ZIEGLER SPECTRA OF VALUATION RINGS

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We show that the Ziegler spectrum Zg_R of a valuation domain R is sober, i.e. every nonempty, irreducible closed set is the closure of a point.

We use the Ziegler spectrum as a tool to prove the following result conjectured by Puninksi, Puninskaya and Toffalori in [PPT07] for valuation domains with dense value group:

Let V be an effectively given valuation domain. Then the following are equivalent:

(i) The theory of V-modules, T_V, is decidable.

(ii) There exists an algorithm which, given $a, b \in V$, answers whether a is in the radical of bV.

We investigate the Ziegler spectrum restricted to the subspace of injectives $\mathrm{Zg}_R|_{\mathrm{inj}}$ for R a valuation ring, a Prüfer ring and the fibre product of two copies of the same valuation ring over the residue field. For these rings, we show that $\mathrm{Zg}_R|_{\mathrm{inj}}$ is sober and compare it with the Hochster dual of the spectrum of R.

Declaration

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Chapter 1

Introduction

A valuation ring is a commutative ring whose ideals form a chain.

The Ziegler spectrum Zg_R of a ring R is a topological space attached to the module category of R. It was defined by Ziegler in [Zie84]. The points of Zg_R are isomorphism classes of indecomposable pure-injective right R-modules and the closed sets correspond to complete theories of modules closed under arbitrary direct sums. The space Zg_R plays a crucial role in understanding the model theory of modules. Many questions in the model theory of modules can be rephrased in terms of Zg_R. In this thesis we investigate Zg_R for R a valuation ring and use it as a tool to prove decidability results.

Chapter 2 contains background material which can be found in more detail in [Lam99], [Pre88] and [Pre09].

The main result in chapter 3 is that the Ziegler spectrum of a valuation domain is sober. We say a topological space is sober if its irreducible closed sets are the closures of points. Soberness for the Ziegler spectrum was shown by Herzog in [Her93] for countable rings. The proof given in [Her93] does not have an obvious generalisation to arbitrary rings.

In the same paper, Herzog used Prest's notion of duality for pp-formulae to give a lattice isomorphism between the lattice of open sets of the right and left Ziegler spectra of a ring. If both the left and ring Ziegler spectra of R are sober this means that up to topologically indistinguishable points, the left and right Ziegler spectra of a ring are homeomorphic. In the situation of commutative rings this in general gives a non-trivial automorphism of Zg_R up to topologically indistinguishable points. For valuation domains we do better than this by giving in 3.4.1 a natural continuous automorphism at the level of points.

We use a formulation of the Ziegler spectrum in terms of equivalence classes of pairs of ideals which is specific to valuation domains. This can be found in [Pun99]. So the proof we give of soberness for the Ziegler spectrum of a valuation domains is unlikely to generalise to arbitrary rings.

We show in 2.3.28 that an arbitrary commutative ring R has sober Ziegler spectrum if and only if each of its localisations at maximal ideals has sober Ziegler spectrum. This result relies on a result due to Prest [Pre09, Theorem 5.5.3] which says that an epimorphism of rings $R \to S$ induces a continuous embedding from Zg_S to Zg_R and the image of this embedding is closed.

Therefore the Ziegler spectrum of a Prüfer domain is sober (cf. 3.6.17).

In chapter 4 we show that, for an effectively given valuation domain V, the following are equivalent:

1. The theory of V-modules is decidable.

2. There is an algorithm which, given $a, b \in V$, answers whether $a \in \mathrm{rad}(bV)$.

This was conjectured for valuation domains with dense value group in [PPT07] where it was shown that if V is an effectively given valuation domain with dense archimedean value group then the theory of modules is decidable (since such a valuation domain has only one non-zero prime ideal, the radical condition is trivial).

The proof of this result goes via the Ziegler spectrum. This method for proving decidability of theories of modules was described in [Zie84] and is the method used in [PPT07].

In chapter 5 we investigate topologies on the set of isomorphism classes of indecomposable injectives, inj_R, over valuation rings, Prüfer rings and fibre products of valuation rings. We consider two topologies, the Ziegler spectrum restricted to injective modules and the $\mathbf{ideals}_R$ topology. The $\mathbf{ideals}_R$ topology is a refinement of the Ziegler topology restricted to injectives. It is defined by giving a basis of open sets

$$\mathcal{O}(I) = \{E \in \mathrm{inj}_R \mid \mathrm{Hom}_R(R/I, E) \neq 0\}$$

where $I \lhd R$. We use a simplified formulation of the Ziegler spectrum restricted to injectives [PR10, Corollary 7.4] which says that $\mathrm{Zg}_R|_{\mathrm{inj}}$ has basis of open sets $\mathcal{O}(I)$ where I is a pp-definable ideal.

The map t taking an indecomposable injective E to the prime ideal consisting of the elements of R which annihilate non-zero elements of E is a continuous map from inj_R to $\mathrm{Spec}^* R$ when either topology is put on inj_R. In the case of a coherent ring $\mathrm{Zg}_R|_{\mathrm{inj}}$ is homeomorphic to $\mathrm{Spec}^* R$ after identifying topologically indistinguishable points in $\mathrm{Zg}_R|_{\mathrm{inj}}$. This is not necessarily the case for non-coherent rings. For example [GP08b] or 5.1.6. We use the map t to compare $\mathrm{Spec}^* R$ with $\mathrm{Zg}_R|_{\mathrm{inj}}$, by which we mean we look at the complexity of the fibres of t. For R a valuation ring, there is at most one fibre containing topologically distinguishable points and that fibre contains at most two pairwise topologically distinguishable points (5.1.5). We give an example of a fibre product of valuation rings where the fibre of the maximal ideal contains 3 pairwise topologically distinguishable points (5.3.8).

We show that inj_R is sober with both the Ziegler topology and the $\mathbf{ideals}_R$ topology for R a Prüfer ring or a fibre product of two copies of the same valuation ring over its residue field, 5.0.18, 5.2.6, 5.3.3, 5.3.4.

Chapter 2

Background

Throughout, we will assume that all rings are unital.

2.1 Injective modules and irreducible ideals

Background material on injectives can be found in [Lam99, Chapter 3].

Definition 2.1.1. *Let R be a ring. We say a module E is injective if for every embedding $i : A \hookrightarrow B$ and map $f : A \to E$ there exists a map $h : B \to E$ such that $h \circ i = f$.*

Proposition 2.1.2. *Let R be a ring and N an R-module. There exists an injective module $\mathrm{E}(N)$ such that N is a submodule of $\mathrm{E}(N)$ and for all injective modules E' and all embeddings $f : N \hookrightarrow E'$ there is an extension of f embedding $\mathrm{E}(N)$ into E'. Moreover, $\mathrm{E}(N)$ is unique up to isomorphism.*

Definition 2.1.3. *Let R be a ring. Let N and $\mathrm{E}(N)$ be as in the above proposition. We call $\mathrm{E}(N)$ the injective hull of N.*

Definition 2.1.4. *Let R be a ring. A right ideal $I \lhd R$ is said to be irreducible if for all right ideals $K, L \lhd R$, $K \cap L = I$ implies either $K = I$ or $L = I$.*

Definition 2.1.5. *Let R be a ring. We say an R-module M is uniform if for all non-zero submodules $N_1, N_2 \subseteq M$, $N_1 \cap N_2 \neq 0$.*

The following theorem is taken from [Lam99] page 84.

Theorem 2.1.6. *For any injective right module M over a ring R, the following conditions are equivalent:*

(i) M *is indecomposable.*

(ii) $M \neq 0$ *and* $M = \mathrm{E}(M')$ *for any non-zero submodule* $M' \subseteq M$*.*

(iii) M *is uniform.*

(iv) $M = \mathrm{E}(U)$ *for some uniform module* U*.*

(v) $M = \mathrm{E}(R/I)$ *for some irreducible right ideal* $I \subsetneq R$*.*

(vi) M *is strongly indecomposable; that is,* $\mathrm{End}(M)$ *is a local ring.*

We denote the set of isomorphism classes of indecomposable injective R-modules by inj_R.

Lemma 2.1.7. *Let R be a commutative ring and E an indecomposable injective R-module. Then for all non-zero $w \in E$, $\mathrm{ann}_R w$ is an irreducible ideal in R.*

Proof. Suppose $w \in E$ and w is non-zero. Then $wR = R/\mathrm{ann}_R w$. Let $I = \mathrm{ann}_R w$. Suppose $K, L \lhd R$ and $K \cap L = I$ then $K \supseteq I$ and $L \supseteq I$ so K/I and L/I are submodules of R/I. Since $K \cap L = I$, $K/I \cap L/I = 0$. Note that a submodule of a uniform module is uniform. Hence, since R/I is uniform, either $K/I = 0$ or $L/I = 0$ so either $K = I$ or $L = I$. □

Definition 2.1.8. *Let R be a commutative ring and $I, J \lhd R$. We define the ideal quotient:*

$$(I : J) := \{r \in R | Jr \subseteq I\}.$$

If $I \lhd R$ and $x \notin I$ then we write $(I : x)$ for $(I : xR)$.

Lemma 2.1.9. *[Nis72] Let R be a commutative ring and let $I, J \lhd R$ be irreducible. Then $\mathrm{E}(R/I) \cong \mathrm{E}(R/J)$ if and only if there exists $r \in R \backslash I$ and $s \in R \backslash J$ such that $(I : r) = (J : s)$.*

Definition 2.1.10. *Let R be a commutative ring and E be an indecomposable injective R-module, then the attached prime of E, denoted* Att$E$ *is the set of all $r \in R$ such that there exists a non-zero $w \in E$ with $wr = 0$.*

Lemma 2.1.11. *Let R be a commutative ring and E be an indecomposable injective R-module. Then* Att$E$ *is a prime ideal.*

Proof. Suppose $r, s \in \mathrm{Att}E$ and $\lambda \in R$. There exists non-zero $w, u \in E$ such that $wr = 0$ and $us = 0$. Then $wr\lambda = 0$ so $r\lambda \in \mathrm{Att}E$. Since E is uniform $wR \cap uR \neq 0$. Take non-zero $v \in wR \cap uR$ then $vr = 0$ since $v \in wR$ and $vs = 0$ since $v \in uR$. Therefore $v(r + s) = 0$ so $r + s \in \mathrm{Att}E$. Hence $\mathrm{Att}E$ is an ideal.

Suppose $s, r \notin \mathrm{Att}E$. Then for any non-zero $w \in E$, $ws \neq 0$ and $wr \neq 0$ therefore for any non-zero $w \in E$, $wrs \neq 0$. Hence $rs \notin \mathrm{Att}E$. So $\mathrm{Att}E$ is a prime ideal. □

Definition 2.1.12. *Let R be a commutative ring and $I \lhd R$ an irreducible ideal then the attached prime of I, denoted $I^{\#}$, is the set of all $x \in R$ such that there exists $g \notin I$ with $gx \in I$.*

Lemma 2.1.13. *Let R be a commutative ring and $I \lhd R$ a irreducible ideal. Then*

$$I^{\#} = \bigcup_{x \notin I} (I : x).$$

Lemma 2.1.14. *Let R be a commutative ring and $I \lhd R$ an irreducible ideal. Then* $\mathrm{AttE}(R/I) = I^{\#}$.

Proof. Suppose $x \in I^{\#}$. Then there exists $r \notin I$ such that $xr \in I$ so $x \in (I : r)$. By lemma 2.1.9, $\mathrm{E}(R/I) \cong \mathrm{E}(R/(I : r))$. Hence $R/(I : r) \hookrightarrow \mathrm{E}(R/I)$. So $x \in \mathrm{Att}E$.

Suppose $x \in \mathrm{Att}E$. Take non-zero $w \in E$ to be such that $wx = 0$. Then $x \in \mathrm{ann}_R w$. By theorem 2.1.6, since E is indecomposable, $E = \mathrm{E}(wR)$. Since $wR \cong R/\mathrm{ann}_R w$, $E \cong \mathrm{E}(R/\mathrm{ann}_R w)$. By lemma 2.1.9, there exists $s \notin \mathrm{ann}_R w$ and $r \notin I$ such that $(I : r) = (\mathrm{ann}_R w : s)$. Therefore $x \in (I : r)$, since $x \in \mathrm{ann}_R w \subseteq (\mathrm{ann}_R w : s)$. So $xr \in I$ and $r \notin I$. Therefore $x \in I^{\#}$. □

Lemma 2.1.15. *Let R be a commutative ring and $I \lhd R$ an irreducible ideal then the attached prime I is a prime ideal.*

Proof. Let $I \lhd R$ be irreducible. By 2.1.14, $\mathrm{AttE}(R/I) = I^{\#}$. By 2.1.11, $\mathrm{AttE}(R/I)$ is a prime ideal. Hence $I^{\#}$ is prime. $\square$

2.2 pp-formulae, pp-types and pure-injectives

Background material for this section can be found in [Pre09] or [Pre88].

If R is a ring then the language of right R-modules, denoted $\mathcal{L}_R$, is $(+, 0, \{r\}_{r\in R})$ where $+$ is a binary function symbol, 0 is a constant symbol and for each $r \in R$, r is a unary function symbol. By abuse of notation we write $\phi(\bar{x}) \in \mathcal{L}_R$ to mean $\phi(\bar{x})$ is an $\mathcal{L}_R$-formula. We write ${}_R\mathcal{L}$ for the language of left R-modules.

Definition 2.2.1. *A formula $\phi(\bar{x}) \in \mathcal{L}_R$ is a pp-formula if it is of the form:*

$$\exists \bar{y} (\bar{y}\bar{x}) A = 0$$

where A is a matrix with entries in R.

We extend the term pp-formula to include formulae equivalent in the theory of R-modules to a pp-formula. We call a pp-formula in n free variables, a pp-n-formula.

Proposition 2.2.2. *Let R be a ring, ϕ a pp-formula and M a right R-module.*

(i) $\phi(M)$ is an additive subgroup of M^n where n is the number of free variables in ϕ.

(ii) If $\phi(x)$ is a pp-1-formula then $\phi(M)$ is a left $\mathrm{End}(M)$-submodule of M.

(iii) If R is commutative and $\phi(x)$ is a pp-1-formula then $\phi(M)$ is an R-submodule of M.

We say pp-formulae ϕ, ψ are equivalent if their solution sets are equal in every R-module, that is, they are equivalent in the theory of R-modules.

Proposition 2.2.3. *The set of equivalence classes of pp-n-formulae endowed with the ordering, $\phi \leq \psi$ if and only if $\phi \to \psi$, is a modular lattice where the supremum of pp-n-formulae ϕ, ψ is*

$$\phi(\bar{x}) + \psi(\bar{x}) = \exists y_1 y_2 (\phi(\bar{y_1}) \wedge \psi(\bar{y_2}) \wedge \bar{x} = \bar{y_1} + \bar{y_2})$$

and infimum is $\phi(\bar{x}) \wedge \psi(\bar{x})$.

Note that, though $+$ is the join in the lattice of pp-n-formulae, it should not be confused with disjunction in $\mathcal{L}_R$.

Definition 2.2.4. *Let R be a ring. An invariants sentence is a sentence in $\mathcal{L}_R$ which expresses the statement $\left|\frac{\phi(\bar{x})}{\psi(\bar{x})}\right| \geq n$ in all modules, for some ϕ, ψ pp-formulae of the same arity and $n \in \mathbb{N}$.*

Theorem 2.2.5 (Baur-Monk Theorem)**.** *[Pre88] Let R be a ring. Then, for every formula $\xi(\bar{x}) \in \mathcal{L}_R$ there is a formula $\tau(\bar{x})$, a boolean combination of pp-formulae and invariants sentences, which is equivalent to $\xi(\bar{x})$ in all R-modules.*

Corollary 2.2.6. *Let R be a ring. Then every sentence $\chi \in \mathcal{L}_R$ is equivalent to a boolean combination of invariants sentences $\left|\frac{\phi(x)}{\psi(x)}\right| \geq n$ where ϕ, ψ are pp-1-formulae.*

Proof. It is enough to show that for any $n \in \mathbb{N}$ greater than 1 and any pair of pp-n-formulae ϕ, ψ, the sentence $\left|\frac{\phi(\bar{x})}{\psi(\bar{x})}\right| \geq m$ can be written as a boolean combination of sentences $\left|\frac{\phi'(\bar{x})}{\psi'(\bar{x})}\right| \geq m'$ where each pair ϕ', ψ' has strictly less that n free variables.

Let ϕ, ψ be pp-n-formulae and N an R-module. Let $f : \frac{\phi(N^n)}{\psi(N^n)} \twoheadrightarrow \frac{\exists y \phi(N^{n-1}, y)}{\exists y \psi(N^{n-1}, y)}$ be the map given by projecting onto the first $(n-1)$-variables, this map is an epimorphism of abelian groups. The kernel of this map is isomorphic to $\frac{\phi(\bar{0}, N)}{\psi(\bar{0}, N)}$.

Hence $\left|\frac{\phi(N^n)}{\psi(N^n)}\right| \geq m$ if and only if $\left|\frac{\phi(\bar{0}, N)}{\psi(\bar{0}, N)}\right| \cdot \left|\frac{\exists y \phi(N^{n-1}, y)}{\exists y \psi(N^{n-1}, y)}\right| \geq m$. Therefore the sentence $\left|\frac{\phi}{\psi}\right| \geq m$ is equivalent to

$$\bigvee_{i=1}^{m} \left(\left|\frac{\phi(\bar{0}, x)}{\psi(\bar{0}, x)}\right| \geq i \wedge \left|\frac{\exists y \phi(\bar{x}, y)}{\exists y \psi(\bar{x}, y)}\right| \geq \llcorner \frac{m}{i} \lrcorner \right)$$

where $\llcorner \frac{m}{i} \lrcorner$ is the least integer greater than $\frac{m}{i}$.

$\square$

For the following lemma see [Pre88, Lemma 2.10].

Lemma 2.2.7. *Suppose $\phi(\bar{x}), \psi(\bar{x})$ are pp-formulae with the same number of free variables and let $\{M_i \mid i \in \mathcal{I}\}$ be any set of modules. Then*

$$\phi(\bigoplus_{i \in \mathcal{I}} M_i) = \bigoplus_{i \in \mathcal{I}} \phi(M_i)$$

and

$$\frac{\phi(\bigoplus_{i\in\mathcal{I}} M_i)}{\psi(\bigoplus_{i\in\mathcal{I}} M_i)} = \bigoplus_{i\in\mathcal{I}} \frac{\phi(M_i)}{\psi(M_i)}.$$

Definition 2.2.8. *Let R be a ring and M an R-module. We say a pp-pair ϕ/ψ is minimal in the theory of M if $\phi(M)$ strictly contains $\psi(M)$ but no pp-definable subgroup of M lies strictly between $\phi(M)$ and $\psi(M)$.*

We now define a map from the set of right pp-n-formulae to the set of left pp-n-formulae of a ring. This map induces a lattice anti-isomorphism between the lattice of equivalence classes of right pp-n-formulae and the lattice of equivalence classes of left pp-n-formulae. It sends the right pp-1-formula $xr = 0$ to the left pp-1-formula $r|x$ and the right pp-1-formula $r|x$ to the left pp-1-formula $rx = 0$.

Definition 2.2.9. *Let R be a ring and let ϕ be a pp-n-formula in the language of right R-modules $\exists \bar{y}(\bar{x}, \bar{y})H = 0$. Then $\mathrm{D}\phi$ is the pp-n-formula in the language of left R-modules $\exists \bar{z} \begin{pmatrix} I & H' \\ 0 & H'' \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{z} \end{pmatrix} = 0$, where $\begin{pmatrix} H' \\ H'' \end{pmatrix} = H$.*

Proposition 2.2.10. *[Pre09][Pre88] For each $n \geq 1$ the operator D is a duality between the lattice of equivalence classes of pp-n-formulae in the language of right R-modules and the lattice of equivalence classes of pp-n-formulae in the language of left R-modules. That is, for every pp-n-formula ϕ we have $\mathrm{D}^2\phi$ equivalent to ϕ and also $\psi \leq \phi$ if and only if $\mathrm{D}\phi \leq \mathrm{D}\psi$.*

Corollary 2.2.11. *For all pp-formulae ϕ, ψ in the same number of free variables we have $\mathrm{D}(\phi + \psi) = \mathrm{D}\phi \wedge \mathrm{D}\psi$ and $\mathrm{D}(\phi \wedge \psi) = \mathrm{D}\phi + \mathrm{D}\psi$.*

Definition 2.2.12. *Let R be a ring and N, M R-modules. Let $f : N \hookrightarrow M$ be an embedding. We say that f is a pure-embedding if for every pp-formula $\phi(\bar{x})$, $f(\phi(N)) = \phi(M) \cap f(N)^n$ where n is the arity of $\phi(\bar{x})$.*

We say N a submodule of M is pure in M if its embedding into M is pure.

Definition 2.2.13. *Let R be a ring. We say a module N is pure-injective if for every pure-embedding $i : A \hookrightarrow B$ and map $f : A \to N$ there exists a map $h : B \to N$ such that $h \circ i = f$.*

Note that injective modules are pure-injective. We denote the set of isomorphism classes of indecomposable pure-injective modules by pinj_R.

Proposition 2.2.14. *Let R be a ring and M be an R-module then there exists a pure-injective module $PE(M)$ such that M is a pure-submodule of $PE(M)$ and for all pure-injectives M' and all pure-embeddings $f : M \hookrightarrow M'$ there is an extension of f embedding $PE(M)$ purely into M'. Moreover, $PE(M)$ is unique up to isomorphism over M.*

Definition 2.2.15. *Let R be a ring. Let M and $PE(M)$ be as in the above proposition. We call $PE(M)$ the pure-injective hull of M.*

Proposition 2.2.16. *[Pre88, Corollary 4.11] Let R be a ring and N an indecomposable pure-injective R-module. Then for any non-zero $a, b \in N$, there exists a pp-formula $\phi(x, y)$ such that $N \models \phi(a, b) \wedge \neg\phi(a, 0)$. We call such a formula a linking formula.*

Definition 2.2.17. *Let R be a ring and M an R-module. The pp-type, $\mathrm{pp}_M(\bar{m})$,of a tuple of elements $\bar{m} \in M$ is the set of pp-formulae it satisfies.*

Lemma 2.2.18. *[Pre09, 3.2.5] Let p be a filter in the lattice of pp-n-formulae. Then there is a module M and an n-tuple $\bar{m} \in M$ such that $\mathrm{pp}^M(\bar{m}) = p$.*

Definition 2.2.19. *We say a pp-type is irreducible if it can be realised in an indecomposable pure-injective module.*

Theorem 2.2.20 (Ziegler's criterion)**.** *[Zie84] Let p be a pp-n-type. Then the following are equivalent:*

(i) For all $\phi, \psi \notin p$ there exists $\sigma \in p$ such that $\phi \wedge \sigma + \psi \wedge \sigma \notin p$.

(ii) The pp-n-type p is irreducible.

Proposition 2.2.21. *[ZHZ78, Theorem 9] Let R be a ring and N an indecomposable pure-injective module. Then N has local endomorphism ring.*

2.3 The Ziegler Spectrum

Background material on the Ziegler spectrum can be found in [Pre09, Chapter 5].

Definition 2.3.1. *[Zie84] Let R be a ring. The (right) Ziegler spectrum, Zg_R, is a topological space with set of points isomorphism classes of indecomposable pure-injective modules and a basis of open sets:*

$$\left(\frac{\phi}{\psi}\right) = \{N \in \mathrm{pinj}_R \mid \phi(N) \supsetneq \psi(N)\}$$

where ϕ, ψ are pp-1-formulae.

We denote the left Ziegler spectrum by ${}_R\mathrm{Zg}$. Throughout this text, we will say a subset X of a topological space $\mathcal{T}$ is compact if every open cover of X has a finite subcover. Note that we do not include Hausdorff in our definition of compact.

Proposition 2.3.2. *[Zie84, 4.9][Pre09, Theorem 5.1.22] Let R be a ring. Then:*

(i) For all pp-1-formulae ϕ, ψ, the open set $\left(\frac{\phi}{\psi}\right)$ is compact.

(ii) All compact open sets are finite unions of sets of the form $\left(\frac{\phi}{\psi}\right)$ for some pp-1-formulae ϕ, ψ.

(iii) The Ziegler spectrum is compact.

One important property of the Ziegler spectrum is that its closed sets correspond to theories of modules closed under arbitrary direct sums. The following definition and lemma explicitly gives this correspondence.

Definition 2.3.3. *Let T be a complete theory of modules closed under arbitrary direct sums. We define $\mathcal{C}(T)$ to be the following set of isomorphism classes of pure injectives:*

$$\{N \in \mathrm{pinj}_R \mid N \textit{ is a direct summand of some model of } T\}.$$

Let $\mathcal{C}$ be a Ziegler closed set. We define $T(\mathcal{C})$ to be the theory axiomatised by

(i) $\left|\frac{\phi}{\psi}\right| = 1$ if $\left(\frac{\phi}{\psi}\right) \cap \mathcal{C} = \emptyset$.

(ii) $\left|\frac{\phi}{\psi}\right| > n$ for all $n \in \mathbb{N}$ if $\left(\frac{\phi}{\psi}\right) \cap \mathcal{C} \neq \emptyset$.

Lemma 2.3.4. *[Pre88, Theorem 4.67] Let R be a ring. If $\mathcal{C}$ is a closed subset of Zg_R and T is a theory of modules closed under arbitrary direct sums then the following statements hold:*

(i) $\mathcal{C}(T)$ *is a closed set.*

(ii) $T(\mathcal{C})$ *is a complete theory of modules closed under products.*

(iii) $\mathcal{C}(T(\mathcal{C})) = \mathcal{C}$.

(iv) $T(\mathcal{C}(T)) = T$.

Definition 2.3.5. *Let $\mathcal{T}$ be a topological space and $X \subseteq \mathcal{T}$. We say X is an irreducible set if for all closed subsets Y, Z of $\mathcal{T}$, $X \subseteq Y \cup Z$ implies $X \subseteq Y$ or $X \subseteq Z$.*

Definition 2.3.6. *We say a topological space is sober if every non-empty irreducible closed set is the closure of a point.*

Note that the above definition is not the usual definition of soberness. The usual definition says a topological space is sober if every irreducible closed set is the closure of a unique point. Thus the usual definition of soberness implies that the space is T_0. As many of the spaces we consider are not T_0, the usual definition of soberness is not appropriate. Moreover, for T_0 spaces the two definitions are equivalent.

Definition 2.3.7. *Let $\mathcal{T}$ be a topological space and X an irreducible closed set of $\mathcal{T}$. We say x is a generic point of X if the closure of x in $\mathcal{T}$ is X.*

The normal definition of a generic point includes its uniqueness, again this is not appropriate for our situation.

Definition 2.3.8. *Let $\mathcal{T}$ be a topological space. We say a point $x \in \mathcal{T}$ specialises to a point $y \in \mathcal{T}$ if y is in the closure of x.*

So if X is an irreducible closed set then a generic point of X is a point which specialises to all points in X.

Lemma 2.3.9. *Let $\mathcal{T}$ be a topological space with basis of open sets $\{\mathcal{W}_i \mid i \in I\}$. Then $x \in \mathcal{T}$ specialises to $y \in \mathcal{T}$ if and only if for all $i \in I$, $y \in \mathcal{W}_i$ implies $x \in \mathcal{W}_i$.*

Lemma 2.3.10. *Let $\mathcal{T}$ be a topological space. Then for all $x \in \mathcal{T}$, the closure of x is an irreducible closed set.*

We say that two points in a topological space are topologically indistinguishable if they are contained in exactly the same open sets. If $\mathcal{T}$ is a topological space, let $\approx$ be the equivalence relation on points of $\mathcal{T}$ such that $x \approx y$ if and only if x and y are topologically indistinguishable in $\mathcal{T}$.

The open sets of any topological space are a complete lattice under inclusion. Recall that a complete lattice is a partially ordered set for which every subset has a supremum (and therefore also an infimum). Let L_1, L_2 be lattices, then a lattice morphism from $f : L_1 \to L_1$ is a poset morphism which preserves meets and joins. Following [Pre09, Section 5.4], we say that two topological spaces are homeomorphic at the level of topology if there is an isomorphism between their lattices of open sets. Note that lattice isomorphisms preserve arbitrary infima and suprema.

The following three statements are taken from [Pre09] and were originally in [Her93].

Theorem 2.3.11. *For any ring R, the right and left Ziegler spectra of R are homeomorphic at the level of topology, the isomorphism being defined by taking the basic open set $\left(\frac{\phi}{\psi}\right)$ to $\left(\frac{\mathrm{D}\psi}{\mathrm{D}\phi}\right)$.*

Proposition 2.3.12. *If $\mathcal{C}$ is an irreducible closed set in Zg_R such that $\mathcal{C}$ has a countable basis of open sets in the relative topology, then $\mathcal{C}$ has a generic point.*

Proof. Suppose $\mathcal{C}$ is an irreducible closed set in Zg_R with countable basis of open sets O_i indexed by $i \in \mathbb{N}$. We define, by induction, a sequence of pp-1-formulae ϕ_i, ψ_i such that $\emptyset \neq \left(\frac{\phi_i}{\psi_i}\right) \cap \mathcal{C} \subseteq O_i$ and $\phi_i \geq \phi_{i+1} \geq \psi_{i+1} \geq \psi_i$ in the lattice of pp-formulae.

Since the open sets $\left(\frac{\phi}{\psi}\right)$ with pp-1-formulae ϕ, ψ, are a basis for Zg_R, we can take ϕ_1, ψ_1 such that $\emptyset \neq \left(\frac{\phi_1}{\psi_1}\right) \cap \mathcal{C} \subseteq O_1$.

Suppose we have already defined ϕ_i, ψ_i, the irreducibility of $\mathcal{C}$ implies $\mathcal{C} \cap \left(\frac{\phi_i}{\psi_i}\right) \cap O_{i+1} \neq \emptyset$.

Let $N \in \mathcal{C} \cap \left(\frac{\phi_i}{\psi_i}\right) \cap O_{i+1}$ and take $c \in N$ such that $c \in \phi_i(N) \backslash \psi_i(N)$. By [Zie84, 4.9], there exists $\phi_{i+1} \in \mathrm{pp}^N(c)$ and $\psi_{i+1} \notin \mathrm{pp}^N(c)$ such that $\phi_i \geq \phi_{i+1} \geq \psi_{i+1} \geq \psi_i$ and $\left(\frac{\phi_{i+1}}{\psi_{i+1}}\right) \cap \mathcal{C} \subseteq O_{i+1}$.

Let Φ be the filter generated by $\{\phi_i \mid i \in \mathbb{N}\}$ and Ψ the ideal generated by $\{\phi_i \mid i \in \mathbb{N}\}$. By [Pre88, Theorem 4.33], there is an irreducible $T(\mathcal{C})$-consistent pp-type p such that $p \cap \Psi = \emptyset$ and $\Phi \subseteq p$. Let N be a indecomposable pure-injective module realising p. Then $N \in \mathcal{C}$ and $N \in \left(\frac{\phi_i}{\psi_i}\right) \subseteq O_i$ for every i. Therefore N is a generic point of $\mathcal{C}$.

□

It is not obvious how to generalise this proof to arbitrary rings as if we had an uncountable basis of open sets indexed by some ordinal, it is not clear how one would define ϕ_i, ψ_i at limit ordinals.

