct

$$\widetilde{\psi} f_*(a_6) = f_*(a_2) \otimes f_*(a_4) + f_*(a_4) \otimes f_*(a_2) = 0.$$

So $f_*(a_6)$ is primitive. Since $Sq_*^1 f_*(a_6) = 0, Sq_*^2 f_*(a_6) = 0$, then $f_*(a_6)$ is $\mathcal{A}$-annihilated. The only $\mathcal{A}$-annihilated and primitive elements of $H_6QMO(4)$ is $e_1^3e_3$ as shown in the following table:

	Sq_*^1	Sq_*^2	$\widetilde{\psi}$
$e_1^2e_2^2$	0	e_1^4	0
$e_1^3e_3$	0	0	0

Hence we deduce that

$$f_*(a_6) = \beta e_1^3 e_3.$$

A basis for $H_8QMO(4)$ is given by $\{e_1e_2^2e_3,\ e_1^2e_2e_4,\ e_1^3e_5,\ e_1^2e_3^2,\ e_2^4,\ e_1^4 \cdot e_1^4\}$. To eliminate the impossible values of $f_*(a_8) \in H_8QMO(4)$, we consider the following table.

	Sq^1_*	Sq^2_*	Sq^4_*	$\widetilde{\psi}$
$e_1e_2^2e_3$	0	$e_1^3e_3$	0	0
$e_1^2e_2e_4$	$e_1^3e_4+e_1^2e_2e_3$	$e_1^3e_3+e_1^2e_2^2$	0	0
$e_1^3e_5$	0	$e_1^3e_3$	0	0
$e_1^2e_3^2$	0	0	0	0
e_2^4	0	0	e_1^4	$e_1^4\otimes e_1^4$
$e_1^4\cdot e_1^4$	0	0	0	0

The reduced coproduct of $f_*(a_8)$ is given by the following:

$$\begin{aligned}\widetilde{\psi}(f_*(a_8)) &= f_*(a_2)\otimes f_*(a_6)+f_*(a_4)\otimes f_*(a_4)+f_*(a_6)\otimes f_*(a_2)\\ &= \alpha^{'}(e_1^4\otimes e_1^4).\end{aligned}$$

Then $f_*(a_8)$ is not primitive if $\alpha^{'}=1$. Hence

$$f_*(a_8)=\alpha^{'}e_2^4+\text{primitive terms.}$$

Since $Sq^1_*(f_*(a_8))=0, Sq^2_*(f_*(a_8))=f_*(a_6)=\beta e_1^3e_3$ and $Sq^4_*(f_*(a_8))=f_*(a_4)=\alpha^{'}e_1^4$, then $f_*(a_8)$ is not $\mathcal{A}$-annihilated if $\alpha^{'}=1$ or $\beta=1$.

Since $Sq^4_*(f_*(a_8))=\alpha^{'}e_1^4$, the coefficient of $e_2^4\in f_*(a_8)$ is $\alpha^{'}$. Hence

$$f_*(a_8)=\alpha^{'}e_2^4 \text{ modulo the other basis elements .}$$

On the other hand $Sq^2_*(f_*(a_8))=f_*(a_6)=\beta e_1^3e_3$, and we have two elements in $H_8QMO(4)$ which are $e_1e_2^2e_3$ and $e_1^3e_5$, with $Sq^2_*e_1e_2^2e_3=Sq^2_*e_1^3e_5=e_1^3e_3$. However, $e_1e_2^2e_3+e_1^3e_5$ is an $\mathcal{A}$-annihilated and primitive element.

Since the coefficient of $e_1e_2^2e_3\in f_*(a_8)$ is β. Then

$$f_*(a_8)=\alpha^{'}e_2^4+\beta e_1e_2^2e_3+\text{modulo annihilated primitive elements.}$$

From the above table we deduce that the $\mathcal{A}$-annihilated and primitive elements in $H_8QMO(4)$ are spanned by the set $\{e_1e_2^2e_3+e_1^3e_5,\ e_1^2e_3^2,\ e_1^4\cdot e_1^4\}$. Hence

$$f_*(a_8)=\alpha^{'}e_2^4+\beta e_1e_2^2e_3+\phi$$

where $\phi = \gamma e_1^2 e_3^2 + \delta e_1^4 \cdot e_1^4 + \epsilon(e_1 e_2^2 e_3 + e_1^3 e_5)$ is the set of an $\mathcal{A}$-annihilated and primitive element in $H_8 QMO(4)$. Hence

$$f_*(a_8) = \alpha' e_2^4 + \beta e_1 e_2^2 e_3 + \gamma e_1^2 e_3^2 + \delta e_1^4 \cdot e_1^4 + \epsilon(e_1 e_2^2 e_3 + e_1^3 e_5).$$

We are almost ready to invoke the $\mathcal{A}$-annihilated and primitive elements for $H_{10} QMO(4)$ as follows

