Evolutionary Finance and Dynamic Games

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Abstract

Evolutionary finance studies financial markets from an evolutionary point of view. A financial market can be interpreted in the context of its evolution: it can be understood as a dynamical system in which frequently interacting investment strategies compete for market capital. We are mainly interested in the long-run performance of investment strategies.

This thesis explores the "Darwinian theory" of portfolio selection. An asset market can be modelled by a game-theoretic evolutionary model in which asset prices are endogenously determined by market clearing condition. A general version of the Kelly rule is shown to allow an investor to "survive" in the asset market. We then investigate the stochastic model with independent and identical distributed states of the world from a different, game-theoretic, angle and examine Nash equilibrium strategies, satisfying equilibrium conditions with probability one. Evolutionary finance and asset market games also provide new angles to present fundamental facts of capital growth theory. Relations between financial growth and the property of "survival" of investment strategies are established in the market selection process.

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning

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Chapter 1 Introduction

1.1 Motivation and background of evolutionary finance

Evolutionary finance studies financial markets from an evolutionary point of view. A financial market, like a living system, can be interpreted in the context of its evolution: market change is the consequence of mutation and selection, which are two important concepts in evolutionary theory (Nowak, 2006). According to the Darwinian theory, selection among species occurs when some species reproduce faster than the other, and mutation emerges when some unusual gene transfer, resulting in different species. The two forces also work in financial markets. A market can be understood as being populated by a group of heterogeneous investors (Evstigneev et al. 2009, p.513). These investors select investment strategies at each trading date. And the investment strategies interact with each other and lead to the wealth dynamics on the investors. On the one hand, the market selection mechanism makes the population of investment strategy simpler since strategies with poor performances will be driven out of the market, but on the other hand, mutation creates new types of investment strategies for fighting against incumbent rules. An analogy between biology and finance has been drawn in the paper of Hens et al. (2005): certain animals are fighting for food or other resources for survival, whilst investors in financial markets are competing for one sort of food which can be viewed as money; species but not individual animals count for evolution, while investment strategies but not individual investors manage wealth dynamics. An asset market therefore can be understood as a dynamical system in which frequently interacting investment strategies compete for market capital.

Evolutionary finance is an interdisciplinary research, involving financial economics, economic theory, mathematical finance, and dynamical systems theory (Evstigneev et al. 2009, p.513). It generally aims at developing the "Darwinian theory" of portfolio selection (Hens et al., 2004). The application of evolutionary idea to economics can be traced back to at least 60 years ago with the publication of Alchian (1950). Alchian writes:

> Realized profits, not maximum profits, are the mark of success and viability. It does not matter through what process of reasoning or motivation such success was achieved. The fact of its accomplishment is sufficient. This is the criterion by which the economic system selects survivors: those who realize positive profits are the survivors; those who suffer losses disappear.

The descriptive approach to financial markets attracted much discussion. In particular, great developments have been made in the 1990s with the publications of Arthur et al.(1997), LeBaron et al. (1999), Farmer and Lo (1999), Blume and Easley (1992), Sandroni (2000). Their research laid the foundations for our line of research. The recent progress in the theory and application of evolutionary finance models has been made with the publications of Evstigneev et al. (2002, 2006, 2008, 2009), Amir et al. (2005, 2008) and Hens et al. (2004, 2005a, 2005b, 2006). Their studies played an inspirational and motivational role in our work on evolutionary finance.

The evolutionary modelling principle does not rely on any notion of utility and its maximization that are very commonly used in traditional economics (Evstigneev et al., 2009, p.510). Instead it mainly focuses on actual wealth dynamics managed by interactive investment strategies and uncertain asset payoffs. Evstigneev et al.(2009, p.511) has commented:"This approach lets actions speak louder than intentions and money speak louder than happiness." Evolutionary finance aims at developing models which are better to describe the dynamic nature of financial markets through the application of Darwinian ideas (Evstigneev et al., 2009, p.510).

The evolutionary approach to study financial markets has quite successfully challenged sophisticated equilibrium concepts and the assumption of a high degree of rationality on investors that play an important role in classic finance and financial economics (Evstigneev et al., 2009, p.511). One of most commonly used equilibrium proposed by Radner (1972) requires market participants have "perfect foresight". In particular, investors have to agree on the price of each asset of the possible future realization of the states of the world. In addition, investors do not always behave as those extreme rationality hypothesis due to some technical limitations in practical markets. Evolutionary finance, in sharp contrast, is concerned only with the observable dynamics of wealth distribution and attempts to make as less restrictions on market behavior as possible[1]. It is thus closer to practical markets than traditional models[2]. The only one equilibrium involved in the dynamical system is the market clearing condition: asset supply equals to asset demand at each trading date. The principle objective of the evolutionary approach consists in developing new models that would constitute a plausible alternative to conventional general equilibrium.

1.2 Evolutionary model

A stochastic dynamic model is employed to describe the evolution of an asset market[3]. This model exhibits the interaction of investment strategies and its effect on changes in the distribution of wealth between investors. The dynamics of the market is modelled in terms of the Marshallian principle of temporary equilibrium[4]. The ideas of Marshall were developed in the framework of mathematical models in economics by Samuelson (1947, p.321-323). He writes about this approach:

> I, myself, find it convenient to visualize equilibrium processes of quite different speed, some very slow compared to others. Within each long run there is a shorter run, and within each shorter run there is a still shorter run, and so forth in an infinite regression. For analytic purposes it is often convenient

[1] Usually investment strategies are defined through myopic mean-variance optimization (Evstigneev et al., 2009, p.511).

[2] Concepts involved in evolutionary finance are observable and can be estimated empirically.

[3] The application of random dynamical systems theory in economics has been believed to be more than a fashionable trend to the description of economic phenomenon (see the survey of Schenk-Hoppé (2001)).

[4] This concept is in much detail analyzed in an economics context by Schlicht (1985).

> to treat slow processes as data and concentrate upon the processes of interest. For example, in a short run study of the level of investment, income, and employment, it is often convenient to assume that the stock of capital is perfectly or sensibly fixed.

Due to the hypothesis of the hierarchy of equilibrium processes in the market, the set of variables in our models can be divided into two groups according to different speeds. The set of investors' portfolios can be temporarily fixed, while the asset prices can be assumed to rapidly reach the unique state of short-run equilibrium.

Evolutionary finance models, generally, can be divided into two classes according to the life span of the assets: short-lived assets and long-lived assets. Short-lived assets refer to those living for one period (i.e., the assets pay random payoffs at the end of the trading date and disappear then, e.g., horse racing bets, one-period options). The model is discussed in detail in Evstigneev et al. (2002) and Amir et al. (2005, 2008). Long-lived assets correspond to the opposite situation in which they live for eternity. These assets pay dividends at each trading date and have their own values so that they can be traded between investors, e.g., stocks (see the discussion about the model in Evstigneev et al. (2008)). Another difference between these two model classes lies in investors' income. For short-lived asset models each investor obtains income from asset payoffs, while in long-lived asset models the resources of investors' budget are not only from asset payoffs but also from capital gains (or loss).

The characteristics of evolutionary finance models, such as heterogeneous investment strategies, dynamic interaction, market selection and stability, are discussed in the survey of Evstigneev et al. (2009). Investment strategies employed to represent investment decisions, play a key role in evolutionary models. In the finance context what matters is not "who does what but how much capital is behind a particular investment style" (Evstigneev et al., 2009, p.513). Heterogeneity of strategies can be viewed as a cornerstone of evolutionary finance, which makes it possible for investors to analyze the performance of different investment types. Investors with the same investment strategies can be regarded as a class of investors[5]. Two forces—selection and mutation therefore work and drive the evolution of the market: investment strategies with better-performance are selected while, at the same time, some new investment types are introduced to the market in competition with old ones for market capital.

The dynamic interaction between heterogeneous investment strategies determines each investor's return (i.e., the performance of an investor is also affected by the market decisions made by the others through asset pricing system). There are thus no optimal investment strategies in evolutionary finance models. One thing that only matters in evolutionary models is questions of survival and extinction in the long run. By this criterion, market selection occurs. Since the selection results can only be observed in the long term, the stability of dynamical systems

[5] For proving theorems in the 2nd and 3rd chapter those investors with the same investment strategies are viewed as an investor.

must be taken into account [6]. The stability lays the foundation for the evolutionary asset-pricing theory[7].

In addition, the performance of each investor in the evolutionary model with long-lived assets is usually related to his/her consumption rate. The consumption rate is the proportion of wealth consumed during each trading date (e.g., in the model with long-lived assets, the consumption rate lies in $(0, 1)$). In the evolutionary model, however, the consumption rate of each investor usually is required to be the same for all the investors because a seemingly worse performance of a portfolio rule in the long run might be simply due to a higher consumption rate of the investor.

1.3 The Kelly rule

The Kelly rule is of importance in studying questions of survival and extinction of portfolio rules. This investment portfolio rule was firstly proposed by Kelly, who drew the model from the real-life situation of gambling for studying the rate of transmission over a communication channel (Kelly, 1956). He discovered that in a pari-mutuel betting market, the gambler who decides to "betting your beliefs" will maximize the exponential rate of growth of his/her capital. His discovery laid the foundation for capital growth theory. And it has been developed and extended by various authors, in particular by Breiman (1961), Algoet and Cover (1988) and

[6] The stability of evolutionary models refers to a steady state that the distribution of wealth is stable even though a mutant is introduced. For the discussion of this question see Hens et al.(2005) and Evstigneev et al. (2008).

[7] Empirical applications in this field have been studied by Hens et al. (2004, 2005, 2006).

Hakansson and Ziemba (1995). Recent studies have shown that the Kelly rule has the remarkable property of survival in evolutionary finance models where survival is equivalent to the fastest growth of wealth (see i.e., Evstigneev et al.(2002, 2008, 2009); Amir et al. (2005,2008)). This section will elaborate the Kelly rule through a horse racing model.

Consider a race with K horses. This horse race is assumed to repeat infinitely and in each of them, only one horse wins. Denote by $p(k) > 0$ the probability of the bet "horse k wins" and let $p = (p(1), p(2), .., .p(K))$. The odds[8] of the bet of "horse k wins" are $1 : W$ ($W > 0$ is a constant) (i.e., the bettor who bets y pounds on the horse will gain Wy when it wins and receive nothing otherwise). Denote by $s_t \in \{1, 2, ..., K\}$ the outcome of the horse race at time t and let $s_t = k$ if horse k wins at time t. The states of the world $s_1, s_2,$ ...are independent and identical distributed with probabilities $P\{s_t = k\} = p(k)$. Define

$$Z_t^k(s_t) = \begin{cases} W & \text{if } s_t = k, \\ 0 & \text{otherwise,} \end{cases}$$

and let $Z_t = (Z_t^1, ..., Z_t^K)$. Given a $K-$dimensional betting strategy $\lambda = (\lambda_1, \lambda_2, ..., \lambda_K)$ ($\lambda_k \geq 0$, and $\sum_k \lambda_k = 1$) and initial wealth $w_0 > 0$, the bettor distributes his/her initial wealth across the K horses in the proportions $\lambda_1, \lambda_2, ..., \lambda_K$ at the beginning and receives payoffs at the end of the race. Suppose the bettor fixes this portfolio rule and always reinvests all his/her payoffs into the next race. Then the wealth of

[8] The odds express the rates obtained when horse k wins.

the bettor after t races is given by

$$w_t = w_0 \langle \lambda, Z_1 \rangle \langle \lambda, Z_2 \rangle \ldots \langle \lambda, Z_t \rangle, \tag{1.1}$$

where the scalar product $\langle \lambda, Z_t \rangle = \sum_k \lambda_k Z_t^k(s_t)$ is $\lambda_k W$ when $s_t = k$. The average logarithmic growth rate over t periods therefore is

$$\frac{1}{t} \ln \left(\frac{w_t}{w_0} \right) = \frac{1}{t} \sum_{d=1}^{t} \ln \langle \lambda, Z_d \rangle \tag{1.2}$$

The strong law of large numbers[9] implies that the $t-$period growth rate (1.2) converges almost surely to

$$\begin{aligned} E \ln \langle \lambda, Z_t \rangle &= \sum_k p(k) \ln \langle \lambda, Z_t \rangle \\ &= \sum_k p(k) \ln \lambda_k W \\ &= \ln W + \sum_k p(k) \ln \lambda_k. \end{aligned} \tag{1.3}$$

as $t \to \infty$.

The maximum of (1.3) is attained at $\lambda^* = (p(1), p(2), ...p(K))$. It follows from

$$\sum_k p(k) \ln p(k) > \sum_k p(k) \ln \lambda_k,$$

where $\lambda = (\lambda_1, \lambda_2, ..., \lambda_K) \neq \lambda^*(\lambda_k > 0$ and $\sum_k \lambda_k = 1)$. And the vector of investment proportions $\lambda^* = (p(1), p(2), ...p(K))$ is called the *Kelly rule*. The Kelly rule λ^* can guarantee the bettor experiences a strictly positive growth rate[10] only if $W \neq K$, because $E \ln \langle \lambda, Z_t \rangle = 0$ when $W = K = 1/\lambda_1 = ... = 1/\lambda_K$.

[9] **(Law of large numbers)** Let $X_1, X_2, ...X_n$ be an independent trials process, with finite expected value $\mu = E(X_i)$ and finite variance $\sigma^2 = Var(X_i)$. Let $S_n = X_1 + X_2 + ... + X_n$. Then for any $\varepsilon > 0$, $P(|S_n/n - \mu| \geq \varepsilon) \to 0$ as $n \to \infty$. Equivalently, $P(|S_n/n - \mu| < \varepsilon) \to 1$ as $n \to \infty$.

[10] (**Theorem**) Let the vector λ^* in the unit simplex maximize the function $U(\lambda) = E \ln \langle \lambda, Z_t(s_t) \rangle$ (s_t are i.i.d. and $U(x)$ does not depend on t). Consider the simple betting strategy $\lambda \neq \lambda^*$and initial wealth $w_0 > 0$. Then we have $w_T^*/w_T \to \infty$ with probability one.

From the horse racing model it can be observed that the survival investment strategy does not depend on the odds. Bettors who bet their wealth across assets according to p will overtake the others who choose different strategies. If the odds equal to the true probabilities of events, it will not produce positive growth in the game. And bettors with the Kelly rule have no growth of wealth and any other bettor' wealth tends to be zero.

Bettors with the Kelly rule survive through the market selection mechanism in the long run, because they have better performance than the others. In a practical market, however, bettors usually do not know the objective probabilities and have to estimate them according to their beliefs. The Kelly rule is thus called as "betting your beliefs". A gambler who has a more accurate estimation of the probability of the event that horse k win will get the faster growth of the wealth than the one with the inferior estimation (Evstigneev et al., 2009, p.518).

Despite the fact that the Kelly rule can do better than any other investment rules and has the remarkable property of survival in asset markets, it still causes some controversy in financial economics. For instance, Samuelson (1979) argued strenuously against it, mainly because he believed one should maximize one's utility function rather than make one's decision based on some other criterion. But he ignored that the approach of investment is not necessarily normative but rather descriptive (Evstigneev, 2009, p.518). Further, if an individual has a logarithmic utility, the Kelly bet will maximize the utility. So there is no conflict between them in this case. In the second chapter, a new form of the Kelly rule is generalized, which is proved to be more applicable in real asset markets.

1.4 Structure of the Thesis

Chapter 2 examines a game-theoretic evolutionary model of an asset market with endogenous equilibrium asset prices. We attempt to identify strategies allowing an investor to survive in the market selection process, i.e., to maintain a positive, bounded away from zero, share of total wealth over the infinite time horizon, irrespective of the portfolio rules used by the other traders. Chapter 3 discusses the evolutionary model from a different, game-theoretic, angle and examine Nash equilibrium strategies, satisfying equilibrium conditions with probability one. We consider a different (stronger) solution concept: almost sure Nash equilibrium. According to our definition of an equilibrium strategy, any unilateral deviation from it leads to a decrease in the random payoff with probability one, and not only to a decrease in the expected payoff. Chapter 4 presents relations between evolutionary finance and capital growth theory. We attempt to present fundamental facts of capital growth theory from a new angle suggested by recent studies on evolutionary finance and asset market games. Chapter 5 summaries this thesis.

Chapter 2 Evolutionary Finance and Dynamic Games

2.1 Introduction

2.1.1 The model and results

This chapter[11] investigates a financial market with long-lived assets and focuses on analyzing its wealth dynamics induced by investment strategies (portfolio rules). We employ a game-theoretic evolutionary model developed by Evstigneev et al. (2006, 2008, 2009) to describe the market. In the evolutionary model the numbers of assets and investors are finite and fixed. The prices of the assets are endogenously determined by a short-run equilibrium of supply and demand. The behavior of the investors is characterized by a strategy profile, leading to the dynamics of the market. Randomness is modelled in terms of a discrete-time stochastic process of "states of the world" with a given probability distribution. Given the realization of this process assets pay dividends at each time. The dividends together with capital gains form investors' budgets, which are partially consumed and partially reinvested. Investors distribute their available budgets across the assets at each trading date according to their investment strategies. The random dynamical system exhibits the process of the evolution of a financial market, in which investors' strategies interact with each other and the interaction results in a

[11] The content of this chapter is based on the paper by R. Amir, I. Evstigneev, T. Hens and L. Xu "Evolutionary finance and dynamic games," Swiss Finance Institute Research Paper No 09-49, January 2010 (the previous version of this Research Paper was entitled "Strategies of survival in dynamic asset market games").

sequence of time-dependent market shares (fractions of total wealth) of each investor.

The main goal of the study is to identify strategies allowing an investor to survive in the market selection process, i.e., to maintain a positive, bounded away from zero, share of total wealth over the infinite time horizon, irrespective of the portfolio rules used by the other traders. A general version of the Kelly rule of "betting your belief" is recommended in this chapter. It turns out that this portfolio rule possesses the remarkable property of unconditional survival. Moreover, the strategy possessing this property is shown essentially unique: any other strategy of this kind (belonging to a certain class) is asymptotically similar to the RES strategy. The result on asymptotic uniqueness may be regarded as an analogue of turnpike theorems[12], stating that all optimal or quasi-optimal paths of economic dynamics converge to each other in the long run.

2.1.2 Evolutionary finance

The approach employed in this study is to apply evolutionary dynamics—mutation and selection—to the analysis of the long-run performance of investment strategies. A stock market can be considered as being populated by a group of heterogeneous investment strategies. These strategies interact with each other and compete for market capital.

The application of evolutionary approach in economics and finance can be

[12] See, e.g., Nikaido (1968) and McKenzie (1986).

traced back at least to 60 years ago. Alchian (1950) argued that realized profits rather than maximum profits are the mark of investment success, which laid the foundation for evolutionary finance. This interdisciplinary research experienced great developments during 1980s and 1990s. Blume and Easley (1992) studied the questions of survival and extinction of portfolio rules in an arrow market, showing that the unique survivor of the market selection process is "betting your beliefs". Arthur et al. (1997) proposed a theory of asset pricing based on heterogeneous agents, presenting a computational platform for analyzing stock markets. Their results have been extended by LeBaron et al. (1999). They mainly focused on time series features of artificial markets. In the review paper of Farmer and Lo (1999), they commented the bright future of the approach to the analysis of financial systems from a biological perspective.

The above studies play an inspirational role in the line of our work. Our approach to evolutionary finance marks a shift from theirs not only in the modelling frameworks and in the specific problems analyzed, but also in the general objectives of work. In particular, we deal with models based on random dynamical systems, rather than on the conventional general equilibrium settings where agents maximize discounted sums of expected utilities. We mainly focus on the wealth dynamics of investors in the market and attempt to find explicit formulas for surviving portfolio rules with the view to making the theory closer to practical applications. In contrast with a number of the above-mentioned papers, we use the rigorous mathematical approach, rather than computer simulations, to justify our conclusions. Considerable efforts are aimed at obtaining results in most general situations, without imposing restrictive assumptions to simplify the analysis. This requires the consideration of models having a rich mathematical structure and exploiting advanced mathematical tools.

In our work, the approach to define the equilibrium concept dispenses with the traditional paradigm of how markets work. One of the most commonly used equilibrium frameworks is that proposed by Radner (1972)—involving agents' plans, prices and price expectations. A well-known drawback of that framework is the necessity of agents' "perfect foresight" to establish an equilibrium. In particular, the market participants have to agree on the future prices for each of the possible future realizations of the states of the world (without knowing which particular state will be realized). The evolutionary approach avoids this assumption and only needs previous observations and the current state of the world to determine investment decisions. Another feature of the approach, in comparison with the conventional frameworks, is the data of the model we assume to be given. We avoid using unobservable agents' characteristics such as individual utilities or subjective beliefs and attempt to constitute a plausible alternative to conventional general equilibrium.

The approach to asset pricing can be viewed as another characteristic of the evolutionary model. The asset prices in the model we deal with are not dependent on commodity money. They are endogenously formed by simultaneous actions of all players through an internal equilibrium in terms of the market clearing price condition. The internal equilibrium can be regarded as a medium of trade through which market capital flows across investors. In this sense, investors may naturally avoid dealing with "end effect" which might be introduced by fiat money[13].

In addition, the evolutionary model is constructed in terms of the Marshallian principle of temporary equilibrium[14]. In the process of the market dynamics there coexists at least two sets of economic variables changing with different speed: the one with slower speed can be temporarily fixed and the other with faster speed can be assumed to rapidly reach the unique state of short-run equilibrium. In the model under consideration the set of investors' portfolios is regarded as changing slower, and the asset prices can be obtained from the market clearing equilibrium at each date.

2.1.3 Evolutionary finance and game theory

Game theory is one of the main general tools in mathematics-based research in economics and finance. It studies behavior in strategic situations, in which an individual's performance depends on not only his/her own decision, but also the choices of others' behavior (see Dutta, 1999, p.4). The model under consideration is a game-theoretic version of the evolutionary model, which analyzes the interaction between investors in a financial market. The study can be linked to the paradigm of market behavior of non-cooperative market games.

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13 See related discussion in the paper of Shubik (1972).

14 The modeling principle is discussed in detail in Evstigneev et al. (2008,2009).

Nash equilibrium is prevalent to study strategic behavior in market games. Our work, however, does not explicitly involve any Nash equilibrium[15] or any specific payoff functions for maximization. What we are concerned with is survival portfolio rules that guarantee almost surely a strictly positive share of market wealth in the long run. All variables involved in the model are observable or can be estimated empirically. This approach therefore is much closer to reality, where typically quantitative information about investors' utilities is lacking.

The solution concept in evolutionary finance also can be linked to various notions of evolutionary stable strategies in evolutionary game theory, including the celebrated concepts in evolutionary game theory by Maynard Smith (1982), asymptotically stable steady states of replicator dynamics processes (Samuelson, 1997), and others. Although these theories concentrate on issues of survival and extinction in selection process, they are typically based on a given static game and random matching in a population of players, in terms of which an evolutionary process leading to survival or extinction of its participants is defined. But our model relies on market primitives, mainly focusing on wealth accumulation of investors in a stochastic dynamic finance model. And the model makes it possible to address directly those questions that are of interest in the context of the modelling of asset market dynamics.

Another model involving concepts of survival and extinction is zero-sum game

[15] For the relationship between evolutionary finance and almost sure Nash equilibrium is discussed in 3rd chapter.

theory. In a survival game, there exits two players starting with a fixed level of wealth $w = w_1 + w_2$($w_1$and w_2 are initial wealth of player 1 and 2, for each). At each time, they play a zero-sum game and part of one's wealth will transit to the other, leading their wealth become $w_1 - b$ and $w_2 + b$ or $w_1 + b$ and $w_2 - b$ respectively. They keep playing this game until one of players loses all of wealth and becomes bankruptcy. The Nash equilibria (or minmax/maxmin strategies) are defined in terms of the probabilities of survival, which can be understood in that context, in contrast with this paper, as avoiding bankruptcy at a random (finite) moment of time.

The structure of the chapter is as follows. Section 2.2 describes the model. Section 2.3 states the main results (Theorems 2.1 and 2.2). Section 2.4 contains the proofs of the results. And the Appendix 2.5 contains technical details of the proofs.

2.2 The model

2.2.1 Asset market

Consider a market with K *assets* and N *investors* ($K \geq 2$ and $N \geq 2$). Market uncertainty is modelled in terms of an exogenous stochastic process $s_1, s_2, ...$, where s_t is a random element of a measurable space S_t. At each date $t = 1, 2, ...$, asset $k = 1, 2, ..., K$ pay dividends $D_{t,k}$. And the dividends $D_{t,k}$ are supposed to be the functions of the history $s^t := (s_1, ..., s_t)$ of states of the world up to date t

$$D_{t,k} = D_{t,k}(s^t) \geq 0 \; (k = 1, ..., K, \; t = 1, 2, ...).$$

The functions $D_{t,k}(s^t) \geq 0$ are measurable and satisfy

$$\sum_{k=1}^{K} D_{t,k}(s^t) > 0 \text{ for all } t, s^t. \tag{2.1}$$

This condition means that at least one asset yields a strictly positive dividend at each date in each random situation. Otherwise, investors will not have motivations to allocate their wealth to the assets in the market. The total amount (the number of units) of asset k available in the market at date t is $V_{t,k}(s^t) > 0$ for all t, s^t, k. For $t = 0$, $V_{t,k}(s^t)$ is a strictly positive constant number, and for $t \geq 1$, $V_{t,k}(s^t)$ is a measurable function of s^t.

The market prices of the assets are denoted by a K dimensional vector

$$p_t = (p_{t,1}, ..., p_{t,K}) \in \mathbb{R}_+^K,$$

where the coordinate $p_{t,k}$ of p_t stands for the price of one unit of asset k at date t. In an asset market, each investor needs to decide what amount of what asset to buy. In other words, investors should select their *portfolios* at each trading date. A portfolio of investor i at date $t = 0, 1, ...$ is characterized by a vector $x_t^i = (x_{t,1}^i, ..., x_{t,K}^i) \in \mathbb{R}_+^K$ where $x_{t,k}^i$ is the amount (the number of physical units) of asset k in the portfolio x_t^i. The coordinates of x_t^i are non-negative, which means short sellings are not allowed. The value of the investor i's portfolio is expressed by the scalar product of asset prices p_t and the investor i's portfolio x_t^i at date t

$$\langle p_t, x_t^i \rangle = \sum_{k=1}^{K} p_{t,k} x_{t,k}^i.$$

The *state of the market* at each date t is characterized by a set of vectors $(p_t, x_t^1, ..., x_t^N)$, where p_t is the price vector and $x_t^1, ..., x_t^N$ are the traders' portfolios.

At time $t = 0$ investor $i = 1, 2, ..., N$ have initial wealth $w_0^i > 0$ that form their budgets at date 0. At time $t \geq 1$, trader i's budget can be characterized by a scalar product $\langle D_t(s^t) + p_t, x_{t-1}^i \rangle$, where $D_t(s^t) := (D_{t,1}(s^t), ..., D_{t,K}(s^t))$ refers to dividends paid by K assets at date t. It consists of two components: the dividends $\langle D_t(s^t), x_{t-1}^i \rangle$ paid by the portfolio x_{t-1}^i and the market value $\langle p_t, x_{t-1}^i \rangle$ of the portfolio x_{t-1}^i expressed in terms of the today's prices p_t. Assume that the budget is partially reinvested and partially consumed. A fraction $\alpha_t := \alpha_t(s^t)$ expresses the *investment rate* and $1 - \alpha_t$ represents the fraction of the budget saved to support investors' life or business at time t. The fraction $1 - \alpha_t$ can be interpreted as the *tax rate* or the *consumption rate*. The investment rate $1 - \alpha_t \in (0, 1)$ is assumed to be the same for all the investors, although it may vary in terms of time and random factors in reality. This assumption is indispensable in this work since we focus on the analysis of the performance of competitive trading strategies in the long run. Without this assumption, an analysis of this kind does not make sense: a seemingly worse performance of a portfolio rule in the long run might be simply due to a higher consumption rate of the investor.

Further, suppose that the function $\alpha_t(s^t)$ is measurable (for $t = 0$ it is constant) and satisfies the following condition:

$$\alpha_t(s^t) < V_{t,k}(s^t)/V_{t-1,k}(s^t). \tag{2.2}$$

This condition holds, in particular, when the total mass $V_{t,k}(s^t)$ of each asset k does not decrease, i.e., when the right-hand side (2.2) is not less than one. But (2.2) does not exclude the situation when $V_{t,k}(s^t)$ decreases at some rate, not faster than α_t.

2.2.2 Investment strategies

In financial markets, investment strategies can be used as guides for investors to make investment decisions. Each trader $i = 1, 2, ..., N$ selects a vector of *investment proportions* $\lambda_t^i = (\lambda_{t,1}^i, ..., \lambda_{t,K}^i)$ at each $t \geq 0$, according to which he/she distributes the available wealth between assets. Vectors λ_t^i belong to the unit simplex

$$\Delta^K := \{(a_1, ..., a_K) \geq 0 : \ a_1 + ... + a_K = 1\}.$$

In terms of the game we deal with, the vectors λ_t^i describe the *investors*'*actions* or *control variables*. Suppose N investors are non-cooperative with each other and select the investment proportions simultaneously and independently at each date $t \geq 0$. Then the model we consider can be viewed as a simultaneous-move N-person dynamic game. For $t \geq 1$, players' actions might depend, generally, on the history $s^t := (s_1, ..., s_t)$ of the process of states of the world and the *history of the game* $(p^{t-1}, x^{t-1}, \lambda^{t-1})$, where $p^{t-1} = (p_0, ..., p_{t-1})$ is the sequence of asset price vectors up to time $t-1$, and

$$x^{t-1} := (x_0, x_1, ..., x_{t-1}), \ x_l = (x_l^1, ..., x_l^N),$$

$$\lambda^{t-1} := (\lambda_0, \lambda_1, ..., \lambda_{t-1}),\ \lambda_l = (\lambda_l^1, ..., \lambda_l^N),$$

are the sets of vectors describing the investors' portfolios and investment proportions at all the dates up to $t-1$. The history of the game assembles information about all the *market history*, including the sequence $(p_0, x_0), ..., (p_{t-1}, x_{t-1})$ of the states of the market and the actions λ_l^i of all the investors $i = 1, ..., N$ at all the dates $l = 0, ..., t-1$. An *investment (trading) strategy* Λ^i of trader i is formed by a vector $\Lambda_0^i \in \Delta^K$ and a sequence of measurable functions with values in Δ^K

$$\Lambda_t^i(s^t, p^{t-1}, x^{t-1},\ \lambda^{t-1}),\ t = 1, 2, ...,$$

specifying a *portfolio rule* according to which trader i selects investment proportions at each date $t \geq 0$. This is a general game-theoretic definition of a strategy, assuming full information about the history of the game which includes the players' previous actions, and the knowledge of all the past and present states of the world.

