COMPUTATIONAL INVESTIGATION INTO FINITE GROUPS

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We briefly discuss the algorithm given in [4] intended for determining the distance between two vertices in a commuting involution graph of a symmetric group.

We develop the algorithm in [8] for computing a subgroup of $N_G(X)$, the normalizer of a 2-subgroup X in a finite group G, examining in particular the issue of when to terminate the randomized procedure. The resultant algorithm is capable of handling subgroups X of order up to 2^9 and is suitable, for example, for matrix groups of large degree (an example calculation is given using 112×112 matrices over $GF(2)$).

We also determine the suborbits of conjugacy classes of involutions in several of the sporadic simple groups—namely Janko's group J_4, the Fischer sporadic groups, and the Thompson and Harada-Norton groups. We use our results to determine the structure of some graphs related to this data.

We include implementations of the algorithms discussed in the computer algebra package Magma, as well as representative elements for the involution suborbits.

Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Copyright Statement

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Chapter 1

Introduction

The use of computational techniques has a long history within group theory. Even before the advent of the modern computer, group-theoretic algorithms were being developed—for example Todd and Coxeter's coset enumeration routine [27] for determining the index of a subgroup was originally intended to be implemented by hand (it appeared in 1936). Recently the introduction of dedicated computer algebra packages, notably GAP [13] and Magma [9], has greatly increased the scope of the discipline. We apply the potential of these resources to some problems in finite group theory.

In Chapter 2 we discuss some background material and set out the notation that will be used in the subsequent chapters.

In Chapter 3 we briefly explain an algorithm in [4] intended for determining the distance between two vertices in the commuting involution graph $\mathcal{C}(G, X)$, where $G \cong S_n$, the symmetric group on n letters, and X is any conjugacy class of involutions. We implement this in Magma and determine that it sometimes returns an incorrect result.

In Chapter 4 we develop a randomised black-box algorithm to compute a subgroup of the normalizer $N_G(X)$ in a finite group G of a 2-subgroup X. The work in [8], in which maximal chains of subgroups of X are exploited to deliver normalizing elements, forms the basis of the algorithm. Via careful selection of these maximal chains, we improve the efficiency and likelihood of success of the routine. Details of a Magma implementation of the algorithm are provided, as well as of example calculations performed to demonstrate the capabilities of the algorithm, and the implementation itself is also available: details of this are given in Appendix A.

In Chapter 5 we investigate computationally the suborbit structure of involution conjugacy classes in several of the sporadic simple groups (and where appropriate their automorphism groups): Janko's group J_4, the three Fischer groups and the Thompson and Harada-Norton groups. Representative elements for the suborbits are given, where possible as words in the standard generators of the group, and elsewhere are provided electronically as explicit matrices or permutations (again see Appendix A for details). We use this information to determine the diameter and disc structures of graphs whose vertex sets are the involution conjugacy classes themselves. For J_4 and Fi_{24} we investigate the commuting involution graph, defined as the graph where two vertices are joined by an edge if and only if they commute. For Th and HN we look at the point-line collinearity graph of a particular minimal parabolic geometry, where two points of the geometry (here two involutions in a particular conjugacy class) are joined by an edge if they are incident on a common line.

Chapter 2

Background and Notation

2.1 Algorithms

Algorithms are mostly presented in 'pseudocode' format, similarly to those found in [16]. So normal coding constructs such as '**if**. . . **then**' statements and '**for**' loops are employed, but individual statements are written in normal mathematical notation rather than any one programming language. We note that the symbol $x \leftarrow y$ means that the variable x is assigned the value y. An example algorithm presented in this style is given in Section 2.1.3.

In the following sections we discuss some relevant considerations to our use and development of algorithms.

2.1.1 Computer implementations

The most widely used packages for computational group theory are Magma [9] and GAP [13]. We mainly use the former for the computation carried out here, and in particular the algorithm produced in Chapter 4 was implemented in Magma, but GAP is sometimes employed, in particular due to the extensive library of character tables from the Atlas [11] which are easily available to its users. The electronic files (see Appendix A) are all in Magma format, though the files associated with Section 5.2 are also given in GAP format.

The example calculations described in Section 4.4.1 were carried out on a Unix machine with 8 GB of memory and a 3.2 GHz processor, running Magma version 2.11-15. All computation described in Chapter 5 was carried out on a Unix machine with 16 GB of memory and a 3.2 GHz processor, with Magma version 2.15-15 and GAP version 4.4.10.

2.1.2 Randomised Algorithms

A *randomised algorithm*, as the name suggests, is an algorithm which at some point in its execution is required to make a random choice: for our purposes this will generally be selecting a random element of a group. The algorithm developed in Chapter 4 relies centrally on such random selections, while the results obtained in Chapter 5 were found only by frequently employing randomised searches through the group elements.

Of course no truly random process is possible in a computer, but this problem is beyond the scope of the current study and we generally assume the ability to select a random integer from within a given (finite) range. Even so, no algorithm is known that will generate uniformly-distributed random elements of a group. However, an excellent algorithm for producing a very close approximation to random group elements is the *product replacement algorithm* designed by Leedham-Green and Soicher (see Section 3.2.2 of [16]). Variations on this procedure are employed in both Magma and GAP for this purpose, and we always assume the ability to select a random element from a finite group in all our computation.

2.1.3 Black-Box Groups

Despite the name, the appellation ‘black box’ in fact denotes not a type of group but a class of algorithms designed to work with a particular type of group representation (or rather, the absence of one). The concept was introduced in [1].

Definition 2.1. *An algorithm is* black-box *on the group G if the elements of G are represented as (not necessarily unique) strings of bounded length in a finite alphabet, and given elements $g, h \in G$, the algorithm can perform only the following tasks*

(i) Output a string representing gh;

(ii) Output a string representing g^{-1};

(iii) Test whether $g = h$.

The advantages and drawbacks of writing an algorithm of this type are obvious: a black-box algorithm works without further adaptation on any type of group representation, but the cost of this is that no information can be gleaned from the particular representation actually used (for example, from the action of a matrix group on its associated module).

We loosen the definition slightly to allow our algorithms to select random group elements as discussed in the previous section (the product-replacement algorithm is black-box in any case), and to allow the calculation of the order of a group element, which is impossible to do efficiently in the pure black-box setting but can be achieved relatively easily in most group representations. This more relaxed definition is fairly common, see for example [15].

As an example, we explain the randomised method presented in [10] for computing elements commuting with a given involution, since it forms the linchpin of the algorithm developed in Chapter 4 and is employed frequently in Chapter 5 for its original purpose of computing the centralizer of an involution. It is based on the following simple result (from Section 2.2 of [10]).

Lemma 2.2. *Suppose* $t \in G$ *is an involution and* $h \in G$ *is an arbitrary element. Let* n *be the order of* $[t, h]$*. Then if* n *is even,* $[t, h]^{\frac{n}{2}}, [t, h^{-1}]^{\frac{n}{2}} \in C_G(t)$ *while if* n *is odd,* $h[t, h]^{\frac{n-1}{2}} \in C_G(t)$*.*

From this observation we build Algorithm 1.

Algorithm 1 CentralizingElement

Input: G a black-box group;
t an involution in G.
$h \leftarrow \text{Random}(G)$
$n \leftarrow \text{Order}([t, h])$
if n is even **then**
  $c \leftarrow [t, h]^{\frac{n}{2}}$ or $[t, h^{-1}]^{\frac{n}{2}}$ {Choose one at random, or return both.}
else
  $c \leftarrow h[t, h]^{\frac{n-1}{2}}$
end if
Output: c, an element commuting with t.

We see that the only calls to the group G made by Algorithm 1 are to select a random element h, compute products and inverses when forming the commutator $[t, h]$ and the centralizing elements, and to determine the order of $[t, h]$. So Algorithm 1 is black-box.

Using Algorithm 1 repeatedly we can build up a set S of elements centralizing our involution t and form the group $H = \langle S \rangle \leq C_G(t)$. However, this approach falls short of producing a fully-fledged algorithm for computing the centralizer of an involution because it is not clear when to terminate the algorithm: how can one test whether $\langle S \rangle = C_G(t)$? In [10], it is proved that the centralizing elements produced by odd-order commutators $[t, h]$ are in some sense uniformly distributed through $C_G(t)$ and this fact can be used to ensure the full centralizer has been generated to within an arbitrarily high probability (creating what is known as a *Monte Carlo* algorithm). However, as the following result (explained in Section 3 of [10]) demonstrates, sometimes such odd-order commutators never arise and in those cases we will never unearth the full centralizer.

Lemma 2.3. *Suppose* $t \in G$ *is an involution, and suppose* $t \in O_2(G)$ *but* $C_G(t) \nleq O_2(G)$*. Then a set* S *of elements formed by Algorithm 1 will never generate* $C_G(t)$*.*

Proof. Let h be an arbitrary element of G. The commutator $[t, h] = tt^h \in O_2(G)$ (since $t \in O_2(G)$ and $O_2(G)$ is normal in G). So it has even order—specifically two-power order. Then Algorithm 1 returns a power of $[t, h]$, which is in $O_2(G)$. So all elements formed will lie in $C_G(t) \cap O_2(G)$, which is not the full centralizer by assumption. □

In Section 4.3 we briefly examine the potential repercussions of this problem for our work.

2.2 Notation

The ATLAS of Finite Group Representations [11], and its online counterpart [29], provide the vast majority of our data concerning the sporadic simple groups (see section 2.3), and as such we employ many of its notational conventions. So S_n represents the symmetric group on n letters, and A_n the alternating group. Elementary abelian groups are denoted by their order in the form p^n, while p^{1+n} (for n even) denotes an extraspecial group of that order (that is, a group whose center is cyclic of order p and has an elementary abelian group of order p^n as its factor group) with the two distinct types distinguished by a subscript $+$ or $-$ symbol. The more general case p^{m+n} denotes an instance of $p^m.p^n$ (see below). An integer m denotes the cyclic group of that order.

We also use the ATLAS names for the sporadic simple groups. The following rules apply for group extensions:

- $A \times B$ denotes the direct product of A and B;
- $A.B$ denotes any group with A as a normal subgroup whose factor group is B;
- $A : B$ denotes specifically that the group is the semidirect product of A and B;
- $A^{\cdot}B$ denotes any case of $A.B$ except a semidirect product.

The ATLAS also includes a standard for naming conjugacy classes, which we adopt. A class of elements is given a name of the form nX, where n is the order of an element of that class, and X is a letter from the alphabet $A, B, C, \ldots$, distinguishing the classes of elements of the same order, and ordering them by increasing length. So for example, class $2A$ is the smallest class of involutions in its group.

2.3 Sporadic simple groups

In Chapter 5 we are concerned with determining the structure of the involutions in various of the *sporadic simple groups*. We recall the famous classification of the finite simple groups.

Theorem 2.4. *Let G be a finite simple group. Then, up to isomorphism, G is either in of the following infinite families:*

- *the cyclic groups C_p (p prime);*
- *the alternating groups A_n on n letters ($n \geq 5$);*
- *the Chevalley and twisted Chevalley groups or the Tits group;*

or G is one of 26 specific 'sporadic simple groups'.

As might be expected from their exceptional status, much about the sporadic groups remains unknown. Although the smaller sporadics such as the Mathieu groups can easily be investigated computationally as they possess small-degree representations and have relatively modest order, some of the larger groups present more of a challenge. However, there are some sources of information available.

2.3.1 Character tables

The character tables for many finite simple groups, including all of the 26 sporadic groups, have been calculated and are available in the ATLAS of Finite Groups [11].

They are also available in GAP [13]. A great deal of information regarding a group can be gleaned from considering its character table, and we make frequent use of this data in Chapter 5

2.3.2 Standard generators

To ensure that results derived computationally are able to be verified independently, it is desirable that the group generators used are easily available to others. To this end, for the sporadic simple groups, Wilson [30] introduced the concept of *standard generators*.

Generators a, b for each of the sporadic simple groups are chosen in such a way that they can be easily defined by stating the classes of a, which is always an involution, of b and ab, and the order of at most one other short word in the generators. For example, Janko's sporadic simple group J_4 has standard generators a, b where $a \in 2A$, $b \in 4A$ and $abab^2$ has order 10. Where a sporadic group G is not equal to its automorphism group, standard generators for $\mathrm{Aut}(G)$ are chosen in the same way and are denoted c, d.

The online ATLAS of Finite Group Representations [29] contains generators in matrix and/or permutation representations for many finite simple groups, including all of the sporadic groups except the Monster group. The generators for the sporadic groups are always standard generators. So where possible, in Chapter 5 we specify elements in the sporadic simple groups in terms of words in the standard generators. For example, while the work on the group HN in Section 5.4 was carried out in the 132-dimensional $GF(4)$-representation, the words provided can equally be used in the degree 1,140,000 permutation representation (or any other representation provided in the online ATLAS).

2.3.3 Involutions in the sporadic groups

In Chapter 5, we study in detail the involution structure of several of the 26 sporadic simple groups. This is in part motivated by the study of *commuting involution graphs* which are defined as follows.

Definition 2.5. *Let G be a group and $t \in G$ an involution. Set $X = t^G$. The* commuting involution graph $\mathcal{C}(G, X)$ *is the undirected graph with X as its vertex set and two distinct vertices $x, y \in X$ joined by an edge if and only if they commute. Let $d(-,-)$ be the usual graph-theoretic distance metric. Where $\mathcal{C}(G, X)$ is connected, we define the* diameter *of $\mathcal{C}(G, X)$ to be the largest value of $d(x, y)$ for any $x, y \in X$, and the i^{th}* disc *of $\mathcal{C}(G, X)$ around t is defined as*

$$\Delta_i(t) = \{x \in X \mid d(t, x) = i\}.$$

Since X is a conjugacy class, $\mathcal{C}(G, X)$ is a vertex-transitive graph, so whatever our choice of t, the discs of the graph are the same sizes, and so we generally begin study of a commuting involution graph by choosing and fixing a base vertex $t \in X$ without loss of generality.

Commuting involution graphs have been studied for a wide variety of finite groups. The diameters of these graphs (they are almost all connected) have been determined for G a symmetric group [4], which we discuss in Chapter 3; a general finite Coxeter group [3]; certain symplectic groups [12] and others. In [2], the diameter and disc sizes of $\mathcal{C}(G, X)$ were determined, using a mixture of computational and theoretical techniques, for all but six pairs (G, X) where G is a sporadic simple group or an automorphism group of such, and X is a conjugacy class of involutions in G. As a result of the present work we will be able to deal with three of the outstanding cases.

Our direct concern, however, is determining the suborbit structure of the involution classes in the groups we consider. That is, we wish to learn about the orbits of the conjugation action of $C_G(t)$ on X for t an arbitrary element of X. The following easy result shows how the two are intimately related.

Lemma 2.6. *Let G be a finite group, X a class of involutions in G and $t \in X$. Then every disc $\Delta_i(t)$ of the commuting involution graph $\mathcal{C}(G, X)$ is a union of $C_G(t)$-orbits of X (as $C_G(t)$ acts by conjugation on X).*

Proof. Let $x \in \Delta_i(t)$. So $d(t, x) = i$, that is there exists a sequence $t = x_0, x_1, \ldots, x_i = x$ such that x_j, x_{j-1} commute for all $j = 1, \ldots, i$, and no shorter such sequence exists. Let $c \in C_G(t)$. Then $t^c = t$ and so we have the sequence $t = x_0^c, x_1^c, \ldots, x_i^c = x^c$ with adjacent terms commuting, and hence $d(t, x^c) \leq i$. But if a shorter sequence existed joining t and x^c then conjugation by c^{-1} would give $d(t, x) < i$, a contradiction. So $x^c \in \Delta_i(t)$ as required. □

For fifteen of the sporadic groups—those furnished with small-degree permutation representations—and their automorphism groups, the suborbits for all involution classes were computed in [5]. Of the remaining eleven, Conway's group Co_1 is dealt with in [6], and another six are determined here: J_4, Fi_{22}, Fi_{23}, ${Fi_{24}}'$, Th and HN.

Chapter 3

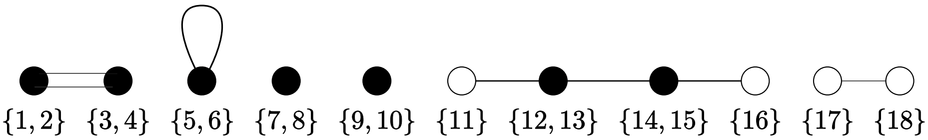
Commuting distance in the symmetric groups

In [4], results are determined regarding the commuting involution graph $\mathcal{C}(G, X)$ where $G \cong S_n$ and X is an arbitrary conjugacy class of involutions (see Definition 2.5 for the definition of this graph). So $X = a^G$ where $a = (12)(34) \ldots (2m-1\ 2m)$ with $2m \leq n$. We summarize the main results regarding the connectedness and diameter of these graphs here. (Theorems 1.1 and 1.2 of [4].)

Theorem 3.1. *The graph $\mathcal{C}(G, X)$ is disconnected if and only if $n = 2m+1$ or $n = 4$ and $m = 1$. If $\mathcal{C}(G, X)$ is connected then its diameter is at most 4, being exactly 4 only when $2m + 2 = n \in \{6, 8, 10\}$.*

A further result given in [4] is an algorithm which it is claimed determines whether two involutions have distance at most two in $\mathcal{C}(G, X)$. Without loss of generality we assume one of these involutions is $t \in X$, our base vertex in $\mathcal{C}(G, X)$, and call the other x. We discuss the algorithm in the following sections, first introducing the concepts and notation used and then sketching the argument that underlies it. In Section 3.3 we uncover circumstances where this procedure delivers an incorrect result, claiming that $d(t, x) \geq 3$ when in fact we can find an element $y \in X \cap C_G(t) \cap C_G(x)$.

Figure 3.1: Example x-graph



$$G \cong S_{18};\ t = (12)(34)(56)(78)(9\ 10)(12\ 13)(14\ 15);$$
$$x = (13)(24)(56)(11\ 12)(13\ 14)(15\ 16)(17\ 18).$$

3.1 Determining $d(t, x)$, $t, x \in X$

Before we begin discussion of the algorithm itself, we introduce the concept of an *x-graph.*

Definition 3.2. *Let $x \in X$. The x-graph $\mathcal{G}_x$ (relative to our fixed involution t) is the graph with the orbits of t as its vertices, with two vertices v_1, v_2 joined by an edge if and only if there exist $\alpha \in v_1$ and $\beta \in v_2$ such that x interchanges α and β. Clearly all the orbits of t have size 1 or 2, and we distinguish these by colouring vertices corresponding to the former white, and to the latter black. A pair of black vertices may be joined by two edges where their associated orbits are $\{\alpha, \beta\}$ and $\{\gamma, \delta\}$ and x interchanges α and γ as well as β and δ.*

An example x-graph is given in Figure 3.1. We note the following properties of the x-graphs (Lemma 2.1 of [4]).

Proposition 3.3. *Suppose $t = (12)(34)\ldots,(2m-1\ 2m) \in S_n$, $X = t^{S_n}$. Then*

(i) For any $x \in X$, the x-graph $\mathcal{G}_x$ has exactly m black vertices (each with valency at most 2); $n - 2m$ white vertices with (each with valency at most 1); and m edges.

(ii) All graphs satisfying the conditions in (i) are realised as the x-graphs of elements $x \in X$.

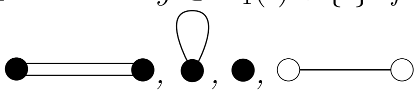
(iii) Two elements $x, y \in X$ are $C_G(t)$-conjugate if and only if their x-graphs $\mathcal{G}_x$ and

$\mathcal{G}_y$ are isomorphic (where an isomorphism here is understood to preserve vertex colours).

An immediate consequence of 3.3(iii) and Lemma 2.6 is that elements with isomorphic x-graphs lie in the same disc of $\mathcal{C}(G, X)$, and so the algorithm for determining whether $d(t, x) \leq 2$ exploits these x-graphs. The general strategy is to use $\mathcal{G}_x$ to construct the x-graph $\mathcal{G}_y$ for an element $y \in X$ commuting with both t and x. If such a y exists then clearly $d(t, x) \leq 2$. In the following section we consider how to find such an element.

3.2 Constructing $y \in X \cap C_G(t) \cap C_G(x)$

Given an element $x \in X$ and its x-graph $\mathcal{G}_x$, we are attempting to construct $\mathcal{G}_y$ for an element $y \in X$ commuting with both t and x, that is an element in $\Delta_1(t) \cap C_G(x)$. We first look at the restrictions placed on $\mathcal{G}_y$ by the condition that $y \in \Delta_1(t)$, in the following easily verified result (Lemma 2.3 of [4]).

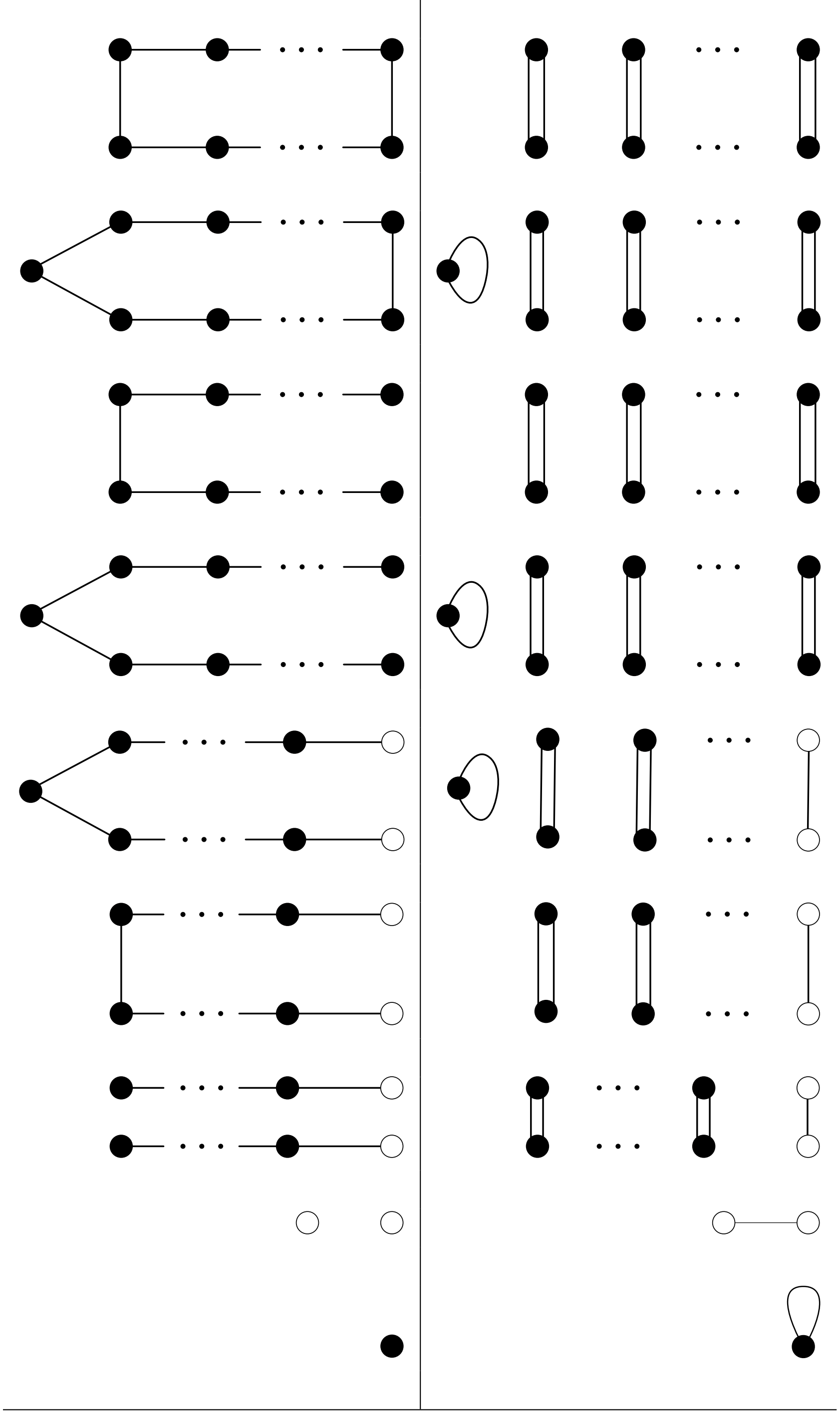
Lemma 3.4. *Suppose $y \in X$. Then $y \in \Delta_1(t) \cup \{t\}$ if and only if every connected component of $\mathcal{G}_y$ is one of*  *and* ○.

So we see that $d(t, x) \leq 2$ if and only if we can construct an x-graph $\mathcal{G}_y$ consisting only of the connected components above, such that y commutes with x.

Since the vertices of $\mathcal{G}_x$ and $\mathcal{G}_y$ both represent the orbits if t, we have a correspondence between the vertex sets of the two graphs. In Table 3.1 we list all possible configurations of edges in $\mathcal{G}_y$ so that x and y commute, next to the corresponding sections of $\mathcal{G}_x$. Clearly for any part of $\mathcal{G}_x$, the corresponding part of $\mathcal{G}_y$ may contain no edges, and such configurations are omitted. Two black chains of equal length can also be paired similarly to the configuration in row 7 of Table 3.1, but this uses the same total number of edges in $\mathcal{G}_y$ as resolving both chains as in rows 2 and 3, so this is omitted also.

Table 3.1: Arrangements of edges in $\mathcal{G}_y$

$\mathcal{G}_x$ part	$\mathcal{G}_y$ part



Given an element $x \in X$, we partition the connected components of $\mathcal{G}_x$ (or in one case, pairs of connected components) into various sets based on the properties of the possible associated components of $\mathcal{G}_y$ in Table 3.1.

Definition 3.5. *Let $\mathcal{C}$ be the set of connected components of $\mathcal{G}_x$, $x \in X$. A component containing of at least one edge and with no circuits is called a* chain*. We partition $\mathcal{C}$ into the following sets:*

$P(x)$ *is a maximal set of pairs of chains, each the same length and with precisely one white vertex (as in row 7 of Table 3.1);*

$U(x)$ *consists of chains with precisely one white vertex that cannot be paired up as above;*

$N(x)$ *consists of all chains with no white vertices;*

$F(x)$ *consists of components having no edges (that is, of isolated vertices);*

$R(x)$ *consists of all other connected components: chains with two white vertices and circuits of black vertices.*

We further set $b(x)$ and $w(x)$ to be the number of black and white vertices respectively in $F(x)$.

To ensure that the element y formed lies in X, we need to ensure $\mathcal{G}_y$ has the same number of edges as $\mathcal{G}_x$, namely m. So we need a method for choosing how to replace each component of $\mathcal{G}_x$ in $\mathcal{G}_y$ so as to preserve the overall number of edges.

We see from Table 3.1 that components in $R(x)$ can be replaced with the associated graphs when constructing $\mathcal{G}_y$ without affecting the number of edges. Components in $N(x)$ and pairs of chains in $P(x)$ can be replaced with the associated subgraphs of $\mathcal{G}_y$ at the cost of increasing the number of edges by 1. Components in $U(x)$ do not appear in the table as they can only be replaced in G_y by the subgraph containing no edges (and of course this can also be done for any other component). Black vertices and pairs of white vertices in $F(x)$ can take edges in $\mathcal{G}_y$.

In Algorithm 2 we present the routine from [4] for determining whether $d(t, x) \leq 2$. The algorithm essentially attempts to construct the graph $\mathcal{G}_y$ for $y \in X$ commuting with t and x: when a step refers to 'cancelling' a component it is understood that it will be replaced in $\mathcal{G}_y$ by the subgraph displayed in Table 3.1, while if a component has edges cancelled from it, that component will be replaced by the subgraph with no edges. Where one component in $P(x) \cup N(x)$ is left unaffected when the algorithm terminates it too will be replaced by the subgraph with no edges, while components in $R(x)$ left unaffected are replaced by the subgraph in Table 3.1. The value $\ell(x)$ computed is then the number of edges that must be added to black vertices or between pairs of white vertices in $F(x)$ so that $\mathcal{G}_y$ has m edges as required. If this value is greater than $b(x)+w(x)/2$ we see that not enough edges can be so added and therefore $d(t, x) > 2$. For brevity, 'component' here refers to a connected component or double chain in $P(x)$

In Proposition 3.6 of [4], it is claimed that Algorithm 2 fails to produce such a y if and only of none exists, that is, that $\ell(x) \leq b(x) + w(x)/2$ if and only if $d(t, x) \leq 2$. In the next section we discover that this is not the case.

3.3 Implementation and Examples

In [4] examples are given of the successful application of this algorithm. We provide a Magma implementation (see Appendix A) which, as well as calculating $\ell(x)$ and hence claiming to determine whether $d(t, x) \leq 2$, also returns an element $y \in \Delta_1(t) \cap C_G(x)$ following the construction described in the previous section. However, testing with this implementation reveals circumstances where Algorithm 2 incorrectly reports that $d(t, x) > 2$. We give two examples.

Example Let $G \cong S_{30}$ and take $t = (12)(34)\ldots(23\ 24)$ and $X = t^G$. We set

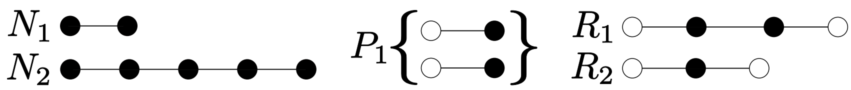
$x = (1\ 25)(3\ 26)(6\ 7)(10\ 11)(12\ 13)(14\ 15)(16\ 17)(19\ 27)(20\ 28)(21\ 29)(22\ 23)(24\ 30).$

Then following Definition 3.5 the connected components of $\mathcal{G}_x$ are

Algorithm 2 CommutingGraphDistance

Input: the connected components of $\mathcal{G}_x$, partitioned into the sets $P(x)$, $N(x)$, $R(x)$, $F(x)$, $U(x)$ and the values $b(x)$ and $w(x)$ given in Definition 3.5.

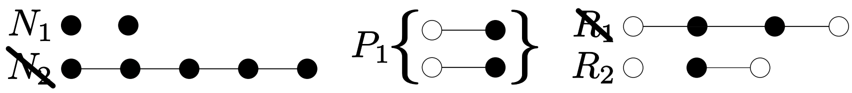
while $U(x)$ contains edges **and** $P(x) \cup N(x) \neq \emptyset$ **do**
  Cancel an edge from a chain in $U(x)$
  Cancel a component or double chain from $P(x) \cup N(x)$ with a maximal number of edges.
end while
if $P(x) \cup N(x) = \emptyset$ **then**
  Set $\ell(x)$ to be the number of edges remaining in $U(x)$.
else
  while $P(x) \cup N(x)$ has one or fewer components left with edges **and** any component in $P(x) \cup N(x)$ with edges has fewer edges than every component in $R(x)$ **do**
    if a component C exists from which some (but not all) edges have been cancelled **then**
      Cancel a further edge from C.
    else
      Cancel an edge from a component in $P(x) \cup N(x) \cup R(x)$ with a minimal number of edges, choosing such a component from $R(x)$ if possible.
    end if
    Cancel a component in $P(x) \cup N(x)$ with a maximal number of edges.
  end while
  if $P(x) \cup N(x) = \emptyset$ **then**
    Set $\ell(x)$ to be the number of edges remaining in the last component from which edges were cancelled.
  else
    Set $\ell(x)$ to be the number of edges remaining in the single component in $P(x) \cup N(x)$, unless edges have been cancelled from a component in $R(x)$ in which case set $\ell(x)$ to be the number of edges remaining in that component.
  end if
end if
$d(t, x) \leq 2$ if and only if $\ell(x) \leq b(x) + w(x)/2$.



,

with $P(x) = \{P_1\}$, $N(x) = \{N_1, N_2\}$, $R(x) = \{R_1, R_2\}$, and $F(x)$ and $U(x)$ empty.

We apply Algorithm 2 to this graph. Since $U(x)$ is empty, the first **while** loop is not executed. With two runs through the second **while** loop, we first cancel the edge in N_1 (the component in $P(x) \cup N(x) \cup R(x)$ with fewest edges) against the component N_2 (the component in $P(x) \cup N(x)$ with most edges). Secondly we cancel an edge in R_1 against the component P_1. We arrive at this position (cancelled components have a strike through their label):



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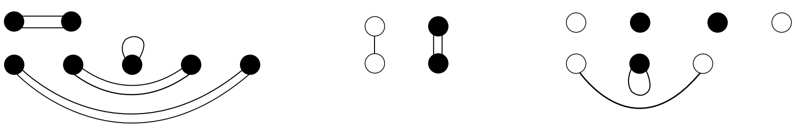
Since $P(x) \cup N(x)$ now contains no edges, the algorithm terminates with $\ell(x)$ set to the number of edges remaining in R_1, the last component from which edges were cancelled. So $\ell(x) = 1 > 0 = b(x) + w(x)/2$ and we conclude that $d(t, x) \geq 3$.

But note that a different choice of component in step 12 from which to cancel edges would give a different result. Cancelling the two edges in P_1 against the two components in $N(x)$ would give a satisfactory resolution, as would cancelling the three edges in R_2 against the three components in $P(x) \cup N(x)$. The latter gives the following graph for $\mathcal{G}_y$:



,

and we see that with

$$y = (13)(24)(58)(67)(13\ 14)(11\ 16)(12\ 15)(9\ 18)(10\ 17)(19\ 20)(25\ 26)(27\ 28)$$

we have $y \in C_G(t) \cap C_G(x) \cap X$.

Example We note that a double chain in $P(x)$ contains the same (even) number of edges as black vertices, and so an element $x \in X$ could have $\mathcal{G}_x$ consisting solely of such components, and likewise any sequence (e_i) of even numbers gives rise to a valid $\mathcal{G}_x$ consisting entirely of double chains P_i in $P(x)$ with P_i having e_i edges.

In this case, Algorithm 2 simply cancels edges from double chains starting at the one with fewest edges against components starting at the one with most edges. Note that in this case $b(x) + w(x)/2 = 0$ and so $d(t, x) \leq 2$ if and only if cancelled edges can be matched exactly against cancelled components.

Suppose $(e_i) = (2, 4, 6, 6, 6)$. Then following Algorithm 2, the three components with six edges are cancelled against the two edges in P_1 and one edge in P_2. Then $\ell(x)$ is set to 3 (the number of remaining edges in P_2) and we conclude that $d(t, x) > 2$. But note that we could simply cancel the four edges in P_2 against the other four components, so in fact $d(t, x) \leq 2$.

Included in the electronic files accompanying the paper version of this thesis (see Appendix A for details), we give a Magma implementation of an updated version of this algorithm. This implementation simply checks all possible choices of sets of components in $P(x) \cup N(x)$ to cancel, and looks for a suitable set of edges in $P(x) \cup N(x) \cup R(x)$ to cancel against them. Since Table 3.1 contains every possible arrangement of edges in $\mathcal{G}_y$, a complete check of possible choices will always deliver the correct result. However, the algorithm will exhaust the available memory when faced with elements having large numbers of components in their x-graphs, typically arising in large-degree groups (degree 1,000 or more).

Chapter 4

A randomised algorithm for computing the normalizer of a 2-subgroup

In [8], Bates and Rowley give an algorithm for computing a subgroup of $N_G(X)$ where X is a p-subgroup of G. The algorithm exploits the method for finding elements centralizing an involution discussed as Algorithm 1 in Section 2.1.3. We begin this chapter with an explanation of Bates and Rowley's algorithm, with our attention restricted to the case $p = 2$. It faces similar termination issues as Algorithm 1. These are resolved in Sections 4.2 and 4.3, whose results also appear in [23]. Throughout this chapter, G is a finite group and X a 2-subgroup of G.