	Sq_*^1	Sq_*^2	Sq_*^4	$\widetilde{\psi}$
$e_1 e_2 e_3 e_4$	$e_1^2 e_3 e_4 + e_1 e_2 e_3^2$	$e_1^2 e_3^2$	0	0
$e_1 e_2^2 e_5$	0	$e_1 e_2^2 e_3 + e_1^3 e_5$	$e_1^3 e_3$	0
$e_1 e_3^3$	0	0	0	0
$e_1^2 e_2 e_6$	$e_1^3 e_6 + e_1^2 e_2 e_5$	$e_1^3 e_5$	$e_1^3 e_3$	0
$e_1^2 e_3 e_5$	0	$e_1^2 e_3^2$	0	0
$e_1^2 e_4^2$	0	$e_1^2 e_3^2$	$e_1^2 e_2^2$	0
$e_1^3 e_7$	0	0	0	0
$e_2^2 e_3^2$	0	$e_1^2 e_3^2$	0	A
$e_2^3 e_4$	$e_1 e_2^2 e_4 + e_2^3 e_3$	$e_1^2 e_2 e_4 + e_2^4 + e_1 e_2^2 e_3$	$e_1^2 e_2^2$	$B + C$
$e_1^4 \cdot e_1^2 e_2^2$	0	$e_1^4 \cdot e_1^4$	0	A
$e_1^4 \cdot e_1^3 e_3$	0	0	0	B
$e_1^3 e_2 \cdot e_1^3 e_2$	0	$e_1^4 \cdot e_1^4$	0	0
$Q^6 e_1^4$	$Q^5 e_1^4$	0	0	0

where

$$\begin{aligned} A &= e_1^4 \otimes e_1^2 e_2^2 + e_1^2 e_2^2 \otimes e_1^4 \\ B &= e_1^4 \otimes e_1^3 e_3 + e_1^3 e_3 \otimes e_1^4 \\ C &= e_1^3 e_2 \otimes e_1^3 e_2. \end{aligned}$$

The reduced coproduct of $f_*(a_{10})$ is obtained as follows

$$\begin{aligned}\widetilde{\psi}(f_*(a_{10})) &= f_*(a_4)\otimes f_*(a_6)+f_*(a_6)\otimes f_*(a_4)\\ &= \alpha^{'}e_1^4\otimes \beta e_1^3e_3+\beta e_1^3e_3\otimes \alpha^{'}e_1^4\\ &= \alpha^{'}\beta(e_1^4\otimes e_1^3e_3+e_1^3e_3\otimes e_1^4)\\ &= \alpha^{'}\beta B.\end{aligned}$$

Then $f_*(a_{10})$ is not primitive if $\alpha^{'}\beta = 1$. Hence

$$f_*(a_{10}) = \alpha^{'}\beta e_1^4\cdot e_1^3e_3 + \text{primitive terms}.$$

Because $Sq_*^1(a_{10}) = 0$, $Sq_*^2 f_*(a_{10}) = 0$, and $Sq_*^4 f_*(a_{10}) = f_*(a_6) = \beta e_1^3e_3$, then $f_*(a_{10})$ is not $\mathcal{A}$-annihilated if $\beta = 1$.

Now if we find all primitive elements in $H_{10}QMO(4)$ which have $Sq_*^1(f_*(a_{10})) = 0$ we deduce that

$$f_*(a_{10}) = \alpha^{'}\beta\ e_1^4\cdot e_1^3e_3 + c_1\ e_1e_2^2e_5 + c_2\ e_1e_3^3 + c_3\ e_1^2e_3e_5 + c_4\ e_1^2e_4^2 + c_5\ e_1^3e_7 +$$

$$c_6(e_1^2e_3e_5 + e_2^2e_3^2 + e_1^4\cdot e_1^2e_2^2 + e_1^3e_2\cdot e_1^3e_2),$$

where $c_i\in\mathbb{Z}/2,\ i=1,\ldots,6$.

By evaluating Sq_*^4 on $f_*(a_{10})$ we get the following.

Since $Sq_*^4(f_*(a_{10})) = f_*(a_6) = \beta\ e_1^3e_3$, then $\beta\ e_1^3e_3 = c_1\ e_1^3e_3 + c_4\ e_1^2e_4^2$ and so

$$c_1 = \beta \ \text{ and }\ c_4 = 0.$$

Hence

$$f_*(a_{10}) = \alpha^{'}\beta e_1^4\cdot e_1^3e_3 + \beta\ e_1e_2^2e_5 + c_2\ e_1e_3^3 + c_3\ e_1^2e_3e_5 + c_5\ e_1^3e_7 +$$

$$c_6\ (e_1^2e_3e_5 + e_2^2e_3^2 + e_1^4\cdot e_1^2e_2^2 + e_1^3e_2\cdot e_1^3e_2).$$

Finally, we note that because $Sq_*^2 f_*(a_{10}) = 0$, then

$$\beta(e_1e_2^2e_3 + e_1^3e_5) + c_3\ e_1^2e_3^2 + c_6\ (e_1^2e_3e_5 + e_2^2e_3^2 + e_1^4\cdot e_1^2e_2^2 + e_1^3e_2\cdot e_1^3e_2) = 0.$$

Therefore, $\beta = c_3 = 0$.

Now after the above calculations we find

$$h(\alpha) = f_*(a_{10}) = c_2\ e_1e_3^3 + c_5\ e_1^3e_7 + c_6(e_1^2e_3e_5 + e_2^2e_3^2 + e_1^4 \cdot e_1^2e_2^2 + e_1^3e_2 \cdot e_1^3e_2).$$

We are now in a position to determine the double point manifold of an immersion F. By using diagram (6.3.3) and referring to Theorem 6.2.1 we find that

$$p_2h(\alpha) = c_6(e_1^4 \cdot e_1^2e_2^2 + e_1^3e_2 \cdot e_1^3e_2) \in H_{10}D_2MO(4).$$

Then

$$\xi_*p_2h(\alpha) = c_6(e_1^6e_2^2 + e_1^6e_2^2) = 0 \in H_{10}MO(8).$$

Hence the double point manifold of the immersion F is a boundary. □

The above theorem then shows that for any immersion $F : M^6 \looparrowright \mathbb{C}P^5$ its double point manifold is a boundary.

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