# PURE-INJECTIVE MODULES OVER TUBULAR ALGEBRAS AND STRING ALGEBRAS

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# Contents

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**Pure-Injective Modules over Tubular Algebras and String Algebras**
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We show that, for any tubular algebra, the lattice of pp-definable subgroups of the direct sum of all indecomposable pure-injective modules of slope $r$ has m-dimension 2 if $r$ is rational, and undefined breadth if $r$ is irrational- and hence that there are no superdecomposable pure-injectives of rational slope, but there are superdecomposable pure-injectives of irrational slope, if the underlying field is countable.

We determine the pure-injective hull of every direct sum string module over a string algebra. If $A$ is a domestic string algebra such that the width of the lattice of pp-formulas has defined breadth, then classify "almost all" of the pure-injective indecomposable $A$-modules.

# Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

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# Chapter 1

# Introduction

We consider two different classes of finite dimensional $K$-algebras: In chapters 3 and 4, we consider tubular algebras, and in chapters 5, 6 and 7, we consider string algebras.

Tubular algebras are defined as being tubular extensions of a tame concealed algebra $A_0$, of extension type either $(2,2,2,2)$, $(3,3,3)$, $(4,4,2)$ or $(6,3,2)$.

Tubular algebras are usually described in terms of their Auslander-Reiten quiver: the set of all finite dimensional indecomposable modules over a tubular algebra $A$ can be partitioned into sets $\mathcal{P}_0 \cup \mathcal{Q}_\infty \cup \bigcup_{\gamma \in \mathbb{Q}_0^\infty} \mathcal{T}_\gamma$- where $\mathcal{P}_0$ is a connected preprojective component, $\mathcal{Q}_\infty$ a connected preinjective component, and each $\mathcal{T}_\gamma$ is a set of pairwise orthogonal, sincere, standard tubes. Furthermore, the components satisfy the following conditions:

- For all $\gamma \in \mathbb{Q}^+$, every tube in $\mathcal{T}_\gamma$ is stable, whereas $\mathcal{T}_0$ and $\mathcal{T}_\infty$ contain non-stable tubes: one tube in $\mathcal{T}_0$ contains a projective module, and one tube in $\mathcal{T}_\infty$ contains an injective module.

- $\mathrm{Hom}(\mathcal{Q}_\infty, \mathcal{T}_\gamma) = \mathrm{Hom}(\mathcal{T}_\gamma \mathcal{P}_0) = \mathrm{Hom}(\mathcal{Q}_\infty, \mathcal{P}_0) = 0$ for all $\gamma \in \mathbb{Q}_0^\infty$.

- $\mathrm{Hom}(\mathcal{T}_\gamma, \mathcal{T}_\delta) = 0$ for all $\gamma > \delta$.

- Given any $\gamma \in \mathbb{Q}_0^\infty$, and any tube in $\mathcal{T}_\gamma$, any homomorphism from a module in $\mathcal{P}_0 \cup \bigcup_{\delta<\gamma} \mathcal{T}_\delta$ to a module in $\mathcal{Q}_\infty \cup \bigcup_{\delta>\gamma} \mathcal{T}_\delta$ factors through a direct sum of modules in $\mathcal{T}_\gamma$.

Any module $M \in$ A-Mod is said to have slope $r$ if $\mathrm{Hom}(M, \bigcup_{\gamma<r} \mathcal{T}_\gamma) = 0$ and $\mathrm{Hom}(\bigcup_{\delta>r} \mathcal{T}_\delta, M) = 0$. Ringel and Reiten proved in [22] that every indecomposable module over a tubular algebra (other than those in $\mathcal{P}_0$ and $\mathcal{Q}_\infty$) has a unique slope. We study the lattice of pp-formulas over this algebra, with the aim of further extending the knowledge of modules over this algebra.

In section 3.5, we describe the pure-injective $A$-modules of which lie in the support of $\mathcal{T}_\gamma$, for any positive rational $\alpha$, and that this set coincides with the set of all indecomposable pure-injective modules of slope $\gamma$. We also prove that the Cantor-Bendixson rank of $\mathrm{Supp}(\mathcal{T}_\gamma)$ is 2, and that the m-dimension of the lattice of pp-definable subgroups of $\bigoplus_{M\in\mathcal{T}_\gamma} M$- which in turn implies that there are no superdecomposable modules of slope $\gamma$.

In chapter 4 we consider the lattice of pp-definable subgroups of $M(r)$- the direct sum of all indecomposable pure-injectives of slope $r$- for any irrational $r$. By theorem 30, a pp-pair is closed on $M(r)$ if and only if there exists $\epsilon > 0$ such that $\phi/\psi$ is closed on all modules in $\bigcup_{r-\epsilon<\gamma} \mathcal{T}_\gamma$.

We consider four specific tubular algebras: $C(4,\lambda)$, $C(6)$, $C(7)$ and $C(8)$. We prove in theorem 31 that ${}_C\mathrm{pp}(M(r))$ is wide for every irrational $r$.

In section 4.7, we extend this result to all tubular algebras through shrinking functors- which are a type of tilting functor between two tubular algebras. We prove that, given any irrational $r$, we can induce from a shrinking functor $\Sigma_T : A \to B$, an embedding from ${}_B\mathrm{pp}/\sim_s$ to ${}_B\mathrm{pp}^k/\sim_r$ (for some irrational $s$ and $k \geq 1$). It follows that if $w({}_B\mathrm{pp}/\sim_s) = \infty$ for all irrational $s$, then $w({}_B\mathrm{pp}/\sim_s) = \infty$ for all irrational $r$.

In [23], Ringel shows that given any tubular algebra $A$, there exists a finite set of tubular algebras $B_1, \dots B_n$ and a series of shrinking functors:

$$A \xrightarrow{\Sigma_1} B_1 \xrightarrow{\Sigma_2} B_2 \cdots \xrightarrow{\Sigma_n} B_n$$

-with $B_n$ being either $C(4,\lambda)$, $C(6)$, $C(7)$ or $C(8)$. It therefore follows that $w({}_A\mathrm{pp}/\sim_r) = \infty$ for all tubular algebras $A$ and all positive irrationals $r$.

It follows that, if the underlying field $K$ is countable, then there exists a pure-injective superdecomposable $A$-module of slope $r$.

We define string algebras at the start of chapter 5. It was proved.... that the finite dimensional indecomposable modules over a string algebra are all string modules or band modules.

Given any infinite word, one can extend the definition of a finite dimensional string module, to define, to give a number of infinite dimensional string modules: In particular, the direct sum string module, $M(w)$, which is of countable dimension over the underlying field $K$, and the direct product string module, $\overline{M}(w)$, which is of uncountable dimension over $K$. It was proved in [13] that every direct sum string module is indecomposable. We show, in proposition 4, that the direct product module $\overline{M}(w)$ is always pure-injective.

In [24], Ringel introduced a number of infinite dimensional string modules over periodic and almost periodic words- which we refer to as Ringel's list- with the intention of proving that it contained all the indecomposable pure-injective modules over a domestic string algebra. It follows from our results that every module on Ringel's list is indeed indecomposable.

In [6], Burke describes some pure-embeddings between direct sum and direct product modules over periodic and almost periodic words. We extend this result to all words- in particular, that for all aperiodic words, $w$, the canonical embedding from $M(w)$ to $\overline{M}(w)$ is a pure embedding.

In [18], Prest and Puninski proved that, for every $\mathbb{N}$-word, $w$, there exists a unique infinite dimensional one-directed indecomposable pure-injective module- which we denote as $M_w$- and that the map $w \mapsto M_w$ defines a bijection between $\mathbb{N}$-words and (isomorphism classes of) infinite dimensional one-directed indecomposable pure-injective modules. If $w$ is periodic or almost periodic, then $M_w$ must be the module on Ringel's list. If $w$ is aperiodic, then we prove, in corollary 29, that $M_w$ is the pure-injective hull of $M(w)$.

In chapter 6, we find necessary and sufficient conditions on an infinite word, $w$, to determine whether or not the direct sum string module $w$ is pure-injective, and to determine whether or not the direct product module $\overline{M}(w)$ is indecomposable.

Specifically, we prove that the direct sum module $M(w)$ is pure-injective (and indeed $\Sigma$-pure-injective) if and only if both $\mathcal{W}_z$ and $\mathcal{U}_z$ (cf. 6.1) satisfy the ascending chain condition. Also, $\overline{M}(w)$ is indecomposable if and only if the poset of standard basis elements $\{z_i : i \in I\}$ satisfies both the descending chain condition and (IC) (cf. 6.1).

It follows from these results that there are aperiodic $\mathbb{N}$-words, $w$ such that neither $M(w)$ nor $\overline{M}(w)$ is both pure-injective and indecomposable- and hence that $M_w$ is neither $M(w)$ nor $\overline{M}(w)$.

In chapter 7, we attempt to extend theorem 40 to two-directed modules. We show in theorem 51- that for every non-periodic $\mathbb{Z}$-word, $u_0^{-1}w_0$, there exists a unique (up to isomorphism) two-directed module $M_w$, containing a fundamental element (cf. (7.1.2)) with right-word $w_0$ and left-word $u_0$.

Furthermore, we prove that there is exactly one two-directed pure-injective indecomposable module containing a fundamental element with right-word $w_0$ and left word $u_0$- giving us a bijective correspondence between non-periodic $\mathbb{Z}$-words and pure-injective indecomposables containing a fundamental element $x$ such that $u_x^{-1}w_x$ is not periodic (where $w_x$ and $u_x$ denote the right-word and left-word of $x$ in $M$). This correspondence implies that if every pure-injective indecomposable $A$-module does contain a fundamental element, then we can classify almost all the indecomposable pure-injective $A$-modules.

We extend the results of [6] by finding the pure-injective hull of every direct sum string module $M(w)$. As in the one-directed case, $H(M(w)) \cong M_w$ whenever $w$ is aperiodic. Again, it follows from these results that $M_w$ is a direct summand of $\overline{M}(w)$. However, unlike in theorem 40, we cannot prove that $M_w \ncong M_{w'}$ for any pair of distinct $\mathbb{Z}$-words, $w$ and $w'$.

It is conjectured that $w(\text{pp}) < \infty$ for every domestic string algebra, $A$. We prove that, if $w({}_A\text{pp}) < \infty$, then every pure-injective indecomposable $A$-module contains a fundamental element. Given such an algebra, it follows that every infinite dimensional indecomposable pure-injective $A$-module is either a module on Ringel's list, or a module obtained from a homogeneous tube, or an "anomaly" (theorem 53).

Given any aperiodic $\mathbb{Z}$-words, $w$ and $w'$, we write $w \preccurlyeq w'$ if every finite subword of $w$ is also a finite subword of $w'$. We prove in section.... that $w \preccurlyeq w'$ if and only if $\text{Supp}(M(w)) \subseteq \text{Supp}(M(w'))$. We also show that there exists distinct $\mathbb{Z}$-words, $w$ and $w'$ such that $M(w) \ncong M(w')$ and $\text{Supp}(M(w)) = \text{Supp}(M(w'))$.

We prove in section 7.5 that $\text{Supp}(M(w)) = \text{Supp}(\overline{M}(w))$ for every aperiodic $\mathbb{Z}$-word or $\mathbb{N}$-word, $w$.

Finally, we show in section 7.6 that there are examples of words $w$ and $u$ such that $\overline{M}(w)$ is a direct summand of $\overline{M}(u)$: Indeed, we construct a pure embedding from $\overline{M}(w)$ to $\overline{M}(u)$ to show this.

# Chapter 2

# Background

First of all, we point out a few conventions, which we use throughout the thesis: we denote by $\mathbb{N}$ the set of all non-negative integers, and by $\mathbb{N}^+$ the set of all strictly positive integers.

We denote by $\mathbb{Q}^+$ the set of all strictly positive rationals, and by $\mathbb{Q}_0^\infty$ the set $\mathbb{Q}^+ \cup \{0\} \cup \{\infty\}$ (where $\infty > q$ for all $q \in \mathbb{Q}$).

We will also assume throughout the thesis that $K$ denotes an algebraically closed field.

## 2.1 Homological algebra

Throughout this section, $R$ will denote any ring, $K$ any field, and $A$ any $K$-algebra.

Given any ring $R$, we denote by $R$-Mod and Mod-$R$ the set of all left $R$-modules and the set of all right $R$-modules respectively. We denote by $R$-mod (respectively, mod-$R$) the set of all finitely presented left $R$-modules (respectively, right $R$-modules).

We will only be working over finite dimensional $K$-algebras. Such rings are Artinian, and hence Noetherian, and so every finitely generated module over such an algebra is finitely presented.

The opposite algebra of $A$, denoted $A^{\text{op}}$, is the $K$-algebra with the same underlying vector space, but with multiplication reversed: i.e. $a \times b$ in $A^{\text{op}}$ is the element $ba$ of $A$.

Of course, $(A^{\text{op}})^{\text{op}}$ is $A$, and every left (respectively, right) $A$-module may be considered as a right (respectively, left) $A^{\text{op}}$-module.

Given any $M \in A$-Mod, the *$K$-dual of $M$*, denoted $DM$, is the $K$-vector space $\text{Hom}_K(A, K)$. We may consider it as a right $A$-module, where for all $a \in A$, $fa : M \to K$ is the map taking every $m \in M$ to $f(am)$.

The $K$-dual $D$ induces a duality between $A$-mod and mod-$A$ (since $D(DM) \cong M$ for all $M \in A$-mod).

A map $f \in \text{Hom}(L, M)$ is called a *section* if there exists $h \in \text{Hom}(M, L)$ such that $hf = 1_L$- any such $h$ is called a *retraction of $f$*. A map $g \in \text{Hom}(M, N)$ is called a *retraction* if there exists $h \in \text{Hom}(N, M)$ such that $fh = 1_N$.

A *chain complex* in $A$-Mod is a sequence of $A$-modules $M_i$ and homomorphisms $f_i$:

$$\cdots \xrightarrow{f_3} M_2 \xrightarrow{f_2} M_1 \xrightarrow{f_1} M_0 \xrightarrow{f_0} 0$$

-such that $f_i f_{i+1} = 0$ for all $i \in \mathbb{N}$. It is called an *exact sequence* if $\mathrm{Im}(f_{i+1}) = \mathrm{Ker}(f_i)$ for all $i \in \mathbb{N}$.

Similarly, a *cochain complex* in $A$-Mod is a sequence of $A$-modules $N_i$ and homomorphisms $f_i$:

$$0 \xrightarrow{g_0} N_0 \xrightarrow{g_1} N_1 \xrightarrow{g_2} N_2 \xrightarrow{g_3} \cdots$$

-such that $g_{i+1} g_i = 0$ for all $i \in \mathbb{N}$. It is called an *exact sequence* if $\mathrm{Im}(g_i) = \mathrm{Ker}(g_{i+1})$ for all $i \in \mathbb{N}$.

A *short exact sequence* is any sequences of modules $L, M, N \in A$-Mod and homomorphisms $f, g$:

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

-such that $f$ is an embedding, $g$ a surjection, and $\mathrm{Im}(f) = \mathrm{Ker}(g)$. A short exact sequence is said to be *split* if $f$ is a section, or equivalently, $g$ is a retraction.

### 2.1.1 Projective and injective modules

A module $P \in R$-Mod is said to be *projective* if, for all $M, N \in R$-Mod, surjections $g : M \twoheadrightarrow N$, and homomorphisms $f \in \mathrm{Hom}(P, N)$, there exists $h \in \mathrm{Hom}(P, M)$ such that $f = gh$.

Dually, a module $E \in A$-Mod is said to be *injective* if, for all $L, M \in R$-Mod, embeddings $g : L \hookrightarrow M$, and homomorphisms $f \in \mathrm{Hom}(L, E)$, there exists $h \in \mathrm{Hom}(M, E)$ such that $f = hg$.

Given any $M \in R$-Mod, the *projective cover of* $M$ is an epimorphism $h_0 : P \twoheadrightarrow M$ such that any submodule $N$ of $P$ with $\mathrm{Ker}(h_0) + N = P$ must in fact be $P$. If a projective cover exists, then it is unique up to isomorphism.

A *minimal projective presentation of* $M$ is an exact sequence:

$$P_1 \xrightarrow{h_1} P_0 \xrightarrow{h_0} M \longrightarrow 0$$

-where $h_0$ is the projective cover of $M$, and $h_1$ is the composition of the projective cover of $\mathrm{Ker}(h_0)$ and the natural embedding $\mathrm{Ker}(h_0) \hookrightarrow M$.

Given any $M \in A$-Mod, a *projective resolution of* $M$ is any sequence:

$$\cdots \xrightarrow{f_3} P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0$$

-and map $f_0 \in \mathrm{Hom}(P_0, M)$ such that the sequence:

$$\cdots \xrightarrow{f_3} P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \longrightarrow 0$$

-is exact. The *projective dimension of* $M$- denoted $\mathrm{pd}M$- is defined to be the minimal $m \in \mathbb{N}$ such that there exists an exact sequence of the form:

$$0 \to P_m \to P_{m-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

-with every $P_i$ being projective. If no such $m$ exists, then $\mathrm{pd}M := \infty$.

Dually, given any $M \in R$-Mod, the *injective envelope of* $M$ is an embedding $h_0 : P \twoheadrightarrow M$ such that $N \cap \mathrm{Im}(h_0) \neq 0$ for all non-zero submodules $N$ of $M$. The *minimal injective copresentation* of a module $M$ is an exact sequence:

$$0 \longrightarrow M \xrightarrow{f_0} E_0 \xrightarrow{f_1} E_1$$

-such that $f_0$ and the map $E_0/\mathrm{Im}(f_0) \hookrightarrow E_1$ induced by $f_1$ are injective envelopes.

Given any $M \in A$-Mod, an *injective resolution of* $M$ is a complex:

$$0 \longrightarrow E_0 \xrightarrow{h_1} E_1 \xrightarrow{h_2} E_2 \xrightarrow{h_3} \ldots$$

-and a map $h_0 \in \mathrm{Hom}(M, E_0)$ such that the sequence:

$$0 \longrightarrow M \xrightarrow{h_0} E_0 \xrightarrow{h_1} E_1 \xrightarrow{h_2} E_2 \xrightarrow{h_3} \ldots$$

-is exact. The *injective dimension of* $M$- denoted $\mathrm{id}M$ is defined to be the minimal $m \in \mathbb{N}$ such that there exists an exact sequence of the form:

$$0 \to M \to E_0 \to E_1 \to \cdots \to E_{m-1} \to E_m \to 0$$

-with every $E_i$ being injective. If no such $m$ exists, then $\mathrm{id}M := \infty$.

The *right global dimension of* $A$ is defined to be $\max\{\mathrm{pd}M : M \in \text{Mod-}A\}$, and the *left global dimension of* $A$ is defined to be $\max\{\mathrm{id}M : M \in A\text{-Mod}\}$. If $A$ is a finite-dimensional $K$-algebra, then the right global dimension of $A$ and the left global dimension of $A$ are equal (see (A.4.9) of SS1). We refer to it as the *global dimension of* $A$.

### 2.1.2 Ext and Tor

Given any $M, N \in A$-Mod, and any $k \geq 1$, $\mathrm{Ext}^k(M,N)$ is defined as follows: take a projective resolution of $M$:

$$\cdots \xrightarrow{f_3} P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \longrightarrow 0$$

Applying the functor $\mathrm{Hom}(\_, N)$, we obtain a cochain complex:

$$\mathrm{Hom}(P_0, N) \xrightarrow{\mathrm{Hom}(f_1,N)} \mathrm{Hom}(P_1, N) \xrightarrow{\mathrm{Hom}(f_2,N)} \mathrm{Hom}(P_2, N) \xrightarrow{\mathrm{Hom}(f_3,N)} \cdots$$

Define:

$$\mathrm{Ext}^k(M,N) := \mathrm{Ker}(\mathrm{Hom}(f_{k+1}, N))/\mathrm{Im}(\mathrm{Hom}(f_k, N))$$

**Theorem 1.** *Given any* $X \in A$-Mod*, and any short exact sequence in* $A$-Mod*:*

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

*There exists a long exact sequence:*

$$\begin{aligned} 0 \quad &\longrightarrow \quad \mathrm{Hom}(N,X) \xrightarrow{\mathrm{Hom}(g,X)} \mathrm{Hom}(M,X) \xrightarrow{\mathrm{Hom}(f,X)} \mathrm{Hom}(L,X) \\ &\longrightarrow \quad \mathrm{Ext}^1(N,X) \longrightarrow \mathrm{Ext}^1(M,X) \longrightarrow \mathrm{Ext}^1(L,X) \\ &\longrightarrow \quad \mathrm{Ext}^2(N,X) \longrightarrow \mathrm{Ext}^2(M,X) \longrightarrow \mathrm{Ext}^2(L,X) \longrightarrow \cdots \end{aligned}$$

*Proof.* See [4, (2.5.2)] □

Given any $M \in A$-Mod, and $N \in \text{Mod-}A$, and $k \geq 1$, $\mathrm{Tor}_k^A$ is defined as follows: given a projective resolution of $M$:

$$\cdots \xrightarrow{f_3} P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \longrightarrow 0$$

-we induce the chain complex:

$$\cdots \xrightarrow{1\otimes f_3} N \otimes_A P_3 \xrightarrow{1\otimes f_2} N \otimes_A P_2 \xrightarrow{1\otimes f_1} N \otimes_A P_0$$

Then:

$$\mathrm{Tor}_k(N, M) := \mathrm{Ker}(1 \otimes f_k)/\mathrm{Im}(1 \otimes f_{k+1})$$

## 2.2 Model theory of modules

Given any ring, $R$, we denote the language of rings by $\mathcal{L}_R$. A formula $\phi(v_1, \ldots, v_n)$ in $\mathcal{L}_R$ called a *pp-formula* if it is of the form:

$$\exists v_{n+1}, \ldots, v_{n+m} \bigwedge_{i=1}^{k} \sum_{j=1}^{n+m} r_{ij} v_j = 0$$

-with $r_{ij} \in R$ for all $i$ and $j$. Fora all $M \in R$-Mod, and pp-formulas $\phi(v_1, \ldots, v_n)$, we define:

$$\phi(M) := \{\overline{m} \in M^n : M \models \phi(\overline{m})\}$$

Any such subset is called a *pp-definable subset of* $M^n$. We say that two pp-formulas $\phi(v_1, \ldots, v_n)$ and $\psi(v_1, \ldots, v_n)$ are *equivalent* if $\phi(M) = \psi(M)$ for all $M \in R$-Mod.

Given any $n \in \mathbb{N}$, there exists a partial ordering on the set of all (equivalence classes of) pp-formulas in the free variables $v_1, \ldots, v_n$, given by:

$$\phi \geq \psi \Longleftrightarrow \phi(M) \supseteq \psi(M) \text{ for all } M \in R\text{-Mod}$$

Furthermore, this poset- denoted ${}_R\mathrm{pp}^n$- is in fact a modular lattice, with the meet operation given by $\phi(\overline{v}) \wedge \psi(\overline{v})$, and the join given by:

$$\exists \overline{w}(\phi(\overline{w}) \wedge \psi(\overline{v} - \overline{w}))$$

In general, we refer to the lattice ${}_R\mathrm{pp}^1$ as ${}_R\mathrm{pp}$.

Given any $M \in R$-Mod, we define ${}_R\mathrm{pp}(M) := \{\phi(M) : \phi(v) \in_R \mathrm{pp}\}$- referred to as the lattice of pp-definable subgroups of $M$ (the lattice operations $\leq, \wedge, \vee$ being $\subseteq, \cap, +$ respectively). Of course, it is a quotient lattice of ${}_R\mathrm{pp}$- with the surjection being the map taking $\phi(v)$ to $\phi(M)$.

Given any $n \in \mathbb{N}^+$, a *pp-$n$-type* is a set of pp-formulas in ${}_R\mathrm{pp}^n$, which is closed under conjunction and logical implication. For example, given any $M \in R$-Mod and $\overline{m} \in M^n$, the *pp-type of $\overline{m}$ in $M$* is:

$$\mathrm{pp}^M(\overline{m}) := \{\phi \in_R \mathrm{pp}^n : \overline{m} \in \phi(M)\}$$

Also, given any $\psi \in {}_R\mathrm{pp}^n$, the *pp-type generated by $\psi$*- denoted $\langle\psi\rangle$ is the set $\{\phi \in_R \mathrm{pp}^n : \psi \leq \phi\}$. It is clearly a pp-type. A pp-$n$-type is said to be *finitely generated* if there exists $\psi \in {}_R\mathrm{pp}^n$ such that the pp-type is equal to $\langle\psi\rangle$.

A map $f \in \mathrm{Hom}(M, N)$ is said to be a *pure embedding* if, for all $n \in \mathbb{N}^+$ and $\overline{m} \in M$:

$$\mathrm{pp}^M(\overline{m}) = \mathrm{pp}^N(f(\overline{m}))$$

In fact, $f$ is a pure embedding if and only if $\mathrm{pp}^M(m) = \mathrm{pp}^N(f(m))$ for all $m \in M$. Note that any pure embedding is an embedding (taking $\phi(v)$ to be $v = 0$ shows this).

**Lemma 1.** *Given any set of modules $\{M_i : i \in I\}$, $\bigoplus M_{i\in I}$ and $\prod_{i\in I} M_i$ are elementarily equivalent.*

*Proof.* [16, (2.23)] □

Given any $n \in \mathbb{N}^+$, an *$n$-pointed module*, denoted $(M, \overline{m})$, is an $R$-module $M$ and an $n$-tuple $\overline{m} = (m_1, \dots, m_n)$ in $M^n$. Given any such module, we define $f_{(M,\overline{m})}$ to be the unique map in $\mathrm{Hom}(R^n, M)$ taking the element $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ of $R^n$ (with $i$th coordinate 1) to $m_i$.

Given any $n$-pointed modules $(M, \overline{m})$ and $(C, \overline{c})$ (where $\overline{m} = (m_1, \dots, m_n)$ and $\overline{c} = (c_1, \dots, c_n)$), a morphism from $(M, \overline{m})$ to $(C, \overline{c})$ is any $f \in \mathrm{Hom}(M, C)$ such that $f(m_i) = c_i$ for all $i \leq n$.

We write $(M, \overline{m}) \geq (C, \overline{c})$ whenever there exists a morphism from $(M, \overline{m})$ to $(C, \overline{c})$. $(M, \overline{m})$ and $(C, \overline{c})$ are said to be equivalent if both $(M, \overline{m}) \geq (C, \overline{c})$ and $(C, \overline{c}) \geq (M, \overline{m})$

The set of equivalence classes of finitely presented $n$-pointed modules, endowed with $\geq$, is a poset. Furthermore, this poset has a lattice structure, where the join of

$(M, \overline{m})$ and $(C, \overline{c})$, is given by $(M \oplus C, (\overline{m}, \overline{c}))$, and the meet is the pushout of $f_{(M,\overline{m})}$ and $f_{(C,\overline{c})}$.

An $n$-pointed finitely presented module $(C, \overline{c})$ is said to be a *free realisation of* $\phi(v_1, \ldots, v_n)$ if $\mathrm{pp}^C(c) = \langle \phi \rangle$.

**Lemma 2.** *Given any $M \in R\text{-mod}$, and any $n$-tuple $\overline{m}$ in $M$, the pp-type of $\overline{m}$ in $M$ is finitely generated.*

*Proof.* See [17, (1.2.6)] □

**Lemma 3.** *Every pp-formula has a free realisation.*

*Proof.* See [16, (8.12)] □

**Theorem 2.** *The lattice of $n$-pointed finitely presented modules is equivalent to ${}_R\mathrm{pp}^n$.*

*Furthermore, the equivalence is obtained by taking every pp-formula to a free realisation, and every pointed module $(M, \overline{m})$ to a generator of $\mathrm{pp}^M(\overline{m})$*

*Proof.* See [17, (3.1.4)] □

**Lemma 4.** *Given any pp-formula $\phi(v_1, \ldots, v_n)$, with free realisation $(C, \overline{c})$, and any $M \in A\text{-Mod}$, the exact sequence:*

$$R^n \xrightarrow{f_{(C,c)}} C \xrightarrow{\pi} \mathrm{Coker}(f_{(C,\overline{c})}) \to 0$$

*-gives rise to the exact sequence of abelian groups:*

$$0 \longrightarrow \mathrm{Hom}(\mathrm{Coker}(f_{(C,\overline{c})}), M) \xrightarrow{(\pi,-)} \mathrm{Hom}(C, M) \xrightarrow{g} \phi(M) \longrightarrow 0$$

*Where $g$ is the map taking any $h \in \mathrm{Hom}(C, M)$ to $h(\overline{c}) = (h(c_1), \ldots, h(c_n))$.*

*Proof.* See [17, (1.2.19)] □

### 2.2.1 Pure-injectives

An $R$-module, $M$ is said to be *pure-injective* if it is injective over pure embeddings: i.e. given any pure embedding $f \in \mathrm{Hom}(L, N)$, any map $g \in \mathrm{Hom}(L, M)$ factors through $f$.

A module is said to be *algebraically compact* if every finitely satisfiable system of linear equations in (possibly infinitely many) free variables, with parameters in $M$, is satisfiable in $M$.

Equivalently (by [17, (4.2.1)]), a module is algebraically compact if every pp-1-type with parameters from $M$ has a solution in $M$.

**Theorem 3.** *An R-module is pure-injective if and only if it is algebraically compact.*

*Proof.* See [17, (4.3.11)] □

A module is said to be *Σ-pure injective* if every direct sum of copies of $M$ is pure-injective.

**Theorem 4.** *A module $M$ is Σ-pure-injective if and only if* $\mathrm{pp}(M)$ *has the descending chain condition.*

*Proof.* See [17, (4.4.5)] □

**Lemma 5.** *Let $A$ be a $K$-algebra, and $M$ any $A$-module, which is of countable dimension over $K$. Then $M$ is Σ-pure-injective if and only if it is pure injective.*

*Proof.* See [17, (4.4.9)] and [17, (4.4.10)] □

**Lemma 6.** *Suppose $M$ is a module, such that -for any $x, y \in M$- there exists a pp-formula $\rho(v, v')$ such that:*

- $(x, y) \in \rho(M)$
- $(x, 0) \notin \rho(M)$

*Then $M$ is indecomposable.*

*Proof.* Let $M = M_1 \oplus M_2$, and pick any non-zero $m_1 \in M_1$ and $m_2 \in M_2$. Let $x = (m_1, m_2)$ and $y = (0, m_2)$. Then the map:

$$M_1 \oplus M_2 \twoheadrightarrow M_1 \hookrightarrow M_1 \oplus M_2$$

(where the maps are the canonical projection and canonical embedding of the direct summand) takes $x$ to $x$, and $y$ to 0. Consequently, given any pp-formula $\rho(v_1, v_2)$:

$$M \models \rho(x, y) \Longrightarrow M \models \rho(x, 0)$$

-so there are no pp-formulas which satisfy the required conditions. □

**Lemma 7.** *Suppose $M$ is a pure injective indecomposable module. Then, for any $x, y \in M$, there exists a pp-formula $\rho(v, v')$ such that:*

- $(x, y) \in \rho(M)$
- $(x, 0) \notin \rho(M)$

*Proof.* See [16, (4.11)] □

Every module with local endomorphism ring is indecomposable: to see this, take any non-indecomposable module, $M_1 \oplus M_2$ and let $f$ be the map:

$$M_1 \oplus M_2 \twoheadrightarrow M_1 \hookrightarrow M_1 \oplus M_2$$

-where there two maps are the projection onto, and the embedding of, the direct summand $M_1$. Then clearly both $f$ and $1 - f$ are non-invertible, so $\mathrm{End}(M_1 \oplus M_2)$ is not local.

**Theorem 5.** *Every indecomposable pure-injective module has local endomorphism ring.*

*Proof.* See [17, (4.3.43)] □

### 2.2.2 Pp-pairs and finitely presented functors

An object $C$ of a category is said to be *finitely presented* if the functor $\mathrm{Hom}(C, \_)$ commutes with direct limits. The following result describes the finitely presented objects in the category of functors from $R$-mod to **Ab** (the category of abelian groups).

**Lemma 8.** *For every finitely presented functor* $F \in (R\text{–mod}, \mathbf{Ab})$*, there exists* $A, B \in R$-mod *and* $f \in \mathrm{Hom}(A, B)$ *such that* $F \simeq \mathrm{Coker}(f, \_)$*.*

*Furthermore, every functor in* $(R\text{-mod}, \mathbf{Ab})$ *of the form* $\mathrm{Coker}(f, \_)$ *(with* $A, B \in R$-mod*) is finitely presented.*

*Proof.* See [17, (10.2.1)] □

We denote by $(R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$ the full subcategory of $(R\text{-mod}, \mathbf{Ab})$ containing all the finitely presented functors.

A *pp-pair* is any pair of pp-formulas $\phi(v)$ and $\psi(v)$ such that $\phi \geq \psi$. We usually write them as $\phi/\psi$. Given any pp-pair $\phi/\psi$, and any $M \in R$-Mod, we say that $\phi/\psi$ is *open on* $M$ if $\phi(M) > \psi(M)$, and *closed on* $M$ otherwise.

A pp-pair is said to be *proper* if there exists $M \in R$-Mod such that $\phi(M) > \psi(M)$. Given any pp-pair $\phi/\psi$ and $M \in R$-Mod, we denote by $(\phi/\psi)(M)$ the group $\phi(M)/\psi(M)$.

Every pp-pair $\phi/\psi$ determines a unique functor $F_{\phi/\psi} : R\text{–Mod} \to \mathbf{Ab}$ which takes any $R$-module $M$ to $(\phi/\psi)(M)$.

Let $\phi/\psi$ and $\phi'/\psi'$ be pp-pairs in $n$ and $m$ free variables (respectively). Suppose that there is a pp-formula $\rho(\overline{x}, \overline{y})$ (where $\overline{x}$ has length $n$ and $\overline{y}$ has length $m$) such that:

$$\rho(\overline{x}, \overline{y}) \wedge \psi(\overline{x}) \leq \psi'(\overline{y})$$

$$\rho(\overline{x}, \overline{y}) \wedge \phi(\overline{x}) \leq \phi'(\overline{y})$$

$$\phi(\overline{x}) \leq \exists \overline{y} \rho(\overline{x}, \overline{y})$$

Then $\rho$ defines a unique map $f : (\phi/\psi)(M) \to (\phi'/\psi')(M)$, for any $M \in R$-Mod, as follows: Given any $a \in \phi(M)$, there exists $\overline{b} \in M$ such that $M \models \rho(\overline{a}, \overline{b})$. Define $f(\overline{a} + \psi(M))$ to be $\overline{b} + \psi'(M)$. This map is well defined: Given any $\overline{b}, \overline{c} \in \phi'(M)$ such that $M \models \rho(\overline{a}, \overline{b})$ and $M \models \rho(\overline{a}, \overline{c})$, we have that:

$$M \models \psi(0) \wedge \rho(0, \overline{b} - \overline{c})$$

-and hence that $\overline{b} - \overline{c} \in \psi'(M)$, so $\overline{b} + \psi'(M) = \overline{c} + \psi'(M)$, as required.

Any two such pp-formulas, $\rho(\overline{x},\overline{y})$ and $\rho'(\overline{x}',\overline{y}')$, satisfying those three conditions are said to be equivalent if, for all $M \in R$-Mod, the map $(\phi/\psi)(M) \to (\phi'/\psi')(M)$ defined by $\rho$ is equal to the map $(\phi/\psi)(M) \to (\phi'/\psi')(M)$ defined by $\rho'$.

We define the category of pp-pairs, denoted ${}_R\mathbb{L}^{\text{eq}+}$, to be the category whose objects are the pp-pairs, and whose morphisms are the equivalence classes of pp-conditions of the form $\rho(\overline{x},\overline{y})$, as described above.

**Theorem 6.** *For any ring, $R$, $(R\text{−mod}, \mathbf{Ab})^{\text{fp}}$ is equivalent to ${}_R\mathbb{L}^{\text{eq}+}$.*

*Proof.* See [17, (10.2.30)]. □

### 2.2.3 Pure-injective hulls

Given any $M \in R$-Mod, the *pure-injective hull of* $M$ is a pure-injective module $H(M)$ and a pure-embedding $f : M \to H(M)$ such that $f$ does not factor through any direct summand of $H(M)$. The module $H(M)$ may also be referred to as the pure-injective hull of $M$.

**Theorem 7.** *Every module $M \in R$-Mod has a pure-injective hull $f : M \to H(M)$. Furthermore, it is unique up to isomorphism: given any second pure injective hull $g : M \to N$ of $M$, there exists an isomorphism $j : H(M) \to N$ such that $jf = g$.*

*Proof.* See [17, (4.3.18)] □

**Theorem 8.** *Every module $M$ is elementarily equivalent to its pure-injective hull- i.e. given any sentence $\sigma$, $M \models \sigma$ if and only if $H(M) \models \sigma$.*

*Proof.* See [26, Cor 4]. □

**Lemma 9.** *Let $f : M \to H(M)$ be a pure-injective hull of $M$. Then given any pure-injective $N \in R$-Mod, and pure embedding $g : M \to N$, there exists $h \in \operatorname{Hom}(H(M), N)$ such that $hf = g$.*

*Furthermore, any $h$ such that $g = hf$ must be pure, and hence a section.*

*Proof.* See [17, (4.3.17)] □

**Theorem 9.** *Given any $M \in R$-Mod, any pp-pair is open on $M$ if and only if it is open on $H(M)$.*

*Proof.* See [17, (4.3.21)] □

A module $M$ is said to be *superdecomposable* if it has no indecomposable direct summands.

**Theorem 10.** *Given any pure-injective module, $M$, there exists a set of indecomposable pure-injective modules $\{N_\lambda : \lambda \in \Lambda\}$ and a superdecomposable pure-injective module $N_c$ such that $M \simeq N_c \oplus H(\bigoplus_\lambda N_\lambda)$*

*Furthermore, $N_c$ and the modules $N_\lambda$ (and their multiplicities) are unique up to isomorphism.*

*Proof.* See [17, (4.4.2)] □

**Lemma 10.** *Let $\{M_i : i \in I\}$ be any collection of $R$-modules. Given any pure-injective indecomposable module $N$, and pure embedding $f : N \to \bigoplus_{i\in I} M_i$, $N$ must be isomorphic to a direct summand of some $M_i$.*

*Proof.* See [17, (4.4.1)] □

## 2.3 Ziegler spectrum

Given any set of pp-pairs $T = \{\phi_i/\psi_i : i \in I\}$, let $\mathrm{Mod}(T)$ denote the subcategory of $R$-Mod whose objects are precisely the $R$-modules $M$ such that $\phi_i(M) = \psi_i(M)$ for all $i \in I$. Any such category is called a *definable subcategory of* $R$-Mod, and the object class of $\mathrm{Mod}(T)$ is called a definable subclass of $R$-Mod.

**Theorem 11.** *Let $\mathcal{Z}$ be a subclass of $R$-Mod. Then $\mathcal{Z}$ is definable if and only if it is closed under direct products, direct limits and pure submodules.*

*Proof.* See [17, (3.4.7)] □

Given any $M \in R$-Mod, the *definable subcategory of* $R$-Mod *generated by* $M$, denoted $\langle M \rangle$, is defined to be intersection of all definable subcategories of $R$-Mod containing $M$.

The left Ziegler Spectrum of a ring $R$- denoted ${}_R\text{Zg}$- is the topological space whose points are the pure-injective indecomposable left $R$-modules, and whose closed sets are the sets of the form:

$$\{X : \phi_i(X) = \psi_i(X) \text{ for all } i \in I\}$$

-for any set $\{\phi_i/\psi_i : i \in I\}$ of pp-pairs.

**Theorem 12.** *Given any proper pp-pair $\phi/\psi$, and any $M \in R$-Mod such that $\phi(M) > \psi(M)$, there exists a pure injective indecomposable module $N$ in $\langle M \rangle$ such that $\phi(N) > \psi(N)$.*

*Proof.* See [28, (4.8)] □

Given any $M \in R$-Mod, define the *support of* $M$- denoted $\text{Supp}(M)$- to be the set of all pure-injective indecomposables in $\langle M \rangle$. Notice that, given any $M, N \in R$-Mod, $\text{Supp}(M) \subseteq \text{Supp}(N)$ if and only if every pp-pair closed on $N$ is closed on $M$.

Given any set $\mathcal{Z}$ of $R$-modules, we define $\text{Supp}(\mathcal{Z})$ to be the set of all pure-injective indecomposable modules $M$ such that every pp-pair closed on every module in $\mathcal{Z}$ is closed on $M$.

### 2.3.1 Cantor-Bendixson rank

Given a topological space $T$, we say a point $p \in T$ is *isolated* if $\{p\}$ is an open set.

In order to define the Cantor-Bendixson rank of $T$, one has to recursively define a topological space $T_\alpha$ for every ordinal $\alpha$, as follows: First of all, let $T_0$ be $T$.

Given $T_\alpha$, let $T_{\alpha+1}$ be the set of all non-isolated points of $T_\alpha$- this is a closed set in $T_\alpha$. Let the topology on $T_{\alpha+1}$ be the topology induced from $T_\alpha$: i.e. the closed subsets of $T_{\alpha+1}$ are those of the form $X \cap T_{\alpha+1}$- where $X$ is a closed subset of $T_\alpha$.

Given a limit ordinal $\gamma$, define $T_\gamma = \bigcap_{\alpha<\gamma} T_\alpha$, and let the closed subsets of $T_\gamma$ be the sets $X \cap T_\gamma$- for every closed subset $X$ of $T$.

We say that a point $p$ in $T$ has *Cantor-Bendixson rank* $\alpha$ if $p \in T_\alpha \backslash T_{\alpha-1}$. We say that $T$ has Cantor-Bendixson rank $\alpha$ if $T_{\alpha-1} \neq \emptyset$ and $T_\alpha = \emptyset$

Given any pp-pair $\phi/\psi$, $(\phi/\psi)$ denotes the set of all indecomposable pure-injective $R$-modules $M$ such that $\phi(M) > \psi(M)$. We say that a pp-pair is *minimal on $M$* if $\phi(M) > \psi(M)$, and there is no pp-formula $\chi$ such that $\phi(M) > \chi(M) > \psi(M)$.

We say that a closed subset $X$ of ${}_R\mathrm{Zg}$ satisfies the *isolation condition* if, for all closed subsets $Y$ of $X$ and all isolated points $N$ of $Y$, there exists a $Y$-minimal pp-pair $\phi/\psi$ such that $(\phi/\psi) \cap Y = \{N\}$.

**Lemma 11.** *Given any closed set $X$ of ${}_R\mathrm{Zg}$, the following are equivalent:*

1. *$X$ satisfies the isolation condition.*

2. *Every $N \in X$ which is isolated in some closed subset of $X$ is isolated in* $\mathrm{Supp}(N)$ *by a minimal pair.*

*Proof.* See [17, (5.3.16)] □

## 2.4 Bound quiver algebras

Let $Q = (Q_0, Q_1)$ denote any finite quiver- where $Q_0$ is the set of vertices, and $Q_1$ the set of arrows. Given any $\alpha \in Q_1$, let $s(\alpha)$ and $t(\alpha)$ denote the source and target of $\alpha$.

A *path of length $n$* in $Q$ is any string $w = \alpha_1\alpha_2 \ldots \alpha_n$ of elements of $Q_1$, such that $s(\alpha_i) = t(\alpha_{i+1})$ for all $i \leq n-1$. We define $s(w) = s(\alpha_n)$ and $t(w) = t(\alpha_1)$.

For each $a \in Q_0$, we define a "path of length 0", $e_a$, such that $s(e_a) = t(e_a) = a$.

Given a field, $K$, the path algebra $KQ$ is defined as follows: As a $K$-vector space, it has basis given by the set of all paths in $Q$: the multiplication of elements in $KQ$ is such that:

$$w \times u = \begin{cases} wu & \text{if } s(w) = t(u) \\ 0 & \text{otherwise} \end{cases}$$

$$w \times e_a = \begin{cases} w & \text{if } s(w) = a \\ 0 & \text{otherwise} \end{cases}$$

$$e_b \times w = \begin{cases} w & \text{if } t(w) = b \\ 0 & \text{otherwise} \end{cases}$$

For each $a \in Q_0$, $e_a$ is an idempotent. And $1 = \sum_{a \in Q_0} e_a$.

A *relation* in $KQ$ is a finite $K$-linear combination of paths in $Q$- all of which have a common source, and a common target. A bound quiver algebra is any $K$-algebra of the form $KQ/\mathcal{I}$, where $\mathcal{I}$ is an ideal generated by finitely many relations.

A *$K$-representation of* $Q$ is any set of $K$-vector spaces $\{M_a : a \in Q_0\}$, and a set of morphisms $\{f_\alpha \in \mathrm{Hom}_K(M_{s(\alpha)}, M_{t(\alpha)}) : \alpha \in Q_1\}$. We say that it is finite dimensional if $\bigoplus_{a \in Q_0} M_a$ is finite dimensional.

Given any path $w = \alpha_1\alpha_2 \ldots \alpha_n$ in $Q$, denote $f_{\alpha_1} f_{\alpha_2} \ldots f_{\alpha_n}$ by $f_w$. We say that a $K$-representation of $Q$ is *bound by* $\mathcal{I}$, if $\sum_i \lambda_i f_{w_i} = 0$ for all relations $\sum_i \lambda_i w_i$ which generate $\mathcal{I}$.

We denote by $\mathrm{Rep}_K(Q, \mathcal{I})$ (and respectively, $\mathrm{rep}_K(Q, \mathcal{I})$) the category of all $K$-representations of $Q$ (respectively, finite dimensional $K$-representations of $Q$) which are bound by $\mathcal{I}$.

**Theorem 13.** *$\mathrm{Rep}_K(Q, \mathcal{I})$ and Mod $KQ/\mathcal{I}$ are equivalent categories.*

*Furthermore, if $Q$ is a finite quiver, then $\mathrm{rep}_K(Q, \mathcal{I})$ and mod $KQ/\mathcal{I}$ are equivalent categories.*

*Proof.* See [1, (III.1.6)] □

A quiver is said to be *acyclic* if there are no cyclic paths in $Q$. The *underlying graph* of a quiver is the (undirected) graph obtained by replacing each arrow in $Q_1$ by an undirected edge.

### 2.4.1 The quiver of an algebra

Let $A$ be any finite dimensional $K$-algebra. An *idempotent* of $A$ is any element $e \in A$ such that $e^2 = e$. Idempotents $e_i$ and $e_j$ are said to be *orthogonal* if $e_i e_j = e_j e_i = 0$. An idempotent $e$ is said to be *primitive* if there is no pair of orthogonal idempotents $e_i$ and $e_j$ in $A$ such that $e = e_i + e_j$. A *central idempotent* of $A$ is any idempotent $e$ of $A$ such that $ae = ea$ for all $a \in A$.

The radical of $A$, denoted $\mathrm{rad}(A)$, is the intersection of all maximal right ideals of $A$.

An algebra $A$ is said to be *connected* if it cannot be written as a direct product of two non-zero algebras- or equivalently, the only central idempotents of $A$ are 0 and 1.

Let $\{e_1, e_2, \dots, e_n\}$ be the complete set of primitive orthogonal idempotents in $A$. We say that $A$ is *basic* if $e_iA \ncong e_jA$ for all $i \neq j$.

The quiver of $A$- denoted $Q_A$- is the quiver with vertex set $\{1, 2, \dots, n\}$, and $\dim_K(e_i(\mathrm{rad}(A)/\mathrm{rad}^2(A))e_j)$ arrows from the vertex $i$ to the vertex $j$.

**Theorem 14.** *For any basic, connected, finite dimensional $K$-algebra $A$, there exists an admissible ideal, $\mathcal{I}$ of $KQ_A$ such that $A \cong KQ_A/\mathcal{I}$.*

*Proof.* See [1, (II.3.7)] □

## 2.5 Auslander-Reiten quivers

Given any $M, N \in A$-mod, a map $f \in \mathrm{Hom}(M, N)$ is said to be *irreducible* if $f$ is neither a section nor a retraction, and given any factorisation:

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "g"'] & & N \\ & L \arrow[ur, "h"'] & \end{tikzcd}$$

-either $g$ must be a section, or $h$ a retraction.

Given any $K$-algebra $A$, the *Jacobson radical* of $A$-mod (denoted $\mathrm{rad}_A$) is the two-sided ideal in the category $A$-mod defined by:

$$\mathrm{rad}_A(X, Y) = \{h \in \mathrm{Hom}_A(X, Y) : 1_X - g \circ h \text{ is invertible for all } g \in \mathrm{Hom}(Y, X)\}$$

-for all $X, Y \in A$-mod. Given any $n \in \mathbb{N}$, define $\mathrm{rad}_A^n$ to be the ideal consisting of all finite sums of maps of the form:

$$X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_{n-1} \xrightarrow{f_n} X_n = Y$$

-with each map $h_i \in \mathrm{rad}_A(X_{i-1}, X_i)$. Notice that $\mathrm{rad}_A^{n+1} \subseteq \mathrm{rad}_A^n$ for all $n \geq 1$. Define $\mathrm{rad}_A^\infty := \bigcap_{n \geq 1} \mathrm{rad}_A^n$.

**Lemma 12.** *Take any* $X, Y \in A$-mod*, and* $f \in \mathrm{Hom}(X, Y)$*. Then* $f$ *irreducible if and only if* $f \in \mathrm{rad}_A(X, Y) \backslash \mathrm{rad}_A^2(X, Y)$*.*

*Proof.* See [1, (IV.1.6)] □

### 2.5.1 Translation quivers

A quiver (finite or infinite) is said to be *locally finite* if, given any $a \in Q_0$, there are only finitely many $\alpha \in Q_1$ with source $a$ and only finitely many $\beta \in Q_1$ with target $a$.

A *translation quiver* is a locally finite quiver, endowed with a a subset $Q_0' \subseteq Q_0$ and an injective map $\tau : Q_0' \to Q_0$ such that, for all $a \in Q_0'$ and $b \in Q_0$ the number of maps from $b$ to $a$ is equal to the number of maps from $\tau a$ to $b$.

### 2.5.2 The Auslander-Reiten quiver

Given any basic, connected, finite dimensional $K$-algebra, $A$, the Auslander-Reiten quiver $\Gamma_A$ is given as follows:

- The vertices of $\Gamma$ are the (isomorphism classes of) indecomposable modules in $A$ mod.
- Given any such modules $M$ and $N$, there are precisely $\dim_K(\mathrm{rad}_K(M, N)) - \dim_K(\mathrm{rad}_K^2(M, N))$ arrows with source $M$ and target $N$.

An almost split exact sequence is a short exact sequence:

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

-where $L$ and $N$ are indecomposable, and $f$ and $g$ are irreducible.

A map $f \in \mathrm{Hom}(L, M)$ is *left minimal* if any $h \in \mathrm{End}(M)$ such that $hf = f$ is an isomorphism. It is *left almost split* if it is not a section, and any map $h \in \mathrm{Hom}(L, X)$, which is not a section, factors through $f$. We say that $f$ is *left minimal almost split* if it is left minimal, and left almost split.

Dually a map $g \in \mathrm{Hom}(M,N)$ is *right minimal* if any $h \in \mathrm{End}(M)$ such that $gh = g$ is an isomorphism. It is *right almost split* if it is not a retraction, and any map $h \in \mathrm{Hom}(X,N)$, which is not a retraction, factors through $g$. We say that $f$ is *right minimal almost split* if it is left minimal, and left almost split.

**Lemma 13.** *Given any short exact sequence:*

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

*The following are equivalent:*

- *The sequence is an almost split exact sequence*
- *$f$ is left minimal almost split*
- *$g$ is right minimal almost split.*

*Proof.* See [1, (IV.1.13)] □

Given any $M \in A$-mod, we denote by $M^t$ the $A^{\mathrm{op}}$-module $\mathrm{Hom}_A(M,A)$. The functor $(\_)^t : A\text{-mod} \to A^{\mathrm{op}}\text{-mod}$ induces an isomorphism between the set of finitely generated projective right $A$-modules, and the set of finitely generated projective left $A$-modules.

Given any $M \in A$-mod which is not projective, take a minimal projective presentation of $M$:

$$P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \longrightarrow 0$$

The functor $(\_)^t$ is left exact. Applying it gives the exact sequence:

$$0 \longrightarrow M^t \xrightarrow{f_0^t} P_0^t \xrightarrow{f_1^t} P_1^t \longrightarrow \mathrm{Coker}(f_1^t) \longrightarrow 0$$

Define $\mathrm{Tr}(M) := \mathrm{Coker}(f_1^t)$. Define $\tau^-(M) := D\mathrm{Tr}(M)$, and $\tau(M) := \mathrm{Tr}D(M)$.

**Lemma 14.** *Let $M$ be any indecomposable module in $A$-mod. If $M$ is not injective, then there exists an almost split exact sequence of the form:*

$$0 \to M \to N \to \tau^- M \to 0$$

*If $M$ is not projective, then there exists an almost split exact sequence of the form:*

$$0 \to \tau M \to N \to M \to 0$$

*Proof.* See [1, (IV.3.1)] □

**Theorem 15.** *Let $M, N \in A$-mod be indecomposable. If $N$ is not injective, then:*

$$\mathrm{Ext}^1_A(M,N) \simeq D\mathrm{Hom}_A(\tau^- N, M)$$

*If $M$ is not projective, then:*

$$\mathrm{Ext}^1_A(M,N) \simeq D\mathrm{Hom}_A(N, \tau M)$$

*If either $M$ is projective, or $N$ injective, then:*

$$\mathrm{Ext}^1_A(M,N) = 0$$

*Proof.* See [1, (IV.2.13)] □

**Theorem 16.** *Let $A$ be a $K$-algebra, and $M, N \in A$-Mod.*

- *If $M \in A$-mod and* $\mathrm{pd}M \leq 1$ *then* $\mathrm{Ext}(M,N) \simeq D\mathrm{Hom}(N, \tau(M))$
- *If $N \in A$-mod and* $\mathrm{id}N \leq 1$ *then* $\mathrm{Ext}(M,N) \simeq D\mathrm{Hom}(\tau^{-1}(N), M)$

*Proof.* See [15] □

A connected component of the Auslander-Reiten quiver is said to be *preprojective* if it is acyclic, and every indecomposable $M \in A$-mod in the component is isomorphic to $\tau^{-n}P$, for some $n \geq 0$ and projective $P \in A$-mod.

Dually, a connected component of the Auslander-Reiten quiver is said to be *preinjective* if it is acyclic, and every indecomposable $M \in A$-mod in the component is isomorphic to $\tau^n E$, for some $n \geq 0$ and injective $E \in A$-mod.

Two components $\Gamma_1$ and $\Gamma_2$ of an Auslander-Reiten quiver are said to be *orthogonal* if $\mathrm{Hom}(M,N) = \mathrm{Hom}(N,M) = 0$ for all $M$ in $\Gamma_1$ and $N$ in $\Gamma_2$.

### 2.5.3 Projective covers in quiver algebras

Let $A$ be any bound quiver algebra, $KQ/\mathcal{I}$, such that $Q_0$ is finite. Then, given any $a \in Q_0$, the stationary path $e_a$ is a primitive idempotent of $A$. Furthermore, every primitive idempotent is equal to $e_a$ for some $a \in Q_0$.

For each $a \in Q_0$, define $P(a) \in A$-mod to be $Ae_a$. As a $K$-vector space, it has basis the set of all paths in $Q$ with source $a$, modulo $\mathcal{I}$.

**Lemma 15.** *Let $A = KQ/\mathcal{I}$, for some finite quiver $Q$. Then given any $a \in Q_0$, $P(a)$ is an indecomposable projective in $A$-Mod.*

*Furthermore, every projective module in $A$-mod is isomorphic to a finite direct sum of copies of modules in $\{P(a) : a \in Q_0\}$.*

*Proof.* See [3]. □

Dually, for each $a \in Q_0$, define $I(a) \in A$-mod to be $D(e_A A^{\mathrm{op}})$.

**Lemma 16.** *Let $A = KQ/\mathcal{I}$, for some finite quiver $Q$. Then given any $a \in Q_0$, $I(a)$ is an indecomposable injective in $A$-Mod.*

*Furthermore, every injective module in $A$-mod is isomorphic to a finite direct sum of copies of modules in $\{I(a) : a \in Q_0\}$.*

To every $a \in Q_0$, we also assign a simple module $S(a)$, with $M_a \simeq K$, and $M_b = 0$ for all $b \in Q_0 \backslash a$. Every simple $A$-module is isomorphic to $S(a)$, for some $a \in Q_0$.

**Lemma 17.** *Let $A = KQ/\mathcal{I}$, for some finite quiver $Q$.*

*Then every $M \in A$-mod has a projective cover.*

*Proof.* See [1, (I.5.8)] □

## 2.6 Torsion pairs and tilting

Given any class $\mathcal{Z}$ of modules in $A$-Mod, we write $\mathrm{Hom}(M, \mathcal{Z}) = 0$ (respectively, $\mathrm{Hom}(\mathcal{Z}, M) = 0$) to mean that $\mathrm{Hom}(M, Z) = 0$ (respectively, $\mathrm{Hom}(Z, M) = 0$) for all $Z \in \mathcal{Z}$.

We define $r(\mathcal{Z})$ to be the class of all $M \in A$-Mod such that $\mathrm{Hom}(\mathcal{Z}, M) = 0$, and $l(\mathcal{Z})$ to be the class of all $M \in A$-Mod such that $\mathrm{Hom}(M, \mathcal{Z}) = 0$.

Let $\mathcal{F}$ and $\mathcal{G}$ be classes of left $A$-modules. We say that $(\mathcal{F}, \mathcal{G})$ is a *torsion pair* if both $l(\mathcal{F}) = \mathcal{G}$ and $r(\mathcal{G}) = \mathcal{F}$. We call $\mathcal{F}$ the *torsionfree class*, and $\mathcal{G}$ the *torsion class*.

**Lemma 18.** *$(\mathcal{F}, \mathcal{G})$ is a torsion pair if and only if the following condition holds:*

- *Hom$(N, M) = 0$ for all $M \in \mathcal{F}$ and $N \in \mathcal{G}$*
- *For all $M \in A$-Mod, there exists a submodule $M'$ of $M$ in $\mathcal{F}$ such that $M/M' \in \mathcal{G}$.*

*Proof.* See lemma 1 of [22]. □

In any torsion pair, the torsionfree class is closed under submodules, and the torsion class is closed under quotient modules.

A torsion pair $(\mathcal{F}, \mathcal{G})$ is said to be *split* if $\mathrm{Ext}^1(\mathcal{F}, \mathcal{G}) = 0$- or, equivalently, if every $M \in R$-Mod can be decomposed into $M' \oplus M''$, with $M' \in \mathcal{F}$ and $M'' \in \mathcal{G}$.

### 2.6.1 Tilting functors

Given any set $\mathcal{Z}$ of $A$-modules, we define add$(\mathcal{Z})$ to be the set of all direct products of direct summands of modules in $\mathcal{Z}$. We also define $\prod \mathcal{Z}$ to be the set of all direct summands of direct products of modules in $\mathcal{Z}$.

Given any $M \in A$-Mod, we define add$(M) = $ add$(\{M\})$.

Given any finite dimensional $K$-algebra $A$, a *tilting $A$-module* is any $T \in A$-mod such that:

- pd$(T) \leq 1$
- Ext$(T, T) = 0$
- There exists an exact sequence:

$$0 \rightarrow_A A \rightarrow T' \rightarrow T'' \rightarrow 0$$

  -with $T'$ and $T''$ in add$(T)$.

Given any algebra $A$, and any tilting module $T \in A$-mod, let $B = \mathrm{End}_A({}_A T)$, and define the functors:

$$\Sigma_T := \mathrm{Hom}_A(T, \_) : A\text{-Mod} \rightarrow B\text{-Mod}$$

$$\Sigma'_T := \mathrm{Ext}_A(T, \_) : A\text{-Mod} \to B\text{-Mod}$$

$$\Upsilon_T :=_A T_B \otimes \_ : B\text{-Mod} \to A\text{-Mod}$$

$$\Upsilon'_T := \mathrm{Tor}_1^B(T, \_) : B\text{-Mod} \to A\text{-Mod}$$

Define two subclasses of $A$-Mod by $\mathcal{F}(T) := \mathrm{Ker}(\Sigma_T)$ and $\mathcal{G}(T) := \mathrm{Ker}(\Sigma'_T)$. Define two subclasses of $B$-Mod by $\mathcal{X}(T) := \mathrm{Ker}(\Upsilon_T)$ and $\mathcal{Y}(T) := \mathrm{Ker}(\Upsilon'_T)$.

**Theorem 17.** *$(\mathcal{F}(T), \mathcal{G}(T))$ is a torsion pair in $A$-Mod, and $(\mathcal{Y}(T), \mathcal{X}(T))$ is a torsion pair in $B$-Mod.*

**Theorem 18.** *$\Sigma_T$ and $\Upsilon_T$ are mutually inverse equivalences between the categories $\mathcal{G}(T)$ and $\mathcal{X}(T)$.*

*Also, $\Sigma'_T$ and $\Upsilon'_T$ are mutually inverse equivalences between the categories $\mathcal{F}(T)$ and $\mathcal{Y}(T)$.*

*Proof.* See [9, (1.4)] □

# Chapter 3

# Tubular Algebras

## 3.1 Tubular algebras

### 3.1.1 Integral quadratic forms

Given any finite dimensional $K$-algebra $A$, the Grothendieck group $K_0(A)$ is defined as follows: Let $F$ be the free group generated by isomorphism classes of modules in $A$-mod. Given any $M \in A$-mod, let $[M]$ denote its image as an element of $F$. Let $E$ be the subgroup of $F$ generated by elements of the form $[Y] - [X] - [Z]$, for every short exact sequence in $A$-mod:

$$0 \to X \to Y \to Z \to 0$$

Then $K_0(A) := F/E$.

Let $A$ be any finite dimensional $K$-algebra. It follows from the Jordan-Holder theorem that $K_0(A)$ is isomorphic to $\mathbb{Z}^n$- where $n$ is the number of non-isomorphic simple $A$-modules.

Let $KQ/\mathcal{I}$ be the bound quiver algebra isomorphic to $A$. Recall that there is exactly one simple $KQ/\mathcal{I}$-module for each vertex of $Q$- so we may label the vertices of $Q$ as $\{1, 2, \dots, n\}$.

Recall, the set of simple modules $\{S(a) : a \in Q_0\}$ from (2.5.3). Given any $M \in A$-mod, let $x_1, \dots, x_n$ be such that $[M] = \sum_{a=1}^{n} x_a[S(a)]$ (as elements of $K_0(A)$)- or equivalently, let $x_a = \dim_K(e_a M)$. Define $(x_1, \dots, x_n)$ to be the *dimension vector* of $M$- which we denote as $\underline{\dim}(M)$.

Given a finite dimensional basic $K$-algebra $A$, let $\{P(a) : a = 1, \dots, n\}$ denote the indecomposable projective $A$-modules. The *Cartan matrix* $C_A$ is defined to be the $n \times n$ matrix whose $i$-$j$-th entry is $\dim_K(\mathrm{Hom}_A(P(i), P(j)))$.

**Lemma 19.** *Let $A$ be any finite dimensional $K$-algebra with finite global dimension. Then $C_A$ has an inverse in $M_n(\mathbb{Q})$.*

*Furthermore, if $C_A$ is upper triangular, and* $\dim_K(\mathrm{End}_A(P(i))) = 1$ *for all indecomposable projectives $P(i)$, then $C_A$ has an inverse in $M_n(\mathbb{Z})$.*

*Proof.* See page 70 of [23]. □

In particular, if $A = KQ/\mathcal{I}$ for some acyclic quiver $Q$, then $C_A$ has an inverse in $M_n(\mathbb{Z})$.

Let $A$ be any finite dimensional $K$-algebra such that $C_A$ is invertible. Define the bilinear form $\langle \_, \_ \rangle : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ by:

$$\langle \overline{x}, \overline{y} \rangle := \overline{x} C^{-T} \overline{y}^T \text{ for all } \overline{x}, \overline{y} \in \mathbb{Z}^n$$

**Lemma 20.** *Let $A$ be a basic algebra of finite global dimension. Then, for all $X, Y \in$ $A$-mod:*

$$\langle \underline{\dim}(X), \underline{\dim}(Y) \rangle = \sum_{n \geq 0} (-1)^n \dim_K(\mathrm{Ext}^n(X, Y))$$

*-where* $\mathrm{Ext}^0(X, Y) := \mathrm{Hom}(X, Y)$.

*Proof.* See [1, (3.1.3)] □

Given any $K$-algebra $A$ such that $C_A$ is invertible in $M_n(\mathbb{Z})$, define $\chi_A : \mathbb{Z}^n \to \mathbb{Z}$ by:

$$\chi_A(\overline{x}) := \langle \overline{x}, \overline{x} \rangle$$

Then $\chi_A$ is an integral quadratic form- i.e. it is of the form:

$$\chi_A : (x_1, \ldots, x_n) \mapsto \sum_{i=1}^{n} x_i^2 + \sum_{i<j} \mu_{ij} x_i x_j$$

-with $\mu_{ij} \in \mathbb{Z}$ for all $i, j$. We say that a quadratic form $\chi : \mathbb{Z}^n \to \mathbb{Z}$ is *positive semi-definite* if $\chi(x_1, \ldots, x_n) \geq 0$ for all $(x_1, \ldots, x_n) \in \mathbb{Z}^n$. We define:

$$\mathrm{rad}(\chi) := \{\overline{x} \in \mathbb{Z}^n : \chi(\overline{x}) = 0\}$$

$\mathrm{rad}_\chi$ is a subgroup of $\mathbb{Z}^n$, and every element of $\mathrm{rad}(\chi)$ is called a *radical vector*. The *radical rank* of $\chi$ is defined to be the rank of $\mathrm{rad}(\chi)$ as a subgroup of $\mathbb{Z}^n$.

Let $\overline{x} = (x_1, \ldots, x_n)$ be any element of $\mathbb{Z}^n$. We say that $\overline{x}$ is *sincere* if $x_i \neq 0$ for all $i$. It is *positive* if $x_i \geq 0$ for all $i$.

The *support* of $\overline{x}$ is the set of all $i \in Q_0$ such that $x_i \neq 0$. We say that $\overline{x}$ is *connected* if and only if the full subquiver of $Q$ on the support of $\overline{x}$ is a connected subquiver of $Q$. We say that $\overline{x}$ is a *root* of $\chi_A$ if $\chi_A(\overline{x}) = 1$.

Let $U$ be any subset of $K_0(A)$, such that $\chi(\overline{x}) \geq 0$ for all $\overline{x} \in U$. Let $\mathcal{Y}$ be any module class in $A$-mod. We say that $\mathcal{Y}$ is *controlled by the restriction of $\chi_A$ to $U$* if:

- For all indecomposable $A$-modules $M$ in $\mathcal{Y}$, $\underline{\dim}(M)$ is either a connected positive root, or a connected positive radical vector of $\chi_A$ in $U$.

- For every connected positive root $\overline{x}$ of $\chi_A$ in $U$, there is one indecomposable $A$-module $M$ (up to isomorphism) in $\mathcal{Y}$ with $\underline{\dim}(M) = \overline{x}$.

- For every connected, positive radical vector $\overline{x} \in U$, there is an infinite family of (isomorphism classes of) indecomposable modules in $\mathcal{Y}$ with dimension vector $\overline{x}$

### 3.1.2 Tubes

Given any translation quiver $\Gamma = (\Gamma_0, \Gamma_1, \tau)$, the *geometric realisation of* $\Gamma$ is defined formally in [5, (4.1)]. Informally, we may define it as follows:

For all non-injective $x \in \Gamma_0$, define $\gamma_x$ to be an arrow from $x$ to $\tau^- x$. Let $\Gamma_1'$ be the set of all such arrows.

Recall that, for all non-injective $x \in \Gamma_0$, and all $y \in \Gamma_0$, the number of maps from $x$ to $y$ equals the number of maps from $y$ to $\tau^- x$- we may therefore assign, to each $\alpha : x \to y$, a unique map $\beta : y \to \tau^- x$- which we shall denote as $\sigma(\alpha)$.

For each arrow $\alpha \in \Gamma_1$, assign a 2-dimensional simplex, $\triangle_\alpha$ to $\alpha$, which is the triangle:

$$\begin{tikzcd} & y \arrow[dr, "\sigma(\alpha)"] & \\ x \arrow[ur, "\alpha"] \arrow[rr, "\gamma_\alpha"] & & \tau^- x \end{tikzcd}$$

We may informally define the geometric realisation of $\Gamma$ to be the "shape" obtained from the set of all triangles $\triangle_\alpha$ by identifying any edges of triangles which correspond to the same arrow in $\Gamma_1$, or in $\Gamma_1'$.

A translation quiver $\Gamma$ is called a *tube* if it contains a cyclic path, and the geometric realisation of $\Gamma$ is $S^1 \times \mathbb{R}_0^+$ (where $S^1$ is the unit circle).

Given any ring $R$, let $\Gamma'$ be a component of $\Gamma(R\text{-mod})$ which is a tube. We say that $\Gamma'$ is a *stable tube* if and only if every $R$-module associated to a vertex of $\Gamma'$ is neither projective nor injective.

Given any $n \in \mathbb{N}^+$, define $\mathbb{Z}\mathbb{A}_\infty/n$ to be the translation quiver with vertex set $\mathbb{Z}_n \times \mathbb{N}^+$, and arrow set:

$$\bigcup_{i \in \mathbb{Z}_n} \bigcup_{j \in \mathbb{N}^+} \{\alpha_{i,j} : (i,j) \to (i,j+1), \beta_{i,j} : (i,j+1) \to (i+1,j)\}$$

-with $\tau(i,j) = (i-1,j)$ for all $(i,j) \in \mathbb{Z}_n \times \mathbb{N}^+$.

**Lemma 21.** *A component $\Gamma'$ of an Auslander-Reiten quiver is a stable tube if and only if it is of the form $\mathbb{Z}\mathbb{A}_\infty/n$ for some $n \in \mathbb{N}^+$.*

*Proof.* See [23, (3.1.0)]. □

Given any stable tube $\Gamma$, which looks like $\mathbb{Z}\mathbb{A}_\infty/n$, we define the *rank* of $\Gamma$ to be $n$. A stable tube is said to be *homogeneous* if it has rank 1. We define the *mouth* of $\Gamma$ to be the vertices in $\{(i,1) : i \in \mathbb{Z}_n\}$.

Given any stable tube of rank $n$, we will normally write the module associated to the vertex $(i,j)$ as $E_i[j]$, and the maps as:

$$f_i^j : E_i[j] \to E_i[j+1]$$

$$g_i^j : E_{i-1}[j+1] \to E_i[j]$$

-for all $i \in \mathbb{Z}_n$ and $j \geq 1$. Notice that, for all $i$, we have an almost split exact sequence:

$$0 \to E_i[1] \xrightarrow{f_i^1} E_i[2] \xrightarrow{g_{i+1}^1} E_{i+1}[1] \to 0$$

And for all $i \in \mathbb{Z}_n$ and $k \geq 2$, we have an almost split exact sequence:

$$0 \to E_i[k] \xrightarrow{(f_i^k, g_{i+1}^{k-1})} E_i[k+1] \oplus E_{i+1}[k-1] \xrightarrow{(g_{i+1}^k, f_{i+1}^{k-1})^t} E_{i+1}[k] \to 0$$

Given any quasisimple module $E_i$ in a stable tube, we denote by $E_i[\infty]$ the direct limit of the sequence:

$$E_i[1] \xrightarrow{f_i^1} E_i[2] \xrightarrow{f_i^2} E_i[3] \xrightarrow{f_i^3} \dots$$

And we denote by $\widehat{E}_i$ the inverse limit of the sequence:

$$\cdots \xrightarrow{g_i^3} E_{i-2}[3] \xrightarrow{g_i^2} E_{i-1}[2] \xrightarrow{g_i^1} E_i[1]$$

**Lemma 22.** *Take any module $E_i[k]$ in a stable tube of rank $n$, and any indecomposable $M \in A$-Mod which is not isomorphic to $E_j[m]$ for any $j \in \mathbb{Z}_n$ and $m \leq n$.*

*Then for all $f \in \mathrm{Hom}(E_i[k], M)$ there exists $g \in \mathrm{Hom}(E_i[k+1], M)$ such that $f = gf_i^k$*

*Proof.* We prove the result by induction on $k$: Assume that we have the result for $k-1$. Consider the almost split exact sequence:

$$0 \to E_i[k] \overset{(f_i^k, g_{i+1}^{k-1})}{\longrightarrow} E_i[k+1] \oplus E_{i+1}[k-1] \overset{(g_{i+1}^k, f_{i+1}^{k-1})^t}{\longrightarrow} E_{i+1}[k] \to 0$$

Since $M \not\cong E_i[k]$, $f$ is not a section, so there exists $h \in \mathrm{Hom}(E_i[k+1], M)$ and $h' \in \mathrm{Hom}(E_{i+1}[k-1], M)$ such that $f = hf_i^k + h'g_{i+1}^{k-1}$. By the induction hypothesis, $h'$ factors through $f_{i+1}^{k-1}$- i.e. there exists $h'' \in \mathrm{Hom}(E_{i+1}[k], M)$ such that $h' = h'' f_{i+1}^{k-1}$. Then:

$$\begin{aligned} f &= hf_i^k + h'g_{i+1}^{k-1} \\ &= hf_i^k + h'' f_{i+1}^{k-1} g_{i+1}^{k-1} \\ &= hf_i^k - h'' g_{i+1}^k f_i^k \end{aligned}$$

-so $f$ factors through $f_i^k$, as required. □

Given any component $\Gamma'$, we say that an indecomposable module $M_0 \in A$-mod (but not in $\Gamma'$) is a *proper predecessor* of $\Gamma'$ if there exists a finite set of modules $M_1, \ldots, M_k \in A$-mod such that $M_k \in \Gamma'$ and $\mathrm{Hom}(M_{i-1}, M_i) \neq 0$ for all $i \leq k$. We say that an indecomposable module $N_0 \in A$-mod (but not in $\Gamma'$) is a *proper successor* of $\Gamma'$ if there exists a finite set of modules $N_1, \ldots, N_k \in A$-mod such that $N_k \in \Gamma'$ and $\mathrm{Hom}(N_i, N_{i-1}) \neq 0$ for all $i \leq k$. $\Gamma'$ is said to be *standard* if no indecomposable $M \in A$-mod is both a proper predecessor and a proper successor of $\Gamma'$

**Lemma 23.** *Let $\mathcal{T}(\rho)$ be any standard stable tube. Given any indecomposable modules $M$,$N$ in $\mathcal{T}(\rho)$, any map in Hom$(M, N)$ is a $K$ linear combination of the identity map (if $M \cong N$) and compositions of irreducible morphisms in the tube (i.e. the ones associated with arrows of the tube).*

*Proof.* See [27, (2.7)] $\square$

**Corollary 1.** *Given any two modules $E_i[m]$ and $E_{i'}[m']$ in a standard stable tube $\mathcal{T}(\rho)$, let $J$ be the set of all $a \geq 0$ such that $m - m' \leq a \leq m - 1$ and $n_\rho | (i' - i + a)$.*

*Then every map in $\mathrm{Hom}(E_i[m], E_{i'}[m'])$ can be written in the form:*

$$\sum_{a \in J} \lambda_a f_{i'}^{m'-1} f_{i'}^{m'-2} \dots f_{i'}^{m-a} g_{i+a}^{m-a} \dots g_{i+2}^{m-2} g_{i+1}^{m-1}$$

*-with all $\lambda_a \in K$.*

*Proof.* Let $h : E_i[m] \to E_{i'}[m']$ be any composition $h_{a'} \dots h_3 h_2 h_1$ of irreducible maps. Let $a$ be the number of maps $h_{j'}$ which are of the form $g_j^n$ (for some $j$ and $n$). Of course, the other $a' - a$ maps take the form $g_j^n$, for some $j$ and $n$.

Notice that, for all $n \geq 2$ and $j$ the almost split exact sequence starting at $E_j[n]$ gives $g_{j+1}^n f_j^n = -f_{j+1}^{n-1} g_{j+1}^{n-1}$.

If $a \geq m$, then we can "re-shuffle" $h$ into a map of the form:

$$h' g_{i-m}^1 f_{i-m-1}^1 g_{i-m-1}^1 g_{i-m-1}^2 \dots g_{i-m-1}^k$$

-for some $h' \in \mathrm{Hom}(E_{i+m+1}[2], E_{i'}[k'])$. The exact sequence starting at $E_{i-m-1}[1]$ gives that $g_{i-m}^1 f_{i-m-1}^1 = 0$- and hence that $h = 0$.

If $a < m$, then- since, we can "re-shuffle" $h$ into a map of the form:

$$\pm f_{i'}^{m'-1} f_{i'}^{m'-2} \dots f_{i'}^{m'-(a'-a)} g_{i+a}^{m-a} \dots g_{i+2}^{m-2} g_{i+1}^{m-1}$$

Notice that:

$$f_{i'}^{m'-1} f_{i'}^{m'-2} \dots f_{i'}^{m'-(a'-a)} \in \mathrm{Hom}(E_{i'}[m' - (a' - a)], E_{i'}[k'])$$

$$g_{i+a}^{m-a} \dots g_{i+2}^{m-2} g_{i+1}^{m-1} \in \mathrm{Hom}(E_i[m], E_{i+a}[m - a])$$

-and so $E_{i'}[m' - (a' - a)] \cong E_{i-a}[m - a]$. Thus $a' = 2a + m' - m$, and $i' - i + a$ is divisible by $n_\rho$.

Lemma 23 completes the proof. $\square$

Notice that the following lemma can be applied to any quasisimple module $E_i[1]$ in a standard stable tube- where the left-minimal almost split map is the map $f_i^1 : E_i[1] \to E_i[2]$:

**Lemma 24.** *Let $L \in A$-mod be indecomposable and such that* $\mathrm{End}(L)) \simeq K$, *and let $f : L \to M$ be left-minimal almost split. Then* $\mathrm{Coker}(f, L)$ *is 1-dimensional as a $K$-vector space, and* $\mathrm{Coker}(f, X) = 0$ *for all indecomposable modules $X$ (other than $L$).*

*Proof.* Given any indecomposable $X$ which is not isomorphic to $L$, any map $g \in \mathrm{Hom}(L, X)$ cannot be a section (since that would imply that $L$ is a direct summand of $X$), and hence factors through $f$.

The identity map in $\mathrm{End}(L)$ does not factor through $f$: since that would imply that there exists $h : M \to L$ such that $hf = 1$- i.e. that $f$ is a section, which contradicts the fact that it is almost split. The fact that $\dim_K(\mathrm{Coker}(f, L)) \leq \dim_K(\mathrm{Hom}(L, L))$ completes the proof. □

Notice that, given any tube, and any $k \geq 2$, the sequence:

$$0 \longrightarrow E_i \overset{f_i^{k-1} \ldots f_i^2 f_i^1}{\longrightarrow} E_i[k] \overset{g_{i+1}^{k-1}}{\longrightarrow} E_{i+1}[k-1] \longrightarrow 0$$

### 3.1.3 Generalised tubes

A *generalised tube* is any collection of modules and morphisms $(M_i, f_i, g_i)_{i \in \mathbb{N}^+}$, where $f_i : M_i \to M_{i+1}$ and $g_i : M_{i+1} \to M_i$ for all $i \in \mathbb{N}^+$, such that the following sequence is exact:

$$0 \longrightarrow M_i \overset{(f_i, g_i)^t}{\longrightarrow} M_{i+1} \oplus M_{i-1} \overset{(g_i, -f_{i-1})}{\longrightarrow} M_i \longrightarrow 0$$

(where $M_0$ is the zero module, and $f_0$ and $g_0$ are zero maps, by convention). Given any generalised tube, let $M_\infty$ denote the direct limit of the sequence:

$$M_1 \overset{f_1}{\longrightarrow} M_2 \overset{f_2}{\longrightarrow} M_3 \overset{f_3}{\longrightarrow} \ldots$$

-and let $\widehat{M}$ denote the inverse limit of the sequence:

$$\cdots \overset{g_3}{\longrightarrow} M_3 \overset{g_2}{\longrightarrow} M_2 \overset{g_1}{\longrightarrow} M_1$$

Of course, any homogeneous tube is a generalised tube. In fact, every stable tube (in the notation of (3.1.2)) gives us a generalised tube: where $M_i = E_1[i] \oplus \cdots \oplus E_{n_\rho}[i]$, and $f_i$ and $g_i$ are given by the maps:

$$f_i : (x_1, \ldots, x_{n_\rho}) \mapsto (f_1^i(x_1), \ldots, f_{n_\rho}^i(x_{n_\rho}))$$

$$g_i : (y_1, \ldots, y_{n_\rho}) \mapsto (g_{n_\rho}^i(y_{n_\rho}), g_1^i(y_1), \ldots, g_{n_\rho-1}^i(y_{n_\rho-1}))$$

Furthermore, $M_\infty \cong E_1[\infty] \oplus \cdots \oplus E_{n_\rho}[\infty]$ and $\widehat{M} \cong \widehat{E}_1 \oplus \cdots \oplus \widehat{E}_{n_\rho}$.

### 3.1.4 Krause's canonical exact sequence

Let $(M_i, f_i, g_i)_{i \in \mathbb{N}^+}$ be any generalised tube. Fix any $j \in \mathbb{N}$. Then for all $i \in \mathbb{N}^+$ we have a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & M_{i+1} \arrow[r, "f_{i+j-1}\cdots f_i"] \arrow[d, "g_i"] & M_{i+j+1} \arrow[r, "g_j\cdots g_{i+j-1}"] \arrow[d, "g_{i+j}"] & M_j \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M_i \arrow[r, "f_{i+j-1}\cdots f_i"] & M_{i+j} \arrow[r, "g_j\cdots g_{i+j-1}"] & M_j \arrow[r] & 0
\end{tikzcd}$$

Taking the inverse limit of such sequences, we obtain an exact sequence of the form:

$$0 \longrightarrow \widehat{M} \xrightarrow{\Phi^j} \widehat{M} \xrightarrow{h_j} M_j \longrightarrow 0$$

-where $\Phi \in \mathrm{Hom}(\widehat{M}, \widehat{M})$ is the kernel of $h_1$. Now, for all $j \in \mathbb{N}^+$, the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & \widehat{M} \arrow[r, "\Phi^j"] \arrow[d, equal] & \widehat{M} \arrow[r, "h_j"] \arrow[d, "\Phi"] & M_j \arrow[r] \arrow[d, "g_j"] & 0 \\
0 \arrow[r] & \widehat{M} \arrow[r, "\Phi^{j+1}"] & \widehat{M} \arrow[r, "h_{j+1}"] & M_{j+1} \arrow[r] & 0
\end{tikzcd}$$

Taking the direct limit of such sequences, we obtain an exact sequence:

$$0 \longrightarrow \widehat{M} \longrightarrow Q \longrightarrow M_\infty \longrightarrow 0$$

This sequence- as originally described by Krause in [14]- will be referred to as the *canonical exact sequence* associated to $(M_i, f_i, g_i)_{i \in \mathbb{N}^+}$.

**Theorem 19.** *Given any generalised tube over a finite dimensional $K$-algebra, consider the canonical exact sequence:*

$$0 \longrightarrow \widehat{M} \longrightarrow Q \longrightarrow M_\infty \longrightarrow 0$$

*Then:*

1. *Every infinite dimensional module in the Ziegler closure of* $\{M_i : i \in \mathbb{N}\}$ *is a direct summand of* $M_\infty \oplus \widehat{M} \oplus Q$.

2. $Q$ *has finite length over its endomorphism ring.*

3. *Every module in the Ziegler-closure of* $M_\infty$ *is a direct summand of* $M_\infty \oplus Q$

4. *Every module in the Ziegler-closure of* $\widehat{M}$ *is a direct summand of* $\widehat{M} \oplus Q$

*Proof.* See [14, (8.10)] □

Given any ring $R$, a module $G \in R$-mod is said to be *generic* (in the sense of [11]) if it is indecomposable, of finite endolength, and is not finitely presented.

**Theorem 20.** *Let* $E_i$ *and* $E_j$ *be any modules lying on the mouth of a stable tube in* $\mathcal{T}_\gamma$. *Then:*

- *The direct limit* $E_i[\infty]$ *is* $\Sigma$*-pure injective and indecomposable.*

- $E_i[\infty] \cong E_j[\infty]$ *if and only if* $E_i \cong E_j$.

- *The Ziegler closure of* $E_i[\infty]$ *consists of* $E_i[\infty]$ *and finitely many generic modules (which are the distinct direct summands of the middle term* $Q$ *of the canonical exact sequence)*

*Proof.* See [17, (15.1.9)] □

**Theorem 21.** *Let* $E_i$ *and* $E_j$ *be any modules lying on the mouth of a stable tube in* $\mathcal{T}_\gamma$. *Then:*

- *The inverse limit* $\widehat{E_i}$ *is pure injective and indecomposable.*

- $\widehat{E_i} \cong \widehat{E_j}$ *if and only if* $E_i \cong E_j$.

- *The Ziegler closure of* $\widehat{E_i}$ *consists of* $\widehat{E_i}$ *and finitely many generic modules (which are the distinct direct summands of the middle term* $Q$ *of the canonical exact sequence)*

*Proof.* See [17, (15.1.9)] □

### 3.1.5 Tubular families

A *tubular family* (indexed by $I$) is any set of tubes $\{T(\rho) : \rho \in I\}$ in a given Auslander-Reiten quiver. It is said to be *stable* if $T(\rho)$ is stable for all $\rho \in I$.

Given a stable tubular family $\mathcal{T} = \{T(\rho) : \rho \in I\}$, let $n_\rho$ be the rank of $\mathcal{T}(\rho)$ for each $\rho \in I$. Define the *type of* $\mathcal{T}$ to be the map $I :\to \mathbb{N}^+$ taking each $\rho \in I$ to $n_\rho$. If $\mathbb{T}$ contains only finitely many non-homogeneous tubes- say $\mathcal{T}(\rho_1), \ldots, \mathcal{T}(\rho_t)$- then we say that $\mathcal{T}$ *has type* $(n_{\rho_1}, \ldots, n_{\rho_t})$- we will usually assume that the tubes are labeled so that $n_{\rho_1} \geq \cdots \geq n_{\rho_t}$.

We say that a module $M$ *lies in* $\mathcal{T}$ (written $M \in \mathcal{T}$) if and only if it lies in one of the tubes in $\mathcal{T}$.

**Lemma 25.** *Let $\mathcal{T}$ be a standard stable tubular family in $A$-*mod*. Then* add$(\mathcal{T})$ *is an abelian category, which is serial, and closed under extensions.*

*Proof.* See [23, (3,1,3)] □

A tubular family is said to be *sincere* if, given any simple $A$-module $S$, there exists a module $T$ in one of the tubes $T(\rho)$ such that $S$ is one of the composition factors of $T$. If $A$ is a bound quiver algebra $KQ/\mathcal{I}$, then this is equivalent to saying that, for all vertices $a \in Q_0$, there exists a module $T$ in some tube such that $e_a T \neq 0$.

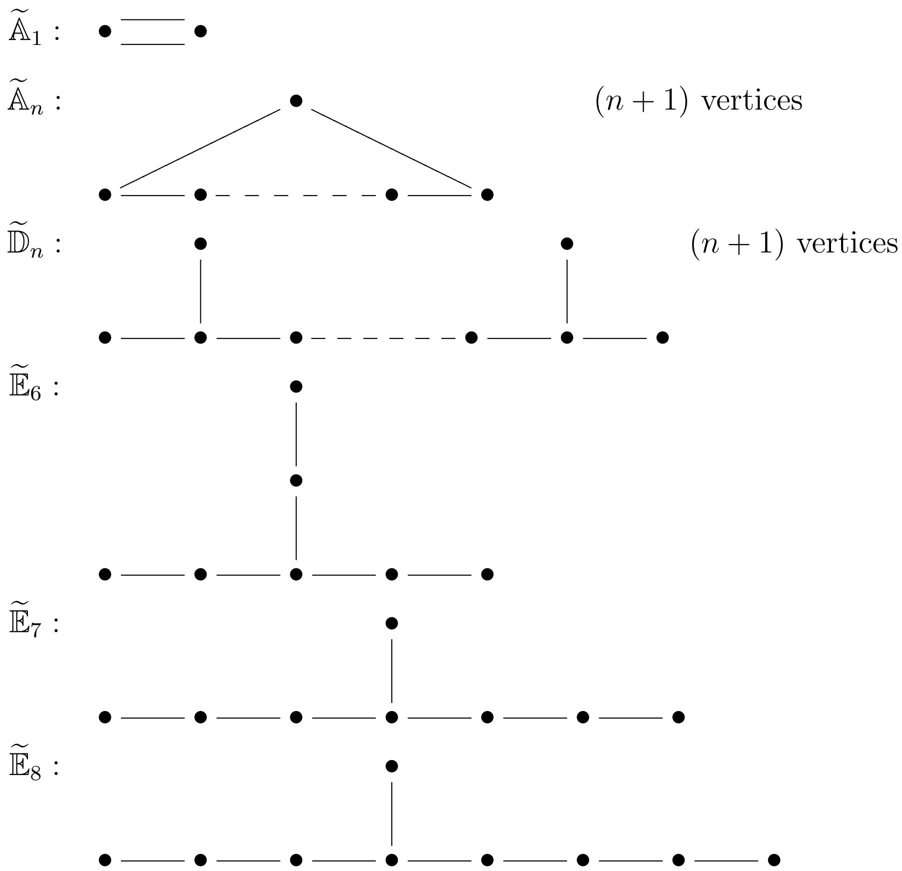
A tubular family $\mathcal{T} = \{\mathcal{T}(\rho) : \rho \in I\}$ is said to be *separating* if there exist subsets $\mathcal{P}$ and $\mathcal{Q}$ of $A$-mod such that:

- $\mathcal{P} \cup \mathcal{T} \cup \mathcal{Q}$ is a partition of the set of all indecomposable modules in $A$-mod.
- $\mathrm{Hom}(\mathcal{Q}, \mathcal{T}) = \mathrm{Hom}(\mathcal{T}, \mathcal{P}) = \mathrm{Hom}(\mathcal{Q}, \mathcal{P}) = 0$.
- $\mathrm{Hom}(\mathcal{T}(\rho), \mathcal{T}(\rho')) = 0$ for all $\rho \neq \rho'$ in $I$.
- Given any $M \in \mathcal{P}$, any $N \in \mathcal{Q}$, and any tube $T(\rho)$ in $\mathcal{T}$, every map $f \in \mathrm{Hom}(M, N)$ can be factored through a module in $\mathrm{add}(\mathcal{T}(\rho))$.