Let

$$1 = X_0 < X_1 < \cdots < X_n = X$$

be a maximal chain of subgroups of X, which we denote $\mathcal{C}$. So $[X_i : X_{i-1}] = 2$ for $i = 1, \ldots, n$. Choose a representative element $x_i \in X_i \setminus X_{i-1}$ for each $i = 1, \ldots, n$, so that $X_i = \langle X_{i-1}, x_i \rangle$. We describe the inductive procedure by which we arrive at a subgroup of $N_G(X)$.

Lemma 4.1. *Let G be a finite group, X a 2-subgroup of G and $\mathcal{C}$ a maximal chain of* *subgroups of X as above. Suppose $M_j \leq N_G(X_j)$ for some $j < n$ (with $X_{j+1} \leq M_j$). Set $\overline{M_j} = M_j/X_j$. Then $N_{M_j}(X_{j+1})$ is the inverse image of $C_{\overline{M_j}}(\overline{x_{j+1}})$.*

Proof. Let $h \in N_{M_j}(X_{j+1})$. Then h normalizes X_{j+1} and X_j (as $h \in M_j \leq N_G(X_j)$). $X_{j+1} = X_j \cup x_{j+1}X_j$, so h must fix the coset $x_{j+1}X_j$, that is, $\overline{h}$ must centralize $\overline{x_j}$. Likewise, if $\overline{h} \in C_{\overline{M_j}}(\overline{x_{j+1}})$, we see that h fixes $x_{j+1}X_j$ and X_j and so must normalize X_{j+1} as required. □

Lemma 4.1 quickly yields the method by which we compute the normalizer. Given a subgroup $M_j \leq N_G(X_j)$ it allows us to find a $M_{j+1} \leq N_{M_j}(X_{j+1})$ simply by computing the centralizer of an involution $\overline{x_{j+1}}$. So beginning with $M_0 = G$, we first compute $M_1 \leq N_G(X_1)$ (this is merely a direct application of Algorithm 1 to find $C_G(x_1) = N_G(X_1)$). This becomes our M_1, and repeating the procedure n times we arrive at a subgroup M_n of $N_G(X_n) = N_G(X)$. However, the subgroup that results is heavily dependent on the choice of chain $\mathcal{C}$: in fact it is contained in $\bigcap_{i=1}^{n} N_G(X_i)$.

We observe that to employ this result we need to be able to use Algorithm 1 to generate the centralizer of an involution in a factor group. However, with an easy modification we avoid the need to explicitly construct the factor group. Where step 2 of Algorithm 1 tests the order of the commutator element formed, in the equivalent step here we wish to find the order of $[x_j, h]$ in the factor group X_j/X_{j-1}, that is, the smallest k such that $[x_j, h]^k \in X_{j-1}$. The resulting procedure is given in Algorithm 3, which can be seen to be a black-box algorithm as long as we have the ability to test for membership of the subgroups X_j. Later we will see that in practice Algorithm 3 is used when X is elementary abelian and of relatively small order, where we can easily form an explicit listing of its elements, so such membership testing is simple.

To ensure we generate as much of the normalizer as possible, we might wish to apply Algorithm 3 to every maximal chain of subgroups of X. However, as the size of X increases the number of maximal chains soon becomes too large for this to be practicable. For example, when X is an elementary abelian 2-subgroup (an important case, as will be seen presently), the numbers of maximal chains are shown in Table

Algorithm 3 ChainStabilizer

Input: G a black box group;
X a 2-subgroup of G;
$x_i \in X$ for $i = 1, \ldots, n$, representative elements of a maximal chain $\mathcal{C}$.

$M_0 \leftarrow G$
for $j = 1$ to m **do**
  $S \leftarrow \emptyset$
  for $i = 1$ to m **do**
    $h \leftarrow \text{Random}(M_{j-1})$
    $k \leftarrow \min\{k \in \mathbb{N} \mid [x_j, h]^k \in X_{j-1}\}$
    if k is even **then**
      $S \leftarrow S \cup \{[x_j, h]^{\frac{k}{2}}, [x_j, h^{-1}]^{\frac{k}{2}}\}$
    else
      $S \leftarrow S \cup \{h[x_j, h]^{\frac{k-1}{2}}\}$
    end if
    $M_j \leftarrow \langle S \rangle$
  end for
end for

Output: M_m a subgroup of the stabilizer of the chain $\mathcal{C}$.

4.1.

We consider what subgroup of $N_G(X)$ would be generated if we ran Algorithm 3 on every maximal chain of X. First we recall the following definition.

Definition 4.2. *Given a group H, the unique largest normal subgroup of H whose factor group has odd order is denoted $O^{2'}(H)$. Equivalently, $O^{2'}(H)$ is the subgroup generated by all Sylow 2-subgroups of H.*

Theorem 4.3. *Applying Algorithm 3 to every maximal chain of X would generate*

Table 4.1: Maximal chains of subgroups of elementary abelian 2-groups

n	Max. chains, X elt. ab. of order 2^n
2	3
3	21
4	315
5	9,765
6	615,195
7	78,129,765
8	19,923,090,075
9	10,180,699,028,325
10	10,414,855,105,976,475

the subgroup $K = O^{2'}(N_G(X)/C_G(X))C_G(X)$.

Proof. Note that elements of $N_G(X)/C_G(X)$ acting by conjugation give automorphisms of X. Given a maximal chain of subgroups $\mathcal{C} = (X_0 < X_1 < \cdots < X_n)$, Algorithm 3 returns a subgroup of $M_{\mathcal{C}} = \bigcap_{i=1}^{n} N_G(X_i)$. Let $g \in M_{\mathcal{C}}$. Since for any $i \in \{1, \ldots, n\}$, g normalizes X_i and X_{i-1}, and since $[X_i, X_{i-1}] = 2$, g must act trivially on the factor group X_i/X_{i-1}. So g stabilizes the chain $\mathcal{C}$ (following the definition at the start of [14], Section 5.3). Then Corollary 5.2.2 of [14] gives us that $M_n/C_G(X)$ must be a 2-group. Hence it is contained in a Sylow 2-subgroup of $N_G(X)/C_G(X)$, and so as $\mathcal{C}$ varies across all maximal chains, we see that the group generated by Algorithm 3 is contained inside K. □

We note that it is a simple matter to determine the centralizer $C_G(X)$ by repeated applications of Algorithm 1 (since it is just $\bigcap_{i=1}^{n} C_G(x_i)$), and we generally assume that we have the group $C_G(X)$.

The rest of this chapter is devoted to increasing the efficiency of this procedure, by carefully selecting sets of maximal chains from which we can generate all of K despite being only a fraction of the full set of chains. We begin as promised by explaining how we may restrict our attention to when X is an elementary abelian 2-subgroup of G.

4.1 Chain of Characteristic Subgroups

Definition 4.4. *A subgroup H of G is called a* characteristic subgroup *if $\phi(H) = H$ for any $\phi \in \mathrm{Aut}(G)$. We write H ch G.*

Suppose we have a chain of subgroups

$$1 = X_{(0)} < X_{(1)} < \cdots < X_{(r)} = X$$

such that $X_{(i-1)}$ is characteristic in $X_{(i)}$ for all $i = 1, \ldots, r$. Let $g \in N_G(X_{(i)})$. Then g acting by conjugation induces an automorphism of $X_{(i)}$. Since $X_{(i-1)}$ is characteristic in $X_{(i)}$, g normalizes $X_{(i-1)}$, that is, $g \in N_G(X_{(i-1)})$. So we have

$$G = N_G(X_{(0)}) \geq N_G(X_{(1)}) \geq \cdots \geq N_G(X_{(r)}) = N_G(X).$$

Since Algorithm 3 is already suited to working in factor groups, we see that it is simple to work along such a chain of characteristic subgroups. That is, we begin by calculating a subgroup $M_{(1)}$ of $N_G(X_{(1)})$ and then use this to find $M_{(2)} \leq N_{M_{(1)}}(X_{(2)})$, and so on. In fact, we note that Algorithm 3 works as stated if the input group X is $X_{(j)}$ and the x_i are representative elements for a chain of subgroups running from $X_{(j-1)}$ to $X_{(j)}$.

4.1.1 Obtaining a chain of characteristic subgroups

To employ the strategy described above, we require a method of generating such a chain of characteristic subgroups. One way of achieving this is using the *Frattini subgroup*.

Definition 4.5. *Given a group H, the Frattini subgroup $\Phi(H)$ is the intersection of all maximal subgroups of H.*

Since any automorphism of H will permute the set of maximal subgroups of H, it is clear that $\Phi(H)$ *ch* H. So given our 2-group X, we can take $X_{(r)} = X$ and $X_{(j-1)} = \Phi(X_{(j)})$ to define our chain of characteristic subgroups. It is also clear that $\Phi(H) \neq H$ and so the chain will terminate with 1. This choice has the advantage that the factor groups $X_{(j)}/X_{(j-1)}$ will be elementary abelian (see Theorem 5.1.3 of [14]), which we will require in the next section.

It remains to consider how to generate $\Phi(Y)$ given a 2-group Y, subject to our requirement to produce a black-box algorithm. Suppose $Y = \langle y_1, \ldots, y_m \rangle$. Then $\Phi(Y) = \langle [y_i, y_j], y_i^2 \mid i, j = 1, \ldots, m \rangle$ (see, for example, the proof of Theorem 5.1.3 in [14]), so we easily obtain generators for $\Phi(Y)$ from the generators for Y.

4.2 X is Elementary Abelian

We now assume our chain of characteristic subgroups has the property described in the previous section that successive factor groups are elementary abelian. So we now consider in detail how to employ Algorithm 3 to generate as large as possible a subgroup of $N_G(X)$ in the restricted case where X is an elementary abelian 2-subgroup of G.

Set $L = GL_n(2)$ (where $|X| = 2^n$) and let V be the natural n-dimensional $GF(2)L$-module. We have that $V \cong X$ as abelian groups, and a maximal chain of subgroups of X corresponds to a maximal flag of subspaces of V. We refer to both interchangeably as 'chains'. Then we have a natural homomorphism from $N_G(X)$ to L, corresponding to its conjugation action on X (and hence V). Henceforth, we denote $N_G(X)$ as N, $C_G(X)$ as C, and $\overline{M}$ where $M \leq N$ denotes the image of M under this homomorphism, so that $\overline{M} \leq L$. Note that C is the kernel of this homomorphism so that $\overline{C} = 1$. We fix this notation for the rest of the chapter.

Recall from Lemma 4.3 that we are attempting to find $K = O^{2'}(N)C$ without needing to Apply Algorithm 3 to every chain. We begin our attack by determining whether we have the trivial case where $K = C$, so $\overline{K} = 1$.

We see that applying Algorithm 3 to a chain $\mathcal{C}$ will only find normalizing elements that fall into the stabilizer in L of the chain. The stabilizers of maximal flags in L are precisely the Sylow 2-subgroups of L, so we obtain a further correspondence between maximal chains in V (or X) and Sylow 2-subgroups of L. From its definition we see that $\overline{K}$ is trivial if and only if it contains no involutions, so we see that our problem of selecting an initial set of chains that will suffice to determine whether $K = C$ is equivalent to the problem of selecting a set $\mathcal{S}$ of Sylow 2-subgroups of L such that every involution in L is contained in at least one member of $\mathcal{S}$. We consider this problem in the next section.

4.2.1 Finding $\mathcal{S}$

We begin with some definitions and simple results that will be useful in uncovering our set $\mathcal{S}$.

Definition 4.6. *Let $L = GL_n(2)$ and V be the natural n-dimensional vector space over $GF(2)$. Let $x \in L$. We define two subspaces of V: the commutator space $[V, x] = \langle v + v^x \mid v \in V\rangle$ and the centralizer $C_V(x) = \{v \in V \mid v^x = v\}$.*

Remarks 4.7. *(i) Suppose $x \in L$ is an involution. Then for any $v \in V$ we see that $(v+v^x)^x = v^x+v^{x^2} = v^x+v$. So $v+v^x \in C_V(x)$ and hence $[V, x] \leq C_V(x)$. Further if x is not an involution then $[V, x] \not\leq C_V(x)$.*

(ii) Let $x \in L$. Consider the homomorphism $\varphi_x : V \to [V, x]$ defined by $\varphi_x(v) = v+v^x$. We see that $\ker(\varphi_x) = C_V(x)$ and so $\dim([V, x]) = \dim(V)-\dim(C_V(x))$.

Definition 4.8. *Let L, V be as above. Let $m = n/2$ if n is even and $(n+1)/2$ if n is odd. Let $\mathcal{V}_m$ denote the set of m-dimensional subspaces of V. For $U \in \mathcal{V}_m$, we define*

$$Q_U = \{x \in L \mid [V, x] \leq U \leq C_V(x)\}.$$

Let $\mathcal{I}(L)$ denote the set of involutions in L. From Remarks 4.7(i), we see that every Q_U consists solely of involutions, while (ii) implies that $\dim([V, x]) \leq m \leq \dim(C_V(x))$, so there exists some $U_x \in \mathcal{V}_m$ with $[V, x] \leq U_x \leq C_V(x)$, and so every $x \in \mathcal{I}(L)$ is contained in at least one Q_U.

If $x \in Q_U$ we see that x fixes U, and so clearly $Q_U \subseteq \mathrm{Stab}_L(U)$. Suppose $x, y \in Q_U$. Then we see that $\langle [V, x], [V, y]\rangle \leq U \leq C_V(x) \cap C_V(y)$. But $[V, xy] \leq \langle [V, x], [V, y]\rangle$ and $C_V(x) \cap C_V(y) \leq C_V(xy)$ so we see that $xy \in Q_U$. All elements of Q_U are involutions and so self-inverse, and so we see that in fact Q_U is a subgroup of $\mathrm{Stab}_L(U)$, and in particular it is an elementary abelian 2-group. Hence Q_U is contained in a Sylow 2-subgroup of $\mathrm{Stab}_L(U)$, which will also be a Sylow 2-subgroup of L (specifically, the stabilizer of a maximal flag containing U). This gives us the following result.

Table 4.2: Sizes of the sets $\mathcal{V}_m$

n	Maximal chains	$\lvert\mathcal{V}_m\rvert$
2	3	3
3	21	7
4	315	35
5	9,765	155
6	615,195	1,395
7	78,129,765	11,811
8	19,923,090,075	200,787
9	10,180,699,028,325	3,309,747
10	10,414,855,105,976,475	109,221,651

Lemma 4.9. *Let $L, V, \mathcal{V}_m$ be as above. For $U \in \mathcal{V}_m$, let T_U be a Sylow 2-subgroup of* $\mathrm{Stab}_L(U)$*. Then*

$$\mathcal{I}(L) = \bigcup_{U \in \mathcal{V}_m} Q_U \subseteq \bigcup_{U \in \mathcal{V}_m} T_U.$$

So we can now create a set $\mathcal{S}$ of Sylow 2-subgroups of L such that $|\mathcal{S}| = |\mathcal{V}_m|$, by selecting one Sylow 2-subgroup of L from inside each $\mathrm{Stab}_L(U)$, $U \in \mathcal{V}_m$ (or equivalently, for every $U \in \mathcal{V}_m$, selecting a chain of subspaces containing U). Table 4.2 compares the sizes of these sets with the number of maximal chains as shown in Table 4.1, demonstrating the benefits of this approach.

We now introduce a certain configuration of subspaces of V, whose properties will allow us to further reduce these numbers.

Definition 4.10. *Suppose $U_1, U_2, U_3, U_4 \in \mathcal{V}_m$ are distinct m-dimensional subspaces of V such that*

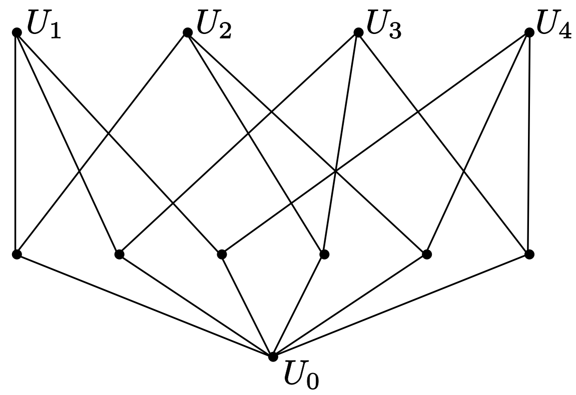
(i) $U_0 = \bigcap_{i \in I} U_i$ has dimension $m - 2$; and

(ii) for each $i \in I$, $U_i \cap U_j$ $(j \in I \setminus \{i\})$ are the three $m-1$-dimensional subspaces of U_i containing U_0

(where $I = \{1, 2, 3, 4\}$). Then the set $\{U_1, U_2, U_3, U_4\}$ is called a crown.

Figure 4.2.1 gives the subspace lattice of a crown. The following is the crucial result motivating our interest in crowns.

Figure 4.1: The subspace lattice of a crown



Theorem 4.11. *Let $L, V, \mathcal{V}_m, Q_U$ be as defined above, and let $\{U_1, U_2, U_3, U_4\}$ be a crown. For $i = 2, 3, 4$, choose $T_i \in \mathrm{Syl}_2(L)$ such that T_i fixes the spaces $U_1 \cap U_i$, U_i and $\langle U_1, U_i \rangle$. Then*

$$Q_{U_1} \subseteq T_2 \cup T_3 \cup T_4.$$

Proof. The somewhat technical proof of the theorem is given in several parts.

(4.11.1) $[Q_{U_1} : Q_{U_1} \cap T_i] = 2$ for any $i \in \{2, 3, 4\}$.

Let $i \in \{2, 3, 4\}$. Since T_i stabilizes $U_1 \cap U_i$, U_i and $\langle U_1, U_i \rangle$, it is the stabilizer of a maximal chain of the form

$$0 = V_0 < \cdots < V_{m-2} < U_1 \cap U_i < U_i < \langle U_1, U_i < V_{m+2} < \cdots < V_n = V,$$

which we denote γ. Since $U_1 \cap U_i \leq C_V(Q_{U_1})$ we have that Q_{U_1} stabilizes all subspaces $V_0, V_1, \ldots, V_{m-1}, U_1 \cap U_i$. And since $[V, Q_{U_1}] \leq U_1$, Q_{U_1} also stabilizes $\langle U_1, U_i \rangle, V_{m+2}, \ldots, V_n$. So $\langle Q_{U_1}, T_i \rangle$ stabilizes the flag $\nu = \gamma \setminus \{U_i\}$. The stabilizer of ν is a minimal parabolic subgroup P, and we have that either $\langle Q_{U_1}, T_i \rangle = P$ or $\langle Q_{U_1}, T_i \rangle = T_i$. Suppose $\langle T_i, Q_{U_1} \rangle = T_i$, that is, $Q_{U_1} \leq T_i$. But $Q_{U_i} \leq T_i$, and Q_{U_i} is weakly closed in T_i with respect to L (that is, no other L-conjugate of Q_{U_i} is a subgroup of T_i. This forces $Q_{U_1} = Q_{U_i}$ and so $U_1 = U_i$, a contradiction. Hence $\langle Q_{U_1}, T_i \rangle = P$, and (4.11.1) follows.

(4.11.2) $T_i \cap T_j \cap Q_{U_1} = T_2 \cap T_3 \cap T_4 \cap Q_{U_1}$ for any $i, j \in \{2, 3, 4\}$ $(i \neq j)$.

Set $Q = Q_{U_1}$, and without loss of generality suppose $i = 2, j = 3$. Let $U_0 = \bigcap_{i=1}^{4} U_i$ and $U^0 = \langle U_1, U_2 \rangle$. We set $R = T_2 \cap T_3 \cap Q$. Then it suffices to show that $R \leq T_4$.

Since $U_1 = C_V(Q)$ and $R \leq Q$ we see that R fixes U_1 and all of its subspaces. Further, $R \leq T_2 \cap T_3$ so that R stabilizes U_2 and U_3 and therefore $U_2 \cap U_3$. Now, there are three $(m-1)$-dimensional subspaces W with $U_0 < W < U_2$, which by the definition of a crown are $U_2 \cap U_1$, $U_2 \cap U_3$ and $U_2 \cap U_4$. We have that R fixes the first two of these and so it must fix the third. Now R must stabilize U_4 as it stabilizes $U_1 \cap U_4$ and $U_2 \cap U_4$, which generate U_4. We know from (4.11.1) that Q and hence R act trivially on V/U^0 and on $U_1 \cap U_4$, and so R acts trivially on the whole maximal chain of which T_4 is the stabilizer. Hence $R \leq T_4$ as required.

(4.11.3) $T_i \cap Q_{U_1} \neq T_i \cap T_j \cap Q_{U_1}$ for any $i, j \in \{2, 3, 4\}$.

Again we suppose $i = 2$ and $j = 3$ without loss of generality, and we define U_0, U^0 as above. Now, $\mathrm{Stab}_L(U_1)$ has subgroups $H_1 \cong GL_{n-m}(2)$ and $H_2 \cong GL_m(2)$ whose natural $GF(2)$-modules are respectively V/U_1 and U_1. We see that H_1 acts trivially on U_1 while H_2 acts trivially on V/U_1. So we can select $x \in \mathcal{I}(L)$ such that $[V, x] = U_1 \cap U_2$ and such that x centralizes U_1 but does not centralizer U^0/U_0. Note that $x \in Q_{U_1}$. Since x centralizes $V/(U_1 \cap U_2)$ and $U_1 \cap U_2$, We have that $x \in T_2$. So it suffices to show that $x \notin T_3$. Suppose x were in T_3. Then x would leave U_3 invariant and so we would have

$$[U_3, x] \leq [V, x] \cap U_3 = U_1 \cap U_2 \cap U_3 = U_0,$$

and so x would centralize U_3/U_0 and hence centralize $\langle U_1, U_3 \rangle / U_0 = U^0/U_0$, contrary to the choice of x. Hence $x \notin T_3$, thus proving (4.11.3).

We are now in a position to prove the theorem. By (4.11.1) and (4.11.2), we see that $[Q_{U_1} : Q_{U_1} \cap T_i] = 2$ for $i = 2, 3, 4$, while $[Q_{U_1} : Q_{U_1} \cap T_i \cap T_j] = 4$ for distinct $i, j \in \{2, 3, 4\}$. By (4.11.2) we have $Q_{U_1} \cap T_i \cap T_j = Q_{U_1} \cap T_2 \cap T_3 \cap T_4$ (for $i \neq j$).

We consider the group

$$Q' = (Q_{U_1} \cap T_2) \cup (Q_{U_1} \cap T_3) \cup (Q_{U_1} \cap T_4).$$

By considering the sizes of the intersections of the $Q_{U_1} \cap T_i$ detailed above it is apparent that we must have $Q' = Q$ and hence the theorem holds.

□

This result yields the following immediate corollary.

Corollary 4.12. *Let $\{U_1, U_2, U_3, U_4\} \subset \mathcal{V}_m$ be a crown. Then for any $i \in \{1, 2, 3, 4\}$ a set $\mathcal{S} = \{T_U \in \mathrm{Syl}_2(\mathrm{Stab}_L(U)) \mid U \in \mathcal{V}_m \setminus \{U_i\}\}$ exists which has the property that every involution in L is contained in some $T \in \mathcal{S}$.*

Proof. For each $j \in \{1, 2, 3, 4\} \setminus \{i\}$ we select T_U to be T_j as defined in Theorem 4.11, and for each $U \in \mathcal{V}_m \setminus \{U_1, U_2, U_3, U_4\}$ we select an arbitrary $T_U \in \mathrm{Syl}_2(\mathrm{Stab}_L(U))$. Now the result follows from Lemma 4.9 and Theorem 4.11. □

So given a crown $\gamma = \{U_1, U_2, U_3, U_4\}$, we can 'discard' one space $U_i \in \gamma$ when forming our set $\mathcal{S}$ from the spaces in $\mathcal{V}_m$, so long as the T_U associated with the other three spaces are selected in accordance with the hypotheses of Theorem 4.11. Clearly this can be extended so that if we have a set of pairwise disjoint crowns we can discard one space from each crown. The following theorem explains how a set of non-disjoint crowns can be used if enough care is taken in its construction.

Theorem 4.13. *Let $\gamma_1, \gamma_2, \ldots, \gamma_r$ be a set of crowns, and in each of these crowns fix a subspace $U_i \in \gamma_i$. Suppose that for any $U \in \gamma_i \cap \gamma_j$ the following conditions hold:*

(i) $U \neq U_i$ and $U \neq U_j$; and

(ii) $U \cap U_i = U \cap U_j$ and $\langle U, U_i \rangle = \langle U, U_j \rangle$.

Then there exists a set $\mathcal{S}$ of Sylow 2-subgroups of L such that $|\mathcal{S}| = |\mathcal{V}_m| - r$ and every involution in L is contained in at least one $T \in \mathcal{S}$.

Proof. Let $\mathcal{U} = \mathcal{V}_m \setminus \{U_i \mid i = 1, \ldots, r\}$. We select a set $\mathcal{S}$ of Sylow 2-subgroups of L with $|\mathcal{S}| = |\mathcal{U}| = |\mathcal{V}_m| - r$ in the following way.

If a space $U \in \mathcal{U}$ occurs in a crown γ_i then we select a Sylow 2-subgroup T_i fixing the spaces $U \cap U_i$, U and $\langle U, U_i \rangle$. Note that the hypotheses of the theorem ensure that T_i meets these requirements for every crown in which it occurs. Where a space $U \in \mathcal{U}$ occurs in none of our crowns, we select an arbitrary $T \in \mathrm{Syl}_2(L)$ fixing U. Now Lemma 4.9 and Theorem 4.11 give that every involution in L is contained in some $T \in \mathcal{S}$. □

So in order to make our set $\mathcal{S}$ as small as possible, we require as large as possible a set of crowns satisfying the hypotheses of Theorem 4.13. Our sets of crowns are obtained by the heuristic method described in Section 4.3.

This set allows us to determine whether the trivial case $K = C$ occurs. In the next section we consider how to determine K when this is not the case.

4.2.2 Determining K

The results in the previous section allow us to determine whether we have the trivial case that $K = C$. This is done by using maximal chains sufficient to find every involution on $\overline{K}$, and so having done this we can generate a subgroup $\overline{M} \leq \overline{K}$ where $\overline{M} = \langle x \in K \mid x^2 = 1 \rangle$. It remains to determine further chains to use that will allow us to generate the entire group K. We begin with the following result which suggests a fast way of arriving at this group.

Lemma 4.14. *Let $S \in \mathrm{Syl}_2(\overline{M})$, and let $\{T_1, \ldots, T_l\}$ be the set of all Sylow 2-subgroups of $\overline{K}$ containing S. Then*

$$K = \langle \overline{M}, T_1, \ldots, T_l \rangle.$$

Proof. Since $\overline{K} = O^{2'}(\overline{N})$ is generated by its Sylow 2-subgroups it suffices to show that an arbitrary Sylow 2-subgroup of $\overline{K}$ is contained in $\langle \overline{M}, T_1, \ldots, T_l \rangle$. Let $T \in$

$\mathrm{Syl}_2(\overline{K})$. Since $\overline{M}$ is generated by the set of involutions in $\overline{K}$ it is normal in $\overline{K}$ (indeed characteristic) and so $T \cap \overline{M} \in \mathrm{Syl}_2(\overline{M})$. Therefore, $T \cap \overline{M}$ is $\overline{M}$-conjugate to S, and hence T is $\overline{M}$-conjugate to a Sylow 2-subgroup of $\overline{K}$ containing S, that is, to one of the T_i. So $T = T_i^m$ for some $m \in \overline{M}$ and some $i \in \{1, \dots, l\}$, hence $T \in \langle \overline{M}, T_1, \dots, T_l \rangle$. □

Our goal then is to find this set of all the Sylow 2-subgroups of $\overline{K}$ containing a chosen $S \in \mathrm{Syl}_2(\overline{M})$. This is achieved via an inductive procedure, starting with $\mathcal{A} = \{S\}$ and at each stage finding a set of chains sufficient to determine all the subgroups of $\overline{K}$ which contain a group in $\mathcal{A}$ as an index 2 subgroup. Working up index 2 at a time in this way we will eventually arrive at $\mathcal{A} = \{T_1, \dots, T_l\}$ as required. The procedure by which we find such a set of maximal flags is described Section 4.3.2.

4.2.3 The algorithm in full

We conclude this section with Algorithm 4, which outlines the complete procedure. The subroutines 'InputChains' and 'Index2Chains' will be described in the next section.

We note that Algorithm 4 accesses information from the group G only when it applies Algorithm 3 to compute the stabilizer of a chain, and this algorithm is black-box as discussed earlier. Information from the subgroups X and $C_G(X)$ is used, but the former is an elementary abelian group so a list of generators gives all the information needed, while the latter is easily obtained by applying Algorithm 1. All other computation is carried out in subgroups of L. So Algorithm 4 too is black-box on G.

Recall that elements of $\overline{K} \leq L$ are found by taking elements of $K \leq N$ and examining their conjugation action on X. In order to be able to produce a generating set for K as an output, the algorithm must ensure elements of K remain associated to the relevant elements of $\overline{K}$, so that a generating set for K can be built from elements corresponding to generators for $\overline{K}$ along with generators for C.

Algorithm 4 TwoGroupNormalizer

Input: G a black-box group;
X an elementary-abelian 2-subgroup of G

$n \leftarrow \log_2 |X|$
$L \leftarrow GL_n(2)$
$\mathcal{S} \leftarrow \text{InputChains}(n)$
$H \leftarrow C_G(X)$
for $\mathcal{C} \in \mathcal{S}$ **do**
    $H \leftarrow \langle H, \text{ChainStabilizer}(\mathcal{C}) \rangle$
end for
$\overline{M} \leftarrow \langle x \in \overline{H} \mid x^2 = 1 \rangle$
if $\overline{M} = 1$ **then**
    $K \leftarrow C_G(X)$; $\overline{K} \leftarrow 1$
else
    $\overline{S} \leftarrow \text{Syl}_2(\overline{M})$
    $\mathcal{A} \leftarrow \{\overline{S}\}$
    repeat
        $\mathcal{A} \leftarrow \mathcal{A}_2$
        $\mathcal{A}_2 \leftarrow \emptyset$
        for $\overline{T} \in \mathcal{A}$ **do**
            $\mathcal{S} \leftarrow \text{Index2Chains}(\overline{T})$
            for $\mathcal{C} \in \mathcal{S}$ **do**
                $H \leftarrow \langle H, \text{ChainStabilizer}(\mathcal{C}) \rangle$
            end for
            $\mathcal{A}_2 \leftarrow \mathcal{A}_2 \cup \{\langle \overline{T}, h \rangle \mid h \in \overline{H}, h^2 \in \overline{T}\}$
        end for
    until $\mathcal{A}_2 = \emptyset$
    $\overline{K} \leftarrow \langle \overline{M}, \mathcal{A} \rangle$
end if

Output: $\overline{K}$;
generators for K

4.3 Implementation of the algorithm

We discuss some of the considerations arising from the implementation of the present algorithm.

4.3.1 Creating the initial chains

We note that the initial set of chains $\mathcal{S}$ used by our algorithm is not dependent on the particular groups G and X in question. So we create these sets for each value of n beforehand and they are stored and used as inputs by the algorithm. Before we describe the method by which we arrive at these input sets, we remark on some properties of crowns relevant to our strategy.

Remarks 4.15. *Let* $\{U_1, U_2, U_3, U_4\}$ *be a crown. Then*

(i) Let $U^0 = \langle U_1, U_2 \rangle$*. Then* $\dim U^0 = m + 1$ *and* $U^0 = \langle U_i, U_j \rangle$ *for any* $i, j \in \{1, 2, 3, 4\}$ *with* $i \neq j$*;*

(ii) If $\{U_1, U_2, U_3, U_4'\}$ *is a crown then* $U_4' = U_4$*, that is, two distinct crowns intersect in at most two subspaces;*

(iii) The spaces U_0, U_i, U_j *for any* $i, j \in \{1, 2, 3, 4\}$ *with* $i \neq j$*, uniquely determine a crown.*

Proof. Let $i, j \in \{1, 2, 3, 4\}$ with $i \neq j$. Then $U_i = \langle U_1 \cap U_i, U_2 \cap U_i \rangle$ and $U_j = \langle U_1 \cap U_j, U_2 \cap U_j \rangle$ and so $\langle U_i, U_j \rangle = \langle U_1, U_2 \rangle$, giving (i). For (ii), let $\{U_1, U_2, U_3, U_4'\}$ be a crown. Then $U_1 \cap U_4'$ is an $(m-1)$-dimensional subspace of U_1 containing U_0, so it must be equal to one of $U_1 \cap U_2$, $U_1 \cap U_3$, $U_1 \cap U_4$. If $U_1 \cap U_4'$ were equal to $U_1 \cap U_2$ or $U_1 \cap U_3$ it would force U_4' to equal U_2 or U_3, which it is not. So $U_1 \cap U_4' = U_1 \cap U_4$ giving that $U_4' = U_4$. Now for (iii), suppose without loss of generality that $i = 2$ and $j = 3$. So fixing U_0, U_1 and U_2 determines five of the six $(m-1)$-dimensional subspaces appearing in the subspace lattice of the crown. This determines a third space in the crown which we suppose is U_3. Then the result follows from (ii). □

We recall our objective is to create a set of crowns as large as possible satisfying the conditions of Theorem 4.13. Note that condition (i) of this theorem is equivalent to requiring that any two crowns that intersect nontrivially have the same space U^0. Note also that condition (ii) precludes the possibility of three crowns $\gamma_i, \gamma_j, \gamma_k$ having a space U common to all three, as this would then require four distinct subspaces U, U_i, U_j, U_k all of dimension m and all lying between $U \cap U_i$ and $\langle U, U_i \rangle$, whereas these spaces have codimension 2 and as such only three distinct subspaces lie between them. We outline the procedure by which our initial sets of crowns were formed.

Algorithm 5 CrownsMaker

Input: V, n, m as defined.

Compute the set $\mathcal{V}_{m+1}$ consisting of all subspaces of V of dimension $m+1$ (by making one arbitrary such space and acting on it by every element of a transversal of its stabilizer across L).
for $U^0 \in \mathcal{V}_{m+1}$ **do**
  We attempt to build as many crowns as possible with $\langle U_1, U_2 \rangle = U^0$. We initialise our set with an arbitrary such crown. Recall we will 'discard' the space U_1 from each crown.
  repeat
    for the spaces U_2, U_3, U_4 in each crown found so far **do**
      Form another crown sharing that space. Choosing new spaces U_1, U_0 allows the whole crown quickly to be determined (see Remarks 4.15(iii)). If the crown formed shares no other space with any crown already in our set, adding it to the set.
    end for
  until A loop of this procedure yields no further suitable crowns
end for

Output: the set of crowns formed.

Table 4.3 shows the savings made in terms of numbers of initial chains by considering crowns.

Once a suitable set of crowns has been found, the initial set of chains is formed. First, every m-space of V is found (again by taking a transversal across a stabilizer). We then make chains for each space U_1, U_2, U_3 in a crown by taking arbitrary subspaces building up to the required spaces $U_i \cap U_4$, U_i and $\langle U_i, U_4 \rangle$, and from there up to V. (Where we have used a space in two crowns, its chain is built up from either of the crowns containing it; by the previous result the chain will be suitable for the

Table 4.3: Numbers of crowns found

n	$\lvert\mathcal{V}_m\rvert$	No. crowns found	$\lvert\mathcal{S}\rvert$	% reduction
3	7	1	6	14.3
4	35	7	28	20.0
5	155	41	114	26.5
6	1395	350	1,045	25.1
7	11,811	3,208	8,603	27.2
8	200,787	54,936	145,851	27.4
9	3,309,747	926,280	2,383,467	28.0

other crown as well). Arbitrary chains are built around the m-spaces not used in any crown to complete the initial set.

After a set of chains has been created, they must be stored. Each space in a chain is stored as a representative vector (i.e. one which together with the previous space generates the space required). Chains are stored grouped together in sets sharing a common 'root'. This reduces the space needed for storage, since the root need only be stored once, and improves the efficiency of the algorithm itself, since if several chains all begin with $X_0 < X_1 < \ldots X_r$ for some $r < n$, when employing Algorithm 3 we clearly need only generate the group M_r once.