Definition 2.3.13. *Let R be a ring and $\mathcal{C}$ a closed subset of Zg_R. Then $\mathcal{C} = \mathrm{Zg}_R \backslash \bigcup \left(\frac{\phi_i}{\psi_i}\right)$ for some set of pp-pairs ϕ_i/ψ_i. Define $D\mathcal{C}$ to be the ${}_R\mathrm{Zg}$ closed subset ${}_R\mathrm{Zg} \backslash \bigcup \left(\frac{D\psi_i}{D\phi_i}\right)$.*

Theorem 2.3.14. *[Her93] If $\mathcal{C}$ is a closed subset of Zg_R and has countable basis of open sets, then $\mathcal{C}/\approx$ is homeomorphic to $\mathrm{D}\mathcal{C}/\approx$. In particular, if R is a countable ring, then $\mathrm{Zg}_R/\approx$ is homeomorphic to ${}_R\mathrm{Zg}/\approx$.*

Definition 2.3.15. *Let R be a commutative ring. By $\mathrm{Spec}R$ we mean the set of prime ideals of R equipped with basis of open sets $D(f) = \{\mathfrak{p} \in \mathrm{Spec}R \mid f \notin \mathfrak{p}\}$ where $f \in R$. For $I \lhd R$, let $V(I)$ denote the set of prime ideals containing I.*

Proposition 2.3.16. *Let R be a commutative ring.*

(i) All closed sets are of the form $V(I)$ for some ideal $I \lhd R$.

(ii) Let $I, J \lhd R$ then $V(I) \cap V(J) = V(I+J)$ and $\mathrm{V}(I) \cup \mathrm{V}(J) = V(IJ)$.

(iii) An open set in $\mathrm{Spec}R$ is compact (recall that we do not include Hausdorff in the definition of compact) if and only if it is the complement of $V(I)$ for some finitely generated $I \lhd V$. Thus $\mathrm{Spec}R$ has a basis of compact open sets stable under intersection i.e the intersection of two compact open sets is compact open.

(iv) Spec$R$ *is* T_0.

(v) Spec$R$ *is sober.*

Proof. Parts (i), (ii) and (v) can be found in many commutative algebra or algebraic geometry text book, for instance in [Bou98]. Part (iv) is obvious.

(iii) For all $f \in R$, the open set $D(f)$ is compact ([Bou98, II §4 Proposition 12]). Thus, since the open sets $D(f)$ are a basis for Spec$R$, all compact open sets are of the form $\bigcup_{i=1}^{n} D(f_i)$ where $f_1, ..., f_n \in R$. Finally,

$$\mathrm{Spec}R \backslash \bigcup_{i=1}^{n} D(f_i) = \bigcap_{i=1}^{n} \mathrm{Spec}R \backslash D(f_i) = \bigcap_{i=1}^{n} V(f_i R) = V(\sum_{i=1}^{n} f_i R).$$

□

Definition 2.3.17. *Let R be a commutative ring. Let* Spec$^*R$ *denote the Hochster dual of* Spec$R$*, that is the topological space got by declaring all compact open sets in* Spec$R$ *as closed.*

Proposition 2.3.18. *The sets $V(I)$ where $I \lhd R$ is finitely generated form a basis for* Spec$^*R$ *and the sets $V(fR)$ where $f \in R$ form a sub-basis for* Spec$^*R$.

Proof. Proposition 2.3.16 (iii) implies that the $V(I)$ where $I \lhd R$ is finitely generated are a sub-basis for Spec$^*R$. By proposition 2.3.16 (ii) if $I_1, ..., I_n \lhd R$ are finitely generated ideals then $\cap_{i=1}^{n} V(I_i) = V(\sum_{i=1}^{n} I_i)$ and $\sum_{i=1}^{n} I_i$ is finitely generated. So the $V(I)$ where $I \lhd R$ is finitely generated are a basis for Spec$^*R$.

In order to show that the $V(fR)$ where $f \in R$ are a sub-basis for Spec$^*R$ we need only observe that for any ideal $I = \langle f_1, ..., f_n \rangle$, $V(I) = \cap_{i=1}^{n} V(f_i R)$. □

We say a topological space is spectral if it is homeomorphic to Spec$R$ for some commutative ring R. In [Hoc69], Hochster showed that if a topological space is spectral then the dual space is also spectral. Hence we have the following proposition.

Proposition 2.3.19. *Let R be a commutative ring. Spec$^*R$ is sober.*

Proposition 2.3.20. *Let R be a commutative ring. The irreducible closed sets of* Spec$R$ *are exactly* $\mathrm{V}(\mathfrak{p})$ *where* $\mathfrak{p} \lhd R$ *is prime and* $\mathfrak{p}$ *is the generic point. The irreducible closed sets of* Spec$^*R$ *are* $\mathrm{W}(\mathfrak{p}) = \{\mathfrak{q} \in \mathrm{Spec}^*R \mid \mathfrak{q} \subseteq \mathfrak{p}\}$ *and* $\mathfrak{p}$ *is the generic point.*

Proof. Let R be a commutative ring. By lemma 2.3.9, a prime ideal $\mathfrak{p}$ specialises to a prime ideal $\mathfrak{q}$ in Spec$R$ if and only if for all $f \in R$, $\mathfrak{q} \in \mathrm{D}(f)$ implies $\mathfrak{p} \in \mathrm{D}(f)$, that is for all $f \in R$, $f \notin \mathfrak{q}$ implies $f \notin \mathfrak{p}$. Therefore $\mathfrak{p}$ specialises to $\mathfrak{q}$ in Spec$R$ if and only if $\mathfrak{p} \subseteq \mathfrak{q}$. Therefore, for any prime ideal $\mathfrak{p}$, the closure in Spec$R$ of $\mathfrak{p}$ is $\mathrm{V}(\mathfrak{p})$. Hence, since Spec$R$ is sober, the irreducible closed sets of Spec$R$ are exactly $\mathrm{V}(\mathfrak{p})$ where $\mathfrak{p} \lhd V$ is prime and $\mathfrak{p}$ is the generic point.

In the dual topology the specialisation relation is reversed. Hence the closure of $\mathfrak{p}$ a prime in Spec$^*R$ is $\mathrm{W}(\mathfrak{p})$. Since Spec$^*R$ is sober, the irreducible closed sets of Spec$^*R$ are exactly of the form $\mathrm{W}(\mathfrak{p})$ for some $\mathfrak{p} \lhd R$ prime and $\mathfrak{p}$ is the generic point. □

Proposition 2.3.21. *[Zie84] Let R be a ring and $C(R)$ the centre of R. Then for any indecomposable pure-injective module N, the set of $r \in C(R)$ whose action on N by multiplication is not bijective is a prime ideal of $C(R)$.*

Proof. Let N be an indecomposable pure-injective module and $\mathfrak{p}$ the set of central elements of R which act non-bijectively on N. Let $f : C(R) \to \mathrm{End}(N)$ where $r \in C(R)$ is mapped to the endomorphism of N given by multiplication of r. Then r acts bijectively if and only if the image of r under f in $\mathrm{End}(N)$ is not in the maximal ideal of $\mathrm{End}(N)$ (unique since $\mathrm{End}(N)$ is local by 2.2.21). Hence $\mathfrak{p}$ is the inverse image of the maximal ideal of $\mathrm{End}(N)$. Therefore $\mathfrak{p}$ is a prime ideal. □

Definition 2.3.22. *Let R be a commutative ring and N an indecomposable pure-injective module. We call the prime ideal in the above proposition the attached prime of N,* Att$N$.

For an indecomposable injective module E over a commutative ring R we have already defined the attached prime for E to be the set of all $r \in R$ which annihilate some non-zero element of E (see 2.1.10). The following lemma shows that these two definitions of attached prime, of an indecomposable injective, coincide.

Lemma 2.3.23. *Let R be a commutative ring. Suppose E is an indecomposable injective module then if the action by multiplication of $r \in R$ on E is not bijective then it is not injective.*

Proof. Let E be an indecomposable injective R-module. Suppose multiplication by $r \in R$ gives an injective map from E to E. Let Er denote the image of E under this map. Since Er is isomorphic to E, Er is an injective module and therefore Er is a direct summand of E. As E is indecomposable, $E = Er$. Hence the action of r on E is bijective. $\square$

For the proof of the following lemma and a more general statement, see [Pre88] Chapter 4 section 4.4.

Lemma 2.3.24. *Let N be an indecomposable pure-injective module. For any non-zero $x \in N$ and $r \in \mathrm{Att}N$, $\mathrm{pp}^N(xr) \supsetneq \mathrm{pp}^N(x)$.*

Proposition 2.3.25. *Let R be a commutative ring. The map taking an indecomposable pure-injective module to its attached prime induces a continuous map from Zg_R to $\mathrm{Spec}^* R$.*

Proof. In order to check that

$$f : \mathrm{Zg}_R \to \mathrm{Spec}^* R,\ \ f : N \to \mathrm{Att}N$$

is continuous, it is enough to check the preimage of subbasic open sets are open. First note that the collection of open sets $V(aR) = \{\mathfrak{p} \in \mathrm{Spec}^* \mid a \in \mathfrak{p}\}$, $a \in R$ are a sub-basis for $\mathrm{Spec}^* R$ so it is enough to check that the pre-image under f of each $V(aR)$ is open. Suppose N is an indecomposable pure-injective module and $a \in R$. Observe that the following 3 statements are equivalent:

(i) $f(N) \in V(aR)$.

(ii) Either there exists $n \in N \backslash \{0\}$ such that $na = 0$ or there exists $n \in N$ such that a does not divide n.

(iii) $N \in \left(\frac{xa=0}{x=0}\right) \cup \left(\frac{x=x}{a|x}\right)$.

Hence for any $a \in R$ the pre-image of $V(a)$ under f is $\left(\frac{xa=0}{x=0}\right) \cup \left(\frac{x=x}{a|x}\right)$ hence f is continuous. $\square$

Theorem 2.3.26. *[Pre09, pg67] Suppose that $f : R \to S$ is an epimorphism of rings. If N is an indecomposable pure injective S-module then as an R-module, N is indecomposable pure-injective. The induced map from Zg_S to Zg_R continuously embeds Zg_S into Zg_R as closed set.*

If R is a ring and $\mathfrak{p} \lhd R$ is prime then we denote the localisation of R at $\mathfrak{p}$ by $R_\mathfrak{p}$.

Lemma 2.3.27. *Let R be a commutative ring, $\mathfrak{p} \lhd R$ be a prime ideal and $f : R \to R_\mathfrak{p}$. Then the image of the map induced by f from $\mathrm{Zg}_{R_\mathfrak{p}}$ to Zg_R is the set of indecomposable pure-injectives with attached prime contained in $\mathfrak{p}$.*

Proof. Suppose N has attached prime $\mathfrak{q} \subseteq \mathfrak{p}$. Then for all $r \notin \mathfrak{p}$, multiplication by r is a bijective map. Hence we may define multiplication by $1/r$ to be the inverse of this map. So N can be endowed with the structure of an $R_\mathfrak{p}$-module.

Suppose N is an $R_\mathfrak{p}$-module. Then N may be viewed via f as an R module. For any $t \notin \mathfrak{p}$, since N is an $R_\mathfrak{p}$-module, the action of t is invertible. Hence $t \notin \mathrm{Att}N_R$. Therefore $\mathfrak{p} \supseteq \mathrm{Att}N_R$. □

Proposition 2.3.28. *Let R be a commutative ring. Then the following are equivalent:*

(i) Zg_R *is sober.*

(ii) For all $\mathfrak{p} \lhd R$ prime, $\mathrm{Zg}_{R_\mathfrak{p}}$ is sober.

(iii) For all $\mathfrak{m} \lhd R$ maximal, $\mathrm{Zg}_{R_\mathfrak{m}}$ is sober.

Proof. (i)⇒(ii) Suppose Zg_R is sober then for any prime ideal $\mathfrak{p} \lhd R$, $\mathrm{Zg}_{R_\mathfrak{p}}$ is homeomorphic to a closed subset of Zg_R and hence is sober. (ii)⇒ (iii) is obvious.

(iii)⇒ (i) Suppose $C \subseteq \mathrm{Zg}_R$ is an irreducible closed set. Then its image $f(C)$ in Spec* is irreducible. Therefore the closure of this set has a generic point $\mathfrak{p}$. Hence $N \in C$ implies $f(N) \subseteq \mathfrak{p}$. Let $\mathfrak{m}$ be a maximal ideal containing $\mathfrak{p}$. Then $N \in C$ implies $f(N) \subseteq \mathfrak{m}$. Therefore by lemma 2.3.27 C is contained in a closed set homeomorphic to $\mathrm{Zg}_{R_\mathfrak{m}}$. Hence, if $\mathrm{Zg}_{R_\mathfrak{m}}$ is sober then C has a generic point. □

Thus, the question of whether Zg_R is sober for all commutative rings reduces to the question of whether Zg_R is sober for all local commutative rings.

2.4 Valuation rings

Definition 2.4.1. *A commutative ring is called a valuation ring if its set of ideals is totally ordered by inclusion. Thus a valuation ring is a local ring. We say a valuation ring is a valuation domain if it has no zero divisors.*

Note that, since the ideals of a valuation ring form a chain, for all $a, b \in V$ either a divides b or b divides a and all ideals in a valuation ring are irreducible.

Lemma 2.4.2. *Let R be a valuation ring. Then $I \subseteq R$ is an ideal if and only if for all $i \in I$ and $r \in R$, $ir \in I$.*

Proof. $\Leftarrow$ Suppose $I \subseteq R$ such that for all $i \in I$ and $r \in R$, $ir \in I$. In order to show that I is an ideal we need to show that if $a, b \in I$ then $a + b \in I$. Suppose $a, b \in I$ then without loss of generality we may assume $a = b\gamma$ for some $\gamma \in R$. Therefore $a + b = b(\gamma + 1)$ hence $a + b \in I$. □

If K is a field we denote $K\backslash\{0\}$ by K^*.

Definition 2.4.3. *Let K be a field and G a totally ordered abelian group. Then a surjective function $v : K^* \to G$ is called a valuation if $v(x.y) = v(x) + v(y)$ and $v(x + y) \geq \inf\{v(x), v(y)\}$.*

Lemma 2.4.4. *Let R be a valuation domain with maximal ideal $\mathfrak{m}$, field of quotients Q and group of units U. Then the canonical map $v : Q^* \to Q^*/U$ is a valuation when Q^*/U is given the ordering $aU \geq bU$ if and only if $ab^{-1} \in R$.*

Let Q be a field and $v : Q^* \to G$ a valuation. Then $R = \{0\} \cup \{x \in Q^* \mid v(x) \geq 0\}$ is a valuation domain with group of units $U = \{x \in Q^* \mid v(x) = 0\}$, so $G \simeq Q^*/U$.

We call Q^*/U the value group of a valuation domain R.

Theorem 2.4.5. *[Kru32] Let k be a field and G a totally ordered abelian group. Then there exists a valuation domain with residue field isomorphic to k and value group isomorphic (as an ordered group) to G.*

In chapter 3 we separate valuation domains into those with dense value group, by which we mean densely ordered, and those with non-dense value group. The following lemma gives two conditions on a valuation domain equivalent to its value group being dense.

Lemma 2.4.6. *Let R be a valuation domain and $\mathfrak{m} \lhd R$ its maximal ideal. The following are equivalent:*

(i) The value group of R is dense.

(ii) The maximal ideal $\mathfrak{m}$ is not finitely generated.

(iii) $\mathfrak{m}^2 = \mathfrak{m}$.

Proof. Let $\mathcal{G}$ be the value group of R, Q the quotient field of R and $v : Q^* \to \mathcal{G}$ the valuation map.

(i)$\Leftrightarrow$(ii) Suppose $\mathfrak{m}$ is finitely generated. Let t generate $\mathfrak{m}$. Then $v(t)$ is greater than 0 and there is no element in $\mathcal{G}$ smaller than $v(t)$ and greater than 0. Hence $\mathcal{G}$ is not dense.

Suppose $\mathcal{G}$ is not dense. Then there exists $g_1, g_2 \in \mathcal{G}$ such that $g_1 < g_2$ and there is no element in $\mathcal{G}$ between g_1 and g_2. Therefore $g_2 - g_1$ is the least strictly positive element of $\mathcal{G}$. Take $r \in v^{-1}(g_2 - g_1)$. Then

$$\mathfrak{m} = \{s \in R \mid v(s) > 0\} = \{s \in R \mid v(s) \geq v(r)\} = rR.$$

So $\mathfrak{m}$ is finitely generated.

(ii)$\Leftrightarrow$ (iii) Suppose $\mathfrak{m}$ is finitely generated by $r \in R$. Then $r \notin r^2R$ so $\mathfrak{m}^2 \neq \mathfrak{m}$.

Suppose $\mathfrak{m}^2 \neq \mathfrak{m}$. Take $r \in \mathfrak{m} \backslash \mathfrak{m}^2$. Suppose $t \notin rR$ then $r = t\lambda$ for some non-unit $\lambda \in R$. Hence $\lambda \in \mathfrak{m}$. Therefore $t \notin \mathfrak{m}$. Hence $\mathfrak{m} = rR$.

□

Chapter 3

The Ziegler Spectrum of a Valuation Domain

Throughout this chapter, V will be a valuation domain and $\mathfrak{m}$ will denote its (unique) maximal ideal.

3.1 Some lemmas

We will start this section by explaining the relationship between ideals of V and particular convex subsets of the value group of V. Let V be a valuation domain with field of fractions Q, value group Γ and valuation map $v : Q^* \to \Gamma$. If I is a proper ideal of V, let $v(I)$ be the set

$$\{\gamma \in \Gamma \mid \text{ there exists } x \in I \backslash \{0\} \text{ with } v(x) = \gamma\}$$

and let $-v(I)$ be the set $\{-\gamma \mid \gamma \in v(I)\}$.
The map S that sends a proper ideal I of V to

$$S(I) = \Gamma \backslash (v(I) \cup -v(I))$$

is an inclusion reversing bijection between proper ideals of V and non-empty symmetric convex subsets of Γ. Under S, prime ideals correspond to convex subgroups of Γ. Note that the zero ideal of V corresponds to the whole of Γ and the maximal ideal $\mathfrak{m}$ corresponds to the trivial subgroup of Γ.

If I is a proper ideal of V then it is readily seen that

$$S(I^{\#}) = \{\gamma \in \Gamma \mid \text{ for all } \delta \in S(I),\ \gamma + \delta \in S(I)\}.$$

Lemma 3.1.1. *If V is a valuation domain and $I \lhd V$ then $x \in I^{\#}$ if and only if $Ix \subsetneq I$.*

Proof. First, note that for any $x, g \in V$, $gx \in Ix$ implies $g \in I$. To see this, suppose $gx \in Ix$. There exists $i \in I$ such that $gx = ix$. Hence, since V is a domain, $g = i$. Therefore $g \in I$.

Suppose $x \in I^{\#}$ then there exists $g \notin I$ such that $gx \in I$. Since $g \notin I$, $gx \notin Ix$. Therefore $I \supsetneq Ix$.

Suppose $Ix \subsetneq I$. If $x \in I$ then $x \in I^{\#}$. So suppose $x \notin I$. Then for all $g \in I$, x divides g. Take $g \in I \backslash Ix$ then $g = xr$ for some $r \in V$ and $r \notin I$ since $g \notin Ix$. Hence $x \in I^{\#}$. □

Lemma 3.1.2. *Suppose $I \lhd V$ then for all $\lambda \in V$, $\lambda \notin I$ if and only if $\lambda\mathfrak{m} \supseteq I$.*

Proof. Suppose $\lambda \notin I$. Then $\lambda V \supseteq I$ since ideals are totally ordered. Suppose $i \in I$ then there exists $r \in V$ such that $i = \lambda r$. If $r \notin \mathfrak{m}$ then r is a unit so $\lambda \in I$. Therefore $r \in \mathfrak{m}$. Hence $\lambda\mathfrak{m} \supseteq I$.

Suppose $\lambda\mathfrak{m} \supseteq I$. As V is local, $\lambda \notin \lambda\mathfrak{m}$. Hence $\lambda \notin I$. □

Lemma 3.1.3. *Let $I \lhd V$. Then for all $\lambda \in V$, $\lambda \notin I$ if and only if $\lambda I^{\#} \supseteq I$.*

Proof. Let $I \lhd V$. Suppose $\lambda \notin I$. Take $i \in I$ then there exists $r \in V$ such that $\lambda r = i$. Therefore $r \in I^{\#}$ so $i \in \lambda I^{\#}$.

Suppose $\lambda I^{\#} \supseteq I$ then $\lambda\mathfrak{m} \supseteq \lambda I^{\#}$ so $\lambda \notin I$. □

Lemma 3.1.4. *Let $I, J \lhd V$. Suppose $J \subsetneq I^{\#}$. Then there exists $a \notin I$ such that $J \subsetneq (I : a)$.*

Proof. Take $t \in I^{\#} \backslash J$ then there exists $a \notin I$ such that $ta \in I$. Therefore $t \in (I : a)$. Since ideals are totally ordered $J \subsetneq tR$ so $J \subsetneq (I : a)$. □

We will use the following lemma through out this chapter and the next. For a proof see [FS01, Chapter II Lemma 4.6].

Lemma 3.1.5. *Let V be a valuation domain and I, J proper ideals of V. Then $I^\# \cap J^\# = (IJ)^\#$.*

3.2 pp-formulae and pp-types over valuation domains

The following lemma is crucial in the reduction of pp-formulae over valuation domains to various special forms. We follow the proof given in [Pun01] for serial rings, we specialise to valuation domains. A more general version of this result was first seen in [Dro75] and [War75].

Theorem 3.2.1 (Drozd's diagonalisation theorem for valuation domains)**.** *Suppose M is an $m \times n$ matrix over a valuation domain V. Then there exist invertible matrices T and S over V such that TMS is diagonal. That is all entries of TMS are zero except for the leading diagonal.*

Proof. We will show the equivalent statement that any matrix over a valuation domain can be made diagonal by a series of invertible row and column operations. By padding with zeroes we may assume that the matrix is square. We proceed by induction on n the number of rows of the matrix. Suppose that the statement of the lemma is true for all matrices of dimension smaller than n. Then consider an $n \times n$ matrix M with entries $m_{i,j}$. Consider the ideal generated by the elements $m_{i,j}$. Since V is a valuation ring, there exist k and l such that $m_{k,l}$ generates this ideal. It is clear that there exists a series of invertible row operations that leave us with a matrix with zeroes down the lth column except for the (k, l)th entry and a series of invertible column operations that leave us with a matrix with zeros across the kth row except for the (k, l)th entry . By the induction hypothesis we may now apply elementary invertible row and column operation to make the (k, l) minor diagonal without effecting the lth column or the kth row. The matrix we now get is diagonal. $\square$

The following two lemmas and their corollaries can be found in [EH95] and [Pun92].

Lemma 3.2.2. *Let V be a valuation ring. Then every pp-formula over V is equivalent to a pp-formula of the form:*

$$\bigwedge_{i=1}^{n}(a_i|\bar{x}\bar{b}_i)$$

Proof. Suppose ϕ is of the form $\exists \bar{y}\ \bar{y}M = \bar{x}B$. By the previous theorem there exists invertible matrices S and T over V such that TMS is diagonal. Let ψ be the formula $\exists \bar{y}\ \bar{y}TMS = \bar{x}BS$. Suppose N is an V-module. Then $\bar{n} \in \phi(N)$ if and only if there exists $\bar{m}$ a tuple in N such that $\bar{m}M = \bar{n}B$ if and only if there exists $\bar{m} \in N$ such that $\bar{m}MS = \bar{n}BS$ since S is invertible if and only if there exists $\bar{m} \in N$ such that $\bar{m}TMS = \bar{n}BS$ since T is invertible. Now ψ is of the form $\bigwedge_{i=1}^{n}(a_i|\bar{x}\bar{b}_i)$. □

Corollary 3.2.3. *Let V be a valuation ring. Then every pp-1-formula over V is equivalent to a pp-formula of the form:*

$$\bigwedge_{i=1}^{n}(a_i|x) + (xb_i = 0)$$

for some $a_i, b_i \in V$.

Proof. It remains to show that a formula of the form $\exists y\ ya = x\lambda$ is equivalent to one of the required form. Let $\theta = \exists y\ ya = x\lambda$ for some $a, \lambda \in V$ and N an V-module. Suppose a divides λ then there exists $t \in V$ such that $\lambda = at$. Then $x \in \theta(N)$ if and only if there exists $y \in N$ such that $(y - xt)a = 0$, this is true for all $x \in N$ so θ is identically true. Suppose λ divides a then there exists $t \in V$ such that $\lambda t = a$. Then $x \in \theta(N)$ if and only if there exists $y \in N$ such that $(yt - x)\lambda = 0$ that is if and only if $x \in \zeta(N)$ where $\zeta = (t|x) + (x\lambda = 0)$. □

Lemma 3.2.4. *Let V be a valuation ring. Then every pp-formula over V is equivalent to a pp-formula of the form:*

$$\sum_{i=1}^{n} \exists y_i \bigwedge_{j=1}^{k}(x_j = y_i r_{ij}) \wedge (y_i s_i = 0)$$

for some $r_{ij}, s_i \in V$.

Proof. Let ϕ be a pp-formula in k-variables over V. Then $D\phi$, the dual of ϕ, is equivalent to a pp-formula of the form $\bigwedge_{i=1}^{n}(a_i|\bar{x}\bar{b}_i)$. Therefore $D^2\phi$ is equivalent to a formula of the form $\sum_{i=1}^{n} \exists y_i \bigwedge_{j=1}^{k}(x_j = y_i r_{ij}) \wedge (y_i s_i = 0)$, as the dual of a formula of the form $(a|\bar{x}\bar{b})$ is $\exists y \bigwedge_{j=1}^{k}(x_j = yb_j) \wedge (ya = 0)$. This is enough since $D^2\phi$ is equivalent to ϕ. □

Corollary 3.2.5. *Let V be a valuation ring. Then every pp-1-formula over V is equivalent to a pp-formula of the form:*

$$\sum_{i=1}^{n}(xa_i = 0 \wedge b_i|x)$$

for some $a_i, b_i \in V$.

Lemma 3.2.6. *[EH95] Let V be a valuation domain and N an indecomposable pure-injective V-module. Then the pp-1-definable subgroups of N are totally ordered.*

Proof. This proof will follow closely the proof (of a more general statement) given in Lemma 11.4 in [Pun01].

Corollary 3.2.5 states that the lattice of pp-1-formulae is generated by the sets of pp-1-formulae $\{xr = 0 \mid r \in V\}$ and $\{s|x \mid s \in V\}$. Each of these sets is a chain in the lattice of pp-1-formulae. It is stated in [Grä03, Theorem 13 Chapter IV] that a modular lattice generated by two chains is distributive. Hence the lattice of pp-1-formulae over a valuation domain is distributive.

We now show that if N is an indecomposable pure-injective V-module then as an $\mathrm{End}(N)$-module, it is uniserial, that is its $\mathrm{End}(N)$-submodules are totally ordered by inclusion. Since the endomorphism ring of any indecomposable pure-injective module is local it is enough to show that the $\mathrm{End}(N)$-submodules are distributive, see [Ste74]. It is stated in [Pre09, 4.3.10] that if N is pure-injective and $\bar{a} \in N$ with $p = pp^N(\bar{a})$ then $p(N) = S\bar{a}$ where $S = \mathrm{End}(N)$. Therefore for any pp-1-types p, q, r we need to show that $p(N) \cap [q(N) + r(N)] = [p(N) \cap q(N)] + [p(N) \cap r(N)]$. In fact this is enough because this proves the result for cyclic $\mathrm{End}(N)$-modules and if the lattice of cyclic $\mathrm{End}(N)$-submodules is distributive then the lattice of all $\mathrm{End}(N)$-submodules is distributive.

Hence we need to show that $p(N)\cap[q(N)+r(N)] \subseteq [p(N)\cap q(N)]+[p(N)\cap r(N)]$ since the other inclusion is true in any module. Suppose $n \in p(N) \cap [q(N) + r(N)]$. Then $n \in \phi(N) \cap [\psi(N) + \vartheta(N)]$ for all $\phi \in p$, $\psi \in q$ and $\vartheta \in r$. Hence $n \in [\phi(N) \cap \psi(N)] + [\phi(N) \cap \vartheta(N)]$ because the lattice of pp-1-formulae is distributive. Since N is pure-injective, hence algebraically compact, this means $n \in [p(N) \cap q(N)] + [p(N) \cap r(N)]$. Hence we have shown that the lattice of End(N)-submodules is distributive hence totally ordered by inclusion.

It remains to note that pp-1-definable subgroups are End(N)-submodules. So the lattice of pp-1-definable subgroups is totally ordered by inclusion. □

Corollary 3.2.7. *Let V be a valuation domain. Then a pp-1-type p is irreducible if and only if for all pp-1-formulae ϕ, ψ, $\phi, \psi \notin p$ implies $\phi + \psi \notin p$.*

Proof. Suppose that p is a pp-1-type and for all pp-1-formulae ϕ, ψ, $\phi, \psi \notin p$ implies $\phi + \psi \notin p$ then by Ziegler's Criterion 2.2.20, p is irreducible.

Suppose p is an irreducible pp-1-type, then p is realised in some indecomposable pure-injective module N. Suppose $a \in N$ realises p. Then $\phi, \psi \notin p$ implies $a \notin \phi(N)$ and $a \notin \psi(N)$ but since N is indecomposable pure-injective, the pp-definable subgroups are totally ordered. Therefore $\phi(N) = \phi(N) + \psi(N)$ or $\psi(N) = \phi(N) + \psi(N)$. So $a \notin \phi(N) + \psi(N)$. Hence $\phi + \psi \notin p$. □

We now give a correspondence between irreducible pp-types over valuation domains and pairs of ideals.

Lemma and Definition 3.2.8. *[Zie84][EH95] Let V be a valuation domain and p an irreducible complete pp-1-type. Let $I_p = \{r \in V \mid xr = 0 \in p\}$ and $J_p = \{r \in V \mid r|x \notin p\}$. Then I_p and J_p are ideals and (I_p, J_p) is called the pair associated to p.*

Proof. Let p be a complete pp-1-type and let I_p and J_p be as defined above. Suppose $r \in I_p$ and $\lambda \in V$. Then $xr = 0 \in p$ implies $xr\lambda = 0 \in p$ since p is closed under implication. Therefore I_p is an ideal. Suppose $r \in J_p$ and $\lambda \in V$. Then $r|x \notin p$ so $r\lambda|x \notin p$ since $r\lambda|x \in p$ implies $r|x \in p$. □

Lemma 3.2.9. *[Zie84][EH95] Let V be a valuation domain. There is a bijective correspondence between the irreducible pp-1-types of V and pairs of proper ideals of V. Under this correspondance an irreducible pp-1-type p is sent to its associated pair (I_p, J_p) and a pair of ideals (I, J) is sent to the (unique) irreducible pp-1-type generated by the formulae $\{xa = 0 | a \in I\} \cup \{b|x|b \notin J\}$.*

Proof. Suppose p is an pp-1-type over V. Let ϕ be a pp-1-formula over V. Then ϕ is equivalent to a formula of the form $\bigwedge_{i=1}^{n}(a_i|x) + (xb_i = 0)$ for some $a_i, b_i \in V$. So $\phi \in p$ if and only if for each i, $(a_i|x) + (xb_i = 0) \in p$. Since p is irreducible this is true if and only if $(a_i|x) \in p$ or $(xb_i = 0) \in p$ that is, if and only if $a_i \notin J_p$ or $b_i \in I_p$. So we have show that a pp-1-type p is uniquely determined by its associated pair of ideal (I_p, J_p).

It remains to show that every pair of proper ideals (I, J) is the associated pair of some irreducible pp-1-type. In 4.4.4, for each pair of proper ideals (I, J) we will give a uniserial module M and a non-zero element $m \in M$ such that m satisfies the formula $xa = 0$ if and only if $a \in I$ and m satisfies the formula $b|x$ if and only if $b \notin J$. By [EH95, Proposition 4.1], the pure-injective hull of a uniserial module is indecomposable. Note that since M embeds purely into its pure-injective hull, the pp-type of m in M is equal to the pp-type of m in the pure-injective hull of M. So for every pair of proper ideals (I, J) there is an indecomposable pure-injective module N and an element $n \in N$ such that n satisfies the formula $xa = 0$ if and only if $a \in I$ and n satisfies the formula $b|x$ if and only if $b \notin J$. Therefore every pair of proper ideals (I, J) is the associated pair of some irreducible pp-1-type. □

3.3 The Ziegler spectrum of a valuation domain

The aim of this section is to formulate the Ziegler spectrum in terms of pairs of ideals under an equivalence relation.