In which case, we say that $\mathcal{T}$ *separates* $\mathcal{P}$ *from* $\mathcal{Q}$.

### 3.1.6 Hereditary algebras and concealed algebras

The *Euclidean diagrams* are the graphs $\widetilde{\mathbb{A}}_n$ (with $n \geq 1$), $\widetilde{\mathbb{D}}_n$ (with $n \geq 4$), $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, and $\widetilde{\mathbb{E}}_8$- which are defined as follows:




A $K$-algebra $A$ is said to be *hereditary* if every submodule of a projective $A$-module is projective.

**Lemma 26.** *Let $Q$ be any acyclic quiver, whose underlying graph is a Euclidean diagram. Then $KQ$ is a representation-infinite hereditary algebra. The Auslander Reiten quiver of $KQ$ can be partitioned into $\mathcal{P} \cup \mathcal{T} \cup \mathcal{Q}$, where:*

- *$\mathcal{P}$ is a connected preprojective component, containing all the projective $KQ$-modules.*
- *$\mathcal{Q}$ is a connected preinjective component, containing all injective $KQ$-modules.*
- *$\mathcal{T}$ is a family of standard stable tubes $\{\mathcal{T}(\rho) : \rho \in \mathbb{P}^1(K)\}$, which separates $\mathcal{P}$ from $\mathcal{Q}$.*

- *Let $n_\rho$ denote the rank of $\mathcal{T}_\rho$ for each $\rho \in \mathbb{P}^1(K)$, and $n$ be the number of vertices in $Q$. Then $\sum_{\rho \in \mathbb{P}^1(K)}(n_\rho - 1) \leq n - 2$. In particular, only finitely many tubes are non-homogeneous.*

- $\mathrm{add}(\mathcal{T})$ *is a serial abelian category.*

*Proof.* See [27, (XI.2)] □

A *concealed algebra of Euclidean type* is any algebra of the form $\mathrm{End}(_AT)$-where $A = KQ$ is any quiver algebra over an acyclic quiver $Q$, whose underlying graph is Euclidean, and $T$ is any preprojective tilting $A$-module (as defined in section 2.6).

$\mathrm{End}(_AT)$ is said to be tame if and only if $A$ is tame.

**Theorem 22.** *Let $B$ be any concealed algebra of Euclidean type- i.e. $B = \mathrm{End}(_AT)$ for some tilting module $_AT$ over a quiver algebra $A = KQ$, where $Q$ is an acyclic quiver, whose underlying graph, $\overline{Q}$, is Euclidean.*

*Then the Auslander Reiten quiver can be partitioned into components $\mathcal{P}$, $\mathcal{T}$ and $\mathcal{Q}$, where:*

- *$\mathcal{P}$ is a preprojective component, containing all the projective $B$-modules.*

- *$\mathcal{Q}$ is a preinjective component, containing all the injective $B$-modules.*

- *$\mathcal{T}$ is a stable tubular family $\{\mathcal{T}(\rho) : \rho \in \mathbb{P}^1(K)\}$, separating $\mathcal{P}$ from $\mathcal{Q}$.*

- *There is a group isomorphism $f : K_0(A) \to K_0(B)$ such that the following diagram commutes:*

$$\begin{tikzcd} K_0(A) \times K_0(A) \arrow[r, "f \times f"] \arrow[dr, "\langle -,- \rangle_A"'] & K_0(B) \times K_0(B) \arrow[d, "\langle -,- \rangle_B"] \\ & \mathbb{Z} \end{tikzcd}$$

  *In particular, $\chi_B f = \chi_A$.*

- *$\chi_B(x) \geq 0$ for all $x \in \mathbb{Z}^n$, and $\mathrm{rad}(\chi_B)$ is a rank 1 subgroup of $K_0(B)$.*

- $\mathrm{gl.dim}(B) \leq 2$, *and* $\mathrm{pd}(X) \leq 1$ *for almost all (isomorphism classes of) indecomposable $B$-modules.*

- $\mathrm{pd}(X) = \mathrm{id}(X) = 1$ *for all modules* $X$ *in* $\mathcal{T}$.
- *The category* $\mathrm{add}(\mathcal{T})$ *is serial, abelian, and closed under extensions.*

*Furthermore, the tubular type of* $\mathcal{T}$ *is:*

- $(\min(p, q), \max(p, q))$, *if* $\overline{Q}$ *is* $\widetilde{\mathbb{A}}_m$*- where* $p$ *and* $q$ *are the number of anticlockwise and clockwise arrows (respectively) in* $Q$.
- $(2, 2, m-2)$ *if* $\overline{Q}$ *is* $\widetilde{\mathbb{D}}_m$ *(with* $m \geq 4$*)*
- $(2, 3, 3)$, *if* $\overline{Q}$ *is* $\widetilde{\mathbb{E}}_6$
- $(2, 3, 4)$, *if* $\overline{Q}$ *is* $\widetilde{\mathbb{E}}_7$
- $(2, 3, 5)$, *if* $\overline{Q}$ *is* $\widetilde{\mathbb{E}}_8$

*Proof.* See [27, (XI.3.3)] and [27, (XII.3.4)] □

An algebra $A$ is called *minimal representation-infinite* if it is representation-infinite, but such that $A/(AeA)$ is representation-finite for all idempotents $e$ of $A$ (other than 0 and 1).

We define an *extended Kronecker quiver* to be any quiver of the form $Q = (Q_0, Q_1)$, where $Q_0 = \{0, 1\}$ and $Q_1 = \{\alpha_1, \ldots, \alpha_t\}$ for some $t \geq 3$, with $\alpha_i : 0 \to 1$ for all $i \leq t$.

**Theorem 23.** *The following are equivalent, for any basic connected algebra, A:*

- *$A$ is minimal representation-infinite, and* $\Gamma(A\text{–}\mathrm{mod})$ *has a preprojective component containing all the projectives.*
- *$A$ is either a concealed algebra of Euclidean type, or the path algebra of an extended Kronecker quiver.*

*Proof.* See [27, (XIV.2.4)] □

A finite dimensional $K$-algebra $A$ is said to be *tame* if, for all $d \in \mathbb{N}$, there is a finite set of $A - K[X]$-bimodules $M_1, \ldots, M_n$ (which are free and of rank $d$ over $K[X]$) such that all but finitely many indecomposable $A$-modules of dimension $d$ are isomorphic to $M_i \otimes_{K[X]} K[X]/\langle X - \lambda \rangle$ for some $\lambda \in K$ and $i \leq n$.

The $K$-algebras over extended Kronecker quiver are well known to not be of tame representation type. Which gives the following result:

**Corollary 2.** *Let $A$ be any basic connected algebra of tame representation type. Then the following are equivalent:*

- *$A$ is minimal representation-infinite, and $\Gamma(A{-}\mathrm{mod})$ has a preprojective component containing all the projectives.*
- *$A$ is a concealed algebra of Euclidean type.*

### 3.1.7 Branches

Let $\mathcal{S}(-1,1)$ denote the set of all finite sequences in $\{1,-1\}$ (including the sequence of length 0, denoted $\emptyset$).

The *complete branch* is an infinite bounded quiver $Q = (Q_0, Q_1)$, with $Q_0 = \{b_s : s \in \mathcal{S}(1,-1)\}$, and $Q_1 = \{\beta_{(s,+1)} : s \in \mathcal{S}(1,-1)\} \cup \{\beta_{(s,-1)} : s \in \mathcal{S}(1,-1)\}$, where:

$$\beta_{(s,-1)} : b_{(s,-1)} \longrightarrow b_s$$

$$\beta_{(s,+1)} : b_s \longrightarrow b_{(s,+1)}$$

-with a relation $\beta_{(s,-1)}\beta_{(s,+1)} = 0$ for every $s \in \mathcal{S}(-1,1)$. Define a *finite branch* to be any finite, full, connected subquiver of the complete branch, containing the vertex $b_\emptyset$. The *length* of any finite branch is the number of vertices in it.

Notice that a finite branch $B$ is uniquely characterised by a finite set of non-empty finite sequences in $+1$ and $-1$: namely, let $\mathcal{S}^B$ be the set of non-empty sequences $a \in \mathcal{S}(-1,+1)$ such that $b_a$ is a vertex of $B$. Then the vertex set of $B$ is $\{b_\emptyset\} \cup \{b_a : a \in \mathcal{S}^B\}$, and the arrow set is $\{\beta_a : a \in \mathcal{S}^B\}$.

Let $KQ/I$ be a bound quiver algebra, and $B$ a finite branch- let $B_0$ denote the vertex set of $B$, and $B_1$ the set of arrows, and $\mathcal{I}_B$ the set of relations. Let $Q \cup B$ denote the quiver whose vertex set is the disjoint union of $Q_0$ and $B_0$, and whose arrow set is the disjoint union of $Q_1$ and $B_1$.

Let $Q'$ be the quiver obtained from $Q \cup B$ by identifying vertex $a$ with the vertex $b_\emptyset$. Every relation in $\mathcal{I}$ or in $\mathcal{I}_B$ gives us a unique relation of the quiver $Q'$: let $\mathcal{I}'$ be the ideal generated by all such relations in $Q'$. We call $KQ'/\mathcal{I}'$ the *algebra obtained from $KQ/I$ by adding the branch $B$ at $a$.*

For example: if $Q$ is the quiver:

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\gamma} 3$$

-with ideal $\mathcal{I} = \langle \gamma\alpha \rangle$, and $B$ is the branch uniquely determined by the set $\{-1, +1\}$:

$$b_{+1} \xleftarrow{\beta_{+1}} b_{\emptyset} \xleftarrow{\beta_{-1}} b_{-1}$$

-with relation $\mathcal{I}_B = \langle \beta_{-1}\beta_{+1} \rangle$, then the algebra obtained by adding $B$ at 2 is the $K$-algebra over the quiver:

$$\begin{tikzcd}
 & 1 \arrow[r, "\alpha"] & b_{\emptyset} \arrow[r, "\gamma"] \arrow[dl, "\beta_{+1}"'] & 3 \\
 b_{+1} & & & b_{-1} \arrow[ul, "\beta_{-1}"']
\end{tikzcd}$$

-with the ideal being $\langle \alpha\gamma, \beta_{-1}\beta_{+1} \rangle$.

Let $B$ be any finite branch. Given any vertices $b_{i_1,\ldots,i_n}$ and $b_{j_1,\ldots,j_m}$, we say that $b_{i_1,\ldots,i_n}$ *depends on* $b_{j_1,\ldots,j_m}$ if $(j_1, \ldots, j_m)$ is an initial subsequence of $(i_1, \ldots, i_n)$- i.e. if $m \leq n$ and $j_k = i_k$ for all $k \in \{1, 2, \ldots, m\}$.

Given any vertex $b_s$ of a finite branch $B$, we define $B(b_s)$ to be full subquiver of $B$ whose vertex set is the set of all vertices in $B$ which depend on $b_s$. Let $\ell_B(b_s)$ be the number of vertices in $B(b_s)$.

Recall that $B$ is a bound quiver $(B, \mathcal{I}_B)$. Let $\ell_B$ be the element of $K_0(KB/\mathcal{I}_B)$ given by:

$$\ell_B = \sum_{b_s \in B} \ell_B(b_s) S(b_s)$$

For example, if $B$ is the branch uniquely characterised by the set $\{-1, +1\}$, then $\ell_B = 3[S(b_\emptyset)] + [S(b_{-1})] + [S(b_{-1})]$.

### 3.1.8 Tubular extensions

Let $A$ be a $K$-algebra, and $X \in A$-mod. The *one-point extension of $A$ by $X$*- which is denoted $A[X]$, is the $K$-algebra:

$$\begin{pmatrix} A & {}_A X_K \\ 0 & K \end{pmatrix}$$

-where addition is matrix addition, and the multiplication of two elements is given by matrix multiplication.

If $A$ is a quiver algebra $KQ/\mathcal{I}$, then $A[X]$ will be (isomorphic to) a quiver algebra $KQ'/\mathcal{I}'$, where $Q'$ is obtained from $Q$ by adding an extra vertex, say 0, and precisely $\dim_K(X)$ arrows from 0 to vertices of $Q$. We call the vertex 0 of $Q'$ the *extension vertex.*

Let $A$ be a $K$-algebra, and $\mathcal{T} = \{\mathcal{T}(\rho) : \rho \in I\}$ a family of pairwise orthogonal stable tubes in $A$-mod. Given any module $E_1$ lying on the mouth of a tube in $\mathcal{T}$, and any finite branch $B_1$, define $A[E_1, B_1]$ to be the algebra obtained from the one-point extension $A[E_1]$ by adding the branch $B_1$ to the extension vertex of $A[E_1]$.

Given any $s \in \mathbb{N}^+$, any pairwise non-isomorphic modules $E_1, \dots, E_s$- each of which lies on the mouth of a tube in $\mathcal{T}$- and any set of finite branches $B_1, \dots, B_s$, define $A[E_i, B_i]_{i=1}^s$ inductively, using the formula:

$$A[E_i, B_i]_{i=1}^{k+1} := (A[E_i, B_i]_{i=1}^{k})[E_{k+1}, B_{k+1}]$$

-for all $k \geq 1$.

Any algebra $A[E_i, B_i]_{i=1}^s$ of this form is called a *tubular extension of $A_0$ using modules in $\mathcal{T}$*. For each module $E_i$, let $\rho_i \in I$ be such that $E_i$ lies in $\mathcal{T}(\rho_i)$. Let $r_i$ be the rank of $\mathcal{T}(\rho_i)$, and define the *extension type of $A[E_i, B_i]_{i=1}^s$ over $A$* to be the map $n : I \to \mathbb{N}^+$, such that:

$$n : \rho \mapsto n_\rho = r_\rho + \sum_{E_i \in \mathcal{T}(\rho)} |B_i|$$

(where $|B_i|$ denotes the number of vertices in $B_i$). If $n_\rho = 1$ for almost all $\rho \in I$, then we write the extension type as $(n_{\rho_1}, \dots, n_{\rho_t})$- where $\{\rho_1, \dots \rho_t\}$ is the set of all $\rho \in I$ such that $n_\rho \neq 1$, and (by convention) $n_{\rho_1} \geq n_{\rho_2} \geq \dots \geq n_{\rho_t}$.

**Theorem 24.** *Let $A_0$ be an algebra with a tubular family $\mathcal{T}$, which separates $\mathcal{P}$ from $\mathcal{Q}$.*

*Let $A = A_0[E_i, B_i]_{i=1}^t$ be any tubular extension of $A_0$ (where $E_i \in \mathcal{T}$). Then we can partition $A$-mod into $\mathcal{P}_0 \cup \mathcal{T}_0 \cup \mathcal{Q}_0$- where:*

- $\mathcal{P}_0$ *is the class of all modules in* $\mathcal{P}$ *(of course, every* $A_0$*-module is also an* $A$*-module).*

- $\mathcal{T}_0$ *is the class of all indecomposable* $M \in A$*-mod such that either* $M|A_0$ *is a non-zero element of* $\mathcal{T}$*, or the support of* $M$ *is contained in some* $B_i$ *and* $\langle \ell_{B_i}, \underline{\dim}(M)\rangle < 0$.

- $\mathcal{Q}_0$ *is the class of all indecomposable* $M \in A$*-mod such that either* $M|A_0$ *is a non-zero element of* $\mathcal{Q}$*, or the support of* $M$ *is contained in some* $B_i$ *and* $\langle \ell_{B_i}, \underline{\dim}(M)\rangle > 0$.

*Furthermore,* $\mathcal{T}_0$ *is a tubular family, which separates* $\mathcal{P}_0$ *from* $\mathcal{Q}_0$*.*

*Proof.* See [23, (4.7.1)] □

A *tubular algebra* is defined to be any tubular extension of a tame concealed algebra $A_0$ (using modules in the separating tubular family as defined in theorem 22), of extension type either $(2,2,2,2)$, $(3,3,3)$, $(4,4,2)$ or $(6,3,2)$.

### 3.1.9 Basic properties of a tubular algebra

Dual to the idea of tubular extension is the idea of tubular coextension. Given any algebra $A_\infty$, with a separating tubular family $\mathcal{T}$, let $\mathcal{T}^*$ be the set of all $A_\infty^{\mathrm{op}}$-modules which are duals of modules in $\mathcal{T}$. It is a separating tubular family of $A_\infty^{\mathrm{op}}$-mod. A *tubular coextension of* $A_\infty$ *using modules from* $\mathcal{T}$ is an algebra $A$ such that:

$$A^{\mathrm{op}} = A_\infty^{\mathrm{op}}[DE_i, K_i^{\mathrm{op}}]_{i=1}^t$$

-where $A_\infty^{\mathrm{op}}[DE_i, K_i^{\mathrm{op}}]_{i=1}^t$ is a tubular extension of $A_\infty^{\mathrm{op}}$ using modules from $\mathcal{T}^*$. The extension type of $A$ is defined to be the extension type of $A_\infty^{\mathrm{op}}[DE_i, K_i^{\mathrm{op}}]_{i=1}^t$.

An algebra $A$ is said to be *cotubular* if it is a tubular coextension of a tame concealed algebra $A_\infty$, of extension type either $(2,2,2,2)$, $(3,3,3)$, $(4,4,2)$ or $(6,3,2)$. Of course, an algebra $A$ is tubular if and only if $A^{\mathrm{op}}$ is cotubular.

**Lemma 27.** *Let* $A_\infty$ *be an algebra with a tubular family* $\mathcal{T}$*, which separates* $\mathcal{P}$ *from* $\mathcal{Q}$*.*

*Let $A$ be a tubular coextension of $A_\infty$ using modules from $T$. Then we can partition $A$-mod into $\mathcal{P}_\infty \cup \mathcal{T}_\infty \cup \mathcal{Q}_\infty$, where $\mathcal{T}_\infty$ is a tubular family, which separates $\mathcal{P}_\infty$ from $\mathcal{Q}_\infty$.*

*Proof.* This is just the dual of theorem 24 □

**Theorem 25.** *An algebra is tubular if and only if it is cotubular.*

*Proof.* See [23, (5.2.3)]. □

**Theorem 26.** *Let $A$ be a tubular algebra. Let $A_0$ and $A_\infty$ be tame concealed algebras, such that $A$ is a tubular extension of $A_0$, and a cotubular extension of $A_\infty$. Then $A_0$ and $A_\infty$ are uniquely determined by $A$.*

*Let $h_0$ and $h_\infty$ be the positive radical generators of $A_0$ and $A_\infty$ respectively. Then* $\operatorname{rad}(\chi_{\mathrm{A}})$ *is a group of rank 2, and the subgroup of* $\operatorname{rad}(\chi_{\mathrm{A}})$ *generated by $h_0$ and $h_\infty$ has finite index in* $\operatorname{rad}(\chi_{\mathrm{A}})$.

*Proof.* See [23, (5.1.1)]. □

Notice that, since $\langle \_, \_ \rangle$ is a bilinear form, and $\langle h_0, h_0 \rangle = \langle h_\infty, h_\infty \rangle = 0$, we have that:

$$\langle h_0, h_\infty \rangle = -\langle h_\infty, h_0 \rangle$$

Define $\iota_0 : K_0(A) \to \mathbb{Z}$ and $\iota_\infty : K_0(A) \to \mathbb{Z}$ by:

$$\iota_0(x) = \langle h_0, x \rangle$$

$$\iota_\infty(x) = \langle h_\infty, x \rangle$$

Given any $x \in K_0(A)$, we define the *index of* $x$ to be the element of $\mathbb{Q}_0^\infty$ given by:

$$-\frac{\iota_0(x)}{\iota_\infty(x)}$$

-which we denote $\iota(x)$. Given any $M \in A$-mod, define the index of $M$ to be $\iota(\underline{\dim}(M))$. For all $\gamma \in \mathbb{Q}^{>0}$, define $\iota_\gamma : K_0(A) \to \mathbb{Q}$ by:

$$\iota_\gamma(x) := \iota_0(x) + \gamma \iota_\infty(x)$$

Note that $x \in \operatorname{Ker}(\iota_\gamma)$ if and only if $\iota(x) = \gamma$.

**Lemma 28.** *Given any $\gamma \in \mathbb{Q}^+$, pick any $a, b \in \mathbb{N}$ such that $\gamma = b/a$. Let $c$ be the greatest common divisor of the coordinates of $ah_0 + bh_\infty$.*

*Then $\mathrm{Ker}(\iota_\gamma) \cap \mathrm{rad}(\chi_\mathrm{A})$ is a subgroup of $K_0(A)$ of rank 1- which is generated by $(a/c)h_0 + (b/c)h_\infty$.*

*Proof.* Take any $x \in \mathrm{Ker}(\iota_\gamma) \cap \mathrm{rad}(\chi_\mathrm{A})$. Since $\mathrm{rad}(\chi)$ is a rank 2 subgroup, and $h_0$ and $h_\infty$ are linearly independent elements of $\mathrm{rad}(\chi)$, there exist $q_1, q_2 \in \mathbb{Q}$ such that $x = q_1 h_0 + q_2 h_\infty$. Then:

$$\begin{aligned} \iota(x) &= -\frac{\langle h_0, q_1 h_0 + q_2 h_\infty \rangle}{\langle h_0, q_1 h_0 + q_2 h_\infty \rangle} \\ &= -\frac{q_2 \langle h_0, h_\infty \rangle}{q_1 \langle h_\infty, h_0 \rangle} \\ &= q_2/q_1 \end{aligned}$$

Then:

$$b/a = \gamma = \iota(x) = q_2/q_1$$

And so every element of $\mathrm{rad}(\chi) \cap \mathrm{Ker}(\iota_\gamma)$ is equal to $q(ah_0 + bh_\infty)$, for some $q \in \mathbb{Q}$- so the subgroup does have rank 1.

Finally, note that every element of the set $\{q(ah_0 + bh_\infty) : q \in \mathbb{Q}\} \cap \mathbb{Z}^n$ must equal $d((a/c)h_0 + (b/c)h_\infty)$ for some $d \in \mathbb{Z}$. □

Define $\mathcal{P}_0$, $\mathcal{T}_0$, and $\mathcal{Q}_0$ to be the module classes as in theorem 24. Dually, define $\mathcal{P}_\infty$, $\mathcal{T}_\infty$, and $\mathcal{Q}_\infty$ to be the module classes as found in lemma 27

Define $\mathcal{P}_\gamma$ (respectively, $\mathcal{T}_\gamma$, $\mathcal{Q}_\gamma$) to be the set of all indecomposable $M \in A$-mod such that $\iota_\gamma(\underline{\dim}(M)) < 0$ (respectively, $\iota_\gamma(\underline{\dim}(M)) = 0$, $\iota_\gamma(\underline{\dim}(M)) > 0$).

**Theorem 27.** *For all $\gamma \in \mathbb{Q}^+$, $\mathcal{T}_\gamma$ is a sincere stable tubular $\mathbb{P}^1(K)$-family of type $\mathbb{T}$, separating $\mathcal{P}_\gamma$ from $\mathcal{Q}_\gamma$.*

*It is controlled by the restriction of $\chi_A$ to Ker$(\iota_\gamma)$.*

*Proof.* See [23, (5.2.2)]. □

**Lemma 29.** *For all $\gamma \in \mathbb{Q}_0^\infty$, $\mathcal{P}_\gamma = \mathcal{P}_0 \cup \bigcup_{\alpha<\gamma} \mathcal{T}_\alpha$, and $\mathcal{Q}_\gamma = \mathcal{Q}_\infty \cup \bigcup_{\beta>\gamma} \mathcal{T}_\beta$*

*Proof.* By [23, p275], $A$-mod can be partitioned as:

$$\mathcal{P}_0 \cup \mathcal{T}_0 \cup (\mathcal{P}_\infty \cap \mathcal{Q}_0) \cup \mathcal{T}_\infty \cup \mathcal{Q}_\infty$$

-and we have the following:

$$\iota_0(\underline{\dim}(X)) > 0 \text{ and } \iota_\infty(\underline{\dim}(X)) < 0 \text{ for all } X \in \mathcal{P}_\infty \cap \mathcal{Q}_0$$

$$\iota_0(\underline{\dim}(X)) \leq 0 \text{ and } \iota_\infty(\underline{\dim}(X)) \leq 0 \text{ for all } X \in \mathcal{P}_0 \cup \mathcal{T}_0$$

$$\iota_0(\underline{\dim}(X)) \geq 0 \text{ and } \iota_\infty(\underline{\dim}(X)) \geq 0 \text{ for all } X \in \mathcal{P}_\infty \cup \mathcal{T}_\infty$$

So, given any $X \in \mathcal{P}_0 \cup \mathcal{T}_0$, $\iota_\gamma(\underline{\dim}(X)) \leq 0 \leq \gamma$. So $\mathcal{P}_0 \cup \mathcal{T}_0 \subseteq \mathcal{P}_\gamma$, and similarly, $\mathcal{Q}_\infty \cup \mathcal{T}_\infty \subseteq \mathcal{Q}_r$.

Now, given any $X \in \mathcal{P}_\infty \cap \mathcal{Q}_0$, we have:

$$\begin{aligned} X \in \mathcal{P}_\gamma &\iff \iota_0(\underline{\dim}(X)) + \gamma\iota_\infty(\underline{\dim}(X)) < 0 \\ &= \iota(\underline{\dim}(X)) = \beta \text{ for some } \beta \in (0,\gamma) \\ &= X \in \mathcal{T}_\alpha \text{ for some } \beta \in (0,\gamma) \end{aligned}$$

And so $\mathcal{P}_\gamma = \mathcal{P}_0 \cup \bigcup_{\alpha<\gamma} \mathcal{T}_\alpha$. The proof for $\mathcal{Q}_\gamma$ follows similarly. □

Given any $r \in \mathbb{R}^+ \backslash \mathbb{Q}^+$, we define:

$$\mathcal{P}_r := \mathcal{P}_0 \cup \bigcup_{\alpha<r} \mathcal{T}_\alpha$$

$$\mathcal{Q}_r := \mathcal{Q}_\infty \cup \bigcup_{\beta>r} \mathcal{T}_\beta$$

Note that $\mathcal{Q}_r \cup \mathcal{P}_r$ is a partition of the set of all indecomposable modules in $A$-mod, and $\mathrm{Hom}(\mathcal{Q}_r, \mathcal{P}_r) = 0$. Note that, given any $r, s \in \mathbb{R}_0^+$:

$$r \leq s \iff \mathcal{P}_r \subseteq \mathcal{P}_s \iff \mathcal{Q}_s \subseteq \mathcal{Q}_r$$

By convention, we set $\mathcal{T}_r := 0$ for all $r \in \mathbb{R}^+ \backslash \mathbb{Q}$.

## 3.2 Slope

Throughout this section, $A$ will be a tubular algebra, and the components $\mathcal{P}_\gamma$, $\mathcal{T}_\gamma$ ,$\mathcal{Q}_\gamma$ (for all $\gamma \in \mathbb{R}_0^\infty$) are as described in the previous section.

**Lemma 30.** *Take any $\gamma \in \mathbb{Q}^+$ and any stable tube $\mathcal{T}(\rho)$ in $\mathcal{T}_\gamma$.*

*Given any $M \in \mathrm{add}(\mathcal{P}_\gamma)$ there exists a module $T \in \mathrm{add}(\mathcal{T}(\rho))$ such that there is an embedding $M \hookrightarrow T$.*

*Dually, given any $N \in \mathrm{add}(\mathcal{Q}_\gamma)$ there exists a module $T' \in \mathrm{add}(\mathcal{T}(\rho))$ such that there is a surjection $T' \twoheadrightarrow N$.*

*Proof.* Let $h : M \hookrightarrow E(M)$ be an injective hull of $M$. Since $\mathcal{T}_\gamma$ separates $\mathcal{P}_\gamma$ from $\mathcal{Q}_\gamma$, there exists $T \in \mathrm{add}(\mathcal{T}(\rho)$, and maps $f : M \to T$ and $g : T \to E(M)$ such that $h = gf$. Since $h$ is an embedding, so must $f$ be.

The other case is proved dually. □

**Corollary 3.** *Take any $\alpha, \beta \in \mathbb{Q}_0^\infty$ with $\alpha < \beta$. Then, for all $M \in A$-*Mod*, $\mathrm{Hom}(M, \mathcal{T}_\beta) = 0$ implies $\mathrm{Hom}(M, \mathcal{T}_\alpha) = 0$, and $\mathrm{Hom}(\mathcal{T}_\alpha, M) = 0$ implies that $\mathrm{Hom}(\mathcal{T}_\beta, M) = 0$.*

*Consequently, for all $\alpha \in \mathbb{Q}_0^\infty$, $l(\mathcal{T}_\alpha) \subseteq l(\mathcal{P}_\alpha)$ and $r(\mathcal{T}_\alpha) \subseteq r(\mathcal{Q}_\alpha)$ (where $r(\_)$ and $l(\_)$ are as defined in section 2.6).*

*Proof.* Take any $X \in \mathcal{T}_\beta$ and any map $f \in \mathrm{Hom}(M, X)$. By lemma 30, we can pick $Y \in \mathrm{add}(\mathcal{T}_\beta)$ such that there exists an embedding $h : X \hookrightarrow Y$. Then $hf \in \mathrm{Hom}(M, \mathcal{T}_\beta) = 0$, and hence $f = 0$ (since $h$ is an embedding). The other case is proved dually. □

Let $A$ be any tubular algebra. Given any $r \in \mathbb{R}_0^\infty$ , we say that a module $M \in A$-Mod *has slope* $r$ if and only if $\mathrm{Hom}(\mathcal{Q}_r, M) = \mathrm{Hom}(M, \mathcal{P}_r) = 0$.

**Lemma 31.** *Given any $M \in A$*-Mod, and any $r \in \mathbb{R}_0^\infty$, the following are equivalent:

1. $M$ has slope $r$.

2. $\mathrm{Hom}(\mathcal{Q}_r, M) = \mathrm{Ext}(\mathcal{P}_r, M) = 0$.

3. There exists $\epsilon > 0$ such that:

$$\mathrm{Hom}(\mathcal{Q}_r \cap \mathcal{P}_{r+\epsilon}, M) = \mathrm{Hom}(M, \mathcal{P}_r \cap \mathcal{Q}_{r-\epsilon}) = 0$$

4. There exists $\epsilon > 0$ such that:

$$\mathrm{Hom}(\mathcal{Q}_r \cap \mathcal{P}_{r+\epsilon}, M) = \mathrm{Ext}(\mathcal{P}_r \cap \mathcal{Q}_{r-\epsilon}, M) = 0$$

(where $\mathcal{Q}_\alpha := \mathcal{Q}_0$ if $\alpha < 0$ and $\mathcal{Q}_{\infty+\epsilon} := \mathcal{Q}_\infty$).

*Proof.* First of all, given any connected component $\Gamma'$ of the AR quiver, theorem 16 gives that:

$$\begin{aligned}\mathrm{Ext}(X, M) = 0 \text{ for all } X \in \Gamma &\iff \mathrm{Hom}(M, \tau X) = 0 \text{ for all } X \in \Gamma \\ &\iff \mathrm{Hom}(M, X) = 0 \text{ for all } X \in \Gamma\end{aligned}$$

(since components of an Auslander Reiten quiver are closed under $\tau$ and $\tau^-$). And so (2) is equivalent to (1), and (3) is equivalent to (4).

Clearly, (1) implies (3). To show the converse, suppose that there exists $\epsilon > 0$ as in (3). Then given any $Y \in \mathcal{Q}_r$ and $f \in \mathrm{Hom}(Y, M)$, we can pick a rational $\beta \in (r, r+\epsilon)$ such that $Y \in \mathcal{Q}_\beta$. By lemma 30, there exists a module $T \in \mathrm{add}(\mathcal{T}_\beta)$ such that there exists a surjection $g : T \twoheadrightarrow Y$. Then $fg \in \mathrm{Hom}(T, M) = 0$ by our assumption. Since $g$ is a surjection, $f$ must be zero. Dually, one can show that $\mathrm{Hom}(M, \mathcal{P}_r) = 0$. $\square$

Of course, if $M$ is finite dimensional and indecomposable, and does not lie in $\mathcal{P}_0$ or $\mathcal{Q}_\infty$, then it lies in $\mathcal{T}_\gamma$ for some unique $\gamma \in \mathbb{Q}_0^\infty$- and, since each tubular family $\mathcal{T}_\beta$ separates $\mathcal{P}_\beta$ from $\mathcal{Q}_\beta$, the slope of $M$ is $\gamma$.

**Theorem 28.** *Let $M \in A$-Mod be any indecomposable module, which does not lie in $\mathcal{P}_0$ or $\mathcal{Q}_\infty$. Then there exists a unique $r \in \mathbb{R}_0^\infty$ such that $M$ has slope $r$.*

*Proof.* See [22], Theorem 6. $\square$

Given any $X \in A$-mod, there exists (by theorem 6) a pp-pair $\phi/\psi$ such that $(\phi/\psi)(M) \cong \mathrm{Hom}(X, M)$ for all $M \in A$-Mod. We denote the sentence $\forall \overline{v}(\phi(\overline{v}) \to \psi(\overline{v})$ by $\mathrm{Hom}(X, \_) = 0$.

Similarly, there exists a pp-pair $\phi/\psi$ such that $(\phi/\psi)(M) \cong \mathrm{Ext}(X, M)$ for all $M \in A$-Mod. We denote the sentence $\forall(\overline{v})(\phi(\overline{v}) \to \psi(\overline{v})$ by $\mathrm{Ext}(X, \_) = 0$.

Given any $r \in \mathbb{R}_0^\infty$, we define the theory $\Phi_r$ by:

$$\Phi_r := \{\mathrm{Ext}(Y, \_) = 0 : Y \in \mathcal{P}_r\} \cup \{\mathrm{Hom}(X, \_) = 0 : X \in \mathcal{Q}_r\}$$

So for all $M \in A$-Mod, $M \models \Phi_r$ if and only if $M$ has slope $r$.

**Lemma 32.** *Given any $r \in \mathbb{R}_0^\infty$, any $A$-module $M$ lies in $l(\mathcal{P}_r)$ if and only if it is generated by $\mathcal{T}_\gamma$, for all rational $\gamma < r$.*

*Proof.* See lemma 11 of [22]. $\square$

**Lemma 33.** *Given any $r \in \mathbb{R}_0^\infty$, $\epsilon > 0$, and any $M \in A$-Mod of slope $r$, there exists a directed system $(M_i, f_{ij})_I$ (with every $M_i \in \mathrm{add}((\mathcal{P}_r \cup \mathcal{T}_r) \cap \mathcal{Q}_{r-\epsilon})$) with direct limit isomorphic to $M$.*

*Proof.* Let $\{M_\lambda : \lambda \in I\}$ be the set of all finite dimensional submodules of $M$ which are isomorphic to a module in $\mathrm{add}((\mathcal{P}_r \cup \mathcal{T}_r) \cap \mathcal{Q}_{r-\epsilon})$. Let $\leq$ be the partial ordering on $I$ such that $i \leq j$ if and only if $M_i$ is a submodule of $M_j$.

Consider the directed system, with modules $\{M_\lambda : \lambda \in I\}$, and morphisms $f_{ij} : M_i \to M_j$: where $f_{ij}$ is the natural inclusion map of $M_i$ into $M_j$

For all $i \in I$, define $h_i : M_i \hookrightarrow M$ be the natural embedding of the submodule $M_i$ into $M$. We claim that $(M, (h_i)_{i \in I})$ is the direct limit of the system.

Firstly, given any $i \leq j$, the following diagram clearly commutes:

$$\begin{tikzcd} M_i \arrow[r, "f_{ij}"] \arrow[dr, "h_i"'] & M_j \arrow[d, "h_j"] \\ & M \end{tikzcd}$$

(since all the maps involved are inclusions of submodules).

Now, given any module $N$, and set of maps $\{g_i : M_i \to N : i \in I\}$ such that $g_i = g_j f_{ij}$ for all $i, j \in I$ such that $i \leq j$, we construct a map $F \in \mathrm{Hom}(M, N)$ such that $F \circ h_i = g_i$ for all $i \in I$.

Pick any $\alpha \in (r - \epsilon, r)$. By lemma 32, there exists a module $\bigoplus_{k \in J} T_k$, with each $T_k \in \mathcal{T}_\alpha$, and a surjection $\Psi : \bigoplus_{k \in J} T_k \twoheadrightarrow M$.

Given any $m \in M$, pick any $t \in \bigoplus_{k\in J} T_k$ such that $\Psi(t) = m$. Let $J'$ be the set of all $k \in J$ such that $t$ has a component in $T_k$, and let:

$$\rho : \bigoplus_{k\in J'} T_k \hookrightarrow \bigoplus_{k\in J} T_k$$

-be the natural embedding of the direct summand. Now, $\mathrm{Im}(\Psi\rho)$ is a finite dimensional submodule of $M$ which is isomorphic to a module in $\mathrm{add}((\mathcal{P}_r \cup \mathcal{T}_r) \cap \mathcal{Q}_{r-\epsilon})$. Let $M_i$ be the relevant submodule in the directed system. We define $F(m)$ to be $g_i\Psi\rho(t)$.

One can check that this map is well defined, and that it satisfies the required conditions. □

## 3.3 Modules in stable tubes

Throughout this section, $\mathcal{T}(\rho)$ will be a standard stable tube of rank $n$- and the modules in $\mathcal{T}(\rho)$ will be denoted $\{E_i[m] : i \in \mathbb{Z}_n, m \in \mathbb{N}^+\}$, and $\gamma$ will denote the slope of all the modules in $\mathcal{T}(\rho)$. We define $E_i[0]$ to be the zero module for all $i \in \mathbb{Z}_n$.

**Lemma 34.** *For all $i \in \mathbb{Z}_n$ and $m \in \mathbb{N}^+$:*

$$\underline{\dim}(E_i[m]) = \sum_{j=1}^{m} \underline{\dim}(E_{i+j}[1])$$

*Proof.* For all $i \in \mathbb{Z}_n$ and $k \in \mathbb{N}^+$, there exists an exact sequence:

$$0 \longrightarrow E_i[k] \longrightarrow E_i[k+1] \oplus E_{i+1}[k-1] \longrightarrow E_{i+1}[k] \longrightarrow 0$$

And so:

$$\underline{\dim}(E_i[k+1]) = \underline{\dim}(E_i[k]) + \underline{\dim}(E_{i+1}[k]) - \underline{\dim}(E_{i+1}[k-1])$$

The result follows by induction on $k$. □

**Lemma 35.** *Take any $\gamma \in \mathbb{Q}^+$ and any tube $\mathcal{T}(\rho)$ in $\mathcal{T}_\gamma$.*

*Given any $M \in A$-*Mod *with slope greater than $\gamma$, there exists a module $T \in \mathcal{T}(\rho)$ such that* $\mathrm{Hom}(T, M) \neq 0$.

*Dually, given any $N \in A$-*Mod *with slope less than $\gamma$, there exists a module $T' \in \mathcal{T}(\rho)$ such that* $\mathrm{Ext}(T', M) \neq 0$.

*Proof.* We will only prove the first half. The proof of the second half follows a similar argument.

Let $r \in \mathbb{R}$ be the slope of $M$. Since $r > \gamma$, then pick any $\epsilon \in (0, r-\gamma)$. We claim that there exists a module $N \in \mathcal{P}_r \cap \mathcal{Q}_{r-\epsilon}$ such that $\mathrm{Hom}(N, M) \neq 0$: if not, then $\mathrm{Hom}(\mathcal{Q}_{r-\epsilon}, M) = 0$ and $\mathrm{Hom}(M, \mathcal{P}_{r-\epsilon}) = 0$ (since $\mathcal{P}_{r-\epsilon} \subseteq \mathcal{P}_r$), and so $M$ has slope $r-\epsilon$- contradicting theorem 28 (since $M$ has slope $r$).

Consequently, we can pick a module $N \in \mathcal{P}_r \cap \mathcal{Q}_{r-\epsilon}$, and a non-zero map $g \in \mathrm{Hom}(N, M)$. By lemma 30, there exists $T' \in \mathrm{add}(\mathcal{T}(\rho))$ and a surjection $f : T \twoheadrightarrow N$. Then $gf \neq 0$, so $\mathrm{Hom}(\mathrm{add}(\mathcal{T}(\rho)), M) = 0$, and so $\mathrm{Hom}(\mathcal{T}(\rho), M) \neq 0$, as required. □

**Lemma 36.** *Take any $\gamma \in \mathbb{Q}^+$, and any homogeneous tube $\mathcal{T}(\rho)$ in $\mathcal{T}_\gamma$. Denote the modules in $\mathcal{T}(\rho)$ by $E[1], E[2], E[3], \ldots$.*

*Then for all $M \in A$-Mod with slope less than $\gamma$, and all $k \geq 1$:*

$$\dim_K(\mathrm{Hom}(M, E[k])) \neq 0$$

*Furthermore, if $\mathrm{Hom}(M, E[k])$ is finite dimensional, then:*

$$\dim_K(\mathrm{Hom}(M, E[k])) = k\dim_K(\mathrm{Hom}(M, E[1]))$$

*Dually, for all $N \in A$-Mod of slope greater than $\gamma$, and all $k \geq 1$:*

$$\dim_K(\mathrm{Hom}(E[1], N)) \neq 0$$

*-and if $\mathrm{Hom}(E[1], N)$ is finite dimensional, then:*

$$\dim_K(\mathrm{Hom}(E[k], N)) = k\dim_K(\mathrm{Hom}(E[1], N))$$

*Proof.* For all $k \in \mathbb{N}^+$, we have an exact sequence:

$$0 \longrightarrow E[k] \longrightarrow E[k-1] \oplus E[k+1] \longrightarrow E[k] \longrightarrow 0$$

Since $\mathrm{Ext}(M, E[k]) = 0$ for all $k \in \mathbb{N}^+$, we induce the exact sequence:

$$\begin{aligned} 0 &\longrightarrow \mathrm{Hom}(M, E[k]) \longrightarrow \mathrm{Hom}(M, E[k+1] \oplus E[k-1]) \longrightarrow \mathrm{Hom}(M, E[k]) \\ &\longrightarrow \mathrm{Ext}(M, E[k]) = 0 \end{aligned}$$

And so:

$$\dim_K(\mathrm{Hom}(M, E[k+1])) = 2\dim_K(\mathrm{Hom}(M, E[k])) - \dim_K(\mathrm{Hom}(M, E[k-1]))$$

By induction we get that:

$$\dim_K(\mathrm{Hom}(M, E[k])) = k\dim_K(\mathrm{Hom}(M, E[1]))$$

Finally, by lemma 35, there exists $k' \in \mathbb{N}$ such that $\mathrm{Hom}(M, E[k']) \neq 0$. Then, for all $k \in \mathbb{N}$:

$$\mathrm{Hom}(M, E[k]) = \frac{k}{k'}\mathrm{Hom}(M, E[k']) \neq 0$$

-as required. □

**Lemma 37.** *For all $k, k' \in \mathbb{N}$, and any indecomposable module $E$ on the mouth of a stable tube:*

$$\dim_K(\mathrm{Hom}(E[k], E[k'])) = \dim_K(\mathrm{Ext}(E[k], E[k'])) = \min(k, k')$$

*Proof.* Follows straight from corollary 1 □

**Corollary 4.** *Let $E_i$ and $E_j$ be any pair of modules on the mouth of a stable tube $\mathcal{T}(\rho)$, and take any $k \geq 1$. Then $\mathrm{Hom}(E_i, E_j[k]) \neq 0$ if and only if $E_i \cong E_j$.*

*Furthermore, $\dim_K(\mathrm{Hom}(E_i, E_i[k])) = 1$ for all $k \in \mathbb{N}^+$.*

*Proof.* Follows straight from corollary 1 □

## 3.4 Lattices and dimension

An equivalence relation $\sim$ on a lattice $L$ is called a *congruence* if, for all $a, b, c \in L$, $a \sim b$ implies both $a+c \sim b+c$ and $a \wedge c \sim b \wedge c$. Given any class $\mathcal{L}$ of modular lattices, which is closed under sublattices and quotient lattices, we define the $\mathcal{L}$-dimension of a modular lattice $L$ as follows:

Let $L_0 := L$. Define, for every non-zero ordinal $\alpha$, a modular lattice $L_\alpha$ and a lattice surjection $\pi_\alpha : L \twoheadrightarrow L_\alpha$ by induction:

Given $L_\alpha$ and $\pi_\alpha : L \to L_\alpha$, define $\sim_{\alpha+1}$ to be the smallest equivalence on $L$ such that, $a \sim_{\alpha+1} b$ whenever the interval $[\pi_\alpha(a),\pi_\alpha(b)]$ in $L_\alpha$ is isomorphic to a lattice in $\mathcal{L}$. Then the quotient lattice $L/\sim_{\alpha+1}$ is a modular lattice, which we denote $L_{\alpha+1}$. Define $\pi_{\alpha+1} : L \twoheadrightarrow L_{\alpha+1}$ to be the natural projection.

For a limit ordinal $\gamma$, define $\sim_\gamma$ to be the congruence on $L$ such that, for all $a, b \in L_0$:

$$a \sim_\gamma b \text{ if and only if } \pi_\alpha(a) \sim_\alpha \pi_\alpha(b) \text{ for some } \alpha < \gamma$$

And define $L_\gamma := L_0/\sim_\gamma$, and $\pi_\gamma : L_0 \to L_0/\sim_\gamma$ to be the obvious surjection.

Let $1_L$ and $0_L$ denote the top and bottom elements of $L$. If $\pi_\alpha(0_L) \neq \pi_\alpha(1_L)$ for all $\alpha$, then we define the $\mathcal{L}$-dimension of $L$ to be $\infty$. Otherwise, let $\alpha$ be minimal such that $\pi_\alpha(0_L) = \pi_\alpha(1_L)$. Then $\alpha$ is not a limit ordinal, so we define the $\mathcal{L}$-dimension of $L$ to be $\alpha - 1$: it is denoted $\mathcal{L}$-dim$(L)$.

Notice that, if $\mathcal{L}'$ is a subclass of $\mathcal{L}$, then $\mathcal{L}'$-dim$(L) \geq \mathcal{L}$-dim$(L)$.

**Lemma 38.** *Let $\mathcal{L}$ be any class of modular lattices, closed under sublattices and quotient lattices. Let $L$ be any modular lattice, and $a, b \in L$. Let $\sim_1$ be the congruence defined on $L$ by $\mathcal{L}$ as above.*

*Then, for all $a, b \in L$, $a \sim_1 b$ if and only if there exists a finite set of elements $c_0, c_1, \dots c_n$ of $L$ such that:*

$$a \wedge b = c_0 \leq c_1 \leq \dots c_{n-1} \leq c_n = a + b$$

*-and every interval $[c_i, c_{i-1}]$ is isomorphic to a lattice in $\mathcal{L}$.*

*Proof.* PSL 290 See [17, (7.1.1)] □

Let $\mathcal{L}_m$ be the class of all 1-point and 2-point lattices. Then the $\mathcal{L}_m$-dimension of $L$ is called the *m-dimension* of $L$- and is denoted mdim$(L)$.

**Lemma 39.** *Let $R$ be any ring, and $X$ any closed subset $X$ of ${}_R\mathrm{Zg}$ which satisfies the isolation condition. Then* $\mathrm{CB}(X) = \mathrm{mdim}(\mathrm{pp}(X))$.

*Proof.* See [17, (5.3.60)]. □

Let $\mathcal{L}_b$ be the class of all totally ordered lattices. Then $\mathcal{L}_b$-dim$(L)$ is called the *breadth* of $L$- and is denoted $w(L)$.

A subposet $P$ of a modular lattice $L$ is said to be *wide* if, given any two points $a > b$ in $P$, there exists $c, d \in P$ such that $c \not\leq d$, and $d \not\leq c$, and the elements $c + d$ and $c \wedge d$ of $L$ satisfy $a \geq c + d > c \wedge d \geq b$.

**Lemma 40.** *Given any modular lattice $L$, the following are equivalent:*

- $w(L) = \infty$.
- *$L$ has a wide subquotient.*
- *$L$ has a wide subposet*

*Proof.* See [17, (7.3.1)] □

**Theorem 29.** *Let $R$ be any ring, and $M$ an $R$-module. Then:*

- *If there exists a superdecomposable pure-injective $R$-module, $N$ with* $\mathrm{Supp}(N) \subseteq \mathrm{Supp}(M)$*, then* $w(\mathrm{pp}(M)) = \infty$.
- *If* $\mathrm{pp}(M)$ *is countable (for example, if $R$ is countable) and* $w(\mathrm{pp}(M))\infty$*, then there exists a superdecomposable pure-injective $R$-module, $N$, with* $\mathrm{Supp}(N) \subseteq \mathrm{Supp}(M)$.

*Proof.* See [28, (7.8)]. □

**Corollary 5.** *Let $A$ be a tubular algebra. Given any $r \in \mathbb{R}_0^\infty$, let $M(r)$ denote the direct sum of all pure-injective indecomposable $A$-modules of slope $r$.*

*If there exists a superdecomposable pure-injective $A$-module $N$ of slope $r$, then the breadth of* $\mathrm{pp}(M(r))$ *is $\infty$.*

*Furthermore, if $A$ is countable, and the breadth of* $\mathrm{pp}(M(r))$ *is $\infty$, then there exists a superdecomposable pure-injective $A$-module $N$ of slope $r$.*

*Proof.* By thm 29, it is enough to prove that, given any $N \in A$-Mod, $\mathrm{supp}(N) \subseteq \mathrm{supp}(M(r))$ if and only if $N$ has slope $r$. Recall (from section 2.3) that $\mathrm{Supp}(N) \subseteq$

Supp($M(r)$) if and only if every pp-pair closed on $M(r)$ is closed on $N$. If Supp($N$) $\subseteq$ Supp($M(r)$), then $N \in$ Supp($M(r)$), so $N$ has slope $r$, by lemma 41.

To prove the other direction, suppose that $N$ has slope $r$ (i.e. that $N \models \Phi_r$), and that a pp-pair $\phi/\psi$ is open on $N$. By theorem 12 there exists a pure-injective indecomposable $M$ in $\langle N \rangle$ such that $\phi(M) > \psi(M)$. Then $M \models \Phi_r$ (since every pp-pair closed on $N$ is closed on $M$), i.e. $M$ slope $r$, and hence must be a direct summand of $M(r)$. Since $\phi/\psi$ is open on $M$, it is open on $M(r)$, as required. □

Given any $r \in \mathbb{R}^+$, we shall attempt to calculate the m-dimension and the breadth of the lattice pp($M(r)$), which will determine whether or not there exists a superdecomposable $A$-module of slope $r$ (if $R$ is countable).

## 3.5 Modules arising from separating tubular families

Throughout this section, $A$ will be any $K$-algebra, such that $A$-mod has a sincere, stable tubular family $\mathcal{T} = \{\mathcal{T}(\rho) : \rho \in \mathbb{P}^1(K)\}$ which separates the set of proper predecessors in $A$-mod (denoted $\mathcal{P}$) from the set of proper successors in $A$-mod (denoted $\mathcal{Q}$).

### 3.5.1 Infinite dimensional modules

Define:

$$\mathcal{C}_{\mathcal{T}} := r(\mathcal{Q}) = \{M \in A\text{-Mod} : \mathrm{Hom}(\mathcal{Q}, M) = 0\}$$

$$\mathcal{D}_{\mathcal{T}} := l(\mathcal{T}) = \{M \in A\text{-Mod} : \mathrm{Hom}(M, \mathcal{T}) = 0\}$$

$$\mathcal{R}_{\mathcal{T}} := r(\mathcal{D}_{\mathcal{T}}) = \{M \in A\text{-Mod} : \mathrm{Hom}(\mathcal{D}_{\mathcal{T}}, M) = 0\}$$

$$\omega_{\mathcal{T}} := \mathcal{C}_{\mathcal{T}} \cap \mathcal{D}_{\mathcal{T}}$$

$$\mathcal{B}_{\mathcal{T}} := l(\mathcal{P}) = \{M \in A\text{-Mod} : \mathrm{Hom}(M, \mathcal{P}) = 0\}$$

$$\mathcal{M}_{\mathcal{T}} := \mathcal{B}_{\mathcal{T}} \cap \mathcal{C}_{\mathcal{T}}$$

Notice that, if $A$ is a tubular algebra, and $\mathcal{T}$ is a tubular family $\mathcal{T}_\gamma$, then $\mathcal{M}_\mathcal{T}$ is the set of all $A$-modules of slope $\gamma$.

**Lemma 41.** *Let $A$ be a tubular algebra, and $r \in \mathbb{R}_0^\infty$. Given any set of $A$-modules $\{M_i : i \in \mathbb{N}\}$, all of which have slope $r$, any module in $\operatorname{Supp}(\bigoplus_{i\in I} M_i)$ also has slope $r$.*

*Proof.* Pick any module $N$ in $\operatorname{Supp}(\bigoplus_{i\in I} M_i)$. For all $i \in \mathbb{N}$, $M_i$ has slope $r$, and so $M_i \models \Phi_r$. Since every sentence in $\Phi_r$ is the "closure of a pp-pair" and every pp-pair closed on all $M_i$ is closed on $N$, we have:

$$N \models \Phi_r$$

-i.e. $N$ has slope $r$. □

Let $E$ be any module lying on the mouth of a tube $\mathcal{T}(\rho)$ in $\mathcal{T}$. Then the direct limit $E[\infty]$ obtained from the ray starting at $E$ is called a *Prüfer module.* Dually, the inverse limit $\widehat{E}$ obtained from the coray ending at $E$ is called an *adic* module.

**Lemma 42.** *There exists exactly one (up to isomorphism) infinitely generated indecomposable module $G_\mathcal{T}$ in $\omega_\mathcal{T}$ such that $\operatorname{End}(G_\mathcal{T})$ is a division ring.*

*Furthermore, $G_\mathcal{T}$ is generic.*

*Proof.* See theorem 2 and corollary 6 of [22]. □

The module described in lemma 42 will be referred to throughout this section as $G_\mathcal{T}$.

**Lemma 43.** *For all quasisimple modules $E$ in $\mathcal{T}$, the Prüfer module $E[\infty]$ lies in $\omega_\mathcal{T}$. Furthermore, every module in $\omega_\mathcal{T}$ is a direct sum of copies of $G_\mathcal{T}$ and Prüfer modules.*

*Proof.* Theorem 4 of [22] □

**Lemma 44.** *Let $E$ and $E'$ be any two quasisimple modules lying in tubes in $\mathcal{T}$. Then:*

$$\operatorname{Hom}(E', E[\infty]) \neq 0 \Longleftrightarrow E \cong E'$$

*Furthermore:*

$$\dim_K(\mathrm{Hom}(E, E[\infty])) = 1$$

*Proof.* By theorem 20, $E[\infty]$ lies in the definable subcategory generated by the set $\{E[1], E[2], E[3], \dots\}$- so any pp-pair closed on the module $\bigoplus_{k\in\mathbb{N}^+} E[k]$ is closed on $E[\infty]$.

By theorem 6 there is a pp-pair $\phi/\psi$ such that $F_{\phi/\psi} \simeq \mathrm{Hom}(E', \_)$. If $E' \ncong E$, then $\mathrm{Hom}(E', E[k]) = 0$ for all $k$ (by corollary 4). So $\phi/\psi$ is closed on $E[k]$ for all $k$, and hence closed on $E[\infty]$- so $\mathrm{Hom}(E', E[\infty]) = 0$, as required.

Since the functor $\mathrm{Hom}(E, \_)$ commutes with direct limits, we have that:

$$\mathrm{Hom}(E, E[\infty]) = \mathrm{Hom}(E, \varinjlim E[k]) = \varinjlim \mathrm{Hom}(E, E[k])$$

By corollary 4, $\dim_K(\mathrm{Hom}(E, E[k])) = 1$ for all $k \geq 1$. It follows that:

$$\dim_K(\mathrm{Hom}(E, E[\infty])) = 1$$

$\square$

Dually, one can prove that:

**Lemma 45.** *Let $E$ and $E'$ be any two quasisimple modules lying in tubes in $\mathcal{T}$. Then:*

$$\mathrm{Hom}(\widehat{E}, E') \neq 0 \iff E \cong E'$$

*Furthermore:*

$$\dim_K(\mathrm{Hom}(\widehat{E}, E)) = 1$$

**Lemma 46.** *Let $E_i[\infty]$ be any Prüfer module, associated to a module $E_i$, in a tube $\mathcal{T}(\rho)$ in $\mathcal{T}$. Then* $\mathrm{Hom}(G_{\mathcal{T}}, E_i[\infty]) = 0$.

*Proof.* See [22], chapter 8. $\square$

**Corollary 6.** $\mathrm{Ext}(\mathcal{T}, G_{\mathcal{T}}) = \mathrm{Hom}(\mathcal{T}, G_{\mathcal{T}}) = 0$.

*Proof.* Of course, $\mathrm{Hom}(\mathcal{T}, G_{\mathcal{T}}) = 0$- since $G_{\mathcal{T}} \in \omega_{\mathcal{T}}$. To show the second result, take any module $E_i[k]$ in a tube $\mathcal{T}(\rho)$ in $\mathcal{T}$, and any map $h_k \in \mathrm{Hom}(E_i[k], G_{\mathcal{T}})$. By repeatedly applying lemma 22, we obtain a series of maps $h_m \in \operatorname{Hom}(E_m[k], G_{\mathcal{T}})$ such that $h_m = h_{m+1} f_i^m$ for all $m \geq k$. Then $h_k$ must factor through the direct limit of the sequence:

$$E_i[k] \xrightarrow{h_k} E_i[k+1] \xrightarrow{h_{k+1}} E_i[k+2] \xrightarrow{h_{k+2}} \dots$$

However, the direct limit of this sequence is the Prüfer module $E_i[\infty]$. By lemma 46, $\operatorname{Hom}(E_i[\infty], G_{\mathcal{T}}) = 0$. It follows that $h_k = 0$, as required. □

**Lemma 47.** *Given any stable tube $\mathcal{T}(\rho)$ in $\mathcal{T}$, let $(M_i, f_i, g_i)$ be the generalised tube associated with $\mathcal{T}(\rho)$. Then the middle term $Q$ of the canonical exact sequence lies in $\omega_{\mathcal{T}}$.*

*Consequently, $Q$ is a direct sum of copies of $G_{\mathcal{T}}$.*

*Proof.* By theorem 20, every indecomposable direct summand of $Q$ lies in the support of $M_\infty$, and hence in the support of $\{M_i : i \in \mathbb{N}^+\}$. By lemma 41, $Q \in \mathcal{M}_{\mathcal{T}}$ (since $M_i \in \mathcal{M}_{\mathcal{T}}$ for all $i \geq 1$): In particular, $Q \in \mathcal{C}_{\mathcal{T}}$.

To prove that $Q \in \mathcal{D}_{\mathcal{T}}$, it is enough to prove that $\operatorname{Hom}(Q, E_i) = 0$ for all quasisimples $E_i$ of $\mathcal{T}(\rho)$. Suppose, for a contradiction, that there exists a non-zero map $f \in \operatorname{Hom}(Q, E_i)$, for some $i$. Let $\{\rho_i : i \in \mathbb{N}^+\}$ be the set of maps such that $(Q, (\rho_j)_{j\in\mathbb{N}^+})$ is the direct limit of the sequence:

$$\widehat{M} \xrightarrow{\Phi} \widehat{M} \xrightarrow{\Phi} \widehat{M} \xrightarrow{\Phi} \dots$$

-so $\rho_j = \rho_{j+1}\Phi$ for all $j \in \mathbb{N}^+$. Since $f \neq 0$, there must exist $j \in \mathbb{N}$ such that $f \circ \rho_j \neq 0$. Then $f \circ \rho_{j+1} \neq 0$ (since $f\rho_{j+1}\Phi = f\rho_j \neq 0$).

By lemma 45, $\operatorname{Hom}(\widehat{M}, E_i) \cong \operatorname{Hom}(\widehat{E}_i, E_i)$-which is a 1-dimensional $K$-vector space. Let $\pi : E_1 \oplus \dots \oplus E_{n_\rho} \twoheadrightarrow E_i$ be the natural projection, and let the maps $h_1, h_2, h_, \dots$ be as in (3.1.4). Since $\pi$ and $h_1$ are surjections, $\pi h_1 \neq 0$, and so every map in $\operatorname{Hom}(\widehat{M}, E_i)$ is a $K$ multiple of $\pi h_1$.

In particular, $f \circ \rho_{j+1} = \lambda\pi h_1$ for some $\lambda \in K$. And so:

$$f\rho_j = f\rho_{j+1}\Phi = \lambda\pi h_1\Phi$$

However, since we have an exact sequence:

$$0 \longrightarrow \widehat{M} \xrightarrow{\Phi} \widehat{M} \xrightarrow{h_1} M_1 \longrightarrow 0$$

-we must have that $f\rho_{j+1} = 0$- giving our required contradiction.

Consequently, $Q \in \omega_{\mathcal{T}}$. By lemma 43 it is a direct sum of copies of $G_{\mathcal{T}}$ (since no direct summands of $Q$ are Prüfer modules). □

### 3.5.2 The pure-injective modules in Supp($\mathcal{T}$)

**Lemma 48.** *$(\mathcal{R}_{\mathcal{T}}, \mathcal{D}_{\mathcal{T}})$ is a split torsion pair.*

*Proof.* See corollary 1 of [22] □

**Lemma 49.** *Let $M$ be a pure-injective module in $\mathcal{M}_{\mathcal{T}} \cap \mathcal{R}_{\mathcal{T}}$. Then there exists a module $M_\rho \in \prod(\mathcal{T}(\rho))$ for all $\rho \in \mathbb{P}^1(K)$ such that:*

$$M \cong \prod_{\rho \in \mathbb{P}^1(K)} M_\rho$$

*Proof.* See [25, (2.2)] □

**Lemma 50.** *The following is a complete list of all the indecomposable pure-injectives in A-Mod which lie in $\mathcal{M}_{\mathcal{T}}$:*

- *The modules in $\mathcal{T}$ (i.e. all the finitely generated ones).*
- *A unique Prüfer module $E[\infty]$ for each indecomposable $E$ lying on the mouth of a tube in $\mathcal{T}$.*
- *An unique adic module $\widehat{E}$ for each indecomposable $E$ lying on the mouth of a tube in $\mathcal{T}$.*
- *The generic module, $G_{\mathcal{T}}$.*

*Proof.* Clearly the set of all indecomposable modules in $A$-mod in $\mathcal{M}_{\mathcal{T}}$ is the set of modules lying in tubes in $\mathcal{T}$. Now, let $M$ be any infinitely generated pure-injective indecomposable in $\mathcal{M}_{\mathcal{T}}$. Since $(\mathcal{R}_{\mathcal{T}}, \mathcal{D}_{\mathcal{T}})$ is a split torsion pair, $M$ must lie in either $\mathcal{R}_{\mathcal{T}}$ or $\mathcal{D}_{\mathcal{T}}$.

If $M \in \mathcal{D}_{\mathcal{T}}$, then $M \in \omega_{\mathcal{T}}$, and so it is either one of the Prüfer modules or $G_{\mathcal{T}}$- by lemma 43.

If $M \in \mathcal{R}_{\mathcal{T}}$, then by lemma 49, there exists modules $M_\rho \in \prod(\mathcal{T}(\rho))$ for all $\rho \in \mathbb{P}^1(K)$ such that:

$$M = \prod_{\rho \in \mathbb{P}^1(K)} M_\rho$$

Pick any $\rho' \in \mathbb{P}^1(K)$ such that $M_\rho \neq 0$ (at least one must exist, since $M \neq 0$). Then:

$$M = M_{\rho'} \oplus \prod_{\rho \neq \rho'} M_\rho$$

Since $M$ is indecomposable, $\prod_{\rho \neq \rho'} M_\rho = 0$- so $M \cong M_{\rho'}$. Since definable categories are closed under direct products and direct summands, $M_\rho$ must lie in the definable category generated by $\mathcal{T}(\rho)$- and hence in the support of $\mathcal{T}_\gamma$. By theorem 19, $M$ must be either a Prüfer, or an adic, or a direct summand of $Q$- which, by lemma 47, must be $G_{\mathcal{T}}$. Since $G_{\mathcal{T}}$ and all relevant Prüfer modules lie in $\omega_{\mathcal{T}}$, $M_\rho$ must be an adic module- which completes the proof. □

**Corollary 7.** *The set of all pure-injective indecomposables of slope $\gamma$ is equal to* supp($\mathcal{T}$).

*Proof.* By lemma 41 every module in the support of $\mathcal{T}$ lies in $\mathcal{M}_{\mathcal{T}}$- and hence is one of the modules listed in lemma 50.

Conversely, any Prüfer module lies in the support of some tube $\mathcal{T}(\rho)$ (by theorem 20), and hence in the support of $\mathcal{T}$- and similarly for the adic modules. Finally, $G_{\mathcal{T}}$ is a direct summand of the middle term $Q$ of a canonical exact sequence associated to a tube $\mathcal{T}(\rho)$ in $\mathcal{T}$, and hence lies in the support of any given Prüfer module from that tube- and hence in the support of $\mathcal{T}_\gamma$. □

### 3.5.3 The CB-rank of Supp($\mathcal{T}$)

**Lemma 51.** *Let $X$ be the Ziegler-closure of the set of all modules in $\mathcal{T}$. Then the CB-ranks of the modules in $X$ are as follows:*

- *The finite dimensional modules (i.e. those in $\mathcal{T}$) have CB-rank 0.*
- *Every Prüfer and adic module has CB-rank 1.*

- *The generic module $G_{\mathcal{T}}$ has CB-rank 2.*

*Furthermore, $X$ satisfies the isolation condition.*

*Proof.* Let $X_0$ denote $X$, and $X_1$ be the set of all non-isolated points in $X$ (with the induced topology), and $X_2$ the set of all non-isolated points in $X_1$. We shall prove that $X_1$ contains precisely the Prüfers, adics, and the generic, and that $X_2$ contains just the generic.

We shall also prove that every point $M$ in $X$ can be isolated in its closure by an $M$-minimal pair. By lemma 11, this is enough to prove that $X$ satisfies the isolation condition.

First of all, every finite dimensional module $E_i[k]$ in $X$ is isolated: Let $f$ denote the left minimal almost split map:

$$E_i[k] \longrightarrow E_i[k+1] \oplus E_{i+1}[k-1]$$

By theorem 6, there exists a pp-pair $\phi/\psi$ such that $F_{\phi/\psi} \simeq \mathrm{Coker}(f, \_)$. By lemma 24, $\phi/\psi$ is closed on every indecomposable module other than $E_i[k]$- so $\{E_i[k]\}$ is indeed a closed set of ${}_A\mathrm{Zg}$. Also, $(\phi/\psi)(E_i[k])$ is a 1-dimensional $K$-vector space over $K$ (by lemma 24), and so $\phi/\psi$ is an $E_i[k]$-minimal pair, isolating $E_i[k]$ in its closure.

Now, any given Prüfer module $E_i[\infty]$ is not isolated: By theorem 20, any closed set containing $\{E_i[k] : k \in \mathbb{N}\}$ must contain $E_i[\infty]$, and so the set $X\backslash\{E_i[\infty]\}$ cannot be closed. Thus $\mathrm{CB}(E_i[\infty]) > 0$.

Let $\phi/\psi$ be a pp-pair such that $F_{\phi/\psi} \simeq \mathrm{Hom}(E_i, \_)$. Then $\mathrm{Hom}(E_i, G_{\mathcal{T}}) = 0$ by corollary 6, and $\mathrm{Hom}(E_i, \widehat{E}_j) = 0$ for all $j$ (since $\widehat{E}_j \in \mathcal{R}_{\mathcal{T}}$), and so, by lemma 44, $(\phi/\psi) \cap X_0 = \{E_i[k] : k \in \mathbb{N}\} \cup \{E_i[\infty]\}$. Thus $(\phi/\psi) \cap X_1 = \{E_i[\infty]\}$- so $E_i[\infty]$ is isolated in $X_1$- and hence has CB-rank 1.

By theorem 19, the Ziegler-closure of $E_i[\infty]$ is $\{E_i[\infty], G_{\mathcal{T}}\}$- so $\phi/\psi$ isolates $E_i[\infty]$ in its closure (since $\mathrm{Hom}(E_i, G_{\mathcal{T}}) = 0$). Furthermore, by lemma 44, $\phi/\psi$ $E_i[\infty]$-minimal.

Similarly, one can show that every adic module is isolated in $X_1$- and hence has CB-rank 1- and also that it is isolated in its Ziegler closure by a minimal pair.

Finally, the generic module $G_{\mathcal{T}}$ is not isolated in $X_1$, or indeed $X_0$- since it lies in the Ziegler closure of any given Prüfer module $E_i[\infty]$ (by theorem 20. Consequently, $X_2 = \{G_{\mathcal{T}}\}$ (since every other module in $X$ has CB-rank less than 2). Thus $\mathrm{CB}(G_{\mathcal{T}}) = 2$.

Since $G_{\mathcal{T}}$ has finite dimension over $\mathrm{End}(G_{\mathcal{T}})$, the lattice of pp-definable subgroups of $G_{\mathcal{T}}$ has no infinite descending chains: so we can pick $\phi \in \mathrm{pp}$ such that $\phi(G_{\mathcal{T}}) \neq 0$, and the pp-pair $\phi/(v = 0)$ is minimal on $G_{\mathcal{T}}$.

Since the Ziegler closure of $G_{\mathcal{T}}$ is $\{G_{\mathcal{T}}\}$, this pp-pair isolates $G_{\mathcal{T}}$ in its closure, as required. □

**Corollary 8.** *The lattice* $\mathrm{pp}(\bigoplus_{M\in\mathcal{T}} M)$ *has m-dimension* 2.

*Consequently, there are no superdecomposable modules in* $\mathcal{M}_{\mathcal{T}}$*.*

*Proof.* By lemma 51, the set $\mathrm{supp}(\mathcal{T})$ has the isolation condition. Thus, by lemma 39 the m-dimension of $\mathrm{pp}(\mathcal{T})$ is equal to the CB rank of $\mathrm{supp}(\mathcal{T})$- which, by lemma 51, is 2. □

## 3.6 Irrational cuts

Throughout this section, $A$ will be a tubular algebra. Given any $r \in \mathbb{R}^+$, we denote by $M(r)$ the direct sum of all pure-injective indecomposable $A$-modules of slope $r$. By corollary 8, we have:

**Proposition 1.** *Given any* $\gamma \in \mathbb{Q}^+$*,* $\mathrm{pp}(M(\gamma))$ *has m-dimension* 2.

*Proof.* We claim that any pp-pair $\phi/\psi$ is closed on $M(\gamma)$ if and only if it's open on $\bigoplus_{M\in\mathcal{T}_\gamma} M$. Of course, one direction is obvious, since $\bigoplus_{M\in\mathcal{T}_\gamma} M$ is a direct summand of $M(\gamma)$. Conversely, every direct summand of $M(\gamma)$ is either a Prüfer, adic, or generic module, and hence lies in $\mathrm{Supp}(\mathcal{T})$, and so any pp-pair closed on $\bigoplus_{M\in\mathcal{T}_\gamma} M$ is closed on $M(\gamma)$.

Consequently, we have an isomorphism between the two lattices, $\mathrm{pp}(M(r))$ and $\mathrm{pp}(\bigoplus_{M\in\mathcal{T}_\gamma}(M))$ (the map taking $\phi(M(\gamma))$ to $\phi(\bigoplus_{M\in\mathcal{T}_\gamma}(M))$ for all pp-formulas $\phi$).

Since $\mathcal{T}_\gamma$ is a sincere, stable, separating tubular family, corollary 8 completes the proof. $\square$

Given any Given any $r \in \mathbb{R}^+ \backslash \mathbb{Q}$, the Auslander-Reiten quiver of $A$ partitions into $\mathcal{P}_r \cup \mathcal{Q}_r$, with $\mathrm{Hom}(\mathcal{Q}_r, \mathcal{P}_r) = 0$. We refer to the modules of slope $r$ as "lying in the irrational cut"- in the sense that they lie between the modules in $\mathcal{P}_r$ and the modules in $\mathcal{Q}_r$.

We wish to determine the m-dimension, and indeed the breadth of $\mathrm{pp}(M(r))$ when $r$ is irrational.

Given any pp-pair, $\phi/\psi$ and any $r \in \mathbb{R}^+$, we say that $\phi/\psi$ is *closed near the left of* $r$ if there exists $\epsilon > 0$ such that $\phi(X) = \psi(X)$ for all $X \in \mathcal{P}_r \cap \mathcal{Q}_{r-\epsilon}$. We say that it is *open near the left of* $r$ if it is not closed near the left of $r$.

We say that $\phi/\psi$ is *closed near the right of* $r$ if there exists $\epsilon > 0$ such that $\phi(X) = \psi(X)$ for all $X \in \mathcal{P}_{r+\epsilon} \cap \mathcal{Q}_r$. We say that it is *open near the right of* $r$ if it is not closed near the right of $r$.

**Lemma 52.** *Let $r \in \mathbb{R}^+$. Let $\phi/\psi$ be any pair which is open near the right of $r$, or open near the left of $r$.*

*Then there exists a pure-injective indecomposable module $M$ of slope $r$ such that $\phi(M) > \psi(M)$.*

*Proof.* We denote by $\mathrm{Th}(A\text{-Mod})$ the theory of left $A$-modules. We claim that the theory:

$$\mathrm{Th}(A\text{-Mod}) \cup \Phi_r \cup \{\exists v(\phi(v) \wedge \neg\psi(v))\}$$

-is finitely satisfiable. By the completeness theorem, this will imply that the theory is satisfiable.

Given any finite subset $\Phi'$ of $\Phi_r$, there are only finitely many $X \in \mathcal{P}_r$ such that $\mathrm{Ext}(X, \_) = 0$ appears in $\Phi'$- so we may pick $\alpha < r$ such that every such $X$ lies in $\mathcal{P}_\alpha$. Similarly, we may pick $\beta > r$ such that $Y \in \mathcal{Q}_\beta$ for every $Y$ such that $\mathrm{Hom}(Y, \_)$ appears in $\Phi'$.

Recall that we are assuming that $\phi/\psi$ is open either near the left of $r$ or near the right of $r$.

1. If it is open near the left of $r$, then there exists a module $M \in \mathcal{P}_r \cap \mathcal{Q}_\alpha$ such that $\phi(M) > \psi(M)$. Then $\mathrm{Ext}(X, M) = 0$ for all sentences of the form $\mathrm{Ext}(X, \_)$ in $\Phi'$ (since $X \in \mathcal{P}_\alpha$). Furthermore, $\mathrm{Hom}(Y, M) = 0$ for all sentences of the form $\mathrm{Hom}(Y, \_) = 0$ in $\Phi'$( since $Y \in \mathcal{Q}_r$), so:

$$M \models \Phi' \cup \{\exists v(\phi(v) \wedge \neg\psi(v))\}$$

2. If $\phi/\psi$ is open near the right of $r$, then there exists a module $M \in \mathcal{P}_\beta \cap \mathcal{Q}_r$ such that $\phi(M) > \psi(M)$. Then $\mathrm{Ext}(X, M) = 0$ for all sentences of the form $\mathrm{Ext}(X, \_)$ in $\Phi'$ (since $X \in \mathcal{P}_r$). Furthermore, $\mathrm{Hom}(Y, M) = 0$ for all sentences of the form $\mathrm{Hom}(Y, \_) = 0$ in $\Phi'$( since $Y \in \mathcal{Q}_\beta$), so:

$$M \models \Phi' \cup \{\exists v(\phi(v) \wedge \neg\psi(v))\}$$

So the theory is indeed finitely satisfiable. Let $N$ be any model of it. Then $N \in A$-Mod. Since $\phi(N) > \psi(N)$, theorem 12 implies that there exists a pure-injective indecomposable $M$ in $\langle N \rangle$ such that $\phi(M) > \psi(M)$. Since $N$ has slope $r$, so does $M$, by lemma 41. □

### 3.6.1 Pp-formulas at an irrational cut

Recall that, given any 1-pointed $A$-module $(C, c)$, we denote by $f_{(C,c)}$ the unique map in $\mathrm{Hom}_A(A, C)$ taking 1 to $c$.

**Proposition 2.** *Let $r$ be any positive irrational and $\phi(v)$ be any pp-formula.*

*Then there exists a pp-formula $\phi' \geq \phi$, with free realisation $(M', m')$, and $\epsilon > 0$ such that:*

- $M' \in \mathrm{add}(\mathcal{P}_{r-\epsilon})$
- $\mathrm{Coker}(f_{(M',m')}) \in \mathrm{add}(\mathcal{Q}_{r+\epsilon})$
- $\phi(X) = \phi'(X)$ *for all indecomposable* $X \in A$-Mod *with slope in* $(r - \epsilon, r + \epsilon)$.
- $\dim_K(\phi(X)) = \dim_K(\phi'(X)) = \dim_K(\mathrm{Hom}(M', X'))$ *for all* $X \in \mathcal{Q}_{r-\epsilon} \cap \mathcal{P}_{r+\epsilon}$.

*Proof.* Let $(N, n)$ be the free realisation of $\phi(v)$. Decompose $N$ as $M \oplus L$, with $M \in \mathrm{add}(\mathcal{P}_r)$ and $N \in \mathrm{add}(\mathcal{Q}_r)$. Let $m \in M$ and $l \in L$ be such that the element $(m, l)$ of $M \oplus L$ corresponds to the element $n$ of $N$. Notice that, for all $X \in \mathcal{P}_r$:

$$\phi(X) = \{f(m) : f \in \mathrm{Hom}(M, X)\}$$

Let $C_R \in \mathrm{add}(\mathcal{Q}_r)$ and $C_L \in \mathrm{add}(\mathcal{P}_r)$ be such that $\mathrm{Coker}(f_{(M,m)}) \cong C_L \oplus C_R$. Let $\pi_L \in \mathrm{Hom}(M, C_L)$ and $\pi_R \in \mathrm{Hom}(M, C_R)$ be such that the natural surjection $M \twoheadrightarrow \mathrm{Coker}(f_\phi)$ is the map:

$$M \xrightarrow{(\pi_L, \pi_R)} C_L \oplus C_R$$

Let $K_L = \mathrm{Ker}(\pi_L)$ and $K_R = \mathrm{Ker}(\pi_R)$. Notice that:

- Since they are both submodules of $M$, $K_L$ and $K_R$ both lie in $\mathrm{add}(\mathcal{P}_r)$.
- Since $\pi_L(m) = \pi_R(m) = 0$ we can think of $m$ as an element of $K_L$, and as an element of $K_R$

Let $i_L : K_L \hookrightarrow M$ and $i_R : K_R \hookrightarrow M$ denote the natural embeddings. Notice that $i_L f_{(K_L,m)} = i_R f_{(K_R,m)} = f_{(M,m)}$, and so:

$$\mathrm{Im}(i_L f_{(K_L,m)}) = \mathrm{Im}(i_R f_{(K_R,m)}) = \mathrm{Im}(f_{(M,m)}) = \langle m \rangle$$

(Where $\langle m \rangle := \{am : a \in A\}$). Since the lattice of submodules of $M$ is modular, the interval:

$$\begin{tikzcd}[column sep=small] & M \arrow[dl, dash] \arrow[dr, dash] & \\ K_L \arrow[dr, dash] & & K_R \arrow[dl, dash] \\ & \langle m \rangle & \end{tikzcd}$$

-gives us that $M/K_R \simeq K_L/\langle m \rangle$, and hence that the following sequence is exact:

$$0 \longrightarrow \langle m \rangle \longrightarrow K_L \xrightarrow{\pi_R i_L} C_R \longrightarrow 0$$

(since $\pi_R i_L(m) = \pi_R(m) = 0$). And so $C_R \cong \mathrm{Coker}(f_{(K_L,m)})$.

Now, let $\phi'$ be a pp-formula which generates $\mathrm{pp}^{K_L}(m)$, and pick $\epsilon > 0$ such that no indecomposable direct summands of $M \oplus C_L \oplus C_R \oplus K_R \oplus K_L$ have slope in $(r - \epsilon, r + \epsilon)$.

Given any $X \in \mathcal{P}_r \cap \mathcal{Q}_{r-\epsilon}$, $\mathrm{Hom}(C_R, X) = 0$, so $\mathrm{Hom}(\mathrm{Coker}(f_{(M,m)}), X) \simeq \mathrm{Hom}(C_L, X)$. Thus:

$$\dim_K(\phi(X)) = \dim_K(\mathrm{Hom}(M, X)) - \dim_K(\mathrm{Hom}(C_L, X))$$

Since $\mathrm{Hom}(\mathrm{Coker}(f_{(K_L,m)}), X) \cong \mathrm{Hom}(C_R, X) = 0$, we have that:

$$\dim_K(\phi'(X)) = \dim_K(\mathrm{Hom}(K_L, X))$$

Applying theorem 1 to the exact sequence $0 \to K_L \to M \to C_L \to 0$ gives an exact sequence:

$$0 \to \mathrm{Hom}(C_L, X) \to \mathrm{Hom}(M, X) \to \mathrm{Hom}(K_L, X) \to \mathrm{Ext}(C_L, X) = 0$$

So:

$$\begin{aligned}
\dim_K(\phi')(X) &= \dim_K(\mathrm{Hom}(K_L, X)) \\
&= \dim_K(\mathrm{Hom}(M, X)) - \dim_K(\mathrm{Hom}(C_L, X)) \\
&= \dim_K(\mathrm{Hom}(M, X)) - \dim_K(\mathrm{Hom}(\mathrm{Coker}(f_{(M,m)}), X)) \\
&= \dim_K(\phi(X))
\end{aligned}$$

So $\phi(X) = \phi'(X)$. Taking $(M', m')$ to be $(K_L, m)$ completes the proof. □

**Corollary 9.** *Let $\phi/\psi$ be any pp-pair, and $r > 0$ any irrational.*

*If $\phi/\psi$ is open near the left of $r$, then there exists $\epsilon > 0$ such that $\phi/\psi$ is open on every module lying in a homogeneous tube in $\mathcal{P}_r \cap \mathcal{Q}_{r-\epsilon}$.*

*Similarly, if $\phi/\psi$ is open near the right of $r$, then there exists $\epsilon > 0$ such that $\phi/\psi$ is open on every module lying in a homogeneous tube in $\mathcal{Q}_r \cap \mathcal{P}_{r+\epsilon}$.*

*Proof.* We shall only prove the first assertion. The second  proved similarly.

Apply proposition 2 to $\phi$ and $\psi$ to obtain pp-formulas $\phi'$ and $\psi'$ with free realisations $(M', m')$ and $(N', n')$, and $\epsilon_1, \epsilon_2$ satisfying the relevant conditions. Let $\epsilon = \min(\epsilon_1, \epsilon_2)$.

Suppose that there exists $\gamma \in (r-\epsilon, r) \cap \mathbb{Q}$, and a module $E[k]$ lying in a homogeneous tube $\mathcal{T}(\rho)$ in $\mathcal{T}_\gamma$, such that $\phi/\psi$ is closed on $E[k]$. We shall prove that $\phi/\psi$ is therefore closed near the left of $r$.

Then $\phi'(E[k]) = \phi(E[k]) = \psi(E[k]) = \psi'(E[k])$, and so:

$$\dim_K(\mathrm{Hom}(M', E[k])) = \dim_K(\mathrm{Hom}(N', E[k]))$$

Then, as in the proof of lemma 36, it follows that, for all $m \in \mathbb{N}^+$:

$$\dim_K(\mathrm{Hom}(M', E[m])) = \dim_K(\mathrm{Hom}(N', E[m]))$$

-thus $\phi'/\psi'$ is closed on every module in $\mathcal{T}(\rho)$.

Now, given any $X \in \mathcal{Q}_\gamma \cap \mathcal{P}_r$, and any $x \in \phi(X) = \phi'(X)$, there exists $f \in \mathrm{Hom}(M', X)$ such that $f(m') = x$. Then $f$ factors through a module $Y \in \mathrm{add}(\mathcal{T}(\rho))$:

$$\begin{tikzcd} M' \arrow[rr, "f"] \arrow[dr, dashed, "\exists g"'] & & X \\ & Y \arrow[ur, dashed, "\exists h"'] & \end{tikzcd}$$

Since $g(m') \in \phi'(Y) = \psi'(Y)$, there exists $g' \in \mathrm{Hom}(N', Y)$ such that $g'(n') = g(m')$. Then $hg'(n') = x$, and so $x \in \psi'(X) = \psi(X)$. Thus $\phi/\psi$ is closed on every module in $\mathcal{Q}_\gamma \cap \mathcal{P}_r$- as required. □

**Proposition 3.** *Let $\phi/\psi$ be any pp-pair, and $r$ any positive irrational.*

*Then, there exists $\epsilon > 0$ and a vector $v \in K_0(A)$ such that $dim_K((\phi/\psi)(X)) = v.\underline{\dim}(X)$ for all $X \in \mathcal{P}_{r+\epsilon} \cap \mathcal{Q}_{r-\epsilon}$.*

*Proof.* Let $M'$, $m'$ and $\epsilon$ be as in proposition 2. Since $M' \in \mathcal{P}_{r-\epsilon}$, it has projective dimension at most 1.....(find reference....): and so there exists an exact sequence:

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M' \longrightarrow 0$$

-with $P_0$ and $P_1$ being projective, and hence in $\mathcal{P}_0 \cup \mathcal{T}_0$. Given any $X \in \mathcal{Q}_{r-\epsilon} \cap \mathcal{P}_{r+\epsilon}$ we can induce an exact sequence:

$$0 \to \mathrm{Hom}(M, X) \to \mathrm{Hom}(P_0, X) \to \mathrm{Hom}(P_1, X) \to \mathrm{Ext}(M', X) = 0$$

Consequently:

$$\dim_K(\phi(X)) = \dim_K(\mathrm{Hom}(M', X)) = \dim_K(\mathrm{Hom}(P_0, X)) - \dim_K(\mathrm{Hom}(P_1, X))$$

Now, label the vertices of $Q$ as $1, 2, \ldots n$, and consider the indecomposable projectives $\{P(a) : a \in Q_0\}$ of $A$-mod. Then there exists $c_1, \ldots, c_n, d_1, \ldots, d_n \in \mathbb{N}$ such that $P_0 \cong \bigoplus_{a=1}^{n} P(a)^{c_a}$ and $P_1 \cong \bigoplus_{a=1}^{n} P(a)^{d_a}$.

Now, given any $X \in A$-mod, let $(x_1, x_2, \ldots, x_n)$ be $\underline{\dim}(X)$. Then for all $a \in Q_0$, $\dim_K(\mathrm{Hom}(P(a), X) = x_a$, and so:

$$\dim_K(\mathrm{Hom}(P_0, X)) = (c_1, \ldots, c_n).(x_1, \ldots, x_n)$$

$$\dim_K(\mathrm{Hom}(P_1, X)) = (d_1, \ldots, d_n).(x_1, \ldots, x_n)$$

Let $v_1 = (c_1 - d_1, \ldots, c_n - d_n)$. Then for all $X \in \mathcal{P}_{r-\epsilon} \cap \mathcal{Q}_{r+\epsilon}$:

$$\dim_K(\phi(X)) = \dim_K(\mathrm{Hom}(M', X)) = (v_1).\underline{\dim}(M)$$

Similarly, there exists $\delta > 0$ and a vector $v_2$ in $\mathbb{Z}^n$ such that, for all $X \in \mathcal{P}_{r-\delta} \cap \mathcal{Q}_{r+\delta}$:

$$\dim_K(\psi(X)) = (v_2).\underline{\dim}(N')$$

Taking $v = v_1 - v_2$ and relabeling $\min(\epsilon, \delta)$ as $\epsilon$ completes the proof. □

### 3.6.2 The lattice of pp-formulas at an irrational cut

**Theorem 30.** *Given any irrational $r \in \mathbb{R}_0^\infty$, let $M(r)$ denote the direct sum of all indecomposable pure-injective $A$-modules of slope $r$. Then, given any pp-pair $\phi/\psi$, the following are equivalent:*

1. *$\phi/\psi$ is closed near the left of $r$*

2. *$\phi/\psi$ is closed near the right of $r$*

3. *$\phi(M(r)) = \psi(M(r))$.*

*Proof.* First of all, lemma 52 gives that (3) implies (2) (and indeed, (1)).

To prove that (1) implies (3)- suppose that $\phi/\psi$ is closed near the left of $r$- i.e. that there exists $\epsilon > 0$ such that $\phi(X) = \psi(X)$ for all $X \in \mathcal{P}_r \cap \mathcal{Q}_{r-\epsilon}$. By lemma 33, there exists a direct system $((M_i), (f_{ij}))$, with each $M_i$ in $\mathrm{add}(\mathcal{P}_r \cap \mathcal{Q}_{r+\epsilon})$, with direct limit $M(r)$. Since pp-formulas commute with direct sums (by [17, (1.2.31)]), we have that:

$$\phi(\varinjlim M_i) = \varinjlim \phi(M_i) = \varinjlim \psi(M_i) = \psi(\varinjlim M_i)$$

-so $\phi/\psi$ is indeed closed on $M(r)$- as required.

Finally, we prove that (2) implies (1). Let $v$ and $\epsilon$ be as in proposition 3. Assume that (2) holds- i.e. there exists $\delta > 0$ such that $\phi/\psi$ is closed on all modules in $\mathcal{Q}_r \cap \mathcal{P}_{r+\delta}$.

We claim that $v.h_0 + \gamma v.h_\infty = 0$ for all $\gamma \in (r, r+\delta) \cap \mathbb{Q}$. Indeed, given any such $\gamma$, pick any $k \in \mathbb{N}^+$ such that $k\gamma \in \mathbb{N}$. By corollary 12, there exists a homogeneous indecomposable module $X$ with $\underline{\dim}(X) = kh_0 + k\gamma h_\infty$. Then $X$ has slope $\gamma$, and so $(\phi/\psi)(X) = 0$, and hence $k(v.h_0 + \gamma v.h_\infty) = 0$, so $v.h_0 + \gamma v.h_\infty = 0$ as claimed.

Since this holds for all $\gamma \in (r, r+\epsilon)$, it follows that $v.h_0 = v.h_\infty = 0$. Now, given any module $X$ in a homogeneous tube in $\mathcal{P}_r \cap \mathcal{Q}_{r-\delta}$, $\underline{\dim}(X) \in \mathrm{rad}(\chi_A)$, and so $\underline{\dim}(X) = bh_0 + b'h_\infty$ for some $b, b' \in \mathbb{N}$. Thus $v\underline{\dim}(X) = 0$, and so $(\phi/\psi)(X) = 0$.