Chains taken from m-spaces contained in a crown are grouped together, since they can be chosen to share a common root up to the space U_0 of the crown. The remaining chains are grouped together by a simple search procedure.

Where Algorithm 4 calls the subroutine 'InputChains' it is to be understood that at this point the initial sets of chains for the relevant values of n are loaded into the algorithm.

4.3.2 Chains for index 2 subgroups

Recall that for Lemma 4.14 we require all the Sylow 2-subgroups of $\overline{K}$ containing the known 2-subgroup S. We aim to find these by an inductive procedure involving finding all the subgroups of $\overline{K}$ that contain a given group as an index 2 subgroup.

That is, at each stage of this process, we require for a given $\overline{R} \leq \overline{K}$ a set of chains corresponding to Sylow 2-subgroups of L which between them contain every element t for which $[\langle \overline{R}, t\rangle : \overline{R}] = 2$. Our strategy for finding these is given in Algorithm 6.

Algorithm 6 Index2Chains

Input: L, V as defined above;
  $\overline{R} \leq L$ a 2-subgroup.
1: $A \leftarrow N_L(\overline{R})$
2: **if** A is a 2-group **then**
3:   **repeat**
4:     $A \leftarrow N_L(A)$
5:   **until** $A \in \mathrm{Syl}_2(L)$
6:   $\mathcal{S}' \leftarrow \{\mathcal{C}$, a chain corresponding to $A\}$
7: **else**
8:   $W_1 \leftarrow C_V(\overline{R})$
9:   $d_1 \leftarrow \dim W_1$
10:   $i \leftarrow 1$
11:   **repeat**
12:     $i \leftarrow i+1$
13:     $W_i \leftarrow C_{V/W_{i-1}}(\overline{R})$
14:     $d_i \leftarrow \dim W_i$
15:   **until** $W_i = V$;
16:   $\mathcal{S}' \leftarrow \emptyset$
17:   Form a new basis for V, starting with a basis for W_1 followed by vectors extending it to a basis of the inverse image of W_2, then in turn the inverse image of $W_3, \ldots, W_i = V$.
18:   **for** all $(\mathcal{C}_j)_{j=1}^{i}$ such that $\mathcal{C}_j \in \mathrm{InputChains}(d_j)$ **do**
19:     Construct a new chain $\mathcal{C}$ for V, corresponding to $\mathcal{C}_j$ on the space W_j for each $j = 1, \ldots, i$.
20:     $\mathcal{S}' \leftarrow \mathcal{S}' \cup \{\mathcal{C}\}$
21:   **end for**
22:   Transform the chains in $\mathcal{S}'$ to the original basis for V.
23: **end if**

Output: the set $\mathcal{S}'$ of chains formed.

We see that if t satisfies $[\langle \overline{R}, t\rangle : \overline{R}] = 2$ then t must lie in $N_L(\overline{R})$ so this group is calculated first. If it is a 2-group then clearly a single chain corresponding to a Sylow 2-subgroup containing $N_L(\overline{R})$ will suffice. Otherwise, a new set of multiple chains must be formed. Any t of the desired form will act as an involution on each of the spaces W_i formed in Algorithm 6, so in steps 16–22 for each of these we use the set of input chains for the relevant dimension d_i, which were designed to find all involutions.

4.3.3 Limitations of the algorithm

We recall from Section 2.1.3 the following limitation to the involution centralizer algorithm that underpins the current work. Suppose $t \in G$ is an involution with $t \in O_2(G)$ but $C_G(t) \not\leq O_2(G)$. Then by Lemma 2.3, Algorithm 1 never produces elements of $C_G(t)$ outside $C_G(t) \cap O_2(G)$. If this happens in an when finding the centralizer of a factor group X_i/X_{i-1} during an application of Algorithm 3, it may mean we fail to generate the full stabilizer of the chain.

The large numbers of chains involved in the algorithm mean that in practice this is unlikely to be an issue, but we wish to do what we can to mitigate the problem. It is possible to diagnose when this unfortunate circumstance may have arisen, because the value k computed at step 6 of Algorithm 3 will always be a power of 2. When this occurs, a remedy may be available.

Recall that the construction of individual chains is mostly arbitrary: only the spaces of dimensions $m-1$, m and $m+1$ are determined when the chain is involved in a crown, and only the space of dimension m when it is not. So if this unfavourable situation arises when calculating the centralizer of X_i/X_{i-1} for i other than one of these values, we can simply produce a different chain around the spaces which are determined, and try again. Only if our problem presents itself at one of the fixed spaces, or if replacement chains repeatedly suffer the same issue, do we need to face the possibility that we have failed to generate the entire stabilizer of the chain. The Magma implementation of the algorithm discussed in the next section includes a routine to carry out this replacement of 'faulty' chains.

4.3.4 Magma implementation

This algorithm has been implemented in the Magma computer algebra package by the author. Appendix A gives details on how the code can be obtained, along with the best sets of initial chains found (with the sizes given in Table 4.3) as well as generators used for the groups in the example calculations outlined in the next section. We note that input data is only given for values of n up to 9. For $n = 10$ and above, MAGMA exhausts its available memory when attempting to perform the construction of the input chains. Specifically, computing the transversal of $\mathrm{Stab}_L(U)$ for an m-dimensional subspace U fails.

This means the algorithm can only be used for groups X with elementary abelian factors of at most order 2^9. In the next section we give details of some example calculations carried out.

4.4 Analysis of the algorithm

4.4.1 Example calculations

Given the black-box nature of our algorithm—dealing with only single group elements at a time—we would expect it to outperform a deterministic algorithm when the group G is large or has a large degree representation. In particular the standard algorithms for matrix groups are significantly less efficient than for permutation groups, whereas we only face a relatively small increase in the cost of the basic group arithmetic. So we give example computations in these sorts of groups. Generators for all the groups G and subgroups X used are available in the electronic files detailed in Appendix A.

1. Given $G \cong Sp_{12}(3)$ (using the natural matrix representation) and selecting an elementary abelian subgroup X of order 2^6, MAGMA calculates the normalizer $N_G(X)$ in 158.2 seconds, while our algorithm calculates K (which in this case is $N_G(X)$) in 138.66 seconds, using 10 random elements for each application of Algorithm 3. Moving to $G \cong Sp_{16}(3)$ with X having order 2^7, the advantages of the present approach become apparent: we calculate K having order $2^{12} \cdot 3^2 \cdot 5 \cdot 7$ in 10014.5 seconds, while the standard MAGMA routine exhausts the available memory and returns no output.

2. We take $G \cong J_4$ using the same representation as in Section 5.2. We note from

that section that we have a maximal subgroup $M \cong 2^{11}.M_{24}$ and within it we find an X elementary abelian of order 2^7. The algorithm, using 20 random elements for each use of Algorithm 3, finds a group $K \leq N_G(X)$ having order 2^{15} in 57538 seconds. The standard MAGMA routine fails to determine the normalizer.

3. We give an example where X is not an elementary abelian subgroup. Set $G \cong S_{20}$ and $X = \Phi(P)$ where $P \in \mathrm{Syl}_2(G)$. Then X has order 2^{12} and employing the Frattini subgroup method described in Section 4.1 we decompose it into a chain of characteristic subgroups with elementary abelian factors having orders 2, 2^5, 2^6. In this small representation both our algorithm and the standard MAGMA function quickly compute the normalizer $N_G(X)$ to be of order 2^{18}. However, if we now consider X as a group of permutation matrices over $GF(2)$ and set $G \cong GL_{20}(2)$, our algorithm (using 1000 random elements) returns a group K of order $2^{31} \cdot 3$ in 760.4 seconds, while the standard normalizer function fails.

4.4.2 Efficiency of the initial sets

Recall that our sets of initial chains were constructed with the intention that (viewed as Sylow 2-subgroups of L) they cover all the involutions in L while being as small as possible. We wish to analyse how efficiently our sets of chains accomplish this task. To decrease redundancy it is desirable that a given involution is contained in as few of the Sylow 2-subgroups in our set as possible. Tables 4.4 and 4.6 give, for $n = 5$ and $n = 6$, details of how many chains each involution finds itself in. The data is also presented for each of the conjugacy classes of involutions in L. We see that most involutions are not contained in too many chains, suggesting that the initial sets are fairly close to being maximally efficient.

In Table 4.4 t_1 is an involution with $C_V(t_1)$ having dimension 4, and t_2 an involution with $C_V(t_2)$ having dimension 3. In Table 4.6 t_1, t_2 and t_3 are involutions

Table 4.4: Efficiency of initial sets for $n = 5$

Chains/inv.	All involutions	t_1^L	t_2^L
1	1323	0	1323
2	2478	0	2478
3	1501	0	1501
4	800	0	800
5	284	0	284
6	88	1	87
7	37	4	33
8	30	27	3
9	46	45	1
10	48	48	0
11	65	65	0
12	90	90	0
13	67	67	0
14	52	52	0
15	34	34	0
16	16	16	0
17	9	9	0
18	1	1	0
19	2	2	0
20	2	2	0
21	2	2	0

with $\dim(C_V(t_i))$ respectively 4, 5 and 3. As an example of how the tables are to be read, we see that for $n = 5$, there are 37 involutions on L which find themselves in (the Sylow 2-subgroups corresponding to) seven of the initial chains. Of these, 4 are L-conjugate to t_1 and 33 are L-conjugate to t_2.

Table 4.6: Efficiency of initial sets for $n = 6$

Chains/inv.	All involutions	t_1^L	t_2^L	t_3^L
1	60704	0	0	60704
2	70698	0	0	70698
3	52608	429	0	52179
4	32491	2286	0	30205
5	18826	5801	0	13025
6	15497	10531	0	4966
7	16685	14891	0	1794
8	18592	18023	0	569
9	19128	18974	0	154
10	17537	17490	0	47
11	15075	15061	0	14
12	11924	11920	0	4
13	8342	8342	0	0
14	5613	5612	0	1
15	3375	3375	0	0
16	1953	1953	0	0
17	1032	1032	0	0
18	555	555	0	0
19	252	252	0	0
20	94	94	0	0
21	58	58	0	0
22	17	17	0	0
23	8	8	0	0
24	5	5	0	0
25	1	1	0	0
26	0	0	0	0
⋮	⋮	⋮	⋮	⋮
44	0	0	0	0
45	1	0	1	0
46	1	0	1	0
47	0	0	0	0
48	2	0	2	0
49	10	0	10	0
50	5	0	5	0
51	13	0	13	0
52	8	0	8	0
53	15	0	15	0
54	26	0	26	0
55	23	0	23	0
56	43	0	43	0
57	39	0	39	0
58	54	0	54	0

Chains/inv.	All involutions	t_1^L	t_2^L	t_3^L
59	51	0	51	0
60	69	0	69	0
61	85	0	85	0
62	97	0	97	0
63	89	0	89	0
64	102	0	102	0
65	107	0	107	0
66	85	0	85	0
67	104	0	104	0
68	117	0	117	0
69	95	0	95	0
70	111	0	111	0
71	84	0	84	0
72	85	0	85	0
73	64	0	64	0
74	52	0	52	0
75	43	0	43	0
76	51	0	51	0
77	46	0	46	0
78	42	0	42	0
79	29	0	29	0
80	30	0	30	0
81	18	0	18	0
82	19	0	19	0
83	9	0	9	0
84	6	0	6	0
85	6	0	6	0
86	4	0	4	0
87	6	0	6	0
88	1	0	1	0
89	0	0	0	0
90	0	0	0	0
91	0	0	0	0
92	1	0	1	0
93	0	0	0	0
94	0	0	0	0
95	0	0	0	0
96	0	0	0	0
97	1	0	1	0
98	0	0	0	0
99	0	0	0	0
100	1	0	1	0

Chapter 5

Involution suborbits in sporadic simple groups

Our aim in this chapter is to uncover the suborbit structure of the involution conjugacy classes for some of the sporadic simple groups, that is, the orbits of X as $C_G(t)$ ($t \in X$) acts on it by conjugation. Along the way we will use this data to build graphs whose vertex sets are (or can be considered to be) the involution classes: in Sections 5.2 and 5.3 the commuting involution graphs, and in Section 5.4 the related point-line collinearity graphs for a certain 2-local geometry. But before we get our hands dirty, we discuss what our approach to the problem will be, and lay out some results that greatly simplify the study.

Throughout this chapter G is a finite group, X a conjugacy class of involutions and t an arbitrary fixed element of X.

5.1 General considerations

In all the groups we consider, our basic strategy is to take random group elements $g \in G$ and hence form random conjugates $x = t^g \in X$, hoping to arrive at a complete set of representatives for the $C_G(t)$-orbits on X. We wish to know the sizes of the orbits, for which we determine $C_{C_G(t)}(x)$ for our representatives x (then the sizes of the orbits are of course just $[C_G(t) : C_{C_G(t)}(x)]$). We also require a way of determining whether an element $x \in X$ is in a suborbit for which we already have a representative. We do this either by direct testing of $C_G(t)$-conjugacy, or where this is impossible by consideration of various suborbit invariants.

The following well-known results show how some light can be shed on these suborbits before we begin our search, by utilising the character table of the group.

Proposition 5.1. *The number of $C_G(t)$-orbits on X is equal to the value $\|\chi\|$ where χ is the character of the permutation action of G on X acting by conjugacy. This number is called the* permutation rank *of G on X.*

It is not always possible to determine the permutation rank and hence the number of $C_G(t)$-orbits, but where we can it is useful information to have to hand. We now consider some further information regarding the $C_G(t)$-orbits that is always available.

Definition 5.2. *Let X be a conjugacy class of a group G, and let $t \in X$. For a second class C of G, we define the following subset of X:*

$$X_C = \{x \in X \mid tx \in C\}.$$

Proposition 5.3. *The size of a set X_C is given by*

$$|X_C| = \frac{|G|}{|C_G(t)||C_G(z)|} \sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(t)^2 \chi(z)}{\chi(1)}.$$

for any $z \in C$.

We see that the value of $|X_C|$ for a given class C is easily calculated from the character table, and indeed GAP contains a function `ClassMultiplicationCoefficient` that will compute it. Our interest in this number is sparked by the following observation.

Lemma 5.4. *Each set X_C is a (possibly empty) union of $C_G(t)$-orbits.*

Proof. If $x \in X$ with $tx \in C$, then for $h \in C_G(t)$ we have $tx^h = t^h x^h = (tx)^h \in C$ and so $x^h \in X_C$. □

Given an involution $x \in X$, we generally write z for the element tx. So we see from Lemma 5.4 that consideration of the class of z is useful when attempting to find the $C_G(t)$-orbits on X. In particular, the class C in which z lies is an invariant of the $C_G(t)$-orbit, and since the sizes of the sets X_C can be simply calculated, we can know when all the suborbits sharing this invariant have been found. Of course this depends on the ability to determine the class of a given element z, and in practice this is not always possible. We deal with such an eventuality on a case-by-case basis.

The following observation provides another useful invariant of a $C_G(t)$-orbit, and can also be used to help find new suborbit representatives.

Remarks 5.5. *Let $x \in X$, and set $z = tx$ and $C = z^G$. For any n, let $y = tz^n$. Then $y = t(tx)^n = xtx \dots tx$ is clearly an involution in X, and $y \in X_D$ where $D = (z^n)^G$. All elements in a particular X_C power in this manner to elements in X_D, and if x_1, x_2 are $C_G(t)$-conjugate then so are $y_1 = t(tx_1)^n$ and $y_2 = t(tx_2)^n$.*

Given an element $x \in X$ and n dividing the order of $z = tx$, we henceforth use the notation $x^{(n)}$ for the element $tz^n \in X$ formed in this manner. We observe that if we have already catalogued the suborbits in X_D then we can use information about the suborbit containing $x^{(n)}$ as further invariants for x, while if we have not found all the orbits in X_D, then examining the elements $x^{(n)}$ for $x \in X_C$ may be a helpful way to discover them.

If C, D are classes in G such that elements $h \in C$ have $h^m \in D$ for some m, we say that the class C 'powers to' D. The Atlas (in paper and online forms) gives the 'power maps' for the groups it lists, so we always know which X_D to consider. We sometimes analogously talk of the $C_G(t)$-orbits themselves powering to other suborbits.

Our final result in this section gives a useful method of finding new orbits when we encounter difficulty computing the sizes of the orbits comprising some X_C.

Remarks 5.6. *Let $z \in C$ for some class C of G with X_C non-empty. We recall the definition of the extended centralizer of z in G, $C_G^*(z) = \{g \in G \mid z^g \in \{z, z^{-1}\}\}$. Then*

(i) *$t, x \in C_G^*(z)$. In fact, since $[C_G^*(z) : C_G(z)] = 2$ and t and x both invert z, we must have $C_G^*(z) = \langle C_G(z), t\rangle = \langle C_G(z), x\rangle$;*

(ii) *Suppose $x \in X_C$. Then $tx \in C$ and so $(tx)^g = z$ for some $g \in G$. Then $t^g, x^g \in C_G^*(z) \cap X$ with $t^g x^g = z$. Similarly if we have $t_1, x_1 \in C_G^*(z) \cap X$ with $t_1 x_1 = z$, then $t_1^h = t$ for some $h \in G$ and $t_1^h x_1^h = t x_1^h = z^h$ so $x_1^h \in X_C$. Crucially, $|C_{C_G(t)}(x_1^h)| = |C_{C_G(z)}(x_1)|$ and this can be calculated inside $C_G^*(z)$.*

Suppose we can, for some $z \in C$, compute $C_G^*(z)$, and that this group is small enough to calculate in directly. Then instead of looking for representatives $x \in X_C$, we can shift our focus to $C_G^*(z)$ and look in this group for pairs (t_1, x_1) with $t_1, x_1 \in X$ and $t_1 x_1 = z$, and from these we can obtain elements in X_C, the sizes of whose $C_G(t)$-orbits are easily determined. To do this, we need to find an $h \in G$ with $t_1^h = t$. The following trick allows us to find such a conjugating element.

Remarks 5.7. *Let t_1, t_2 be involutions in a group G. Recall that any two involutions generate a dihedral group and that, where their product has odd order, the dihedral group contains a single conjugacy class of involutions. Hence Algorithm 7 will produce an element conjugating t_1 to t_2.*

Where we have $tx = z$, we can sometimes compute $C_G(z)$ using a randomised procedure based on Algorithm 1, explained in [7].

In the following sections we describe how the $C_G(t)$-orbits were found for various sporadic simple groups, beginning with Janko's group J_4.

Algorithm 7 InvolutionConjugator

Input: G a finite group;
t_1,t_2 conjugate involutions in G.

```
repeat
    g ← Random(G)
    r ← t_1^g
until |t_1 r| and |t_2 r| are odd
D_1 ← ⟨t_1, r⟩
D_2 ← ⟨t_2, r⟩
Find h_1 ∈ D_1 such that t_1^{h_1} = r.
Find h_2 ∈ D_2 such that r^{h_2} = t_2.
```

Output: $g = h_1 h_2$, an element conjugating t_1 to t_2.

5.2 Janko's group J_4

Janko was responsible for the discovery of four of the 26 sporadic simple groups. The first of these, J_1, was also the first of the 'modern' sporadics to be found, in 1965 (the five Mathieu groups had been known for over a century at this time). The fourth, J_4, was the last of the sporadics to be unearthed and was found in 1975 [17]. Janko established the order and local structure of the group, and in 1980 Norton proved its existence and uniqueness using machine calculations [19]. A computer-free construction followed in 1999.

We compute the suborbits for both involution conjugacy classes in Janko's sporadic simple group J_4. The representatives for the class $2B$ are used to determine the commuting involution graph on that class. The results in this section also appear in [22]. We begin by describing the computational setting.

5.2.1 Computation with J_4

The smallest available representation of J_4 in [29] is as 112×112 matrices over $GF(2)$. For the rest of this section G denotes J_4 in this representation. Computation with such large matrices is difficult, and performing any calculation that requires knowledge of the entire group (such as directly constructing centralizers or normalizers) will fail in anything but very small order subgroups, that is $H < G$ with $|H|$ around 10^6 or less.

We can however perform basic group arithmetic, take random elements, calculate orders and so on. We can also look at the action of elements on the associated 112-dimensional $GF(2)$-module, which we henceforth denote by V.

Given an element $g \in G$ we cannot in general determine which class of G contains g. However examining the order of g and $\dim(C_V(g))$ will usually specify the class, and if not will limit us to only a few possibilities. We deal with any ambiguities as they arise.

When $t \in 2A$, its centralizer $C_G(t)$ has order 21,799,895,040 and when $t \in 2B$, $|C_G(t)| = 1,816,657,920$. Magma can construct a base and strong generating set for these subgroups, so calculation within $C_G(t)$ is possible, albeit slow. However, computation involving elements outside of $C_G(t)$, such as finding $C_{C_G(t)}(x)$ or testing for $C_G(t)$-conjugacy between two elements of X, is undesirably slow given the large amount of such computation a randomised search requires. In the following section we outline some of the tricks employed to get around this issue.

5.2.2 Suborbit Invariants

Our general strategy for both classes $2A$ and $2B$ will be to take random conjugates x of t, searching for representative elements for the $C_G(t)$-orbits, which we distinguish by means of various suborbit invariants more readily computed than $C_{C_G(t)}(x)$. Of course there is no guarantee that two distinct $C_G(t)$-orbits will be distinguishable by any of the suborbit invariants that we consider, and indeed this is often the case for the class $2B$, so in Section 5.2.4 we introduce a technique that will in some circumstances allow us to test $C_G(t)$-conjugacy in a more direct fashion.

We employ all the general suborbit invariants discussed in the previous section as well the following ones which employ knowledge of the module V or some other specific property of the group at hand.

Lemma 5.8. *Let $x \in X$ and let $d_x = \dim(C_V(t) \cap C_V(x))$. Then $d_x = d_y$ for any $y \in x^{C_G(t)}$.*

Proof. Suppose $h \in C_G(t)$. Then $C_V(t) \cap C_V(x^h) = C_V(t^h) \cap C_V(x^h) = (C_V(t) \cap C_V(x))^h$, and so these spaces have the same dimension. □

As will be seen in the following sections, for both $t \in 2A$ and $t \in 2B$ we can construct a normal subgroup Q of $C_G(t)$ containing t, small enough to allow direct computation within it. In particular for $x \in X$ we may calculate $C_Q(x)$. Clearly $|C_Q(x)|$ is a $C_G(t)$-orbit invariant. The following results give further invariants that can be found employing Q, and a strategy that often allows us to calculate $C_{C_G(t)}(x)$ for $x \in X$, and hence the length of the orbit containing x.

Remarks 5.9. *For $x \in X$ let q_{2A}, q_{2B} be the number of $2A$-, respectively $2B$-elements in $C_Q(x)$. We observe that these values are further $C_G(t)$-orbit invariants.*

We note that elements in $2A$ and $2B$ are distinguished by the dimension of their fixed spaces so these invariants are easily calculated. The group Q can however be put to much more productive use, as the following results demonstrate.

Definition 5.10. *Let $Q \trianglelefteq C_G(t)$ as above. Set $\Omega = Q$ and define $\varphi : C_G(t) \to \mathrm{Sym}(\Omega)$ to be given by the conjugation action of $C_G(t)$ on Q. We define S to be the image of $C_G(t)$ under φ.*

Lemma 5.11. *For $x \in X$, let $S_x = \mathrm{Stab}_S(C_Q(x))$. Then*

(i) $|S_x|$ is a $C_G(t)$-orbit invariant, and

(ii) $C_{C_G(t)}(x)$ is contained in $\varphi^{-1}(S_x)$.

Proof. If $c \in C_G(t)$ then $C_Q(x^c) = C_Q(x)^c$ and so we see that $|S_x|$ will be the same for both, giving us (i). Now suppose $h \in C_{C_G(t)}(x)$. Since $C_Q(x) \leq C_{C_G(t)}(x)$ it must be centralized by h and so $\varphi(h) \in S_x$ as required. □

By examining the action on $\Omega = Q$ of the generators of $C_G(t)$ we can construct the homomorphism φ explicitly in Magma. Then since S is a permutation group of relatively small degree, computation in S is easy. So finding S_x (which is generally small) and computing $C_{C_G(t)}(x)$ in its inverse image gives us an invaluable method for finding the size of the suborbit while avoiding working in the large matrix group.

We now give details of how the $C_G(t)$-orbits in in the two involution conjugacy classes were determined, beginning with the smaller and easier case.

5.2.3 $X = 2A$

We may calculate in GAP the rank of the permutation action following Proposition 5.1, and on doing so we discover that X breaks into twenty $C_G(t)$-orbits.

For an involution $t \in 2A$, the group $C_G(t)$ has shape $2^{1+12}.3.M_{22} : 2$ and is maximal in G, hence generators for it can be found at [29]. From this it is easy to construct $Q = O_{2,3}(C_G(t)) \cong 2^{1+12}.3$: we simply take random $h \in C_G(t)$ until we find ones with order 21 or 33, and then h^7 or respectively h^{11} is in Q (see [18] for more details). The unique non-identity central element of Q is then our t.

Searching among random conjugates of t, we soon discover that the twenty $C_G(t)$ orbits of X can be distinguished using only the invariants $|C_Q(x)|$; q_{2A} and q_{2B}; and the class of $z = tx$ (that is, which set X_C contains the orbit). It only remains to determine the sizes of these orbits.

Of the 14 non-empty X_C, four contain more than one $C_G(t)$-orbits: X_{2A}, X_{2B}, X_{4B} and X_{4C}. We use the techniques described in Definition 5.10 and Lemma 5.11 to determine the sizes of the ten suborbits in these sets. Magma can construct the homomorphism φ explicitly, allowing us to take the inverse image of subgroups of S to get subgroups of $C_G(t)$.

Since $z = tx$ has even order $2m$ in all four of these cases, and since z is inverted by both t and x, we have that $z^m \in C_{C_G(t)}(x)$. We see also that $C_{C_G(t)}(x) \leq C_G(z) \leq C_G(z^m)$, and so $\varphi(C_{C_G(t)}(x) \leq C_S(\varphi(z^m))$. Combining this with Lemma 5.11, we compute $S_x \cap C_S(\varphi(z^m))$ for each representative x whose orbit size we wish to find, and take its inverse image. In all cases these subgroups are small enough that we can

Table 5.1: The $C_G(t)$-orbits for $G \cong J_4$, $t \in 2A$

C	Orbit size	$\|C_Q(x)\|$	q_{2A}	q_{2B}
$1A$	1	$2^{13}.3$	1387	2772
$2A$	$2^5.3^2.5.7.11$	2^8	107	84
	$2.3^2.7.11$	2^{12}	747	1364
$2B$	$2^7.3^2.5.11$	2^7	71	56
	$2^4.3.5.7.11$	$2^9.3$	139	180
$3A$	$2^{14}.3^2.5.11$	1	0	0
$4A$	$2^8.3^2.5.7.11$	2^6	33	30
$4B$	$2^{12}.3^3.5.7.11$	2^2	1	2
	$2^{11}.3^3.5.7.11$	2^2	3	0
	$2^{10}.3^2.5.7.11$	2^3	3	4
	$2^{10}.3^2.5.7.11$	2^4	9	6
$4C$	$2^{11}.3^2.5.7.11$	2^3	3	4
	$2^{11}.3^2.5.7.11$	2^3	7	0
$5A$	$2^{15}.3^2.5.7.11$	1	0	0
$6B$	$2^{15}.3^3.5.7.11$	1	0	0
$6C$	$2^{14}.3^2.5.7.11$	1	0	0
$8C$	$2^{15}.3^3.5.7.11$	1	0	0
$10A$	$2^{16}.3^3.5.7.11$	1	0	0
$11B$	$2^{20}.3^3.5.7$	1	0	0
$12B$	$2^{17}.3^2.5.7.11$	1	0	0

then directly compute $C_{C_G(t)}(x)$ and thus find the size of the orbit containing x.

These results are summarised in Table 5.1. In the table, each row corresponds to a $C_G(t)$-orbit, which are grouped by virtue of which sets X_C they lie in. For each suborbit we give its size and the values of the invariants used to distinguish them. We now look at the larger class of involutions $2B$.

5.2.4 $X = 2B$

For $X = 2B$, we see from the ATLAS that $C_G(t)$ has shape $2^{11}.M_{22} : 2$. This group is contained in a maximal subgroup M of shape $2^{11}.M_{24}$, whose generators are available from [29]. Given M it is simple to obtain generators for $Q = O_2(C_G(t)) \cong 2^{11}$ (a randomly chosen $h \in M$ which has even order $2m \geq 16$ must have $h^m \in Q$ by the orders of elements of M_{24}). With Q found we choose and fix $t \in Q \cap 2B$, and now generators for $C_G(t)$ are easily found using Algorithm 1. (Note that termination issues do not arise here as $|C_G(t)|$ is known.)

Following Proposition 5.1 we find using GAP that the number of $C_G(t)$-orbits is 119, and 47 of the sets X_C are non-empty, so clearly this class poses a much more significant challenge. We begin by defining the following additional $C_G(t)$-orbit invariant.

Let φ, S be as described in Definition 5.10 and Lemma 5.11. From the above we see that $S \cong M_{22} : 2$. Suppose $x \in X$ with $z = tx$ having even order $2m$. Then $z^m \in C_G(t)$. So we can ask where $w = \varphi(z^m)$ is in S. Clearly w is either trivial (if $z^m \in Q$) or is an involution in S. Examining [11]the ATLAS we see that $S \cong M_{22} : 2$ has three conjugacy classes of involutions. So w is in (using bars to distinguish classes of S from those of G) $\overline{1A}, \overline{2A}, \overline{2B}$ or $\overline{2C}$, and this is a further invariant of a $C_G(t)$-orbit.

Sometimes we encounter $C_G(t)$-orbits that agree on all the invariants we consider, so in order to conclude that they are are indeed different orbits, we must find a way of directly determining $C_G(t)$-conjugacy between their representatives. Suppose $x_1, x_2 \in X$ are representatives we suspect lie in different $C_G(t)$-orbits. We aim to form a subset $Y \subset C_G(t)$ such that any element conjugating x_1 to x_2 must lie in Y, and with $|Y|$ sufficiently small that simply testing every $y \in Y$ is feasible. We employ the following easy result.

Lemma 5.12. *Let $x_1, x_2 \in X$. Set $H_i = \varphi(C_{C_G(t)}(x_i))$ (for $i = 1, 2$). If H_1, H_2 are not S-conjugate then x_1, x_2 are not $C_G(t)$-conjugate. Suppose they are conjugate by an element $s \in S$. Then for any $h \in C_G(t)$ with $x_1^h = x_2$, we must have $\varphi(h) \in N_S(H_1)s$.*

A further improvement can be made. Recall that $S \subseteq \mathrm{Sym}(\Omega)$ with $\Omega = Q$, and the group action corresponding to conjugation on Q. Since any $h \in C_G(t)$ conjugating x_1 to x_2 must conjugate $C_Q(x_1)$ to $C_Q(x_2)$, if these groups are non-trivial we need only consider those elements on $N_S(H_1)s$ that map $C_Q(x_1)$ to $C_Q(x_2)$ (considered as subsets of Ω). Call the subset of $N_S(H_1)s$ where this happens Z. Then we take $\varphi^{-1}(Z)$ as our set Y. This routine constitutes Algorithm 8. We can successfully employ this algorithm wherever we have two representatives x_1 and x_2 and have already found $C_{C_G(t)}(x_i)$ $(i = 1, 2)$, so long as either the size of $N_S(H_1)$ or the added restriction where $C_Q(x_1)$ is non-trivial result in the set Y being small enough for an element-by-element check to be feasible.

Algorithm 8 inSameOrbit

Input: $x_1, x_2 \in X$;
$C_{C_G(t)}(x_1), C_{C_G(t)}(x_2)$;
S, Q, φ as defined above.

$H_1 \leftarrow \varphi(C_{C_G(t)}(x_1))$
$H_2 \leftarrow \varphi(C_{C_G(t)}(x_2))$
if H_1, H_2 are not S-conjugate **then**
    sameorbit $\leftarrow$ `false`
else
    Let $g \in S$ be such that $H_1^g = H_2$
    $N \leftarrow N_S(H_1)$
    $Z \leftarrow \{h \in Ng \mid h \text{ maps } C_Q(x_1) \text{ to } C_Q(x_2)\}$
    $Y \leftarrow \varphi^{-1}(Z)$
    sameorbit $\leftarrow$ `false`
    for $y \in Y$ **do**
        for $q \in Q$ **do**
            if $x_1^{qy} = x_2$ **then**
                sameorbit $\leftarrow$ `true`
            end if
        end for
    end for
end if

Output: sameorbit, having value `true` if and only if x_1, x_2 are $C_G(t)$-conjugate.

Our strategy is to attempt to break each of the non-empty X_C into its constituent $C_G(t)$-orbits. We begin with classes C of elements of small order and work up. This has the advantage that when considering a class X_C, we have already catalogued the orbits in X_D for each class D powered to from C, so given a representative $x \in X_C$ we can use the suborbit of $x^{(n)}$ for n dividing the order of elements in C as a further suborbit invariant for x. (In practice we generally consider $x^{(2)} = tz^2 = t^x$.)

We now briefly describe how each of the non-empty sets X_C can be broken down into $C_G(t)$-orbits. Discussion is omitted for classes C where $C_G(z) = \langle z \rangle$ for $z \in C$, where calculations are trivial.

X_{2A}, X_{2B} When $z = tx$ has order two we have $x \in C_G(t) \cap X$, and t and x generate a Klein-four group in $C_G(t)$. A list of such groups can be found in [18], which gives us the suborbits in X_{2A} and X_{2B}.

X_{3A} Representatives $x \in X_{3A}$ can be found with $|C_Q(x)| = 1$ and 4. When $|C_Q(x)| = 4$ it is easy to find $C_{C_G(t)}(x)$ by first calculating $\varphi^{-1}(S_x)$ (see Lemma 5.11). Taking an $x \in X_{3A}$ with $C_Q(x)$ trivial, we can find $C_{C_G(t)}(x)$ by first computing $C_G(z)$ using the method explained in [7]. Since this group has order 2,661,120 (and contains $C_{C_G(t)}(x)$), the calculation is then trivial. We see that the two orbit lengths now known total $|X_{3A}|$ and so we are done.

X_{4A}, X_{4B} Elements in $4A$ and $4B$ cannot be distinguished by the dimensions of their centralizers in V so we must for the moment deal with the two sets together. Random searching gives a list of 13 suborbit representatives with different invariants (including the size of the orbit which we can calculate in $\varphi^{-1}(S_x)$). However from the orbit sizes we know that this list does not include all the $C_G(t)$-orbits in $X_{4A} \cup X_{4B}$. We deal with this problem momentarily. Since X_{4A} is the smaller we aim to determine which of our orbits comprise it first.

We assume for the moment that our list of 13 suborbits contains all the orbits in X_{4A}. From the power maps in [29] we see that for elements $g \in G$ of order $4m$ with $m \in \{5, 10, 11\}$, we have $g^m \in 4A$. So we search for $x \in X$ with tx having such an order. Then $x^{(m)} \in X_{4A}$ (see Lemma 5.5). Matching the invariants of these elements against our list of 13 suborbit representatives we find two that we can conclude are certainly in X_{4A}. Now, we know $|X_{4A}|$, and it is not divisible by 5, so it must contain at leat one orbit of size not divisible by 5. Only one of our thirteen orbits has this property (and it is not one of the two already placed in X_{4A}) so it is a third orbit in X_{4A}. Subtracting the sizes of these three suborbits from $|X_{4A}|$ gives $2^6.3^2.5.7.11$, which is the size of the smallest orbit for which we have a representative, so we conclude that this orbit too is in X_{4A}.