Definition 3.3.1. *[Zie84][EH95] Let V be a valuation domain and $I_1, J_1, I_2, J_2 \lhd R$. Then we say $(I_1, J_1) \sim (I_2, J_2)$ if either of the following hold:*

1. *There exists $a \notin I_1$ such that $(I_1 : a) = I_2$ and $J_1 a = J_2$.*

2. *There exists $a \notin J_1$ such that $I_1 a = I_2$ and $(J_1 : a) = J_2$.*

We will show that $\sim$ is an equivalence relation on the set of pairs of proper ideals of V and that $(I, J) \sim (K, L)$ if and only if the irreducible pp-type corresponding to (I, J) and the irreducible pp-type corresponding to (K, L) (under the correspondence in lemma 3.2.9), are realised in the same indecomposable pure-injective module.

Lemma 3.3.2. *Let V be a valuation domain. The binary relation $\sim$ is symmetric and reflexive.*

Proof. It is clear that $\sim$ is reflexive. We now show that $\sim$ is symmetric. Suppose that $(I_1, J_1) \sim (I_2, J_2)$. First suppose that the first condition in definition 3.3.1 holds. Then there exists $a \notin I_1$ such that $(I_1 : a) = I_2$ and $J_1 a = J_2$. Since V is a valuation domain $I_1 = I_2 a$, $a \notin J_2$ and $J_1 = (J_2 : a)$. Hence $(I_2, J_2) \sim (I_1, J_1)$. Next suppose that the second condition in definition 3.3.1 holds. Then there exists $a \notin J_1$ such that $I_1 a = I_2$ and $(J_1 : a) = J_2$. Since V is a valuation domain $J_1 = J_2 a$, $a \notin I_2$ and $I_1 = (I_2 : a)$. Hence $(I_2, J_2) \sim (I_1, J_1)$. Therefore $\sim$ is symmetric. □

Lemma 3.3.3. *Let V be a valuation domain and N be an indecomposable pure-injective V-module. Let $a, b \in N$, $p = pp_N(a)$, $q = pp_N(b)$ and (I_p, J_p) be the pair associated to p. Suppose that $b = a\lambda$ for some $\lambda \notin I_p$. Then the pair associated to q is $((I_p : \lambda), J_p.\lambda)$, so in particular $(I_p, J_p) \sim (I_q, J_q)$.*

Proof. For any $r \in V$, $br = 0$ if and only if $a\lambda r = 0$ if and only if $r \in (I_p : \lambda)$. Therefore $I_q = (I_p : \lambda)$.

We now show that $J_p.\lambda = J_q$. Take $r \in V$ such that r doesn't divide a. Then $r\lambda$ doesn't divide $b = a\lambda$ since $\lambda \neq 0$. Therefore $J_p\lambda \subseteq J_q$.

Take $r \in J_q$. Then r doesn't divide b, so r doesn't divide λ. Hence there exists $\gamma \in V$ such that $\lambda\gamma = r$. Note that γ does not divide a since if γ divided a then $r = \lambda\gamma$ would divide $b = a\lambda$. Therefore $\gamma \in J_p$. Hence $r \in J_p\lambda$. So $J_q \subseteq J_p\lambda$. □

Lemma 3.3.4. *[EH95][Zie84] Let p, q be irreducible pp-1-types. Then p and q are realised in the same indecomposable pure injective module if and only if their corresponding pairs of ideals are such that $(I_p, J_p) \sim (I_q, J_q)$.*

Proof. Suppose that p and q are realised in the same indecomposable pure injective N, p is realised by $a \in N$ and q is realised by $b \in N$. Then there exists a linking formula $\phi(x_1, x_2)$ (2.2.16) such that $(a, b) \in \phi(N)$ and $(a, 0) \notin \phi(N)$. By lemma 3.2.4 ϕ is equivalent to a pp-formula of the form:

$$\sum_{i=1}^{n} \exists y_i \bigwedge_{j=1}^{2} (x_j = y_i r_{ij}) \wedge (y_i s_i = 0)$$

for some $r_{ij}, s_i \in R$. Let $\rho_i(x_1, x_2) = \exists y_i \bigwedge_{j=1}^{2} (x_j = y_i r_{ij}) \wedge (y_i s_i = 0)$. Then

$$N \models \phi(x_1, x_2) \leftrightarrow \sum_{i=1}^{n} \rho_i(x_1, x_2)$$

so

$$N \models \exists x_2 \phi(x_1, x_2) \leftrightarrow \exists x_2 \sum_{i=1}^{n} \rho_i(x_1, x_2)$$

hence

$$N \models \exists x_2 \phi(x_1, x_2) \leftrightarrow \sum_{i=1}^{n} \exists x_2 \rho_i(x_1, x_2).$$

Since $N \models \exists x_2 \phi(a, x_2)$ and by lemma 3.2.6, the pp-1-definable subgroups of N in one variable form a chain, there exists an i such that $N \models \exists x_2 \rho_i(a, x_2)$. We may assume $i = 1$. But this means there is a $c \in N$ such that $N \models \rho_1(a, c)$. Hence $N \models \neg\rho_1(a, 0)$ since $(a, 0) \notin \phi(N)$ and $N \models \phi(a, c)$.

Now observe that since $N \models \rho_1(a, c)$, either a is a multiple of c or c is a multiple of a. Therefore by lemma 3.3.3 the pp-type of c has associated pair (I_c, J_c) such that $(I_p, J_p) \sim (I_c, J_c)$. It now remains to show that c has pp-type q. Suppose that $N \models \theta(b)$, then since the lattice of pp-1-subgroups of N is a chain and $b \notin \phi(0, N)$, $\theta(N) \supseteq \phi(0, N)$. Note that $N \models \phi(0, c - b)$ so since $c = b + (c - b)$, $c \in \theta(N)$. Similarly if $c \in \theta(N)$ then $b \in \theta(N)$.

We now prove the converse. Suppose p, q are irreducible pp-1-types such that $(I_p, J_p) \sim (I_q, J_q)$. Since p is irreducible, it is realised in an indecomposable pure-injective module N. Suppose $n \in N$ realises p.

Case 1: There exists $\gamma \notin I_p$ such that $(I_p : \gamma) = I_q$ and $J_p\gamma = I_q$.

By lemma 3.3.3, $n\gamma$ has pp-type q.

Case 2: There exists $\gamma \notin J_p$ such that $I_p\gamma = I_q$ and $(J_p : \gamma) = J_q$.

Since $\gamma \notin J_p$, $\gamma | n$. Let $m \in N$ be such that $m\gamma = n$ and suppose the pp-type of m has associated pair (K, L). By 3.3.3, $(K : \gamma) = I_p$ and $L\gamma = I_p$. Therefore $I_p\gamma = K$ and $(I_p : \gamma) = L$ so m has pp-type q. $\square$

Lemma 3.3.4 implies that $\sim$ is a transitive relation. So lemma 3.3.2 and 3.3.4 together imply that $\sim$ is an equivalence relation.

Lemma 3.3.5. *Let $I \lhd V$. Then*

(i) If $x \notin I$ then $(I : x)^{\#} = I^{\#}$.

(ii) If $x \neq 0$ then $(Ix)^{\#} = I^{\#}$.

Proof. (1) Fix $x \notin I$. Suppose $v \in (I : x)^{\#}$ then there exists $s \notin (I : x)$ (here we are using that $x \notin I$) such that $vs \in (I : x)$. Therefore $sx \notin I$ and $vsx \in I$ so $v \in I^{\#}$.

Suppose $v \in I^{\#}$ then there exists $s \notin I$ such that $vs \in I$. If $v \in (I : x)$ then $v \in (I : x)^{\#}$. So suppose $v \notin (I : x)$. Hence $vx \notin I$. Therefore there exists $t \in V$ such that $vxt = vs$ since $vs \in I$. Hence $xt = s$. So $t \notin (I : x)$ and $vxt \in I$ so $vt \in (I : x)$. Therefore $v \in (I : x)^{\#}$.

(2)If $x \neq 0$ then $x \notin Ix$ and $(Ix : x) = I$. Therefore by (1) $(Ix)^{\#} = I^{\#}$. $\square$

Note that this means if $I, J, K, L \lhd V$ and $(I, J) \sim (K, L)$ then $I^{\#} = K^{\#}$ and $J^{\#} = L^{\#}$. Also, note that if $(I, J) \sim (K, L)$ then $IJ = KL$ since for any $x \notin I$, $(I : x)Jx = (I : x)xJ = IJ$.

Proposition 3.3.6. *Let p be a pp-type realised in an indecomposable pure-injective module N and (I_p, J_p) be the pair associated to p. Then* $\mathrm{Att}N = I_p^{\#} \cup J_p^{\#}$.

Proof. Suppose the action by multiplication of $r \in V$ on N is not injective. Then there exists $n \in N$ such that $nr = 0$. Let (I_n, J_n) be the pair associated to $pp^N(n)$ and note $r \in I_n$. Since p and $pp^N(n)$ are realised in the same indecomposable pure-injective, $(I_n, J_n) \sim (I_p, J_p)$. Therefore $r \in I_n^{\#} = I_p^{\#}$.

Suppose the action by multiplication of $r \in V$ on N is not surjective. There there exists $n \in N$ such that r does not divide n. Let (I_n, J_n) be the pair associated to $pp^N(n)$ and note $r \in J_n$. As above, $r \in J_n^{\#} = J_p^{\#}$. Hence we have shown that if $r \in \mathrm{Att}N$ then $r \in I_p^{\#} \cup J_p^{\#}$.

Suppose $r \in I_p^{\#} \cup J_p^{\#}$. If $r \in I_p^{\#}$ then there exists an $x \notin I_p$ such that $r \in (I_p : x)$. Therefore, there exists $n \in N$ with $((I_p : x), J_p x)$ the associated pair of $pp^N(n)$. Hence $nr = 0$. If $r \in J_p^{\#}$ then there exists $x \notin J_p$ such that $r \in (J_p : x)$. Therefore, there exists $n \in N$ with $(I_p x, (J_p : x))$ the associated pair of $pp^N(n)$. Hence r does not divide n. □

Recall that a pp-type is irreducible if it can be realised in a indecomposable pure-injective module (definition 2.2.19) and that, by lemma 3.2.9, every pair of proper ideals (I, J) corresponds to an irreducible pp-1-type. For an arbitrary ring R, if p is an irreducible pp-n-type and N, M are indecomposable pure-injective modules realising p then $N \cong M$. A proof of this fact can be found in [Pre09, Corollary 4.3.47.]. In the case of valuation domains, since $\sim$ is symmetric, it is implied by lemma 3.3.4.

Definition 3.3.7. *Let $I, J \lhd V$. Denote by $N(I, J)$ the (unique) indecomposable pure-injective module in which the pp-type corresponding to (I, J) is realised.*

By lemma 3.3.4, $N(I, J) \cong N(K, L)$ if and only if $(I, J) \sim (K, L)$.

Lemma 3.3.8. *Let $l, m, n \in \mathbb{N}$ and $a_i, b_i, c_j, d_j \in V$ for $0 < i \leq l$ and $0 < j \leq m$. Let $\phi_i = (xa_i = 0 \wedge b_i | x)$ and $\psi_j = (xc_j = 0 + d_j | x)$. Suppose ϕ is the pp-1-formula $\sum_{i=1}^{l} \phi_i$ and ψ is the pp-1-formula $\bigwedge_{j=1}^{m} \psi_j$ then for all indecomposable pure-injective modules N the following are equivalent:*

1. *$|\phi(N)/\psi(N)| = n$.*
2. *There exists $0 < h \leq l$ and $0 < k \leq m$ such that $|\phi_h(N)/\psi_k(N)| = n$ and $|\phi_i(N)/\psi_j(N)| \leq n$ for all $0 < i \leq l$ and $0 < j \leq m$.*

Proof. Let N be an indecomposable pure-injective V-module. By 3.2.6, the pp-1-definable subgroups of N are totally ordered. Therefore

$$|\phi(N)/\psi(N)| = \max_{i,j} \{|\phi_i(N)/\psi_j(N)|\}.$$

□

Lemma 3.3.9. *[Pun99] Let V be a valuation domain. Let $a, b, c, d \in V$. Let ϕ be the pp-1-formula $xa = 0 \wedge b|x$ and let ψ be the pp-1-formula $xc = 0 + d|x$. The following are equivalent:*

1. *For all indecomposable pure-injective modules N, $|\phi(N)/\psi(N)| = 1$.*

2. *$c \in aV$ or $b \in dV$ or $c = 0$ or $b = 0$.*

Proof. (2)⇒(1) Suppose $c = 0$ then for any V-module the pp-subgroup defined by $xc = 0 + d|x$ is the whole module. Therefore (1) holds. Suppose $b = 0$ then for any V-module the pp-subgroup defined by $xa = 0 \wedge b|x$ is 0. Therefore (1) holds.

Suppose $c \in aV$ then $c = at$ for some $t \in V$ then for all modules the pp-subgroup defined by $xa = 0$ is contained in the pp-subgroup defined by $xat = 0$ hence (1) holds. Suppose $b \in dV$ then $b = dt$ for some $t \in V$ then for all modules the pp-subgroup defined by $b|x$ is contained in the pp-subgroup defined by $d|x$ hence (1) holds.

(1)⇒(2) Suppose for all indecomposable pure-injective modules N, $|\phi(N)/\psi(N)| = 1$. Then since every module is elementary equivalent to a direct sum of indecomposable pure-injective modules

$$T_V \models (xa = 0 \wedge b|x) \rightarrow (xc = 0 + d|x).$$

Suppose $c \neq 0$ and $b \neq 0$. If a is a unit then $c \in aV$ for all $c \in V$.

Suppose a not a unit. Consider the module V/abV. The image of b in V/abV satisfies $xa = 0 \wedge b|x$ hence satisfies $xc = 0 + d|x$. Since V/abV is uniserial either $bc \in abV$ hence $c \in aV$ or there exists $y \in V$ such that $dy - b \in abV$ hence $dy = b(at + 1)$ for some $t \in V$. We assume $a \in \mathfrak{m}$ therefore $at + 1$ is a unit so $b \in dV$. □

Lemma 3.3.10. *[Pun99] Let V be a commutative valuation domain. The collection of sets*

$$\mathcal{W}_{a,b,g,h} = \left(\frac{xag = 0 \wedge b|x}{(xa = 0) + (bh|x)}\right)$$

where $a, b \neq 0$ and $g, h \in \mathfrak{m}$ form a basis of Zg_V.

Proof. Let ϕ, ψ be pp-1-formulae. Then by corollary 3.2.5 ϕ is equivalent to a formula $\sum_{i=1}^{n} a_i x = 0 \wedge b_i | x$ for some $n \in \mathbb{N}$ and $a_i, b_i \in V$ and by corollary 3.2.3 ψ is equivalent to a formula $\bigwedge_j c_j x = 0 + d_j | x$ for some $c_j, d_j \in V$. By 3.2.6 the pp-definable subgroups of an indecomposable pure-injective N are totally ordered hence $N \in (\phi/\psi)$ if and only if $N \in \left(\frac{a_i x = 0 \wedge b_i | x}{c_j x = 0 + d_j | x}\right)$ for some i, j. So

$$\left(\frac{\phi}{\psi}\right) = \bigcup_{i,j} \left(\frac{x a_i = 0 \wedge b_i | x}{x c_j = 0 + d_j | x}\right).$$

By lemma 3.3.9 $\left(\frac{a_i x = 0 \wedge b_i | x}{c_j x = 0 + d_j | x}\right)$ is empty unless c_j divides a_i, b_i divides d_j, $b_i, c_j \neq 0$ and $a_i/c_j, d_j/b_i \in \mathfrak{m}$. Therefore the open sets of the form $\mathcal{W}_{a,b,g,h}$ with $a, b \in V \backslash \{0\}$ and $g, h \in \mathfrak{m}$ are a basis for Zg_V. □

Lemma 3.3.11. *Let N be an indecomposable pure-injective module over V. The following are equivalent:*

(i) $N \in \mathcal{W}_{a,b,g,h}$.

(ii) There is a pp-1-type realised in N with associated pair (I, J) such that $a \notin I$, $b \notin J$, $ag \in I$ and $bh \in J$.

Proof. Suppose $N \in \mathcal{W}_{a,b,g,h}$. There exists an element $n \in N$ such that $nag = 0$, $b|n$, $na \neq 0$ and $bh \nmid n$. Let $p = pp_N(n)$ and (I_p, J_p) be the pair associated to p. Then $a \notin I_p$, $b \notin J_p$, $ag \in I_p$ and $bh \in J_p$.

Let $n \in N$ with pp-type p and let (I_p, J_p) be the pair associated to p. Suppose that $a \notin I_p$, $b \notin J_p$, $ag \in I_p$ and $bh \in J_p$ then $xag = 0 \wedge b|x \in p$ and since the pp-type of n is irreducible $(xa = 0) + (bh|x) \in p$ implies $xa = 0 \in p$ or $bh|x \in p$. Therefore $(xa = 0) + (bh|x) \notin p$. Hence $N \in \mathcal{W}_{a,b,g,h}$. □

Remark 3.3.12. *Let (I, J) be a pair of proper ideals of V. From here on, we will identify the $\sim$ equivalence class of (I, J) with the indecomposable pure-injective module $N(I, J)$. We will say $(I, J) \in \mathcal{W}_{a,b,g,h}$ to mean $N(I, J) \in \mathcal{W}_{a,b,g,h}$. By lemma 3.3.11, $N(I, J) \in \mathcal{W}_{a,b,g,h}$ if and only if there exists (K, L) a pair of ideals with $(K, L) \sim (I, J)$ and $a \notin K$, $b \notin L$, $ag \in K$ and $bh \in L$.*

The above remark reformulates the Ziegler spectrum of a valuation domain in terms of pairs of ideals in V under the equivalence relation $\sim$.

Corollary 3.3.13. *Let V be a valuation domain, $a, b \in V\backslash\{0\}$ and $g, h \in \mathfrak{m}$. Let (I, J) be a pair of ideals in V. Then $(I, J) \in \mathcal{W}_{a,b,g,h}$ if and only if one of the following holds:*

1. *There exists $\gamma \notin I$ such that $a \notin (I : \gamma)$, $b \notin J\gamma$, $ag \in (I : \gamma)$ and $bh \in J\gamma$.*

2. *There exists $\gamma \notin J$ such that $a \notin I\gamma$, $b \notin (J : \gamma)$, $ag \in I\gamma$ and $bh \in (J : \gamma)$.*

Proof. Suppose $(I, J) \in \mathcal{W}_{a,b,g,h}$. By lemma 3.3.11, there exists (K, L) a pair of ideals realised in $N(I, J)$ such that $a \notin K$, $b \notin L$, $ag \in K$ and $bh \in L$. Since (I, J) and (K, L) are realised in the same indecomposable pure-injective module, $(I, J) \sim (K, L)$. Therefore, by definition of $\sim$, either there exists $\gamma \notin I$ such that $K = (I : \gamma)$ and $L = J\gamma$ or there exists $\gamma \notin J$ such that $K = I\gamma$ and $L = (J : \gamma)$. Thus, either there exists $\gamma \notin I$ such that $a \notin (I : \gamma)$, $b \notin J\gamma$, $ag \in (I : \gamma)$ and $bh \in J\gamma$ or there exists $\gamma \notin J$ such that $a \notin I\gamma$, $b \notin (J : \gamma)$, $ag \in I\gamma$ and $bh \in (J : \gamma)$.

Conversely, first suppose that there exists $\gamma \notin I$ such that $a \notin (I : \gamma)$, $b \notin J\gamma$, $ag \in (I : \gamma)$ and $bh \in J\gamma$. Then $((I : \gamma), J\gamma) \sim (I, J)$. So by lemma 3.3.11, $(I, J) \in \mathcal{W}_{a,b,g,h}$. Now suppose that $\gamma \notin J$ such that $a \notin I\gamma$, $b \notin (J : \gamma)$, $ag \in I\gamma$ and $bh \in (J : \gamma)$. Then $(I\gamma, (J : \gamma)) \sim (I, J)$. So be lemma 3.3.11, $(I, J) \in \mathcal{W}_{a,b,g,h}$. □

3.4 Duality for the Ziegler spectrum of a valuation domains

In this section we give an automorphism of Zg_V which induces the lattice isomorphism D given in theorem 2.3.11.

Proposition 3.4.1. *The map $t : \mathrm{Zg}_R \to \mathrm{Zg}_R : N(I, J) \mapsto N(J, I)$ is a well-defined homeomorphism. Moreover, t induces the lattice isomorphism $D : \mathrm{Zg}_R \to \mathrm{Zg}_R : \left(\frac{\phi}{\psi}\right) \mapsto \left(\frac{\mathrm{D}\psi}{\mathrm{D}\phi}\right)$ given in theorem 2.3.11.*

Proof. First we note that t is well defined since $(I,J) \sim (K,L)$ if and only if $(J,I) \sim (L,K)$.

Claim: For any $a,b \in V\backslash\{0\}$, $g,h \in \mathfrak{m}$ and pair of ideals (I,J), $(I,J) \in \mathcal{W}_{a,b,g,h}$ if and only if $(J,I) \in \mathcal{W}_{b,a,h,g}$.

Suppose $(I,J) \in \mathcal{W}_{a,b,g,h}$ then there exists (K,L) such that $(I,J) \sim (K,L)$ and $a \notin K$, $ag \in K$, $b \notin L$ and $bh \in L$. Therefore $(L,K) \in \mathcal{W}_{b,a,h,g}$ and $(J,I) \sim (L,K)$ so $(J,I) \in \mathcal{W}_{b,a,h,g}$. The reverse direction is by symmetry.

Therefore t is a homeomorphism and

$$N(I,J) \in \left(\frac{xag = 0 \wedge b|x}{xa = 0 + bh|x}\right) \text{ if and only if } N(J,I) \in \left(\frac{xbh = 0 \wedge a|x}{xb = 0 + ag|x}\right).$$

Noting lemma 3.3.9, this means that for any $\alpha, \beta, \delta, \gamma \in V$,

$$\left(\frac{x\alpha = 0 \wedge \beta|x}{x\gamma = 0 + \delta|x}\right) \mapsto \left(\frac{x\delta = 0 \wedge \gamma|x}{x\beta = 0 + \alpha|x}\right).$$

It remains to show that for each pp-pair $\left(\frac{\phi}{\psi}\right) \mapsto \left(\frac{\mathrm{D}\psi}{\mathrm{D}\phi}\right)$. Take ϕ, ψ pp-1-formulae. By lemma 3.2.5 we can find a pp-formula $\sum_{i=1}^{n}(x\alpha_i = 0 \wedge \beta_i|x)$ equivalent to ϕ and by lemma 3.2.3 a pp-formulae $\bigwedge_{j=1}^{m}(x\gamma_j = 0 + \delta_j|x)$ equivalent to ψ. So $\left(\frac{\phi}{\psi}\right) = \cup_{i,j}\left(\frac{x\alpha_i=0\wedge\beta_i|x}{x\gamma_j=0+\delta_j|x}\right)$. By lemma 2.2.10 $\mathrm{D}\phi$ is equivalent to $\mathrm{D}(\sum_{i=1}^{n}(x\alpha_i = 0 \wedge \beta_i|x))$ which is equivalent to $\bigwedge_{i=1}^{n}(x\beta_i = 0 + \alpha_i|x)$ and $\mathrm{D}\psi$ is equivalent to $\sum_{j=1}^{m}(x\delta_j = 0 \wedge \gamma_j|x)$. So $\left(\frac{\mathrm{D}\psi}{\mathrm{D}\phi}\right) = \cup_{i,j}\left(\frac{(x\delta_j=0\wedge\gamma_j|x)}{(x\beta_i=0+\alpha_i|x)}\right)$. Hence $\left(\frac{\phi}{\psi}\right) \mapsto \left(\frac{\mathrm{D}\psi}{\mathrm{D}\phi}\right)$.

□

3.5 Description of the open sets

The aim of this section is to get a more manageable characterisation of when a pair $(I,J) \in \mathcal{W}_{a,b,g,h}$. That is we will replace the existential quantifiers in corollary 3.3.13 with simple conditions on pairs of ideals (I,J) invariant under $\sim$.

The following lemma reduces the number of coefficients needed to describe a basic open set.

Lemma 3.5.1. *Let V be a valuation domain. Let $a,b,g,h \in V$, $a,b \neq 0$ and $g,h \in \mathfrak{m}$. Then $\mathcal{W}_{a,b,g,h} = \mathcal{W}_{a.b,1,g,h} = \mathcal{W}_{1,a.b,g,h}$.*

Proof. First note that $\mathcal{W}_{a,b,g,h} = \mathcal{W}_{a.b,1,g,h}$ implies $\mathcal{W}_{a,b,g,h} = \mathcal{W}_{1,a.b,g,h}$ because

$$(I, J) \in \mathcal{W}_{a,b,g,h} \text{ if and only if } (J, I) \in \mathcal{W}_{b,a,h,g} = \mathcal{W}_{ab,1,h,g}$$

and

$$(J, I) \in \mathcal{W}_{ab,1,h,g} \text{ if and only if } (I, J) \in \mathcal{W}_{1,ab,g,h}.$$

Now we prove the first equality. Suppose $(I, J) \in \mathcal{W}_{a,b,g,h}$. Take $(K, L) \sim (I, J)$ with $a \notin K$, $ag \in K$, $b \notin L$ and $bh \in L$. Since $b \notin L$, $(K, L) \sim (Kb, (L : b))$. Now $ab \notin Kb$ (as $a \notin K$), $abg \in Kb$ (as $ag \in K$) and $h \in (L : b)$ (as $bh \in L$). So $(I, J) \sim (Kb, (L : b)) \in \mathcal{W}_{a.b,1,g,h}$.

Conversely, suppose $(I, J) \in \mathcal{W}_{a.b,1,g,h}$. Take $(K, L) \sim (I, J)$ with $ab \notin K$, $abg \in K$ and $h \in L$. Then $b \notin K$, $a \notin (K : b)$, $ag \in (K : b)$, $b \notin Lb$ and $bh \in Lb$. So $(I, J) \sim ((K : b), Lb) \in \mathcal{W}_{a,b,g,h}$. □

The following 4 lemmas will be used in the proof of proposition 3.5.6.

Lemma 3.5.2. *Let $J \lhd V$ and $a, b \in V$. Then $a \notin Jb$ if and only if $b \in (a\mathfrak{m} : J)$.*

Proof. By definition, $b \in (a\mathfrak{m} : J)$ if and only if $Jb \subseteq a\mathfrak{m}$ if and only if $a \notin Jb$. □

Lemma 3.5.3. *Let $\lambda, g, h \in V$, $\lambda \neq 0$ and $g, h \in \mathfrak{m}$. Let (I, J) be a pair of V. Then $(I, J) \in \mathcal{W}_{1,\lambda,g,h}$ implies $\lambda \notin I.J$, $\lambda gh \in I.J$, $g \in I^{\#}$ and $h \in J^{\#}$.*

Proof. Suppose $(I, J) \in \mathcal{W}_{1,\lambda,g,h}$. By corollary 3.3.13, either there exists $\gamma \notin I$ such that $g \in (I : \gamma)$, $\lambda \notin J.\gamma$ and $\lambda h \in J.\gamma$ or there exists $\gamma \notin J$ such that $g \in I.\gamma$, $\lambda \notin (J : \gamma)$ and $\lambda h \in (J : \gamma)$. In either case lemma 3.3.5 implies that $g \in I^{\#}$ and $h \in J^{\#}$. If $\gamma \notin I$ then it is clear that $(I : \gamma).J.\gamma = I.J$ since $(I : \gamma).\gamma = I$. Similarly, if $\gamma \notin J$ then $I.\gamma(J : \gamma) = I.J$. Therefore in either of the above cases $\lambda \notin I.J$ and $\lambda gh \in I.J$. □

Lemma 3.5.4. *Let $I, J \lhd V$. Then $I.J \subseteq K$ if and only if $I \subseteq (K : J)$. Equivalently, for valuation domains, $I.J \supsetneq K$ if and only if $I \supsetneq (K : J)$.*

Proof. Suppose $I.J \subseteq K$. Take $x \in I$. Then $x.J \subseteq I.J \subseteq K$. So $x \in (K : J)$.

Suppose $I \subseteq (K : J)$. Take $i \in I$ and $j \in J$. Then $i.j \in i.J \subseteq K$. So $I.J \subseteq K$. □

Lemma 3.5.5. *Suppose that $J \lhd V$ and $\lambda, h \in V$ such that $\lambda h \in J$ and $h \in J^{\#}$. Then $(\lambda h\mathfrak{m} : J) \subsetneq (\lambda\mathfrak{m} : J)$.*

Proof. If $\lambda\mathfrak{m} \supseteq J$ then $(\lambda\mathfrak{m} : J) = V$ and $\lambda h\mathfrak{m} \subsetneq J$ so $(\lambda h\mathfrak{m} : J) \neq V$. Otherwise $\lambda \in J$. Since $h \in J^{\#}$, $Jh \subsetneq J$. Take $a \in J \backslash Jh$. Then $a\mathfrak{m} \supseteq Jh \supseteq \lambda hV$. Let $t \in \mathfrak{m}$ be such that $at = \lambda h$. Since $a \notin Jh$, $a\lambda \notin J\lambda h$. So $a\lambda \notin Jat$. Hence $\lambda \notin Jt$. So $\lambda\mathfrak{m} \supseteq Jt$. Since $a \in J$, $a\lambda h \in J\lambda h = Jat$. So $\lambda h \in Jt$. Hence $\lambda h\mathfrak{m} \subsetneq tJ$. Therefore $t \in (\lambda\mathfrak{m} : J)$ and $t \notin (\lambda h\mathfrak{m} : J)$. □

Proposition 3.5.6. *Let $\lambda, g, h \in V$, $\lambda \neq 0$ and $g, h \in \mathfrak{m}$. Let (I, J) be a pair of ideals in V. Then the following are equivalent:*

(i) $(I, J) \in \mathcal{W}_{1,\lambda,g,h}$.

(ii) $g \in I^{\#}$, $h \in J^{\#}$, $\lambda gh \in IJ$ *and* $(I, J) \in \mathcal{W}_{1,\lambda,0,0}$.

Proof. (ii)⇒(i). We split the proof into two cases:

Case 1: $\lambda h \in J$.

In order to show that $(I, J) \in \mathcal{W}_{1,\lambda,g,h}$ we must find $x \notin I$ such that $g \in (I : x)$, $\lambda \notin J.x$ and $\lambda h \in J.x$. This follows from corollary 3.3.13 and $\lambda h \in J$.

We can rewrite $g \in (I : x)$ as $x \in (I : g)$, $\lambda \notin J.x$ as $x \in (\lambda\mathfrak{m} : J)$ by lemma 3.5.2 and $\lambda h \in J.x$ as $x \notin (\lambda h\mathfrak{m} : J)$ by lemma 3.5.2. As ideals are totally ordered, it is enough to show that the following strict inequalities hold:

(1) $I \subsetneq (I : g)$

(2) $I \subsetneq (\lambda\mathfrak{m} : J)$

(3) $(\lambda h\mathfrak{m} : J) \subsetneq (I : g)$

(4) $(\lambda h\mathfrak{m} : J) \subsetneq (\lambda\mathfrak{m} : J)$

(1) is true since $g \in I^{\#}$ and (4) holds by 3.5.5 using $h \in J^{\#}$.

(3) By (ii) $\lambda gh \in I.J$, which implies that $\lambda gh\mathfrak{m} \subsetneq I.J$. If $g \in I$ then $(I : g) = V$ so $\lambda h\mathfrak{m} \subsetneq (I : g)J$. Otherwise $g \notin I$. Suppose for a contradiction that $\lambda h\mathfrak{m} \supseteq (I : g)J$.