Among general portfolio rules, we will distinguish those for which Λ_t^i depends only on s^t, and not on the market history $(p^{t-1}, x^{t-1}, \lambda^{t-1})$. This class of portfolio rules plays an important role in the present work: the survival strategy we construct belongs to this class.

2.2.3 Dynamic equilibrium

Suppose at the very beginning $t = 0$ each trader i has selected some investment proportions $\lambda_0^i = (\lambda_{0,1}^i, ..., \lambda_{0,K}^i) \in \Delta^K$. Then each of them has the amount $\alpha_0 \lambda_{0,k}^i w_0^i$ invested in asset k and the total amount invested in asset k is $\alpha_0 \sum_{i=1}^N \lambda_{0,k}^i w_0^i$.

It is assumed that at each trading date t the market reaches to market clearing equilibrium (asset supply is equal to asset demand). Prices obtained from this equilibrium are called market clearing equilibrium prices. And the equilibrium price $p_{0,k}$ of each asset k is determined by the following equation

$$p_{0,k}V_{0,k} = \alpha_0 \sum_{i=1}^{N} \lambda_{0,k}^i w_0^i, \ k = 1, 2, ..., K. \tag{2.3}$$

The left-hand side $p_{0,k}V_{0,k}$ indicates the total value of all the assets of the type k in the market (recall that the amount of each asset k at date 0 is $V_{0,k}$). On the right-hand side of (2.3) $\alpha_0 \sum_{i=1}^{N} \lambda_{0,k}^i w_0^i$ represents the total amount of money invested in asset k by all the investors.

The portfolios $x_0^i = (x_{0,1}^i, ..., x_{0,K}^i)$ of each investor i are determined by the investment proportions $\lambda_0^i = (\lambda_{0,1}^i, ..., \lambda_{0,K}^i)$ at date 0 by the formula

$$x_{0,k}^i = \frac{\alpha_0 \lambda_{0,k}^i w_0^i}{p_{0,k}}, \ k = 1, 2, ..., K, \ i = 1, ..., N. \tag{2.4}$$

This formula states that the current market value $p_{0,k}x_{0,k}^i$ of the kth position of the portfolio x_0^i of investor i is equal to the fraction $\lambda_{0,k}^i$ of the i's investment budget $\alpha_0 w_0^i$. It can be verified that the total demand is equal to the total supply by aggregating (2.4) over N investors

$$\sum_{i=1}^{N} x_{0,k}^i = V_{0,k} = \sum_{i=1}^{N} \frac{\alpha_0 \lambda_{0,k}^i w_0^i}{p_{0,k}}. \tag{2.5}$$

Assume that all the traders have decided their investment proportion vectors $\lambda_t^i = (\lambda_{t,1}^i, ..., \lambda_{t,K}^i)$ at date $t \geq 1$. The market clearing prices p_t are implicitly determined by

$$p_{t,k}V_{t,k} = \alpha_t \sum_{i=1}^{N} \lambda_{t,k}^i \langle D_t(s^t) + p_t, x_{t-1}^i \rangle,\ k = 1, ..., K. \tag{2.6}$$

The above equations implicitly determine the price $p_{t,k}$ of asset k at date t. It can be shown that under assumption (2.2) there always exists a non-negative and unique vector p_t satisfying these equations (for any s^t and any feasible x_{t-1}^i and $\lambda_{t,k}^i$)—see Proposition 2.1 in Section 2.4.

The investors' budgets $\alpha_t \langle D_t(s^t) + p_t, x_{t-1}^i \rangle$ of traders $i = 1, 2, ..., N$ are distributed between assets in the proportions $\lambda_{t,k}^i$, so that the kth position of the trader i's portfolio $x_t^i = (x_{t,1}^i, ..., x_{t,K}^i)$ is

$$x_{t,k}^i = \frac{\alpha_t \lambda_{t,k}^i \langle D_t(s^t) + p_t, x_{t-1}^i \rangle}{p_{t,k}},\ k = 1, ..., K,\ i = 1, ..., N. \tag{2.7}$$

Analogously, by summing up equations (2.7) over investor $i = 1, ..., N$, we also have

$$\sum_{i=1}^{N} x_{t,k}^i = \frac{\sum_{i=1}^{N} \alpha_t \lambda_{t,k}^i \langle D_t(s^t) + p_t, x_{t-1}^i \rangle}{p_{t,k}} = \frac{p_{t,k}V_{t,k}}{p_{t,k}} = V_{t,k}. \tag{2.8}$$

Given a strategy profile $(\Lambda^1, ..., \Lambda^N)$ of investors and their initial endowments $w_0^1, ..., w_0^N$, we can generate a path of the market game by setting

$$\lambda_0^i = \Lambda_0^i,\ i = 1, ..., N, \tag{2.9}$$

$$\lambda_t^i = \Lambda_t^i(s^t, p^{t-1}, x^{t-1},\ \lambda^{t-1}),\ t = 1, 2, ...,\ i = 1, ..., N \tag{2.10}$$

and by defining p_t and x_t^i recursively according to equations (2.3)–(2.7). The random dynamical system described defines step by step the vectors of investment proportions $\lambda_t^i(s^t)$, the equilibrium prices $p_t(s^t)$ and the investors' portfolios $x_t^i(s^t)$ as measurable vector functions of s^t for each moment of time $t \geq 0$ (for $t = 0$ these vectors are constant). Thus we obtain a random path of the game

$$(p_t(s^t); x_t^1(s^t), ..., x_t^N(s^t); \lambda_t^1(s^t), ..., \lambda_t^N(s^t)), \tag{2.11}$$

as a vector stochastic process in $\mathbb{R}_+^K \times \mathbb{R}_+^{KN} \times \mathbb{R}_+^{KN}$.

Note that equations (2.4) and (2.7) make sense only if $p_{t,k} > 0$ for all k, or equivalently, if the aggregate demand for each asset (under the equilibrium prices) is strictly positive. Those strategy profiles which guarantee that the recursive procedure described above leads at each step to strictly positive equilibrium prices will be called *admissible*. In what follows, we will deal only with such strategy profiles. The hypothesis of admissibility guarantees that the random dynamical system under consideration is well-defined. Under this hypothesis, we obtain by induction that on the equilibrium path all the portfolios $x_t^i = (x_{t,1}^i, x_{t,2}^i, ...x_{t,K}^i)$ are non-zero and the wealth

$$w_t^i := \langle D_t + p_t, x_{t-1}^i \rangle \tag{2.12}$$

of each investor is strictly positive. Thus for every equilibrium states of the market $(p_t, x_t^1, ..., x_t^N)$, we have $p_t > 0$ and $x_t^i \neq 0$.

A simple sufficient condition is provided to guarantee a strategy profile to be admissible. This condition will hold for all the strategy profiles under consideration in this chapter, and in this sense it does not restrict generality. Suppose at least one trader, say trader 1, uses a strictly positive portfolio rule (distributes his/her money into all the assets in strictly positive proportions $\lambda^1_{t,k}$). Then a strategy profile containing this portfolio rule is admissible. Indeed, for $t = 0$, we get from (2.3) that $p_{0,k} \geq \alpha_0 V_{0,k}^{-1}\, \lambda^1_{0,k} w^1_0 > 0$ and from (2.4) that $x^1_0 = (x^1_{0,1}, ..., x^1_{t,k}) > 0$. Assuming that $x^1_{t-1} > 0$ and arguing by induction, we obtain $\langle D_t + p_t, x^i_{t-1}\rangle \geq \langle D_t, x^i_{t-1}\rangle > 0$ in view of (2.1), which in turn yields $p_t > 0$ and $x^1_t > 0$ by virtue of (2.6) and (2.7), as long as $\lambda^1_{t,k} > 0$.

2.2.4 Comments on the model

The model we deal with describes an asset market with long-lived dividend paying assets. It employs investment proportions to characterize investors' behavior. In the investment process investors actively select investment proportions at date t in terms of the market information and history prior to trading date t. This approach reflects the principle of active portfolio management (antipodal to a passive, buy-and hold strategy). The "less active" strategy in the framework was discussed by Evstigneev et al. (2008), in which traders use fixed-mix investment strategies—allocating their wealth in constant, time-independent, proportions—rebalancing the portfolios with the view to adjusting the weights of different assets in accordance with changing relative prices.

The evolutionary model under consideration in this chapter allows investors to select investment proportions to distribute their wealth and maintain these proportions over each of the time periods $(t-1,t]$. This can be linked to portfolio rebalancing which is a quite common fund management approach in practical markets. As the asset prices change in terms of market clearing condition at each time t, each investor's portfolio will be rebalanced on a periodic basis. In practical markets, the period of maintaining a type of asset allocation may be a day, a month or when a substantial deviation (exceeding some fixed percentage) from the given proportions occurs owing to changes in asset prices.

The investment proportions selected by an investor specify his/her asset allocations at each trading date. And his/her portfolio is rebalanced during a periodic time. This approach is convenient and efficient for traders to manage their wealth. But it still has difficulty in representing some portfolio rules that are quite naturally defined in terms of "physical units" of assets (e.g., the buy-and-hold strategy) in the framework of investment proportions. It is not a problem when asset prices are known. In particular, the prices in the evolutionary model are endogenous. Further, although the buy-and-hold strategy makes investors have higher returns than the others, it has been shown that in a volatile market, it is quite often inferior to *any* completely diversified constant-proportions strategy involving periodic portfolio rebalancing (Dempster et al., 2008). In our setting, the investors who select the buy-and-hold strategy would be driven out of the market if the numbers of assets are increased (e.g., when $\gamma_t = \gamma > 1$, see (2.15) below), irrespective of the dynamics of their financial values.

The model at hand, in its present form, does not aim at comparing the performance of active and passive investment strategies. Its purpose is different: to reflect—in quantitative terms—the process of active trading characteristic for contemporary financial industry and to develop a framework more suitable in the present context than the conventional general equilibrium theory. Extensions of the model focusing on other theoretical and applied questions will constitute the subject of further research.

2.3 The main results

2.3.1 The notion of survival

Consider an admissible strategy profile of the investors $(\Lambda^1, ..., \Lambda^N)$ and initial endowments $w_0^i, i = 1, 2, ..., N$. Then the path (2.11) of the random dynamical system can be generated through (2.3) to (2.7). Let w_t^i $(t \geq 0)$ be the investor i's wealth available for consumption and investment at date t. If $t = 0$, the initial endowment w_0^i of investor i is a constant number. If $t > 0$, then $w_t^i = w_t^i(s^t)$ is a measurable function of s^t given by formula (2.12). As we have noted above, $w_t^i(s^t) > 0$.

We are primarily interested in the long-run behavior of the *relative wealth* or the *market shares* $r_t^i := w_t^i / W_t$ of the traders, where $W_t := \sum_{i=1}^N w_t^i$ is the *total market wealth*. Recall that we are concerned with the property of survival of investment strategies, rather than comparing the performances between different types of strategies.

Definition 2.1. We shall say that the portfolio rule Λ^1 (or investor 1 using it) *survives* with probability one if $\inf_{t\geq 0} r_t^1 > 0$ (a.s.).

This means that for almost all realizations of the process of states of the world $s_1, s_2, ...$, the market share of the first investor is bounded away from zero by a strictly positive random constant. Alternatively, survival can be defined by the requirement that $\liminf_{t\to\infty} r_t^1 > 0$, which is equivalent, as long as the numbers r_t^1 are strictly positive, to the condition that $\inf_{t\geq 0} r_t^1 > 0$.

Definition 2.2 A portfolio rule Λ^1 is defined as a *survival strategy* if investor 1 using it survives with probability one regardless of what portfolio rules are used by the other investors.

Alternatively, the notion of a survival strategy can be reformulated in terms of the wealth processes $w_t^i (i = 1, 2, ..., N)$. Survival of a portfolio rule Λ^1 used by investor 1 means that $w_t^1 \geq c \sum_{i=1}^N w_t^i$ (a.s.), where c is a strictly positive random constant. Indeed, since we define survival by the condition that $\inf_{t\geq 0} r_t^1 > 0$ (a.s.), we have

$$r_t^1 \geq \inf_{t\geq 0} r_t^1 > 0 \text{ (a.s.)}.$$

Let $\inf_{t\geq 0} r_t^1 = c (c > 0)$, we obtain

$$r_t^1 = \frac{w_t^1}{\sum_{i=1}^N w_t^i} \geq c \text{ (a.s.)},$$

and

$$w_t^1 \geq c \sum_{i=1}^N w_t^i \text{ (a.s.)}. \tag{2.13}$$

The above inequality holds if and only if

$$w_t^i \leq C w_t^1, i = 1, ..., N, \text{ (a.s.)}, \tag{2.14}$$

where C is some strictly positive random constant. By observing (2.13), we find

$$w_t^1 \geq c w_t^i \text{ (a.s.)},$$

because $w_t^i > 0$. Put c to the left-hand side of the above inequality, we get

$$w_t^i \leq \frac{1}{c} w_t^1, i = 1, ..., N, \text{(a.s.)}.$$

Let $1/c = C$, we obtain (2.14).

Further, by summing up (2.14) over $i = 2, ..., N$ and the inequality $w_t^1 \leq w_t^1$, we have

$$\sum_{i=1}^{N} w_t^i \leq [(N-1)C + 1] \, w_t^1, i = 1, ..., N \text{ (a.s.)}.$$

Put $c = 1/\left[(N-1)C + 1\right]$, then we get (2.13).

Property (2.14) indicates that the wealth of any investor i using any strategy Λ^i cannot grow asymptotically faster than the wealth of investor 1 who uses the strategy Λ^1. Thus, the strategy Λ^1 is competitive: it cannot be beaten (in terms of the asymptotic growth rate of wealth) in competition with any set of strategies used by the investor 1's rivals.

2.3.2 The Kelly rule and its generalizations

Assume that the total mass of each asset grows (or decreases) at the same rate $\gamma_t = \gamma_t(s^t) > 0$

$$V_{t,k}/V_{t-1,k} = \gamma_t (t \geq 1). \tag{2.15}$$

Thus

$$V_{t,k} = \gamma_t ... \gamma_1 V_k, \tag{2.16}$$

where $V_k > 0$ $(k = 1, 2, ..., K)$ are the initial amounts of the assets. In the case of real dividened-paying assets—involving long-term investments in the real economy (e.g., real estate, transportation, media, infrastructure, etc.)—the above assumption means that the economic system under consideration is on a *balanced growth path*.

Define the *relative dividends* of the assets $k = 1, ..., K$ by

$$R_{t,k} = R_{t,k}(s^t) := \frac{D_{t,k}(s^t)V_{t-1,k}(s^{t-1})}{\sum_{m=1}^{K} D_{t,m}(s^t)V_{t-1,m}(s^{t-1})},\ k = 1, ..., K,\ t \geq 1, \tag{2.17}$$

and put $R_t(s^t) = (R_{t,1}(s^t), ..., R_{t,K}(s^t))$. By virtue of (2.16), we have

$$R_{t,k}(s^t) := \frac{D_{t,k}(s^t)V_k}{\sum_{m=1}^{K} D_{t,m}(s^t)V_m}. \tag{2.18}$$

Further, define

$$\rho_t := \alpha_t / \gamma_t,$$

$$\rho_t^l := \begin{cases} 1 - \rho_{t+l}, & \text{if } l = 1, \\ \rho_{t+1}\rho_{t+2...}\rho_{t+l-1}(1 - \rho_{t+l}) & \text{if } l > 1, \end{cases} \tag{2.19}$$

and assume that

$$\rho_t < 1 - \kappa, \tag{2.20}$$

where κ is a strictly positive constant. Consider the portfolio rule Λ^* with the vectors of investment proportions $\lambda_t^*(s^t) = (\lambda_{t,1}^*(s^t), ..., \lambda_{t,K}^*(s^t))$ given by

$$\lambda_{t,k}^* = E_t \sum_{l=1}^{\infty} \rho_t^l R_{t+l,k}, \tag{2.21}$$

where $E_t(\cdot) = E_t(\cdot|s^t)$ is the conditional expectation given s^t. If $t = 0$, then $E_t(\cdot) = E_0(\cdot)$ stands for the unconditional expectation $E(\cdot)$. In view of (2.20), the series of random variables

$$\sum_{l=1}^{\infty} \rho_t^l = (1 - \rho_{t+1}) + \rho_{t+1}(1 - \rho_{t+2}) + \rho_{t+1}\rho_{t+2}(1 - \rho_{t+3}) + ...$$

converges uniformly[16], and its sum is equal to one. Therefore the series of random vectors $\sum_{l=1}^{\infty} \rho_t^l R_{t+l,k}$ in (2.21) converges uniformly[17] to a random vector belonging the unit simplex [18]$\triangle^K$, and so $\lambda_{t,k}^*$ is well-defined.

The expected flow of discounted future relative dividends is used to specify the portfolio rule in (2.21). According to this portfolio rule, investors will distribute wealth across assets. The discount factors ρ_t^l are defined in terms of the investment rate α_t and the growth rate γ_t in terms of formula (2.19). It should be emphasized that the investment proportions $\lambda_{t,k}^*(s^t)$ prescribed by the portfolio rule Λ^* generally depend on time t and the sequence of exogenous states of the world $s^t = (s_1, ...s_t)$, but do not depend on the history of the game $(p^{t-1}, x^{t-1}, \lambda^{t-1})$, so

16 (**Uniformly Convergency**) Suppose S is a set and $f_n\colon S \to R$ are real-valued functions for every natural number n. We say that the sequence (f_n) is uniformly convergent with limit $f\colon S \to R$ if for every $\varepsilon > 0$, there exists a natural number N such that for all x in S and all $n \geq N$, $|f_n(x) - f(x)| < \varepsilon$.

17 Weierstrass M-test is used to prove this argument. In mathematics, the Weierstrass M-test is an analogue of the comparison test for infinite series, and applies to a series whose terms are themselves functions with real or complex values.
(**Weierstrass M-test**) Suppose (f_n) is a sequence of real- or complex-valued functions defined on a set A, and that there exist positive constants M_n such that $|f_n(x)| \leq M_n$ for all $n \geq 1$ and all x in A. Suppose further that the series $\sum_{n=1}^{\infty} M_n$ converges. Then, the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on A.

18 By summing up $\sum_{l=1}^{\infty} \rho_t^l R_{t+l,k}$ over $k = 1, ..., K$, we have

$$\sum_{k=1}^{K}\sum_{l=1}^{\infty} \rho_t^l R_{t+l,k} = \sum_{l=1}^{\infty} \rho_t^l \sum_{k=1}^{K} R_{t+l,k} = \sum_{l=1}^{\infty} \rho_t^l = 1.$$

that the strategy Λ^* is basic.

The strategy Λ^* is a generalization of the Kelly portfolio rule of "betting your beliefs", playing an important role in capital growth theory—see Kelly (1956), Breiman (1961), Algoet and Cover (1988), and Hakansson and Ziemba (1995). Since the conditional expectations in (2.21) are taken with respect to the probability measure on the space of paths of the process of states of the world known to all the market participants (*rational expectations hypothesis*), it is natural to call the generalized the Kelly portfolio rule Λ^* *rational expectations strategy (RES).*

If $\rho = \rho_t$ is constant, then formula (2.21) can be written as

$$\lambda^*_{t,k} = E_t \sum_{l=1}^{\infty} [(1-\rho)\rho^{l-1} R_{t+l,k}].$$

Further, if the random elements s_t are independent and identically distributed and the relative dividends $R_{t,k}(s^t) = R_k(s_t)$ depend only on the current state s_t and do not explicitly depend on t, then $E_t R_k(s_{t+l}) = E_t R_k(s_t)(l \geq 1)$, and so

$$\lambda^*_{t,k} = ER_k(s_t), \tag{2.22}$$

which means that the strategy Λ^* is formed by the sequence of constant vectors $(ER_1(s_t), ..., ER_K(s_t))$ (independent of t and s^t). Note that it does not include the factor ρ. In the general case, however, even when ρ is constant, one has to take into account the expected discounted sum of all the relative dividends at all the future dates after t.

2.3.3 The Kelly rule is a survival strategy

Assume that for all k and t we have

$$E_t R_{t+1,k} > 0 \text{ (a.s.).} \tag{2.23}$$

This assumption implies that the conditional expectation in (2.21), which is not less than $E_t(\kappa R_{t+1,k})$ is strictly positive a.s.. Indeed, according to the inequality (2.20), we have

$$\lambda^*_{t,k} = E_t \sum_{l=1}^{\infty} \rho_t^l R_{t+l,k} \geq E_t \rho_t^1 R_{t+l,k} > E_t(\kappa R_{t+1,k}) > 0 \text{ (a.s.).}$$

So we can select a version of this conditional expectation that is strictly positive for all s^t. This version will be used in the definition of the strategy Λ^*.

A central result is as follows.

Theorem 2.1 *The portfolio rule* Λ^* *is a survival strategy.*

An analogous result has been established in the framework of a model with one-period, "short-lived" assets[19] in Amir et al. (2008). That framework may be regarded as a limiting case as $\rho \to 0$ (with constant $\rho_t = \rho$) of the one considered in the present chapter. It has been proved that investors who use the Kelly rule ($\lambda^*_{t,k} = E_t R_{t+1,k}$) survive in the case of short lived assets. These proportions are limits of those in (2.21) as $\rho \to 0$. The analysis of the model with long-lived assets in which general strategies are allowed and no assumptions on the process of states of the world are imposed is much more demanding. It requires a substan-

[19] Models of this kind were considered by Blume and Easley (1992), Amir et al. (2005), and others; see surveys in Blume and Easley (2008) and Evstigneev et al. (2009).

tial generalization of the concept of the Kelly portfolio rule, taking into account the discounting of the future dividends, and it is based on new techniques (relying upon stochastic Lyapunov functions) designed for the analysis of random dynamical systems arising in connection with the dynamic market games at hand.

As we have noted above, if the states of the world s_t are i.i.d. and the functions $R_{t,k}(s) = R_k(s)$ do not depend on t, then the investment proportions $\lambda_k^* = ER_k(s_t)$ of the strategy Λ^* are constant: they depend neither on time nor on the states of the world (such strategies are called *simple*). A version of the asset market model with long-lived assets in which all the investors use only simple portfolio rules and the states of the world are i.i.d. is considered in Evstigneev et al. (2008). It is shown in that context that the strategy Λ^* not only survives, but also outperforms all other simple strategies. Those investors who use Λ^* dominate the market, i.e., gather in the limit total market wealth, while those who use simple strategies distinct from Λ^* vanish: their market shares tend to zero with probability one. This is not so in the model considered in the present paper, where general, not necessarily simple, portfolio rules are allowed. Here, Λ^*-investors survive, i.e., keep market shares bounded away from zero a.s., but they do not necessarily dominate the market.

2.3.4 Asymptotic uniqueness of the survival strategy.

Theorem 2.1 shows that the strategy Λ^* is a survival strategy in the model under consideration. The strategy Λ^* belongs to the class of basic portfolio rules: the investment proportions $\lambda_t^*(s^t)$ depend only on the history s^t of the process of states of the world, and do not depend on the market history. The following theorem shows that in this class the survival strategy $\Lambda^* = (\lambda_t^*)$ is essentially unique: any other basic survival strategy is asymptotically similar to Λ^*.

Theorem 2.2 *If $\Lambda = (\lambda_t)$ is a basic survival strategy, then*

$$\sum_{t=0}^{\infty} ||\lambda_t^* - \lambda_t||^2 < \infty \text{ (a.s.).}$$

Here, we denote by $||\cdot||$ the Euclidean norm[20] in a finite-dimensional space. Theorem 2.2 is akin to various *turnpike* results in the theory of economic dynamics, expressing the idea that all optimal or asymptotically optimal paths of an economic system follow in the long run essentially the same route: the turnpike (See, e.g., Arkin and Evstigneev (1987), p.12-27). Survival strategies Λ can be characterized by the property that the wealth w_t^j of any investor j cannot grow infinitely faster (with strictly positive probability) than the wealth of investor i using Λ. The class of such investment strategies is similar to the class of "good" paths of economic dynamics, as introduced by Gale (1967)—paths that cannot be "infinitely worse" than the turnpike. Theorem 2.2 is a direct analogue of Gale's turnpike theorem for good paths (Gale, 1967, Theorem 8); for a stochastic version of this result see Arkin and Evstigneev (1987, Chapter 4, Theorem 6[21]).

20 (**Euclidean norm**) On R^n, the intuitive notion of Euclidean norm of the vector $x = (x_1, x_2, ..., x_n)$ can be defined by the formula

$$||x|| := \sqrt{x_1^2 + x_2^2 + ... + x_n^2}.$$

21 (**Convergence of good infinite programmes to the turnpike**) If $\{z_t\}$ is a good *programme*, then

Note that the class of basic strategies is *sufficient* in the following sense. Any sequence of vectors $r_t = (r_t^1, ..., r_t^N)$ $(r_t = r_t(s^t))$ of market shares generated by some strategy profile $(\Lambda^1, ..., \Lambda^N)$ can be generated by a strategy profile $(\lambda_t^1(s^t), ..., \lambda_t^N(s^t))$ consisting of basic portfolio rules. The corresponding vector functions $\lambda_t^i(s^t)$ can be defined recursively by (2.9) and (2.10), using (2.3)-(2.7). Thus it is sufficient to prove Theorem 2.1 only for basic portfolio rules; this will imply that the portfolio rule (2.21) survives in competition with any, not necessarily basic strategies. Such considerations cannot be automatically applied to the problem of asymptotic characterization of general survival strategies. This problem remains open; it indicates an interesting direction for further research.

2.4 Proofs

In this section the program of proving Theorems 2.1 and 2.2 is established step by step. We begin with some auxiliary propositions whose proofs are routine and relegated to the Appendix 2.5. Based on these auxiliary results, we present at the end of the section the final steps of the proofs of Theorems 2.1 and 2.2. The first proposition establishes the existence and uniqueness of an equilibrium price vector at each date $t \geq 0$.

Proposition 2.1 *Let assumption (2.2) hold. Let $x_{t-1} = (x_{t-1}^1, ..., x_{t-1}^N)$ be a set of vectors $x_{t-1}^i \in \mathbb{R}_+^K$ satisfying (2.8). Then for any s_t there exists a unique*

$$\lim_{t \to \infty} E|\overline{z}_t - z_t| = 0.$$

And a sequence of vectors $\{z_1, z_2, ...\}$, $z_t := (x_{t-1}, y_t)$—finite or infinite—is called a programme if $z_t \in Q_t$, $y_t \geq x_t$ $(t \geq 1)$.

solution $p_t \in \mathbb{R}^K_+$ to equations (2.6). This solution is measurable with respect to all the parameters involved in (*2.6*).

This proposition guarantees our dynamic model well-defined. It indicates the equilibrium price vector always exists at each trading date t no matter what the state of world is. In the next proposition, we derive a system of equations governing the dynamics of the market shares of the investors given their admissible strategy profile $(\Lambda^1, ..., \Lambda^N)$. Consider the path (2.11) of the random dynamical system generated by $(\Lambda^1, ..., \Lambda^N)$ and the sequence of vectors $r_t = (r_t^1, ...r_t^N)$, where r_t^i is the investor i's market share at date t.

Proposition 2.2 *The following equations hold:*

$$r_{t+1}^i = \sum_{k=1}^{K} [\rho_{t+1} \langle \lambda_{t+1,k}, r_{t+1} \rangle + (1 - \rho_{t+1}) R_{t+1,k}] \frac{\lambda_{t,k}^i r_t^i}{\langle \lambda_{t,k}, r_t \rangle}, i = 1, ..., N, \ t \geq 0. \tag{2.24}$$

The above proposition avoids the complex process of the evolution of an asset market and obtains market shares of investors recursively. The next proposition shows that it is sufficient to prove Theorem 2.1 when $N = 2$, i.e., the general model can be reduced to the case of two investors. Define

$$\tilde{\lambda}_{t,k}^2 = \frac{\lambda_{t,k}^2 r_t^2 + ... + \lambda_{t,k}^N r_t^N}{1 - r_t^1}. \tag{2.25}$$

Note that $1 - r_t^1 = r_t^2 + ... + r_t^N > 0$ and so $\tilde{\lambda}_{t,k}^2$ is well-defined. Furthermore,

$$\sum_{k=1}^{K} \tilde{\lambda}_{t,k}^2 = \frac{r_t^2 + ... + r_t^N}{1 - r_t^1} = 1,$$

which means that the vector $\tilde{\lambda}_t^2 := (\tilde{\lambda}_{t,1}^2, ..., \tilde{\lambda}_{t,K}^2)$ belongs to the unit simplex Δ^K.