The sizes of the $C_G(t)$-orbits we now have lying in X_{4B} do not total $|X_{4B}|$. This suggests that two orbits in X_{4B} have the same size and the same invariants and so our search does not distinguish between them. We use the strategy outlined in Lemma 5.12 to locate these orbits. Locating this fourteenth orbit also verifies the assumption we made when placing suborbits in X_{4A}.

X_{4C} All six suborbits in X_{4C} are easily distinguished by their invariants and sizes (and in every case $\varphi^{-1}(S_x)$ for a representative element x is small enough to allow direct computation of $C_{C_G(t)}(x)$ in it).

X_{5A} A representative $x \in X_{5A}$ can be found with $|C_Q(x)| = 2^3$, allowing its size to easily be determined. However the sizes of any other orbits in X_{5A} are not so quickly found. Set $z = tx$. It is clear from its order that $\langle t, z, C_{C_G(t)}(x)\rangle = C_G^*(z)$. Now we employ the trick in Remark 5.6 to locate a second orbit of different size, and obtain a representative element using Lemma 5.7. These two orbits together comprise X_{5A}.

X_{6A}, X_{6B}, X_{6C} All three classes of order 6 elements can be distinguished by the dimensions of their fixed spaces. Further, all of their suborbits are distinguished by their invariants and sizes (which can be determined by the methods from Definition 5.10 and Lemma 5.11).

X_{8A}, X_{8B}, X_{8C} Elements g from all three classes of order 8 have the same value of $\dim C_V(g)$, so we face some difficulty determining which of these sets a suborbit representative lies in. We do know from the power maps that the square of an $8A$-element is in $4A$ while the squares of $8B$- and $8C$-elements are in $4B$. So if we have $x \in X_{8A}$ then $x^{(2)} \in X_{4A}$ while if $x \in X_{8B} \cup X_{8C}$ then $x^{(2)} \in X_{4B}$. Since we have determined the suborbits in X_{4A} and X_{4B} and can easily distinguish the ones in X_{4A}, this gives us a way of determining which orbits lie in X_{8A}.

The 'powering down' of orbits in Remark 5.5 also gives us a way of determining the orbits in X_{8B}. We search for representatives $y \in X$ with ty having order 24. Then $y^{(3)} \in X_{8B}$. Doing this we find two suborbits which together comprise X_{8B}.

Any remaining suborbit is now in X_{8C} an random searching soon yields representatives for the four suborbits that comprise it. (We note that two of these orbits can only be distinguished by considering which orbit of X_{4B} the element $x^{(2)}$ lies in.)

X_{10A} We begin by taking the group $C^*_G(z_1)$ for $z_1 \in 5A$ found in our calculations for X_{5A}. Searching in this group we find a $z \in 10A$ such that $z^2 = z_1$. We then compute $C^*_G(z) \leq C^*_G(z_1)$. As with X_{5A}, we now find conjugates of our $C_G(t)$-orbits following Remark 5.6, and having determined that X_{10A} breaks into two suborbits, obtain the correct representatives using Lemma 5.7.

X_{10B} Random search gives a representative $x \in X_{10B}$ with $|C_Q(x)| = 2^2$ allowing us to find $C_{C_G(t)}(x)$ using Lemma 5.11. Now we again use the groups $C_G(z_1), C^*_G(z_1)$, $z_1 \in 5A$ from above. We choose $t_1 \in C^*_G(z_1) \cap X$ such that t_1 inverts z_1, and observe that for any $s \in X \cap C_G(z_1) \cap C_G(t_1)$, $z_1 s \in 10B$. So we search for such s with $t_1 z_1 s \in X$, so that similarly to Remark 5.6 $t_1 z_1 s$ can be conjugated to a representative $x \in X_{10B}$. Doing this yields three representatives all of which have $C_Q(x)$ trivial.

X_{11A} Suppose $x \in X_{11A}$. Set $z = tx \in 11A$. From [18] we know that $N_G(z)$ is a maximal subgroup of G, and so we can obtain a G-conjugate of it from [29]. Call this group N_1. Then we may find $t', x' \in X \cap N_1$ (by random search) so that $z' = t'x' \in 11A$ with $\langle z' \rangle \trianglelefteq N_1$. Computation reveals that $|C_{C_G(z')}(x')| = 22$ and since $|X_{11A}| = |C_G(t)|/22$, we conclude that X_{11A} is a single $C_G(t)$-orbit.

X_{11B} With N_1 and t' as above we can find an element $x'' \in 2B \cap N_1$ with $z'' = t'x'' \in 11B$. We find $C_{N_1}(z'')$ and discover by its order that it is equal to $C_G(z'')$. We find that $C_{C_G(z'')}(t') = 2$ and then by the size of X_{11B} conclude that X_{11B} is a single $C_G(t)$-orbit.

X_{12A}, X_{12B} We cannot distinguish between $12A$- and $12B$-elements by the dimensions of their fixed spaces, but we can find an $x \in X_{12B}$ by taking $x = y^{(2)}$ for $y \in X_{24A}$. Suppose we have $x \in X_{12A} \cup X_{12B}$. Set $z = tx$. Then we can find $C_G(z)$

by first computing $C_G(z^4)$ ($z^4 \in 3A$) as before. Now $C_G^*(z) = \langle C_G(z), t\rangle$ and $|C_G^*(z)| = 384$, so it is easy to check every element of $C_G^*(z)$, and so we form the following subset:

$$\mathcal{I} = \{w \mid w \in 2B, wz \in 2B, w \in C_G^*(z) \setminus C_G(z)\}.$$

We discover that for our known $12B$-element, $|\mathcal{I}| = 84$ while for other z it is 72, so we can now distinguish between $12A$- and $12B$-elements. Random searching now delivers the suborbits, though we must resort to using Algorithm 8 to distinguish some.

X_{12C} Elements in $12C$ are determined by the dimension of their fixed space, and $|C_{C_G(t)}(x)|$ for representatives x are easily calculated by working in either $C_G(z^4)$ (found as with X_{3A} since $z^4 \in 3A$) or in $\varphi^{-1}(C_S(\varphi(z^6)))$. We find that X_{12C} breaks into six $C_G(t)$-orbits, though we need to employ the method of Lemma 5.12 to tell several of these suborbits apart.

X_{20A}, X_{20B} We again begin with our group $C_G(z_1)$ with $z \in 5A$. We find an $f \in C_G(z_1)$ with order 4 and take $z = fz_1$ to be our $20A$-element. (Note from the Atlas that $20A$ and $20B$ are 'master' and 'slave' classes forming an algebraically conjugate family of classes with $20A$-elements cubing to $20B$-elements, and so it is arbitrary which class of order-20 elements we denote $20A$ and which $20B$). Now we find $C_G^*(z) = C_{C_G^*(z_1)}(z)$ and work here employing Remark 5.6. Doing so we see that X_{20A} consists of two suborbits, distinguished by their size. Clearly X_{20B} will have the same suborbit structure and to find representatives we take z^3 as our $20B$-element and repeat the above procedure.

X_{40A}, X_{40B} As with $20A$, $20B$, classes $40A$ and $40B$ are identical so our choice of naming is arbitrary (except that to line up with the power maps in the Atlas we might wish a $40B$-element to cube to a $20A$-element). Working again in $C_G^*(z_1)$ we quickly discover that X_{40A} and X_{40B} each split into two $C_G(t)$-orbits.

In Tables 5.2 to 5.9 we collate the data on the $C_G(t)$-orbits. Again each row corresponds to one suborbit, and the suborbits are grouped according to the sets X_C.

Table 5.2: Suborbits in class $2B$ of $G \cong J_4$ (i)

C	Orbit size	Class of $\overline{w}$	$\lvert C_Q(x)\rvert$	q_{2A}	q_{2B}	d_x
$1A$	1	$\overline{1A}$	2^{11}	1771	276	56
$2A$	$2^6.3^2.7.11$	$\overline{2C}$	2^6	51	12	31
	$2^4.3.5.7.11$	$\overline{2A}$	2^7	91	36	32
	$3.7.11$	$\overline{1A}$	2^{11}	1771	276	36
$2B$	$2^5.3^2.5.7.11$	$\overline{2A}$	2^7	91	36	28
	$2^5.3.5.7.11$	$\overline{2B}$	2^7	91	36	28
	$2^2.11$	$\overline{1A}$	2^{11}	1771	276	36
$3A$	$2^{13}.3^2.7.11$	-	1	0	0	20
	$2^{12}.3.5.11$	-	2^3	7	0	20
$5A$	$2^{15}.3^2.5.7.11$	-	1	0	0	12
	$2^{13}.3.5.11$	-	2^3	7	0	12
$11A$	$2^{18}.3^2.5.7$	-	1	0	0	1
$11B$	$2^{18}.3^2.5.7.11$	-	1	0	0	6
$23A$	$2^{19}.3^2.5.7.11$	-	1	0	0	1
$29A$	$2^{19}.3^2.5.7.11$	-	1	0	0	0
$31A$	$2^{19}.3^2.5.7.11$	-	1	0	0	1
$31B$	$2^{19}.3^2.5.7.11$	-	1	0	0	1
$31C$	$2^{19}.3^2.5.7.11$	-	1	0	0	1
$37A$	$2^{19}.3^2.5.7.11$	-	1	0	0	2
$37B$	$2^{19}.3^2.5.7.11$	-	1	0	0	2
$37C$	$2^{19}.3^2.5.7.11$	-	1	0	0	2
$43A$	$2^{19}.3^2.5.7.11$	-	1	0	0	0
$43B$	$2^{19}.3^2.5.7.11$	-	1	0	0	0
$43C$	$2^{19}.3^2.5.7.11$	-	1	0	0	0

Each table gives the suborbits in certain of the X_C. A column headed $x^{(n)}$ gives the suborbit that the orbit powers to, with an entry of the form C_i, representing the i^{th} listed suborbit in X_C.

5.2.5 The Commuting Involution Graph on Class $2B$

Having found our representatives for the $C_G(t)$-orbits on $X = 2B$, we wish to determine which disc of the commuting involution graph $\mathcal{C}(G, X)$ each lies in, and thus discover the diameter and disc sizes of the graph. In Lemma 2.2 of [2] some results are given that allow us to determine the disc locations of may of the sets X_C from the power maps in the Atlas. We summarise these here.

Table 5.3: Suborbits in class $2B$ of $G \cong J_4$ (ii)

C	Orbit size	Class of $\overline{w}$	$\|C_Q(x)\|$	q_{2A}	q_{2B}	$\|S_x\|$	d_x
$4A$	$2^{12}.3^2.7.11$	$\overline{2C}$	2	1	0	3840	16
	$2^{10}.3.5.7.11$	$\overline{2A}$	2^3	7	0	768	17
	$2^6.3^2.5.7.11$	$\overline{1A}$	2^7	91	36	192	20
	$2^7.3^2.7.11$	$\overline{1A}$	2^6	51	12	320	20
$4B$	$2^{12}.3^2.5.7.11$	$\overline{2C}$	2	1	0	3840	16
	$2^{10}.3^2.5.7.11$	$\overline{2A}$	2^2	3	0	768	16
	$2^{10}.3^2.5.7.11$	$\overline{2A}$	2^2	3	0	128	17
	$2^{10}.3^2.5.7.11$	$\overline{2A}$	2^2	3	0	128	17
	$2^{10}.3^2.5.7.11$	$\overline{2A}$	2^2	3	0	128	16
	$2^9.3^2.5.7.11$	$\overline{2A}$	2^2	3	0	768	18
	$2^8.3^2.5.7.11$	$\overline{1A}$	2^6	51	12	32	19
	$2^8.3^2.5.7.11$	$\overline{2A}$	2^3	7	0	768	19
	$2^8.3.5.7.11$	$\overline{1A}$	2^7	91	36	192	18
	$2^8.3.5.7.11$	$\overline{2A}$	2^3	7	0	768	18
$4C$	$2^{12}.3^2.5.7.11$	$\overline{2A}$	2^2	3	0	128	14
	$2^{12}.3^2.5.7.11$	$\overline{2B}$	2^2	3	0	128	14
	$2^{11}.3^2.5.7.11$	$\overline{2A}$	2^2	3	0	128	14
	$2^{11}.3^2.5.7.11$	$\overline{2B}$	2^2	3	0	768	14
	$2^9.3.5.7.11$	$\overline{1A}$	2^7	91	36	168	18
	$2^9.3.5.7.11$	$\overline{2B}$	2^3	7	0	768	18

Table 5.4: Suborbits in class $2B$ of $G \cong J_4$ (iii)

C	Orbit size	Class of $\overline{w}$	$\|C_Q(x)\|$	q_{2A}	q_{2B}	d_x	$x^{(2)}$
$8A$	$2^{15}.3^2.5.7.11$	$\overline{2A}$	2	1	0	9	$4A_2$
	$2^{15}.3^2.5.7.11$	$\overline{2C}$	1	0	0	9	$4A_1$
	$2^{13}.3^2.5.7.11$	$\overline{1A}$	2	1	0	10	$4A_4$
	$2^{13}.3^2.5.7.11$	$\overline{1A}$	2^2	3	0	10	$4A_3$
	$2^{13}.3^2.5.7.11$	$\overline{2C}$	1	0	0	10	$4A_1$
	$2^{13}.3^2.5.7.11$	$\overline{2C}$	1	0	0	10	$4A_1$
	$2^{13}.3^2.7.11$	$\overline{1A}$	2	1	0	10	$4A_4$
	$2^{13}.3^2.7.11$	$\overline{2C}$	1	0	0	10	$4A_1$
$8B$	$2^{15}.3^2.5.7.11$	$\overline{2A}$	1	0	0	8	$4B_5$
	$2^{15}.3^2.5.7.11$	$\overline{2C}$	1	0	0	8	$4B_1$
$8C$	$2^{16}.3^2.5.7.11$	$\overline{2C}$	1	0	0	8	$4B_1$
	$2^{14}.3^2.5.7.11$	$\overline{2A}$	1	0	0	9	$4B_2$
	$2^{14}.3^2.5.7.11$	$\overline{2A}$	1	0	0	9	$4B_6$
	$2^{13}.3^2.5.7.11$	$\overline{2A}$	1	0	0	10	$4B_6$

Table 5.5: Suborbits in class $2B$ of $G \cong J_4$ (iv)

C	Orbit size	Class of $\overline{w}$	$\lvert C_Q(x)\rvert$	q_{2A}	q_{2B}	d_x	$x^{(2)}$
$6A$	$2^{13}.3^2.7.11$	$\overline{2C}$	1	0	0	10	$3A_1$
$6B$	$2^{15}.3^2.5.7.11$	$\overline{2C}$	1	0	0	11	$3A_1$
	$2^{13}.3^2.5.7.11$	$\overline{2A}$	1	0	0	12	$3A_1$
	$2^{13}.3^2.5.7.11$	$\overline{2A}$	2^2	3	0	12	$3A_2$
	$2^{13}.3^2.5.7.11$	$\overline{2C}$	2^2	3	0	11	$3A_2$
	$2^{13}.3^2.5.7.11$	$\overline{2C}$	1	0	0	11	$3A_1$
$6C$	$2^{15}.3^2.5.7.11$	$\overline{2A}$	1	0	0	10	$3A_1$
	$2^{13}.3^2.5.7.11$	$\overline{2A}$	2^2	3	0	10	$3A_2$
	$2^{13}.3^2.5.7.11$	$\overline{2A}$	1	0	0	10	$3A_1$
	$2^{13}.3^2.5.7.11$	$\overline{2B}$	1	0	0	10	$3A_1$
	$2^{12}.3.5.7.11$	$\overline{2B}$	2^3	7	0	10	$3A_2$

Table 5.6: Suborbits in class $2B$ of $G \cong J_4$ (v)

C	Orbit size	Class of $\overline{w}$	$\lvert C_Q(x)\rvert$	q_{2A}	q_{2B}	d_x	$x^{(2)}$
$12A$	$2^{16}.3^2.5.7.11$	$\overline{2C}$	1	0	0	6	$6B_1$
	$2^{16}.3^2.5.7.11$	$\overline{2C}$	1	0	0	6	$6B_1$
	$2^{15}.3^2.5.7.11$	$\overline{2A}$	1	0	0	7	$6B_2$
	$2^{15}.3^2.5.7.11$	$\overline{2A}$	1	0	0	7	$6B_3$
$12B$	$2^{16}.3^2.5.7.11$	$\overline{2C}$	1	0	0	6	$6B_1$
	$2^{16}.3^2.5.7.11$	$\overline{2C}$	1	0	0	6	$6B_1$
	$2^{15}.3^2.5.7.11$	$\overline{2A}$	1	0	0	7	$6B_3$
	$2^{15}.3^2.5.7.11$	$\overline{2A}$	1	0	0	7	$6B_2$
	$2^{15}.3^2.5.7.11$	$\overline{2A}$	1	0	0	6	$6B_3$
$12C$	$2^{16}.3^2.5.7.11$	$\overline{2A}$	1	0	0	5	$6C_1$
	$2^{16}.3^2.5.7.11$	$\overline{2A}$	1	0	0	5	$6C_1$
	$2^{16}.3^2.5.7.11$	$\overline{2A}$	1	0	0	5	$6C_1$
	$2^{16}.3^2.5.7.11$	$\overline{2B}$	1	0	0	5	$6C_4$
	$2^{16}.3^2.5.7.11$	$\overline{2B}$	1	0	0	5	$6C_4$
	$2^{16}.3^2.5.7.11$	$\overline{2B}$	2	1	0	5	$6C_3$

Table 5.7: Suborbits in class $2B$ of $G \cong J_4$ (vi)

C	Orbit size	Class of $\overline{w}$	$\|C_Q(x)\|$	q_{2A}	q_{2B}	$\|S_x\|$	d_x	$x^{(2)}$
$10A$	$2^{16}.3^2.5.7.11$	$\overline{2C}$	1	0	0	887040	7	$5A_1$
	$2^{15}.3^2.5.7.11$	$\overline{2A}$	1	0	0	887040	8	$5A_1$
$10B$	$2^{16}.3^2.5.7.11$	$\overline{2A}$	1	0	0	887040	6	$5A_1$
	$2^{16}.3^2.5.7.11$	$\overline{2A}$	1	0	0	887040	6	$5A_1$
	$2^{15}.3^2.5.7.11$	$\overline{2A}$	1	0	0	887040	6	$5A_1$
	$2^{15}.3^2.5.7.11$	$\overline{2A}$	4	3	0	48	6	$5A_2$
$20A$	$2^{17}.3^2.5.7.11$	$\overline{2C}$	1	0	0	887040	4	$10A_1$
	$2^{16}.3^2.5.7.11$	$\overline{2A}$	1	0	0	887040	5	$10A_2$
$20B$	$2^{17}.3^2.5.7.11$	$\overline{2C}$	1	0	0	887040	4	$10A_1$
	$2^{16}.3^2.5.7.11$	$\overline{2A}$	1	0	0	887040	5	$10A_2$
$40A$	$2^{18}.3^2.5.7.11$	$\overline{2C}$	1	0	0	887040	3	$20B_1$
	$2^{18}.3^2.5.7.11$	$\overline{2A}$	1	0	0	887040	3	$20B_2$
$40B$	$2^{18}.3^2.5.7.11$	$\overline{2C}$	1	0	0	887040	3	$20A_1$
	$2^{18}.3^2.5.7.11$	$\overline{2A}$	1	0	0	887040	3	$20A_2$

Table 5.8: Suborbits in class $2B$ of $G \cong J_4$ (vii)

C	Orbit size	Class of $\overline{w}$	$\|C_Q(x)\|$	q_{2A}	q_{2B}	$\|S_x\|$	d_x	$x^{(2)}$
$22A$	$2^{18}.3^2.5.7.11$	$\overline{2C}$	1	0	0	887040	1	$11A$
$33A$	$2^{18}.3^2.5.7.11$	-	1	0	0	887040	0	-
	$2^{18}.3^2.5.7.11$	-	1	0	0	887040	0	-
$33B$	$2^{18}.3^2.5.7.11$	-	1	0	0	887040	0	-
	$2^{18}.3^2.5.7.11$	-	1	0	0	887040	0	-
$44A$	$2^{18}.3^2.5.7.11$	$\overline{2C}$	1	0	0	887040	1	$22A$
	$2^{18}.3^2.5.7.11$	$\overline{2C}$	1	0	0	887040	1	$22A$
$66A$	$2^{18}.3^2.5.7.11$	$\overline{2C}$	1	0	0	887040	0	$33B_1$
	$2^{18}.3^2.5.7.11$	$\overline{2C}$	1	0	0	887040	0	$33B_2$
$66B$	$2^{18}.3^2.5.7.11$	$\overline{2C}$	1	0	0	887040	0	$33A_1$
	$2^{18}.3^2.5.7.11$	$\overline{2C}$	1	0	0	887040	0	$33A_2$

Table 5.9: Suborbits in class $2B$ of $G \cong J_4$ (viii)

C	Orbit size	Class of $\overline{w}$	$\|C_Q(x)\|$	q_{2A}	q_{2B}	$\|S_x\|$	d_x	$x^{(2)}$
$15A$	$2^{18}.3^2.5.7.11$	-	1	0	0	887040	4	-
	$2^{18}.3^2.5.7.11$	-	1	0	0	887040	4	-
$16A$	$2^{17}.3^2.5.7.11$	$\overline{2C}$	1	0	0	887040	5	$8C_1$
	$2^{17}.3^2.5.7.11$	$\overline{2A}$	1	0	0	887040	5	$8C_4$
$24A$	$2^{17}.3^2.5.7.11$	$\overline{2A}$	1	0	0	887040	3	$x2B_5$
	$2^{17}.3^2.5.7.11$	$\overline{2C}$	1	0	0	887040	3	$12B_1$
$24B$	$2^{17}.3^2.5.7.11$	$\overline{2A}$	1	0	0	887040	3	$12B_5$
	$2^{17}.3^2.5.7.11$	$\overline{2C}$	1	0	0	887040	3	$12B_1$
$30A$	$2^{18}.3^2.5.7.11$	$\overline{2C}$	1	0	0	887040	2	$15A_1$
	$2^{18}.3^2.5.7.11$	$\overline{2C}$	1	0	0	887040	2	$15A_2$

Proposition 5.13. *Let G be a group and $t \in G$ an involution. Set $X = t^G$ and let $\Delta_i(t)$ denote the i^{th} disc around t in the commuting involution graph $\mathcal{C}(G, X)$. For C a class of elements of G with order m such that X_C is non-empty we have*

(i) $X_C \subseteq \Delta_1(t)$ if and only if $m = 2$, that is if C is a class of involutions;

(ii) If m is even ($m \geq 4$) and the the $(m/2)^{th}$ powers of elements of C are in X then $X_C \subseteq \Delta_2(t)$;

(iii) If m is odd and no class D of elements of order $2m$ exists in G such that elements of D have their m^{th} powers in X, then no $C_G(t)$-orbits in X_C lie in $\Delta_1(t)$ or $\Delta_2(t)$.

We set $G = J_4$, $t \in X = 2A$ as in the previous section. Proposition 5.13(i) gives us the obvious fact that $\Delta_1(t) = X_{2A} \cup X_{2B}$. From (ii) we can deduce that $X_{4C} \cup X_{6C} \cup X_{10B} \cup X_{12C} \subseteq \Delta_2(t)$, while (iii) gives us that all X_C with C a class of elements of odd order greater than 10 have all their suborbits at distance 3 or greater from t.

As also mentioned in Lemma 2.2 of [2], for any representative $x \in X$, if $C_{C_G(t)}(x) \cap X \neq \emptyset$ then $d(t, x) \leq 2$. Recall we have the invariant q_{2B} giving the size of $C_Q(x) \cap X \subseteq C_{C_G(t)}(x) \cap X$, so we can use this to quickly place some further orbits. For the X_C

not covered above, this gives us that two orbits in X_{4A} and two orbits in X_{4B} are in $\Delta_2(t)$.

For the remaining orbits we must check more directly. For each suborbit representative x we compute $C_{C_G(t)}(x)$ and check whether it contains any $2B$-elements. This gives us all the suborbits in $\Delta_2(t)$. It remains to check whether the other orbits all lie in $\Delta_3(t)$ or whether the graph has diameter 4 or more.

For each representative x with $d(t,x) \geq 3$, we repeatedly take random elements y of $C_G(x)$ using Algorithm 1 until we get a $2B$-element of order 3, 4, 5, 6, 8, 10, 12 or 16. From the above we know that $d(t,y) = 2$ and hence that $d(t,x) = 3$. Luckily, this procedure is successful for every remaining orbit representative. This completes our study of the graph. The results are summarised in the following theorem, which concludes our study of J_4.

Theorem 5.14. *Let $G = J_4$, $t \in X = 2B$ and let $\mathcal{C}(G,X)$ be the commuting involution graph on X. The diameter of $\mathcal{C}(G,X)$ is 3 and its disc structure is as follows:*

- $\Delta_0(t)$ *has size 1 and consists of* $X_{1A} = \{t\}$.
- $\Delta_1(t)$ *has size 173,987 and consists of* $X_{2A} \cup X_{2B}$.
- $\Delta_2(t)$ *has size 9,988,198,176 and consists of the sets* X_{3A}, X_{4A}, X_{4B}, X_{4C}, X_{5A}, X_{6A}, X_{6B}, X_{6C}, X_{8A}, X_{8B}, X_{8C}, X_{10A}, X_{10B}, X_{11B}, X_{12A}, X_{12B}, X_{12C}, X_{16A}, X_{24A}, X_{24B}, *and of the smaller of the two orbits in* X_{20A} *and in* X_{20B}.
- $\Delta_3(t)$ *has size 37,778,227,200 and consists of the sets* X_{11A}, X_{15A}, X_{22A}, X_{23A}, X_{29A}, X_{30A}, X_{31A}, X_{31B}, X_{31C}, X_{33A}, X_{33B}, X_{37A}, X_{37B}, X_{37C}, X_{40A}, X_{40B}, X_{43A}, X_{43B}, X_{43C}, X_{44A}, X_{66A}, X_{66B}, *and of the larger of the two orbits in* X_{20A} *and in* X_{20B}.

5.3 The Fischer groups

The sporadic simple groups Fi_{22}, Fi_{23} and $Fi_{24}{}'$ arose from Fischer's classification of the *3-transposition groups*, which are defined as follows.

Definition 5.15. *A 3-transposition group is a finite group H with a conjugacy class Y (the 3-transpositions, often shortened to just 'transpositions') such that*

(i) Y is a class of involutions;

(ii) $H = \langle Y \rangle$; and

(iii) for any distinct $x, y \in Y$, either x and y commute, or the order of xy is 3.

The most obvious examples of 3-transposition groups are the symmetric groups S_n, where the class of 3-transpositions is the normal class of transpositions $(12)^{S_n}$. Fischer classified all 3-transposition groups (subject to certain technical conditions) and the three exceptional groups in this classification were the Fischer groups Fi_{22}, Fi_{23} and Fi_{24}—originally called $M(22)$, $M(23)$ and $M(24)$ to emphasise their relation to the Mathieu groups. The first two of these also gave new sporadic simple groups, and while Fi_{24} itself is non-simple, its derived subgroup $Fi_{24}{}'$ gave a third sporadic.

These groups are 3-transposition groups with respect to their smallest involution conjugacy classes: in Atlas notation $2A$ for Fi_{22} and Fi_{23}, and $2C$ for Fi_{24} since the Atlas lists the classes in the simple group $Fi_{24}{}'$ first.

Elements in the remaining involution conjugacy classes of the Fischer groups can be expressed as products of pairwise commuting elements, so that for example in Fi_{22}, class $2B$ consists of products of two commuting transpositions and is often called the class of *bi-transpositions*, and similarly $2C$ consists of products of three pairwise commuting transpositions, called *tri-transpositions* (note this is not the same as 3-transpositions).

The Fischer 3-transposition groups were originally found by study of their commuting involution graphs on the class of 3-transpositions, so this class is already well-understood (see Section 5.7 of [28] for details) and it is easy to see that each such class consists of three suborbits equal to the sets $X_{1A} = \{t\}$, X_{2B} and X_{3A}. However, the other classes are not so transparent. We calculate the $C_G(t)$-orbits, and from these the disc sizes for the two largest classes $2B$, $2D$ in Fi_{24}. For the other class, $2A$, and for the groups Fi_{22} and its automorphism group $Fi_{22}:2$, and for Fi_{23} (where the disc sizes are already known), we deduce the involution suborbits and find representative elements for them. We also look briefly at the group $3.Fi_{24}$.

5.3.1 Computation in the Fischer groups

The Fischer groups can be represented as permutation groups of relatively small degree, so many of the computational problems faced in the previous section do not arise here: in particular, we are able to compute orbit sizes and test $C_G(t)$-conjugacy directly. However, these tasks are still relatively expensive computationally and so we will aim to minimize their use when finding the suborbits.

Table 5.10 summaries the relevant information gleaned from the character tables, as well as listing the degree of the smallest available permutation representation for each group. A number in brackets denotes that the permutation rank could not be deduced from the character table, but for completeness we include the correct figure as determined by our computation.

5.3.2 Finding the suborbits

Let G be one of the Fischer sporadic simple groups or one of their automorphism groups, and X a G-conjugacy class of involutions. Fix an element $t \in X$.

Since $C_G(t)$-conjugacy testing is possible in the Fischer groups, we do not as in the previous section expend a great amount of effort calculating a large number of invariants for $C_G(t)$-orbit representatives x, but we do consider two easily-calculated

Table 5.10: Permutation ranks in the Fischer groups

G	X	Perm. Rank	$\lvert\{C \mid X_C \neq \emptyset\}\rvert$	Degree
Fi_{22}	$2A$	3	3	3510
	$2B$	(14)	11	
	$2C$	(136)	57	
$Fi_{22}:2$	$2D$	(4)	4	
	$2E$	(56)	31	
	$2F$	(74)	38	
Fi_{23}	$2A$	3	3	31671
	$2B$	(12)	10	
	$2C$	(303)	92	
Fi_{24}'	$2A$	13	11	306936
	$2B$	233	104	
Fi_{24}	$2C$	(3)	3	
	$2D$	(232)	84	

invariants: the cycle structure of $z = tx$, and the lengths of the orbits (in the permutation action) of $\langle t, x\rangle$.

Determining the conjugacy class of a given element may not be easy, so it is helpful to coalesce the X_C for all classes of elements of a given order. We set $X_n = (X_{nA} \cup X_{nB} \cup \dots)$. So the sizes of the sets X_n follow immediately from the known sizes of the X_C.

Algorithm 9 outlines a basic procedure by which we might attempt to find the representatives. For an element $x \in X$ we take Invariants(x) to denote an ordered pair of the two invariants mentioned above, while sizesX_n denotes a sequence of integers $[|X_1|, |X_2|, \dots]$.

However, this simple approach would prove very slow, since we often need to find a large number of $C_G(t)$-orbits, some of which may be very small. We use the following tricks to speed up the search for the suborbits.

Algorithm 9 SuborbitRepresentatives

Input: G a finite group;
t an involution in G;
sizesX_n the data from the character table described above.

reps:=$\{t\}$
currentsizesX_n := $[1, 0, 0, \dots]$
while currentsizes$X_n \neq$ sizesX_n **do**
repeat
$g \leftarrow$ Random(G)
$x \leftarrow t^g$
$k \leftarrow |x|$
until currentsizes$X_n[k]$ $<$sizes$X_n[k]$
newsuborbit $\leftarrow$ `true`
for $y \in$ reps with Invariants(x)=Invariants(y) **do**
if x is $C_G(t)$-conjugate to y **then**
newsuborbit $\leftarrow$ `false`
end if
end for
if newsuborbit **then**
reps $\leftarrow$ reps $\cup\{x\}$
currentsizes$X_n[k] \leftarrow$ currentsizes$X_n[k] + [C_G(t) : C_{C_G(t)}(x)]$
end if
end while

Output: reps, the set of suborbit representatives.

5.3.3 Powering Suborbits

Recall from Remarks 5.5 that for a representative x, and n dividing the order of $z = tx$, we can form a new representative $x^{(n)} = tz^n$, and that for two representatives x, y of the same $C_G(t)$-orbit, $x^{(n)}$ and $y^{(n)}$ are also $C_G(t)$-conjugate (and all other $C_G(t)$-conjugates of $x^{(n)}$ can be formed in this way).

In the previous chapter this was used to give us additional suborbit invariants, but here it is better used to unearth new suborbits. When we have found a representative x for a new suborbit, we form and check $x^{(n)}$ for each n dividing the order of z. This allows us much more easily to alight on suborbits which are small and so hard to find by random searching, but which can be powered down to from much larger orbits. For example, when $G \cong Fi_{22}$ and X is the class $2C$, there is a suborbit in X_{2C} of size 9, which we would be very unlikely to find in a random search, however, a representative is easily found as $x^{(4)}$ for x in an orbit of size 55296 contained in X_{8A}.

5.3.4 Inverse suborbits

Our suborbit representatives are formed by taking random elements $g \in G$ and setting $x = t^g$. Now consider the element $x' = t^{g^{-1}}$. We note the following properties:

Lemma 5.16. *Let $t \in X$, $g \in G$. Set $x = t^g$ and $x' = t^{g^{-1}}$. Then*

(i) the $C_G(t)$-orbits containing x and x' are of the same size;

(ii) x and x' lie in the same X_C; and

(iii) if $y = t^h$ is $C_G(t)$-conjugate to x then $y' = t^{h^{-1}}$ is $C_G(t)$-conjugate to x'.

Proof. We have that $C_{C_G(t)}(x) = C_G(t) \cap C_G(t^g)$ and $C_{C_G(t)}(x') = C_G(t) \cap C_G(t^{g^{-1}})$, so clearly $C_{C_G(t)}(x)^{g^{-1}} = C_{C_G(t)}(x')$. Then $|C_{C_G(t)}(x)| = |C_{C_G(t)}(x')|$ and so their orbits have the same size, giving (i). For (ii), we note that $(tx)^{g^{-1}t} = tx'$. Now let $c \in C_G(t)$ so that $x^c = y$. So $t^{gc} = t^h$, that is, $gch^{-1} \in C_G(t)$. Then we see that $x'^{(gch^{-1})} = t^{g^{-1}(gch^{-1})} = t^{ch^{-1}} = t^{h^{-1}} = y'$, giving (iii). $\square$

So when we find a representative x of a new $C_G(t)$-orbit, we can form the element x', and we need only to check it for $C_G(t)$-conjugacy against x to see if it represents a new suborbit (if it were in a previously discovered suborbit $\mathcal{O}$ then when that orbit were found we would have uncovered the orbit $x^{C_G(t)}$ by the same process). In some of the classes we consider a large number of the $C_G(t)$-orbits fall into pairs of 'inverse' orbits in this manner so use of this technique can save much computational time.

Algorithm 10 builds these refinements into the approach from Algorithm 9. Implementing these algorithms in Magma, we find that with $G \cong Fi_{22}$ and $X = 2C$ Algorithm 9 takes an average of 638 seconds to locate all 136 suborbits, while the improved version takes on average 58 seconds, locating 39 pairs of inverse orbits. If $G \cong Fi_{22} : 2$ and $X = 2F$, the basic algorithm takes on average 55 seconds to find 74 suborbits while the new version takes just 8 seconds, locating 17 inverse pairs.

5.3.5 Finding representative words

The most efficient way to store representatives is to give for each suborbit a word in the standard generators a, b (or c, d in an automorphism group) giving an element $g \in G$ such that $x = t^g$ is a representative for that orbit. This also has the advantage that representatives can be found using any representation of the group. We briefly describe how we obtain these words. The two largest classes in Fi_{24}, for which we determine the commuting involution graphs, are too large for the search for words to be feasible. In the electronic files (see Appendix A) we give for these cases t, generators for $C_G(t)$ and representatives for each $C_G(t)$-orbit on X in the form of base images relative to a particular base and strong generating set for G (see Chapter 4 of [16] for a discussion of this concept, of vital importance to computation in permutation groups).