Then $\lambda gh\mathfrak{m} \supseteq IJ$, a contradiction since $\lambda gh \in IJ$. Therefore $\lambda h\mathfrak{m} \subsetneq (I : g)J$ so by 3.5.4, $(\lambda h\mathfrak{m} : J) \subsetneq (I : g)$.

(2) By (ii) $(I, J) \in \mathcal{W}_{1,\lambda,0,0}$. So either $\lambda \notin J$ or there exists $\gamma \notin I$ such that $\lambda \notin \gamma.J$. If $\lambda \notin J$ then $\lambda\mathfrak{m} \supseteq J$. So $(\lambda\mathfrak{m} : J) = V$. Therefore (2) holds. If there exists $\gamma \notin I$ such that $\lambda \notin \gamma.J$ then $\lambda\mathfrak{m} \supseteq \gamma.J$ i.e. there exists $\gamma \in (\lambda\mathfrak{m} : J)$ not in I. So (2) holds.

Case 2: $\lambda h \notin J$

Again, by corollary 3.3.13 and $\lambda h \notin J$, in order to show that $(I, J) \in \mathcal{W}_{1,\lambda,g,h}$ we must find $x \notin J$ such that $g \in I.x$, $\lambda \notin (J : x)$ and $\lambda h \in (J : x)$. That is $x \notin J$ such that $x \notin (g\mathfrak{m} : I)$, $x \notin (J : \lambda)$ and $x \in (J : \lambda h)$. So it is enough to show that the following strict inequalities hold:

(1) $(J : \lambda h) \supsetneq (g\mathfrak{m} : I)$

(2) $(J : \lambda h) \supsetneq (J : \lambda)$

(1) By hypothesis $\lambda gh \in I.J$. Therefore $\lambda gh\mathfrak{m} \subsetneq I.J$. Hence $g\mathfrak{m} \subsetneq I.(J : \lambda h)$, therefore by proposition 3.5.4 $(g\mathfrak{m} : I) \subsetneq (J : \lambda h)$.

(2) The second is clear since $\lambda h \notin J$ and $h \in J^{\#}$.

(i)$\Rightarrow$(ii). Now suppose $(I, J) \in \mathcal{W}_{1,\lambda,g,h}$. There exists (I', J') such that $(I, J) \sim (I', J')$ and $g \in I'$, $\lambda \notin J'$ and $\lambda h \in J'$. Therefore $(I', J') \in \mathcal{W}_{1,\lambda,0,0}$, so $(I, J) \in \mathcal{W}_{1,\lambda,0,0}$ and by proposition 3.5.3 $\lambda gh \in I.J$, $g \in I^{\#}$ and $h \in J^{\#}$.

$\square$

From proposition 3.5.6 we can deduce that if $I, J \lhd V$ and $I^{\#} = J^{\#}$ then (I, J) and (J, I) are topologically indistinguishable since $\mathcal{W}_{1,\lambda,0,0} = \mathcal{W}_{\lambda,1,0,0}$.

It remains to consider when a pair $(I, J) \in \mathcal{W}_{1,\lambda,0,0}$. In order to do this we first group ideals into 4 distinct classes. We start by showing that if $I \lhd V$ and $(I^{\#})^2 \neq I^{\#}$ then $I = aI^{\#}$ for some $a \in V$.

Lemma 3.5.7. *Suppose that $\mathfrak{p}$ is a prime ideal and $\mathfrak{p}^2 \neq \mathfrak{p}$. Then if $I \lhd V$ with $I^{\#} = \mathfrak{p}$ there exists $a \in V\backslash\{0\}$ such that $I = a\mathfrak{p}$.*

Proof. Suppose $I \lhd V$, $I^{\#} = \mathfrak{p}$ and $\mathfrak{p}^2 \neq \mathfrak{p}$. Take $k \in \mathfrak{p} \backslash \mathfrak{p}^2$. Since $I^{\#} = \mathfrak{p}$ there exists $a \notin I$ such that $k \in (I : a)$. Given any $t \in \mathfrak{p}$, either $t \in kV$ so $t \in (I : a)$ or $k = tc$ for some $c \in V$. Note that $c \notin \mathfrak{p}$ since $k \notin \mathfrak{p}^2$. Hence $t \in (I : a)$ since $(I : a)^{\#} = I^{\#} = \mathfrak{p}$. Therefore $(I : a) = \mathfrak{p}$. So $I = a\mathfrak{p}$. □

Definition 3.5.8. *Let $\mathfrak{p} \lhd V$ be a prime ideal and let $a \in \mathfrak{p}$ be non-zero. We define*

$$I_a^{\mathfrak{p}} = \{b \in V \mid \textit{there exists } r \notin \mathfrak{p} \textit{ such that } br \in aV\}.$$

The ideal $I_a^{\mathfrak{p}}$ is the pre-image of the ideal generated by a in $V_{\mathfrak{p}}$. Note that for any valuation domain V and any $a \in \mathfrak{m} \backslash \{0\}$, $I_a^{\mathfrak{m}} = aV$. The following lemmas give properties of the ideals $I_a^{\mathfrak{p}}$.

Lemma 3.5.9. *Let $\mathfrak{p} \lhd V$ be a prime ideal and let $a \in \mathfrak{p}$ be non-zero. Then $I_a^{\mathfrak{p}}$ is an ideal with attached prime $\mathfrak{p}$.*

Proof. Suppose $b \in I_a^{\mathfrak{p}}$ and $r \in V$. Then there exists $k \notin \mathfrak{p}$ such that $bk \in aV$. Therefore $(br)k \in aV$. So $br \in I_a^{\mathfrak{p}}$. Hence, by 2.4.2, $I_a^{\mathfrak{p}}$ is an ideal.

We now show that $I_a^{\mathfrak{p}}$ has attached prime $\mathfrak{p}$. Suppose that $b \notin I_a^{\mathfrak{p}}$, $c \in V$ and $bc \in I_a^{\mathfrak{p}}$. Then there exists $k \notin \mathfrak{p}$ such that $bck \in aV$ but since $b \notin I_a^{\mathfrak{p}}$, $ck \in \mathfrak{p}$. Therefore $c \in \mathfrak{p}$.

Suppose $c \in \mathfrak{p}$. Then either $c \in aV$ (hence $c \in I_a^{\mathfrak{p}}$) or $c \notin aV$. Suppose $c \notin aV$. Then $a = c\gamma$ for some $\gamma \in V$. Suppose, for a contradiction, that $\gamma \in I_a^{\mathfrak{p}}$. Then there exists $t \notin \mathfrak{p}$ such that $\gamma t \in aV = c\gamma V$. Hence $t \in cV$. A contradiction since $c \in \mathfrak{p}$. Therefore $c \in (I_a^{\mathfrak{p}})^{\#}$.

□

Lemma 3.5.10. *Let $\mathfrak{p} \lhd V$ be a prime ideal, $a \in \mathfrak{p}$ and $\lambda \in V$. Then*

$$\lambda \notin I_a^{\mathfrak{p}} \textit{ if and only if } a \in \lambda\mathfrak{p}.$$

Proof. It is clear that $a \in \lambda\mathfrak{p}$ implies $\lambda \notin I_a^{\mathfrak{p}}$. Suppose $\lambda \notin I_a^{\mathfrak{p}}$. Then $\lambda \notin aV$. So $a = \lambda\gamma$ and $\gamma \in \mathfrak{p}$. Hence $a \in \lambda\mathfrak{p}$. □

Lemma 3.5.11. *Let $\mathfrak{p} \lhd V$ be a prime ideal and $a, b \in \mathfrak{p}$ be non-zero. Then $I_a^{\mathfrak{p}} \cdot b = I_a^{\mathfrak{p}}.I_b^{\mathfrak{p}} = I_{ab}^{\mathfrak{p}}$.*

Proof. <u>Claim:</u> $I^{\mathfrak{p}}_a \cdot b = I^{\mathfrak{p}}_a I^{\mathfrak{p}}_b$.

First note that $b \in I^{\mathfrak{p}}_b$. So $I^{\mathfrak{p}}_a \cdot b \subseteq I^{\mathfrak{p}}_a I^{\mathfrak{p}}_b$. Suppose $x \in I^{\mathfrak{p}}_a I^{\mathfrak{p}}_b$. Then $x = x_1 x_2$ for some $x_1 \in I^{\mathfrak{p}}_a$ and $x_2 \in I^{\mathfrak{p}}_b$. So there exists $\gamma_1 \notin \mathfrak{p}$ such that $x_1\gamma_1 \in aV$ and $\gamma_2 \notin \mathfrak{p}$ such that $x_2\gamma_2 \in bV$. Since $x_1 \in \mathfrak{p}$ and $\gamma_2 \notin \mathfrak{p}$ there exists $\mu \in V$ such that $\gamma_2\mu = x_1$. Therefore $\mu \in I^{\mathfrak{p}}_a$ since $\mu\gamma_1\gamma_2 = x_1\gamma_1 \in aV$ and $\gamma_1\gamma_2 \notin \mathfrak{p}$. Hence $x = x_1x_2 = \mu.(\gamma_2 x_2) \in I^{\mathfrak{p}}_a \cdot bV$. So $I^{\mathfrak{p}}_a \cdot b \supseteq I^{\mathfrak{p}}_a I^{\mathfrak{p}}_b$.

<u>Claim:</u> $I^{\mathfrak{p}}_a \cdot b = I^{\mathfrak{p}}_{ab}$.

Suppose $x \in I^{\mathfrak{p}}_{ab}$. Then $b|x$. To see this, suppose for a contradiction that $b = x\gamma$ for some $\gamma \in V$. Then, since $x \in I^{\mathfrak{p}}_{ab}$, there exists $t \notin \mathfrak{p}$ such that $xt \in abV = ax\gamma V$. Hence $t \in a\gamma V$, contradicting $t \notin \mathfrak{p}$. Hence b divides x.

Let $\mu \in V$ be such that $x = b\mu$. It remains to show that $\mu \in I^{\mathfrak{p}}_a$. Since $b\mu \in I^{\mathfrak{p}}_{ab}$ there exists $k \notin \mathfrak{p}$ such that $b\mu k \in abV$ therefore $\mu k \in aV$ so $\mu \in I^{\mathfrak{p}}_a$. Hence $x \in I^{\mathfrak{p}}_a \cdot b$.

Suppose $x \in I^{\mathfrak{p}}_a$. Then there exists $k \notin \mathfrak{p}$ such that $xk \in aV$. Therefore $xbk \in abV$. So $xb \in I^{\mathfrak{p}}_{ab}$.

□

Lemma 3.5.12. *Let $\mathfrak{p} \lhd V$ be a prime ideal and $a \in \mathfrak{p}$. Then $I^{\mathfrak{p}}_a \mathfrak{p} = a\mathfrak{p}$.*

Proof. The inclusion $I^{\mathfrak{p}}_a \mathfrak{p} \supseteq a\mathfrak{p}$ holds as $I^{\mathfrak{p}}_a \supseteq aV$. Suppose $t \in I^{\mathfrak{p}}_a$. There exists $\gamma \notin \mathfrak{p}$ such that $t\gamma \in aV$. Take any $p \in \mathfrak{p}$. Then $p = \gamma r$ for some $r \in \mathfrak{p}$. Therefore $tp = t\gamma r \in a\mathfrak{p}$. So $I^{\mathfrak{p}}_a \mathfrak{p} \subseteq a\mathfrak{p}$. □

Lemma 3.5.13. *Let $\mathfrak{p} \lhd V$ and $I \lhd V$ such that $\mathfrak{p}^2 = \mathfrak{p}$ and $I^{\#} = \mathfrak{p}$. Then $I\mathfrak{p} \subsetneq I$ if and only if $I = I^{\mathfrak{p}}_a$ for some $a \in \mathfrak{p}$.*

Proof. $\Rightarrow$ Suppose $a \in I \backslash I\mathfrak{p}$. We will now show that $I = I^{\mathfrak{p}}_a$.

Take $t \in I$. Then either $t \in aV$ or $a = tr$ for some $r \in V$. If $t \in aV$ then $t \in I^{\mathfrak{p}}_a$. Suppose $a = tr$. Then $r \notin \mathfrak{p}$ since $a \notin I\mathfrak{p}$. Hence $t \in I^{\mathfrak{p}}_a$.

Now suppose $t \in I^{\mathfrak{p}}_a$. There exists $\gamma \notin \mathfrak{p}$ such that $t\gamma \in aV$. Hence $t\gamma \in I$. Since $\gamma \notin \mathfrak{p}$ and $\mathfrak{p} = I^{\#}$, $t \in I$.

$\Leftarrow$ Suppose $a \in \mathfrak{p}$. Then $I^{\mathfrak{p}}_a \mathfrak{p} = a\mathfrak{p}$. Clearly $a\mathfrak{p} \subsetneq I^{\mathfrak{p}}_a$ since $a \notin a\mathfrak{p}$ and $a \in I^{\mathfrak{p}}_a$. □

Definition 3.5.14. *We say that $I \lhd V$ with $I^{\#} = \mathfrak{p}$ is a proper cut if it is not equal to $I_a^{\mathfrak{p}}$ for any $a \in \mathfrak{p}\backslash\{0\}$ or $b\mathfrak{p}$ for any $b \in V\backslash\{0\}$.*

We now given an example of a proper cut. Let V be a valuation domain with value group $\mathbb{Q}$ under addition. Such a valuation domain exists by 2.4.5. Let v be the valuation map. Then $I \lhd V$ is a proper cut if and only if $I = \{r \in V \mid v(r) > c\}$ for some strictly positive irrational real number c.

We now split the question of when a pair of ideals (I, J) lies in $\mathcal{W}_{1,\lambda,0,0}$ into the following cases:

1. $I^{\#} \neq J^{\#}$ (lemma 3.5.16 and discussion directly below that).

2. $I^{\#} = J^{\#} = \mathfrak{p}$ and exactly one of the following conditions

   (i) $\mathfrak{p} \neq \mathfrak{p}^2$ (lemma 3.5.18, noting lemma 3.5.7).

   (ii) $\mathfrak{p} = \mathfrak{p}^2$, $I = t\mathfrak{p}$ and $J = s\mathfrak{p}$ for some non-zero $t, s \in V$ (lemma 3.5.18).

   (iii) $\mathfrak{p} = \mathfrak{p}^2$, $I = I_a^{\mathfrak{p}}$ and $J = I_b^{\mathfrak{p}}$ for some non-zero $a, b \in \mathfrak{p}$ (lemma 3.5.19).

   (iv) $\mathfrak{p} = \mathfrak{p}^2$, $I = I_a^{\mathfrak{p}}$ and $J = t\mathfrak{p}$ for some non-zero $a \in \mathfrak{p}$ and non-zero $t \in V$ or $I = t\mathfrak{p}$ and $J = I_a^{\mathfrak{p}}$ for some non-zero $t \in V$ and some non-zero $a \in \mathfrak{p}$ (lemma 3.5.21).

   (v) $\mathfrak{p} = \mathfrak{p}^2$ and I or J is a proper cut (lemma 3.5.22).

Lemma 3.5.15. *Suppose $I, J \lhd V$ such that $J \supsetneq I^{\#}$. Then $IJ = I$.*

Proof. Suppose $I, J \lhd V$ such that $J \supsetneq I^{\#}$. Suppose $x \in I$. Take $y \in J\backslash I^{\#}$. Then there exists $r \in V$ such that $yr = x$. Therefore $r \in I$ since $y \notin I^{\#}$. So $x \in IJ$. Hence $IJ \supseteq I$. The other inclusion is true for all ideals so $IJ = I$. □

Lemma 3.5.16. *Let $\lambda \in V$ be non-zero and (I, J) a pair in V such that $I^{\#} \subsetneq J$. The following are equivalent:*

(i) $(I, J) \in \mathcal{W}_{1,\lambda,0,0}$.

(ii) $(I, J^{\#}) \in \mathcal{W}_{1,\lambda,0,0}$.

(iii) $\lambda \notin I.J.$

Proof. First note that if $I, J \lhd V$ are such that $J \supseteq I^{\#}$ then $x \notin I$ implies $Jx \supseteq I$. To see this, suppose $x \notin I$ and take $i \in I$. There exists $\gamma \in V$ such that $i = x\gamma$. Then $\gamma \in J$ since $\gamma \in I^{\#}$. So $i \in Jx$.

(i)⇒(ii). Suppose that $(I, J) \in \mathcal{W}_{1,\lambda,0,0}$. Then there exists $x \notin I$ such that $\lambda \notin J.x$ but then $\lambda \notin I$. Hence $(I, J^{\#}) \in \mathcal{W}_{1,\lambda,0,0}$ since $\lambda \notin J^{\#}.\lambda$.

(ii)⇒(i). Suppose $(I, J^{\#}) \in \mathcal{W}_{1,\lambda,0,0}$. Then there exists $x \notin I$ such that $\lambda \notin J^{\#}.x$. Therefore $\lambda \notin J.x$. Hence $(I, J) \in \mathcal{W}_{1,\lambda,0,0}$.

(i)⇔(iii). Suppose $\lambda \notin IJ$. So $\lambda \notin I$ since $IJ = I$ (by lemma 3.5.15). So $(I, J) \sim ((I : \lambda), J\lambda))$ and $\lambda \notin J\lambda$. Hence $(I, J) \in \mathcal{W}_{1,\lambda,0,0}$. The reverse implication is part of proposition 3.5.3. □

Note that for any ideals $I, J \lhd V$ with $I^{\#} \subsetneq J^{\#}$ and any $\lambda \in V\backslash\{0\}$,

$$(I, J) \in \mathcal{W}_{1,\lambda,0,0} \text{ if and only if } \lambda \notin IJ.$$

To see this, note that there exists $x \notin J$ such that $I^{\#} \subsetneq (J : x)$. Therefore $(Ix, (J : x)) \in \mathcal{W}_{1,\lambda,0,0}$ if and only if $\lambda \notin Ix(J : x) = IJ$. Since $(I, J) \sim (Ix, (J : x))$, $(I, J) \in \mathcal{W}_{1,\lambda,0,0}$ if and only if $\lambda \notin IJ$.

Corollary 3.5.17. *Suppose (I, J) is a pair in V such that $I^{\#} \subsetneq J^{\#}$. Then there exists $T \lhd V$ with $T^{\#} = I^{\#}$ such that $(T, J^{\#})$ is topologically indistinguishable from (I, J).*

Proof. Suppose $I^{\#} \subsetneq J^{\#}$. Then there exists $x \notin J$ such that $I^{\#} \subsetneq (J : x)$. Note that $(Ix)^{\#} = I^{\#}$ and $(J : x)^{\#} = J^{\#}$, by lemma 3.3.5. Therefore, by 3.5.16, for any $\lambda \in V\backslash\{0\}$, $(Ix, (J : x)) \in \mathcal{W}_{1,\lambda,0,0}$ if and only if $(Ix, J^{\#}) = (Ix, (J : x)^{\#}) \in \mathcal{W}_{1,\lambda,0,0}$.

By lemma 3.5.15, $Ix(J : x) = Ix(J : x)^{\#} = IxJ^{\#}$. Hence, by proposition 3.5.6, for any $\lambda \in V\backslash\{0\}$ and $g, h \in \mathfrak{m}$, $(Ix, (J : x)) \in \mathcal{W}_{1,\lambda,g,h}$ if and only if $(Ix, J^{\#}) \in \mathcal{W}_{1,\lambda,g,h}$. Since $(I, J) \sim (Ix, (J : x))$, (I, J) is topologically indistinguishable from $(Ix, J^{\#})$.

□

Lemma 3.5.18. *Suppose $\mathfrak{p} \lhd V$ is prime and $\lambda, t_1, t_2 \in V\backslash\{0\}$. Then the following are equivalent:*

(i) $(t_1\mathfrak{p}, t_2\mathfrak{p}) \in \mathcal{W}_{1,\lambda,0,0}$.

(ii) $\lambda \notin t_1t_2\mathfrak{p}$.

Proof. (ii)⇒(i). First note that $(t_1\mathfrak{p}, t_2\mathfrak{p}) \sim (\mathfrak{p}, t_1t_2\mathfrak{p})$. So if $\lambda \notin t_1t_2\mathfrak{p}$ then $(t_1\mathfrak{p}, t_2\mathfrak{p}) \in \mathcal{W}_{1,\lambda,0,0}$.

(i)⇒ (ii). Suppose $(t_1\mathfrak{p}, t_2\mathfrak{p}) \in \mathcal{W}_{1,\lambda,0,0}$. Then there exists $\gamma \notin t_1\mathfrak{p}$ such that $\lambda \notin \gamma t_2\mathfrak{p}$. If $\gamma \notin t_1\mathfrak{p}$ then $\gamma\mathfrak{p} \supseteq t_1\mathfrak{p}$. Therefore $\gamma t_2\mathfrak{p} \supseteq t_1t_2\mathfrak{p}$. Hence $\lambda \notin t_1t_2\mathfrak{p}$. □

Lemma 3.5.19. *Suppose that $\mathfrak{p} \lhd V$ is a prime ideal and $\mathfrak{p}^2 = \mathfrak{p}$. If $a, b \in \mathfrak{p}$ then $(I_a^{\mathfrak{p}}, I_b^{\mathfrak{p}}) \in \mathcal{W}_{1,\lambda,0,0}$ if and only if $\lambda \notin I_a^{\mathfrak{p}}I_b^{\mathfrak{p}}$.*

Proof. By lemma 3.5.3 $(I_a^{\mathfrak{p}}, I_b^{\mathfrak{p}}) \in \mathcal{W}_{1,\lambda,0,0}$ implies $\lambda \notin I_a^{\mathfrak{p}}I_b^{\mathfrak{p}}$. Suppose $\lambda \notin I_a^{\mathfrak{p}}I_b^{\mathfrak{p}} = I_{ab}^{\mathfrak{p}}$. Either $\lambda \notin I_b^{\mathfrak{p}}$ so $(I_a^{\mathfrak{p}}, I_b^{\mathfrak{p}}) \in \mathcal{W}_{1,\lambda,0,0}$ or $\lambda \in I_b^{\mathfrak{p}}$. Suppose $\lambda \in I_b^{\mathfrak{p}}$. By lemma 3.5.10 this means $ab \in \lambda\mathfrak{p}$ and $b \notin \lambda\mathfrak{p}$. Let $k \in V$ be such that $\lambda\mathfrak{p} = kb\mathfrak{p}$. Such a k exists since either $\lambda\mathfrak{p} = b\mathfrak{p}$ or $b\mathfrak{p} \supsetneq \lambda\mathfrak{p}$ hence $\lambda \in b\mathfrak{p}$. Then $a \in k\mathfrak{p}$ since $ab \in \lambda\mathfrak{p} = kb\mathfrak{p}$. As $\mathfrak{p} = \mathfrak{p}^2$ there exists $\gamma_1, \gamma_2 \in \mathfrak{p}$ such that $a = k\gamma_1\gamma_2$. So $a \in k\gamma_1\mathfrak{p}$ and $k\gamma_1 \in k\mathfrak{p}$. So by lemma 3.5.10 $k\gamma_1 \notin I_a^{\mathfrak{p}}$ and $\lambda \notin I_{k\gamma_1 b}^{\mathfrak{p}} = I_b^{\mathfrak{p}}k\gamma_1$ since $k\gamma_1 b \in \lambda\mathfrak{p}$. Therefore $(I_a^{\mathfrak{p}}, I_b^{\mathfrak{p}}) \in \mathcal{W}_{1,\lambda,0,0}$. □

Lemma 3.5.20. *Let (I, J) be a pair in V such that $I^\# = J^\#$. Then for all $\lambda \in V\backslash\{0\}$, $\lambda I^\# \supsetneq IJ$ implies $(I, J) \in \mathcal{W}_{1,\lambda,0,0}$.*

Proof. Suppose $I, J \lhd V$, $I^\# = J^\# = \mathfrak{p}$ and $\lambda\mathfrak{p} \supsetneq IJ$. If $\lambda \notin J$ then $(I, J) \in \mathcal{W}_{1,\lambda,0,0}$. So suppose $\lambda \in J$. Take $p \in \mathfrak{p}$ such that $\lambda p \notin IJ$. So $\lambda p\mathfrak{m} \supseteq IJ$. Hence, by 3.5.4, $(\lambda p\mathfrak{m} : J) \supseteq I$ and by 3.5.5 $(\lambda\mathfrak{m} : J) \supsetneq (\lambda p\mathfrak{m} : J)$. Take $x \in (\lambda\mathfrak{m} : J)\backslash I$. Then $\lambda \notin Jx$ and $x \notin I$. So $(I, J) \in \mathcal{W}_{1,\lambda,0,0}$. □

Suppose $\mathfrak{p}^2 = \mathfrak{p}$, $I = I_a^{\mathfrak{p}}$ and $J = t\mathfrak{p}$ for some non-zero $a \in \mathfrak{p}$ and non-zero $t \in V$. Then $t \notin J$, $(J : t) = \mathfrak{p}$ and by lemma 3.5.11, $I.t = I_{at}^{\mathfrak{p}}$. Therefore $(I, J) \sim (I_{at}^{\mathfrak{p}}, \mathfrak{p})$. The following lemma characterises when a pair of the form $(I_a^{\mathfrak{p}}, \mathfrak{p})$ lies in $\mathcal{W}_{1,\lambda,0,0}$ where $\mathfrak{p}^2 = \mathfrak{p}$ and $a \in \mathfrak{p}\backslash\{0\}$.

Lemma 3.5.21. *Let $\mathfrak{p} \lhd V$ be a prime ideal such that $\mathfrak{p}^2 = \mathfrak{p}$ and $a \in \mathfrak{p}\backslash\{0\}$. Let $\lambda \in V\backslash\{0\}$. Then the following are equivalent:*

(i) $(I_a^{\mathfrak{p}}, \mathfrak{p}) \in \mathcal{W}_{1,\lambda,0,0}$.

(ii) $\lambda\mathfrak{p} \supsetneq I_a^{\mathfrak{p}}\mathfrak{p}$.

(iii) $\lambda \notin I_a^{\mathfrak{p}}$.

Proof. (ii)⇒(i) is by lemma 3.5.20.

(iii)⇒(ii) By lemma 3.5.10, $\lambda \notin I_a^{\mathfrak{p}}$ implies $a \in \lambda\mathfrak{p}$. By lemma 3.5.12 $I_a^{\mathfrak{p}}\mathfrak{p} = a\mathfrak{p} \subsetneq \lambda\mathfrak{p}$.

(i)⇒(iii) Suppose $(I_a^{\mathfrak{p}}, \mathfrak{p}) \in \mathcal{W}_{1,\lambda,0,0}$. Then there exists $\gamma \notin I_a^{\mathfrak{p}}$ such that $\lambda \notin \gamma\mathfrak{p}$. Hence $\gamma\mathfrak{p} \supseteq I_a^{\mathfrak{p}}$. So $\lambda \notin I_a^{\mathfrak{p}}$.

□

Lemma 3.5.22. *Suppose that (I, J) is a pair in V, $\mathfrak{p} = I^{\#} = J^{\#}$ and $\mathfrak{p} = \mathfrak{p}^2$. If either I or J corresponds to a proper cut then $(I, J) \in \mathcal{W}_{1,\lambda,0,0}$ if and only if $\lambda\mathfrak{p} \supsetneq I.J$.*

Proof. Let $I, J \lhd V$ with $I^{\#} = J^{\#} = \mathfrak{p}$ be such that either I or J is a proper cut. Suppose that $(I, J) \in \mathcal{W}_{1,\lambda,0,0}$. Then $\lambda \notin I.J$. So $\lambda\mathfrak{p} \supseteq I.J$. If $I.J \neq \lambda\mathfrak{p}$ then we are done so suppose for a contradiction that $\lambda\mathfrak{p} = I.J$. Since $(I, J) \in \mathcal{W}_{1,\lambda,0,0}$ there exists $\gamma \notin I$ such that $\lambda \notin J\gamma$. Therefore $I.J = \lambda\mathfrak{p} \supseteq J\gamma$ and $J\gamma \supseteq I.J$ since $\gamma \notin I$ hence $J\gamma = I.J$. This means that $(I : \gamma)J = J$. Take any $t \notin (I : \gamma)$. Then $tV \supseteq (I : \gamma)$. So $tJ \supseteq (I : \gamma)J = J$. Therefore $t \notin \mathfrak{p}$. Hence $(I : \gamma) = \mathfrak{p}$. So $I = \mathfrak{p}\gamma$. But $IJ = J\gamma = \lambda\mathfrak{p}$. So neither I or J is a proper cut, a contradiction. Therefore $\lambda\mathfrak{p} \supsetneq IJ$.

The converse is by lemma 3.5.20.

□

Definition 3.5.23. *We say that a pair (I, J) is a normal point if for all $\lambda \in V\backslash\{0\}$, $\lambda \notin IJ$ implies $(I, J) \in \mathcal{W}_{1,\lambda,0,0}$. Otherwise we call a pair (I, J) abnormal.*

Lemma 3.5.24. *Let $\mathfrak{p} \lhd V$ be prime and let $I, J \lhd V$ be such that $I^{\#} = J^{\#} = \mathfrak{p}$ and (I, J) is abnormal. Then for all $\lambda \in V\backslash\{0\}$ the following are equivalent:*

(i) $(I,J) \in \mathcal{W}_{1,\lambda,0,0}$.

(ii) $\lambda\mathfrak{p} \supsetneq IJ$.

Proof. (ii)$\Rightarrow$ (i) By lemma 3.5.20, $\lambda\mathfrak{p} \supsetneq IJ$ implies $(I,J) \in \mathcal{W}_{1,\lambda,0,0}$.

(i)$\Rightarrow$(ii) Let $a \in V$ be such that $a \notin IJ$ and $(I,J) \notin \mathcal{W}_{1,a,0,0}$. Such an a exists since (I,J) is abnormal. Suppose $(I,J) \in \mathcal{W}_{1,\lambda,0,0}$. There exists $\gamma \notin I$ such that $\lambda \notin J\gamma$. Hence $\lambda\mathfrak{p} \supseteq J\gamma$. Since $(I,J) \notin \mathcal{W}_{1,a,0,0}$, $a \in J\gamma$. Hence $a \in \lambda\mathfrak{p}$. Therefore $\lambda\mathfrak{p} \supsetneq a\mathfrak{p}$. Since $a \notin IJ$, $a\mathfrak{p} \supseteq IJ$. So $\lambda\mathfrak{p} \supsetneq a\mathfrak{p} \supseteq IJ$. $\square$

Lemma 3.5.25. *Let $\mathfrak{p} \lhd V$ be a prime ideal such that $\mathfrak{p}^2 = \mathfrak{p}$ and (I,J) an abnormal point with $I^{\#} = J^{\#} = \mathfrak{p}$. Then there exists non-zero $a \in \mathfrak{p}$ such that (I,J) and $(I_a^{\mathfrak{p}}, \mathfrak{p})$ are topologically indistinguishable.*

Proof. Let $\mathfrak{p} \lhd V$ be a prime ideal such that $\mathfrak{p}^2 = \mathfrak{p}$ and let $I, J \lhd V$ be such that $I^{\#} = J^{\#} = \mathfrak{p}$. First we show that (I,J) abnormal implies that there exists $a \in \mathfrak{p}$ such that $IJ = a\mathfrak{p}$. Suppose (I,J) is abnormal. By definition of abnormal, there exists $a \notin IJ$ such that $(I,J) \notin \mathcal{W}_{1,a,0,0}$. Since $a \notin IJ$ and $(IJ)^{\#} = \mathfrak{p}$, $a\mathfrak{p} \supseteq IJ$. By lemma 3.5.20 and because $(I,J) \notin \mathcal{W}_{1,a,0,0}$, we have $a\mathfrak{p} \subseteq IJ$.

By lemma 3.5.24 and since $IJ = a\mathfrak{p} = I_a^{\mathfrak{p}}\mathfrak{p}$, for all $\lambda \in V\backslash\{0\}$, $(I,J) \in \mathcal{W}_{1,\lambda,0,0}$ if and only if $(I_a^{\mathfrak{p}}, \mathfrak{p}) \in \mathcal{W}_{1,\lambda,0,0}$.