Thus the sequence of vectors $\tilde{\lambda}_t^2 = \tilde{\lambda}_t^2(s^t)$ defines a portfolio rule, which will be denoted by $\tilde{\Lambda}$. Define

$$\tilde{r}_t^1 = r_t^1,\ \tilde{r}_t^2 = 1 - r_t^1,\ \tilde{r}_t = (\tilde{r}_t^1, \tilde{r}_t^2),\ \tilde{\lambda}_{t,k}^1 = \lambda_{t,k}^1,\ \tilde{\lambda}_{t,k} = (\tilde{\lambda}_{t,k}^1, \tilde{\lambda}_{t,k}^2).$$

Proposition 2.3 *We have*

$$\tilde{r}_{t+1}^i = \sum_{k=1}^{K} [\rho_{t+1} \langle \tilde{\lambda}_{t+1,k}, \tilde{r}_{t+1} \rangle + (1 - \rho_{t+1}) R_{t+1,k}] \frac{\tilde{\lambda}_{t,k}^i \tilde{r}_t^i}{\langle \tilde{\lambda}_{t,k}, \tilde{r}_t \rangle},\ i = 1, 2,\ t \geq 0.$$

Thus in the model with two investors $i = 1, 2$ using the strategies Λ and $\tilde{\Lambda}$, respectively, the market share $\tilde{r}_t^1$ of the first investor coincides with r_t^1 (coming from the original model) and the market share $\tilde{r}_t^2$ of the second is equal to $1 - r_t^1$.

Consider the model with two traders ($N = 2$) using strategies $\Lambda^i = (\lambda_{t,k}^i(s^t)),\ i = 1, 2$, and denote by z_t the ratio r_t^1 / r_t^2 of their market shares.

Proposition 2.4 *The process z_t is governed by the following random dynamical system:*

$$z_{t+1} = z_t \frac{\sum_{k=1}^{K} [\rho_{t+1} \lambda_{t+1,k}^2 + (1 - \rho_{t+1}) R_{t+1,k}] \dfrac{\lambda_{t,k}^1}{\lambda_{t,k}^1 z_t + \lambda_{t,k}^2}}{\sum_{k=1}^{K} [\rho_{t+1} \lambda_{t+1,k}^1 + (1 - \rho_{t+1}) R_{t+1,k}] \dfrac{\lambda_{t,k}^2}{\lambda_{t,k}^1 z_t + \lambda_{t,k}^2}}. \tag{2.26}$$

In the next proposition, we derive an equation which can be used as an equivalent definition of the portfolio rule Λ^*.

Proposition 2.5 *The portfolio rule $\Lambda^* = (\lambda_{t,k}^*)$ satisfies*

$$E_t[\rho_{t+1} \lambda_{t+1,k}^* + (1 - \rho_{t+1}) R_{t+1,k}] = \lambda_{t,k}^* \text{ (a.s.)}. \tag{2.27}$$

It can be shown (by using a contraction principle) that Λ^* is a unique solution to (2.27), but this fact will not be needed in what follows.

For proving Theorem 2.1, we firstly simplify the random dynamic model with N players to that with two players. Proposition 2.3 shows that without loss of generality, it is sufficient to consider two dimensional random dynamical system. Secondly, we attempt to use one variable $z_t = r_t^1/r_t^2$ to describe the evolution of the market shares of the two investors. Proposition 2.4 provides one dimensional system to describe the wealth dynamics of the two investors.

Assume that investor 1 plays the investment proportions $\lambda_{t,k}^1 = \lambda_{t,k}^*(s^t)$ prescribed by the portfolio rule Λ^* and investor 2 uses investment proportions $\lambda_{t,k}^2 = \lambda_{t,k}(s^t)$ specified by some other portfolio rule Λ. The ratio z_t of the market shares of the two investors can be obtained from (2.26). Our goal is to show that the random sequence (z_t) defined recursively by (2.26) is bounded away from zero a.s.. To this end we introduce the following change of variables

$$y_t^k = \lambda_{t,k}/z_t, \; k = 1, ..., K, \tag{2.28}$$

and define $y_t := (y_t^1, ..., y_t^K)$. We examine the dynamics of the random vectors $y_t = y_t(s^t)$ implied by the system (2.26). The norm $|y_t| := \sum_k |y_t^k|$ of the vector $y_t \geq 0$ is equal to $\sum_k |\lambda_{t,k}/z_t| = 1/z_t$, and what we need is to show that $1/|y_t|$ is bounded away from zero a.s.. To prove this, we construct a stochastic Lyapunov function—a function of y_t which forms a non-negative supermartingale (ζ_t) along a path (y_t) of the system at hand (see Lemma 2.3 below). By applying the supermartingale convergence theorem, we prove that the stochastic process ζ_t converges a.s., which implies that it is bounded a.s.. The proof of Theorem 2.1 is completed by showing that the boundness of ζ_t implies that $z_t = 1/|y_t|$ is bounded away from zero.

Three lemmas are introduced to realize the plan of the proof of Theorem 2.1. These three lemmas contain inequalities involving the variables y_t^k defined by (2.28). Define the non-negative random variables

$$Y_t := \ln(1 + |y_t|) = -\ln r_t^1, \tag{2.29}$$

$$Z_{t,k} := \ln\left(1 + \frac{y_t^k}{\lambda_{t,k}^*}\right) = \ln\left(1 + \frac{r_t^2 \lambda_{t,k}}{r_t^1 \lambda_{t,k}^*}\right), Z_t := \sum_{k=1}^{K} \lambda_{t,k}^* Z_{t,k}, \tag{2.30}$$

and put

$$U_t := Y_t - Z_t. \tag{2.31}$$

Lemma 2.1 *The following inequality holds:*

$$\rho_{t+1} Z_{t+1} + (1 - \rho_{t+1}) Y_{t+1} \le \sum_{k=1}^{K} [\rho_{t+1} \lambda_{t+1,k}^* + (1 - \rho_{t+1}) R_{t+1,k}] Z_{t,k}. \tag{2.32}$$

Lemma 2.2 *We have*

$$U_t = \sum_{k=1}^{K} \lambda_{t,k}^* \ln \frac{\lambda_{t,k}^*}{r_t^1 \lambda_{t,k}^* + r_t^2 \lambda_{t,k}} \ge 0. \tag{2.33}$$

From the above results, we derive the following fact.

Lemma 2.3 *The random sequence*

$$\zeta_t := \rho_t Z_t + (1 - \rho_t) Y_t \tag{2.34}$$

is a non-negative supermartingale satisfying

$$\zeta_t - E_t \zeta_{t+1} \ge (1 - \rho_t) U_t. \tag{2.35}$$

The following two lemmas will be used in proof of Theorem 2.2.

Lemma 2.4 *Let* ζ_t *be a supermartingale such that* $\inf_t E\zeta_t > -\infty$. *Then the series of non-negative random variables* $\sum_{t=0}^{\infty}(\zeta_t - E_t\zeta_{t+1})$ *converges a.s..*

Lemma 2.5 *For any vectors* $(a_1, ..., a_K) > 0$ *and* $(b_1, ..., b_K) > 0$ *satisfying* $\sum a_k = \sum b_k = 1$, *the following inequality holds*

$$\sum_{k=1}^{K} a_k \ln a_k - \sum_{k=1}^{K} a_k \ln b_k \geq \frac{1}{4}\sum_{k=1}^{K}(a_k - b_k)^2. \tag{2.36}$$

Proof of theorem 2.1. Since ζ_t is a non-negative supermartingale[22], the sequence ζ_t converges[23] a.s., and hence it is bounded above a.s. by some random constant C. This implies (see (2.34) and (2.20)) that $(1-\rho_t)Y_t \leq \zeta_t \leq C$ a.s., and so

$$-\ln r_t^1 = Y_t \leq \zeta_t/(1-\rho_t) \leq B \text{ (a.s.)},$$

where $B := C/\kappa$. Therefore $r_t^1 \geq e^{-B}$a.s.. □

Proof of theorem 2.2. Let $\Lambda = (\lambda_t)$ be a basic survival strategy. Suppose that investors $i = 1, 2, ..., N-1$ use the strategy $\Lambda^* = (\lambda_t^*)$ and investor N uses Λ. By summing up equations (2.24) with $\lambda_t^i = \lambda_t^*$ over $i = 1, ..., N-1$, we obtain

$$\begin{aligned}&\widehat{r}_{t+1}^1\\ = \ &\sum_{k=1}^{K}\left\{\rho_{t+1}\left[\lambda_{t+1,k}^*\widehat{r}_{t+1}^1 + \lambda_{t+1,k}\left(1-\widehat{r}_{t+1}^1\right)\right] + (1-\rho_{t+1})R_{t+1,k}\right\}\frac{\lambda_{t,k}^*\widehat{r}_t^1}{\lambda_{t,k}^*\widehat{r}_t^1 + \lambda_{t,k}\left(1-\widehat{r}_t^1\right)},\end{aligned}$$

[22] (**Supermartingale**) Let $(\Omega, \mathcal{F}, P)$ be a given probability space and $(\mathcal{F}_n)$ be a family of $\sigma-$algebras $\mathcal{F}_n$, $n \geq 0$, such that $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq ... \subseteq \mathcal{F}_n$. A stochastic sequence $X = (X_n, \mathcal{F}_n)$ is a supermartingale if for all $n \geq 0$,

$$E|X_n| < \infty,$$

and

$$E(X_{n+1}|\mathcal{F}_n) \leq X_n.$$

[23] (**Supermartingal convergence corollary**) If X_n is a non-negative supermartingale, then with probability 1 the limit $\lim X_n$ exists and is finite.

where $\widehat{r}_t^1 := r_t^1 + ... + r_t^{N-1}$ is the market share of the group of investors $i = 1, 2, ..., N-1$ and $1 - \widehat{r}_t^1 = r_t^N$ is the market share of investor N. We used here the fact that

$$\langle \lambda_{t,k}, r_t \rangle = \sum_{i=1}^{N} \lambda_{t,k}^i r_t^i = \sum_{i=1}^{N-1} \lambda_{t,k}^* r_t^i + \lambda_{t,k} r_t^N =$$
$$\lambda_{t,k}^* \sum_{i=1}^{N-1} r_t^i + \lambda_{t,k} r_t^N = \lambda_{t,k}^* \widehat{r}_t^1 + \lambda_{t,k} \left(1 - \widehat{r}_t^1\right).$$

Further, we have

$$\begin{aligned} & 1 - \widehat{r}_{t+1}^1 \\ = & \sum_{k=1}^{K} \left\{ \rho_{t+1} \left[\lambda_{t+1,k}^* \widehat{r}_{t+1}^1 + \lambda_{t+1,k} \left(1 - \widehat{r}_{t+1}^1\right) \right] + (1 - \rho_{t+1}) R_{t+1,k} \right\} \frac{\lambda_{t,k} \left(1 - \widehat{r}_t^1\right)}{\lambda_{t,k}^* \widehat{r}_t^1 + \lambda_{t,k} \left(1 - \widehat{r}_t^1\right)}. \end{aligned}$$

Thus the dynamics of the market shares $\widehat{r}_t^1 = r_t^1 + ... + r_t^{N-1}$, $1 - \widehat{r}_t^1 = r_t^N$ is exactly the same as the dynamics of the market shares $\widehat{r}_t^1, \widehat{r}_t^2 = 1 - \widehat{r}_t^1$ of two investors $i = 1, 2$ $(N = 2)$ using the strategies $(\lambda_t^1) = (\lambda_t^*)$ and $(\lambda_t^2) = (\lambda_t)$, respectively. Since (λ_t) is a survival strategy, the random sequence $r_t^N = 1 - \widehat{r}_t^1 = \widehat{r}_t^2$ is bounded away from zero almost surely.

Since investor 1 uses the strategy Λ^*, by virtue of Lemma 2.3 the sequence ζ_t defined by (2.34) is a non-negative supermartingale, and inequality (2.35) holds. Since ζ_t is greater than 0, in view of Lemma 2.4, the series $\sum_{t=0}^{\infty} (\zeta_t - E_t \zeta_t)$ of non-negative random variables converges a.s.. The inequality

$$\zeta_t - E_t \zeta_t \geq (1 - \rho_t) \sum_{k=1}^{K} \lambda_{t,k}^* \ln \frac{\lambda_{t,k}^*}{\widehat{r}_t^1 \lambda_{t,k}^* + \widehat{r}_t^2 \lambda_{t,k}} < \infty \text{ (a.s.)}$$

established in Lemmas 2.2 and 2.3 and assumption (2.20) imply that

$$\sum_{t=1}^{\infty}\sum_{k=1}^{K}\lambda_{t,k}^{*}\ln\frac{\lambda_{t,k}^{*}}{\widehat{r}_t^1\lambda_{t,k}^{*}+\widehat{r}_t^2\lambda_{t,k}}<\infty \text{ (a.s.).} \tag{2.37}$$

Finally, according to Lemma 2.5, we observe that

$$\sum_{k=1}^{K}\lambda_{t,k}^{*}\ln\frac{\lambda_{t,k}^{*}}{\widehat{r}_t^1\lambda_{t,k}^{*}+\widehat{r}_t^2\lambda_{t,k}}=$$

$$\sum_{k=1}^{K}\lambda_{t,k}^{*}\ln\lambda_{t,k}^{*}-\sum_{k=1}^{K}\lambda_{t,k}^{*}\ln\left(\widehat{r}_t^1\lambda_{t,k}^{*}+\widehat{r}_t^2\lambda_{t,k}\right)\geq$$

$$\frac{1}{4}\sum_{k=1}^{K}\left[\lambda_{t,k}^{*}-\left(1-\widehat{r}_t^2\right)\lambda_{t,k}^{*}-\widehat{r}_t^2\lambda_{t,k}\right]^2=\frac{1}{4}\sum_{k=1}^{K}\left(\widehat{r}_t^2\lambda_{t,k}^{*}-\widehat{r}_t^2\lambda_{t,k}\right)^2=$$

$$\frac{1}{4}\left(\widehat{r}_t^2\right)^2\sum_{k=1}^{K}(\lambda_{t,k}^{*}-\lambda_{t,k})^2=\frac{1}{4}\left(\widehat{r}_t^2\right)^2||\lambda_{t,k}^{*}-\lambda_{t,k}||^2, \tag{2.38}$$

where the sequence $\widehat{r}_t^2$ is bounded away from zero a.s., as long as (λ_t) is a survival strategy. Therefore

$$\widehat{r}_t^2\geq c>0 \text{ (a.s.),} \tag{2.39}$$

where c is a random constant. From relation (2.37)-(2.39) we conclude that the series $\sum_{t=0}^{\infty}||\lambda_{t,k}^{*}-\lambda_{t,k}||^2$ converges a.s., which completes the proof of Theorem 2.2. □

2.5 Appendix

Proof of Proposition 2.1. Fix some t and s^t and consider the operator transforming a vector $p=(p_1,...,p_K)\in\mathbb{R}_+^K$ into the vector $q=(q_1,...,q_K)\in\mathbb{R}_+^K$ with coordinates

$$q_k=\alpha_t V_{t,k}^{-1}\sum_{i=1}^{N}\lambda_{t,k}^{i}\langle D_t+p,x_{t-1}^{i}\rangle.$$

This operator is contracting in the norm $||p||_V=\sum_k|p_k|V_{t-1,k}$. Indeed, by virtue of (2.2) we have

$$\widetilde{\alpha} := \max_{k=1,\dots,K} \left\{ \alpha_t V_{t,k}^{-1} V_{t-1,k} \right\} < 1$$

$$||q - q^{'}||_V = \sum_{k=1}^{k} |q_k - q_k^{'}| V_{t-1,k} \leq$$

$$\alpha_t \sum_{k=1}^{K} V_{t,k}^{-1} V_{t-1,k} \sum_{i=1}^{N} \lambda_{t,k}^i |\langle p - p', x_{t-1}^i \rangle| \leq \widetilde{\alpha} \sum_{i=1}^{N} \sum_{k=1}^{K} \lambda_{t,k}^i |\langle p - p', x_{t-1}^i \rangle| =$$

$$\widetilde{\alpha} \sum_{i=1}^{N} |\langle p - p', x_{t-1}^i \rangle| \leq \widetilde{\alpha} \sum_{i=1}^{N} \sum_{m=1}^{K} |p_m - p'_m| x_{t-1,m}^i =$$

$$\widetilde{\alpha} \sum_{m=1}^{K} \sum_{i=1}^{N} |p_m - p'_m| x_{t-1,m}^i = \widetilde{\alpha} \sum_{m=1}^{K} |p_m - p'_m| V_{t-1,m} = \widetilde{\alpha} ||p - p^{'}||_V,$$

where the last but on equality follows from (2.8). By using the contraction principle, we obtain the existence, uniqueness and measurability of a solution to (2.6).

□

Proof of proposition 2.2. From (2.6) and (2.7) we get

$$p_{t,k} = \alpha_t V_{t,k}^{-1} \sum_{i=1}^{N} \lambda_{t,k}^i \langle p_t + D_t, x_{t-1}^i \rangle \tag{2.40}$$

$$= \alpha_t V_{t,k}^{-1} \sum_{i=1}^{N} \lambda_{t,k}^i w_t^i = \alpha_t V_{t,k}^{-1} \langle \lambda_{t,k}, w_t \rangle,$$

$$x_{t,k}^i = \frac{\alpha_t \lambda_{t,k}^i \langle D_t(s^t) + p_t, x_{t-1}^i \rangle}{p_{t,k}} = \tag{2.41}$$

$$\frac{\alpha_t \lambda_{t,k}^i \langle D_t(s^t) + p_t, x_{t-1}^i \rangle}{\alpha_t V_{t,k}^{-1} \sum_{i=1}^{N} \lambda_{t,k}^i \langle p_t + D_t, x_{t-1}^i \rangle} = \frac{V_{t,k} \lambda_{t,k}^i w_t^i}{\langle \lambda_{t,k}, w_t \rangle},$$

where $t \geq 1$, $w_t := (w_t^1, ..., w_t^N)$ and $\lambda_{t,k} := (\lambda_{t,k}^1, ..., \lambda_{t,k}^N)$. The analogous formulas for $t = 0$,

$$p_{0,k} = \alpha_0 V_{0,k}^{-1} \langle \lambda_{0,k}, w_0 \rangle, \; x_{0,k}^i = \frac{V_{0,k} \lambda_{0,k}^i w_0^i}{\langle \lambda_{0,k}, w_0 \rangle}, \tag{2.42}$$

follow from (2.3) and (2.4). Consequently, we have

$$w^i_{t+1} = \sum_{k=1}^{K}(p_{t+1,k} + D_{t+1,k})x^i_{t,k} =$$

$$\sum_{k=1}^{K}(\alpha_{t+1}V^{-1}_{t+1,k}\langle\lambda_{t+1,k}, w_{t+1}\rangle + D_{t+1,k})\frac{V_{t,k}\lambda^i_{t,k}w^i_t}{\langle\lambda_{t,k}, w_t\rangle} =$$

$$\sum_{k=1}^{K}(\alpha_{t+1}V^{-1}_{t+1,k}V_{t,k}\langle\lambda_{t+1,k}, w_{t+1}\rangle + D_{t+1,k}V_{t,k})\frac{\lambda^i_{t,k}w^i_t}{\langle\lambda_{t,k}, w_t\rangle},\ t \geq 0. \tag{2.43}$$

By summing up these equations over $i = 1, ..., N$, we obtain

$$W_{t+1} = \sum_{k=1}^{K}\left(\alpha_{t+1}V^{-1}_{t+1,k}V_{t,k}\langle\lambda_{t+1,k}, w_{t+1}\rangle + D_{t+1,k}V_{t,k}\right)\frac{\sum_{i=1}^{N}\lambda^i_{t,k}w^i_t}{\langle\lambda_{t,k}, w_t\rangle} =$$

$$\sum_{k=1}^{K}(\alpha_{t+1}V^{-1}_{t+1,k}V_{t,k}\langle\lambda_{t+1,k}, w_{t+1}\rangle + D_{t+1,k}V_{t,k}).$$

As long as

$$V_{t+1,k}/V_{t,k} = \nu_{t+1} > 0 \tag{2.44}$$

(see(2.2)), we have

$$W_{t+1} = \sum_{k=1}^{K}(\alpha_{t+1}V^{-1}_{t+1,k}V_{t,k}\langle\lambda_{t+1,k}, w_{t+1}\rangle + D_{t+1,k}V_{t,k}) =$$

$$\alpha_{t+1}\nu^{-1}_{t+1}W_{t+1} + \sum_{k=1}^{K}D_{t+1,k}V_{t,k}.$$

This implies the formula

$$W_{t+1} = \frac{1}{1 - \alpha_{t+1}\nu^{-1}_{t+1}}\sum_{m=1}^{K}D_{t+1,m}V_{t,m}, \tag{2.45}$$

where $\alpha_{t+1}\nu^{-1}_{t+1} = \rho_{t+1}$.From (2.43) and (2.44), we find

$$w^i_{t+1} = \sum_{k=1}^{K}(\alpha_{t+1}V^{-1}_{t+1,k}V_{t,k}\langle\lambda_{t+1,k}, w_{t+1}\rangle + D_{t+1,k}V_{t,k})\frac{\lambda^i_{t,k}w^i_t}{\langle\lambda_{t,k}, w_t\rangle}\ t \geq 0.$$

Dividing both sides of this equation by W_{t+1} and using (2.45), we get

$$r^i_{t+1} = \sum_{k=1}^{K}\left[\rho_{t+1}\langle\lambda_{t+1,k}, r_{t+1}\rangle + (1-\rho_{t+1})\frac{D_{t+1,k}V_{t,k}}{\sum_{m=1}^{K} D_{t+1,m}V_{t,m}}\right]\frac{\lambda^i_{t,k}w^i_t/W_t}{\langle\lambda_{t,k}, w_t\rangle/W_t},$$

which yields (2.24) by virtue of (2.15) and (2.18). □

Proof of proposition 2.3. In view of (2.24) and (2.25) we have

$$r^1_{t+1} = \sum_{k=1}^{K}\left\{\rho_{t+1}\left[\lambda^1_{t+1,k}r^1_{t+1} + (1-r^1_{t+1})\widetilde{\lambda}^2_{t+1,k}\right] + (1-\rho_{t+1})R_{t+1,k}\right\}\frac{\lambda^1_{t,k}r^1_t}{\lambda^1_{t,k}r^1_t + \widetilde{\lambda}^2_{t,k}(1-r^1_t)}.$$

By summing up equations (2.24) over $i = 2, ..., N$, we find

$$\widetilde{r}^2_{t+1} = 1 - r^1_{t+1} = \sum_{k=1}^{K}[\rho_{t+1}\langle\lambda_{t+1,k}, r_{t+1}\rangle + (1-\rho_{t+1})R_{t+1,k}]\frac{\widetilde{\lambda}^2_{t,k}(1-r^1_t)}{\lambda^1_{t,k}r^1_t + \widetilde{\lambda}^2_{t,k}(1-r^1_t)}.$$

Thus we obtain

$$\widetilde{r}^2_{t+1} = \sum_{k=1}^{K}\left\{\rho_{t+1}\left[\lambda^1_{t+1,k}r^1_{t+1} + (1-r^1_{t+1})\widetilde{\lambda}^2_{t+1,k}\right] + (1-\rho_{t+1})R_{t+1,k}\right\}\frac{\widetilde{\lambda}^2_{t,k}(1-r^1_t)}{\lambda^1_{t,k}r^1_t + \widetilde{\lambda}^2_{t,k}(1-r^1_t)},$$

which completes the proof. □

Proof of proposition 2.4. By using (2.24) with $N = 2$, we get

$$r^i_{t+1} = \sum_{k=1}^{K}\left\{\rho_{t+1}\left[\lambda^i_{t+1,k}r^i_{t+1} + (1-r^i_{t+1})\lambda^j_{t+1,k}\right] + (1-\rho_{t+1})R_{t+1,k}\right\}\frac{\lambda^i_{t,k}r^i_t}{\lambda^i_{t,k}r^i_t + \lambda^j_{t,k}r^j_t},$$

where $i, j \in \{1, 2\}$ and $i \neq j$. Setting $C^{ij}_{t,k} := \lambda^i_{t,k}r^i_t/(\lambda^i_{t,k}r^i_t + \lambda^j_{t,k}r^j_t)$, we have

$$r^i_{t+1} = \sum_{k=1}^{K}\left[\rho_{t+1}\lambda^i_{t+1,k}r^i_{t+1} + \rho_{t+1}(1-r^i_{t+1})\lambda^j_{t+1,k} + (1-\rho_{t+1})R_{t+1,k}\right]C^{ij}_{t,k} =$$

$$\sum_{k=1}^{K}\rho_{t+1}\left(\lambda^i_{t+1,k} - \lambda^j_{t+1,k}\right)r^i_{t+1}C^{ij}_{t,k} + \sum_{k=1}^{K}\left[\rho_{t+1}\lambda^j_{t+1,k} + (1-\rho_{t+1})R_{t+1,k}\right]C^{ij}_{t,k},$$

which implies

$$r^i_{t+1}\left[1 + \rho_{t+1}\sum_{k=1}^{K}\left(\lambda^j_{t+1,k} - \lambda^i_{t+1,k}\right)C^{ij}_{t,k}\right] = \sum_{k=1}^{K}\left[\rho_{t+1}\lambda^j_{t+1,k} + (1-\rho_{t+1})R_{t+1,k}\right]C^{ij}_{t,k}.$$

Thus

$$\frac{r^i_{t+1}}{r^j_{t+1}} = \frac{A^{ij}_{t+1}/B^{ij}_{t+1}}{A^{ji}_{t+1}/B^{ji}_{t+1}},$$

where

$$A_{t+1}^{ij} := \sum_{k=1}^{K} \left[\rho_{t+1}\lambda_{t+1,k}^{j} + (1-\rho_{t+1})R_{t+1,k}\right] C_{t,k}^{ij},$$

$$B_{t+1}^{ij} := 1 + \rho_{t+1} \sum_{k=1}^{K} \left(\lambda_{t+1,k}^{j} - \lambda_{t+1,k}^{i}\right) C_{t,k}^{ij}.$$

Observe that $B_{t+1}^{ij} = B_{t+1}^{ji}$. Indeed,

$$B_{t+1}^{ij} - B_{t+1}^{ji} = \rho_{t+1} \sum_{k=1}^{K} \left[\left(\lambda_{t+1,k}^{j} - \lambda_{t+1,k}^{i}\right) C_{t,k}^{ij} - \left(\lambda_{t+1,k}^{i} - \lambda_{t+1,k}^{j}\right) C_{t,k}^{ji}\right] =$$

$$\rho_{t+1} \sum_{k=1}^{K} \left(\lambda_{t+1,k}^{j} - \lambda_{t+1,k}^{i}\right) = 0$$

because $C_{t,k}^{ji} + C_{t,k}^{ij} = 1$. Consequently,

$$\frac{r_{t+1}^{1}}{r_{t+1}^{2}} = \frac{A_{t+1}^{12}}{A_{t+1}^{21}} = \frac{r_t^1}{r_t^2} \frac{\sum_{k=1}^{K} \left[\rho_{t+1}\lambda_{t+1,k}^{2} + (1-\rho_{t+1})R_{t+1,k}\right] \dfrac{\lambda_{t,k}^{1}}{\lambda_{t,k}^{1} r_t^1/r_t^2 + \lambda_{t,k}^{2}}}{\sum_{k=1}^{K} \left[\rho_{t+1}\lambda_{t+1,k}^{1} + (1-\rho_{t+1})R_{t+1,k}\right] \dfrac{\lambda_{t,k}^{2}}{\lambda_{t,k}^{1} r_t^1/r_t^2 + \lambda_{t,k}^{2}}},$$

which yields (2.26). □

Proof of proposition 2.5. By virtue of (2.21), we have

$$E_t(\rho_{t+1}\lambda_{t+1,k}^{*}) = E_t\left(\rho_{t+1} E_{t+1} \sum_{l=1}^{\infty} \rho_{t+1}^{l} R_{t+l+1,k}\right) =$$

$$E_t\left(E_{t+1} \sum_{l=1}^{\infty} \rho_{t+1}\rho_{t+1}^{l} R_{t+l+1,k}\right) = E_t \sum_{l=1}^{\infty} \rho_{t+1}\rho_{t+1}^{l} R_{t+l+1,k},$$

and so

$$E_t[\rho_{t+1}\lambda_{t+1,k}^{*} + (1-\rho_{t+1})R_{t+1,k}] = E_t\left[\sum_{l=1}^{\infty} \rho_{t+1}\rho_{t+1}^{l} R_{t+l+1,k} + (1-\rho_{t+1})R_{t+1,k}\right] =$$

$$E_t\left(\sum_{l=1}^{\infty} \rho_t^{l+1} R_{t+l+1,k} + \rho_t^1 R_{t+1,k}\right) = E_t \sum_{l=1}^{\infty} \rho_t^{l} R_{t+l,k} = \lambda_{t,k}^{*}$$

because $1 - \rho_{t+1} = \rho_t^1$ and

$$\rho_{t+1}\rho_{t+1}^{l} = \rho_{t+1}\rho_{t+2 \ldots}\rho_{t+l}(1-\rho_{t+l+1}) = \rho_t^{l+1}$$

for $l \geq 1$. □

Proof of Lemma 2.1. From formula (2.26) with $\lambda^1_{t,k} = \lambda^*_{t,k}$ and $\lambda^2_{t,k} = \lambda_{t,k}$, we get

$$\sum_{k=1}^{K}[\rho_{t+1}\lambda^*_{t+1,k} + (1-\rho_{t+1})R_{t+1,k}]\frac{\lambda_{t,k}}{\lambda^*_{t,k}z_t + \lambda_{t,k}} =$$
$$\sum_{k=1}^{K}\left[\rho_{t+1}\frac{\lambda_{t+1,k}}{z_{t+1}} + (1-\rho_{t+1})\frac{R_{t+1,k}}{z_{t+1}}\right]\frac{\lambda^*_{t,k}z_t}{\lambda^*_{t,k}z_t + \lambda_{t,k}}.$$

By using the notation $y^k_t = \lambda_{t,k}/z_t$ and the fact that $|y_t| = 1/z_t$, we write

$$\sum_{k=1}^{K}[\rho_{t+1}\lambda^*_{t+1,k} + (1-\rho_{t+1})R_{t+1,k}]\frac{y^k_t}{\lambda^*_{t,k} + y^k_t} =$$
$$\sum_{k=1}^{K}\left[\rho_{t+1}y^k_{t+1} + (1-\rho_{t+1})R_{t+1,k}|y_{t+1}|\right]\frac{\lambda^*_{t,k}}{\lambda^*_{t,k} + y^k_t},$$

which implies

$$\rho_{t+1}\sum_{k=1}^{K}\frac{\lambda^*_{t,k}y^k_{t+1} - \lambda^*_{t+1,k}y^k_t}{\lambda^*_{t,k} + y^k_t} + (1-\rho_{t+1})\sum_{k=1}^{K}R_{t+1,k}\frac{\lambda^*_{t,k}|y_{t+1}| - y^k_t}{\lambda^*_{t,k} + y^k_t} = 0. \quad (2.46)$$