From the above we have obtained a set of $C_G(t)$-orbit representatives. For each representative x we check $x^{(d)}$ for every divisor d of the order of $z = tx$ to see what suborbit it lies in. Hence we obtain full information on the powering of suborbits.

Algorithm 10 SuborbitRepresentatives

Input: G a finite group;
t an involution in G;
sizesX_n the data from the character table described above.

reps:=$\{t\}$
currentsizes$X_n := [1, 0, 0, \ldots]$
while currentsizes$X_n \neq$ sizesX_n **do**
    repeat
        $g \leftarrow$ Random(G)
        $x \leftarrow t^g$
        $k \leftarrow |tx|$
    until currentsizes$X_n[k]$ <sizes$X_n[k]$
    newsuborbit $\leftarrow$ `true`
    for $y \in$ reps with Invariants(x)=Invariants(y) **do**
        if x is $C_G(t)$-conjugate to y **then**
            newsuborbit $\leftarrow$ `false`
        end if
    end for
    if newsuborbit **then**
        reps $\leftarrow$ reps $\cup\{x\}$
        currentsizes$X_n[k] \leftarrow$ currentsizes$X_n[k] + [C_G(t) : C_{C_G(t)}(x)]$
        $x' \leftarrow t^{g^{-1}}$
        if x' is not $C_G(t)$-conjugate to x **then**
            reps $\leftarrow$ reps $\cup\{x'\}$
            currentsizes$X_n[k] \leftarrow$ currentsizes$X_n[k] + [C_G(t) : C_{C_G(t)}(x')]$
        end if
        for n dividing the order of z **do**
            $x^{(n)} \leftarrow tz^n$
            newsuborbit $\leftarrow$ `true`
            for $y \in$ reps with Invariants($x^{(n)}$)=Invariants(y) **do**
                if $x^{(n)}$ is $C_G(t)$-conjugate to y **then**
                    newsuborbit $\leftarrow$ `false`
                end if
            end for
            if newsuborbit **then**
                reps $\leftarrow$ reps $\cup\{x^{(n)}\}$
                currentsizes$X_n[k] \leftarrow$ currentsizes$X_n[k] + [C_G(t) : C_{C_G(t)}(x^{(n)})]$
                if $x^{(n)'}$ is not $C_G(t)$-conjugate to $x^{(n)}$ **then**
                    reps $\leftarrow$ reps $\cup\{x^{(n)'}\}$
                    currentsizes$X_n[k] \leftarrow$ currentsizes$X_n[k]$
                        $+[C_G(t) : C_{C_G(t)}(x^{(n)'})]$
                end if
            end if
        end for
    end if
end while

Output: reps, the set of suborbit representatives.

We also know which suborbits are inverses of each other. Hence we can create a shorter list of representatives from which we can obtain representative elements for all suborbits by taking powers and inverses.

Now we simply form each word g in a and b up to some feasible length and evaluate t^g to see which suborbit representative it is $C_G(t)$-conjugate to, aiming to find a word conjugating t to each of our reduced list. If we do not find a word for a particular suborbit, we either take random longer words hoping to find a suitable candidate, or replace one of the generators by a short word that generates the whole group with the other generator, and repeat the process (that is, make words in, say, $a' = a$ and $b' = ab$).

The following sections present the results of the computations on each of the involution conjugacy classes of each of the Fischer sporadic groups and their automorphism groups. For $({Fi_{24}}', 2B)$ and $(Fi_{24}, 2D)$ we give the sizes of the $C_G(t)$-orbits and their locations in the commuting involution graph, thus determining its disc structure. The disc locations of the $C_G(t)$-orbits were determined using the same procedure as in Section 5.2.5. For the other cases, where the structure of the graph is already known, we present the information on the $C_G(t)$-orbits, and give representatives as words in the standard generators.

5.3.6 $G \cong Fi_{24}$

Theorem 5.17. *Let $\mathcal{C}(G, X)$ be the commuting involution graph of $G \cong {Fi_{24}}'$ on $X = 2B$. Then for $t \in X$,*

- $\Delta_0(t) = \{t\}$ *has size 1 and is composed of one $C_G(t)$-orbit.*
- $\Delta_1(t)$ *has size 3,324,762 and is composed of six $C_G(t)$-orbits.*
- $\Delta_2(t)$ *has size 3,755,093,739,776 and is composed of one hundred and seventy-four $C_G(t)$-orbits.*
- $\Delta_3(t)$ *has size 4,064,208,224,256 and is composed of fifty-two $C_G(t)$-orbits.*

The $C_G(t)$-orbits comprising the graph are as detailed in Table 5.11.

Theorem 5.18. *Let $\mathcal{C}(G, X)$ be the commuting involution graph of $G \cong Fi_{24}$ on $X = 2D$. Then for $t \in X$,*

- *$\Delta_0(t) = \{t\}$ has size 1 and is composed of one $C_G(t)$-orbit.*
- *$\Delta_1(t)$ has size 3,682,503 and is composed of six $C_G(t)$-orbits.*
- *$\Delta_2(t)$ has size 822,139,288,316 and is composed of one hundred and ninety-one $C_G(t)$-orbits.*
- *$\Delta_3(t)$ has size 2,021,240,770,560 and is composed of thirty-six $C_G(t)$-orbits.*

The $C_G(t)$-orbits comprising the graph are as detailed in Table 5.12.

In Tables 5.11 and 5.12 each row corresponds to one $C_G(t)$-orbit, except that where more than one $C_G(t)$-orbit of the same size occurs in the same X_C, we collapse their entries into one row, so a symbol $4B_{1,2}$ means that there are two $C_G(t)$-orbits of the given size in X_{4B}.

5.3.7 $G \cong 3.Fi_{24}$

We briefly consider the group $G \cong 3.Fi_{24}$ with $X = 2D$, the class corresponding to its namesake in Fi_{24}. This is of particular interest because it occurs as a maximal subgroup of the Monster group $\mathbb{M}$, and hence is linked with the so-called 'Monster graph' (see the next section where two more associated cases are considered for more details). The group has size $2^{22}.3^{17}.5^2.7^3.11.13.17.23.29$ and the smallest-degree permutation representation available is of degree 920808, so computation in this group is difficult. Also the number of $C_G(t)$-orbits is evidently very large.

Our methods have not uncovered all of the suborbits, but in Appendix A we give representative elements for the 435 that have been found. Only suborbits in X_6 and X_{12} are missing.

Table 5.11: $C_G(t)$-orbits for $G \cong Fi'_{24}$, $t \in X = 2B$

$C_G(t)$-orbit	Orbit Size	(factored)	Disk
t	1	1	Δ_0
$2A_1$	24192	$2^7 \cdot 3^3 \cdot 7$	Δ_1
$2A_2$	45360	$2^4 \cdot 3^4 \cdot 5 \cdot 7$	Δ_1
$2B_1$	3402	$2 \cdot 3^5 \cdot 7$	Δ_1
$2B_2$	816480	$2^5 \cdot 3^6 \cdot 5 \cdot 7$	Δ_1
$2B_3$	2177280	$2^8 \cdot 3^5 \cdot 5 \cdot 7$	Δ_1
$3A_1$	258048	$2^{12} \cdot 3^2 \cdot 7$	Δ_2
$3B_1$	917504	$2^{17} \cdot 7$	Δ_2
$3C_1$	1032192	$2^{14} \cdot 3^2 \cdot 7$	Δ_2
$3C_2$	10321920	$2^{15} \cdot 3^2 \cdot 5 \cdot 7$	Δ_2
$3D_1$	165150720	$2^{19} \cdot 3^2 \cdot 5 \cdot 7$	Δ_2
$3E_1$	278691840	$2^{15} \cdot 3^5 \cdot 5 \cdot 7$	Δ_2
$4A_1$	1306368	$2^8 \cdot 3^6 \cdot 7$	Δ_2
$4A_2$	4354560	$2^9 \cdot 3^5 \cdot 5 \cdot 7$	Δ_2
$4A_3$	104509440	$2^{12} \cdot 3^6 \cdot 5 \cdot 7$	Δ_2
$4A_4$	139345920	$2^{14} \cdot 3^5 \cdot 5 \cdot 7$	Δ_2
$4B_{1,2}$	52254720	$2^{11} \cdot 3^6 \cdot 5 \cdot 7$	Δ_2
$4B_{3,4}$	209018880	$2^{13} \cdot 3^6 \cdot 5 \cdot 7$	Δ_2
$4C_{1,2}$	78382080	$2^{10} \cdot 3^7 \cdot 5 \cdot 7$	Δ_2
$4C_{3,4,5}$	313528320	$2^{12} \cdot 3^7 \cdot 5 \cdot 7$	Δ_2
$4C_6$	1254113280	$2^{14} \cdot 3^7 \cdot 5 \cdot 7$	Δ_2
$5A_1$	334430208	$2^{16} \cdot 3^6 \cdot 7$	Δ_2
$5A_2$	836075520	$2^{15} \cdot 3^6 \cdot 5 \cdot 7$	Δ_2
$6A_1$	34836480	$2^{12} \cdot 3^5 \cdot 5 \cdot 7$	Δ_2
$6A_2$	55738368	$2^{15} \cdot 3^5 \cdot 7$	Δ_2
$6A_3$	69672960	$2^{13} \cdot 3^5 \cdot 5 \cdot 7$	Δ_2
$6B_1$	278691840	$2^{15} \cdot 3^5 \cdot 5 \cdot 7$	Δ_2
$6C_1$	123863040	$2^{17} \cdot 3^3 \cdot 5 \cdot 7$	Δ_2
$6C_2$	371589120	$2^{17} \cdot 3^4 \cdot 5 \cdot 7$	Δ_2
$6D_1$	209018880	$2^{13} \cdot 3^6 \cdot 5 \cdot 7$	Δ_2
$6D_2$	836075520	$2^{15} \cdot 3^6 \cdot 5 \cdot 7$	Δ_2
$6E_1$	1114767360	$2^{17} \cdot 3^5 \cdot 5 \cdot 7$	Δ_2
$6F_1$	139345920	$2^{14} \cdot 3^5 \cdot 5 \cdot 7$	Δ_2
$6F_{2,3,4}$	278691840	$2^{15} \cdot 3^5 \cdot 5 \cdot 7$	Δ_2
$6F_5$	1114767360	$2^{17} \cdot 3^5 \cdot 5 \cdot 7$	Δ_2
$6G_1$	4459069440	$2^{19} \cdot 3^5 \cdot 5 \cdot 7$	Δ_2
$6H_1$	4459069440	$2^{19} \cdot 3^5 \cdot 5 \cdot 7$	Δ_2
$6I_{1,2,3}$	836075520	$2^{15} \cdot 3^6 \cdot 5 \cdot 7$	Δ_2
$6I_4$	3344302080	$2^{17} \cdot 3^6 \cdot 5 \cdot 7$	Δ_2
$6J_1$	13377208320	$2^{19} \cdot 3^6 \cdot 5 \cdot 7$	Δ_2
$6K_1$	5016453120	$2^{16} \cdot 3^7 \cdot 5 \cdot 7$	Δ_2
$6K_2$	20065812480	$2^{18} \cdot 3^7 \cdot 5 \cdot 7$	Δ_2

$C_G(t)$-orbit	Orbit Size	(factored)	Disk
$7A_1$	63700992	$2^{18} \cdot 3^5$	Δ_3
$7A_2$	6688604160	$2^{18} \cdot 3^6 \cdot 5 \cdot 7$	Δ_3
$7B_{1,2}$	11466178560	$2^{20} \cdot 3^7 \cdot 5$	Δ_2
$8A_{1,2}$	278691840	$2^{15} \cdot 3^5 \cdot 5 \cdot 7$	Δ_2
$8A_{3,4}$	836075520	$2^{15} \cdot 3^6 \cdot 5 \cdot 7$	Δ_2
$8A_{5,6}$	2508226560	$2^{15} \cdot 3^7 \cdot 5 \cdot 7$	Δ_2
$8A_{7,8}$	10032906240	$2^{17} \cdot 3^7 \cdot 5 \cdot 7$	Δ_2
$8B_1$	10032906240	$2^{17} \cdot 3^7 \cdot 5 \cdot 7$	Δ_2
$8B_2$	20065812480	$2^{18} \cdot 3^7 \cdot 5 \cdot 7$	Δ_2
$8C_{1,2,3,4}$	10032906240	$2^{17} \cdot 3^7 \cdot 5 \cdot 7$	Δ_2
$9A_1$	2972712960	$2^{20} \cdot 3^4 \cdot 5 \cdot 7$	Δ_2
$9B_1$	445906944	$2^{18} \cdot 3^5 \cdot 7$	Δ_3
$9B_2$	2229534720	$2^{18} \cdot 3^5 \cdot 5 \cdot 7$	Δ_3
$9C_1$	4459069440	$2^{19} \cdot 3^5 \cdot 5 \cdot 7$	Δ_2
$9D_1$	5945425920	$2^{21} \cdot 3^4 \cdot 5 \cdot 7$	Δ_3
$9E_{1,2}$	8918138880	$2^{20} \cdot 3^5 \cdot 5 \cdot 7$	Δ_2
$9F_1$	26754416640	$2^{20} \cdot 3^6 \cdot 5 \cdot 7$	Δ_3
$10A_{1,2}$	5016453120	$2^{16} \cdot 3^7 \cdot 5 \cdot 7$	Δ_2
$10A_3$	6688604160	$2^{18} \cdot 3^6 \cdot 5 \cdot 7$	Δ_2
$10A_4$	668860416	$2^{17} \cdot 3^6 \cdot 7$	Δ_2
$10A_5$	10032906240	$2^{17} \cdot 3^7 \cdot 5 \cdot 7$	Δ_2
$10B_{1,2,3}$	5016453120	$2^{16} \cdot 3^7 \cdot 5 \cdot 7$	Δ_2
$10B_{4,5}$	10032906240	$2^{17} \cdot 3^7 \cdot 5 \cdot 7$	Δ_2
$11A_1$	80263249920	$2^{20} \cdot 3^7 \cdot 5 \cdot 7$	Δ_3
$12A_1$	1672151040	$2^{16} \cdot 3^6 \cdot 5 \cdot 7$	Δ_2
$12A_2$	3344302080	$2^{17} \cdot 3^6 \cdot 5 \cdot 7$	Δ_2
$12B_{1,2}$	1114767360	$2^{17} \cdot 3^5 \cdot 5 \cdot 7$	Δ_2
$12B_{3,4}$	3344302080	$2^{17} \cdot 3^6 \cdot 5 \cdot 7$	Δ_2
$12C_1$	5016453120	$2^{16} \cdot 3^7 \cdot 5 \cdot 7$	Δ_2
$12C_2$	10032906240	$2^{17} \cdot 3^7 \cdot 5 \cdot 7$	Δ_2
$12D_{1,2,3,4}$	1672151040	$2^{16} \cdot 3^6 \cdot 5 \cdot 7$	Δ_2
$12D_{5,6}$	5016453120	$2^{16} \cdot 3^7 \cdot 5 \cdot 7$	Δ_2
$12E_{1,2}$	3344302080	$2^{17} \cdot 3^6 \cdot 5 \cdot 7$	Δ_2
$12E_{3,4}$	6688604160	$2^{18} \cdot 3^6 \cdot 5 \cdot 7$	Δ_2
$12F_{1,2}$	13377208320	$2^{19} \cdot 3^6 \cdot 5 \cdot 7$	Δ_2
$12G_{1,2}$	13377208320	$2^{19} \cdot 3^6 \cdot 5 \cdot 7$	Δ_2
$12H_{1,2}$	20065812480	$2^{18} \cdot 3^7 \cdot 5 \cdot 7$	Δ_2
$12I_1$	20065812480	$2^{18} \cdot 3^7 \cdot 5 \cdot 7$	Δ_2
$12I_2$	40131624960	$2^{19} \cdot 3^7 \cdot 5 \cdot 7$	Δ_2
$12J_1$	20065812480	$2^{18} \cdot 3^7 \cdot 5 \cdot 7$	Δ_2
$12J_2$	40131624960	$2^{19} \cdot 3^7 \cdot 5 \cdot 7$	Δ_2
$12K_{1,2}$	10032906240	$2^{17} \cdot 3^7 \cdot 5 \cdot 7$	Δ_2
$12K_{3,4}$	20065812480	$2^{18} \cdot 3^7 \cdot 5 \cdot 7$	Δ_2
$12L_{1,2,3,4}$	6688604160	$2^{18} \cdot 3^6 \cdot 5 \cdot 7$	Δ_2
$12L_{5,6}$	20065812480	$2^{18} \cdot 3^7 \cdot 5 \cdot 7$	Δ_2
$12M_{1,2}$	40131624960	$2^{19} \cdot 3^7 \cdot 5 \cdot 7$	Δ_2

$C_G(t)$-orbit	Orbit Size	(factored)	Disk
$13A_1$	8918138880	$2^{20} \cdot 3^5 \cdot 5 \cdot 7$	Δ_3
$13A_2$	80263249920	$2^{20} \cdot 3^7 \cdot 5 \cdot 7$	Δ_3
$14A_{1,2}$	6688604160	$2^{18} \cdot 3^6 \cdot 5 \cdot 7$	Δ_3
$14A_3$	13377208320	$2^{19} \cdot 3^6 \cdot 5 \cdot 7$	Δ_3
$14A_4$	40131624960	$2^{19} \cdot 3^7 \cdot 5 \cdot 7$	Δ_3
$14B_{1,2}$	80263249920	$2^{20} \cdot 3^7 \cdot 5 \cdot 7$	Δ_2
$15A_{1,2}$	6688604160	$2^{18} \cdot 3^6 \cdot 5 \cdot 7$	Δ_3
$15B_1$	53508833280	$2^{21} \cdot 3^6 \cdot 5 \cdot 7$	Δ_3
$15C_{1,2}$	26754416640	$2^{20} \cdot 3^6 \cdot 5 \cdot 7$	Δ_2
$16A_{1,2,3,4}$	40131624960	$2^{19} \cdot 3^7 \cdot 5 \cdot 7$	Δ_2
$17A_1$	160526499840	$2^{21} \cdot 3^7 \cdot 5 \cdot 7$	Δ_3
$18A_1$	40131624960	$2^{19} \cdot 3^7 \cdot 5 \cdot 7$	Δ_2
$18B_{1,2}$	13377208320	$2^{19} \cdot 3^6 \cdot 5 \cdot 7$	Δ_2
$18C_1$	80263249920	$2^{20} \cdot 3^7 \cdot 5 \cdot 7$	Δ_2
$18D_{1,2}$	13377208320	$2^{19} \cdot 3^6 \cdot 5 \cdot 7$	Δ_3
$18D_{3,4}$	20065812480	$2^{18} \cdot 3^7 \cdot 5 \cdot 7$	Δ_3
$18E_{1,2}$	80263249920	$2^{20} \cdot 3^7 \cdot 5 \cdot 7$	Δ_2
$18F_1$	80263249920	$2^{20} \cdot 3^7 \cdot 5 \cdot 7$	Δ_3
$20A_{1,2}$	13377208320	$2^{19} \cdot 3^6 \cdot 5 \cdot 7$	Δ_2
$20A_{3,4}$	40131624960	$2^{19} \cdot 3^7 \cdot 5 \cdot 7$	Δ_2
$20B_{1,2,3,4,5,6}$	20065812480	$2^{18} \cdot 3^7 \cdot 5 \cdot 7$	Δ_2
$21A_1$	80263249920	$2^{20} \cdot 3^7 \cdot 5 \cdot 7$	Δ_3
$21B_1$	160526499840	$2^{21} \cdot 3^7 \cdot 5 \cdot 7$	Δ_3
$21C_{1,2}$	80263249920	$2^{20} \cdot 3^7 \cdot 5 \cdot 7$	Δ_2
$21D_{1,2}$	80263249920	$2^{20} \cdot 3^7 \cdot 5 \cdot 7$	Δ_2
$22A_1$	80263249920	$2^{20} \cdot 3^7 \cdot 5 \cdot 7$	Δ_3
$24A_{1,2}$	40131624960	$2^{19} \cdot 3^7 \cdot 5 \cdot 7$	Δ_2
$24B_{1,2}$	40131624960	$2^{19} \cdot 3^7 \cdot 5 \cdot 7$	Δ_2
$24C_{1,2}$	80263249920	$2^{20} \cdot 3^7 \cdot 5 \cdot 7$	Δ_2
$24D_{1,2}$	80263249920	$2^{20} \cdot 3^7 \cdot 5 \cdot 7$	Δ_2
$24E_{1,2,3,4}$	40131624960	$2^{19} \cdot 3^7 \cdot 5 \cdot 7$	Δ_2
$24F_{1,2,3,4}$	40131624960	$2^{19} \cdot 3^7 \cdot 5 \cdot 7$	Δ_2
$24G_{1,2,3,4}$	40131624960	$2^{19} \cdot 3^7 \cdot 5 \cdot 7$	Δ_2
$26A_{1,2}$	80263249920	$2^{20} \cdot 3^7 \cdot 5 \cdot 7$	Δ_3
$27A_1$	160526499840	$2^{21} \cdot 3^7 \cdot 5 \cdot 7$	Δ_3
$27B_1$	160526499840	$2^{21} \cdot 3^7 \cdot 5 \cdot 7$	Δ_3
$27B_1$	160526499840	$2^{21} \cdot 3^7 \cdot 5 \cdot 7$	Δ_3
$28A_{1,2}$	80263249920	$2^{20} \cdot 3^7 \cdot 5 \cdot 7$	Δ_3
$29A_1$	160526499840	$2^{21} \cdot 3^7 \cdot 5 \cdot 7$	Δ_3
$29B_1$	160526499840	$2^{21} \cdot 3^7 \cdot 5 \cdot 7$	Δ_3
$30A_{1,2}$	20065812480	$2^{18} \cdot 3^7 \cdot 5 \cdot 7$	Δ_3
$30A_{3,4}$	40131624960	$2^{19} \cdot 3^7 \cdot 5 \cdot 7$	Δ_3
$30B_{1,2}$	80263249920	$2^{20} \cdot 3^7 \cdot 5 \cdot 7$	Δ_2
$33A_1$	160526499840	$2^{21} \cdot 3^7 \cdot 5 \cdot 7$	Δ_3
$33B_1$	160526499840	$2^{21} \cdot 3^7 \cdot 5 \cdot 7$	Δ_3

$C_G(t)$-orbit	Orbit Size	(factored)	Disk
$35A_1$	160526499840	$2^{21} \cdot 3^7 \cdot 5 \cdot 7$	Δ_3
$36A_{1,2}$	40131624960	$2^{19} \cdot 3^7 \cdot 5 \cdot 7$	Δ_2
$36B_{1,2}$	40131624960	$2^{19} \cdot 3^7 \cdot 5 \cdot 7$	Δ_2
$36C_{1,2}$	80263249920	$2^{20} \cdot 3^7 \cdot 5 \cdot 7$	Δ_2
$36D_{1,2}$	80263249920	$2^{20} \cdot 3^7 \cdot 5 \cdot 7$	Δ_3
$39A_1$	160526499840	$2^{21} \cdot 3^7 \cdot 5 \cdot 7$	Δ_3
$39B_1$	160526499840	$2^{21} \cdot 3^7 \cdot 5 \cdot 7$	Δ_3
$39C_1$	160526499840	$2^{21} \cdot 3^7 \cdot 5 \cdot 7$	Δ_3
$39D_1$	160526499840	$2^{21} \cdot 3^7 \cdot 5 \cdot 7$	Δ_3
$42A_1$	80263249920	$2^{20} \cdot 3^7 \cdot 5 \cdot 7$	Δ_3
$42B_{1,2}$	80263249920	$2^{20} \cdot 3^7 \cdot 5 \cdot 7$	Δ_2
$42C_{1,2}$	80263249920	$2^{20} \cdot 3^7 \cdot 5 \cdot 7$	Δ_2
$45A_1$	160526499840	$2^{21} \cdot 3^7 \cdot 5 \cdot 7$	Δ_3
$45B_1$	160526499840	$2^{21} \cdot 3^7 \cdot 5 \cdot 7$	Δ_3
$60A_{1,2}$	80263249920	$2^{20} \cdot 3^7 \cdot 5 \cdot 7$	Δ_3

5.3.8 Words for suborbit representatives

The tables for the $C_G(t)$-orbits for $G \cong Fi_{22}$ and Fi_{23} are somewhat lengthy, so are displayed in Appendix B, where we also explain how they are intended to be read.

5.4 The Thompson and Harada-Norton groups

We determine the suborbit structure for classes of involutions in the sporadic simple groups Th and HN. Then, as further motivation for our study into involution suborbits, we use this information to determine the point-line collinearity graphs for the minimal parabolic geometries associated with these groups. Study of these geometries for the sporadic groups originated in [20] and much recent work on the structure of their point-line collinearity graphs has been undertaken, for example in [25] and [26]. Study on the graph associated with the Monster group begins in [21] but much is still unknown about the graph's structure. The two graphs determined here occur as full subgraphs of the 'Monster graph' so will be of help in its continued study. The work in this section appears in [24]

Our study of these graphs will be from a group-theoretic rather than geometrical

Table 5.12: $C_G(t)$-orbits for $G \cong Fi_{24}$, $t \in X = 2D$

$C_G(t)$-orbit	Orbit Size	(factored)	Disk
$1A_1$	1	1	Δ_0
$2A_1$	2079	$3^3 \cdot 7 \cdot 11$	Δ_1
$2A_2$	38016	$2^7 \cdot 3^3 \cdot 11$	Δ_1
$2A_3$	62370	$2 \cdot 3^4 \cdot 5 \cdot 7 \cdot 11$	Δ_1
$2B_1$	187110	$2 \cdot 3^5 \cdot 5 \cdot 7 \cdot 11$	Δ_1
$2B_2$	997920	$2^5 \cdot 3^4 \cdot 5 \cdot 7 \cdot 11$	Δ_1
$2B_3$	2395008	$2^7 \cdot 3^5 \cdot 7 \cdot 11$	Δ_1
$3A_1$	8448	$2^8 \cdot 3 \cdot 11$	Δ_2
$3A_2$	221760	$2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$3B_1$	1261568	$2^{14} \cdot 7 \cdot 11$	Δ_2
$3C_1$	1892352	$2^{13} \cdot 3 \cdot 7 \cdot 11$	Δ_2
$3C_2$	17031168	$2^{13} \cdot 3^3 \cdot 7 \cdot 11$	Δ_2
$3D_1$	37847040	$2^{15} \cdot 3 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$3D_2$	113541120	$2^{15} \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$3E_1$	48660480	$2^{15} \cdot 3^3 \cdot 5 \cdot 11$	Δ_2
$4A_1$	23950080	$2^8 \cdot 3^5 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$4A_2$	57480192	$2^{10} \cdot 3^6 \cdot 7 \cdot 11$	Δ_2
$4A_3$	71850240	$2^8 \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$4A_4$	95800320	$2^{10} \cdot 3^5 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$4B_{1,2}$	4790016	$2^8 \cdot 3^5 \cdot 7 \cdot 11$	Δ_2
$4B_{3,4}$	23950080	$2^8 \cdot 3^5 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$4B_{5,6}$	71850240	$2^8 \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$4B_{7,8}$	287400960	$2^{10} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$4C_1$	287400960	$2^{10} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$4C_2$	574801920	$2^{11} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$	Δ_3
$4C_3$	1724405760	$2^{11} \cdot 3^7 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$5A_1$	43794432	$2^{14} \cdot 3^5 \cdot 11$	Δ_2
$5A_2$	1532805120	$2^{14} \cdot 3^5 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$6A_{1,2}$	2128896	$2^{10} \cdot 3^3 \cdot 7 \cdot 11$	Δ_2
$6A_3$	9123840	$2^{11} \cdot 3^4 \cdot 5 \cdot 11$	Δ_3
$6A_{4,5}$	23950080	$2^8 \cdot 3^5 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$6A_{6,7}$	95800320	$2^{10} \cdot 3^5 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$6B_{1,2}$	153280512	$2^{13} \cdot 3^5 \cdot 7 \cdot 11$	Δ_2
$6C_1$	170311680	$2^{14} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$6C_2$	510935040	$2^{14} \cdot 3^4 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$6D_1$	23950080	$2^8 \cdot 3^5 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$6D_2$	35925120	$2^7 \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$6D_{3,4,5,6}$	95800320	$2^{10} \cdot 3^5 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$6D_7$	574801920	$2^{11} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$6E_1$	1532805120	$2^{14} \cdot 3^5 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$6F_{1,2}$	85155840	$2^{13} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$6F_{3,4,5,6}$	766402560	$2^{13} \cdot 3^5 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$6G_1$	2043740160	$2^{16} \cdot 3^4 \cdot 5 \cdot 7 \cdot 11$	Δ_3
$6G_{2,3}$	3065610240	$2^{15} \cdot 3^5 \cdot 5 \cdot 7 \cdot 11$	Δ_2

$C_G(t)$-orbit	Orbit Size	(factored)	Disk
$6H_1$	2043740160	$2^{16} \cdot 3^4 \cdot 5 \cdot 7 \cdot 11$	Δ_3
$6H_{2,3,4}$	3065610240	$2^{15} \cdot 3^5 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$6H_5$	9196830720	$2^{15} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$6I_{1,2,3,4,5}$	766402560	$2^{13} \cdot 3^5 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$6I_6$	6897623040	$2^{13} \cdot 3^7 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$6K_1$	3065610240	$2^{15} \cdot 3^5 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$7A_1$	1839366144	$2^{15} \cdot 3^6 \cdot 7 \cdot 11$	Δ_2
$7A_2$	9196830720	$2^{15} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$7B_1$	10510663680	$2^{18} \cdot 3^6 \cdot 5 \cdot 11$	Δ_3
$8A_{1,2,3,4}$	2299207680	$2^{13} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$8A_{5,6,7,8}$	6897623040	$2^{13} \cdot 3^7 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$8B_1$	27590492160	$2^{15} \cdot 3^7 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$8C_{1,2}$	9196830720	$2^{15} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$	Δ_3
$9A_1$	1362493440	$2^{17} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$9B_1$	340623360	$2^{15} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$9B_2$	3065610240	$2^{15} \cdot 3^5 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$9C_1$	340623360	$2^{15} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$9C_2$	3065610240	$2^{15} \cdot 3^5 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$9D_1$	2724986880	$2^{18} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$9E_1$	1362493440	$2^{17} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$9E_2$	12262440960	$2^{17} \cdot 3^5 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$9F_1$	6131220480	$2^{16} \cdot 3^5 \cdot 5 \cdot 7 \cdot 11$	Δ_3
$9F_2$	12262440960	$2^{17} \cdot 3^5 \cdot 5 \cdot 7 \cdot 11$	Δ_3
$9F_3$	55180984320	$2^{16} \cdot 3^7 \cdot 5 \cdot 7 \cdot 11$	Δ_3
$10A_1$	919683072	$2^{14} \cdot 3^6 \cdot 7 \cdot 11$	Δ_2
$10A_{2,3}$	4598415360	$2^{14} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$10A_4$	9196830720	$2^{15} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$10A_5$	13795246080	$2^{14} \cdot 3^7 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$10B_{1,2}$	4598415360	$2^{14} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$10B_3$	27590492160	$2^{15} \cdot 3^7 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$11A_1$	110361968640	$2^{17} \cdot 3^7 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$12A_1$	191600640	$2^{11} \cdot 3^5 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$12A_2$	574801920	$2^{11} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$12A_3$	4598415360	$2^{14} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$12B_{1,2}$	1532805120	$2^{14} \cdot 3^5 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$12B_{3,4}$	4598415360	$2^{14} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$12C_1$	574801920	$2^{11} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$12C_2$	1149603840	$2^{12} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$	Δ_3
$12C_{3,4,5}$	1724405760	$2^{11} \cdot 3^7 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$12C_6$	4598415360	$2^{14} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$12D_{1,2}$	153280512	$2^{13} \cdot 3^5 \cdot 7 \cdot 11$	Δ_2
$12D_{3,4,5,6}$	2299207680	$2^{13} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$12D_{7,8}$	6897623040	$2^{13} \cdot 3^7 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$12E_1$	3065610240	$2^{15} \cdot 3^5 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$12E_2$	27590492160	$2^{15} \cdot 3^7 \cdot 5 \cdot 7 \cdot 11$	Δ_2

$C_G(t)$-orbit	Orbit Size	(factored)	Disk
$12F_{1,2}$	3065610240	$2^{15}\cdot 3^5\cdot 5\cdot 7\cdot 11$	Δ_2
$12F_{3,4}$	9196830720	$2^{15}\cdot 3^6\cdot 5\cdot 7\cdot 11$	Δ_2
$12G_{1,2}$	18393661440	$2^{16}\cdot 3^6\cdot 5\cdot 7\cdot 11$	Δ_2
$12H_{1,2}$	27590492160	$2^{15}\cdot 3^7\cdot 5\cdot 7\cdot 11$	Δ_2
$12J_{1,2}$	27590492160	$2^{15}\cdot 3^7\cdot 5\cdot 7\cdot 11$	Δ_2
$12K_{1,2}$	3065610240	$2^{15}\cdot 3^5\cdot 5\cdot 7\cdot 11$	Δ_2
$12K_{3,4,5,6}$	9196830720	$2^{15}\cdot 3^6\cdot 5\cdot 7\cdot 11$	Δ_2
$12K_{7,8}$	27590492160	$2^{15}\cdot 3^7\cdot 5\cdot 7\cdot 11$	Δ_2
$12L_{1,2}$	18393661440	$2^{16}\cdot 3^6\cdot 5\cdot 7\cdot 11$	Δ_3
$12L_{3,4,5,6}$	27590492160	$2^{15}\cdot 3^7\cdot 5\cdot 7\cdot 11$	Δ_2
$13A_{1,2}$	36787322880	$2^{17}\cdot 3^6\cdot 5\cdot 7\cdot 11$	Δ_3
$14A_1$	18393661440	$2^{16}\cdot 3^6\cdot 5\cdot 7\cdot 11$	Δ_2
$14A_{2,3}$	27590492160	$2^{15}\cdot 3^7\cdot 5\cdot 7\cdot 11$	Δ_2
$14A_4$	55180984320	$2^{16}\cdot 3^7\cdot 5\cdot 7\cdot 11$	Δ_2
$14B_1$	73574645760	$2^{18}\cdot 3^6\cdot 5\cdot 7\cdot 11$	Δ_3
$15A_1$	613122048	$2^{15}\cdot 3^5\cdot 7\cdot 11$	Δ_2
$15A_2$	27590492160	$2^{15}\cdot 3^7\cdot 5\cdot 7\cdot 11$	Δ_2
$15B_1$	73574645760	$2^{18}\cdot 3^6\cdot 5\cdot 7\cdot 11$	Δ_2
$15C_1$	12262440960	$2^{17}\cdot 3^5\cdot 5\cdot 7\cdot 11$	Δ_2
$15C_2$	110361968640	$2^{17}\cdot 3^7\cdot 5\cdot 7\cdot 11$	Δ_2
$17A_1$	220723937280	$2^{18}\cdot 3^7\cdot 5\cdot 7\cdot 11$	Δ_3
$18A_{1,2}$	9196830720	$2^{15}\cdot 3^6\cdot 5\cdot 7\cdot 11$	Δ_2
$18B_1$	3065610240	$2^{15}\cdot 3^5\cdot 5\cdot 7\cdot 11$	Δ_2
$18B_{2,3}$	9196830720	$2^{15}\cdot 3^6\cdot 5\cdot 7\cdot 11$	Δ_2
$18B_4$	27590492160	$2^{15}\cdot 3^7\cdot 5\cdot 7\cdot 11$	Δ_2
$18C_1$	36787322880	$2^{17}\cdot 3^6\cdot 5\cdot 7\cdot 11$	Δ_2
$18D_1$	6131220480	$2^{16}\cdot 3^5\cdot 5\cdot 7\cdot 11$	Δ_2
$18D_{2,3}$	9196830720	$2^{15}\cdot 3^6\cdot 5\cdot 7\cdot 11$	Δ_2
$18D_4$	55180984320	$2^{16}\cdot 3^7\cdot 5\cdot 7\cdot 11$	Δ_2
$18E_{1,2}$	36787322880	$2^{17}\cdot 3^6\cdot 5\cdot 7\cdot 11$	Δ_2
$18F_1$	36787322880	$2^{17}\cdot 3^6\cdot 5\cdot 7\cdot 11$	Δ_3
$18F_{2,3}$	55180984320	$2^{16}\cdot 3^7\cdot 5\cdot 7\cdot 11$	Δ_3
$18G_{1,2}$	36787322880	$2^{17}\cdot 3^6\cdot 5\cdot 7\cdot 11$	Δ_3
$18G_{3,4}$	110361968640	$2^{17}\cdot 3^7\cdot 5\cdot 7\cdot 11$	Δ_3
$20A_{1,2}$	18393661440	$2^{16}\cdot 3^6\cdot 5\cdot 7\cdot 11$	Δ_2
$20A_{3,4}$	55180984320	$2^{16}\cdot 3^7\cdot 5\cdot 7\cdot 11$	Δ_2
$20B_1$	110361968640	$2^{17}\cdot 3^7\cdot 5\cdot 7\cdot 11$	Δ_3
$21A_1$	18393661440	$2^{16}\cdot 3^6\cdot 5\cdot 7\cdot 11$	Δ_3
$21A_2$	55180984320	$2^{16}\cdot 3^7\cdot 5\cdot 7\cdot 11$	Δ_3
$21B_1$	73574645760	$2^{18}\cdot 3^6\cdot 5\cdot 7\cdot 11$	Δ_2
$22A_1$	110361968640	$2^{17}\cdot 3^7\cdot 5\cdot 7\cdot 11$	Δ_2
$24A_{1,2}$	9196830720	$2^{15}\cdot 3^6\cdot 5\cdot 7\cdot 11$	Δ_2
$24A_{3,4}$	27590492160	$2^{15}\cdot 3^7\cdot 5\cdot 7\cdot 11$	Δ_2
$24B_{1,2}$	9196830720	$2^{15}\cdot 3^6\cdot 5\cdot 7\cdot 11$	Δ_2
$24B_{3,4}$	27590492160	$2^{15}\cdot 3^7\cdot 5\cdot 7\cdot 11$	Δ_2
$24D_{1,2,3,4}$	55180984320	$2^{16}\cdot 3^7\cdot 5\cdot 7\cdot 11$	Δ_2

$C_G(t)$-orbit	Orbit Size	(factored)	Disk
$26A_{1,2}$	110361968640	$2^{17} \cdot 3^7 \cdot 5 \cdot 7 \cdot 11$	Δ_3
$27A_1$	73574645760	$2^{18} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$	Δ_3
$28A_{1,2}$	110361968640	$2^{17} \cdot 3^7 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$30A_{1,2}$	27590492160	$2^{15} \cdot 3^7 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$30A_{3,4}$	55180984320	$2^{16} \cdot 3^7 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$30B_{1,2}$	110361968640	$2^{17} \cdot 3^7 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$33A_1$	220723937280	$2^{18} \cdot 3^7 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$33B_1$	220723937280	$2^{18} \cdot 3^7 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$35A_1$	220723937280	$2^{18} \cdot 3^7 \cdot 5 \cdot 7 \cdot 11$	Δ_3
$36B_{1,2}$	36787322880	$2^{17} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$36C_{1,2}$	110361968640	$2^{17} \cdot 3^7 \cdot 5 \cdot 7 \cdot 11$	Δ_2
$39A_{1,2}$	73574645760	$2^{18} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$	Δ_3
$42A_{1,2}$	55180984320	$2^{16} \cdot 3^7 \cdot 5 \cdot 7 \cdot 11$	Δ_3
$42A_3$	110361968640	$2^{17} \cdot 3^7 \cdot 5 \cdot 7 \cdot 11$	Δ_3
$60A_{1,2}$	110361968640	$2^{17} \cdot 3^7 \cdot 5 \cdot 7 \cdot 11$	Δ_2

viewpoint and so we define the graph we wish to investigate in these terms.