It follows from corollary 9 that $\phi/\psi$ is closed near the left of $r$- which completes the proof

□

We refer to any pp pair satisfying the conditions of theorem 30 as being *closed near* $r$. We say that a pp-pair is *open near* $r$ if it is not closed near $r$.

Notice that theorem 30 does not hold if $r$ is rational- for example, take a stable tube $\mathcal{T}(\rho)$ in $\mathcal{T}_r$ and let $E[1]$ be any quasisimple in it. Let $f_1$ be the irreducible map in $\mathrm{Hom}(E[1], E[2])$. By theorem 6, there exists a pp-formula $\phi/\psi$ which is equivalent to $\mathrm{Cok}(f_1, \_)$- where $(f_1, \_)$. By lemma 24, $\phi/\psi$ is open on $E_1$ and closed on all other modules in $A$-Mod- and hence is open on a module in $\mathcal{T}_r$, but is closed near the left and near the right of $r$.

Given any $r \in \mathbb{R}_0^\infty$, let $\sim_r$ be the relation on $\mathrm{pp}_R$ such that $\phi \sim_r \psi$ if and only if there exists $\epsilon > 0$ such that $\phi(X) = \psi(X)$ for all $X \in \mathcal{Q}_{r-\epsilon} \cap (\mathcal{P}_r \cup \mathcal{T}_r)$. It is clearly a congruence on $\mathrm{pp}_R$. Of course, if $\phi/\psi$ is a pp-pair and $r \notin \mathbb{Q}$, then $\phi \sim_r \psi$ if and only if $\phi/\psi$ is closed near the left of $r$.

**Corollary 10.** *Given any $r \in \mathbb{R}^+ \backslash \mathbb{Q}$, let $M(r)$ be the direct sum of all indecomposable pure-injectives of slope $r$.*

*Then the lattices $\mathrm{pp}(M(r))$ and ${}_A\mathrm{pp}/\sim_r$ are naturally isomorphic.*

*Proof.* Define a map from $\mathrm{pp}(M(r))$ to ${}_A\mathrm{pp}/\sim_r$, taking any pp-definable subgroup $\phi(M(r))$ to the equivalence class of $\phi$ in ${}_A\mathrm{pp}/\sim_r$. By theorem 30 it is an isomorphism. One can easily check it is a well defined map. $\square$

# Chapter 4

# Modules of Irrational Slope

In the last chapter we proved that, given any $r \in \mathbb{R}^+ \backslash \mathbb{Q}$, the lattices $\mathrm{pp}(M(r))$ and $\mathrm{pp}/\sim_r$ are equivalent- where $M(r)$ is the direct sum of all pure-injective indecomposables in $A$-mod of slope $r$. We prove in this chapter that the breadth of this lattice is undefined.

We prove the result, first of all, for a few specific tubular algebras- $C(4,\lambda)$, $C(6)$, $C(7)$ and $C(8)$- and then show how the result can, through tilting functors, be extended to all tubular algebras.

## 4.1 Modules in tubular families

Throughout this section $A$ will be any tubular algebra, and $\mathbb{T} = (n_1, \ldots, n_t)$ will be the tubular type of $A$. $\gamma$ will denote any positive rational. The set of tubes in $\mathcal{T}_\gamma$ will be denoted $\{\mathcal{T}(\rho) : \rho \in \mathbb{P}^1(K)\}$. For each $\rho \in \mathbb{P}^1(K)$, $n_\rho$ will denote the rank of the tube $\mathcal{T}(\rho)$. By theorem 27, there exist pairwise distinct $\rho_1, \ldots, \rho_t \in \mathbb{P}^1(K)$ such that $n_{\rho_s} = n_s$ for all $s \in \{1, 2, \ldots, t\}$, and $n_\rho = 1$ for all $\rho \notin \{\rho_1, \ldots, \rho_t\}$.

Given any stable tube $\mathcal{T}(\rho)$ of rank $k$, the quasisimple modules will normally be denoted $\{E_i^\rho : i \in \mathbb{Z}_k\}$- such that $\tau^-(E_i^\rho) = E_{i+1}^\rho$ for all $i \in \mathbb{Z}$.

**Lemma 53.** *Given any stable tube $\mathcal{T}(\rho)$, any indecomposable quasisimple module $E_i$ in $\mathcal{T}(\rho)$, and any $k \geq 1$:*

$$\chi_A(\underline{\dim}(E_i[k])) = \begin{cases} 0 & \textit{if } n_\rho | k \\ 1 & \textit{otherwise} \end{cases}$$

*Proof.* Since $E[k]$ lies in a stable tube, it has projective dimension 1 (by theorem...). Thus, by lemma 20 and theorem 16:

$$\begin{aligned} \chi_A(E_i[k]) &= \dim_K(\mathrm{Hom}(E_i[k], E_i[k])) - \dim_K(\mathrm{Ext}(E_i[k], E_i[k])) \\ &= \dim_K(\mathrm{Hom}(E_i[k], E_i[k])) - \dim_K(\mathrm{Hom}(E_i[k], \tau E_i[k])) \\ &= \dim_K(\mathrm{Hom}(E_i[k], E_i[k])) - \dim_K(\mathrm{Hom}(E_i[k], E_{i-1}[k])) \end{aligned}$$

It follows from corollary 1 that $\dim_K(\mathrm{Hom}(E_i[k], E_i[k]))$ is the number of elements $a$ of $\{1, 2, \ldots, k\}$ such that $n_\rho | (a-k)$. Similarly, $\dim_K(\mathrm{Hom}(E_i[k], E_{i-1}[k]))$ is the number of elements $a$ in $\{1, 2, \ldots, k\}$ such that $n_\rho | (a - 1 - k)$. The result follows straight from these facts. $\square$

**Lemma 54.** *Let $V$ be the subgroup of $K_0(A)$ generated by the set $\{\underline{\dim}(M) : M \in \mathcal{T}_\gamma\}$. Then the rank of $V$ is $1 - t + \sum_{s=1}^{t} n_s$.*

*Proof.* By [23, (5.3.2')], the rank of $V$ is at least $1 - t + \sum_{s=1}^{t} n_s$. Pick any $a, b \in \mathbb{N}$ such that $b/a = \gamma$. By lemma 28, $\mathrm{rad}(\chi_A) \cap \mathrm{Ker}(\iota_\gamma)$ is a subgroup of $K_0(A)$ of rank 1- in fact every element of it is equal to $q(ah_0 + bh_\infty)$ for some $q \in \mathbb{Q}$. Let $C$ be the set:

$$C := \{ah_0 + bh_\infty\} \cup \bigcup_{s=1}^{t} \{\underline{\dim}(E_j^{\rho_s}) : 1 \le j \le n_s - 1\}$$

Since $|C| \le 1 - t + \sum_{s=1}^{t} n_s$, it will be enough to prove that $V$ is spanned (over $\mathbb{Q}$) by $C$.

Given any $\rho \in I$, the elements $\underline{\dim}(E_1^\rho), \ldots \underline{\dim}(E_{n_\rho - 1}^\rho)$ lie in $C$. Furthermore, by lemma 34:

$$\sum_{i=1}^{n_\rho} \underline{\dim}(E_i^\rho[1]) = \underline{\dim}(E_1^\rho[n_\rho])$$

By lemma 53, $\underline{\dim}(E_1^\rho[n_\rho]) \in \mathrm{rad}(\chi)$, and since $\underline{\dim}(E_1^\rho[n_\rho]) \in \mathrm{Ker}(\iota_\gamma)$, there exists $q \in \mathbb{Q}$ such that:

$$\underline{\dim}(E_1^\rho[n_\rho]) \in \mathrm{rad}(\chi) = q(ah_0 + bh_\infty)$$

-thus $\underline{\dim}(E_{n_\rho}^\rho)$ lies in the $\mathbb{Q}$-span of $C$.

Finally, every indecomposable module in $\mathcal{T}_\gamma$ is isomorphic to $E_i^\rho[k]$ for some $\rho \in I$, $k \in \mathbb{N}$, and $i \in \mathbb{Z}_{n_\rho}$. By lemma 34:

$$\underline{\dim}(E_i^\rho[k]) = \sum_{j=1}^{k} \underline{\dim}(E_{i+j-1}^\rho[1])$$

-and hence lies in the $\mathbb{Q}$-span of $C$, as required. $\square$

Given an element $x \in K_0(A)$, let $\langle x \rangle$ denote the subgroup of $K_0(A)$ generated by $x$. We say that $x$ is *primitive* if and only if the quotient lattice $K_0(A)/\langle x \rangle$ is torsionfree- i.e. if and only if there is no $y \in K_0(A)$ and integer $n \ge 2$ such that $ny = x$.

**Lemma 55.** *Assume that every $X \in \mathcal{T}_\gamma$ has both projective dimension and injective dimension* 1*. Let $U$ be any subgroup of $K_0(A)$ of rank $1 - t + \sum_{s=1}^{t} n_s$, such that $\underline{\dim}(M) \in U$ for all $M \in \mathcal{T}_\gamma$. Then the following are equivalent:*

1. *For all connected positive $x \in K_0(A)$ such that $\chi_A(x) \in \{0, 1\}$, there exists an indecomposable module $M \in \mathcal{T}_\gamma$ with $x = \underline{\dim}(M)$.*
2. *For all $s \in \{1, 2, \dots, t\}$, $\sum_{i=1}^{n_s} \underline{\dim}(E_i^{(\rho_s)})$ is primitive in $U$, and the subgroup of $U$ generated by $\{x \in U : \chi_A(x) \in \{0, 1\}\}$ is the subgroup of $U$ generated by $\{\underline{\dim}(M) : M \in \mathcal{T}_\gamma\}$.*

*Proof.* See [23, (5.3.3)]. □

**Corollary 11.** *Pick any $a, b \in \mathbb{N}$ such that $\gamma = b/a$. Let $c$ be the greatest common divisor of all the coordinates of $ah_0 + bh_\infty$. Then, given any stable tube $\mathcal{T}(\rho_s)$:*

$$\sum_{i=1}^{n_s} \underline{\dim}(E_i^{(\rho_s)}) = (a/c)h_0 + (b/c)h_\infty$$

*Proof.* Let $U$ be the subgroup of $K_0(A)$ generated by $\{\underline{\dim}(M) : M \in \mathcal{T}_\gamma\}$. By lemma 54, it has rank $1 - t + \sum_{s=1}^{t} n_s$. Notice that $U$ is a subgroup of $\mathrm{Ker}(\gamma)$.

Given any connected positive $x \in U$ such that $\chi(x) \in \{0, 1\}$, there exists an indecomposable module $M \in \mathcal{T}_\gamma$ such that $\underline{\dim}(M) = x$ (by theorem 27).

Thus, by lemma 55, $\sum_{i=1}^{n_s} \underline{\dim}(E_i^{(\rho_s)})$ is primitive in $U$. As in the proof of lemma 54, $\sum_{i=1}^{n_s} \underline{\dim}(E_i^{(\rho_s)}) \in \mathrm{rad}(\chi_A) \cap \mathrm{Ker}(\chi_C)$.

By lem 28:

$$\mathrm{rad}(\chi_A) \cap \mathrm{Ker}(\iota_\gamma) = \{(d/c)(ah_0 + bh_\infty) : d \in \mathbb{Z}\} \subseteq U$$

Since $\sum_{i=1}^{n_s} \underline{\dim}(E_i^{(\rho_s)})$ is primitive in $U$, it must be primitive in $\mathrm{rad}(\chi_A) \cap \mathrm{Ker}(\iota_\gamma)$, and so:

$$\sum_{i=1}^{n_s} \underline{\dim}(E_i^{(\rho_s)}) = \frac{1}{c}(ah_0 + bh_\infty)$$

-as required. □

**Corollary 12.** *Given any $a, b \in \mathbb{N}^+$, there exists a homogeneous indecomposable module with dimension vector $ah_0 + bh_\infty$.*

*Proof.* Since $ah_0 + bh_\infty \in \operatorname{rad}(\chi) \cap \operatorname{Ker}(\iota)$, there exist infinitely many (isomorphism classes of) $A$-modules with dimension vector $ah_0 + bh_\infty$.

One can easily check (from lemma 34) that there can only be finitely many indecomposable modules in any given tube with dimension vector $ah_0 + bh_\infty$. Since there are only finitely many non-homogeneous tubes, the result follows. $\square$

## 4.2 The tubular algebras, $C(4, \lambda)$, $C(6)$, $C(7)$ and $C(8)$

We now introduce the bound quiver algebras, $C(4, \lambda)$, $C(6)$, $C(7)$ and $C(8)$, as well as calculating their characteristic $\chi$, and quoting a few other properties from [23, (5.6)]. Indeed, they are tubular algebras, by [23, (5.6.1)].

### 4.2.1 $C(4, \lambda)$

Given any $\lambda \in K \backslash \{0, 1\}$, $C(4, \lambda)$ denotes the algebra over the quiver:

$$\begin{tikzcd}
1 & & 4 \arrow[dl, "\alpha_{12}"'] & \\
& 3 \arrow[ul, "\beta"'] \arrow[dl, "\gamma"] & & 6 \arrow[ul, "\alpha_{11}"'] \arrow[dl, "\alpha_{21}"] \\
2 & & 5 \arrow[ul, "\alpha_{22}"] &
\end{tikzcd}$$

-subject to the relations $\beta(\alpha_{12}\alpha_{11} - \alpha_{22}\alpha_{21})$ and $\gamma(\alpha_{12}\alpha_{11} - \lambda\alpha_{22}\alpha_{21})$. The tubular type of $C(4, \lambda)$ is $(2, 2, 2, 2)$. The Cartan matrix is:

$$C_{C(4,\lambda)} = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

And $(C^{-1})^T$ is:

$$C^{-T} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & -1 & -1 & 1 \end{pmatrix}$$

The characteristic is:

$$\begin{aligned} \chi_{C(4,\lambda)}(x_1,\ldots,x_6) &= \frac{1}{2}(x_1-x_2)^2 + (x_3 + \frac{1}{2}(x_1+x_2+x_4+x_5))^2 \\ &+ (x_6 + \frac{1}{2}(x_1+x_2-x_4-x_5))^2 + \frac{1}{2}(x_4-x_5)^2 \end{aligned}$$

Also, $h_0 = (1,1,2,1,1,0)$ and $h_\infty = (0,0,1,1,1,1)$, and so the index of any element $(x_1,\ldots,x_6) \in K_0(A)$ is:

$$\iota(x_1,\ldots,x_6) = \frac{x_4+x_5-x_2-x_1}{x_3-x_6}$$

Also, $\langle h_0, h_\infty \rangle = 2$.

### 4.2.2 $C(6)$

$C(6)$ is the algebra with underlying quiver:

$$\begin{tikzcd} & & & 4 \arrow[dl, "\alpha_3"'] & 5 \arrow[l, "\alpha_2"'] & \\ 1 & 2 \arrow[l, "\gamma'"'] & 3 \arrow[l, "\gamma"'] & & & 8 \arrow[ul, "\alpha_1"'] \arrow[dl, "\beta_1"] \\ & & & 6 \arrow[ul, "\beta_3"] & 7 \arrow[l, "\beta_2"] & \end{tikzcd}$$

-with relations $\gamma(\alpha_3\alpha_2\alpha_1 - \beta_3\beta_2\beta_1) = 0$.

The tubular type of $C(6)$ is $(3,3,3)$, and the Cartan matrix is:

$$C_{C(6)} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$C^{-T} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & -1 & 1 \end{pmatrix}$$

And so the characteristic is:

$$\begin{aligned} \chi_{C(6)} &= (x_1 - \frac{1}{2}x_2)^2 + \frac{3}{4}(x_2 - \frac{2}{3}(x_3 - x_8))^2 + (x_4 - \frac{1}{2}(x_3 + x_5))^2 \\ &+ (x_6 - \frac{1}{2}(x_3 + x_7))^2 + \frac{3}{4}(x_5 - \frac{1}{3}(2x_8 + x_3))^2 + \frac{3}{4}(x_7 - \frac{1}{3}(2x_8 + x_3))^2 \end{aligned}$$

So $h_0 = (1,2,3,2,1,2,1,0)$ and $h_\infty = (0,0,1,1,1,1,1,1)$. And the index of any element $(x_1,\ldots,x_8)$ of $K_0(C(6))$ is given by:

$$\iota(x_1,\ldots,x_8) = \frac{x_4 + x_5 + x_6 + x_7 - x_1 - x_2 - x_3}{x_3 - x_8}$$

Also, $\langle h_0, h_\infty \rangle = 3$

### 4.2.3 $C(7)$

$C(7)$ is the algebra with quiver:

$$\begin{tikzcd}
 & & 3 \arrow[dl, "\gamma_4"'] & 4 \arrow[l, "\gamma_3"'] & 5 \arrow[l, "\gamma_2"'] & \\
1 & 2 \arrow[l, "\gamma"'] & & & & 9 \arrow[ul, "\gamma_1"'] \arrow[dl, "\beta_1"] \\
 & & 6 \arrow[ul, "\beta_4"] & 7 \arrow[l, "\beta_3"] & 8 \arrow[l, "\beta_2"] &
\end{tikzcd}$$

-with the relation $\gamma(\alpha_4\alpha_3\alpha_2\alpha_1 - \beta_4\beta_3\beta_2\beta_1) = 0$.

The tubular type is $(4, 4, 2)$, and the Cartan matrix is:

$$C_{C(7)} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$C^{-T} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 1 \end{pmatrix}$$

So the characteristic $\chi_{C(7)} : K_0(C(7)) \to \mathbb{Z}$ is given by:

$$\begin{aligned}
\chi &= \frac{2}{3}(x_3 - \frac{1}{4}(x_9 + 3x_2))^2 + \frac{2}{3}(x_6 - \frac{1}{4}(x_9 + 3x_2))^2 + (x_1 - \frac{1}{2}(x_2 - x_9))^2 \\
&+ \frac{3}{4}(x_7 - \frac{1}{3}(x_9 + 2x_6))^2 + \frac{3}{4}(x_4 - \frac{1}{3}(x_9 + 2x_3))^2 \\
&+ (x_8 - \frac{1}{2}(x_7 + x_9))^2 + (x_5 - \frac{1}{2}(x_4 + x_9))^2
\end{aligned}$$

Also, $h_0 = (2,4,3,2,1,3,2,1,0)$ and $h_\infty = (0,1,1,1,1,1,1,1,1)$, so the index of any element $(x_1, x_2, \ldots, x_9)$ is given by:

$$\iota(x_1, \ldots, x_9) = \frac{-2x_1 - 2x_2 + \sum_{i=3}^{8} x_i}{x_2 - x_9}$$

Also, $\langle h_0, h_\infty \rangle = 4$.

### 4.2.4 $C(8)$

$C(8)$ is the algebra with quiver:

$$\begin{tikzcd}
 & & 3 \arrow[dl, "\alpha_3"'] & & 4 \arrow[ll, "\alpha_2"'] & & \\
1 & 2 \arrow[l, "\gamma"'] & & & & & 10 \arrow[ull, "\alpha_1"'] \arrow[d, "\beta_1"] \\
 & & 5 \arrow[ul, "\beta_6"] & 6 \arrow[l, "\beta_5"] & 7 \arrow[l, "\beta_4"] & 8 \arrow[l, "\beta_3"] & 9 \arrow[l, "\beta_2"]
\end{tikzcd}$$

-with relation $\gamma(\alpha_3\alpha_2\alpha_1 - \beta_6\beta_5\beta_4\beta_3\beta_2\beta_1) = 0$. The tubular type of $C(8)$ is $(6,3,2)$, and its Cartan matrix is:

$$C_{C(8)} = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

$$C^{-T} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

The characteristic is:

$$\begin{aligned} \chi_{C(8)} &= (x_1 - \frac{1}{2}(x_2 - x_{10}))^2 + \frac{3}{4}(x_3 - \frac{1}{3}(x_{10} + 2x_2))^2 + (x_4 - \frac{1}{2}(x_3 + x_{10}))^2 \\ &+ \frac{3}{5}(x_5 - \frac{1}{6}(x_{10} + 5x_2))^2 \frac{5}{8}(x_6 - \frac{1}{5}(x_{10} + 4x_5))^2 + \frac{2}{3}(x_7 - \frac{1}{4}(x_{10} + 3x_6))^2 \\ &+ \frac{3}{4}(x_8 - \frac{1}{3}(x_{10} + 2x_7))^2 + (x_9 - \frac{1}{2}(x_8 + x_{10}))^2 \end{aligned}$$

Also, $h_0 = (3,6,4,2,5,4,3,2,1,0)$, $h_\infty = (0,1,1,1,1,1,1,1,1,1)$ and the index of any element $(x_1,\ldots,x_{10})$ of $K_0(C(8))$ is:

$$\iota(x_1,\ldots,x_{10}) = \frac{-9x_1 - 3x_2 + 8x_3 + 4x_4 + 5x_5 + 4x_6 + 3x_7 + 2x_8 + x_9}{x_2 - x_{10}}$$

Also, $\langle h_0, h_\infty \rangle = 6$.

## 4.3 Indecomposables over $C(4,\lambda)$, $C(6)$, $C(7)$ and $C(8)$

Throughout this section $C$ will denote one of the four tubular algebras, $C(4,\lambda)$, $C(6)$, $C(7)$ or $C(8)$. And $K_0(C)$ will be identified as $\mathbb{Z}^n$- where $n$ is the number of vertices of the quiver associated to $C$.

In order to study the lattice ${}_C\mathrm{pp}/\sim_r$ (where $r$ is irrational), we need a few results regarding the dimension vectors of indecomposable $C$-modules.

### 4.3.1 The dimension vectors of $C$-modules

**Lemma 56.** $\mathrm{rad}(\chi) = \{ah_0 + bh_\infty : a, b \in \mathbb{Z}\}$

*Proof.* By theorem 26, $\mathrm{rad}(\chi)$ is a rank 2 subgroup of $K_0(C)$, and $h_0, h_\infty \in \mathrm{rad}(\chi)$ are linearly independent elements of it. Consequently, we can write any $(x_1, \ldots, x_n) \in \mathrm{rad}(\chi)$ as a $\mathbb{Q}$-linear combination of $h_0$ and $h_\infty$:

$$(x_1, \ldots, x_n) = q_1 h_0 + q_2 h_\infty$$

Notice that the $(n-1)$-th and $n$-th coordinate of $h_0$ are 1 and 0 respectively, and the $(n-1)$-th and $n$-th coordinate of $h_\infty$ are both 1.

By projecting onto the $(n-1)$-th coordinate and the $n$-th coordinate of $\mathbb{Q}^n$, we get:

$$q_1 + q_2 = x_{n-1}$$

$$q_2 = x_n$$

Since $x_n$ and $x_{n-1}$ lie in $\mathbb{Z}$, so must $q_1$ and $q_2$- which completes the proof. □

**Lemma 57.** *For all $x \in K_0(C)$:*

$$\chi_C(x + h_0) = \chi_C(x + h_\infty) = \chi_C(x - h_0) = \chi_C(x - h_\infty) = \chi_C(x)$$

*Proof.* For all four of the tubular algebras, $\chi_C$ takes the form:

$$\chi(x_1, \ldots, x_n) = \sum_{j=1}^{k} p_j(x_1, \ldots, x_n)^2$$

-where each $p_j(x_1, \ldots, x_n)$ is a (homogeneous) polynomial in $\mathbb{Q}[x_1, \ldots, x_n]$, of degree 1.

Since $\chi(h_0) = 0$, it follows that $p_j(h_0) = 0$ for all $j$. And so, for example, $p_j(x + h_0) = p_j(x)$. It follows that:

$$\sum_{j=1}^{k} p_j(x + h_0)^2 = \sum_{j=1}^{k} p_j(x)^2$$

-i.e. $\chi(x) = \chi(x + h_0)$. One can similarly show the other results. □

Of course, since $\chi_A : \mathbb{Z}^n \to \mathbb{Z}$ is just a polynomial in $n$ variables, we may also consider it as a map from $\mathbb{Q}^n$ to $\mathbb{Q}$.

**Lemma 58.** *There exists $p \in \mathbb{N}$ such that, given any $(x_1, \dots, x_n) \in \mathbb{Q}^n$, with $x_{n-1} = x_n = 0$ and $\chi(x_1, x_2, \dots, x_n) = 1$:*

$$|x_i| \leq p \textit{ for all } i \in \{1, 2, \dots n-2\}$$

*Proof.* I'm only proving this for $C(4,\lambda)$- the other cases are proved similarly. Recall that, for any $x = (x_1, \dots, x_6) \in \mathbb{Q}^6$:

$$\begin{aligned}\chi_{C(4,\lambda)}(x_1, \dots, x_6) &= \frac{1}{2}(x_1 - x_2)^2 + (x_3 + \frac{1}{2}(x_1 + x_2 + x_4 + x_5))^2 \\ &+ \frac{1}{2}(x_4 - x_5)^2 + (x_6 + \frac{1}{2}(x_1 + x_2 - x_4 - x_5))^2\end{aligned}$$

So if $x_5 = x_6 = 0$, and $\chi(x) = 1$, then:

$$(\frac{1}{2}(x_1 + x_2 - x_4))^2 \leq 1$$

$$(x_3 + \frac{1}{2}(x_1 + x_2 + x_4))^2 \leq 1$$
$$\frac{1}{2}(x_1 - x_2)^2 \leq 1$$
$$\frac{1}{2}(x_4)^2 \leq 1$$

Consequently, $|x_4| \leq \sqrt{2}$, and $|x_1 - x_2| \leq \sqrt{2}$. Furthermore, since $(x_1 + x_2 - x_4)^2/4 \leq 1$, we have that:

$$|x_1 + x_2| \leq 2 + |x_4| < 4$$

It follows that $|x_1| \leq 3$ and $|x_2| \leq 3$. Finally, since $(x_3 + (x_1 + x_2 + x_4)/2)^2 \leq 1$, we have that $|x_3| \leq 4$- which completes the proof. □

Define $\Omega \subseteq K_0(C)$ to be the set of all elements $(x_1, \dots, x_n)$ of $K_0(C)$, such that $x_n = x_{n-1} = 0$ and $\chi_C(x_1, \dots, x_n) = 1$.

**Lemma 59.** *The set $\Omega$ is finite, and we have a bijective correspondence between the set $\{x \in K_0(C) : \chi(x) = 1\}$ and the set:*

$$\{ah_0 + bh_\infty + y : a \in \mathbb{Z}, b \in \mathbb{Z}, y \in \Omega\}$$

*Proof.* Take any $(x_1, \dots, x_n) \in K_0(C)$ such that $\chi(x_1, \dots, x_n) = 1$, and let $y = (x_1, \dots, x_n) - (x_{n-1} - x_n)h_0 - x_n h_\infty$. By lemma 57, $\chi(y) = 1$, and the last two coordinates of $y$ are clearly 0.

The finiteness follows straight from lemma 58 $\square$

Define $\mu : K_0(A) \to \mathbb{N}$ to be the map such that $\mu(\underline{\dim}(M)) = \dim_K(M)$ for all $M \in A$-mod.

**Lemma 60.** *Take any coprime $a, b \in \mathbb{N}^+$ such that $b > -n_\rho\langle h_\infty, y\rangle$ for all $y \in \Omega$ and $\rho \in \mathbb{P}^1(K)$. Let $\gamma = b/a$. We can pick $p \geq 0$ such that, for all $y \in \Omega$:*

$$|\frac{1}{\langle h_0, h_\infty\rangle}(\mu(\langle h_\infty, y\rangle h_0 - \langle h_0, y\rangle h_\infty)) + y| \leq p$$

*Let $\mathcal{T}(\rho)$ be any non-homogeneous stable tube in $\mathcal{T}_\gamma$. Let $E$ be any quasisimple in $\mathcal{T}(\rho)$. Then:*

$$\dim_K(E) \geq \frac{1}{\langle h_0, h_\infty\rangle}\mu(ah_0 + bh_\infty) - p$$

*Furthermore, if $n_\rho = \langle h_0, h_\infty\rangle$, then:*

$$|\dim_K(E) - \frac{1}{\langle h_0, h_\infty\rangle}\mu(ah_0 + bh_\infty)| \leq p$$

*Proof.* Let $E_1, E_2, \dots, E_{n_\rho}$ denote the quasisimples of $\mathcal{T}(\rho)$. By lemma 59, there exists (for each $i$) unique $c_i, d_i \in \mathbb{N}$ and $y_i \in \Omega$ such that:

$$\underline{\dim}(E_i) = c_i h_0 + d_i h_\infty + y_i$$

Since the slope of $E_i$ is $b/a$ we have:

$$b/a = \iota(\underline{\dim}(E_i)) = \frac{d_i\langle h_0, h_\infty\rangle + \langle h_0, y_i\rangle}{c_i\langle h_0, h_\infty\rangle - \langle h_\infty, y_i\rangle}$$

Let $k_i = \gcd(d_i\langle h_0, h_\infty\rangle + \langle h_0, y_i\rangle, c_i\langle h_0, h_\infty\rangle - \langle h_\infty, y_i\rangle)$ (noting that both things are indeed non-zero). Then:

$$k_i b = d_i\langle h_0, h_\infty\rangle + \langle h_0, y_i\rangle$$

$$k_i a = c_i\langle h_0, h_\infty\rangle - \langle h_\infty, y_i\rangle$$

So:

$$\underline{\dim}(E_i) = \frac{1}{\langle h_0, h_\infty\rangle}\left((k_i a + \langle h_\infty, y_i\rangle)h_0 + (k_i b - \langle h_0, y_i\rangle)h_\infty\right)$$

Recall, from corollary 11, that $ah_0 + bh_\infty = \sum_{i=1}^{n_\rho} \underline{\dim}(E_i)$. By considering the last coordinate in $\mathbb{Z}^n$ of this equation, we get:

$$b = \sum_{i=1}^{n_\rho} d_i = \sum_{i=1}^{n_\rho} d_i = \sum_{i=1}^{n_\rho} \frac{k_i b - \langle h_0, y_i\rangle}{\langle h_0, h_\infty\rangle}$$

So:

$$b\left(\langle h_0, h_\infty\rangle - \sum_{i=1}^{n_\rho} k_i\right) = -\sum_{i=1}^{n_\rho}\langle h_\infty, y_i\rangle$$

Since $b > -n_\rho\langle h_\infty, y\rangle \geq 0$ for all $y \in \Omega$, we must have:

$$\langle h_0, h_\infty\rangle = \sum_{i=1}^{n_\rho} k_i$$

Recall that we are trying to prove two statements: Firstly, if $n_\rho = \langle h_0, h_\infty\rangle$, then $k_i = 1$ for all $i \leq k$, so:

$$d_i = \frac{1}{n_\rho}(b - \langle h_0, y_i\rangle)$$
$$c_i = \frac{1}{n_\rho}(a + \langle h_\infty, y_i\rangle)$$

(for all $i \leq n_\rho$). Thus:

$$\begin{aligned} & |\dim_K(E_i) - \frac{1}{n_\rho}(\mu(ah_0 + bh_\infty)| \\ = \ & |\frac{1}{n_\rho}(\langle h_0, y_i\rangle\mu(h_0) - \langle h_0, y_i\rangle\mu(h_\infty) + \mu(y_i))| \\ \leq \ & p \end{aligned}$$

Secondly, if $n_\rho \neq \langle h_0, h_\infty\rangle$, then $n_\rho < \langle h_0, h_\infty\rangle$ (no stable tube has rank greater than $\langle h_0, h_\infty\rangle$). Then:

$$\begin{aligned} & \dim_K(E_i) - \frac{1}{\langle h_0, h_\infty\rangle}(\mu(ah_0 + bh_\infty)) \\ = \ & \frac{1}{\langle h_0, h_\infty\rangle}\left(\mu((k_i a + \langle h_\infty, y_i\rangle)h_0) + \mu((k_i b - \langle h_0, y_i\rangle)h_\infty) - \mu(ah_0 + bh_\infty)\right) \\ \geq \ & \frac{1}{\langle h_0, h_\infty\rangle}\left(\mu((a + \langle h_\infty, y_i\rangle)h_0) + \mu((b - \langle h_0, y_i\rangle)h_\infty) - \mu(ah_0 + bh_\infty)\right) \\ \geq \ & -p \end{aligned}$$

-as required.

□

### 4.3.2 Pp-pairs near an irrational cut

**Lemma 61.** *Given any $r_1, r_2 \in \mathbb{R}$ such that $0 < r_1 < r_2$, and any $\gamma_1, \gamma_2 \in \mathbb{Q}$, there are only finitely many pairs $(a, b) \in \mathbb{N}^2$ such that:*

$$\frac{b}{a} \leq r_1 < r_2 \leq \frac{b+\gamma_1}{a+\gamma_2}$$

*Proof.* Given any $a, b \in \mathbb{N}$:

$$\frac{b+\gamma_1}{a+\gamma_2} - \frac{b}{a} = \frac{\gamma_1 - (b/a)\gamma_2}{a+\gamma_2}$$

Let $S := \{(a, b) \in \mathbb{N}^2 : \frac{b}{a} \leq r_1 < r_2 \leq \frac{b+\gamma_1}{a+\gamma_2}\}$.

Let $s = |\gamma_1| + r_2|\gamma_2|$. Then for all $a, b$ such that $b/a \leq s$:

$$\frac{b+\gamma_1}{a+\gamma_2} = \frac{b}{a} + \frac{\gamma_1 - (b/a)\gamma_2}{a+\gamma_2} \leq r_1 + \frac{s}{a+\gamma_2}$$

Consequently, we can pick $a' \in \mathbb{N}$ large enough such that $(a, b) \notin S$, for all $a \geq a'$ and $b \in \mathbb{N}$.

Finally, given any $a \leq a'$, there are only finitely many $b \in \mathbb{N}$ such that $b/a \leq r_1$. It follows that $S$ is finite.

□

**Lemma 62.** *Given any $r_1, r_2 \in \mathbb{R}$ such that $0 < r_1 < r_2$, and any $\gamma_1, \gamma_2 \in \mathbb{Q}$, there are only finitely many pairs $(a, b) \in \mathbb{N}^2$ such that:*

$$\frac{b+\gamma_1}{a+\gamma_2} \leq r_1 < r_2 \leq \frac{b}{a}$$

*Proof.* Let $S := \{(a, b) \in \mathbb{N}^2 : \frac{b+\gamma_1}{a+\gamma_2} \leq r_1 < r_2 \leq \frac{b}{a}\}$. We claim that there exists $k \in \mathbb{N}$ such that:

$$\sup\{(b/a)\gamma_2 - \gamma_1 : (a, b) \in S\} \leq k$$

Indeed, if $\gamma_2 \leq 0$, then let $k = -\gamma_1$. Whereas, if $\gamma_2 > 0$, then, $(a, b) \in S$ implies that:

$$b < -\gamma_1 + r_1(a + \gamma_2)$$

And hence that:

$$(b/a)\gamma_2 - \gamma_1 \leq (\gamma_2/a)(-\gamma_1 + r_1(a + \gamma_2)) - \gamma_1$$

-and one can clearly see that there exists $k \in \mathbb{N}$ such that, for all $a \in \mathbb{N}$:

$$(\gamma_2/a)(-\gamma_1 + r_1(a+\gamma_2)) - \gamma_1 \leq k$$

Now, given any such $k$, we have that, for all $(a,b) \in S$:

$$\frac{(b/a)\gamma_2 - \gamma_1}{a+\gamma_2} \leq \frac{k}{a+\gamma_2}$$

Now, pick any $a_0$ large enough such that $k/(a+\gamma_2) \leq r_2 - r_1$ for all $a \geq a_0$. Then, for all $a \geq a_0$ and $b \in \mathbb{N}$:

$$\frac{b}{a} - \frac{b+\gamma_1}{a+\gamma_2} = \frac{(b/a)\gamma_2 - \gamma_1}{a+\gamma_2} \leq \frac{k}{a+\gamma_2} < r_2 - r_1$$

-and so $(a,b) \notin S$.

Finally, for all $a < a_0$, there are only finitely many $b \in \mathbb{N}$ such that $\frac{b+\gamma_1}{a+\gamma_2} \leq r_1$, and hence only finitely many $b$ such that $(a,b) \in S$- which completes the proof. □

**Corollary 13.** *Take any $\gamma_1, \gamma_2 \in \mathbb{Q}^+$, any irrational $r > 0$, and any $\epsilon > 0$.*

*Then there exists $\delta \in (0,\epsilon)$ such that, for all $a, b \in \mathbb{N}$, and $y \in \Omega$:*

$$\iota(ah_0 + bh_\infty + \gamma) \in (r-\delta, r+\delta) \Longrightarrow \iota(ah_0 + bh_\infty) \in (r-\epsilon, r+\epsilon)$$

*Proof.* Recall that $\langle h_0, h_\infty \rangle = -\langle h_\infty, h_0 \rangle$. For all $a, b \in \mathbb{N}$, and $y \in \Omega$:

$$\begin{aligned} \iota(ah_0 + bh_\infty + \gamma) &= -\frac{\langle h_0, ah_0 + bh_\infty + y \rangle}{\langle h_\infty, ah_0 + bh_\infty + y \rangle} \\ &= -\frac{\langle h_0, bh_\infty \rangle + \langle h_0, y \rangle}{\langle h_\infty, ah_0 \rangle + \langle h_\infty, y \rangle} \\ &= \frac{b + (\langle h_0, y \rangle)/(\langle h_0, bh_\infty \rangle)}{a - (\langle h_\infty, y \rangle)/(\langle h_0, bh_\infty \rangle)} \end{aligned}$$

Let $\gamma_1 = \langle h_0, y \rangle)/(\langle h_0, bh_\infty \rangle$ and let $\gamma_2 = -(\langle h_\infty, y \rangle)/(\langle h_0, bh_\infty \rangle)$, and pick any $\epsilon' \in (0,\epsilon)$. Then by lemma 62 , there are only finitely $(a,b) \in \mathbb{N}^2$ and $y \in \Omega$ such that:

$$\frac{b+\gamma_1}{a-\gamma_2} \leq r + \epsilon' < r + \epsilon \leq b/a$$

Similarly, by lemma 61, there are only finitely $(a,b) \in \mathbb{N}^2$ and $y \in \Omega$ such that:

$$b/a \leq r - \epsilon < r - \epsilon' \leq \frac{b+\gamma_1}{a-\gamma_2} \leq r + \epsilon'$$

Consequently, we can pick $\delta \in (0, \epsilon')$ such that, for all $a, b \in \mathbb{N}$ and $y \in \Omega$:

$$\iota(ah_0 + bh_\infty + y) \in (r - \delta, r + \delta) \Longrightarrow r - \epsilon < b/a < r + \epsilon$$

□

**Lemma 63.** *Let $\phi/\psi$ be any pp-pair which is open near $r$ (cf (3.6.2)). Then there exists $\epsilon > 0$ such that $\phi/\psi$ is open on every $C$-module in $\mathcal{P}_r \cap \mathcal{Q}_{r-\epsilon}$.*

*Proof.* By corollary 9 there exists $\epsilon' > 0$ such that $\phi/\psi$ is open on every module lying in a homogeneous tube in $\mathcal{P}_{r+\epsilon'} \cap \mathcal{Q}_{r-\epsilon'}$.

By proposition 3 there exists $\epsilon > 0$ and a vector $v \in \mathbb{Z}^n$ (i.e. $K_0(C)$) such that $\dim_K(\phi/\psi)(X) = v.\underline{\dim}(X)$ for all $X \in \mathcal{P}_{r+\epsilon} \cap \mathcal{Q}_{r-\epsilon}$. We may assume that $\epsilon \leq \epsilon'$, and that $\epsilon \in \mathbb{Q}$.

We claim that $v.h_0 + \gamma v.h_\infty > 0$ for all $\gamma \in (r - \epsilon, r + \epsilon) \cap \mathbb{Q}$: to see this, take any $a \in \mathbb{N}^+$ large enough such that $a\gamma \in \mathbb{N}$. By corollary 12 there exists a homogeneous indecomposable module $X \in A$-mod, with $\underline{\dim}(X) = ah_0 + a\gamma h_\infty$. Then $X \in (\mathcal{Q}_{r-\epsilon}, \mathcal{P}_{r+\epsilon})$, and so $\phi/\psi$ is open on $X$, and so $a(v.h_0 + \gamma.h_\infty) > 0$, as required.

By corollary 13, there exists $\delta \in (0, \epsilon)$ such that, for all $a, b \in \mathbb{N}$ and $y \in \Omega$:

$$\iota(ah_0 + bh_\infty + y) \in (r - \delta, r + \delta) \Longrightarrow \iota(ah_0 + bh_\infty) \in (r - \epsilon, r + \epsilon)$$

Now, let:

$$s = \min(v.h_0 + (r - \epsilon)v.h_\infty, v.h_0 + (r + \epsilon)v.h_\infty)$$

Notice that $s \in \mathbb{R}\backslash\mathbb{Q}$ (since $r \pm \epsilon \in \mathbb{R}\backslash\mathbb{Q}$), and that $s = \inf\{v.h_0 + \gamma v.h_\infty : \gamma \in (r - \epsilon, r + \epsilon)\}$.Thus $s > 0$.

Now, pick any $a' \in \mathbb{N}$ such that $a' > -(v.y)/s$ for all $y \in \Omega$. We can pick $\delta' > 0$ small enough such that $\iota(ah_0 + bh_\infty + y) \notin (r - \delta', r)$ for all $y \in \Omega$, $b \in \mathbb{N}$ and $a \leq a'$.

We claim that $\phi/\psi$ is open on every $X \in \mathcal{Q}_{r-\delta'} \cap \mathcal{P}_r$. Indeed, given any such $X$, let $a, b \in \mathbb{N}$ and $y \in \Omega$ be such that $\underline{\dim}(X) = ah_0 + bh_\infty + y$. Then $a \geq a'$ (by our choice of $\epsilon'$), and $b/a \in (r-\delta, r+\delta)$ (by our choice of $\delta$), and so:

$$\begin{aligned}\dim_K(\phi(X)) - \dim_K(\psi(X)) &= v.(ah_0 + bh_\infty + y)\\ &= av.(h_0 + (b/a)h_\infty) + v.y\\ &\geq a's + v.y\\ &> 0\end{aligned}$$

So $\phi/\psi$ is open on $X$, as required. Relabeling $\delta'$ as $\epsilon$ completes the proof. $\square$

## 4.4 $\mathbf{rad}^{+}(\chi)$

Define $\mathrm{rad}^{+}(\chi)$ to be the set $\{ah_0 + bh_\infty : a, b \in \mathbb{N}^2 \backslash \{(0,0)\}\}$. Let $\iota : \mathrm{rad}^{+}(\chi) \to \mathbb{Q}^{+} \cup \{\infty\}$ be the map $\iota : ah_0 + bh_\infty \mapsto b/a$. Let $\mu : \mathrm{rad}^{+}(\chi) \to \mathbb{N}$ be the map such that $\mu(\underline{\dim}(M)) = \dim_K(M)$ for any $M$ with $\underline{\dim}(M) \in \mathrm{rad}^{+}(\chi)$.

**Lemma 64.** *Take any $x, y \in rad^{+}(\chi)$ such that $\iota(x) < \iota(y)$. Then:*

$$\iota(x) < \iota(x+y) < \iota(y)$$

$$\lim_{n\to\infty}\iota(x+ny) = \iota(y)$$

*Proof.* These can be easily checked. $\square$

We define a pre-order on $\mathrm{rad}^{+}(\chi)$ by:

$$x \leq y \iff \iota(x) < \iota(y) \text{ or } (\iota(x) = \iota(y) \text{ and } \mu(x) \leq \mu(y))$$

It is in fact a total order: If $ah_0 + bh_\infty \leq a'h_0 + b'h_\infty$ and $a'h_0 + b'h_\infty \leq ah_0 + bh_\infty$ then one can easily check that $a = a'$ and $b = b'$.

**Lemma 65.** *Given any $r \in \mathbb{R}^{+}\backslash\mathbb{Q}$, $k \in \mathbb{N}$, and any $\epsilon > 0$, there exists $x \in \mathrm{rad}^{+}(\chi)$ such that $r - \epsilon < \iota(x) < r$, and, for all $y \in \mathrm{rad}^{+}(\chi)$:*

$$\iota(x) < \iota(y) < r \implies \mu(y) > \mu(x) + k$$

*Proof.* First of all, given any $k' \geq 1$, consider the set:

$$\{x \in \mathrm{rad}^+(\chi) : \mu(x) \leq k', \iota(x) < r\}$$

There exists $k' \geq k$ such that this set is non-zero. It is clearly finite, so we can pick an element $x_0 = a_0 h_0 + b_0 h_\infty$ which is maximal in this set (w.r.t. the total order on $\mathrm{rad}^+(\chi)$). Notice that, for all $y \in \mathrm{rad}^+(\chi)$:

$$\iota(x) < \iota(y) < r \implies \mu(y) > \mu(x_0)$$

Suppose, for a contradiction, that for all $x \in \mathrm{rad}^+(\chi)$ with $r - \epsilon < \iota(x) < r$, there exists $y \in \mathrm{rad}^+(\chi)$ with $\iota(x) < \iota(y) < r$ and $\mu(y) \leq \mu(x) + k$. Then we can recursively define non-empty sets $S_1, S_2, S_3, \ldots$, and elements $x_i = (a_i h_0 + b_i h_\infty) \in S_i$ by:

$$S_{i+1} = \{y \in \mathrm{rad}^+(\chi) : \mu(y) < \mu(x_i) + k, \iota(x_i) < \iota(y) < r\}$$

$$x_{i+1} = \max(S_{i+1})$$

Define $c_i := a_i - a_{i-1}$ and $d_i = b_i - b_{i-1}$ for all $i \geq 1$: So $x_i - x_{i-1} = c_i h_0 + d_i h_\infty$. Notice that, for all $i$:

- $0 \leq \mu(c_i h_0 + d_i h_\infty) \leq k$ (by our choice of $x_{i-1}$)
- $c_i$ and $d_i$ can't both be negative (since $0 \leq \mu(c_i h_0 + d_i h_\infty)$)
- $d_i \geq 0$- Suppose for a contradiction that $d_i < 0$. Then $c_i \geq 0$ (by above), and so:

  $$\iota(x_i) = b_i/a_i = (d_i + b_{i-1})/(c_i + a_{i-1}) < b_{i-1}/a_{i-1} = \iota(x_{i-1})$$

  -contradicting the definition of $S_i$.

- $c_i \geq 0$: Suppose for a contradiction, that $c_i < 0$. Then:

  $$\frac{b_{i-1}}{a_{i-1}} < \frac{b_{i-1}}{a_{i-1} - 1} \leq \frac{b_{i-1} + d_i}{a_{i-1} + c_i} = \iota x_i < r$$

  -and so $(a_{i-1} - 1)h_0 + b_{i-1} h_\infty \in S_{i-1}$-contradicting our choice of $x_i$.

- $d_i/c_i > \iota(x_{i-1})$- since $d_i/c_i \leq \iota(x_{i-1})$ would imply that:

  $$b_i/a_i = (b_{i-1} + d_i)/(a_{i-1} + c_i) \leq b_{i-1}/a_{i-1}$$

  (by lemma 64), which contradicts the fact that $x_i = a_i h_0 + b_i h_\infty \in S_i$.

Of course, it follows from our choice of $x_0$ that $d_i/c_i > r$.

Now, let $A$ be the finite set:

$$A := \{y \in \mathrm{rad}^{+}(\chi) : \mu(y) \leq k, \iota(y) > r\}$$

And define, for all $n$:

$$A_n := \{\sum_{i=1}^{n} y_i : y_i \in A \text{ for all } i \leq n\}$$

Notice that $S_i \subseteq \{x_0 + z : z \in A_i\}$ for all $i$. We claim that there exists $n$ such that:

$$\iota(x_0 + z) > r \text{ for all } z \in A_n$$

-this will give our required contradiction.

To prove this, let $z_0 \in A$ be such that $\iota(z_0)$ is minimal (if there is more than one, then pick the one such that $\mu(z_0)$ is minimal too). Let $e_0, f_0 \in \mathbb{N}$ be such that $e_0 h_0 + f_0 h_\infty = z_0$.

By lemma 64, we can find $N$ such that $\iota(x_0 + Nz_0) > r$. Take any $z = eh_0 + fh_\infty \in A_{Nf_0}$. Then $f \geq Nf_0$. Let $q = f/Nf_0 \geq 1$. Notice that:

$$r < \iota(x_0 + Nz_0) = \frac{b_0 + Nf_0}{a_0 + Ne_0} \leq \frac{b_0 + qNf_0}{a_0 + qNe_0}$$

(since $(b + Nf_0)/(a + Ne_0) \leq f_0/e_0$ ). Also:

$$\frac{f_0}{e_0} \leq \frac{f}{e} = \frac{qNf_0}{e}$$

So $e \leq qNe_0$. And so:

$$r < \frac{b_0 + qNf_0}{a_0 + qNe_0} \leq \frac{b + f}{a + e}$$

-so $\iota(x_0 + z) > r$ for all $z \in A_n$- thus proving the claim. □

**Lemma 66.** *Let $C$ be either $C(4,\lambda)$, $C(6)$, $C(7)$ or $C(8)$. Given any irrational $r>0$, and any $\epsilon>0$, and any $d\geq 1$, there exists a tube $\mathcal{T}(\rho)$ of rank $\langle h_0, h_\infty\rangle$, and index in $(r-\epsilon, r)$ such that, given any quasisimple $E$ of $\mathcal{T}(\rho)$, and any indecomposable $N\in C$-*mod*:*

$$\iota(\underline{\dim}(N)) \in (\iota(\underline{\dim}(E)), r) \Longrightarrow \dim_K(N) \geq \dim_K(E)+d$$

*Proof.* Let $p$ be the bound from lemma 60. Pick any $k\geq 1$ large enough such that:

$$\frac{1}{\langle h_0, h_\infty\rangle}(k-p)-p\geq d$$

By lemma 65, there exist coprime $a, b\in\mathbb{N}$, such that $r-\epsilon<b/a<r$, and given any $a', b'\in\mathbb{N}$:

$$b/a < b'/a' < r \Longrightarrow \mu(a'h_0+b'h_\infty) > \mu(ah_0+bh_\infty)+k$$

Pick $\mathcal{T}(\rho)$ to be any tube of index $b/a$ and rank $\langle h_0, h_\infty\rangle$.

Now, take any indecomposable $N\in C$-mod, with slope in $(b/a, r)$. Of course, $N\cong E'[j]$ for some quasisimple $E[j']$. Let $a', b'$ be coprime integers such that $b'/a'$ is the slope of $E'$.

By lemma 65 and lemma 60:

$$\begin{aligned}
\dim_K(E'[j]) &\geq \dim_K(E'[1])\\
&\geq \frac{1}{\langle h_0, h_\infty\rangle}(\mu(a'h_0+b'h_\infty))-p\\
&\geq \frac{1}{\langle h_0, h_\infty\rangle}(\mu(ah_0+bh_\infty)+k)-p\\
&\geq \frac{1}{\langle h_0, h_\infty\rangle}(\langle h_0, h_\infty\rangle\dim_K(E)-p+k)-p\\
&\geq \dim_K(E)+d
\end{aligned}$$

□

## 4.5 The width of ${}_C\mathrm{pp}/\sim_r$

We assume throughout this section, that $C$ is one of the four tubular algebras: $C(4,\lambda)$, $C(6)$, $C(7)$, $C(8)$.

**Lemma 67.** *Given any $r \in \mathbb{R}^+ \backslash \mathbb{Q}$, $d \in \mathbb{N}$, $\epsilon > 0$ and any pp-pair $\phi/\psi$ which is open near $r$, there exists $\gamma \in (r-\epsilon, r)$ and a stable tube $\mathcal{T}(\rho)$ in $\mathcal{T}_\gamma$ of rank $\langle h_0, h_\infty \rangle$ such that:*

- *$\phi/\psi$ is open on every module in the tube.*
- *Given any quasisimple $E$ in $\mathcal{T}(\rho)$, and any $X \in \mathcal{Q}_\gamma \cap \mathcal{P}_r$:*

$$\dim_K(X) > \dim_K(E) + d$$

*Proof.* By lemma 63, there exists $\delta > 0$ such that $\phi/\psi$ is open on every module in $\mathcal{P}_r \cap \mathcal{Q}_{r-\delta}$. By lemma 66, there exists $\gamma \in (r-\delta, r) \cap \mathbb{Q}$ satisfying the required conditions. □

**Theorem 31.** *Let $C$ be either $C(4, \lambda)$, $C(6)$, $C(7)$ or $C(8)$. Then, given any $r \in \mathbb{R}^+ \backslash \mathbb{Q}$, the lattice ${}_C\mathrm{pp}/\sim_r$ is wide.*

*Proof.* Take any pp-pair $\phi/\psi$ such that $\phi \nsim_r \psi$. Apply proposition 2 to $\phi$ (and $N$) to obtain a pointed module $(M, m)$ (respectively $(N, n)$) and $\epsilon_1 > 0$ (respectively, $\epsilon_2 > 0$). Let $\epsilon = \min(\epsilon_1, \epsilon_2)$.

Let $d = \dim_K(N)$, and apply lemma 67 to find $\gamma \in (r-\epsilon, r) \cap \mathbb{Q}$ and a tube $\mathcal{T}(\rho)$. Since $\mathcal{T}(\rho)$ is a non-homogeneous tube (it has rank $\langle h_0, h_\infty \rangle > 1$), we can pick any two non-isomorphic modules, $E$ and $E'$, on the mouth of $\mathcal{T}(\rho)$.

Pick any $x \in \phi(E) \backslash \psi(E)$, and any $x' \in \phi(E) \backslash \psi(E')$, and let $\theta$ and $\theta'$ be generators of $\mathrm{pp}^E(x)$ and $\mathrm{pp}^{E'}(x')$ respectively. We shall prove that the images of $\psi + \theta$ and $\psi + \theta'$ in $\mathrm{pp}/\sim_r$ are incomparable.

First of all, notice that every quotient module of $E$ (other than $E$ itself) has dimension less than $\dim_K(E)$, and hence lies in $\mathrm{add}(\mathcal{Q}_r)$. In particular, every non-zero map from $E$ to a module in $(\mathcal{Q}_\gamma \cup \mathcal{T}_\gamma) \cap \mathcal{P}_r$ is an embedding.

As a non-trivial quotient of $E$, $\mathrm{Coker}(f_{(E,x)})$ must lie in $\mathrm{add}(\mathcal{Q}_r)$. So, given any $Z \in \mathcal{Q}_\gamma \cap \mathcal{P}_r$:

$$\begin{aligned} \dim_K(\theta(Z)) &= \dim_K(\mathrm{Hom}(E, Z)) - \dim_K(\mathrm{Hom}(\mathrm{Coker}(f_{(E,x)}), Z)) \\ &= \dim_K(\mathrm{Hom}(E, Z)) \end{aligned}$$

Similarly:

$$\dim_K(\theta'(Z)) = \dim_K(\mathrm{Hom}(E', Z))$$

Now, we suppose for a contradiction, that $\theta+\psi \leq_r \theta'+\psi$. Then $\theta \sim_r \theta \wedge +(\psi+\theta')$- i.e. there exists $\delta > 0$ such that, for all $X \in \mathcal{Q}_{r-\delta} \cap \mathcal{P}_r$:

$$\theta(X) = (\theta \wedge (\psi + \theta'))(X)$$

Since $\delta$ can be arbitrarily small, we may assume that $\delta < \epsilon$. We can find a free realisation for $\theta \wedge (\psi + \theta')$, by considering the pushout, $L$, of the maps $f_{(E,x)}$ and $f_{(N\oplus E',(n,x'))}$:

$$\begin{tikzcd} C \arrow[r, "f_{(E,x)}"] \arrow[d, "f_{(N\oplus E',(n,x'))}"'] & E \arrow[d, dashed, "\exists g"] \\ N \oplus E' \arrow[r, dashed, "\exists g'"] & L \end{tikzcd}$$

Let $l = g(x)$. Then $(L, l)$ is a free realisation of $\theta \wedge (\psi + \theta')$. Notice that there exists a surjection $E \oplus N \oplus E' \twoheadrightarrow L$ with non-zero kernel, and so:

$$\dim_K(L) < \dim_K(E) + \dim_K(E') + \dim_K(N)$$

We claim that every map $h$ from $E$ to a module $Z \in \mathcal{P}_r \cap \mathcal{Q}_{r-\delta}$ factors through $g$: Indeed, since $h(x) \in \theta(Z) = \theta \wedge (\psi+\theta')(Z)$, there must exist a map $h' : L \to Z$ taking $l$ to $h(x)$. Furthermore, since $(h'g - h)(x) = 0$, it must factor through $\mathrm{Coker}(f_{(E,x)})$. However, since $\mathrm{Hom}(\mathrm{Coker}(f_{(E,x)}), Z) = 0$, it must be zero. So $h = h'g$, as required.

Given any direct summand $Y$ of $L$, let $\pi_Y : L \twoheadrightarrow Y$ denote the projection onto $Y$. We can split $L$ into $L' \oplus L'' \oplus L'''$, where:

- $L'$ is the direct sum of all indecomposable direct summands $Y$ of $L$ in $\mathcal{P}_r$ such that $\pi_Y g \neq 0$.
- $L''$ is the direct sum of all indecomposable direct summands $Y$ of $L$ in $\mathcal{P}_r$ such that $\pi_Y g = 0$.
- $L'''$ is the direct sum of all indecomposable direct summands of $L$ in $\mathcal{Q}_r$.

Let $\pi' : L \twoheadrightarrow L'$, $\pi'' : L \twoheadrightarrow L''$, and $\pi''' : L \twoheadrightarrow L'''$ be the natural surjections onto the direct summands. We claim that $\mathrm{Coker}(\mathrm{Hom}(\pi' g, Z)) = 0$ for all $Z \in \mathcal{Q}_{r-\epsilon} \cap \mathcal{P}_r$.

To prove this, take any $h : E \to Z$. Since $\mathrm{Coker}(\mathrm{Hom}(g, Z)) = 0$, $h$ factors through $g$- i.e. there exist maps $h', h'', h'''$ such that the following diagram commutes:

$$\begin{tikzcd} E \arrow[r, "(\pi' g, \pi'' g, \pi''' g)"] \arrow[dr, "h"'] & L' \oplus L'' \oplus L''' \arrow[d, dashed, "(h', h'', h''')"] \\ & Z \end{tikzcd}$$

Since $X \in \mathcal{Q}_{r-\epsilon}$, $\mathrm{Hom}(L''', Z) = 0$, so $h''' = 0$. Also $\pi'' g = 0$ (by our choice of $L''$). Thus $h = h'\pi' g$, as required.

Define $f' := \pi' g$- we have proved that $\mathrm{Coker}(f', Z) = 0$ for all $Z \in \mathcal{P}_r \cap \mathcal{Q}_{r-\delta}$. Also, $f'$ is an embedding (as shown at the start of the proof). Consider the exact sequence:

$$0 \longrightarrow E \xrightarrow{f'} L' \longrightarrow \mathrm{Coker}(f') \longrightarrow 0$$

Now, since $L' \in \mathrm{add}(\mathcal{P}_r)$, we can pick $\gamma' \in (r - \delta, r) \cap \mathbb{Q}$ such that $L' \in \mathrm{add}(\mathcal{P}_{\gamma'})$. Take any module $Z$ in a homogeneous tube in $\mathcal{T}_{\gamma'}$. Then $\mathrm{Ext}(L', Z) = 0$, and theorem 1 gives us an exact sequence:

$$\begin{array}{rl} 0 \longrightarrow & \mathrm{Hom}(\mathrm{Coker}(f'), Z) \longrightarrow \mathrm{Hom}(L', Z) \xrightarrow{\mathrm{Coker}(f',\text{-})} \mathrm{Hom}(E, Z) \\ \longrightarrow & \mathrm{Ext}(\mathrm{Coker}(f'), Z) \longrightarrow \mathrm{Ext}(L', Z) = 0 \end{array}$$

So $\mathrm{Ext}(\mathrm{Coker}(f'), Z) \cong \mathrm{Coker}(f', Z) = 0$ (by the claim above). Lemma 36 therefore implies that $\mathrm{Coker}(f')$ has no direct summands in $\mathcal{Q}_{\gamma'}$, In particular, $\mathrm{Coker}(f') \in \mathrm{add}(\mathcal{P}_r)$.

Now, notice that:

$$\begin{aligned} \dim_K(\mathrm{Coker}(f')) &= \dim_K(L') - \dim_K(E) \\ &\leq \dim_K(L) - \dim_K(E) \\ &< \dim_K(N) + \dim_K(E') + \dim_K(E) - \dim_K(E) \end{aligned}$$

Since every module in $\mathcal{P}_r \cap \mathcal{Q}_\gamma$ has $K$-dimension at least $\dim_K(N) + \dim_K(E')$ (by our choice of $\gamma$, using lemma 67), $\mathrm{Coker}(f')$, cannot have any direct summands in $\mathcal{P}_r \cap \mathcal{Q}_{r-\epsilon}$, and hence $\mathrm{Coker}(f') \in \mathrm{add}(\mathcal{T}_\gamma)$.

The surjection $L' \twoheadrightarrow \mathrm{Coker}(f')$ implies that $L'$ must have a direct summand $Y$ in $\mathcal{T}_\gamma$. By our definition of $L'$, $\mathrm{Hom}(E, Y) \neq 0$, and so $Y \cong E[k]$ for some $k \geq 1$ (by corollary 4). Indeed, we will say that $Y = E[k]$.

Recall that $Y$ is a direct summand of $L$, and $\pi_Y : L \twoheadrightarrow Y$ denotes the natural projection onto $Y$. By our choice of $L'$, $\pi_Y g \neq 0$. By corollary 4, $\dim_K(\mathrm{Hom}(E, E[k])) = 1$, and so $\pi_Y g$ is equal to (a non-zero scalar multiple of) the embedding $\rho$ in the short exact sequence:

$$0 \longrightarrow E[1] \xrightarrow{\rho} E[k] \xrightarrow{\pi} \tau^- E[k-1] \longrightarrow 0$$

-as described in at the end of (3.1.2). We will assume that $\pi_Y g = \rho$. Then $y = \rho(x) \neq 0$. Now, the map:

$$N \oplus E' \xrightarrow{g} L \xrightarrow{\pi_Y} Y$$

-takes $(n, x')$ to $y$. Since $\mathrm{Hom}(E', Y) = 0$ (by corollary 4), the map $\pi_Y g i_N : N \to Y$ (where $i_N : N \hookrightarrow N \oplus E'$ is the natural embedding) takes $n$ to $y$.

Now, $\pi\pi_Y g i_N(n) = \pi(y) = \pi\rho(x) = 0$, and hence factors through $\mathrm{Coker}(f_{(N,n)}$. Since $\mathrm{Coker}(f_{(N,n)} \in \mathrm{add}(\mathcal{Q}_r)$, we have that $\pi_Y g i_N \in \mathrm{Ker}(\pi) = \mathrm{Im}(\rho)$, and hence factors through $\rho$:

$$\begin{tikzcd} N \arrow[rr, "\pi_Y g i_N"] \arrow[dr, dashed, "f''"] & & E[k] \\ & E[1] \arrow[ur, "\rho"] & \end{tikzcd}$$

Since $\rho$ is an embedding, and $\rho(x) = \rho f''(n)$, it follows that $f''(n) = x$.

Since $E \in \mathcal{P}_r \cap \mathcal{Q}_{r-\epsilon}$, our choice of $N$, $n$ and $\epsilon$ (cf. proposition 2) give that $x \in \psi(E)$- which contradicts our choice of $x$.

Thus $\theta + \psi \nleq_r \theta' + \psi$, and similarly $\theta + \psi \ngeq_r \theta' + \psi$- and so the lattice is indeed wide. $\square$

## 4.6 Other classes of tubular algebras

In order to extend theorem 31 to all tubular algebras, we first need to define a few more types of tubular algebra:

### 4.6.1 Canonical algebras

Given any $t \geq 2$, take any $n_1, \ldots, n_t \geq 1$. Let $\Delta(n_1, n_2, \ldots, n_t)$ denote the quiver:

$$\begin{tikzcd}
 & a_1^{(1)} \arrow[r, "\alpha_2^{(1)}"] & a_2^{(1)} \arrow[r] & \cdots \arrow[r, "\alpha_{n_1-1}^{(1)}"] & a_{n_1-1}^{(1)} \arrow[ddr, "\alpha_{n_1}^{(1)}"] & \\
 & a_1^{(1)} \arrow[r, "\alpha_2^{(2)}"] & a_2^{(1)} \arrow[r] & \cdots \arrow[r, "\alpha_{n_2-1}^{(2)}"] & a_{n_1-1} \arrow[dr, "\alpha_{n_2}^{(2)}"] & \\
0 \arrow[uur, "\alpha_1^{(1)}"] \arrow[ur, "\alpha_1^{(2)}"'] \arrow[dr, "\alpha_1^{(t)}"'] & \vdots & \vdots & & \vdots & \omega \\
 & a_1^{(1)} \arrow[r, "\alpha_2^{(t)}"] & a_2^{(1)} \arrow[r] & \cdots \arrow[r, "\alpha_{n_t-1}^{(t)}"] & a_{n_1-1} \arrow[ur, "\alpha_{n_t}^{(t)}"'] &
\end{tikzcd}$$

For all $i \leq t$, let $\alpha^{(i)}$ be shorthand for the path $\alpha_{n_i}^{(i)} \ldots \alpha_2^{(i)} \alpha_1^{(i)}$. Let $V$ be the $t$-dimensional vector space, with basis $\{\alpha^{(1)}, \ldots, \alpha^{(t)}\}$. A *generic subspace* $\mathcal{I}$ of $V$ is any $(t-2)$-dimensional subspace such that, given any $m, n \in \{1, 2, \ldots, t\}$, $\mathcal{I} \cap \langle \alpha^{(n)}, \alpha^{(m)} \rangle = \{0\}$. Of course, any generic subspace of $V$ is also an ideal in $K\Delta(n_1, \ldots n_t)$.

A bound quiver algebra $KQ/\mathcal{I}$ is called a *canonical algebra* if $Q$ is a quiver of the form, $\Delta(n_1, \ldots, n_t)$, and $\mathcal{I}$ an ideal of $KQ$ given by a generic subspace.

Given any canonical algebra $C = K\Delta(n_1, \ldots, n_t)/\mathcal{I}$, define:

$$\mathcal{P} = \{M \in C\text{-mod} : \dim_K(e_\omega M) - \dim_K(e_0 M) < 0\}$$

$$\mathcal{T} = \{M \in C\text{-mod} : \dim_K(e_\omega M) - \dim_K(e_0 M) = 0\}$$

$$\mathcal{Q} = \{M \in C\text{-mod} : \dim_K(e_\omega M) - \dim_K(e_0 M) > 0\}$$

**Lemma 68.** *Given any canonical algebra $C$, $\mathcal{T}$ is a sincere, stable tubular family, indexed by $\mathbb{P}^1(K)$, which separates $\mathcal{P}$ from $\mathcal{Q}$.*

We define a canonical tubular extension of a canonical algebra $C$ to be any tubular extension of $C$ (as described in (3.1.8)) using modules from $\mathcal{T}$.

### 4.6.2 Bush algebras

Recall the set $\mathcal{S}(+1, -1)$ of finite sequences of $+1$ and $-1$. We say that a sequence $a \in \mathcal{S}(-1, +1)$ is *strictly positive* (or *strictly negative*) if it has length at least one, and every element of it is $+1$ (respectively, $-1$).

Let $B^{(1)}, \ldots, B^{(t)}$ be finite branches. Recall from (3.1.7) that, for each branch $B^{(s)}$, there exists a finite subset $\mathcal{S}^{(s)} \subset \mathcal{S}\{-1,+1\}$ such that $\{b^{(s)}_{\emptyset}\} \cup \{b^{(s)}_a : a \in \mathcal{S}^{(s)}\}$ is the vertex set of $B^{(s)}$, and $\{\beta^{(s)}_a : a \in \mathcal{S}^{(s)}\}$ the arrow set of $B^{(s)}$. Let $n_s = |\mathcal{S}^{(s)}| + 1$ (the number of vertices of $B^{(s)}$). Label the vertex $b^{(s)}_{\emptyset}$ of $B^{(s)}$ as $\omega$.

Let $Q = (Q_0, Q_1)$ be the graph with $Q_0$ containing one vertex, $\omega$, and $Q_1$ being empty. Let $A$ the algebra obtained from $KQ$ by adding the branches $\{B^{(1)}, \ldots, B^{(t)}\}$ to $\omega$. Any such algebra $A$ is called a *bush algebra*, and the *branching type* of $A$ is $(n_1, \ldots, n_t)$.

For example, taking $B^{(1)}$ to be the branch with vertex set $B^{(1)}_0 = \{b_{\emptyset}, b_{-1}, b_{+1}\}$, and $B^{(2)}$ to be the branch with vertex set $B^{(2)}_0 = \{b^{(2)}_{\emptyset}, b^{(2)}_{-1}, b^{(2)}_{+1}, b^{(2)}_{+1,-1}, b^{(2)}_{+1,+1}\}$, then the bush algebra obtained has underlying quiver:

$$\begin{tikzcd}
b^{(1)}_{-1} & & b^{(1)}_{+1} \arrow[dl, "\beta^{(1)}_{+1}"] & \\
& b_{\emptyset} \arrow[ul, "\beta^{(1)}_{-1}"'] \arrow[dl, "\beta^{(2)}_{-1}"'] & & \\
b^{(2)}_{-1} & & b^{(2)}_{+1} \arrow[ul, "\beta^{(2)}_{+1}"] \arrow[dl, "\beta^{(2)}_{+1,-1}"'] & \\
& b^{(2)}_{+1,-1} & & b^{(2)}_{+1,+1} \arrow[ul, "\beta^{(2)}_{+1,+1}"]
\end{tikzcd}$$

-and the ideal $\mathcal{I}$ is $\langle \beta^{(1)}_{-1}\beta^{(1)}_{+1}, \beta^{(2)}_{-1}\beta^{(2)}_{+1}, \beta^{(2)}_{+1,-1}\beta^{(2)}_{+1,+1} \rangle$.

Given any bush algebra, an $A$-module is called a *coordinate module* if its $K$-representation satisfies the following:

- The vector space associated to $\omega$ is a 2-dimensional $K$-vector space (denoted $M_{\omega}$)

- There exists a pairwise different set of 1-dimensional subspaces $U_1, U_2, \ldots U_t$ of $M_{\omega}$ such that, given any vertex $b^{(s)}_a$ of $B$, the $K$-vector space associated to it is $U^{(s)}$ if $a$ is strictly positive, and $M_{\omega}/U^{(s)}$ if it is strictly negative, and 0 otherwise.

- The $K$-homomorphism associated to any arrow of the form $\beta^{(s)}_{+1}$ is the natural embedding $U^{(s)} \hookrightarrow M_{\omega}$

- The $K$-homomorphism associated to any arrow of the form $\beta_a^{(s)}$ (where $a$ is strictly positive, and has length at least 2) is the identity map $U^{(s)} \to U^{(s)}$.
- The $K$-homomorphism associated to any arrow of the form $\beta_{-1}^{(s)}$ is the natural projection $M_\omega \twoheadrightarrow M/U^{(s)}$.
- The $K$-homomorphism associated to any arrow of the form $\beta_a^{(s)}$ (where $a$ is a strictly negative sequence of length at least 2) is the identity map $M/U^{(s)} \to M/U^{(s)}$.
- The $K$-homomorphism associated to any other arrow is the zero map.

For example, if $A$ is the branch algebra as defined above, then the coordinate modules are the $A$ modules with representation:

$$\begin{tikzcd}
(K \oplus K)/\langle(1,\lambda)\rangle & & \langle(1,\lambda)\rangle \arrow[dl, hook'] & \\
& K \oplus K \arrow[ul, two heads] \arrow[dl] & & \\
(K \oplus K)/\langle(1,\mu)\rangle & & \langle(1,\mu)\rangle \arrow[ul, hook'] \arrow[dl] & \\
& 0 & & \langle(1,\mu)\rangle \arrow[ul, "1"']
\end{tikzcd}$$

-for some distinct $\lambda, \mu \in \mathbb{P}^1(K)$ (where the maps to and from $K \oplus K$ are just the embeddings of subspaces, and canonical projections onto the factor spaces respectively).

**Lemma 69.** *An algebra $C$ is a canonical tubular extension of a canonical algebra $C_0$ if and only $C$ is a coextension of a bush algebra $C_\infty$ by a coordinate module.*

*If so, then the extension type of $C$ over $C_0$ equals the branching type of $C_\infty$.*

*Proof.* By [23, (4.8.1)]. □

## 4.7 Shrinking functors

Let $A$ be any tubular algebra. By theorem 26, there is a unique tame concealed algebra $A_0$ such that $A$ is a tubular extension $A_0[E_i, K_i]_{i=1}^t$ of $A_0$- where $E_1, \ldots, E_t$ are elements of the separating tubular family $\mathcal{T}$ in $A_0$-mod, and $K_1, \ldots, K_t$ are branches.

A tilting module $T \in A$-mod is called a *shrinking module* if there exists a preprojective tilting $A_0$-module $T_0$ and a projective $A$-module $T_p$ such that $T \cong T_0 \oplus T_p$.

Given any shrinking module ${}_AT$, the functor $\mathrm{Hom}(T, \_) : A\text{-mod} \to B\text{-mod}$ (where $B := \mathrm{End}({}_AT)$) is called a *left shrinking functor*.

We define $B_0 := \mathrm{End}(A_0)$.

**Lemma 70.** *Let* $\Sigma_0$ *be the functor* $\mathrm{Hom}(T_0, \_) : A_0\text{-mod} \to B_0\text{-mod}$. *Then* $B = B_0[\Sigma_0(E_i), K_i]_{i=1}^t$.

*Proof.* By [23, (4.7.4)]. □

**Theorem 32.** *If* $A$ *is a tubular algebra, and* ${}_AT$ *a shrinking* $A$*-module, then* $B = \mathrm{End}({}_AT)$ *is a tubular algebra.*

*Proof.* By [23, (5.5.1)]. □

**Lemma 71.** *There exists a linear transformation* $\sigma_T : K_0(A) \to K_0(B)$, *such that:*

$$\sigma_T(\underline{\dim}(M)) = \underline{\dim}(\Sigma_T M) - \underline{\dim}(\Sigma'_T M)$$

*-where* $\Sigma'_T := \mathrm{Ext}(T, \_)$.

*Proof.* By [23, (4.1.7)]. □

Let $h_0^A$ and $h_\infty^A$ be the positive radical generators in $\mathrm{rad}(\chi_A)$ as in theorem 26. Since $B$ is also a tubular algebra, there are positive radical generators $h_0^B$ and $h_\infty^B$ in $\mathrm{rad}(h_B)$ (as in the theorem).

Define $\iota_0^A := \langle h_0^A, \_\rangle : K_0(A) \to \mathbb{Z}$ and $\iota_0^B := \langle h_0^B, \_\rangle : K_0(B) \to \mathbb{Z}$.

**Lemma 72.** $\sigma_T(h_0^A) = h_0^B$. *Furthermore,* $\sigma(h_\infty^A)$ *is in* $\mathrm{rad}(\chi_B)$, *and so there exist* $n_0, n_\infty \in \mathbb{Q}_0^+$ *such that:*

$$\sigma(h_\infty^A) = n_0 h_0^B + n_\infty h_\infty^B$$

*Proof.* For the first assertion, see page 290 of [23]. The second is by [23, (5.4.a)]. □

Define $\overline{\sigma} : \mathbb{Q}_0^\infty \to \mathbb{Q}_0^\infty$ by:

$$\overline{\sigma}(\gamma) := \frac{n_\infty \gamma}{n_0\gamma + 1}$$

-where $n_0$ and $n_\infty$ are as in lemma 72. Notice that $\overline{\sigma} : \mathbb{Q}_0^\infty \to \{\delta \in \mathbb{Q}_0^\infty : 0 \leq \delta \leq n_\infty/n_0\}$ is an order preserving bijection. $\Sigma_T$ is said to be a *proper* shrinking functor if $n_0 \neq 0$.

**Lemma 73.** *Let $_AT$ be a shrinking module. Suppose that an indecomposable $M \in$ $A$-mod doesn't lie in $\mathcal{G}(T)$ (cf (2.6.1)). Then $M$ is a preprojective $A_0$-module.*

*Proof.* See [23, (5.4.1)] $\square$

**Lemma 74.** $\Sigma_T$ *defines an equivalence from $\mathcal{P}_\infty^A \cap \mathcal{G}(_AT)$ onto $\mathcal{P}_{\overline{\sigma}\infty}^B$*

*Proof.* By [23, (5.4.2')]- noting that $\mathcal{T}_{\overline{\sigma}\infty}^B$ is indeed a separating tubular family. $\square$

**Theorem 33.** *Given any $\gamma \in \mathbb{Q}_0^\infty$, $\Sigma_T$ induces an equivalence of categories from $\mathcal{T}_\gamma$ onto $\mathcal{T}_{\overline{\sigma}(\gamma)}^B$.*

*Proof.* See [23, (5.4.3)] $\square$

**Corollary 14.** *Given any $r \in \mathbb{R}_0^\infty$, $\Sigma_T$ gives an equivalence between $\mathcal{P}_r^A \cap \mathcal{G}(T)$ and $\mathcal{P}_{\overline{\sigma}(r)}^B$.*

### 4.7.1 Inducing an embedding of pp-lattices

First of all, note the following:

**Lemma 75.** *Let $\phi/\psi$ be any pp-pair in any algebra, $A$, and let $(M, \overline{m})$ and $(N, \overline{n})$ be the free realisations of $\phi$ and $\psi$ respectively.*

*Then, given any module $X$, $\phi/\psi$ is closed on $X$ if and only if, for every $h \in$ Hom$(M, X)$ there exists $h' \in$ Hom$(N, X)$ such that the following diagram commutes:*

$$\begin{array}{ccc} A^k & \xrightarrow{f_{(M,m)}} & M \\ \downarrow{\scriptstyle f_{(N,n)}} & & \downarrow{\scriptstyle h} \\ N & \xrightarrow{h'} & X \end{array}$$

*Proof.* This can easily be checked. $\square$

Throughout the rest of this subsection, $A$ will be a tubular algebra, $_AT$ a shrinking module, and $B = \text{End}(_AT)$ (which is also a tubular algebra, by 32. We fix an irrational $r > 0$.

Recall (from (2.6.1)) that there exists a functor $\Upsilon_T$ given by:

$$ {}_AT_B \otimes \_ : B\text{-Mod} \to A\text{-Mod} $$

-and that $\Upsilon_T$ and $\Sigma_T$ induce some mutually inverse equivalences (as in theorem 18). In fact:

**Lemma 76.** *$\Sigma_T$ induces an equivalence of categories between $\mathcal{P}_r^A \cap \mathcal{G}({}_AT)$ and $\mathcal{P}^B_{\overline{\sigma}(r)}$- with the mutually inverse functors being $\Upsilon_T$ and $\Sigma_T$.*

*Proof.* Fairly easy, given lemma 74 and theorem 33. □

We denote by $M(r)$ the direct sum of all indecomposable pure-injective $A$-modules of slope $r$, and by $N(\overline{\sigma}(r))$ the set of all indecomposable pure-injective $B$-modules of slope $\overline{\sigma}(r)$. We aim to show that $w({}_B\mathrm{pp}(M(\overline{\sigma}(r)))) = \infty$ implies $w({}_B\mathrm{pp}(M(r))) = \infty$.

We denote by $\sim_{\overline{\sigma}(r)}$ the equivalence relation on ${}_B\mathrm{pp}$ (and hence on the lattice of 1-pointed finitely presented $B$-modules) such that $\phi \sim_{\overline{\sigma}(r)} \psi$ if and only if there exists $\epsilon > 0$ such that $\phi(Y) = \psi(Y)$ for all $Y \in \mathcal{P}^B_{\overline{\sigma}(r)} \cap \mathcal{Q}^B_{\overline{\sigma}(r-\epsilon)}$.

Let $t_1, \ldots, t_k$ be any generating set for $T$ (as a $K$-module). Notice that, given any $Y \in A$-mod, and any maps $f, g \in \mathrm{Hom}(T, Y)$:

$$ f = g \iff f(t_i) = g(t_i) \text{ for all } i \leq k $$

We define a map from ${}_B\mathrm{pp}^1$ to ${}_A\mathrm{pp}^k$ as follows: Given any $\phi(v) \in_B \mathrm{pp}^1$, let $(C, c)$ be a free realisation of $\phi(v)$, and let $g_{(C,c)} \in \mathrm{Hom}(_B, _B C)$ be the unique map taking 1 to $c$.

Consider the $k$-pointed $A$-module:

$$ (\Upsilon_T C, ((\Upsilon_T g_{(C,c)})(t_1), \ldots, (\Upsilon_T g_{(C,c)})(t_k))) $$

We define $\Upsilon_T(\phi)$ to be any pp-formula $\psi \in_A \mathrm{pp}^k$ such that:

$$ \langle \phi \rangle = \mathrm{pp}^{\Upsilon_T C}((\Upsilon_T g_{(C,c)})(t_1), \ldots, (\Upsilon_T g_{(C,c)})(t_k)) $$

We shall show this map induces an embedding from ${}_B\mathrm{pp}^1/\sim_{\overline{\sigma}(r)}$ into ${}_A\mathrm{pp}^k/\sim_r$.

**Lemma 77.** *Given any $\phi, \psi \in_B$ pp:*

$$\Upsilon(\phi) \sim_r \Upsilon(\psi) \Longrightarrow \phi \sim_{\overline{\sigma}(r)} \psi$$

*Proof.* Let $({}_BC, c)$ and $({}_BD, d)$ be the free realisations of $\phi$ and $\psi$ respectively.

Suppose that $\Upsilon(\phi) \sim_r \Upsilon(\psi)$. Pick any $\epsilon > 0$ such that $(\Upsilon(\phi))(X) = (\Upsilon(\psi))(X)$ for all $X \in \mathcal{P}^A_r \cap \mathcal{Q}^A_{r-\epsilon}$. We claim that $\phi(Y) = \psi(Y)$ for all $Y$ in $\mathcal{P}^B_{\overline{\sigma}(r)} \cap \mathcal{Q}^B_{\overline{\sigma}(r-\epsilon)}$.

Take any $y \in \phi(Y)$- so there exists a map $h \in \mathrm{Hom}_B(C, Y)$ taking $c$ to $y$. Consider the map:

$$\Upsilon_T B \stackrel{\Upsilon_T g_{(C,c)}}{\longrightarrow} \Upsilon_T C \stackrel{\Upsilon_T h}{\longrightarrow} \Upsilon_T X$$

Let $x_i = (\Upsilon_T g_{(C,c)})(\Upsilon_T h)(t_i)$ for each $i \leq k$. Then:

$$(x_1, \ldots, x_n) \in (\Upsilon(\phi))(\Upsilon_T Y)$$

(by our definition of $\Upsilon_T(\phi)$. Of course, $\Upsilon_T Y \in \mathcal{P}^A_r \cap \mathcal{Q}^A_{r-\epsilon}$, so, by our assumption:

$$(\Upsilon(\phi))(\Upsilon_T Y) = (\Upsilon(\psi))(\Upsilon_T Y)$$

- thus there exists a map $f \in \mathrm{Hom}(\Upsilon_T D, \Upsilon Y)$ such that $(f)(\Upsilon_T g_{(D,d)})(t_i) = x_i$ for all $i \leq k$.

Then $(f)(\Upsilon_T g_{(D,d)})(t_i) = x_i = (\Upsilon_T g_{(C,c)})(\Upsilon_T h)(t_i)$ for all $i \leq n$, and hence $(f)(\Upsilon_T g_{(D,d)}) = (\Upsilon_T g_{(C,c)})(\Upsilon_T h)$ (since $t_1, \ldots, t_k$ generated $T$).

Of course, the equivalence of categories gives that:

$$\Sigma_T((\Upsilon_T h)(\Upsilon_T g_{(C,c)})) = \Sigma_T \Upsilon_T (h g_{(C,c)}) = h g_{(C,c)}$$

Also the equivalence of categories implies that $f = \Upsilon_T \Sigma_T f$. And so:

$$\begin{aligned}
\Sigma_T((f)(\Upsilon_T g_{(D,d)})) &= \Sigma_T((\Upsilon_T \Sigma_T f)(\Upsilon_T g_{(D,d)})) \\
&= \Sigma_T \Upsilon_T ((\Sigma_T f)(g_{(D,d)})) \\
&= (\Sigma_T f)(g_{(D,d)})
\end{aligned}$$

□

**Corollary 15.** *The map $\phi \mapsto \Upsilon_T(\phi)$ induces a lattice embedding of from ${}_Bpp^1/\sim_{\overline{\sigma}(r)}$ to ${}_App^k/\sim_r$. Consequently, if $w({}_Bpp^1/\sim_{\overline{\sigma}(r)}) = \infty$ then $w({}_App^k/\sim_r) = \infty$, and hence $w({}_App^1/\sim_r) = \infty$*

*Proof.* By lemma 77, the induced map is an embedding. One can easily check that it is a well defined lattice homomorphism.

The fact that $w({}_A\mathrm{pp}^k/\sim_r) = \infty$ implies $w({}_A\mathrm{pp}^1/\sim_r) = \infty$ follows from [17, (7.3.8)]. □

### 4.7.2 Shrinking functors between different tubular algebras

**Lemma 78.** *Given any canonical tubular algebra $A$, there exists a proper left shrinking functor from $A$-mod to $C$-mod- where $C$ is either $C(6)$, $C(7)$, $C(8)$ or $C(4,\lambda)$ (for some $\lambda \in K\backslash\{0,1\}$.*

*Proof.* See [23, (5.7.1)]. □

**Corollary 16.** *Given any canonical tubular algebra, $A$, and any $r \in \mathbb{R}^+\backslash\mathbb{Q}$, the lattice $pp_A/\sim_r$ has infinite breadth.*

*Proof.* By lemma 78 there exists a proper shrinking functor from $A$ to either $C(4,\lambda)$, $C(6)$, $C(7)$ or $C(8)$. The result follows, by corollary 15 and theorem 31. □

**Lemma 79.** *Let $B_0$ be a tame concealed bush algebra, and $M$ a coordinate module for $B$. Let $B = B_0[M]$- note that, by lemma..., $B^{op}$ is a canonical tubular extension of a canonical tubular algebra.*

*If $B$ is not a canonical algebra, then there exists a proper left shrinking functor from $B$-mod to $C$-mod- for some canonical tubular algebra $C$*

*Proof.* See [23, (5.7.2)] □

**Corollary 17.** *Let $B_0$ be a tame concealed bush algebra, and $M$ a coordinate module for $B$. Let $B = B_0[M]$.*

*Then, given any $r \in \mathbb{R}^+\backslash\mathbb{Q}$, the lattice $pp_B/\sim_r$ has infinite breadth.*

*Proof.* If $B$ is a canonical tubular algebra, then corollary 16 gives the required result. If not, then by lemma 79 there exists a proper shrinking functor from $B$-mod to $C$-mod- for some canonical tubular algebra $C$.

The result follows, by corollary 15 and corollary 16. □

**Lemma 80.** *Given any tubular algebra $A$, there exists a proper left shrinking functor from $A$-*mod *to $B$-*mod- *for some canonical tubular extension $B$ of a tame concealed, canonical algebra.*

*Proof.* See [23, (5.7.1)] $\square$

**Theorem 34.** *Given any tubular algebra $A$, and any $r \in \mathbb{R}^+ \backslash \mathbb{Q}$, the lattice $pp_A / \sim_r$ has infinite breadth.*

*And consequently, if $K$ is countable, then there exists a superdecomposable pure-injective $A$-module of slope $r$.*

*Proof.* Follows from lemma 80, corollary 15 and corollary 17. $\square$

Recall, from theorem 28, that every indecomposable pure-injective module over a tubular algebra has unique slope. If $K$ is countable, then this result does not extend to superdecomposable modules: For example, given any positive irrationals $r, s$ such that $r > s$, theorem 34 gives us a pure-injective superdecomposable modules $M$ and $N$ of slope $r$ and $s$ respectively. By lemma 35, $\mathrm{Hom}(\mathcal{T}_\gamma, N) \neq 0$ (and hence $\mathrm{Hom}(\mathcal{T}_\gamma, M \oplus N) \neq 0$) for all $\gamma < r$. Thus $M \oplus N$ cannot have slope less than $r$.

Similarly, $\mathrm{Hom}(M, \mathcal{T}_\delta) \neq 0$ (and hence $\mathrm{Hom}(M \oplus N, \mathcal{T}_\delta) \neq 0$) for all $\delta > s$, and so $M \oplus N$ cannot have slope greater than $s$. Hence it is a pure-injective superdecomposable module, which doesn't have slope.

This raises the question of whether or not every pure-injective superdecomposable module can be expressed as a direct sum of modules, each of which has slope. We leave this question open.

# Chapter 5

# String Algebras

# 5.1 String algebras

A *string algebra* is a bound quiver algebra $KQ/\mathcal{I}$ (over a finite quiver, $Q$) satisfying the following conditions:

- For all $a \in Q_0$ there are at most two arrows with source $a$, and at most two with target $a$.
- Given any $\alpha \in Q_1$, there is at most one $\beta \in Q_1$ such that $s(\beta) = t(\alpha)$ and $\beta\alpha \notin \mathcal{I}$
- Given any $\alpha \in Q_1$, there is at most one $\gamma \in Q_1$ such that $t(\gamma) = s(\alpha)$ and $\alpha\gamma \notin \mathcal{I}$
- There exists $N \in \mathbb{N}$ such that any $Q$-path of length at least $N$ lies in $\mathcal{I}$

For example, the path algebra of the Kronecker quiver is a string algebra, as is the path algebra of the quiver:

$$\begin{tikzcd} a \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & b \arrow[r, "\gamma"] & c \arrow[r, shift left, "\delta"] \arrow[r, shift right, "\epsilon"'] & d \end{tikzcd}$$

-and ideal $\mathcal{I} = \langle \delta\gamma, \gamma\alpha \rangle$. Another example is the Gelfand-Ponomarev algebra, $G_{m,n}$ (for all $m, n \geq 2$)- which has underlying quiver:

$$\begin{tikzcd} a \arrow[loop left, "\alpha"] \arrow[loop right, "\beta"] \end{tikzcd}$$

-and ideal $\langle \alpha^m, \beta^n, \alpha\beta, \beta\alpha \rangle$.

## 5.1.1 Finite dimensional string modules

For every $\alpha \in Q_1$, we define a formal inverse $\alpha^{-1}$, with $s(\alpha^{-1}) = t(\alpha)$ and $t(\alpha^{-1}) = s(\alpha)$. We define $Q_1^{-1}$ to be the set of all such inverse arrows. We define $(\alpha^{-1})^{-1}$ to be $\alpha$, for all $\alpha^{-1} \in Q_1^{-1}$.