Consequently, using proposition 3.5.6, (I,J) and $(I_a^{\mathfrak{p}}, \mathfrak{p})$ are topologically indistinguishable. $\square$

Note that (I,J) abnormal implies $I^{\#} = J^{\#}$ by lemma 3.5.16. Suppose (I,J) is a point with $I^{\#} = J^{\#} = \mathfrak{p}$. Then $\mathfrak{p}^2 \neq \mathfrak{p}$ implies (I,J) is abnormal. Lemma 3.5.20 implies that if (I,J) is abnormal then $IJ = a\mathfrak{p}$ for some $a \in V\backslash\{0\}$. Moreover, if $IJ = a\mathfrak{p}$ for some $a \in V\backslash\{0\}$ then (I,J) is abnormal if and only if $(I,J) \notin \mathcal{W}_{1,a,0,0}$.

Finally note that if $\mathfrak{p}^2 = \mathfrak{p}$ then for any non-zero $a, b \in \mathfrak{p}$, the point $(I_a^{\mathfrak{p}}, I_b^{\mathfrak{p}})$ is normal and for any non-zero $a, b \in V$ the point $(a\mathfrak{p}, b\mathfrak{p})$ is normal (cf. lemma 3.5.19 and lemma 3.5.18).

We now give some examples of normal and abnormal points in Zg_V for particular valuation domains.

Example 3.5.26. *Suppose V is a valuation domain with value group $\mathbb{Z}$. The maximal ideal $\mathfrak{m}$ of V is finitely generated. Let k generate $\mathfrak{m}$.*

1. *For all $n, m \in \mathbb{N}$, $(k^nV, k^mV) \sim (\mathfrak{m}, k^{m+n-2}\mathfrak{m})$ and (k^nV, k^mV) is an abnormal point.*

2. *The points $(I, 0)$, $(0, I)$ and $(0, 0)$ are normal for all proper $I \lhd V$.*

Proof. First note, such a valuation domain exists by 2.4.5. Also note that $\mathfrak{m}$ is the only non-zero prime ideal in V and since $\mathbb{Z}$ is not dense, $\mathfrak{m}^2 \neq \mathfrak{m}$ and $\mathfrak{m}$ is finitely generated. All proper non-zero ideals of V are of the form k^nV for some $n \in \mathbb{N}$.

(1) For any $n, m \in \mathbb{N}$, $(k^nV, k^mV) \sim (\mathfrak{m}, k^{m+n-2}\mathfrak{m})$ since $(k^nV : k^{n-1}) = \mathfrak{m}$ and $k^mV.k^{n-1} = k^{n+m-2}\mathfrak{m}$. By lemma 3.5.18, $(\mathfrak{m}, k^{m+n-2}\mathfrak{m}) \notin \mathcal{W}_{1,k^{m+n-1},0,0}$, since $k^{m+n-1} \in k^{m+n-2}\mathfrak{m} = k^{m+n-1}V$. Therefore, since $k^{m+n-1} \notin \mathfrak{m}^2k^{m+n-2} = k^{m+n}V$, $(\mathfrak{m}, k^{m+n-2}\mathfrak{m})$ is abnormal. Hence the points (k^nV, k^mV) where $m, n \in \mathbb{N}$ are abnormal.

(2) The points $(I, 0)$, $(0, I)$ are normal by lemma 3.5.16. The point $(0, 0)$ is normal since $(0, 0) \in \mathcal{W}_{1,l,\lambda,g,h}$ if and only if $g = h = 0$. □

Example 3.5.27. *Suppose V is a valuation domain with value group $\mathbb{Q}$ under addition. Let v be the valuation map. Suppose $q \in \mathbb{Q}$ is strictly positive and $s \in V$ is such that $v(s) = q$. Let $I_{\geq q}$ be the ideal $\{r \in V \mid v(r) \geq q\}$, note that this ideal is generated by s. Suppose $c, d \in \mathbb{R}$ are strictly positive and irrational. Let I_c be the ideal $\{r \in V \mid v(r) > c\}$ and I_d be the ideal $\{r \in V \mid v(r) > d\}$.*

1. *The point $(I_{\geq q}, \mathfrak{m})$ is abnormal.*

2. *If $c + d$ is irrational then (I_c, I_d) is normal.*

3. *If $c + d$ is rational then (I_c, I_d) is abnormal.*

4. *If $c+d$ is rational then (I_c, I_d) is topologically indistinguishable from $(I_{\geq c+d}, \mathfrak{m})$.*

Proof. First note, such a valuation domain exists by 2.4.5. Also note that $\mathfrak{m}$ is the only non-zero prime ideal in V and since $\mathbb{Q}$ is dense, we have $\mathfrak{m}^2 = \mathfrak{m}$.

(1) Since $I_{\geq q}\mathfrak{m} = s\mathfrak{m}$, we have $s \notin I_{\geq q}\mathfrak{m}$. By lemma 3.5.21 and as $s \in I_{\geq q}$, $(I_{\geq q}, \mathfrak{m}) \notin \mathcal{W}_{1,s,0,0}$. Therefore $(I_{\geq q}, \mathfrak{m})$ is abnormal.

(2) First note that the ideals I_c and I_d have attached prime $\mathfrak{m}$. Since $I_cI_d = I_{c+d}$, there does not exist $a \in V$ such that $I_{c+d} = a\mathfrak{m}$. Therefore, (I_c, I_d) is normal.

(3) Let $r \in V$ be such that $v(r) = c + d$. We now show that $I_cI_d = I_{c+d} = r\mathfrak{m}$, where $I_{c+d} = \{t \in V \mid v(t) > c + d\}$. If $x \in V$ and $v(x) \geq c$ then $v(x) > c$, since c is irrational and $v(x)$ is rational. Therefore, if $x \in I_c$ and $y \in I_d$ then $v(xy) = v(x) + v(y) > c + d = v(r)$. So $xy \in r\mathfrak{m}$. If $x \in r\mathfrak{m}$ then $v(x) > r = c + d$. So, since $\mathbb{Q}$ is dense, there exists strictly positive $a, b \in \mathbb{Q}$ such that $a > c$, $b > d$ and $a + b = v(x)$. We can now pick $y, z \in V$ such that $v(y) = a$, $v(z) = b$ and $x = yz$. Therefore $y \in I_c$ and $z \in I_d$, so $x \in I_cI_d$. Hence $I_cI_d = r\mathfrak{m}$.

Suppose, for a contradiction, that $(I_c, I_d) \in \mathcal{W}_{1,r,0,0}$. Then there exists $\gamma \notin I_c$ such that $r \notin \gamma I_d$. This means that $v(\gamma) \leq c$ and $c + d = v(r) \leq v(\gamma) + d$. So $v(\gamma) = c$. But this is a contradiction, since $v(\gamma)$ is rational and c is irrational. Therefore $(I_c, I_d) \in \mathcal{W}_{1,r,0,0}$. Hence (I_c, I_d) is abnormal, since $r \notin I_cI_d = r\mathfrak{m}$.

(3) Note that if $c + d$ is rational then (I_c, I_d) is topologically indistinguishable from $(I_{\geq c+d}, \mathfrak{m})$ (cf. lemma 3.5.25). □

3.6 The Ziegler spectrum of a valuation domain is sober

The aim of this section is to show that if V is a valuation domain then every irreducible closed set in Zg_V is the closure of a point.

Lemma 3.6.1. *Let $\mathcal{T}$ be a topological space and C an irreducible closed set in $\mathcal{T}$. Then for all $\mathcal{U}_1$ and $\mathcal{U}_2$ open sets in $\mathcal{T}$, $C \cap \mathcal{U}_1 \neq \emptyset$ and $C \cap \mathcal{U}_2 \neq \emptyset$ implies $\mathcal{U}_1 \cap \mathcal{U}_2 \cap X \neq \emptyset$.*

Proof. Suppose C is an irreducible closed set and $\mathcal{U}_1, \mathcal{U}_2$ are open sets such that $\mathcal{U}_1 \cap \mathcal{U}_2 \cap C = \emptyset$. Then $C \subseteq \mathcal{T} \backslash (\mathcal{U}_1 \cap \mathcal{U}_2) = (\mathcal{T} \backslash \mathcal{U}_1) \cup (\mathcal{T} \backslash \mathcal{U}_2)$. So, since C is irreducible, either $C \subseteq \mathcal{T} \backslash \mathcal{U}_1$ or $C \subseteq \mathcal{T} \backslash \mathcal{U}_2$. Therefore $C \cap \mathcal{U}_1 = \emptyset$ or $C \cap \mathcal{U}_2 = \emptyset$. □

Lemma 3.6.2. *Let $t, s \in \mathfrak{m}$. Then $\mathcal{W}_{1,ts,0,0} \cap \mathcal{W}_{1,1,t,s} = \emptyset$.*

Proof. For any pair of ideals (I, J), $(I, J) \in \mathcal{W}_{1,ts,0,0}$ implies $ts \notin IJ$ and $(I, J) \in \mathcal{W}_{1,1,t,s}$ implies $ts \in IJ$. Therefore $\mathcal{W}_{1,ts,0,0} \cap \mathcal{W}_{1,1,t,s} = \emptyset$. □

Lemma 3.6.3. *Let C be an irreducible closed set of Zg_V. Then there exists $T \lhd V$ such that $T = IJ$ for all normal points $(I, J) \in C$.*

Proof. Let $(I, J), (K, L) \in C$ and suppose both are normal points. Suppose, for a contradiction, that $IJ \subsetneq KL$. Take $\lambda \in KL$, $\lambda \notin IJ$. Then $\lambda = kl$ for some $k \in K$ and $l \in L$. Therefore $(I, J) \in \mathcal{W}_{1,\lambda,0,0}$, $(K, L) \in \mathcal{W}_{1,1,k,l}$ and $\mathcal{W}_{1,\lambda,0,0} \cap \mathcal{W}_{1,1,k,l} = \emptyset$, contradicting the irreducibility of C. Hence $IJ = KL$. □

Lemma 3.6.4. *Suppose C is an irreducible closed set containing at least one normal point. Let T be as in lemma 3.6.3. If (I, J) is an abnormal point contained in C then*

$$T(IJ)^{\#} \subseteq IJ \subseteq T.$$

Proof. Let (K, L) be a normal point in C and (I, J) an abnormal point in C. Note that $I^{\#} = J^{\#}$.

Suppose that $T = KL \subsetneq IJ$. Take $i \in I$ and $j \in J$ such that $ij \notin T$. Then $(I, J) \in \mathcal{W}_{1,1,i,j}$, $(K, L) \in \mathcal{W}_{1,ij,0,0}$ and $\mathcal{W}_{1,1,i,j} \cap \mathcal{W}_{1,ij,0,0} = \emptyset$, contradicting the irreducibility of C. Hence $T \supseteq IJ$.

Suppose $IJ \subsetneq T(IJ)^{\#}$. Then there exists $\mu \in T$ such that $IJ \subsetneq \mu(IJ)^{\#}$. So $(I, J) \in \mathcal{W}_{1,\mu,0,0}$, by lemma 3.5.20 and there exists $k \in K$ and $l \in L$ such that $kl = \mu$. Hence $(K, L) \in \mathcal{W}_{1,1,k,l}$. But $\mathcal{W}_{1,1,k,l} \cap \mathcal{W}_{1,\mu,0,0} = \emptyset$, contradicting the irreducibility of C. Therefore $T(IJ)^{\#} \subseteq IJ$. □

Corollary 3.6.5. *Suppose C is an irreducible closed set containing at least one normal point. Let T be as in 3.6.3. Then each normal point in C specialises to each abnormal point in C.*

Proof. Suppose $(I, J) \in C$ abnormal with $I^{\#} = J^{\#} = \mathfrak{p}$. In order to show that a normal point $(K, L) \in C$ specialises to (I, J), it is enough to show for all basic open sets $\mathcal{W}_{1,\lambda,g,h}$, if $(I, J) \in \mathcal{W}_{1,\lambda,g,h}$ then $(K, L) \in \mathcal{W}_{1,\lambda,g,h}$.

Suppose $(I,J) \in \mathcal{W}_{1,\lambda,g,h}$. Then $\lambda\mathfrak{p} \supsetneq IJ \supseteq \lambda ghV$ and $g,h \in \mathfrak{p}$. By lemma 3.6.4, $IJ \supseteq T\mathfrak{p}$ so $\lambda\mathfrak{p} \supsetneq T\mathfrak{p}$. Hence $\lambda \notin T$. Again, by 3.6.4 $\lambda gh \in IJ$ implies $\lambda gh \in T$. Now suppose, for a contradiction, that $\mathfrak{p} \supsetneq T^{\#}$. Then $T(IJ)^{\#} = T\mathfrak{p} = T$. So $IJ = T$, by lemma 3.6.4. Hence $T^{\#} = \mathfrak{p}$. So $T^{\#} \supseteq \mathfrak{p}$. Therefore $g,h \in \mathfrak{p}$ implies $g,h \in T^{\#}$. If $(K,L) \in C$ is a normal point then $K^{\#} \supseteq T^{\#}$ and $L^{\#} \supseteq T^{\#}$. So $g \in K^{\#}$ and $h \in L^{\#}$. Hence $(K,L) \in \mathcal{W}_{1,\lambda,g,h}$. So (K,L) specialises to (I,J). □

Lemma 3.6.6. *Let C be an irreducible closed set containing at least one normal point. Let T be as in lemma 3.6.3. Then one of the following is true:*

- *For all normal $(I,J) \in C$ either $I^{\#} = T^{\#}$ and $J^{\#} = T^{\#}$ or (I,J) is topologically indistinguishable from $(T,\mathfrak{p})$ for some prime ideal $\mathfrak{p} \supsetneq T^{\#}$.*
- *For all normal $(I,J) \in C$ either $I^{\#} = T^{\#}$ and $J^{\#} = T^{\#}$ or (I,J) is topologically indistinguishable from $(\mathfrak{p},T)$ for some prime ideal $\mathfrak{p} \supsetneq T^{\#}$.*

Proof. Let C be an irreducible closed set containing at least one normal point and let T be as in lemma 3.6.3. Note that if C is an irreducible closed set and $(I,J) \in C$ then $I^{\#} \supseteq T^{\#}$, $J^{\#} \supseteq T^{\#}$ and either $I^{\#} = T^{\#}$ or $J^{\#} = T^{\#}$ since $I^{\#} \cap J^{\#} = (IJ)^{\#} = T^{\#}$.

Suppose, for a contradiction, that there exists $(I,J) \in C$ and $(K,L) \in C$ both normal points such that $I^{\#} \supsetneq T^{\#}$ and $L^{\#} \supsetneq T^{\#}$. Then $I^{\#} \cap L^{\#} \supsetneq T^{\#}$. Take $t \in I^{\#} \cap L^{\#} \backslash T^{\#}$ and $\mu \in T$. Then $\mu = tr$ for some $r \in T^{\#}$. So $(I,J) \in \mathcal{W}_{1,1,t,r}$ and $(K,L) \in \mathcal{W}_{1,1,r,t}$. Hence $C \cap \mathcal{W}_{1,1,r,t} \cap \mathcal{W}_{1,1,t,r} \neq \emptyset$. But if $(M,N) \in C \cap \mathcal{W}_{1,1,r,t} \cap \mathcal{W}_{1,1,t,r}$ then $t \in M^{\#}$ and $t \in N^{\#}$ so $N^{\#} \supsetneq T^{\#}$ and $M^{\#} \supsetneq T^{\#}$ hence (M,N) is an abnormal point. So by lemma 3.6.4, $T(NM)^{\#} \subseteq NM \subseteq T$ but $T(NM)^{\#} = T$ so $MN = T$, a contradiction.

Therefore if $(I,J),(K,L) \in C$ and both are normal points such that $I^{\#} \cup J^{\#} \supsetneq T^{\#}$ and $K^{\#} \cup L^{\#} \supsetneq T^{\#}$ then either $I^{\#} \supsetneq T^{\#}$, $K^{\#} \supsetneq T^{\#}$ and $J^{\#} = L^{\#} = T^{\#}$ or $J^{\#} \supsetneq T^{\#}$, $L^{\#} \supsetneq T^{\#}$ and $I^{\#} = K^{\#} = T^{\#}$.

It remains to show that if $(I,J) \in C$ a normal point and $I^{\#} \supsetneq T^{\#}$ then (I,J) is topologically indistinguishable from $(I^{\#},T)$. Note that $(I,J) \in \mathcal{W}_{1,\lambda,g,h}$ if and only if $\lambda \notin IJ = T$, $\lambda gh \in IJ = T$, $g \in I^{\#}$ and $h \in J^{\#} = T^{\#}$ and $(I^{\#},T) \in \mathcal{W}_{1,\lambda,g,h}$ if and only if $\lambda \notin I^{\#}T = T$, $\lambda gh \in I^{\#}T = T$, $g \in I^{\#}$ and $h \in T^{\#}$. Therefore (I, J) and $(I^{\#}, T)$ are topologically indistinguishable. $\square$

Lemma 3.6.7. *Let $\{\mathfrak{p}_i \mid i \in \mathfrak{I}\}$ be a set of prime ideals of V and let T be an ideal of V such that $T^{\#} \subsetneq \mathfrak{p}_i$ for each $i \in \mathfrak{I}$. If C is a closed set in Zg_V such that $(T, \mathfrak{p}_i) \in C$ for all $i \in \mathfrak{I}$ then $(T, \cup_{i\in\mathfrak{I}}\mathfrak{p}_i) \in C$*

Proof. Suppose $\lambda \in V$ non-zero and $g, h \in \mathfrak{m}$ are such that $(T, \cup_{i\in\mathfrak{I}}\mathfrak{p}_i) \in \mathcal{W}_{1,\lambda,g,h}$. Since $(T, \cup_{i\in\mathfrak{I}}\mathfrak{p}_i)$ is a normal point $\lambda \notin T$, $\lambda gh \in T$, $g \in T^{\#}$ and $h \in \cup_{i\in\mathfrak{I}}\mathfrak{p}_i$. Therefore $h \in \mathfrak{p}_i$ for some $i \in \mathfrak{I}$ so $(T, \mathfrak{p}_i) \in \mathcal{W}_{1,\lambda,g,h}$. $\square$

Definition 3.6.8. *Let $\mathfrak{p}, \mathfrak{q} \lhd V$ be prime ideals. Then*

$$X_{\mathfrak{p},\mathfrak{q}} = \{(I, J) \in \mathrm{Zg}_V \mid I^{\#} = \mathfrak{p} \text{ and } J^{\#} = \mathfrak{q}\}.$$

Lemma 3.6.9. *Let $\mathfrak{p} \lhd V$ be a prime ideal. Suppose C is an irreducible closed set in Zg_V. Then all normal points in $X_{\mathfrak{p},\mathfrak{p}} \cap C$ are topologically indistinguishable.*

Proof. Suppose $\mathfrak{p} \lhd V$ is a prime ideal. Let T be as in lemma 3.6.3. Suppose $(I, J) \in C \cap X_{\mathfrak{p},\mathfrak{p}}$ is a normal point in Zg_V. Then for all $\lambda \in V\backslash\{0\}$ and $g, h \in \mathfrak{m}$, $(I, J) \in \mathcal{W}_{1,\lambda,g,h}$ if and only if $\lambda \notin IJ = T$, $\lambda gh \in IJ = T$ and $g, h \in I^{\#} = J^{\#} = \mathfrak{p}$. Since whether (I, J) is in $\mathcal{W}_{1,\lambda,g,h}$ depends only on T and $\mathfrak{p}$, all normal points in $C \cap X_{\mathfrak{p},\mathfrak{p}}$ must be topologically indistinguishable. $\square$

Proposition 3.6.10. *Let C be an irreducible closed set containing at least one normal point. Then C has a generic point.*

Proof. Let T be as in lemma 3.6.3. Without loss of generality we may assume that for all normal points $(I, J) \in C$ either $I^{\#} = T^{\#}$ and $J^{\#} = T^{\#}$ or (I, J) topologically indistinguishable from $(T, \mathfrak{p})$ where $T^{\#} \subsetneq \mathfrak{p}$ is a prime ideal (see lemma 3.6.6).

First suppose all normal $(I, J) \in C$ have $I^{\#} = T^{\#}$ and $J^{\#} = T^{\#}$. By lemma 3.6.9, C contains at most one normal point (up to topological indistinguishability). By lemma 3.6.5 this normal points specialises to all abnormal points in C. Hence C has a generic point.

Now suppose there exists at least one point $(T, \mathfrak{p}) \in C$ with $\mathfrak{p} \supsetneq T^{\#}$. Let $\mathfrak{I}$ index prime ideals $\mathfrak{p}_i$ such that $(T, \mathfrak{p}_i) \in C$ with $\mathfrak{p}_i \supsetneq T^{\#}$. Then by lemma 3.6.7 $(T, \cup_{i\in\mathfrak{I}}\mathfrak{p}) \in C$. It remains to show that $(T, \cup_{i\in\mathfrak{I}}\mathfrak{p})$ specialises to all points in C. This follows for abnormal points by 3.6.4. Suppose $(I, J) \in C$ is a normal point. Then either $I^{\#} = T^{\#}$ and $J^{\#} = T^{\#}$ or (I, J) is topologically indistinguishable from $(T, \mathfrak{p}_i)$ for some $i \in \mathfrak{I}$. Therefore if $(I, J) \in \mathcal{W}_{1,\lambda,g,h}$ then $\lambda \notin T$, $\lambda gh \in T$, $g \in I^{\#}$ and $h \in J^{\#}$. So $g \in T^{\#}$ and $h \in \mathfrak{p}_i$ for some $i \in \mathfrak{I}$. Therefore $h \in \cup_{i\in\mathfrak{I}}\mathfrak{p}_i$. So, noting that $T \cdot (\cup_{i\in\mathfrak{I}}\mathfrak{p}_i) = T$ and $(T, \cup_{i\in\mathfrak{I}}\mathfrak{p}_i)$ is a normal point, $(T, \cup_{i\in\mathfrak{I}}\mathfrak{p}_i) \in \mathcal{W}_{1,\lambda,g,h}$. Therefore $(T, \cup_{i\in\mathfrak{I}}\mathfrak{p}_i)$ specialises to (I, J). Hence C has generic point $(T, \cup_{i\in\mathfrak{I}}\mathfrak{p}_i)$. □

Lemma 3.6.11. *Let C be an irreducible closed set of Zg_V containing only abnormal points. Then for each prime ideal $\mathfrak{p} \lhd V$, all points in $C \cap X_{\mathfrak{p},\mathfrak{p}}$ are topologically indistinguishable.*

Proof. Suppose $\mathfrak{p} \lhd V$ is a prime ideal such that $\mathfrak{p}^2 = \mathfrak{p}$ and $C \cap X_{\mathfrak{p},\mathfrak{p}} \neq \emptyset$. Suppose, for a contradiction, $(I, J), (K, L) \in C \cap X_{\mathfrak{p},\mathfrak{p}}$ with $IJ \subsetneq KL$. As noted earlier (see proof of 3.5.25) since (I, J) and (K, L) are abnormal there exists $a, b \in V$ such that $IJ = a\mathfrak{p}$ and $KL = b\mathfrak{p}$. Since $\mathfrak{p}^2 = \mathfrak{p}$ there exists $c \in \mathfrak{p}$ such that $a\mathfrak{p} \subsetneq c\mathfrak{p} \subsetneq b\mathfrak{p}$. Therefore $(I, J) \in \mathcal{W}_{1,c,0,0}$ and since $\mathfrak{p} = \mathfrak{p}^2$ there exists $\gamma_1, \gamma_2 \in \mathfrak{p}$ such that $\gamma_1\gamma_2 = c$ so $(K, L) \in \mathcal{W}_{1,1,\gamma_1,\gamma_2}$. But $\mathcal{W}_{1,c,0,0} \cap \mathcal{W}_{1,1,\gamma_1,\gamma_2} = \emptyset$, contradicting the irreducibility of C. Therefore $IJ = KL$. So, using proposition 3.5.6 and lemma 3.5.24, (I, J) is topologically indistinguishable from (K, L).

Suppose $\mathfrak{p} \lhd V$ is a prime ideal such that $\mathfrak{p}^2 \neq \mathfrak{p}$ and $C \cap X_{\mathfrak{p},\mathfrak{p}} \neq \emptyset$. Choose $k \in \mathfrak{p} \backslash \mathfrak{p}^2$. Let $a, b \in V$ be such that $(\mathfrak{p}, a\mathfrak{p}), (\mathfrak{p}, b\mathfrak{p}) \in C$. First suppose $a\mathfrak{p} \supsetneq ak\mathfrak{p} \supsetneq b\mathfrak{p}$. Then $ak^2 \notin b\mathfrak{p}$ and $ak^2 \in a\mathfrak{p}^2$. Hence $(\mathfrak{p}, b\mathfrak{p}) \in \mathcal{W}_{1,ak^2,0,0}$ and $(\mathfrak{p}, a\mathfrak{p}) \in \mathcal{W}_{1,1,k,ak}$. But this contradicts irreducibility of C since $\mathcal{W}_{1,ak^2,0,0} \cap \mathcal{W}_{1,1,k,ak} = \emptyset$.

Next suppose that $b\mathfrak{p} = ak\mathfrak{p}$. Then $(\mathfrak{p}, ak\mathfrak{p}) \in \mathcal{W}_{1,ak,0,0}$ and $(\mathfrak{p}, a\mathfrak{p}) \in \mathcal{W}_{1,1,k,ak}$. Suppose $(I, J) \in \mathcal{W}_{1,1,k,ak} \cap \mathcal{W}_{1,ak,0,0} \cap C$ is abnormal. Then $I^{\#} = J^{\#} = \mathfrak{q}$ for some prime ideal $\mathfrak{q}$ and $\mathfrak{q} \supsetneq \mathfrak{p}$. Suppose $\mathfrak{q}^2 = \mathfrak{q}$. Then (I, J) is topologically indistinguishable from $(I^{\mathfrak{q}}_{\gamma}, \mathfrak{q})$ for some $\gamma \in \mathfrak{q}$.

<u>Claim</u>: Either $\gamma\mathfrak{q} \supsetneq a\mathfrak{p}^2$ or $a\mathfrak{p}^2 \supsetneq I^{\mathfrak{q}}_{\gamma}$.

Suppose $I^{\mathfrak{p}}_{\gamma} \supseteq a\mathfrak{p}^2$. Then $\gamma\mathfrak{q} = I^{\mathfrak{q}}_{\gamma}\mathfrak{q} \supseteq a\mathfrak{p}^2\mathfrak{q} = a\mathfrak{p}^2$. Therefore $\gamma\mathfrak{q} \supsetneq a\mathfrak{p}^2$.

If $\gamma\mathfrak{q} \supsetneq a\mathfrak{p}^2$ take $\mu \in \gamma\mathfrak{q}$, $\mu \notin a\mathfrak{p}^2$ and $\mu_1, \mu_2 \in \mathfrak{q}$ such that $\mu_1\mu_2 = \mu$. So $(\mathfrak{p}, ak\mathfrak{p}) \in \mathcal{W}_{1,\mu,0,0}$ and $(I^{\mathfrak{q}}_{\gamma}, \mathfrak{q}) \in \mathcal{W}_{1,1,\mu_1,\mu_2}$. But $\mathcal{W}_{1,1,\mu_1,\mu_2} \cap \mathcal{W}_{1,\mu,0,0} = \emptyset$, contradicting irreducibility of C. If $a\mathfrak{p}^2 \supsetneq I^{\mathfrak{q}}_{\gamma}$ take $\mu \in a\mathfrak{p}^2$, $\mu \notin I^{\mathfrak{q}}_{\gamma}$ and $\mu_1, \mu_2 \in \mathfrak{p}$ such that $\mu_1\mu_2 = \mu$. Then $(\mathfrak{p}, a\mathfrak{p}) \in \mathcal{W}_{1,1,\mu_1,\mu_2}$ and $(I^{\mathfrak{q}}_{\gamma}, \mathfrak{q}) \in \mathcal{W}_{1,\mu,0,0}$. But $\mathcal{W}_{1,1,\mu_1,\mu_2} \cap \mathcal{W}_{1,\mu,0,0} = \emptyset$, contradicting irreducibility of C.

Next suppose $\mathfrak{q}^2 \neq \mathfrak{q}$. Then (I, J) is topologically indistinguishable from a point of the form $(\mathfrak{q}, c\mathfrak{q})$ for some $c \in \mathfrak{q}$.

<u>Claim</u>:Either $c\mathfrak{q}^2 \supsetneq a\mathfrak{p}^2$ or $a\mathfrak{p}^2 \supsetneq c\mathfrak{q}$

Suppose $a\mathfrak{p}^2 \subseteq c\mathfrak{q}$. Then $a\mathfrak{p}^2\mathfrak{q} \subseteq c\mathfrak{q}^2$ but since $\mathfrak{q} \supsetneq \mathfrak{p}$, $a\mathfrak{p}^2\mathfrak{q} = a\mathfrak{p}^2$. Therefore $a\mathfrak{p}^2 \subseteq c\mathfrak{q}^2$ since $\mathfrak{q} \supsetneq \mathfrak{p}$. Hence $a\mathfrak{p}^2 \subsetneq c\mathfrak{q}^2$, again since $\mathfrak{q} \supsetneq \mathfrak{p}$.

If $c\mathfrak{q}^2 \supsetneq a\mathfrak{p}^2$ take $t \in c\mathfrak{q}^2$, $t \notin a\mathfrak{p}^2 = ak\mathfrak{p}$ and $t_1, t_2 \in \mathfrak{q}$ such that $t_1t_2 = t$. So $(\mathfrak{p}, ak\mathfrak{p}) \in \mathcal{W}_{1,t,0,0}$ and $(\mathfrak{q}, c\mathfrak{q}) \in \mathcal{W}_{1,1,t_1,t_2}$. But $\mathcal{W}_{1,t,0,0} \cap \mathcal{W}_{1,1,t_1,t_2} = \emptyset$, contradicting irreducibility of C. If $a\mathfrak{p}^2 \supsetneq c\mathfrak{q}$ take $t \in a\mathfrak{p}^2$, $t \notin c\mathfrak{q}$ and $t_1, t_2 \in \mathfrak{p}$ such that $t_1t_2 = t$. So $(\mathfrak{q}, c\mathfrak{q}) \in \mathcal{W}_{1,t,0,0}$ and $(\mathfrak{p}, a\mathfrak{p}) \in \mathcal{W}_{1,1,t_1,t_2}$. But $\mathcal{W}_{1,t,0,0} \cap \mathcal{W}_{1,1,t_1,t_2} = \emptyset$, contradicting irreducibility of C. Therefore all points in $C \cap X_{\mathfrak{p},\mathfrak{p}}$ are topologically indistinguishable. $\square$

Lemma 3.6.12. *Suppose $\mathfrak{p} \supsetneq \mathfrak{q}$ are prime ideals in V, $\gamma \in V$ and $J \lhd V$ with $J^{\#} = \mathfrak{q}$. Then $\gamma\mathfrak{p} \supseteq J$ implies $\gamma\mathfrak{p}^2 \supseteq J$.*

Proof. Suppose $j \in J$. Take $t \in \mathfrak{p} \backslash \mathfrak{q}$. Then $j = tj'$ for some $j' \in J$. Hence $j' \in \gamma\mathfrak{p}$ so $j = j't \in \gamma\mathfrak{p}^2$. $\square$

Lemma 3.6.13. *Let $\mathfrak{p}, \mathfrak{q} \lhd V$ be prime ideals with $\mathfrak{p} \supsetneq \mathfrak{q}$. Suppose $(I, \mathfrak{p}) \in X_{\mathfrak{p},\mathfrak{p}}$ is an abnormal point and $I \subseteq \mathfrak{q}$. Then there exists $(J, \mathfrak{q}) \in X_{\mathfrak{q},\mathfrak{q}}$ such that $(I, \mathfrak{p}) \in \mathrm{cl}(J, \mathfrak{q})$.*

Proof. We split the proof into 4 cases:

<u>Case 1</u>: $\mathfrak{p}^2 = \mathfrak{p}$ and $\mathfrak{q}^2 = \mathfrak{q}$.