We have

$$\frac{\lambda^*_{t,k}y^k_{t+1} - \lambda^*_{t+1,k}y^k_t}{\lambda^*_{t,k} + y^k_t} = \lambda^*_{t+1,k}\frac{\lambda^*_{t,k}y^k_{t+1}/\lambda^*_{t+1,k} - y^k_t}{\lambda^*_{t,k} + y^k_t} = \lambda^*_{t+1,k}\frac{y^k_{t+1}/\lambda^*_{t+1,k} - y^k_t/\lambda^*_{t,k}}{1 + y^k_t/\lambda^*_{t,k}} =$$

$$\lambda^*_{t+1,k}\left(\frac{1 + y^k_{t+1}/\lambda^*_{t+1,k}}{1 + y^k_t/\lambda^*_{t,k}} - 1\right) \geq \lambda^*_{t+1,k}\ln\frac{1 + y^k_{t+1}/\lambda^*_{t+1,k}}{1 + y^k_t/\lambda^*_{t,k}}, \quad (2.47)$$

where the last relation follows from the inequality $a - 1 \geq \ln a (a > 0)$.By using (2.47), we find

$$\sum_{k=1}^{K}\frac{\lambda^*_{t,k}y^k_{t+1} - \lambda^*_{t+1,k}y^k_t}{\lambda^*_{t,k} + y^k_t} \geq \sum_{k=1}^{K}\lambda^*_{t+1,k}\left[\ln\left(1 + \frac{y^k_{t+1}}{\lambda^*_{t+1,k}}\right) - \ln\left(1 + \frac{y^k_t}{\lambda^*_{t,k}}\right)\right] =$$

$$\sum_{k=1}^{K}\lambda^*_{t+1,k}(Z_{t+1,k} - Z_{t,k}) = Z_{t+1} - \sum_{k=1}^{K}\lambda^*_{t+1,k}Z_{t,k}. \quad (2.48)$$

Further, we have

$$\frac{\lambda^*_{t,k}|y_{t+1}| - y^k_t}{\lambda^*_{t,k} + y^k_t} = \frac{\lambda^*_{t,k}|y_{t+1}| + \lambda^*_{t,k}}{\lambda^*_{t,k} + y^k_t} - 1 \geq$$
$$\ln \frac{\lambda^*_{t,k}|y_{t+1}| + \lambda^*_{t,k}}{\lambda^*_{t,k} + y^k_t} = \ln \frac{|y_{t+1}| + 1}{1 + y^k_t/\lambda^*_{t,k}},$$

and so

$$\sum_{k=1}^K R_{t+1,k} \frac{\lambda^*_{t,k}|y_{t+1}| - y^k_t}{\lambda^*_{t,k} + y^k_t} \geq \sum_{k=1}^K R_{t+1,k} \ln \frac{1 + |y_{t+1}|}{1 + y^k_t/\lambda^*_{t,k}} =$$
$$\ln(1 + |y_{t+1}|) - \sum_{k=1}^K R_{t+1,k} \ln(1 + y^k_t/\lambda^*_{t,k}) = Y_{t+1} - \sum_{k=1}^K R_{t+1,k} Z_{t,k}$$

(see (2.29) and (2.30)), which yields

$$\sum_{k=1}^K R_{t+1,k} \frac{\lambda^*_{t,k}|y_{t+1}| - y^k_t}{\lambda^*_{t,k} + y^k_t} \geq Y_{t+1} - \sum_{k=1}^K R_{t+1,k} Z_{t,k} \tag{2.49}$$

By combining (2.46), (2.47) and (2.49), we find

$$0 \geq \rho_{t+1} \left(Z_{t+1} - \sum_{k=1}^K \lambda^*_{t+1,k} Z_{t,k} \right) + (1 - \rho_{t+1}) \left(Y_{t+1} - \sum_{k=1}^K R_{t+1,k} Z_{t,k} \right) =$$
$$\rho_{t+1} Z_{t+1} + (1 - \rho_{t+1}) Y_{t+1} - \sum_{k=1}^K [\rho_{t+1} \lambda^*_{t+1,k} + (1 - \rho_{t+1}) R_{t+1,k}] Z_{t,k}$$

which proves (2.32). □

Proof of Lemma 2.2. To prove the first relation in (2.33) we proceed as follows:

$$U_t = Y_t - Z_t = -\ln r^1_t - \sum_{k=1}^K \lambda^*_{t,k} \ln \left(1 + \frac{r^2_t \lambda_{t,k}}{r^1_t \lambda^*_{t,k}} \right) =$$
$$\sum_{k=1}^K \lambda^*_{t,k} \ln \frac{1}{r^1_t} + \sum_{k=1}^K \lambda^*_{t,k} \ln \frac{r^1_t \lambda^*_{t,k}}{r^1_t \lambda^*_{t,k} + r^2_t \lambda_{t,k}} = \sum_{k=1}^K \lambda^*_{t,k} \ln \frac{\lambda^*_{t,k}}{r^1_t \lambda^*_{t,k} + r^2_t \lambda_{t,k}}.$$

The last relation in (2.33) follows from the elementary inequality $\sum_{k=1}^K a_k \ln a_k - \sum_{k=1}^K a_k \ln b_k \geq 0$, which is presented in a somewhat refined form in Lemma 2.5.

□

Proof of Lemma 2.3. It is clear that $\zeta_t \geq 0$. Indeed, from (2.34), we have

$$\zeta_t := \rho_t Z_t + (1-\rho_t)Y_t =$$

$$\rho_t \sum_{k=1}^{K} \lambda^*_{t,k} \ln\left(1 + \frac{r_t^2 \lambda_{t,k}}{r_t^1 \lambda^*_{t,k}}\right) - (1-\rho_t)\ln r_t^1 \geq 0,$$

because $\ln\left(1 + r_t^2\lambda_{t,k}/r_t^1\lambda^*_{t,k}\right) \geq 0$ and $\ln r_t^1 \leq 0$. By taking the conditional expectation $E_t(\cdot)$ of both sides of inequality (2.32) and using (2.27), we obtain

$$E_t\zeta_{t+1} \leq \sum_{k=1}^{K} Z_{t,k} E_t[\rho_{t+1}\lambda^*_{t+1,k} + (1-\rho_{t+1})R_{t+1,k}] = \sum_{k=1}^{K} Z_{t,k}\lambda^*_{t,k} = Z_t. \quad (2.50)$$

In view of (2.31), we get

$$E_t\zeta_{t+1} + (1-\rho_t)U_t \leq Z_t + (1-\rho_t)Y_t - (1-\rho_t)Z_t =$$

$$\rho_t Z_t + (1-\rho_t)Y_t = \zeta_t,$$

which proves (2.35). Thus $E_t\zeta_{t+1} \leq \zeta_t - (1-\rho_{t+1})U_t \leq \zeta_t$ because $U_t \geq 0$ (see Lemma 2.4). The last inequality implies $E\zeta_t \leq E\zeta_0 = \zeta_0 < +\infty$. Since $\zeta_t \geq 0$, we have $E|\zeta_t| < \infty$, and so ζ_t is a non-negative supermartingale. □

Proof of Lemma 2.4. The random variables $\eta_t := \zeta_t - E_t\zeta_{t+1}$ are non-negative by the definition of a supermartingale. Further, we have

$$\sum_{t=0}^{T-1} E\eta_t = \sum_{t=0}^{T-1}(E\zeta_t - E\zeta_{t+1}) = E\zeta_0 - E\zeta_T,$$

and so the sequence $\sum_{t=0}^{T-1} E\eta_t$ is bounded because $\sup_T(-E\zeta_T) = -\inf E\zeta_T < +\infty$. According to the property of non-negative series[24], the series of the expectations $\sum_{t=0}^{\infty} E\eta_t$ of the non-negative random variables η_t converges, which implies

[24] When a_n is a non-negative real number for every n, the sequence S_n of partial sums is non-decreasing. It follows that a series $\sum a_n$ with non-negative terms converges if and only if the sequence S_n of partial sums is bounded.

$\sum_{t=0}^{\infty} \eta_t < \infty$ a.s. because $\sum_{t=0}^{\infty} E\eta_t = E\sum_{t=0}^{\infty} \eta_t$ (the last equality holds[25] for any sequence $\eta_t \geq 0$). $\square$

Proof of Lemma 2.5. We have $\ln x \leq x - 1$, which implies $(\ln x)/2 \leq \sqrt{x} - 1$, and so $-\ln x \geq 2 - 2\sqrt{x}$. By using this inequality, we get

$$\sum_{k=1}^{K} a_k(\ln a_k - \ln b_k) = -\sum_{k=1}^{K} a_k \ln \frac{b_k}{a_k} \geq \sum_{k=1}^{K} a_k \left(2 - 2\frac{\sqrt{b_k}}{\sqrt{a_k}}\right) =$$

$$2 - 2\sum_{k=1}^{K} \sqrt{a_k b_k} = \sum_{k=1}^{K} \left(a_k - 2\sqrt{a_k b_k} + b_k\right) = \sum_{k=1}^{K} \left(\sqrt{a_k} - \sqrt{b_k}\right)^2.$$

This yields (2.36) because $\left(\sqrt{a_k} - \sqrt{b_k}\right)^2 \geq (a_k - b_k)^2/4$ for $0 \leq a_k, b_k \leq 1$[26]. $\square$

[25] If (Z_k) is a sequence of non-negative random variables such that $\sum E(Z_k) < \infty$, then $\sum Z_k < \infty$ (a.s.) and $Z_k \to 0$ (a.s.).

[26] $\left(\sqrt{a_k} - \sqrt{b_k}\right)^2 \geq \left(\sqrt{a_k} - \sqrt{b_k}\right)^2 \left(\sqrt{a_k} + \sqrt{b_k}\right)^2/4$
$= (a_k - b_k)^2/4$, because $0 \leq a_k, b_k \leq 1$ and $\left(\sqrt{a_k} + \sqrt{b_k}\right)^2/4 \leq 1$.

Chapter 3 Almost sure Nash equilibrium strategies in evolutionary models of asset markets

3.1 Introduction

This chapter[27] examines a stochastic model of a financial market with long-lived dividend-paying assets and endogenous market clearing asset prices. This model is a version of that proposed in Evstigneev et al.(2006), and then analyzed primarily in the context of evolutionary finance (for a survey of the field see Evstigneev et al. (2009)). The main focus in evolutionary finance is on questions of "survival and extinction" of investment strategies (portfolio rules). In this chapter we analyze the model from a different perspective and treat its decision-theoretic framework as a game in which the payoffs of the players (investors) are defined in terms of the growth rates of their relative wealth. We show that in the game under consideration the Kelly (1956) portfolio rule of "betting your beliefs" forms with probability one a unique symmetric Nash equilibrium strategy.

Game-theoretic models of asset markets dealing with relative wealth of investors have been put forth by Bell and Cover (1980,1988). In the one-shot, two-person zero-sum models, each investor wishes to outperform any other investor. The solution concept used in their models is a Nash equilibrium defined in terms

27 This chapter is based on the material of the paper by W. Bahsoun, I. Evstigneev and L. Xu "Almost sure Nash equilibrium strategies in evolutionary models of asset markets," Working paper No 10-08 of the Mathematics Department of the University of Loughborough, February 2010.

of the expectations of random payoff functions. It is shown that anyone who deviates the log-optimal strategy results in a fall in the expected payoff. In this chapter, we consider a different (stronger) solution concept: almost sure Nash equilibrium. Any unilateral deviation from the Kelly rule leads to a decrease in the random payoff with probability one.

Another work which can be linked to this chapter is the paper of Alós-Ferrer and a Ania (2004). Both of us employ a game-theoretic asset market model and focus on the performance of investment strategies. But their model is limited in finitely many states of the world and allows for redundant assets. Whilst our model deals with infinitely states of the world and does not allow for redundant assets. Another difference between our works lies in the solution concept. They define a pure-strategy Nash equilibrium in terms of expected payoff as a game solution concept, but we employ a (stronger) solution concept—almost sure Nash equilibrium with respect to the random payoff.

The present chapter focuses on optimality almost surely, which is characteristic for capital growth theory (Kelly (1956), Breiman (1961), Algoet and Cover (1988), Hakansson and Ziemba (1995), Maclean et al. (2010)). Most of the previous research deals with asset market models with exogenous asset prices. Results related to evolutionary finance may be regarded as analogues, and in certain cases as generalizations, of those pertaining to classical models of capital growth. The main difference between the two modelling frameworks lies in the fact that in the former the accumulation of wealth of each investor might depend (via the endogenous price formation mechanism) not only on his/her strategy, but also on the strategies used by the other investors. Therefore in the present context a game-theoretic model, rather than a single-agent optimization framework, is a suitable setting for the analysis of questions related to capital growth.

This chapter is organized as follows. Section 3.2 is the model description, Section 3.3 states the main results and Section 3.4 provides the proof of the main theorem. The Appendix 3.5 contains the proof of a technical lemma.

3.2 The model

We consider a general asset market with $K \geq 2$ *assets* and $N \geq 2$ *investors* (*traders*) acting in the market. The market is influenced by random factors modelled in terms of independent identically distributed random elements $s_1, s_2, ...$ in a measurable space S. At each trading date $t = 1, 2, ...$ one unit of asset $k = 1, 2, ..., K$ yields nonnegative dividends $D_k(s_t) \geq 0$ depending on the "state of the world" s_t at date t. The dividends $D_k(s_t)$ are measurable and satisfy

$$\sum_{k=1}^{K} D_k(s) > 0 \text{ for all } s. \tag{3.1}$$

This condition means that in each random situation at least one asset yields a strictly positive dividend. The total volume (the number of units) of asset k traded in the market at date t is

$$V_{t,k} = V_{t,k}(s^t) > 0,$$

where $s^t := (s_1, ..., s_t)$ is the history of the process (s_t) from time 1 to time t. For

$t = 0$, $V_{t,k}$ is a constant number, and for $t \geq 1$, $V_{t,k}(s^t)$ is a measurable function of s^t.

Denote by $p_t \in \mathbb{R}^K_+$ the vector of market prices of the assets. For each $k = 1, ..., K$, the coordinate $p_{t,k}$ of $p_t = (p_{t,1}, ..., p_{t,K})$ stands for the price of one unit of asset k at date t. A *portfolio* of investor i at date $t = 0, 1, ...$ is specified by a vector $x^i_t = (x^i_{t,1}, ..., x^i_{t,K}) \in \mathbb{R}^K_+$ where $x^i_{t,k}$ is the amount (the number of units) of asset k. It means the numbers of units of all assets purchased by investor i. The scalar product $\langle p_t, x^i_t \rangle = \sum_{k=1}^K p_{t,k} x^i_{t,k}$ expresses the value of the investor i's portfolio x^i_t at date t in terms of the prices $p_{t,k}$.

At date $t = 0$ investor i has initial endowments $w^i_0 > 0$ that can be viewed as his/her budget at date 0. Investor i's *budget* (wealth) at date $t \geq 1$ consists of two components: the dividends $\langle D_t, x^i_{t-1} \rangle$ paid by the portfolio x^i_{t-1} and the market value $\langle p_t, x^i_{t-1} \rangle$ of the portfolio x^i_{t-1} expressed in terms of the today's prices p_t

$$w^i_t := \langle D_t + p_t, x^i_{t-1} \rangle, \tag{3.2}$$

where

$$D_t := D(s_t) := (D_1(s_t), ..., D_K(s_t)).$$

Let $\alpha_t = \alpha_t(s^{t-1})$ be a fraction of the budget invested into assets. Suppose that the *investment rate* $0 < \alpha_t(s^{t-1}) < 1$ is the same for all the traders, although in general it may depend on time and random factors. We assume that α_t is *predictable*: it depends on the history s^{t-1} of the process (s_t) up to time $t-1$ (not t). The number

$1 - \alpha_t$ can represent a fraction of money used for supporting investors's life or business, which can also be understood as *the tax rate* or *the consumption rate*. The assumption that $1 - \alpha_t$ is the same for all the investors is quite natural in the former case. In the latter case it is indispensable since we focus in this work on the analysis of the comparative performance of trading strategies (portfolio rules) in the long run. Without this assumption, an analysis of this kind does not make sense: a seemingly worse performance of a portfolio rule in the long run might be simply due to a higher consumption rate of the investor.

Suppose that the function $\alpha_t(s^{t-1})$ is measurable (for $t = 0, 1$ it is constant) and not greater than the supply growth rate of each asset k, i.e., that satisfies the following condition

$$\alpha_t(s^{t-1}) < V_{t,k}(s^t)/V_{t-1,k}(s^{t-1}). \tag{3.3}$$

This condition holds, in particular, when the total mass $V_{t,k}(s^t)$ of each asset k does not decrease, i.e., when the right-hand side of (3.3) is not less than one. But (3.3) does not exclude the situation when $V_{t,k}$ decreases at some rate, not faster than α_t.

An *investment strategy* (*portfolio rule*) is specified by a vector of *investment proportions* $\lambda_t^i = (\lambda_1^i, ..., \lambda_K^i) \in \Delta^K$ according to which he/she plans to distribute the available budget between assets at each date t. Vector λ_t^i belongs to the unit simplex

$$\Delta^K := \{(a_1, ..., a_K) \geq 0 : \; a_1 + ... + a_K = 1\}.$$

Strategies of this kind are called *fixed-mix*, or *constant proportions*, portfolio rules: they prescribe to select investment proportions at time 0 and keep them fixed over the whole infinite time horizon.

The class of fixed-mix investment strategies is quite widely used in financial theory and practice, playing an important role in portfolio theory; see, e.g., Perold and Sharpe (1988) and Browne (1998). Investors continuously rebalance their portfolios in order to keep fixed constant investment proportions. Under certain conditions, strategies with constant investment proportions lead to the growth of the portfolio value ("volatility pumping"— Luenberger (1998)). From the theoretical standpoint, this class of portfolio rules provides a convenient laboratory for the analysis of questions we are interested in. It makes it possible to formalize in a clear and compact way the concept of the *type* of an investor which determines the performance of his/her portfolio rule in the long run.

In the model at hand, the asset market evolves in time, remaining in the state of a dynamic *short-run (temporary) equilibrium*[28]. The notion of market equilibrium is defined as follows. Suppose each investor $i = 1, ..., N$ has selected a strategy— a vector of investment proportions $\lambda^i = (\lambda^i_1, ..., \lambda^i_K) \in \Delta^K$ at each date t. Then

[28] In this paper, we use the term "equilibrium" in two different meanings. Here it is related to market equilibrium: a situation when asset supply is equal to asset demand. Later, the same term will appear in a game-theoretic context. It will pertain to Nash equilibrium strategies in a certain dynamic game.

at date $t \geq 0$, the amount of wealth allocated to asset k by trader i is $\alpha_t \lambda_k^i w_t^i$. By summing up all traders wealth invested in asset k, we have the total amount demand of asset k, $\alpha_t \sum_{i=1}^N \lambda_k^i w_t^i$, where w_t^i is investor i's budget at time t. At each trading date t, the market satisfies the market clearing condition: asset supply is equal to asset demand, making it possible to determine the equilibrium price $p_{t,k}$ of each asset k from the following equations

$$p_{t,k} V_{t,k} = \alpha_t \sum_{i=1}^{N} \lambda_k^i w_t^i, \; k = 1, ..., K. \tag{3.4}$$

The left-hand side of (3.4) is the total market value of the supply of asset k at date t (recall that the amount of each asset k at date t is $V_{t,k}$). The right-hand side represents the total wealth invested in asset k by all the investors. Equilibrium implies the equality in (3.4). The portfolios $x_t^i = (x_{t,1}^i, ..., x_{t,K}^i)$ are determined by the investment proportions $\lambda_1^i, ..., \lambda_K^i$ chosen by the traders at time t by the formulas

$$x_{t,k}^i = \frac{\alpha_t \lambda_k^i w_t^i}{p_{t,k}}, \; k = 1, ..., K, \; i = 1, ..., N. \tag{3.5}$$

Note that for $t \geq 1$, investor i's budget can be expressed by (3.2) and the price vector p_t is determined implicitly as the solution to the system of equations (3.4), which can be written as

$$p_{t,k} V_{t,k} = \alpha_t \sum_{i=1}^{N} \lambda_k^i \langle D_t + p_t, x_{t-1}^i \rangle, \; k = 1, ..., K. \tag{3.6}$$

It can be shown that under assumption (3.3) a non-negative vector p_t satisfying these equations exists and is unique (for any s^t and any feasible x_{t-1}^i and λ^i)—see

Proposition 2.1).

Equations (3.5) make sense only if $p_{t,k} > 0$, or equivalently, if the aggregate demand for each asset (under the equilibrium prices) is strictly positive. We say a strategy profile $(\lambda^1, ..., \lambda^N)$ is *admissible* if it guarantees that each asset k has a strictly positive equilibrium price through the above recursive procedure described step by step from (3.4)–(3.5). In what follows, we will deal only with such strategy profiles so as to guarantee the random dynamical system under consideration is well-defined.

A path of market dynamics can be generated recursively according to a strategy profile $(\lambda^1, ..., \lambda^N)$ of all the investors and their initial wealth $w_0^1, ..., w_0^N$

$$(p_t; x_t^1, ..., x_t^N), \tag{3.7}$$

where $p_t = p_t(s^t)$ is the price vectors and $x_t^i = x_t^i(s^t)$ is investor i's portfolio (i.e., the amounts of units of all assets purchased by investor i). Since the hypothesis of admissibility is throughout this chapter, we obtain that all the portfolios $x_t^i = (x_{t,1}^i, ..., x_{t,K}^i)$ are non-zero and the wealth $w_t^i = \langle D_t + p_t, x_{t-1}^i \rangle$ of each investor is strictly positive from the above equilibrium path. Further, by summing up equations (3.5) over $i = 1, ..., N$, we find that

$$\sum_{i=1}^{N} x_{t,k}^i = \frac{\sum_{i=1}^{N} \alpha_t \lambda_k^i w_t^i}{p_{t,k}} = \frac{p_{t,k} V_{t,k}}{p_{t,k}} = V_{t,k} \tag{3.8}$$

(the market clears) for every asset k and each date $t \geq 1$. Thus for every equilibrium state of the market $(p_t, x_t^1, ..., x_t^N)$, we have $p_t > 0$, $x_t^i \neq 0$ and (3.8).

We shall provide a simple sufficient condition for making a strategy profile admissible which can guarantee asset prices are strictly positive and market dynamics are well defined. Suppose that some trader, say trader 1, selects a *completely mixed* portfolio rule to invest into all the assets, i.e., each coordinate of the investment strategy vector of trader 1 at time $t = 0, 1, ...$ must be strictly positive. Then a strategy profile containing this portfolio rule is admissible. Indeed, for $t = 0$, we get from (3.4) that $p_{0,k} \geq \alpha_0 V_{0,k}^{-1} \lambda_k^1 w_0^1 > 0$ and from (3.5) that $x_0^1 = (x_{0,1}^1, ..., x_{0,K}^1) > 0$ (coordinatewise). Assuming that $x_{t-1}^1 > 0$ and arguing by induction, we obtain $\langle D_t + p_t, x_{t-1}^1 \rangle \geq \langle D_t, x_{t-1}^1 \rangle > 0$ in view of (3.1), which in turn yields $p_t > 0$ and $x_t^1 > 0$ by virtue of (3.4) and (3.5), as long as $\lambda_k^1 > 0$.

3.3 The main results

Let $(\lambda^1, ..., \lambda^N)$ be an admissible strategy profile of the investors. Given this strategy profile and initial endowments, a path of market dynamics (3.7) can be generated in accordance with the equations (3.4)–(3.5). As above, let w_t^i denote the investor i's wealth available at date $t \geq 0$. If $t = 0$, then the initial endowment w_0^i of investor i is assumed as a constant number. If $t \geq 1$, then $w_t^i = w_t^i(s^t)$ is defined as a measurable function of s^t given by formula (3.2). As we have noted above, $w_t^i(s^t) > 0$.

We are primarily interested in the long-run behavior of the *relative wealth* of the investors. The relative wealth of investor $i,\ i = 1, 2, ...N$ are given by

$$r_t^i = \frac{w_t^i}{W_t},$$

where $W_t := \sum_{i=1}^N w_t^i$ is the *total market wealth.* Given a strategy profile $(\lambda^1, ..., \lambda^N)$ the performance of a strategy λ^i used by investor i can be characterized by the ratio between investor i's relative wealth and the coalition $\{j : j \neq i\}$ of i's rivals in the game under consideration

$$\xi^i := \limsup_{t\to\infty} \frac{1}{t} \ln \frac{r_t^i}{\sum_{j\neq i} r_t^j}. \tag{3.9}$$

The random variable $\xi^i = \xi^i(s^\infty; \lambda^1, ..., \lambda^N)$ depends on the strategy profile $(\lambda^1, ..., \lambda^N)$ and on the whole history $s^\infty := (s_1, s_2, ...)$ of states of the world from time 1 to ∞, playing the role of the (random) *payoff function* of player i. Further, this payoff function reflects the fact that the performance of investor i is influenced not only by his/her investment strategy, but also his/her rivals'.

Definition 3.1. We shall say that a strategy $\bar{\lambda}$ forms a *symmetric Nash equilibrium almost surely (a.s.)* if

$$\xi^i(s^\infty; \bar{\lambda}, ..., \bar{\lambda}) \geq \xi^i(s^\infty; \bar{\lambda}, ..., \lambda, ..., \bar{\lambda}) \text{ (a.s.)} \tag{3.10}$$

for every i, each strategy λ of investor i and each set of initial endowments $w_0^1 > 0, ..., w_0^N > 0$. The Nash equilibrium is called *strict* if the inequality in (3.10) is strict.

The strategy profile $\left(\bar{\lambda}, ..., \bar{\lambda}\right)$ is admissible (recall that we consider only admissible strategy profiles) if and only if the vector $\bar{\lambda}$ is strictly positive. This observation is immediate from (3.6).

Assume that the total mass $V_{t,k}$ of each asset k grows (or decreases) at the same rate $\gamma_t = \gamma_t(s^{t-1}) > 0$:

$$V_{t,k}/V_{t-1,k} = \gamma_t \; (t \geq 1). \tag{3.11}$$

Thus

$$V_{t,k}(s^{t-1}) = \gamma_t(s^{t-1})...\gamma_2(s_1)\gamma_1 V_k, \tag{3.12}$$

where $V_k > 0$ ($k = 1, 2, ..., K$) are the initial amounts of the assets. The growth rate process γ_t (like the investment rate process α_t) is predictable: γ_t depends only on the history s^{t-1} of the states of the world up to time $t-1$. In the case of dividend-paying assets involving investments in the real economy, assumption (3.11) means that the economic system under consideration is on a *balanced growth path*.

Define the *relative dividends* of the assets $k = 1, ..., K$ by

$$R_k(s_t) = \frac{D_k(s_t)V_k}{\sum_{m=1}^{K} D_m(s_t)V_m} \tag{3.13}$$

It follows from (3.12) that

$$R_{t,k} = \frac{D_{t,k}V_{t-1,k}}{\sum_{m=1}^{K} D_{t,m}V_{t-1,m}}.$$

where $R_{t,k} = R_k(s_t)$ and $D_{t,k} = D_k(s_t)$. Indeed, from (3.12), we have

$$\begin{aligned} R_k(s_t) &= \frac{D_{t,k}V_{t-1,k}}{\sum_{m=1}^{K} D_{t,m}V_{t-1,m}} \\ &= \frac{D_{t,k}\gamma_{t-1}(s^{t-2})...\gamma_2(s_1)\gamma_1 V_k}{\sum_{m=1}^{K} D_{t,m}\gamma_{t-1}(s^{t-2})...\gamma_2(s_1)\gamma_1 V_m} \\ &= \frac{D_k(s_t)V_k}{\sum_{m=1}^{K} D_m(s_t)V_m}. \end{aligned}$$

Define

$$\lambda_k^* = ER_k(s_t),\ k = 1, 2, ..., K \tag{3.14}$$

and put $\lambda^* = (\lambda_1^*, ..., \lambda_K^*)$. The investment strategy specified by (3.14) may be regarded as a generalization of the Kelly portfolio rule of "betting your beliefs". In this context it takes on the form of a *rational expectations strategy* (see Chapter 2). It is expressed in terms of the expected relative dividends, according to which investors distribute wealth across assets in accordance with the proportions of the expected relative dividends (which do not depend on t because the random elements s_t are i.i.d.).

Assume that the following conditions hold.

(R1) For each k, the expectation $ER_k(s_t)$ is strictly positive.

(R2) The functions $R_1(s), ..., R_K(s)$ are linearly independent with respect to the probability distribution of s_t, i.e., the equality $\sum \beta_k R_k(s_t) = 0$ holding a.s. for some constants β_k implies that $\beta_1 = ... = \beta_K = 0$.

(R3) There exist constants $0 < \rho' < \rho'' < 1$ such that the process

$$\rho_t(s^{t-1}) := \alpha_t(s^{t-1})/\gamma_t(s^{t-1})$$

satisfies $\rho' \leq \rho_t(s^{t-1}) \leq \rho''$.

Condition (R1) implies that the vector λ^* has strictly positive coordinates. Hypothesis (R2) can be interpreted as the absence of *redundant assets*. Condition (R3) states that the discount factor ρ_t cannot be too close to 0 and 1. Under these assumptions, the following theorem holds.