Define a *letter* to be any element of $Q_1 \cup Q_1^{-1}$. Every letter in $Q_1$ is said to be *direct*, and every letter in $Q_1^{-1}$ is said to be *inverse*. A *finite word* is any finite string of letters $l_1 l_2 l_3 \ldots l_n$, such that:

- $t(l_{i+1}) = s(l_i)$ for all $i \in \{1, 2, \dots, n-1\}$
- $l_i \neq l_{i+1}^{-1}$ for all $i \in \{1, 2, \dots, n-1\}$
- No substring of it (i.e. string of the form $l_j l_{j+1} \dots l_k$ with $j \geq 1$ and $k \leq n$) lies in $\mathcal{I}$.
- There are no substrings $l_j l_{j+1} \dots l_k$ such that $l_k^{-1} \dots l_{j+1}^{-1} l_j^{-1}$ lies in $\mathcal{I}$

The *length* of a finite word $l_1 \dots l_n$ is $n$. We refer to $l_1$ and $l_n$ as the first and last letters, respectively, of $l_1 \dots l_n$. We define $t(l_1 \dots l_n) := t(l_1)$ and $s(l_1 \dots l_n) := s(l_n)$.

For any such word $l_1 \dots l_n$, we define $w^{-1} := l_n^{-1} \dots l_1^{-1}$. Notice that $w^{-1}$ is also a word.

For each $a \in Q_0$, we define two more words, $1_{a,+1}$ and $1_{a,-1}$, of length zero- such that $s(1_{a,1}) = s(1_{a,-1}) = t(1_{a,1}) = t(1_{a,-1}) = a$. Furthermore, we define $(1_{a,1})^{-1} = 1_{a,-1}$ and $(1_{a,-1})^{-1} = 1_{a,1}$. We define $\mathcal{W}$ to be the set of all finite words for $KQ/\mathcal{I}$ (including the words of length zero).

Given any word $D = l_1 \dots l_n$, a *subword* of $D$ is any word of the form $l_k \dots l_m$, for some $k \geq 1$ and $m \leq n$ such that $k < m$. We call it an *initial subword* if $k = 1$.

**Lemma 81.** *$D \neq D^{-1}$, for all $D \in \mathcal{W}$.*

*Proof.* Write $D$ as $l_0 l_1 l_2 \dots l_n$. Assume, for a contradiction, that $D = D^{-1}$. So $l_i = l_{n-i}^{-1}$ for all $i \in \{0, 1, 2, \dots, n\}$,.

If $n$ is even, say $n = 2k$, then $l_k = l_{n-k}^{-1}$- which, since $n - k = k$, is clearly a contradiction. However, if $n$ is odd, say $n = 2k+1$, then $l_k = l_{n-k}^{-1} = l_{k+1}^{-1}$- contradicting the definition of a word. □

Given any finite word $w = l_1 \dots l_n$ (with $n \geq 0$), let $M(w)$ be an $n+1$-dimensional $K$-vector space with basis $z_0, z_1, z_2, \dots, z_n$. We endow it with an $A$-module structure as follows: For all $a \in Q_0$ and $i \in \{0, 1, \dots, n\}$, define:

$$e_a z_i = \begin{cases} z_i & \text{if } i > 0 \text{ and } s(l_i) = a \\ z_0 & \text{if } i = 0 \text{ and } t(l_1) = 0 \\ 0 & \text{otherwise} \end{cases}$$

For all $\alpha \in Q_1$, define:

$$\alpha z_i = \begin{cases} z_{i-1} & \text{if } l_{i-1} = \alpha \\ z_{i+1} & \text{if } l_i = \alpha^{-1} \\ 0 & \text{otherwise} \end{cases}$$

Any module of this form is called a (finite dimensional) *string module*. And we call the set $\{z_0, z_1, \ldots, z_n\}$ the *standard basis of* $M(w)$.

**Theorem 35.** *For all $w \in \mathcal{W}$, $M(w)$ is an indecomposable $A$-module.*

*Furthermore, for all $u, w \in \mathcal{W}$, $M(w) \cong M(u)$ if and only if either $w = u$ or $w = u^{-1}$.*

*Proof.* See [8] page 161. □

### 5.1.2 Finite dimensional band modules

Given any $D = (l_1 \ldots l_n) \in \mathcal{W}$, and $k \in \mathbb{N}^+$, define $D^k$ to be the string of letters $l'_1 l'_2 \ldots l'_{kn}$ such that $l'_{i(\text{mod } n)} = l_i$ for all $i \leq nk$- note that it is not necessarily a word.

A word $D$ is said to be *cyclic* if $D^k \in \mathcal{W}$ for all $k \in \mathbb{N}$. A cyclic word $D$ is *primitive* if there is no $C \in \mathcal{W}$ and $k \geq 2$ such that $D = C^k$. Any primitive cyclic word is called a *band*.

**Lemma 82.** *Let $D$ be any band. Then $D$ does not equal any (non-trivial) cyclic permutation of $D$.*

*Furthermore, $D$ does not equal any cyclic permutation of $D^{-1}$.*

*Proof.* Write $D$ as $l_1 \ldots l_n$. First of all, suppose that $D$ is a cyclic permutation of $D^{-1}$: i.e. there exists $k$ such that:

$$l_1 \ldots l_n = l_k^{-1} \ldots l_1^{-1} l_n^{-1} \ldots l_{k+1}^{-1}$$

Then, in particular, $l_1 \ldots l_k = l_k^{-1} \ldots l_1^{-1}$- contradicting lemma 81

Now, suppose that $D$ is a non-trivial cyclic permutation of itself- i.e there exists $k \in \{0, 1, \ldots n-1\}$ such that:

$$l_1 \ldots l_n = l_{k+1} \ldots l_n l_1 \ldots l_k$$

-so, for all $i \in \mathbb{Z}_n$, $l_i = l_{i+k}$. Let $m$ be the highest common factor of $n$ and $k$. Elementary number theory gives us that, for every $i \in \{1, 2, \ldots m\}$:

$$l_i = l_{i+m} = l_{i+2m} = \cdots = l_{n-2m+i} = l_{n-m+i}$$

And so $y = (l_1 \ldots l_m)^{n/m}$- contradicting the definition of a band. $\square$

Given any band, $D = l_1 \ldots l_n$, and any indecomposable $N \in K[T, T^{-1}]$-mod, we define an $A$-module $M({}_{K[T,T^{-1}]}N, D)$ as follows: First of all, recall that any $N \in K[T, T^{-1}]$-mod is uniquely determined by a finite dimensional $K$-vector space, $K^m$, and an automorphism $\phi \in \mathrm{Aut}_K(K^m)$ (which is the action of multiplying by $T$). Let $V_0, V_1, \ldots V_{n-1}$ be copies of $K_m$, and define $M({}_{K[T,T^{-1}]}N, D)$ to be the module with underlying vector space $\bigoplus_{i=0}^{k-1} V_i$. The $A$-module structure is defined as follows:

Given any $a \in Q_0$, define $e_a$ to be the identity map on all $V_i$ such that $t(l_{i+1}) = a$, and zero on all the other $V_i$. And for all $\alpha \in Q_1$, define $\alpha$ to be the map such that, for any $V_i$ and any $x \in V_i$:

$$\alpha x = \begin{cases} x \text{ (as an element of } V_{i-1}) & \text{if } i \neq 0 \text{ and } l_i = \alpha \\ \phi(x) \text{ (as an element of } V_{n-1}) & \text{if } i = 0 \text{ and } l_{n-1} = \alpha \\ x \text{ (as an element of } V_{i+1}) & \text{if } i \neq n-1 \text{ and } l_{i+1} = \alpha^{-1} \\ \phi^{-1}(x) \text{ (as an element of } V_1) & \text{if } i = n-1 \text{ and } l_n = \alpha^{-1} \\ 0 & \text{otherwise} \end{cases}$$

We will normally denote this module as $M(D, m, \phi)$. In fact, since $m$ is the $K$-dimension of $\mathrm{Im}(\phi)$, we need not specify $m$ in the notation- so we may refer to the module as $M(D, \phi)$. Any module of this form is called a *band module.*

**Theorem 36.** *For all indecomposable module* $(K^m, \phi)$ *in* $K[T, T^{-1}]$-mod and all bands $C$, $M(C, \phi)$ is an indecomposable $A$-module.

Moreover $M(C, \phi)$ and $M(D, \psi)$ are isomorphic if and only if one of the following is true:

- $C$ is a cyclic permutation of $D$, and the $K[T, T^{-1}]$-modules corresponding to $\phi$ and $\psi$ are isomorphic.

- $C$ is a cyclic permutation of $D^{-1}$, and the $K[T,T^{-1}]$-modules corresponding to $\phi$ and $\psi^{-1}$ are isomorphic.

*Proof.* See [8] page 161. □

**Theorem 37.** *Every indecomposable* $M \in A$-mod is either a string module $M(C)$, or a band $M(D,\phi)$.

Furthermore, no finite dimensional string module is isomorphic to a (finite dimensional) band module.

*Proof.* See [8] page 161. □

## 5.2 Infinite words

We define an $\mathbb{N}$-word to be any infinite string of letters $l_1 l_2 \dots l_n l_{n+1} \dots$ such that $l_1 \dots l_n \in \mathcal{W}$ for all $n \in \mathbb{N}^+$. An $\mathbb{N}$-word $l_1 l_2 l_3 \dots$ is said to be *periodic* if there exists $k \geq 1$ such that $l_n = l_{n+k}$ for all $n \in \mathbb{N}$.

Every periodic $\mathbb{N}$-word can be written in the form $D^\infty$-for some unique band $D$. We say that a periodic $\mathbb{N}$-word $D^\infty$ is *contracting* (respectively, *expanding*) if the last letter of $D$ lies in $Q_1$ (respectively, in $Q_1^{-1}$).

An $\mathbb{N}$-word $l_1 l_2 l_3 \dots$ is said to be *almost periodic* if there exists $k \geq 1$ such that $l_{k+1} l_{k+2} l_{k+3} \dots$ is periodic, but $l_k l_{k+1} l_{k+2} l_{k+3} \dots$ is not. A $\mathbb{N}$-word is said to be *aperiodic* if it is not periodic or almost periodic.

For any almost periodic $\mathbb{Z}$-word, $w$, there exists a unique band $D$, and a unique $k \geq 0$ such that $l_k^\infty$ is not periodic, and $l_1 \dots l_{k-1} l_k D^\infty$. It is said to be *contracting* (respectively, *expanding*) if $D^\infty$ is contracting (respectively, expanding). Notice that both $l'_n l'_1$ and $l_k l'_1$ are both words, and so $l'_n$ and $l'_1$ can't both be direct (by the definition of a string algebra), and similarly, can't both be inverse.

We define a $\mathbb{Z}$-word to be any 2-sided infinite string of the form:

$$\dots l_{-m-1} l_m \dots l_{-1} l_0 l_1 \dots l_n l_{n+1} \dots$$

-such that, for all $m, n \in \mathbb{N}$, $l_{-n} \dots l_{-1} l_0 l_1 \dots l_m \in \mathcal{W}$. We say that a $\mathbb{Z}$-word is:

- *periodic*- if there exists $k \in \mathbb{N}^+$ such that $l_{n+k} = l_n$ for all $n \in \mathbb{Z}$.

- *almost periodic*- if is not periodic, but is of the form $u^{-1}w$, where $u$ and $w$ are almost periodic $\mathbb{N}$-words.

- *half-periodic*- if it is of the form $u^{-1}w$, with one of the $\mathbb{N}$-words $w, u$ being almost periodic, and the other being aperiodic.

- *aperiodic*- if it is none of the above.

Any almost periodic $\mathbb{Z}$-word, $w$ can be written in the unique form $u^{-1}l_1 \dots l_m v$- for some periodic $\mathbb{N}$-words $u, v$, and finite word $l_1 \dots l_m$ such that $l_m v$ and $l_1^{-1}u$ are almost periodic. We say $w$ is:

- *contracting*- if both $u$ and $v$ are contracting.

- *expanding*- if both $u$ and $v$ are expanding.

- *mixed*- if $u$ is contracting, and $v$ expanding.

- *negative mixed*- if $u$ is expanding, and $v$ contracting.

**Lemma 83.** *The following are equivalent for any string algebra A:*

1. *There are only finitely many bands*

2. *Every* $\mathbb{N}$*-word is periodic or almost periodic*

3. *Every* $\mathbb{Z}$*-word is periodic or almost periodic*

*Proof.* Proposition 2 of [24]. □

A string algebra is said to be *domestic* if it satisfies the conditions of lemma 83, and *non-domestic* otherwise.

### 5.2.1 Infinite dimensional string modules

We will be dealing a lot with $K$-vector spaces of the form:

$$\prod_{i \in I} V_i$$

-where $V_i$ is a 1-dimensional $K$-vector space. Elements of such a space are usually written in the form $(x_i)_{i \in I}$ (with each $x_i$ in $V_i$)- however, we will write such an element as $\sum_{i \in I} x_i$ throughout the chapters on string algebras.

Let $w$ be any $\mathbb{N}$-word, $l_1 l_2 l_3 \ldots$, or $\mathbb{Z}$-word, $\ldots l_{-1} l_0 l_1 l_2 \ldots$. Define the *index set* $I$ of $w$ to be $\mathbb{N}$ in the former case, and $\mathbb{Z}$ in the latter.

We define an infinite dimensional $A$-module $\overline{M}(w)$- referred to as the *direct product module over* $w$- as follows: For each $i \in I$, let $V_i$ be a one dimensional $K$-vector space, and let $z_i$ be any non-zero element of $V_i$. We define $\overline{M}(w)$ to be the $A$-module with underlying $K$-vector space $\prod_{i \in I} V_i$ , such that, for all $a \in Q_0$ and $i \in I$:

$$e_a z_i = \begin{cases} z_i & \text{if } t(l_{i+1}) = a \\ 0 & \text{otherwise} \end{cases}$$

-and such that $e_a(\sum_{i \in I} \lambda_i z_i) = \sum_{i \in I} \lambda_i e_a z_i$ for all elements $\sum_{i \in I} \lambda_i z_i$ of $\overline{M}(w)$. Also, for all $\alpha \in Q_1$:

$$\alpha z_i = \begin{cases} z_{i-1} & \text{if } l_i = \alpha \\ z_{i+1} & \text{if } l_{i+1} = \alpha^{-1} \\ 0 & \text{otherwise} \end{cases}$$

-and $\alpha(\sum_{i \in I} \lambda_i z_i) = \sum_{i \in I} \lambda_i \alpha z_i$ for all elements $\sum_{i \in I} \lambda_i z_i$ of $\overline{M}(w)$.

We define the *direct sum string module*- denoted $M(w)$- to be the submodule of $\overline{M}(w)$ with underlying $K$-vector space $\bigoplus_{i \in I} V_i$.

If $w$ is a $\mathbb{Z}$-word, then we define $M^+(w)$ and $M^-(w)$ to be the submodules of $\overline{M}(w)$ with underlying $K$-vector space $\prod_{i \geq 0} V_i \oplus \bigoplus_{i<0} V_i$ and $\bigoplus_{i \geq 0} V_i \oplus \prod_{i<0} V_i)$ respectively.

We define a *string module over* $w$ to be any module of the form $M(w)$, $\overline{M}(w)$, $M^+(w)$ or $M^-(w)$. Notice that, for any string module $M$ over $w$, $M(w)$ is a submodule of $M$, and $M$ is a submodule of $\overline{M}(w)$. We refer to the embeddings $M(w) \hookrightarrow M$

and $M \hookrightarrow \overline{M}(w)$ (corresponding to the inclusion of submodules) as canonical embeddings.

We call the set $\{z_i : i \in I\}$ the *standard basis of* $M(w)$. In fact, given any string module $M$ over $w$, we refer to $\{z_i : i \in I\}$ as the standard basis of $M$ (even though it is not strictly a basis of, for example, $\overline{M}(w)$).

### 5.2.2 Some isomorphisms between string modules

We say that $w$ is an *infinite word* if it is either a $\mathbb{Z}$-word, an $\mathbb{N}$-word, or the inverse of an $\mathbb{N}$-word. We call $w$ a *word* if and only if it is a finite word, or an infinite word.

Given any two words $u$ and $w$, we say that $u$ *is a subword of* $w$ if there exist words $u'$ and $u''$ such that $w = u'uu''$ (of course, $u''$ would have to be a finite word or $\mathbb{N}$-word, and $u'$ a finite word or inverse of an $\mathbb{N}$-word).

Given any two words, $w = \dots l_{-1}l_0l_1l_2\dots$, and $w' = \dots l_{-2}l_{-1}l_0l_1l_2\dots$ (with index sets $I$ and $J$ respectively), we write $w = w'$ whenever there exists $k \in \mathbb{Z}$ such that $\{i+k : i \in I\} = J$ and $l_i = l'_{i+k}$ for all $i \in I$.

Of course, given any two such words, $M(w) \cong M(w')$, $\overline{M}(w) \cong \overline{M}(w')$, $M^-(w) \cong M^-(w')$ and $M^+(w) \cong M^+(w')$, via the map $\sum_{i\in I} \lambda_i z_i \mapsto \sum_{i\in I} \lambda_i y_{i+k}$ (where $\{z_i : i \in I\}$ and $\{y_j : j \in J\}$ are the standard bases of $M(w)$ and $M(w')$ respectively).

Also, we write $w = (w')^{-1}$ whenever there exists $k \in \mathbb{Z}$ such that $\{k-i : i \in I\} = J$ and $l_i = l'_{k-i}$ for all $i \in I$. Of course, given any two such words, $M(w) \cong M(w')$, $\overline{M}(w) \cong \overline{M}(w')$, $M^-(w) \cong M^+(w')$ and $M^+(w) \cong M^-(w')$, via the map $\sum_{i\in I} \lambda_i z_i \mapsto \sum_{i\in I} \lambda_i y_{k-i}$.

### 5.2.3 Ringel's List

In [24] Ringel focusses on the following set of modules:

- A module $M(w)$, for every contracting periodic or almost periodic $\mathbb{N}$-word, $w$.
- A module $\overline{M}(w)$, for every expanding periodic or almost periodic $\mathbb{N}$-word, $w$.
- A module $M(w)$, for every contracting almost periodic $\mathbb{Z}$-word, $w$.

- A module $\overline{M}(w)$, for every expanding almost periodic $\mathbb{Z}$-word, $w$.
- A module $M^+(w)$ for every mixed almost periodic $\mathbb{Z}$-word, $w$.
- A module $M^-(w)$ for every negative mixed almost periodic $\mathbb{Z}$-word, $w$.

We will refer to the set of all such modules as *Ringel's list.* Since $M^+(w) \cong M^-(w^{-1})$ for any mixed almost periodic $\mathbb{Z}$-word, $w$, we may ignore the modules on that list over negative mixed words.

**Theorem 38.** *Every module on Ringel's list is pure-injective.*

*Proof.* See [24]. □

It was suspected that every module on Ringel's list is also indecomposable. Indeed, this result (theorem 41) does follow from our results in the next chapter.

It was conjectured that every infinite dimensional pure injective indecomposable module over a string algebra is a string module over some infinite word $w$: and further, that every infinite dimensional indecomposable pure-injective module over a domestic string algebra is isomorphic to a module on Ringel's list.

**Proposition 4.** *Let $w$ be any $\mathbb{N}$-word or $\mathbb{Z}$-word over a string algebra $A$. Then $\overline{M}(w)$ is pure-injective.*

*Proof.* Consider the opposite algebra $A^{\mathrm{op}} = \mathrm{Hom}_K(A, K)$. It will be enough to prove that $\overline{M}(w)$ is the $K$-dual of an $A^{\mathrm{op}}$-module- since any such module is a pure-injective $A$-module- by [17, (4.3.29)].

Given any $x \in A$, let $f_x$ denote the corresponding element in $\mathrm{Hom}(A, K)$. One can easily check that $A^{\mathrm{op}}$ is a string algebra, with $\{f_{e_a} : a \in Q_0\}$ being primitive orthogonal idempotents (i.e. the stationary paths), and $\{f_\alpha : \alpha \in Q_1\}$ the set of arrows (i.e. the paths of length 1).

Given any arrow $f_\alpha$ in $A^{\mathrm{op}}$, denote the inverse letter associated to $f_\alpha$ as $f_{\alpha^{-1}}$. This gives an obvious bijection between the letters of $A$ and the letters of $A^{\mathrm{op}}$- where each $l$ corresponds to $f_l$.

If $l_1 l_2 l_3 \dots$ is a word in $A$, then one can easily check that $f_{l_1} f_{l_2} f_{l_3} \dots$ is a word in $A^{\mathrm{op}}$.

Consider the string module $M(f_{l_1} f_{l_2} f_{l_3} \dots)$ over $A^{\mathrm{op}}$. One can easily check that $\mathrm{Hom}(M(f_{l_1} f_{l_2} f_{l_3} \dots), K)$ is isomorphic (as an $A$-module) to $\overline{M}(w)$. $\square$

**Theorem 39.** *Let $w$ be any infinite word. Then $M(w)$ is indecomposable.*

*Proof.* See [13]. $\square$

## 5.3 Some pp-formulas over string algebras

### 5.3.1 Partially ordering the set of words

Let $\mathcal{W}'$ denote the set of all finite words and $\mathbb{N}$-words. Of course, $\mathcal{W}'$ can be partitioned into $\bigcup_{a \in Q_0} \mathcal{W}'_a$- where $\mathcal{W}'_a$ is the set of all $w \in \mathcal{W}'$ such that $t(w) = a$. Furthermore, we partition each $\mathcal{W}'_a$ into two sets $H_1(a)$ and $H_{-1}(a)$, as follows:

By definition of a string algebra, there exists at most two direct letters in $Q_1$ with target $a$, and at most two inverse letters in $Q_1^{-1}$ with target $a$. We can arbitrarily place each of these (at most four) letters in either $H_1(a)$ or $H_{-1}(a)$, to satisfy the following criteria:

- $H_s(a)$ contains at most one direct letter, and at most one inverse letter (for each $s \in \{-1, +1\}$.
- Given any $l_1, l_2 \in H_1(a)$, the string $l_1^{-1} l_2$ is not a word
- Given any $l_1, l_2 \in H_{-1}(a)$, the string $l_1^{-1} l_2$ is not a word

By the definition of a word (and of a string algebra), there is always at least one way of doing this. Now place $1_{a,+1} \in H_1(a)$ and $1_{a,-1} \in H_{-1}(a)$. Given any $w \in \mathcal{W}'$ of non-zero length, we place $w$ in whatever subset we placed its first letter in.

Given any $s \in \{-1, +1\}$ and $w \in H_s(a)$, we define $1_{a,s} w$ to be $w$. Now we define a total order on $H_s(a)$ such that $C < D$ if and only if one of the following holds:

- $D = C \alpha E$ for some word $E$ and $\alpha \in Q_1$

- $C = D\beta^{-1}F$ for some word $F$ and $\beta \in Q_1$
- $C = E_1\gamma^{-1}F_1$ and $D = E_1\delta F_2$ for some words $E_1, F_1, F_2$ and $\gamma, \delta \in Q_1$

**Lemma 84.** *Given any $u, w_1, \ldots, w_n \in H_s(a)$ such that $u < w_i$ for all $i$, there exists a finite word $D$ such that $u < D \leq w_i$ for all $i$.*

*Proof.* Let $w = \min\{w_i : 1 \leq i \leq n\}$. If $w$ is finite, then let $D = w$. If not, then there are two possibilities:

Firstly, if $u$ is an initial subword of $w$, then (since $u < w$) there exists $\alpha \in Q_1$ and a word $w'$ such that $w = u\alpha w'$- in which case set $D = u\alpha$.

Secondly, if $u$ is not an initial subword of $w$, then (since $u < w$) there exists $\alpha, \beta \in Q_1$, and words $D$, $u'$ and $w'$ such that $u = D\beta^{-1}u'$ and either $w = D\alpha w'$ or $w = D$. Then $u < D \leq w$, as required. $\square$

**Lemma 85.** *Given any $w = l_1l_2l_3 \cdots \in H_1(a)$, let $D = l'_1l'_2 \ldots l'_k$ be any finite word (with $k \geq 1$) in $H_1(a)$ which is not an initial subword of $w$. Then $D > w$ implies $l_1 \ldots l_{k-1} \geq w$.*

*Similarly, $D < w$ implies $l_1 \ldots l_{k-1} \leq w$.*

*Proof.* Firstly, if $w = l'_1 \ldots l'_n$ for some $n \leq k-1$, then $l'_{n+1}$ must be a direct letter (since $D > w$), and so:

$$l'_1 \ldots l'_n l'_{n+1} \ldots l'_{k-1} \geq l'_1 \ldots l'_n = w$$

-as required. Secondly, if $w$ is not an initial subword of $D$, then there must exist $n < k$ such that $l_1 \ldots l_n = l'_1 \ldots l'_n$, and $l_{n+1} \in Q_1^{-1}$, and $l'_{n+1} \in Q_1$. And so:

$$l'_1 \ldots l'_n l'_{n+1} \ldots l'_{k-1} \geq l'_1 \ldots l'_n > w$$

-as required. $\square$

**Lemma 86.** *Given any $a \in Q_0$ and $s \in \{-1, +1\}$, let $w_1, w_2, w_3, \ldots$ be any strictly descending infinite sequence of words in $H_s(a)$. Then there exists $w \in H_s(a)$ such that $w \leq w_i$, and is maximal such- i.e. $w \geq u$ for all $u \in H_1(a)$ such that $w_i > u$ for all $i \in \mathbb{N}$.*

*Furthermore, $w$ is an $\mathbb{N}$-word, and given any finite initial subword $D$ of $w$, there exists $k \geq 1$ such that, $D$ is an initial subword of $w_i$, for all $i \geq k$.*

*Proof.* We can write each $w_i$ as $l_{i,1}l_{i,2}l_{i,3}\dots$- where each $l_{i,j}$ is either a direct letter, an inverse letter, or "zero" (in the case that $w_i$ is finite, of length less than $j$).

We define, recursively, letters $l_1, l_2, l_3, \dots$ as follows: First of all, there are at most two possible words in $H_1(S)$, which we denote $\alpha$ and $\beta^{-1}$. Define:

$$l_1 := \begin{cases} \alpha & \text{if } l_{i,1} = \alpha \text{ for all } i \in \mathbb{N}^+ \\ \beta^{-1} & \text{otherwise} \end{cases}$$

Notice that, if $l_{i,1} = \beta^{-1}$ for any $i$, then $l_{i+1,1} = \beta^{-1}$ (since $w_{i+1} \leq w_i$). Furthermore, if $w_{i,1} = 1_{a,s}$ for some $i$, then $l_{i+1,1} = \beta^{-1}$. Define $k_1 \in \mathbb{N}$ to be minimal such that $l_{k_1,1} = l_1$.

Now, assume that we have defined a word $l_1 \dots l_{n-1}$, and a smallest possible $k_{n-1}$ such that $l_{i,1}l_{i,2}\dots l_{i,n-1} = l_1 \dots l_{n-1}$ for all $i \geq k_{n-1}$. As before, let $\alpha, \beta \in Q_1$ be such that $l_{n-1}\alpha$ and $l_{n-1}\beta^{-1}$ are words. Define:

$$l_n := \begin{cases} \alpha & \text{if } l_{i,n} = \alpha \text{ for all } i \geq k_{n-1} \\ \beta^{-1} & \text{otherwise} \end{cases}$$

Define $k_n \in \mathbb{N}$ to be minimal such that $k_n \geq k_{n-1}$ and $l_{k_n,n} = l_n$. Notice that:

$$l_{k_n,1}l_{k_n,2}\dots l_{k_n,n} = l_1 l_2 \dots l_n$$

Define $w := l_1 l_2 l_3 \dots$.

To show that $w_n > w$ for all $i \geq 0$, it is enough to prove that $w_{k_i} \geq w$, for all $i$ (since $w_{k_i} > w_{k_{i+1}} > w_{k_{i+2}} > \dots$). Suppose, for a contradiction, that $w_{k_i} < w$ for some $i$. Then there must exist $m$ such that:

$$l_1 l_2 \dots l_m = l_{k_i,1}, l_{k_i,2} \dots l_{k_i,m}$$

-with $l_{m+1} = \alpha$ and $l_{k_i,m+1} = \beta^{-1}$ for some $\alpha, \beta \in Q_1$. However, given any $j \geq k_i$:

$$l_1 l_2 \dots l_m = l_{k_j,1}, l_{k_j,2} \dots l_{k_j,m}$$

-and , since $l_{k_i,m+1} = \beta^{-1}$, $l_{j,m+1}$ must be $\beta^{-1}$ (otherwise $w_j > w_{k_i}$). Thus, by its definition, $l_{m+1} = \beta^{-1}$- giving our required contradiction.

To show it is a greatest lower bound, take any $u > w$. Let $n \in \mathbb{N}$ be such that $l_1 l_2 \dots l_n$ is the longest possible common initial subword of $u$ and $w$- notice that $u \geq l_1 \dots l_n$, and $l_{n+1}$ must be an inverse letter.

Now, consider the word $w_{k_{n+1}}$. Then:

$$l_{n+1,1} l_{n+1,2} \dots l_{n+1,n+1} = l_1 l_2 \dots l_{n+1} < l_1 l_2 \dots l_n \leq u$$

-as required. □

We refer to the word $w$ as defined in lemma 86 as $\varinjlim w_i$.

**Corollary 18.** *Given any $a \in Q_0$ and $s \in \{-1, +1\}$, any subset $Y \subseteq H_s(a)$ has a unique infimum,* inf(Y)*- i.e. a word $w \in H_1(S)$ which is maximal in $H_s(a)$ such that $w \leq u$ for all $u \in Y$.*

*Furthermore, if* sup(Y) *is not an element of $Y$, then it is an $\mathbb{N}$-word.*

*Proof.* If $Y$ has a minimal element, then the result is obvious. So we assume it does not.

For each $n \in \mathbb{N}$, let $D_n$ be the minimal word of length at most $n$ such that there exists $u \in Y$ with $u \leq D_n$. Of course, $D_1 \geq D_2 \geq D_3 \geq \dots$, and there is no $n \geq 0$ such that $D_k = D_n$ for all $k \geq n$ (this would imply that $D_n \in Y$ and it is a minimal element of $Y$).

Define $\inf(Y) = \varinjlim D_n$. By lemma 86, it is an $\mathbb{N}$-word, and one can easily check that it satisfies the required conditions. □

Of course, we have similar results regarding upper bounds:

**Lemma 87.** *Given any $a \in Q_0$ and $s \in \{-1, +1\}$, let $w_1, w_2, w_3, \dots$ be any strictly ascending infinite sequence of words in $H_s(a)$. Then there exists $w \in H_s(a)$ such that $w \geq w_i$, and is minimal such- i.e. $w \leq u$ for all $u \in H_1(a)$ such that $w_i < u$ for all $i \in \mathbb{N}$.*

*Furthermore, $w$ is an $\mathbb{N}$-word, and given any finite initial subword $D$ of $w$, there exists $k \geq 1$ such that, $D$ is an initial subword of $w_i$, for all $i \geq k$.*

We denote the word defined in lemma 87 as $\varinjlim w_i$.

**Corollary 19.** *Given any* $a \in Q_0$ *and* $s \in \{-1, +1\}$, *any subset* $Y \subseteq H_s(a)$ *has a unique supremum,* $\sup(\mathrm{Y})$*- i.e. a word* $w \in H_1(S)$ *which is minimal in* $H_s(a)$ *such that* $w \geq u$ *for all* $u \in Y$.

*Furthermore, if* $\sup(\mathrm{Y})$ *is not an element of* $Y$, *then it is an* $\mathbb{N}$*-word.*

### 5.3.2 pp-definable subsets obtained from words

Given any $M \in A$-Mod, subset $X \subseteq M$ of $M$, and $\alpha \in Q_1$ we define:

$$\alpha X := \{\alpha x : x \in X\}$$

$$\alpha^{-1} X := \{m \in M : \alpha m \in X\}$$

So, given any finite word $D = l_1 l_2 \ldots l_n$, we can induct this notation to define:

$$DX := l_1(l_2(\ldots(l_n(X))\ldots))$$

Of course, if $X \subseteq Y$, then $DX \subseteq DY$.

**Lemma 88.** *Given any* $D \in \mathcal{W}$, *suppose there exists* $\alpha, \beta \in Q_1$ *such that* $D\beta \in \mathcal{W}$ *and* $D\alpha^{-1} \in \mathcal{W}$. *Then* $\alpha$ *and* $\beta$ *are unique, and for all* $M \in A$-Mod:

$$D\beta M \subseteq DM \subseteq D\alpha^{-1} M$$

$$D\beta 0 \subseteq D0 \subseteq D\alpha^{-1} 0$$

$$D\beta(M) \subseteq D\alpha^{-1}(0)$$

*Proof.* If $D$ is of non-zero length, then the uniqueness of $\alpha$ and $\beta$ (if they exist) follows from the definition of a string algebra, as does the fact that $\alpha\beta \in \mathcal{I}$- and hence that $\alpha\beta M = \{0\}$.

If $D$ has zero length- without loss of generality $D = 1_{a,+1}$- then the uniqueness of $C$ and $D$, and the fact that $\alpha\beta \in \mathcal{I}$ follows from the definition of $H_{+1}(a)$.

The remaining assertions follow straight from the definition, and the fact that $\alpha\beta M = \{0\}$. □

Let $D = l_1 \dots l_n$ be any finite word. For any $x \in M$, we define $Dx$ to be $D\{x\}$. There exists a pp-formula $\phi(v, v')$ such that, for all $M \in A$-Mod:

$$\phi(M) = \{(x_1, x_2) \in M^2 : x_1 \in Dx_2\}$$

-namely, the formula:

$$\exists v_1, \dots, v_{n-1}((l_n v' = v_{n-1}) \wedge (l_1 v_1 = v) \wedge \bigwedge_{i=2}^{n-1} l_i v_i = v_{i-1})$$

(Where, if $l_i \in Q_1^{-1}$, say $l_i = \beta^{-1}$, then $l_i v_i = v_{i-1}$ refers to the pp-formula $v_i = \beta v_{i-1}$). We shall refer to this pp-formula as $v \in Dv'$. Of course, for all $M \in A$-Mod, and $x \in M$:

$$x \in DM \iff M \models \exists v'(x \in Dv')$$

Notice that $v \in Dv'$ and $v' \in D^{-1}v$ are logically equivalent, and that $0 \in D0$.

### 5.3.3 Subword notation

For a finite word $w = l_{m+1} \dots l_n$, we define $w_k := l_{k+1} \dots l_n$ and $u_k = l_k^{-1} \dots l_{m+2}^{-1} l_{m+1}^{-1}$ , for all $k$ in the index set of $w$.

If $w$ is an $\mathbb{N}$-word, $l_1 l_2 l_3, \dots$, then we define $w_k := l_{k+1} l_{k+2} l_{k+3} \dots$ and $u_k := l_k^{-1} \dots l_2^{-1} l_1^{-1}$ for all $k \in \mathbb{N}^+$.

If $w$ is a a $\mathbb{Z}$-word, $\dots l_{-2} l_{-1} l_0 l_1 l_2 l_3, \dots$, then we define $w_k := l_{k+1} l_{k+2} l_{k+3} \dots$ and $u_k := l_k^{-1} l_{k-1}^{-1} l_{k-2}^{-1} \dots$ for all $k \in \mathbb{Z}$.

Given any word $w$, and any $i$ in the index set of $w$, we define:

$$\hat{w}_i := \begin{cases} w_i & \text{if } w_i \in H_1(a) \text{ for some } a \in Q_0 \\ u_i & \text{if } w_i \in H_{-1}(a) \text{ for some } a \in Q_0 \end{cases}$$

$$\hat{u}_i := \begin{cases} u_i & \text{if } u_i \in H_{-1}(a) \text{ for some } a \in Q_0 \\ w_i & \text{if } u_i \in H_1(a) \text{ for some } a \in Q_0 \end{cases}$$

Of course, for any word $w$, and any $k$, $u_k^{-1} w_k = w$.

For example, if we take $A$ to be the string algebra over the following quiver:

$$\begin{tikzcd} a \arrow[r, bend left=15, "\alpha"] \arrow[r, bend right=15, "\beta"'] & b \arrow[l, bend right=70, "\gamma"'] \end{tikzcd}$$

-with ideal $\mathcal{I} = \langle \beta\gamma, \gamma\alpha \rangle$, then we could take $H_1(b)$ to be the set of all words starting with $\alpha$ or $\gamma^{-1}$ (and $1_{+1}(b)$), and $H_{-1}$ to be the set of all words starting with $\beta$ (and $1_{-1}(b)$). Take $w$ to be the word:

$$l_1 l_2 l_3 l_4 l_5 l_6 = \alpha\gamma\beta\alpha^{-1}\beta\alpha^{-1}$$

Then $w_0 = w \in H_1(b)$, and $w_2 = \beta\alpha^{-1}\beta\alpha^{-1} \in H_{-1}(b)$. So $\hat{w}_0 = w$, and $\hat{u}_0 = 1_{-1}(b)$. Also $\hat{u}_2 = \beta\alpha^{-1}\beta\alpha^{-1}$ and $\hat{w}_2 = \gamma^{-1}\alpha^{-1}$.

### 5.3.4 Results about pp-formulas defined by words

**Lemma 89.** *Let $w$ be any word. If there exist distinct $i, j$ in the index set of $w$, such that $\hat{w}_i = \hat{w}_j$, and $\hat{u}_i = \hat{u}_j$, then $w$ is a periodic $\mathbb{Z}$-word.*

*Proof.* Without loss of generality, we may assume that $i < j$. There are two possibilities: Either $w_i = u_j$ and $w_j = u_i$, or $w_i = w_j$ and $u_i = u_j$.

Suppose we have the former case. By considering the first $j - i$ letters of $w_i$ and $u_j$, we have:

$$l_{i+1} l_{i+2} \dots l_j = l_j^{-1} \dots l_{i+2}^{-1} l_{i+1}^{-1}$$

-contradicting lemma 81.

We must therefore have that $w_j = w_i = l_{i+1} \dots l_j w_j$. Consequently, both $w_i$ and $w_j$ are $\mathbb{N}$-words, and $l_k = l_{k+j-i}$ for all $k \geq i + 1$: so $w_i$ is indeed periodic, and $w_i = (l_{i+1} \dots l_j)^\infty$.

Similarly, $u_i = u_j = (l_j^{-1} l_{j-1}^{-1} \dots l_{i+1}^{-1})^\infty$- completing the proof. □

**Lemma 90.** *Let $w = l_1 l_2 l_3 \dots$ be any periodic $\mathbb{N}$-word. Let $E = l_1 \dots l_n$ be the unique band such that $w = E^\infty$. Then, for all $i \in \mathbb{N}$, $E$ is not an initial subword of $u_i$, and $E$ is an initial subword of $w_i$ if and only if $i \in \mathbb{N}$.*

*Proof.* Follows from lemma 82. □

Given any word $w$, $M$ any string module over $w$ ,and $X$ any subset of the standard basis $\{z_i : i \in I\}$ of $M$, the *$K$-span of $X$ in $M$* is the $K$-vector subspace:

$$\mathrm{sp}_K^M(X) := \{\sum_{z_i \in X} \lambda_i z_i : \lambda_i \in K\} \cap M$$

For example, if $w = l_1 l_2 l_3 \dots$ is an $\mathbb{N}$-word, and $X = \{z_i : i \geq n\}$ for some $n \in \mathbb{N}$, then $\mathrm{sp}_K^{\overline{M}(w)}(X) = \prod_{i \geq n} K z_i$, and $\mathrm{sp}_K^{M(w)} = \bigoplus_{i \geq n} K z_i$.

Notice that, given any subsets $X_0, X_1, X_2, X_3, \dots$ of the index set:

$$\mathrm{sp}_K(\bigcup_{j \in \mathbb{N}} X_j) = \sum_{j \in \mathbb{N}} \mathrm{sp}_K(X_j)$$

**Lemma 91.** *Let $M$ be any string module over a word $w$, with standard basis $\{z_i : i \in I\}$. Let $x = \sum_{i \in I'} \lambda_i z_i$ be any element of $M$, such that $\lambda_i \neq 0$ for all $i \in I'$.*

*Then $x \in \beta M$ if and only if, for all $i \in I'$ exactly one of the set $\{l_{i+1}, l_i^{-1}\}$ is $\beta$.*

*Proof.* First of all, suppose that $\beta \in \{l_{i+1}, l_i^{-1}\}$ for every $i$ such that $\lambda_i \neq 0$.

Let $I_1 = \{i : l_{i+1} = \beta\}$, and $I_2 = \{i : l_i = \beta^{-1}\}$. Then:

$$\beta(\sum_{i \in I_1} \lambda_i z_{i+1}) + \beta(\sum_{i \in I_2} \lambda_i z_{i-1}) = x$$

To show the other direction, assume that $x \in \beta M$. Pick any $y = \sum_i \mu_i z_i$ in $M$ such that $x = \beta y$. Notice that, for all $i$:

$$\beta \mathrm{sp}_K(z_i) = \mathrm{sp}_K(\beta z_i) \subseteq \mathrm{sp}_K(z_{i-1}, z_{i+1})$$

Suppose, for a contradiction, that there exists $j \in I$ such that $\lambda_j \neq 0$, and neither $l_{j+1}$ nor $l_j^{-1}$ is $\beta$. Then $\beta z_{j+1} \neq z_j$, and so $\beta z_{j+1} \in \mathrm{sp}_K(z_{j+2})$. Similarly, $\beta z_{j-1} \in \mathrm{sp}_K(z_{j-2})$.

Also, for all $i \notin \{j+1, j-1\}$, $\beta z_i \in \mathrm{sp}_K(z_{i-1}, z_{i+1})$, so for all $i$:

$$\beta z_i \in \mathrm{sp}_K(\{z_k : k \neq j\})$$

-and so:

$$\beta y \in \mathrm{sp}_K(\{z_k : k \neq j\})$$

However, $\beta y = x \notin \mathrm{sp}_K(\{z_k : k \neq j\})$ (since $\lambda_j \neq 0$)- giving our required contradiction. □

**Lemma 92.** *Let $M$ be any string module over a word $w$, with standard basis $\{z_i : i \in I\}$. Then, for all $i, j \in I$ such that $i < j$:*

$$M \models z_i \in l_{i+1} l_{i+2} \dots l_j (z_j)$$

*Proof.* Clearly $M \models z_k \in l_{k+1}z_{k+1}$ for all $k \in \{i, i+1, \ldots, j-1\}$. The result follows by induction. ∎

**Lemma 93.** *Let $w$ be any word, and $M$ any string module over $w$, with standard basis $\{z_i : i \in I\}$.*

*Let $x = \sum_k \lambda_k z_k$ and $y = \sum_k \mu_k z_k$ be any two elements of $M$. Suppose that, for some $n$:*

$$M \models y \in l_n x$$

*Then $\lambda_n = \mu_{n-1}$.*

*Proof.* We may assume without loss of generality that $l_n$ is direct- if it is inverse, then we can instead write:

$$M \models x \in l_n^{-1} y$$

-with $l_n^{-1}$ being a direct letter (and consider $M$ as a module over $w^{-1}$).

Let $\alpha \in Q_1$ be such that $l_n = \alpha$. Then $l_{n-1} \neq \alpha^{-1}$ (by definition of a word), so $\alpha z_{n-2} \in \mathrm{sp}_K(z_{n-3})$. Also, for all $k \notin \{n-2, n\}$:

$$\alpha z_k \in \mathrm{sp}(z_{k-1}, z_{k+1}) \subseteq \mathrm{sp}_K(z_j : j \neq n-1)$$

And so:

$$\alpha(x - \lambda_n z_n) = \sum_{j \neq n} \lambda_j \alpha z_j \in sp_K(\{z_j : j \neq n-1\})$$

Thus:

$$\begin{aligned}
\lambda_n z_{n-1} - \mu_{n-1} z_{n-1} &= \alpha \lambda_n z_n - \mu_{n-1} z_{n-1} \\
&= -\alpha x + \alpha \lambda_n z_n + y - \mu_{n-1} z_{n-1} \\
&= -\alpha(x - \lambda_n z_n) + \sum_{i \in I \setminus \{n-1\}} \mu_i z_i \\
&\in sp_K(\{z_m : m \neq n-1\})
\end{aligned}$$

Thus $\lambda_n - \mu_{n-1} = 0$, as required. ∎

The following result follows by induction on $m$:

**Corollary 20.** *Let $w$ be any word, and $M$ any string module over $w$, with standard basis $\{z_i : i \in I\}$.*

*Let $C = l_{n+1} \dots l_{n+m}$, be any subword of $w$, and $x = \sum_k \lambda_k z_k$ and $y = \sum_k \mu_k z_k$ any elements of $M$ such that:*

$$M \models y \in Cx$$

*Then $\lambda_{n+m} = \mu_n$.*

### 5.3.5 Pre-Subwords and Post-Subwords

Given any finite word $w = l_1 \dots l_k$, a *pre-subword* is any subword $l_{m+1} \dots l_n$ such that either $m = 0$ or $l_m \in Q_1^{-1}$, and either $n = k$ or $l_{n+1} \in Q_1$.

Given any $\mathbb{N}$-word $w = l_1 l_2 l_3 \dots$, every subword of $w$ is either of the form $w_k = l_{k+1} l_{k+2} \dots$ or of the form $l_{m+1} \dots l_n$. In the former case, it is a *pre-subword* if and only if either $k = 0$ or $l_k \in Q_1^{-1}$. In the latter case, it is a *pre-subword* if $l_{n+1} \in Q_1$, and either $m = 0$ or $l_m \in Q_1^{-1}$

Finally a *pre-subword of a $\mathbb{Z}$-word*, $w = \dots l_{-1} l_0 l_1 l_2 \dots$ is any subword of the form $w$, or $w_k$ for some $k \in \mathbb{Z}$ such that $l_k \in Q_1^{-1}$, or $u_k^{-1}$ for some $k$ such that $l_{k+1} \in Q_1$, or $l_{m+1} \dots l_n$ such that $l_m \in Q_1^{-1}$ and $l_{n+1} \in Q_1$.

**Lemma 94.** *Let $M$ be any string module over a word $w$, and let $\{z_i : i \in I\}$ be the standard basis of $M$.*

*Let $u$ be any subword of $w$, and let $I'$ be the index set of $u$. Then $sp_K(\{z_i : i \in I'\})$ is an $A$-submodule of $M$ if and only if $u$ is a pre-subword. And if so, then the submodule is isomorphic to some string module over $u$.*

We refer to the module as defined in lemma 94 as the submodule of $M$ defined by $u$. The map from it to $M$ corresponding to the inclusion of the submodule will be called the canonical embedding.

For example if $w = \dots l_{-1} l_0 l_1 l_2 \dots$ is a $\mathbb{Z}$-word, and $u = l_{k+1} l_{k+2} \dots$ a pre-subword of $w$, then the submodule obtained from $M^+(w)$ has underlying $K$-vector space $\prod_{i \geq k} K z_i$- and is isomorphic to $\overline{M}(u)$.

The definition of a *post-subword* of a word $w$ is the same as the definition of a pre-subword, but with every every occurrence of $Q_1$ replaced by $Q_1^{-1}$, and vice versa. For example, $l_1 \dots l_m$ is a post subword of $l_0 l_1 \dots l_m l_{m+1}$ if and only if $l_0 \in Q_1$ and $l_{m+1} \in Q_1^{-1}$.

**Lemma 95.** *Given any string module $M$ over $w$, and any subword $u$ of $w$, let $I'$ be the index set of $u$. Then $u$ is a post-subword if and only if there exists a well-defined homomorphism $g : M \to \overline{M}(u)$ given by:*

$$g : \sum_{i \in I} \lambda_i z_i = \sum_{i \in I'} \lambda_i y_i$$

*(where $\{y_i : i \in I'\}$ is the standard basis of $\overline{M}(u)$).*

*Furthermore, if $g$ does exist, then the image of $g$ is isomorphic to a string module over $u$.*

Given any post-subword $u$ of $w$, we refer to the the string module $\mathrm{Im}(g)$ as defined in lemma 95 as the quotient module of $M$ defined by $u$- and we refer to the projection of $M$ onto $\mathrm{Im}(g)$ as the canonical projection.

For example, given any $\mathbb{N}$-word, $w$, with post-subword $u = l_{n+1} l_{n+2} l_{n+3} \dots$, let $\{z_i : i \geq 0\}$ and $\{y_i : i \geq k\}$ denote the standard bases of $M(w)$ and $M(u)$ respectively. Then there exists a well defined homomorphism $f : M(w) \twoheadrightarrow M(u)$ given by:

$$f(\sum_{i \in \mathbb{N}} \lambda_i z_i) = \sum_{i \geq n} \lambda_i y_i$$

Similarly, there exists a well defined homomorphism from $\overline{M}(w)$ to $\overline{M}(u)$ (which is defined the same way).

## 5.4 Comparing infinite and finite strings

In order to study the model theory of string modules, one often wishes to consider the pp-type of a given element of a given module. If the underlying word of a string module is infinite and aperiodic, then this can be a fairly daunting prospect. The results of this section show that we only need to look at a certain finite substring of $w$, in order to determine whether a given pp-formula lies in the pp-type.

Given any aperiodic word $w$, and any $x \in M(w)$, there is clearly a finite pre-subword $E$ of $w$ such that $x$ lies in the subword $M(E)$ of $M(w)$. We shall prove that, for any $m \in \mathbb{N}$, we can pick a finite pre-subword ${}^{(m)}E^{(m)}$ of $w$ such that $E$ is a subword of ${}^{(m)}E^{(m)}$, and:

$$M({}^{(m)}E^{(m)}) \models \phi(x) \Longleftrightarrow M(w) \models \phi(x) \Longleftrightarrow \overline{M}(w) \models \phi(x)$$

-for all pp-formulas $\phi(v)$ which contain at most $m$ equations. We will also find a post-subword ${}^{(m+)}E^{(m+)}$ such that:

$$M({}^{(m+)}u^{(m+)}) \models \phi(\pi(x)) \Longleftrightarrow M(w) \models \phi(x) \Longleftrightarrow \overline{M}(w) \models \phi(x)$$

(Where $\pi : \overline{M}(w) \twoheadrightarrow M({}^{(m+)}E^{(m+)})$ is the canonical projection, as defined after lemma 95.)

Throughout this section, $w$ will be any word, and $\{z_i : i \in I\}$ will be the standard basis of $M(w)$. We call a standard basis element $z_i$ of $M(w)$ a *trough* provided $l_{i+1}$ (if it exists) is a direct letter and $l_i$ (if it exists) is an inverse letter. Similarly, we say that $z_i$ is a *peak* if $l_{i+1}$ (if it exists) is inverse and $l_i$ (if it exists) is direct. By lemma 94, every trough $z_i$ gives a submodule of $M(w)$, with underlying vector space $Kz_i$- we shall refer to this submodule as $Kz_i$.

We say that two troughs $z_i$ and $z_j$ in $w$ (with $i < j$) are *adjacent* if there is no trough $z_k$ with $i < k < j$.

Notice that the distance between in between two adjacent troughs in any word $w$ is bounded- i.e. there exists $N \in \mathbb{N}$ such that $|j - i| \leq n$ for all pairs of adjacent troughs $z_i$ and $z_j$: this is because $l_{i+1} \dots l_j = ED^{-1}$, for some words $E$, $D$ consisting of only direct letters letters. By the definition of a string algebra, there exists $N' \in \mathbb{N}$ such that there are no paths in $A$ of length greater than $N'$. Consequently $j - i \leq 2N'$. Of course, this implies that every infinite word has infinitely many troughs.

Given any trough $T = z_i$, let $w_T$ denote the subword $w_i$ of $w$, $u_T$ the subword $u_i$ of $w^{-1}$. By lemma 94, $\overline{M}(u_T)$ and $\overline{M}(w_T)$ are submodules of $\overline{M}(w)$, with standard bases $\{z_j : j \leq i\}$ and $\{z_j : j \geq i\}$ respectively, such that:

$$\overline{M}(u_T) \cap \overline{M}(w_T) = K z_i$$

$$\overline{M}(u_T) + \overline{M}(w_T) = \overline{M}(w)$$

We say that two troughs $T$ and $T'$ are *comparable* provided there exists $a \in Q_1$ and $s \in \{-1, +1\}$ such that both $w_T$ and $w_{T'}$ lie in $H_s(a)$. We define a pre-order $\leq_{\mathcal{T}}$ on the set of of $w$ troughs by:

$$T \leq_{\mathcal{T}} T' \Longleftrightarrow T \text{ and } T' \text{ are comparable and } w_T \leq w_{T'}$$

**Lemma 96.** *Every set of* $2m|Q_0|+1$ *troughs in* $w$ *contains a subset* $\{T_{i_0}, T_{i_1}, \ldots, T_{i_m}\}$ *such that:*

$$T_{i_0} \leq_{\mathcal{T}} T_{i_1} \leq_{\mathcal{T}} \cdots \leq_{\mathcal{T}} T_{i_m}$$

*Furthermore, if* $w_{T_{i_j}}$ *is not a periodic* $\mathbb{N}$*-word, for all* $j$*, then we may choose the troughs* $T_i$ *such that:*

$$T_{i_0} <_{\mathcal{T}} T_{i_1} <_{\mathcal{T}} \cdots <_{\mathcal{T}} T_{i_m}$$

*Proof.* First of all, we can partition the set into:

$$\bigcup_{a \in Q_0} \bigcup_{s=-1,+1} \mathcal{X}_a^s$$

- where $\mathcal{X}_a^s$ is the set of those troughs $T$ such that $e_a T = T$ and $w_T \in H_s(a)$.

Then there must exist $a \in Q_0$ and $s \in \{-1, +1\}$ such that $|\mathcal{X}_a^s| \geq m + 1$. Since $w_T \in H_s(a)$ for all $T \in \mathcal{X}_s(a)$, they must be pairwise comparable.

Furthermore, if $w_{T_i} = w_{T_j}$ (for any distinct $i$ and $j$) implies that both $w_{T_i}$ and $w_{T_j}$ are periodic (as in the proof of lemma 89), which proves the last assertion. □

Take any trough $z_c$ in $w$, and consider the subword $u_c^{-1} = \ldots l_{c-2} l_{c-1} l_c$ of $w$. Given any $m \in \mathbb{N}$, we define the subwords $(u_c^{-1})^{(m)}$ and $(u_c^{-1})^{(m+)}$ of $w$ as follows:

Pick $N \geq c$ to be minimal such that the set $\{z_i : c \leq i \leq N\}$ contains $m+1$ pairwise comparable troughs- if no such $N$ exists (i.e. because the word $w_c$ is finite, and not long enough) then define both $(u_c^{-1})^{(m)}$ and $(u_c^{-1})^{(m+)}$ to be $w$. Otherwise, denote these troughs as $T_0, T_1, \ldots, T_m$. For each trough $T_i$, let $t_i$ be such that $z_{t_i} = T_i$. We assume that the troughs are labeled such that $t_i < t_{i+1}$ for all $i$.

Of course, $t_m = N$. If $w_N$ is periodic, then define both $(u_c^{-1})^{(m)}$ and $(u_c^{-1})^{(m+)}$ to be $w$.

If $w_N$ is not periodic, then, for each distinct pair $i, j \in \{0, 1, \ldots, m\}$, the words $w_{t_i}$ and $w_{t_j}$ are distinct- let $d_{i,j}$ be the length of the longest possible common initial subword of $w_{t_i}$ and $w_{t_j}$.

Now, let $k \in \mathbb{Z}$ be minimal such that $z_k$ is a trough, and:

$$k > \max\{t_i + d_{i,j} : i, j \in \{0, 1, \ldots, m\}, i \neq j\}$$

Then define $(u_c^{-1})^{(m)}$ to be the pre-subword $u_k^{-1}$ of $w$, and define $u_c^{(m+)}$ to be $u_{k_1}^{-1}$- where $k_1 > k$ is maximal such that $l_{k+1} \ldots l_{k_1}$ is a string of direct letters.

For an example over the string algebra $G_{3,3}$ (defined at the start of the chapter), first of all, set $H_1(a)$ to be the set of all words with first letter $\alpha$ or $\beta^{-1}$, and $H_{-1}(a)$ to be the set of all words with first letter $\alpha^{-1}$ or $\beta$. Let $w$ be the $\mathbb{Z}$-word with $w_0 = \alpha\beta^{-1}\beta^{-1}\alpha\beta^{-1}(\alpha\alpha\beta^{-1})^\infty$ and $u_0 = (\beta^{-1}\alpha)^\infty$. We shall show how to find the subword $(u_0^{-1})^{(1)}$ of $w$.

Of course, $z_0$ is a trough, and $z_3$ is an adjacent comparable trough. So $t_0 = 0$ and $t_1 = 3$. The longest possible common initial subword of $w_0$ and $w_3$ is $\alpha\beta^{-1}$, so $d_{0,1} = 2$. Then $k > t_1 + d_{0,1} = 3 + 2$, is minimal such that $z_k$ is a trough. Then $k = 8$, and so $u_0^{(1)} = u_8^{-1} = \ldots l_6 l_7 l_8$. And since $l_9 l_{10} l_{11} = \alpha\alpha\beta^{-1}$, $u_0^{(1+)} = u_{10}^{-1}$.

**Lemma 97.** *Let $(u_c^{-1})^{(m)}$ and $T_0, T_1, \ldots T_m$ be as above. Then given any distinct $i, j$ such that $T_i \leq T_j$, there exists a map $f \in Hom(\overline{M}(w_{T_i}), \overline{M}(w))$, with image contained in $\overline{M}((u_c^{-1})^{(m)})$, such that $f(T_i) = T_j$.*

*Proof.* First of all, if $w_{T_i} = w_{T_j}$, then $\overline{M}(w_{T_i})$ and $\overline{M}(w_{T_j})$ are isomorphic via the map $g : \overline{M}(w_{T_i}) \to \overline{M}(w_{T_j})$ defined by:

$$g : \sum_{k \in \mathbb{N}^+} \lambda_{t_i+k} z_{t_i+k} \mapsto \sum_{k \in \mathbb{N}^+} \lambda_{t_j+k} z_{t_j+k}$$

Since $w_{T_i} = w_{T_j}$, they must both be periodic, and so the subword $(u_c^{-1})^{(m)}$ of $w$ must be $w$ itself. Since $w_{T_j}$ is a pre-subword of $w$, there exists a canonical embedding $\overline{M}(w_{T_j}) \hookrightarrow \overline{M}(w)$.

Now, if $w_{T_i} \neq w_{T_j}$, then let $d_{i,j}$ be as above: so $l_{t_i+1} \dots l_{t_i+d_{i,j}} = l_{t_j+1} \dots l_{t_j+d_{i,j}}$, and $l_{t_i+d_{i,j}+1} \neq l_{t_j+d_{i,j}+1}$. Since $w_{T_i} < w_{T_j}$, $l_{t_i+d_{i,j}+1} \in Q_1^{-1}$ and $l_{t_j+d_{i,j}+1} \in Q_1$, and so $l_{t_i+1} \dots l_{t_i+d_{i,j}}$ is a post-subword of $w_{T_j}$, and $l_{t_j+1} \dots l_{t_j+d_{i,j}}$ a pre-subword of $w_{T_j}$, and hence of $w$ (since $z_{t_j}$ is a trough).

Consider the map:

$$\overline{M}(w_{T_i}) \twoheadrightarrow M(l_{t_i+1} \dots l_{t_i+d_{i,j}}) \to M(l_{t_j+1} \dots l_{t_j+d_{i,j}}) \hookrightarrow \overline{M}(w)$$

-where the first map is the canonical projection (as in lemma 95), the third map is the canonical embedding (as defined after lemma 94), and the second map is the isomorphism as described in (5.2.2).

Of course, this map takes $z_{t_i}$ to $z_{t_j}$, and $l_{t_j+1} \dots l_{t_j+d_{i,j}}$ is a subword of $(u_c^{-1})^{(m)}$ (by the definition of $(u_c^{-1})^{(m)}$), and so the image of this map lies in $\overline{M}((u_c^{-1})^{(m)})$. $\square$

In our example, $w_0 < w_3$, and so there exists a map $f : \overline{M}(w_0) \to \overline{M}(w)$ given by:

$$f : \sum_{i \geq 0} \lambda_i z_i = \lambda_0 z_3 + \lambda_1 z_4 + \lambda_2 z_5$$

-which clearly has image contained in the $K$-span of $\{z_i : i \leq 8\}$- and and hence in $\overline{M}((u_0^{-1})^{(m)})$. -which takes $T_1 = z_3$ to $T_0 = z_0$, as in lemma 97.

### 5.4.1 Comparing pp-types

**Lemma 98.** *Let $w$ be any word, $z_n$ a trough of $w$, and $m \in \mathbb{N}$. Then, for any $x \in \overline{M}(u_n^{-1})$, and any $\phi(v) \in pp_A$ with at most $m$ equations:*

$$\overline{M}(w) \models \phi(x) \iff \overline{M}((u_n^{-1})^{(m)}) \models \phi(x)$$

*In particular, if $(u_n^{-1})^{(m)}$ is either a finite word, or the inverse of an $\mathbb{N}$-word, then:*

$$\overline{M}(w) \models \phi(x) \iff M^{-}(w) \models \phi(x) \iff \overline{M}((u_n^{-1})^{(m)}) \models \phi(x)$$

*Proof.* Of course, there exists a canonical embedding $\overline{M}((u_n^{-1})^{(m)}) \hookrightarrow \overline{M}(w)$, so:

$$\overline{M}((u_n^{-1})^{(m)} \models \phi(x) \Longrightarrow \overline{M}(w) \models \phi(x)$$

Assume from now on that $\overline{M}(w) \models \phi(x)$. Write $\phi(v)$ as $\exists v_1, \dots v_n \psi(v_1, \dots, v_n v)$, where $\psi$ is the formula:

$$\bigwedge_{j=1}^{m} (\sum_{i=1}^{n} r_{ij} v_i = r_j v)$$

Now, take any witnesses $x_1, \dots x_n$ to the statement $\overline{M}(w) \models \phi(x)$. We shall use the maps as described in lemma 97 to "patch together" a set of witnesses $y_1, \dots, y_n$ to the statement:

$$\overline{M}((u_n^{-1})^{(m)}) \models \phi(x)$$

Let $T_0, T_1, \dots, T_m$ be the pairwise comparable troughs of $w$, as in the definition of $(u_c^{-1})^{(m)}$. For each such trough $T_s$, we can write every $x_i$ (not necessarily uniquely) as $x_i^{\leq T_s} + x_i^{> T_s}$, where the former lies in $\overline{M}(u_{T_s}^{-1})$, and the latter in $\overline{M}(w_{T_s})$.

Now, since $x_1, \dots x_n$ are witnesses to $\overline{M}(w) \models \phi(z_0)$, they must satisfy (for all $j \leq m$):

$$\sum_{i=1}^{n} r_{ij}(x_i) = r_j x$$

Consequently:

$$\sum_{i=1}^{n} r_{ij} x_i^{> T_s} = - \sum_{i=1}^{n} r_{ij} x_i^{\leq T_s} + r_j x$$

Since the left hand side lies in $\overline{M}(w_{T_s})$, and the right hand side in $\overline{M}(u_{T_s}^{-1})$, both sides must lie in $KT_s$- so both sides equal $\rho_{js} T_p$, for some $\rho_{js} \in K$.

Having done this for every $j \in \{1, \dots m\}$, consider the set of vectors in $K^m$:

$$\{(\rho_{1p}, \dots \rho_{ms}) : 0 \leq s \leq m\}$$

This set must be linearly dependent over $K$, so we can pick $\mu_0, \mu_1, \dots \mu_m$ (not all zero) such that $\sum_{s \in S} \mu_s \rho_{js} = 0$ for every $j \in \{1, \dots, m\}$.

Now, recall that we have a total ordering on $\{T_s : 0 \leq s \leq m\}$. Pick the largest $T_k$ with respect to this ordering, such that $\mu_k$ is nonzero. By lemma 97, there must exist maps $f_s \in Hom(\overline{M}(w_{T_s}), \overline{M}(w_{T_k}))$, for every $s \in S \backslash \{k\}$, taking $T_s$ to $T_k$ - and each one must have image contained in $\overline{M}((u_c^{-1})^{(m)})$.

We may assume that $\mu_k = 1$, and hence that $\rho_{jk} + \sum_{s\neq k} \mu_p \rho js = 0$. Now, for every $i \leq n$, define:

$$y_i := x_i^{\leq T_k} + \sum_{s\neq k} \mu_s f_s(x_i^{>T_s})$$

First of all, notice that every $y_i$ lies in $\overline{M}((u_c^{-1})^{(m)})$. Also, for every $j \in \{1, \dots m\}$ we have:

$$\begin{aligned}
\sum_{i=1}^{n} r_{ij} y_i &= \sum_{i=1}^{n} r_{ij}(x_i^{\leq T_k}) + \sum_{i=1}^{n} r_{ij} \sum_{s\neq k} \mu_s f_s(x_i^{>T_s}) \\
&= \sum_{i=1}^{n} r_{ij} x_i^{\leq T_k} + \sum_{s\neq k} \mu_s f_s(\sum_{i=1}^{n} r_{ij} x_i^{>T_s}) \\
&= \sum_{i=1}^{n} r_{ij} x_i^{\leq T_k} + \sum_{s\neq k} \mu_s f_s(\rho_{js} T_s) \\
&= r_j x + \rho_{jk} T_k + \sum_{s\neq k} \mu_s \rho_{js} T_k \\
&= r_j x + \sum_{s\in S} \mu_s \rho_{js} T_k \\
&= r_j x
\end{aligned}$$

Since $y_1, \dots y_n$ lie in $\overline{M}((u_n^{-1})^{(m)})$ and satisfy $\psi(y_1, \dots y_n, x)$, we have that:

$$\overline{M}((u_n^{-1})^{(m)}) \models \phi(x)$$

-as required. □

Given any subword of $w$ of the form $w_k = l_{k+1} l_{k+2}$ (with $z_k$ a trough), the subword ${}^{(m)}w_k$ of $w$ is defined symmetrically: i.e. take the subword $w_k^{-1} = \dots l_{k+2}^{-1} l_{k+1}^{-1}$ of $w^{-1}$, and consider the subword $(w_k^{-1})^{(m)}$ of $w^{-1}$. If it is of the form $\dots l_{j+2}^{-1} l_{j+1}^{-1}$ for some $j \leq k$, then define ${}^{(m)}w_k$ to be $l_{j+1} l_{j+2} \dots$. Otherwise, define ${}^{(m)}w_k$ to be $w$.

Now, given any word $w$, and any finite pre-subword $E = l_{k+1} \dots l_n$ of $w$, such that $z_k$ and $z_n$ are troughs in $w$, we define ${}^{(m)}(E)^{(m)}$ to be the subword:

$$({}^{(m)}(l_{k+1} \dots l_n))^{(m)}$$

**Corollary 21.** *Let $E = l_{k+1} \dots l_m$ be any pre-subword of $w$, such that $z_k$ and $z_n$ are troughs. Then for all $x \in M(E)$, and pp-formulas $\phi(v)$ with at most $m$ equations:*

$$\overline{M}(w) \models \phi(x) \Longleftrightarrow \overline{M}({}^{(m)}E^{(m)}) \models \phi(x)$$

*In particular, if* ${}^{(m)}E^{(m)}$ *is a finite word, then:*

$$\overline{M}(w) \models \phi(x) \iff M(w) \models \phi(x) \iff M({}^{(m)}E^{(m)}) \models \phi(x)$$

### 5.4.2 Comparing words with similar subwords

Suppose we have two words $w$ and $w'$, and a pre-subword $E$ of $w$ such that ${}^{(m)}E^{(m)}$ is a pre-subword of $w'$. We may consider $M({}^{(m)}E^{(m)})$- and hence $M(E)$ as a submodule of $M(w')$. We prove, in this section, that for all $x \in M(E)$:

$$M(w) \models \phi(x) \iff M({}^{(m)}E^{(m)}) \models \phi(x) \iff M(w') \models \phi(x)$$

-for any $\phi(v) \in$ pp with at most $m$ equations.

**Lemma 99.** *Let* $w = \ldots l_{-1}l_0l_1l_2\ldots$ *and* $w' = \ldots l'_{-1}l'_0l'_1l'_2\ldots$ *be any two words, with index sets* $I$ *and* $I'$ *respectively. Take any* $i \in I \cap I'$ *such that* $z_i$ *and* $z'_i$ *are troughs, and any* $m \in \mathbb{N}$.

*Suppose that* $(u_i^{-1})^{(m)} = u_j^{-1}$*, for some* $j \geq i$*, and that:*

$$l_{i+1}\ldots l_jl_{j+1} = l'_{i+1}\ldots l'_{i+1}\ldots l'_jl'_{j+1}$$

*Then* $((u'_i)^{-1})^{(m)} = (u'_j)^{-1} = \ldots l'_{j-2}l'_{j-1}l'_j$

*Proof.* This follows straight from the way that $(u_i^{-1})^{(m)}$ is constructed: Let $N \geq i$ be minimal such that the set $\{z_i, z_{i+1}, \ldots, z_N\}$ contains $m+1$ comparable troughs. Label the troughs in $\{z_k : i \leq k \leq N\}$ as $\{z_{t_s} : 0 \leq s \leq m'\}$- of course, $m' \geq m$.

Notice that $N \leq j$ (by the definition of $(u_i^{-1})^{(m)}$), and so:

$$l_{i+1}\ldots l_Nl_{N+1} = l'_{i+1}\ldots l'_Nl'_{N+1}$$

Therefore the set $\{z'_{t_s} : 0 \leq s \leq m'\}$ is precisely the set of troughs of $w'$ in the set $\{z'_i, z'_{i+1}, \ldots, z'_N\}$. Furthermore, any two given troughs $z'_{t_s}$ and $z'_{t_r}$ are comparable (under the ordering troughs in $w'$) if and only if $z'_{t_s}$ and $z'_{t_r}$ are comparable (under the ordering of troughs in $w$).

So, given any two comparable troughs, $z'_{t_s}$ and $z'_{t_r}$ in $w'$, $z_{t_s}$ and $z_{t_r}$ are comparable in $w$. Let $d_{r,s}$ be as in the definition of $(u_i^{-1})^{(m)}$. Then:

$$l_{t_s+1}\ldots l_{t_s+d_{r,s}} = l_{t_r+1}\ldots l_{t_r+d_{r,s}}$$

-and $l_{t_s+d_{r,s}+1} \neq l_{t_r+d_{r,s}+1}$. By definition of $(u_i^{-1})^{(m)}$:

$$\max(t_s + d_{r,s} + 1, t_r + d_{r,s} + 1) \leq j$$

And so:

$$l'_{t_s+1} \dots l'_{t_s+d_{r,s}} = l'_{t_r+1} \dots l'_{t_r+d_{r,s}}$$

-and $l'_{t_s+d_{r,s}+1} \neq l'_{t_r+d_{r,s}+1}$. The result follows. □

Lemma 98 and lemma 99 give the following two results:

**Corollary 22.** *Let $w, w', k, i, j$ and $m$ be as in lemma 99. Then, for any $x \in M(u_i^{-1})$:*

$$M(w) \models \phi(x) \iff M((u_i^{-1})^{(m)}) \models \phi(x) \iff M(w') \models \phi(x)$$

**Corollary 23.** *Given any word $w$, any subword of the form $u_k^{-1}$, and any $x \in \overline{M}(u_k^{-1})$:*

$$\overline{M}(w) \models \phi(x) \iff \overline{M}((u_k^{-1})^{(m)}) \models \phi(x) \iff \overline{M}((u_k^{-1})^{(m+)}) \models \phi(x)$$

*Also, given any finite subword $E$ of $w$ and $x \in M(E)$:*

$$\overline{M}(w) \models \phi(x) \iff \overline{M}({}^{(m)}E^{(m)}) \models \phi(x) \iff \overline{M}({}^{(m+)}E^{(m+)}) \models \phi(x)$$

### 5.4.3 A further comment on these subwords

Take any $\mathbb{Z}$-word, $w$, and any $i \in \mathbb{Z}$ such that $u_i$ is not periodic. We shall prove in this subsection, that for all sufficiently small $j < i$, the subword $(u_j^{-1})|^{(m)}$ of $w$ lies "strictly to the left of $z_i$"- and hence that $z_i$ is not contained in the submodule $M((u_j^{-1})^{(m)})$ of $M(w)$.

Consequently, given given an aperiodic $\mathbb{N}$-word or $\mathbb{Z}$-word, $w$, and any $m \in \mathbb{N}$ and $z_i$, there are only finitely many $j \in \mathbb{Z}$ such that $z_i \in M({}^{(m)}z_j^{(m)})$.

First of all, given a cyclic word $l_1 \dots l_n$, and any $k \in \mathbb{N}^+$, we define $(l_1 \dots l_n)^{k/n}$ to be the word $l'_1 \dots l'_k$- where $l'_{i \bmod n} = l_i$ for all $i \in \{1, \dots, k\}$. We also define ${}^{k/n}(l_1 \dots l_k)$ to be $(((l_1 \dots l_k)^{-1})^{k/n})^{-1}$. For example:

$$(\alpha\gamma\beta^{-1})^{7/3} = \alpha\gamma\beta^{-1}\alpha\gamma\beta^{-1}\alpha$$

$${}^{4/3}(\alpha\gamma\beta^{-1}) = \beta^{-1}\alpha\gamma\beta^{-1}$$

**Lemma 100.** *Let $l_{n+1}l_{n+2}l_{n+3}\dots$ be any finite word or $\mathbb{N}$-word. Take any $m > n$ and $k \in \mathbb{N}$ such that:*

$$l_{n+1}\dots l_{n+k} = l_{m+1}\dots l_{m+k}$$

*Then $l_{n+1}\dots l_{n+k} = (l_{n+1}\dots l_m)^q$- where $q = k/(m-n)$*

*Proof.* Let $d = m - n$. Since $l_{n+1}\dots l_{n+k} = l_{m+1}\dots l_{m+k}$, it follows that $l_{n+i} = l_{n+d+i}$ for all $i \leq k$. The result follows. □

**Corollary 24.** *Let $w = \dots l_{-1}l_0l_1l_2\dots$ be any $\mathbb{Z}$-word, such that $u_0^{-1}$ is not periodic. Then for all $i \in \mathbb{Z}$ such that $z_i$ is a trough, there exists $c < i$ such that, for all $j \leq c$, $(u_j^{-1})^{(m)}$ is of the form $u_k^{-1}$ for some $k < i$.*

*And hence that $z_i \notin M((u_j^{-1})^{(m)})$.*

*Proof.* Let $N = 2m|Q_0|N'$- where $N'$ is the maximal possible distance between two troughs in $w$.

Relabeling if necessary, we may assume that $i > 0$ and $z_0$ is a trough of $w$. Then for all positive $n \leq N$, let $k_n \in \mathbb{N}$ be maximal such that:

$$(l_0^{-1}l_{-1}^{-1}\dots l_{-n+1}^{-1})^{k_n/n} = l_0^{-1}l_{-1}^{-1}\dots l_{-k_n+1}^{-1}$$

($k_n$ exists, since $u_0^{-1}$ is not periodic). Pick any $c < \min\{-k_n : 1 \leq n \leq N\}$ such that $z_c$ is a trough.

Now, given any $j < c$, let $T_0, T_1, \dots, T_m$ be the comparable troughs in $w$ as in the definition of $(u_j^{-1})^{(m)}$: note that, given any $j_1, j_2 \leq s$, $|t_i - t_j| \leq N$.

Given any distinct $j_1, j_2 \leq m$ let $d_{j_1,j_2}$ be as in the definition of $(u_j^{-1})^{(m)}$. It will be enough to prove that $\max(t_{j_1} + d_{j_1,j_2}, t_{j_2} + d_{j_1,j_2}) < 0$.

Assume without loss of generality, that $j_1 < j_2$. Suppose, for a contradiction, that $t_{j_1} + d_{j_1,j_2} \geq 0$. By the definition of $d_{j_1,j_2}$:

$$t_{j_1+1}\dots t_{j_1+d_{j_1,j_2}} = t_{j_2+1}\dots t_{j_2+d_{j_1,j_2}}$$

-and so, by lemma 100, there exists $q \in \mathbb{Q}^+$ such that:

$$l_{t_{j_2}+1}\dots l_{t_{j_2}+d_{j_1,j_2}} = (l_{t_{j_1}+1}\dots l_{t_{j_2}})^q$$

-and hence a rational $q' \leq q$ such that:

$$l_{t_{j_2}+1} \dots l_{-1} l_0 = (l_{t_{j_1}+1} \dots l_{t_{j_2}})^{q'}$$

-and so there exists a cyclic permutation $E$ of $l_{t_{j_1}+1} \dots l_{t_{j_2}}$ such that:

$$l_{t_{j'}+1} \dots l_{-1} l_0 =^{q'} E$$

Since$l_{t_{j_1}+1} \dots l_{t_{j_2}}$, and hence $E$, has length at most $N$, we have contradicted our choice of $c$- completing the proof. □

**Corollary 25.** *Take any aperiodic $\mathbb{Z}$-word, $w$, and $m \in \mathbb{N}$. Then for all $i \in \mathbb{Z}$ there are only finitely many $j \in \mathbb{Z}$ such that $z_i$ lies in the submodule $M({}^{(m)}z_j^{(m)})$ of $M(w)$.*

*Similarly, given any $i \in \mathbb{Z}$, there are only finitely many $j \in \mathbb{Z}$ such that the canonical projection $M(w) \twoheadrightarrow M({}^{(m+)}z_j^{(m+)})$ takes $z_i$ to $0$.*

## 5.5 Simple String Maps

In [10] Crawley-Boevey describes the the homomorphisms between any two direct sum string modules $M(w)$ and $M(u)$, in terms maps called windings. We extend this idea to any pair of string modules $M$ and $N$, by defining what we call "simple string maps".

If we restrict to maps between direct sum modules, then every simple string map is a winding, and every winding is a simple string map.

The set of all simple string maps are defined as follows:

1. If $w$ is a word, and $M$ a string module over $w$, then the canonical embedding of the subword $M(w)$ of $M$ into $M$ is a simple string map.

2. If $w$ is a word, and $M$ a string module over $w$, then $M$ is a subword of $\overline{M}(w)$, and the canonical embedding of $M$ into $\overline{M}(w)$ is a simple string map.

3. If $w$ is a word, and $M$ a string module over $w$, and $u$ is a pre-subword of $w$, then the natural embedding of the submodule of $M$ defined by $u$ into $M$ (see lemma 94) is a simple string map.

4. If $w$ is a word, and $M$ a string module over $w$, and $u$ is a post-subword of $w$, then the natural projection of $M$ onto the quotient module of $M$ defined by $u$ (see lemma 95) is a simple string map.

5. If $w = u$, then the obvious isomorphisms $M(w) \to M(u)$, $M^-(w) \to M^-(u)$, $M^+(w) \to M^+(u)$, and $\overline{M}(w) \to \overline{M}(u)$- (as described in 5.2.2)- are simple string maps.

6. If $w = u^{-1}$, then the four isomorphisms $M(w) \to M(u)$, $M^-(w) \to M^+(u)$, $M^+(w) \to M^-(u)$, and $\overline{M}(w) \to \overline{M}(u)$- as described in (5.2.2)- are simple string maps.

7. If $f : M \to N$ and $g : L \to M$ are simple string maps, then so is $gf : L \to M$

Let $M$ and $N$ be any string modules over words $w$ and $u$ respectively. Let $\{z_i : i \in I\}$ and $\{y_j : j \in J\}$ be the standard bases of $M$ and $N$ respectively.

Given any non-zero simple string map $f$, there exists $s \in \{-1, +1\}$, $k \in \mathbb{Z}$ and $a, b \in I \cup \{-\infty, +\infty\}$ (with $a + 1 < b$) such that for all elements $\sum_{i \in I} \lambda_i z_i$ of $M(w)$.

$$f(\sum_{i \in I} \lambda_i z_i) = \sum_{i \in I' \cap (a,b)} \lambda_i y_{si+k}$$

Furthermore, any simple string map in $\mathrm{Hom}(M, N)$ is uniquely determined by such an $a, b, s$ and $k$.

Given any simple string map, these elements $a, b \in I \cup \{-\infty, +\infty\}$ define a unique subword $l_{a+2}l_{a+3} \ldots l_{b-2}l_{b-1}$ of $w$. For a couple of examples: If $w$ is a $\mathbb{Z}$-word, $b = +\infty$ and $a \in I$, then the subword is the $\mathbb{N}$-word $l_{a+2}l_{a+3} \ldots$; if $w$ is a finite word, $a = -\infty$, and $b = +\infty$, then the subword is $w$ itself.

One can easily check that $l_{a+1} \in Q_1$ (if $a \in I$): Suppose, for a contradiction, that $l_{a+1} = \alpha^{-1} \in Q_1^{-1}$. Then:

$$f(z_{a+1}) = f(\alpha z_a) = \alpha f(z_a) = \alpha 0 = 0$$

-contradicting the choice of $a$. Similarly, $l_b \in Q_1^{-1}$ (if $b \in I$) and so $l_{a+2}l_{a+3} \ldots l_{b-1}$ is a post-subword of $w$.

Furthermore, if $s = 1$ then $l'_{k+a+2} \dots l'_{k+b-1}$ is a pre-subword of $u$, and:

$$l'_{k+a+2} \dots l'_{k+b-1} = l_{a+2} l_{a+3} \dots l_{b-1}$$

Similarly, if $s = -1$, then $l'_{k-b+1} \dots l'_{k-a-2}$ is a pre-subword of $u$, and:

$$l'_{k-b+1} \dots l'_{k-a-2} = l_{a+2} l_{a+3} \dots l_{b-1}$$

The following lemma follows straight from these conditions:

**Lemma 101.** *Let $M$ and $N$ be any two string modules, with standard bases $\{z_i : i \in I\}$ and $y_j : j \in J\}$ respectively. Let $f, g \in$*Hom$(M, N)$ be any simple string maps, such that $f(z_i) = g(z_i) \neq 0$ for some $i \in I$.

Then $f = g$.

## 5.6 Pure embeddings between string modules

### 5.6.1 Periodic and almost-periodic results

Let $w = \dots l_0 l_1 l_2 \dots$ be any $\mathbb{N}$-word or $\mathbb{Z}$-word such that, for some $s \in \mathbb{Z}$, $l_s$ is direct (if it exists), and $l_{s+1} l_{s+2} l_{s+3} \dots = D^\infty$, for some band $D$ (of length $n$) with inverse last letter.

Let $w' = \dots l'_{-2} l'_{-1} l'_0 l'_1 l'_2 \dots$ be the periodic $\mathbb{Z}$-word, such that $l'_{kn+1} \dots l'_{kn+n} = D$ for all $k \in \mathbb{Z}$.

Let $h_D : M(w) \to M({}^\infty D^\infty)$ denote the simple string map uniquely determined by the post-subword $l_{s+1} l_{s+2} l_{s+3} \dots$ of $w$ and the pre-subword $l'_1 l'_2 l'_3 \dots$ of $w'$.

In [6] the following pure-embeddings were found between string modules over periodic $\mathbb{N}$-words:

**Lemma 102.** *For any contracting periodic or almost periodic $\mathbb{Z}$-word or $\mathbb{N}$-word, $w$, the canonical embedding $M(w) \hookrightarrow \overline{M}(w)$ is pure.*

**Lemma 103.** *Let $w = l_1 l_2 l_3 l_4 \dots$ be any expanding periodic or almost periodic $\mathbb{N}$-word. Let $f \in$* Hom$(M(w), \overline{M}(w))$ be the canonical embedding, and $h_D$ be as defined above. Then the map:

$$(f, h_D) : M(w) \longrightarrow \overline{M}(w) \oplus M({}^{\infty}D^{\infty})$$

-is a pure embedding.

If $w = \ldots l_0 l_1 l_2 \ldots$ is a $\mathbb{Z}$-word such that $l_1$ is inverse, and $l_0^{-1} l_{-1}^{-1} l_{-2}^{-1} \cdots = E^{\infty}$, for some band $E$ with inverse last letter, then we can define, as above, a simple string map $g_E : M(w) \to M({}^{\infty}E^{\infty})$, using the post-subword $u_0^{-1}$ of $w$.

**Lemma 104.** *Let $w = {}^{\infty} E l_1 \ldots l_s D^{\infty}$ be any almost periodic $\mathbb{Z}$-word. Let $f : M(w) \to \overline{M}(w)$ be the natural embedding. Then:*

- *If $w$ is contracting, then $f$ is a pure-embedding.*
- *If $w$ is expanding, then the map:*

$$(f, g_E, h_D)^t : M(w) \longrightarrow \overline{M}(w) \oplus M({}^{\infty}E^{\infty}) \oplus M({}^{\infty}D^{\infty})$$

  *-is a pure embedding.*

- *If $w$ is mixed (i.e. $D^{\infty}$ is expanding and $(E^{-1})^{\infty}$ contracting) then the map:*

$$(f, h_D)^t : M(w) \longrightarrow \overline{M}(w) \oplus M({}^{\infty}D^{\infty})$$

  *-is a pure embedding.*

### 5.6.2 Aperiodic and half-periodic results

Our results from section 5.4 extend Burke's results to all infinite words, $w$:

**Proposition 5.** *Suppose that $w$ is an aperiodic $\mathbb{N}$-word or $\mathbb{Z}$-word. Then the natural embedding $M(w) \hookrightarrow \overline{M}(w)$ is pure.*

*Proof.* Take any $x \in M(w)$, and any pp-formula $\phi(v)$. Pick any troughs $z_k$ and $z_n$ in $w$ (with $k \leq n$) such that $x$ lies in the submodule $M(l_{k+1} \ldots l_n)$ of $M(w)$.

Pick any $\phi \in \mathrm{pp}^{\overline{M}(w)}(x)$, and let $m$ be the number of equations in $\phi$. Since $w$ is aperiodic, ${}^{(m)}(l_{k+1} \ldots l_n)^{(m)}$ is a finite word, and so, by corollary 21, $x \in \phi(M(w))$. $\square$

**Proposition 6.** *Let $w$ be any contracting half-periodic $\mathbb{Z}$-word. Then the natural embedding $M(w) \to \overline{M}(w)$ is pure.*

*Proof.* Let $s \in \mathbb{Z}$ be such that $w_s = D^\infty$ (for some band $D$) and $w_{s-1}$ is not periodic. Take any pp-formula $\phi(v)$, and let $m$ be the number of equations in $\phi$. Pick any $x \in M(w)$ such that $x \in \phi(\overline{M}(w))$.

Pick any trough $z_n$ in $w$ such that $n \leq s$ and $x \in M(w_n)$. By lemma 98:

$$\overline{M}({}^{(m)}w_n) \models \phi(x)$$

Since $u_n^{-1}$ is aperiodic, ${}^{(m)}w_n$ is an $\mathbb{N}$-word- i.e. there exists $k \leq n$ such that ${}^{(m)}w_n$ is the subword $w_k$ of $w$. Of course $w_k = l_{k+1} \dots l_s D^\infty$ is a contracting almost periodic $\mathbb{Z}$-word, and so, by lemma 102, the canonical embedding:

$$M({}^{(m)}w_n) \longrightarrow \overline{M}({}^{(m)}w_n)$$

-is pure, so $x \in \phi(M({}^{(m)}w_n))$. This completes the proof, since $M({}^{(m)}w_n)$ is a submodule of $M(w)$. □

**Proposition 7.** *Let $w = \dots l_{s-1} l_s D^\infty$ be any expanding half periodic $\mathbb{Z}$-word. Then the map:*

$$(f, h_D) : M(w) \to \overline{M}(w) \oplus ({}^\infty D^\infty)$$

*(where $f$ is the canonical embedding, and $h_D$ is as defined above) is a pure embedding.*

*Proof.* Take any pp-formula $\phi(v)$, and let $m$ be the number of equations in $\phi$. Pick any $x \in M(w)$ such that $x \in \phi(\overline{M}(w))$ and $g_D(x) \in \phi(M({}^\infty D^\infty))$.

Pick any trough $z_n$ in $w$ such that $n \leq s$ and $x \in M(w_n)$. By lemma 98:

$$\overline{M}({}^{(m)}w_n) \models \phi(x)$$

Since $u_n^{-1}$ is an aperiodic $\mathbb{N}$-word, there exists $k \leq n$ such that ${}^{(m)}w_n$ is the subword $w_k$ of $w$. Of course $w_k = l_{k+1} \dots l_s D^\infty$ is an expanding almost periodic $\mathbb{Z}$-word, and so, by lemma 103, the map:

$$M({}^{(m)}w_n) \longrightarrow \overline{M}({}^{(m)}w_n) \oplus (M^\infty D^\infty)$$

-is a pure embedding. Thus $x \in \phi(M({}^{(m)}w_n))$- which completes the proof, since $M({}^{(m)}w_n)$ is a submodule of $M(w)$. □

## 5.7 Pp-formulas obtained from finite words

Take any $D \in \mathcal{W}$, and let $a = t(D)$. There is at most one $\gamma \in Q_1$ such that $D\gamma^{-1} \in \mathcal{W}$. The pp-formula $(.D)(v)$ (as defined in [18]) is the pp-formula:

$$(.D)(v) := \begin{cases} (v = e_a v) \wedge v \in D\gamma^{-1}(0) & \text{if such a } \gamma \text{ exists} \\ (v = a_a v) \wedge v \in D(M) & \text{otherwise} \end{cases}$$

Similarly there exists at most one $\alpha \in Q_1$ such that $\alpha D \in \mathcal{W}$. We define:

$$(1.D)(v) := \begin{cases} (.D)(v) \wedge \alpha v = 0 & \text{if such an } \alpha \text{ exists} \\ (.D)(v) & \text{otherwise} \end{cases}$$

Also, there exists at most one $\beta \in Q_1$ such that $\beta^{-1}D \in \mathcal{W}$. We define:

$$({}^{+}1.D)(v) := \begin{cases} (.D)(v) \wedge v \in \beta M & \text{if such a } \beta \text{ exists} \\ v = 0 & \text{otherwise} \end{cases}$$

Too illustrate these pp-formulas, one may look at their free realisations (as described in [18]):

- Let $E$ be the longest possible string of direct letters such that $ED \in \mathcal{W}$. Then $(M(ED), z)$ is a free realisation of $(.D)(v)$ (where $z$ is the standard basis element which lies "in between $E$ and $D$").

- The free realisation of $(1.D)(v)$ is the pointed module $(M(D), z)$, where $z$ is the basis element of $M(D)$ which lies "furthest to the left".