If $(I, \mathfrak{p})$ is an abnormal point there exists $\gamma \in \mathfrak{p}$ such that $I = I^{\mathfrak{p}}_{\gamma}$. Note that $\gamma \in \mathfrak{q}$ and let $J = I^{\mathfrak{q}}_{\gamma}$. Then $(J, \mathfrak{q}) \in \mathcal{W}_{1,\lambda,g,h}$ if and only if $\lambda \notin J$, $\lambda gh \in J\mathfrak{q}$ and $g, h \in \mathfrak{q}$.

Note that $I_\gamma^{\mathfrak{q}} \supseteq I_\gamma^{\mathfrak{p}}$ so $\lambda \notin I_\gamma^{\mathfrak{p}}$ and $I\mathfrak{p} = \gamma\mathfrak{p} \supseteq \gamma\mathfrak{q} = J\mathfrak{q}$ so $\lambda gh \in \gamma\mathfrak{p}$. Since $\mathfrak{p} \supseteq \mathfrak{q}$, $g, h \in \mathfrak{p}$. Hence $(I, \mathfrak{p}) \in \mathcal{W}_{1,\lambda,g,h}$.

<u>Case 2</u>: $\mathfrak{p}^2 = \mathfrak{p}$ and $\mathfrak{q}^2 \neq \mathfrak{q}$.

If $(I, \mathfrak{p})$ is an abnormal point there exists $\gamma \in V$ such that $I_\gamma^{\mathfrak{p}} = I$. Note that $I_\gamma^{\mathfrak{p}} \subseteq \mathfrak{q}$ implies $\gamma \in \mathfrak{q}$. If $\gamma \notin \mathfrak{q}^2$ let $\gamma' = 1$ and $k = \gamma$, otherwise take $k \in \mathfrak{q}\backslash\mathfrak{q}^2$, then there exists $\gamma' \in V$ such that $k\gamma' = \gamma$. Note that, in either case, $\gamma'\mathfrak{q}^2 = \gamma\mathfrak{q}$ and $\gamma \in \gamma'\mathfrak{q}$. Let $J = \gamma'\mathfrak{q}$.

Suppose $(J, \mathfrak{q}) \in \mathcal{W}_{1,\lambda,g,h}$. Then $\lambda \notin \gamma'\mathfrak{q}$, $\lambda gh \in \gamma'\mathfrak{q}^2 = \gamma\mathfrak{q}$ and $g, h \in \mathfrak{q}$. Hence $\lambda gh \in \gamma\mathfrak{p}$ since $\gamma\mathfrak{p} \supseteq \gamma\mathfrak{q}$ and $g, h \in \mathfrak{p}$ since $\mathfrak{p} \supseteq \mathfrak{q}$. It remains to show that $\lambda \notin I_\gamma^{\mathfrak{p}}$, equivalently, $\gamma \in \lambda\mathfrak{p}$. But $\lambda \notin \gamma'\mathfrak{q}$ and $\gamma \in \gamma'\mathfrak{q}$. Therefore $\gamma \in \lambda\mathfrak{q}$. So $\gamma \in \lambda\mathfrak{p}$ since $\lambda\mathfrak{q} \subseteq \lambda\mathfrak{p}$. Hence $(I, \mathfrak{p}) \in \mathcal{W}_{1,\lambda,g,h}$.

<u>Case 3</u>: $\mathfrak{p}^2 \neq \mathfrak{p}$ and $\mathfrak{q}^2 = \mathfrak{q}$.

If $(I, \mathfrak{p})$ is an abnormal point there exists $\gamma \in V$ such that $I = \gamma\mathfrak{p}$. Let $J = I_\gamma^{\mathfrak{q}}$. Then $(J, \mathfrak{q}) \in \mathcal{W}_{1,\lambda,g,h}$ implies that $\lambda \notin I_\gamma^{\mathfrak{q}}$, $\lambda gh \in \gamma\mathfrak{q}$ and $g, h \in \mathfrak{q}$. Therefore $\lambda \notin \gamma\mathfrak{p}$ since $I_\gamma^{\mathfrak{q}} \supseteq \gamma\mathfrak{p}$, $\lambda gh \in \gamma\mathfrak{p}^2$ since $\gamma\mathfrak{q} \subseteq \gamma\mathfrak{p}^2$ and $g, h \in \mathfrak{p}$ since $\mathfrak{q} \subseteq \mathfrak{p}$. Hence $(I, \mathfrak{p}) \in \mathcal{W}_{1,\lambda,g,h}$.

<u>Case 4</u>: $\mathfrak{p}^2 \neq \mathfrak{p}$ and $\mathfrak{q}^2 \neq \mathfrak{q}$.

If $(I, \mathfrak{p})$ is an abnormal point there exists $\gamma \in V$ such that $I = \gamma\mathfrak{p}$, in fact $\gamma \in \mathfrak{q}$ since $I \subseteq \mathfrak{q}$. If $\gamma \in \mathfrak{q}^2$ take $k \in \mathfrak{q}\backslash\mathfrak{q}^2$ then $\gamma = \gamma' k$ for some $\gamma' \in V$, otherwise let $k = \gamma$ and $\gamma' = 1$. Let $J = \gamma'\mathfrak{q}$. Note that $\gamma\mathfrak{q} = \gamma' k\mathfrak{q}$ hence $\gamma'\mathfrak{q} \supseteq \gamma\mathfrak{p} = \gamma' k\mathfrak{p}$ since $k \in \mathfrak{q}$.

Then $(J, \mathfrak{q}) \in \mathcal{W}_{1,\lambda,g,h}$ implies $\lambda \notin J$, $\lambda gh \in J\mathfrak{q}$ and $g, h \in \mathfrak{q}$. Therefore $\lambda \notin \gamma\mathfrak{p}$ since $J \supseteq \gamma\mathfrak{p}$, $\lambda gh \in \gamma\mathfrak{p}^2$ since $\gamma\mathfrak{p}^2 \supseteq J\mathfrak{q} = \gamma'\mathfrak{q}^2 = \gamma\mathfrak{q}$ (by lemma 3.6.12) and $g, h \in \mathfrak{p}$ since $\mathfrak{q} \subseteq \mathfrak{p}$. Hence $(I, \mathfrak{p}) \in \mathcal{W}_{1,\lambda,g,h}$.

$\square$

Proposition 3.6.14. *Let C be an irreducible closed set containing only abnormal points. Then C has a generic point.*

Proof. Let $\mathfrak{I}$ be a totally ordered set indexing the prime ideals $\mathfrak{p}_i$ with $C \cap X_{\mathfrak{p}_i,\mathfrak{p}_i} \neq \emptyset$ such that $i \geq j$ if and only if $\mathfrak{p}_i \supseteq \mathfrak{p}_j$. For all $i \in \mathfrak{I}$, let $J_i = I_{a_i}^{\mathfrak{p}_i}$ for some $a_i \in \mathfrak{p}_i$ where $(I_{a_i}^{\mathfrak{p}_i}, \mathfrak{p}_i) \in C$ if $\mathfrak{p}_i^2 = \mathfrak{p}_i$ and $J_i = a_i\mathfrak{p}_i$ where $(a_i\mathfrak{p}_i, \mathfrak{p}_i) \in C$ otherwise. We can do this since an abnormal point (I,J) with $I^{\#}=J^{\#}=\mathfrak{p}$ is topologically indistinguishable from $(I_a^{\mathfrak{p}},\mathfrak{p})$ for some $a\in\mathfrak{p}$ if $\mathfrak{p}^2=\mathfrak{p}$ and if $\mathfrak{p}^2\neq\mathfrak{p}$ then $(I,J)\sim(\mathfrak{p},a\mathfrak{p})$ for some $a\in V$.

<u>Claim:</u> If $i\geq j$ then $J_i\subseteq J_j$.

Suppose $i>j$ and $J_i\supsetneq J_j$. Take $\gamma\in\mathfrak{p}_i\backslash\mathfrak{p}_j$ and $t\in J_j$. Then $t=\gamma\mu$ for some $\mu\in J_j$. Hence $\mu\in J_i$ so $t=\gamma\mu\in J_i\mathfrak{p}_i$. Therefore $J_i\mathfrak{p}_i\supseteq J_j$. Hence $J_i\mathfrak{p}_i\supsetneq J_j$ since $\mathfrak{p}_i\neq\mathfrak{p}_j$. Take $\lambda\in J_i\mathfrak{p}_i\backslash J_j$, $a\in J_i$ and $b\in\mathfrak{p}_i$ such that $ab=\lambda$. Then $(J_i,\mathfrak{p}_i)\in\mathcal{W}_{1,1,a,b}$ and $(J_j,\mathfrak{p}_j)\in\mathcal{W}_{1,\lambda,0,0}$, contradicting irreducibility of C since $\mathcal{W}_{1,1,a,b}\cap\mathcal{W}_{1,\lambda,0,0}=\emptyset$. Therefore if $i\geq j$ then $J_i\subseteq J_j$.

<u>Claim:</u> $(\cap_{i\in\mathfrak{I}}J_i,\cup_{i\in\mathfrak{I}}\mathfrak{p}_i)\in C$

Let $\lambda\in V\backslash\{0\}$ and $g,h\in\mathfrak{m}$. Suppose $(\cap_{i\in\mathfrak{I}}J_i,\cup_{i\in\mathfrak{I}}\mathfrak{p}_i)\in\mathcal{W}_{1,\lambda,g,h}$. Then

$$(\cap_{i\in\mathfrak{I}}J_i,\cup_{i\in\mathfrak{I}}\mathfrak{p}_i)\in\mathcal{W}_{1,\lambda,0,0},\ \lambda gh\in(\cap_{i\in\mathfrak{I}}J_i)(\cup_{i\in\mathfrak{I}}\mathfrak{p}_i),\ g\in(\cap_{i\in\mathfrak{I}}J_i)^{\#}\text{ and }h\in\cup_{i\in\mathfrak{I}}\mathfrak{p}_i.$$

We aim to show that there exists $i\in\mathfrak{I}$ such that $(J_i,\mathfrak{p}_i)\in\mathcal{W}_{1,\lambda,g,h}$.

First note that $(\cap_{i\in\mathfrak{I}}J_i)^{\#}\subseteq\cup_{i\in\mathfrak{I}}\mathfrak{p}_i$. To see this, suppose that $x\in(\cap_{i\in\mathfrak{I}}J_i)^{\#}$. Then there exists $\gamma\notin(\cap_{i\in\mathfrak{I}}J_i)$ such that $x\gamma\in(\cap_{i\in\mathfrak{I}}J_i)$. So $x\gamma\in J_i$ for all $i\in\mathfrak{I}$ and $\gamma\notin J_k$ for some $k\in\mathfrak{I}$. Hence $x\in J_k^{\#}$.

Therefore $g\in(\cap_{i\in\mathfrak{I}}J_i)^{\#}$ implies $g\in\mathfrak{p}_i$ for some $i\in\mathfrak{I}$. Note also that $h\in\cup_{i\in\mathfrak{I}}\mathfrak{p}_i$ implies $h\in\mathfrak{p}_i$ for some $i\in\mathfrak{I}$.

Suppose, for a contradiction, that $(J_i,\mathfrak{p}_i)\notin\mathcal{W}_{1,\lambda,0,0}$ for all $i\in\mathfrak{I}$. Then $\lambda\in J_i$ for all $i\in\mathfrak{I}$. So $\lambda\in\cap_{i\in\mathfrak{I}}J_i$. Since $(\cap_{i\in\mathfrak{I}}J_i,\cup_{i\in\mathfrak{I}}\mathfrak{p}_i)\in\mathcal{W}_{1,\lambda,g,h}$, $\lambda\notin(\cap_{i\in\mathfrak{I}}J_i)(\cup_{i\in\mathfrak{I}}\mathfrak{p}_i)$. Therefore $\cap_{i\in\mathfrak{I}}J_i\supsetneq(\cap_{i\in\mathfrak{I}}J_i)(\cup_{i\in\mathfrak{I}}\mathfrak{p}_i)$. So either $\cup_{i\in\mathfrak{I}}\mathfrak{p}_i\subsetneq(\cap_{i\in\mathfrak{I}}J_i)^{\#}$ contradicting $(\cap_{i\in\mathfrak{I}}J_i)^{\#}\subseteq\cup_{i\in\mathfrak{I}}\mathfrak{p}_i$ or $(\cap_{i\in\mathfrak{I}}J_i)^{\#}=\cup_{i\in\mathfrak{I}}\mathfrak{p}_i$. If $(\cup_{i\in\mathfrak{I}}\mathfrak{p}_i)^2=\cup_{i\in\mathfrak{I}}\mathfrak{p}_i$ then $(\cap_{i\in\mathfrak{I}}J_i)=I_r^{\cup_{i\in\mathfrak{I}}\mathfrak{p}_i}$ for some $r\in\cup_{i\in\mathfrak{I}}\mathfrak{p}_i$, by 3.5.13. Otherwise $(\cup_{i\in\mathfrak{I}}\mathfrak{p}_i)^2\neq\cup_{i\in\mathfrak{I}}\mathfrak{p}_i$. Hence $(\cap_{i\in\mathfrak{I}}J_i,\cup_{i\in\mathfrak{I}}\mathfrak{p}_i)\in\mathcal{W}_{1,\lambda,0,0}$ implies $\lambda\notin\cap_{i\in\mathfrak{I}}J_i$. Therefore $\lambda\notin J_j$ for some $j\in\mathfrak{I}$. Since $i\geq j$ implies $J_i\subseteq J_j$, there exists $j\in\mathfrak{I}$ such that $\lambda\notin J_i$ for all $i\geq j$.

If $\lambda gh\in(\cap_{i\in\mathfrak{I}}J_i)(\cup_{i\in\mathfrak{I}}\mathfrak{p}_i)$ then $\lambda gh=ts$ for some $t\in(\cap_{i\in\mathfrak{I}}J_i)$ and $s\in(\cup_{i\in\mathfrak{I}}\mathfrak{p}_i)$. Therefore there exists $i\in\mathfrak{I}$ such that $s\in\mathfrak{p}_j$ for all $j\geq i$. So since $t\in J_j$ for all $j\in\mathfrak{I}$, $\lambda gh=ts\in J_j\mathfrak{p}_j$ for all $j\geq i$.

Therefore there exists $i \in \mathfrak{I}$ such that $(J_i, \mathfrak{p}_i) \in \mathcal{W}_{1,\lambda,g,h}$. Hence for all closed sets C, $(J_i, \mathfrak{p}_i) \in C$ for all $i \in \mathfrak{I}$ implies $(\cap_{i\in\mathfrak{I}} J_i, \cup_{i\in\mathfrak{I}} \mathfrak{p}_i) \in C$.

It remains to show that $(\cap_{i\in\mathfrak{I}} J_i, \cup_{i\in\mathfrak{I}} \mathfrak{p}_i)$ is a generic point for C. This follows from 3.6.13 and 3.6.11.

□

Theorem 3.6.15. *Let V be a valuation domain. Then* Zg_V *is sober.*

Proof. Follows directly from proposition 3.6.14 and proposition 3.6.10. □

Definition 3.6.16. *An integral domain R is called a Prüfer domain if its localisations at all maximal ideals are valuation domains.*

Theorem 3.6.17. *Let R be a Prüfer domain. Then* Zg_R *is sober.*

Proof. By lemma 2.3.28 Zg_R is sober if and only if $\mathrm{Zg}_{R_\mathfrak{m}}$ is sober for all maximal ideals $\mathfrak{m} \lhd R$. So by theorem 3.6.15, Zg_R is sober. □

Chapter 4

Decidability

Convention: Throughout this chapter we will use a naive notion of cardinality. That is, if X, Y are sets then $|X| = |Y|$ means, if either X or Y is of finite cardinality then their cardinality is equal.

Suppose R is a commutative ring and $I \lhd R$. The radical of I is the following set

$$\mathrm{rad}(I) = \{r \in R \mid \text{ there exists } n \in \mathbb{N} \text{ such that } r^n \in I\}$$

.

Note that for any commutative ring R and $I \lhd R$, $\mathrm{rad}(I)$ is the intersection of all prime ideals containing I.

The following statement was conjectured for valuation domain with dense value group in [PPT07].

Theorem 4.0.1. *Let V be an effectively given valuation domain. Then the following are equivalent:*

1. *The theory of V-modules, T_V, is decidable*
2. *There is an algorithm which, given $a, b \in V$ decides whether $a \in \mathrm{rad}(b)$.*

The aim of this chapter is to prove the above theorem. The key step in proving this theorem is to show that there is an algorithm which answers whether one Ziegler basic open set $(\frac{\phi}{\psi})$ is contained in a finite union $\bigcup_{i=1}^{n}(\frac{\phi_i}{\psi_i})$ of Ziegler basic open sets.

We largely follow the structure of the proof given in [PPT07]. In fact, the only ingredient needed to extend the proof given in [PPT07] for archimedean valuation domains to valuation domains with dense value group is an algorithm to effectively decide when a basic Ziegler open set is contained in a finite union of other basic Ziegler open sets. Both our decidability proof and the proof given in [PPT07] are inspired by a remark in [Zie84] (immediately before example 9.5).

Definition 4.0.2. *A valuation domain V is said to be effectively given if it has a bijection with $\mathbb{N}$ such that the maximal ideal is a recursive set and addition and multiplication are recursive functions. Note that this of course implies we can decide equality of ring elements.*

4.1 Necessary conditions for the theory of modules of a commutative ring to be decidable

It only makes sense to talk about decidability of the theory of V-modules for countable rings, as otherwise the language is uncountable.

Lemma 4.1.1. *Let V be an effectively given valuation domain. Then there is an algorithm which decides, given $a, b \in V$ whether $a|b$ and if so gives the quotient.*

Proof. Since V is effectively given, we have a bijection between V and $\mathbb{N}$ and multiplication is a recursive function. Take the first element λ_1 on the list of elements of V, ask whether $a.\lambda_1 = b$, if not ask whether $b\lambda_1 = a$. Continue with all elements λ_i until we find an $i \in \mathbb{N}$ such that $a\lambda_i = b$ or $b\lambda_i = a$. We will find such an i because for all $a, b \in V$ either $a|b$ or $b|a$.

If this process ends by finding an i such that $a\lambda_i = b$ then $a|b$ and λ_i is the quotient.

If this process ends by finding an i such that $b\lambda_i = a$, check if $\lambda_i \in \mathfrak{m}$. We can do this since $\mathfrak{m}$ is a recursive set. If $\lambda_i \in \mathfrak{m}$ then a does not divide b. If $\lambda_i \notin \mathfrak{m}$ then a divides b and it remains to find the quotient. To do this simply search through the list of $\mu \in V$ until we find a μ such that $a\mu = b$. $\square$

Proposition 4.1.2. *Let R be a countable commutative ring with decidable theory of modules. Then multiplication and addition are recursive function and there is an algorithm which, given $a, b \in V$, answers whether $a|b$.*

Proof. Let $r, s, t \in R$. Then $r + s = t$ if and only if

$$T_R \models \forall x(xr + xs = xt).$$

Let $r, s, t \in R$. Then $rs = t$ if and only if

$$T_R \models \forall x \exists y(xr = y \wedge ys = xt).$$

Let $r, s \in R$. Then $r|s$ if and only if

$$T_R \models \forall x \exists y(yr = xs).$$

□

Proposition 4.1.3. *Let R be a countable commutative ring with a decidable theory of modules. Then there is an algorithm which, given $a, b \in V$ decides whether $a \in$ $\mathrm{rad}(bR)$.*

Proof. Claim:

$$T_R \models \exists x(x \neq 0 \wedge xb = 0) \rightarrow \exists y(y \neq 0 \wedge xa = 0)$$

if and only if

$$a \in \mathrm{rad}(bR).$$

First suppose that $a \in \mathrm{rad}(b)$, so there exists an $n \in \mathbb{N}$ such that $a^n \in bV$. Suppose N is an R-module and $x \in N$ such that $x \neq 0$ and $xb = 0$. Then $xa^n = 0$. Take m least such that $xa^m = 0$, then $(xa^{m-1})a = 0$ and $xa^{m-1} \neq 0$.

Now suppose that

$$T_R \models \exists x(x \neq 0 \wedge xb = 0) \rightarrow \exists y(y \neq 0 \wedge xa = 0).$$

Let $\mathfrak{p} \lhd R$ be a prime ideal such that $b \in \mathfrak{p}$. Then $1 + \mathfrak{p} \in R/\mathfrak{p}$ is annihilated by b and non-zero. Hence there exists $y \in V \backslash \mathfrak{p}$ such that $ay \in \mathfrak{p}$. Therefore $a \in \mathfrak{p}$. So $a \in \mathfrak{p}$ for every prime ideal $\mathfrak{p}$ containing b. Hence $a \in \mathrm{rad}(bV)$ since $\mathrm{rad}(bV)$ is the intersection of all prime ideals containing b. □

4.2 Algorithms

In this section we show that if V is an effectively given valuation domain with an algorithm which, given $a, b \in V$, answers whether $a \in \mathrm{rad}(bV)$ then there exists an algorithm which given $n \in \mathbb{N}$, a pp-pair ϕ/ψ and n pp-pairs ϑ_i/ξ_i, answers whether

$$\left(\frac{\phi}{\psi}\right) \subseteq \bigcup_{i=1}^{n} \left(\frac{\vartheta_i}{\xi_i}\right).$$

For any $n \in \mathbb{N}$, pp-1-formulae ϕ, ψ and pp-1-formulae ϑ_i, ξ_i for $0 < i \leq n$, $T_R \models \neg\left(\left|\frac{\phi}{\psi}\right| > 1 \wedge \bigwedge_{i=1}^{n} \left|\frac{\vartheta_i}{\xi_i}\right| = 1\right)$ is equivalent to $\left(\frac{\phi}{\psi}\right) \subseteq \bigcup_{i=1}^{n} \left(\frac{\vartheta_i}{\xi_i}\right)$. Hence, decidability of T_R implies we can effectively decide whether $\left(\frac{\phi}{\psi}\right) \subseteq \bigcup_{i=1}^{n} \left(\frac{\vartheta_i}{\xi_i}\right)$.

Lemma 4.2.1. *Let V be an effectively given valuation domain. There exists an algorithm which, given a pp-1-formula ϕ, produces a formula of the form $\sum_{i=1}^{n}(xa_i = 0 \wedge b_i|x)$ equivalent to ϕ.*

Proof. Since V is effectively given its theory of modules is recursively axiomatisable, so we have an algorithm which lists sentences true in all V-modules. By lemma 3.2.5, we know that there exists a formula of the form $\sum_{i=1}^{n}(xa_i = 0 \wedge b_i|x)$ equivalent to ϕ. Hence we need only look down the list of sentences true in all V-modules until we find one of the form:

$$\forall x \left(\phi(x) \leftrightarrow \sum_{i=1}^{n}(xa_i = 0 \wedge b_i|x)\right)$$

for some $n \in \mathbb{N}$ and $a_i, b_i \in V$. $\square$

Lemma 4.2.2. *Let V be an effectively given valuation domain. There exists an algorithm which, given a pp-1-formula ϕ, produces a formula of the form $\bigwedge_{i=1}^{n}(xa_i = 0 + b_i|x)$ equivalent to ϕ.*

Proof. Since V is effectively given its theory of modules is recursively axiomatisable, so we have an algorithm which lists sentences true in all V-modules. By lemma 3.2.3, we know that there exists a formula of the form $\bigwedge_{i=1}^{n}(xa_i = 0 + b_i|x)$ equivalent to ϕ. Hence we need only look down the list of sentences true in all V-modules until we find one of the form:

$$\forall x \left(\phi(x) \leftrightarrow \bigwedge_{i=1}^{n}(xa_i = 0 + b_i|x)\right)$$

for some $n \in \mathbb{N}$ and $a_i, b_i \in V$. $\square$

The algorithms described above would be rather inefficient. There is a possibly more efficient algorithm which would be based on diagonalising a matrix over a valuation domain. The proof of 3.2.1 clearly shows that diagonalisation is an effective process. Plus, taking the dual of a pp-formula is also clearly effective. The above two lemmas were proved this way in [PPT07].

Recall that, lemma 3.3.10, the sets $\mathcal{W}_{a,b,g,h}$ were originally sets of isomorphism classes of indecomposable pure-injective modules.

Corollary 4.2.3. *Let V be an effectively given valuation domain. Then there exists an algorithm which, given ϕ/ψ a pp-pair, returns the symbol $\emptyset$ if $\left(\frac{\phi}{\psi}\right)$ is empty and otherwise returns $n \in \mathbb{N}$, $a_i, b_i \in V\backslash\{0\}$ and $g_i, h_i \in \mathfrak{m}$ such that*

$$\left(\frac{\phi}{\psi}\right) = \bigcup_{i=1}^{n} \mathcal{W}_{a_i,b_i,g_i,h_i}.$$

Proof. By lemma 4.2.1 we can effectively rewrite ϕ as $\sum_{i=1}^{n}(a_i x = 0 \wedge b_i|x)$ for some $n \in \mathbb{N}$ and $a_i, b_i \in V$ for $0 < i \leq n$ and by lemma 4.2.2 we can effectively rewrite ψ as $\bigwedge_{j=1}^{m}(c_j x = 0 + d_j|x)$ for some $m \in \mathbb{N}$ and $c_j, d_j \in V$ for $0 < j \leq m$. Lemma 3.2.6 states that the pp-definable subgroups of an indecomposable pure-injective module are totally ordered. Therefore, for any indecomposable pure-injective module N, $N \in \left(\frac{\phi}{\psi}\right)$ if and only if $N \in \left(\frac{xa_i=0\wedge b_i|x}{xc_j=0+d_j|x}\right)$ for some $0 < i \leq n$ and $0 < j \leq m$. Hence

$$\left(\frac{\phi}{\psi}\right) = \bigcup_{i,j}\left(\frac{xa_i = 0 \wedge b_i|x}{xc_j = 0 + d_j|x}\right).$$

By lemma 3.3.9, $\left(\frac{x\alpha=0\wedge\beta|x}{x\gamma=0+\delta|x}\right)$ is empty if and only if $\alpha \notin \gamma\mathfrak{m}$, $\delta \notin \beta\mathfrak{m}$, $\beta = 0$ or $\gamma = 0$. Hence for each $0 < i \leq n$ and $0 < j \leq m$ either $\left(\frac{xa_i=0\wedge b_i|x}{xc_j=0+d_j|x}\right)$ is empty and this can be effectively checked or $\left(\frac{xa_i=0\wedge b_i|x}{xc_j=0+d_j|x}\right) = \mathcal{W}_{c_j,b_i,a_i/c_j,d_j/b_i}$ and a_i/c_j and d_j/b_i can be effectively calculated.

$\square$

Definition 4.2.4. *Suppose $x, y \in V$. We define $< x, y >$ as*

$$< x, y >= \begin{cases} y/x & \text{if } x|y, \\ x/y & \text{otherwise} \end{cases}$$

Definition 4.2.5. *Let $t \in V$. Denote by $\mathfrak{p}_t$ the radical of tV.*

Lemma 4.2.6. *For any $t \in V$, $\mathfrak{p}_t$ is a prime ideal and hence is the smallest prime ideal containing t.*

Proof. Recall that the radical of an ideal is the intersection of all prime ideals containing it. Since V is a valuation domain these prime ideals are totally ordered. The intersection of any chain of prime ideals is prime. Hence the radical of any ideal in a valuation domain is prime. $\square$

Lemma 4.2.7. *Let n be a natural number, $\lambda \in V\backslash\{0\}$, $g, h \in \mathfrak{m}$ and for each natural number $0 < i \leq n$ let $\mu_i \in V\backslash\{0\}$, $a_i, b_i \in \mathfrak{m}$. If there exists (I, J) a normal point such that $(I, J) \in \mathcal{W}_{1,\lambda,g,h}$ and $(I, J) \notin \bigcup_{i=1}^{n} \mathcal{W}_{1,\mu_i,a_i,b_i}$ then there exists a point $(K, L) \in \mathcal{W}_{1,\lambda,g,h}$ and $(K, L) \notin \bigcup_{i=1}^{n} \mathcal{W}_{1,\mu_i,a_i,b_i}$ such that $K^\# = \mathfrak{p}_r$ $L^\# = \mathfrak{p}_s$ where $r =< x, y >\in \mathfrak{m}$ and $s =< u, w >\in \mathfrak{m}$ and x, y, u, w are taken from the set*

$$\{\mu_i a_i b_i \ , \ \mu_i \mid 0 < i \leq n\} \cup \{1 \ , \ \lambda \ , \ g \ , \ h \ , \ \lambda gh\}\,.$$

Proof. First, recall that for any normal point (I, J), $(I, J) \notin \mathcal{W}_{1,\mu,a,b}$ if and only if either $\mu \in IJ$, $\mu ab \notin IJ$, $a \notin I^\#$ or $b \notin J^\#$. Therefore, if $(I, J) \notin \bigcup_{i=1}^{n} \mathcal{W}_{1,\mu_i,a_i,b_i}$ then for each $0 < i \leq n$, either $\mu_i \in IJ$, $\mu_i a_i b_i \notin IJ$, $a_i \notin I^\#$ or $b_i \notin J^\#$.

We now choose $a, b, \mu, d \in V$ as follows:

Suppose there exists $0 < i \leq n$ such that $a_i \notin I^\#$. Let a be such that $a \notin I^\#$, $a_i = a$ for some $0 < i \leq n$ and a_i divides a for all $a_i \notin I^\#$. If, for all $0 < i \leq n$, $a_i \in I^\#$ then let $a = 1$. Note, this means for any ideal K if $a \notin K$ and $a_i \notin I^\#$ then $a_i \notin K$.

Suppose there exists $0 < i \leq n$ such that $b_i \notin J^\#$. Let b be such that $b \notin J^\#$, $b = b_i$ for some $0 < i \leq n$ and b_i divides b for all $0 < i \leq n$ such that $b_i \notin J^\#$. If for all $0 < i \leq n$, $b_i \in J^\#$ then let $b = 1$. Note, this means for any ideal K if $b \notin K$ and $0 < i \leq n$ is such that $b_i \notin J^\#$ then $b_i \notin K$.

Suppose there exists $0 < i \leq n$ such that $\mu_i \in IJ$. Let $\mu = \mu_i$ for some $0 < i \leq n$ such that $\mu_i \in IJ$ and μ divides μ_i for all $0 < i \leq n$ with $\mu_i \in IJ$. If for all $0 < i \leq n$,

$\mu_i \notin IJ$, let $\mu = 0$. Note, this means for any ideal K, if $\mu \in K$ and $0 < i \leq n$ is such that $\mu_i \in IJ$ then $\mu_i \in K$.

Suppose there exists $0 < i \leq n$ such that $\mu_i a_i b_i \notin I.J$. Let $d = \mu_i a_i b_i$ for some $0 < i \leq n$ such that $\mu_i a_i b_i \notin IJ$ and $\mu_i a_i b_i$ divides d for all $0 < i \leq n$ with $\mu_i a_i b_i \notin IJ$. If for all $0 < i \leq n$, $\mu_i a_i b_i \in IJ$, let $d = 1$. Note, this means for any ideal K, if $d \notin K$ and $0 < i \leq n$ is such that $\mu_i a_i b_i \notin IJ$ then $\mu_i a_i b_i \notin K$.

Note that for any point (K, L), if $\mu \neq 0$ then $(K, L) \notin \mathcal{W}_{1,\mu,0,0}$, $d \notin KL$, $a \notin K^\#$ and $b \notin L^\#$ implies $(K, L) \notin \mathcal{W}_{1,\mu_i,a_i,b_i}$ for all $0 < i \leq n$ and if $\mu = 0$ then $d \notin KL$, $a \notin K^\#$ and $b \notin L^\#$ implies $(K, L) \notin \mathcal{W}_{1,\mu_i,a_i,b_i}$ for all $0 < i \leq n$.