Theorem 3.1 *The portfolio rule* λ^* *is a unique strategy forming a symmetric Nash equilibrium a.s.. This equilibrium is strict.*

This result implies the following property of the portfolio rule λ^*. If all the investors except one, say investor i, use the strategy λ^* and i uses any other strategy λ distinct from λ^*, then the relative wealth $r_t^i / \sum_{j \neq i} r_t^j$ of i tends to zero at the exponential rate $\xi^i < 0$ (a.s.). In other words, the coalition of the Kelly investors drives the non-Kelly one out of the market. We note that this result (without an exponential estimate of the convergence rate) can be derived from Theorem 1 in Evstigneev et al. (2008) under the assumptions that the state space S is finite and all the strategies under consideration are completely mixed.

3.4 Proofs

For the proof of Theorem 3.1 we begin with a system of equations governing the dynamics of the *market shares* $r_t^i := w_t^i / \sum_j w_t^j$. Consider the path (3.7) of the random dynamical system generated by $(\lambda^1, ..., \lambda^N)$ and the sequence of vectors $r_t = (r_t^1, ..., r_t^N)$ of the market shares of the investors at date t. Proposition 2.2 states the following equations hold

$$r_{t+1}^i = \sum_{k=1}^{K} [\rho_{t+1} \langle \lambda_k, r_{t+1} \rangle + (1 - \rho_{t+1}) R_{t+1,k}] \frac{\lambda_k^i r_t^i}{\langle \lambda_k, r_t \rangle}, \tag{3.15}$$

$i = 1, ..., N$, $t \geq 0$. From the above equations we observe that the market share of investor i evolves in terms of the interaction of the strategies $\lambda^1, ..., \lambda^N$.

Secondly, it suffices to prove Theorem 3.1 in an asset market with only two investors. The above system can be reduced to the case of two investors. And the ratio of the market shares of investors 1 and 2 at date t is

$$z_t := r_t^1/r_t^2 = w_t^1/w_t^2,$$

where investor 1 and investor 2 select the Kelly rule $\lambda^* = (\lambda_1^*, ..., \lambda_K^*)$ and $\lambda = (\lambda_1, ..., \lambda_K)$, respectively. The dynamics of z_t are described by the following equation

$$z_{t+1} = z_t \frac{\sum_{k=1}^K [\rho_{t+1}\lambda_k + (1-\rho_{t+1})R_{t+1,k}] \frac{\lambda_k^*}{\lambda_k^* z_t + \lambda_k}}{\sum_{k=1}^K [\rho_{t+1}\lambda_k^* + (1-\rho_{t+1})R_{t+1,k}] \frac{\lambda_k}{\lambda_k^* z_t + \lambda_k}}. \tag{3.16}$$

For a proof of (3.16), see the proof of proposition 2.4 in Appendix 2.5.

Consider any measurable relative dividend vector function $R(s) = (R_1(s), ..., R_K(s))$ on S satisfying (R1) and (R2). For any $\lambda = (\lambda_1, ..., \lambda_K) \in \Delta^K$, $\rho' \leq \rho \leq \rho''$ (see (R3)) and $\kappa \in (0, 1]$, define

$$F_\rho(\lambda, \kappa; s) := \frac{\sum_{k=1}^K [\rho\lambda_k + (1-\rho)R_k(s)] \frac{\lambda_k^*}{\lambda_k^* \kappa + \lambda_k(1-\kappa)}}{\sum_{k=1}^K [\rho\lambda_k^* + (1-\rho)R_k(s)] \frac{\lambda_k}{\lambda_k^* \kappa + \lambda_k(1-\kappa)}}, \tag{3.17}$$

where $\lambda_k^* = ER_k(s)$ ($E(\cdot)$ is the unconditional expectation with respect to the given probability P on S). The function $F_\rho(\lambda, \kappa; s)$ is well-defined and takes on finite strictly positive values.

Lemma 3.1 *For any $\lambda \in \Delta^K$ distinct from λ^*there exist constants $H > 0$ and $\delta > 0$ such that*

$$E \min\{H, \ln F_\rho(\lambda, \kappa; s)\} \geq \delta \tag{3.18}$$

for all $\kappa \in (0, 1]$ and all $\rho \in [\rho', \rho'']$.

Lemma 3.1 plays a key role in the proof of Theorem 3.1. The proof of this lemma is routine, but rather lengthy, and we relegate it to the Appendix 3.5.

Proof of Theorem 3.1. To demonstrate that λ^* forms a strict symmetric Nash equilibrium a.s. it is sufficient to consider the case of two investors 1 and 2, using λ^* and λ, and show that

$$\lim\inf_{t\to\infty} \frac{1}{t} \ln z_t > 0 \text{ (a.s.)}, \tag{3.19}$$

where z_t is the ratio of the market shares of 1 and 2.

To prove that the problem reduces to the case $N = 2$, let us first observe that by virtue of symmetry, it is sufficient to verify the property (3.10) for $i = N$. Suppose investors $i = 1, 2, ..., N-1$ use λ^* and investor N uses $\lambda \neq \lambda^*$. Then the total market share $r_t^* := r_t^1 + ... + r_t^{N-1}$ of $i = 1, 2, ..., N-1$ satisfies

$$r_{t+1}^* = \sum_{k=1}^{K} \left[\rho_{t+1}(\lambda_k^* r_{t+1}^* + \lambda_k r_{t+1}^N) + (1-\rho_{t+1}) R_{t+1,k}\right] \frac{\lambda_k^* r_t^*}{\lambda_k^* r_t^* + \lambda_k r_t^N}. \tag{3.20}$$

This relation is obtained by summing up equations (3.15) over $i = 1, 2, ..., N-1$. At the same time, by virtue of (3.15), we have

$$r_{t+1}^N = \sum_{k=1}^{K} [\rho_{t+1}(\lambda_k^* r_{t+1}^* + \lambda_k r_{t+1}^N) + (1-\rho_{t+1}) R_{t+1,k}] \frac{\lambda_k r_t^N}{\lambda_k^* r_t^* + \lambda_k r_t^N}. \tag{3.21}$$

Thus the vector (r_t^*, r_t^N) evolves in time as the vector $(\tilde{r}_t^1, \tilde{r}_t^2)$ of market shares of two investors using the strategies λ^* and λ, respectively. If (3.19) holds, then

$$\xi^N(\lambda^*, ..., \lambda^*, \lambda) = \lim\sup_{t\to\infty} \frac{1}{t} \ln \frac{w_t^N}{w_t^*} =$$
$$\lim\sup_{t\to\infty} \frac{1}{t} \ln \frac{r_t^N}{r_t^*} = \lim\sup_{t\to\infty} \frac{1}{t} \ln \frac{\tilde{r}_t^2}{\tilde{r}_t^1} =$$
$$\lim\sup_{t\to\infty} \left(-\frac{1}{t} \ln z_t\right) = -\liminf_{t\to\infty} \left(\frac{1}{t} \ln z_t\right) < 0 = \xi^N(\lambda^*, ..., \lambda^*, \lambda^*) \text{ (a.s.)},$$

where the last equality holds because the market shares of all the investors remain constant, as long as all of them use the same strategy. Indeed, if all N investors use the same strategy, say λ^*, $r_t^i, i = 1, 2, ..., N$ has the following equation from (3.15)

$$\begin{aligned} r_{t+1}^i &= \sum_{k=1}^K [\rho_{t+1}\langle\lambda_k^*, r_{t+1}\rangle + (1-\rho_{t+1})R_{t+1,k}]\frac{\lambda_k^* r_t^i}{\langle\lambda_k^*, r_t\rangle} \\ &= r_t^i \sum_{k=1}^K [\rho_{t+1}\langle\lambda_k^*, r_{t+1}\rangle + (1-\rho_{t+1})R_{t+1,k}]\frac{\lambda_k^*}{\lambda_k^* \sum_{i=1}^N r_t^i} \\ &= r_t^i \sum_{k=1}^K [\rho_{t+1}\lambda_k^* + (1-\rho_{t+1})R_{t+1,k}] \\ &= r_t^i = ... = r_0^i. \end{aligned}$$

Let us verify (3.19). Put $G_t = \ln(z_t/z_{t-1})$. Then

$$\sum_{t=1}^T G_t = \sum_{t=1}^T (\ln z_t - \ln z_{t-1}) = \ln z_T - \ln z_0.$$

Therefore it suffices to prove that $\liminf_{T\to\infty} T^{-1}\sum_{t=1}^T G_t > 0$ a.s.. For any constant H define $G_t^H := \min\{G_t, H\}$. Since $G_t^H \le G_t$ it is sufficient to prove that

$$\liminf_{T\to\infty} \frac{1}{T}\sum_{t=1}^T G_t^H > 0 \text{ (a.s.)} \tag{3.22}$$

for some H.

Observe that

$$G_{t+1} = \ln\frac{z_{t+1}}{z_t} = \ln\frac{\sum_{k=1}^K [\rho_{t+1}\lambda_k + (1-\rho_{t+1})R_{t+1,k}]\frac{\lambda_k^*}{\lambda_k^* z_t + \lambda_k}}{\sum_{k=1}^K [\rho_{t+1}\lambda_k^* + (1-\rho_{t+1})R_{t+1,k}]\frac{\lambda_k}{\lambda_k^* z_t + \lambda_k}} =$$

$$\ln\frac{\sum_{k=1}^K [\rho_{t+1}\lambda_k + (1-\rho_{t+1})R_k(s_{t+1})]\frac{\lambda_k^*}{\lambda_k^* r_t^1 + \lambda_k(1-r_t^1)}}{\sum_{k=1}^K [\rho_{t+1}\lambda_k^* + (1-\rho_{t+1})R_k(s_{t+1})]\frac{\lambda_k}{\lambda_k^* r_t^1 + \lambda_k(1-r_t^1)}} = \ln F_{\rho_{t+1}}(\lambda, r_t^1; s_{t+1}), \tag{3.23}$$

where $r_t^1 = r_t^1(s^t)$ and $\rho_{t+1} = \rho_{t+1}(s^t)$ (recall that the process ρ_t is predictable).

By virtue of Lemma 3.1, there exist $H > 0$ and $\delta > 0$ such that $E_t G_{t+1}^H \geq \delta$, where $E_t(\cdot) = E(\cdot|s^t)$ is the conditional expectation given s^t and

$$G_{t+1}^H(s^{t+1}) = \min\{H, \ln F_{\rho_{t+1}(s^t)}(\lambda, r_t^1(s^t); s_{t+1})\}.$$

When computing $E_t G_{t+1}^H$we fix s^t and take the unconditional expectation of G_{t+1}^H with respect to s_{t+1}, which is justified because s^t and s_{t+1} are independent.

Finally, we have

$$\frac{1}{T}\sum_{t=1}^{T} G_t^H = \frac{1}{T}\sum_{t=1}^{T} E_{t-1}G_t^H + \frac{1}{T}\sum_{t=1}^{T}(G_t^H - E_{t-1}G_t^H).$$

Since G_t^H is uniformly bounded, we can apply to the process $B_t^H := G_t^H - E_{t-1}G_t^H$ the strong law of large numbers for martingale differences (see, e.g., Hall and Heyde (1980)), which yields $\frac{1}{T}\sum_{t=1}^{T} B_t^H \to 0$ (a.s.). Therefore $\liminf T^{-1}\sum_{t=1}^{T} G_t^H \geq \delta$, which proves (3.22).

Suppose a strategy $\lambda \neq \lambda^*$ forms a symmetric Nash equilibrium with probability one. Then

$$0 = \xi^N(s^\infty; \lambda, ..., \lambda) \geq \xi^N(s^\infty; \lambda, ..., \lambda, \lambda^*) \text{ (a.s.),} \tag{3.24}$$

where

$$\xi^N(s^\infty; \lambda, ..., \lambda, \lambda^*) = \limsup_{t\to\infty} \frac{1}{t} \ln \frac{r_t^N}{1 - r_t^N}.$$

By interchanging λ and λ^* in formulas (3.20) and (3.21), we obtain that the vector $(r_t^1 + ... + r_t^{N-1}, r_t^N)$ evolves in time as the vector $(\hat{r}_t^1, \hat{r}_t^2)$ of market shares of two investors using the strategies λ and λ^*, respectively. As we have proved above, this implies

$$\liminf_{t\to\infty} \frac{1}{t} \ln \frac{r_t^N}{1 - r_t^N} > 0 \text{ (a.s.)}.$$

Therefore $\xi^N(s^\infty; \lambda, ..., \lambda, \lambda^*) > 0$ (a.s.), because $\limsup \geq \liminf$, which yields the inequality "$<$" in (3.24). This is a contradiction. □

3.5 Appendix

Proof of Lemma 3.1. We observe that the function $F_\rho(\lambda, \kappa; s)$ is bounded below

$$F_\rho(\lambda, \kappa; s) \geq c^2, \tag{3.25}$$

where $c := \min_k \lambda_k^* (> 0)$. Indeed, $F_\rho(\lambda, \kappa; s) = A/B$, where

$$A = \sum_{k=1}^{K} [\rho\lambda_k + (1-\rho)R_k(s)] \frac{\lambda_k^*}{\lambda_k^*\kappa + \lambda_k(1-\kappa)},$$
$$B = \sum_{k=1}^{K} [\rho\lambda_k^* + (1-\rho)R_k(s)] \frac{\lambda_k}{\lambda_k^*\kappa + \lambda_k(1-\kappa)}.$$

We observe

$$A \geq \sum_{k=1}^{K} [\rho\lambda_k + (1-\rho)R_k(s)] \min_k \frac{\lambda_k^*}{\lambda_k^*\kappa + \lambda_k(1-\kappa)}$$
$$\geq \min_k \frac{\lambda_k^*}{\lambda_k^*\kappa + \lambda_k(1-\kappa)} \geq \frac{\min_k \lambda_k^*}{\max_k[\lambda_k^*\kappa + \lambda_k(1-\kappa)]} \geq \min_k \lambda_k^*,$$

since $\max[\lambda_k^*\kappa + \lambda_k(1-\kappa)] \leq 1$.

And B is bounded. Indeed, we have the following inequalities

$$B \geq \sum_{k=1}^{K} \rho\lambda_k^* \frac{\lambda_k}{\lambda_k^*\kappa + \lambda_k(1-\kappa)} \geq \rho' c \sum_{k=1}^{K} \lambda_k \geq \rho' c,$$

and

$$B \leq \max_k \frac{\lambda_k}{\lambda_k^*\kappa + \lambda_k(1-\kappa)} \leq \max_k \frac{1}{\lambda_k^*\kappa + (1-\kappa)} \leq \frac{1}{\min_k \lambda_k^*} = \frac{1}{c}, \tag{3.26}$$

since $0 \leq \lambda_k \leq 1$, and $\lambda_k^*\kappa + (1-\kappa) \geq \lambda_k^*$.

Assume that at least one of the coordinates of the vector λ is zero, so that the set $\mathbf{K} := \{k : \lambda_k = 0\}$ is not empty. Then

$$A = (1-\rho)\frac{1}{\kappa}\sum_{k\in\mathbf{K}} R_k(s) + \sum_{k\notin\mathbf{K}}[\rho\lambda_k + (1-\rho)R_k(s)]\frac{\lambda_k^*}{\lambda_k^*\kappa + \lambda_k(1-\kappa)}$$

and

$$B = \sum_{k\notin\mathbf{K}}^{K}[\rho\lambda_k^* + (1-\rho)R_k(s)]\frac{\lambda_k}{\lambda_k^*\kappa + \lambda_k(1-\kappa)}.$$

By virtue of (R1), there exists $\theta > 0$ such that $\sum_{k\in\mathbf{K}} R_k(s) \geq \theta$ for all s in a set $\bar{S}$ with $P(\bar{S}) > 0$. We have

$$(1-\rho^{''})\frac{\theta}{\kappa}\mathbf{1}_{\bar{S}}(s) \leq (1-\rho)\frac{1}{\kappa}\sum_{k\in\mathbf{K}} R_k(s) \leq A$$
$$\leq \max_{k\notin\mathbf{K}} \frac{\lambda_k^*}{\lambda_k^*\kappa + \lambda_k(1-\kappa)} + (1-\rho)\frac{1}{\kappa} \leq \frac{1}{\min_{k\notin\mathbf{K}}\lambda_k}\frac{1}{\kappa} + \frac{1}{\kappa}.$$

Therefore

$$\frac{d_1}{\kappa}\mathbf{1}_{\bar{S}}(s) \leq A \leq \frac{D_1}{\kappa},$$

where $\mathbf{1}_{\bar{S}}(s)$ is the indicator function of the set $\bar{S}$, $d_1 := (1-\rho'')\theta$ and $D_1 := 1 + (\min_{k\notin\mathbf{K}}\lambda_k)^{-1}$.

Also, we can see B is bounded. Indeed, we have

$$\rho^{'} c \min_{k\notin\mathbf{K}} \lambda_k \leq \rho^{'} c \sum_{k\notin\mathbf{K}}^{K} \frac{\lambda_k}{\lambda_k^*\kappa + \lambda_k(1-\kappa)} \leq \sum_{k\notin\mathbf{K}}^{K} \rho\lambda_k^* \frac{\lambda_k}{\lambda_k^*\kappa + \lambda_k(1-\kappa)}$$
$$\leq B \leq \max \frac{\lambda_k}{\lambda_k^*\kappa + \lambda_k(1-\kappa)} \leq \frac{1}{c}.$$

Therefore

$$d_2 \leq B \leq D_2,$$

where $d_2 := \rho' c \min_{k\notin\mathbf{K}} \lambda_k$ and $D_2 := c^{-1}$ (see (3.26)). And

$$\frac{d}{\kappa}\mathbf{1}_{\bar{S}}(s) \leq F_\rho(\lambda, \kappa; s) \leq \frac{D}{\kappa}, \tag{3.27}$$

where $d := d_1/D_2$ and $D := D_1/d_2$.

From the first of these inequalities and (3.25) we obtain

$$(\ln d - \ln \kappa)\mathbf{1}_{\bar{S}}(s) + (2\ln c)(1 - \mathbf{1}_{\bar{S}}(s)) \le \ln F_\rho(\lambda, \kappa; s), \tag{3.28}$$

and so

$$E \min[H, \ln F_\rho(\lambda, \kappa; s)] \ge 2\ln c + \min(H, \ln d - \ln \kappa) P(\bar{S}) \tag{3.29}$$

for any $H > 0$.

Put

$$\overline{\kappa} := \exp\left[\ln d - \frac{1 - 2\ln c}{P(\bar{S})}\right], \ H := \ln D - \ln \overline{\kappa}, \tag{3.30}$$

and if $0 < \kappa < \overline{\kappa}$, then we have

$$E \min[H, \ln F_\rho(\lambda, \kappa; s)] \ge 2\ln c + \min(H, \ln d - \ln \kappa) P(\bar{S})$$

$$\ge 2\ln c + (\ln d - \ln \kappa) P(\bar{S}) \ge 1 \tag{3.31}$$

by virtue of (3.29), (3.30) and the inequality $d \le D$ (following from (3.27)). If $\kappa \ge \overline{\kappa}$, then $\ln F_\rho(\lambda, \kappa; s) \le \ln D - \ln \kappa \le \ln D - \ln \overline{\kappa} = H$, and so $\min[H, \ln F_\rho(\lambda, \kappa; s)] = \ln F_\rho(\lambda, \kappa; s)$. Thus in order to complete the proof of the lemma in the case when λ has zero coordinates it remains to show that

$$\inf_{\kappa \in [\bar{\kappa}, 1],\ \rho \in [\rho', \rho'']} E \ln F_\rho(\lambda, \kappa; s) > 0 \tag{3.32}$$

for each $\bar{\kappa} \in (0, 1]$.

By virtue of (3.25) and (3.27), the function $\ln F_\rho(\lambda, \kappa; s)$ is continuous with respect to $(\rho, \kappa) \in [\rho', \rho''] \times [\bar{\kappa}, 1]$ for each s and uniformly bounded. Therefore the function $E \ln F_\rho(\lambda, \kappa; s)$ is continuous on the compact set $[\rho', \rho''] \times [\bar{\kappa}, 1]$ and hence it attains its minimum on this set. Thus, in order to establish (3.32) it is sufficient to prove that $E \ln F_\rho(\lambda, \kappa; s) > 0$ for each $\rho \in [0, 1)$ and $\kappa \in (0, 1]$.

According to (3.23), we have

$$\begin{aligned} & E \ln F_\rho(\lambda, \kappa; s) \\ = \quad & E \ln \sum_{k=1}^{K} [\rho\lambda_k + (1-\rho)R_k(s)] \frac{\lambda_k^*}{\lambda_k^*\kappa + \lambda_k(1-\kappa)} \\ & -E \ln \sum_{k=1}^{K} [\rho\lambda_k^* + (1-\rho)R_k(s)] \frac{\lambda_k}{\lambda_k^*\kappa + \lambda_k(1-\kappa)}. \end{aligned}$$

By applying Jensen's inequality twice, we find

$$E \ln \sum_{k=1}^{K} [\rho\lambda_k + (1-\rho)R_k(s)] \frac{\lambda_k^*}{\lambda_k^*\kappa + \lambda_k(1-\kappa)} \geq$$

$$\rho E \ln \sum_{k=1}^{K} \lambda_k \frac{\lambda_k^*}{\lambda_k^*\kappa + \lambda_k(1-\kappa)} + (1-\rho) E \ln \sum_{k=1}^{K} R_k(s) \frac{\lambda_k^*}{\lambda_k^*\kappa + \lambda_k(1-\kappa)} \geq$$

$$\rho \ln \sum_{k=1}^{K} \lambda_k \frac{\lambda_k^*}{\lambda_k^*\kappa + \lambda_k(1-\kappa)} + (1-\rho) \sum_{k=1}^{K} \lambda_k^* \ln \frac{\lambda_k^*}{\lambda_k^*\kappa + \lambda_k(1-\kappa)}, \qquad (3.33)$$

where the second inequality holds because $\sum_{k=1}^{K} R_k(s) = 1$ and $R_k(s) \geq 0$. Note that all the expression in the last chain of relations are well-defined and finite because $\kappa > 0$ and $\lambda_k^* > 0$. By applying Jensen's inequality again, we find

$$E \ln \sum_{k=1}^{K} [\rho\lambda_k^* + (1-\rho)R_k(s)] \frac{\lambda_k}{\lambda_k^*\kappa + \lambda_k(1-\kappa)} <$$

$$\ln \sum_{k=1}^{K} [\rho\lambda_k^* + (1-\rho) E R_k(s)] \frac{\lambda_k}{\lambda_k^*\kappa + \lambda_k(1-\kappa)} = \ln \sum_{k=1}^{K} \frac{\lambda_k^*\lambda_k}{\lambda_k^*\kappa + \lambda_k(1-\kappa)}. \quad (3.34)$$

(Here $-\infty \leq E \ln(\cdot) < +\infty$.) The inequality in (3.34) is strict because there is no constant γ such that

$$\sum_{k=1}^{K}[\rho\lambda_k^* + (1-\rho)R_k(s)]\frac{\lambda_k}{\lambda_k^*\kappa + \lambda_k(1-\kappa)} = \gamma\,\text{(a.s.)}. \tag{3.35}$$

Indeed, if (3.35) holds, then

$$\sum_{k=1}^{K}[\rho\lambda_k^* + (1-\rho)R_k(s)]\left[\frac{\lambda_k}{\lambda_k^*\kappa + \lambda_k(1-\kappa)} - \gamma\right] = 0\,\text{(a.s.)}.$$

Observe that at least one of the numbers $\gamma_k := \lambda_k[\lambda_k^*\kappa + \lambda_k(1-\kappa)]^{-1} - \gamma$ is not equal to zero. Indeed, otherwise $\lambda_k = \gamma[\lambda_k^*\kappa + \lambda_k(1-\kappa)]$ for all k. By summing up these equations over k, we have $\gamma = 1$, which yields $\lambda_k = \lambda_k^*\kappa + \lambda_k(1-\kappa)$, $\lambda_k^*\kappa = \lambda_k\kappa$, and $\lambda_k^* = \lambda_k$ (recall that $\kappa \neq 0$). This is a contradiction because $\lambda \neq \lambda^*$. Thus

$$\sum_{k=1}^{K}[\rho\lambda_k^* + (1-\rho)R_k(s)]\gamma_k = 0\,\text{(a.s.)}, \tag{3.36}$$

where $\gamma = (\gamma_1, ..., \gamma_K) \neq 0$. Consequently, $\sum_{k=1}^{K} R_k(s)\gamma_k = b\,\text{(a.s.)}$, where b is some constant. This constant is not zero because the functions $R_k(s)$ are linearly independent. By setting $\gamma_k' := \gamma_k/b$, we obtain that the non-zero vector $\gamma' = (\gamma_1', ..., \gamma_K')$ satisfies $\sum_{k=1}^{K} R_k(s)\gamma_k' = 1\,\text{(a.s.)}$, which yields $\sum_{k=1}^{K} R_k(s)(\gamma_k' - 1) = 0\,\text{(a.s.)}$. In view of the linear independence of $R_k(s)$, this implies $\gamma_1' = ... = \gamma_K' = 1$. Since $\gamma_k = b\gamma_k' = b$, we obtain that the left-hand side of (3.36) is equal to $b \neq 0$, which is a contradiction.

From (3.33) and (3.34) we get

$$E\ln F_\rho(\lambda,\kappa;s) > (1-\rho)\left[\sum_{k=1}^{K}\lambda_k^*\ln\frac{\lambda_k^*}{\lambda_k^*\kappa + (1-\kappa)\lambda_k} - \ln\sum_{k=1}^{K}\frac{\lambda_k\lambda_k^*}{\lambda_k^*\kappa + \lambda_k(1-\kappa)}\right]. \tag{3.37}$$

Denote the expression in the square brackets in (3.37) by $\Phi_\kappa(\lambda)$. It is proved in Evstigneev et al. (2002), p. 337-338, that if $\lambda > 0$, the following

$$\Phi_\kappa(\lambda) \geq 0 \text{ for each } \kappa \in [0,1] \tag{3.38}$$

holds. Therefore $\Phi_\kappa(\lambda(1-\varepsilon)+\varepsilon\lambda^*) > 0$ for each $\varepsilon > 0$. The function $\Phi_\kappa(\lambda)$ is finite and continuous on Δ^K (because $\lambda_k^* > 0$ and $\kappa > 0$). Consequently, $\Phi_\kappa(\lambda) = \lim_{\varepsilon\downarrow 0}\Phi_\kappa(\lambda(1-\varepsilon)+\varepsilon\lambda^*) \geq 0$. By using (3.37), we obtain that $E\ln F_\rho(\lambda,\kappa) > 0$ for all $\rho \in [0,1)$ and $\kappa \in (0,1]$. This completes the proof of the lemma in the case when the vector λ has zero coordinates.

Now assume that $\lambda_k > 0$ for each k. Then the function $\ln F_\rho(\lambda,\kappa;s)$ is well-defined, finite, continuous with respect to (ρ,κ) on the set $[\rho',\rho'']\times[0,1]$ (including $\kappa = 0$) and uniformly bounded. The lower bound for this function is $2\ln c$ (see (3.25)) and the upper bound is obtained from the inequalities

$$\ln F_\rho(\lambda,\kappa;s) \leq \ln\frac{\max_k \frac{\lambda_k^*}{\lambda_k^*\kappa+\lambda_k(1-\kappa)}}{\min_k \frac{\lambda_k}{\lambda_k^*\kappa+\lambda_k(1-\kappa)}} \leq \ln(\min_k \lambda_k)^{-2}.$$

To complete the proof it is sufficient to show that the infimum in (3.32) with $\bar\kappa = 0$ is strictly positive (then δ can be defined as this infimum and H as $2|\ln c| + 2|\ln\min\lambda_k|$). In view of the continuity of $E\ln F_\rho(\lambda,\kappa;s)$ this will be proved if we establish the inequality $E\ln F_\rho(\lambda,\kappa;s) > 0$ for each $\rho \in [0,1)$ and $\kappa \in [0,1]$. If $\kappa > 0$, this inequality is proved by exactly the same arguments as above—by deriving relations (3.33), (3.34), (3.37) and using (3.38). If $\kappa = 0$, we change the above arguments as follows: instead of strict, we establish non-strict inequalities in

(3.34) and show that the right-hand side of (3.37) is strictly positive, which follows from the relation

$$\Phi_0(\lambda) = \sum_{k=1}^{K} \lambda_k^* \ln \frac{\lambda_k^*}{\lambda_k} = \sum_{k=1}^{K} \lambda_k^* \ln \lambda_k^* - \sum_{k=1}^{K} \lambda_k^* \ln \lambda_k > 0.$$

Chapter 4 Growth-Optimal Investments and Asset Market Games

4.1 Introduction

The theory of growth-optimal investments or *capital growth theory* is a fascinating subject, having a rich and in a sense dramatic history. The central question in this field is: how to invest in order to achieve the highest (asymptotic) growth rate of wealth in the long run? The first publication on capital growth theory was that by Kelly (1956), who considered the case of Arrow securities (the payoff of security i is 1 if the "state of the world" is i and 0 otherwise), interpreted as a "horse race model". It was shown that the growth optimal investment strategy could be found by the maximization of the expected logarithm of the portfolio return. In the case of the horse race model, this has led to the famous Kelly portfolio rule—"betting your beliefs"— allocating wealth in the proportions equal to the probabilities of winning. Kelly arrived at his results from information theory, and his paper was entitled "A new interpretation of information rate". The history of Kelly's discovery is described in various papers and books, including popular ones (see, e.g., Poundstone 2005). This discovery has been developed and extended by various authors, in particular by Breiman (1961), Algoet and Cover (1988) and Hakansson and Ziemba (1995). The paper by Algoet and Cover (1988) contains the most advanced and general mathematical treatment of capital growth theory.

When speaking of those who contributed to capital growth theory, one must necessarily mention the name of Claude Shannon—the famous founder of the mathematical theory of information. Although he did not publish on investment-related issues, his ideas, expressed in his lectures on investment problems, strongly influenced his collaborators: Kelly, Breiman, Cover and others, whose publications initiated the strand of literature on growth optimal investments. For the history of these ideas and a related discussion see Cover (1998).