- Let $E$ be a longest possible string of direct letters such that $E\beta^{-1}D$ is a word. Let $z$ be the element of the standard basis of $M(E\beta^{-1}D)$ which lies "between $E\beta^{-1}$ and $D$". Then $(M(E\beta^{-1}D), z)$ is a free realisation of $({}^{+}.D)(v)$.

For example, working over $G_{3,3}$- if $D$ is the word $\alpha\beta^{-1}$, then a free realisation of $(.D)(v)$ is $(M(\alpha^{-1}\alpha^{-1}\beta), z_1)$, where the string module looks like:

$$\begin{tikzcd}
 & & Kz_2 \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\
 & Kz_1 \arrow[dl, "\alpha"'] & & Kz_3 \\
Kz_0 & & &
\end{tikzcd}$$

A free realisation of $(1.D)(v)$ is $(M(\alpha^{-1}\beta), z_0)$:

$$\begin{tikzcd}
& Kz_1 \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\
Kz_0 & & Kz_2
\end{tikzcd}$$

And a free realisation of $({}^{+}1.D)(v)$ is $M(\alpha^{-1}\alpha^{-1}\beta\alpha^{-1}\beta), z_3)$:

$$\begin{tikzcd}
& & Kz_2 \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & & Kz_4 \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\
& Kz_1 \arrow[dl, "\alpha"'] & & Kz_3 & & Kz_5 \\
Kz_0 & & & & &
\end{tikzcd}$$

Given any $C, D \in \mathcal{W}$ such that $C^{-1}D \in \mathcal{W}$, we define:

$$(C^{-1}.D)(v) := (.C)(v) \wedge (.D)(v)$$

The free realisation of $(C^{-1}.D)(v)$ is $(M(C^{-1}D), z)$- where let $z$ denotes the standard basis element of $M(C^{-1}D)$ which lies "in between $C^{-1}$ and $D$": for example, if $C$ is $\beta\alpha^{-2}$ and $D$ is $\beta^{-1}\alpha$, then a free realisation of $(C.D)(v)$ is $(M(\alpha^2\beta^{-2}\alpha), z_3)$, where the string module looks like:

$$\begin{tikzcd}
& & Kz_2 \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & & & \\
& Kz_1 \arrow[dl, "\alpha"'] & & Kz_3 \arrow[dr, "\beta"] & & Kz_5 \arrow[dl, "\alpha"'] \\
Kz_0 & & & & Kz_4 &
\end{tikzcd}$$

### 5.7.1 Links to simple string maps

Throughout this section, $w$ will be any word, and $M$ a string module over $w$, with standard basis $\{z_i : i \in I\}$.

**Lemma 105.** *Let $C = (l'_{m+1} \dots l'_1 l'_0)^{-1}$ and $D = l_1 \dots l_n$ be any finite words such that $C^{-1}D \in \mathcal{W}$. Let $z'_m, z'_{m+1}, \dots, z'_{n-1}, z'_n$ be the standard basis of $M(C^{-1}D)$.*

*Take any $x \in M$, and write it in the form $\sum_{i \in I_0} \lambda_i z_i$- where $\lambda_i \neq 0$ for all $i \in I_0$. Then the following are equivalent:*

1. $\hat{u}_i \geq C$ *and* $\hat{w}_i \geq D$ *for all* $i \in I_0$.

2. *For all* $i \in I_0$*, there exists a simple string map* $M(C^{-1}D) \to M$ *taking* $z'_0$ *to* $z_i$

3. *There exists* $f \in Hom(M(C^{-1}D), M)$*, which is a* $K$*-linear combination of simple string maps, taking* $z'_0$ *to* $x$.

4. $x \in (C^{-1}.D)(M)$.

*Proof.* Clearly (3) implies (4), as $(M(l'_{m+1} \dots l'_n), z'_0)$ is a free realisation of $(C.D)(v)$. We shall prove that (4) implies (1), (1) implies (2), and (2) implies (3).

Assume that (4) holds, and suppose, for a contradiction, that there exists $i \in I_0$ such that (without loss of generality) $\hat{w}_i < D$. We may assume (without loss of generality) that $\hat{w}_i = w_i$. Let $k \leq m$ be maximal such that $l_{i+1} \dots l_{i+k} = l'_1 \dots l'_k$.

Suppose, first of all, that $k = n$. Since $w_i < D$, $l_{i+k+1}$ must be an inverse letter- say $\alpha^{-1}$. Then $D\alpha^{-1} = l_{i+1} \dots l_{i+k} l_{i+k+1} \in \mathcal{W}$, so $(.D)(v)$ is the pp-formula $v \in D\alpha^{-1}(0)$ (by definition). But if $x \in (.D)(M)$, then:

$$M \models x \in l_{i+1} \dots l_{i+k} l_{i+k+1}(0)$$

Since $\lambda_i \neq 0$, this contradicts corollary 20.

Now, if $k < n$, then $l'_{k+1}$ must be a direct letter- say $\beta$ (since $w_i < D$). Then:

$$x \in (C.D)(M) \subseteq (.D)(M) \subseteq DM \subseteq l'_1 \dots l'_k \beta(M)$$

Pick any $y \in M$ such that $x \in l'_1 \dots l'_k(y)$ and $y \in \beta M$. By corollary 20, $y$ must have $z_{i+k}$-coefficient $\lambda_i$. Since $l_{i+k+1} \neq \beta$ (by our choice of $k$), lemma 91 gives that $l_{i+k} = \beta^{-1}$. Then $k \not\geq 1$- since that would imply that $\beta^{-1}\beta = l_{i+k} l'_{k+1} = l'_k l'_{k+1} \in \mathcal{W}$- so $k = 0$.

But if $k = 0$, then $l_i^{-1} \in H_{-1}(a)$ (since $u_i \in H_{-1}(S)$) and $\beta \in H_1(a)$ (since $D \in H_1(a)$)- giving our required contradiction.

Now, assume that (1) holds. Take any $i \in I_0$. Assume without loss of generality that $w_i \in H_1(a)$. Let $j$ and $k$ be maximal such that $l'_1 \dots l'_k = l_{i+1} \dots l_{i+k}$ and $l'_{-j} \dots l'_0 = l_{i-j} \dots l_i$.

Then $l'_{-j} \dots l'_k$ is a post-subword of $C^{-1}D$ and $l_{i-j} \dots l_{i+k}$ is a pre-subword of $w$. Consider the map:

$$M(C^{-1}D) \twoheadrightarrow M(l'_{-j} \dots l'_k) \to M(l_{i-j} \dots l_{i+k}) \hookrightarrow M$$

-where the first map is the natural projection onto the quotient module, the third map is the natural embedding of the submodule, and the second map is isomorphism as in (5.2.2). This map clearly takes $z'_0$ to $z_i$.

Finally, assume that (2) holds. For each $i \in I_0$, let $f_i$ be the simple string map such that $f_i(z'_0) = z_i$. Let $f = \sum_{i \in I_0} \lambda_i f_i$- one can easily verify that $f$ is a well defined homomorphism: For example, if $M$ is $M(w)$, then $I_0$ must be finite, and so, for all $y \in M(C^{-1}D)$, $\sum_{i \in I_0} f_i(y)$ is a $K$-linear combination of finitely many $z_j$- and hence is a well defined element of $M(w)$. □

**Corollary 26.** *Let $\phi(v)$ be any pp-formula of the form $(C.D)(v)$, $(.D)(v)$ $(1.D)$, or $(^{+}1.D)(v)$.*

*Take any element $x = \sum_{i \in I} \lambda_i z_i$ of $M$. Then $x \in \phi(M)$ if and only if $z_i \in \phi(M)$ for all $i \in I$ such that $\lambda_i \neq 0$.*

*Proof.* If $\phi(v)$ is $(C.D)(v)$, then this follows straight from lemma 105

By considering their free realisations, it is easy to see that any pp-formula of the form $(.D)(v)$, $(1.D)(v)$ or $(^{+}1.D)(v)$ is equivalent to one of the form $(C.D)(v)$- the result follows. □

**Lemma 106.** *Take any $a \in Q_0, s \in \{-1, +1\}$, and $C, D \in H_s(a)$. Then $(.D) \to (.C)$ if and only if $C \leq D$.*

*Proof.* Recall the free realisations of $(.C)(v)$ and $(.D)(v)$, as described above: Let $E_C$ and $E_D$ be the longest possible strings of direct letters such that $E_C C \in \mathcal{W}$ and $E_D D \in \mathcal{W}$. Let $z$ and $y$ denote the standard basis elements of $M(E_C C)$ and $M(E_D D)$ such that $(M(E_C C), z)$ (respectively, $M(E_D D), y)$) is a free realisation of $(.C)(v)$ (respectively $(.D)(v)$).

If $(.D) \to (.C)$, then $y \in (.D)(M(E_D D)) \subseteq (.C)(M(E_D D))$, and so $C \leq D$ by lemma 105.

Conversely, suppose that $C \leq D$. Then $E_D C \in \mathcal{W}$: to see this, let $F$ be the longest possible initial subword of $C$ which contains only direct letters. Since $C \leq D$, $F$ must also be an initial subword of $D$ ,and so $E_D F \in \mathcal{W}$ (since it is a subword of $E_D D$). Thus $E_D F$ doesn't lie in the ideal $\mathcal{I}$ of $KQ$. It follows, by definition, that $E_D C$ is a word.

Consequently, $E_D^{-1}$ is an initial subword of $E_C^{-1}$, and hence $E_D^{-1} \leq E_C^{-1}$. Thus, by lemma 105, there exists a map from $M(E_C C)$ to $M(E_D D)$ taking $z$ to $y$. □

Notice that, given any finite word, $D$, we can express $DM$ in terms of the pp-formulas of the form $(.C)(v)$: for example, if the first letter of $D$ is direct, then $DM = (.DE)(M)$, where $E$ is the longest possible string of inverse letters such that $DE \in \mathcal{W}$.

The following corollary follows straight from this fact, and corollary 26:

**Corollary 27.** *Let $x = \sum_{i \in I} \lambda_i z_i$ be any element of a string module $M$ over a word $w$ (where $\lambda_i \neq 0$ for all $i \in I$). Then, for any finite word $D$:*

$$x \in DM \iff z_i \in DM \text{ for all } i \in I$$

### 5.7.2 Homomorphisms between string modules

**Lemma 107.** *Given any words $w$ and $w'$, any homomorphism from $M(w')$ to $M(w)$ is a $K$-linear combination of simple string maps.*

*Proof.* See [10]. □

Our results from the last section give a slight extension of this:

**Lemma 108.** *Let $D = l_1 \dots l'_k$ be any finite word, and $w$ any word. Then, for any string module $M$ over $w$, any $f \in$* $\mathrm{Hom}(M(D), M)$ is a $K$-linear combination of simple string maps.

*Proof.* We prove the result by induction on $k$: assume we have the result for all $n < k$, and take any word $l'_1 \dots l'_k$ of length $k$.

Let $z'_0, z'_1, \dots, z'_k$ be a standard basis of $M(l'_1 \dots l'_k)$. Pick $m \le k$ to be maximal such that $z'_m$ is a peak. Of course, $l'_{m+i}$ is inverse for all $i$ such that $1 \le i \le k-m$. Say, $l_{m+i} = \alpha_i^{-1} \in Q_1^{-1}$ for all such $i$.

Let $C = (l'_1 \dots l'_m)^{-1}$ and $D = l'_{m+1} \dots l'_k$. Then $z'_m \in (C^{-1}.D)(M(l'_1 \dots l'_k))$, so $f(z'_m) \in (C^{-1}.D)(M)$. By lemma 105, there exists $g \in \mathrm{Hom}(M(l'_1 \dots l'_k), M)$- which is a $K$-linear combination of simple string maps- such that $g(z'_m) = f(z'_m)$.

Then $(f-g)(z'_m) = 0$, and furthermore:

$$(f-g)(z'_{m+i}) = (f-g)(\alpha_i \dots \alpha_1 z_m) = \alpha_i \dots \alpha_1 (f-g)(z_m) = 0$$

-for all $i \ge 1$. Consequently, $(f-g)$ factors through the canonical projection $\pi : M(l'_1 \dots l'_k) \twoheadrightarrow M(l'_1 \dots l'_{m-1})$:

$$\begin{tikzcd} M(l'_1 \dots l'_k) \arrow[r, "\pi"] \arrow[dr, "f-g"'] & M(l'_1 \dots l'_{m-1}) \arrow[d, "\exists h"] \\ & M(w) \end{tikzcd}$$

By induction, $h$ is a $K$-linear combination of simple string maps- and hence so is $h\pi$. Thus so is $f = g + h\pi$. $\square$

Let $M$ be any string module over a word $w$, and let $\{z_i : i \in I\}$ be its standard basis. Let $N$ be any string module over a pre-subword $u$ of $w$- of course, it has standard basis $\{z_i : i \in I'\}$, for some $I' \subseteq I$.

Given any simple string map $f : N \to M$, we say that $f$ is a *right shift* if, for all $i$, $f(z_i)$ is either zero, or $z_j$, for some $j > i$. Similarly, we say that $f$ is a *left shift* if, for all $i$, $f(z_i)$ is either zero, or $z_j$, for some $j < i$.

**Lemma 109.** *Let $w, u, M, N, I$ and $I'$ be as above. Then, for any simple string map $f : N \to M$, exactly one of the following holds:*

- *$f$ is a right shift.*
- *$f$ is a left shift.*
- *$f(z_i) = z_i$ for all $i \in I'$ (i.e. $f$ is the canonical embedding of the submodule, as in lemma 94).*

*Proof.* First of all, if $f(z_i) = z_i$ for any $i \in I'$, then by lemma 101, $f$ must be the canonical embedding (as defined after lemma 94).

Assume from now on, that $f(z_i) \neq z_i$ for all $i$. Recall that there exists $a, b \in I' \cup \{-\infty, \infty\}$ and $k \in \mathbb{Z}$ such that either:

$$f(z_i) = \begin{cases} z_{k+i} & \text{if } a < i < b \\ 0 & \text{otherwise} \end{cases}$$

-or:

$$f(z_i) = \begin{cases} z_{k-i} & \text{if } a < i < b \\ 0 & \text{otherwise} \end{cases}$$

In the first case, $k$ must be non-zero. Then $f$ is a right shift if $k > 0$, and a left shift if $k < 0$.

Assume, that we have the second case. Suppose, for a contradiction, that there exists $i, j \in I'$ such that $a < i < j < b$ and $k - i > k - j$. Let $m = \max\{i \in I' : i < k - i\}$. Then $m + 1 < b$ (otherwise $k - i > i$ for all $i \in (a, b)$, contradicting our assumption), and so $f(z_{m+1}) = z_{k-(m+1)}$.

Of course, $m + 1 \geq k - (m + 1)$, by our choice of $m$, and so $m + 1 > k - (m + 1)$ (since $f(z_{m+1}) \neq z_{m+1}$), and so $k - m = m + 1$ and $k - (m + 1) = m$. Assume, without loss of generality, that $l_{m+1}$ is direct- say $l_{m+1} = \alpha$. Then:

$$\alpha z_m = \alpha f(z_{m+1}) = f(\alpha z_{m+1}) = f(z_m) = z_{m+1}$$

-and so $l_{m+1}$ must be $\alpha^{-1}$- which is clearly a contradiction. □

### 5.7.3 Simple string endomorphisms

Let $M$ be any string module over a word $w$, with standard basis $\{z_i : i \in I\}$. We define the binary relation $\leq_w$ on the set $\{z_i : i \in I\}$ by:

$$z_i \leq_w z_j \iff \hat{w}_i \leq \hat{w}_j \text{ and } \hat{u}_i \leq \hat{u}_j$$

If $w$ is not a periodic $\mathbb{Z}$-word, then, by lemma 89, there are no distinct $i, j \in I$ such that both $\hat{u}_i = \hat{u}_j$ and $\hat{w}_i = \hat{w}_j$- and so $\leq_w$ is a well defined partial ordering of $\{z_i : i \in I\}$.

**Lemma 110.** *Let $w$ be any word, $f \in \mathrm{End}(M(w))$ any simple string map, and $i \in I$ such that $f(z_i) \neq 0$. Then $f(z_i) \geq_w z_i$.*

*Furthermore, if $w$ is not a periodic $\mathbb{Z}$-word and $f$ is not the identity, then $f(z_i) >_w z_i$.*

*Proof.* For all finite words $D \leq \hat{w}_i$, $z_i \in (.D)(M)$, and so $z_j = f(z_i) \in (.D)(M)$, so $D \leq \hat{w}_j$. It follows that $\hat{w}_i \leq \hat{w}_j$, and similarly, $\hat{u}_i \leq \hat{w}_j$.

To prove the second assertion, lemma 109 gives that $f(z_i) = z_j$ for some $j \neq i$, and either $\hat{w}_i \neq \hat{w}_j$ or $\hat{u}_i \neq \hat{u}_j$ (by lemma 89). Thus $z_j > z_i$, as required. □

## 5.8 1-Sided Modules

Given any $M \in A$-Mod, any $a \in Q_1$, and any non-zero $m \in e_a M$, the set $\{D \in H_1(a) : D \text{ is finite, and } m \in (.D)(M)\}$ is downwards-closed (by lemma 106). We define the *right-word of $m$ in $M$* to be the supremum of it.

Similarly, we define the *left-word of $m$ in $M$* to be the supremum of $\{C \in H_{-1}(a) : C \text{ is finite, and } m \in (.C)(M)\}$.

**Lemma 111.** *Let $M$ be any string module over a word $w$, with standard basis $\{z_i : i \in I\}$. Then, for all $i \in I$, $z_i$ has right-word $\hat{w}_i$, and left word $\hat{u}_i$.*

*Proof.* Follows straight from lemma 105. □

**Lemma 112.** *Take any pure-injective $M \in A$-Mod, and any $m_0 \in M$. Let $w = l_1 l_2 l_3 \ldots$ be the right word of $m_0$ in $M$.*

*If $m_0 \in (1.D)(M)$ for some $D \leq w$, then there exists $f \in Hom_A(M(w), M)$ such that $f(z_0) = m_0$ (where $\{z_i : i \in \mathbb{N}\}$ is the standard basis of $M(w)$).*

*Proof.* Let $r \in A$ denote the unique $\alpha \in Q_1$ such that $\alpha^{-1} \in H_{-1}(a)$ (if such an $\alpha$ exists, and 0 otherwise). Since $m_0 \in (1.D)(M)$, $rm_0 = 0$.

Consequently, it will be enough to find a set $\{m_i : i \in \mathbb{N}^+\}$ such that $l_i m_i = m_{i-1}$ for all $i \in \mathbb{N}^+$: given such a set, we let $f$ be the unique map such that $f : z_i \mapsto m_i$ for all $i \in \mathbb{N}$.

Since $M$ is pure-injective, and hence algebraically compact, it will be enough to show that the set:

$$\{l_1 v_1 = m_0\} \cup \bigcup_{i \geq 2} l_i v_i = v_{i-1}\}$$

-is finitely satisfiable in $M$. Given any finite subset $X$ of it, pick any trough $z_k$ of $w$ such that no equation of the form $l_i v_i = v_{i-1}$ with $i > k$ lies in $X$. Let $C = l_1 \dots l_k$. Since $C < w$:

$$M \models (.C)(m_0)$$

(by the definition of right-word). Consequently there exists $m_1, \dots, m_k \in M$ such that $l_i m_i = m_{i-1}$ for all $i \leq k$- as required. □

A module $M \in A$-Mod is said to be *one-directed* (as defined in [18]) if there exists a finite word $D$ such that the pp-pair $(1.D)/({}^{+}1.D)$ is open on $M$. $M$ is said to be *two-directed* if it is not one-directed.

**Lemma 113.** *Any $M \in A$-Mod is two-directed if and only if, for all $a \in Q_1$ and $m \in e_a M$, both the right-word and the left word of $m$ in $M$ are $\mathbb{N}$-words.*

*Proof.* See [18] □

Let $M$ be any string module over a word, $w$, with standard basis $\{z_i : i \in I\}$. Take any $a \in Q_0$, and $x \in e_a M$, and write $x$ as $\sum_{i \in I_0} \lambda_i z_i$- where $\lambda_i \neq 0$ for all $i \in I_0$. It follows from lemma 105, that $x$ has right-word $\inf\{\hat{w}_i : i \in I_0\}$, and left-word $\inf\{\hat{u}_i : i \in I_0\}$.

Consequently, every string module over a finite word or $\mathbb{N}$-word is one-directed. Furthermore, every string module over a $\mathbb{Z}$-word is two-directed: Any element $x = \sum_{i \in I_0} \lambda_i z_i$ of $M$ has right-word $\sup\{\hat{w}_i : i \in I_0\}$, which is an $\mathbb{N}$-word, by lemma 87, and similarly, the left word is an $\mathbb{N}$-word, so the above lemma implies that $M$ is two-directed.

**Lemma 114.** *Take any $a \in Q_0$, $s \in \{-1, +1\}$, and finite word $D \in H_s(a)$. Let $\phi(v)$ be any pp-formula such that:*

$$(1.D) < \phi < ({}^{+}1.D)$$

*Then there exists $E > D$ such that $\phi$ is equivalent to $({}^{+}1.D) + (1.E)$.*

*Proof.* See [18, (4.4)] $\square$

Let $M$ be a one-directed module. Take any $m_0 \in (1.D)(M) \backslash ({}^{+}1.D)(M)$ (where $D$ is a finite word). Assume without loss of generality that $D \in H_1(a)$ for some $a \in Q_1$. Let $w$ be the right word of $m_0$ in $M$. We say that $m_0$ is *homogeneous* in $M$ if, for all $D \leq w$ and $E \in H_1(a)$:

$$m_0 \in (({}^{+}1.D) + (1.E))(M) \Longleftrightarrow E \leq w$$

**Lemma 115.** *Take any $a \in Q_0$ and $w \in H_1(a)$. Let $M$ be either $M(w)$ or $\overline{M}(w)$, with standard basis $\{z_i : i \in I\}$. Then $z_0$ is homogeneous in $M$, with right word $w$.*

*Proof.* We only need to prove that $z_0$ is homogeneous: Suppose, for a contradiction, that $z_0 \in ({}^{+}1.D)+(1.E)$ for some $D \leq w$ and $E > w$. Then there exists $x \in (1.E)(M)$ such that $z_0 - x \in ({}^{+}1.D)$. Since $z_0 \notin (1.E)(M)$, $x$ has $z_0$-coefficient 0- by corollary 26.

Recall (from the definition) that $({}^{+}1.D)(M)$ is either $(.D)(M) \cap \beta M$ (if there exists $\beta \in Q_1 \cap H_{-1}(a)$) or $v = 0$ (if not). Of course, $z_0 \neq 0$. Furthermore, $z_0$ has left-word $1_{a,-1}$, and hence $z_0 \notin \beta M \subseteq ({}^{+}1.D)(M)$. Since $z_0 - x$ has $z_0$-coefficient 1, it follows from corollary 26 that $z_0 - x \notin ({}^{+}1.D)(M)$- giving our required contradiction. $\square$

**Theorem 40.** *Given any $a \in Q_0$, and $w \in H_{+1}(a)$ (respectively, $H_{-1}(a)$), there exists a unique (up to isomorphism) pure-injective indecomposable one-directed $M_w \in$ $A$-*Mod *containing a homogeneous element $m_0$ with right-word (respectively, left-word) $w$.*

*Furthermore, every indecomposable pure-injective one-directed module is isomorphic to $M_w$, for some finite word or* $\mathbb{N}$-*word, $w$.*

*Proof.* See [18, (5.4)] $\square$

Notice that, if $w$ is a finite word, then $M(w)$ is one-directed, pure-injective, and indecomposable, and $z_0$ satisfies the conditions required of $m_0$ in theorem 40. Thus $M_w \cong M(w)$. Also, we have the following result for $\mathbb{N}$-words:

**Corollary 28.** *Let $w$ be any $\mathbb{N}$-word, such that $M(w)$ is pure-injective. Then $\overline{M}(w)$ is not indecomposable.*

*Proof.* First of all, notice that $z_0$ is a homogeneous element of both $M(w)$ and $\overline{M}(w)$, with right-word (or left-word) $w$- by lemma 115. Consequently, if $\overline{M}(w)$ was indecomposable, then both $M(w)$ and $\overline{M}(w)$ would be indecomposable and pure-injective (by theorem 39 and proposition 4), and so $M(w)$ and $\overline{M}(w)$ would be isomorphic (by theorem 40).

Since $M(w)$ is of countable dimension over $K$, and $\overline{M}(w)$ of uncountable dimension, they cannot, however, be isomorphic. $\square$

By lemma 112, there exists $f \in \mathrm{Hom}(M(w), M_w)$ such that $f(z_0) = m_0$ (where $\{z_i : i \in I\}$ is the standard basis of $M(w)$). We shall prove that, if $w$ is an aperiodic $\mathbb{N}$-word, then $f$ is a pure embedding.

In fact, we will prove a slightly more general result- which will be needed for some of the proofs in chapter 7. First of all, we will need the following result:

**Lemma 116.** *Let $w = l_1 l_2 \dots$ be any $\mathbb{N}$-word, and $M$ be either $M(w)$ or $\overline{M}(w)$. Let $k > 0$ be such that $z_k$ is a trough in $w$, and let $i, j \in \mathbb{N}$ be such that:*

$$z_j \in ((l_1 \dots l_i).(l_{i+1} \dots l_k))(M)$$

*If $l_{i+1} \dots l_k$ is not an initial subword of either $w_j$ or $u_j$, then there exists a simple string map $h \in \mathrm{End}(M)$ such that $h(z_i) = z_j$, and $h(z_n) = 0$ for all $n \geq k$.*

*Proof.* Assume, without loss of generality, that $l_{i+1} \dots l_k \in H_1(a)$ for some $a \in Q_0$. Since $z_j \in ((l_1 \dots l_i).(l_{i+1} \dots l_k))(M)$, lemma 105 implies that $l_{i+1} \dots l_k \leq \hat{w}_j$ and $(l_1 \dots l_i)^{-1} \leq \hat{u}_i$.

Since $(l_{i+1} \dots l_k)$ is not an initial subword of $\hat{w}_j$, lemma 85 gives that $\hat{w}_j \geq l_{i+1} \dots l_{k-1}$, and so:

$$z_j \in ((l_1 \dots l_i).(l_{i+1} \dots l_{k-1}))(M)$$

Let $y_0, \dots y_{k-1}$ be the standard basis of $M(l_1 \dots l_{k-1})$. By lemma 105, there exists a simple string map $h : M(l_1 \dots l_{k-1}) \to M(w)$ taking $y_i$ to $z_j$.

Since $z_k$ is a trough, $l_1 \dots l_{k-1}$ is a post-subword of $w$. Let $\pi$ be the canonical projection $M(w) \twoheadrightarrow M(l_1 \dots l_{k-1})$. Then $h\pi$ is a simple string map, and satisfies the required conditions. $\square$

### 5.8.1 1-Sided Modules over Aperiodic Words

**Lemma 117.** *Let $w$ be an aperiodic $\mathbb{N}$-word (without loss of generality, $w \in H_1(a)$ for some $a \in Q_0$). Let $M$ be a one-directed module, containing a homogeneous element $m_0$ with right word $w$, such that $m_0 \in (1.D)(M) \backslash ({}^{+}1.D)(M)$ for some $D \leq w$.*

*Then* $\mathrm{pp}^{M(w)}(z_0) = \mathrm{pp}^{M}(m_0)$.

*Proof.* Of course, the map in lemma 112 gives that $\mathrm{pp}^{M(w)}(z_0) \subseteq \mathrm{pp}^{M}(m_0)$.

To show the converse, take any $\phi(v) \in \mathrm{pp}^{M}(m_0)$. We must prove that $z_0 \in \phi(M(w))$. Let $m \in \mathbb{N}$ be the number of equations in $\phi$. Since $w$ is aperiodic, the subword $z_0^{(m)}$ is finite, so we may pick $k \in \mathbb{N}$ such that $z_k$ is a trough, and $z_0^{(m+)}$ is an initial subword of $l_1 l_2 \dots l_{k-1}$.

Pick any $k' \in \mathbb{N}$ large enough such that:

- For every $i \in \{2, 3, \dots, k\}$, $l_1 \dots l_{k'}$ is not an initial subword of $w_i$ or $u_i$
- $z_{k+k'}$ is a trough in $w$.

Let $D$ denote $l_1 \dots l_{k+k'}$. Since $z_{k+k'}$ is a trough, $D < w$. Define $\psi(v)$ to be $\phi(v) \wedge (1.D)(v)$. Of course, $(1.D) \geq \psi + ({}^{+}1.D) \geq ({}^{+}1.D)$. Furthermore, $\psi + ({}^{+}1.D) > ({}^{+}1.D)$- since $m_0 \in \psi(M) \backslash ({}^{+}1.D)(M)$. By lemma 114, there exists $E > D$ such that $\psi + ({}^{+}1.D)$ is equivalent to $(1.E) + ({}^{+}1.D)$. Of course:

$$m_0 \in \psi(M) \subseteq ((1.E) + ({}^{+}1.D))(M)$$

Since $m_0$ is homogeneous in $M$, $E < w$- and so $M(w) \models (1.E)(z_0)$. Thus:

$$z_0 \in ((1.E) + ({}^{+}1.D))(M(w)) = (\psi + ({}^{+}1.D))(M(w))$$

Pick any $x \in M(w)$ such that:

$$M(w) \models \psi(z_0 - x) \wedge ({}^{+}1.D)(x)$$

We can write $x$ uniquely as $\sum_{i\in I}\lambda_i z_i$ (where $\lambda_i \neq 0$ for all $i\in I$). By corollary 26, $z_i \in ({}^{+}1.D)(M(w))$ for all $i\in I$. In particular, $0\notin I$. Now, partition $I$ into $I_L = I\cap\{1,\dots k-1\}$ and $I_R = I\cap\{i\in\mathbb{N} : i\geq k\}$. Let $x_L = \sum_{i\in I_L}\lambda_i z_i$ and $x_R = \sum_{i\in I_R}\lambda_i z_i$.

Given any $i\in I_L$, $z_i\in({}^{+}1.D)(M(w))\subseteq(1.D)(M(w))$. By our choice of $k'$, $D$ is not an initial subword of $\hat{w}_i$ so by lemma 116, there exists a simple string map $g_i\in \mathrm{End}(M(w))$ taking $z_0$ to $z_i$ , such that $g_i(z_j) = 0$ for all $j\geq k+k'$.

Notice that $g_i$ is a right shift (by lemma 109, since $g_i(z_0) = z_i$). Let $g = \sum_{i\in I_L}\lambda_i z_i$ (note that $g(z_0) = x_L$). Then $g(z_j)\in sp_K\{z_n : n > j\}$ for all $j\in\mathbb{N}$, and $g(z_j) = 0$ for all $j\geq k+k'$, so $g^{k+k'} = 0$. So:

$$(\sum_{n=0}^{k+k'} g^n)(z_0 - x_L) = (\sum_{n=0}^{k+k'} g^n)(1-g)(z_0) = (1-g^{k+k'})(z_0) = z_0$$

Now, let $\pi : M(w)\twoheadrightarrow M(z_0^{(m+)})$ denote the canonical projection. Since $z_0^{(m+)}$ is an initial subword of $w_k$, it follows that $\pi(x_R) = 0$, and hence that $\pi_k(g^n(x_R)) = 0$ for all $n\geq 0$. Thus:

$$\pi_{k-1}(\sum_{n=0}^{k+k'} g^n)(z_0 - x_L - x_R) = \pi_{k+k'-1}(z_0) = \pi(z_0)$$

Since $M(w)\models\psi(z_0 - x_L - x_R)$, we have that:

$$\pi(z_0)\in\psi(M(z_0^{(m+)})\subseteq\phi(M(z_0^{(m+)}))$$

And hence $z_0\in\phi(M(w))$, by corollary 23 $\square$

**Proposition 8.** *Let $w, a, M$ and $m_0$ be as in lemma 117. Then any map $f : M(w)\to M$ taking $z_0$ to $m_0$ is a pure-embedding.*

*Proof.* Take any $x\in M(w)$, and any $\phi\in\mathrm{pp}^M(f(x))$. We must show that $x\in\phi(M(w))$.

Write $x$ as $\sum_{i\in I}\lambda_i z_i$, where $\lambda_i\neq 0$ for all $i\in I$. Let $m$ be the number of equations in $\phi$. Pick any $d\geq\max\{i : i\in I\}$ such that $z_d$ is a trough. Then $x$ lies in the submodule $M(l_1\dots l_d)$ of $M(w)$, so- by corollary 23- it will be enough to prove that $\pi(x) \in \phi(M((l_1 \dots l_d)^{(m+)}))$- where $\pi : M(w) \twoheadrightarrow M((l_1 \dots l_d)^{(m+)})$ is the canonical projection.

Since $w$ is aperiodic, we can pick $k \in \mathbb{N}$ such that:

- $(l_1 \dots l_d)^{(m+)}$ is an initial subword of $l_1 \dots l_{k-1}$
- For all distinct $i, j \leq d$, $l_{i+1} \dots l_k$ is not an initial subword of $u_j$ or $w_j$
- $z_k$ is a trough.

Now pick any $k' \geq 0$ large enough such that:

- for all $i, i' \leq k$ such that $i \neq i'$, $l_i \dots l_{i+k'} \neq l_{i'} \dots l_{i'+k'}$.
- $z_{k+k'}$ is a trough in $w$.

Let $D = l_1 \dots l_{k+k'}$, and let $\psi(v_0)$ be the pp-formula:

$$\exists v_1, \dots, v_{k+k'} \left( \chi(v_0, v_1, \dots, v_{k+k'}) \wedge \phi(\sum_{i \in I} \lambda_i v_i) \right)$$

-where $\chi(v_0, v_1, \dots, v_{k+k'})$ is a pp-formula generating $\mathrm{pp}^{M(D)}(z_0, z_1, \dots, z_{k+k'})$.

Of course, $m_0 \in \psi(M)$ (we could take $f(z_1), f(z_2), \dots f(z_{k+k'})$ to be witnesses for $v_1, v_2, \dots, v_{k+k'}$). Thus, by lemma 117, $z_0 \in \psi(M(w))$- i.e. there exists $x_1, \dots x_{k+k'}$ in $M(w)$ such that:

$$M(w) \models \chi(z_0, x_1, x_2, \dots, x_{k+k'}) \wedge \phi(\lambda_0 z_0 + \sum_{i \in I \setminus \{0\}} \lambda_i x_i)$$

By the definition of $\chi$, there exists $f \in \mathrm{Hom}(M(D), M(w))$ taking $z_0$ to $z_0$, and $z_i$ to $x_i$ for all $i \in \{1, 2, \dots, k+k'\}$.

We shall construct a map $h' \in \mathrm{End}(M(w))$ such that $h'(z_0 + \sum_{i \in I \setminus \{0\}} \lambda_i x_i) = \sum_{i \in I} \lambda_i z_i + x'$- where $x'$ lies in the submodule $M(w_k)$ of $M(w)$. This will be enough to complete the proof, since it will imply that $\pi(x') = 0$ (by our choice of $k$), and hence that:

$$\pi(\sum_{i \in I} \lambda_i z_i) = \pi h'(\lambda_0 z_0 + \sum_{i \in I \setminus \{0\}} \lambda_i x_i) \in M((l_1 \dots l_d)^{(m+)})$$

Since $z_{k+k'}$ is a trough, $M(D)$ is a submodule of $M(w)$. Let $\rho : M(D) \hookrightarrow M(w)$ denote the canonical embedding. By lemma 108, $f - \rho$ is a $K$-linear combination of finitely many simple string maps:

$$f - \rho = \sum_{j \in J} \mu_j f_j$$

Since $(f - \rho)(z_0) = 0$, each such map is either a left shift or a right shift. Let $J_0$ be the set of all $j \in J$ such that $f_j(z_i) \neq 0$ for some $i \leq k + k'$. Of course, for all $i \leq k + k'$:

$$\sum_{j \in J_0} \mu_j f_j(z_i) = \sum_{j \in J} \mu_j f_j(z_i) = x_i - z_i$$

Partition $J_0$ into $J_L \cup J_R \cup J_{RR}$, where:

- $j \in J_L$ if and only if it is a left shift.
- $j \in J_R$ if and only if it is a right shift and $f_j(z_i) \in \{z_{i+1}, \ldots, z_k\}$ for some $i \leq d$.
- $j \in J_R$ if and only if it is a right shift and $f_j(z_i) \notin \{z_0, z_1, \ldots, z_k\}$ for all $i \leq d$.

Given any $j \in J_L$, pick any $i \leq d$ such that $f_j(z_i) = z_{i'}$ for some $i' \leq i$. Then $z_{i'} \in ((l_1 \ldots l_i).(l_{i+1} \ldots l_{k+k'}))(M(w))$ and $l_{i+1} \ldots l_k$ is not an initial subword of $w_{i'}$ or $u'_i$ (by our choice of $k$).

By lemma 116, there exists a simple string map $h_j : M(w) \to M(w)$ such that $h(z_i) = z_{i'}$, and $h_j(z_n) = 0$ for all $n \geq k$. By lemma 101), $h_j \rho = g_j$.

Similarly, given any $j \in J_R$, pick any $i \leq d$ such that $f_j(z_i) = z_{i'}$ for some $i' \leq k$. Then $z_{i'} \in ((l_1 \ldots l_i).(l_{i+1} \ldots l_{k+k'}))(M(w))$ and $l_{i+1} \ldots l_{k+k'}$ is not an initial subword of $w_{i'}$ or $u_{i'}$ (by our choice of $k'$), so there exists (by lemma 116) a simple string map $h_j : M(w) \to M(w)$ such that $h(z_i) = z_{i'}$, and $h_j(z_n) = 0$ for all $n \geq k$. By lemma 101 $h_j \rho = g_j$.

Define $h \in \mathrm{End}(M(w))$ to be the map:

$$h = \sum_{j \in J_L \cup J_R} \mu_j h_j$$

First, we claim that $h^{(k+k'+1)} = 0$: suppose, for a contradiction, that $h^{(k+k'+1)}(z_i) \neq 0$ for some $i$- so there exists $j_1, j_2, \ldots, j_{k+k'+1} \in J_L \cup J_R$ such that $h_{j_{k+k'+1}} \ldots h_{j_2} h_{j_1}(z_i) \neq 0$.

Since $h_{j_{n+1}}h_{j_n}\dots h_{j_1}(z_i)\neq 0$, it follows that $h_{j_n}\dots h_{j_2}h_{j_1}(z_i)\in\{z_0,z_1,\dots z_{k+k'-1}\}$ for all $n<k+k'$. And so:

$$z_i <_w h_{j_1}(z_i) <_w h_{j_2}h_{j_1}(z_i) <_w \cdots <_w h_{j_2}h_{j_1}(z_i)$$

-giving us $k+k'+1$ distinct elements of $\{z_0,z_1,\dots z_{k+k'-1}\}$- which is clearly a contradiction.

Now, given any $i\leq k$, let $x_i'=h(z_i)$ and $x_i''=\sum_{j\in J_{RR}}\mu_j f_j(z_i)$. Of course, $z_i+x_i'+x_i''=x_i$. Notice that $x_i''\in\mathrm{sp}_K\{z_k,z_{k+1},z_{k+2},\dots\}$. Now:

$$(\sum_{n=0}^{k+k'}(-1)^n h^n)(x_i'+z_i)=(\sum_{n=0}^{k+k'}(-1)^n h^n)(h+1)(z_i)=z_i$$

Notice that, given any $n\geq k$, $h_j(z_n)=0$ for all $j\in J_L$, and that $h_j(z_n)\in sp_K\{z_k,z_{k+1},z_{k+2},\dots\}$ for all $j\in J_R$. It follows that, for all $n\in\mathbb{N}$, $h^n(x_i'')\in\mathrm{sp}_K\{z_k,z_{k+1},z_{k+2},\dots\}$, and hence that:

$$\pi(\sum_{n=0}^{k+k'}(-1)^n h^n)(x_i'')=0$$

Define $h'=\sum_{n=0}^{k+k'}(-1)^n h^n$. Then, for all $x\leq d$:

$$\begin{aligned}\pi h'(x_i) &= \pi(\sum_{n=0}^{k+k'}(-1)^n h^n)(z_i+x_i')+\pi(\sum_{n=0}^{k+k'}(-1)^n h^n)(x_i'')\\ &= \pi(z_i)\end{aligned}$$

So $\pi h'(\sum_{i\in I}\lambda_i x_i)=\pi(\sum_{i\in I}\lambda_i z_i)$, and hence:

$$\pi(\sum_{i\in I}\lambda_i z_i)\in\phi(M((l_1\dots l_d)^{(m+)})$$

-as required. □

**Corollary 29.** *Given any aperiodic $\mathbb{N}$-word, $w$, let $M_w$ and $m_0\in M_w$ be as in theorem 40. By lemma 112, there exists $f\in\mathrm{Hom}(M(w),M_w)$ such that $f(z_0)=m_0$.*

*Then $f:M(w)\hookrightarrow M_w$ is the pure-injective hull of $M(w)$. Furthermore, $M_w$ is a direct summand of $\overline{M}(w)$*

*Proof.* By proposition 8 it is indeed a pure embedding. Furthermore, since $M_w$ is indecomposable (by theorem 40), $f$ cannot be factored through a direct summand of $M_w$.

□

**Corollary 30.** *Let $w$ be any $\mathbb{N}$-word. Then $M_w$ is a direct summand of $\overline{M}(w)$.*

*Proof.* If $w$ is expanding, then $\overline{M}(w)$ satisfies the conditions of theorem 40 (for its indecomposability, see corollary 32- and so $\overline{M}(w) \cong M_w$, by the theorem.

If $w$ is contracting, then $M(w)$ satisfies the conditions of theorem 40, and so $M(w) \cong M_w$, by the theorem. The canonical embedding $M(w) \hookrightarrow \overline{M}(w)$ is pure (by lemma 102), and hence split- so $M_w$ is indeed a direct summand of $\overline{M}(w)$.

Finally, if $w$ is contracting or aperiodic, then by proposition 5, the canonical embedding $M(w) \hookrightarrow \overline{M}(w)$ is pure. Thus by lemma 9, $M_w$ is a direct summand of $\overline{M}(w)$. □

### 5.8.2 1-Sided Modules over Contracting Words

Throughout this section, $w = l_1 l_2 l_3 \dots$ will be a contracting periodic or almost periodic $\mathbb{N}$-word, and $M$ will be a 1-directed module, containing a homogeneous element $m_0 \in (1.D)(M) \backslash ({}^{+}1.D)(M)$ (for some $D \leq w$) which has right word $w$.

Let $s \in \mathbb{N}$ be minimal such that $l_{s+1} l_{s+2} \dots$ is periodic. Then there exists a unique $n > 0$ such that $l_{s+1} \dots l_{s+n}$ is a band- we let $C = l_{s+1} \dots l_{s+n}$.

For all $i \in \{1, 2, \dots, ,\}$, we denote by $C_i$ the cyclic permutation of $C$ with first letter $l_{s+i}$.

There exists (as in [24]) a simple string map $\Phi_w : M(w) \to M(w)$, defined by:

$$\Phi_w : \sum_{i \in \mathbb{N}} \lambda_i z_i \mapsto \sum_{i \geq s} \lambda_{i+s} z_i$$

-we refer to it as the *Ringel shift*.

Notice that, given any simple string map $f \in \operatorname{End}(M(w))$ which is not a power of $\Phi_w$ (or the identity), $\operatorname{Im}(f)$ is finite dimensional: Indeed if $\operatorname{Im}(f)$ is infinite dimensional, then $f$ is uniquely determined by an infinite post-subword $w_i$ and an infinite pre-subword $w_j$ of $w$ such that $w_i = w_j$. It follows that either $i = j$ (i.e. $f$ is the identity) or $i = s + kn$ (for some $k \in \mathbb{N}^{+}$) and $j = s$- and hence that $f = \Phi_w^k$.

**Lemma 118.** *Let $w$ be any contracting $\mathbb{N}$-word, and $M$ a 1-directed module, containing a homogeneous element $m_0 \in (1.D)(M) \backslash ({}^{+}1.D)(M)$ (for some $D \leq w$) which has right word $w$.*

*Then* $\mathrm{pp}^{M(w)}(z_0) = \mathrm{pp}^M(m_0)$.

*Proof.* Take any $\phi \in \mathrm{pp}^M(m_0)$. We must show that $z_0 \in \phi(M(w))$.

First of all, we claim that there exists a trough $z_k$ of $w$ such that, for all $i > 0$ with $z_i \in (1.(l_1 \dots l_k))(M(w))$, $l_1 \dots l_k$ is not an initial subword of $\hat{w}_i$.

Recall from lemma 111 that, for all words $D \in H_1(a)$, and $i \geq 1$, $z_i \in (1.D)(M(w))$ if and only if $\hat{w}_i \geq D$, and $\hat{u}_i \geq 1_{a,-1}$ (and hence that its first letter is direct).

First of all, if $w$ is periodic, then pick any $k \geq n$ such that $z_k$ is a trough. Given any $i > 0$, it follows from lemma 82 that $C$ is an initial subword of $\hat{w}_i$ if and only if $i \in n\mathbb{N}$. However, for any such $i$, the first letter of $\hat{u}_i^{-1}$ is the first letter of $C^{-1}$- which is inverse, since $w$ is contracting.

If $w$ is not periodic (i.e. $s \geq 1$), then we claim that there exists $k > 0$ such that, for all $i \in \mathbb{N}^+$, $l_1 \dots l_k$ is not an initial subword of $\hat{w}_i$. Indeed, pick any $k > 0$ such that:

- $k > 2s$
- For all $i > 0$, $l_1 \dots l_k$ is not an initial subword of $w_i$- this is possible, since $w_i \neq w$ for all $i > 0$, and the set $\{w_i : i > 0\}$ contains only finitely many different words.
- $l_1 \dots l_k$ is not an initial subword of $(C')^\infty$, for any cyclic permutation $C'$ of $C^{-1}$.
- $z_k$ is a trough

Given any $i \geq 1$, $l_1 \dots l_k$ is not an initial subword of $w_i$ (by our choice of $k$). Furthermore if $i < 2k$, then $l_1 \dots l_k$ cannot be an initial subword of $u_i$ (it follows from lemma 81), and if $i \geq 2k > k + s$, then the initial subword of $u_i$ of length $k$ is equal to $(C')^{k/n}$, for some cyclic permutation $C'$ of $C^{-1}$- and so it cannot equal $l_1 \dots l_k$ (by our choice of $k$). So $l_1 \dots l_k$ is not an initial subword of $\hat{w}_i$, as required.

Given any such $k$, let $D = l_1 \dots l_k$, and let $\psi(v)$ be $\phi(v) \wedge (1.D)(v)$. Then $(1.D) \geq \psi + ({}^+1.D) > ({}^+1.D)$, so by lemma 114 there exists $E \in H_1(a)$ such that $\psi + ({}^+1.D)$ is equivalent to $(1.E) + ({}^+1.D)$.

Since $m_0 \in \phi(M)$, we must have $E < w_0$. Thus:

$$z_0 \in (1.E)(M(w)) \subseteq (\psi + (^+1.D))(M(w))$$

Pick any $x \in (^+1.D)(M(w))$ such that $z_0 - x \in \psi(M(w))$. Write $x$ as $\sum_{i \in I} \lambda_i z_i$- where $\lambda_i \neq 0$ for all $i \in I$.

By corollary 26, $z_i \in (^+1.D)(M(w))$ for all $i \in I$- in particular, $i \neq 0$, and $D \leq \hat{w}_i$. By our choice of $k$, $D$ is not an initial subword of $\hat{w}_i$, so lemma 116 implies that there exists a simple string map $f_i \in \mathrm{End}(M(w))$ taking $z_0$ to $z_i$, and such that $f_i(z_j) = 0$ for all $j \geq k$.

Let $f = \sum_{i \in I} \lambda_i f_i$. Then $f(z_j) \in sp_K\{z_{j+1}, z_{j+2}, \ldots\}$ for all $j \in \mathbb{N}$, and $f(z_j) = 0$ for all $j \geq k$, so $f^{k+1} = 0$, and hence:

$$\sum_{j=0}^{k} f^j(z_0 - x) = \sum_{j=0}^{k} f^j(1-f)(z_0) = z_0$$

Since $z_0 - x \in \psi(M(w)) \subseteq \phi(M(w))$:

$$z_0 \in \psi(M(w)) \subseteq \phi(M(w))$$

□

**Proposition 9.** *Let $w, M, m_0$ and $z_0$ be as in lemma 118. Let $f : M(w) \to M$ be any map taking $z_0$ to $m_0$. Then $f$ is a pure-embedding.*

*Proof.* Take any element $x = \sum_{i \in \mathbb{N}} \lambda_i z_i$ of $M(w)$. Pick any $k > s$ such that $z_k$ is a trough, and $\lambda_i = 0$ for all $i \geq k$- so $x = \sum_{i=0}^{k} \lambda_i z_i$. Take any $\phi \in \mathrm{pp}^M(f(x))$. We must show that $x \in \phi(M(w))$.

We claim that there exists $k'$ such that for all $i, j \leq k$, $l_{i+1} \ldots l_{i+k'}$ is not an initial subword of $u_j$, and it's an initial subword of $w_j$ if and only if $w_i = w_j$: Indeed, pick any $k'$ such that:

- $k' > s + n$
- $l_{i+1} \ldots l_{i+k} \neq l_{j+1} \ldots l_{j+k}$ for all distinct $i, j \leq s$.
- For all $i \leq s$, $l_{i+1} \ldots l_{i+k}$ is not an initial subword of $(C')^\infty$, for any cyclic permutation $C'$ of $C^{-1}$.

- $z_{k+k'}$ is a trough.

One can easily check it satisfies the required condition.

Let $D = l_1 \dots l_{k+k'}$. It is a pre-subword of $w$, and so $M(D)$ is a submodule of $M(w)$, with standard basis $z_0, z_1, \dots, z_{k+k'}$. Let $\chi(v_0, v_1, \dots, v_{k+k'})$ be a pp-formula which generates $\mathrm{pp}^{M(D)}(z_0, z_1, \dots, z_{k+k'})$. Let $\psi(v_0)$ be:

$$\exists v_1, \dots v_{k+k'} \left( \chi(v_0, v_1, v_2, \dots) \wedge \phi(\sum_{i=0}^{k} \lambda_i v_i) \right)$$

Of course, $m_0 \in \psi(M)$ (we could take $f(z_1), f(z_2), \dots, f(z_{k+k'})$ to be witnesses to it), so $z_0 \in \psi(M(w))$, by lemma 118- i.e. there exists $x_1, \dots x_{k+k'} \in M(w)$ such that:

$$M(w) \models \phi(\lambda_0 z_0 + \sum_{i=1}^{k} x_i) \wedge \chi(z_0, x_1, x_2, \dots x_{k+k'})$$

-so there exists a map $g : M(D) \to M(w)$ such that $g(z_0) = z_0$ and $g(z_i) = x_i$ for all $i \in \{1, 2, \dots, k + k'\}$.

Let $\rho : M(D) \hookrightarrow M(w)$ be the canonical embedding as defined after lemma 94. By lemma 108, $g - \rho$ is a $K$-linear combination of (distinct) simple string maps:

$$g - \rho = \sum_{j \in J} \mu_j g_j$$

Let $J'$ be the set of all $j \in J$ such that $g_j(z_i) \neq 0$, for some $i \leq k$, and let $g' = \sum_{j \in J'} \mu_j g_j(z_i)$. Then $g'(z_0) = z_0$, and $g'(z_i) = g(z_i) = x_i - z_i$ for all $i$ such that $1 \leq i \leq k$.

We claim that, for all $j \in J'$, there exists a simple string map $h_j \in \mathrm{End}(M(w))$ such that $g_j = h_j \rho$.

Given any $j \in J'$, take any $i \leq k$ such that $g_j(z_i) \neq 0$. By lemma 101, it's enough to find a simple string map $h_j$ such that $h_j(z_i) = g_j(z_i)$.

Let $i' \in \mathbb{N}$ be such that $g_j(z_i) = z_{i'}$. Assume, without loss of generality, that $w_i = \hat{w}_i$- i.e. that $w_i \in H_1(a)$ for some $a \in Q_0$.

If $l_{i+1} \dots l_{k+k'}$ is an initial subword of $\hat{w}_{i'}$, then $w_i = w_{i'}$ (by our choice of $k'$) and so $i' - i \in n\mathbb{Z}$. Since $u_i = (l_1 \dots l_i)^{-1} \leq u_b$, it follows that $i' \leq i$, and so $\Phi_w^{(i-i')/n}(z_i) = z_{i'} = g(z_i)$, as required.

If $l_{i+1}\dots l_{k+k'}$ is not an initial subword of $\hat{w}_{i'}$, then lemma 116 gives our required $h_j$.

Now, define $h := -\sum_{j\in J'}\mu_j h_j \in \mathrm{End}(M(w))$. Of course, $(1-h)(z_0) = z_0$, and $(1-h)(z_i) = x_i$ for all $i \in \{1, 2, \dots, k\}$. We claim that $\sum_{n=1}^{\infty} h^n$ is a well-defined endomorphism of $M(w)$: it's enough to prove that, for all $i \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that $h^N(x_i) = 0$.

Partition $J$ into $J_1 \cup J_2$- where $j \in J_1$ if and only if $\mathrm{Im}(h_j)$ is finite dimensional. Notice that if $j \in J_2$, then $h_j$ is a finite power of $\Phi$- and hence is a left shift.

Given any $j \in J_1$, $\mathrm{Im}(h_j)$ is finite dimensional- so (since $J_1$ is finite) we can pick $N$ such that, for all $i \in \mathbb{N}$ and $j \in J_1$, $h_j(z_i) \in \{0, z_0, z_1, z_2, \dots z_{N-1}\}$.

Also, for all $j \in J_2$ and $i < N$, $h_j(z_i) \in \{0, z_0, z_1, z_2, \dots z_{N-1}\}$. Consequently, $h^N(z_i) = 0$- otherwise there would be $j_0, j_1 \dots j_N \in J$ such that $g_{j_N} \dots g_2 g_1(z_i) \neq 0$, and hence a descending chain:

$$z_i >_w g_1(z_i) >_w \cdots >_w g_{j_N} \dots g_2 g_1(z_i)$$

-with each element in $\{z_0, z_1, \dots, z_{n-1}\}$- which is clearly a contradiction.

Now, given any $i > N$, and $j \in J$, either $g_j(z_i) = 0$, or $g_j(z_i) = \Phi^d(z_i) = z_{i-dn}$ for some $d \in \mathbb{N}^+$. It follows that there exists $N' \in \mathbb{N}$ such that $h^{N_1}(z_i) \in \mathrm{sp}_K\{z_0, z_1, \dots z_{N-1}\}$. Thus $h^{N_1+N}(z_i) = 0$, as required.

Of course:

$$\sum_{n=0}^{\infty} h^n(\sum_i \lambda_i x_i) = \sum_{n=0}^{\infty} h^n(\sum_i \lambda_i)(1-h)(z_i) = \sum_i \lambda_i z_i$$

Since $M(w) \models \phi(\sum_i \lambda_i x_i)$, we have that:

$$M(w) \models \phi(x)$$

□

### 5.8.3 1-Sided Modules over Expanding Words

Throughout this section, $w = l_1 l_2 l_3 \dots$ will be a expanding periodic or almost periodic $\mathbb{N}$-word.

Let $s \in \mathbb{N}$ be minimal such that $l_{s+1}l_{s+2}\dots$ is periodic. Then there exists a unique $n > 0$ such that $l_{s+1}\dots l_{s+n}$ is a band. We denote it $E$.

For all $i \in \{1, 2, \dots, n\}$, we denote by $E_i$ the cyclic permutation of $E$ with first letter $l_{s+i}$.

There exists (as in [24]) a simple string map $\Phi_w : \overline{M}(w) \to \overline{M}(w)$, defined by:

$$\Phi_w : \sum_{i\in\mathbb{N}} \lambda_i z_i \mapsto \sum_{i\geq s} \lambda_i z_{i+n}$$

-we refer to it as the *Ringel shift*.

Notice that, given any simple string map $f \in \mathrm{End}(M(w))$ either $f$ is a positive power of $\Phi$, or the identity, or $\mathrm{Im}(f)$ is finite dimensional.

**Lemma 119.** *Let $w$ be any expanding periodic or almost periodic $\mathbb{N}$-word. Let $R$ be the set of all $f \in \mathrm{End}(\overline{M}(w))$ which are a $K$-linear combination of simple string maps which are right shifts.*

*Then $R$ is a local ring.*

*Proof.* $R$ is closed under addition, and multiplication- since the composition of any two simple string right shifts is a simple string right shift. Thus $R$ is indeed a ring, with $0_R$ and $1_R$ being the zero map and the identity map of $\mathrm{End}(\overline{M}(w))$.

Now, take any $f$ in $R$. We can write it uniquely as $\lambda 1_E - \sum_{i\in I} \lambda_i f_i$- where each $f_i$ is a simple string right shift, and $\lambda_i \neq 0$ for all $i \in I$.

We need to show that either $f$ or $1 - f$ is invertible. We may therefore assume that $\lambda \neq 0$. By multiplying through by $\lambda^{-1}$, we may assume that $\lambda = 1$.

Let $g = 1 - f$. We claim that $\sum_{n=0}^{\infty} g^n$ is a well defined endomorphism of $w$. If so, then it is an element of $R$, and $(\sum_{n=0}^{\infty} g^n)f = f(\sum_{n=0}^{\infty} g^n) = 1$, as required.

Take any $x \in \overline{M}(w)$. We must show that $\sum_{n=0}^{\infty} g^n(x)$ is a well defined element of $\overline{M}(w)$. It will be enough to prove that, given any $k \in \mathbb{N}$, the $z_k$ coefficient of $\sum_n g^n(x)$ is an element of $K$.

Given any $n \in \mathbb{N}$, define:

$$I_n = \{f \in R : f(x) \in sp_K\{z_{n+1}, z_{n+2}, z_{n+3}, \dots\}\}$$

One can easily check that it is an ideal of $R$, and that $I_n I_m \subseteq I_{n+m}$ for all $m, n \in \mathbb{N}$.

Now, since $g \in I_1$, $g^n \in I_n$ for all $n \geq 1$. So the $z_k$-coefficient of $\sum_{n=k+1}^{\infty} g^n(x)$ is zero. So the $z_k$-coefficient of $\sum_{n=0}^{\infty} g^n(x)$ is the $z_k$-coefficient of $\sum_{n=0}^{k} g^n(x)$- which is clearly a well defined element of $K$. $\square$

**Lemma 120.** *Let $w$ be any expanding periodic $\mathbb{N}$-word, and $\{f_i \in \operatorname{End}(\overline{M}(w)) : i \in I\}$ be any set of simple string maps, such that $f_i(z_0) = z_i$ for all $i \in I$.*

*Then every map of the form $\sum_{i \in I} \lambda_i f_i$ is a well defined endomorphism of $\overline{M}(w)$.*

*Proof.* Assume that $w \in H_1(S)$. Given any $i \in I$, if $w_i \in H_1(S)$, then $f_i$ is a simple string map taking every $z_j$ to either $z_{j+i}$ or zero.

If $w_i \in H_{-1}(S)$, then $w > u_i^{-1}$, and $f_i$ takes every $z_j$ to $z_{j-i}$ or zero. Of course, if $j \leq 2i$, then $j - i \leq i$, and so $f_i(z_j) = 0$ (by lemma 109).

So, for any $i \in I$, $\operatorname{Im}(f_i) \in \operatorname{sp}_K\{z_j : j > i/2\}$. Thus, given any $k \in \mathbb{N}$, and any $x \in \overline{M}(w)$, the $z_k$-coefficient of $\sum_{i \in I} \lambda_i f_i(x)$ is the $z_k$-coefficient of $\sum_{i \leq 2k} \lambda_i f_i(x)$

Of course, $\sum_{i \leq 2k} \lambda_i f_i$ is a well defined endomorphism, so we are done.

$\square$

**Lemma 121.** *Let $w = l_1 l_2 l_3 \ldots$ be a expanding periodic or almost periodic $\mathbb{N}$-word. Suppose that $M$ is a one-directed module, and contains a homogeneous element $m_0 \in (1.D)(M) \backslash (^+1.D)(M)$ (for some $D \leq w$) which has right word $w$.*

*Then $\operatorname{pp}^{\overline{M}(w)}(z_0) = \operatorname{pp}^M(m_0)$.*

*Proof.* Take any $\phi \in \operatorname{pp}^M(m_0)$. We must show that $z_0 \in \phi(\overline{M}(w))$.

First of all, we claim that there exists a trough $z_k$ of $w$ such that, for all $i > 0$ with $z_i \in (1.(l_1 \ldots l_k))(M(w))$, there exists a simple string map in $\operatorname{End}(\overline{M}(w))$ taking $z_0$ to $z_i$.

If $w$ is not periodic, then as in the proof of lemma 118, there exists $k \in \mathbb{N}^+$ such that, for all $i > 0$, $l_1 \ldots l_k$ is not an initial subword of $\hat{w}_i$. Then lemma 116 gives the required simple string map.

If $w$ is periodic, pick any $k \geq n$ such that $z_k$ is a trough. Then, given any $i > 0$ such that $z_i \in (1.l_1 \ldots l_k)(\overline{M}(w))$, the first letter of $\hat{u}_i$ is inverse, and $l_1 \ldots l_k \leq \hat{w}_i$. If $l_1 \ldots l_k$ is an initial subword of $\hat{w}_i$, then it follows from lemma 82 that $i \in n\mathbb{N}$- and so $\Phi^{i/n}$ is the required simple string map. If $l_1 \dots l_k$ is not an initial subword of $\hat{w}_i$, then lemma 116 gives the required simple string map .

Now, let $\psi(v)$ be $\phi(v) \wedge (1.D)(v)$. Then $(1.D) \geq \psi + ({}^{+}1.D) > ({}^{+}1.D)$, so by lemma 114 there exists $E \in H_1(a)$ such that $\psi + ({}^{+}1.D)$ is equivalent to $(1.E) + ({}^{+}1.D)$.

Since $m_0 \in \phi(M)$, we must have $E < w_0$. Thus:

$$z_0 \in (1.E)(M(w)) \subseteq (\psi + ({}^{+}1.D))(M(w))$$

Pick any $x \in ({}^{+}1.D)(M(w))$ such that $z_0 - x \in \psi(M(w))$. Write $x$ as $\sum_{i \in I} \lambda_i z_i$- where $\lambda_i \neq 0$ for all $i \in I$. Of course, $x \in ({}^{+}1.D)(\overline{M}(w))$, and $z_0 - x \in \psi(\overline{M}(w))$ too.

By corollary 26, $z_i \in ({}^{+}1.D)(\overline{M}(w))$ for all $i \in I$- in particular, $i \neq 0$. Since $z_i \in (1.D)(\overline{M}(w))$, our choice of $k$ gives that there exists a simple string right shift $h_i \in \mathrm{End}(\overline{M}(w))$ taking $z_0$ to $z_i$.

Define $h = \sum_{i \in I} \lambda_i h_i$. By lemma 120, it is a well defined element of the subring $R$ of $\mathrm{End}(\overline{M}(w))$, and so by lemma 119, there exists $g \in R$ such that $gh = 1_{\mathrm{End}(\overline{M}(w))}$

Then $g(z_0 - x) = gh(z_0) = z_0$, so $z_0 \in \phi(\overline{M}(w))$ (since $z_0 - x \in \phi(\overline{M}(w))$). □

**Proposition 10.** *Let $w, M, m_0, z_0$ be as in lemma 121. Let $f : M(w) \to M$ be any map taking $z_0$ to $m_0$. Let $h_E : M(w) \to M({}^{\infty}E^{\infty})$ be the map as defined before lemma 103.*

*Then $(f, h_E) : M(w) \to M \oplus M({}^{\infty}E^{\infty})$ is a pure-embedding.*

*Proof.* Take any element $x = \sum_{i \in \mathbb{N}} \lambda_i z_i$ of $M(w)$. Pick any $k > s$ such that $z_k$ is a trough, and $\lambda_i = 0$ for all $i \geq k$- so $x = \sum_{i=0}^{k} \lambda_i z_i$.

Take any pp-formula $\phi$ such that $f(x) \in \phi(M)$ and $h_E(x) \in \phi(M^{\infty}E^{\infty})$. By lemma 103, it's enough to prove that $x \in \phi(\overline{M}(w))$.

As in the proof of proposition 9, we can pick $k'$ such that for all $i, j \leq k$, $l_{i+1} \dots l_{i+k'}$ is not an initial subword of $u_j$, and it's an initial subword of $w_j$ if and only if $w_i = w_j$- and also such that $z_{k+k'}$ is a trough.

Let $D = l_1 \dots l_{k+k'}$. It is a pre-subword of $w$, and so $M(D)$ is a submodule of $M(w)$, with standard basis $z_0, z_1, \dots, z_{k+k'}$. Let $\chi(v_0, v_1, \dots, v_{k+k'})$ be a pp-formula which generates $\mathrm{pp}^{M(D)}(z_0, z_1, \ldots, z_{k+k'})$. Let $\psi(v_0)$ be:

$$\exists v_1, \ldots v_{k+k'} \left( \chi(v_0, v_1, v_2, \ldots) \wedge \phi(\sum_{i=0}^{k} \lambda_i v_i) \right)$$

Of course, $m_0 \in \psi(M)$ (we could take $f(z_1), f(z_2), \ldots, f(z_{k+k'})$ to be witnesses to it), so $z_0 \in \psi(M(w))$, by lemma 118- i.e. there exists $x_1, \ldots x_{k+k'} \in M(w)$ such that:

$$M(w) \models \phi(\sum_{i=0}^{k} x_i) \wedge \chi(z_0, x_1, x_2, \ldots x_{k+k'})$$

-so there exists a map $g : M(D) \to M(w)$ such that $g(z_0) = z_0$ and $g(z_i) = x_i$ for all $i \in \{0, 1, \ldots, k + k'\}$.

We shall, from now on, consider $g$ as a map in $\mathrm{Hom}(M(D), \overline{M}(w))$ (by simply composing it with the canonical embedding of $M(w)$ into $\overline{M}(w)$).

Let $\rho : M(D) \hookrightarrow \overline{M}(w)$ be the canonical embedding as defined after lemma 94. By lemma 108, $g - \rho$ is a $K$-linear combination of (distinct) simple string maps:

$$g - \rho = \sum_{j \in J} \mu_j g_j$$

We claim that, for all $j \in J$, there exists a simple string map $h_j \in \mathrm{End}(\overline{M}(w))$ such that $h_j \rho = g_j$:

Take any $i$ such that $g_j(z_i) \neq 0$. Then $g_j(z_i) = z_{i'}$ for some $i'$. Notice that $z_i' \in ((l_1 \ldots l_i), (l_{i+1} \ldots l_{k+k'}))(M(w))$.

If $l_{i+1} \ldots l_{i+k+k'}$ is not an initial subword of $w_{i'}$, then lemma 116 gives the required map. If it is an initial subword, then $w_i = w_i'$ (by our choice of $k'$), and so $i - i' \in n\mathbb{Z}$. Since $u_i < i'$, $i - i'$ must be negative, and so $\Phi_w^{(i'-i)/n}(z_i) = z_{i'}$. Thus $\Phi_w^{(i'-i)/n} \rho = g_j$, by lemma 101.

Now, define $h := -\sum_{j \in J'} \mu_j h_j \in \mathrm{End}(\overline{M}(w))$. Of course, $(1 - h)(z_0) = z_0$, and $(1 - h)(z_i) = x_i$ for all $i \in \{1, 2, \ldots, k\}$. We shall prove that $\sum_{n' \geq 0} h^{n'}$ is a well defined endomorphism of $\overline{M}(w)$.

Let $N$ be maximal such that $h_j(z_N) \neq 0$ for some $j \in J$ such that $h_j$ is not a power of $\Phi_w$. We claim that $h^N$ is a $K$-linear combination of right shifts. Notice that, for all $j \in J$ and $i > N$, $h_j(z_i)$ is either 0, or $z_{i+dn}$, for some $d \in \mathbb{N}^+$

Given any $i \leq N$, we have that $h^{N+1}(z_i) \in \mathrm{sp}_K\{z_{N+1}, z_{N+2}, \dots\}$: If not, then there must be $j_0, j_1 \dots j_N \in J$ such that $h_{j_N} \dots h_{j_2} h_{j_1}(z_i) \neq 0$, and hence a descending chain:

$$z_i >_w h_{j_1}(z_i) >_w \cdots >_w h_{j_N} \dots h_{j_2} h_{j_1}(z_i)$$

-with each element in $\{z_0, z_1, \dots, z_N\}$- which is clearly a contradiction.

Of course, given any $j \in J$ and $i > N$, $g_j(x_i) \in \mathrm{sp}_K\{z_{i+1}, z_{i+2}, \dots\}$ (since either $g_j(x_i) = 0$, or $j$ is a power of $\Phi_w$). Thus, given any $x \in \overline{M}(w)$, and any $n' > N$, $\mathrm{Im}(h^{n'}) \subseteq \mathrm{sp}_K\{z_{n'}, z_{n'+1}, z_{n'+2}, \dots\}$. Consequently, as in the proof of lemma 120, $\sum_{n'>N} h^{n'}$ is a well defined element of $R$- and so $\sum_{n'\geq 0} h^{n'} \in \mathrm{End}(\overline{M}(w))$. Then:

$$\sum_{n'\geq 0} h^{n'}(x_i) = \sum_{n'\geq 0} h^{n'}(1-h)(z_i) = z_i$$

-and similarly $\sum_{n'\geq 0} h^{n'}(z_0) = z_0$. Since $M(w) \models \phi((1-h)(x))$, we have that $\overline{M}(w) \models \phi(x)$- which completes the proof. $\square$

# Chapter 6

# Indecomposable Pure-Injective Modules

Let $w$ be any $\mathbb{N}$-word or non-periodic $\mathbb{Z}$-word. Recall from theorem 39 and proposition 4 that $M(w)$ is indecomposable, and $\overline{M}(w)$is pure-injective. In this chapter, we determine what conditions on $w$ determine whether or not $M(w)$ is pure-injective, and what conditions on $w$ determine whether or not $\overline{M}(w)$ is indecomposable.

## 6.1 Indecomposable Direct Product Modules

Given any word, $w$, we define $\mathcal{W}_w := \{\hat{w}_i : i \in I\}$ and $\mathcal{U}_w := \{\hat{u}_i : i \in I\}$. Of course, these are subsets of $\bigcup_{a \in Q_0} H_1(a)$ and $\bigcup_{a \in Q_0} H_{-1}(a)$ respectively, and so we can define partial orders on them both. For example, the partial order on $\mathcal{W}_w$ will be defined by:

$$\hat{w}_i \leq \hat{w}_j \iff \hat{w}_i, \hat{w}_j \in H_1(a) \text{ for some } a \in Q_0 \text{ and } w_i \leq w_j$$

If $w$ is not a periodic $\mathbb{Z}$-word, then we define a partial order on the standard basis $\{z_i : i \in I\}$ of $M(w)$ by:

$$z_i \leq z_j \text{ if and only if } \hat{w}_i \leq \hat{w}_j \text{ and } \hat{u}_i \leq \hat{u}_j$$

-indeed, this is equal to the partial order $\leq_w$ as defined in (5.7.3).

Given any subset $J \subseteq I$, we say that the set $\{z_i : i \in J\}$ satisfies the "Indecomposability Criterion"-or (IC)- if:

- Given any $a \in Q_0$, we can partition any subset of $\{j \in J : z_i \in e_a M(w)\}$ into $I_L \cup \{i_0\} \cup I_R$, where $\inf\{\hat{u}_i : i \in I_L\} > \hat{u}_{i_0}$ and $\inf\{\hat{w}_i : i \in I_R\} > \hat{w}_{i_0}$.

We say that $w$ satisfies (IC) if and only if $\{z_i : i \in I\}$ satisfies (IC).

We shall prove that- given any $\mathbb{N}$-word or non-periodic $\mathbb{Z}$-word, $w$- $\overline{M}(w)$ is indecomposable if and only if $w$ satisfies (IC) and the poset $\{z_i : i \in I\}$ has no infinite descending chains.

**Lemma 122.** *Let $w$ be any $\mathbb{N}$-word or $\mathbb{Z}$-word (other than a periodic $\mathbb{Z}$-word). Let $\{z_i : i \in I\}$ be the standard basis for $\overline{M}(w)$. Then, for any subset $J \subseteq I$:*

- *If $\{z_j : j \in J\}$ satisfies* (IC), then so does $\{z_{j_1}\} \cup \{z_j : j \in J\}$, for any $j_1 \in I$

- If $\{z_j : j \in J\}$ satisfies (IC), then so does $\{z_{j_k} : k \geq 1\} \cup \{z_j : j \in J\}$, for any ascending chain $z_{j_1} < z_{j_2} < z_{j_3} < \ldots$ in $\{z_i : i \in I\}$.

*Proof.* Take any subset $J_0$ of $J$. By our assumption, it can be partitioned into $J_L \cup \{j_0\} \cup J_R$ of $J_0$, as in the definition of (IC). Let:

$$w' = \inf\{\hat{w}_j : j \in J_R\} > \hat{w}_{j_0}$$

$$u' = \inf\{\hat{u}_j : j \in J_L\} > \hat{u}_{j_0}$$

There are three different cases to consider: Firstly, if $\hat{w}_{j_1} > \hat{w}_{j_0}$, then let $J'_R = J_R \cup \{j_1\}$. Then:

$$\inf\{\hat{w}_j : j \in J'_R\} = \min(w', \hat{w}_{j_1}) > \hat{w}_{j_0}$$

-so the partition $J_L \cup \{j_0\} \cup J_R$ of $J_0 \cup \{j_1\}$ satisfies the definition of (IC). Furthermore, if we define $J''_R = J'_R \cup \{z_{j_k} : k \geq 2\}$, then the partition $\{j_0\} \cup J_L \cup J''_R$ of $J_0 \cup \{z_{j_k} : k \in \mathbb{N}^+\}$ satisfies the definition of (IC). If $\hat{u}_{j_1} > \hat{u}_{j_0}$ then the result is proved symmetrically.

Finally, suppose that both $\hat{w}_{j_1} \leq \hat{w}_{j_0}$ and $\hat{u}_{j_1} \leq \hat{u}_{j_0}$. $w$ is not a periodic $\mathbb{Z}$-word, so by lemma 89, $\hat{w}_{j_1} < \hat{w}_{j_0}$ (without loss of generality). Let $J'_R = J_R \cup \{j_0\}$. Then:

$$\inf\{\hat{w}_j : j \in J'_R\} = \hat{w}_{j_0} > \hat{w}_{j_1}$$

And so $J_L \cup \{j_1\} \cup J'_R$ is a partition of $J_0 \cup \{j_1\}$ satisfying the conditions required of (IC). Furthermore, since $z_{j_2} > z_{j_1}$ we have, without loss of generality, that $\hat{w}_{j_2} > \hat{w}_{j_1}$, and that, for all $k \geq 2$:

$$\hat{u}_{j_k} \geq \hat{w}_{j_2} > \hat{w}_{j_1}$$

And so, setting $J''_R = J'_R \cup \{z_{j_k} : k \geq 2\}$, the partition $J_L \cup \{j_1\} \cup J''_R$ of $j_0 \cup \{j_k : k \geq 1\}$ satisfies the conditions required of (IC). $\square$

**Corollary 31.** *Let $w = \ldots l_{-2}l_{-1}l_0D^\infty$ be any expanding half-periodic $\mathbb{Z}$-word. Let $\{z_i : i \in \mathbb{Z}\}$ be the standard basis of $M(w)$.*

*Then $w$ satisfies* (IC) *if and only if, for all $i \in I_0$, the set $\{z_i : i \leq i_0\}$ satisfies* (IC).

*Proof.* Of course, if $w$ satisfies (IC), then so does $\{z_i : i \leq 0\}$.

To prove the converse, let $s \in \mathbb{Z}$ be minimal such that $w_s$ is periodic, and let $n \in \mathbb{N}^+$ be minimal such that $w_s = (l_{s+1} \dots l_{s+n})^\infty$. Since $w$ is expanding half-periodic, $l_s \in Q_1$, and $l_{s+n} \in Q_1^{-1}$. Consequently, for all $i$ such that $s \leq i < s+n$:

$$z_i < z_{k+i} < z_{2k+i} < z_{3k+i} < \dots$$

We can partition $\{z_j : j \in \mathbb{Z}\}$ into:

$$\{z_j : j < s\} \cup \bigcup_{0 \leq i \leq n-1} \{z_{s+i+mn} : m \in \mathbb{N}\}$$

So, if $\{z_i : i \leq s-1\}$ satisfies (IC), then, by lemma 122, so does $w$. □

### 6.1.1 Words satisfying (IC) and the descending chain condition

Recall that we refer to every element of $\overline{M}(w)$ in the form $\sum_{i \in I} \lambda_i z_i$- where there may be infinitely many non-zero $\lambda_i$. This is the element corresponding to the element $(\lambda_i z_i)_{i \in I}$ of $\prod_{i \in I} K z_i$.

**Proposition 11.** *Let $w$ be any $\mathbb{Z}$-word or $\mathbb{N}$-word, which satisfies* (IC), such that the poset $\{z_i : i \in I\}$ contains no infinite descending chains.

Then $\overline{M}(w)$is indecomposable.

*Proof.* Take any two elements $x, y \in \overline{M}(w)$. Pick any $a_x, a_y \in Q_0$ such that $e_{a_x} x \neq 0$ and $e_{a_y} y \neq 0$. Write $e_{a_x} x$ as $\sum_{i \in I_x} \lambda_i z_i$ and $e_{a_y} y$ as $\sum_{i \in I_y} \mu_i z_i$- where $\lambda_i \neq 0$ for all $i \in I_x$ and $\mu_i \neq 0$ for all $i \in I_y$.

Partition $I_x$ into $I_L \cup \{i_0\} \cup I_R$, as in the definition of (IC). Relabeling the standard basis of $w$, we may assume that $i_0 = 0$. Let $D$ be the longest possible common initial subword of $\hat{w}_0$ and $\inf\{\hat{w}_i : i \in I_R\}$. Since $\hat{w}_0 < \inf\{\hat{w}_i : i \in I_R\}$, there are two cases to consider:

- If $\hat{w}_0 = D$, then there exists $\alpha \in Q_1$ such that $D\alpha$ is an initial subword of $\inf\{\hat{w}_i : i \in I_R\}$. Define $\phi_1(v)$ to be the pp-formula such that:

$$\phi_1(M) = D\alpha M$$

Notice that $z_0 \notin \phi_1(\overline{M}(w))$, and that $z_i \in \phi_1(\overline{M}(w))$ for all $i \in I_R$.

- If $\hat{w}_0 < D$, then let $\phi_1$ be $(.D)(M)$. Then $z_0 \notin \phi_1(\overline{M}(w))$ (by lemma 105), and for all $i \in I_R$, $\hat{w}_i \geq D$, so $z_i \in \phi_1(\overline{M}(w))$ (by lemma 105).

Notice that, given any element $m = \sum_i \nu_i z_i$ of $\overline{M}(w)$:

$$m \in \phi_1(\overline{M}(w)) \text{ if and only if } \nu_i z_i \in \phi_1(\overline{M}(w)) \text{ for all } i \text{ such that } \nu_i \neq 0$$

-by corollary 27, or corollary 26.

We can similarly find a pp-formula $\phi_2(v)$ such that $z_0 \notin \phi_2(\overline{M}(w))$, $z_i \in \phi_2(\overline{M}(w))$ for all $i \in I_L$, and given any element $m = \sum_i \nu_i z_i$ of $\overline{M}(w)$:

$$m \in \phi_2(\overline{M}(w)) \text{ if and only if } \nu_i z_i \in \phi_2(\overline{M}(w)) \text{ for all } i \text{ such that } \nu_i \neq 0$$

Similarly, we may partition $I_y$ into $J_L \cup \{j_0\} \cup J_R$, and find pp-formulas $\psi_1(v)$ and $\psi_2(v)$ such that $z_{j_0} \notin \psi_1(\overline{M}(w))$, $z_{j_0} \notin \psi_2(\overline{M}(w))$, and:

$$z_i \in \psi_1(\overline{M}(w)) \text{ for all } i \in J_R$$

$$z_i \in \psi_2(\overline{M}(w)) \text{ for all } i \in J_L$$

-and also, for all elements $m = \sum_i \nu_i z_i$ of $\overline{M}(w)$, and $k \in \{1, 2\}$:

$$m \in \psi_k(\overline{M}(w)) \text{ if and only if } \nu_i z_i \in \psi_k(\overline{M}(w)) \text{ for all } i \text{ such that } \nu_i \neq 0$$

We may assume, without loss of generality, that $j_0 \geq 0$. Let $\rho(v_1, v_2)$ be the pp-formula:

$$\begin{aligned} \exists v_3, v_4, v_5, v_6 \quad & (\phi_1(v_3) \wedge \phi_2(v_4) \wedge \psi_1(v_5) \wedge \psi_2(v_6) \\ & \wedge (v_1 - v_3 - v_4) \in l_1 \dots l_{j_0}(v_2 - v_5 - v_6)) \end{aligned}$$

We claim that this satisfies the conditions required of lemma 6. Indeed, taking $v_3 = \sum_{i \in I_R} \lambda_i z_i$, $v_4 = \sum_{i \in I_L} \lambda_i z_i$, $v_5 = \sum_{i \in J_R} \mu_i z_i$, $v_4 = \sum_{i \in J_L} \mu_i z_i$, we have:

$$\overline{M}(w) \models \rho(x, y)$$

Suppose, for a contradiction, that $\overline{M}(w) \models \rho(x, 0)$. Let $m_3, m_4, m_5, m_6 \in \overline{M}(w)$ be any witnesses to it.

Since $m_5 \in \psi_1(\overline{M})$, its $z_{j_0}$ component must be zero. Also $m_6 \in \psi_1(\overline{M})$, so its $z_{j_0}$ component must be zero.

Since $M \models (x - m_3 - m_4) \in l_1 \dots l_{j_0-1}(-m_5 - m_6)$, it follows from corollary 20 that $x - m_3 - m_4$ has $z_0$ component zero.

However, $m_3$ and $m_4$ must have $z_0$-component zero. And therefore, so must $x$- giving our required contradiction. So $\overline{M}(w) \models \rho(x, 0)$, and hence $\overline{M}(w)$ is indecomposable, by lemma 6. □

**Corollary 32.** *Let $w$ be any $\mathbb{N}$-word or $\mathbb{Z}$-word, which is expanding periodic or expanding almost periodic. Then $\overline{M}(w)$is indecomposable.*

*Proof.* By proposition 11, it's enough to prove that $w$ has (IC), and $\{z_i : i \in I\}$ has no infinite descending chains. We will take the case where $w$ is an $\mathbb{N}$-word, $l_1 l_2 l_3 \dots$- the proof for a $\mathbb{Z}$-word is similar.

Recall that there exists unique $s \in \mathbb{N}$ and $n \in \mathbb{N}^+$ such that $D = l_{s+1} \dots l_{s+n}$ is a band, $l_{s+1} l_{s+2} \dots = D^\infty$, $l_{s+n} \in Q_1^{-1}$, and $l_s \in Q_1$ (if $s \geq 1$).

Since $w$ is an expanding, we have that, for all $i$ such that $s < i \leq s + k$:

$$z_i < z_{k+i} < z_{2k+i} < z_{3k+i} < \dots$$

We can partition $\mathbb{N}$ into:

$$\{z_1, \dots, z_n\} \cup \bigcup_{i=1}^{k} \{z_{i+mk} : m \geq 1\}$$

Thus $\{z_i : i \in \mathbb{N}\}$ has no infinite descending chains, and, by lemma 122 the finite set $\{z_1, \dots, z_n\}$ satisfies (IC), and hence so does $\{z_i : i \in \mathbb{N}\}$, as required. □

These arguments can also be applies to mixed $\mathbb{Z}$-words, to prove that $M^+(w)$ is indecomposable:

**Proposition 12.** *Let $w$ be any mixed $\mathbb{Z}$-word. Then $M^+(w)$ is indecomposable.*

*Proof.* First of all, given any subset $J \subseteq \mathbb{Z}$ containing only finitely many negative elements, the set $\{z_i : i \in J\}$ satisfies (IC), and has the descending chain condition- one can show this by mimicking the proof of corollary 32.

Now, given any two elements $x$ and $y$ of $M^+(w)$, one can mimic the proof of proposition 11 to find a pp-formula $\rho(v_1, v_2)$ such that:

$$M^+(w) \models \rho(x, y) \wedge \neg\rho(x, 0)$$

-which completes the proof, by lemma 6. □

**Theorem 41.** *Every module on Ringel's list is indecomposable.*

*Proof.* It follows straight from from theorem 39, corollary 32 and proposition 12. □

### 6.1.2 Words not satisfying the descending chain condition

We shall prove in this section, that $\overline{M}(w)$ is not indecomposable, for all words $w$ such that the set of standard basis elements of $\overline{M}(w)$ contains an infinite descending chain.

**Lemma 123.** *If $w$ is one of the following words:*

- *A contracting periodic or almost periodic $\mathbb{N}$-word.*
- *A contracting almost periodic $\mathbb{Z}$-word*
- *A mixed almost periodic $\mathbb{Z}$-word*
- *A contracting half-periodic $\mathbb{Z}$-word.*

*Then $\overline{M}(w)$ is not indecomposable.*

*Proof.* Write $w$ as either $l_1 l_2 l_3 \ldots$ or $\ldots l_{-1} l_0 l_1 l_2 \ldots$, depending on whether it is an $\mathbb{N}$-word or a $\mathbb{Z}$-word. Let $s$ be minimal such that $l_{s+1} l_{s+2} l_{s+3} \ldots$ is a periodic $\mathbb{N}$-word, and let $n \geq 1$ be minimal such that $l_{s+1} l_{s+2} l_{s+3} \cdots = (l_{s+1} \ldots l_{s+n})^\infty$.

Notice that $l_{s+n} \in Q_1$, and $l_s$ (if it exists) lies in $Q_1^{-1}$ (if $w$ is a mixed word, then we consider $w^{-1}$ rather than $w$- in order for it to satisfy this property).

Consequently, $w_{s+n} = l_{s+n+1} l_{s+n+2} l_{s+n+3} \ldots$ is a post-subword of $w$, and $w_s = l_{s+1} l_{s+2} l_{s+3} \ldots$ a pre-subword of $w$. Since $w_s = w_{s+n}$, there exists a simple string map $\Phi \in \mathrm{End}(\overline{M}(w))$, defined by:

$$\Phi : \sum_{i \in I} \lambda_i z_i \mapsto \sum_{i \geq s} \lambda_{i+n} z_i$$

-where $\{z_i : i \in I\}$ is the standard basis of $\overline{M}(w)$.

Suppose for a contradiction, that $\overline{M}(w)$ is indecomposable. Since $\overline{M}(w)$ is pure-injective (by proposition 4), there exists- by lemma 6- a pp-formula $\rho(v_1, v_2)$ such that:

$$\overline{M}(w) \models \rho(\sum_{k\in\mathbb{N}} z_{kn+s}, z_s) \wedge \neg\rho(x, 0)$$

And hence:

$$\overline{M}(w) \models \phi(\Phi(\sum_{k\in\mathbb{N}} z_{kn+s}), \Phi(z_s))$$

However, $\Phi(\sum_{k\in\mathbb{N}} z_{kn+s}) = \sum_{k\in\mathbb{N}} z_{kn+s}$ and $\Phi(z_s) = 0$, which gives our required contradiction. $\square$

**Corollary 33.** *Let $w$ be any $\mathbb{Z}$-word or $\mathbb{N}$-word. Suppose that there exists a sequence $i_1, i_2, i_3, \ldots$ such that:*

$$\hat{w}_{i_1} = \hat{w}_{i_2} = \hat{w}_{i_3} = \ldots$$

$$\hat{u}_{i_1} > \hat{u}_{i_2} > \hat{u}_{i_3} > \ldots$$

*Then $\overline{M}(w)$ is not indecomposable.*

*Proof.* It suffices to prove that $w$ is one of the words described in lemma 123. Let $a \in Q_0$ be such that $z_{i_k} \in e_a\overline{M}(w)$ for all $k \in \mathbb{N}$.

We can pick a subsequence $j_1, j_2, \ldots$ of $i_1, i_2, \ldots$ which is either strictly ascending or strictly descending, and such that either $w_{j_k} \in H_1(a)$ for all $k \in \mathbb{N}$ or $w_{j_k} \in H_{-1}(a)$ for all $k \in \mathbb{N}$. We assume, without loss of generality, that $w_{j_k} \in H_1(a)$ for all $k \in \mathbb{N}$.

Since $w_{j_k} = w_{j_1}$ for all $k \in \mathbb{N}$, there must exist a band $D$, such that $w_{j_k} = D^\infty$ for all $k$. Note that $w$ cannot be a periodic $\mathbb{Z}$-word- since it would imply that $u_{j_k} = u_{j_1}$ for all $k \in \mathbb{N}$. Let $s < j_0$ be minimal such that $w_s = l_{s+1}l_{s+2}l_{s+3}\ldots$ is periodic. Then $w_s = E^\infty$ for some cyclic permutation $E$ of $D$.

If $E^\infty$ was expanding, then there would be a map $\Phi : \overline{M}(w) \to \overline{M}(w)$ given by:

$$\Phi : \sum_i \lambda_i z_i \mapsto \sum_{i\geq s} \lambda_i z_{i+n}$$

-where $n$ is the length of $D$. And so $\Phi^{(j_1-j_0)/n}(z_{j_0}) = z_{j_1}$- contradicting the fact that $z_{i_0} > z_{i_1}$. Consequently, $E$ must be contracting, and so $w$ (or $w^{-1}$) is indeed one of the four types of word as described in lemma 123. $\square$

**Lemma 124.** *Let $w$ be any $\mathbb{N}$-word or non-periodic $\mathbb{Z}$-word. Suppose that the poset $\{z_i : i \geq 0\}$ contains an infinite descending chain. Then $\overline{M}(w)$is not indecomposable.*

*Proof.* Suppose that an infinite descending chain exists:

$$z_{i_0} > z_{i_1} > z_{i_2} > \dots$$

Let $a \in Q_0$ be such that $z_{i_k} \in e_a(\overline{M}(w))$ for all $k$. First of all, consider the chains:

$$\hat{w}_{i_1} \geq \hat{w}_{i_2} \geq \hat{w}_{i_3} \geq \dots$$

$$\hat{u}_{i_1} \geq \hat{u}_{i_2} \geq \hat{u}_{i_3} \geq \dots$$

If either of them is eventually stationary, we can apply corollary 33. If not, then by picking a suitable subsequence, we may assume that $\hat{w}_{i_{k+1}} < \hat{w}_{i_k}$ and $\hat{u}_{i_{k+1}} < \hat{u}_{i_k}$ for all $k \in \mathbb{N}$.