We now choose $p_1, p_2, t \in V$ as follows:

If d divides λ, let $p_1 = \lambda$, otherwise let $p_1 = d$. Note that for any ideal K, $p_1 \notin K$ implies $\lambda \notin K$ and $d \notin K$. If μ divides λgh, let $p_2 = \mu$, otherwise let $p_2 = \lambda gh$. Note that for any ideal K, $p_2 \in K$ implies $\mu \in K$ and $\lambda gh \in K$. Since $p_1 \notin IJ$ and $p_2 \in IJ$, there exists $t \in (IJ)^\# = I^\# \cap J^\#$ such that $p_2 = p_1 t$.

First observe that $a \notin \mathfrak{p}_t$ and $a \notin \mathfrak{p}_g$ since $t \in I^\#$, $g \in I^\#$ and $a \notin I^\#$. Similarly, $b \notin \mathfrak{p}_t$ and $b \notin \mathfrak{p}_h$.

We split the rest of the proof into two cases.

<u>Case 1:</u> $\mathfrak{p}_g \cup \mathfrak{p}_t \neq \mathfrak{p}_h \cup \mathfrak{p}_t$ or $\mathfrak{p}_g \cup \mathfrak{p}_t = \mathfrak{p}_h \cup \mathfrak{p}_t$ and $(\mathfrak{p}_g \cup \mathfrak{p}_t)^2 = \mathfrak{p}_g \cup \mathfrak{p}_t$

Then $(\mathfrak{p}_g \cup \mathfrak{p}_t, p_1(\mathfrak{p}_h \cup \mathfrak{p}_t))$ is a normal point and $(\mathfrak{p}_g \cup \mathfrak{p}_t).(\mathfrak{p}_h \cup \mathfrak{p}_t) = (\mathfrak{p}_g \cup \mathfrak{p}_t) \cap (\mathfrak{p}_h \cup \mathfrak{p}_t)$ so $t \in (\mathfrak{p}_g \cup \mathfrak{p}_t).(\mathfrak{p}_h \cup \mathfrak{p}_t)$.

The point $(\mathfrak{p}_g \cup \mathfrak{p}_t, p_1(\mathfrak{p}_h \cup \mathfrak{p}_t)) \in \mathcal{W}_{1,\lambda,g,h}$ since $g \in \mathfrak{p}_g \cup \mathfrak{p}_t$; $h \in \mathfrak{p}_h \cup \mathfrak{p}_t$; $p_1 \notin p_1(\mathfrak{p}_g \cup \mathfrak{p}_t).(\mathfrak{p}_h \cup \mathfrak{p}_t)$ implies $\lambda \notin p_1(\mathfrak{p}_g \cup \mathfrak{p}_t).(\mathfrak{p}_h \cup \mathfrak{p}_t)$ and $p_2 = p_1 t \in p_1(\mathfrak{p}_g \cup \mathfrak{p}_t).(\mathfrak{p}_h \cup \mathfrak{p}_t)$ implies $\lambda gh \in p_1(\mathfrak{p}_g \cup \mathfrak{p}_t).(\mathfrak{p}_h \cup \mathfrak{p}_t)$. It remains to show $(\mathfrak{p}_g \cup \mathfrak{p}_t, p_1(\mathfrak{p}_h \cup \mathfrak{p}_t)) \notin \mathcal{W}_{1,\mu_i,a_i,b_i}$ for all i.

We have shown that $a \notin \mathfrak{p}_g \cup \mathfrak{p}_t$, $b \notin \mathfrak{p}_h \cup \mathfrak{p}_t$. Since $p_1 \notin p_1(\mathfrak{p}_g \cup \mathfrak{p}_t) \cdot (\mathfrak{p}_h \cup \mathfrak{p}_t)$, $d \notin p_1(\mathfrak{p}_g \cup \mathfrak{p}_t) \cdot (\mathfrak{p}_h \cup \mathfrak{p}_t)$. Since $p_2 \in p_1(\mathfrak{p}_g \cup \mathfrak{p}_t) \cdot (\mathfrak{p}_h \cup \mathfrak{p}_t)$, $\mu \in p_1(\mathfrak{p}_g \cup \mathfrak{p}_t) \cdot (\mathfrak{p}_h \cup \mathfrak{p}_t)$. Therefore either $\mu = 0$ or $(\mathfrak{p}_g \cup \mathfrak{p}_t, p_1(\mathfrak{p}_h \cup \mathfrak{p}_t)) \notin \mathcal{W}_{1,\mu,0,0}$ since $(\mathfrak{p}_g \cup \mathfrak{p}_t, p_1(\mathfrak{p}_h \cup \mathfrak{p}_t))$ is a normal point. Therefore, for all $0 < i \leq n$, $(\mathfrak{p}_g \cup \mathfrak{p}_t, p_1(\mathfrak{p}_h \cup \mathfrak{p}_t)) \notin \mathcal{W}_{1,\mu_i,a_i,b_i}$.

<u>Case 2:</u> $\mathfrak{p} = \mathfrak{p}_g \cup \mathfrak{p}_t = \mathfrak{p}_h \cup \mathfrak{p}_t$ and $\mathfrak{p}^2 \neq \mathfrak{p}$

Observe that for any $\pi \in V$, in order to show that $(\mathfrak{p}, \pi\mathfrak{p}) \in \mathcal{W}_{1,\lambda,g,h}$ and $(\mathfrak{p}, \pi\mathfrak{p}) \notin \mathcal{W}_{1,\mu_i,a_i,b_i}$ we must show that $\lambda \notin \pi\mathfrak{p}$, $\mu \in \pi\mathfrak{p}$, $\lambda gh \in \pi\mathfrak{p}^2$ and $d \notin \pi\mathfrak{p}^2$.

It is clear that $\lambda \notin p_1\mathfrak{p}$, $\mu \in p_1\mathfrak{p}$ and $d \notin p_1\mathfrak{p}^2$. If $\lambda gh \in p_1\mathfrak{p}^2$, let $\pi = p_1$. Otherwise, $\lambda gh \notin p_1\mathfrak{p}^2$. Then $\lambda \notin p_1 V$ since $gh \in \mathfrak{p}^2$. Therefore $p_1 = \lambda\gamma$ for some $\gamma \in \mathfrak{m}$.

Now $\lambda gh \notin \lambda\gamma\mathfrak{p}^2$ so $gh \notin \gamma\mathfrak{p}^2$. So $\gamma \in \mathfrak{p}$ since $gh \in \mathfrak{p}^2$.

Since $\gamma \in \mathfrak{p}$, there exists $\tau \in \mathfrak{p}\backslash\mathfrak{p}^2$ and $k \in V$ such that $\gamma = \tau k$. This means that $k\mathfrak{p}^2 = \gamma\mathfrak{p}$. Hence $\lambda k\mathfrak{p}^2 = \lambda\gamma\mathfrak{p} = p_1\mathfrak{p}$. Therefore $\lambda gh \in \lambda k\mathfrak{p}^2 = p_1\mathfrak{p}$ since $p_2 \in p_1\mathfrak{p}$.

It remains to show that $\lambda \notin \lambda k\mathfrak{p}$, $\mu \in \lambda k\mathfrak{p}$ and $d \notin \lambda k\mathfrak{p}^2$. Since $p_2 \in p_1\mathfrak{p}$, $\mu \in p_1\mathfrak{p} = \lambda k\mathfrak{p}^2 \subseteq \lambda k\mathfrak{p}$. Since $p_1 \notin p_1\mathfrak{p}$, $d \notin \lambda k\mathfrak{p}^2 = p_1\mathfrak{p}$ and $\lambda \notin \lambda k\mathfrak{p}$. Let $\pi = \lambda k$.

Therefore there exists a $\pi \in V$ such that $(\mathfrak{p}, \pi\mathfrak{p}) \in \mathcal{W}_{1,\lambda,g,h}$ and $(\mathfrak{p}, \pi\mathfrak{p}) \notin \mathcal{W}_{1,\mu_i,a_i,b_i}$ for all $0 < i \leq n$.

Finally note that $\mathfrak{p}_t \cup \mathfrak{p}_g = \mathfrak{p}_r$ and $\mathfrak{p}_t \cup \mathfrak{p}_h = \mathfrak{p}_s$ for some $r =< x, y >$ and $s =< u, v >$ where x, y, v, u are taken from the set:

$$\{1\ ,\ \lambda\ ,\ g\ ,\ h\} \cup \{\mu_i\ ,\ \mu_i a_i b_i \mid 0 < i \leq n\}\,.$$

□

Lemma 4.2.8. *Let n be a natural number, $\lambda \in V\backslash\{0\}$ and $g, h \in \mathfrak{m}$ and for each natural number $0 < i \leq n$ let $\mu_i \in V\backslash\{0\}$ and $a_i, b_i \in \mathfrak{m}$. If there exists (I, J) an abnormal point such that $(I, J) \in \mathcal{W}_{1,\lambda,g,h}$ and $(I, J) \notin \bigcup_{i=1}^n \mathcal{W}_{1,\mu_i,a_i,b_i}$ then there exists a point $(K, L) \in \mathcal{W}_{1,\lambda,g,h}$ and $(K, L) \notin \bigcup_{i=1}^n \mathcal{W}_{1,\mu_i,a_i,b_i}$ such that $K^\# = \mathfrak{p}_r$ $L^\# = \mathfrak{p}_s$ where $r =< x, y >$ and $s =< u, w >$ and x, y, u, w are taken from the set*

$$\{\mu_i a_i b_i\ ,\ \mu_i \mid 0 < i \leq n\} \cup \{1\ ,\ \lambda\ ,\ g\ ,\ h\ ,\ \lambda gh\}\,.$$

Proof. First note that since (I, J) is abnormal $I^\# = J^\#$, let $\mathfrak{p} = I^\#$.

We now choose $a, b, \mu, d \in V$ as follows:

Suppose there exists $0 < i \leq n$ such that $a_i \notin \mathfrak{p}$. Let a be such that $a \notin \mathfrak{p}$, $a_i = a$ for some $0 < i \leq n$ and a_i divides a for all $a_i \notin \mathfrak{p}$. If for all $0 < i \leq n$, $a_i \in \mathfrak{p}$ then let $a = 1$. Note, this means for any ideal K if $a \notin K$ and $a_i \notin \mathfrak{p}$ then $a_i \notin K$.

Suppose there exists $0 < i \leq n$ such that $b_i \notin \mathfrak{p}$. Let b be such that $b \notin \mathfrak{p}$, $b = b_i$ for some $0 < i \leq n$ and b_i divides b for all $0 < i \leq n$ such that $b_i \notin \mathfrak{p}$. If for all $0 < i \leq n$, $b_i \in \mathfrak{p}$ then let $b = 1$. Note, this means for any ideal K, if $b \notin K$ and $b_i \notin \mathfrak{p}$ then $b_i \notin K$.

Suppose there exists $0 < i \leq n$ such that $(I, J) \notin \mathcal{W}_{1,\mu_i,0,0}$. Let $\mu = \mu_i$ for some $0 < i \leq n$ such that $(I, J) \notin \mathcal{W}_{1,\mu_i,0,0}$ and μ divides μ_i for all $0 < i \leq n$ such $(I, J) \notin \mathcal{W}_{1,\mu_i,0,0}$.

Note, this means for any pair $(K, L) \notin \mathcal{W}_{1,\mu,0,0}$, if $0 < i \leq n$ is such that $(I, J) \notin \mathcal{W}_{1,\mu_i,0,0}$ then $(K, L) \notin \mathcal{W}_{1,\mu_i,0,0}$. If for all $0 < i \leq n$, $(I, J) \in \mathcal{W}_{1,\mu_i,0,0}$, let $\mu = 0$.

Suppose there exists $0 < i \leq n$ such that $\mu_i a_i b_i \notin I.J$. Let $d = \mu_i a_i b_i$ for some $0 < i \leq n$ such that $\mu_i a_i b_i \notin I.J$ and $\mu_i a_i b_i$ divides d for all $\mu_i a_i b_i \notin I.J$. Note, this means for any ideal K, if $d \notin K$ and $0 < i \leq n$ is such that $\mu_i a_i b_i \notin I.J$ then $\mu_i a_i b_i \notin K$. If for all $0 < i \leq n$, $\mu_i a_i b_i \in I.J$, let $d = 1$.

If $\mu \in IJ$ then precede as in the proof of lemma 4.2.7. Otherwise, $\lambda\mathfrak{p} \supsetneq d\mathfrak{p} = \mu\mathfrak{p} = IJ \supseteq \lambda ghV$ and $\mu \neq 0$. Note that $\mu \in \mathfrak{p}$ since $\mathfrak{p} \supseteq \lambda\mathfrak{p} \supsetneq \mu\mathfrak{p}$.

We now choose $t \in V$ and $\gamma \in V$ as follows:

Let $t \in V$ be such that $d = \lambda t$ and $\gamma \in V$ such that $\lambda gh = \mu\gamma$. Note that since $t, \gamma, g, h \in \mathfrak{p}$, $a, b \notin \mathfrak{p}_t \cup \mathfrak{p}_\gamma \cup \mathfrak{p}_g \cup \mathfrak{p}_h$. Let $\mathfrak{q} = \mathfrak{p}_t \cup \mathfrak{p}_\gamma \cup \mathfrak{p}_g \cup \mathfrak{p}_h$. Then either $\mathfrak{p} = \mathfrak{q}$ so $d\mathfrak{q} = \mu\mathfrak{q}$ or $\mathfrak{q} \subsetneq \mathfrak{p}$ so $d\mathfrak{q} = d\mathfrak{p}\mathfrak{q} = \mu\mathfrak{p}\mathfrak{q} = \mu\mathfrak{q}$.

Note that $t =< d, \lambda >$ and $d = 1$ or $d = \mu_i a_i b_i$ for some $0 < i \leq n$. Note that $\gamma =< \lambda gh, \mu >$ and $\mu = \mu_i$ for some $0 < i \leq n$. Therefore $\mathfrak{q} = \mathfrak{p}_r$ for some $r =< x, y >$ where x, y are taken from the set:

$$\{1\ ,\ \lambda\ ,\ g\ ,\ h\ ,\ \lambda gh\} \cup \{\mu_i a_i b_i\ ,\ \mu_i \mid 0 < i \leq n\}\,.$$

<u>Case 1</u>: $\mathfrak{q} = \mathfrak{q}^2$.

Consider $(I_d^{\mathfrak{q}}, \mathfrak{q})$. Then $\lambda \notin I_d^{\mathfrak{q}}$ (since $d \in \lambda\mathfrak{q}$), $\lambda gh \in I_d^{\mathfrak{q}}\mathfrak{q} = d\mathfrak{q}$, $g \in \mathfrak{q}$ and $h \in \mathfrak{q}$ so $(I_d^{\mathfrak{q}}, \mathfrak{q}) \in \mathcal{W}_{1,\lambda,g,h}$ by lemmas 3.5.6 and 3.5.21.

Then $\mu \in I_d^{\mathfrak{q}}$ (since $d \notin \mu\mathfrak{q}$), $d \notin I_d^{\mathfrak{q}} \cdot \mathfrak{q} = d\mathfrak{q}$, $a \notin \mathfrak{q}$ and $b \notin \mathfrak{q}$. Therefore $(I_d^{\mathfrak{q}}, \mathfrak{q}) \notin \mathcal{W}_{1,\mu_i,a_i,b_i}$ for all $0 < i \leq n$.

<u>Case 2</u>: $\mathfrak{q} \neq \mathfrak{q}^2$.

If $d \notin \mathfrak{q}^2$, consider $(\mathfrak{q}, \mathfrak{q})$. Then $\lambda \notin \mathfrak{q}$ since $d = \lambda t$ and $t \in \mathfrak{q}$. Hence $(\mathfrak{q}, \mathfrak{q}) \in \mathcal{W}_{1,\lambda,g,h}$ since $\lambda \notin \mathfrak{q}$, $g \in \mathfrak{q}$ and $\lambda h \in \mathfrak{q}$.

Since $\mu \in \mathfrak{q}$, $d \notin \mathfrak{q}^2$, $a \notin \mathfrak{q}$ and $b \notin \mathfrak{q}$, by lemmas 3.5.6 and 3.5.18, $(\mathfrak{q}, \mathfrak{q}) \notin \mathcal{W}_{1,\mu_i,a_i,b_i}$ for all $0 < i \leq n$.

If $d \in \mathfrak{q}^2$, take $\tau \in \mathfrak{q} \backslash \mathfrak{q}^2$ then there exists $k \in V$ such that $d = \tau k$. Hence $d\mathfrak{q} = k\mathfrak{q}^2$. Consider $(\mathfrak{q}, k\mathfrak{q})$.

Then $\lambda gh \in k\mathfrak{q}^2 = d\mathfrak{q}$. Since $\tau \notin \mathfrak{q}^2$, $\tau \notin t\mathfrak{q}$ because $t \in \mathfrak{q}$. Therefore $\lambda\tau \notin \lambda t\mathfrak{q}$ hence $\lambda\tau \notin d\mathfrak{q}$. Therefore $\lambda\tau \notin \tau k\mathfrak{q}$ so $\lambda \notin k\mathfrak{q}$.

Therefore, by lemma 3.5.6 and lemma 3.5.18, $(\mathfrak{q}, k\mathfrak{q}) \in \mathcal{W}_{1,\lambda,g,h}$ since $g, h \in \mathfrak{q}$.

It remains to show that $(\mathfrak{q}, k\mathfrak{q}) \notin \mathcal{W}_{1,\mu_i,a_i,b_i}$ for all $0 < i \leq n$. By lemmas 3.5.6 and 3.5.18, it is enough to show that $\mu \in k\mathfrak{q}$ and $d \notin k\mathfrak{q}^2$ since $a, b \notin \mathfrak{q}$. Suppose $\mu \notin k\mathfrak{q}$ then $\mu\mathfrak{q} \supseteq k\mathfrak{q} \supsetneq k\tau\mathfrak{q} = d\mathfrak{q}$, a contradiction. Therefore $\mu \in k\mathfrak{q}$. Since $k\mathfrak{q}^2 = d\mathfrak{q}$, $d \notin k\mathfrak{q}^2$. Therefore $(\mathfrak{q}, k\mathfrak{q}) \notin \mathcal{W}_{1,\mu_i,a_i,b_i}$ for all $0 < i \leq n$.

□

Lemma 4.2.9. *Let V be an effectively given valuation domain. Suppose $\mathfrak{p}, \mathfrak{q} \lhd V$ are prime ideals and that $\mathfrak{p} \subsetneq \mathfrak{q}$. Suppose there is an algorithm that given $a \in V$, answers whether $a \in \mathfrak{p}$ and an algorithm that given $b \in V$, answers whether $b \in \mathfrak{q}$. Then for any natural number n there is an algorithm that given $\lambda, \mu_1, ...\mu_n \in V\backslash\{0\}$ and $g, h, a_1, ..., a_n, b_1, ..., b_n \in \mathfrak{m}$, answers whether $\mathcal{W}_{1,\lambda,g,h} \cap X_{\mathfrak{p},\mathfrak{q}} \subseteq \bigcup_{i=1}^{n} \mathcal{W}_{1,\mu_i,a_i,b_i} \cap X_{\mathfrak{p},\mathfrak{q}}$.*

Proof. First note that, by lemma 4.1.2, if there is an algorithm that, given $a \in V$, answers whether $a \in \mathfrak{p}$ then there is an algorithm that, given $a, b \in V$, answers whether $a \in b\mathfrak{p}$.

Suppose n is a fixed natural number. First we will describe an algorithm that, given $\lambda, \mu_1, ...\mu_n \in V\backslash\{0\}$, $g, a_1, ..., a_n \in \mathfrak{p}$ and $h, b_1, ..., b_n \in \mathfrak{q}$, answers whether $\mathcal{W}_{1,\lambda,g,h} \cap X_{\mathfrak{p},\mathfrak{q}} \subseteq \bigcup_{i=1}^{n} \mathcal{W}_{1,\mu_i,a_i,b_i} \cap X_{\mathfrak{p},\mathfrak{q}}$. It is enough to describe such an algorithm as if $a \notin \mathfrak{p}$ or $b \notin \mathfrak{q}$ then $\mathcal{W}_{1,\mu,a,b} \cap X_{\mathfrak{p},\mathfrak{q}} = \emptyset$ for all $\mu \in V\backslash\{0\}$.

STEP 1 Let $t_0 = \lambda$ and $j = 0$.

STEP 2 If there does not exist $0 < i \leq n$ such that $\mu_i \notin t_j\mathfrak{p}$ and $\mu_i a_i b_i \in t_j\mathfrak{p}$ then FALSE. Otherwise, let i be the least i such that $\mu_i \notin t_j\mathfrak{p}$ and $\mu_i a_i b_i \in t_j\mathfrak{p}$. If

$\lambda gh \notin \mu_i a_i b_i \mathfrak{p}$ then TRUE. Otherwise, set $t_{j+1} = \mu_i a_i b_i$ and $s_{j+1} = \mu_i$. Then go back to the start of **STEP 2** with j increased by 1.

It is obvious that the above algorithm terminates since, for all values of j which occur, $t_j \mathfrak{p} \supsetneq t_{j+1}\mathfrak{p}$ and the values of t_j are taken from a finite set.

Suppose the above algorithm returns FALSE. We must show there exists a point $(I, J) \in X_{\mathfrak{p},\mathfrak{q}}$ such that $(I, J) \in \mathcal{W}_{1,\lambda,g,h}$ and $(I, J) \notin \mathcal{W}_{1,\mu_i,a_i,b_i}$ for all $0 < i \leq n$.

Suppose the algorithm returns FALSE when $j = 0$. Then $(\mathfrak{p}, \lambda\mathfrak{q}) \in \mathcal{W}_{1,\lambda,g,h}$ since $\lambda \notin \lambda\mathfrak{q}$, $g \in \mathfrak{p}$ and $\lambda h \in \lambda\mathfrak{q}$. Note that, since $\mathfrak{p} \subsetneq \mathfrak{q}$, $\mathfrak{p}\cdot\mathfrak{q} = \mathfrak{p}$. For all $0 < i \leq n$, either $\mu_i \in \lambda\mathfrak{p} = \lambda\mathfrak{p}\mathfrak{q}$ or $\mu_i a_i b_i \notin \lambda\mathfrak{p} = \lambda\mathfrak{p}\mathfrak{q}$. Therefore $(\mathfrak{p}, \lambda\mathfrak{q}) \notin \mathcal{W}_{1,\mu_i,a_i,b_i}$ for all $0 < i \leq n$.

Suppose the algorithm returns FALSE when $j \neq 0$. Then $\lambda gh \in t_j\mathfrak{p} = t_j\mathfrak{p}\mathfrak{q}$ and $\lambda\mathfrak{p} \supsetneq t_j\mathfrak{p} = t_j\mathfrak{p}\mathfrak{q}$ so $\lambda \notin t_j\mathfrak{p}\mathfrak{q}$ therefore $(t_j\mathfrak{p}, \mathfrak{q}) \in \mathcal{W}_{1,\lambda,g,h}$ and there does not exist an $0 < i \leq n$ such that $\mu_i \notin t_j\mathfrak{p} = t_j\mathfrak{p}\mathfrak{q}$ and $\mu_i a_i b_i \in t_j\mathfrak{p} = t_j\mathfrak{p}\mathfrak{q}$ so $(t_j\mathfrak{p}, \mathfrak{q}) \notin \mathcal{W}_{1,\mu_i,a_i,b_i}$ for all $0 < i \leq n$.

Suppose the above algorithm returns TRUE. We must show $\mathcal{W}_{1,\lambda,g,h} \cap X_{\mathfrak{p},\mathfrak{q}} \subseteq \bigcup_{i=1}^{n} \mathcal{W}_{1,\mu_i,a_i,b_i} \cap X_{\mathfrak{p},\mathfrak{q}}$. Suppose $(I, J) \in \mathcal{W}_{1,\lambda,g,h}$, $I^{\#} = \mathfrak{p}$ and $J^{\#} = \mathfrak{q}$. Then (I, J) is normal since $\mathfrak{p} \neq \mathfrak{q}$. Therefore $\lambda \notin IJ$ and $\lambda gh \in IJ$. It is enough to show that there exists an j such that $s_j \notin IJ$ and $t_j \in IJ$.

Let k be the value of j at which the algorithm terminates, then there is an i such that $\lambda gh \notin \mu_i a_i b_i \mathfrak{p}$ and $\mu_i a_i b_i \in t_k\mathfrak{p}$. Using $\lambda gh \in IJ$ and $\lambda gh \notin \mu_i a_i b_i \mathfrak{p}$ we get $\mu_i a_i b_i \in IJ$. Note $\mu_i \notin t_k\mathfrak{p}$ so either $t_k \in IJ$ or $\mu_i \notin IJ$. If $\mu_i \notin IJ$ and $\mu_i a_i b_i \in IJ$ then we are done. So suppose $t_k \in IJ$. Hence if $s_k \notin IJ$ we are done. So assume $s_k \in IJ$. Observe that $s_1 \notin IJ$ since $\lambda \notin IJ$ and $s_1 \notin \lambda\mathfrak{p}$. Therefore there exists a j such that $s_j \notin IJ$ and $s_{j+1} \in IJ$. Note that $s_{j+1} \in IJ$ implies $t_j \in IJ$ since $s_{j+1} \notin t_j\mathfrak{p}$. Hence $s_j \notin IJ$ and $t_j \in IJ$.

$\square$

Definition 4.2.10. *Let $a, b \in V$ and $\mathfrak{p} \lhd V$ be prime. We write*

$$a <_{\mathfrak{p}} b \text{ if and only if } b \in a\mathfrak{p}$$

and

$$a =_{\mathfrak{p}} b \text{ if and only if } a\mathfrak{p} = b\mathfrak{p}.$$

Definition 4.2.11. *Let $\mathfrak{p} \lhd V$ be prime, $t \in V$ and $s \in \mathfrak{p}$. We define*

$$(t, st)_{\mathfrak{p}} := \{r \in V \mid t <_{\mathfrak{p}} r <_{\mathfrak{p}} st\}$$

and

$$[t, st]_{\mathfrak{p}} := \{r \in V \mid t \leq_{\mathfrak{p}} r \leq_{\mathfrak{p}} st\}$$

Proposition 4.2.12. *Suppose $\mathfrak{p} \lhd V$ is prime, $n \in \mathbb{N}$, $\lambda, \mu_1, ..., \mu_n \in V \backslash \{0\}$ and $g, h, a_1, ..., a_n, b_1, ..., b_n \in \mathfrak{p}$. Then the following are equivalent:*

1.
$$(\lambda, \lambda gh)_{\mathfrak{p}} \subseteq \bigcup_{i=1}^{n} (\mu_i, \mu_i a_i b_i)_{\mathfrak{p}}$$

2.
$$\mathcal{W}_{1,\lambda,g,h} \cap X_{\mathfrak{p},\mathfrak{p}} \subseteq \bigcup_{i=1}^{n} \mathcal{W}_{1,\mu_i,a_i,b_i} \cap X_{\mathfrak{p},\mathfrak{p}}.$$

Proof. (1)$\Rightarrow$ (2) Suppose $(\lambda, \lambda gh)_{\mathfrak{p}} \subseteq \cup_{i=1}^{n} (\mu_i, \mu_i a_i b_i)_{\mathfrak{p}}$.

Suppose first that (I, J) is a normal point in $X_{\mathfrak{p},\mathfrak{p}}$, recall that this means $\mathfrak{p}^2 = \mathfrak{p}$. Suppose for a contradiction that $(I, J) \in \mathcal{W}_{1,\lambda,g,h}$ and $(I, J) \notin \mathcal{W}_{1,\mu_i,a_i,b_i}$ for all $0 < i \leq n$. Then for all $0 < i \leq n$ either $\mu_i \in IJ$ or $\mu_i a_i b_i \notin IJ$.

Let $k_1 \in V$ be such that

$$k_1 V = \bigcup_{\mu_i \in IJ} \mu_i V$$

or 0 if $\mu_i \notin IJ$ for all i. So $\mu_i \in IJ$ implies $\mu_i \in k_1 V$ hence $\mu_i \geq_{\mathfrak{p}} k_1$

Let $k_2 \in V$ be such that

$$k_2 \mathfrak{p} = \bigcap_{\mu_i a_i b_i \notin IJ} \mu_i a_i b_i \mathfrak{p}$$

or 1 if $\mu_i a_i b_i \in IJ$ for all i. So $\mu_i a_i b_i \notin IJ$ implies $\mu_i a_i b_i \mathfrak{p} \supseteq k_2 \mathfrak{p}$ hence $k_2 \geq_{\mathfrak{p}} \mu_i a_i b_i$.

$$\textbf{Claim: } [k_2, k_1]_{\mathfrak{p}} \cap \bigcup_{i=1}^{n} (\mu_i, \mu_i a_i b_i)_{\mathfrak{p}} = \emptyset.$$

Suppose that $k_2 \leq_{\mathfrak{p}} l \leq_{\mathfrak{p}} k_1$, that is $k_2 \mathfrak{p} \supseteq l\mathfrak{p} \supseteq k_1 \mathfrak{p}$. Suppose for some $0 < i \leq n$ $\mu_i <_{\mathfrak{p}} l$. Then $\mu_i \notin IJ$; for otherwise $k_1 <_{\mathfrak{p}} l$. Therefore $\mu_i a_i b_i \notin IJ$. So $k_2 \geq_{\mathfrak{p}} \mu_i a_i b_i$. Therefore $l \notin (\mu_i, \mu_i a_i b_i)_{\mathfrak{p}}$.

Claim: $[k_2, k_1]_\mathfrak{p} \cap (\lambda, \lambda gh)_\mathfrak{p} \neq \emptyset$.

Let $d \in V$ be such that $dV = k_1V \cup \lambda ghV$. So $d \in IJ$ since $k_1 \in IJ$ and $\lambda gh \in IJ$. Let $p \in V$ be such that $p\mathfrak{p} = \lambda\mathfrak{p} \cap k_2\mathfrak{p}$. Since $\lambda \notin IJ$ and $k_2 \notin IJ$, $IJ \subseteq p\mathfrak{p}$. Therefore $dV \subseteq IJ \subseteq p\mathfrak{p}$. Hence, there exists $\delta \in \mathfrak{p}$ such that $d = p\delta$. Since $\mathfrak{p}^2 = \mathfrak{p}$, there exists $\delta_1, \delta_2 \in \mathfrak{p}$ such that $\delta_1\delta_2 = \delta$. Now $d \in p\delta_1\mathfrak{p}$ and $p\delta_1 \in p\mathfrak{p}$. Hence $k_1 \in p\delta_1\mathfrak{p}$, $\lambda gh \in p\delta_1\mathfrak{p}$, $p\delta_1 \in \lambda\mathfrak{p}$ and $p\delta_1 \in k_2\mathfrak{p}$. So $p\delta_1 \in (\lambda, \lambda gh)_\mathfrak{p}$ and $p\delta_1 \in (k_1, k_2)_\mathfrak{p}$.

Combining these two claims contradicts $(\lambda, \lambda gh)_\mathfrak{p} \subseteq \cup_{i=1}^n (\mu_i, \mu_i a_i b_i)_\mathfrak{p}$. Therefore $(I, J) \in \mathcal{W}_{1,\mu_i,a_i,b_i}$ for some $0 < i \leq n$.