Cover's (1998) biographical note on Shannon contains interesting recollections about a discussion between Shannon and another famous scholar, Paul Samuelson. Cover writes:

> ... In the mid 1960s, Shannon gave a lecture on maximizing the growth rate of wealth and gave a geometric Wiener example.
>
> At about this time, Shannon and Samuelson (a Nobel Prize winner-to-be in economics) held a number of evening discussion meetings on information theory and economics. It is not clear what was said in these meetings, but Samuelson seems to have become set in his views. He published several papers arguing strongly against maximizing the expected logarithm as an acceptable investment criterion. (It happens that maximizing the expected logarithm is the prescription for the growth-rate optimal portfolio.)
>
> For example, Samuelson (1969) wrote: Our analysis enables us to dispel a fallacy that has been borrowed into portfolio theory from information theory of the Shannon type. Samuelson goes on to argue that growth rate optimal policies do not achieve maximum utility unless one has a logarithmic utility for money. Of course this is the case, but it does not deny the fact that log optimal wealth has an objective property: it has a better growth rate than that achieved by any other strategy. Since growth rate optimal policies achieve a demonstrably desirable goal, growth rate optimal portfolios

should only have a utility interpretation as an afterthought. In fact, Samuelson (1979) wrote a paper entitled "Why we should not make mean log of wealth big though years to act are long." This is a two page paper in words of one syllable that makes the point that maximizing the expected log of wealth is not appropriate. The growth optimal portfolio literature has been slow to develop. It is possible that Samuelson's eloquent admonitions had their effect.

In this discussion, Samuelson and those who followed his views later, presumed (implicitly or explicitly) that the problem of growth-optimal investments was equivalent to the problem of the maximization of logarithmic utility functionals. By and large this presumption was true in those models which were considered at the time of the above discussion—half a century ago. More recent studies have shown that this is not the case in more advanced and realistic models, e.g., those describing financial markets with frictions—transaction costs and trading constraints (Bahsoun et al. 2009). And of course this is not true in evolutionary finance models (with endogenous equilibrium prices), where survival is equivalent to the fastest growth of wealth and where the problem of the identification of survival strategies cannot be reduced to any single-agent optimization problem, in particular, to the maximization of logarithmic utilities.

In this chapter we revisit the classical capital growth theory with exogenous asset prices, viewing it from new angles and using achievements of evolutionary finance and game theory. This makes it possible to throw a new light on known results, obtain their new versions and generalizations, and moreover, consider absolutely new questions leading to new insights. Moreover, we hope that the novel treatment of classical capital growth theory might be helpful for developing methods of analysis which could find applications in fields of current research that emerged as its variants and generalizations.

4.2 Growth-optimal investments

4.2.1 Model description

In the model under consideration, we are given a stochastic process $s_1, s_2, ...$ with values in a measurable space S representing random factors influencing the economic system. The random element $s_t \in S$ represents the *state of the world* at date $t = 0, 1, ...$ There are K assets in the market. Each asset is characterized by its (gross) *return*

$$A_{t,k} = A_{t,k}(s^t) \geq 0 \; (t \geq 1)$$

over a time period $t-1, t$, depending on the sequence

$$s^t := (s_1, ..., s_t)$$

of states of the world up to time t. The functions $A_{t,k}(s^t)$, as well as all the other functions of s^t we consider in what follows, are assumed to be measurable. The return $A_{t,k}$ on asset k is expressed through the asset prices $p_{t,k} > 0$ and $p_{t-1,k} > 0$ (if they are given) by the formula

$$A_{t,k} = \frac{p_{t,k}}{p_{t-1,k}}. \tag{4.1}$$

Note that the return specified above is different from the one in finance. We are interested in gross return rather than the rate of return.

An investor (trader) acting in the market possesses an initial endowment $w_0 > 0$ at date 0. At each date $t-1$ ($t \geq 1$), the trader makes an investment *decision* specified by a vector of investment proportions

$$\lambda_{t-1} = (\lambda_{t-1,1}, ..., \lambda_{t-1,K}) \geq 0, \sum_{k=1}^{K} \lambda_{t-1,k} = 1, \tag{4.2}$$

according to which the investor's *wealth* w_{t-1} available at date $t-1$ is distributed across the assets $k = 1, 2, ..., K$. The set of vectors λ_{t-1} in the K-dimensional Euclidean space $\mathbb{R}^K$ satisfying (4.2) (the unit simplex) will be denoted by Δ^K. Given the decision λ_{t-1}, the investor wealth w_t at date t can be computed according to the formula

$$w_t = \langle A_t, \lambda_{t-1} \rangle w_{t-1}, \tag{4.3}$$

where

$$A_t = (A_{t,1}, ..., A_{t,K})$$

stands for the vector of asset returns and $\langle A_t, \lambda_{t-1} \rangle$ denotes the scalar product

$$\langle A_t, \lambda_{t-1} \rangle = \sum_{k=1}^{K} A_{t,k} \lambda_{t-1,k}.$$

This scalar product expresses the return on the portfolio constructed according to the investment proportions $\lambda_{t-1} = (\lambda_{t-1,1}, ..., \lambda_{t-1,K})$. The formula (4.3) comes from the following equations

$$x_{t-1,k} = \frac{w_{t-1} \lambda_{t-1,k}}{p_{t-1,k}},$$

and

$$w_t = \sum_{k=1}^{K} w_{t,k} = \sum_{k=1}^{K} p_{t,k} x_{t-1,k} = \sum_{k=1}^{K} p_{t,k} \frac{w_{t-1} \lambda_{t-1,k}}{p_{t-1,k}} =$$

$$\sum_{k=1}^{K} A_{t,k} w_{t-1} \lambda_{t-1,k} = w_{t-1} \sum_{k=1}^{K} A_{t,k} \lambda_{t-1,k} = \langle A_t, \lambda_{t-1} \rangle w_{t-1}.$$

An investor's *strategy* (*portfolio rule*) Λ is a rule prescribing what decision to make at each date $t \geq 0$ and in each random situation, characterized by the state of the world at date t and all the previous dates. Formally, Λ is defined by a sequence of measurable functions

$$\lambda_t(s^t), t \geq 0,$$

with values in Δ^K. (We write $\lambda_0(s^0)$ for a constant vector λ_0.) If the investor possessing the initial endowment $w_0 > 0$ at date 0 has chosen a strategy $\Lambda = (\lambda_t)_{t=0}^{\infty}$, then his/her wealth at each date $t \geq 1$ can be computed recursively by using formula (4.3).

We are primarily interested in those strategies which guarantee the (asymptotically) fastest growth of wealth.

Definition 4.1 We shall say that a portfolio rule $\Lambda^* = (\lambda_t^*)_{t=0}^{\infty}$ is *growth-optimal* (or *asymptotically optimal*) if for any other portfolio rule $\Lambda = (\lambda_t)_{t=0}^{\infty}$ there exists a random variable $C > 0$ such that

$$w_t \leq C w_t^* \text{(a.s.) for all } t, \tag{4.4}$$

where (w_t^*) and (w_t) are the wealth processes generated by the strategies Λ^*, Λ and any initial endowments $w_0^* > 0, w_0 > 0$, respectively.

Property (4.4) expresses the fact that the wealth of an investor using any strategy Λ cannot grow asymptotically faster than the wealth of an investor employing the strategy Λ^*. It is clear from (4.3) that (4.4) holds for any initial endowments $w_0^* > 0, w_0 > 0$ if it holds for some pair of strictly positive initial endowments w_0^*, w_0, for example $w_0^* = w_0 = 1$. Indeed. If $w_0^* = w_0 = 1$ and (4.3) holds, then we have

$$\langle A_t, \lambda_{t-1}\rangle\langle A_{t-1}, \lambda_{t-2}\rangle...\langle A_1, \lambda_0\rangle \leq C\langle A_t, \lambda_{t-1}^*\rangle\langle A_{t-1}, \lambda_{t-2}^*\rangle...\langle A_1, \lambda_0^*\rangle.$$

Put $w_0^*C'/w_0 = C$ ($w_0^* > 0$ and $w_0 > 0$), we get

$$\langle A_t, \lambda_{t-1}\rangle\langle A_{t-1}, \lambda_{t-2}\rangle...\langle A_1, \lambda_0\rangle \leq \frac{C'}{w_0} w_0^*\langle A_t, \lambda_{t-1}^*\rangle\langle A_{t-1}, \lambda_{t-2}^*\rangle...\langle A_1, \lambda_0^*\rangle.$$

Thus we get

$$w_0\langle A_t, \lambda_{t-1}\rangle\langle A_{t-1}, \lambda_{t-2}\rangle...\langle A_1, \lambda_0\rangle \leq C'\langle A_t, \lambda_{t-1}^*\rangle\langle A_{t-1}, \lambda_{t-2}^*\rangle...\langle A_1, \lambda_0^*\rangle w_0^*,$$

which make (4.4) holds for any initial endowments $w_0^* > 0, w_0 > 0$.

4.2.2 Log-optimal portfolio rules

We will provide a method for constructing growth optimal investment strategies based on certain optimization problems involving expected logarithms of portfolio returns. We shall assume that for any $t \geq 1$ and s^t

$$\sum_{k=1}^{K} A_{t,k}(s^t) > 0. \tag{4.5}$$

This condition means that at each date and in each random situation at least one asset yields strictly positive return. Under this condition, we can define the *normalized (relative) returns* on assets $k = 1, 2, ..., K$ by

$$R_{t,k}(s^t) = \frac{A_{t,k}(s^t)}{\sum_{m=1}^{K} A_{t,m}(s^t)}. \tag{4.6}$$

Definition 4.2. Consider a portfolio rule $\Lambda^* = (\lambda_t^*)_{t=0}^{\infty}$.We shall say that Λ^* is *log-optimal* if for any $t \geq 0$ and any measurable vector function $\lambda_t(s^t)$ with values in the unit simplex Δ^K the following inequality holds

$$E \ln\langle \lambda_t, R_{t+1}\rangle \leq E \ln\langle \lambda_t^*, R_{t+1}\rangle. \tag{4.7}$$

According to this definition, the vector of investment proportions $\lambda_t^*(\cdot)$ maximizes (for each t) the expected logarithm of the return $\langle \lambda_t, R_{t+1}\rangle$ on the portfolio defined by the vector of investment proportions λ_t.

Note that the expectations in (4.7) are well-defined and take values in $[-\infty, 0]$. Indeed, we have $\langle \lambda_t, R_{t+1}\rangle \leq 1$ because both vectors λ_t and R_{t+1} belong to Δ^K. Further, observe that for the vector $\lambda_t := (1/K, ..., 1/K)$ the expectation $E \ln\langle \lambda_t, R_{t+1}\rangle = \ln(1/K)$ is finite, which implies (see (4.7)) that $E \ln\langle \lambda_t^*, R_{t+1}\rangle > -\infty$. This, in turn, implies that with probability one the random variables $\langle \lambda_t^*, R_{t+1}\rangle$ and $\langle \lambda_t^*, A_{t+1}\rangle$ are strictly positive.

The following technical comment is in order. When analyzing properties of a log-optimal portfolio rule λ_t^*, it will be convenient to assume that the random variables $\langle \lambda_t^*(s^t), R_{t+1}(s^{t+1})\rangle$ and $\langle \lambda_t^*(s^t), A_{t+1}(s^{t+1})\rangle$ are strictly positive for *all* s^{t+1}, and not only for almost all s^{t+1}. This can be achieved by a suitable change of the given random vectors $A_{t+1}(s^{t+1})$ on the set of those s^{t+1} where $\langle \lambda_t^*(s^t), A_{t+1}(s^{t+1})\rangle = 0$ (e.g., by redefining $A_{t+1}(s^{t+1})$ as $(K^{-1}, ..., K^{-1})$ for such s^{t+1}). This change will not lead to a loss of generality because all the properties of log-optimal portfolio rules we establish are formulated in terms of assertions holding with probability one.

Remark 4.1. Algoet and Cover (1988) define the notion of a log-optimal portfolio rule λ_t^* in terms of the original, not normalized, asset returns A_t by the relation

$$E \ln \frac{\langle \lambda_t, A_{t+1} \rangle}{\langle \lambda_t^*, A_{t+1} \rangle} \leq 0 \tag{4.8}$$

holding for all $\lambda_t(s^t)$, implicitly assuming that $\langle \lambda_t^*, A_{t+1} \rangle > 0$ (otherwise the fraction in (4.8) does not make sense). The inequality (4.8) is assumed to hold for all λ_t for which the expectation involved is well-defined. This definition is equivalent to that we deal with in this paper. Indeed, since

$$\frac{\langle \lambda_t, A_{t+1} \rangle}{\langle \lambda_t^*, A_{t+1} \rangle} = \frac{\langle \lambda_t, R_{t+1} \rangle}{\langle \lambda_t^*, R_{t+1} \rangle}$$

under assumption (4.5), condition (4.8) is equivalent to

$$E \ln \frac{\langle \lambda_t, R_{t+1} \rangle}{\langle \lambda_t^*, R_{t+1} \rangle} \leq 0. \tag{4.9}$$

If λ_t^* is log-optimal in the sense of Definition 4.2, i.e., condition (4.7) holds, then (4.9) holds as well because (as we have noted) $E \ln \langle \lambda_t^*, R_{t+1} \rangle > -\infty$, and so

$$E \ln \langle \lambda_t, R_{t+1} \rangle - E \ln \langle \lambda_t^*, R_{t+1} \rangle = E \ln \frac{\langle \lambda_t, R_{t+1} \rangle}{\langle \lambda_t^*, R_{t+1} \rangle}.$$

Conversely, observe that for $\lambda_t := (1/K, ..., 1/K)$,

$$\frac{\langle \lambda_t, A_{t+1} \rangle}{\langle \lambda_t^*, A_{t+1} \rangle} = \frac{\langle \lambda_t, R_{t+1} \rangle}{\langle \lambda_t^*, R_{t+1} \rangle} = \frac{1/K}{\langle \lambda_t^*, R_{t+1} \rangle} \geq 1/K,$$

and so for this vector λ_t, the expectation in (4.9) is well-defined, (4.9) yields

$$0 \geq E \ln \frac{1/K}{\langle \lambda_t^*, R_{t+1} \rangle} = \ln 1/K - E \ln \langle \lambda_t^*, R_{t+1} \rangle,$$

and so

$$E \ln\langle \lambda_t^*, R_{t+1}\rangle \geq \ln 1/K > -\infty.$$

Thus the expectation $E \ln\langle \lambda_t^*, R_{t+1}\rangle$ is finite, and consequently, (4.9) implies (4.7).

Theorem 4.1 *A log-optimal portfolio rule exists.*

The result below (Theorem 4.2) provides two other, equivalent, definitions of a log-optimal portfolio rule. Let us write $E_t(\cdot) = E(\cdot|s^t)$ for the conditional expectation with respect to s^t. Let $\pi_t(s^t, dx)$ denote the *conditional distribution* of the random vector R_{t+1} given s^t. By the definition, $\pi_t(s^t, B)$ is a probability measure on Borel sets $B \subseteq \Delta^K$ for each s^t, a measurable function of s^t for each B, and

$$E_t f(s^t, R_{t+1}) = \int\limits_{\Delta^K} \pi_t(s^t, dx) f(s^t, x)$$

for any jointly measurable function $f(s^t, x)$ for which the expressions on both sides of the above formula are well-defined. The existence of conditional distributions is proved in Arkin and Evstigneev (1987), Appendix II, Theorem 1.

Put

$$\phi_t(s^t, a) := \int\limits_{\Delta^K} \pi_t(s^t, dx) \ln\langle a, x\rangle. \tag{4.10}$$

The jointly measurable function defined by (4.10) is the *regular conditional expectation*[29] of the random variable $\ln\langle a, R_{t+1}\rangle$ given s^t, characterized by the property

$$\phi_t(s^t, a(s^t)) = E_t \ln\langle a(s^t), R_{t+1}\rangle \text{ (a.s.)} \tag{4.11}$$

[29] See I.V. Evstigneev (1986), Regular conditional expectations of random variables depending on parameters.

holding for any measurable mapping $a(s^t)$ with values in Δ^K.

Theorem 4.2 *Let* $\Lambda^* = (\lambda_t^*(s^t))_{t=0}^{\infty}$ *be a portfolio rule. The following conditions are equivalent:*

(a) Λ^* *is log-optimal;*

(b) the inequality

$$E_t \ln\langle \lambda_t, R_{t+1}\rangle \le E_t \ln\langle \lambda_t^*, R_{t+1}\rangle \text{ (a.s.)} \tag{4.12}$$

holds for every function $\lambda_t(s^t)$ *with values in* Δ^K*;*

(c) with probability one, we have

$$\max_{a\in\Delta^K} \phi_t(s^t, a) = \phi_t(s^t, \lambda_t^*(s^t)). \tag{4.13}$$

Although a log-optimal portfolio rule need not be unique, its return $\langle \lambda_t^*, R_{t+1}\rangle$ is a uniquely defined random variable. The following theorem is valid.

Theorem 4.3 *If* $\lambda_t'(s^t)$ *and* $\lambda_t''(s^t)$ *are two vectors of investment proportions maximizing the function* $E \ln\langle \lambda_t, R_{t+1}\rangle$*, then* $\langle \lambda_t', R_{t+1}\rangle = \langle \lambda_t'', R_{t+1}\rangle$ *a.s..*

The following theorem provides a sufficient condition for the uniqueness of the log-optimal portfolio rule. Speaking of uniqueness, we do not distinguish between two random vectors if they coincide with probability one.

Theorem 4.4 *If the support of the conditional distribution* $\pi_t(s^t, dx)$ *of the random vector* R_{t+1} *given* s^t *is not contained in any (proper) hyperplane of the space* $\mathcal{R}^K$ *for almost all* s^t*, then a log-optimal portfolio rule is unique.*

The condition stated in the above theorem expresses the fact that the conditional distribution of R_{t+1} given s^t is non-degenerate. This assumption can be equivalently formulated as follows: if $E_t\langle\lambda'_t - \lambda_t, R_{t+1}\rangle^2 = 0$ a.s., then $\lambda'_t = \lambda_t$ a.s. Indeed, if

$$0 = E_t\langle\lambda'_t - \lambda_t, R_{t+1}\rangle^2 = \int \pi_t(s^t, dx)\langle\lambda'_t(s^t) - \lambda_t(s^t), x\rangle^2$$

and $\lambda'_t(s^t) \neq \lambda_t(s^t)$ with strictly positive probability, then for a set of histories s^t having strictly positive measure, the support of the distribution $\pi_t(s^t, dx)$ is contained in the proper hyperplane

$$\{x : \langle\lambda'_t(s^t) - \lambda_t(s^t), x\rangle = 0\}.$$

Later, we will need a stronger version of the above property (see (ND) below), the meaning of which will be explained in more detail.

4.2.3 Asymptotic optimality and log-optimality

The central result of this section is as follows.

Theorem 4.5 *A log-optimal portfolio rule is asymptotically optimal.*

With any strategy $(\lambda_t)_{t=0}^{\infty}$, we associate the sequence of portfolio returns $(\langle\lambda_t, R_{t+1}\rangle)_{t=0}^{\infty}$ which it generates. It can be shown that for any asymptotically optimal strategy, its sequence of returns $(\langle\lambda_t, R_{t+1}\rangle)_{t=0}^{\infty}$ is asymptotically similar to the sequence $(\langle\lambda^*_t, R_{t+1}\rangle)_{t=0}^{\infty}$ generated by the log-optimal portfolio rule $(\lambda^*_t)_{t=0}^{\infty}$. The following theorem holds.

Theorem 4.6 *If* $(\lambda_t)_{t=0}^{\infty}$ *is an asymptotically optimal strategy, then*

$$\sum_{t=0}^{\infty}(\langle\lambda^*_t, R_{t+1}\rangle - \langle\lambda_t, R_{t+1}\rangle)^2 < \infty \text{ (a.s.)}. \tag{4.14}$$

This result implies that $\langle \lambda_t^*, R_{t+1} \rangle - \langle \lambda_t, R_{t+1} \rangle \to 0$, i.e. the random variables $\langle \lambda_t^*, R_{t+1} \rangle$ and $\langle \lambda_t, R_{t+1} \rangle$ converge to each other a.s. as $t \to \infty$. Moreover, the rate of this convergence is fast enough, so that the sum in (4.14) is a.s. finite.

As we have seen, the log-optimal portfolio rule is unique when the conditional distribution of the asset returns is non-degenerate. Under a somewhat stronger non-degeneracy condition, we can show that any asymptotically optimal strategy is similar in an asymptotic sense to the log-optimal one, $\Lambda^* = (\lambda_t^*(s^t))_{t=0}^{\infty}$.

Theorem 4.7 *Let the following condition hold:*

(ND) For any two vectors of investment proportions $\lambda_t(s^t)$ and $\lambda_t'(s^t)$,

$$||\lambda_t - \lambda_t'||^2 \leq BE_t \langle \lambda_t - \lambda_t', R_{t+1} \rangle^2 \text{ (a.s.)}, \tag{4.15}$$

where $B > 0$ is a non-random constant. Then for any asymptotically optimal strategy $\Lambda = (\lambda_t(s^t))_{t=0}^{\infty}$, we have

$$\sum_{t=0}^{\infty} ||\lambda_t^* - \lambda_t||^2 < \infty \text{ (a.s.)} \tag{4.16}$$

Here, we denote by $|| \cdot ||$ the Euclidean norm in $\mathcal{R}^K$. By virtue of Theorem 4.7, any asymptotically optimal strategy (λ_t) gets closer to the log-optimal one (λ_t^*) a.s., and moreover, the distance between λ_t^* and λ_t tends to zero sufficiently fast, so that the series in (4.16) converges. Theorem 4.7 is a direct analogue of the "turnpike theorem" (Theorem 2.2) obtained in Chapter 2.

Some comments regarding condition (ND) are in order. Suppose that the returns $\langle \lambda_t', R_{t+1} \rangle$ and $\langle \lambda_t, R_{t+1} \rangle$ corresponding to two portfolios $\lambda_t'(s^t)$ and $\lambda_t(s^t)$ coincide for $\pi_t(s^t, dx)$-almost all values of R_{t+1} and almost all s^t (recall that $\pi_t(s^t, dx)$ is the conditional distribution of R_{t+1} given s^t). Then $E_t\langle\lambda_t' - \lambda_t, R_{t+1}\rangle^2 = 0$, and so by virtue of (4.15), λ_t and λ_t' coincide a.s.. This consequence of assumption (ND) was used in Theorem 4.4 (see the comments after the statement of that theorem). Assumption (ND) represents a stronger requirement: it states that if the returns $\langle\lambda_t', R_{t+1}\rangle$ and $\langle\lambda_t, R_{t+1}\rangle$ are close to each other in the sense of the conditional L^2 norm, then the corresponding portfolios λ_t' and λ_t must be close to each other for almost all s^t. It has to be emphasized that the property described is uniform with respect to time t and history s^t, since the constant B in (4.15) does not depend on t and s^t.

4.3 Growth-optimal investments: proofs of the results

Let us begin with Theorem 4.2

Proof of Theorem 4.2 If λ_t^* satisfies (c), then

$$\phi_t(s^t, \lambda_t(s^t)) \leq \phi_t(s^t, \lambda_t^*(s^t)) \text{ (a.s.)}$$

for every measurable function $\lambda_t(s^t)$ with values in Δ^K. By virtue of (4.11), we have $\phi_t(s^t, \lambda_t(s^t)) = E_t \ln\langle\lambda_t(s^t), R_{t+1}\rangle$ (a.s.) and $\phi_t(s^t, \lambda_t^*(s^t)) = E_t \ln\langle\lambda_t^*(s^t), R_{t+1}\rangle$ (a.s.), which proves (4.12). By taking the expectations of both sides of (4.12), we obtain (a). Thus we have proved the implications (c)⇒(b)⇒(a).

Before proving the remaining implication (a)⇒(c), let us show that a measurable vector function $\lambda_t^*(s^t)$ satisfying (4.13) for almost all s^t exists. For each fixed

s^t, if $\Delta^K \ni a_k \to a$, then

$$\limsup \int \pi_t(s^t, dx) \ln\langle a_k, x\rangle \leq \int \pi_t(s^t, dx) \ln\langle a, x\rangle$$

by Fatou's lemma (recall that $\ln\langle a_k, x\rangle \leq 0$), and so the function $\phi_t(s^t, a)$ defined by (4.10) is upper semicontinuous. Consequently, the set $A_t(s^t)$ of those points $a \in \Delta^K$ where it attains its maximum is not empty. Since $\phi_t(s^t, a)$ is jointly measurable, we can select for each s^t a vector $\lambda_t^*(s^t)$ in Δ^K such that $\lambda_t^*(s^t) \in A_t(s^t)$ for almost all s^t and the function $\lambda_t^*(s^t)$ is measurable. This fact is a consequence of Aumann's measurable selection theorem (see, e.g., Arkin and Evstigneev 1987, Appendix I, Corollary 3). Thus $\lambda_t^*(s^t)$ is the vector of investment proportions satisfying (4.13) for almost all s^t.

Consider another (possibly distinct from $\lambda_t^*(s^t)$) vector of log-optimal investment proportions $\hat{\lambda}_t(s^t)$. Then

$$E \ln\langle \hat{\lambda}_t, R_{t+1}\rangle = E \ln\langle \lambda_t^*, R_{t+1}\rangle,$$

and so

$$EE_t \ln\langle \hat{\lambda}_t, R_{t+1}\rangle = EE_t \ln\langle \lambda_t^*, R_{t+1}\rangle,$$

which implies

$$E\phi_t(s^t, \hat{\lambda}_t(s^t)) = E\phi_t(s^t, \lambda_t^*(s^t)).$$

The expectations of the non-positive random variables $\phi_t(s^t, \lambda_t^*(s^t))$ and $\phi_t(s^t, \hat{\lambda}_t(s^t))$ are finite, consequently, we get

$$E\left[\phi_t(s^t, \lambda_t^*(s^t)) - \phi_t(s^t, \hat{\lambda}_t(s^t))\right] = 0.$$

But the random variable $\phi_t(s^t, \lambda_t^*(s^t)) - \phi_t(s^t, \hat{\lambda}_t(s^t))$ is non-negative by the definition of $\lambda_t^*(s^t)$. Thus

$$\phi_t(s^t, \hat{\lambda}_t(s^t)) = \phi_t(s^t, \lambda_t^*(s^t)) = \max_{a \in \Delta^K} \phi_t(s^t, a) \text{ (a.s.)},$$

which proves (c). □

Proof of Theorem 4.1 In the course of the proof of Theorem 4.2, we constructed a measurable vector function $\lambda_t^*(s^t)$ for which assertion (c) holds. The implication (c)⇒(a) shows that the vector of investment proportions $\lambda_t^*(s^t)$ is log-optimal. This proves Theorem 4.1. □

Proof of Theorem 4.3 Since both portfolio rules (λ_t') and (λ_t'') are log-optimal, we have

$$E \ln\langle \lambda_t', R_{t+1} \rangle = E \ln\langle \lambda_t'', R_{t+1} \rangle > -\infty \text{ (a.s.)}.$$

Define

$$\lambda_t := \frac{\lambda_t' + \lambda_t''}{2}.$$

By virtue of concavity of the function $\ln x$, we obtain

$$\ln\langle \lambda_t, R_{t+1} \rangle \geq \frac{\ln\langle \lambda_t', R_{t+1} \rangle + \ln\langle \lambda_t'', R_{t+1} \rangle}{2}, \tag{4.17}$$

with strict inequality when

$$\langle \lambda_t', R_{t+1} \rangle \neq \langle \lambda_t'', R_{t+1} \rangle. \tag{4.18}$$

Suppose (4.18) holds with strictly positive probability. Then we have a strict inequality in (4.17) with strictly positive probability. Therefore

$$E \ln\langle \lambda_t, R_{t+1} \rangle > E \frac{\ln\langle \lambda_t', R_{t+1} \rangle + \ln\langle \lambda_t'', R_{t+1} \rangle}{2} = E \ln\langle \lambda_t'', R_{t+1} \rangle$$

(because all the expectations involved are finite). Thus the portfolio rule λ_t yields a greater expected logarithmic return than λ_t'', which is a contradiction. The contradiction obtained proves that $\langle \lambda_t', R_{t+1} \rangle = \langle \lambda_t'', R_{t+1} \rangle$ (a.s.). $\square$

Proof of Theorem 4.4 Consider two log-optimal portfolio rules (λ_t') and (λ_t''). As before, let $\pi_t(s^t, dx)$ be the conditional distribution of the random vector R_{t+1} given s^t. As we have proved in Theorem 4.3, $\langle \lambda_t', R_{t+1} \rangle = \langle \lambda_t'', R_{t+1} \rangle$ (a.s.). Therefore for almost all s^t the equality $\langle \lambda_t'(s^t), x \rangle = \langle \lambda_t''(s^t), x \rangle$ holds for almost all $x \in \mathcal{R}^K$ with respect to the conditional distribution $\pi_t(s^t, dx)$[30]. Consequently, for these s^t the support of the probability measure $\pi_t(s^t, \cdot)$ is contained in the set

$$\{x \in \mathcal{R}^K : \langle \lambda_t'(s^t) - \lambda_t''(s^t), x \rangle = 0\}.$$

If $P\{\lambda_t'(s^t) \neq \lambda_t''(s^t)\} > 0$, then with strictly positive probability the support of $\pi_t(s^t, \cdot)$ is contained in a proper hyperplane in $\mathcal{R}^K$.Thus if with probability one the support of $\pi_t(s^t, \cdot)$ is not contained in a proper hyperplane in $\mathcal{R}^K$, then the vector of log-optimal proportions is essentially unique. $\square$

Proof of Theorem 4.5. Let $\Lambda^* = (\lambda_t^*(s^t))_{t=0}^{\infty}$ be a log-optimal portfolio rule and $\Lambda = (\lambda_t(s^t))_{t=0}^{\infty}$ any other portfolio rule. Denote by (w_t^*) and (w_t) the wealth processes generated by Λ^*, Λ and some initial endowments $w_0^* > 0$ and $w_0 > 0$. We first note that $w_t^* > 0$ (a.s.), which follows from formula (4.3) because

[30] Indeed, the equality $|\langle \lambda_t' - \lambda_t'', R_{t+1} \rangle| = 0$ (a.s.) implies

$$0 = E_t |\langle \lambda_t' - \lambda_t'', R_{t+1} \rangle| =$$
$$\int \pi_t(s^t, dx) |\langle \lambda_t'(s^t) - \lambda_t''(s^t), x \rangle|$$

for almost all s^t, which yields the assertion.