As in the proof of corollary 33, we may- by picking a suitable subsequence- assume that the sequence $i_0, i_1, i_2, \dots$ is strictly increasing or strictly decreasing, and that there exists $s \in \{-1, +1\}$ such that $\hat{w}_{i_k} \in H_s(a)$ for all $k \in \mathbb{N}$.

We assume, without loss of generality, that $i_0, i_1, i_2, \dots$ is strictly ascending, and $\hat{w}_{i_k} \in H_1(a)$ for all $k \in \mathbb{N}$. We define, recursively, a subsequence $j_0, j_1, j_2, \dots$ of $i_0, i_1, i_2, \dots$ and finite words $C_k, D_k$ (for every $k \geq 0$) such that:

1. $j_0 < j_1 < j_2 < j_3 < \dots$
2. $D_k$ is an initial pre-subword of $w_{j_k}$, and an initial post-subword of $w_{j_n}$ for all $n > k$.
3. $C_k$ is an initial pre-subword of $u_{j_k}$, and an initial post-subword of $u_{j_n}$ for all $n > k$.
4. For all $k \geq 0$, $j_{k+1} - j_k > c_k + d_k$ (where $c_k$ denotes the length of $C_k$, and $d_k$ the length of $D_k$).

To do this, consider the descending chains $w_{i_0}, w_{i_1}, w_{i_2}, \dots$ and $u_{i_0}, u_{i_1}, u_{i_2}, \dots$. Write $\varinjlim w_{i_k}$ as $l'_1 l'_2 l'_3 \dots$, and $\varinjlim u_{i_k}$ as $(l'_0)^{-1}(l'_{-1})^{-1}(l'_{-2})^{-1} \dots$.

Assume that- for some $n \in \mathbb{N}^+$, we have found $j_k, D_k$ and $C_k$ for all $k \leq n$, such that:

1. $j_0 < j_1 < j_2 < j_3 < \cdots < j_n$
2. For all $m \leq n$, $D_m$ is an initial pre-subword of $w_{j_m}$, and an initial post-subword of $\varinjlim w_{i_k}$.
3. For all $m \leq n$, $C_m$ is an initial pre-subword of $u_{j_m}$, and an initial post-subword of $\varinjlim u_{i_k}$
4. For all $m \leq n$, $j_m - j_{m-1} > c_{m-1} + d_{m-1}$ (where $c_{m-1}$ denotes the length of $C_{m-1}$, and $d_{m-1}$ the length of $D_{m-1}$).

Let $k \geq 0$ be such that $i_k = j_n$. Consider the descending chain $z_{i_{k+1}} > z_{i_{k+2}} > z_{i_{k+3}} > \ldots$.

As in the proof of lemma 86, there exists $k' \geq k$ such that: $l'_1 \ldots l'_{d_k} l'_{d_k+1}$ is an initial subword of $w_{i_{k'}}$ and $(l'_0)^{-1}(l'_{-1})^{-1} \ldots (l'_{-c_k+1})^{-1}(l'_{-c_k})^{-1}$ is an initial post-subword of $u_{i_{k'}}$. Furthermore, we may pick $k'$ large enough such that $i_{k'} - j_n > c_n + d_n$.

Define $j_{n+1}$ to be this $i'_k$. Let $d_{n+1}$ be maximal such that $l'_1 \ldots l'_{d_{n+1}}$ is an initial subword of $w_{j_{n+1}}$, and let $D_{n+1} = l'_1 \ldots l'_{d_{n+1}}$.

Similarly, let $c_{n+1}$ be maximal such that $(l'_0)^{-1} \ldots (l'_{-c_{n+1}+1})^{-1}$ is an initial subword of $u_{j_{n+1}}$, and let $C_{n+1} = (l'_0)^{-1} \ldots (l'_{-c_{n+1}+1})^{-1}$. Then $j_{n+1}$, $D_{n+1}$ and $C_{n+1}$ clearly satisfy the required conditions.

Having defined the sequence, consider, for each $k \geq 0$, the finite string module $M(C_k^{-1}D_k)$. Let $y^{(k)}$ denote the standard basis of $M(C_k^{-1}D_k)$ with left-word $C_k$ and right-word $D_k$. Since $C_k$ and $D_k$ are initial post-subwords of $u_{j_{k+1}}$ and $w_{j_{k+1}}$ respectively, there exists a canonical projection:

$$\overline{M}(w) \twoheadrightarrow M(C_k^{-1}D_k)$$

-taking $z_{j_{k+1}}$ to $y^{(k)}$. Since $C_k$ and $D_k$ are initial pre-subwords of $u_{j_k}$ and $w_{j_k}$ respectively, there exists a canonical embedding:

$$M(C_k^{-1}D_k) \hookrightarrow \overline{M}(w)$$

-taking $y^{(k)}$ to $z_{j_k}$. Combining these two maps, we have a map $f_k$:

$$\overline{M}(w) \twoheadrightarrow M(C^{-1}D) \hookrightarrow \overline{M}(w)$$

-which takes $z_{j_{k+1}}$ to $z_{j_k}$. Notice that, for every $k$, $\mathrm{Im}(f_k) = \mathrm{sp}_K\{z_i : j_k - c_k \leq i \leq j_k + d_k\}$. Therefore, $\mathrm{Im}(f_k) \cap \mathrm{Im}(f_{k'}) = \{0\}$ for all $k' \neq k$, and so the map $f := \sum_{k\geq 0} f_k$ is a well defined endomorphism of $\overline{M}(w)$. Furthermore, $f_k(z_{i_{k'}}) = 0$ for all $k' \neq k$. So:

$$f(\sum_{k\geq 0} z_{i_k}) = \sum_{k\geq 0} z_{i_k}$$

Now, assume for a contradiction, that $M$ is indecomposable. Consider the elements $z_{i_0}$ and $\sum_{k\geq 0} z_{i_k}$. Since $\overline{M}(w)$is pure injective, lemma 7, gives a pp-formula $\rho(v_1, v_2)$ such that:

$$M \models \rho(z_{i_0}, \sum_{k\geq 0} z_{i_k}) \wedge \neg\rho(0, \sum_{k\geq 0} z_{i_k})$$

However, this implies that:

$$M \models \rho(f(z_{i_0}), f(x))$$

-giving our required contradiction. □

### 6.1.3 Words not satisfying (IC)

Let $w$ be any $\mathbb{Z}$-word or $\mathbb{N}$-word. Given any $i \in I$ and $m \in \mathbb{N}$, recall the post-subword ${}^{(+m)}z_i^{(m+)}$ of $w$, as defined in section 5.4. Let $\pi_i^m : \overline{M}(w) \twoheadrightarrow \overline{M}({}^{(+m)}z_i^{(m+)})$ denote the canonical projection.

**Lemma 125.** *If $w$ is an aperiodic $\mathbb{Z}$-word or $\mathbb{N}$-word, then for all $j \in I$ and $m \geq 1$, there are only finitely many $i \in I$ such that $\pi_j^m(z_i) \neq 0$.*

*If $w$ is a half periodic $\mathbb{Z}$-word, then given any $j \in \mathbb{Z}$ and $m \in \mathbb{N}$, there are only finitely many $i < 0$ in $I$ such that $\pi_j^m(z_i) \neq 0$.*

*Proof.* It's a straightforward extension of corollary 25. □

Given any word $w$ with standard basis $\{z_i : i \in I\}$, any subset $J \subseteq I$, and any $i \in J$, we say $z_i$ is *$J$-minimal* if $z_j \not< z_i$ for all $j \in J$.

Given any $\mathbb{N}$-word or $\mathbb{Z}$-word, $w$, and any standard basis elements $z_{j_1}, z_{j_2}, z_{j_3}, \ldots$ of $\overline{M}(w)$ and $z_j$, we say that the sequence $z_{j_1}, z_{j_2}, z_{j_3}, \ldots$ *right converges on* $z_j$ if:

- $\hat{u}_{j_k} \leq \hat{u}_{j_{k+1}}$ and $\hat{u}_{j_k} \leq \hat{u}_j$ for all $k \in \mathbb{N}^+$

- $\hat{w}_{j_1} > \hat{w}_{j_2} > \hat{w}_{j_3} > \ldots$, with $\varinjlim \hat{w}_{j_k} = \hat{w}_j$.

Similarly, we say that $z_{j_1}, z_{j_2}, z_{j_3}, \ldots$ *left converges on* $z_j$ if $\hat{w}_{j_k} \leq \hat{w}_{j_{k+1}}$ and $\hat{w}_{j_k} \leq \hat{w}_j$ for all $k \geq 1$, and $\hat{u}_{j_1} > \hat{u}_{j_2} > \hat{u}_{j_3} > \ldots$, with $\varinjlim \hat{u}_{j_k} = \hat{u}_j$.

**Lemma 126.** *Let $w$ be any $\mathbb{N}$-word or non-periodic $\mathbb{Z}$-word, with standard basis $\{z_i : i \in I\}$, which contains no infinite descending chains. Let $I_0$ be any subset of $I$, such that $\{z_i : i \in I_0\}$ doesn't satisfy* (IC).

*Then, given any $j \in I_0$ such that $z_j$ is $I_0$-minimal, there exists $i_1, i_2, i_3, \ldots$ in $I_0$ such that each $z_{i_k}$ is $I_0$-minimal, and the sequence $z_{i_1}, z_{i_2}, z_{i_3}, \ldots$ either left-converges or right converges on $z_j$.*

*Proof.* If $w$ is a $\mathbb{Z}$-word, then both $\hat{w}_j$ and $\hat{u}_j$ are $\mathbb{N}$-words, so we may pick descending chains of finite words $D_1 > D_2 > D_3 > \ldots$ and $C_1 > C_2 > C_3 > \ldots$ with such that $\varinjlim D_n = \hat{w}_j$ and $\varinjlim C_n = \hat{u}_j$ respectively.

If $w$ is an $\mathbb{N}$-word, then (without loss of generality) $\hat{w}_j$ is an $\mathbb{N}$-word, and $\hat{u}_j$ a finite word, so we pick a descending chain of finite words $D_1 > D_2 > D_3 > \ldots$ such that $\varinjlim D_n = \hat{w}_j$, and we let $C_n = \hat{u}_j$ for all $n \in \mathbb{N}^+$.

Given any $n \in \mathbb{N}^+$, we can partition the set $I \backslash \{j\}$ into sets $I_1^n \cup I_2^n \cup I_3^n \cup I_4^n \cup I_5^n \cup I_6^n$, where:

$$I_1^n := \{i \in I_0 \backslash \{j\} : \hat{w}_i > D_n\}$$

$$I_2^n := \{i \in I_0 \backslash \{j\} : \hat{w}_i \leq D_n \text{ and } \hat{u}_i > C_n\}$$

$$I_3^n := \{i \in I_0 \backslash \{j\} : \hat{u}_i \leq \hat{u}_j \text{ and } \hat{w}_j < \hat{w}_i \leq D_n\}$$

$$I_4^n := \{i \in I_0 \backslash \{j\} : \hat{w}_i \leq \hat{w}_j \text{ and } \hat{u}_j < \hat{u}_i \leq C_n\}$$

$$I_5^n := \{i \in I_0 \backslash \{j\} : \hat{u}_j < \hat{u}_i \leq C_n \text{ and } \hat{w}_j < \hat{w}_i \leq D_n$$

$$I_6^n := \{i \in I_0 \backslash \{j\} : \hat{u}_i \leq \hat{u}_j \text{ and } \hat{w}_i \leq \hat{w}_j\}$$

Of course, $I_6^n = \emptyset$ for all $n$- since $z_j$ is minimal with respect to $I$, and $w$ is not a periodic $\mathbb{Z}$-word. Furthermore, there must exist $n_0 \in \mathbb{N}$ such that $I_5^{n_0} = 0$ - if not, then we could easily find an infinite descending chain $z_{i_1} > z_{i_2} > z_{i_3} > \ldots$ in $I$-contradicting our assumption.

Now, for all $n > n_0$, either $I_3^n \neq \emptyset$ or $I_4^n \neq \emptyset$: Suppose, for a contradiction, that $I_3^n = I_4^n = \emptyset$ for some $n \geq n_0$- then $I_0$ can be partitioned into $I_1^n \cup I_2 \cup \{j\}$. Since $\hat{w}_i > D_n$ for all $i \in I_1^n$, it follows that $\inf\{\hat{w}_i : i \in I_1^n\} \geq D_n$. If $\hat{w}_j$ is infinite, then $D_n > \hat{w}_j$. Whereas, if $\hat{w}_j$ is finite, then $\hat{w}_i \neq \hat{w}_j$ for all $i \neq j$, and so $\inf\{\hat{w}_i : i \in I_1^n\} > \hat{w}_j$ (by lemma 86).

Similarly, $\inf\{\hat{u}_i : i \in I_2^n\} > \hat{u}_j$, and so the partition $I_1^n \cup I_2^n \cup \{j\}$ is a partition of $I_0$ as in the definition of (IC)- giving our contradiction.

Now, as $I_3^n \supseteq I_3^{n+1}$ and $I_4^n \supseteq I_4^{n+1}$ for all $n$, we must have (without loss of generality), that $I_3^n \neq 0$ for all $n > n_0$.

Pick any $i \in I_3^{n_0}$, and take $i_1 \in I_0$ such that $z_{i_1} \leq z_i$, and $z_{i_1}$ is minimal with respect to $I_0$. Of course, $i_1 \notin I_6^{n_0}$, and so $i_1 \in I_3^{n_0}$.

Now take any $n_1 > n_0$ such that $i_1 \notin I_3^{n_1}$. Repeating the argument, we can find $i_2 \in I_3^{n_1}$ such that $z_{i_2}$ is minimal with respect to $I_0$.

Inducting this argument will give us a set $j_1, j_2, j_3, \ldots$, such that:

- $u_{j_k} \leq u_j$ for every $k$.
- Every $u_{j_k}$ is minimal with respect to $I_0$.
- For every $k$, $\hat{w}_j < \hat{w}_{j_k} < D_k$.

Since $\varinjlim D_n = \hat{w}_j$, $\varinjlim \hat{w}_{i_k} = \hat{w}_j$. Now, we can pick a subsequence $i_1, i_2, i_3, \ldots$ of $j_1, j_2, j_3, \ldots$ such that $\hat{u}_{i_1}, \hat{u}_{i_2}, \hat{u}_{i_3}, \ldots$ is either non-decreasing or non-increasing. Since $\{z_i : i \in I\}$ contains no infinite descending chains, the sequence must be non-decreasing: thus $z_{i_1}, z_{i_2}, z_{i_3}, \ldots$ right-converges on $z_j$, as required.

Note that, if we had $I_4^n \neq 0$ for all $n > n_0$, then we would have found a chain which left-converges on $z_j$. $\square$

**Lemma 127.** *Let $w$ be any aperiodic $\mathbb{N}$-word, or any aperiodic or half-periodic $\mathbb{Z}$-word. Take any $j, i_1, i_2, i_3, \cdots \in I$, such that $z_{i_1}, z_{i_2}, z_{i_3}, \ldots$ right-converges on $z_j$.*

*Take any pp-formula $\phi(v)$ such that $z_{i_k} \in \phi(\overline{M}(w))$ for all $k \in \mathbb{N}^+$. Then $z_j \in \phi(\overline{M}(w))$.*

*Proof.* We may assume, without loss of generality, that $\hat{w}_j = w_j$. Let $m$ be the number of equations in $\phi$.

First of all, if $\hat{w}_j$ is aperiodic, then there exists $k \in \mathbb{N}$ such that $(u_j^{-1})^{(m)}$ is the subword $u_{j+k}^{-1}$ of $w$. Since $\varinjlim \hat{w}_{j_k} = w_j$, we can pick $d \in \mathbb{N}^+$ such that $l_{j+1} \dots l_{j'} l_{j+k+1}$ is an initial subword of $\hat{w}_{j_d}$.

Assume, without loss of generality, that $w_{j_d} \in H_1(a)$ for some $a \in Q_0$- so $\hat{w}_{j_d} = w_{j_d}$. Since $z_{j_d} \in \phi(\overline{M}(w))$, lemma 98 gives that:

$$z_{j_d} \in \phi(\overline{M}((u_{j_d}^{-1})^{(m)}))$$

By lemma 99, $(u_{j_d}^{-1})^{(m)}$ is the subword $u_{j_d+k}^{-1}$ of $w$. Now, as $u_{j_d} \leq u_j$ and $l_{j_d+1} \dots l_{j_d+k} = l_{j+1} \dots l_{j+k} < w_j$, there exists a simple string map:

$$f : \overline{M}((u_{j_d}^{-1})^{(m)}) \to \overline{M}(w)$$

-such that $f(z_{j_d}) = z_j$. So $z_j \in \phi(\overline{M}(w))$, as required.

Now, if $\hat{w}_j$ is not aperiodic, then $w$ must be half-periodic- so $\hat{u}_j$ must be an aperiodic $\mathbb{N}$-word- and so there exists $d \in \mathbb{Z}$ such that ${}^{(m)}w_j$ is the subword $w_d = l_{d+1} l_{d+2} \dots$ of $w$.

We assume, for now, that $(l_d l_{d+1} \dots l_j)^{-1}$ is an initial pre-subword of $\varinjlim \hat{u}_{j_k}$, then (by lemma 87) there exists $n \in \mathbb{N}^+$ such that, for all $k \geq n$, $(l_d l_{d+1} \dots l_j)^{-1}$ is an initial subword of $\hat{u}_{j_k}$- and so $(l_{d+1} \dots l_j)^{-1}$ is an initial pre-subword of $\hat{u}_{j_k}$. Thus, by lemma 98:

$$\overline{M}(w) \models \phi(z_j) \iff \overline{M}(l_{d+1} \dots l_j \hat{w}_{j_k}) \models \phi(z_{j_k})$$

Given any $k \geq n$, since $\hat{w}_{j_{k+1}} < \hat{w}_{j_k}$, there exists a simple string map:

$$g_k : \overline{M}(l_{d+1} \dots l_j \hat{w}_{j_{k+1}}) \to \overline{M}(l_{d+1} \dots l_j \hat{w}_{j_k})$$

-taking $z_{j_{k+1}}$ to $z_{j_k}$. Note that $\mathrm{Im}(g_k)$ is finite dimensional. Now consider the chain:

$$\dots \xrightarrow{g_{n+2}} \overline{M}(l_{d+1} \dots l_j \hat{w}_{j_{n+2}}) \xrightarrow{g_{n+1}} \overline{M}(l_{d+1} \dots l_j \hat{w}_{j_{n+1}}) \xrightarrow{g_n} \overline{M}(l_{d+1} \dots l_j \hat{w}_{j_n})$$

Since $\varinjlim w_{j_k} = w_j$, the inverse limit of this sequence is $\overline{M}(l_{d+1} \dots l_j \hat{w}_j)$, endowed with maps $h_k \in \mathrm{Hom}(\overline{M}(l_{d+1} \dots l_j \hat{w}_j), \overline{M}(l_{d+1} \dots l_j \hat{w}_{j_k}))$ for each $k \geq n$: $h_k$ being the simple string map taking $z_j$ to $z_{j_k}$.

Let $(C, c)$ be a free realisation of $\phi(v)$. For each $k > n$, consider the set of maps:

$$S_k := \{g_{k-1} \dots g_n f : f \in \mathrm{Hom}(C, \overline{M}(l_{d+1} \dots l_j \hat{w}_{j_k})), \text{ such that } f(c) \in \mathrm{sp}_K(z_{j_n})\}$$

Such a set is non-empty, since $z_{j_k} \in \phi(\overline{M}(l_{d+1} \dots l_j \hat{w}_{j_k}))$ for all $k > n$.

Furthermore, $S_k$ is a $K$-vector space, which is finitely generated (since $\mathrm{Im}(g_n)$ is finitely generated), and $S_{k+1} \subseteq S_k$ for all $k > n$, and so:

$$\bigcap_{k>n} S_k \neq \emptyset$$

Consequently, there exists a series of maps $(f_k \in \mathrm{Hom}(C, \overline{M}(l_{d+1} \dots l_j \hat{w}_{j_k}))_{k \geq n}$ such that $f_k(c) = z_{j_k}$ and $g_k f_{k+1} = f_k$ for all $k \geq n$- and hence there exists a map $f : C \to \overline{M}(l_{d+1} \dots l_j \hat{w}_j)$ such that $f_k = h_k f$ for all $k \geq n$. It follows that $f(c) = z_j$- and so $z_j \in \phi(\overline{M}(l_{d+1} \dots l_j \hat{w}_j))$, as required.

Now, if $(l_d l_{d+1} \dots l_j)^{-1}$ is not an initial subword of $\varinjlim \hat{u}_{j_k}$, then let $d' \leq d$ be maximal such that $(l_{d'+1} \dots l_j)^{-1}$ is an initial subword of $\varinjlim \hat{u}_{j_k}$. Let $E = l_{d'+1} \dots l_j$. Since $\varinjlim \hat{u}_{j_k} < (l_d l_{d+1} \dots l_j)^{-1}$, it follows that there exists $n \in \mathbb{N}^+$ such that:

$$\hat{u}_{j_k} < E^{-1} \text{ for all } k \geq n$$

So there exists a canonical projection $\pi_k : \overline{M}(w) \twoheadrightarrow \overline{M}(E^{-1} w_{j_k})$, which gives that:

$$\overline{M}(E^{-1} w_{j_k}) \models \phi(\pi_k(z_j))$$

As above, we can construct a sequence

$$\dots \xrightarrow{g_{n+2}} \overline{M}(E^{-1} \hat{w}_{j_{n+2}}) \xrightarrow{g_{n+1}} \overline{M}(E^{-1} \hat{w}_{j_{n+1}}) \xrightarrow{g_n} \overline{M}(E^{-1} \hat{w}_{j_n})$$

Since $\pi_k(z_{j_k}) \in \phi(\overline{M}(E^{-1} w_{j_k}))$ for all $k \geq n$, the same argument as above gives that:

$$z_j \in \phi(\overline{M}(E^{-1} \hat{w}_j))$$

-and, since this is a submodule of $\overline{M}({}^{(m)}\hat{w}_j)$, and hence of $\overline{M}(w)$:

$$\overline{M}(w) \models \phi({}^{(m)}\hat{w}_j)$$

-as required. □

**Lemma 128.** *Let $\phi(v)$ be any pp-formula with $m$ equations. Let $w$ be any aperiodic $\mathbb{Z}$-word, and $I_0$ any subset of $I$ such that $z_i \in \phi(\overline{M}(w))$ for all $i \in I_0$. Then:*

$$\sum_{i \in I_0} z_i \in \phi(\overline{M}(w))$$

*Similarly, if $w$ is an expanding half-periodic $\mathbb{Z}$-word, and $I_0$ any subset of $I \cap \{i \in \mathbb{Z} : i \leq 0\}$ such that $z_i \in \phi(\overline{M}(w))$ for all $i \in I_0$, then:*

$$\sum_{i \in I_0} z_i \in \phi(\overline{M}(w))$$

*Proof.* Similar to the proof of lemma 171. □

**Proposition 13.** *Let $w$ be word, other than a periodic $\mathbb{Z}$-word. If $w$ doesn't satisfy* (IC), *then $\overline{M}(w)$is not indecomposable.*

*Proof.* Since every finite word satisfies (IC), $w$ must be a $\mathbb{N}$-word, or a $\mathbb{Z}$-word. If the poset $\{z_i : i \in \mathbb{Z}\}$ has an infinite descending chain, then we may apply lemma 124. We assume therefore that it does not. If $w$ is periodic, or almost periodic, then it cannot be expanding (by corollary 32)- so lemma 123 gives that $\overline{M}(w)$ is not indecomposable.

We therefore assume, from now on, that $M(w)$ is either aperiodic, or is a half-periodic $\mathbb{Z}$-word. Assume, for a contradiction, that $\overline{M}(w)$is indecomposable. Take any subset $I_0 \subseteq I$ which cannot be partitioned as in the definition of (IC). Note that, if $w$ is half-periodic, then it must be expanding (since we are assuming that $\{z_i : i \in I\}$ has d.c.c.)- and so, by corollary 31, we may assume that $i \leq 0$ for all $i \in I_0$.

We claim that there exists an infinite subset $J \subseteq I_0$, such that:

- For all $j \in J$, $z_j$ is $I_0$-minimal.
- For all $m \in \mathbb{N}$, there are only finitely many distinct $i, j \in J$ such that $\pi_j^m(z_i) \neq 0$ (where $\pi_j^m : \overline{M}(w) \twoheadrightarrow \overline{M}({}^{(m+)}z_j^{(m+)})$ is the canonical projection associated from the post-subword, as in lemma 95).
- For every $j \in J$, there exists $j_1, j_2, j_3, \ldots$ in $J$, such that $z_{j_1}, z_{j_2}, z_{j_3}, \ldots$ either right-converges on $z_j$, or left-converges on $z_j$.

We shall first explain why such a set implies that $\overline{M}(w)$ is not indecomposable- and then prove that it exists.

Let $J$ be any set satisfying the claim. Assume, for a contradiction, that $\overline{M}(w)$is indecomposable. Pick any $i \in J$. Relabeling $w$ if necessary, we may assume that $i = 0$. Since $\overline{M}(w)$ is pure-injective, there exists- by lemma 7- a pp-formula $\rho(v_1, v_2)$ satisfying:

$$\overline{M}(w) \models \rho(z_0, \sum_{j \in J} z_j) \wedge \neg\rho(0, \sum_{j \in J} z_j)$$

Let $m$ be the number of equations in $\rho$. Define:

$$J' := \{j \in J : \exists j' \in J \backslash \{j\} \text{ such that } \pi_j^m(z'_j) \neq 0\}$$

By the conditions of $J$, $J'$ is finite. Since $J$ is infinite, $J \backslash J' \neq 0$. Given any non-zero $i \in J \backslash J'$, consider the natural projection $\pi_i^m : \overline{M}(w) \twoheadrightarrow \overline{M}({}^{(m+)}z_i^{(m+)})$. Of course, $\pi_j^m(\sum_{j \in J} z_j) = \pi_i^m(z_i)$, so:

$$\overline{M}({}^{(+m)}z_i^{(m+)}) \models \rho(0, \pi_i^m(z_i))$$

So, by corollary 23, we have that $\overline{M}(w) \models \rho(z_i, 0)$. Thus lemma 128 gives:

$$\overline{M}(w) \models \rho(0, \sum_{j \in J \backslash J'} z_j)$$

Now, given any $j \in J'$, there exists a sequence $j_1, j_2, \ldots$ of elements of $J$ such that- without loss of generality- $z_{j_1}, z_{j_2}, z_{j_3}, \ldots$ right converges on $z_j$. Thus, by lemma 127, $\overline{M}(w) \models \rho(0, z_j)$. Since $J'$ is finite, we have that:

$$\overline{M}(w) \models \rho(0, \sum_{j \in J'} z_j)$$

And so $\models \rho(0, \sum_{j \in J} z_j)$, giving our required contradiction.

All that remains, therefore, is to show that such a set exists. Let $I_0$ be any subset of $I$ which cannot be partitioned as in the definition of (IC). Note that, if $w$ is a half periodic $\mathbb{Z}$-word, then it must be expanding (since $w$ has no infinite descending chain) so we may use the subset $\{i \in I_0 : i < 0\}$ instead of $I_0$: by proposition 11, it cannot be partitioned (as in the definition of (IC)). Furthermore, by corollary 125, for all $i \in I_0$ and $m \geq 1$, there are only finitely many $j \leq 0$ in $I_0$ such that $\pi_j^m(z_i) \neq 0$.

Given any $n \in \mathbb{N}^+$, let $\mathcal{S}_n$ denote the set of all sequences of elements of $\mathbb{N}^+$ such that the sum of all the terms is a given sequence is $n$. For example:

$$\mathcal{S}_3 = \{(1,1,1),(1,2),(2,1),(3)\}$$

Also, we let $\mathcal{S}_0 := \{0\}$. Given any sequence $s \in \mathcal{S}_n$, of length $k$, and any $t \in \mathbb{N}^+$, we denote by $s, t$ the sequence in $\mathcal{S}_{n+t}$ of length $k+1$, such that $s$ is an initial subsequence, and whose last term is $t$. For example, if $s$ is $(1,2,1)$, and $t = 4$, then $s, t$ is $(1,2,1,4)$.

We shall define, recursively, for every $n \in \mathbb{N}$, a set $J_n = \{i_s : s \in \mathcal{S}_n\} \subset I_0$, and-for every $s \in \bigcup_{0 \leq k \leq n} \mathcal{S}_k$- a sequence $y_s^1, y_s^2, y_s^3, \ldots$ in $\{z_i : i \in I_0\}$ such that:

- $z_{i_s}$ is $I_0$-minimal for all $s \in \mathcal{S}_n$.
- $y_s^{n'}$ is $I_0$ minimal, for all $s \in \bigcup_{k \leq n} \mathcal{S}_k$, and $n' \in \mathbb{N}^+$
- For every $k \leq n$ and $s \in \mathcal{S}_k$, the sequence $z_{i_{s,1}}, z_{i_{s,2}}, \ldots, z_{i_{s,k-n}}, y_s^1, y_s^2, y_s^3, \ldots$ either right converges or left converges on $x_s$
- Given any $n < m$ and any $s \in \mathcal{S}_n$, $s' \in \bigcup_{k \geq 1} \mathcal{S}_k$:

$$\pi^n_{i_s}(x_{s'}) = \pi^n_{i_{s'}}(x_s) = 0$$

For the $n = 0$ case, pick any $i_0 \in I_0$ such that $z_{i_0}$ is $I_0$-minimal. By lemma 126, there exists a sequence $y_0^1, y_0^2, y_0^3, \ldots$ of $I_0$-minimal elements in $\{z_i : i \in I_0\}$, which either left converges or right converges on $z_{i_0}$.

Now, suppose that, for some $n$, we have sets $J_0, J_1, J_2, \ldots J_n$, and a sequence $y_s^1, y_s^2, y_s^3, \ldots$ for every $s \in \bigcup_{0 \leq k \leq n} \mathcal{S}_k$, satisfying the given conditions.

Notice that any element of $\mathcal{S}_n$ can be written uniquely in the form $(s, m)$- where $1 \leq m \leq n$ and $s \in \mathcal{S}_{n-m}$. Furthermore:

$$\mathcal{S}_{n+1} = \{(s, m, 1) : (s, m) \in \mathcal{S}_n\} \cup \{(s, m+1) : (s, m) \in \mathcal{S}_n\}$$

-except when $n = 0$, in which case $\mathcal{S} = \{(1)\}$.

Of course, there are- by corollary 125- only finitely many $j \in I_0 \backslash \bigcup_{k \leq n} J_k$ such that:

$$\pi_j^{n+1}(z_{i_s}) \neq 0 \text{ for some } s \in \bigcup_{k \leq n} J_k$$

-and also only finitely many $j \in I_0 \backslash \bigcup_{k \leq n} J_k$ such that:

$$\pi_{i_s}^{n+1}(z_j) \neq 0 \text{ for some } s \in \bigcup_{k \leq n} J_k$$

Consequently, given any element of $\mathcal{S}_n$ of the form $(s, m)$ (with $m \geq 1$, $s \in \mathcal{S}_{n-m}$), consider the sequence $y_s^1, y_s^2, y_s^3, \dots$. We can therefore pick $k$ such that $y_s^k \notin \{z_{i_{s'}} : s' \in \bigcup_{j \leq n} \mathcal{S}\}$, and such that:

$$\pi_{s'}^{n+1}(y_s^k) = \pi_{y_s^k}^{n+1}(z_{i_{s'}}) = 0 \text{ for all } s' \in \bigcup_{k \leq n} J_k$$

(where $\pi_{y_s^k}^{n+1}$ means the map $\pi_j^{n+1}$- where $j$ is the element of $I_0$ such that $y_s^k = z_j$). Define $i_{s,m+1}$ to be the $j \in I_0$ such that $y_s^k = z_j$. Relabel $y_s^{k+1}$ as $y_s^1$, and $y_s^{k+2}$ as $y_s^{m+2}$, and so on. Of course, the sequence $y_s^1, y_s^2, y_s^3, \dots$ still (either right or left) converges on $x_s$.

Also, by lemma 126, we can pick a sequence $y_{s,m+1}^1, y_{s,m+1}^2, y_{s,m+1}^3, \dots$ which either left converges or right converges on $z_{i_{s,m+1}}$.

Now, if $n \neq 0$, then consider the sequence $y_{s,m}^1, y_{s,m}^2, y_{s,m}^3, \dots$ which either left converges or right converges on $z_{s,m}$. Again, we can pick $k \geq 1$ such that:

$$\pi_{s'}^{n+1}(y_{s,m}^k) = \pi_{y_{s,m}^k}^{n+1}(z_{i_{s'}}) = 0 \text{ for all } s' \in \bigcup_{k \leq n} J_k$$

-and we define $i_{s,m,1}$ to be such that $z_{i_{s,m,1}} = y_{s,m}^k$. Relabel $y_{s,m}^{k+1}$ as $y_{s,m}^1$, and $y_{s,m}^{k+2}$ as $y_{s,m}^2$, and so on. Notice that $y_{s,m}^1, y_{s,m}^2, y_{s,m}^3, \dots$ still (left or right) converges on $z_{i_{s,m}}$.

Also, pick any sequence $y_{s,m,1}^1, y_{s,m,1}^2, y_{s,m,1}^3, \dots$ of $I_0$-minimal elements which either left converges or right converges on $x_{s,m,1}$.

We can do this for every element of $J_n$, taking care to ensure that:

$$\pi_{i_s}^{n+1}(z_{i_{s'}}) = \pi_{i_{s'}}(z_{i_s}) = 0 \text{ for all } s, s' \in \mathcal{S}_{n+1}$$

-which will give us an element $i_s$ for every $s \in \mathcal{S}_{n+1}$, and a sequence $y_s^1, y_s^2, y_s^3, \dots$ for every $s \in \bigcup_{0 \leq k \leq n+1} \mathcal{S}_k$ satisfying the required conditions.

Having done this for every $n \in \mathbb{N}$, one can easily check that the set $\bigcup_{n\in\mathbb{N}} J_n$ satisfies the conditions required of $J$. $\square$

## 6.2 Pure-injective direct sum string modules

Given any $\mathbb{N}$-word or non-periodic $\mathbb{Z}$-word, $w$, we define $\mathcal{W}_w := \{\hat{w}_i : i \in I\}$ and $\mathcal{U}_w := \{\hat{u}_i : i \in I\}$. Of course, these are subsets of $\bigcup_{a\in Q_0} H_1(a)$ and $\bigcup_{a\in Q_0} H_{-1}(a)$ respectively, and so we can define partial orders on them both. For example, the partial order on $\mathcal{W}_w$ will be defined by:

$$\hat{w}_i \leq \hat{w}_j \Longleftrightarrow \hat{w}_i, \hat{w}_j \in H_1(a) \text{ for some } a \in Q_0 \text{ and } w_i \leq w_j$$

We shall prove that $M(w)$ is pure-injective if and only if both $\mathcal{W}_w$ and $\mathcal{U}_w$ satisfy the ascending chain condition.

**Proposition 14.** *Let $w$ be any $\mathbb{N}$-word or $\mathbb{Z}$-word. If either of the posets $\{\hat{w}_i : i \in \mathbb{Z}\}$ or $\{\hat{u}_i : i \in \mathbb{Z}\}$ contains an infinite ascending chain, then $M(w)$ is not pure-injective.*

*Proof.* Suppose, without loss of generality, that $\{\hat{w}_i : i \in \mathbb{Z}\}$ contains an infinite ascending chain:

$$\hat{w}_{i_1} < \hat{w}_{i_2} < \hat{w}_{i_3} < \dots$$

For each $n \in \mathbb{N}^+$, pick a finite word $D_n$ such that $\hat{w}_{i_n} < D_{i_n} < \hat{w}_{i_{n+1}}$. Notice that $(.D_{n+1})(v) \to (.D_n)(v)$ for all $n \in \mathbb{N}$, and that:

$$z_{i_n} \in (.D_n)(M(w)) \backslash (.D_{n+1})(M(w))$$

Thus we have an infinite descending chain of pp-definable subgroups of $M(w)$:

$$(.D_1)(M(w)) > (.D_2)(M(w)) > (.D_3)(M(w)) > \dots$$

Thus $M(w)$ is not $\Sigma$-pure-injective (by theorem 4), and so- since $w$ is of countable dimension over $K$, it is not pure-injective (by lemma 5).

$\square$

The rest of this chapter is devoted to proving the converse of proposition 14:

**Theorem 42.** *Suppose that $w$ is an $\mathbb{N}$-word, or a non-periodic $\mathbb{Z}$-word, such that both $\mathcal{W}_w$ and $\mathcal{U}_w$ have the ascending chain condition.*

*Then $M(w)$ is totally transcendental- i.e. the lattice $\mathrm{pp(M(w))}$ contains no infinite descending chains.*

*And consequently $M(w)$ is pure injective.*

## 6.3 Maps between string and band modules

First of all, we need a little background on band modules.

### 6.3.1 Tubes in the AR-quiver of a string algebra

Recall that every finite dimensional $K[T, T^{-1}]$-module can be written as $(M, \phi)$- where $M$ is a finite dimensional $K$-vector space, and $\phi$ is an automorphism of $M$.

Every indecomposable finite dimensional $K[T, T^{-1}]$-module is isomorphic to a module of the form $(K^n, J_{n,\lambda})$- where $n \in \mathbb{N}$, $\lambda \in K \backslash \{0\}$, and $J_{n,\lambda}$ is the $n \times n$ Jordan matrix, with every entry on the diagonal being $\lambda$ (which is an indecomposable automorphism of $K^n$) . Furthermore, $(K^n, J_{n,\lambda}) \cong (K^m, J_{m,\mu})$ if and only if $m = n$ and $\lambda = \mu$.

It is known (see [8]) that the Auslander-Reiten quiver of $K[T, T^{-1}]$ consists of a family of orthogonal homogeneous stable tubes $\{\mathcal{T}_\lambda : \lambda \in K \backslash \{0\}\}$, where, for each $\lambda$, the unique ray in $\mathcal{T}_\lambda$ is given by:

$$(K, J_{1,\lambda}) \xrightarrow{f_1} (K^2, J_{2,\lambda}) \xrightarrow{f_2} (K^3, J_{3,\lambda}) \xrightarrow{f_3} \dots$$

**Lemma 129.** *Given any band, $D = l_1 \dots l_m$, there exists a functor:*

$$F_D : K[T, T^{-1}]\text{-mod} \to A\text{-mod}$$

*-taking each module $(V, \phi)$ to the band module $M(D, n, \phi)$.*

*Furthermore, $F_D$ preserves almost-split exact sequences, and takes every homogeneous tube in $K[T, T^{-1}]-$mod to a homogeneous tube in $A$-mod.*

*Proof.* See [8, p164] □

We shall write each band module $M(D,n,J_{n,\lambda})$ as $S^D_\lambda[n]$, and the irreducible maps between band modules as $f_{D,n,\lambda} : S^D_\lambda[n] \hookrightarrow S^D_\lambda[n+1]$ and $g_{D,n,\lambda} : S^D_\lambda[n+1] \twoheadrightarrow S^D_\lambda[n]$.

When it is clear which tube we are talking about, we will refer to the modules as just $S[n]$, and the morphisms as $f_n$ and $g_n$. Furthermore, we will write the map:

$$f_{n+k-1} \dots f_{n+1} f_n : S^D_\lambda[n] \hookrightarrow S^D_\lambda[n+k]$$

-as $f^{(k)}$, and the map:

$$g_n g_{n+1} \dots g_{n+k-1} : S^D_\lambda[n+k] \twoheadrightarrow S^D_\lambda[n]$$

-as $g^{(k)}$.

Recall that every band module has underlying $K$-vector space $\bigoplus_{i=0}^{m-1} V_i$, where $V_i = K^n$ for all $i$. Let $e_{i,1}, \dots, e_{i,n}$ be the canonical basis for each $V_i$. Then we refer to the set $\{e_{i,j} : 0 \leq i \leq m-1, 1 \leq j \leq n\}$ as the *standard basis of* $S^D_\lambda[n]$.

## 6.3.2 Maps between string and band modules

The homomorphisms between band modules and direct sum string modules have been determined in [12]. We present an equivalent definition, which is more consistent with the notion of simple string maps, as defined in section 5.5.

Fix any non-zero $\lambda \in K$ and band $D$. This gives us a unique tube in the AR quiver: we shall denote its elements as $S[n]$ and its irreducible morphisms as $f_n$ and $g_n$.

Consider the string module $M({}^\infty D^\infty)$. Let $\{z_0 : i \in \mathbb{Z}\}$ be a standard basis for it (such that $z_0$ has right word $D^\infty$). We can define a map $\pi^1_D : M({}^\infty D^\infty) \twoheadrightarrow S^D_\lambda[1]$ such that:

$$\pi_1 : (z_{im+j}) \mapsto \lambda^{-i} z_{1,j}$$

-for all $i \in \mathbb{Z}$ and $j$ such that $0 \leq j \leq m-1$. One can easily check that it is well defined.

**Lemma 130.** *There exists, for every $n$, a map $\pi_n : M({}^\infty D^\infty) \twoheadrightarrow S[n]$ such that:*

$$\pi_1 = g_1 g_2 \dots g_{n-1} \pi_n$$

*Proof.* This can be proved by induction on $n$, using the dual result of lemma 22, noting that every $\pi_n$ cannot be a retraction, since $M({}^\infty D^\infty)$ is indecomposable (by theorem 39).

□

Given any direct sum string module $M(w)$, we define a *simple string map* from $M(w)$ to $S_\lambda^D[n]$ to be any map of the form:

$$M(w) \xrightarrow{f} M({}^\infty D^\infty) \overset{\pi_j}{\twoheadrightarrow} S[j] \overset{f^{(n-j)}}{\hookrightarrow} S[n]$$

-where $f$ is a simple string map, and $1 \leq j \leq n$.

**Theorem 43.** *Every homomorphism from a string module $M(w)$ to a band module $S[n]$ is a $K$-linear combination of simple string maps.*

*Proof.* See [12] □

By a dual argument, we can define a series of maps:

$$\{i_n \in \mathrm{Hom}(S[n], \overline{M}({}^\infty D^\infty)) : n \in \mathbb{N}\}$$

-such that, for all $n$:

$$i_n f_{n-1} \dots f_2 f_1 = i_1$$

(Each map $i_n$ is the $K$-dual of the map $\pi_n$ over the opposite algebra $\Lambda^{op}$.

We define a *simple string map* from a band module $S_\lambda^D[n]$ to a string module $M(w)$ to be any map of the form:

$$S[n] \overset{g^{(n-j)}}{\twoheadrightarrow} S[j] \overset{i_j}{\hookrightarrow} \overline{M}({}^\infty D^\infty) \xrightarrow{h} M(w)$$

-where $h$ is a simple string map (with finite dimensional image), and $j \leq n$.

**Theorem 44.** *Let $f : S_\lambda^D[n] \to M(w)$ be any homomorphism from a band module to a direct sum string module.*

*Then $f$ is a $K$-linear combination of finitely many simple string maps.*

*Proof.* See [12] □

Finally, a *simple string map* map from a band module $S^D_\lambda[n]$ to a band module $S^C_\mu[m]$ is any map $f$ which takes one of the following two forms:

1. A map of the form:

$$S^D_\lambda[n] \xrightarrow{g^{(n-k)}} S^D_\lambda[k] \xrightarrow{f^{(m-k)}} S^D_\mu[m]$$

   (which can only happen if $C = D$ and $\lambda = \mu$- i.e. both band modules lie in the same tube).

2. A map of the form:

$$S^D_\lambda[n] \overset{g^{(n-k)}}{\twoheadrightarrow} S^D_\lambda[k] \overset{i_k}{\hookrightarrow} \overline{M}({}^\infty D^\infty) \xrightarrow{h} M({}^\infty C^\infty) \overset{\pi_j}{\twoheadrightarrow} S^C_\mu[j] \overset{f^{(m-j)}}{\hookrightarrow} S^C_\mu[m]$$

   -for some $k \leq n$, $j \leq m$, and simple string map $h$.

Notice that, in the second case, $h$ must be a simple string map of the form:

$$\overline{M}({}^\infty D^\infty) \twoheadrightarrow M(E) \hookrightarrow M({}^\infty C^\infty)$$

-where $E$ is a post-subword of ${}^\infty D^\infty$, and a pre-subword of ${}^\infty C^\infty$, and the maps are the canonical projection and canonical embedding as defined after lemma 95 and lemma 94 respectively.

It follows that, if $h'$ is a simple string map of the second kind, then given any standard basis element $z$ of $S^D_\lambda[n]$, the right-word (and left-word) of $h'(z)$ in $S^C_\mu[m]$ is strictly greater than the right-word (respectively, left-word) of $z$ in $S^D_\lambda[n]$.

**Theorem 45.** *Any map between two band modules is a finite $K$-linear combination of simple string maps.*

*Proof.* See [12] $\square$

### 6.3.3 Results about simple string maps

**Lemma 131.** *Let $M$ and $N$ be any direct sum string modules or band modules, and $f, g \in \mathrm{Hom}(M, N)$ be simple string maps.*

*Let $z$ be any standard basis element of $M$ such that $f(z) \neq 0$. Then $f = g$ if and only if $f(z) = g(z)$.*

*Proof.* Lemma 101 gives the case when $M$ and $N$ are both string modules. The other cases also follow from this lemma, by considering what the simple string maps look like. $\square$

Let $M$ and $N$ be any band modules or direct sum string modules. We define $\mathrm{Hom}'(M,N)$ to be the $K$-vector subspace of $\mathrm{Hom}(M,N)$ consisting of all maps $f$ which are a $K$-linear combination of finitely many simple string maps. Notice that, if $M$ is a finite dimensional string module or band module, then $\mathrm{Hom}(M,N) = \mathrm{Hom}'(M,N)$: Indeed, every $f \in \mathrm{Hom}(M,N)$ is a $K$-linear combination of distinct simple string maps $\sum_{j\in J}\lambda_j f_j$. Given any standard basis element $z$ of $M$, there are only finitely many different $j \in J$ such that $\lambda_j f_j(z)$ is non-zero- otherwise $\sum_{j\in J}\lambda_j f_j$ would be an infinite sum of different basis elements of $N$- which cannot happen in a band module or a direct sum string module. Since $M$ has only finitely many basis elements, it follows that there are only finitely many $j \in J$ such that $\lambda_j$ is finite.

Define $\mathrm{End}'(M)$ to be $\mathrm{Hom}'(M,N)$. Notice that, if $M$ is a band module, then $\mathrm{End}'(M) = \mathrm{End(M)}$, and so it is a ring. Also, if $M$ is a string module, then the composition of two simple string maps in $\mathrm{End}'(M)$ is a simple string map, so $\mathrm{End}'(M)$ is a ring.

### 6.3.4 A variant of König's Lemma

Given any $X_1, X_2, Y_1, Y_2 \in A$-Mod, any map $f : X_1 \oplus Y_1 \to X_2 \oplus Y_2$ can be written in the form:

$$\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} : \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} \to \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix}$$

-we define *the restriction of $f$ from $X_1$ to $X_2$* to be the map $f_{11} : X_1 \to X_2$.

For each $i \in \mathbb{N}^+$, let $M_i = \bigoplus_{j=1}^{n_i} X_{i,j}$, where $X_{i_1}, \ldots, X_{i,n_i}$ are indecomposable. Take any infinite sequence:

$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \xrightarrow{f_3} \ldots$$

We define an *indecomposable subchain* of this sequence, to be any sequence of the form:

$$X_{1,k_1} \xrightarrow{g_1} X_{2,k_2} \xrightarrow{g_2} X_{3,k_3} \xrightarrow{g_3} \dots$$

-with $k_i \in \{1, 2, \dots, n_i\}$ for each $i \in \mathbb{N}^+$, and with each map $g_i : X_{i,k_i} \to X_{i+1,k_{i+1}}$ being the restriction of $f_i$ from $X_{i,k_i}$ to $X_{i+1,k_{i+1}}$.

In the interests of easing notation, we shall usually write an indecomposable subchain as $X_1, X_2, \dots$- where $X_i = X_{i,k_i}$ (for some $k_i$) for each $i \in \mathbb{N}^+$.

We define a *finite indecomposable subchain* (of length $n$) of the sequence to be any sequence:

$$X_{1,k_1} \xrightarrow{g_1} X_{2,k_2} \xrightarrow{g_2} X_{3,k_3} \xrightarrow{g_3} \dots X_{n,k_n}$$

-with $k_i \in \{1, 2, \dots, n_i\}$ for each $i \leq n$, and with each map $g_i : X_{i,k_i} \to X_{i+1,k_{i+1}}$ being the restriction of $f_i$ from $X_{i,k_i}$ to $X_{i+1,k_{i+1}}$.

We shall usually write a finite indecomposable subchain as $X_1, X_2, \dots$- where $X_i = X_{i,k_i}$ (for some $k_i$) for each $i \in \mathbb{N}^+$.

The following result is a variant of König's lemma, written in terms of these sequences:

**Lemma 132.** *For each $i \in \mathbb{N}^+$, let $M_i = \bigoplus_{j=1}^{n_i} X_{i,j}$, for some indecomposable modules $X_{i_1}, \dots, X_{i,n_i}$. Take any infinite sequence:*

$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \xrightarrow{f_3} \dots$$

*Let $(\dagger)$ be any unary predicate on the set of all finite indecomposable subchains of this sequence, such that,:*

$$(\dagger)(X_{1,k_1}, \dots X_{n,k_n}) \implies (\dagger)(X_{1,k_1}, X_{2,k_2}, \dots, X_{n-1,k_{n-1}})$$

*-for every finite indecomposable subchain $(X_{1,k_1}, X_{2,k_2}, \dots, X_{n,k_n})$.*

*Suppose that, for all $n \geq 1$, the sequence has a finite indecomposable subchain $X_{1,k_1}, X_{2,k_2}, \dots, X_{n,k_n}$ such that $(\dagger)(X_{1,k_1}, X_{2,k_2}, \dots, X_{n,k_n})$.*

*Then there exists an infinite indecomposable subchain:*

$$X_{1,k_1} \to X_{2,k_2} \to X_{3,k_3} \to \dots$$

*-such that $(\dagger)(X_{1,k_1}, X_{2,k_2}, \dots, X_{k_m}$ for all $m \geq 1$.*

*Proof.* Given any $n > 0$, let $\mathcal{C}_n$ be the set of all indecomposable subchains of length $n$ satisfying (†). By our assumption, $\mathcal{C}_n \neq \emptyset$ for all $n$.

Given any finite indecomposable subchain $X_{1,k_1}, X_{2,k_2}, \ldots X_{m,k_m}$, and any $n \geq 0$, let $\mathcal{C}_n(X_{1,k_1}, X_{2,k_2}, \ldots X_{m,k_m})$ be the set of all indecomposable subchains of length $m+n$ whose first $m$ modules are $X_{1,k_1}, X_{2,k_2}, \ldots X_{m,k_m}$, and which satisfy (†).

We shall recursively define a sequence:

$$X_{1,k_1} \to X_{2,k_2} \to X_{3,k_3} \to \ldots$$

-such that, for all $m$ and all $n \geq 0$, $\mathcal{C}_n(X_{1,k_1}, X_{2,k_2}, \ldots, X_{m,k_m}) \neq \emptyset$: and hence that $(\dagger)(X_{1,k_1}, \ldots X_{m,k_m})$ for all $m \geq 1$.

Assume that, for some $m \geq 0$, we have found $X_{1,k_1}, X_{2,k_2}, \ldots, X_{m,k_m}$ satisfying the condition.

Then, for all $n > 0$, $\mathcal{C}_n(X_{1,k_1}, \ldots X_{m,k_m})$ partitions into:

$$\bigcup_{j \leq n_{m+1}} \mathcal{C}_{n-1}(X_{1,k_1}, \ldots X_{m,k_m}, X_{m+1,j})$$

So there exists $j \leq n_{m+1}$ such that $\mathcal{C}_{n-1}(X_{1,k_1}, \ldots X_{m,k_m}, X_{k+1,j}) \neq \emptyset$ for all $n > 0$. We define $k_{m+1}$ to be any such $j$- completing the induction.

□

## 6.4 Spanning sets and almost-invertible maps

We assume from now on that $w$ is an $\mathbb{N}$-word or $\mathbb{Z}$-word. Notice that, given any finite word $D$, there is at most one $i \in I$ such that $\hat{w}_i = D$.

Define $\mathcal{Z}_w$ to be the basis set $\{z_i : i \in I\}$ of $M(w)$, and define two maps $w : \mathcal{Z}_w \to \mathcal{W}_w$, $u : \mathcal{Z}_w \to \mathcal{U}_w$ by $w(z_i) := \hat{w}_i$ and $u(z_i) := \hat{u}_i$.

We aim to prove that if both $\mathcal{W}_w$ and $\mathcal{U}_w$ have the ascending chain condition, then $M(w)$ is totally transcendental- i.e. it has no infinite descending chains of pp-definable subgroups. In order to do this, we need a fair amount of groundwork.

Recall that, given any descending sequence $z_{i_1} > z_{i_2} > z_{i_3} > \ldots$, $\varinjlim \hat{w}_{i_k}$ and $\varinjlim \hat{u}_{i_k}$ are either finite words or $\mathbb{N}$-words. We claim that they are both $\mathbb{N}$-words:

First of all, if the sequence $\hat{w}_{i_1} \geq \hat{w}_{i_2} \geq \hat{w}_{i_3} \geq \dots$ is eventually stationary, then the limit cannot be a finite word (by our observation above. Whereas, if the sequence is not eventually stationary, then by lemma 86, $\varinjlim w(z_{i_k})$ is an $\mathbb{N}$-word.

We define $\overline{\mathcal{Z}}$ to be the "closure of $\mathcal{Z}$ under limits of descending chains"- i.e. a smallest possible set containing every element of $\mathcal{Z}$, such that, for every infinite descending chain in $\mathcal{Z}$:

$$z_{i_1} > z_{i_2} > z_{i_3} > \dots$$

-there exists an element $z \in \overline{\mathcal{Z}}$ such that $w(z) = \varinjlim w(z_{i_k})$ and $u(z) = \varinjlim u(z_{i_k})$.

**Lemma 133.** *Let $z$ be any element of $\overline{\mathcal{Z}} \backslash \mathcal{Z}$. Then $u(z)^{-1}w(z)$ is a word. Furthermore, if we label $w(z) = l'_1 l'_2 l'_3 \dots$ and $u(z) = (l'_0)^{-1}(l'_{-1})^{-1}(l'_{-2})^{-1}\dots$, then, given any $j \in \mathbb{Z}$, there exists $z' \in \overline{\mathcal{Z}} \backslash \mathcal{Z}$ such that:*

$$u(z') = (l'_j)^{-1}(l'_{j-1})^{-1}(l'_{j-2})^{-1}\dots$$

$$w(z') = l'_{j+1} l'_{j+2} l'_{j+3} \dots$$

*Proof.* Since $z \in \overline{\mathcal{Z}} \backslash \mathcal{Z}$, there exists an infinite descending chain:

$$z_{i_1} > z_{i_2} > z_{i_3} > \dots$$

-such that $\varinjlim w(z_{i_k}) = w(z)$ and $\varinjlim u(z_{i_k}) = u(z)$. Consider the chains:

$$w(z_{i_1}) \geq w(z_{i_2}) \geq w(z_{i_3}) \geq \dots$$

$$u(z_{i_1}) \geq u(z_{i_2}) \geq u(z_{i_3}) \geq \dots$$

Given any $j \geq 0$, there exists- by lemma 86- $k \in \mathbb{N}^+$ such that $l'_1 \dots l'_j$ and $(l'_{-j+1} \dots l'_{-1} l'_0)^{-1})^{-1}$ are initial subwords of $w(z_{i_n})$ and $u(z_{i_n})$ respectively (for all $n \geq k$). Thus $l'_{-j+1} \dots l'_{-1} l'_0 l'_1 \dots l'_j$ is indeed a word for all $j$, and hence so is $u(z)^{-1}w(z)$.

It also follows that:

- $z_{i_n+j} > z_{i_{n+1}+j} > z_{i_{k+2}+j} > \dots$
- $\varinjlim w(z_{i_k+j}) \} = l'_{j+1} l'_{j+2} l'_{j+3} \dots$
- $\varinjlim u(z_{i_k+1}) \} = (l'_j)^{-1}(l'_{j-1})^{-1}(l'_{j-2})^{-1}(l'_{j-3})^{-1} \dots$

And so there must exist $z' \in \overline{\mathcal{Z}}$ such that $u(z') = (l'_j)^{-1}(l'_{j-1})^{-1}(l'_{j-2})^{-1}\dots$ and $w(z') = l'_{j+1}l'_{j+2}l'_{j+3}\dots$, as required. $\square$

One can easily check that the set $\{w(z) : z \in \overline{\mathcal{Z}}\}$ (respectively $\{u(z) : z \in \overline{\mathcal{Z}}\}$) has the ascending chain condition if and only if $\mathcal{W}_w$ (respectively, $\mathcal{U}_w$) does.

### 6.4.1 The spanning set of a pp-definable subgroup

Given any $z \in \overline{\mathcal{Z}}$, $(u(z))^{-1}w(z)$ is a word. We denote by $M(z)$ the string module $M((u(z))^{-1}w(z))$. Define:

$$\mathcal{A} := \{M(z) : z \in \overline{\mathcal{Z}}, M(z) \text{ is not a periodic } \mathbb{Z}\text{-word}\}$$

$$\mathcal{P} := \{M(z) : z \in \overline{\mathcal{Z}}, M(z) \text{ is a periodic } \mathbb{Z}\text{-word}\}$$

$$\mathcal{B} := \{S^D_\lambda[n] : n \geq 1, \lambda \in K\backslash\{0\}, \exists z \in \overline{\mathcal{Z}} \text{ with } w(z) = D^\infty \text{ and } u(z) = (D^{-1})^\infty\}$$

And define $\mathcal{M} := \mathcal{A} \cup \mathcal{P} \cup \mathcal{B}$. We write add$(\mathcal{M})$ to mean the set of all finite direct sums of modules in $\mathcal{M}$.

Notice that, for all $M \in \mathcal{M}$, and standard basis elements $x$ of $M$, there exists $z \in \overline{\mathcal{Z}}$ such that $x$ has right-word $w(z)$ and left-word $u(z)$ in $M$ (by lemma 133).

Define $E := \mathrm{End}'(M(w))$. Given any pointed module $(M, m)$, with $M \in \mathrm{add}(\mathcal{M})$, we define:

$$(M, m)(M(w)) := \{f(m) : f \in \mathrm{Hom}'(M, M(w))\}$$

It is clearly an $E$-submodule of $M(w)$.

We define an $\mathcal{M}$-*sequence* to be any collection $(M_i, f_i, m_i)_{i\in\mathbb{N}^+}$- where $M_i \in \mathrm{add}(\mathcal{M})$, $f_i \in \mathrm{Hom}'(M_i, M_{i+1})$, and $m_i \in M_i$ for all $i \geq 1$, and $f_i(m_i) = m_{i+1}$ for all $i$. Such a sequence will usually be written in the form:

$$(M_1, m_1) \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \xrightarrow{f_3} \dots$$

An $\mathcal{M}$-sequence is said to be *eventually stationary* (respectively *eventually zero*) on $M(w)$ if there exists $k \geq 1$ such that $(M_j, m_j)(M(w)) = (M_k, m_k)(M(w))$ (respectively $(M_j, m_j)(M(w)) = 0$) for all $j \geq k$.

Given a second $\mathcal{M}$-sequence, $(N_i, g_i, n_i)_{i\in\mathbb{N}^+}$, we say the two sequences are *equivalent* if $(M_k, m_k)(M(w)) = (N_k, n_k)(M(w))$ for all $k \geq 1$.

Given any pp-formula $\phi(v)$, we say that a pointed module $(M, m)$ is a *spanning set* for $\phi(M(w))$, provided $M \in \mathcal{M}$, $m \in \phi(M)$, and $(M, m)(M(w)) = \phi(M(w))$.

Given any descending chain of pp-definable subgroups of $M(w)$:

$$\phi_1(M(w)) \geq \phi_2(M(w)) \geq \phi_3(M(w)) \geq \dots$$

- a *spanning sequence* for $\phi_1(M(w)), \phi_2(M(w)), \dots$ is defined to be any $\mathcal{M}$-sequence, $(M_i, f_i, m_i)_{i\in\mathbb{N}^+}$ such that, for all $k \geq 1$, $(M_k, m_k)(M(w))$ is a spanning set for $\phi_k(M(w))$.

We shall prove the following two results:

**Theorem 46.** *Let $w$ be any $\mathbb{N}$-word, or non-periodic $\mathbb{Z}$-word, such that $\mathcal{W}_w$ and $\mathcal{U}_w$ have the ascending chain condition.*

*Then every $\mathcal{M}$-sequence is eventually stationary on $M(w)$.*

**Theorem 47.** *Let $w$ be any $\mathbb{N}$-word, or non-periodic $\mathbb{Z}$-word, such that $\mathcal{W}_w$ and $\mathcal{U}_w$ have the ascending chain condition.*

*Then every descending chain of pp-definable subgroups of $M(w)$ has a spanning sequence.*

Of course, it follows from these two results that if $w$ is an $\mathbb{N}$-word, or non-periodic $\mathbb{Z}$-word, such that $\mathcal{W}_w$ and $\mathcal{U}_w$ have the ascending chain condition, then $M(w)$ is totally transcendental.

### 6.4.2 Almost-invertible morphisms

**Lemma 134.** *Let $M$ and $N$ be any pair of modules in $\mathcal{M}$. Given any $m \in M$, and any $f \in \mathrm{Hom}(M, N)$, there exists $g \in \mathrm{Hom}'(M, N)$ such that $g(m) = f(m)$.*

*Proof.* If $M \in \mathcal{B}$, then $f \in \mathrm{End}(M, N) = \mathrm{Hom}'(M, N)$ (as in (6.3.3)) as required. Assume, therefore, that $M \in \mathcal{A} \cup \mathcal{P}$- and hence that $M$ is a direct sum string module. Let $\{f_j : j \in \mathbb{N}\}$ be the set of all simple string maps in Hom(M, N).

Let $J' \subseteq \mathbb{N}$ be as large as possible such that the subset $\{f_j(m) : j \in J'\}$ of $N$ is linearly independent over $K$. Given any $j \in J \backslash J'$, $\lambda_j f_j(m)$ lies in the $K$-span of $\{f_j(m) : j \in J'\}$ (otherwise, the set $\{f_j(m) : j \in J'\} \cup \{j\}$ would be linearly independent, contradicting, the maximality of $J$).

Consequently, there exists $\mu_j \in K$ for all $j \in J'$ such that:

$$f(m) = \sum_{j \in J'} \mu_j f_j(m)$$

Since $N$ is either a direct sum string module or a band module, and all the $f_j(m)$ are linearly independent, only finitely many of the $\mu_j$ can be non-zero. Let $J' := \{j \in J : \mu_j \neq 0\}$. Setting $g = \sum_{j \in J'} \mu_j f_j \in \mathrm{Hom}'(M, N)$ completes the proof. □

Given any periodic $\mathbb{Z}$-word, ${}^{\infty}D^{\infty}$ (where $D$ is a band, of length $n$), there exists a simple string map in $\mathrm{End}(M({}^{\infty}D^{\infty})$ taking every standard basis element $y_i$ to $y_{i+n}$. We shall refer to this map as $\Phi$. Of course, it is invertible, and we refer to its inverse as $\Phi^{-1}$.

**Lemma 135.** *Given any $M \in \mathcal{M}$, and any simple string map $f \in \mathrm{End}'(M)$, the following are equivalent:*

- *$f$ is an isomorphism.*
- *For all standard basis elements $z$ of $M$, $f(z)$ is fundamental (cf 7.1.2) in $M$, with right-word $w(z)$ and left-word $u(z)$*
- *$f$ is the identity map if $M \in \mathcal{A} \cup \mathcal{B}$, or a power of $\Phi$, if $M \in \mathcal{P}$.*

*Proof.* These can easily be checked, using lemma 110 and lemma 156, and the definition of simple string maps. □

Given any $M \in \mathcal{M}$, a map $h \in \mathrm{End}'(M)$ is *almost-invertible* if it cannot be expressed as a $K$-linear combination of finitely many non-invertible simple string maps.

Of course, any simple string map $f \in \mathrm{End}(M)$ is invertible if and only if it is almost-invertible.

### 6.4.3 Facts about almost-invertible maps

In general, given any $M, N \in \mathcal{M}$, we say that a map $f \in \mathrm{Hom}'(M, N)$ is *almost invertible* if and only if one of the following occurs:

- $M \in \mathcal{A} \cup \mathcal{P}$, $N = M$, and $f$ is an almost invertible map in $\mathrm{End}'(M)$.
- $M \in \mathcal{B}$- say, $M = S_\lambda^D[n]$, and $N = S_\lambda^D[n+k]$ for some $k \geq 0$, and $f = hf^{(k)}$ for some invertible map $h \in \mathrm{End}'(N)$.

The concept of an almost invertible map may seem somewhat arbitrarily defined, but they have a practical property: Given any $M, N \in \mathcal{M}$, $x \in M$, and $f \in \mathrm{Hom}'(M, N)$:

$$(M, x)(M(w)) = (N, f(x))(M(w)) \text{ if } f \text{ is almost-invertible}$$

In fact, given any $g \in \mathrm{End}'(M)$:

$$(M, x)(M(w)) = (M, g(x))(M(w)) \text{ if and only if } g \text{ is almost-invertible}$$

-although these results won't actually be proved.

**Lemma 136.** *Take any $L, M, N \in \mathcal{M}$, and any maps $f, h \in \mathrm{Hom}'(L, M)$, $g \in \mathrm{Hom}'(M, N)$.*

*If $gf$ is almost invertible, then so must both $f$ and $g$ be.*

*Also, if $f + h$ is almost invertible, then at least one of $f$ and $h$ must be.*

*Proof.* The second assertion follows straight from the definition of an almost invertible map.

The first assertion can be checked case by case. For example, if $L \in \mathcal{A}$, then $gf$ is almost invertible if and only if $N = L$ and $gf$ is invertible. Then the map:

$$L \xrightarrow{f} M \xrightarrow{g} N \xrightarrow{(gf)^{-1}} L$$

-implies that $L$ is a direct summand of $M$. Since $M$ is indecomposable (by theorem 39, or theorem 36), $M$ must be isomorphic to $L$, and $g$ and $f$ are invertible- and hence almost-invertible. $\square$

Consequently, given any $M \in \mathcal{M}$, the set of all non-almost-invertible $f \in \mathrm{End}'(M)$ is a two-sided ideal in $\mathrm{End}'(M)$.

**Lemma 137.** *Take any $M, N \in \mathcal{M}$, and any simple string map $f : M \to N$ which is not almost-invertible. If $f$ is not one of the following two types of map:*

- *A map of the form $f^{(i)} g^{(i+k)} : S_\lambda^D[n] \to S_\lambda^D[n-k]$*
- *A map of the form $M({}^\infty D^\infty) \overset{\pi_i}{\twoheadrightarrow} S_\lambda^D[i] \overset{f^{(n-i)}}{\hookrightarrow} S_\lambda^D[n]$.*

*-then, for all standard basis elements $z$ of $M$, $f(z)$ is a $K$-linear combination of finitely many standard basis elements $y$ of $N$- all of which satisfy $y > z$ (under the ordering of $\overline{\mathcal{Z}}$).*

*Proof.* If $M$ and $N$ are string modules, then $f(y)$ is a standard basis of $N$, so we may consider both $z$ and $f(z)$ as elements of $\overline{\mathcal{Z}}$. Then $M \cong M(z)$ and $N \cong M(f(z))$. It follows that $z \leq f(z)$ (the proof is similar to that of lemma 110).

Now suppose, for a contradiction, that both $w(f(z)) = w(z)$ and $u(f(z)) = u(z)$, then $u(z)^{-1}w(z) = u(f(z))^{-1}w(f(z))$, and so $M(z) = M(f(z))$, and- by lemma 110- $f$ is either the identity map, or a power of the shift map $\Phi$ (if $M(z) \in \mathcal{P}$)- and hence is invertible, giving our required contradiction. Thus $f(z) > z$ as required.

One can check the other cases similarly, using this fact, and the definitions of simple string maps. □

**Lemma 138.** *Suppose that we have a $\mathcal{M}$-sequence:*

$$(X_1, x_1) \overset{h_1}{\to} X_2 \overset{h_2}{\to} X_3 \to \dots$$

*-where each $X_i$ in indecomposable, and each $h_i$ is a non-invertible simple string map.*

*Then the sequence is eventually zero.*

*Proof.* It will be enough to prove that, for any standard basis element $z$ of $X_1$, there exists $n \geq 1$ such that $h_n \dots h_2 h_1(z) = 0$. Define $n_1, n_2, \dots$ as follows:

$$n_1 := 1$$

$$n_{i+1} := \min\{k > n_i : h_{k-1} \text{ is not of the form } f^{(i)}g^{(j)} \text{ or of the form } \pi_i g^{(j)} \}$$

-note that such a set is always non-empty: since any chain of maps of the form:

$$S[n] \stackrel{f^{(j_1)}g^{(i_1)}}{\longrightarrow} S[n+j_1-i_1] \stackrel{f^{(j_2)}g^{(i_2)}}{\longrightarrow} S[n+j_1-i_1+j_2-i_2] \stackrel{f^{(j_3)}g^{(i_3)}}{\longrightarrow} \dots$$

(with each $i_k > 0$) will eventually be zero.

For all $i \geq 1$, define $Y_i := X_{n_i}$ and $g_i := h_{n_{i+1}-1} \dots h_{n_i+1}h_{n_i}$. Notice that, for all $i$, $g_i$ takes any standard basis element $z'$ of $Y_i$ to a $K$-linear combination of standard basis elements $z'_1, \dots, z'_n$ of $Y_{i+1}$, each satisfying $z'_j > z'$.

Assume, for a contradiction, that $h_n \dots h_2h_1(z) \neq 0$ for all $n$. Then let $z^{(1)} := z$, and define $z^{(2)}, z^{(3)}, \dots$ inductively, as follows: Given $z^{(n)}$ such that for all $m \geq n$, $g_m \dots g_{n+1}g_n(z^{(n)}) \neq 0$, write $g_n(z^{(n)})$ in terms of the standard basis of $Y_{i+1}$:

$$g_n(z^{(n)}) = \sum_{j \in J} \lambda_j y_j$$

(where $J$ is finite, and $\lambda_j \neq 0$ for all $j \in J$). By our assumption, at least one of the $y_j$ must satisfy $g_m \dots g_{n+2}g_{n+1}(y_j) \neq 0$ for all $m \geq n$. Let $z^{(n+1)}$ be any such $y_j$.

Now, we have that:

$$z^{(1)} < z^{(2)} < \dots$$

-contradicting the fact that $\overline{\mathcal{Z}}$ has no infinite ascending chains. □

**Corollary 34.** *Take any $\mathcal{M}$-sequence:*

$$(M_1, m_1) \stackrel{g_1}{\to} M_2 \stackrel{g_2}{\to} M_3 \to \dots$$

*-where each $M_i$ is indecomposable, and each $g_i$ is not almost invertible.*

*Then there exists $n \in \mathbb{N}$ such that $g_n \dots g_2g_1(m) = 0$.*

*Proof.* Write each map $g_i$ as $\sum_{j=1}^{n_j} \lambda_{i,j} f_{i,j}$- with each $f_{i,j}$ being a non-invertible simple string map, and each $\lambda_{i,j}$ being non-zero. We can define a $\mathcal{M}$-sequence:

$$M_1 \xrightarrow{h_1} M_2^{(n_1)} \xrightarrow{h_2} M_3^{(n_1n_2)} \xrightarrow{h_3} M_4^{(n_1n_2n_3)} \longrightarrow \dots$$

-such that, for all $n \geq 2$, the set of all indecomposable subchains of length $n$ is in bijective correspondence with the set of all chains of the form:

$$M_1 \xrightarrow{f_{1,j_1}} M_2 \xrightarrow{f_{2,j_2}} \dots \xrightarrow{f_{n-1,j_{n-1}}} M_n$$

(with $1 \leq j_k \leq n_k$ for all $k \leq n$). Given any finite subchain $X_1, \ldots, X_n$, let $(\dagger)(X_1, \ldots, X_n)$ be the condition:

$$f_{n-1,j_{n-1}} \ldots f_{2,j_2} f_{1,j_1}(m_1) \neq 0$$

Given any infinite indecomposable subchain $X_1, X_2, X_3, \ldots$, there exists $n$ such that $X_1, \ldots, X_n$ does not satisfy $(\dagger)$ (by lemma 138).

Thus, by lemma 132, there exists $n \in \mathbb{N}^+$ such that $g_{n-1} \ldots g_2 g_1(m_1) = 0$. $\square$

### 6.4.4 Inverses of almost-invertible morphisms

**Lemma 139.** *Given any $M \in \mathcal{A} \cup \mathcal{B}$, every almost invertible map in* $\mathrm{End}'(M)$ *has an inverse in* $\mathrm{End}(M)$.

*Proof.* By lemma 135, the only invertible simple string map in $\mathrm{End}(M)$ is the identity map. Consequently, every almost invertible map in $\mathrm{End}'M$ can be written in the form $\lambda(1-g)$, where $g \in \mathrm{End}'(M)$ is a finite combination non-invertible maps- i.e. $g$ is non-almost-invertible.

It's enough to prove that $\sum_{i=0}^{\infty} g^i$ is a well defined endomorphism of $M$, since:

$$\lambda(1-g)(\lambda^{-1}\sum_{i=0}^{\infty} g^i) = (\lambda^{-1}\sum_{i=0}^{\infty} g^i)\lambda(1-g) = 1_M$$

Given any $x \in M$, we can find $n \geq 1$ such that $g^n(x) = 0$, by corollary 34. Then $\sum_{i=0}^{\infty} g^i(x) = \sum_{i=0}^{n} g^i(x)$, which is a well defined element of $M$. So $\sum_{i=0}^{\infty} g^i$ is indeed a well defined endomorphism- completing the proof. $\square$

**Corollary 35.** *Take any $M \in \mathcal{A}$, and $N$ in $\mathcal{M}$, and almost invertible $f \in \mathrm{End}'(M)$.*

*Given any $m \in M$, and any $g \in \mathrm{Hom}'(M, N)$, there exists $h \in \mathrm{Hom}'(M, N)$ such that $hf(m) = g(m)$*

*Proof.* By lemma 139 $f$ has an inverse $f^{-1} \in \mathrm{End}(M)$. The map $gf^{-1} \in \mathrm{Hom}(M, N)$ takes $f(m)$ to $g(m)$- so, by lemma 134, there exists $h \in \mathrm{Hom}'(M, N)$ taking $f(m)$ to $g(m)$, as required. $\square$

**Corollary 36.** *Let $S_\lambda^D[n]$ be any module in $\mathcal{B}$. Let $f \in \mathrm{Hom}'(S_\lambda^D[n], S_\lambda^D[n+k])$ be any almost invertible map. Take any $M \in \mathcal{M}$ which is not isomorphic to $S_\lambda^D[i]$ for any $i < n+k$.*

*Then for any $g \in \mathrm{Hom}'(S_\lambda^D[n], M)$, there exists $h \in \mathrm{Hom}'(S_\lambda^D[n+k], M)$ such that $g = hf$.*

*Proof.* By repeatedly applying lemma 22, there exists $h' \in \mathrm{Hom}(S[n+k], M)$ such that $g = h'f^{(k)}$.

By the definition of almost-invertible, there exists an invertible $\rho \in \mathrm{End}'(S_\lambda^D[n+k])$ such that $f = \rho f^{(k)}$.

Let $h = h'\rho^{-1}$ ($\rho^{-1}$ exists, by lemma 139). Then $hf = h'\rho^{-1}\rho f^{(k)} = g$. Since $\mathrm{Hom}'(S[n+k], M) = \mathrm{Hom}(S[n+k], M)$, we are done. □

## 6.5 Infinite almost-invertible chains

Given any $\mathcal{M}$-sequence:

$$(M_1, m_1) \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \to \dots$$

We say that an indecomposable direct summand $X_1$ of $M_1$ *admits infinitely many almost invertible maps* if there exists, for every $n$, a direct summand $Z$ of $M_n$ such that the restriction of $f_{n-1} \dots f_2 f_1$ from $X_1$ to $Z$ is almost invertible.

**Lemma 140.** *Suppose we have a $\mathcal{M}$-sequence:*

$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \to \dots$$

*-and that an indecomposable direct summand $X_1$ of $M_1$ admits infinitely many almost invertible maps.*

*Then there exists, for every $n \geq 2$, an indecomposable direct summand $X_n$ of $M_n$ such that:*

- *For all $n \geq 1$, the restriction of $f_n$ from $X_n$ to $X_{n+1}$ is almost invertible.*

- *For all $n \geq 1$, the restriction of $f_{n-1} \dots f_2 f_1$ from $X_1$ to $X_n$ is almost invertible.*

*Proof.* We wish to apply lemma 132, with $(\dagger)(X_1, X_2, \ldots, X_n)$ being the conjunction of the following two conditions:

1. The restriction of $f_{n-1} \ldots f_2 f_1$ from $X_1$ to $X_n$ is almost invertible

2. for all $j < n$, the restriction of $f_j$ from $X_j$ to $X_{j+1}$ is almost invertible

By lemma 132, it's enough to prove that, for all $n \geq 1$, there exists $X_1, X_2, \ldots, X_n$ such that $(\dagger)(X_1, X_2, \ldots, X_n)$ holds.

Indeed, given any $n$, there exists a direct summand $X_n$ of $M_n$ such that the restriction of $f_{n-1} \ldots f_1$ from $X_1$ to $X_n$ is almost invertible (since $X_1$ admits infinitely many almost invertible chains).

Furthermore, this map is the sum of all maps of the form $h_{n-1} \ldots h_2 h_1$ corresponding to finite indecomposable subchains:

$$X_1 \stackrel{h_1}{\to} Y_2 \stackrel{h_2}{\to} Y_3 \stackrel{h_3}{\to} \cdots \stackrel{h_{n-2}}{\to} X_{n-1} \stackrel{h_{n-1}}{\to} X_n$$

Then, by lemma 136, at least one such map is almost invertible. And for that finite indecomposable subchain, each $h_i$ must be almost invertible. □

We define any subchain $X_1, X_2, X_3, \ldots,$ satisfying the properties of lemma 140 to be an *almost invertible subchain* of the $\mathcal{M}$-sequence.

The contrapositive of lemma 140 gives us:

**Corollary 37.** *Suppose that we have a $\mathcal{M}$-sequence:*

$$(M_1, x) \stackrel{f_1}{\to} M_2 \stackrel{f_2}{\to} M_3 \to \ldots$$

*-which does not admit an infinite almost invertible subchain. Then there exists $n \geq 1$ such that for all direct summands $X$ of $M_1$ and $Z$ of $M_n$, the restriction of $f_{n-1} \ldots f_2 f_1$ from $X$ to $Z$ is not almost invertible.*

**Lemma 141.** *Suppose that an $\mathcal{M}$-sequence:*

$$(M_1, x) \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \ldots$$

*-is such that, for all $n$, the sequence:*

$$(M_n, f_{n-1}\dots f_1(x)) \xrightarrow{f_n} M_{n+1} \xrightarrow{f_{n+1}} M_{n+2}\dots$$

*-does not admit an infinite almost invertible chain.*

*Then there exists $n$ such that $f_n \dots f_2 f_1(x) = 0$*

*Proof.* Let $n_1 := 1$, and define $n_2, n_3, n_4, \dots \in \mathbb{N}^+$ inductively, as follows: Given any $n_k$, write $M_{n_k}$ as a direct sum of indecomposables:

$$M_{n_k} \simeq \bigoplus_{i=1}^{n} Y_i$$

Since each $Y_i$ does not admit an infinite almost invertible subchain, there exists- by corollary 37- an $n_{k+1} > n_k$ such that, for all direct summands $Z$ of $M_{n_{k+1}}$, the restriction of $f_{n_{k+1}-1}\dots f_{n_k+1} f_{n_k}$ from $Y_i$ to $Z$ is not almost invertible.

Now consider the $\mathcal{M}$-sequence:

$$(M_{n_1}, x) \xrightarrow{g_1} M_{n_2} \xrightarrow{g_2} M_{n_3} \xrightarrow{g_3} \dots$$

Where $g_k := f_{n_{k+1}-1}\dots f_{n_k}$ for all $k \geq 1$. Given any finite indecomposable subchain $X_1, \dots, X_n$, let $(\dagger)(X_1, \dots, X_n)$ be the statement:

$$g_{n-1}\dots g_2 g_1(x) \neq 0$$

Given any infinite indecomposable subchain $X_1, X_2, \dots,$ corollary 34 implies that there exists $n$ such that $X_1 \dots X_n$ doesn't satisfy $(\dagger)$. So, by lemma 132 there exists $n$ such that no infinite indecomposable subchain of length $k$ satisfies $(\dagger)$, and so $g_k \dots g_2 g_1 g_0(x) = 0$. Thus:

$$f_{n_{k+1}}\dots f_2 f_1(x) = 0$$

-as required. ☐

## 6.6 Periodic string modules

Suppose that we have an $\mathcal{M}$-sequence:

$$(M_1, m_1) \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \xrightarrow{f_3} \dots$$

-such that, for all $n \geq 1$, no direct summands of $M_n$ in $\mathcal{A} \cup \mathcal{P}$ admit an infinite almost invertible chain.

We shall prove, in this section, that this sequence is eventually stationary on $M(w)$.

### 6.6.1 Power series rings

Take any band $D = l'_1 \dots l'_m$. Let $w' = \dots l'_{-2} l'_{-1} l'_0 l'_1 l'_2 \dots$ be the $\mathbb{Z}$-word such that $l'_k = l'_{k \bmod m}$ for all $k \in \mathbb{Z}$, and let $\{y_i : i \in \mathbb{Z}\}$ be the standard basis of $M(w')$. We shall denote $M(w')$ as $M({}^\infty D^\infty)$.

Given any $x_1 \in M({}^\infty D^\infty)$, we define:

$$\langle x_1 \rangle (M(w)) := (M({}^\infty D^\infty), x_1)(M(w))$$

Given any $x_1, \dots, x_k$ in $M({}^\infty D^\infty)$, define:

$$\langle x_1, x_2, \dots, x_k \rangle (M(w)) := \langle x_1 \rangle (M(w)) + \dots + \langle x_k \rangle (M(w))$$

These are $E$-submodules of $M(w)$ (where $E := \mathrm{End}'(M(w))$).

We consider the power series rings $K[[T]][T^{-1}]$ and $K[T][[T^{-1}]]$: note that both of these are fields. Let $V$ be an $m$-dimensional $K[[T]][T^{-1}]$-vector space, with basis $e_0, \dots e_{m-1}$, and $V'$ be an $m$-dimensional $K[T][[T^{-1}]]$-vector space, with basis $e'_0, \dots, e'_{m-1}$.

Define a $K$-linear maps $F : M({}^\infty D^\infty) \to V$ and $F' : M({}^\infty D^\infty) \to V'$ by:

$$F\left(\sum_{a \in \mathbb{Z}} \sum_{b=0}^{m-1} \lambda_{am+b} y_{am+b}\right) := \sum_{a \in \mathbb{Z}} \sum_{b=0}^{m-1} \lambda_{am+b} e_b T^a$$

$$F'\left(\sum_{a \in \mathbb{Z}} \sum_{b=0}^{m-1} \lambda_{am+b} y_{am+b}\right) := \sum_{a \in \mathbb{Z}} \sum_{b=0}^{m-1} \lambda_{am+b} e'_b T^a$$

We can also define $K$-linear maps $G : V \to M^+({}^\infty D^\infty)$ and $G' : V' \to M^-({}^\infty D^\infty)$ by:

$$G\left(\sum_{a \in \mathbb{Z}} \sum_{b=0}^{m-1} \lambda_{am+b} T^a e_b\right) := \sum_{a \in \mathbb{Z}} \sum_{b=0}^{m-1} \lambda_{am+b} y_{am+b}$$

$$G'\left(\sum_{a \in \mathbb{Z}} \sum_{b=0}^{m-1} \lambda_{am+b} T^a e_b\right) := \sum_{a \in \mathbb{Z}} \sum_{b=0}^{m-1} \lambda_{am+b} y_{am+b}$$

Of course, the canonical embedding $M({}^\infty D^\infty) \hookrightarrow M^+({}^\infty D^\infty)$ is equal, as a $K$-linear map, to $GF$, and similarly for $G'F'$

For $x_1, \dots, x_k \in M({}^\infty D^\infty)$, we define $\langle F(x_1), \dots, F(x_k)\rangle$ to be the $K[[T]][T^{-1}]$-subspace of $V$ generated by $\{F(x_1), \dots, F(x_n)\}$, and we define $\langle F'(x_1), \dots, F'(x_k)\rangle$ to be the $K[[T^{-1}]][T]$-subspace of $V'$ generated by $\{F'(x_1), \dots, F'(x_n)\}$

**Lemma 142.** *Let $x, x_1, x_2, \dots x_n$ be any elements of $M({}^\infty D^\infty)$. If both $F(x) \in \langle F(x_1), \dots, F(x_k)\rangle$ and $F'(x) \in \langle F'(x_1), \dots, F'(x_k)\rangle$, then:*

$$\langle x\rangle(M(w)) \subseteq \langle x_1, x_2, \dots, x_k\rangle(M(w))$$

*Proof.* Assume that $F(x) \in \langle F(x_1), \dots, F(x_k)\rangle$ and $F'(x) \in \langle F'(x_1), \dots, F'(x_k)\rangle$. It will be enough, by lemma 134- to prove that, given any simple string map $g : M({}^\infty D^\infty) \to M(w)$, there exists $g_1, g_2, \dots, g_k \in \mathrm{Hom}'(M({}^\infty D^\infty), M(w))$ such that $\sum_{i=1}^k g_i(x_i) = g(x)$.

Recall- from section 5.5- that every simple string map from $M({}^\infty D^\infty)$ to $M(w)$ looks like:

$$M({}^\infty D^\infty) \twoheadrightarrow M(C) \hookrightarrow M(w)$$

-where $C$ is a post subword of ${}^\infty D^\infty$ and a pre-subword of $w$, and the two maps are the natural projection and the natural embedding. Since ${}^\infty D^\infty$ is not a subword of $w$ or $w^{-1}$, the map is not an embedding, and so it must factor through one of the following two canonical projections:

$$M({}^\infty D^\infty) = M(w') \twoheadrightarrow M(w'_j) \text{ for some } j \in \mathbb{Z}$$

$$M({}^\infty D^\infty) = M(w') \twoheadrightarrow M((u'_j)^{-1}) \text{ for some } j \in \mathbb{Z}$$

We may assume, without loss of generality, that it is the latter, and that $j = 0$. We refer to the module $M((u'_0)^{-1})$ as $M({}^\infty D)$, and the canonical projection $M(w') \to M((u'_0)^{-1})$ as $\pi : M({}^\infty D^\infty) \to M({}^\infty D)$. It suffices to find $g_1, \dots, g_k \in \mathrm{Hom}'(M({}^\infty D^\infty), M({}^\infty D)$ such that $\sum_{i=1}^k g_i(x_i) = \pi(x)$

Since $F(x) \in \langle F(x_1), \dots, F(x_k)\rangle$, there exists $a_1, \dots, a_k \in k[[T]][T^{-1}]$ such that $\sum_{i=1}^k a_i F(x_i) = F(x)$. Write each $a_i$ as: $\sum_{j\geq n_i} \lambda_{ij} T^j$.

Let $\Phi : M({}^{\infty}D^{\infty}) \to M({}^{\infty}D^{\infty})$ be the simple string map taking every $y_j$ to $y_{j+m}$. For every $i \le k$, let $J_i := \{j \ge n_i : \pi\Phi^j(x_i) \ne 0\}$- noting that $J_i$ is finite.

Define $g_i' \in \mathrm{End}'(M({}^{\infty}D^{\infty}))$ by:

$$g_i' := \sum_{j \in J_i} \lambda_{ij}\Phi^j$$

Of course, $\pi g_i' \in \mathrm{Hom}'(M({}^{\infty}D^{\infty}), M((u_0')^{-1}))$.

Let $h : M({}^{\infty}D^{\infty}) \hookrightarrow M^+({}^{\infty}D^{\infty})$ denote the canonical embedding, and let $\pi' : M^+({}^{\infty}D^{\infty}) \twoheadrightarrow M({}^{\infty}D)$ be the canonical projection. Of course, $\pi = \pi' h$.

For each $i \in \{1, 2, \ldots, n\}$ define $h_i \in \mathrm{Hom}(M({}^{\infty}D^{\infty}), M^+({}^{\infty}D^{\infty}))$ by:

$$h_i := \sum_{j \ge N} \lambda_{ij} h\Phi^j$$

-where $N = \max\{\mathrm{n_i : i \le k}\}$. Then:

$$\pi'(h_i(x_i) - hg_i')(x_i) = \pi' \sum_{j > \max J_i} \lambda_{ij} h(\Phi^j(x_j)) = 0$$

-by definition of $J_i$. So:

$$\begin{aligned}
\sum_{i=1}^{k} \pi g_i'(x_i) &= \sum_{i=1}^{k} \pi' h g_i'(x_i) \\
&= \pi' \sum_{i=1}^{k} h_i(x_i) \\
&= \pi' \sum_{i=1}^{k} G(a_i F(x_i)) \\
&= \pi' G \sum_{i=1}^{k} (a_i F(x_i)) \\
&= \pi' GF(x) \\
&= \pi(x)
\end{aligned}$$

-as required. □

Given any $\mathrm{End}'(M({}^{\infty}D^{\infty}))$-submodule $M_1$ of $M({}^{\infty}D^{\infty})$, we define $V(M_1)$ to be the $K[[T]][T^{-1}]$-subspace of $V$ given by:

$$V(M_1) := \bigcup_{x \in M_1} \langle F(x) \rangle (V)$$

And we define $V'(M_1)$ to be the $K[[T^{-1}]][T]$-subspace of $V'$:

$$V'(M_1) := \bigcup_{x \in M_1} \langle F(x) \rangle (V)$$

-and we define $[M_1](M(w))$ to be the $E$-submodule of $M(w)$:

$$[M_1](M(w)) := \{f(m) : m \in M, f \in \mathrm{Hom}'(M_1, M(w))\}$$

**Corollary 38.** *Let $M_1 \supseteq M_2 \supseteq M_3 \supseteq M_4 \supseteq \ldots$ be any descending chain of* $\mathrm{End}'(M({}^\infty D^\infty))$*-submodules of $M({}^\infty D^\infty)$.*

*Then there exists $k \geq 1$ such that $[M_n](M(w)) = [M_k](M(w))$ for all $n \geq k$.*

*Proof.* Consider the descending chain of $K[[T]][T^{-1}]$-subspaces of $V$:

$$V(M_1) \geq V(M_2) \geq V(M_3) \geq \ldots$$

Since $V$ is finite dimensional (over $K[[T]][T^{-1}]$), there exists $k \geq 1$ such that $V(M_j) = V(M_k)$ for all $j \geq n$.