5.4.1 Point-line collinearity graphs of minimal parabolic geometries

We begin with the definition of a point-line collinearity graph for a geometry.

Definition 5.19. *Let Γ be a geometry with incidence relation $*$, Γ_0 the set of points and Γ_1 the set of lines. Then the point-line collinearity graph of Γ has Γ_0 as its vertex set, and $x, y \in \Gamma_0$ adjacent if and only if there exists $l \in \Gamma_1$ with $x * l$ and $y * l$.*

We wish to investigate this graph for the minimal parabolic geometries of the Thompson and Harada-Norton groups. So we define the following graph which is equivalent to the point-line collinearity graphs for our two cases.

Definition 5.20. *Let G be isomorphic to the Thompson group or to the Harada-Norton group. Let X be a conjugacy class of involutions in G, with $X = 2A$ if $G \cong Th$ and $X = 2B$ if $G \cong HN$. We define the graph $\mathcal{G}(G, X)$ to have X as its vertex set and two vertices $x, y \in X$ adjacent if and only if $x \in O_2(C_G(y))$.*

We note that this adjacency condition, while not necessarily symmetric in x and y for a general group and conjugacy class, is symmetric in our cases, where it arises from geometrical considerations.

The similarity of $\mathcal{G}(G, X)$ to the commuting involution graph on X is obvious. Again we fix $t \in X$. Since $O_2(C_G(t)) \trianglelefteq C_G(t)$, we see that by the same argument as in the proof of Lemma 2.6, discs around t are again unions of $C_G(t)$-orbits on X, so we once more begin by determining these suborbits. We provide representative elements for the suborbits in the form of words in the standard generators of G conjugating t to such representatives. These were obtained using the same techniques as described in the previous section.

But as well as the orbit and disc sizes, we wish to compute more intricate data regarding the connections between the $C_G(t)$-orbits. This data will be encoded in a collapsed adjacency matrix.

Definition 5.21. *Let $\mathcal{G}(G, X)$ be the graph described in Definition 5.20. Denote the $C_G(t)$-orbits of X by $\mathcal{O}_1, \ldots, \mathcal{O}_n$, and let $x_i \in \mathcal{O}_i$ for $i = 1, \ldots, n$. The collapsed adjacency matrix for $\mathcal{G}(G, X)$ is the $n \times n$ matrix which has as its $(i, j)^{\text{th}}$ entry the total number of edges in $\mathcal{G}(G, X)$ running from x_i to vertices in $\mathcal{O}_j$, that is, the value $|\Delta_1(x_i) \cap \mathcal{O}_j|$.*

Sections 5.4.2 and 5.4.3 describe how the $C_G(t)$-orbits were identified for $G \cong Th$ and $G \cong HN$ respectively, and Subsection 5.4.4 give details of how the collapsed adjacency matrices were determined and displays the matrices themselves, along with the words found for suborbit representatives.

5.4.2 Suborbits in the Thompson group

The smallest-degree representation of Th available in [29] is as 248×248 matrices over $GF(2)$, and for $t \in 2A$, $|C_G(t)| = 92,897,280$, so we face similar computational difficulties as with J_4 in Section 5.2. Therefore, we employ similar techniques.

Let a, b be the standard generators of G. So a is in class $2A$, b is in class $3A$, ab has order 19, and $\langle a, b \rangle = G$. We set $t = a$, $X = t^G = 2A$, then since $C_G(t)$ is a maximal subgroup, we use the straight line program provided in [29] to obtain generators for $C_G(t)$.

Calculations with the character table reveal that X consists of thirty-eight $C_G(t)$-orbits across twenty-nine non-empty X_C. So we know that at least twenty of the X_C are single suborbits and at most nine split into more than one suborbit.

The group $C_G(t)$ has shape $2_+^{1+8}.A_9$. Similarly to Section 5.2, we set $Q = O_2(C_G(t)) \cong 2_+^{1+8}$, so we can employ Lemma 5.11 to help determine orbit sizes. We now describe how the orbit sizes were determined. (We note that the class of an element is easily determined by its order and the dimension of its fixed space.)

- Seven classes C with non-empty X_C have $C_G(h) = \langle h \rangle$ for $h \in C$, so it is trivial for $x \in X_C$ to compute $C_{C_G(t)}(x) \leq C_G(z)$ where $z = tx$.
- Let $x \in X_{13A}$. We have that $|X_C| = |C_G(t)|/3$, giving us that $|C_{C_G(t)}(x)| \geq 3$. On the other hand, $C_{C_G(t)}(x) \leq C_G(z)$ where $z = tx \in 13A$ and $|C_G(z)| = 39$. Since 13 does not divide $|C_G(t)|$ we know that $|C_{C_G(t)}(x)|$ is 1 or 3. Hence we conclude $C_{C_G(t)}(x) = 3$ and that X_{13A} is a single $C_G(t)$-orbit.
- Where C is a class of elements of even order we employ Lemma 5.11 to compute $C_{C_G(t)}(x)$ for $x \in X_C$ and so determine the orbit sizes.
- The above leaves only the sets X_{5A}, X_{7A} unanalysed. However, we now observe that we have already found nine sets X_C that split into two $C_G(t)$-orbits, so we conclude that these two sets are each single suborbits.

5.4.3 Suborbits in the Harada-Norton group

The smallest representation of HN available is as 132×132 matrices over $GF(4)$. This is a somewhat large representation but crucially $C_G(t)$ for $t \in 2B$ is small, having order 3,686,400, and so direct calculation in $C_G(t)$ is possible. So we can find the suborbit invariants using Algorithm 9, although using a different set of invariants, namely the order of z and the dimension of its fixed space, and the value d_x as defined in Lemma 5.8.

We note that MAGMA's command `IsConjugate` sometimes gives an incorrect negative result when testing $C_G(t)$-conjugacy in matrix groups. However since $C_G(t)$ has shape $2^{1+8}.(A_5 \times A_5) : 2$, it is simple to isolate $Q = O_2(C_G(t)) \cong 2^{1+8}$ and then conjugacy testing can be verified following the process described in Lemma 5.12.

Although we have focused our attention on the class $2B$ due to its use in determining the point-line collinearity graph, exactly the same method described here can be used to find $C_G(t)$-orbit representatives for the class $2A$ and, in $\mathrm{Aut}(HN)$ the class $2C$. There are respectively 9 and 88 suborbits in these classes, and the relevant data and representatives are provided in the accompanying electronic materials (see Appendix A).

5.4.4 Determining the collapsed adjacency matrices

Recall that to find the collapsed adjacency matrix for $\mathcal{G}(G, X)$ we must determine for each pair $\mathcal{O}_i, \mathcal{O}_j$ of $C_G(t)$-orbits, how many edges join a chosen element in $\mathcal{O}_i$ to any element in $\mathcal{O}_j$. Clearly it suffices to determine for each of our suborbit representatives x what suborbits each of its neighbours lies in, that is, to find the suborbit location of every $y \in \Delta_i(x)$.

We begin by determining an explicit list of the elements of $\Delta_1(t)$. Since by definition $\Delta_1(t) = (X \cap O_2(C_G(t))) \backslash \{t\}$ this is easily found—it is just $(X \cap Q) \backslash \{t\}$, and Q has already been determined. We have for each representative $x \in X$ a word in the standard generators giving an element $g \in G$ such that $t^g = x$. So $\Delta_1(x) = \Delta_1(t)^g$. It only remains to determine what suborbit each $y \in \Delta_1(x)$ lies in. This is easy in HN since $C_G(t)$-conjugacy testing is possible (subject to the caveat mentioned in the previous section), while in Th we rely on the suborbit invariants and occasionally resort to the method of Lemma 5.12.

Once the collapsed adjacency matrix is completed it is trivial to determine which disc each $C_G(t)$-orbit is in, and so we arrive at these results.

Theorem 5.22. *Suppose that $G \cong Th$ and $\mathcal{G}$ is the point-line collinearity graph of the characteristic 2 minimal parabolic geometry for G. We identify $V(\mathcal{G})$ with the $2A$ conjugacy class of G, which we denote X. Then $\mathcal{G}$ has diameter 5, and for $t \in X$ we have*

(i) $|\Delta_1(t)| = 270$ *with* $\Delta_1(t)$ *a* $C_G(t)$*-orbit;*

(ii) $|\Delta_2(t)| = 64800$, $\Delta_2(t)$ *consisting of two* $C_G(t)$*-orbits;*

(iii) $|\Delta_3(t)| = 15060480$, $\Delta_3(t)$ *consisting of six* $C_G(t)$*-orbits;*

(iv) $|\Delta_4(t)| = 858497006$, $\Delta_4(t)$ *consisting of twenty-six* $C_G(t)$*-orbits; and*

(v) $|\Delta_5(t)| = 103219200$, $\Delta_5(t)$ *consisting of two* $C_G(t)$*-orbits.*

The collapsed adjacency matrix for this graph is as displayed in Figure 5.1, and the suborbit sizes and representative elements are as listed in Table 5.13.

Theorem 5.23. *Suppose that $G \cong HN$ and $\mathcal{G}$ is the point-line collinearity graph of the characteristic 2 minimal parabolic geometry for G. We identify $V(\mathcal{G})$ with the $2B$ conjugacy class of G, which we denote X. Then $\mathcal{G}$ has diameter 5 and for $t \in X$ we have*

(i) $|\Delta_1(t)| = 150$ *with* $\Delta_1(t)$ *a* $C_G(t)$*-orbit;*

(ii) $|\Delta_2(t)| = 17760$, $\Delta_2(t)$ *consisting of three* $C_G(t)$*-orbits;*

(iii) $|\Delta_3(t)| = 1638400$, $\Delta_3(t)$ *consisting of eight* $C_G(t)$*-orbits;*

(iv) $|\Delta_4(t)| = 68721664$, $\Delta_4(t)$ *consisting of fifty-five* $C_G(t)$*-orbits; and*

(v) $|\Delta_5(t)| = 3686400$, $\Delta_5(t)$ *consisting of three* $C_G(t)$*-orbits.*

The collapsed adjacency matrix for this graph is as displayed in Figure 5.2, and the suborbit sizes and representative elements are as listed in Table 5.14.

In the following collapsed adjacency matrices and tables, the orbits are ordered first in increasing order of distance from t and then in increasing order of size, so that for example $\Delta_3^1(t)$ denotes the smallest $C_G(t)$-orbit in $\Delta_3(t)$. Figure 5.1 gives the collapsed adjacency matrix for $G \cong Th$ broken into four columns, while Figure 5.2 presents the somewhat lager matrix for $G \cong HN$ in twelve pieces, scanning across the table in two rows.

Following the collapsed adjacency matrices, we tabulate details of the $C_G(t)$-orbits, giving their sizes and words in the standard generators which allow representative elements to be obtained. In the columns giving these words, a symbol g_i refers to a word defined elsewhere in the table. A symbol $g_i \to n$ denotes that $x^{(n)}$ is a representative for that orbit where $x = t^{g_i}$.

Figure 5.1: Collapsed adjacency matrix for $G \cong Th$

	Δ_0^1	Δ_1^1	Δ_2^1	Δ_2^2	Δ_3^1	Δ_3^2	Δ_3^3	Δ_3^4	Δ_3^5	Δ_3^6
Δ_0^1	0	270	0	0	0	0	0	0	0	0
Δ_1^1	1	29	112	128	0	0	0	0	0	0
Δ_2^1	0	1	13	16	0	16	96	0	0	128
Δ_2^2	0	1	14	15	16	0	0	112	112	0
Δ_3^1	0	0	0	9	9	0	0	126	126	0
Δ_3^2	0	0	1	0	0	1	12	0	0	16
Δ_3^3	0	0	1	0	0	2	11	0	0	16
Δ_3^4	0	0	0	1	2	0	0	21	14	8
Δ_3^5	0	0	0	1	2	0	0	14	13	0
Δ_3^6	0	0	1	0	0	2	12	8	0	23
Δ_4^1	0	0	0	0	0	9	0	0	0	0
Δ_4^2	0	0	0	0	0	0	0	0	0	27
Δ_4^3	0	0	0	0	0	1	0	0	8	0
Δ_4^4	0	0	0	0	0	0	5	0	0	0
Δ_4^5	0	0	0	0	0	0	9	0	0	0
Δ_4^6	0	0	0	0	0	0	0	2	2	1
Δ_4^7	0	0	0	0	0	0	0	4	0	5
Δ_4^8	0	0	0	0	0	0	0	0	2	3
Δ_4^9	0	0	0	0	0	0	0	6	0	3
Δ_4^{10}	0	0	0	0	0	2	0	2	0	1
Δ_4^{11}	0	0	0	0	0	0	0	0	2	3
Δ_4^{12}	0	0	0	0	0	0	1	0	0	0
Δ_4^{13}	0	0	0	0	0	1	0	6	2	3
Δ_4^{14}	0	0	0	0	0	0	3	0	2	0
Δ_4^{15}	0	0	0	0	0	0	0	2	2	1
Δ_4^{16}	0	0	0	0	0	0	1	0	0	0
Δ_4^{17}	0	0	0	0	0	0	0	0	2	3
Δ_4^{18}	0	0	0	0	0	0	2	2	0	1
Δ_4^{19}	0	0	0	0	0	1	0	0	2	0
Δ_4^{20}	0	0	0	0	0	0	1	0	2	0
Δ_4^{21}	0	0	0	0	0	0	2	2	0	1
Δ_4^{22}	0	0	0	0	0	0	0	0	2	3
Δ_4^{23}	0	0	0	0	0	0	1	0	0	0
Δ_4^{24}	0	0	0	0	0	0	1	0	0	0
Δ_4^{25}	0	0	0	0	0	0	1	0	2	0
Δ_4^{26}	0	0	0	0	0	0	0	2	0	1
Δ_5^1	0	0	0	0	0	0	0	0	0	0
Δ_5^2	0	0	0	0	0	0	0	0	0	0

	Δ_4^1	Δ_4^2	Δ_4^3	Δ_4^4	Δ_4^5	Δ_4^6	Δ_4^7	Δ_4^8	Δ_4^9
Δ_0^1	0	0	0	0	0	0	0	0	0
Δ_1^1	0	0	0	0	0	0	0	0	0
Δ_2^1	0	0	0	0	0	0	0	0	0
Δ_2^2	0	0	0	0	0	0	0	0	0
Δ_3^1	0	0	0	0	0	0	0	0	0
Δ_3^2	8	0	8	0	0	0	0	0	0
Δ_3^3	0	0	0	8	16	0	0	0	0
Δ_3^4	0	0	0	0	0	8	16	0	24
Δ_3^5	0	0	8	0	0	8	0	8	0
Δ_3^6	0	4	0	0	0	4	20	12	12
Δ_4^1	0	0	9	0	0	0	0	0	0
Δ_4^2	0	0	0	0	0	0	27	27	0
Δ_4^3	1	0	0	0	0	8	0	8	0
Δ_4^4	0	0	0	0	10	0	0	0	0
Δ_4^5	0	0	0	9	9	0	0	0	0
Δ_4^6	0	0	2	0	0	0	2	8	3
Δ_4^7	0	1	0	0	0	2	4	3	12
Δ_4^8	0	1	2	0	0	8	3	2	0
Δ_4^9	0	0	0	0	0	3	12	0	6
Δ_4^{10}	2	0	2	0	0	7	2	6	3
Δ_4^{11}	0	1	2	0	0	2	9	5	6
Δ_4^{12}	0	0	0	1	2	4	4	4	4
Δ_4^{13}	1	0	3	0	0	5	9	2	12
Δ_4^{14}	0	0	2	5	6	4	0	4	0
Δ_4^{15}	0	0	2	0	0	5	6	4	7
Δ_4^{16}	0	0	0	1	2	8	0	8	0
Δ_4^{17}	0	1	2	2	0	6	3	9	0
Δ_4^{18}	0	0	0	4	4	3	4	2	5
Δ_4^{19}	1	0	3	0	0	6	2	6	2
Δ_4^{20}	0	0	2	1	2	2	6	2	6
Δ_4^{21}	0	0	0	2	4	3	6	2	7
Δ_4^{22}	0	1	2	0	0	6	5	9	2
Δ_4^{23}	0	0	0	1	2	4	4	4	4
Δ_4^{24}	0	0	0	1	2	4	4	4	4
Δ_4^{25}	0	0	2	1	2	4	4	4	4
Δ_4^{26}	0	0	0	0	0	5	6	4	7
Δ_5^1	0	0	0	0	0	0	9	0	9
Δ_5^2	0	0	0	3	0	3	3	3	3

	Δ_4^{10}	Δ_4^{11}	Δ_4^{12}	Δ_4^{13}	Δ_4^{14}	Δ_4^{15}	Δ_4^{16}	Δ_4^{17}	Δ_4^{18}	Δ_4^{19}
Δ_0^1	0	0	0	0	0	0	0	0	0	0
Δ_1^1	0	0	0	0	0	0	0	0	0	0
Δ_2^1	0	0	0	0	0	0	0	0	0	0
Δ_2^2	0	0	0	0	0	0	0	0	0	0
Δ_3^1	0	0	0	0	0	0	0	0	0	0
Δ_3^2	64	0	0	64	0	0	0	0	0	96
Δ_3^3	0	0	8	0	48	0	16	0	32	0
Δ_3^4	8	0	0	48	0	24	0	0	24	0
Δ_3^5	0	8	0	16	24	24	0	24	0	24
Δ_3^6	4	12	0	24	0	12	0	36	12	0
Δ_4^1	72	0	0	72	0	0	0	0	0	108
Δ_4^2	0	27	0	0	0	0	0	81	0	0
Δ_4^3	8	8	0	24	24	24	0	24	0	36
Δ_4^4	0	0	5	0	50	0	10	20	40	0
Δ_4^5	0	0	9	0	54	0	18	0	36	0
Δ_4^6	7	2	6	10	12	15	24	18	9	18
Δ_4^7	2	9	6	18	0	18	0	9	12	6
Δ_4^8	6	5	6	4	12	12	24	27	6	18
Δ_4^9	3	6	6	24	0	21	0	0	15	6
Δ_4^{10}	14	0	6	22	6	9	24	12	9	36
Δ_4^{11}	0	2	6	10	6	18	0	15	6	12
Δ_4^{12}	4	4	0	4	10	12	18	8	12	12
Δ_4^{13}	11	5	3	24	6	21	0	6	12	21
Δ_4^{14}	2	2	5	4	27	8	14	20	24	10
Δ_4^{15}	3	6	6	14	8	10	8	10	9	14
Δ_4^{16}	8	0	9	0	14	8	25	16	12	16
Δ_4^{17}	4	5	4	4	20	10	16	24	14	14
Δ_4^{18}	3	2	6	8	24	9	12	14	16	6
Δ_4^{19}	12	4	6	14	10	14	16	14	6	19
Δ_4^{20}	0	8	7	10	12	18	2	6	10	12
Δ_4^{21}	3	4	8	10	14	13	12	4	17	8
Δ_4^{22}	4	7	6	6	10	14	16	23	6	16
Δ_4^{23}	4	4	9	4	10	12	18	8	12	12
Δ_4^{24}	4	4	9	4	10	12	18	8	12	12
Δ_4^{25}	2	6	7	8	14	16	10	10	10	14
Δ_4^{26}	5	4	8	10	4	15	16	8	11	12
Δ_5^1	0	9	9	9	0	18	0	0	9	9
Δ_5^2	3	3	6	3	18	9	12	21	21	9

	Δ_4^{20}	Δ_4^{21}	Δ_4^{22}	Δ_4^{23}	Δ_4^{24}	Δ_4^{25}	Δ_4^{26}	Δ_5^{1}	Δ_5^{2}
Δ_0^1	0	0	0	0	0	0	0	0	0
Δ_1^1	0	0	0	0	0	0	0	0	0
Δ_2^1	0	0	0	0	0	0	0	0	0
Δ_2^2	0	0	0	0	0	0	0	0	0
Δ_3^1	0	0	0	0	0	0	0	0	0
Δ_3^2	0	0	0	0	0	0	0	0	0
Δ_3^3	16	32	0	16	16	32	0	0	0
Δ_3^4	0	24	0	0	0	0	48	0	0
Δ_3^5	24	0	24	0	0	48	0	0	0
Δ_3^6	0	12	36	0	0	0	24	0	0
Δ_4^1	0	0	0	0	0	0	0	0	0
Δ_4^2	0	0	81	0	0	0	0	0	0
Δ_4^3	24	0	24	0	0	48	0	0	0
Δ_4^4	10	20	0	10	10	20	0	0	60
Δ_4^5	18	36	0	18	18	36	0	0	0
Δ_4^6	6	9	18	12	12	24	30	0	18
Δ_4^7	18	18	15	12	12	24	36	6	18
Δ_4^8	6	6	27	12	12	24	24	0	18
Δ_4^9	18	21	6	12	12	24	42	6	18
Δ_4^{10}	0	9	12	12	12	12	30	0	18
Δ_4^{11}	24	12	21	12	12	36	24	6	18
Δ_4^{12}	14	16	12	18	18	28	32	4	24
Δ_4^{13}	15	15	9	6	6	24	30	3	9
Δ_4^{14}	12	14	10	10	10	28	8	0	36
Δ_4^{15}	18	13	14	12	12	32	30	4	18
Δ_4^{16}	2	12	16	18	18	20	32	0	24
Δ_4^{17}	6	4	23	8	8	20	16	0	42
Δ_4^{18}	10	17	6	12	12	20	22	2	42
Δ_4^{19}	12	8	16	12	12	28	24	2	18
Δ_4^{20}	17	16	12	14	14	40	24	6	18
Δ_4^{21}	16	12	8	16	16	28	30	4	18
Δ_4^{22}	12	8	16	12	12	28	24	2	18
Δ_4^{23}	14	16	12	9	18	28	32	4	24
Δ_4^{24}	14	16	12	18	9	28	32	4	24
Δ_4^{25}	20	14	14	14	14	27	24	4	18
Δ_4^{26}	12	15	12	16	16	24	29	4	24
Δ_5^{1}	27	18	9	18	18	36	36	0	27
Δ_5^{2}	9	9	9	12	12	18	24	3	54

Figure 5.2: Collapsed adjacency matrix for $G \cong HN$

	Δ_0^1	Δ_1^1	Δ_2^1	Δ_2^2	Δ_2^3	Δ_3^1	Δ_3^2	Δ_3^3	Δ_3^4	Δ_3^5	Δ_3^6	Δ_3^7
Δ_0^1	0	150	0	0	0	0	0	0	0	0	0	0
Δ_1^1	1	5	32	48	64	0	0	0	0	0	0	0
Δ_2^1	0	5	5	0	20	0	0	120	0	0	0	0
Δ_2^2	0	1	0	5	0	0	16	32	16	16	0	0
Δ_2^3	0	1	2	0	11	16	0	24	0	0	48	48
Δ_3^1	0	0	0	0	6	6	0	9	0	0	9	0
Δ_3^2	0	0	0	1	0	0	1	0	2	2	0	0
Δ_3^3	0	0	1	2	2	2	0	9	0	0	14	0
Δ_3^4	0	0	0	1	0	0	2	0	1	2	0	0
Δ_3^5	0	0	0	1	0	0	2	0	2	1	0	0
Δ_3^6	0	0	0	0	2	1	0	7	0	0	18	8
Δ_3^7	0	0	0	0	1	0	0	0	0	0	4	9
Δ_3^8	0	0	0	1	0	0	0	2	0	0	0	0
Δ_4^1	0	0	0	0	0	0	0	0	0	0	30	0
Δ_4^2	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^3	0	0	0	0	0	0	9	0	0	0	0	0
Δ_4^4	0	0	0	0	0	0	6	3	0	0	0	0
Δ_4^5	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^6	0	0	0	0	0	0	0	5	0	0	0	0
Δ_4^7	0	0	0	0	0	0	0	5	0	0	0	0
Δ_4^8	0	0	0	0	0	0	0	0	0	5	0	0
Δ_4^9	0	0	0	0	0	0	0	0	5	0	0	0
Δ_4^{10}	0	0	0	0	0	5	0	5	0	0	5	10
Δ_4^{11}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{12}	0	0	0	0	0	1	0	0	0	0	3	0
Δ_4^{13}	0	0	0	0	0	0	2	1	0	0	4	0
Δ_4^{14}	0	0	0	0	0	0	0	1	2	0	0	0
Δ_4^{15}	0	0	0	0	0	2	0	1	0	0	0	2
Δ_4^{16}	0	0	0	0	0	0	0	1	0	0	2	2
Δ_4^{17}	0	0	0	0	0	0	0	0	1	0	0	0
Δ_4^{18}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{19}	0	0	0	0	0	0	1	0	0	0	0	0
Δ_4^{20}	0	0	0	0	0	0	0	1	0	0	2	2
Δ_4^{21}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{22}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{23}	0	0	0	0	0	0	0	1	0	2	2	0

	Δ_3^8	Δ_4^1	Δ_4^2	Δ_4^3	Δ_4^4	Δ_4^5	Δ_4^6	Δ_4^7	Δ_4^8	Δ_4^9	Δ_4^{10}	Δ_4^{11}
Δ_0^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_1^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_2^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_2^2	64	0	0	0	0	0	0	0	0	0	0	0
Δ_2^3	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^1	0	0	0	0	0	0	0	0	0	0	36	0
Δ_3^2	0	0	0	8	8	0	0	0	0	0	0	0
Δ_3^3	8	0	0	0	4	0	8	8	0	0	8	0
Δ_3^4	0	0	0	0	0	0	0	0	0	8	0	0
Δ_3^5	0	0	0	0	0	0	0	0	8	0	0	0
Δ_3^6	0	2	0	0	0	0	0	0	0	0	4	0
Δ_3^7	0	0	0	0	0	0	0	0	0	0	4	0
Δ_3^8	15	0	2	0	2	2	0	0	0	0	0	2
Δ_4^1	0	0	0	0	20	0	0	0	0	0	0	0
Δ_4^2	25	0	0	0	0	25	0	0	0	0	0	0
Δ_4^3	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^4	6	2	0	0	0	0	0	0	0	0	0	0
Δ_4^5	5	0	5	0	0	0	0	0	0	0	0	0
Δ_4^6	0	0	0	0	0	0	5	0	0	0	0	0
Δ_4^7	0	0	0	0	0	0	0	5	0	0	0	0
Δ_4^8	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^9	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{10}	0	0	0	0	0	0	0	0	0	0	15	0
Δ_4^{11}	5	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{12}	6	1	0	0	1	0	0	0	0	0	0	0
Δ_4^{13}	6	0	0	0	3	0	0	0	0	0	0	0
Δ_4^{14}	2	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{15}	0	0	0	0	0	0	0	0	0	0	2	0
Δ_4^{16}	2	0	0	0	0	0	2	0	0	0	0	0
Δ_4^{17}	0	0	0	0	0	4	0	0	0	1	0	0
Δ_4^{18}	1	0	0	0	0	0	0	0	0	0	0	6
Δ_4^{19}	0	0	0	1	0	0	0	0	0	0	4	0
Δ_4^{20}	2	0	0	0	0	0	0	2	0	0	0	0
Δ_4^{21}	1	0	0	0	0	0	0	0	4	0	0	0
Δ_4^{22}	1	0	0	0	0	0	0	0	0	4	0	0
Δ_4^{23}	2	0	0	0	0	0	2	0	0	0	2	0

	Δ_4^{12}	Δ_4^{13}	Δ_4^{14}	Δ_4^{15}	Δ_4^{16}	Δ_4^{17}	Δ_4^{18}	Δ_4^{19}	Δ_4^{20}	Δ_4^{21}	Δ_4^{22}	Δ_4^{23}
Δ_0^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_1^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_2^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_2^2	0	0	0	0	0	0	0	0	0	0	0	0
Δ_2^3	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^1	12	0	0	72	0	0	0	0	0	0	0	0
Δ_3^2	0	8	0	0	0	0	0	8	0	0	0	0
Δ_3^3	0	4	8	8	8	0	0	0	8	0	0	8
Δ_3^4	0	0	16	0	0	8	0	0	0	0	0	0
Δ_3^5	0	0	0	0	0	0	0	0	0	0	0	16
Δ_3^6	4	8	0	0	8	0	0	0	8	0	0	8
Δ_3^7	0	0	0	4	4	0	0	0	4	0	0	0
Δ_3^8	4	6	4	0	4	0	2	0	4	2	2	4
Δ_4^1	20	0	0	0	0	0	0	0	0	0	0	0
Δ_4^2	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^3	0	0	0	0	0	0	0	9	0	0	0	0
Δ_4^4	2	9	0	0	0	0	0	0	0	0	0	0
Δ_4^5	0	0	0	0	0	20	0	0	0	0	0	0
Δ_4^6	0	0	0	0	10	0	0	0	0	0	0	10
Δ_4^7	0	0	0	0	0	0	0	0	10	0	0	0
Δ_4^8	0	0	0	0	0	0	0	0	0	20	0	0
Δ_4^9	0	0	0	0	0	5	0	0	0	0	20	0
Δ_4^{10}	0	0	0	10	0	0	0	20	0	0	0	10
Δ_4^{11}	0	0	0	0	0	0	30	0	0	0	0	0
Δ_4^{12}	11	3	0	6	0	0	0	12	0	0	0	0
Δ_4^{13}	2	6	0	4	0	0	4	0	0	4	4	0
Δ_4^{14}	0	0	3	6	4	4	0	0	0	0	4	0
Δ_4^{15}	2	2	6	11	4	0	0	8	4	0	0	0
Δ_4^{16}	0	0	4	4	5	0	4	4	6	0	0	6
Δ_4^{17}	0	0	4	0	0	4	0	0	4	0	4	0
Δ_4^{18}	0	2	0	0	4	0	4	0	0	2	1	4
Δ_4^{19}	4	0	0	8	4	0	0	12	4	0	0	4
Δ_4^{20}	0	0	0	4	6	4	0	4	5	0	0	0
Δ_4^{21}	0	2	0	0	0	0	2	0	0	4	0	4
Δ_4^{22}	0	2	4	0	0	4	1	0	0	0	4	0
Δ_4^{23}	0	0	0	0	6	0	4	4	0	4	0	17

	Δ_4^{24}	Δ_4^{25}	Δ_4^{26}	Δ_4^{27}	Δ_4^{28}	Δ_4^{29}	Δ_4^{30}	Δ_4^{31}	Δ_4^{32}	Δ_4^{33}	Δ_4^{34}	Δ_4^{35}
Δ_0^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_1^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_2^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_2^2	0	0	0	0	0	0	0	0	0	0	0	0
Δ_2^3	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^2	0	0	0	0	0	16	0	0	32	0	0	0
Δ_3^3	0	8	8	8	0	8	8	0	0	0	0	0
Δ_3^4	0	0	0	0	0	0	16	0	0	0	0	0
Δ_3^5	0	0	0	16	8	0	0	0	0	16	0	0
Δ_3^6	0	8	8	0	0	8	8	0	0	0	0	0
Δ_3^7	0	4	4	0	0	0	0	0	8	0	0	8
Δ_3^8	2	4	4	4	0	16	4	2	0	0	4	4
Δ_4^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^2	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^3	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^4	0	6	6	0	0	6	0	0	24	0	0	0
Δ_4^5	20	0	0	0	20	20	0	0	0	0	0	0
Δ_4^6	0	0	10	0	0	10	0	0	0	0	0	0
Δ_4^7	0	10	0	0	0	10	10	0	0	0	0	0
Δ_4^8	0	0	0	0	5	0	0	0	0	30	0	0
Δ_4^9	0	0	0	0	0	0	0	0	0	0	40	0
Δ_4^{10}	0	0	0	0	0	0	10	0	0	0	0	0
Δ_4^{11}	5	10	10	0	0	0	0	30	0	0	0	0
Δ_4^{12}	0	6	6	0	0	0	0	0	6	0	0	12
Δ_4^{13}	0	6	6	0	0	2	0	4	4	8	0	0
Δ_4^{14}	0	2	0	2	4	2	2	0	4	4	8	8
Δ_4^{15}	0	0	0	6	0	0	0	0	0	4	8	4
Δ_4^{16}	0	2	0	0	4	2	0	0	0	4	4	0
Δ_4^{17}	4	0	0	4	0	4	0	0	4	0	8	0
Δ_4^{18}	0	2	8	0	0	0	0	12	10	12	2	2
Δ_4^{19}	0	0	0	0	0	4	4	0	4	4	4	4
Δ_4^{20}	0	0	2	4	0	2	6	4	0	0	0	8
Δ_4^{21}	2	0	10	4	4	0	0	1	4	8	0	0
Δ_4^{22}	2	10	0	0	0	0	4	2	4	0	18	6
Δ_4^{23}	0	0	4	2	0	0	2	0	4	16	0	0