$\langle R_t, \lambda^*_{t-1}\rangle > 0$ and hence $\langle A_t, \lambda^*_{t-1}\rangle > 0$ (recall that $\langle \lambda^*_t(s^t), R_{t+1}(s^{t+1})\rangle$ and $\langle \lambda^*_t(s^t), A_{t+1}(s^{t+1})\rangle$ are assumed to be strictly positive for *all* s^{t+1}). Further, we define $r^*_t := w^*_t/(w^*_t + w_t)$ and observe that r^*_t evolves according to the random dynamical system

$$r^*_t = \frac{\langle R_t, \lambda^*_{t-1}\rangle r^*_{t-1}}{\langle R_t, \lambda_{t-1}\rangle(1 - r^*_{t-1}) + \langle R_t, \lambda^*_{t-1}\rangle r^*_{t-1}}. \tag{4.19}$$

Indeed, by using (4.3), we write

$$r^*_t = \frac{w^*_t}{w^*_t + w} = \frac{\langle A_t, \lambda^*_{t-1}\rangle w^*_{t-1}}{\langle A_t, \lambda^*_{t-1}\rangle w^*_{t-1} + \langle A_t, \lambda_{t-1}\rangle w_{t-1}} =$$

$$\frac{\langle R_t, \lambda^*_{t-1}\rangle w^*_{t-1}}{\langle R_t, \lambda^*_{t-1}\rangle w^*_{t-1} + \langle R_t, \lambda_{t-1}\rangle w_{t-1}} = \frac{\langle R_t, \lambda^*_{t-1}\rangle r^*_{t-1}}{\langle R_t, \lambda^*_{t-1}\rangle r^*_{t-1} + \langle R_t, \lambda_{t-1}\rangle (1 - r^*_{t-1})}.$$

The initial state r^*_0 of the random dynamical system (4.19) is $w^*_0/(w^*_0 + w_0) > 0$.

By using formula (4.19), we get

$$\ln r^*_t = \ln\langle R_t, \lambda^*_{t-1}\rangle + \ln r^*_{t-1} -$$

$$\ln[\langle R_t, \lambda_{t-1}\rangle(1 - r^*_{t-1}) + \langle R_t, \lambda^*_{t-1}\rangle r^*_{t-1}]$$

(where all the logarithms are finite). From this we obtain

$$E_{t-1} \ln r^*_t = \ln r^*_{t-1} +$$

$$E_{t-1}\{\ln\langle R_t, \lambda^*_{t-1}\rangle - \ln[\langle R_t, \lambda_{t-1}\rangle(1 - r^*_{t-1}) + \langle R_t, \lambda^*_{t-1}\rangle r^*_{t-1}]\} =$$

$$E_{t-1} \ln\langle R_t, \lambda^*_{t-1}\rangle - E_{t-1} \ln[\langle R_t, \lambda_{t-1}\rangle(1 - r^*_{t-1}) + \langle R_t, \lambda^*_{t-1}\rangle r^*_{t-1}\rangle], \tag{4.20}$$

where the last equality is valid because $E_{t-1} \ln\langle R_t, \lambda^*_{t-1}\rangle > -\infty$. By using (4.20) and (4.12), we find

$$E_{t-1} \ln r^*_t \geq \ln r^*_{t-1}. \tag{4.21}$$

This inequality by induction gives $E \ln r_t^* \geq \ln r_0^* > -\infty$ because r_0^* is a strictly positive non-random number. Since $\ln r_t^* \leq 0$, we conclude that $E|\ln r_t^*| < \infty$. Consequently, the random sequence $\ln r_t^*$ is a non-positive submartingale, and hence it has a.s. a finite limit

$$l := \lim \ln r_t^*.$$

Therefore

$$r_t^* \to e^l > 0 \text{ (a.s.)},$$

and so

$$c := \inf r_t^* > 0 \text{ (a.s.)}.$$

From this we find $(r_t^*)^{-1} \leq 1 + (1/c)$, which yields

$$1 + (1/c) \geq (w_t^* + w_t)/w_t^* = 1 + w_t/w_t^*,$$

Finally, we get $w_t \leq (1/c)w_t^*$, which completes the proof. □

Proof of Theorem 4.6 Let $\Lambda = (\lambda_t)$ be an asymptotically optimal strategy and $\Lambda^* = (\lambda_t^*)$ the log-optimal one. Let w_t^*, w_t and r_t^* denote the same random variables as in the proof of Theorem 4.5. In the course of the proof of Theorem 4.5 we have shown that the sequence $\ln r_t^*$ is a non-positive submartingale satisfying (4.20). By virtue of Lemma 4.1 (see below), the series $\sum E(E_{t-1} \ln r_t^* - \ln r_{t-1}^*)$ converges a.s.. Consequently, we get

$$\sum_{t=1}^{\infty} E(\ln\langle R_t, \lambda_{t-1}^*\rangle - \ln\langle R_t, \mu_{t-1}\rangle) < \infty, \text{(a.s.)} \qquad (4.22)$$

where

$$\mu_{t-1} := \lambda^*_{t-1} r^*_{t-1} + \lambda_{t-1}(1 - r^*_{t-1}).$$

The following relations are valid

$$\frac{1}{2}E(\ln\langle R_t, \lambda^*_{t-1}\rangle - \ln\langle R_t, \mu_{t-1}\rangle) =$$
$$E\ln\langle R_t, \lambda^*_{t-1}\rangle - \frac{1}{2}E(\ln\langle R_t, \lambda^*_{t-1}\rangle + \ln\langle R_t, \mu_{t-1}\rangle) \geq$$
$$E\ln\left\langle R_t, \frac{\lambda^*_{t-1} + \mu_{t-1}}{2}\right\rangle - \frac{1}{2}E(\ln\langle R_t, \lambda^*_{t-1}\rangle + \ln\langle R_t, \mu_{t-1}\rangle) =$$

$$E\left(\ln\frac{\alpha+\beta}{2} - \frac{\ln\alpha + \ln\beta}{2}\right), \tag{4.23}$$

where $\alpha := \langle R_t, \lambda^*_{t-1}\rangle$ and $\beta := \langle R_t, \mu_{t-1}\rangle$. In this chain of relations, the first inequality holds because λ^*_{t-1} is the vector of log-optimal proportions, and so $E\ln\langle R_t, \lambda^*_{t-1}\rangle \geq E\ln\langle R_t, (\lambda^*_{t-1} + \mu_{t-1})/2\rangle$.

We are going to use the elementary inequality

$$\ln\frac{x+y}{2} - \frac{\ln x + \ln y}{2} \geq \frac{1}{4}(x-y)^2 \tag{4.24}$$

holding for $x, y \in (0, 1]$. To prove it we write

$$\ln\frac{2\sqrt{xy}}{x+y} \leq \frac{2\sqrt{xy}}{x+y} - 1$$

and observe that the left hand side of (4.24) can be estimated as follows

$$\ln\frac{x+y}{2} - \frac{\ln x + \ln y}{2} = -\ln\frac{2\sqrt{xy}}{x+y} \geq$$
$$1 - \frac{2\sqrt{xy}}{x+y} = \frac{x+y-2\sqrt{xy}}{x+y} = \frac{\left(\sqrt{x}-\sqrt{y}\right)^2}{x+y} \geq$$
$$\frac{1}{2}\left(\sqrt{x}-\sqrt{y}\right)^2 \geq \frac{1}{4}(x-y)^2,$$

which yields (4.24). In this chain of relations, the last inequality holds because

$\left(\sqrt{x}-\sqrt{y}\right)^2 \geq \frac{1}{2}(x-y)^2$. Indeed, we have the following relations

$$2 \geq \left(\sqrt{x}+\sqrt{y}\right)^2,$$

$$2\left(\sqrt{x}-\sqrt{y}\right)^2 \geq \left(\sqrt{x}-\sqrt{y}\right)^2\left(\sqrt{x}+\sqrt{y}\right)^2,$$

and

$$\left(\sqrt{x}-\sqrt{y}\right)^2 \geq \frac{1}{2}(x-y)^2,$$

where the first inequality holds because $0 \leq x, y \leq 1$.

By using relations (4.23) and inequality (4.24) with $x = \alpha$ and $y = \beta$, we obtain

$$\mu_{t-1} := \lambda^*_{t-1} r^*_{t-1} + \lambda_{t-1}(1 - r^*_{t-1}),$$

$$2E(\ln\langle R_t, \lambda^*_{t-1}\rangle - \ln\langle R_t, \mu_{t-1}\rangle) \geq E(\langle R_t, \lambda^*_{t-1}\rangle - \langle R_t, \mu_{t-1}\rangle)^2 =$$

$$E[\langle R_t, \lambda^*_{t-1} - \lambda^*_{t-1} r^*_{t-1} - \lambda_{t-1}(1 - r^*_{t-1})\rangle]^2 =$$

$$E[\langle R_t, \lambda^*_{t-1} - \lambda_{t-1}\rangle(1 - r^*_{t-1})]^2. \tag{4.25}$$

By combining (4.25) and (4.22), we find

$$\sum_{t=1}^{\infty} E[\langle R_t, \lambda^*_{t-1} - \lambda_{t-1}\rangle(1 - r^*_{t-1})]^2 < \infty, \text{(a.s.)} \tag{4.26}$$

and so

$$\sum_{t=1}^{\infty} [\langle R_t, \lambda^*_{t-1} - \lambda_{t-1}\rangle(1 - r^*_{t-1})]^2 < \infty \text{ (a.s.)}. \tag{4.27}$$

(If the sum of a series of expectations of non-negative random variables is finite, then the series of these random variables converges a.s..) This implies

$$\sum_{t=1}^{\infty} \langle R_t, \lambda^*_{t-1} - \lambda_{t-1}\rangle^2 < \infty \text{ (a.s.)}, \tag{4.28}$$

since (λ_t) is an asymptotically optimal strategy. Indeed, the asymptotic optimality of (λ_t) implies that $w_t^* \leq Cw_t$, where $C > 0$ as a random constant, and from this we obtain

$$1 - r_{t-1}^* = 1 - \frac{w_t^*}{w_t^* + w_t} = \frac{w_t}{w_t^* + w_t} \geq \frac{w_t}{Cw_t + w_t} = \frac{1}{C+1}. \tag{4.29}$$

Thus $1 - r_{t-1}^*$ is bounded away from zero a.s. by a strictly positive random constant, and so (4.28) follows from (4.27). This proves (4.14).□

Proof of Theorem 4.7 By using the fact of convergence of the series (4.22), we get

$$\sum_{t=1}^{\infty} E\{E_{t-1}[\langle R_t, \lambda_{t-1}^* - \lambda_{t-1}\rangle(1 - r_{t-1}^*)]^2\} < \infty, \text{(a.s.)}$$

and so

$$\sum_{t=1}^{\infty} E_{t-1}[\langle R_t, \lambda_{t-1}^* - \lambda_{t-1}\rangle(1 - r_{t-1}^*)]^2 < \infty \text{ (a.s.)}.$$

Consequently,

$$\sum_{t=1}^{\infty}(1 - r_{t-1}^*)^2 E_{t-1}\langle R_t, \lambda_{t-1}^* - \lambda_{t-1}\rangle^2 < \infty \text{ (a.s.)},$$

where the random variables $1 - r_{t-1}^*$ are bounded away from zero by a strictly positive constant (see (4.29)), which implies that

$$\sum_{t=1}^{\infty} E_{t-1}\langle R_t, \lambda_{t-1}^* - \lambda_{t-1}\rangle^2 < \infty \text{ (a.s.)}. \tag{4.30}$$

By combining (4.15) and (4.30), we obtain (4.16). □

In the course of the proof of Theorem 4.6, we used the following lemma.

Lemma 4.1 *Let ξ_t be a submartingale such that $\sup_t E\xi_t < \infty$. Then the sum $\sum_{t=0}^{\infty} E(E_t\xi_{t+1} - \xi_t)$ is finite and the series of non-negative random variables*

$\sum_{t=0}^{\infty}(E_t\xi_{t+1} - \xi_t)$ *converges a.s.*

To conclude this section we provide an example where condition (ND) is satisfied.

Proposition 4.1 *Let the process s_t be formed by a sequence of independent identically distributed random elements of the space S distributed according to some probability measure $Q(ds)$. If the vectors of asset returns $R_t(s^t)$ depend only on the current state s_t and do not explicitly depend on t,*

$$R_t(s^t) = R(s_t) = (R_1(s_t), ..., R_K(s_t)),$$

and if the functions $R_1(s), ..., R_K(s)$ are linearly independent $\bmod Q$ then, condition (ND) holds.

If $R_1(s), ..., R_K(s)$ are measurable functions on a space S with measure Q, they are said to be linearly independent $\bmod Q$ if the relation

$$b_1 R_1(s) + ... + b_K R_K(s) = 0$$

holding for Q-almost all s for some numbers $b_1, ..., b_K$ implies that $b_1 = ... = b_K = 0$.

The proofs of Lemma 4.1 and Proposition 4.1 are relegated to the Appendix 4.6.

4.4 Asset market games

4.4.1 Investment strategies: a game-theoretic approach

In this section, we will consider game-theoretic models of an asset market, for the analysis of which we shall apply the previous results on growth-optimal investments. Consider a market in which K assets $k = 1, 2, ..., K$ are traded. We will use the same framework as the one introduced in Section 4.2, in which a random process $s_t \in S$, $t = 0, 1, ...$, of "states of the world" is given, and the asset returns $A_{t,k} = A_{t,k}(s^t) \geq 0$ $(t \geq 1)$ depend on the history $s^t := (s_1, ..., s_t)$ of this process up to time t.

Suppose there are $N \geq 2$ investors (traders) acting in the market. Every investor $i = 1, 2, ..., N$ possesses an initial endowment $w_0^i > 0$ at date 0. At each date $t - 1$ ($t \geq 1$), every trader i makes an investment decision specified by a vector of investment proportions $\lambda_{t-1}^i = (\lambda_{t-1,1}^i, ..., \lambda_{t-1,K}^i) \in \Delta^K$. Given the decision λ_{t-1}^i, investor i's wealth w_t^i at date t can be expressed through his/her wealth w_{t-1}^i at date $t - 1$ and the vector A_t of asset returns by formula (4.3).

In this section we will use a more general notion of an investment strategy than that considered before.

Definition 4.3 An investor i's *strategy* (*portfolio rule*) Λ^i is defined by a vector $\Lambda_0^i \in \Delta^K$ and a sequence of measurable functions

$$\Lambda_t^i(s^t, \lambda^{(t-1)}),\ t \geq 1, \tag{4.31}$$

with values in Δ^K, where

$$\lambda^{(t-1)} := (\lambda_l^j)_{l=0,1,...,t-1;\ j=1,...,N}$$

is the set of the decisions made by all the investors $j = 1, ..., N$ at all the dates $l = 0, 1, ..., t - 1$.

All the functions under consideration are assumed to be jointly measurable with respect to their arguments (with the Borel measurable structure on Δ^K). According to the above definition, a strategy is a rule prescribing what decision to make at each date $t \geq 0$ and in each random situation s^t, depending on the history of all the previous actions of all the investors. This is the most general notion of a (pure) strategy in the framework of dynamic stochastic games.

In the class of general strategies, we distinguish those for which the functions $\Lambda_t^i(\cdot)$ do not depend on $\lambda^{(t-1)}$:

$$\Lambda_t^i(s^t, \lambda^{(t-1)}) = \Lambda_t^i(s^t);$$

they will be called *basic*. Such strategies play an important role in this context: the solutions of the games we shall deal with will belong to this class. Note that up to now (in Sections 4.2-4.4) we considered only this class of strategies; here we consider a new, more general class, and in order to distinguish it from the old one, we introduce the term "basic".

Suppose all the investors have chosen their strategies Λ^i, $i = 1, ..., N$. Then the strategy profile $(\Lambda^1, ..., \Lambda^N)$ determines recursively by the formulas

$$\lambda_0^i := \Lambda_0^i, \tag{4.32}$$

$$\lambda_t^i(s^t) := \Lambda_t^i(s^t, \lambda^{(t-1)}(s^{t-1})) \tag{4.33}$$

the vectors $\lambda_t^i(s^t)$ of investment proportions of all the traders at each date $t \geq 0$.

In turn, formulas

$$w_t^i = \langle A_t, \lambda_{t-1}^i \rangle w_{t-1}^i, \tag{4.34}$$

together with the initial endowments $w_0^i,\ i = 1, ..., N$ define step by step from $t-1$ to t investor i's wealth $w_t^i(s^t)$ for each date $t = 0, 1, ...$ and each random situation s^t. The random dyanmical system thus defined describes the market dynamics modelled in terms of the stochastic game under consideration. The wealth process $w_t^i(s^t)$, $t = 0, 1, ...$, characterizes the *outcome of the game* for player i. In general, investment decisions of each trader might depend on the previous actions of all the others, and so the wealth process (w_t^i) of each investor i depends on the whole strategy profile $(\Lambda^1, ..., \Lambda^N)$, and not only on his/her strategy Λ^i. But of course, if all the players use only basic strategies, their wealth processes are determined only by their own decisions.

To complete the description of a game-theoretic model we have to define a *solution concept*, that would specify the goals of the players/investors and criteria which would allow one to judge whether (or to what extent) these goals are achieved. We will consider in this work several solution concepts, centering around the idea of growth-optimal investments. The first one is described in the following definition.

Definition 4.4 Let us say that a strategy Λ^1 is *competitive* if it possesses the following property. Suppose some investor, say investor 1, uses this strategy, and the others use some other strategies $\Lambda^2, ..., \Lambda^N$. Then for each $i = 2, ..., N$, there exists a random constant $C^i > 0$ such that

$$w_t^i \leq C^i w_t^1 \text{ a.s. for all } t. \tag{4.35}$$

Relation (4.35) means that with probability one, the wealth w_t^i of any investor i cannot grow faster than the wealth of investor 1 using the strategy Λ^1. In this sense, the portfolio rule Λ^1 guarantees that the trader employing it cannot be beaten in competition with the rivals, irrespective of what strategies the rivals use. If (4.35) holds for some $C^i > 0$, we shall write $(w_t^i) \preceq (w_t^1)$, which defines a preference relation on the set of sequences of non-negative random variables.

Our results will be based on the following simple observation.

Proposition 4.2 *A basic strategy is competitive if and only if it is asymptotically optimal.*

By using Proposition 4.2 and the results in subsection 4.2, we obtain the following.

Theorem 4.8 *A log-optimal strategy is competitive. A basic competitive strategy is asymptotically unique in the sense of Theorems 4.6 and 4.7.*

Remark 4.2 We emphasize that the asymptotic uniqueness results contained in the above theorem are established only for the class of basic strategies. There are examples showing that these results cannot be extended to the class of general, game-theoretic strategies in a straightforward way. The problem of asymptotic uniqueness (or more generally, characterization) of general portfolio rules, as defined in definition 4.3, remains open.

Consider a version of the above game in which the outcome for each player i is specified not in terms of i's (absolute) wealth w_t^i, but in terms of his/her relative wealth, or *market share,* defined by

$$r_t^i := \frac{w_t^i}{w_t^1 + w_t^2 + ... + w_t^N},\ i = 1, ..., N. \tag{4.36}$$

From now on, we will impose the following assumption.

Assumption 4.1 In what follows, we will consider only those strategies (portfolio rules) which generate vectors of investment proportions satisfying

$$\langle R_t(s^t), \lambda_{t-1}^i(s^{t-1})\rangle > 0 \tag{4.37}$$

for all t, i and almost all s^t.

This assumption will guarantee that with probability one the wealth of each investor at each moment of time is strictly positive (bankruptcy is excluded). As we noted, a log-optimal portfolio rule satisfies this condition. Under the above assumption, the market shares r_t^i are well-defined (a.s.) since the denominator of the fraction in (4.36) is strictly positive with probability one. To define r_t^i on the remaining set of measure zero, where $w_t^1 + w_t^2 + ... + w_t^N = 0$, we put $r_t^i = 1/N$.

Definition 4.5 A portfolio rule Λ^1 of investor 1 is called a *survival strategy* if for any strategies $\Lambda^2, ..., \Lambda^N$ of investors $i = 2, ..., N$, we have

$$\inf_t r_t^1 > 0 \text{ (a.s.)}. \tag{4.38}$$

According to this definition, the trader using the portfolio rule Λ^1 *survives* almost surely, i.e., keeps with probability one a strictly positive and bounded away from zero share of market wealth over an infinite time horizon, irrespective of the strategies of all the other traders.

Remark 4.3 It is easily seen that the definitions of a survival strategy and a competitive strategy are equivalent. Indeed, suppose a strategy of investor 1 is competitive. Then for any strategy profile in which investor 1 uses this strategy, we have $w_t^i \leq C^i w_t^1$ (a.s.) for some $C^i > 0$, and so

$$Nw_t^1 \geq \sum_{i=1}^{N} (C^i)^{-1} w_t^i, \text{(a.s.)}$$

where $C^1 := 1$. Consequently,

$$w_t^1 \geq N^{-1} \sum_{i=1}^{N} c^i w_t^i \geq N^{-1} \min c^i \sum_{i=1}^{N} w_t^i \text{ (a.s.).}$$

where $c^i := (C^i)^{-1}$. Thus $r_t^1 \geq N^{-1} \min c^i$ (a.s.). Conversely, suppose a strategy of investor 1 guarantees survival. Then $r_t^1 \geq c > 0$ (a.s.), where c is a strictly positive random constant. The last inequality implies

$$w_t^1 \geq c \sum_{i=1}^{N} w_t^i \geq c w_t^i \text{ for each } i \text{ (a.s.),} \tag{4.39}$$

which yields (4.35) for $C^i = c^{-1}$. Thus a survival strategy is competitive.

In view of the equivalence of survival and competitive strategies, we can reformulate Theorem 4.1 as follows.

Theorem 4.9 *A log-optimal portfolio rule is a survival strategy. A basic survival strategy is asymptotically unique in the sense of Theorems 4.6 and 4.7.*

4.4.2 Games defined in terms of utilities of market shares

The game solution concepts considered above (competitive and survival strategies) were defined in terms of the long-run performance of investment strategies. Now we will follow a more conventional approach, characterizing the performance of strategies in terms of numerical criteria—expected utilities. For each $i = 1, 2, ..., N$, let $u^i(r)$ be a concave, continuous and increasing function defined on $[0, 1]$, taking values in $[-\infty, +\infty)$ and finite on $(0, 1]$. Denote the class of all such functions by $\mathcal{U}$. Let $r_0^i > 0, i = 1, 2, ..., N$, be initial market shares of the investors. They can be expressed through the initial endowments w_0^i by

$$r_0 = (r_0^1, ..., r_0^N),\ r_0^i = w_0^i/(w_0^1 + ... + w_0^N). \tag{4.40}$$

Consider a strategy profile $\Lambda^1, ..., \Lambda^N$ of the N investors and the random sequences $(r_t^i),\ i = 1, 2, ..., N$, of their market shares generated by this strategy profile. Recall that the strategy profile generates a sequence of vectors of investment proportions $\lambda_t^i(s^t)$ according to formulas (4.32) and (4.33). In turn, the investment proportions $\lambda_t^i(s^t)$ and initial endowments $w_0^i > 0$ define the sequences $w_t^i(s^t)$ (wealth of investor i at time t) and $r_t^i(s^t)$ (the market share of i at time t) by formula (4.36). It follows from (4.36) and (4.34) that the dynamics of the vectors of market shares $r_t = (r_t^1, ..., r_t^N)$ is governed by the random dynamical system

$$r_t^i = \frac{\langle R_t, \lambda_{t-1}^i \rangle r_{t-1}^i}{\langle R_t, \lambda_{t-1}^1 \rangle r_{t-1}^1 + \langle R_t, \lambda_{t-1}^2 \rangle r_{t-1}^2 + ... + \langle R_t, \lambda_{t-1}^N \rangle r_{t-1}^N} \text{ (a.s.)},$$

which can be written as

$$r_t = f_t(s^t, r_{t-1}) \text{ (a.s.)}, \tag{4.41}$$

where $f_t = (f_t^1, ..., f_t^N)$ and

$$f_t^i(s^t, r_{t-1}) = \frac{\langle R_t, \lambda_{t-1}^i \rangle r_{t-1}^i}{\langle R_t, \lambda_{t-1}^1 \rangle r_{t-1}^1 + \langle R_t, \lambda_{t-1}^2 \rangle r_{t-1}^2 + ... + \langle R_t, \lambda_{t-1}^N \rangle r_{t-1}^N}$$

if the denominator of this fraction is strictly positive, and $f_t(s^t, r_{t-1}) = (1/N, ..., 1/N)$ otherwise. The initial state of this system is the vector $r_0 = (r_0^1, ..., r_0^N)$ given by (4.40).

To derive (4.41), we assume that

$$W_t := \sum_{i=1}^{N} w_t^i = \sum_{i=1}^{N} \langle A_t, \lambda_{t-1}^i \rangle w_{t-1}^i > 0$$

for all t (which is true with probability one) and proceed by induction using the relation $w_t^i = \langle A_t, \lambda_{t-1}^i \rangle w_{t-1}^i$ (see (4.3)). Since $W_t > 0$, we can write

$$r_t^i = w_t^i / W_t = \frac{\langle A_t, \lambda_{t-1}^i \rangle w_{t-1}^i}{\langle A_t, \lambda_{t-1}^1 \rangle w_{t-1}^1 + \langle A_t, \lambda_{t-1}^2 \rangle w_{t-1}^2 + ... + \langle A_t, \lambda_{t-1}^N \rangle w_{t-1}^N} \tag{4.42}$$

Here A_t can be replaced by R_t because we can divide the nominator and denominator of the fraction (4.42) by $\sum_k A_{t,k} > 0$ (see (4.5)). Therefore

$$r_t^i = w_t^i / W_t = \frac{\langle R_t, \lambda_{t-1}^i \rangle w_{t-1}^i}{\langle R_t, \lambda_{t-1}^1 \rangle w_{t-1}^1 + \langle R_t, \lambda_{t-1}^2 \rangle w_{t-1}^2 + ... + \langle R_t, \lambda_{t-1}^N \rangle w_{t-1}^N}.$$

Dividing the nominator and denominator by $W_{t-1} = \sum_i w_{t-1}^i > 0$, we replace w_{t-1}^i by r_{t-1}^i, which leads to formula (4.41).

Define

$$r_\infty^i := \liminf_t r_t^i \tag{4.43}$$

and

$$F_\infty^i := Eu^i(r_\infty^i). \tag{4.44}$$

The expectations appearing here are well-defined and take values in $[-\infty, +\infty)$ because the functions u^i are bounded above. We will consider a stochastic dynamic game over an infinite time horizon in which

$$F_\infty^i = F_\infty^i(\Lambda^1, ..., \Lambda^N) = Eu^i(r_\infty^i), i = 1, ..., N, \tag{4.45}$$

will be the payoff functions of the players.

We can also consider a version of the above game in which the time horizon is finite: $t = 0, 1, ..., T$. In this case strategies (4.31) have to be defined only for $t = 0, ..., T-1$. The payoff functions of the players in the finite-horizon case are given by

$$F_T^i = F_T^i(\Lambda^1, ..., \Lambda^N) = Eu^i(r_T^i), i = 1, ..., N. \tag{4.46}$$

We will denote the games defined above by $\mathcal{G}_\infty$ and $\mathcal{G}_T$ ($T < \infty$), respectively.

Theorem 4.10 *The log-optimal investment strategy Λ^* forms a symmetric robust Nash equilibrium in each of the games $\mathcal{G}_T$ ($1 \le T \le \infty$).*

We use the term *robust equilibrium* to emphasize that one and the same strategy Λ^* possesses the Nash equilibrium property for the whole class of expected utilities of the investors and all strictly positive vectors of their initial endowments.

Proof of Theorem 4.10. Let $\Lambda^* = (\lambda_t^*(s^t))$ be the log-optimal strategy. According to the general definition of a symmetric Nash equilibrium, we have to prove that

$$F_T^N(\Lambda^*, ..., \Lambda^*, \Lambda) \le F_T^N(\Lambda^*, ..., \Lambda^*),\ 1 \le T \le +\infty, \tag{4.47}$$

for any strategy Λ of investor N. The dynamical system describing the game is symmetric[31], and so the validity of inequality (4.47) for $i = N$ will automatically imply the analogous inequality for each $i = 1, ..., N$. For the strategy profile $\Lambda^*, ..., \Lambda^*, \Lambda$, the total market share r_t^* of the group of investors $i = 1, 2, ..., N-1$ can be recursively computed according to the formulas

$$r_t^* = \sum_{i=1}^{N-1} r_t^i = \sum_{i=1}^{N-1} \frac{\langle R_t, \lambda_{t-1}^* \rangle r_{t-1}^i}{\langle R_t, \lambda_{t-1}^* \rangle r_{t-1}^1 + ... + \langle R_t, \lambda_{t-1}^* \rangle r_{t-1}^{N-1} + \langle R_t, \lambda_{t-1}^N \rangle r_{t-1}^N} =$$

$$\frac{\langle R_t, \lambda_{t-1}^* \rangle r_{t-1}^*}{\langle R_t, \lambda_{t-1}^* \rangle r_{t-1}^* + \langle R_t, \lambda_{t-1}^N \rangle (1 - r_{t-1}^*)} \text{ (a.s.)},$$

where $(\lambda_t(s^t))$ is the sequence of vectors of investment proportions for trader N generated by the strategy profile $\Lambda^*, ..., \Lambda^*, \Lambda$. The number $1 - r_t^*$ is the market share r_t^N of the investor N using the strategy $(\lambda_t(s^t))$. In the course of the proof of Theorem 4.2.5, we have proved that the random sequence $\ln r_t^*$ is a non-positive submartingale, and hence it has a.s. a finite limit $\ln r_\infty^*$. Further, the submartingale inequality $E_t(\ln r_{t+1}^*) \ge \ln r_t^*$ implies

$$E_t r_{t+1}^* = E_t \exp(\ln r_{t+1}^*) \ge \exp[E_t(\ln r_{t+1}^*)] \ge$$

$$\exp(\ln r_t^*) = r_t^*.$$

by virtue of Jensen's inequality (since the function $\exp$ is convex and increasing).