Similarly, there exists $k' \geq 1$ such that $V'(M_j) = V'(M_{k'})$ for all $j \geq n$. Assume, without loss of generality, that $k \geq k'$. It follows, from lemma 142, that $[M_j](M(w)) = [M_k](M(w))$ for all $j \geq k$. $\square$

### 6.6.2 Implications for $\mathcal{M}$-sequences

Suppose that we have an $\mathcal{M}$-sequence:

$$(M_1, m_1) \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \xrightarrow{f_3} \ldots$$

-such that, for all $n \geq 1$, no direct summands of $M_n$ in $\mathcal{A} \cup \mathcal{B}$ admit an infinite almost invertible chain.

Write each $M_i$ as $A_i \oplus N_i$ - where $A_i$ is the direct sum of all summands of $M_i$ which admit an infinite almost invertible chain- of course, $A_i \in \mathrm{add}(\mathcal{P})$.

By repeatedly applying corollary 38 and corollary 37, we can find a subsequence $k_1 < k_2 < k_3 < \ldots$ of $\mathbb{N}$ (with $k_1 = 1$) such that, for all $j \geq 1$:

- Given any indecomposable direct summand $N$ of $A_{k_j}$, and any $i \geq k_{j+1}$:

$$[(M_i, m_i)(N)](M(w)) = [(M_{k_{j+1}}, m_{k_{j+1}})(N)](M(w))$$

- Given any indecomposable direct summands $Y$ and $Z$ of $N_{k_j}$ and $A_{k_{j+1}} \oplus N_{k_{j+1}}$ respectively, the restriction of $f_{k_{k+1}-1} \dots f_{k_j}$ from $Y$ to $Z$ is not almost invertible.

We wish to prove that the sequence:

$$(A_{k_1} \oplus N_{k_1}, m_1) \overset{f_{k_2-1} \dots f_{k_1+1} f_{k_1}}{\longrightarrow} A_{k_2} \oplus N_{2_1} \overset{f_{k_3-1} \dots f_{k_2+1} f_{k_2}}{\longrightarrow} A_{k_3} \oplus N_{k_3} \overset{f_{k_3-1} \dots f_{k_2+1} f_{k_2}}{\longrightarrow} \dots$$

-is eventually stationary on $M(w)$. In the interests of easing notation, we relabel it as:

$$(A_1 \oplus N_1, (a_1, n_1)) \xrightarrow{F_1} A_2 \oplus N_2 \xrightarrow{F_2} A_3 \oplus N_3 \xrightarrow{F_3} \dots$$

Write each map $F_k$ as:

$$\begin{pmatrix} f_k & g_k \\ \rho_k & h_k \end{pmatrix} : \begin{pmatrix} A_k \\ N_k \end{pmatrix} \to \begin{pmatrix} A_{k+1} \\ N_{k+1} \end{pmatrix}$$

Decompose $A_i$ as $L_i \oplus B_i$- where every direct summand of $L_i$ is (isomorphic to) a direct summand of $A_{i-1}$, and every direct summand of $B_i$ is not. For example, if:

$$A_i \cong M({}^\infty D^\infty) \text{ and } A_{i+1} \cong M({}^\infty D^\infty) \oplus M({}^\infty D^\infty) \oplus M({}^\infty C^\infty)$$

(with $M({}^\infty C^\infty) \ncong M({}^\infty D^\infty)$ ), then: $L_{i+1} = M({}^\infty D^\infty) \oplus M({}^\infty D^\infty)$ and $B_{i+1} = M({}^\infty C^\infty)$.

Notice that, for all $k \geq 2$, and all $j \geq k$:

$$[(A_j \oplus N_j, (a_j, n_j))(L_k)](M(w)) = [(A_k \oplus N_k, (a_k, n_k))(L_k)](M(w))$$

We define $i_{B_k}, i_{L_k}, \pi_{B_k}, \pi_{L_k}$ to be the following canonical embeddings and projections:

$$i_{B_k} : B_k \hookrightarrow A_k$$

$$i_{L_k} : L_k \hookrightarrow A_k$$

$$\pi_{B_k} : A_k \twoheadrightarrow B_k$$

$$\pi_{L_k} : A_k \twoheadrightarrow L_k$$

**Lemma 143.** *The $\mathcal{M}$ sequence:*

$$(A_1 \oplus N_1, (a_1, n_1)) \overset{F_1}{\to} A_2 \oplus N_2 \overset{F_2}{\to} A_3 \oplus N_3 \overset{F_3}{\to} \dots$$

*(as defined above) is equivalent to the $\mathcal{M}$-sequence:*

$$(A_1 \oplus N_1, (a_1, n_1)) \overset{G_1}{\to} A_2 \oplus N_2 \overset{G_2}{\to} L_2 \oplus A_3 \oplus N_3 \overset{G_3}{\to} L_2 \oplus L_3 \oplus A_4 \oplus N_4 \overset{G_4}{\to} \dots$$

*Where $G_1 = F_1$ and, for all $k \geq 2$, $G_k$ is the map:*

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \pi_{L_k} & 0 \\ 0 & f_k i_{B_k} \pi_{B_k} & g_k \\ 0 & \rho_k i_{B_k} \pi_{B_k} & h_k \end{pmatrix} : \begin{pmatrix} L_2 \oplus \dots L_{k-1} \\ A_k \\ N_k \end{pmatrix} \to \begin{pmatrix} L_2 \oplus \dots \oplus L_{k-1} \\ L_k \\ A_{k+1} \\ N_{k+1} \end{pmatrix}$$

*Proof.* First of all, define $a_k \in A_k$ and $n_k \in N_k$ to be such that:

$$F_k(a_k, n_k) = \begin{pmatrix} f_k & g_k \\ \rho_k & h_k \end{pmatrix} (a_k, n_k)^t = (a_{k+1}, n_{k+1})$$

Also, define $a'_k \in A_k$, $n'_k \in N_k$ and $l_k \in L_k$ to be such that $a'_1 = a_1$, $n'_1 = n_1$ and, for all $k \geq 1$:

$$G_k(l_2, \dots l_{k-1}, a'_k, n'_k) = (l_2, \dots, l_{k-1}, l_k, a'_{k+1} n'_{k+1})$$

We have to prove that, for all $k$:

$$(A_k \oplus N_k, (a_k, n_k))(M(w)) = (L_2 \oplus \dots L_{k-1} \oplus A_k \oplus N_k, (l_2, \dots l_{k-1}, a'_k, n'_k))(M(w))$$

First of all, we prove by induction on $k \geq 1$, that there exist maps:

$$\tau_k \in \mathrm{Hom}'(L_2 \oplus \dots L_{k-1}, A_k)$$

$$\tau'_k \in \mathrm{Hom}'(L_2 \oplus \dots L_{k-1}, N_k)$$

-taking $(l_2, \dots l_{k-1})$ to $(a_k - a'_k)$ and $(n_k - n'_k)$ respectively. Note that the $k = 1$ case is vacuous.

Assume that $\tau_k$ and $\tau'_k$ exist. Then:

$$\begin{aligned} a_{k+1} - a'_{k-1} &= f_k(a_k) + g_k(n_k) - f_k(i_{B_k} \pi_{B_k}(a'_k)) - g_k(n'_k) \\ &= f_k(a_k - a'_k) + i_{L_k} \pi_{L_k}(a'_k) + g_k(n_k - n'_k) \\ &= f_k(\tau_k(l_2, \dots l_{k-1})) + i_{L_k}(l_k) + g_k(\tau'(l_2, \dots, l_{k-1})) \end{aligned}$$

So define $\tau_{k+1}$ to be:

$$\begin{pmatrix} f_k\tau_k + g_k\tau_k', & i_{L_k} \end{pmatrix} : \begin{pmatrix} L_2 \oplus \cdots \oplus L_{k-1} \\ L_k \end{pmatrix} \to (A_{k+1})$$

-we can define $\tau'_{k+1}$ similarly.

Consequently, the map:

$$\begin{pmatrix} \tau_k & 1 & 0 \\ \tau_k' & 0 & 1 \end{pmatrix} : \begin{pmatrix} L_2 \oplus \cdots \oplus L_{k-1} \\ A_k \\ N_k \end{pmatrix} \to \begin{pmatrix} A_k \\ N_k \end{pmatrix}$$

takes $(l_2, \ldots l_{k-1}, a_k', n_k')$ to $(a_k, n_k)$. And so:

$$(A_k \oplus N_k, (a_k, n_k))(M(w) \subseteq (L_2 \oplus \cdots \oplus L_{k-1} \oplus A_k \oplus N_k, (l_2, \ldots, l_k, a_k', n_k'))(M(w))$$

We now prove, by induction on $k$, that:

$$(L_2 \oplus \cdots \oplus L_{k-1} \oplus A_k \oplus N_k, (l_2, \ldots, l_k, a_k', n_k'))(M(w)) \subseteq (A_k \oplus N_k, (a_k, n_k))(M(w)$$

Assume that we have the result for all $j \leq k$. Then, for all $j \leq k$:

$$\begin{aligned} (L_j, l_j)(M(w)) &\subseteq [(A_j, a_j')(L_j)](M(w)) && (6.1) \\ &\subseteq [(A_j \oplus N_j, (a_j, n_j))(L_j)](M(w)) && (6.2) \\ &\subseteq [(A_{k+1} \oplus N_{k+1}, (a_j, n_j))(L_j)](M(w)) && (6.3) \end{aligned}$$

(1) holds because $l_j = \pi_{L_j}(a_j')$, (2) holds by the induction hypothesis, and (3) follows from the observation just before the start of the lemma.

Now, given any $h \in \mathrm{Hom}'(A_{k+1}, M(w))$, consider the map:

$$(-h\tau_{k+1}, h) : \begin{pmatrix} L_2 \oplus \cdots \oplus L_k \\ A_{k+1} \end{pmatrix} \to M(w)$$

It takes $(l_2, \ldots, l_k, a_k)$ to $-h(a_{k+1} - a'_{k+1}) + h(a_{k+1}) = h(a'_{k+1})$, and so:

$$(A_{k+1}, a'_{k+1})(M(w)) \subseteq (L_2 \oplus \cdots \oplus L_k \oplus A_{k+1}, (l_2, \ldots, l_k, a_{k+1}))(M(w))$$

Thus:

$$(A_{k+1}, a'_{k+1})(M(w)) \subseteq (A_{k+1} \oplus N_{k+1}, (a_{k+1}, n_{k+1}))(M(w))$$

-as required. Similarly, one can show that, for all $k$:

$$(N_k, n'_k)(M(w)) \subseteq (A_k \oplus N_k, (a_k, n_k))(M(w))$$

-which completes the proof.

□

**Lemma 144.** *Let $Y$ and $Z$ be direct summands of $B_k \oplus N_k$ and $B_{k+2} \oplus N_{k+2}$ respectively. Then the restriction of $G_{k+1}G_k$ from $Y$ to $Z$ is not almost invertible.*

*Proof.* It is enough to prove that, given any direct summand $Z'$ of $A_{k+1} \oplus N_{k+1}$, at least one of the following holds:

- The restriction of $G_k$ from $Y$ to $Z'$ is not almost invertible
- The restriction of $G_{k+1}$ from $Z'$ to $Z$ is not almost invertible.

If $Y$ is a direct summand of $N_k$, then the restriction of $G_k$ from $Y$ to $Z'$ is equal to the restriction of $F_k$ from $Y$ to $Z'$. This is not almost invertible, by one of the properties of $N_k$.

If $Y$ is a direct summand of $A_k$, then there are three possibilities:

1. If $Z'$ is a direct summand of $N_{k+1}$ then the restriction of $G_{k+1}$ from $Z'$ to $Z$ is not almost-invertible (as above).
2. If $Z'$ is a direct summand of $N_{k+1}$, and $Z' \cong Y$, then $Z'$ must be a direct summand of $L_{k+1}$, so $\pi_{B_{k+1}}$ takes $Z'$ to zero. Thus $f_{k+1}i_{B_{k+1}}\pi_{B_{k+1}}$ and $\rho_{k+1}i_{B_{k+1}}\pi_{B_{k+1}}$ both take $Z'$ to zero- so the restriction of $G_{k+1}$ from $Z'$ to $Z$ is zero.
3. If If $Z'$ is a direct summand of $N_{k+1}$, and $Z' \ncong Y$, then there cannot be any almost invertible maps in $\mathrm{Hom}'(Y, Z')$.

□

**Corollary 39.** *Suppose we have a $\mathcal{M}$-sequence:*

$$(M_1, m_1) \xrightarrow{F_1} M_2 \xrightarrow{F_2} M_3 \xrightarrow{F_3} \dots$$

*-such that, for all $n \geq 1$, only direct summands of $M_n$ in $\mathcal{P}$ admit an infinite almost-invertible chain.*

*Then the sequence is eventually stationary on $M(w)$.*

*Proof.* It is enough to show that the sequence:

$$(A_1 \oplus N_1, (a_1, n_1)) \xrightarrow{G_1} A_2 \oplus N_2 \xrightarrow{G_2} L_2 \oplus A_3 \oplus N_3 \xrightarrow{G_3} L_2 \oplus L_3 \oplus A_4 \oplus N_4 \xrightarrow{G_4} \dots$$

-as defined in lemma 143 is eventually stationary.

First of all, consider the $\mathcal{M}$-sequence:

$$(A_1 \oplus N_1, (a'_1, n'_1)) \xrightarrow{H_1} A_3 \oplus N_3 \xrightarrow{H_2} A_5 \oplus N_5 \xrightarrow{H_3} \dots$$

-where $H_i$ is the restriction of $G_{2i-1}$ from $A_{2i-1}$ to $A_{2i+1}$. Notice that, for all $k$:

$$H_k \dots H_1(a'_1, n'_1) = (a'_{2k+1}, n'_{2k+1})$$

By lemma 144, no direct summand of any $A_{2i-1} \oplus N_{2i-1}$ admits an infinite almost invertible chain- so by lemma 141, there exists $k$ such that $(a'_j, n'_j) = (0, 0)$ for all $j \geq k+1$. Furthermore, $l_j = 0$ for all $j \geq k+1$.

So, given any $j \geq k+1$:

$$\begin{aligned} & (L_2 \oplus \dots \oplus L_{j-1} \oplus A_j \oplus N_j, (l_2, \dots, l_{j-1}, a'_j, n'_j))(M(w)) \\ = \quad & (L_2 \oplus \dots \oplus L_{j-1} \oplus A_j \oplus N_j, (l_2, \dots, l_{k-1}, l_k, 0, 0, \dots 0))(M(w)) \\ = \quad & (L_2 \oplus \dots \oplus L_k, (l_2, \dots, l_k, ))(M(w)) \end{aligned}$$

-so the sequence is indeed eventually stationary on $M(w)$. □

## 6.7 $\mathcal{M}$-sequences are eventually stationary

In order to prove that every $\mathcal{M}$-sequence is eventually stationary, we show how any $\mathcal{M}$-sequence can be "split" into two $\mathcal{M}$-sequences, such that one looks like the ones in corollary 39, and the other is equivalent to an $\mathcal{M}$-sequence of the form:

$$(M, m) \xrightarrow{1_M} M \xrightarrow{1_M} M \xrightarrow{1_M} \dots$$

Since both such sequences are eventually stationary, the result will follow.

**Lemma 145.** *Take any $\mathcal{M}$-sequence:*

$$(M_1, m_1) \xrightarrow{F_1} M_2 \xrightarrow{F_2} M_3 \to \dots$$

*Suppose that $M_1$ has a direct summand $X_1$ in $\mathcal{A} \cup \mathcal{B}$ which admits an infinite almost invertible sequence:*

$$X_1 \to X_2 \to \dots$$

*-and that, if $X_1 \in \mathcal{B}$, then- writing each $X_i$ as $S^D_\lambda[k_i]$- every $M_{i+1}$ has no direct summands of the form $S^D_\lambda[n]$ with $n < k_i$.*

*Then there exists an equivalent spanning sequence of the form:*

$$(X_1 \oplus Y_1, (x_1, y_1)) \to X_2 \oplus Y_2 \to X_3 \oplus Y_3 \to \dots$$

*-where $X_i \oplus Y_i \simeq M_i$ for all $i$, and each map looks like:*

$$\begin{pmatrix} f'_i & g'_i \\ \rho'_i & h'_i \end{pmatrix} : \begin{pmatrix} X_i \\ Y_i \end{pmatrix} \to \begin{pmatrix} X_{i+1} \\ Y_{i+1} \end{pmatrix}$$

*-where $f'_i$is almost invertible, and $\rho'_i(x_i) = 0$.*

*Proof.* First of all, we can write each module $(M_i, m_i)$ as $(X_i \oplus Y_i, (x_i, y_i))$, and each $F_i$ as:

$$\begin{pmatrix} f_i & g_i \\ \rho_i & h_i \end{pmatrix} : \begin{pmatrix} X_i \\ Y_i \end{pmatrix} \to \begin{pmatrix} X_{i+1} \\ Y_{i+1} \end{pmatrix}$$

$f_i$ is the restriction of $F_1$ from $X_1$ to $X_2$- by our assumption, it is almost invertible. So, by corollary 35 or corollary 36, there exists a map $\tau \in \mathrm{Hom}'(X_2, Y_2)$ such that $\tau f_1(x_1) = \rho_1(x_1)$.

For all $i \geq 2$, let $x_i \in X_i$ and $n_i \in N_i$ be such that $F_{i-1}(x_{i-1}, y_{i-1}) = (x_i, y_i)$. Let $x'_2 := x_2$ and $y'_2 := h(y_1) - \tau g_1(y_1)$. Notice that the map:

$$F := \begin{pmatrix} 1 & 0 \\ -\tau & 1 \end{pmatrix} : \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix} \to \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix}$$

-is invertible, and takes $(x_2, y_2)$ to $(x'_2, y'_2)$. So :

$$(X_2 \oplus Y_2, (x_2, y_2))(M(w)) = (X_2 \oplus Y_2, (x'_2, y'_2))(M(w))$$

Now consider the sequence:

$$(X_1 \oplus Y_1, (x_1, y_1)) \stackrel{FF_1}{\to} X_2 \oplus Y_2 \stackrel{F_2F^{-1}}{\to} X_3 \oplus Y_3 \stackrel{F_3}{\to} X_4 \oplus Y_4 \stackrel{F_4}{\to} \dots$$

It is equivalent to the original spanning sequence. Also, $FF_1$ is the map:

$$\begin{pmatrix} f_1 & g_1 \\ \rho_1 - \tau f_1 & h_1 - \tau g_1 \end{pmatrix}$$

-with $\rho_1 - \tau f_1(x_1) = 0$.

Finally, given any $n \geq 3$, we claim that the restriction of $F_{n-1} \dots F_2F^{-1}$ from $X_2$ to $X_n$ is almost invertible: if we have that, then we can induct the argument, to find the the remaining maps.

Let $G$ denote the restriction of $F_{n-1} \dots F_2$ from $X_2$ to $X_n$, and $H$ the restriction of $F_{n-1} \dots F_3$ from $Y_2$ to $X_n$.

The restriction of $F_n \dots F_1$ from $X_1$ to $X_n$ is given by $Gf_1 + H\rho_1$. By our assumptions, this is almost invertible.

Now, since $\rho_1 - \tau f_1(x_1) = 0$, the map:

$$(Gf_1 + H\rho_1) - (Gf_1 + H\tau f_1)$$

-is not an embedding, and hence is not almost invertible. Thus, by lemma 136, $Gf_1 + H\tau f_1$ is almost invertible. By lemma 136, $G + H\tau$ is almost invertible.

Since $G + H\tau$ is the restriction of $F_{n-1} \dots F_2F^{-1}$ from $X_2$ to $X_n$, we are done. □

### 6.7.1 Rearranging band modules

**Lemma 146.** *Take any $m, n, k \geq 1$, any non-zero $\lambda \in K$, and any band $D$. Then, for all $g \in \mathrm{Hom}'(S^D_\lambda[n], S^D_\lambda[m])$, there exists $h \in \mathrm{Hom}'(S^D_\lambda[n+k], S^D_\lambda[m+k])$ such that $f^{(k)}g = hf^{(k)}$.*

*Moreover, $h$ is almost invertible if and only if $g$ is.*

*Proof.* Assume that $g$ is a simple string map. There are two possibilities for what $g$ looks like:

- Suppose $g$ is a map of the form:

$$S[n] \xrightarrow{f} \overline{M}({}^\infty D^\infty) \xrightarrow{f'} M({}^\infty D^\infty) \to S[m]$$

-where $f'$ is a simple string map with finite dimensional image. Since $M({}^\infty D^\infty)$ is indecomposable, it follows from corollary 36) that $f'f$ factors through $f^{(k)} \in \mathrm{Hom}'(S[n], S[n+k])$. Consequently, so does $g$, and so does $f^{(k)}g$. Let $h \in \mathrm{Hom}'(S^D_\lambda[n+k], S^D_\lambda[m+k])$ be such that $f^{(k)}g = hf^{(k)}$.

- Suppose that $g$ is a map of the form $f^{(i+m-n)}g^{(i)}$, for some $i < n$. Then, by considering the almost split exact sequences in the tube look like, we get:

$$f^{(k)}f^{(i+m-n)}g^{(i)} = (-1)^k f^{(i+m-n)}g^{(i)}f^{(k)}$$

Finally, since $f^{(k)}$ is almost invertible, and $hf^{(k)} = f^{(k)}g$, lemma 136 gives that $g$ is almost invertible if and only if $h$ is. □

Any $M \in \mathrm{add}(\mathcal{M})$ can be uniquely decomposed in the form:

$$\bigoplus_{j\in J_0} M_j \oplus \bigoplus_{i\in I_0} S^{D_i}_{\lambda_i}[n_i]$$

-where $I_0$ and $J_0$ are finite sets, and $M_j \in \mathcal{A} \cup \mathcal{P}$ for all $\in J_0$. We define $M[+k]$ to be the module:

$$\bigoplus_{j\in J_0} M_j \oplus \bigoplus_{i\in I_0} S^{D_i}_{\lambda_i}[n_i+k]$$

We denote by $f^{(k)} : M \hookrightarrow M[+k]$ the unique map such that:

- For all $j \in J_0$, the restriction of $f^{(k)}$ from $M_j$ to $M_j$ is the identity.

- For all $i \in I_0$, the restriction of $f^{(k)}$ from $S^{D_i}_{\lambda_i}[n_i]$ to $S^{D_i}_{\lambda_i}[n_i+k]$ is the map $f^{(k)}$ associated with that tube.

- For all other pairs of indecomposable direct summands $X$ (of $M$) and $Y$ (of $M[+k]$), the restriction of $f^{(m)}$ from $X$ to $Y$ is zero.

Given any such map, and any $x \in M$, we shall refer to $f^{(k)}(x)$ as $x$.

**Lemma 147.** *Take any $M, N \in \operatorname{add}(\mathcal{M})$, any $m \in M$, and any $k \geq 1$.*

*Then for all $g \in \operatorname{Hom}'(M, N)$ there exists $h \in \operatorname{Hom}'(M[+k], N[+k])$ such that the following diagram commutes:*

$$\begin{tikzcd}
M \arrow[r, "g"] \arrow[d, "f^{(k)}"] & N \arrow[d, "f^{(k)}"] \\
M[+k] \arrow[r, "h"] & N[+k]
\end{tikzcd}$$

*Proof.* Assume that both $M$ and $N$ are indecomposable. If $M \in \mathcal{A} \cup \mathcal{P}$, then $M[+k] \simeq M$- so the result is vacuous.

Suppose, therefore, that $M \in \mathcal{B}$. If $N \in \mathcal{B}$, then we apply lemma 146. If $N \in \mathcal{A} \cup \mathcal{P}$ then we apply corollary 36. □

Of course, taking $N = M(w)$ in lemma 147 gives:

**Corollary 40.** *Take any $M \in \operatorname{add}(\mathcal{M})$, and any $m \in M$. Then, for all $k \in \mathbb{N}$:*

$$(M, m)(M(w)) = (M[+k], m)(M(w))$$

**Lemma 148.** *Suppose that an $\mathcal{M}$-sequence:*

$$(M_1, m_1) \xrightarrow{F_1} M_2 \xrightarrow{F_2} M_3 \xrightarrow{F_3} \ldots$$

*-admits an infinite almost invertible chain of the form:*

$$S_\lambda^D[k_1] \to S_\lambda^D[k_2] \to S_\lambda^D[k_3] \to \ldots$$

*Let $X_i$ be such that $S[k_i] \oplus X_i \cong M_i$ for all $i$. Then there exists an equivalent $\mathcal{M}$-sequence of the form:*

$$\begin{aligned}
(S_\lambda^D[k_1] \oplus X_1, (s_1, x_1)) &\xrightarrow{G_1} S_\lambda^D[k_2] \oplus X_2[+k_2] \\
&\xrightarrow{G_2} S_\lambda^D[k_2 + k_3] \oplus X_3[+k_2 + k_3] \\
&\xrightarrow{G_3} S_\lambda^D[k_2 + k_3 + k_4] \oplus X_4[+k_2 + k_3 + k_4] \xrightarrow{G_4} \ldots
\end{aligned}$$

*-such that the following indecomposable subchain is almost invertible:*

$$S[k_1] \to S[k_2] \to S[k_2 + k_3] \to S[k_2 + k_3 + k_4] \to \ldots$$

*Proof.* Let $m_j = F_{j-1} \dots F_1(m_1)$ for all $j \geq 1$. Given any $n \geq 1$, let $Y_n$ be such that $M_n \cong S^D_\lambda[k_n] \oplus Y_n$. Let $\rho_n$ be the map:

$$\begin{pmatrix} 1 & 0 \\ 0 & f^{(k_n)} \end{pmatrix} : S^D_\lambda[k_n] \oplus Y_n \hookrightarrow S^D_\lambda[k_n] \oplus Y_n[+k_n]$$

For all $j > n$, let $\rho_j$ be the map:

$$\begin{pmatrix} f^{(k_n)} & 0 \\ 0 & f^{(k_n)} \end{pmatrix} : S^D_\lambda[k_j] \oplus Y_j \hookrightarrow S^D_\lambda[k_j + k_n] \oplus Y_n[+k_n] \text{ for all } j > n$$

By lemma 147, there exists, for every $j \geq n$, a map:

$$H_j \in \mathrm{Hom}'(S^D_\lambda[k_j] \oplus Y_j[+k_n], S^D_\lambda[k_{j+1} + k_n] \oplus Y_{j+1}[+k_n]) \text{ for all } j > n$$

-such that $H_j \rho_j = \rho_{j+1} F_j$. Consider the $\mathcal{M}$-sequence:

$$\begin{aligned}
(M_1, m_1) \xrightarrow{F_1} \dots \xrightarrow{F_{n-2}} M_{n-1} \quad &\xrightarrow{\rho_n F_{n-1}} \quad S^D_\lambda[k_n] \oplus M_n[+k_n] \\
&\xrightarrow{H_n} \quad S^D_\lambda[k_{n+1} + k_n] \oplus M_{n+1}[+k_n] \\
&\xrightarrow{H_{n+1}} \quad S^D_\lambda[k_{n+2} + k_n] \oplus M_{n+2}[+k_n] \\
&\xrightarrow{H_{n+2}} \quad S^D_\lambda[k_{n+3} + k_n] \oplus M_{n+3}[+k_n] \to \dots
\end{aligned}$$

Given any $j \geq n$, $H_{j-1} \dots H_n \rho_n = \rho_j F_{j-1} \dots F_n$, and so:

$$H_{j-1} \dots H_n \rho_n F_{n-1} \dots F_1(m_1) = \rho_j F_{j-1} \dots F_n F_{n-1} \dots F_1(m_1)$$

It follows from corollary 40 that this $\mathcal{M}$-sequence is equivalent to the original one.

Finally, since $\rho_j$ and the restriction of $F_{j-1} \dots F_n F_{n-1} \dots F_1$ from $S^D_\lambda[k_1]$ to $S^D_\lambda[k_j]$ is almost invertible, so is the restriction of $H_{j-1} \dots H_n \rho_n F_{n-1} \dots F_1$ from $S^D_\lambda[k_1]$ to $S^D_\lambda[k_j + k_n]$. Thus the subchain:

$$S^D_\lambda[k_1] \to \dots \to S^D_\lambda[k_n] \to S^D_\lambda[k_{n+1} + k_n] \to S^D_\lambda[k_{n+2} + k_n] \to \dots$$

-is an almost invertible subchain. $\square$

### 6.7.2 The proof

Let $\bigoplus_{i=1}^{k} M_k$ and $\bigoplus_{i=1}^{k} N_k$ be modules in $\mathrm{add}(\mathcal{A} \cup \mathcal{B})$. We say that a map $f \in \mathrm{Hom}'(\bigoplus_{i=1}^{k} M_k, \bigoplus_{i=1}^{k} N_k)$ is *almost invertible* if it looks like:

$$\begin{pmatrix} f_1 & 0 & \dots & 0 \\ 0 & f_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & f_k \end{pmatrix} : \begin{pmatrix} M_1 \\ M_2 \\ \vdots \\ M_k \end{pmatrix} \to \begin{pmatrix} N_1 \\ N_2 \\ \vdots \\ N_k \end{pmatrix}$$

-with each $f_i$ being almost invertible. Notice that, for all $m \in \bigoplus_{i=1}^{k} M_k$:

$$(\bigoplus_{i=1}^{k} M_k, m)(M(w)) = (\bigoplus_{i=1}^{k} N_k, f(m))(M(w))$$

**Theorem 48.** *Every $\mathcal{M}$-sequence is eventually stationary on $M(w)$.*

*Proof.* Take any $\mathcal{M}$-sequence:

$$(M_1, m_1) \xrightarrow{F_1} M_2 \xrightarrow{F_2} M_3 \xrightarrow{F_3} \dots$$

By applying lemma 145 and lemma 148 repeatedly, we can obtain an equivalent spanning sequence of the form:

$$(I_{1,1} \oplus N_1, (x_1, n_1)) \to I_{2,1} \oplus I_{2,2} \oplus N_2 \to I_{3,1} \oplus I_{3,2} \oplus I_{3,3} \oplus M_3 \to \dots$$

-such that, for all $k \geq 1$:

- $I_{k,j} \in \mathrm{add}(\mathcal{A} \cup \mathcal{B})$ (for all $j \leq k$)
- Every map looks like:

$$\begin{pmatrix} f_{k,1} & 0 & 0 & \dots & 0 & g_{k,1} \\ 0 & f_{k,2} & 0 & \dots & 0 & g_{k,2} \\ 0 & 0 & f_{k,3} & \dots & 0 & g_{k,3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & f_{k,k} & g_{k,k} \\ 0 & 0 & 0 & \dots & 0 & \rho_k \\ 0 & 0 & 0 & \dots & 0 & h_k \end{pmatrix} : \begin{pmatrix} I_{k,1} \\ I_{k_2} \\ I_{k,3} \\ \vdots \\ I_{k,k} \\ N_k \end{pmatrix} \to \begin{pmatrix} I_{k+1,1} \\ I_{k+1_2} \\ I_{k+1,3} \\ \vdots \\ I_{k+1,k} \\ I_{k+1,k+1} \\ N_{k+1} \end{pmatrix}$$

  -with each $f_{k,i}$ being almost invertible.

- No direct summand of $N_k$ in $\mathcal{A} \cup \mathcal{B}$ admits an invertible chain in the sequence:

$$N_k \xrightarrow{h_k} N_{k+1} \xrightarrow{h_{k+1}} N_{k+2} \xrightarrow{h_k+2} \dots$$

Let $x_{1,1} \in I_{1,1}$ $n_1 \in N_1$ be such that $m_1$ is the element $(x_{1,1}, n_1)$. For all $k \geq 1$, define $x_{k,1} \in I_{k,1}, \dots, x_{k,k} \in I_{k,k}$ and $n_k \in N_k$ to be such that, for all $k$:

$$F_k(x_{k,1}, \dots, x_{k,k}, n_k) = (x_{k+1,1}, \dots, x_{k+1,k+1}, n_{k+1})$$

By corollary 38, there exists $k \geq 1$ such that:

$$(N_k, n_k)(M(w)) = (N_j, n_j)(M(w)) \text{ for all } j \geq k$$

Take any $j \geq k$. We claim that:

$$\begin{aligned} &(\bigoplus_{i \leq j} I_{j,i} \oplus N_j, (x_{j,1}, \dots, x_{j,j}, n_j))(M(w)) \\ = \quad &(\bigoplus_{i \leq k} I_{j,i} \oplus N_k, (\tau_1(x_{k,1}, \dots \tau_k(x_{k,k}), n_k))(M(w)) \end{aligned}$$

-where $\tau_i = f_{j-1,i} \dots f_{k+1,i} f_{k,i}$ for all $i \leq k$. Notice that $\tau_i$ is almost invertible, and so:

$$(I_{k,i}, x_{k,i})(M(w)) = (I_{j,i}, \tau_i(x_{k,i}))(M(w))$$

-so proving the claim will complete the proof.

To prove it, note that for all $i \in \{k+1, \dots j\}$, the restriction of $G_{j-1} \dots G_{i+1} G_i$ from $N_i$ to $I_{j,i}$ takes $n_i$ to $x_{j,i}$. And so:

$$\begin{aligned} (I_{j,i}, x_{j,i})(M(w)) &\subseteq (N_i, n_i)(M(w)) \\ &= (N_j, n_j)(M(w)) \end{aligned}$$

Also, given any $i < k$, the restriction of $G_{i-1} \dots G_k$ from $N_k$ to $I_{j,i}$ takes $n_k$ to $x_{j,i} - \tau(x_{k,i})$. And so:

$$\begin{aligned} (I_{j,i} \oplus N_j, (x_{j,i}, n_j))(M(w)) &= (I_{j,i} \oplus N_k, (x_{j,i}, n_k))(M(w)) \\ &= (I_{j,i} \oplus N_k, (\tau(x_{k,i}), n_k))M(w) \\ &= (I_{j,i} \oplus N_j, (\tau(x_{k,i}), n_j))(M(w)) \end{aligned}$$

Putting these together proves our claim.

□

## 6.8 Finding spanning sequences

Throughout this section, $w$ will be any $\mathbb{N}$-word or non-periodic $\mathbb{Z}$-word, such that $\mathcal{W}_w$ and $\mathcal{U}_w$ have the ascending chain condition. The standard basis of $M(w)$ will be denoted $\{z_i : i \in I\}$.

**Lemma 149.** *Given any $a \in Q_0$, take any $C \in H_{-1}(a)$ and $D \in H_1(a)$, and any $J \subseteq \mathbb{Z}$ such that $w(z_i) \geq D$ and $u(z_i) \geq C$ for all $i \in J$.*

*Then there exists $z' \in \overline{\mathcal{Z}}$ such that:*

- $u(z') = \mathrm{inf\{u(z_i) : i \in J\} \geq C}$
- $w(z') \geq D$
- $\mathrm{inf\{u(z_i) : i \in J, z_i \not\geq z'\} > u(z')}$

*Proof.* Let $u' := \mathrm{inf\{u(z_i) : i \in I\}}$. Define:

$$\mathcal{D} := \{z_j : j \in J, u(z_j) = u'\}$$

Define $\mathcal{C}$ to be the set of all descending chains in $\mathcal{Z}_w$:

$$z_{k_1} > z_{k_2} > z_{k_3} > \dots$$

-such that $k_j \in J$ for all $j \in \mathbb{N}$, and $\underrightarrow{\lim}\, u(z_{k_j}) = u'$. Notice that $\mathcal{C}$ and $\mathcal{D}$ cannot both be zero.

Given any $z_i, z_j \in \mathcal{D}$, $z_i \geq z_j$ if and only if $w(z_i) \geq w(z_j)$- and so $\mathcal{D}$ is totally ordered with respect to the ordering on $\mathcal{Z}$, and contains no infinite ascending chains.

- If $\mathcal{D}$ is non-zero and finite, then define:

  $$y_1 := \min\{z_i : i \in I, u(z_i) = u'\}$$

  Of course, $D \leq w(y_1)$.

- If $\mathcal{D}$ is non-zero and infinite, then- since it is totally ordered, and $\mathcal{W}_w$ contains no infinite ascending chains- we can label this set as $\{z_{k_j} : j \in \mathbb{N}\}$- with $z_{k_j} > z_{k_{j+1}}$ for all $j \in \mathbb{N}$. Then there exists $z \in \overline{\mathcal{Z}}$ such that $u(z) = \underrightarrow{\lim}\, z_{k_j} = u'$ and $w(z) = \underrightarrow{\lim}\, w(z_{k_j}) \geq D$.

Now, If $\mathcal{C}$ is non-empty, then define:

$$w' := \inf\{\varinjlim w(z_{k_j}) : z_{k_1} > z_{k_2} > z_{k_3} > \dots \text{ is a chain in } \mathcal{C}\}$$

One can easily check that there exists a chain $z_{k_1} > z_{k_2} > z_{k_3} > \dots$ in $\mathcal{Z}$ such that $u(z_{k_j}) = u'$ for all $j$, and $\varinjlim w(z_{k_j}) = w'$- and so there exists $z \in \overline{\mathcal{Z}}$ such that $w(z) = w'$ and $u(z) = u'$. Define $y_2$ to be this $z$.

Define:

$$z' := \begin{cases} y_1 & \text{if } \mathcal{C} \text{ is zero or } y_1 \leq y_2 \\ y_2 & \text{if } \mathcal{C} \text{ is zero or } y_2 \leq y_1 \end{cases}$$

One can easily check that:

$$\inf\{u(z_i) : i \in J, z_i \not\geq z'\} > u'$$

-and hence satisfies all the required conditions.

□

**Corollary 41.** *Given any $a \in Q_0$, let $C \in H_{-1}(a)$ and $D \in H_1(a)$ be any finite words. Then there exists a finite subset $\{z^{(1)}, \dots, z^{(n)}\}$ of $\overline{\mathcal{Z}}$ such that:*

$$\{z_i \in \mathcal{Z} : z_i \in (C^{-1}.D)(M(w))\} = \bigcup_{k=1}^{n} \{z_i \in \mathcal{Z} : z_i \geq z^{(k)}\}$$

*Furthermore, $w(z^{(k)}) > D$ and $u(z^{(k)}) > C$ for all $k \leq n$*

*Proof.* Let $I_0 = \{i \in \mathbb{Z} : z_i \in (C^{-1}.D)(M(w))\}$. By repeatedly applying corollary 41, we can find subsets $I_0 \supseteq I_1 \supseteq I_2 \supseteq I_3, \dots$ and elements $z^{(1)}, z^{(2)}, z^{(3)}, \dots$ of $\overline{\mathcal{Z}}$, such that, for all $n \geq 1$:

- $w(z^{(n)}) \geq D$
- $u(z^{(n)}) = \mathrm{inf\{u(z_i) : i \in I_{n-1}\} \geq C}$
- $I_n = \{z_i : i \in I_{n-1}, z_i \not\geq z^{(n)}\}$
- $\mathrm{inf\{u(z_i) : i \in I_n\} > u(z^{(n)})}$.

Notice that, for all $n$, $u(z^{(n)}) > u(z^{(n-1)})$. Since $\mathcal{U}_w$ contains no infinite ascending chains, there exists $n$ such that $I_n = 0$.

Since $u(z^{(k)}) \geq C$ and $w(z^{(k)}) \geq D$ for all $k \leq n$, we clearly have:

$$\{z_i \in \mathcal{Z} : z_i \in (C^{-1}.D)(M(w))\} \subseteq \bigcup_{k=1}^{n} \{z_i \in \mathcal{Z} : z_i \geq z^{(k)}\}$$

Furthermore, given any $i \in I$, let $k \geq 0$ be such that $i \in I_k \backslash I_{k+1}$. Then $z^{(k)} \leq z_i$. Thus:

$$\{z_i \in \mathcal{Z} : z_i \in (C^{-1}.D)(M(w))\} \supseteq \bigcup_{k=1}^{n} \{z_i \in \mathcal{Z} : z_i \geq z^{(k)}\}$$

-as required. □

**Corollary 42.** *Given any $C \in H_{-1}(S)$ and $D \in H_1(S)$, let $z^{(1)}, \ldots, z^{(n)}$ be as defined in corollary 41*

*Then for all $z \in \overline{\mathcal{Z}}$ such that $w(z) \geq D$ and $u(z) \geq C$, there exists $k \leq n$ such that $z^{(k)} \leq z$.*

*Proof.* We may assume that $z \notin \mathcal{Z}$- and so there exists a descending chain of elements of $\mathcal{Z}$:

$$z_{i_1} > z_{i_2} > z_{i_3} > \ldots$$

-such that $\varinjlim w(z_{i_k}) = w(z)$ and $\varinjlim u(z_{i_k}) = u(z)$. For all $k \geq 1$, define:

$$J_k := \{j \in \{1, 2, \ldots, k\} : z^{(j)} \leq z_{i_k}\}$$

Of course, all $J_k$ are non-zero, and:

$$J_1 \supseteq J_2 \supseteq J_3 \supseteq \ldots$$

-so $\bigcap_{k \geq 1} J_k$ is non-empty. Pick any $j$ in it. Then $w(z^{(j)}) \leq w(z_{i_k})$ for all $k$, so $w(z^{(j)}) \leq \varinjlim w(z_{i_k}) = w(z)$. Similarly, $u(z^{(j)}) \leq u(z)$, completing the proof. □

Given any pp-formula $\phi(v)$ be any pp-formula, we say that $(B, b)$ is a *basis for* $\phi(M(w))$ if it is a spanning set for $\phi(M(w))$, and it also satisfies the following condition:

- Given any $M \in \mathcal{M}$, and $m \in \phi(M)$, there exists $f \in \mathrm{Hom}'(B, M)$ taking $m$ to $m'$.

We aim to prove that a basis exists for every pp-definable subgroup of $M(w)$.

Notice that, if $(M, m)$ is a basis for $\phi(M(w))$, and $(N, n)$ is a basis for $\psi(M(w))$, then $(M \oplus N, (m, n))$ is a basis for $(\phi + \psi)(M(w))$. It is therefore, enough to prove that a basis exists for any $\phi(M(w))$- where $\phi(v)$ is a pp-formula with free realisation $(X, x)$- with $X \in A$-mod being indecomposable.

### 6.8.1 Pp-formulas freely realised in string modules

Suppose we have a pp-formula $\phi(v)$, with free realisation $(M(l_1 \ldots l_k), x)$- for some finite word $l_1 \ldots l_k$. Given any $j \in \{0, 1, \ldots, k\}$, define $D_j = l_{j+1} \ldots l_k$, and $C_j = (l_1 \ldots l_j)$, and consider the set:

$$\{i \in \mathbb{Z} : z_i \in (C_j.D_j)(M(w))\}$$

By corollary 41, we can find a finite subset $Z_j$ of $\overline{\mathcal{Z}}$ such that, for all $i \in \mathbb{Z}$:

$$z_i \in (C_j.D_j)(M(w)) \iff z_i \geq z \text{ for some } z \in Z_j$$

For every $z \in Z_j$, let $M_{j,z}$ be the module $M(z)$, and denote by $z$ the standard basis element of $M_{j,z}$, with right word $w(z)$ and left word $u(z)$.

Since $D_j \leq w(z)$ and $C_j^{-1} \leq u(z)$, there exists a simple string map:

$$f_{j,z} : M(l_1 \ldots l_k) \to M(u(z)^{-1}w(z))$$

-taking $z_j$ to $z$. Define $m_{j,z} := f_{j,z}(x)$, and:

$$M_\phi := \bigoplus_{j=0}^{k} \bigoplus_{z \in Z_j} M_{j,z}$$

Let $m_\phi$ be the element of $M_\phi$ whose $M_{j,z}$-component is $m_{j,z}$ (for every $j$ and $z$). Of course, $m_\phi \in \phi(M_\phi)$.

**Lemma 150.** *Let $\phi(v)$ be any pp-formula with free realisation $(M(l_1 l_2 \ldots l_k), x)$ - for some finite word $l_1 \ldots l_k$.*

*Then $(M_\phi, m_\phi)$ (as defined above) is a basis of $\phi(M(w))$.*

*Proof.* Take any $M \in \mathcal{M}$ and $m \in \phi(M)$. We must prove there exists a map in $\mathrm{Hom}'(M_\phi, M)$ taking $m_\phi$ to $m$.

Since $m \in \phi(M)$, there exists a map $g' \in \mathrm{Hom}'(M(l_1 \dots l_k), M)$ such that $g'(n) = m'$. It is therefore enough to prove that, for every simple string map $g : M(l_1 \dots l_k) \to M$, there exists a simple string map $h : M_\phi \to M$ taking $m_\phi$ to $g(x)$.

We claim that we only need to prove this for every $M \in \mathcal{A} \cup \mathcal{P}$: Indeed, any simple string map from $M(l_1 \dots l_k)$ to a band module $S^D_\lambda[n]$ in $\mathcal{B}$ looks like:

$$M(l_1 \dots l_k) \xrightarrow{g''} M({}^\infty D^\infty) \overset{\pi}{\twoheadrightarrow} S^D_\lambda[n'] \overset{f(n-n')}{\hookrightarrow} S^D_\lambda[n]$$

-where $g''$ is a simple string map, and $n' \leq n$. Of course, if $S^D_\lambda[n] \in \mathcal{B}$, then $M({}^\infty D^\infty) \in \mathcal{P}$. Therefore, if we can prove the result for $\mathcal{P}$, then we have it for $\mathcal{B}$.

We may therefore assume that $M \in \mathcal{A} \cup \mathcal{P}$. Let $\{y_j : 0 \leq j \leq n\}$ be the standard basis of $M(l_1 \dots l_k)$. Since $g \neq 0$, there exists $j \in \{0, 1, 2, \dots k\}$ such that $f(y_j) \neq 0$.

Since $g$ is a simple string map, $g(y_j)$ must be a standard basis element of $M$. By lemma 133, $g(y_j)$ may be thought of as an element $z$ of $\overline{\mathcal{Z}}$. Note that $M \cong M(z)$. Of course, $y_j \in (C_j^{-1}.D_j)(M(l_1 \dots l_k))$, and so:

$$z = g(y_j) \in (C_j^{-1}.D_j)(M)$$

-so, by corollary 42, there exists $y \in Z_j$ such that $y \leq z$. Consider the direct summand $M_{j,y}$ of $M_\phi$. Since $y \leq z$, there exists a simple string map $h : M_{j,y} \to M$ taking $z_{j,y}$ to $z$.

Now, let $\pi$ be the projection onto the direct summand:

$$\pi : M_\phi \twoheadrightarrow M_{j,y}$$

Since $\pi$, $h$ and $f_{j,y}$ are simple string maps, so is $h\pi f_{j,y}$. Since $h\pi f_{j,y}(z_j) = z = g(z_j)$, lemma 131 gives that $h\pi f_{j,y} = g$, and hence that $h\pi(m_\phi) = g(x)$ , as required. □

### 6.8.2 Pp-formulas freely realised in band modules

**Lemma 151.** *Let $D$ be any band such that:*

$$\inf\{\mathrm{w(z) : z \in \overline{\mathcal{Z}}, w(z) \geq D^\infty, u(z) \geq (D^{-1})^\infty}\} = \mathrm{D^\infty}$$

*Then there exists* $z \in \overline{\mathcal{Z}}$ *such that* $w(z) = D^\infty$ *and* $u(z) = (D^{-1})^\infty$.

*Proof.* We can pick a sequence $k_1, k_2, k_3, \dots \in I$ such that:

- $w(z_{k_{j+1}}) < w(z_{k_j})$ for all $j \geq 1$
- $\varinjlim w(z_{k_j}) = D^\infty$.
- $u(z_{k_{j+1}}) \leq u(z_{k_j})$ for all $j \geq 1$

We define, recursively, a subsequence $n_1, n_2, n_3, \dots$ of $k_1, k_2, k_3, \dots$ as follows: Let $n_1 := k_1$. Having found any $n_i$, let $m \in \mathbb{N}$ be such that $n_i = k_m$. Of course:

$$D^2 w(z_{k_m}) > D^2 D^\infty = D^\infty$$

Since $\varinjlim w(z_{k_j}) = D^\infty$, there must exists $j > m$ such that $D^\infty < w(z_{k_j}) < D^2 w(z_{n_i})$. Define:

$$n_{i+1} := \begin{cases} n_k + N & \text{if } \hat{w}_{n_k} = l_{n_k+1} l_{n_k+2} \dots \\ n_k - N & \text{if } \hat{u}_{n_k} = l_{n_k+1} l_{n_k+2} \dots \end{cases}$$

-where $N$ is the length of $D$. Notice that:

- $D^\infty < w(z_{n_{i+1}}) < Dw(z_{n_i})$
- $(D^{-1})^\infty < u(z_{n_{i+1}}) < D^{-1} u(z_{n_i})$

Consequently, for all $i \geq 1$, $D^{i-1}$ is an initial subword of $w(z_{m_i})$ and $(D^{-1})^{i-1}$ is an initial subword of $u(z_{m_i})$. Thus $\varinjlim w(z_{n_i}) = D^\infty$ and $\varinjlim u(z_{n_i}) = (D^{-1})^\infty$. The result follows, by the definition of $\overline{\mathcal{Z}}$. $\square$

**Lemma 152.** *Suppose that* $\phi(v)$ *is a pp-formula with free realisation of the form* $(S^D_\lambda[n], x)$. *Then* $\phi(M(w))$ *has a basis.*

*Proof.* First of all, if there exists $z \in \overline{\mathcal{Z}}$ such that $w(z) = D^\infty$ and $u(z) = (D^{-1})^\infty$, then we define $(M_\phi, m_\phi)$ to be $(S^D_\lambda[n], x)$. This is clearly a basis of $\phi(M(w))$.

Assume, from now on, that no such $z$ exists. Let $m$ be the length of $D$, and let $D_0, D_2 \dots D_{m-1}$ be the cyclic permutations of $D$ (with $D = D_0$). For each $j \in \{0, 1, 2, \dots, m-1\}$, define $C_j := (D_j)^{-1}$, and:

$$A_j := \{i \in \mathbb{Z} : w(z_i) > D_j^\infty, u(z_i) > C_j^\infty\}$$

By lemma 151, $\inf\{\mathrm{w(z_i) : i \in A_j}\} > \mathrm{D}^\infty$. Let $(D_j)^{q_j}$ (with $q_j \in \mathbb{Q}$) be the longest possible common initial subword of $(D_j)^\infty$, and $\inf\{\mathrm{w(z_i) : i \in A_j}\}$. Notice that $D_j^\infty < D_j^{q_j} < \inf\{\mathrm{w(z_i) : i \in A_j}\}$.

Similarly, let Let $C_j^{p_j}$ (with $p_j \in \mathbb{Q}$) be the longest possible common initial subword of $C_j^\infty$, and $\inf\{\mathrm{u(z_i) : i \in A_j}\}$. Notice that $C_j^\infty < C_j^{p_j} < \inf\{\mathrm{u(z_i) : i \in A_j}\}$.

Let $\{y_i : i \in \mathbb{Z}\}$ be a standard basis for $\overline{M}({}^\infty D^\infty)$, such that $w(y_0) = D_0^\infty$. For each $j \in \{0, 1, \ldots, m-1\}$, let $\pi_j : \overline{M}({}^\infty D^\infty) \twoheadrightarrow M((C_j^{p_j})^{-1} D^{q_j})$ denote the natural surjection such that $\pi_j(z_j)$ has right-word $D_j^{q_j}$ in $M((C_j^{p_j})^{-1} D_j^{q_j})$.

Now, for each $k \leq n$ and $j \leq m$, let $h_{j,k}$ be the map:

$$S_\lambda^D[n] \overset{g^{(n-k)}}{\twoheadrightarrow} S_\lambda^D[k] \overset{i_k}{\hookrightarrow} \overline{M}({}^\infty D^\infty) \overset{\pi_j}{\twoheadrightarrow} M((C_j^{p_j})^{-1} D_j^{p_j})$$

(where $i_k$ is the map as defined before lemma 44).

For each $j$ and $k$ such that $0 \leq j \leq m-1$ and $1 \leq k \leq n$, define $M_{j,k} := M((C_j^{p_j})^{-1} D_j^{q_j})$. Define:

$$B := \bigoplus_{0 \leq j \leq m-1} \bigoplus_{1 \leq k \leq n} M_{j,k}$$

-and let $b \in B$ be the element whose $M_{j,k}$ component is $h_{j,k}(x)$, for every $j$ and $k$. By lemma 150, $(B, b)(M(w))$ has a basis. in order to prove that this basis is also a basis for $\phi(M(w))$, it will be enough to prove that, given any $M \in \mathcal{M}$, any simple string map in $\mathrm{Hom}'(S_\lambda^D[n], M)$ factors through some $h_{j,k}$.

First of all, given any $M \in \mathcal{A} \cup \mathcal{P}$, any simple string map in $\mathrm{Hom}'(S_\lambda^D[n], M)$ looks like:

$$S_\lambda^D[n] \overset{g^{(n-k)}}{\twoheadrightarrow} S_\lambda^D[k] \overset{i_k}{\hookrightarrow} \overline{M}({}^\infty D^\infty) \overset{h}{\to} M$$

-for some $k \leq n$ and simple string map $h$. Pick any $j' \in \mathbb{Z}$ such that $h(y_{j'}) \neq 0$. Since $h$ is a simple string map, $h(y_{j'})$ is a basis element of $M$, and so we may consider it an element $z$ of $\overline{\mathcal{Z}}$.

Let $j \in \{0, 1, \ldots, m-1\}$ and $s \in \mathbb{Z}$ be such that $j' = j + sm$. Let $\Phi \in \mathrm{End}'(\overline{M}({}^\infty D^\infty)$ be the shift map taking $y_0$ to $y_m$. Then, the simple string map:

$$\overline{M}({}^\infty D^\infty) \overset{\Phi^s}{\to} \overline{M}({}^\infty D^\infty) \overset{h}{\to} M$$

-takes $y_j$ to $z$. Since $M$ is a direct sum string module, Im(h$\Phi^s$) must be finite dimensional, and it follows that $D_j^\infty < w(z)$ and $C_j^\infty < u(z)$, and so :

$$w(z) > D_j^{q_j} > D_j^\infty$$

$$u(z) > C_j^{p_j} > C_j^\infty$$

-and consequently, $h\Phi^s$ can be factored through $\pi_j$- so $h\Phi^s g^{(n-k)}$ factors through $h_{j,k}$, as required.

Now, given any $S_\mu^C[n'] \in \mathcal{B}$, $C$ is not a cyclic permutation of $D$, and so any simple string map from $S_\lambda^D[n]$ to $S_\mu^C[n']$ must look like:

$$S_\lambda^D[n] \overset{g^{(n-k)}}{\twoheadrightarrow} S_\lambda^D[k] \overset{i_k}{\hookrightarrow} \overline{M}({}^\infty D^\infty) \xrightarrow{f} M({}^\infty C^\infty) \overset{\pi_{k'}}{\twoheadrightarrow} S_\mu^C[k'] \overset{f^{(n'-k')}}{\hookrightarrow} S_\mu^C[n']$$

Then $M({}^\infty C^\infty) \in \mathcal{P}$, and so we can factor the simple string map $fi_kg^{(n-k)}$ through some $h_{j,k}$.

□

**Corollary 43.** *Let $\phi(v)$ be any pp-formula. Then $\phi(M(w))$ has a basis.*

*Proof.* Follows straight from lemma 150 and lemma 152.

□

Of course, theorem 47 follows straight from this corollary: Take any descending chain of pp-definable subgroups:

$$\phi_1(M(w)) \geq \phi_2(M(w)) \geq \phi_3(M(w)) \geq \dots$$

We may assume that $\phi_i \geq \phi_{i+1}$ for all $i$ (by replacing every $\phi_i$ with $\phi_1 \wedge \cdots \wedge \phi_i$). For each $i \in \mathbb{N}^+$, let $(M_i, m_i)$ be a basis for $\phi_i(M(w))$. By the definition of a basis, there exists a map $f_i \in \text{Hom}'(M_i, M_{i+1})$ taking $m_i$ to $m_{i+1}$. Thus the $\mathcal{M}$-sequence:

$$(M_1, m_1) \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \xrightarrow{f_1} \dots$$

-is a spanning sequence for the descending chain.

## 6.9 Examples

Let $w$ be any $\mathbb{N}$-word or non-periodic $\mathbb{Z}$-word. Recall from theorem 39 and proposition 4 that $M(w)$ is indecomposable, and $\overline{M}(w)$is pure-injective.

One can easily check that there is no $\mathbb{N}$-word or non-periodic $\mathbb{Z}$-word, $w$ such that both $\mathcal{W}_w$ and $\mathcal{U}_w$ have the ascending chain condition, and $\mathcal{Z}_w$ has the descending chain condition: and hence there is no $\mathbb{N}$-word or non-periodic $\mathbb{Z}$-word, $w$, such that both $M(w)$ and $\overline{M}(w)$ are indecomposable and pure-injective. Note that we already had this in the $\mathbb{N}$-word case, from corollary 28.

We now present a few examples of $\mathbb{N}$-words, to illustrate the different possibilities that can occur.

First of all, if $w$ is a contracting $\mathbb{N}$-word or $\mathbb{Z}$-word, then $\mathcal{U}_w$ and $\mathcal{W}_w$ have the ascending chain condition, so $\mathcal{Z}_w$ does not have the descending chain condition: so $M(w)$ is pure injective, and $\overline{M}(w)$ is not indecomposable.

If $w$ is an expanding $\mathbb{N}$-word or $\mathbb{Z}$-word, then $\mathcal{Z}_w$ has the descending chain condition, so at least one of $\mathcal{W}_w$ and $\mathcal{U}_w$ doesn't have the ascending chain condition: so $\overline{M}(w)$ is indecomposable, but $M(w)$ is not pure injective.

For some aperiodic examples, let $A$ be the Gelfand-Ponomarev algebra $G_{3,3}$ (cf. section 5.1. Let $C = \alpha\beta^{-1}$, and $D = \alpha\alpha\beta^{-1}\beta^{-1}$. One can check that if $w$ is the word:

$$DCDC^3DC^5DC^7\dots$$

-then both $\mathcal{U}_w$ and $\mathcal{W}_w$ have the ascending chain condition, so $M(w)$ is pure-injective, and $\overline{M}(w)$ is not indecomposable. Conversely, if $w$ is:

$$CDCD^3CD^5CD^7\dots$$

-then $\mathcal{Z}_w$ has the descending chain condition, and $w$ satisfies (IC), and so $\overline{M}(w)$ is indecomposable, and $M(w)$ is not pure-injective.

Finally, if $w$ is the word:

$$CDC^3D^3C^5D^5C^7D^7C^9D^9\dots$$

-then $\mathcal{Z}_w$ does not have the descending chain condition, and at least one of $\mathcal{W}_w$ and $\mathcal{U}_w$ doesn't have the ascending chain condition. And so $M(w)$ is not pure-injective, and $\overline{M}(w)$ is not indecomposable.

# Chapter 7

# Two-Directed Modules

## 7.1 Two-directed modules

Having determined what the one-directed modules over a string algebra look like, we turn our attention to the two-directed modules. We first of all need a few more results about some of the pp-formulas which were defined in chapter 5.

### 7.1.1 Left-words and right-words

**Lemma 153.** *Take any $D \in \mathcal{W}$, and $\beta \in Q_1$ such that $D\beta \in \mathcal{W}$. Then, for all $M \in A$-*Mod*, $D\beta M \subseteq (.D)(M)$.*

*Furthermore, if $M$ is two-directed, then $D\beta M = (.D)(M)$.*

*Proof.* Let $a \in Q_0$ and $s \in \{-1, +1\}$ be such that $D^{-1} \in H_s(a)$. Then, $\beta \in H_{-s}(a)$, since $D\beta \in \mathcal{W}$.

First of all, suppose that there exists an inverse letter $\alpha^{-1}$ in $H_{-s}(a)$. Then $(.D)(M) = D\alpha^{-1}(0)$ (by definition). Given any $x \in D\beta(M)$, pick any $y \in \beta M$ such that $x \in Dy$. Then $\alpha y \in \alpha\beta M = 0$ (since $\alpha^{-1}, \beta \in H_{-s}(a)$), so $x \in D\alpha^{-1}(0)$, as required.

Now, if $H_{-s}(a) \cap Q_1^{-1} = \emptyset$, then $(.D)(M) = D(M)$. So $D\beta M \subseteq DM = (.D)(M)$.

For the second assertion, suppose that $M$ is two-directed. Given any $x \in (.D)(M)$, there exists $y$ such that $x \in Dy$, and $\alpha y = 0$ for any $\alpha^{-1} \in H_{-s}(a) \cap Q_1^{-1}$. Let $E$ be the longest possible string of inverse letters such that $D^{-1}E \in \mathcal{W}$.

Then $D^{-1}M = (.D^{-1}E)(M)$, so $y \in (.D^{-1}E)(M)$. Since $\alpha y = 0$ for any $\alpha \in H_{-s}(a) \cap Q_1^{-1}$, it follows that:

$$y \in (1.D^{-1}E)(M) = (^{+}1.D^{-1}E)(M)$$

(the equality holds, since $M$ is two-directed). By definition:

$$(^{+}1.D^{-1}E)(M) = (1.D^{-1}E)(M) \cap \beta M$$

So $y \in \beta M$, and hence $x \in D\beta M$, as required. □

**Lemma 154.** *Let $w = l_1 l_2 l_3 \dots$ be any be any $\mathbb{N}$-word in $H_1(a)$ for some $a \in Q_0$. Let $M$ be any module, and $m$ any element of $e_a(M)$. Then $m$ has right-word greater than or equal to $w$ if and only if $m \in l_1 l_2 \dots l_n M$ for all $n \in \mathbb{N}$.*

*Proof.* Let $u$ be the right-word of $m$ in $M$. If $u \geq w$, then given any subword $l_1 \ldots l_n$ of $w$, pick any $k \geq n$ such that $l_{k+1} \in Q_1$. Then $l_1 \ldots l_k < w \leq u$, and so $m \in (.(l_1 \ldots l_k))(M)$ (by definition of the right-word of $m$), and so $m \in l_1 \ldots l_k(M) \subseteq l_1 \ldots l_n(M)$, as required.

Conversely, suppose that $m \in l_1 \ldots l_n(M)$ for all $n \in \mathbb{N}$. Pick any ascending chain $k_1 < k_2 < k_3 < \ldots$ in $\mathbb{N}$ such that $l_{k_i+1} \in Q_1$ for all $i \in \mathbb{N}^+$, and let $D_i = l_1 \ldots l_{k_i}$. Then, by lemma 153:

$$m \in D_i l_{k_i+1} M \subseteq (.D_i)(M) \text{ for all } n \in \mathbb{N}^+$$

Since $D_1 < D_2 < D_3 < \ldots$ and $\varinjlim D_i = w$, it follows that $w \leq u$. □

**Lemma 155.** *Take any* $M \in A$-Mod, $a \in Q_0$, and any $x_0, y_0 \in e_a(M)$, with right-words $w$ and $u$ respectively in $M$. Then $x_0 + y_0$ has right-word greater than or equal to $\min(u, w)$ in $M$.

In fact, if $u \neq w$, then $x_0 + y_0$ has right-word $\min(u, w)$.

*Proof.* To prove the first assertion, it's enough (by the definition of right-word) to prove that $x_0 + y_0 \in (.E)(M)$ for all $E \leq \min(u, w)$. Indeed, given any such $E$, $x_0 \in (.E)(M)$ (since $E \leq w$) and $x_0 \in (.E)(M)$ (since $E \leq u$), and so $x_0 + y_0 \in (.E)(M)$, as claimed.

To prove the second result: Let $v$ be the right-word of $x_0+y_0$ and assume, without loss of generality, that $u < w$. Then $v \geq \min(u, w) = u$. Also, since $y_0 = (x_0+y_0)-x_0$, the first assertion gives that $u \geq \min(v, w) = v$ (since $u < w$), and so $u = v$, as required. □

### 7.1.2 Fundamental elements

Given any string module $M$, and any $a \in Q_0$, take any $m \in e_a M$. Let $w$ and $u$ be the right-word and left-word, respectively, of $m$ in $M$. We say $m_0$ is *fundamental in* $M$ if there is no $x \in M$ satisfying:

- $x$ has left-word in $M$ greater than $u$

- $m - x$ has right-word in $M$ greater than $w$

Notice that the statement "$x$ is fundamental in $M$, with right-word $w$ and left-word $u$" can be defined by a infinite conjunction of pp-formulas and negations of pp-formulas:

$$\bigwedge_{C \leq u, D \leq w} \left( (C^{-1}.D)(v) \wedge \bigwedge_{E > u, F > w} \neg((E^{-1}.D) + (C^{-1}.F))(v) \right)$$

The concept of a fundamental element is an extension of the notion of a maximal element, as defined over finite dimensional modules by Baratella and Prest (see [2, (4.1)])

We use fundamental elements as tools to link pure-injective modules over string algebras to string modules. We shall prove first of all, that every standard basis element of a string module $M(w)$ is fundamental in $M(w)$. We shall then prove that (in almost all cases), the pp-type of a fundamental element $m_0$ of a pure-injective module $M$ is uniquely determined by its right-word and left-word $M$.

For an example of a fundamental element, take any $\mathbb{N}$-word, $w$, and consider the one-directed pure-injective indecomposable $M_w$ as defined in theorem 40. Then the homogeneous element $m_0$ as described in the theorem is fundamental in $M_w$.

**Lemma 156.** *Let $w$ be a word, and $M$ be a string module over $w$ (i.e. either $M(w)$, $\overline{M}(w)$, $M^+(w)$ or $M^-(w)$). Then every standard basis element $z_i$ of $M$ is fundamental in $M$, with right-word $\hat{w}_i$ and left-word $\hat{u}_i$.*

*Consequently, given any pure-embedding $f : M \to N$, $f(z_i)$ is fundamental in $N$, with right-word $\hat{w}_i$, and left-word $\hat{u}_i$*

*Proof.* Lemma 111 gives the right-word and left-word of $z_i$ in $M$. Suppose, for a contradiction, that it is not fundamental: Then there exists $C \leq u_i$, $D \leq w_i$, $E > u_i$ and $F > w_i$ such that:

$$M \models ((C^{-1}.F) + (E^{-1}.D))(z_i)$$

Pick any $x \in (C^{-1}.F)(M)$ such that $z_i - x \in (E^{-1}.D)(M)$. Since $x \in (.F)(M)$, $x$ must have $z_i$-coefficient 0, by lemma 105. However, since $z_i - x \in (.E)(M)$, lemma 105 gives that $z_i - x$ must have $z_i$-coefficient 0- giving our required condition.

Since $f$ is a pure embedding, we have that, for all pp-formulas of the form $(C^{-1}.D)(v)$:

$$z_i \in (C^{-1}.D)(M) \iff f(z_i) \in (C^{-1}.D)(N)$$

The result clearly follows. $\square$

**Lemma 157.** *Let $w = l_{-2}l_{-1}l_0l_1l_2\dots$ and $w' = l'_{-2}l'_{-1}l'_0l'_1l'_2\dots$ be any $\mathbb{Z}$-words. Then $M(w) \cong M(w')$ if and only if either $w = w'$ (i.e. there exists $k$ such that $l_i = l'_{i+k}$ for all $i \in \mathbb{Z}$) or $w = w'$ (i.e. there exists $k$ such that $l_i = l'_{k-i}$ for all $i \in \mathbb{Z}$).*

*Proof.* Let $\{z_i : i \in \mathbb{Z}\}$ and $\{y_i : i \in \mathbb{Z}\}$ be the standard bases of $M(w)$ and $M(w')$ respectively. One direction is clear- for example, if $l_i = l'_{i+k}$ for all $i \in \mathbb{Z}$, then there exists a simple string map in $\mathrm{Hom}(M(w), M(w'))$ taking every $z_i$ to $y_{i+k}$, with inverse given by the map taking each $y_i$ to $z_{i-k}$.

Conversely, assume that $w \neq w'$ and $w \neq (w')^{-1}$. Then, given any map $f : M(w) \to M(w')$, write $f(z_0)$ as $\sum_{i \in I_0} \lambda_i y_i$- where $I_0 \subset \mathbb{Z}$ is a finite subset such that $\lambda_i \neq 0$ for all $i \in I_0$.

Given any finite word $E \leq \hat{w}_0$, $z_0 \in (.E)(M(w))$, and so $f(z_0) \in (.E)(M(w))$. Thus, by corollary 105, $E \leq \hat{w}'_i$ for all $i \in I$. Thus the right-word of $y_i$ in $M(w')$ (which is $w'_i$, by lemma 111) is greater than or equal to $\hat{w}_0$.

Also, since $w \neq w'$ and $w^{-1} \neq w'$, we cannot have both $\hat{w}'_i = \hat{w}_0$ and $\hat{u}'_i = \hat{u}_0$ for any $i \in I$.

Consequently, we can partition $I_0$ into $I_1 \cup I_2$, where $\hat{w}'_i > \hat{w}_0$ and $\hat{u}'_i \geq \hat{u}_0$ for all $i \in I_1$, and $\hat{u}'_i > \hat{u}_0$ and $\hat{w}'_i \geq \hat{w}_0$ for all $i \in I_2$.

By lemma 155, $\sum_{I_1} \lambda_i z'_i$ has right-word greater than or equal to $\min\{\hat{w}'_i : i \in I_1\} > \hat{w}_0$, and $\sum_{I_2} \lambda_i z'_i$ has left-word greater than or equal to $\min\{\hat{u}'_i : i \in I_1\} > \hat{u}_0$. Thus $f(z_0)$ is not fundamental, and so $f$ is not pure, by lemma 156. Thus $f$ is not an isomorphism, as required. $\square$

**Lemma 158.** *Let $w = \dots l_{-2}l_{-1}l_0l_1l_2\dots$ be any $\mathbb{Z}$-word, and $M$ any two-directed pure-injective $A$-module, containing a fundamental element $m_0$, with right-word $\hat{w}_0$ and left-word $\hat{u}_0$. Let $\{m_i \in M : i \in \mathbb{Z}\backslash\{0\}\}$ be any set such that $l_i m_i = m_{i-1}$ for all $i \in \mathbb{Z}$ (such a set exists, since $M$ is pure injective).*

*Then, for every $i \in \mathbb{Z}$, $m_i$ is fundamental in $M$, with right-word $\hat{w}_i$, and left word $\hat{u}_i$.*

*Proof.* By symmetry, it's enough to prove the result for all $i \geq 0$. We proceed by induction on $i \in \mathbb{N}$. Assume the statement is true for $m_1, m_2, \ldots, m_{i-1}$. We assume, without loss of generality, that $l_{i+1}l_{i+2}\cdots \in H_1(a)$, for some $a \in Q_1$.

Of course, $m_i \in l_{i+1}l_{i+2}\ldots l_k M$ for all $k \geq i$- so it has right-word greater than or equal to $w_i$. Furthermore, for all finite words $E > w_i$, $l_1 \ldots l_i E > w_0$, so $m_0 \notin (.l_1 \ldots l_i)(M)$. It follows that $m_i \notin (.E)(M)$ (since $m_i \in (.E)(M)$ would imply that $m_0 \in (.l_1 \ldots l_i E)(M)$). Thus, the right-word of $m_i$ in $M$ is indeed $\hat{w}_i$

Notice that, if $l_i$ is direct (say, $\alpha$), then $\alpha m_i = m_{i-1}$ has left word $l_{i-1}^{-1}l_{i-2}^{-1}\ldots$, so it follows that the left-word of $m_i$ is just $\alpha^{-1}l_{i-1}^{-1}l_{i-2}^{-1}\ldots$, as required.

We assume, from now on, that $l_i = \gamma^{-1}$, for some direct letter $\gamma$. Of course, the left-word of $m_i$ is greater than or equal to $u_i$. Now suppose, for a contradiction, that $m_i$ has left-word $u' > u_i$. Since the first letter of $u_i$ is $\gamma$, so must the first letter of $u'$ be. Let $u''$ be such that $u' = \gamma u''$. Of course, $u'' > u_{i-1}$.

Since $M$ is pure-injective, there exists $x \in M$ such that $\gamma x = m_i$, and $x$ has left-word greater ran or equal to $u''$- and hence greater than $u_{i-1}$. Then $\gamma(m_{i-1} - x) = 0$, and so $m_{i-1} - x$ has right-word greater than $\gamma^{-1}w_i$- contradicting the fact that $m_{i-1}$ is fundamental.

It remains to show that $m_i$ is fundamental: Suppose, for a contradiction, that $m_i = m_i' + m_i''$, where $m_i'$ has left word $u' > u_i$, and $m_i''$ has right-word $w' > w_i$. First of all, if the first letter of $u'$ is not $l_i^{-1}$, then it must be direct- say, $\delta$- and $l_i^{-1}$ inverse- say, $\gamma^{-1}$: then $\gamma m_i' \in \gamma\delta M = 0$, and so:

$$\gamma m_i'' = \gamma m_i - \gamma m_i' = m_{i-1}$$

-and hence that $m_i$ has right-word greater than or equal to $\gamma w'$. Since $\gamma w' > \gamma w_i = w_{i-1}$, we have a contradiction.

Secondly, if the first letter of $u'$ is $l_i^{-1}$, then let $u''$ be such that $u' = l_i^{-1}u''$. There are two cases to consider:

If $l_i$ is direct- say, $l_i = \alpha$- then $\alpha m'_i$ has left-word $u'' > u_{i-1}$, and $\alpha m''_i$ has right-word greater than or equal to $\alpha w' > w_{i-1}$- which implies that $m_{i-1} = \alpha m''_i + \alpha m'_i$ is not fundamental.

Assume now that $l_i$ is inverse- say, $l_i = \beta^{-1}$. Then $u' = \beta u''$, for some $u'' > u_{i-1}$. Since $M$ is pure-injective, we can pick $m'_{i-1} \in M$ with left-word greater than or equal to $u''$, such that $\beta m'_{i-1} = m'_i$.

Now, $\beta(m_{i-1} - m'_{i-1}) = m_i - m'_i = m''_i$, so $m_{i-1} - m'_{i-1}$ has right-word greater than or equal to $\beta^{-1} w' > w_{i-1}$ - contradicting the fact that $m_{i-1}$ is fundamental. □

**Lemma 159.** *Let $x$ be a fundamental element of a module $M$ with left word $u$ and right-word $w$. If $y$ is fundamental in $M$ with left word $u$ and right-word $w' > w$, then $x + y$ is a fundamental, with left word $u$ and right-word $w$.*

*Proof.* First of all, lemma 155 tells us that the right word of $x+y$ is $\min(w, w') = w$, and the left word, $u'$ of $x + y$ satisfies $u' \geq u$.

Now, if $u' > u$, then we have that $x = (x + y) - y$- with the left word of $x + y$ being $u' > u$, and the right-word of $y$ being $w' > w$- contradicting the fact that $x$ is fundamental. So $u' = u$.

To show that $x+y$ is fundamental- suppose, for a contradiction, that it is not- i.e. there exists $x', y' \in M$ such that $x'$ has left-word $u'' > u$, $y'$ has right-word $w'' > w$, and $x + y = x' + y'$. Then:

$$x = x' + y' - y$$

The right-word of $y' - y$ is greater than or equal to $\min(w'', w') = w' > w$, and the left word of $x'$ is $u'' > u$- thus $x$ is not fundamental- giving our required contradiction. □

**Corollary 44.** *Given any module $M$, and $a \in Q_0$, let $m_1, \dots, m_k \in e_a M$ be any fundamental elements of $M$, with each $m_i$ having left word $u'_i$ and right-word $w'_i$. Suppose that, for all distinct $i, j \leq n$, either $w'_i \neq w'_j$ or $u'_i \neq u'_j$.*

*Then $\sum_{i=1}^{k} m_i$ is a non-zero element of $M$, with left-word $\min\{u'_i : i \leq k\}$ and right-word $\max\{w'_i : i \leq k\}$.*

*Proof.* Let $J := \{i \leq k : w'_i \leq w'_j$ for all $j \leq k\}$. Then there exists a unique $j_0 \in J$ such that $u_{j'_0} \leq u'_j$ for all $j \in J$.

By lemma 159, $\sum_{j\in J} m_j$ is fundamental in $M$, with right-word $w'_{j_0}$. Since $w'_j > w'_{j_0}$ for all $j \notin J$, it follows from lemma 155 that $\sum_{j\leq k} m_j = \sum_{j\in J} m_j + \sum_{j\notin J} m_j$ has right-word $w'_{j_0} = \min\{w'_j : j \leq n\}$.

Similarly, $\sum_{j\leq k} m_j$ has left-word $\min\{u'_i : i \leq k\}$. □

### 7.1.3 Fundamental and homogeneous elements

Let $w = \dots l_{-1}l_0l_1l_2\dots$ be any $\mathbb{Z}$-word. Let $M$ be any two-directed pure-injective $A$-module, containing a fundamental element $m_0$, with left-word $u_0$ and right-word $w_0$. Then there exists- as in lemma 112- a map $f \in \mathrm{Hom}(M(w), M)$ such that $f(z_0) = m_0$. For all $i \in \mathbb{Z}$, lemma 158 implies that $f(z_i)$ is fundamental in $M$, with left-word $\hat{u}_i$ and right-word $\hat{w}_i$.

Given any $M \in A$-Mod, we say that $m_0 \in M$ is a *trough* if and only if $m \in e_a(M)$ for some $a \in Q_0$, and $\alpha m = 0$ for all $\alpha \in Q_1$.

Given any trough $m_0 \in e_aM$, there exists a unique map $f \in \mathrm{Hom}(e_aA, M)$ taking $e_a$ to $m_0$. Of course, $\mathrm{Im}(f)$ is a 1-dimensional $K$-vector subspace of $M$, which is generated by $m_0$. We denote the cokernel of $f$ by $M/\langle m_0\rangle$.

**Lemma 160.** *Let $w = \dots l_{-1}l_0l_1l_2\dots$ be any non-periodic $\mathbb{Z}$-word, such that $l_1 \in Q_1$ and $l_0 \in Q_1^{-1}$. Suppose that $M$ is a two-directed pure-injective, with elements $m_i \in M$ for each $i \in \mathbb{Z}$ such that $l_im_i = m_{i-1}$ for all $i$, and such that $m_0$ is fundamental in $M$, with right-word $w_0$ and left word $u_0$ (and so $m_0$ is a trough).*

*Assume that $\hat{w}_1 = w_1$- i.e. $w_1 \in H_1(a)$ for some $a \in Q_0$. Then given any $D, E \in H_1(a)$ such that $D \leq w_1$:*

$$M/\langle m_0\rangle \models ((^+1.D) + (1.E))(\overline{m}_1) \text{ if and only if } E \leq w_1$$

*(Where $\overline{m}_1$ corresponds to the image of $m_1$ in $M/\langle m_0\rangle$).*

*Proof.* Since $l_1$ is direct, we shall denote it $\alpha$. One direction of the proof is clear: If $E \leq w_1$, then $m_1 \in (.E)(M)$, and hence $m_1 \in (1.E)(M/\langle m_0\rangle)$ (since $\alpha\overline{m}_1 = \overline{m}_0 = 0$ in $M/\langle m_0\rangle$).

To show the other direction, suppose- for a contradiction- that:

$$M/\langle m_0\rangle \models ((^{+}1.D) + (1.E))(\overline{m}_1)$$

-for some finite word $E > w_1$. Let $C$ be the longest possible common initial subword of $E$ and $w_1$. Then $C > w_1$ and $C \leq E$, so $(1.E)(v) \to (1.C)(v)$. Thus:

$$M/\langle m_0\rangle \models ((^{+}1.D) + (1.C))(\overline{m}_1)$$

We can write $C$ in the (unique) form:

$$C_1 D_1^{-1} C_2 D_2^{-1} \dots C_n D_n^{-1}$$

-where every $C_i$ and $D_i$ is a word containing only direct letters- and that, for all $i \in \{1, 2, \dots, n-1\}$, $C_{i+1}$ and $D_i$ have length at least 1. Let $c_i$ be the length of $C_i$, and $d_i$ the length of $D_i$ for every $i$.

Define, for every $i \in \{0, \dots, n\}$, $N_i := \sum_{j=1}^{i}(c_j + d_j)$. So

$$C_1 D_1^{-1} \dots C_i D_i^{-1} = l_1 l_2 \dots l_{N_i} \text{ for all } i \in \{0, \dots, n\}$$

Since $C > w_1$, $l_{N_n+1} \in Q_1^{-1}$- we shall denote it $\gamma^{-1}$. By our assumption, there exists $x \in M$ such that:

$$M/\langle m_0\rangle \models (^{+}1.D)(\overline{m}_1 - \overline{x}) \wedge (1.C)(\overline{x})$$

If there exists $\beta \in Q_1$ such that $\beta^{-1}D$ is a word, then $\overline{m}_1 - \overline{x} \in (^{+}1.D)(M/\langle m_0\rangle)$ implies that $\overline{m}_1 - \overline{x} \in \beta(M/\langle m_0\rangle)$- i.e. there exists $y \in M$ and $\lambda \in K$ such that $m_1 - x = \beta y + \lambda m_0$.

If no such $\beta$ exists, then $\overline{m}_1 - \overline{x} \in (^{+}1.D)(M/\langle m_0\rangle)$ implies that $\overline{m}_1 - \overline{x} = 0$- i.e. there exists $\lambda \in K$ such that $m_1 - x = \lambda m_0$.

In either case, $\alpha m_0 = 0$ (since $m_0$ is a trough), and so $\alpha x = \alpha m_1 = m_0$ (since $\alpha\beta = 0$ if such a $\beta$ exists).

Now, recall that $M/\langle m_0\rangle \models (1.C)(\overline{x})$- i.e.:

$$M/\langle m_0\rangle \models \exists v_1 \dots v_n \left( C_1 v_1 = \overline{x} \wedge \bigwedge_{i=2}^{n} (C_i v_i = D_{i-1} v_{i-1}) \wedge \gamma D_n v_n = 0 \right)$$

Pick any $x_1, x_2, \dots, x_n$ in $M$ such that their images in $M/\langle m_0\rangle$ are witnesses to that pp-formula. Then there exists $\lambda_0, \lambda_1, \dots, \lambda_n \in K$ such that:

$$M \models C_1x_1 = x + \lambda_0 m_0$$

$$M \models C_ix_i = D_{i-1}x_{i-1} + \lambda_{i-1}m_0 \text{ for every } i \in \{2, \dots n\}$$

$$M \models \gamma D_n x_n = \lambda_n m_0$$

We claim that, for every $i \in \{1, \dots, n\}$ there exists $y_i \in M$, a finite subset $J_i \subset \mathbb{Z}\backslash\{N_i\}$, and non-zero elements $\{\mu_j : j \in J_i\}$ of $K$, such that:

1. $D_ix_i = m_{N_i} + \sum_{j\in J_i} \mu_j m_j + y_i$
2. $y_i \in D_iC_i^{-1} \dots D_1C_1^{-1}\alpha^{-1}(0)$
3. $m_j \notin D_iC_i^{-1} \dots D_1C_1^{-1}\alpha^{-1}(0)$ for all $j \in J_i$

We shall prove this by induction: For $i = 1$, we have that $\alpha C_1x_1 = \alpha(x+\lambda_0m_0) = m_0$. Let $y' = x_1 - m_{c_1}$. Then:

$$D_1x_1 = m_{N_1} + D_1y'$$

And, since $\alpha C_1 y' = 0$, we have that $y' \in C_1^{-1}\alpha^{-1}(0)$, and so we set $y_1 = D_1y' \in D_1C_1^{-1}(0)$ and $J_1 = \emptyset$, as required.

Assume now that the claim holds for $i$. Then:

$$\begin{aligned} C_{i+1}x_{i+1} &= \lambda_i m_0 + D_ix_i \\ &= \lambda_i m_0 + m_{N_i} + \sum_{j\in J_i} \mu_j m_j + y_i \end{aligned}$$

If $m_0 \in D_iC_i^{-1} \dots D_1C_1^{-1}\alpha^{-1}(0)$, then replace $y_i$ by $y_i + \lambda_i m_i$. If not, then we can replace $\mu_0$ by $\mu_0 + \lambda_0$ (if $0 \in J_i$), or define $\mu_0 = \lambda_0$, and replace $J_i$ by $J_i \cup \{0\}$ (if $0 \notin J$)- so that we now have:

$$\begin{aligned} C_{i+1}x_{i+1} &= m_{N_i} + \sum_{j\in J} \mu_j m_j + y_i \\ &= C_{i+1}m_{N_i+c_{i+1}} + \sum_{j\in J_i} \mu_j m_j + y_i \end{aligned}$$

-with $y_i \in D_iC_i^{-1} \dots D_1C_1^{-1}\alpha^{-1}(0)$, and $m_j \notin D_iC_i^{-1} \dots D_1C_1^{-1}\alpha^{-1}(0)$ for all $j \in J_i$.

We claim that $m_j \in C_{i+1}(M)$, for all $j \in J_i$: suppose not, for a contradiction. Assume, without loss of generality, that $C_{i+1} \in H_1(b)$, for some $b \in Q_0$. Then let $J' := \{j \in J_i : \hat{w}_j \leq \hat{w}_{j'} \text{ for all } j' \in J_i\}$, and pick $j_0 \in J'$ such that $\hat{u}_{j_0}$ is minimal- note that it is unique, by lemma 89.

By lemma 159, $\sum_{j\in J'} \mu_j m_j$ is fundamental in $M$, with right-word $\hat{w}_{j_0}$, and left word $\hat{u}_{j_0}$. However, we have that:

$$\sum_{j\in J'} \mu_j m_j = \left( (C_{i+1}x_{i+1} - m_{N_i+c_{i+1}}) - \sum_{j\in J_i\backslash J'} \mu_j m_j \right) + y_i$$

Since $m_{j_0} \notin D_i C_i^{-1} \dots D_1 C_1^{-1} \alpha^{-1}(0)$ and $y_i \in D_i C_i^{-1} \dots D_1 C_1^{-1} \alpha^{-1}(0)$, $y_i$ must have left-word strictly greater than the left-word of $\sum_{j\in J'} \mu_j m_j$.

Furthermore, $\hat{w}_j > \hat{w}_{j_0}$, for all $j \in J_i \backslash J'$, so the right-word of $\sum_{j\in J_i\backslash J'} \mu_j m_j$ is greater than $\hat{w}_{i_0}$ (by lemma 155). Also, since $\sum_{j\in J'} \mu_j m_j \notin C_{i+1}(M)$, the right-word of $C_{i+1}(x_{i+1} - m_{N_i+c_{i+1}})$ must be greater than that of $\sum_{j\in J'} \mu_j m_j$. So the right-word of:

$$C_{i+1}(x_{i+1} - m_{N_i+c_{i+1}}) - \sum_{j\in J_i\backslash J'} \mu_j m_j$$

-in $M$ is greater than $\hat{w}_{j_0}$- contradicting the fact that $\sum_{j\in J'} \mu_j m_j$ is fundamental in $M$. Thus proving that $m_j \in C_{i+1}(M)$ for all $j \in J_i$.

Now, given any $j \in J_i$, the right-word of $m_j$ in $M$ is $\hat{w}_j$- which is either $w_j$ or $u_j$. Since $m_j \in C_{i+1}(M)$, it follows that $C_{i+1}$ must be an initial subword of either $w_j$ or $u_j$ (by lemma 154), and so either $C_{i+1} m_{j+c_{i+1}} = m_j$ or $C_{i+1} m_{j-c_{i+1}} = m_j$. Define the sets:

$$J_+ = \{j + c_{i+1} : j \in J_i, C_{i+1} m_{j+c_{i+1}} = m_j\}$$

$$J_- = \{j - c_{i+1} : j \in J_i, C_{i+1} m_{j-c_{i+1}} = m_j\}$$

Given any $j \in J_+$, let $\mu'_j = \mu_{j-c_{i+1}}$, and given any $j \in J_-$, let $\mu'_j = \mu_{j+c_{i+1}}$. So:

$$C_{i+1}\Big( \sum_{j\in J_+\cup J_-} \mu'_j m_j \Big) = \sum_{j\in J_i} \mu_j m_j$$

Now, given any $j \in J_+$, $D_{i+1} m_j$ is either 0 or $m_{j+d_{i+1}}$- and in the latter case, $m_{j+d_{i+1}} \notin D_{i+1} C_{i+1}^{-1} D_i \dots D_1 C_1^{-1} \alpha^{-1}(0)$ (it follows from lemma 158).

Applying a similar argument to all $j \in J_-$ it follows that:

$$D_{i+1}(\sum_{j \in J_- \cup J_+} \mu' m_j) = \sum_{j \in J_{i+1}} \mu'' m_j$$

-for some finite set $J_{i+1}$, with $m_j \notin D_{i+1}C_{i+1}^{-1}D_i \ldots D_1 C_1^{-1}\alpha^{-1}(0)$ for all $j \in J_{i+1}$.

Now, define:

$$y' := x_{i+1} - m_{N_i + c_{i+1}} - \sum_{j \in J_- \cup J_+} \mu'_j m_j$$

Of course, $C_{i+1}y' = y_i$, and so $y' \in C_{i+1}^{-1}D_iC_i^{-1}\ldots D_1C_1^{-1}\alpha^{-1}(0)$. Now let $y_{i+1} = D_{i+1}y'$. Then:

$$\begin{aligned} D_{i+1}x_{i+1} &= m_{N_{i+1}} + D_{i+1}\sum_{j \in J_+ \cup J_-} \mu' m_j + y_{i+1} \\ &= m_{N_{i+1}} + \sum_{j \in J_{i+1}} \mu'' m_j + y_{i+1} \end{aligned}$$

-which completes the induction.