	Δ_4^{36}	Δ_4^{37}	Δ_4^{38}	Δ_4^{39}	Δ_4^{40}	Δ_4^{41}	Δ_4^{42}	Δ_4^{43}	Δ_4^{44}	Δ_4^{45}	Δ_4^{46}	Δ_4^{47}
Δ_0^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_1^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_2^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_2^2	0	0	0	0	0	0	0	0	0	0	0	0
Δ_2^3	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^2	16	0	0	0	0	0	0	0	0	0	0	16
Δ_3^3	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^4	0	16	0	32	0	0	0	0	0	0	0	0
Δ_3^5	0	0	0	0	0	0	0	0	32	0	0	0
Δ_3^6	0	0	8	0	8	8	8	0	0	0	0	0
Δ_3^7	0	0	8	8	0	0	8	0	8	0	8	0
Δ_3^8	0	0	4	4	8	4	4	4	4	4	0	0
Δ_4^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^2	0	0	0	0	0	0	0	50	0	50	0	0
Δ_4^3	18	0	0	0	36	0	0	0	0	0	36	18
Δ_4^4	0	24	0	0	0	0	0	0	0	0	0	0
Δ_4^5	0	0	0	0	0	0	0	10	0	10	0	0
Δ_4^6	0	20	0	0	0	0	10	0	0	0	0	20
Δ_4^7	20	0	10	0	0	10	0	0	0	0	0	0
Δ_4^8	0	0	20	0	0	0	0	0	0	0	0	0
Δ_4^9	0	10	0	0	0	0	20	0	0	0	0	0
Δ_4^{10}	0	0	0	0	0	0	0	0	0	0	20	0
Δ_4^{11}	0	0	20	10	0	0	20	0	10	0	0	0
Δ_4^{12}	0	0	0	0	6	0	0	0	0	0	6	0
Δ_4^{13}	8	0	0	8	12	0	0	0	8	0	4	8
Δ_4^{14}	4	4	2	4	0	6	0	8	8	8	4	8
Δ_4^{15}	4	4	0	4	8	8	0	0	4	0	8	4
Δ_4^{16}	4	0	4	4	4	12	0	0	4	0	8	4
Δ_4^{17}	0	10	0	0	4	4	0	8	4	12	4	8
Δ_4^{18}	8	4	12	4	0	0	12	0	0	4	0	0
Δ_4^{19}	2	0	4	0	8	0	4	4	0	4	12	2
Δ_4^{20}	4	8	0	4	4	4	4	0	4	0	8	4
Δ_4^{21}	12	0	4	0	4	4	0	0	8	4	2	0
Δ_4^{22}	0	4	0	8	4	4	4	4	0	0	2	12
Δ_4^{23}	4	0	16	0	4	4	0	0	8	0	0	0

	Δ_4^{48}	Δ_4^{49}	Δ_4^{50}	Δ_4^{51}	Δ_4^{52}	Δ_4^{53}	Δ_4^{54}	Δ_4^{55}	Δ_5^{1}	Δ_5^{2}	Δ_5^{3}
Δ_0^1	0	0	0	0	0	0	0	0	0	0	0
Δ_1^1	0	0	0	0	0	0	0	0	0	0	0
Δ_2^1	0	0	0	0	0	0	0	0	0	0	0
Δ_2^2	0	0	0	0	0	0	0	0	0	0	0
Δ_2^3	0	0	0	0	0	0	0	0	0	0	0
Δ_3^1	0	0	0	0	0	0	0	0	0	0	0
Δ_3^2	0	0	0	0	0	32	0	0	0	0	0
Δ_3^3	0	0	0	0	0	0	0	0	0	0	0
Δ_3^4	0	0	16	0	0	0	32	0	0	0	0
Δ_3^5	0	16	0	0	0	0	0	32	0	0	0
Δ_3^6	8	0	0	0	0	0	0	0	0	0	0
Δ_3^7	0	0	0	0	8	16	16	16	0	0	0
Δ_3^8	4	0	0	4	4	0	0	0	0	0	0
Δ_4^1	0	0	0	0	0	0	0	0	0	80	0
Δ_4^2	0	0	0	0	0	0	0	0	0	0	0
Δ_4^3	0	0	0	0	0	0	0	0	12	12	0
Δ_4^4	0	24	0	0	0	0	0	0	12	8	12
Δ_4^5	0	0	0	0	0	0	20	20	0	0	0
Δ_4^6	10	0	0	0	0	20	0	0	0	0	20
Δ_4^7	0	20	0	0	0	20	0	0	0	0	20
Δ_4^8	0	10	0	40	0	0	0	20	0	0	0
Δ_4^9	0	0	30	0	0	0	20	0	0	0	0
Δ_4^{10}	0	0	0	0	0	20	0	0	0	0	20
Δ_4^{11}	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{12}	0	0	0	0	12	0	12	12	6	4	18
Δ_4^{13}	0	0	8	0	0	8	8	8	0	0	0
Δ_4^{14}	4	0	0	0	0	12	12	0	0	0	4
Δ_4^{15}	8	4	4	8	4	4	0	0	0	4	4
Δ_4^{16}	4	8	0	0	8	4	0	16	0	8	0
Δ_4^{17}	0	0	6	4	4	16	4	8	4	0	4
Δ_4^{18}	4	4	0	0	0	0	0	16	0	4	6
Δ_4^{19}	0	0	4	4	4	4	4	4	0	4	16
Δ_4^{20}	12	0	4	4	0	4	16	0	0	8	0
Δ_4^{21}	4	4	0	18	6	8	0	16	2	0	4
Δ_4^{22}	4	0	8	0	0	8	16	0	2	0	4
Δ_4^{23}	0	4	0	4	8	4	4	8	0	4	4

	Δ_0^1	Δ_1^1	Δ_2^1	Δ_2^2	Δ_2^3	Δ_3^1	Δ_3^2	Δ_3^3	Δ_3^4	Δ_3^5	Δ_3^6	Δ_3^7
Δ_4^{24}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{25}	0	0	0	0	0	0	0	1	0	0	2	2
Δ_4^{26}	0	0	0	0	0	0	0	1	0	0	2	2
Δ_4^{27}	0	0	0	0	0	0	0	1	0	2	0	0
Δ_4^{28}	0	0	0	0	0	0	0	0	0	1	0	0
Δ_4^{29}	0	0	0	0	0	0	2	1	0	0	2	0
Δ_4^{30}	0	0	0	0	0	0	0	1	2	0	2	0
Δ_4^{31}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{32}	0	0	0	0	0	0	2	0	0	0	0	2
Δ_4^{33}	0	0	0	0	0	0	0	0	0	1	0	0
Δ_4^{34}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{35}	0	0	0	0	0	0	0	0	0	0	0	2
Δ_4^{36}	0	0	0	0	0	0	1	0	0	0	0	0
Δ_4^{37}	0	0	0	0	0	0	0	0	1	0	0	0
Δ_4^{38}	0	0	0	0	0	0	0	0	0	0	1	2
Δ_4^{39}	0	0	0	0	0	0	0	0	2	0	0	2
Δ_4^{40}	0	0	0	0	0	0	0	0	0	0	1	0
Δ_4^{41}	0	0	0	0	0	0	0	0	0	0	1	0
Δ_4^{42}	0	0	0	0	0	0	0	0	0	0	1	2
Δ_4^{43}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{44}	0	0	0	0	0	0	0	0	0	2	0	2
Δ_4^{45}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{46}	0	0	0	0	0	0	0	0	0	0	0	2
Δ_4^{47}	0	0	0	0	0	0	1	0	0	0	0	0
Δ_4^{48}	0	0	0	0	0	0	0	0	0	0	1	0
Δ_4^{49}	0	0	0	0	0	0	0	0	0	1	0	0
Δ_4^{50}	0	0	0	0	0	0	0	0	1	0	0	0
Δ_4^{51}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{52}	0	0	0	0	0	0	0	0	0	0	0	2
Δ_4^{53}	0	0	0	0	0	0	1	0	0	0	0	2
Δ_4^{54}	0	0	0	0	0	0	0	0	1	0	0	2
Δ_4^{55}	0	0	0	0	0	0	0	0	0	1	0	2
Δ_5^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^2	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^3	0	0	0	0	0	0	0	0	0	0	0	0

	Δ_3^8	Δ_4^1	Δ_4^2	Δ_4^3	Δ_4^4	Δ_4^5	Δ_4^6	Δ_4^7	Δ_4^8	Δ_4^9	Δ_4^{10}	Δ_4^{11}
Δ_4^{24}	1	0	0	0	0	4	0	0	0	0	0	1
Δ_4^{25}	2	0	0	0	1	0	0	2	0	0	0	2
Δ_4^{26}	2	0	0	0	1	0	2	0	0	0	0	2
Δ_4^{27}	2	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{28}	0	0	0	0	0	4	0	0	1	0	0	0
Δ_4^{29}	8	0	0	0	1	4	2	2	0	0	0	0
Δ_4^{30}	2	0	0	0	0	0	0	2	0	0	2	0
Δ_4^{31}	1	0	0	0	0	0	0	0	0	0	0	6
Δ_4^{32}	0	0	0	0	2	0	0	0	0	0	0	0
Δ_4^{33}	0	0	0	0	0	0	0	0	3	0	0	0
Δ_4^{34}	1	0	0	0	0	0	0	0	0	4	0	0
Δ_4^{35}	1	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{36}	0	0	0	1	0	0	0	2	0	0	0	0
Δ_4^{37}	0	0	0	0	2	0	2	0	0	1	0	0
Δ_4^{38}	1	0	0	0	0	0	0	1	2	0	0	2
Δ_4^{39}	1	0	0	0	0	0	0	0	0	0	0	1
Δ_4^{40}	2	0	0	2	0	0	0	0	0	0	0	0
Δ_4^{41}	1	0	0	0	0	0	0	1	0	0	0	0
Δ_4^{42}	1	0	0	0	0	0	1	0	0	2	0	2
Δ_4^{43}	1	0	1	0	0	1	0	0	0	0	0	0
Δ_4^{44}	1	0	0	0	0	0	0	0	0	0	0	1
Δ_4^{45}	1	0	1	0	0	1	0	0	0	0	0	0
Δ_4^{46}	0	0	0	2	0	0	0	0	0	0	2	0
Δ_4^{47}	0	0	0	1	0	0	2	0	0	0	0	0
Δ_4^{48}	1	0	0	0	0	0	1	0	0	0	0	0
Δ_4^{49}	0	0	0	0	2	0	0	2	1	0	0	0
Δ_4^{50}	0	0	0	0	0	0	0	0	0	3	0	0
Δ_4^{51}	1	0	0	0	0	0	0	0	4	0	0	0
Δ_4^{52}	1	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{53}	0	0	0	0	0	0	1	1	0	0	1	0
Δ_4^{54}	0	0	0	0	0	1	0	0	0	1	0	0
Δ_4^{55}	0	0	0	0	0	1	0	0	1	0	0	0
Δ_5^1	0	0	0	2	3	0	0	0	0	0	0	0
Δ_5^2	0	1	0	1	1	0	0	0	0	0	0	0
Δ_5^3	0	0	0	0	1	0	2	2	0	0	2	0

	Δ_4^{12}	Δ_4^{13}	Δ_4^{14}	Δ_4^{15}	Δ_4^{16}	Δ_4^{17}	Δ_4^{18}	Δ_4^{19}	Δ_4^{20}	Δ_4^{21}	Δ_4^{22}	Δ_4^{23}
Δ_4^{24}	0	0	0	0	0	4	0	0	0	2	2	0
Δ_4^{25}	2	3	2	0	2	0	2	0	0	0	10	0
Δ_4^{26}	2	3	0	0	0	0	8	0	2	10	0	4
Δ_4^{27}	0	0	2	6	0	4	0	0	4	4	0	2
Δ_4^{28}	0	0	4	0	4	0	0	0	0	4	0	0
Δ_4^{29}	0	1	2	0	2	4	0	4	2	0	0	0
Δ_4^{30}	0	0	2	0	0	0	0	4	6	0	4	2
Δ_4^{31}	0	2	0	0	0	0	12	0	4	1	2	0
Δ_4^{32}	1	1	2	0	0	2	5	2	0	2	2	2
Δ_4^{33}	0	2	2	2	2	0	6	2	0	4	0	8
Δ_4^{34}	0	0	4	4	2	4	1	2	0	0	9	0
Δ_4^{35}	2	0	4	2	0	0	1	2	4	0	3	0
Δ_4^{36}	0	2	2	2	2	0	4	1	2	6	0	2
Δ_4^{37}	0	0	2	2	0	5	2	0	4	0	2	0
Δ_4^{38}	0	0	1	0	2	0	6	2	0	2	0	8
Δ_4^{39}	0	2	2	2	2	0	2	0	2	0	4	0
Δ_4^{40}	1	3	0	4	2	2	0	4	2	2	2	2
Δ_4^{41}	0	0	3	4	6	2	0	0	2	2	2	2
Δ_4^{42}	0	0	0	0	0	0	6	2	2	0	2	0
Δ_4^{43}	0	0	4	0	0	4	0	2	0	0	2	0
Δ_4^{44}	0	2	4	2	2	2	0	0	2	4	0	4
Δ_4^{45}	0	0	4	0	0	6	2	2	0	2	0	0
Δ_4^{46}	1	1	2	4	4	2	0	6	4	1	1	0
Δ_4^{47}	0	2	4	2	2	4	0	1	2	0	6	0
Δ_4^{48}	0	0	2	4	2	0	2	0	6	2	2	0
Δ_4^{49}	0	0	0	2	4	0	2	0	0	2	0	2
Δ_4^{50}	0	2	0	2	0	3	0	2	2	0	4	0
Δ_4^{51}	0	0	0	4	0	2	0	2	2	9	0	2
Δ_4^{52}	2	0	0	2	4	2	0	2	0	3	0	4
Δ_4^{53}	0	1	3	1	1	4	0	1	1	2	2	1
Δ_4^{54}	1	1	3	0	0	1	0	1	4	0	4	1
Δ_4^{55}	1	1	0	0	4	2	4	1	0	4	0	2
Δ_5^{1}	3	0	0	0	0	6	0	0	0	3	3	0
Δ_5^{2}	1	0	0	3	6	0	3	3	6	0	0	3
Δ_5^{3}	3	0	2	2	0	2	3	8	0	2	2	2

	Δ_4^{24}	Δ_4^{25}	Δ_4^{26}	Δ_4^{27}	Δ_4^{28}	Δ_4^{29}	Δ_4^{30}	Δ_4^{31}	Δ_4^{32}	Δ_4^{33}	Δ_4^{34}	Δ_4^{35}
Δ_4^{24}	0	2	2	0	4	4	0	0	0	4	12	0
Δ_4^{25}	2	9	10	0	0	4	4	8	2	0	0	4
Δ_4^{26}	2	10	9	2	0	4	0	2	2	8	0	4
Δ_4^{27}	0	0	2	3	4	2	0	0	4	0	0	0
Δ_4^{28}	4	0	0	4	4	4	0	0	4	6	4	4
Δ_4^{29}	4	4	4	2	4	11	0	0	6	0	0	4
Δ_4^{30}	0	4	0	0	0	0	17	4	4	0	4	8
Δ_4^{31}	0	8	2	0	0	0	4	4	10	0	0	0
Δ_4^{32}	0	1	1	2	2	3	2	5	6	6	2	2
Δ_4^{33}	2	0	4	0	3	0	0	0	6	7	0	0
Δ_4^{34}	6	0	0	0	2	0	2	0	2	0	9	6
Δ_4^{35}	0	2	2	0	2	2	4	0	2	0	6	9
Δ_4^{36}	2	2	0	4	4	0	0	0	0	4	2	4
Δ_4^{37}	2	0	2	0	0	0	2	2	4	0	2	4
Δ_4^{38}	0	3	3	0	0	1	0	6	4	4	0	2
Δ_4^{39}	1	2	0	4	2	0	4	0	8	0	4	6
Δ_4^{40}	0	2	2	0	2	0	2	0	0	4	0	4
Δ_4^{41}	4	1	3	2	0	1	0	2	4	0	4	4
Δ_4^{42}	0	3	3	1	0	1	8	6	4	6	4	2
Δ_4^{43}	2	0	4	4	6	0	0	2	4	2	4	2
Δ_4^{44}	1	0	2	2	0	0	0	2	8	6	2	0
Δ_4^{45}	2	4	0	4	4	0	0	0	4	6	2	6
Δ_4^{46}	4	1	1	2	2	7	0	0	1	0	8	6
Δ_4^{47}	2	0	2	2	0	0	2	4	0	4	6	2
Δ_4^{48}	4	3	1	3	2	1	2	0	4	2	6	2
Δ_4^{49}	2	2	0	2	5	0	0	2	4	10	6	2
Δ_4^{50}	2	4	0	2	0	0	8	6	6	6	8	6
Δ_4^{51}	6	0	0	4	4	0	0	1	2	8	2	2
Δ_4^{52}	0	2	2	4	0	2	0	1	2	6	2	12
Δ_4^{53}	0	1	1	3	4	5	1	0	1	3	4	3
Δ_4^{54}	3	0	1	0	2	2	2	4	6	3	8	6
Δ_4^{55}	3	1	0	3	1	2	1	0	6	5	1	3
Δ_5^{1}	0	3	3	0	6	3	0	0	6	0	6	0
Δ_5^{2}	6	0	0	0	0	0	3	3	6	6	3	3
Δ_5^{3}	4	3	3	2	2	5	2	3	2	2	0	8

	Δ_4^{36}	Δ_4^{37}	Δ_4^{38}	Δ_4^{39}	Δ_4^{40}	Δ_4^{41}	Δ_4^{42}	Δ_4^{43}	Δ_4^{44}	Δ_4^{45}	Δ_4^{46}	Δ_4^{47}
Δ_4^{24}	4	4	0	2	0	8	0	4	2	4	8	4
Δ_4^{25}	4	0	6	4	4	2	6	0	0	8	2	0
Δ_4^{26}	0	4	6	0	4	6	6	8	4	0	2	4
Δ_4^{27}	8	0	0	8	0	4	2	8	4	8	4	4
Δ_4^{28}	8	0	0	4	4	0	0	12	0	8	4	0
Δ_4^{29}	0	0	2	0	0	2	2	0	0	0	14	0
Δ_4^{30}	0	4	0	8	4	0	16	0	0	0	0	4
Δ_4^{31}	0	4	12	0	0	4	12	4	4	0	0	8
Δ_4^{32}	0	4	4	8	0	4	4	4	8	4	1	0
Δ_4^{33}	4	0	4	0	4	0	6	2	6	6	0	4
Δ_4^{34}	2	2	0	4	0	4	4	4	2	2	8	6
Δ_4^{35}	4	4	2	6	4	4	2	2	0	6	6	2
Δ_4^{36}	5	0	4	6	0	4	4	8	4	0	0	2
Δ_4^{37}	0	5	2	6	4	6	6	4	4	4	4	12
Δ_4^{38}	4	2	10	0	4	3	10	2	4	2	2	4
Δ_4^{39}	6	6	0	7	0	6	4	2	8	6	4	4
Δ_4^{40}	0	4	4	0	13	4	4	8	0	8	10	0
Δ_4^{41}	4	6	3	6	4	4	4	6	4	4	2	4
Δ_4^{42}	4	6	10	4	4	4	10	2	0	2	2	4
Δ_4^{43}	8	4	2	2	8	6	2	5	6	8	0	0
Δ_4^{44}	4	4	4	8	0	4	0	6	7	2	4	6
Δ_4^{45}	0	4	2	6	8	4	2	8	2	5	0	8
Δ_4^{46}	0	4	2	4	10	2	2	0	4	0	12	0
Δ_4^{47}	2	12	4	4	0	4	4	0	6	8	0	5
Δ_4^{48}	4	6	4	4	4	2	3	4	6	6	2	4
Δ_4^{49}	12	4	6	4	4	6	2	4	6	4	4	0
Δ_4^{50}	4	10	6	6	4	2	4	6	0	2	0	4
Δ_4^{51}	6	6	4	2	0	6	0	2	4	4	8	2
Δ_4^{52}	2	2	2	0	4	2	2	6	6	2	6	4
Δ_4^{53}	4	2	1	4	8	3	1	6	4	6	10	4
Δ_4^{54}	4	2	3	5	3	4	4	4	4	3	1	6
Δ_4^{55}	6	6	4	4	3	3	3	3	5	4	1	4
Δ_5^{1}	6	6	0	0	0	6	0	6	0	6	0	6
Δ_5^{2}	6	0	12	3	3	3	12	3	3	3	0	6
Δ_5^{3}	2	2	4	0	4	6	4	2	0	2	2	2

	Δ_4^{48}	Δ_4^{49}	Δ_4^{50}	Δ_4^{51}	Δ_4^{52}	Δ_4^{53}	Δ_4^{54}	Δ_4^{55}	Δ_5^{1}	Δ_5^{2}	Δ_5^{3}
Δ_4^{24}	8	4	4	12	0	0	12	12	0	8	8
Δ_4^{25}	6	4	8	0	4	4	0	4	2	0	6
Δ_4^{26}	2	0	0	0	4	4	4	0	2	0	6
Δ_4^{27}	6	4	4	8	8	12	0	12	0	0	4
Δ_4^{28}	4	10	0	8	0	16	8	4	4	0	4
Δ_4^{29}	2	0	0	0	4	20	8	8	2	0	10
Δ_4^{30}	4	0	16	0	0	4	8	4	0	4	4
Δ_4^{31}	0	4	12	2	2	0	16	0	0	4	6
Δ_4^{32}	4	4	6	2	2	2	12	12	2	4	2
Δ_4^{33}	2	10	6	8	6	6	6	10	0	4	2
Δ_4^{34}	6	6	8	2	2	8	16	2	2	2	0
Δ_4^{35}	2	2	6	2	12	6	12	6	0	2	8
Δ_4^{36}	4	12	4	6	2	8	8	12	2	4	2
Δ_4^{37}	6	4	10	6	2	4	4	12	2	0	2
Δ_4^{38}	4	6	6	4	2	2	6	8	0	8	4
Δ_4^{39}	4	4	6	2	0	8	10	8	0	2	0
Δ_4^{40}	4	4	4	0	4	16	6	6	0	2	4
Δ_4^{41}	2	6	2	6	2	6	8	6	2	2	6
Δ_4^{42}	3	2	4	0	2	2	8	6	0	8	4
Δ_4^{43}	4	4	6	2	6	12	8	6	2	2	2
Δ_4^{44}	6	6	0	4	6	8	8	10	0	2	0
Δ_4^{45}	6	4	2	4	2	12	6	8	2	2	2
Δ_4^{46}	2	4	0	8	6	20	2	2	0	0	2
Δ_4^{47}	4	0	4	2	4	8	12	8	2	4	2
Δ_4^{48}	4	6	0	4	4	6	6	8	2	2	6
Δ_4^{49}	6	5	0	2	4	4	12	4	2	0	2
Δ_4^{50}	0	0	7	0	0	6	10	6	0	4	2
Δ_4^{51}	4	2	0	9	6	8	2	16	2	2	0
Δ_4^{52}	4	4	0	6	9	6	6	12	0	2	8
Δ_4^{53}	3	2	3	4	3	10	7	7	2	0	7
Δ_4^{54}	3	6	5	1	3	7	10	9	2	2	1
Δ_4^{55}	4	2	3	8	6	7	9	10	2	2	1
Δ_5^{1}	6	6	0	6	0	12	12	12	4	6	3
Δ_5^{2}	3	0	6	3	3	0	6	6	3	5	3
Δ_5^{3}	6	2	2	0	8	14	2	2	1	2	12

Table 5.13: Suborbits for $G \cong Th$, $X = 2A$, $t = a \in X$

$C_G(t)$-orbit	z class	Size	Conjugating Word
$\Delta_0^1(t)$	$1A$	1	-
$\Delta_1^1(t)$	$2A$	270	t^{g_1}
$\Delta_2^1(t)$	$2A$	30240	bab^2abab^2ab
$\Delta_2^2(t)$	$4A$	34560	$g_1 = b^2abab^2ab^2abababab^2ab$
$\Delta_3^1(t)$	$3A$	61440	bab^2ab^2ab
$\Delta_3^2(t)$	$4A$	483840	$b^2ab^2abab^2abab^2abab$
$\Delta_3^3(t)$	$4B$	2903040	$bab^2ababab^2abab^2(ab)^3$
$\Delta_3^4(t)$	$6B$	3870720	$b^2(ab)^9$
$\Delta_3^5(t)$	$8A$	3870720	$babab^2ab$
$\Delta_3^6(t)$	$8A$	3870720	bab^2abab
$\Delta_4^1(t)$	$3C$	430080	$(b^2a)^3babab$
$\Delta_4^2(t)$	$3B$	573440	$babab^2ab^2abab$
$\Delta_4^3(t)$	$6A$	3870720	$babab$
$\Delta_4^4(t)$	$5A$	4644864	b^2ab
$\Delta_4^5(t)$	$9A$	5160960	$bab^2(ab)^3$
$\Delta_4^6(t)$	$6C$	15482880	$bab^2ababab^2ab$
$\Delta_4^7(t)$	$7A$	15482880	$(ba)^5b$
$\Delta_4^8(t)$	$9C$	15482880	bab
$\Delta_4^9(t)$	$9C$	15482880	$b^2abab^2(ab)^3$
$\Delta_4^{10}(t)$	$12C$	15482880	$g_2 = (b^2a)^2(ba)^4b$
$\Delta_4^{11}(t)$	$12C$	15482880	g_2^{-1}
$\Delta_4^{12}(t)$	$10A$	23224320	b
$\Delta_4^{13}(t)$	$13A$	30965760	$bab^2a(ba)^3b$
$\Delta_4^{14}(t)$	$12D$	46448640	$(b^2a)^2(ba)^5b$
$\Delta_4^{15}(t)$	$14A$	46448640	b^2abab^2abab
$\Delta_4^{16}(t)$	$18A$	46448640	$bab^2ab^2abab^2ab$
$\Delta_4^{17}(t)$	$18B$	46448640	$g_3 = b^2(ab)^6$
$\Delta_4^{18}(t)$	$18B$	46448640	g_3^{-1}
$\Delta_4^{19}(t)$	$20A$	46448640	$g_4 = (b^2a)^2(ba)^3b^2ab$
$\Delta_4^{20}(t)$	$20A$	46448640	g_4^{-1}
$\Delta_4^{21}(t)$	$28A$	46448640	$g_5 = bab^2(ab)^5$
$\Delta_4^{22}(t)$	$28A$	46448640	g_5^{-1}
$\Delta_4^{23}(t)$	$36A$	46448640	$g_6 = b^2(ab)^3ab^2ab$
$\Delta_4^{24}(t)$	$36A$	46448640	g_6^{-1}
$\Delta_4^{25}(t)$	$19A$	92897280	b^2abab
$\Delta_4^{26}(t)$	$21A$	92897280	$b^2(ab)^3$
$\Delta_5^1(t)$	$9B$	10321920	$ba(b^2a)^4babab$
$\Delta_5^2(t)$	$27A$	92897280	$b^2(ab)^4$

Table 5.14: Suborbits for $G \cong HN$, $X = 2B$, $t = (abab^2ab)^{10} \in X$

$C_G(t)$-orbit	z class	Size	Conjugating Word
$\Delta_0^1(t)$	$1A$	1	-
$\Delta_1^1(t)$	$2B$	150	$g_1 \to 4$
$\Delta_2^1(t)$	$2A$	960	$g_2 \to 10$
$\Delta_2^2(t)$	$2B$	7200	$g_3 \to 5$
$\Delta_2^3(t)$	$4A$	9600	$g_1 \to 2$
$\Delta_3^1(t)$	$3A$	25600	$g_4 \to 4$
$\Delta_3^2(t)$	$4A$	115200	$g_5 \to 5$
$\Delta_3^3(t)$	$4B$	115200	$g_2 \to 5$
$\Delta_3^4(t)$	$4C$	115200	$g_6 \to 5$
$\Delta_3^5(t)$	$4C$	115200	$g_7 \to 5$
$\Delta_3^6(t)$	$6A$	230400	$g_8 \to 2$
$\Delta_3^7(t)$	$8B$	460800	$g_1 = bab^2a(ba)^2b^2a(ba)^3$
$\Delta_3^8(t)$	$8B$	460800	$(ba)^3b^2a(ba)^2b^2aba$
$\Delta_4^1(t)$	$5A$	15360	$g_2 \to 4$
$\Delta_4^2(t)$	$5B$	36864	$g_9 \to 5$
$\Delta_4^3(t)$	$3B$	102400	$g_{10} \to 10$
$\Delta_4^4(t)$	$6A$	153600	$g_{11} \to 2$
$\Delta_4^5(t)$	$10A$	184320	$(ba)^4(b^2a)^3(ba)^2b^2ab^2$
$\Delta_4^6(t)$	$5E$	184320	$g_{12} \to 2$
$\Delta_4^7(t)$	$5E$	184320	$g_{13} \to 2$
$\Delta_4^8(t)$	$5CD$	184320	$g_{10} \to 6$
$\Delta_4^9(t)$	$5CD$	184320	$g_{14} \to 6$
$\Delta_4^{10}(t)$	$5E$	184320	$g_3 \to 2$
$\Delta_4^{11}(t)$	$10A$	184320	$((ab)^2ab^2)^2(ab)^4ab^2(ab)^2$
$\Delta_4^{12}(t)$	$10B$	307200	$g_2 \to 2$
$\Delta_4^{13}(t)$	$6B$	460800	$g_4 \to 2$
$\Delta_4^{14}(t)$	$10F$	921600	$g_{12} = (ba)^2(b^2a)^2b^2$
$\Delta_4^{15}(t)$	$12A$	921600	$g_8 = (ba)^3b^2(ab)^2$
$\Delta_4^{16}(t)$	$12B$	921600	$g_4 = ab^2(ab)^4$
$\Delta_4^{17}(t)$	$10DE$	921600	$g_{10} \to 3$
$\Delta_4^{18}(t)$	$20AB$	921600	$g_5 = babab^2$
$\Delta_4^{19}(t)$	$6C$	921600	$g_{10} \to 5$
$\Delta_4^{20}(t)$	$12B$	921600	$(ab)^4ab^2$
$\Delta_4^{21}(t)$	$20AB$	921600	$bab^2a(ba)^4b$
$\Delta_4^{22}(t)$	$20AB$	921600	$(ab)^2ab^2(ab)^4$
$\Delta_4^{23}(t)$	$10GH$	921600	$g_3 = (ba)^4(b^2a)^2(ba)^3$
$\Delta_4^{24}(t)$	$10C$	921600	$g_5 \to 2$

$C_G(t)$-orbit	z class	Size	Conjugating Word
$\Delta_4^{25}(t)$	$10GH$	921600	$(ab)^5(ab^2)^2$
$\Delta_4^{26}(t)$	$10GH$	921600	$bab^2a(ba)^6b^2$
$\Delta_4^{27}(t)$	$10F$	921600	$g_{13} = (ba)^3b^2ab^2$
$\Delta_4^{28}(t)$	$10DE$	921600	$g_{14} \to 3$
$\Delta_4^{29}(t)$	$12A$	921600	$g_{11} = (ba)^6b$
$\Delta_4^{30}(t)$	$10GH$	921600	$(ab^2)^2(ab)^5$
$\Delta_4^{31}(t)$	$20AB$	921600	bab^2ab^2
$\Delta_4^{32}(t)$	$30A$	1843200	$g_{17} = ba(b^2a)^3(ba)^3b$
$\Delta_4^{33}(t)$	$12C$	1843200	$(ba)^8b$
$\Delta_4^{34}(t)$	$30BC$	1843200	$g_{10} \to 7$
$\Delta_4^{35}(t)$	$15BC$	1843200	$g_{10} \to 14$
$\Delta_4^{36}(t)$	$20DE$	1843200	$g_7 \to 3$
$\Delta_4^{37}(t)$	$20DE$	1843200	$g_7 = (ba)^2b^2aba$
$\Delta_4^{38}(t)$	$22A$	1843200	$g_{16} = (ab)^4ab^2(ab)^2a$
$\Delta_4^{39}(t)$	$15BC$	1843200	$g_{14} \to 2$
$\Delta_4^{40}(t)$	$20C$	1843200	$g_2 = (ba)^3b^2a(ba)^5b^2$
$\Delta_4^{41}(t)$	$11A$	1843200	$g_{16} \to 2$
$\Delta_4^{42}(t)$	$22A$	1843200	$g_{18} = (ba)^2b^2a(ba)^3b$
$\Delta_4^{43}(t)$	$30BC$	1843200	$g_{14} = (ba)^4b$
$\Delta_4^{44}(t)$	$15BC$	1843200	$g_{10} \to 2$
$\Delta_4^{45}(t)$	$30BC$	1843200	$g_{10} = (ab)^2(ab^2)^2(ab)^5$
$\Delta_4^{46}(t)$	$15A$	1843200	$g_{17} \to 2$
$\Delta_4^{47}(t)$	$20DE$	1843200	$g_6 = bab^2(ab)^2$
$\Delta_4^{48}(t)$	$11A$	1843200	$g_{18} \to 2$
$\Delta_4^{49}(t)$	$20DE$	1843200	$g_6 \to 3$
$\Delta_4^{50}(t)$	$12C$	1843200	$(ab)^2ab^2(ab)^5$
$\Delta_4^{51}(t)$	$30BC$	1843200	$g_{19} = (ba)^7b^2(ab)^2$
$\Delta_4^{52}(t)$	$15BC$	1843200	$g_{19} \to 2$
$\Delta_4^{53}(t)$	$21A$	3686400	$g_{20} = bab^2a(ba)^4$
$\Delta_4^{54}(t)$	$25AB$	3686400	$g_9 = bab$
$\Delta_4^{55}(t)$	$25AB$	3686400	$g_9 \to 2$
$\Delta_5^{1}(t)$	$7A$	614400	$g_{20} \to 3$
$\Delta_5^{2}(t)$	$9A$	1228800	$bab^2a(ba)^2b^2a(ba)^4b$
$\Delta_5^{3}(t)$	$14A$	1843200	$(ba)^2(b^2a)^2bab$

Appendix A

Electronic material

Included with the paper version of this thesis is a CD-ROM containing the following electronic materials relating to the work in this thesis.

1. We include MAGMA implementations of the algorithms discussed in Chapter 3 for calculating distance in the commuting involution graph of a symmetric group.

2. For the algorithm developed in Chapter 4, we provide a MAGMA implementation, along with the input data required by the algorithm, and generators for the groups used in the example calculations in Section 4.4.1 so that these may be verified if desired. A 'readme' file is provided with more details. This material is also available as supplementary materials published alongside [23].

3. In Section 5.2, the suborbit structure of the two conjugacy classes in the group J_4 are computed. We provide representatives for these suborbits in the form of 112×112 matrices, in both GAP and MAGMA formats. It is intended that these files will also be published along with the final version of [22].