[31] However, the utilities and the initial endowments are not necessarily symmetric.

Consequently,

$$E_t(1 - r^*_{t+1}) \leq 1 - r^*_t,$$

and so the market share of $r^N_t = 1 - r^*_t$ investor N forms a supermartingale with values in $[0, 1)$. Consequently,

$$E_t u^N(r^N_{t+1}) \leq u^N(E_t r^N_{t+1}) \leq u^N(r^N_t) \tag{4.48}$$

because $u^N : [0, 1] \to [-\infty, +\infty)$ is a concave and increasing function. Note that all the conditional expectations are well-defined (but may be equal to $-\infty$) because $u^N(\cdot) \leq u^N(1)$. If $T < +\infty$, then from (4.48), we get

$$F^N_T(\Lambda^*, ..., \Lambda^*, \Lambda) = Eu^N(r^N_T) \leq$$

$$Eu^N(r^N_0) = F^N_T(\Lambda^*, ..., \Lambda^*, \Lambda^*)$$

The last equality holds since $r^i_t = r^i_0$, as long as all the investors use the same strategy. Suppose now that $T = +\infty$. By using the fact that r^N_t is a bounded supermartingale, we obtain that the limit $r^N_\infty = \lim r^N_t$ exists a.s. and

$$E_0 r^N_\infty = E_0 \lim r^N_t = \lim E_0 r^N_t \leq r^N_0, \tag{4.49}$$

where the last inequality holds because r^N_t is a supermartingale. The function u^Nconcave and increasing, and so (4.49) implies

$$E_0 u^N(r^N_\infty) \leq u^N(E_0 r^N_\infty) \leq u^N(r^N_0).$$

Thus

$$F^N_\infty(\Lambda^*, ..., \Lambda^*, \Lambda) = Eu^N(r^N_\infty) \leq$$

$$Eu^N(r^N_0) = F^N_\infty(\Lambda^*, ..., \Lambda^*, \Lambda^*),$$

which completes the proof. □

4.4.3 Subgames and subgame-perfect robust Nash equilibria

We have shown that the strategy $\Lambda^* = (\lambda_t^*(s^t))$ forms a symmetric robust Nash equilibrium in each of the games $\mathcal{G}_T$ ($T \leq \infty$). Our next goal is to show that it is a unique *subgame-perfect* symmetric robust Nash equilibrium in each of these games. Fix some moment of time $l > 0$, a sequence of states of the world $s^l = (s_1, ..., s_l)$ up to time l and a history

$$\lambda^{(l-1)} := (\lambda_l^j)_{l=0,1,...,l-1;\ j=1,...,N}$$

of actions (investment decisions) of all the investors $j = 1, ..., N$ from time 0 to time $l - 1$. Denote by $P_l(s^l, d\sigma^{l+1})$ the conditional distribution[32] of the sequence of states of the world

$$\sigma^{l+1} := (s_{l+1}, s_{l+2}, ...)$$

given the history s^l of the process (s_t) from time 1 to time l. Consider the *subgame*

$$\mathcal{G}_T^l = \mathcal{G}_T^l(s^l, \lambda^{(l-1)})\ (0 < l < T \leq \infty)$$

of the original game defined as follows. The game (defined for each s^l and $\lambda^{(l-1)}$) starts at time l. The market shares $r_l^i = r_l^i(s^l, \lambda^{(l-1)})$ of the players at time l (the

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[32] By definition, $P_l(s^l, d\sigma^{l+1})$ has the following properties:
(i) for each s^l, $P_l(s^l, \cdot)$ is a probability measure on the space of sequences σ^{l+1};
(ii) for each measurable set Γ in the space of these sequences, $P_l(\cdot, \Gamma)$ is a measurable function of s^l;
(iii) for every bounded measurable function $\phi(\sigma)$ of $\sigma = (s_0, s_1, ...)$, we have

$$E[\phi(\sigma)|s^l] = \int P_l(s^l, d\sigma^{l+1})\phi(\sigma) \text{ (a.s.)}$$

where $\sigma = (s^l, \sigma^{l+1})$. We assume that such a conditional distribution exists. The existence can be guaranteed if the space of states of the world is standard Borel.

initial moment of time for the subgame) are determined by the fixed sequence of states of the world s^l and the fixed history of investment decisions $\lambda^{(l-1)}$. Decisions at dates $l, l+1, ...$ of the N players $i = 1, 2, ..., N$ are vectors $\lambda_t^i,\ t \geq l$, of investment proportions. In the subgame $\mathcal{G}_T^l$, an investor i's *strategy* (*portfolio rule*) Λ^i is defined by a vector $\Lambda_l^i \in \Delta^K$ and a sequence of measurable functions

$$\Lambda_t^i(s_{l+1}^t, \lambda^{(l,t-1)}),\ t > l,$$

with values in Δ^K, where

$$s_{l+1}^t = (s_{l+1}, ..., s_t)$$

is the history of states of the world from time $l+1$ to time t and

$$\lambda^{(l,t-1)} := (\lambda_m^j)_{m=l,...,t-1;\ j=1,...,N}$$

is the set of the decisions made by all the investors $j = 1, ..., N$ at all the dates $m = l+1, ..., t-1$. As in the original game, those strategies for which the functions $\Lambda_t^i(\cdot)$ do not depend on $\lambda^{(l,t-1)}$

$$\Lambda_t^i(s_{l+1}^t, \lambda^{(l,t-1)}) = \Lambda_t^i(s_{l+1}^t),$$

will be called *basic*.

A strategy profile $(\Lambda^1, ..., \Lambda^N)$ of the investors determines recursively by the formulas

$$\lambda_l^i := \Lambda_l^i, \tag{4.50}$$

$$\lambda_t^i(s_{l+1}^t) := \Lambda_t^i(s_{l+1}^t, \lambda^{(l,t-1)}(s_{l+1}^{t-1})) \tag{4.51}$$

the vectors $\lambda_t^i(s_{l+1}^t)$ of investment proportions of all the traders at each date $t \geq l$. In turn, formulas (4.41) together with the initial market shares $r_l^i > 0, i = 1, ..., N$, at date l define step by step (according to the random dynamical system (4.41)) investor i's market share $r_t^i(s^t)$ for each date $t = l+1, l+2, ...$ and each random situation s_{l+1}^t. As before, we define $r_\infty^i := \liminf_t r_t^i$ and put

$$F_\infty^i := \mathbf{E}_l u^i(r_\infty^i),$$

where $\mathbf{E}_l$ is the expectation of the random variable

$$u^i(r_\infty^i(\sigma^{l+1})),\ \sigma^{l+1} := (s_{l+1}, s_{l+2}, ...),$$

with respect to the conditional distribution $P_l(s^l, d\sigma^{l+1})$, that is

$$\mathbf{E}_l[u^i(r_\infty^i)|s^l] = \int P_l(s^l, d\sigma^{l+1}) u^i(r_\infty^i(\sigma^{l+1})) \text{ (a.s.)}$$

The payoff functions of the players in the subgame $\mathcal{G}_\infty^l$ are defined by

$$F_\infty^i = F_\infty^i(\Lambda^1, ..., \Lambda^N) = \mathbf{E}_l u^i(r_\infty^i),\ i = 1, ..., N.$$

For the analogous game $\mathcal{G}_T^l$ over a finite time horizon $T < \infty$, we define the payoff functions by

$$F_T^i = F_T^i(\Lambda^1, ..., \Lambda^N) = \mathbf{E}_l u^i(r_T^i),\ i = 1, ..., N.$$

Definition 4.6 We say that a basic strategy $\Lambda = (\lambda_t(s^t))$ forms a symmetric *subgame-perfect robust Nash equilibrium* in the game $\mathcal{G}_T$ $(1 \leq T \leq \infty)$ if for any moment of time l, any almost all histories of states of the world s^l and for any history of the players' actions $\lambda^{(l-1)} = (\lambda_l^j)_{l=0,1,...,l-1;\ j=1,...,N}$, the strategy ${}^l\Lambda$

defined by

$$\lambda_l(s^l), \lambda_{l+1}(s^l, s_{l+1}), ..., \lambda_t(s^l, s^t_{l+1}), ...$$

(where s^l is held fixed, "frozen"), forms a symmetric robust Nash equilibrium in the game $\mathcal{G}^l_T = \mathcal{G}^l_T(s^l, \lambda^{(l-1)})$.

Having in mind the above definitions of the subgames $\mathcal{G}^l_T$ and subgame-perfect strategies, we can obtain the following refinement of Theorem 4.10.

Theorem 4.11 *The log-optimal investment strategy* $\Lambda^* = (\lambda^*_t(s^t))^\infty_{t=0}$ *forms a subgame-perfect symmetric robust Nash equilibrium in each of the games* $\mathcal{G}_T$ *(*$1 \leq T \leq \infty$*).*

Proof of Theorem 4.11 Observe that the subgame $\mathcal{G}^l_T$ has a similar structure as the game $\mathcal{G}_T$. The random dynamical system which governs the evolution of investors' market shares r^i_t is the same: it is given by equations (4.41). The distinctions are only in the initial moment of time l, the initial market shares $r^i_l = r^i_l(s^l, \lambda^{(l-1)})$ (which are determined by the previous play and random factors), and a different probability measure on the space of paths $\sigma^{l+1} := (s_{l+1}, s_{l+2}, ...)$ of the stochastic process $s_{l+1}, s_{l+2}, ...$, which is given by the conditional distribution $P_l(s^l, d\sigma^{l+1})$. Thus, in order to prove part (a) of Theorem 4.11 it is sufficient, in view of Theorem 4.10, to show that for almost all s^l the strategy ${}^l\Lambda^* = (\lambda^*_t(s^l, s^t_{l+1}))$ is log-optimal with respect to the conditional distribution $P_l(s^l, d\sigma^{l+1})$, that is

$$\mathbf{E}_l \ln\langle\lambda_l, R_{l+1}(s^l, s_{l+1})\rangle \leq \mathbf{E}_l \ln\langle\lambda^*_l(s^l), R_{l+1}(s^l, s_{l+1})\rangle, \tag{4.52}$$

for every vector λ_l of investment proportions and

$$\mathbf{E}_l \ln\langle \lambda_t(s_{l+1}^t), R_{t+1}(s^l, s_{l+1}^{t+1})\rangle \le \mathbf{E}_l \ln\langle \lambda_t^*(s^l, s_{l+1}^t), R_{t+1}(s^l, s_{l+1}^{t+1})\rangle,\ t > l, \tag{4.53}$$

for every measurable vector $\lambda_t(s_{l+1}^t)$ of investment proportions. Inequality (4.52) follows from the fact that

$$\mathbf{E}_l \ln\langle a, R_{l+1}(s^l, s_{l+1})\rangle = \phi_l(s^l, a) \tag{4.54}$$

(see (4.10)) and assertion (c) of Theorem 4.2. Inequality (4.53) is proved analogously by using the rule of iterations of conditional expectations, which yields

$$\mathbf{E}_l \ln\langle \lambda_t(s_{l+1}^t), R_{t+1}(s^l, s_{l+1}^{t+1})\rangle = \mathbf{E}_l[\mathbf{E}_t \ln\langle \lambda_t(s_{l+1}^t), R_{t+1}(s^t, s_{t+1})\rangle],$$

$$\mathbf{E}_l \ln\langle \lambda_t^*(s^l, s_{l+1}^t), R_{t+1}(s^l, s_{l+1}^{t+1})\rangle = \mathbf{E}_l[\mathbf{E}_t \ln\langle \lambda_t^*(s^l, s_{l+1}^t), R_{t+1}(s^t, s_{t+1})\rangle].$$

4.5 Numeraire portfolios (benchmark strategies)

In what follows, we shall consider only basic portfolio rules (omitting the word "basic" everywhere). Following Long (1990), we shall say that a strategy $(\overline{\lambda}_t)$ is a *numeraire portfolio* if $\overline{w}_t > 0$ (a.s.) and for any other strategy (λ_t), the process $w_t/\overline{w}_t$ is a supermartingale. Here, we denote by $\overline{w}_t$ and w_t the wealth processes generated by the strategies $(\overline{\lambda}_t)$ and (λ_t), respectively. In another terminology, which we will follow, numeraire portfolios are termed *benchmark strategies* (see Platen and Heath, 2006).

Theorem 4.12 *A portfolio rule is a benchmark strategy if and only if it is a log-optimal strategy.*

Proof of theorem 4.12 "Only if". Consider any strategy (λ_t) for which $\langle \lambda_{t-1}, R_t \rangle > 0$ (a.s.). By the definition of a benchmark strategy, the process $w_t/\overline{w}_t$ is a supermartingale, and so

$$E_{t-1}\frac{w_t}{\overline{w}_t} \leq \frac{w_{t-1}}{\overline{w}_{t-1}}, \tag{4.55}$$

which implies

$$\frac{w_{t-1}}{\overline{w}_{t-1}} E_{t-1}\frac{\langle \lambda_{t-1}, R_t \rangle}{\langle \overline{\lambda}_{t-1}, R_t \rangle} = E_{t-1}\frac{\langle \lambda_{t-1}, R_t \rangle w_{t-1}}{\langle \overline{\lambda}_{t-1}, R_t \rangle \overline{w}_{t-1}} \leq \frac{w_{t-1}}{\overline{w}_{t-1}}. \tag{4.56}$$

Here, $w_{t-1}/\overline{w}_{t-1} > 0$ because $\langle \lambda_{t-1}, R_t \rangle > 0$, and consequently, the last inequality is equivalent to

$$E_{t-1}\frac{\langle \lambda_{t-1}, R_t \rangle}{\langle \overline{\lambda}_{t-1}, R_t \rangle} \leq 1. \tag{4.57}$$

By using Jensen's inequality, we find:

$$E_{t-1} \ln \frac{\langle \lambda_{t-1}, R_t \rangle}{\langle \overline{\lambda}_{t-1}, R_t \rangle} \leq \ln E_{t-1}\frac{\langle \lambda_{t-1}, R_t \rangle}{\langle \overline{\lambda}_{t-1}, R_t \rangle} \leq 0,$$

and, by taking the expectations, we get

$$E \ln \frac{\langle \lambda_{t-1}, R_t \rangle}{\langle \overline{\lambda}_{t-1}, R_t \rangle} \leq 0. \tag{4.58}$$

If $\lambda_{t-1} = (1/K, ..., 1/K)$, then this inequality becomes

$$E[\ln(1/K) - \ln\langle \overline{\lambda}_{t-1}, R_t \rangle] \leq 0,$$

from which we can conclude that $E \ln\langle \overline{\lambda}_{t-1}, R_t \rangle > -\infty$ and

$$E \ln\langle \lambda_{t-1}, R_t \rangle \leq E \ln\langle \overline{\lambda}_{t-1}, R_t \rangle. \tag{4.59}$$

Inequality (4.59) has been obtained under the assumption that $\langle \lambda_{t-1}, R_t \rangle > 0$. Let us now consider any vector of proportions $\lambda_{t-1}(s^{t-1})$, and for each $m = 1, 2, ...$

define

$$\lambda_{t-1}^{(m)} = \frac{1}{m}\overline{\lambda}_{t-1} + \left(1 - \frac{1}{m}\right)\lambda_{t-1}.$$

Here, $\left\langle \lambda_{t-1}^{(m)}, R_t \right\rangle \geq \langle \overline{\lambda}_{t-1}, R_t \rangle / m > 0$ (a.s.) because, as we have shown, $E \ln \langle \overline{\lambda}_{t-1}, R_t \rangle > -\infty$. Consequently, we can apply (4.59) to $\lambda_{t-1}^{(m)}$, which gives

$$E \ln \langle \overline{\lambda}_{t-1}, R_t \rangle \geq E \ln \left\langle \lambda_{t-1}^{(m)}, R_t \right\rangle = E \ln \left\langle \frac{1}{m}\overline{\lambda}_{t-1} + \left(1 - \frac{1}{m}\right)\lambda_{t-1}, R_t \right\rangle \geq \frac{1}{m} E \ln \langle \overline{\lambda}_{t-1}, R_t \rangle + \left(1 - \frac{1}{m}\right) E \ln \langle \lambda_{t-1}, R_t \rangle$$

by virtue of Jensen's inequality. By passing to the limit as $m \to \infty$, we obtain that $E \ln \langle \overline{\lambda}_{t-1}, R_t \rangle \geq E \ln \langle \lambda_{t-1}, R_t \rangle$, and so λ_{t-1} is the vector of log-optimal proportions.

"If." Suppose $(\overline{\lambda}_t)$ is a log-optimal strategy, i.e., (4.58) holds. We know that for a log-optimal strategy $\langle \overline{\lambda}_{t-1}, R_t \rangle > 0$ and hence $\overline{w}_t > 0$. Consider any vector of investment proportions λ_{t-1} such that all its coordinates are not less than some $\varepsilon > 0$. By virtue of (4.12), we have

$$E_{t-1} \ln \frac{\langle \theta \lambda_{t-1} + (1-\theta)\overline{\lambda}_{t-1}, R_t \rangle}{\langle \overline{\lambda}_{t-1}, R_t \rangle} \leq 0$$

for every $\theta \in (0,1)$. From the above inequality, we obtain

$$0 \geq \theta^{-1} E_{t-1} \ln \left(1 - \theta + \theta \frac{\langle \lambda_{t-1}, R_t \rangle}{\langle \overline{\lambda}_{t-1}, R_t \rangle}\right) = E_{t-1} \left\{ \theta^{-1} \ln \left[1 + \theta \left(\frac{\langle \lambda_{t-1}, R_t \rangle}{\langle \overline{\lambda}_{t-1}, R_t \rangle} - 1\right)\right] \right\}.$$

By using the Fatou lemma, we write

$$0 \geq \liminf_{\theta \to 0} E_{t-1} \left\{ \theta^{-1} \ln \left[1 + \theta \left(\frac{\langle \lambda_{t-1}, R_t \rangle}{\langle \overline{\lambda}_{t-1}, R_t \rangle} - 1\right)\right] \right\} \geq$$

$$E_{t-1}\liminf_{\theta\to 0}\left\{\theta^{-1}\ln\left[1+\theta\left(\frac{\langle\lambda_{t-1},R_t\rangle}{\langle\overline{\lambda}_{t-1},R_t\rangle}-1\right)\right]\right\}=$$
$$E_{t-1}\frac{\langle\lambda_{t-1},R_t\rangle}{\langle\overline{\lambda}_{t-1},R_t\rangle}-1.$$

To justify the use of the Fatou lemma, we observe that

$$\frac{1}{\theta}\ln\frac{\langle\theta\lambda_{t-1}+(1-\theta)\overline{\lambda}_{t-1},R_t\rangle}{\langle\overline{\lambda}_{t-1},R_t\rangle}\geq$$
$$\frac{1}{\theta}[\theta\ln\langle\lambda_{t-1},R_t\rangle+(1-\theta)\ln\langle\overline{\lambda}_{t-1},R_t\rangle-\ln\langle\overline{\lambda}_{t-1},R_t\rangle]=$$
$$\ln\langle\lambda_{t-1},R_t\rangle-\ln\langle\overline{\lambda}_{t-1},R_t\rangle\geq\ln\langle\lambda_{t-1},R_t\rangle\geq\ln\varepsilon.$$

Thus we have obtained (4.57) under the additional assumption that all the coordinates of λ_{t-1} are not less than $\varepsilon>0$. Now consider any vector of proportions $\lambda_{t-1}(s^{t-1})$ and define

$$\lambda_{t-1}^{[m]}=\frac{1}{m}b+(1-\frac{1}{m})\lambda_{t-1},$$

where $b=(1/K,...,1/K)$. By applying (4.57) to $\lambda_{t-1}^{[m]}$ and using the Fatou lemma, we obtain

$$1\geq\liminf_{m\to\infty}E_{t-1}\frac{\left\langle\lambda_{t-1}^{[m]},R_t\right\rangle}{\langle\overline{\lambda}_{t-1},R_t\rangle}\geq$$
$$E_{t-1}\liminf_{m\to\infty}\frac{\left\langle\lambda_{t-1}^{[m]},R_t\right\rangle}{\langle\overline{\lambda}_{t-1},R_t\rangle}=E_{t-1}\frac{\langle\lambda_{t-1},R_t\rangle}{\langle\overline{\lambda}_{t-1},R_t\rangle},$$

which yields (4.57) for any λ_{t-1}. Property (4.57) implies (4.56) and (4.55), and so (since $w_0/\overline{w}_0$ is a positive constant) $w_t/\overline{w}_t$ is a supermartingale. □

Suppose that the asset returns $A_{t,k}$ given in the model are defined in terms of strictly positive asset prices $p_{t,k}$ by formula (4.1). Then the following proposition holds.

Proposition 4.3 *A strategy* $(\overline{\lambda}_t)$ *generating a strictly positive wealth process* $\overline{w}_t$

is a benchmark strategy if and only if for every asset $k = 1, ..., K$ *the process*

$$\frac{p_{t,k}}{\overline{w}_t} \tag{4.60}$$

is a supermartingale.

4.6 Appendix

Proof of Lemma 4.1 We have $\zeta_t := E_t\xi_{t+1} - \xi_t \geq 0$ by the definition of a submartingale. Further, we have

$$\sum_{t=0}^{T-1} E\zeta_t = \sum_{t=0}^{T-1}(E\xi_{t+1} - E\xi_t) = E\xi_T - E\xi_0,$$

and so the sequence $\sum_{t=0}^{T-1} E\zeta_t$ is bounded because $\sup_T E\xi_T < \infty$[33]. Therefore the series of the expectations $\sum_{t=0}^{\infty} E\zeta_t$ of the non-negative random variables ζ_t converges, which implies $\sum_{t=0}^{\infty} \zeta_t < \infty$ a.s. because $E\sum_{t=0}^{\infty} \zeta_t = \sum_{t=0}^{\infty} E\zeta_t$. $\square$

Proof of Proposition 4.1 Since the random elements s_t are independent, we have

$$E_t\langle\lambda_t' - \lambda_t, R_{t+1}\rangle^2 = \int Q(ds)\langle\lambda_t'(s^t) - \lambda_t(s^t), R(s)\rangle^2.$$

In order to prove (ND), it is sufficient to show that

$$\inf_{b\neq 0} \Phi(b) := \inf_{b\neq 0} \frac{\int Q(ds)\langle b, R(s)\rangle^2}{||b||^2} > 0.$$

Suppose the contrary: there exists a sequence of vectors $b^n \in \mathcal{R}^K$ such that $\Phi(b^n) \to 0$. Since $\Phi(b^n) = \Phi(b^n/||b^n||)$, we may assume that $||b^n|| = 1$. By passing to a convergent subsequence, we may further assume that $b^n \to b = (b_1, ..., b_K)$ with $||b|| = 1$. Then, by using the Lebesgue theorem, we obtain

$$\Phi(b^n) = \int Q(ds)\langle b^n, R(s)\rangle^2 \to \int Q(ds)\langle b, R(s)\rangle^2,$$

and so

$$\int Q(ds)\langle b, R(s)\rangle^2 = 0,$$

33 It follows that a series $\sum a_n$ with non-negative terms converges if and only if the sequence $S_N = \sum_{n=1}^{N} a_n$ of partial sums is bounded.

which implies

$$\langle b, R(s)\rangle = b_1 R_1(s) + ... + b_K R_K(s) = 0$$

for Q-almost all s. Since the functions $R_1(s), ..., R_K(s)$ are linearly independent $\bmod Q$, we get $b_1 = ... = b_K = 0$. But this cannot be true since $||b|| = 1$. A contradiction. □

Proof of proposition 4.2. Consider a basic strategy Λ^1 defined by a sequence (λ_t^1) of vectors of investment proportions. Suppose it is asymptotically optimal in the sense of Definition 4.1. Let $\Lambda^2, ..., \Lambda^N$ be any strategies of investors $2, ..., N$. Let $(\lambda_t^2), ..., (\lambda_t^N)$ be the sequences of vectors of proportions generated by the strategy profile $\Lambda^1, \Lambda^2, ..., \Lambda^N$. Then for any i, the wealth process (w_t^i) can be defined recursively by (4.34), which means that it is generated by the basic strategy (λ_t^i). By using the property of asymptotic optimality of (λ_t^1) we obtain that (4.35) holds for some random constant $C^i > 0$.

Conversely, let $\Lambda^1 = (\lambda_t^1)$ be a basic competitive strategy in the dynamic game described above. Consider the strategy profile $\Lambda^2, ..., \Lambda^N$ of player 1's rivals such that the strategies $\Lambda^2, ..., \Lambda^N$ are the same and coincide with some (arbitrary) basic strategy (λ_t). By virtue of (4.35), the wealth processes (w_t) and (w_t^1) generated by $(\lambda_t), (\lambda_t^1)$ and starting from the given $w_0 > 0, w_0^1 > 0$ satisfy $w_t \leq C w_t^1$ a.s. for some random constant C. This implies (see Definition 4.1 and the comments after it) that Λ^1 is asymptotically optimal. □

Proof of proposition 4.3. Assume that $(\overline{\lambda}_t)$ is a benchmark strategy. Consider the sequence of vectors of proportions $\lambda_{t,k} = e_k$ (where e_k is the vector whose coordinates are 0 except the ith coordinate which is 1). Then the value w_t of the portfolio generated by this strategy is given recursively by

$$w_t = \langle A_t, \lambda_{t-1} \rangle w_{t-1} = A_{t,k} w_{t-1} = \frac{p_{t,k}}{p_{t-1,k}} w_{t-1}$$

and so

$$w_t = p_{t,k}/p_{0,k}.$$

Consequently, the process

$$\frac{p_{t,k}}{p_{0,k}\overline{w}_t}$$

is a supermartingale, and thus $p_{t,k}/\overline{w}_t$ is a supermartingale too, because $p_{0,k}$ is a constant.

Conversely, suppose that the process (4.60) is a supermartingale. Consider any strategy (λ_t) and the corresponding wealth process (w_t). We have

$$E_{t-1}\frac{w_t}{\overline{w}_t} = E_{t-1}\frac{\langle A_t, \lambda_{t-1} \rangle w_{t-1}}{\langle A_t, \overline{\lambda}_{t-1} \rangle \overline{w}_{t-1}} = \frac{w_{t-1}}{\overline{w}_{t-1}} E_{t-1}\frac{\langle A_t, \lambda_{t-1} \rangle}{\langle A_t, \overline{\lambda}_{t-1} \rangle}.$$

Here,

$$E_{t-1}\frac{\langle A_t, \lambda_{t-1} \rangle}{\langle A_t, \overline{\lambda}_{t-1} \rangle} = \sum_{k=1}^{K} E_{t-1}\frac{A_{t,k}\lambda_{t-1,k}}{\langle A_t, \overline{\lambda}_{t-1} \rangle},$$

and so it is sufficient to verify that

$$E_{t-1}\frac{A_{t,k}}{\langle A_t, \overline{\lambda}_{t-1} \rangle} \leq 1$$

for each k. This inequality can be written as

$$E_{t-1}\frac{p_{t,k}/p_{t-1,k}}{\langle A_t, \overline{\lambda}_{t-1} \rangle} \leq 1,$$

which is equivalent to

$$E_{t-1}\frac{p_{t,k}}{\langle A_t, \overline{\lambda}_{t-1}\rangle} \leq p_{t-1,k}.$$

The last inequality, in turn, is equivalent to the following one

$$E_{t-1}\frac{p_{t,k}}{\overline{w}_t} \leq \frac{p_{t-1,k}}{\overline{w}_{t-1}}$$

because $\overline{w}_t = \langle A_t, \overline{\lambda}_{t-1}\rangle \overline{w}_{t-1}$. But this relation is valid as long as the process (4.60) is a supermartingale. □

Chapter 5 Conclusion

This thesis analyzes asset markets in the context of evolutionary finance. A stock market can be understood as a heterogeneous population of frequently interacting portfolio rules in competition for market capital. The general approach is to apply evolutionary dynamics—mutation and selection—to the analysis of the long-run performance of investment strategies. Our aim is to contribute to a "Darwinian theory" of portfolio selection.

A game-theoretic evolutionary model of an asset market with endogenous equilibrium asset prices was investigated in second chapter. The assets pay dividends at each time period that are partially consumed and partially reinvested. Only one equilibrium involved in this model is market clearing equilibrium which are used as endogenously generating asset prices. The investors select general, adaptive strategies according to which their wealth are distributed among the assets. We were mainly concerned with the long-run performance of the strategies with the goal of identifying survival strategies, i.e., allowing an investor to survive in the long run.

Relations between evolutionary finance and Nash equilibrium were explored and presented in Chapter 3. We considered a long-lived dividend-paying asset market with independent and identical distributed states of the world. It turned out that in the game under consideration the Kelly rule of "betting your beliefs" forms with probability one a unique symmetric Nash equilibrium strategy.

Fundamental facts of capital growth theory were examined from a new angle suggested by recent studies on evolutionary finance and asset market games in Chapter 4. This new view makes it possible to establish relations between financial growth and the property of "survival" of investment strategies in the market selection process. This study stressed the role of log-optimal investments as a means for achieving asymptotically optimal growth with probability one.

Generally, evolutionary finance, in comparison with conventional paradigm—maximizing individual utility function, has an advantage in expressing the dynamic nature of an asset market. And the robustness of the evolutionary models lays a foundation for establishing a new portfolio selection theory. But many challenging topics within this research field need to be done in the future. As highlighted in the chapters, all the results obtained are established in the class of basic strategies. Whether or not these results can pertain to general strategies is still an open question.

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