As a result of the induction, we have:

$$D_n x_n = m_{N_n} + \sum_{j \in J} \mu_j m_j + y_n$$

-with $y_n \in C^{-1}\alpha^{-1}(0)$ and $m_j \notin C^{-1}\alpha^{-1}(0)$ for all $j \in J_{n+1}$. Recall that $l_{N_n+1} = \gamma^{-1}$, and that $\gamma D_n x_n = \lambda_n m_0$. Thus:

$$\lambda_n m_0 = m_{N_n+1} + \sum_{j \in J} \mu_j \gamma m_j + \gamma y_n$$

If $\lambda_0 m_0 \in \gamma C^{-1}\alpha^{-1}(0)$, then we may replace $y_n$ by $y_n - \lambda_n m_0$, and still have $y_n \in C^{-1}\alpha^{-1}(0)$. If $\lambda_0 m_0 \notin \gamma C^{-1}\alpha^{-1}(0)$ then we may replace $\mu_0$ by $\mu_0 - \lambda_n$. So we now have:

$$0 = m_{N_n+1} + \sum_{j \in J_n} \mu_j \gamma m_j + \gamma y_n$$

Note that $\gamma y_n \in \gamma C^{-1}\alpha^{-1}(0)$. Also, given any $j \in J_n$, either $\gamma m_j = 0$, or $\gamma m_j \notin \gamma C^{-1}\alpha^{-1}(0)$ (since $m_j$ is fundamental).

Thus there exists a finite, non-empty set $J'$, such that $m_j \notin \gamma C^{-1}\alpha^{-1}(0)$ for all $j \in J'$, and non-zero elements $\nu_j$ for all $j \in J'$ such that:

$$\gamma y_i = \sum_{j \in J'} \nu_j m_j$$

However, the left hand side lies in $\gamma C^{-1}\alpha^{-1}(0)$, since $m_j \notin \gamma C^{-1}\alpha^{-1}(0)$ for all $j \in J'$, it follows from corollary 44 that $\sum_{j\in J'} \nu_j m_j \notin \gamma C^{-1}\alpha^{-1}(0)$- giving our desired contradiction. □

### 7.1.4 The pp-type of a fundamental element

**Lemma 161.** *Let $w = \dots l_{-2}l_{-1}l_0l_1l_2\dots$ be any aperiodic or half-periodic $\mathbb{Z}$-word.*

*Let $M$ be any pure-injective $A$-module. Suppose there exists a fundamental $m_0 \in M$ with left word $u_0$, and right-word $w_0$. Then:*

- *If $w_0$ is aperiodic, then any map $f : M(w) \to M$ taking $z_0$ to $m_0$ is a pure-embedding.*
- *If $w$ is contracting half-periodic, then any map $f : M(w) \to M$ taking $z_0$ to $m_0$ is a pure-embedding.*
- *If $w$ is expanding half-periodic, then given any map $f : M(w) \to M$ taking $z_0$ to $m_0$, the map $(f, h_D) : M(w) \to M \oplus M({}^\infty D^\infty)$ is a pure-embedding (where $h_D$ is the map as defined in lemma 103).*

*Proof.* Let $m_i = f(z_i)$ for all $i \neq 0$. By lemma 158 each $m_i$ is fundamental in $M$, with left word $\hat{u}_i$ and right-word $\hat{w}_i$.

Take any $x = \sum_{i\in\mathbb{Z}} \lambda_i z_i$ in $M(w)$ and any pp-formula $\phi(v) \in \mathrm{pp}^M(f(x))$ (with $h_D(x) \in \phi({}^\infty D^\infty)$ if $w$ is expanding half-periodic). Let $m$ be the number of equations in $\phi(v)$. We must show that $x \in \phi(M(w))$.

Pick any trough $z_{i_0}$ such that $\lambda_i = 0$ for all $i \leq i_0$. Then $M(w_{i_0})$ is a submodule of $M(w)$, which contains $x$. Since $u_0$ is aperiodic, there exists $i_1 < i_0$ such that ${}^{(m)}w_{i_0}$ is the pre-subword $w_{i_1}$ of $w$. Pick any $k < i_1$ such that $z_k$ is a trough, and also such that $w_{k+1}$ is not periodic.

Since $w_{i_1}$ is a pre-subword of $w_{k+1}$, we have a map:

$$M(w_k) \hookrightarrow M(w) \xrightarrow{f} M \twoheadrightarrow M/\langle z_k\rangle$$

-where the first map is the canonical embedding, and the last map is the natural projection onto the quotient module. Of course, this map takes $z_k$ to 0, so we can factor it through $M(w_k)/\langle z_k\rangle \cong M(w_{k+1})$:

$$\begin{tikzcd} M(w_k) \arrow[r] \arrow[dr, "\pi"'] & M(w) \arrow[r, "f"] & M \arrow[r, two heads] & M/\langle z_k\rangle \\ & M(w_{k+1}) \arrow[urr, dashed, "\exists g"'] & & \end{tikzcd}$$

We may assume, without loss of generality, that $w_{k+1} \in H_1(a)$ for some $a \in Q_0$. Lemma 160 implies that, for any initial pre-subword $D$ of $w_{k+1}$, and any finite word $E \in H_1(a)$:

$$M/\langle z_k\rangle \models ((^+1.D) + (1.E))(m_1) \text{ if and only if } E < w_{k+1}$$

If $w$ is aperiodic, then $w_{k+1}$ is aperiodic, and proposition 8 gives us that:

$$M(w_{k+1}) \models \phi(x)$$

Similarly, if $w$ is contracting half-periodic, then $w_{k+1}$ is contracting almost periodic, and so, by proposition 9:

$$M(w_{k+1}) \models \phi(x)$$

And if $w$ is expanding half-periodic, then $w_{k+1}$ is expanding almost periodic. And since $M(^\infty D^\infty) \models \phi(h_D(x))$, proposition 10 gives that:

$$M(w_{k+1}) \models \phi(x)$$

Since $^{(m)}w_{i_0}$ is $w_{i_1}$, which is a pre-subword of $w_{k+1}$, lemma 22 gives that:

$$M(^{(m)}w_{i_0}) \models \phi(x)$$

And so $M(w) \models \phi(x)$, as required. □

### 7.1.5 Extending lemma 161 to almost periodic $\mathbb{Z}$-words

We can extend lemma 161 to all almost periodic words. To do so, we need a slight variant of corollary 23.

**Lemma 162.** *Let $w$ be any contracting or mixed almost periodic $\mathbb{Z}$-word- assume it is of the form $^\infty E l_1 l_2 l_3 \ldots$, where $E$ is a band of length $N$, and $l_1^{-1}(E^{-1})^\infty$ is not periodic.*

*Then, given any $x \in M(w)$ and $m \in \mathbb{N}$, there exists $d < 0$ such that $x$ lies "to the right of $z_d$" (i.e. $x$ has $z_i$-coefficient $0$ for all $i \leq d$), $w_d$ is a post-subword of $w$, and for all pp-formulas $\phi(v)$ with at most $m$ equations:*

$$x \in \phi(M(w)) \Longleftrightarrow \pi(x) \in \phi(M(w_d))$$

*-where $\pi : M(w) \twoheadrightarrow M(w_d)$ is the canonical projection.*

*Proof.* First of all we define $d$: Let $z_{t_0}$ denote any trough, with $t_0 < 0$ such that $x \in M(w_t)$. Pick any $c \in \mathbb{N}$ such that $-cN < t_0$. Given any $m \in \mathbb{N}$, pick any $d < -cN(m+1)$ such that $w_d$ is a post-subword of $w$.

Of course, $x \in \phi(M(w))$ implies that $\pi(x) \in \phi(M(w_d))$. For the converse, we use a similar argument to the proof of lemma 98. Write $\phi(v)$ as $\exists v_1, \ldots v_n \psi(v_1, \ldots, v_n v)$, where $\psi$ is the pp-formula:

$$\bigwedge_{j=1}^{m} (\sum_{i=1}^{n} r_{ij} v_i = r_j v)$$

We assume that $\pi(x) \in \phi(M(w_d))$- and hence that there exists $x_1, \ldots, x_n$ in $M(w)$ such that, for all $j \leq m$:

$$\sum_{i=1}^{n} r_{ij} \pi(x_i) = r_j \pi(x)$$

Let $y'_j = \sum_{i=1}^{n} r_{ij} x_i - r_j x$. Of course, $\pi(y'_j) = 0$, so $y'_j$ must have $z_k$-coefficient 0, for all $k \geq d$.

Given any $s$ such that $1 \leq s \leq m$, let $t_s = t_0 - smN$. Since $u_0$ is periodic, $z_{t_s}$ is a trough, and so $M(w_{t_s})$ and $M(u_{t_s}^{-1})$ are submodules of $M(w)$, such that $M(w_{t_s}) + M(u_{t_s}^{-1}) = M(w)$ and $M(w_{t_s}) \cap M(u_{t_s}^{-1}) = K z_{t_s}$.

Thus $x_i = x_i^{\leq T_s} + x_i^{> T_s}$, for some $x_i^{\leq T_s} \in M(u_{t_s}^{-1})$ and $x_i^{> T_s} \in M(w_{t_s})$. And so, for all $j \leq m$, we have:

$$\sum_{i=1}^{n} r_{ij} x_i^{> T_s} - r_j x = -\sum_{i=1}^{n} r_{ij} x_i^{\leq T_s} + y'_j$$

Since the right hand side lies in $M(u_{t_s}^{-1})$, and the left hand side in $M(w_{t_s})$, both sides equal $\rho_{js} z_{t_s}$, for some $\rho_{j,s} \in K$.

As in the proof of 98, we can pick $\{\mu_s \in K : 0 \leq s \leq m\}$ (not all zero) such that $\sum_{s=0}^{m} \mu_s \rho_{js} = 0$ for every $j \in \{1, \ldots, m\}$.

Since $(E^{-1})^\infty$ is a contracting $\mathbb{N}$-word, there exists a simple string map $\Phi \in \mathrm{End}(M(w))$, such that, for all $i \leq -N$, $\Phi(z_i) = z_{i+N}$, and for all $i > -N$, $\Phi(z_i) = 0$. Notice that, for all $j$, $r_j x \in M(w_{t_0})$, and so $\Phi^c(r_j x) = 0$ (since $t_0 > -cN$).

Now, let $k$ be minimal such that $\mu_k \neq 0$. For all $i \in \{1, 2, \ldots, n\}$, define:

$$y_i = \mu_k x_i^{>T_k} + \sum_{s>k} \mu_s \Phi^{(s-k)c}(x_i^{>T_s})$$

Then:

$$\begin{aligned}
\sum_i r_{ij} y_i &= \sum_i r_{ij} \left( \mu_k x_i^{>T_k} + \sum_{s>k} \mu_s \Phi^{(s-k)c}(x_i^{>T_s}) \right) \\
&= \sum_i r_{ij} \mu_k x_i^{>T_k} + \sum_{s>k} \mu_s \Phi^{((s-k)c}(\sum_i r_{ij} x_i^{>T_s}) \\
&= \mu_k(-r_j x + \rho j s z_{t_s} + \sum_{s>k} \mu_s \Phi^{(s-k)c}(-r_j x + \rho_{jk} z_{t_k}) \\
&= -\mu_k r_j x + \rho j s z_{t_s} + \sum_{s>k} \mu_s \rho_{jk} z_{t_s} \\
&= -\mu_k r_j x
\end{aligned}$$

Thus $M(w) \models \phi(-\mu_k x)$, and so $M(w) \models \phi(x)$, as required. □

**Lemma 163.** *Let $w$ be any expanding almost periodic $\mathbb{Z}$-word- assume it is of the form ${}^\infty E l_1 l_2 l_3 \ldots$, where $E$ is a band of length $N$, and $l_1^{-1}(E^{-1})^\infty$ is not periodic.*

*Then, given any $x \in M(w)$ and $m \in \mathbb{N}$, there exists $d < 0$ such that $x$ lies "to the right of $z_d$" (i.e. $x$ has $z_i$-coefficient $0$ for all $i \leq d$), $w_d$ is a post-subword of $w$, and for all pp-formulas $\phi(v)$ with at most $m$ equations:*

$$x \in \phi(M(w)) \iff \pi(x) \in \phi(M(w_d)) \text{ and } g_E(x) \in \phi(M({}^\infty E^\infty))$$

*-where $\pi : M(w) \twoheadrightarrow M(w_d)$ is the canonical projection, and $g_E$ is the map as defined in (5.6.1).*

*Proof.* The proof is very similar to that of lemma 162.

First of all, pick any $t_0 < 0$ such that $z_{t_0}$ is a trough, and $x \in M(w_{t_0})$. Pick any $c \in \mathbb{N}$ such that $-cN < t_0$. Given any $m \in \mathbb{N}$, pick any $d < -cN(m+1)$ such that $w_d$ is a post-subword of $w$.

Of course:

$$x \in \phi(M(w)) \Longrightarrow \pi(x) \in \phi(M(w_d)) \text{ and } g_E(x) \in \phi(M({}^\infty E^\infty))$$

To prove the converse, assume that $\pi(x) \in \phi(M(w_d))$ and $g_E(x) \in \phi(M({}^\infty E^\infty))$. It suffices, by lemma 103, to prove that $x \in \phi(\overline{M}(w))$. Write $\phi(v)$ in the form $\exists v_1, \ldots v_n \psi(v_1, \ldots, v_n v)$, where $\psi$ is the formula:

$$\bigwedge_{j=1}^{m} \left( \sum_{i=1}^{n} r_{ij} v_i = r_j v \right)$$

Since $\pi(x) \in \phi(M(w_d))$, there exists $x_1, \ldots, x_n$ in $M(w)$ such that, for all $j \leq m$:

$$\sum_{i=1}^{n} r_{ij} \pi(x_i) = r_j \pi(x)$$

Let $y'_j = \sum_{i=1}^{n} r_{ij} x_i - r_j x$. Since $\pi(y'_j) = 0$, $y'_j$ must have $z_k$-coefficient 0, for all $k \geq d$.

Given any $s$ such that $1 \leq s \leq m$, let $t_s = t_0 - smN$. Mimicking the proof of lemma 162, we can write each $x_i$ as $x_i^{\leq T_s} + x_i^{>T_s}$, with $x_i^{\leq T_s} \in M(u_{t_s}^{-1})$ and $x_i^{>T_s} \in M(w_{t_s})$. Furthermore, there exists $\rho_{js}$ (for all $j$ and $s$) such that:

$$\sum_{i=1}^{n} r_{ij} x_i^{>T_s} - r_j x = -\sum_{i=1}^{n} r_{ij} x_i^{\leq T_s} + y'_j = \rho_{js} z_{t_s}$$

-and we can pick $\{\mu_s \in K : 0 \leq s \leq m\}$ (not all zero) such that $\sum_{s=0}^{m} \mu_s \rho_{js} = 0$ for every $j \in \{1, \ldots, m\}$.

Since $(E^{-1})^\infty$ is an expanding $\mathbb{N}$-word, there exists a simple string map $\Phi \in \mathrm{End}(\overline{M}(w))$, defined by:

$$\Phi : \sum_{k \in \mathbb{Z}} \lambda_k z_k \mapsto \sum_{k \leq 0} \lambda_k z_{k-N}$$

Now, let $k \leq m$ be minimal such that $\mu_k \neq 0$. For all $i \in \{1, 2, \ldots, n\}$, define:

$$y_i = \mu_k x_i^{>T_k} + \sum_{s > k} \mu_s \Phi^{(s-k)c}(x_i^{>T_s})$$

Then:

$$
\begin{aligned}
\sum_i r_{ij} y_i &= \sum_i r_{ij}\left(\mu_k x_i^{>T_k} + \sum_{s>k} \mu_s \Phi^{(s-k)c}(x_i^{>T_s})\right)\\
&= \sum_i r_{ij}\mu_k x_i^{>T_k} + \left(\sum_{s>k} \mu_s \Phi^{(s-k)c}(\sum_i r_{ij} x_i^{>T_s})\right)\\
&= \mu_k(-r_j x + \rho j s z_{t_s}) + \sum_{s>k} \mu_s \Phi^{(s-k)c}(-r_j x + \rho_{jk} z_{t_k})\\
&= -\mu_k r_j x + \rho j s z_{t_s} - \sum_{s>k} \mu_s \Phi^{(s-k)} r_j x + \sum_{s>k} \mu_s \rho_{jk} z_{t_s}\\
&= -r_j(\sum_{k\le s\le m} \mu_k \Phi^{cN(k-s)})(x)
\end{aligned}
$$

Thus $\overline{M}(w) \models \phi(\sum_{k\le s\le m} \mu_k \Phi^{cN(k-s)}(x))$. Now, the map $\sum_{k\le s\le m} \mu_k \Phi^{cN(k-s)}$ is invertible (by a similar proof to that of lemma 120), and so $\overline{M}(w) \models \phi(x)$, as required. □

**Corollary 45.** *Let $w = {}^{\infty}D l_1 l_2 \ldots l_s E^{\infty}$ be any almost-periodic $\mathbb{Z}$-word (where $l_s E^{\infty}$ and $l_1^{-1}(D^{-1})^{\infty}$ are not periodic). Let $M$ be any two-directed pure-injective indecomposable module, containing a fundamental element $m_0$ with right-word $w_0$ and left-word $u_0$. Then, given any $f \in \mathrm{Hom}(M(w), M)$ taking $z_0$ to $m_0$:*

- *If $w$ is contracting, then $f$ is pure.*
- *If $w$ is mixed, then $(f, h_E) : M(w) \to M \oplus M({}^{\infty}E^{\infty})$ is pure.*
- *If $w$ is expanding, then $(f, g_D, h_E) : M(w) \to M \oplus M({}^{\infty}E^{\infty}) \oplus M({}^{\infty}E^{\infty})$ is pure.*

*Proof.* The proof mimics that of lemma 161, using lemma 162 or lemma 163 instead of corollary 23. □

## 7.2 Pure-injective hulls of string modules

### 7.2.1 Periodic and almost periodic modules

We now turn our attention to periodic $\mathbb{Z}$-words. Given any band $D$, the *shift ring* of $M({}^{\infty}D^{\infty})$ is the subring $S$ of $\mathrm{End}(M({}^{\infty}D^{\infty}))$ generated by the Ringel shift map

$\Phi \in \mathrm{End}(M({}^{\infty}D^{\infty}))$- which is the simple string map taking each standard basis element $z_i$ to $z_{i+k}$ ($k$ being the length of $D$). Notice that it is isomorphic to $K[T, T^{-1}]$, and so we may consider $M({}^{\infty}D^{\infty})$ as a right module over $K[T, T^{-1}]$.

Given a band, $D$, recall, from (6.3.1), the functor $F_D : K[T, T^{-1}]\text{-Mod} \to A\text{-Mod}$, which takes indecomposable finitely generated modules to band modules. It is in fact isomorphic to the functor from $K[T, T^{-1}]$-Mod to $A$-Mod which takes every module ${}_{K[T,T^{-1}]}M$ to ${}_{A}M({}^{\infty}D^{\infty}) \otimes_{K[T,T^{-1}]} M$ (for an explanation, see [6, p4]).

Given any prime ideal $P$ of $S$, we shall denote by $\chi_P$ the map:

$$M({}^{\infty}D^{\infty}) \overset{(m\otimes 1)}{\longrightarrow} M({}^{\infty}D^{\infty}) \otimes S \overset{(m\otimes \chi'_P)}{\longrightarrow} M({}^{\infty}D^{\infty}) \otimes H(S_{(P)})$$

-where $\chi'_P$ is the composition of the embedding of $S$ into its localisation, $S_{(P)}$, and the pure-injective hull $S_{(P)} \hookrightarrow H(S_{(P)})$ of $S_{(P)}$.

We define $\chi : M(w) \longrightarrow \prod_{P\in\mathcal{P}} F_D(H(S_{(P)}))$ to be the map such that for every $x \in M(w)$ and $P \in \mathcal{P}$, the component of $\chi(x)$ in $F_D(H(S_{(P)}))$ is $\chi_P(x)$.

**Theorem 49.** *Let $w$ be be any periodic $\mathbb{Z}$-word, ${}^{\infty}D^{\infty}$. Then the pure-injective hull of $M(w)$ is:*

$$M(w) \overset{\chi}{\longrightarrow} \prod_{P\in\mathcal{P}} F_D(H(S_{(P)}))$$

*-where $\mathcal{P}$ is the set of all non-zero prime ideals of the shift ring, $S$.*

*Furthermore, $\prod_{P\in\mathcal{P}} F_D(H(S_{(P)}))$ is a direct summand of any pure-injective model of the theory of $M({}^{\infty}D^{\infty})$.*

*Proof.* See [6, (2.13)] □

It is known that, for any $P \in \mathcal{P}$, $F_D(H(S_{(P)})) = M({}^{\infty}D^{\infty}) \otimes_{K[T,T^{-1}]} H(S_{(P)})$ is pure-injective and indecomposable. Indeed, as a direct summand of the pure-injective hull of $M({}^{\infty}D^{\infty})$, it must be pure-injective.

To see that it is indecomposable, consider $F_D$ as the functor described in (6.3.1). Then $F_D(S_{(P)})$ is a representation of $Q_A$ with the module $H(S_{(P)})$ placed on every vertex. Since $H(S_{(P)})$ is indecomposable, it follows that $F_D(H(S_{(P)}))$ must also be.

**Theorem 50.** *For an expanding periodic $\mathbb{N}$-word, $w = l_1 \ldots l_s D^\infty$, the pure-injective hull of $M(w)$ is:*

$$(f_1, \chi h_D) : M(w) \longrightarrow \overline{M}(w) \oplus \prod_{P \in \mathcal{P}} F_D(H(S_{(P)}))$$

*-where $f_1$ is the canonical embedding, $h_D$ is as in lemma 103, and $\chi$ as in theorem 49.*

*For a mixed almost periodic $\mathbb{Z}$-word, $w = {}^\infty E l_1 \ldots l_s D^\infty$, the pure-injective hull of $M(w)$ is:*

$$(f_1, \chi h_D) : M(w) \longrightarrow M^+(w) \oplus \prod_{P \in \mathcal{P}} F_D(H(S_{(P)}))$$

*-where $f_1$ is the canonical embedding.*

*For an expanding almost periodic $\mathbb{Z}$-word, $w = {}^\infty E l_1 \ldots l_s D^\infty$, the pure-injective hull of $M(w)$ is:*

$$(f_1, \chi g_E, \chi h_D) : M(w) \longrightarrow \overline{M}(w) \oplus \prod_{P \in \mathcal{P}} F_E(H(S_{(P)})) \oplus \prod_{P \in \mathcal{P}} F_D(H(S_{(P)}))$$

*-where $f_1$ is the canonical embedding, and $h_D$ and $g_E$ are as defined in lemma 104.*

*Proof.* See [6, (3.5)] and [6, (3.7)]. □

### 7.2.2 Aperiodic and half periodic modules

**Proposition 15.** *Let $w$ be any aperiodic or contracting half-periodic $\mathbb{Z}$-word. Then the pure-injective hull of $M(w)$ is a two-directed indecomposable module.*

*Proof.* Since $M(w)$ is a two-directed module, every pp-pair of the form $(1.D)/({}^+1.D)$ is closed on $M(w)$, and hence on $H(M(w))$, by theorem 9.

Now assume- for a contradiction- that $H(M(w))$ is not indecomposable. Write the pure-injective hull as:

$$M(w) \overset{(f_1, f_2)}{\hookrightarrow} M_1 \oplus M_2$$

-with $M_1$ and $M_2$ being non-zero, and pure-injective. It will be enough to show that we can factor this map through either $f_1$ or $f_2$- because this will contradict the minimality condition of pure-injective hulls.

Of course, for all $D < w_0$, $z_0 \in (.D)(M(w))$, and so $f_1(z_0) \in (.D)(M_1)$ and $f_2(z_0) \in (.D)(M_2)$. Similarly, for all $C < u_0$, $f_1(z_0) \in (.C)(M_1)$ and $f_2(z_0) \in (.C)(M_2)$.

We claim that, either $f_1(z_0)$ is fundamental in $M_1$, with left word $u_0$, and right-word $w_0$, or $f_2(z_0)$ is fundamental in $M_2$, with left word $u_0$, and right-word $w_0$. If neither holds, then there must exist $x \in M_1$, with left word $u' > u_0$ such that $f_1(z_0)-x$ has right-word $w' > w_0$ in $M_1$, and similarly, $y \in M_2$, with left word $u'' > u_0$ such that $f_2(z_0) - y$ has right-word $w'' > w_0$ in $M_2$.

Then $(x, y) \in M_1 \oplus M_2$ has left word $\min(u', u'') > u_0$ in $M_1 \oplus M_2$, and $(f_1(z_0) - x, f_2(z_0) - y)$ has right-word $\min(w', w'') > w_0$ in $M_1 \oplus M_2$- but since $(f_1(z_0), f_2(z_0))$ is fundamental in $M_1 \oplus M_2$ with right-word $w_0$ and left-word $u_0$ (by lemma 156), we have our contradiction.

We may therefore assume, without loss of generality, that $f_1(z_0)$ is fundamental in $M_1$, with left-word $u_0$ and right-word $w_0$. Thus, by lemma 161, $f_1$ is a pure-embedding.

Since $M_2$ is pure-injective, $f_2$ must factor through the pure-embedding $f_1$:

$$\begin{tikzcd} & M_2 \\ M(w) \arrow[ur, "f_2"] \arrow[r, "f_1"'] & M_1 \arrow[u, dashed, "\exists g"'] \end{tikzcd}$$

So we can factor $(f_1, f_2)$ through $f_1$, as required. $\square$

**Lemma 164.** *The pure injective hull of $\bigoplus_{P\in\mathcal{P}} F_D(H(S_{(P)}))$ is the canonical embedding:*

$$\bigoplus_{P\in\mathcal{P}} F_D(H(S_{(P)})) \hookrightarrow \prod_{P\in\mathcal{P}} F_D(H(S_{(P)}))$$

*Proof.* By theorem 49 the map:

$$\bigoplus_{P\in\mathcal{P}} F_D(H(S_{(P)})) \hookrightarrow \prod_{P\in\mathcal{P}} F_D(H(S_{(P)}))$$

-is a pure-embedding, and since $\prod_{P\in\mathcal{P}} F_D(H(S_{(P)}))$ is pure-injective, lemma 9 implies that $H(\bigoplus_{P\in\mathcal{P}} F_D(H(S_{(P)})))$ is a direct summand of $\prod_{P\in\mathcal{P}} F_D(H(S_{(P)})))$.

By theorem 49, it remains to show that $H(\bigoplus_{P\in\mathcal{P}} F_D(H(S_{(P)})))$ models the theory of $M({}^\infty D^\infty)$.

The pure-injective hull in theorem 49 implies that $\prod_{P\in\mathcal{P}} F_D(H(S_{(P)}))$ models the theory of $(M({}^\infty D^\infty))$ (by theorem 8).

Also, the canonical embedding:

$$\bigoplus_{P\in\mathcal{P}} F_D(H(S_{(P)})) \hookrightarrow \prod_{P\in\mathcal{P}} F_D(H(S_{(P)}))$$

-is an elementary embedding, by lemma 1.

Thus $\bigoplus_{P\in\mathcal{P}} F_D(H(S_{(P)}))$ models the theory of $(M({}^\infty D^\infty))$. Hence, by theorem 8, so does $H(\bigoplus_{P\in\mathcal{P}} F_D(H(S_{(P)})))$- as required. □

Of course, $K[T,T^{-1}]$ is a principal ideal domain, and- since we are assuming that $K$ is algebraically closed- every prime ideal $P$ of $K[T,T^{-1}]$ can be written as $\langle T-\lambda\rangle$, for some unique $\lambda \in K\backslash\{0\}$. We denote by $\phi_\lambda(v)$ the pp-formula:

$$\exists v_1 \exists v_2 \, (v = v_1 + v_2 \wedge v_1 \in D(\lambda v_2))$$

**Lemma 165.** *Let $w = \ldots l_{s-2} l_{s-1} l_s D^\infty$ be any expanding half-periodic $\mathbb{Z}$-word. Let $f : M(w) \hookrightarrow \overline{M}(w)$ denote the canonical embedding of the submodule. Then, given any prime ideal $P_0 \in \mathcal{P}$, any map of the form:*

$$M(w) \xrightarrow{(f,g)} \overline{M}(w) \oplus \prod_{P\in\mathcal{P}\backslash\{P_0\}} F_D(H(S_{(P)}))$$

*-is not a pure-embedding.*

*Proof.* First of all, recall that there are simple string map $\Phi_w \in \mathrm{End}(M(w)))$ and $\Phi'_w \in \mathrm{End}(\overline{M}(w))$, which take take every $z_i$ with $i \geq s$ to $z_{i+n}$ ($n$ being the length of $D$), and every $z_i$ with $i < s$ to zero.

Write $P_0$ as $\langle T-\lambda\rangle$ and let $\phi_\lambda(v)$ be the pp-formula as defined above.

First of all, for all $P \in \mathcal{P}\backslash\{P_0\}$, every element of $H(S_{(P)})$ is divisible by $T-\lambda$, and hence so is every element of $M({}^\infty D^\infty) \otimes H(S_{(P)})$ (considering it as a left module over the shift ring $S \simeq K[T,T^{-1}]$). Thus $\phi_\lambda(F_D(H(S_{(P)})) = F_D(H(S_{(P)}))$.

It remains to prove that $z_s \notin \phi_\lambda(M(w))$ and $z_s \in \phi_\lambda(\overline{M}(w))$. To prove the latter, notice that $z_s - \lambda z_{s+n} \in \phi_\lambda(\overline{M}(w))$ . Thus $(1-\lambda\Phi'_w)(z_s) \in \phi_\lambda(\overline{M}(w))$, and so $z_s \in \phi_\lambda(\overline{M}(w))$ (since $\Phi'_w$ is invertible).

Finally, suppose, for a contradiction, that $z_s \in \phi_\lambda(M(w))$. Let $x, y \in M(w)$ be such that $z_s = x + y$, and $M(w) \models x \in D(\lambda y)$. Note that $z_s \notin D^{-1}(M(w))$ (by the properties of an expanding word), and $y \in D^{-1}(M(w))$- so $y$ must have $z_s$-coefficient 0 (by corollary 27). Thus $x$ has $z_s$-coefficient 1.

Now, let $j \in \mathbb{N}$ be maximal such that $x$ has non-zero $z_{s+jn}$-coefficient (note that such a $j$ exists). Then, by corollary 20, $y$ must have non-zero $z_{s+(j+1)n}$-coefficient (since $x \in D(\lambda y)$). And hence so does $z_s$- giving our required contradiction. Thus $z_s \notin \phi_y(M(w))$, and so the map is indeed not a pure-embedding. □

**Proposition 16.** *Let $w = \ldots l_{-2}l_{-1}l_0D^\infty$ be any expanding half-periodic $\mathbb{Z}$-word. Then the pure-injective hull of $M(w)$ is:*

$$M(w) \stackrel{(f,\chi h_D)}{\longrightarrow} M_w \oplus \prod_{P\in\mathcal{P}} F_D(H(S_{(P)}))$$

*Where $M_w$ is an indecomposable two-directed direct summand of $\overline{M}(w)$, and $f$ is a map such that $f(z_0)$ is fundamental in $M$, with right-word $D^\infty$ and left-word $u_0$.*

*Proof.* By lemma 161 and theorem 49, the map:

$$M(w) \longrightarrow \overline{M}(w) \oplus M({}^\infty D^\infty) \longrightarrow \overline{M}(w) \oplus \prod_{P\in\mathcal{P}} F_D(H(S_{(P)}))$$

-is a pure embedding. Since $\overline{M}(w)$ and $\prod_{P\in\mathcal{P}} F_D(H(S_{(P)}))$ are pure-injective (by proposition 4 and theorem 49), there exists (by lemma 9) a pure-embedding (and hence section) $g$ such that the following diagram commutes:

$$\begin{tikzcd} M(w) \arrow[r] \arrow[dr] & H(M(w)) \arrow[d, dashed, "\exists g"] \\ & \overline{M}(w) \oplus \prod_{P\in\mathcal{P}} F_D(H(S_{(P)})) \end{tikzcd}$$

By theorem 10 there exist indecomposable modules $\{M_i : i \in I_1\}$, $\{N_i : i \in I_2\}$, $\{L_i : i \in I_3\}$, and superdecomposable (or zero) modules $L_c, M_c, N_c$ such that:

$$H(M(w)) \cong H(\bigoplus_{i\in I_1} M_i) \oplus M_c$$

$$\overline{M}(w) \cong H(\bigoplus_{i\in I_2} N_i) \oplus N_c$$

$$M(w) \cong H(\bigoplus_{i\in I_3} L_i) \oplus L_c$$

-and the indecomposable modules are unique up to isomorphism. It follows from lemma 164 that $\bigoplus_{i\in I_3} L_i \cong \bigoplus_{P\in\mathcal{P}} F_D(H(S_{(P)}))$, and that $L_c = 0$. Since $H(M(w))$ is a direct summand of $\overline{M}(w) \oplus H(\bigoplus_{P\in\mathcal{P}} F_D(H(S_{(P)}))$, it follows that:

$$H(M(w)) \cong M_c \oplus H(\bigoplus_{i\in J_2} N_i) \oplus H(\bigoplus_{P\in\mathcal{P}'} F_D(H(S_{(P)})))$$

-for some subsets $J_2 \subseteq I_2$ and $\mathcal{P}' \subseteq \mathcal{P}$. It follows from lemma 165 that $\mathcal{P}' = \mathcal{P}$, and so there exists a direct summand $M$ of $\overline{M}(w)$ such that the pure-injective hull of $M(w)$ is of the form:

$$(f, \chi h_D) : M(w) \longrightarrow M \oplus \prod_{P\in\mathcal{P}} F_D(H(S_{(P)}))$$

Notice that $\chi h_D(z_0)$ has left-word $(D^{-1})^\infty$ in $\prod_{P\in\mathcal{P}} F_D(H(S_{(P)}))$. It follows from lemma 156 and lemma 155 that $f(z_0)$ must be fundamental in $M$, with left-word $u_0$ and right-word $D^\infty$.

We claim that $M$ is indecomposable: Suppose, for a contradiction, that $M \cong M_1 \oplus M_2$ (with both summands being non-zero). Let $f_1 \in \mathrm{Hom}(M(w), M_1)$ and $f_2 \in \mathrm{Hom}(M(w), M_2)$ be the maps such that $f = (f_1, f_2)$. As in the proof of proposition 15, we have- without loss of generality- that $f_1(z_0)$ is fundamental in $M_1$ with left-word $u_0$ and right-word $w_0$.

By lemma 161, the map $M(w) \to M_1 \oplus \prod F_D(H(S_{(P)}))$ is a pure embedding, and hence $f_2$ can be factored through it- contradicting the minimality condition of a pure-embedding. Thus $M$ is indeed indecomposable, which completes the proof. □

**Theorem 51.** *For any infinite word $w$ (other than a periodic $\mathbb{Z}$-word), there exists a unique indecomposable pure-injective $A$-module, $M_w$, containing a fundamental element $m_0$, with right-word $w_0$ and left-word $u_0$.*

*Furthermore, this module is a direct summand of $\overline{M}(u_0^{-1}w_0)$.*

*Proof.* If $M$ is an $\mathbb{N}$-word, then theorem 40 gives the required result. Assume, therefore, that $w$ is a $\mathbb{Z}$-word.

If $w$ is aperiodic, contracting half-periodic, or contracting almost periodic, then the pure-injective hull of $M(w)$ is indecomposable- by proposition 15. Furthermore, given any other module $N$ and element $n_0$ of $N$ satisfying the conditions, there exists a pure-embedding from $M(w)$ to $N$ taking $z_0$ to $n_0$- and so $M \cong N$, by theorem 7 is a direct summand of $N$ (by lemma 9). So $M \cong N$, as required.

Also, the pure embedding $M(w) \hookrightarrow \overline{M}(w)$ (from proposition 5) and lemma 9 imply that $M_w$ is a direct summand of $\overline{M}(w)$.

Now, suppose that $w$ is expanding half-periodic- write it as $u_s^{-1}D^\infty$, for some band $D$. By lemma 16, the pure-injective hull of $M(w)$ is:

$$M(w) \stackrel{(f,\chi h_D)}{\longrightarrow} M \oplus \prod_{P \in \mathcal{P}} F_D(H(S_{(P)}))$$

-for some indecomposable $M$ and map $f$.

Lemma 156 implies that $(f(z_s), h_D\chi(z_s))$ is fundamental, with right-word $w_s$ and left-word $u_s$. Since $h_D\chi(z_s)$ has left-word $(D^{-1})^\infty > u_s$, it follows that $f(z_s)$ must be fundamental, with right-word $w_s$ and left-word $u_s$.

To prove the uniqueness, take any module $N$ and $n_0 \in N$ satisfying the required conditions. Then, by lemma 161 there exists a pure-embedding of the form:

$$M(w) \stackrel{(g,\chi h_D)}{\longrightarrow} N \oplus \prod_{P \in \mathcal{P}} F_D(H(S_{(P)}))$$

-where $g$ is a map taking $z_0$ to $n_0$. By lemma 9, $H(M(w))$ (and, in particular, $M$) is a direct summand of $N \oplus \prod_{P \in \mathcal{P}} F_D(H(S_{(P)}))$. It follows that $M$ is a direct summand of $N$- and hence that $M \cong N$.

Similar arguments give the remaining cases. $\square$

Recall that if $w$ is an $\mathbb{N}$-word, then by theorem 40, $M_u \cong M_w$ if and only if $w = u$. It would make sense to assume that, for a $\mathbb{Z}$-word, $w$, $M_w \cong M_u$ if and only if $u = w$ or $u = w^{-1}$.

However, we cannot prove that such a condition holds. We can provide some conditions on $w$ and $u$ which imply that $M_u \ncong M_w$. We also have no examples of "different" words $w$ and $u$ such that $M_w \cong M_u$.

## 7.3 Words with similar sets of finite subwords

Let $w = \dots l_{-2}l_{-1}l_0l_1l_2\dots$ and $w' = \dots l'_{-2}l'_{-1}l'_0l'_1l'_2\dots$ be any $\mathbb{Z}$-words. We write $w \preccurlyeq w'$ if every finite subword of $w$ is a subword of $w'$ or $(w')^{-1}$- i.e. for all finite subwords $l_{k+1}\dots l_{k+n}$ of $w$, there exists a finite subword $l'_{m+1}\dots l'_{m+k}$ of $w'$ such that either $l_{k+1}\dots l_{k+n} = l'_{m+1}\dots l'_{m+k}$ or $l_{k+1}\dots l_{k+n} = (l'_{m+1}\dots l'_{m+k})^{-1}$.

We prove in this section, that for any pair of aperiodic $\mathbb{Z}$-words, $w$ and $w'$, $\mathrm{Supp}(M(w') \subseteq \mathrm{Supp}(M(w))$ if and only if $w' \preccurlyeq w$.

We write $w \sim w'$ whenever both $w \preccurlyeq w'$ and $w' \preccurlyeq w$. We prove, in proposition 18, that there do in fact exist $\mathbb{Z}$-words, $w$ and $w'$, such that $w \neq w'$, $w^{-1} \neq w'$ and $w \sim w'$- and hence that $M(w) \not\cong M(w')$ and $\mathrm{Supp}(M(w)) = \mathrm{Supp}(M(w'))$.

### 7.3.1 $\mathrm{Supp}(M(w')) \subseteq \mathrm{Supp}(M(w))$ implies $w' \preccurlyeq w$

**Lemma 166.** *Let $w = \dots l_{-1}l_0l_1l_2\dots$ and $w' = \dots l'_{-1}l'_0l'_1l'_2\dots$ be any $\mathbb{Z}$-words, such that $\mathrm{Supp}(M(w')) \subseteq \mathrm{Supp}(M(w))$.*

*Then $w' \preccurlyeq w$.*

*Proof.* Assume that $\mathrm{Supp}(M(w')) \subseteq \mathrm{Supp}(M(w))$. Let $\{z_i : i \in \mathbb{Z}\}$ and $\{y_i : i \in \mathbb{Z}\}$ denote the standard bases of $M(w)$ and $M(w')$ respectively.

Suppose, for a contradiction, that $w'$ has a finite subword which is not equal to any subwords of $w$ or $w^{-1}$. Since it can be chosen to be arbitrarily long, we may assume that it is of the form $l'_m\dots l'_k$, where $m < 0 < k$, and $y_m$ and $y_k$ are troughs.

Let $C = (l'_{m+1}\dots l'_{-1}l'_0)^{-1}$, and $D = l'_1\dots l'_k$. Also, let $E = (l'_{m+2}\dots l'_{-1}l'_0)^{-1}$, and $F = l'_1\dots l'_{k-1}$. Let $\phi(v)$ be $(C^{-1}.D)(v)$, and $\psi(v)$ be $((C^{-1}.F) + (E^{-1}.D))(v)$. We shall prove that $\phi/\psi$ is open on $M(w')$, but closed on $M(w)$.

We may assume, without loss of generality, that $w_0 \in H_1(a)$ and $u_0 \in H_{-1}(a)$. Of course, $y_j \in (C^{-1}.D)(M(w'))$, but since $y_j$ is fundamental with right-word $w_j < F$ and left-word $u_j < E$, it follows that $y_j \notin ((C^{-1}.F) + (E^{-1}.D))(M(w'))$. So the pp-pair is indeed open on $M(w')$.

Now, take any $x \in (C^{-1}.D)M(w)$. Write $x$ in the form $\sum_{i\in I_0} \lambda_i z_i$, where $\lambda_i \neq 0$ for all $i \in I_0$. By corollary 26, $z_i \in (C^{-1}.D)(M(w))$ for all $i \in I_0$.

Given any $i \in I_0$, either $D$ is not an initial subword of $\hat{w}_i$, or $C$ is not an initial subword of $\hat{u}_i$ (otherwise $w$ will have a subword equal to $C^{-1}D$). We can therefore partition $I_0$ into sets $I_1$ and $I_2$ such that:

- for every $i \in I_1$, $D$ is not an initial subword of $\hat{w}_i$
- For every $i \in I_2$, $C^{-1}$ is not an initial subword of $\hat{u}_i$.

For all $i \in I_1$, lemma 85 implies that $F \leq \hat{w}_i$, and thus, by lemma 105:

$$M(w) \models (C^{-1}.F)(\sum_{i \in I_1} \lambda_i z_i)$$

Similarly, $M(w) \models (E^{-1}.D)(\sum_{i \in I_2} \lambda_i z_i)$, and so:

$$M(w) \models ((C^{-1}.E) + (F^{-1}.D))(x)$$

-so the pp-pair is indeed closed on $M(w)$- giving our required contradiction. □

**Corollary 46.** *Suppose that $w$ and $w'$ are $\mathbb{Z}$-words such that $w \nsim w'$. Then $M_w \ncong M_{w'}$.*

*Proof.* Since $w \nsim w'$, there exists (without loss of generality) a finite subword $l'_m \dots l'_k$ of $w'$ which is not a finite subword of $w$.

Let $\phi/\psi$ be the pp-pair as constructed in the proof of lemma 166. It is closed on $M(w)$, and hence on $H(M(w))$ (by theorem). It is therefore closed on the direct summand $M_w$ of $H(M(w))$.

Now, $M_{w'}$ contains a fundamental element $x$, with left word $u'_0$ and right-word $w'_0$ in $M_u$. Then $x \in \phi(M(w')) \backslash \psi(M(w'))$ (by considering the definition of fundamental in terms of pp-formulas).

Since $\phi/\psi$ is open on $M_{w'}$ and closed on $M_w$, we therefore have $M_w \ncong M_{w'}$. □

**Corollary 47.** *Let $w$ and $u$ be any two almost periodic $\mathbb{Z}$-words such that $w \neq u$ and $w \neq u^{-1}$. Then $M_u \ncong M_w$.*

*Proof.* One can easily check that $w \nsim u$ for any such $w$ and $u$. The result follows, from corollary 46 □

### 7.3.2 $w' \preccurlyeq w$ implies $\mathrm{Supp}(M(w')) \subseteq \mathrm{Supp}(M(w))$

**Lemma 167.** *Suppose that $w$ and $w'$ are aperiodic $\mathbb{Z}$-words such that $w' \preccurlyeq w$.*

*Then* $\mathrm{Supp}(M(w')) \subseteq \mathrm{Supp}(M(w))$.

*Proof.* Suppose that $\phi/\psi$ is a pp-pair which is closed on $M(w)$. In order to prove it is closed on $M(w')$, take any $x \in \phi(M(w'))$. Let $m_\phi$ and $m_\psi$ be the number of atomic formulas in $\phi$ and $\psi$ respectively- and let $m = \max(m_\phi, m_\psi)$.

Take any pre-subword $l'_{c+1} \dots l'_{c+d}$ of $w'$ such that $x \in M(l'_{c+1} \dots l'_{c+d})$.

Let $i, j \in \mathbb{N}$ be such that $l'_{c-i+1} l'_{c-i+2} \dots l'_{c+d+j}$ is the subword ${}^{(m)}l'_{c+1} \dots l'^{(m)}_{c+d}$ of $w'$. By corollary 21:

$$M({}^{(m)}(l'_{c+1} \dots l'_{c+d})^{(m)}) \models \phi(x)$$

Since $w' \preccurlyeq w$, there exists $b \in \mathbb{Z}$ such that (without loss of generality):

$$l_{b-i} l_{b-i+1} l_{b-i+2} \dots l_{b+d+j} l_{b+d+j+1} = l_{c-i} l'_{c-i+1} l'_{c-i+2} \dots l'_{c+d+j} l_{c+d+j+1}$$

In particular, $l_{b+1} \dots l_{b+d}$ is a pre-subword of $w$, and $l_{b+1} \dots l_{b+d} = l'_{c+1} \dots l'_{c+d}$ so we may consider $x$ as an element of $M(l_{b+1} \dots l_{b+d})$. Furthermore, by lemma 99:

$${}^{(m)}(l_{b+1} \dots l_{b+d})^{(m)} = {}^{(m)}(l'_{c+1} \dots l'_{c+d})^{(m)}$$

We may consider $x$ as an element of $M(w)$- by considering the canonical embedding:

$$M({}^{(m)}(l_{b+1} \dots l_{b+d})^{(m)}) \hookrightarrow M(w)$$

Since $x \in \phi(M({}^{(m)}(l_{b+1} \dots l_{b+d})^{(m)}))$, we therefore have that:

$$x \in \phi(M(w)) = \psi(M(w))$$

Thus, by corollary 21, $x \in \psi(M({}^{(m)}(l_{b+1} \dots l_{b+d})^{(m)}))$, and so:

$$M({}^{(m)}l'_{c+1} \dots l'^{(m)}_{c+d}) \models \psi(x)$$

Thus $M(w') \models \psi(x)$, as required.

∎

### 7.3.3 Words with the same set of finite subwords

First of all, we need to make a clear distinction between unlabeled words and labeled words: We may consider a labeled $\mathbb{Z}$-word to be a map $\sigma : \mathbb{Z} \to Q_1 \cup Q_1^{-1}$, such that the string of letters $\ldots\sigma(-1)\sigma(0)\sigma(1)\sigma(2)\sigma(3)\ldots$ is a $\mathbb{Z}$-word.

Given any two labeled words $\sigma$ and $\tau$, we write $\sigma = \tau$ if and only if $\sigma(i) = \tau(i)$ for all $i \in \mathbb{Z}$.

We define $\approx$ to be the equivalence relation on labeled $\mathbb{Z}$-words, such that $\sigma \approx \tau$ if and only if there exists $k \in \mathbb{Z}$ and $c \in \{-1, +1\}$ such that $\sigma(i) = \tau(is + k)$ for all $i \in \mathbb{Z}$.

We refer to any equivalence class in the set of labeled words modulo $\approx$ as an unlabeled word. Notice that every equivalence class contains countably many unlabeled words.

**Proposition 17.** *Let $w$ and $u$ be any labeled $\mathbb{Z}$-words. Then $M(w) \cong M(w')$ if and only if $w \approx w'$.*

*Proof.* We have proved, in (5.2.2), that $w \approx w'$ implies $M(w) \cong M(w')$. To show the converse, take any labeled $\mathbb{Z}$-words, $w' = \ldots l_{-1}; l'_0 l'_1 l'_2 \ldots$ and $w = \ldots l_{-1} l_0 l_1 l_2 \ldots$, such that $M(w) \cong M(w')$. Let $f : M(w') \to M(w)$ be an isomorphism.

Let $\{z_i : i \in \mathbb{Z}\}$ and $\{z'_i : i \in \mathbb{Z}\}$ be the standard bases of $M(w)$ and $M(w')$ respectively. Write $f(z'_0)$ as $\sum_{i\in I} \lambda_i z_i$- where $\lambda_i \neq 0$ for all $i \in I$.

We claim that there is at least one $i \in I$ such that $\hat{w}_i = \hat{w}'_0$ and and $\hat{u}_i = \hat{u}'_0$. If not, then we can partition $I$ into $I_1 \cup I_2$, where $\hat{w}_i \neq \hat{w}'_0$ for all $i \in I_1$, and $\hat{u}_i = \hat{u}'_0$ for all $i \in I_2$. It follows that $\hat{w}_i > \hat{w}'_0$ for all $i \in I_1$ (the proof is similar to that of lemma 110)). Thus, by lemma 155, the right-word of $\sum_{i\in I_1} \lambda_i z_i$ in $M(w)$ is $\min\{\hat{w}_i : i \in I_1\}$, which is greater than $\hat{w}'_0$. Similarly, the left-word of $\sum_{i\in I_2} \lambda_i z_i$ in $M(w')$ is greater than $\hat{u}_0$, and so $f$ is not a pure embedding (by lemma 156)- which contradicts the fact that every isomorphism is a pure embedding.

Consequently, there does exists $i \in \mathbb{Z}$ such that $\hat{w}_i = \hat{w}'_0$ and $\hat{u}_i = \hat{u}'_0$. It follows that $w \approx w'$. □

**Proposition 18.** *There exist $\mathbb{Z}$-words, $w$ and $w'$, such that $M(w) \not\cong M(w')$, and* $\operatorname{Supp}(M(w)) = \operatorname{Supp}(M(w'))$.

*Proof.* We shall take $A$ to be the Gelfand-Ponomarev algebra, $G_{3,3}$. Note that there are only four different bands over $G_{3,3}$ (up to cyclic permutation and taking inverses)- we may consider these bands to be $C = \alpha\beta^{-1}$, $D = \alpha\alpha\beta^{-1}$, $E = \alpha\beta^{-1}\beta^{-1}$ and $F = \alpha\alpha\beta^{-1}\beta^{-1}$. Notice that, given any $n \geq 2$, there is a finite combination of $C, D, E$ and $F$, of length exactly $n$.

Note that any combination of $C, D, E$ and $f$ is a word, and that for every $\mathbb{Z}$-word, $w$, either $w$ or $w^{-1}$ can be written as a combination of these four bands.

Consider the set of all binary sequences $\{(a_i)_{i\in\mathbb{N}} : a_i \in \{0,1\}$ for all $i \in \mathbb{N}\}$. We shall construct an injective map from this set to the set of all labeled $\mathbb{Z}$-words.

Consider the set of all finite combinations of $C, D, E$ and $F$. It is clearly countable, and so we label these words as $W_1, W_2, W_3, \ldots$. Note that, given any finite word $B$, there exists $i \in \mathbb{N}$ such that either $B$ or $B^{-1}$ is a subword $W_i$.

Now, given any sequence $(a_i)_{i\in\mathbb{N}}$, we define a labeled word $\sigma$, as follows:

- Pick any aperiodic $\mathbb{N}$-word, $u_0$, with first letter $\beta$, and define $\sigma(i)$ for all $i \leq 0$ to be such that $u_0 = \sigma(0)^{-1}\sigma(1)^{-1}\sigma(2)^{-1}\ldots$.

- For all $n \geq 1$, define $\sigma(3^n - 2)$, $\sigma(3^n - 1)$ and $\sigma(3^n)$ to be such that $\sigma(3^n - 2)\sigma(3^n - 1)\sigma(3^n) = D$ if $a_n = 0$, and $\sigma(3^n - 2)\sigma(3^n - 1)\sigma(3^n) = E$ if $a_n = 1$.

- Place the finite subwords $W_1, W_2, W_3, \ldots$, one by one, into any available "gaps": i.e. given the word $W_i$, let $n_i$ be the length of $W_i$, and pick any $k \in \mathbb{N}$ large enough such that $n_i \leq 3^k - 5$. Then set $\sigma(3^k + 1), \ldots, \sigma(3^k + n_i)$ to be such that $\sigma(3^k + 1)\ldots\sigma(3^k + n_i) = W_i$.

- Fill in the remaining gaps in the labeled word, using any combinations of $C, D, E$ and $F$- note that we can do this, since every remaining gap is of length greater than or equal to 2.

Notice that, given any two distinct binary sequences $(a_i)_i$ and $(b_i)_i$, there exists some $j$ such that $a_j \neq b_j$, and so the labeled words $\sigma$ and $\tau$ obtained from these sequences will have to be different- since $\sigma(3^j - 1)$ and $\tau(3^j - 1)$ must necessarily be different. So we have our injective map from the set of all binary sequences to the set of all labeled words.

Also, given any two of these words, $w$ and $u$, every possible finite word is a subword of both $w$ and $u$- and so $w \sim u$.

Finally, this map gives us uncountably many different labeled $\mathbb{Z}$-words, each of which contains every single finite word as a subword. Given any one of these words, $w$, there are only countably many other finite words $u$ such that $w \approx u$, and so there must be at least one word $u$ in the set such that $u$ and $w$ do not lie in the same $\approx$-equivalence class. Thus, by proposition 17, $M(w) \ncong M(u)$- as required. □

## 7.4 Algebras with a pp-lattice of defined width

### 7.4.1 The link between fundamental elements and the width of the pp-lattice

The following result has been conjectured (for example, in [19, (1.5)]):

**Conjecture 1.** *If $A$ is a domestic string algebra, then $w({}_A\mathrm{pp}) < \infty$- and hence there are no superdecomposable pure-injective $A$-modules.*

This is certainly not the case for non-domestic string algebras:

**Theorem 52.** *Let $A$ be a non-domestic string algebra. Then $w({}_A\mathrm{pp}) = \infty$.*

*Proof.* See [20, (4.1)] □

Indeed, in [21] Puninski has even shown how to construct a superdecomposable pure-injective over a particular non-domestic string algebra.

Domestic string algebras can have arbitarily large (but finite) m-dimension: i.e. given any $n \in \mathbb{N}$, [7] shows that the CB-rank of the Ziegler Spectrum of the domestic string algebra $\Lambda_n$ is $n + 1$- and hence so is the m-dimension of ${}_{\Lambda_n}\mathrm{pp}$.

In this section, we shall prove that, given any string algebra $A$ such that the width of the lattice ${}_A\mathrm{pp}$ is defined (and hence that $A$ is domestic), every pure-injective $A$-module contains a fundamental element.

**Lemma 168.** *Let $A$ be any string algebra. Suppose a two-directed pure-injective module $M \in A$-Mod contains no fundamental elements. Given any $m \in M$, let $w_0 = l_1 l_2 l_3 \dots$ and $u_0 = l_0^{-1} l_{-1}^{-1} l_{-2}^{-1} \dots$ be the right-word and left-word of $m$ in $M$.*

*Then, given any $w'' > w_0$, there exists $x \in M$ with right-word $w'$ such that $w_0 < w' < w''$, and $m - x$ has left-word greater than $u_0$.*

*Proof.* Let $\mathcal{X}$ be the set of all finite words $C \in H_{-1}(a)$ such that $m \in ((.C)+(.D))(M)$ for some finite word $D > w_0$. Since $m$ is not fundamental in $M$, $\mathcal{X} \neq \emptyset$, and hence has a supremum.

We claim that $\sup(\mathcal{X})$ is an $\mathbb{N}$-word: If it were a finite word, say $C'$, then by lemma 87, $C' \in \mathcal{X}$, so we could pick $x \in (.C')(M)$ such that $m - x \in (.D)(M)$ for some $D > w_0$. Since $M$ is two-directed, the left-word of $x$ in $M$ is an $\mathbb{N}$-word- say, $u''$- and $C' < u''$, since $x \in (.C')(M)$. Pick any finite word $C''$ such that $C' < C'' < u''$. Then $x \in (.C'')(M)$, and so:

$$m = x + (m - x) \in ((.C'') + (.D))(M)$$

And so $C'' \in \mathcal{X}$- contradicting the fact that $C'' > C' = \sup(\mathcal{X})$.

Let $u' = \sup(\mathcal{X})$. As it is an $\mathbb{N}$-word, we can pick initial pre-subwords $C_1, C_2, \dots$ of $u'$, such that the length of $C_{n+1}$ is greater than than the length of $C_n$ for all $n$. Of course, $C_1 < C_2 < C_3 < \dots$, and $\varinjlim C_n = u'$.

For each $n \in \mathbb{N}^+$, the set $\{D > w : ((.C_n) + (.D))(M) \neq \{0\}\}$ is non-empty, since $C_n \in \mathcal{X}$. Let $w(n)$ denote the supremum of it. Of course, $w(n) > w_0$, and $w(n) \geq w(n+1)$ for all $n \in \mathbb{N}^+$.

We claim that $\varinjlim w(n) = w_0$: Suppose not, for a contradiction. Pick any $D \in \mathcal{W}$ such that $w_0 < D < \varinjlim w(n)$, and consider the conjunction of pp-formulas:

$$\bigwedge_{n \in \mathbb{N}^+} ((.C_n)(v) \wedge (.D)(m - v))$$

It is finitely satisfiable: Given any finite subset $I \subseteq \mathbb{N}^+$, let $n = \max(I)$. Then $D < w(n)$, and so $M \models ((.C_n) + (.D))(m)$, as required.

Since $w$ is algebraically compact, the system is satisfiable: i.e. there exists $x \in M$ such that $x \in (.C_n)(M)$ for all $n$, and $m - x \in (.D)(M)$. Then the left-word of $x$ is at least $\sup(\mathcal{X})$, and the right-word of $m - x$ is greater than $D$- and hence greater than $w_0$. Thus, by lemma 155, the right-word of $x$ is $w_0$.

Since $x$ is not fundamental, there exists $y \in M$ with left-word greater than the left-word of $x$ (and hence greater than $u'$), such that the right-word of $x-y$ is greater than $w_0$.

Of course, $m = y + (m - x) + (x - y)$. However, $(m - x) + (x - y)$ has right-word greater than $w_0$ (because both $m - x$ and $x - y$ do)- so we can pick $C > w'$ such that $(m - x) + (x - y) \in (.C)(M)$. Furthermore, the left word of $y$ is greater than $u' = \sup\mathcal{X}$- so we can pick $C > u'$ such that $y \in (.C)(M)$. Thus:

$$m \in ((.C) + (.D))(M)$$

-so $D \in \mathcal{X}$- giving our required contradiction.

Now, given any $w'' > w$, we can pick $n$ such that $w'' > w(n)$ (since $\varinjlim w(n) = w$). Since $C_n \in \mathcal{X}$, there exists $D > w$ such that $m \in ((.C_n) + (.D))(M)$. Pick any $x \in (.C_n)(M)$ such that $m - x \in (.D)(M)$.

Then $m - x$ must have right-word less than or equal to $w(n)$, and hence less than $w''$, as required. □

**Proposition 19.** *Let $A$ be any string algebra. If there exists an $A$-module $M$ with no fundamental elements, then the lattice ${}_A\mathrm{pp}(M)$ contains a wide subposet.*

*Proof.* Pick any $a \in Q_0$, such that $e_a(M) \neq 0$. Given any $x \in e_a(M)$, we will denote by $w_x$ the right-word of $x$ in $M$, and by $u_x$ the left-word of $x$ in $M$.

First of all, we claim that, given any $m \in M$, and any $\mathbb{N}$-words, $u' > u_m$ and $w' > w_m$, we can $x, y \in M$, and finite words $C_y, C_x, D_y, D_x$ such that $w_m < D_x < w_x < D_y < w_y < w'$ and $u_m < C_y < u_y < C_x < u_x < u'$.

Indeed, by lemma 168, we can pick $x' \in M$ such that $w_{x'} = w_m$ and $w' < u_{x'} < u'$. Similarly, we can pick $y' \in M$ such that $u_{y'} = u_m$ and $w < w_{y'} < u'$. Now, by applying lemma 168 to $y'$, we can pick $x \in M$ such that $u_x = u_{x'}$ and $w'_x < w_x < w_{y'}$. Similarly, we can pick $y \in M$ such that $w_y = w_{y'}$ and $u_{y'} < u_y < u_{x'}$. Then:

$$w_m = w'_x < w_x < w'_y = w_y$$

$$u_m = u_{y'} < u_y < u_{x'} = u_x$$

Picking any finite words $C_x, C_y, D_x, D_y$ such that $w_m < D_x < w_x < D_y < w_y$ and $u_m < C_y < u_y < C_x < u_x$ completes the proof of the claim.

Define a pre-order on the set $e_aM \times H_1(a) \times H_{-1}(a)$ by:

$$(x, C, D) \leq (y, E.F) \iff w_x \leq w_y,\ u_x \leq u_y,\ C \leq E \text{ and } D \leq F$$

Now, we recursively define a series of subsets $B_0, B_1, B_2, \ldots$ of $M \times H_{-1}(a) \times H_1(a)$, such that, for all $n \in \mathbb{N}$:

- $D_x < w_x$ and $C_x < u_x$, for all $(x, C_x, D_x) \in B_n$
- $w_x \neq w_y$ and $u_x \neq u_y$ for all elements $(x, C_x, D_x)$ and $(y, C_y, D_y)$ of of $B_n$.
- Given any $(x, C_x, D_x)$ in $B_n$, there is no element $(y, C_y, D_y)$ of $B_n$ (other than $(x, C_x, D_x)$) such that $D_x \leq w_y \leq w_x$ or $C_x \leq u_y \leq u_x$.
- $B_n$ contains at least one comparable pair: i.e. there exists $(x, C_x, D_x)$ and $(y, C_y, D_y)$ in $B_n$ such that $(x, C_x, D_x) < (y, C_y, D_y)$.
- $B_{n-1} \subset B_n$.
- Given any elements $(x, C_x, D_x)$ and $(y, C_y, D_y)$ of $B_{n-1}$, such that $(x, C_x, D_x) < (y, C_y, D_y)$, there exist elements $(z, C_z, D_z)$ and $(z', C_{z'}, D_{z'})$ of $B_n$ such that $w_x < w_z < w_{z'} < w_y$ and $u_x < u_{z'} < u_z < u_y$.

Notice that given any such sets $B_0, B_1, B_2, \ldots$, the set $\bigcup_{n \in \mathbb{N}} B_n$ is partially ordered by $\leq$, and that the last condition implies that every non-trivial interval contains two incomparable elements.

To define $B_0$, take any $m \in e_a(M)$. By the above claim, there exists $m' \in e_a(M)$ such that $w_{m'} > w_m$ and $u_{m'} > u_m$. Pick any finite words $D_m, D_{m'}, C_m, C'_m$ such that $D_m < w_m < D_{m'} < w_{m'}$ and $C_m < u_m < C_{m'} < u_{m'}$. Then define $B_0 := \{(m, C_m, D_m), (m', C_{m'}, D_{m'})\}$. It clearly satisfies the first four conditions (and the last two are vacuous).

Now, given any $n \in \mathbb{N}$, such that $B_n$ is defined, let $\{(m_i, C_{m_i}, D_{m_i}) : 1 \leq i \leq k\}$ be the set of elements of $B_n$. By applying the claim, there exists $x_1, y_1 \in M$, and $C_{x_1}, C_{y_1}, D_{x_1}, D_{y_1}$ such that:

$$w_{m_1} < D_{x_1} < w_{x_1} < D_{y_1} < w_{y_1} < \min\{D_{m_j} : D_{m_j} > w_{m_1}\}$$

$$u_{m_1} < C_{y_1} < u_{y_1} < C_{x_1} < u_{x_1} < \min\{C_{m_j} : C_{m_j} > u_{m_1}\}$$

Again, by the claim, there exists $x_2, y_2 \in M$, and $C_{x_2}, C_{y_2}, D_{x_2}, D_{y_2}$ such that:

$$w_{m_2} < D_{x_2} < w_{x_2} < D_{y_2} < w_{y_2} < \min\{D_{m_j} : D_{m_j} > w_{m_2}\}$$

-and such that $w_{y_2} < D_{x_1}$ if $D_{x_1} > w_{m_2}$, and $w_{y_2} < D_{y_1}$ if $D_{y_1} > w_{m_2}$, and also such that:

$$u_{m_2} < C_{y_2} < u_{y_2} < C_{x_2} < u_{x_2} < \min\{C_{m_j} : C_{m_j} > u_{m_2}\}$$

-and such that $u_{x_2} < C_{x_1}$ if $C_{x_1} > u_{m_2}$, and $u_{x_2} < C_{y_1}$ if $C_{y_1} > u_{m_2}$.

Repeating this argument will give a pair $x_i, y_i \in M$ for every $m_i$. Let $B_{n+1}$ be the set of all $m_i$, $x_i$, and $y_i$, for all $i \leq k$. It clearly satisfies the required conditions- in particular, given any $j, k \leq k$ such that $w_{m_i} < w_{m_j}$ and $u_{m_i} < u_{m_j}$, the elements $x_i$ and $x_j$ satisfy the last condition required of $B_{n+1}$.

Having defined the sets $B_n$ for every $n \in \mathbb{N}$, it follows from the conditions that the set:

$$\{(C_x^{-1}.D_x)(v) : (x, C_x, D_x) \in \bigcup B_n\}$$

-is a wide subposet of $\mathrm{pp}_A$. $\square$

Note that it is possible to have a pure-injective indecomposable module $M \in A$-Mod such that $\mathrm{pp}_A(M)$ contains a wide poset: for example, if we take $A$ to be $G_{3,3}$, and $w$ to be a $\mathbb{Z}$-word such that every finite word $w$ is a subword of it. Then $M_w$ is indecomposable and pure-injective. One can construct a wide sub-poset of ${}_A\mathrm{pp}$ containing only pp-formulas of the form $(C^{-1}.D)(v)$, with $D \in H_1(a)$ and $C \in$

$H_{-1}(a)$. Eveluating each element of this poset on $M_w$ will give a wide sub-poset of $\mathrm{pp}(M_w)$.

### 7.4.2 Domestic string algebras with $w(\mathbf{pp}) < \infty$

**Theorem 53.** *Let $A$ be any string algebra, such that $w({}_A\mathrm{pp}) < \infty$. Then every pure-injective indecomposable $A$-module must be exactly one of the following:*

1. *A finite dimensional string module*
2. *A finite dimensional band module*
3. *A module from Ringel's list*
4. *A Prüfer module, adic module, or generic module associated with a homogeneous tube of $\Gamma_A$ corresponding to a band module.*
5. *Some other infinite dimensional module $M$ containing a fundamental element $m_0$, with right-word $D^\infty$, and left-word $(D^{-1})^\infty$, for some band, $D$.*

*Proof.* Take any indecomposable pure-injective $M \in A$-Mod. If $M$ is finite dimensional, then the result follows from theorem 36 and theorem 35.

If $M$ is infinite dimensional and one-directed, then the result follows from theorem 40, and the fact that every module on Ringel's list is pure-injective and indecomposable.

Finally, if $M$ is two-directed, then by proposition 19 $M$ contains a fundamental element $m_0$. Let $w_0$ and $u_0$ denote the right-word and left-word (respectively) of $m_0$ in $M$. If $u_0^{-1}w_0$ is not periodic, then it must be almost-periodic. There exists a string module on Ringel's list with underlying word $u_0^{-1}w_0$. Thus, by theorem 51, $M$ is isomorphic to this module.

If $u_0^{-1}w_0$ is a periodic $\mathbb{Z}$-word, then there's nothing to prove. □

Of course, this result would extend to all domestic string algebras if conjecture 1 is true. We leave open precisely what modules on the fifth item on the list contains.

We suspect that it may only contain a few anomalies, or possibly even be empty. Although we have no proof for such a statement.

It's worth pointing out exactly which modules on Ringel's list are isomorphic:

**Proposition 20.** *Let $M$ and $N$ be any modules on Ringel's list. Then $M \cong N$ if and only if one of the following holds:*

1. *There exists an $\mathbb{N}$-word, $w$, such that $M = N = M(w)$.*
2. *There exist contracting almost periodic $\mathbb{Z}$-words, $w$ and $u$, such that $M = M(w)$, $N = M(u)$, and either $w = u$ or $w = u^{-1}$.*
3. *There exist expanding almost periodic $\mathbb{Z}$-words, $w$ and $u$, such that $M = \overline{M}(w)$, $N = \overline{M}(u)$, and either $w = u$ or $w = u^{-1}$.*
4. *There exist mixed almost periodic $\mathbb{Z}$-words, $w$ and $u$, such that $M = M^{+}(w)$, $N = M^{+}(u)$, and either $w = u$ or $w = u^{-1}$.*

*Proof.* The $\mathbb{N}$-word case follows straight from theorem 40. The $\mathbb{Z}$-word cases follow from corollary 47. □

## 7.5 The equivalence of $M(w)$ and $\overline{M}(w)$

We shall prove, in this section, that $\mathrm{Supp}(M(w)) = \mathrm{Supp}(\overline{M}(w))$ for any aperiodic $\mathbb{Z}$-word or $\mathbb{N}$-word, $w$. One direction is fairly straightforward:

**Lemma 169.** $\mathrm{Supp}(M(w)) \subseteq \mathrm{Supp}(\overline{M}(w))$ *for any aperiodic $\mathbb{N}$-word or $\mathbb{Z}$-word, $w$.*

*Proof.* Take any pp-pair $\phi/\psi$ which is closed on $\overline{M}(w)$. Given any $x \in \phi(M(w))$, take a pre-subword $E$ of $w$ such that $x \in M(E)$. Let $m = \max(m_\phi, m_\psi)$- where $m_\phi$ and $m_\psi$ are the number of equations in $\phi$ and $\psi$ respectively.

Of course, $x \in \phi(\overline{M}(w))$, and so $x \in \psi(\overline{M}(w))$. Thus $x \in \psi(M({}^{(m)}E^{(m)})$ by corollary 21, and so $m \in \psi(M(w))$ as required. □

In order to prove the converse we take any pp-pair $\phi/\psi$ which is closed on $M(w)$. Given any $x \in \phi(\overline{M}(w))$, we show how $x$ can be split up into an infinite sum $\sum_{i \in \mathbb{I}} x_i$

(where $I$ is the index set of $w$), such that each $x_i$ lies in $\phi(M(w))$. Our assumption therefore gives that each $x_i$ lies in $\psi(M(w))$, and we show that this implies that $\sum_{i\in\mathbb{I}} \in \psi(\overline{M}(w))$.

**Lemma 170.** *Let $w$ be any aperiodic $\mathbb{N}$-word or $\mathbb{Z}$-word, and $\phi(v)$ be any pp-formula. Let $m$ be the number of equations in $\phi(v)$. Then given any $x \in \phi(\overline{M}(w))$ and trough $z_c$ in $w$, there exists*

$$y \in \phi(\overline{M}(w_c))$$

*-such that:*

$$x - y \in \phi(\overline{M}((u_c^{-1})^{(m)}))$$

*Proof.* The proof uses similar arguments to that of lemma 98. Write the pp-formula $\phi(v)$ as $\psi(v, v_1, \ldots, v_n)$- where $\psi(v, v_1, \ldots, v_n)$ is the formula:

$$\bigwedge_{j=1}^{m}(\sum_{i=1}^{n} r_{ij}v_i = r_j v)$$

Let $x_1, \ldots, x_n$ be any witnesses of the statement $\overline{M}(w) \models \phi(x)$.

Let $T_0, T_1, \ldots, T_m$ be the pairwise comparable troughs, as in the definition of $(u_c^{-1})^{(m)}$. For each $s \le m$, there exists $x_i^{\le T_s} \in \overline{M}(u_{T_s}^{-1})$ and $x_i^{>T_s} \in \overline{M}(w_{T_s})$ such that $x = x_i^{\le T_s} + x_i^{>T_s}$.

Also, for all $i \le n$, there exists $x_i^{\le T_s} \in \overline{M}(u_{T_s}^{-1})$ and $x_i^{>T_s} \in \overline{M}(w_{T_s})$ such that $x_i = x_i^{\le T_s} + x_i^{>T_s}$. For all $j \le m$ we have:

$$\sum_{i=1}^{n} r_{ij}x_i = r_j x$$

And so:

$$\sum_{i=1}^{n} r_{ij}x_i^{>T_s} - r_j x_i^{>T_s} = r_j x_i^{\le T_s} - \sum_{i=1}^{n} r_{ij}x_i^{\le T_s}$$

Since the left hand side lies in $\overline{M}(w_{T_s})$, and the right hand side in $\overline{M}(u_{T_s}^{-1})$, both sides must lie in $KT_s$- i.e. they both equal $\rho_{js}T_s$- for some $\rho_{js} \in K$

Having done this for every $j \in \{1, \ldots m\}$, there exists $\{\mu_s \in K : s \in S\}$ (not all zero) such that $\sum_{s=0}^{m} \mu_s \rho_{js} = 0$ for every $j \in \{1, \ldots, m\}$.

Let $T_k$ be minimal (with respect to the ordering on troughs) such that $\mu_k \neq 0$. By lemma 97, there must exist maps $f_k \in \mathrm{Hom}(M(w^{>T_s}), M(w_{T_k}))$, for every $s \in S\backslash\{k\}$, taking $T_s$ to $T_k$ - and each one must have image contained in $M(z_c^{(m)})$.

We may assume that $\mu_k = 1$. Now define:

$$y := x^{>T_k} + \sum_{s \neq k} \mu_s f_s(x_i^{>T_s})$$

$$y_i := x_i^{>T_k} + \sum_{s \neq k} \mu_s f_s(x_i^{>T_s})$$

Notice that $y \in \overline{M}(w_c)$ and that:

$$x - y = x^{\leq T_k} - \sum_{s \neq k} \mu_s f_s(x_i^{>T_s}) \in \overline{M}((u_c^{-1})^{(m)})$$

(since $\mathrm{Im}(f_s) \in \overline{M}((u_c^{-1})^{(m)})$ for all $s \neq k$, by lemma 97).

Similarly, for every $i \leq n$, $y_i \in \overline{M}(w_c)$ and $x_i - y_i \in \overline{M}((u_c^{-1})^{(m)})$. Finally, for every $j \leq m$:

$$\begin{aligned}
& \sum_{i=1}^{n} r_{ij} y_i - r_j y \\
= \quad & (\sum_{i=1}^{n} r_{ij} x_i^{>T_k}) - r_j x^{>T_k} + \sum_{s \neq k} \left( \sum_{i=1}^{n} r_{ij} \mu_s f_s(x_i^{>T_s}) - r_j \mu_s f_s(x_i^{>T_s}) \right) \\
= \quad & \rho_{jk} T_k + \sum_{s \neq k} \mu_s f_s \left( (\sum_{i=1}^{n} r_{ij} x_i^{>T_s}) - r_j x_i^{>T_s} \right) \\
= \quad & \rho_{jk} T_k + \sum_{s \neq k} \mu_s f_s(\rho_{js} T_s) \\
= \quad & (\sum_{s \in S} \mu_s \rho_{js}) T_s \\
= \quad & 0
\end{aligned}$$

So $\overline{M}(w) \models \psi(y, y_1, y_2, \ldots, y_n)$ and $\overline{M}(w) \models \psi(x-y, x_1-y_1, \ldots, x_n-y_n)$. And thus:

$$y \in \phi(\overline{M}(w_c))$$

$$x - y \in \phi(\overline{M}((u_c^{-1})^{(m)}))$$

□

For the rest of this section we let $\{t_i : i \in I\}$ be the subset of $I$ (the index set of $w$) such that $\{z_{t_i} : i \in I\}$ is the set of all troughs in $w$, and $t_i < t_{i+1}$ for all $i \in I$

**Corollary 48.** *Let $w$ be any aperiodic $\mathbb{N}$-word or $\mathbb{Z}$-word, and $\phi(v)$ be any pp-formula. Let $m$ be the number of equations in $\phi(v)$.*

*Given any $x \in \phi(\overline{M}(w))$, and any $m \in \mathbb{N}$, we can find a set $\{x_i : i \in \mathbb{I}\}$ (I being the index set of $w$) such that:*

- $x_i \in \phi\left(\overline{M}(w_{t_i}) \cap \overline{M}((u_{t_{i+1}}^{-1})^{(m)})\right)$ *for every* $i \in \mathbb{Z}$
- $x = \sum_{i\in\mathbb{Z}} x_i$

*Proof.* It's just a case of repeatedly applying lemma 170: For example, if $w$ is an $\mathbb{N}$-word, then applying the lemma, with $c = t_0$ gives that $x = x_0 + y_0$, for some $x \in \phi(\overline{M}((u_{t_0}^{-1})^{(m)}))$ and $y_0 \in \phi(\overline{M}(w_{t_0}))$.

Then applying lemma 170 to $y_0$, with $c = t_1$, gives that $y_0 = x_1 + y_1$, for some $x_1 \in \phi(\overline{M}((u_{t_1}^{-1})^{(m)})))$ and $y_1 \in \phi(\overline{M}(w_{t_1}))$.

Note that, since $y_0 \in \phi(\overline{M}(w_{t_0}))$ and $y_1 \in \phi(\overline{M}(w_{t_1})) \subseteq \phi(\overline{M}(w_{t_0}))$, $x_1 = y_0 - y_1$ must also lie in $\phi(\overline{M}(w_{t_0}))$.

Continuing up the word will give the required result. $\square$

Note that $w_{t_i}$ is an $\mathbb{N}$-word, and that $(u_{t_{i+1}}^{-1})^{(m)}$ is the inverse of an $\mathbb{N}$-word (or a finite word, if $w$ is an $\mathbb{N}$-word), so the "overlap" of them (as subwords of $w$) is a finite subword of $w$: namely, the subword ${}^{(0)}(l_{t_i+1} \dots l_{t_{i+1}})^{(m)}$.

Thus $\overline{M}(w_{t_i}) \cap \overline{M}((u_{t_{i+1}}^{-1})^{(m)})$ is a finite dimensional submodule of $\overline{M}(w)$, and indeed it is equal to $M({}^{(0)}(l_{t_i+1} \dots l_{t_{i+1}})^{(m)})$.

**Lemma 171.** *Let $\phi(v)$ be any pp-formula with at most $m$ equations.*

*Suppose we have a set $\{x_k \in M({}^{(m)}(l_{t_k+1} \dots l_{t_{k+1}})^{(m)}) : k \in I\}$ such that $\overline{M}(w) \models \phi(x_k)$*

*Then $\overline{M}(w) \models \phi(\sum_{k\in I} x_k)$.*

*Proof.* Write $\phi(v)$ as $\exists v_1 \dots \exists v_n \psi(v, v_1, \dots, v_n)$, where $\psi$ is:

$$\bigwedge_{j=1}^{m}(\sum_{i=1}^{n} r_{ij}v_i = r_j v)$$

Applying corollary 21 to the subword ${}^{(m)}(l_{t_k+1}\dots l_{t_{k+1}})^{(m)}$ of $w$ gives that:

$$M({}^{(m)}({}^{(m)}(l_{t_k+1}\dots l_{t_{k+1}})^{(m)})^{(m)}) \models \psi(x_k)$$

Let $y_{k,1},\dots y_{k,n}$ be any witnesses to that statement.

Given any $d \in I$ it follows from lemma 25 that there are only finitely many $k \in I$ such that $z_d \in M({}^{(m)}({}^{(m)}(l_{t_k+1}\dots l_{t_{k+1}})^{(m)})^{(m)})$. It follows that, for every $i \le n$, $\sum_{k\in I} y_{k,i}$ is a well defined element of $\overline{M}(w)$.

Furthermore, for every $j \in \{1,\dots,m\}$:

$$\begin{aligned}\sum_{i=1}^{n} r_{ij}\sum_{k\in\mathbb{Z}} y_{k,i} &= \sum_{k\in\mathbb{Z}}\sum_{i=1}^{n} r_{ij}y_{k,i}\\ &= r_j\sum_{k\in\mathbb{Z}} x_k\end{aligned}$$

So $\overline{M}(w) \models \psi(\sum_k x_k, \sum_k y_{k,1}, \sum_k y_{k,2},\dots,\sum_k y_{k,n}$, and so $\overline{M}(w) \models \phi(\sum_k x_k)$, as required. □

**Proposition 21.** $\mathrm{Supp}(M(w)) = \mathrm{Supp}(\overline{M}(w))$ *for every infinite aperiodic word, $w$.*

*Proof.* Lemma 169 gives one direction. To show the converse, take any pp-pair $\phi/\psi$ which is closed on $M(w)$. Pick $m > 0$ such that $\phi$ and $\psi$ have at most $m$ equations. Take any $x \in \phi(\overline{M}(w))$. We must show that $x \in \psi(\overline{M}(w))$.

By corollary 48 we can find a set of elements $\{x_k \in \phi(M({}^{(0)}(l_{t_i+1}\dots l_{t_{i+1}})^{(m)})) : i \in \mathbb{Z}\}$ such that $x = \sum_{k\in\mathbb{Z}} x_k$.

Of course, we may consider each $x_k$ as an element of $M(w)$. Then, for every $k$, $x_k \in \phi(M(w)) = \psi(M(w))$. Since $x_k \in M({}^{(0)}(l_{t_i+1}\dots l_{t_{i+1}})^{(m)})$, corollary 21 gives:

$$M({}^{(m)}({}^{(0)}(l_{t_i+1}\dots l_{t_{i+1}})^{(m)})^{(m)}) \models \psi(x_k)$$

Thus, by lemma 171 $\sum_k x_k \in \psi(\overline{M}(w))$, as required. □

**Corollary 49.** $\mathrm{Supp}(\overline{M}(w)) \subseteq \mathrm{Supp}(\overline{M}(w'))$ *for all aperiodic $\mathbb{Z}$-words $w$ and $w'$ such that $w \preccurlyeq w'$ then*

*Proof.* This follows directly from lemma 167 and proposition 21 □

## 7.6 Direct summands: an example

We have shown that if $w \preccurlyeq w'$ then $\mathrm{Supp}(\overline{M}(w)) \subseteq \mathrm{Supp}(\overline{M}(w'))$. We will show how, in some circumstances, we have that $\overline{M}(w)$ is in fact a direct summand of $\overline{M}(w')$.

We will take $A$ to be any Gelfand-Ponomarev algebra $G_{m,n}$ (see section 5.1 for a definition). Notice that every word is made up of either the two letters $\alpha$ and $\beta^{-1}$, or the two letters $\alpha^{-1}$ and $\beta$. We may take $H_1(a)$ (respectively $H_{-1}(a)$) to be the set of all words which start with either $\alpha$ or $\beta^{-1}$ (respectively, $\alpha^{-1}$ or $\beta$).

Notice that, given any word $D$, there exists an almost periodic $\mathbb{N}$-word, $w$, such that $Dw \geq Du$ for all $\mathbb{N}$-words, $w$. For example, if $D = 1_{a,1}$ (the word of zero length), then $w = (\alpha^n\beta^{-1})^\infty$.

Take any aperiodic $\mathbb{Z}$-word, $w$, such that $\overline{M}(w)$ is indecomposable- we know that such words do exist, by section 6.9. We assume that $w_0 \in H_1(a)$. Pick any series $D_1, D_2, D_3, \ldots$ of initial post-subwords of $w_0$ of increasing length. Of course, $D_k$ is an initial post-subword of $D_{k+1}$ for all $k \in \mathbb{N}^+$, and $\varinjlim D_k = w_0$.

Similarly, pick any series $C_1, C_2, C_3, \ldots$ of initial post-subwords of $u_0$ of increasing length. Then $C_k$ is an initial post-subword of $Ck + 1$ for all $k \in \mathbb{N}^+$, and $\varinjlim C_k = u_0$.

Now, for each $k$, we can find finite words $E_k$ and $F_k$ such that:

- $D_kF_k$ is not an initial subword of $w_0$
- $D_kF_k > w_0$
- The last letter of $F_k$ is inverse (and hence must be $\beta^{-1}$).
- $C_kE_k$ is not an initial subword of $u_0$
- $C_kE_k > u_0$
- The last letter of $E_k$ is inverse (and hence must be $\alpha^{-1}$).

To see this, notice that $D_k(\alpha^{n-1}\beta^{-1})^\infty$ is a word, and $D_k(\alpha^{n-1}\beta^{-1})^\infty \geq w_0$. Since $w_0$ is aperiodic, then we pick any $N$ sufficiently large such that $D_k(\alpha^{n-1}\beta^{-1})^N$ is not an initial subword of $w_0$, and this finite word satisfies the required conditions.

Now, define $w'$ to be the $\mathbb{N}$-word:

$$E_1^{-1}C_1^{-1}D_1D_1E_2^{-1}C_2^{-1}D_2F_2E_3^{-1}C_3^{-1}D_3F_3\ldots$$

For each $k \geq 1$, there is a simple string map $f'_k : M(w) \to \overline{M}(w')$ which takes $z_0$ to the standard basis element of $M(w')$ with left-word $C_kE_kF_k^{-1}D_{k-1}^{-1}\ldots$ and right-word $D_kF_kE_{k+1}^{-1}C_{k+1}^{-1}\ldots$. Furthermore, the image of $f_k$ is contained in the submodule $M(E_k^{-1}C_k^{-1}D_kF_k)$ of $M(w')$.

For each $k$, let $f_k$ be the composition of $f'_k$ and the canonical embedding $M(w') \hookrightarrow \overline{M}(w')$. Let $f : M(w) \to \overline{M}(w')$ be $\sum_{k\geq 1} f_k$. It is a well defined map. We claim that it is a pure-embedding.

To see this, take any $x \in M(w)$, and any $\phi \in \mathrm{pp}^{\overline{M}(w')}(f(x))$. Let $N$ be the number of equations in $\phi$. Pick a pre-subword $E$ of $w$ such that $x \in M(E)$. Let $\pi M(w) \twoheadrightarrow M({}^{(N+)}E^{(N+)})$ denote the canonical projection.

Since $w$ is aperiodic, we can pick $k$ sufficiently large such that ${}^{(N+)}E^{(N+)}$ is a proper subword of $C_k^{-1}D_k$. Then ${}^{(N+)}E^{(N+)}$ is a post-subword of $C_k^{-1}D_k$, and hence of $w'$, and so there exists a canonical projection $h$ from $\overline{M}(w')$ onto $M({}^{(N+)}E^{(N+)})$ taking $f(z_0)$ to $\pi(z_0)$. Furthermore, $hf = \pi$.

Since $f(x) \in \phi(\overline{M}(w'))$, it follows that $\pi(x) = hf(x) \in \phi(M({}^{(N+)}E^{(N+)}))$, and so $x \in \phi(M(w))$, by corollary 23.

Since $f$ is pure, lemma 9 implies the pure injective hull of $M(w)$ (which is $\overline{M}(w)$, since it is indecomposable) is a direct summand of $\overline{M}(w')$.

# Chapter 8

# Conclusions

In this section, we re-iterate some of the main results that we obtained in chapters 3 to 7, as well as outlining ways in which these results could be extended.

In chapter 3, we showed that the m-dimension of $\mathrm{pp}(\bigoplus_{M \in \mathcal{T}_\gamma} M)$ is 2, for every positive rational $\gamma$. However, we did not deal with the case of $\gamma = 0$ and $\gamma = \infty$. It is suspected that similar results apply to these tubes.

In chapter 4 we proved that, for every $r \in \mathbb{R}^+ \backslash \mathbb{Q}$, $w(\mathrm{pp}(M(r))) = \infty$ (where $M(r)$ is the direct sum of all indecomposable pure-injectives of slope $r$)- and hence that, if the underlying field $K$ is countable, then there exists a superdecomposable pure-injective $A$-module $M$ of slope $r$. We also showed that there are superdecomposable pure-injective modules which do not have slope. It remains an open question, as to whether or not every pure-injective superdecomposable over a string algebra can be decomposed as a direct sum (or a direct product) of modules, all of which have well-defined slope.

In chapter 5, we showed that for every aperiodic $\mathbb{N}$-word, $w$, the unique pure-injective indecomposable module $M_w$ as defined in theorem 40 is the pure-injective hull of $M(w)$, and hence a direct summand of $\overline{M}(w)$. In chapter 6, we found necessary and sufficient conditions on an infinite word, $w$, to determine whether or not $M(w) \cong M_w$ and whether or not $\overline{M}(w) \cong M_w$. We show that there are cases where $M_w \not\cong M(w)$ and $M_w \not\cong \overline{M}(w)$.

A possible extension of this work would be to try and determine what this unique module is, for any such word, $w$. However, there are uncountably many different $\mathbb{N}$-words words over a non-domestic string algebra, and by their nature, they are somewhat more difficult to describe then the almost periodic words, so it is hard to see how someone would be able to determine exactly what these modules are.

In chapter 7, we attempted to extend theorem 40 to 2-sided modules. We showed in theorem 51- that for every non-periodic $\mathbb{Z}$-word, $u_0^{-1}w_0$, there exists a unique (up to isomorphism) 2-sided module $M_w$, containing a fundamental element $m_0$ of right-word $w_0$ and left-word $u_0$. Furthermore, we showed the links between this module and the pure-injective hull of the direct sum string module $M(w)$. However, we cannot provide a proof that $M_w \not\cong M_u$ whenever $w \neq u$ and $w \neq u^{-1}$.

In theorem 53 we classified "almost all" the indecomposable pure-injective modules over a certain class of string algebras (namely, the string algebras $A$ such that $w({}_A\mathrm{pp}) < \infty$). Describing the modules on the fifth point of the list would give a complete classification. Furthermore, if conjecture 1 is true, then we would have the result for every domestic string algebra. This conjecture still remains open.

We also leave open the question of whether or not every indecomposable pure-injective module contains a fundamental element- if such a result was true, then we would have a near-complete classification of all the pure-injective indecomposable modules over a string algebra (similar to theorem 53).

Finally, we showed in section 7.6 that there are examples of words $w$ and $u$ such that $\overline{M}(w)$ is a direct summand of $\overline{M}(u)$, and we constructed a pure embedding from $\overline{M}(w)$ to $\overline{M}(u)$ to prove this. We suspect that this construction can be extended to more pairs of words. Of course, $\overline{M}(w)$ can only be a direct summand of $\overline{M}(u)$ if $\mathrm{Supp}(\overline{M}(w)) \subseteq \mathrm{Supp}(\overline{M}(u))$, and hence if $w \preccurlyeq u$. We leave it open as to whether such a a construction can be be provided for any $u$ and $w$ such that $w \preccurlyeq u$.

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