4. In Section 5.3 we determine the suborbits of the involution conjugacy classes of the three Fischer groups (and their automorphism groups). These are mostly given as words in the standard generators listed in the following appendix, but

for the other cases, classes $2B$ and $2D$ in Fi_{24}, we give representative elements as base images relative to a given base for the group.

5. In Section 5.4 we determine the suborbits of the involution classes $2A$ in the Thompson group and $2B$ in the Harada-Norton group, using this information to determine the point-line collinearity graphs of a certain geometry. But the group HN has a second class of involutions, $2A$, and its automorphism group a third, $2C$. We give representatives for the suborbits in these classes.

Appendix B

Suborbit tables for the Fischer groups

In each of the following tables, each row corresponds to a $C_G(t)$-orbit (note that a row may extend to more than one line if there is a long entry in the third column; each orbit size listed in the second column marks the start of a new row). The first column gives the class C of G for which the suborbit lies in X_C. The second column gives the size of the suborbit and the third gives information on how to obtain a representative element for that suborbit. This is either a word in the standard generators of G giving an element g so that t^g is a representative element; or a list of symbols C_i' or D_i for D a class of G: the former meaning that the suborbit is the inverse of the i^{th} listed suborbit in X_C and the latter meaning the suborbit can be powered to from the i^{th} listed suborbit in X_D.

$G \cong Fi_{22}$, $X = 2B$, $t = (ababab^3)^{12}$.

C	Orbit size	Representative
$1A$	1	-
$2B$	270	$4A_1$
	360	$4E_1$ $6D_1$
	1152	$4E_2$ $6I_1$
$2C$	4320	$4D_1$
$3A$	1024	$6D_1$
$3C$	40960	$6I_1$
$4A$	34560	$b^1ab^1ab^3a$
$4D$	138240	b^2a
$4E$	69120	ab^1a
	69120	$4E_1'$
$5A$	442368	b^1a
$6D$	46080	b^2
$6I$	368640	b

$G \cong Fi_{22}$, $X = 2C$, $t = ((ababab^3)^2ab^3abab^3)^6$.

C	Orbit size	Representative
$1A$	1	t
$2A$	48	$6A_1$ $6A_2$ $6B_1$ $6E_1$ $6E_2$ $10A_1$ $10A_2$ $14A_1$ $14A_2$
$2B$	9	$4A_1$ $4C_2$ $4E_1$ $4E_4$
	216	$4A_3$ $4B_1$ $4C_1$ $4E_5$ $6D_1$ $6D_2$
	288	$4B_3$ $4E_7$ $4E_10$ $6D_4$ $6I_1$ $10B_1$
	576	$4A_2$ $4B_2$ $4B_4$ $4C_3$ $4E_2$ $4E_3$ $4E_6$ $4E_8$ $4E_9$ $6C_1$ $6D_3$ $6D_5$ $6I_2$ $6I_3$ $10B_2$
$2C$	432	$2C_2'$ $6F_1$
	432	$2C_1'$ $6F_2$
	432	$b^5ab^4abababababab^4$
	432	$4D_1$ $4D_2$ $4D_3$

	3456	$4D_4$ $6F_3$ $6F_4$ $6G_1$ $6H_1$ $6J_1$ $6J_2$ $6K_1$
$3A$	768	$6A_1$ $6D_2$ $6D_3$ $6D_4$ $6F_2$ $6F_4$ $15A_1$
	1536	$6A_2$ $6D_1$ $6D_5$ $6F_1$ $6F_3$ $15A_2$ $21A_1$
$3B$	8192	$6B_1$ $6C_1$ $6G_1$ $9A_1$ $9B_1$ $9B_2$
$3C$	12288	$6E_2$ $6I_3$ $6J_2$
	24576	$6E_1$ $6H_1$ $6I_1$ $6I_2$ $6J_1$
$3D$	98304	$6K_1$ $9C_1$
$4A$	1152	$8A_2$ $8B_1$
	9216	$8A_1$ $8A_3$ $8B_2$ $8B_4$ $12A_1$ $12A_2$ $12C_5$
	13824	$8A_4$ $8B_3$ $12C_1$ $12C_2$ $12C_3$ $12C_4$
$4B$	6912	$4B'_2$ $12D_2$
	6912	$4B'_1$ $12D_1$
	13824	$4B'_4$ $12D_3$ $12J_1$ $20A_2$
	13824	$4B'_3$ $12D_4$ $12J_2$ $20A_1$
$4C$	6912	$ababab^4ababab^2$
	6912	$4C'_1$
	27648	$8C_1$ $12H_1$ $12H_2$
$4D$	13824	$8D_3$
	13824	$8D_2$
	13824	$8D_5$
	110592	$8D_1$ $8D_4$ $8D_6$ $12E_1$ $12E_2$ $12F_1$ $12F_2$
		$12G_1$ $12G_2$ $12K_1$ $12K_2$
$4E$	1152	b^2ab^6abab
	1152	$4E'_1$
	6912	$4E'_4$
	6912	$b^3ab^2ab^4ab^2a$
	13824	$4E'_6$ $12I_2$
	13824	$4E'_5$ $12I_1$
	13824	$abab^6abab^2$
	13824	$4E'_7$
	27648	$4E_10'$ $12I_4$

	27648	$4E'_9$ $12I_3$
$5A$	147456	$10A_2$ $10B_1$ $15A_1$
	221184	$10A_1$ $10B_2$ $15A_2$
$6A$	9216	$6A'_2$ $30A_1$
	9216	$6A'_1$ $30A_2$
$6B$	24576	$18C_1$ $18C_2$
$6C$	73728	$12A_1$ $12A_2$ $12H_1$ $12H_2$ $18D_1$
$6D$	13824	$12C_1$ $12C_3$ $12D_2$ $12I_2$
	13824	$12C_2$ $12C_4$
	18432	$12D_1$ $12I_4$
	18432	$12D_3$ $12I_3$
	55296	$12C_5$ $12D_4$ $12I_1$
$6E$	73728	$6E'_2$
	73728	bab^6
$6F$	13824	b^5aba
	13824	$6F'_1$
	55296	b^5ab
	55296	$6F'_3$
$6G$	221184	$12E_1$ $12E_2$ $12F_1$ $12F_2$
$6H$	221184	$12G_1$ $12G_2$
$6I$	73728	$12J_1$
	73728	ab^2ab^3a
	110592	$12J_2$
$6J$	221184	$6J'_2$
	221184	$ababab^4a$
$6K$	884736	$12K_1$ $12K_2$
$7A$	294912	$14A_1$
	884736	$14A_2$ $21A_1$
$8A$	55296	b^2ab^6
	55296	$8A'_1$
	110592	$8A'_4$ $24B_1$

	110592	$8A'_3\ 24B_2$
$8B$	18432	ab^2ab^7
	18432	$8B'_1$
	110592	$8B'_4\ 24A_2$
	110592	$8B'_3\ 24A_1$
$8C$	221184	$ab^2ab^3ab^2$
$8D$	221184	$8D'_5$
	221184	ab^3ab^2
	221184	$8D'_6$
	221184	$8D'_2$
	221184	$abab^2a$
	221184	b^3ab^6a
$9A$	294912	$18D_1$
$9B$	294912	$18C_2$
	884736	$18C_1$
$9C$	1769472	$baba$
$10A$	442368	$10A'_2\ 30A_2$
	442368	$10A'_1\ 30A_1$
$10B$	442368	$20A_2$
	442368	$20A_1$
$12A$	73728	ab^3ab^2ab
	73728	$12A'_1$
$12C$	110592	ab^5aba
	110592	$12C'_1$
	110592	$24A_2$
	110592	$24B_2$
	221184	$24A_1\ 24B_1$
$12D$	110592	$12D'_2$
	110592	ab^3a
	221184	$12D'_4$
	221184	b^2abab^2a

$12E$	442368	$12E'_2$
	442368	b
$12F$	442368	$12F'_2$
	442368	ba
$12G$	442368	bab^5
	442368	$12G'_1$
$12H$	221184	ab^4a
	221184	bab^2abab
$12I$	110592	$12I'_2$
	110592	b^5abab^2a
	221184	$12I'_4$
	221184	b^2abab^4
$12J$	442368	$12J'_2$
	442368	bab^3ab
$12K$	884736	ab
	884736	$12K'_1$
$13A$	1769472	b^5a
$13B$	1769472	ab^3
$14A$	884736	$14A'_2$
	884736	b^4a
$15A$	884736	$30A_1$
	884736	$30A_2$
$18C$	884736	$abab^3a$
	884736	$18C'_1$
$18D$	884736	b^4
$20A$	884736	bab
	884736	$20A'_1$
$21A$	1769472	aba
$24A$	442368	$24A'_2$
	442368	b^6
$24B$	442368	$24B'_2$

	442368	$abab$
$30A$	884736	bab^2
	884736	$30A_1'$

$G \cong Fi_{22}:2$, $X = 2D$, $t = d^9$.

C	Orbit size	Representative
$1A$	1	t
$2B$	1575	$4A_1$
$3C$	22400	$cdcdc$
$4A$	37800	cd^4c

$G \cong Fi_{22}:2$, $X = 2E$, $t = (d^6cdc)^9$.

C	Orbit size	Representative
$1A$	1	t
$2B$	27	$4A_1$ $4A_2$ $4E_2$
	540	$4C_1$ $4E_4$ $6D_2$ $6I_3$ $10B_3$
	1080	$4A_3$ $4C_2$ $4E_1$ $4E_3$ $6C_1$ $6D_1$ $6D_3$ $6I_1$
		$6I_2$ $6I_4$ $10B_1$ $10B_2$
$2C$	3240	$4D_1$ $6H_1$
$3A$	2304	$6D_1$ $6D_2$ $6D_3$ $15A_1$
$3B$	5120	$6C_1$ $9A_1$ $9B_1$
$3C$	5760	$6I_2$
	11520	$6H_1$ $6I_1$ $6I_3$ $6I_4$
$4A$	2160	$8B_2$ $8B_4$
	3240	$8A_1$
	17280	$8A_2$ $8B_1$ $8B_3$ $12A_1$ $12A_2$ $12B_1$
$4C$	25920	$8C_1$
	51840	$8C_2$ $12G_1$
$4D$	103680	$12F_1$
$4E$	8640	$4E_2'$
	8640	d^2cdcd^5c

	51840	$4E'_4\ 12H_1$
	51840	$4E'_3\ 12H_2$
$5A$	27648	$10B_1$
	414720	$10B_2\ 10B_3\ 15A_1$
$6C$	138240	$12A_1\ 12A_2\ 12G_1\ 18C_1$
$6D$	34560	$12H_1$
	34560	$12H_2$
	103680	$12B_1$
$6H$	103680	$12F_1$
$6I$	103680	$6I'_2$
	103680	cd
	138240	cd^8c
	138240	$cdcd$
$7A$	552960	d^2cd^3
$8A$	103680	cd^2cdcd^2
	103680	$8A'_1$
$8B$	34560	$d^2cd^4cdcd^3$
	34560	$8B'_1$
	103680	$8B'_4$
	103680	d^5cd^2
$8C$	414720	$16A_2$
	414720	$16A_1$
$9A$	552960	$18C_1$
$9B$	552960	cd^3c
$10B$	414720	$10B'_2$
	414720	cd^2cd
	829440	$dcdcdcd^3c$
$11A$	3317760	dcd^2cd
$12A$	138240	d^2c
	138240	$12A'_1$
$12B$	414720	cd^3

$12F$	829440	d^2cd^2cdc
$12G$	829440	d^2
$12H$	414720	$dcdcd$
	414720	$12H_1'$
$15A$	1658880	d^3
$16A$	1658880	d^2cd
	1658880	$16A_1'$
$18C$	1658880	dcd

$G \cong Fi_{22} : 2$, $X = 2F$, $t = (d^3c)^{15}$.

C	Orbit size	Representative
$1A$	1	t
$2B$	63	$4E_1$ $6I_1$
	315	$4A_1$ $4A_2$ $4E_2$ $4E_4$ $6D_1$ $6D_2$ $6D_5$
	945	$4A_3$ $4C_1$ $4E_3$ $6C_1$ $6D_3$ $6D_4$ $6D_6$ $6I_2$ $6I_3$ $6I_4$ $10B_1$
$2C$	3780	$4D_1$ $6F_1$ $6F_2$ $6H_1$ $6H_2$ $6K_1$
$3A$	56	$6D_1$ $6D_4$
	240	$6D_2$ $6F_1$
	2520	$6D_3$ $6D_5$ $6D_6$ $6F_2$ $15A_1$ $21A_1$
$3B$	4480	$6C_1$ $9A_1$ $9A_2$ $9B_1$
$3C$	2240	$6H_1$ $6I_1$ $6I_3$
	20160	$6H_2$ $6I_2$ $6I_4$
$3D$	53760	$6K_1$ $9C_1$
$4A$	3780	$8B_2$ $12B_1$ $12C_1$ $12C_2$
	11340	$8A_2$ $12B_2$ $12C_3$ $12C_4$
	15120	$8A_1$ $8B_1$ $12A_1$ $12A_2$ $12B_3$
$4C$	45360	$8C_1$ $12G_1$
$4D$	60480	$12J_1$ $12J_2$
$4E$	15120	$d^2cd^3cd^3cdcd^3cd$
	15120	$4E_1'$

	45360	$4E_4'$ $12H_2$ $12H_3$
	45360	$4E_3'$ $12H_1$ $12H_4$
$5A$	241920	$10B_1$ $15A_1$
$6C$	120960	$12A_1$ $12A_2$ $12G_1$ $18C_1$ $18C_2$
$6D$	2520	$12C_1$ $12H_4$
	15120	$12B_1$ $12C_2$ $12C_3$
	15120	$d^2cd^4cdcd^3cd^2$
	15120	$6D_3'$
	22680	$12B_2$ $12C_4$ $12H_1$
	90720	$12B_3$ $12H_2$ $12H_3$
$6F$	60480	$6F_2'$
	60480	$d^2cd^4cd^2cd^2cd$
$6H$	60480	$6H_2'$
	60480	d^2cd^4cdcd
$6I$	20160	dcd^2cd^2cd
	60480	dcd^2cdcd^2cdcd
	60480	$6I_2'$
	181440	d^2cd^6cd
$6K$	483840	$12J_1$ $12J_2$
$7A$	1451520	$21A_1$
$8A$	181440	$8A_2'$ $24B_2$
	181440	$8A_1'$ $24B_1$
$8B$	181440	$8B_2'$ $24A_2$
	181440	$8B_1'$ $24A_1$
$8C$	362880	d^2cdcd^4cd
$9A$	80640	$18C_2$
	725760	$18C_1$
$9B$	161280	$dcdcdcd^2cdcdcd$
$9C$	967680	$dcdcdcdcdcd$
$10B$	725760	dcd
$11A$	2903040	d^2cdcd^2

$12A$	120960	dcd^5cd^4cd
	120960	$12A'_1$
$12B$	60480	$24A_1$
	181440	$24B_1$
	362880	$24A_2$ $24B_2$
$12C$	30240	dcd^7cd^3cdcd
	30240	$12C'_1$
	90720	$dcdcd^4cdcdcd$
	90720	$12C'_3$
$12G$	725760	$d^2cd^5cd^2cd$
$12H$	181440	$d^2cdcd^2cd^3cd$
	181440	$12H'_1$
	181440	$12H'_4$
	181440	dcd^2cdcd
$12J$	483840	dcd^2cd^2cdcd
	483840	$12J'_1$
$15A$	1451520	d^2cd^5cd
$18C$	725760	$18C'_2$
	725760	dcd^4cd^2cd
$21A$	2903040	dcd^3cd
$24A$	725760	$24A'_2$
	725760	dcd^3cdcd
$24B$	725760	$24B'_2$
	725760	d^2cdcd

$G \cong Fi_{23}$, $X = 2B$, $t = a$.

C	Orbit size	Representative
$1A$	1	t
$2B$	1386	$4C_1$ $6C_1$
	12672	$4C_2$ $6K_1$
$2C$	62370	$4B_1$

$3A$	5632	$6C_1$
$3C$	630784	$6K_1$
$4B$	7983360	$babab$
$4C$	1596672	$babab^2abababab$
	1596672	$4C_1'$
$5A$	14598144	b
$6C$	709632	$b^2ab^2abab^2ababab^2abababab$
$6K$	28385280	bab

$G \cong Fi_{23}$, $X = 2C$, $t = ((b^2a)^3ba)^6$.

C	Orbit size	Representative
$1A$	1	-
$2A$	180	$6A_1$ $6A_2$ $6A_3$ $6B_1$ $6E_1$ $6E_2$ $6E_3$ $6E_4$ $6J_1$ $6J_2$ $10A_1$ $10A_2$ $10A_3$ $10A_4$ $14A_1$ $14A_2$ $14A_3$ $14A_4$ $26A_1$ $26A_2$ $26B_1$ $26B_2$
$2B$	540	$4A_2$ $4C_1$ $4C_3$ $4C_5$ $4C_10$ $6C_2$ $6K_2$ $10B_2$
	3456	$4A_1$ $4A_3$ $4C_2$ $4C_4$ $4C_6$ $4C_7$ $4C_8$ $4C_12$ $6C_1$ $6C_6$ $6D_2$ $6K_1$ $6K_3$ $6K_4$ $10B_1$ $10B_6$ $14B_3$ $22A_1$
	4320	$4A_4$ $4C_9$ $4C_11$ $6C_3$ $6C_4$ $6C_5$ $6D_1$ $6K_5$ $6K_6$ $6K_7$ $10B_3$ $10B_4$ $10B_5$ $14B_1$ $14B_2$ $14B_4$
$2C$	810	$4B_1$ $4B_2$ $4B_3$ $4D_2$
	12960	$2C_3'$ $6I_2$ $6M_1$ $10C_2$
	12960	$2C_2'$ $6I_3$ $6M_2$ $10C_1$
	12960	$4B_4$ $4B_5$ $4D_1$ $6I_1$ $6I_4$

	103680	$4B_6$ $4B_7$ $4D_3$ $6F_1$ $6G_1$ $6G_2$ $6H_1$ $6I_5$ $6I_6$ $6I_7$ $6L_1$ $6L_2$ $6L_3$ $6L_4$ $6M_3$ $6M_4$ $6M_5$ $6M_6$ $6N_1$ $6N_2$ $6O_1$ $6O_2$ $10C_3$ $10C_4$
$3A$	1536	$6A_1$ $6C_1$ $6C_2$ $6C_3$ $6I_1$ $6I_3$ $6I_6$ $15A_1$
	23040	$6A_2$ $6A_3$ $6C_4$ $6C_5$ $6C_6$ $6I_2$ $6I_4$ $6I_5$ $6I_7$ $15A_2$ $15A_3$ $21A_1$ $21A_2$ $39A_1$ $39B_1$
$3B$	81920	$6B_1$ $6D_1$ $6D_2$ $6F_1$ $6H_1$ $9A_1$ $9B_1$ $9B_2$ $9B_3$ $9C_1$ $9C_2$ $9D_1$ $9D_2$
$3C$	73728	$6E_1$ $6K_3$ $6K_6$ $6L_1$ $6M_5$
	122880	$6E_4$ $6G_2$ $6K_1$ $6K_2$ $6K_5$ $6L_2$ $6L_3$ $6M_2$ $6M_4$ $15B_1$
	368640	$6E_2$ $6E_3$ $6G_1$ $6K_4$ $6K_7$ $6L_4$ $6M_1$ $6M_3$ $6M_6$ $15B_2$
$3D$	983040	$6J_2$ $6N_2$ $6O_1$ $9E_1$
	2949120	$6J_1$ $6N_1$ $6O_2$ $9E_2$
$4A$	103680	$4A_2'$ $12A_3$ $12J_1$ $20A_1$
	103680	$4A_1'$ $12A_4$ $12J_3$ $20A_2$
	414720	$4A_4'$ $12A_1$ $12A_5$ $12D_1$ $12J_4$ $12J_5$ $20A_4$ $28A_2$
	414720	$4A_3'$ $12A_2$ $12A_6$ $12D_2$ $12J_2$ $12J_6$ $20A_3$ $28A_1$
$4B$	103680	$8A_2$ $8B_4$
	103680	$8A_4$ $8B_5$
	103680	$8A_6$ $8B_2$
	414720	$8A_8$ $8B_10$ $12C_1$ $12G_4$
	1244160	$8A_9$ $8B_8$ $12C_2$ $12G_1$ $12G_2$ $12G_3$
	1658880	$8A_7$ $8A_10$ $8B_1$ $8B_3$ $8B_6$ $12B_1$ $12B_2$ $12E_1$ $12E_3$ $12G_5$ $12H_1$ $12H_2$ $12M_1$ $12M_2$

	1658880	$8A_1$ $8A_3$ $8A_5$ $8B_7$ $8B_9$ $12B_3$ $12B_4$
		$12C_3$ $12E_2$ $12E_4$ $12H_3$ $12H_4$ $12M_3$
		$12M_4$
$4C$	103680	$(abab^2)^2abab(ab^2)^4(ab)^5(ab^2)^2(ab)^4a$
	103680	$4C'_1$
	207360	$4C'_4$ $12I_1$
	207360	$4C'_3$ $12I_2$
	414720	$4C'_6$ $12N_1$
	414720	$4C'_5$ $12N_2$
	1244160	$4C_10'$ $12I_8$ $12N_5$ $20B_2$
	1244160	$4C_11'$ $12I_5$ $12I_6$
	1244160	$4C_12'$ $12I_9$ $12N_4$ $20B_3$
	1244160	$4C'_7$ $12I_7$ $12N_6$ $20B_1$
	1244160	$4C'_8$ $12I_3$ $12I_4$
	1244160	$4C'_9$ $12I_10$ $12N_3$ $20B_4$
$4D$	1244160	$4D'_2$
	1244160	$(ba)^5b^2ababab^2(ab)^3ab^2a$
	9953280	$8C_1$ $8C_2$ $12F_1$ $12F_2$ $12K_1$ $12K_2$
		$12K_3$ $12K_4$ $12L_1$ $12L_2$ $12L_3$ $12L_4$
		$12O_1$ $12O_2$ $12O_3$ $12O_4$
$5A$	1327104	$10A_1$ $10B_2$ $10B_3$ $10C_1$ $15A_1$
	6635520	$10A_4$ $10B_4$ $10B_6$ $10C_3$ $15A_2$ $15B_1$
		$35A_1$
	6635520	$10A_2$ $10A_3$ $10B_1$ $10B_5$ $10C_2$ $10C_4$
		$15A_3$ $15B_2$
$6A$	69120	$6A'_2$ $30B_3$
	69120	$6A'_1$ $30B_4$
	276480	$30B_1$ $30B_2$ $42A_1$ $42A_2$
$6B$	1474560	$18A_1$ $18A_2$ $18B_1$ $18B_2$ $18B_3$ $18B_4$
$6C$	110592	$12A_1$ $12I_2$ $12I_6$
	138240	$12A_4$ $12I_1$ $12I_7$

	414720	$6C_4'\ 30A_1$
	414720	$6C_3'\ 30A_2$
	829440	$12A_2\ 12A_6\ 12I_3\ 12I_4\ 12I_9\ 30A_4$
	1658880	$12A_3\ 12A_5\ 12I_5\ 12I_8\ 12I_10\ 30A_3$
$6D$	2211840	$12D_2\ 18E_2\ 18E_3\ 18E_4$
	2211840	$12D_1\ 18E_1\ 18F_1\ 18F_2$
$6E$	1105920	$6E_2'$
	1105920	$ab(ab^2)^2(ab)^3ab^2(ababab^2)^2a$
	2211840	$bababab^2ab^2ab^2abababab^2$
	2211840	$6E_3'$
$6F$	6635520	$12B_1\ 12B_2\ 12B_3\ 12B_4\ 12F_1\ 12F_2$
$6G$	3317760	$6G_2'\ 30C_1$
	3317760	$6G_1'\ 30C_2$
$6H$	13271040	$12E_1\ 12E_2\ 12E_3\ 12E_4\ 12K_1\ 12K_2\ 12K_3\ 12K_4\ 18C_1\ 18C_2\ 18D_1\ 18G_1\ 18G_2$
$6I$	414720	$12C_1\ 12G_3$
	829440	$6I_3'$
	829440	$ababab^2ab^2(ab)^4(ab^2)^3ab(ab^2)^2ababa$
	1244160	$12C_2\ 12G_1\ 12G_2\ 12G_4$
	1658880	$bab^2ababab^2ababababab^2abababababa$
	1658880	$6I_5'$
	9953280	$12C_3\ 12G_5$
$6J$	8847360	$6J_2'\ 18H_2$
	8847360	$6J_1'\ 18H_1$
$6K$	1105920	$12J_4\ 12N_2$
	2211840	$12J_3\ 12N_1\ 12N_6$
	3317760	$12J_5\ 12N_5$
	3317760	$12J_1\ 12N_3$
	3317760	$6K_6'$
	3317760	$abababab^2$

	6635520	$12J_2\ 12J_6\ 12N_4$
$6L$	3317760	$6L'_2$
	3317760	$abab^2ab^2abababab^2ababab^2ab^2a$
	6635520	$12H_1\ 12H_2\ 12L_2\ 12L_4$
	19906560	$12H_3\ 12H_4\ 12L_1\ 12L_3$
$6M$	3317760	$bab^2abab^2ab^2ab^2abababab^2a$
	3317760	$6M'_1$
	6635520	$6M'_4$
	6635520	$abab^2abababababababab^2$
	9953280	$6M'_6$
	9953280	$ab^2ab^2ab^2ababababababababab^2$
$6N$	26542080	$bababab^2ababab^2a$
	26542080	$6N'_1$
$6O$	26542080	$12M_1\ 12M_2\ 12O_1\ 12O_2$
	79626240	$12M_3\ 12M_4\ 12O_3\ 12O_4$
$7A$	2654208	$14A_2\ 14B_2$
	26542080	$14A_1\ 14A_3\ 14B_4\ 21A_1$
	39813120	$14A_4\ 14B_1\ 14B_3\ 21A_2\ 35A_1$
$8A$	3317760	$8A'_4$
	3317760	$8A'_3$
	3317760	$bababab^2abababab^2ababab^2abab^2a$
	3317760	$babab^2ab^2ab^2ab^2abababababab^2$
	9953280	$8A'_6$
	9953280	$babab^2abababababababab^2a$
	19906560	$8A'_8\ 24C_3$
	19906560	$8A'_7\ 24C_4$
	19906560	$8A_10'\ 24C_2$
	19906560	$8A'_9\ 24C_1$
$8B$	3317760	$8B'_2$
	3317760	$abababab^2ababababab^2ab^2ababab^2$
	9953280	$8B'_4$

	9953280	$bab^2ab^2abababababababab^2$
	9953280	$bababab^2abababab^2$
	9953280	$8B'_5$
	19906560	$8B_10'$ $24A_4$
	19906560	$8B'_9$ $24A_3$
	19906560	$8B'_8$ $24A_2$
	19906560	$8B'_7$ $24A_1$
$8C$	39813120	$8C'_2$ $24B_1$ $24B_3$
	39813120	$8C'_1$ $24B_2$ $24B_4$
$9A$	17694720	$18D_1$ $27A_1$
$9B$	4423680	$18B_1$ $18E_1$ $18E_2$
	13271040	$18B_4$ $18E_3$
	26542080	$18B_2$ $18B_3$ $18E_4$
$9C$	8847360	$18A_2$ $18C_1$ $18F_1$
	26542080	$18A_1$ $18C_2$ $18F_2$
$9D$	17694720	$18G_2$
	53084160	$18G_1$
$9E$	53084160	$18H_1$
	159252480	$18H_2$
$10A$	13271040	$10A'_2$ $30B_3$
	13271040	$10A'_1$ $30B_4$
	19906560	$10A'_4$ $30B_1$
	19906560	$10A'_3$ $30B_2$
$10B$	13271040	$20A_1$ $20B_4$
	13271040	$20A_2$ $20B_1$
	19906560	$10B'_4$ $30A_1$
	19906560	$10B'_3$ $30A_2$
	39813120	$20A_3$ $20B_3$ $30A_4$
	39813120	$20A_4$ $20B_2$ $30A_3$
$10C$	19906560	$10C'_2$
	19906560	$bababab^2abababab^2ab^2a$

	39813120	$10C_4'\ 30C_2$
	39813120	$10C_3'\ 30C_1$
$11A$	159252480	$22A_1$
$12A$	3317760	$12A_2'$
	3317760	$abab^2abababababab^2ababab^2a$
	3317760	$12A_4'$
	3317760	$bab^2abababab^2ababab^2$
	9953280	$12A_6'\ 60A_2$
	9953280	$12A_5'\ 60A_1$
$12B$	6635520	$12B_2'$
	6635520	$bab^2abababababababab^2$
	19906560	$12B_4'$
	19906560	$abab^2abababababab^2abab^2a$
$12C$	3317760	$24A_1$
	9953280	$24A_3$
	39813120	$24A_2\ 24A_4$
$12D$	26542080	$12D_2'\ 36B_1$
	26542080	$12D_1'\ 36B_2$
$12E$	13271040	$12E_3'\ 36A_2$
	13271040	$12E_4'$
	13271040	$12E_1'\ 36A_1$
	13271040	$abababab^2ab^2ababab^2$
$12F$	39813120	$24B_2\ 24B_3$
	39813120	$24B_1\ 24B_4$
$12G$	9953280	$12G_3'$
	9953280	$24C_2$
	9953280	$babab^2ababab^2ababab^2$
	9953280	$24C_4$
	39813120	$24C_1\ 24C_3$
$12H$	13271040	$12H_2'$
	13271040	$bab^2ab^2ababab^2ab^2abababab^2a$

	39813120	$12H'_4$
	39813120	$bab^2ab^2abababa$
$12I$	3317760	$12I'_2$
	3317760	$abababab^2abababab^2ab^2ababab^2$
	9953280	$abababab^2abababababab^2abab^2$
	9953280	$abababab^2ababab^2ab^2ab^2$
	9953280	$12I'_3$
	9953280	$12I'_4$
	19906560	$12I'_8$
	19906560	$babababababababab^2$
	19906560	$12I_10'$
	19906560	$bababababababab^2ab^2ab^2a$
$12J$	13271040	$12J'_3$
	13271040	$12J'_4$
	13271040	$bababab^2ababababab^2abab^2a$
	13271040	$ababab^2ababab^2ab^2ababab^2a$
	39813120	$abababab^2abab^2$
	39813120	$12J'_5$
$12K$	39813120	$bababab^2ababab^2ababab^2$
	39813120	$12K'_3$
	39813120	$bab^2ab^2ab^2ab^2ababababababa$
	39813120	$12K'_1$
$12L$	39813120	$bab^2ab^2abababab^2abab^2a$
	39813120	$bab^2ab^2ab^2abababab^2a$
	39813120	$12L'_2$
	39813120	$12L'_1$
$12M$	26542080	$abab^2abababababab^2a$
	26542080	$12M'_1$
	79626240	b
	79626240	$12M'_3$
$12N$	13271040	$babababababababab^2$

	13271040	$12N_1'$
	39813120	$bab^2ab^2abab^2ababab^2a$
	39813120	$12N_3'$
	39813120	$abab^2abababababab^2$
	39813120	$12N_5'$
$12O$	79626240	$12O_3'$
	79626240	$ababab^2ababab^2$
	79626240	$bab^2abababab^2ab^2ab^2a$
	79626240	$12O_2'$
$13A$	53084160	$26A_2$
	159252480	$26A_1$ $39A_1$
$13B$	53084160	$26B_2$
	159252480	$26B_1$ $39B_1$
$14A$	26542080	$bab^2ababab^2ab^2abababab a$
	26542080	$14A_1'$
	79626240	$14A_4'$ $42A_2$
	79626240	$14A_3'$ $42A_1$
$14B$	39813120	$bababab^2ababababababa$
	39813120	$14B_1'$
	79626240	$28A_2$
	79626240	$28A_1$
$15A$	13271040	$30A_1$ $30B_3$
	39813120	$30A_2$ $30A_3$ $30B_2$
	79626240	$30A_4$ $30B_1$ $30B_4$
$15B$	53084160	$30C_2$
	159252480	$30C_1$
$17A$	318504960	$bababab^2ab^2abab^2$
$18A$	26542080	$18A_2'$
	26542080	$babab^2ab^2ab^2ababab ab^2$
$18B$	26542080	$babababab ab^2$
	26542080	$18B_1'$

	79626240	$babab^2abababababab^2$
	79626240	$18B_3'$
$18C$	79626240	$babababab^2ab^2a$
	79626240	$18C_1'$
$18D$	159252480	$36A_1$ $36A_2$
$18E$	26542080	$36B_1$
	39813120	$bababab^2abababababa$
	39813120	$18E_2'$
	79626240	$36B_2$
$18F$	26542080	$abababab^2ab^2abab^2$
	79626240	$ababababababab^2ab^2ababab^2$
$18G$	159252480	$bab^2abababababa$
	159252480	$18G_1'$
$18H$	159252480	$bab^2ababab^2$
	159252480	$18H_1'$
$20A$	26542080	$bab^2abababab^2ab^2abababab^2$
	26542080	$20A_1'$
	79626240	$20A_4'$ $60A_1$
	79626240	$20A_3'$ $60A_2$
$20B$	79626240	$20B_2'$
	79626240	$bababab^2ab^2a$
	79626240	$bababab^2abababab^2$
	79626240	$20B_3'$
$21A$	159252480	$42A_2$
	159252480	$42A_1$
$22A$	159252480	$ababab^2abababab^2$
$24A$	79626240	$24A_4'$
	79626240	$24A_3'$
	79626240	$babababab^2a$
	79626240	$ababababab^2ab^2ab^2$
$24B$	79626240	$24B_2'$

	79626240	$bab^2abab^2abababab^2a$
	79626240	$bababab^2a$
	79626240	$24B_3'$
$24C$	79626240	$24C_2'$
	79626240	$bab^2abababababab^2$
	79626240	$babababababab^2$
	79626240	$24C_3'$
$26A$	159252480	$bababab^2$
	159252480	$26A_1'$
$26B$	159252480	$ababab^2abababab^2$
	159252480	$26B_1'$
$27A$	318504960	$bababab^2ab^2$
$28A$	159252480	$abab^2abababab^2$
	159252480	$28A_1'$
$30A$	39813120	$30A_2'$
	39813120	$babababab^2abab^2a$
	79626240	$60A_2$
	79626240	$60A_1$
$30B$	79626240	$abab^2ababab^2$
	79626240	$30B_1'$
	79626240	$babababab^2ab^2abab^2$
	79626240	$30B_3'$
$30C$	159252480	$30C_2'$
	159252480	$bab^2abab^2abab^2$
$35A$	318504960	$babababab^2a$
$36A$	159252480	$36A_2'$
	159252480	bab^2ab^2ababa
$36B$	159252480	$36B_2'$
	159252480	$abab^2abab^2$
$39A$	318504960	$bab^2ababab^2$
$39B$	318504960	$babab^2$

$42A$	159252480	$bababababab^2ab^2a$
	159252480	$42A_1'$
$60A$	159252480	$bababab^2$
	159252480	$60A_1'$

$G \cong Fi_{24}{}'$, $X = 2A$, $t = a$.

C	Orbit size	Representative
$1A$	1	-
$2A$	720	$4B_2$, $6F_1$
	123552	$4B_2$, $6A_1$
$2B$	1216215	$4A_1$, $6A_1$
$3A$	56320	$6A_1$
$3C$	20500480	$6F_1$
$3E$	60825600	$(ba)^3b^2(ab)^3$
$4A$	389188800	$(ba)^4b$
$4B$	88957440	$(b^2a)^2ba(b^2a)^2b(ab)^3$
	88957440	$4B_1'$
$5A$	1423319040	b^2ab
$6A$	19768320	$b^2(ab)^5(ab^2)^2ab$
$6F$	2767564800	bab

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