Suppose that (I, J) is an abnormal point, $\mathfrak{p}^2 = \mathfrak{p}$ and $(I, J) \in \mathcal{W}_{1,\lambda,g,h}$. Then (I, J) is topologically indistinguishable from $(I_\gamma^\mathfrak{p}, \mathfrak{p})$ for some $\gamma \in \mathfrak{p}$ by lemma 3.5.25. Therefore $(I_\gamma^\mathfrak{p}, \mathfrak{p}) \in \mathcal{W}_{1,\lambda,g,h}$ so $\lambda \notin I_\gamma^\mathfrak{p}$ and $\lambda gh \in \gamma\mathfrak{p}$. Hence $\gamma \in \lambda\mathfrak{p}$ and $\lambda gh \in \gamma\mathfrak{p}$, that is $\lambda <_\mathfrak{p} \gamma <_\mathfrak{p} \lambda gh$. Therefore $\gamma \in (\mu_i, \mu_i a_i b_i)_\mathfrak{p}$ for some i. Hence $\mu_i <_\mathfrak{p} \gamma <_\mathfrak{p} \mu_i a_i b_i$ so $\mu_i \notin I_\gamma^\mathfrak{p}$ and $\mu_i a_i b_i \in \gamma\mathfrak{p}$. Therefore $(I_\gamma^\mathfrak{p}, \mathfrak{p}) \in \mathcal{W}_{1,\mu_i,a_i,b_i}$. So $(I, J) \in \mathcal{W}_{1,\mu_i,a_i,b_i}$.

Suppose that (I, J) is an abnormal point, $\mathfrak{p}^2 \neq \mathfrak{p}$ and $(I, J) \in \mathcal{W}_{1,\lambda,g,h}$. Then we may assume $I = \mathfrak{p}$ and $J = \gamma\mathfrak{p}$ for some $\gamma \in V$. If $(\mathfrak{p}, \gamma\mathfrak{p}) \in \mathcal{W}_{1,\lambda,g,h}$ then $\lambda \notin \gamma\mathfrak{p}$ and $\lambda gh \in \gamma\mathfrak{p}^2$. Take $k \in \mathfrak{p}\backslash\mathfrak{p}^2$. Then $\gamma k \in \lambda\mathfrak{p}$ and $\lambda gh \in \gamma k\mathfrak{p}$, so $\lambda <_\mathfrak{p} \gamma k <_\mathfrak{p} \lambda gh$. Therefore $\gamma k \in (\mu_i, \mu_i a_i b_i)_\mathfrak{p}$ for some $0 < i \leq n$. Hence $\gamma k \in \mu_i\mathfrak{p}$. Therefore $\gamma\mathfrak{p} \subseteq \mu_i\mathfrak{p}$, so $\mu_i \notin \gamma\mathfrak{p}$ and $\mu_i a_i b_i \in \gamma k\mathfrak{p} = \gamma\mathfrak{p}^2$. Hence $(\mathfrak{p}, \gamma\mathfrak{p}) \in \mathcal{W}_{1,\mu_i,a_i,b_i}$ by lemmas 3.5.18 and 3.5.6.

(2)$\Rightarrow$(1) Suppose $\mathcal{W}_{1,\lambda,g,h} \cap X_{\mathfrak{p},\mathfrak{p}} \subseteq \cup_{i=1}^n \mathcal{W}_{1,\mu_i,a_i,b_i} \cap X_{\mathfrak{p},\mathfrak{p}}$.

<u>Case 1</u>: $\mathfrak{p}^2 = \mathfrak{p}$.

Take $\gamma \in (\lambda, \lambda gh)_\mathfrak{p}$. Then $\gamma \in \lambda\mathfrak{p}$. Hence $\lambda \notin I_\gamma^\mathfrak{p}$ and $\lambda gh \in \gamma\mathfrak{p}$. So $(I_\gamma^\mathfrak{p}, \mathfrak{p}) \in \mathcal{W}_{1,\lambda,g,h}$. Therefore $(I_\gamma^\mathfrak{p}, \mathfrak{p}) \in \mathcal{W}_{1,\mu_i,a_i,b_i}$ for some $0 < i \leq n$. Hence $\mu_i \notin I_\gamma^\mathfrak{p}$, so $\gamma \in \mu_i\mathfrak{p}$ and $\mu_i a_i b_i \in \gamma\mathfrak{p}$. Therefore $\gamma \in (\mu_i, \mu_i a_i b_i)_\mathfrak{p}$.

<u>Case 2</u>: $\mathfrak{p} \neq \mathfrak{p}^2$.

Take $\gamma \in (\lambda, \lambda gh)_\mathfrak{p}$ and let $k \in \mathfrak{p}\backslash\mathfrak{p}^2$. Then $\gamma \in \lambda\mathfrak{p}$ and $\lambda gh \in \gamma\mathfrak{p}$ (hence $\gamma \in \mathfrak{p}$). First suppose that $\gamma \in \mathfrak{p}^2$. Then $\gamma = kt$ for some $t \in V$. So $\lambda \notin t\mathfrak{p}$ and $\lambda gh \in t\mathfrak{p}^2 = \gamma\mathfrak{p}$. Therefore $(\mathfrak{p}, t\mathfrak{p}) \in \mathcal{W}_{1,\lambda,g,h}$. So there exists an $0 < i \leq n$ such that $(\mathfrak{p}, t\mathfrak{p}) \in \mathcal{W}_{1,\mu_i,a_i,b_i}$.

Hence $\mu_i \notin t\mathfrak{p}$. So $\gamma \in \mu_i\mathfrak{p}$ and $\mu_i a_i b_i \in t\mathfrak{p}^2 = \gamma\mathfrak{p}$. Therefore $\gamma \in (\mu_i, \mu_i a_i b_i)_{\mathfrak{p}}$. Now suppose that $\gamma \in \mathfrak{p}\backslash\mathfrak{p}^2$. Then $\gamma\mathfrak{p} = \mathfrak{p}^2$ and $\lambda \notin \mathfrak{p}$. Therefore $(\mathfrak{p}, \mathfrak{p}) \in \mathcal{W}_{1,\lambda,g,h}$. So there exists an $0 < i \leq n$ such that $(\mathfrak{p}, \mathfrak{p}) \in \mathcal{W}_{1,\mu_i,a_i,b_i}$. So $\mu_i \notin \mathfrak{p}$ and $\mu_i a_i b_i \in \mathfrak{p}^2$. Therefore $\gamma \in \mu_i\mathfrak{p} = \mathfrak{p}$ and $\mu_i a_i b_i \in \gamma\mathfrak{p} = \mathfrak{p}^2$. So $\gamma \in (\mu_i, \mu_i a_i b_i)_{\mathfrak{p}}$.

□

Corollary 4.2.13. *Let V be an effectively given valuation domain. Suppose $\mathfrak{p} \lhd V$ is a prime ideal. Suppose there is an algorithm that given $a \in V$, answers whether $a \in \mathfrak{p}$. Then for any natural number n there is an algorithm that given $\lambda, \mu_1, ...\mu_n \in V\backslash\{0\}$ and $g, h, a_1, ..., a_n, b_1, ..., b_n \in \mathfrak{m}$, answers whether*

$$\mathcal{W}_{1,\lambda,g,h} \cap X_{\mathfrak{p},\mathfrak{p}} \subseteq \bigcup_{i=1}^{n} \mathcal{W}_{1,\mu_i,a_i,b_i} \cap X_{\mathfrak{p},\mathfrak{p}}.$$

Proof. If $g \notin \mathfrak{p}$ or $h \notin \mathfrak{p}$ then $\mathcal{W}_{1,\lambda,g,h} \cap X_{\mathfrak{p},\mathfrak{p}} = \emptyset$. So $\mathcal{W}_{1,\lambda,g,h} \cap X_{\mathfrak{p},\mathfrak{p}} \subseteq \bigcup_{i=1}^{n} \mathcal{W}_{1,\mu_i,a_i,b_i} \cap X_{\mathfrak{p},\mathfrak{p}}$.

Suppose $g, h \in \mathfrak{p}$. Then $(\mathfrak{p}, \lambda\mathfrak{p}) \in \mathcal{W}_{1,\lambda,g,h}$ since $g \in \mathfrak{p}$, $\lambda \notin \lambda\mathfrak{p}$ and $\lambda h \in \mathfrak{p}$. If, for all $0 < i \leq n$, either $a_i \notin \mathfrak{p}$ or $b_i \notin \mathfrak{p}$ then $\bigcup_{i=1}^{n} \mathcal{W}_{1,\mu_i,a_i,b_i} \cap X_{\mathfrak{p},\mathfrak{p}} = \emptyset$. Hence $\mathcal{W}_{1,\lambda,g,h} \cap X_{\mathfrak{p},\mathfrak{p}} \nsubseteq \bigcup_{i=1}^{n} \mathcal{W}_{1,\mu_i,a_i,b_i} \cap X_{\mathfrak{p},\mathfrak{p}}$.

Now suppose $g, h \in \mathfrak{p}$ and there exists $0 < i \leq n$ such that $a_i, b_i \in \mathfrak{p}$. Let $\mathcal{J}$ be the set of all $0 < i \leq n$ such that $a_i, b_i \in \mathfrak{p}$. Then $\mathcal{W}_{1,\lambda,g,h} \cap X_{\mathfrak{p},\mathfrak{p}} \subseteq \bigcup_{i=1}^{n} \mathcal{W}_{1,\mu_i,a_i,b_i} \cap X_{\mathfrak{p},\mathfrak{p}}$ if and only if $\mathcal{W}_{1,\lambda,g,h} \cap X_{\mathfrak{p},\mathfrak{p}} \subseteq \bigcup_{i\in\mathcal{J}} \mathcal{W}_{1,\mu_i,a_i,b_i} \cap X_{\mathfrak{p},\mathfrak{p}}$.

By proposition 4.2.12, $\mathcal{W}_{1,\lambda,g,h} \cap X_{\mathfrak{p},\mathfrak{p}} \subseteq \bigcup_{i\in\mathcal{J}} \mathcal{W}_{1,\mu_i,a_i,b_i} \cap X_{\mathfrak{p},\mathfrak{p}}$ if and only if $(\lambda, \lambda gh)_{\mathfrak{p}} \subseteq \bigcup_{i\in\mathcal{J}} (\mu_i, \mu_i a_i b_i)_{\mathfrak{p}}$.

The existence of an algorithm which, given $a \in V$, answers whether $a \in \mathfrak{p}$ means, since V is effectively given, there exists an algorithm which, given $a, b \in V$, answers whether $a \in b\mathfrak{p}$. Therefore, there is an algorithm which given $\lambda, \mu_1, ..., \mu_k \in V\backslash\{0\}$ and $g, h, a_1, ..., a_k, b_1, ..., b_k \in \mathfrak{p}$, answers whether $(\lambda, \lambda gh)_{\mathfrak{p}} \subseteq \bigcup_{i\in\mathcal{J}} (\mu_i, \mu_i a_i b_i)_{\mathfrak{p}}$. □

Lemma 4.2.14. *Let $n \in \mathbb{N}$. Let V be an effectively given valuation domain such that there exists an algorithm which, given $a, b \in V$, answers whether $a \in \mathrm{rad}(bV)$. Then there exists an algorithm which, given $a, b, \alpha_i, \beta_i \in V\backslash\{0\}$ and $g, h, \gamma_i, \delta_i \in \mathfrak{m}$ for each*

$0 < i \leq n$, *answers whether*

$$\mathcal{W}_{a,b,g,h} \subseteq \bigcup_{i=1}^{n} \mathcal{W}_{\alpha_i,\beta_i,\gamma_i,\delta_i}.$$

Proof. First note for any $a, b \in V\backslash\{0\}$ and $g, h \in \mathfrak{m}$, $\mathcal{W}_{a,b,g,h} = \mathcal{W}_{1,ab,g,h}$. Suppose $n \in \mathbb{N}$, $\lambda, \mu_i \notin V\backslash\{0\}$ and $g, h, a_i, b_i \in \mathfrak{m}$. Let $T = \{< u, v > \in \mathfrak{m} \mid u, v \in \{1\ ,\ \lambda\ ,\ g\ ,\ h\ ,\ \mu_i a_i b_i\ ,\ \mu_i \mid 0 < i \leq n\}\}$. Note that T is a finite set and there is an algorithm which, given λ, g, h and μ_i, a_i, b_i for $0 < i \leq n$, computes T since the function $<\ ,\ >$ and multiplication of ring elements is recursive.

Then in order to check whether

$$\mathcal{W}_{1,\lambda,g,h} \subseteq \bigcup_{i=1}^{n} \mathcal{W}_{1,\mu_i,a_i,b_i}$$

by lemma 4.2.7 and lemma 4.2.8 it is enough to check

$$\mathcal{W}_{1,\lambda,g,h} \cap X_{\mathfrak{p},\mathfrak{q}} \subseteq \bigcup_{i=1}^{n} \mathcal{W}_{1,\mu_i,a_i,b_i} \cap X_{\mathfrak{p},\mathfrak{q}}$$

for $\mathfrak{p} = \mathrm{rad} tV$ and $\mathfrak{q} = \mathrm{rad} sV$ for each $t, s \in T$.

By lemma 4.2.9 and corollary 4.2.13 there exists an algorithm determining the truth of the above statement. □

Theorem 4.2.15. *Let V be an effectively given valuation domain with an algorithm which, given $a, b \in V$, answers whether $a \in \mathrm{rad}(bV)$. Let $n \in \mathbb{N}$. Then there is an algorithm which, given ϕ/ψ a pp-pair and ϑ_i/ξ_i a pp-pair for each $0 < i \leq n$, answers whether:*

$$\left(\frac{\phi}{\psi}\right) \subseteq \bigcup_{i=1}^{n} \left(\frac{\vartheta_i}{\xi_i}\right).$$

Proof. By corollary 4.2.3, given a pp-pair ϕ/ψ we can effectively check whether $\left(\frac{\phi}{\psi}\right)$ is non-empty.

Again using corollary 4.2.3, given a pp-pair ϕ/ψ, if $\left(\frac{\phi}{\psi}\right)$ is non-empty we can effectively find $a_j, b_j \in V\backslash\{0\}$ and $g_j, h_j \in \mathfrak{m}$ such that:

$$\left(\frac{\phi}{\psi}\right) = \bigcup_{j} \mathcal{W}_{a_j,b_j,g_j,h_j}$$

and for each i, if $\left(\frac{\vartheta_i}{\xi_i}\right)$ is non-empty we can effectively find $\alpha_{i,k}, \beta_{i,k} \in V\backslash\{0\}$ and $\gamma_{i,k}, \delta_{i,k} \in \mathfrak{m}$ such that:

$$\left(\frac{\vartheta_i}{\xi_i}\right) = \bigcup_{i,k} \mathcal{W}_{\alpha_{i,k},\beta_{i,k},\gamma_{i,k},\delta_{i,k}}$$

Therefore it is enough to check for each j whether:

$$\mathcal{W}_{a_j,b_j,g_j,h_j} \subseteq \bigcup_{i,k} \mathcal{W}_{\alpha_{i,k},\beta_{i,k},\gamma_{i,k},\delta_{i,k}}.$$

By lemma 4.2.14 there exists an algorithm which determines the truth of the above statement.

□

4.3 Valuation domains with infinite residue field

In this section we prove theorem 4.0.1 for valuation domains with infinite residue field. This case is significantly easier than the case of valuation domains with finite residue field since for all pp-formulae ϕ, ψ and all modules M, $\left|\frac{\phi(M)}{\psi(M)}\right|$ is either 1 or infinite (corollary 4.3.2).

Lemma 4.3.1. *Let V be a valuation domain with infinite residue field. Then all non-zero V-modules have infinitely many elements.*

Proof. First note that for any $I \lhd V$, V/I is infinite since V/I surjectively maps onto $V/\mathfrak{m}$. Suppose M is a non-zero V-module. Take non-zero $m \in M$. Then $mV \cong V/\mathrm{ann}_V m$. Therefore mV is infinite, so M is infinite. □

Corollary 4.3.2. *Let V be a valuation domain with infinite residue field. Then for all V-modules N and all pp-pairs ϕ/ψ either $|\phi(N)/\psi(N)| = 1$ or $|\phi(N)/\psi(N)|$ is infinite.*

Theorem 4.3.3. *Let V be an effectively given valuation domain with infinite residue field. Then the following are equivalent:*

1. *The theory of V-modules, T_V, is decidable.*

2. *There exists an algorithm which, given $a, b \in V$, answers whether $a \in \operatorname{rad}(bV)$.*

Proof. Let V be an effectively given commutative valuation domain with infinite residue field and an algorithm which, given $a, b \in V$, answers whether $a \in \operatorname{rad}(bV)$. First note that since V is effectively given, T_V is recursively axiomatised. Hence we have an algorithm which produces a list of sentences true in all V-modules. Since T_V is not a complete theory, in order to show T_V is decidable we need to effectively produce a list of sentences which are false in some module. Equivalently, we need to effectively produce a list of sentences which are true in at least one module.

By the Baur-Monk theorem every sentence is equivalent to a boolean combination of invariants sentences.

Since T_V is effectively axiomatised, if χ is a sentence in $\mathcal{L}_V$ then we can effectively find a sentence θ, a boolean combination of these invariants sentences, which is equivalent to χ in T_V. That is, we simply look down the list of sentences true in all modules until we find one of the form $\chi \leftrightarrow \theta$ where θ is of the correct form. The Baur-Monk theorem ensures that we will find such a sentence.

Thus we may assume $\chi = \bigvee \sigma_h$, a finite disjunction of conjunctions of invariants sentences and their negations. Suppose $M_h \models \sigma_h$ for some h, then $M_h \models \bigvee \sigma_h$. Therefore we may assume χ is a conjunction of invariants sentences and their negations. Since V has infinite residue field, if ϕ/ψ is a pp-pair and $n \in \mathbb{N}$ then the invariants sentence $|\phi/\psi| > n$ is equivalent to the invariants sentence $|\phi/\psi| > 1$. So we may assume χ is a conjunction of the following sentences:

1. $|\phi_i/\psi_i| > 1$.

2. $|\vartheta_j/\xi_j| = 1$.

where $n, m \in \mathbb{N}$, ϕ_i, ψ_i are pp-1-formulae for $0 < i \leq n$ and ϑ_j, ξ_j are pp-1-formulae for $0 < j \leq m$.

We may now assume that (1) contains at most one sentence, otherwise it is enough to find a V-module M_i for each $0 < i \leq n$ which satisfies $|\phi_i/\psi_i| > 1$ and $|\vartheta_j/\xi_j| = 1$ for all $0 < j \leq m$ as then $\bigoplus_i M_i$ satisfies $|\phi_i/\psi_i| > 1$ and $|\vartheta_j/\xi_j| = 1$ for all $0 < i \leq n$ and $0 < j \leq m$.

Recall that every module is elementary equivalent to a direct sum of indecomposable pure-injective modules. Suppose $M = \bigoplus_k N_k$ is a direct sum of indecomposable pure-injectives. Then, for any pp-pair ϕ/ψ, $|\phi(M)/\psi(M)| > 1$ if and only if there exists a k such that $|\phi(N_k)/\psi(N_k)| > 1$. For, any pp-pair ϕ/ψ, $|\phi(M)/\psi(M)| = 1$ if and only if for all k, $|\phi(N_k)/\psi(N_k)| = 1$. Therefore if there exists a module satisfying $|\phi_1/\psi_1| > 1$ and $|\vartheta_j/\xi_j| = 1$ for all $0 < j \le m$ then there exists an indecomposable pure-injective module satisfying $|\phi_1/\psi_1| > 1$ and $|\vartheta_j/\xi_j| = 1$. Hence this becomes the question of whether:

$$(\phi_1/\psi_1) \subseteq \bigcup_{j=1}^{m} (\vartheta_j/\xi_j).$$

By lemma 4.2.15 we can effectively answer this question.

The other direction is by lemma 4.1.3.

□

4.4 Valuation domains with finite residue field

In this section we prove theorem 4.0.1 for the case of valuation domains with finite residue field and dense value group.

In this section we will describe exactly the indecomposable pure-injective modules N for which there are pp-formulae ϕ, ψ such that $\left|\frac{\phi(N)}{\psi(N)}\right|$ is finite and not equal to 1 (lemma 4.4.14). We will then go on to show that for such modules N, given pp-formulae ϕ, ψ we can effective calculate $\left|\frac{\phi(N)}{\psi(N)}\right|$ (corollaries 4.4.16 and 4.4.18).

The main tool used in this section is that every irreducible pp-1-type over a valuation domain is realised in a uniserial module (lemma 4.4.4) and thus, since the pure-injective hull of a uniserial module is indecomposable ([EH95]), every indecomposable pure-injective module is elementary equivalent to a uniserial module. Because uniserial modules are in general much simpler than indecomposable pure-injective modules, this allows us, given pp-formulae ϕ, ψ to effectively calculate $\left|\frac{\phi(M)}{\psi(M)}\right|$ when M is uniserial.

Lemma 4.4.1. *Let V be a valuation domain. Suppose $u, v, s, t \in \mathfrak{m}\backslash\{0\}$. Then*

$(uV, vV) \sim (sV, tV)$ *if and only if* $uvV = stV$.

Proof. $\Leftarrow$ Suppose $uvV = stV$. Without loss of generality we may assume $uV \subsetneq sV$. Then there exists $\mu \in \mathfrak{m}$ such that $u = s\mu$. Therefore $(uV : \mu) = sV$ and $uv\mu V = st\mu V = utV$. Hence $v\mu V = tV$. Therefore $(sV, tV) = ((uV : \mu), v\mu V) \sim (uV, vV)$.

$\Rightarrow$ We have noted (paragraph below lemma 3.3.5) that for any ideals $I, J, K, L \lhd V$, $(I, J) \sim (K, L)$ implies $IJ = KL$. $\square$

Lemma 4.4.2. *Let V be a valuation domain. Suppose $u, v, s, t \in V\backslash\{0\}$. Then $(u\mathfrak{m}, v\mathfrak{m}) \sim (s\mathfrak{m}, t\mathfrak{m})$ if and only if $uv\mathfrak{m}^2 = st\mathfrak{m}^2$ if and only if $uv\mathfrak{m} = st\mathfrak{m}$.*

Proof. Suppose $uv\mathfrak{m}^2 = st\mathfrak{m}^2$. Then either $\mathfrak{m}^2 = \mathfrak{m}$, so $uv\mathfrak{m} = st\mathfrak{m}$ or $\mathfrak{m}$ is finitely generated by k. If $\mathfrak{m}$ is finitely generated by k then $uvk\mathfrak{m} = uv\mathfrak{m}^2 = st\mathfrak{m}^2 = stk\mathfrak{m}$. Hence $uv\mathfrak{m} = st\mathfrak{m}$. So $uv\mathfrak{m}^2 = st\mathfrak{m}^2$ if and only if $uv\mathfrak{m} = st\mathfrak{m}$.

Suppose $uv\mathfrak{m} = st\mathfrak{m}$. Then $(s\mathfrak{m}, t\mathfrak{m}) \sim (\mathfrak{m}, st\mathfrak{m})$ and $(u\mathfrak{m}, v\mathfrak{m}) \sim (\mathfrak{m}, uv\mathfrak{m})$. Hence $(u\mathfrak{m}, v\mathfrak{m}) \sim (s\mathfrak{m}, t\mathfrak{m})$. $\square$

Definition 4.4.3. *Let V be a valuation domain, Q its quotient field and suppose $J \lhd V$. We define*

$$[\mathfrak{m} : J] := \{x \in Q \mid Jx \subseteq \mathfrak{m}\}.$$

Note that $[\mathfrak{m} : J]$ is a V-submodule of Q.

It is noted in [Zie84] that every indecomposable pure-injective module over a valuation domain is the pure-injective hull of a uniserial module. Hence every irreducible pp-type is realised in a uniserial module. The following lemma explicitly gives a uniserial module realising $p(I, J)$ for each $I, J \lhd V$.

Lemma 4.4.4. *Let $I, J \lhd V$ The pp-type $p(I, J)$ is realised in the following uniserial module:*

$$\frac{[\mathfrak{m} : J]}{I}.$$

Proof. The quotient field Q of V is uniserial as a V-module. Hence $[\mathfrak{m} : J]$ is uniserial. Therefore $\frac{[\mathfrak{m}:J]}{I}$ is uniserial. Let a be the image of 1 in $\frac{[\mathfrak{m}:J]}{I}$. Then for all $r \in V$, $ar = 0$ if and only if $r \in I$. Suppose that $r \in V$ and $r|a$. Then there exists $y \in [\mathfrak{m} : J]$ such that $yr - 1 \in I$. Therefore $1 \in \mathfrak{m} + [\mathfrak{m} : J]r$. Hence $1 \in [\mathfrak{m} : J]r$, so $1 = xr$ for some $x \in [\mathfrak{m} : J]$. So $J = Jxr \subseteq r\mathfrak{m}$. Therefore, by lemma 3.1.2, $r \notin J$. □

The following definition extends the notion of attached prime for ideals to arbitrary proper non-zero V-submodules of Q the quotient field of V (i.e. fractional ideals).

Definition 4.4.5. *Let I be a proper non-zero submodule of Q the quotient field of V. The attached prime $I^{\#}$ of I is the set of $r \in V$ such that $Ir \subsetneq I$. Note that as in the case of ideals $I^{\#}$ is a prime ideal in V.*

Lemma 4.4.6. *Let $J \lhd V$, $b, x \in V$. Then $[b\mathfrak{m} : J]x = [bx\mathfrak{m} : J]$.*

Proof. Suppose $t \in [b\mathfrak{m} : J]x$. Then $t = \gamma x$ for some $\gamma \in [b\mathfrak{m} : J]$. Hence $tJ = \gamma xJ \subseteq bx\mathfrak{m}$. Suppose $t \in [bx\mathfrak{m} : J]$. Then $tJ \subseteq bx\mathfrak{m}$, so $t/xJ \subseteq b\mathfrak{m}$. Hence $t \in [b\mathfrak{m} : J]x$. □

Proposition 4.4.7. *Let $J \lhd V$ and $x \in Q$ non-zero. Then $[\mathfrak{m} : J] = x\mathfrak{m}$ implies $J = (1/x)V$.*

Proof. First we show that $1/x \in J$. Suppose $1/x \notin J$. Since Q is uniserial $(1/x)\mathfrak{m} \supseteq J$, so $\mathfrak{m} \supseteq Jx$. A contradiction since $x \notin [\mathfrak{m} : J]$.

Suppose $y \in J$. Then $xy\mathfrak{m} \subseteq \mathfrak{m}$ so $xy \in V$. Therefore $y \in (1/x)V$.

□

Proposition 4.4.8. *Let $J \lhd V$ and $x \in Q$ non-zero. Then $[\mathfrak{m} : J] = xV$ implies $J = (1/x)\mathfrak{m}$.*

Proof. Since $x \in [\mathfrak{m} : J]$, $xJ \subseteq \mathfrak{m}$. So $J \subseteq (1/x)\mathfrak{m}$.

Suppose, for a contradiction, that $t \in \mathfrak{m}$ and $t/x \notin J$. Then $(t/x)\mathfrak{m} \supseteq J$. Hence $\mathfrak{m} \supseteq (x/t)J$. So $x/t \in [\mathfrak{m} : J] = xV$. Hence $t \notin \mathfrak{m}$, a contradiction. Therefore $J = (1/x)\mathfrak{m}$.

□

Lemma 4.4.9. *Let $J \lhd V$, $b \in V$ and $x \in J^{\#}$. Then $[bx\mathfrak{m} : J] \subsetneq [b\mathfrak{m} : J]$.*

Proof. Since $x \in J^{\#}$, $J \supsetneq Jx$. Take $a \in J \backslash Jx$. Then, since $a \notin Jx$, $a\mathfrak{m} \supseteq Jx$. Therefore $\mathfrak{m} \supseteq J(x/a)$. So $b\mathfrak{m} \supseteq J(bx/a)$. Hence $bx/a \in [b\mathfrak{m} : J]$. Since $a \in J$, $bx \in (bx/a)J$. Therefore $bx\mathfrak{m} \subsetneq (bx/a)J$. Hence $bx/a \notin [bx\mathfrak{m} : J]$.

So since Q is uniserial, $[bx\mathfrak{m} : J] \subsetneq [b\mathfrak{m} : J]$. □

Proposition 4.4.10. *Let V be a valuation domain, Q the quotient of V, $a \in V$ and $I \supseteq J$ V-submodules of Q. Then*

$$\frac{I}{J} \cong \frac{Ia}{Ja}.$$

Proof. Let $f : \frac{I}{J} \rightarrow \frac{Ia}{Ja}$ be the map induced by multiplication by a. The map f is well defined since if $x \in J$ then $xa \in Ja$ and a homomorphism since V is commutative. The map is injective since if $xa \in Ja$ then $x \in J$. The map is clearly surjective. Hence f is an isomorphism. □

Lemma 4.4.11. *Let V be a valuation domain with finite residue field consisting of q elements and dense value group. Then, up to isomorphism, $V/\mathfrak{m}$ is the only finite non-zero uniserial V-module. Moreover, any non-zero V-module of finite size is of size q^n for some $n \in \mathbb{N}$.*

Proof. First note that $V/\mathfrak{m}$ is the only finite non-zero cyclic V-module since V has dense value group. Suppose M is a finite non-zero uniserial module. Then all cyclic submodules of M are isomorphic to $V/\mathfrak{m}$. Since $V/\mathfrak{m}$ is simple and M is uniserial, M is isomorphic to $V/\mathfrak{m}$, i.e. if M has two non-zero cyclic submodules N_1 and N_2 then either $N_1 \supseteq N_2$ or $N_2 \supseteq N_1$ but since both N_1 and N_2 are simple $N_1 = N_2$.

We now prove the second claim. Suppose M is a non-zero V-module of finite size. Then, since $V/\mathfrak{m}$ is the only finite non-zero cyclic V-module, every cyclic submodule of M is isomorphic to $V/\mathfrak{m}$. Since $V/\mathfrak{m}$ is simple and M is finite, we can pick pairwise non-equal $m_1, ..., m_n \in M\backslash\{0\}$ such that for each $0 < i, j \leq n$ $m_iV \cap m_jV = \emptyset$ unless $i = j$ and $m_1V + ... + m_nV = M$. For each $0 < i \leq n$, $|m_iV| = q$. Therefore $|M| = q^n$. □

Lemma 4.4.12. *Let V be a valuation domain with dense value group and finite residue field of size q. Then for all pp-1-formulae ϕ, ψ and all indecomposable pure-injective modules N, $\left|\frac{\phi(N)}{\psi(N)}\right|$ is either 1, q or infinite.*

Proof. Suppose ϕ, ψ are pp-1-formulae and N an indecomposable pure-injective module. By lemma 4.4.4 and comments just before, there is a uniserial module M elementary equivalent to N. Hence, if $\left|\frac{\phi(N)}{\psi(N)}\right|$ is of finite size greater than q then $\left|\frac{\phi(M)}{\psi(M)}\right|$ is of finite size greater than q. But, since M uniserial, $\frac{\phi(M)}{\psi(M)}$ is uniserial. This can only have finite size 1 or q by lemma 4.4.11. $\square$

Lemma 4.4.13. *Let V be a valuation domain and finite residue field consisting of q elements. Let ϕ be the pp-fomula $(xag = 0 \wedge b|x)$ and let ψ be the pp-formula $(xa = 0 + bh|x)$ where $a, b \in V\backslash\{0\}$ and $g, h \in \mathfrak{m}$. Then for any $I, J \lhd V$, if $a \notin I$, $ag \in I$, $b \notin J$ and $bh \in J$ then*

$$\left|\frac{\phi(N(I,J))}{\psi(N(I,J))}\right| = \ min \ \left\{\left|\frac{[I:ag]}{[I:a]}\right| \ , \ \left|\frac{[I:ag]}{[bh\mathfrak{m}:J]}\right| \ , \ \left|\frac{[b\mathfrak{m}:J]}{[I:a]}\right| \ , \ \left|\frac{[b\mathfrak{m}:J]}{[bh\mathfrak{m}:J]}\right|\right\}.$$

Proof. By lemma 4.4.4 we know that $p(I,J)$ is realised in $\frac{[\mathfrak{m}:J]}{I}$. Since $M = \frac{[\mathfrak{m}:J]}{I}$ is uniserial, its pure-injective hull is indecomposable (see [EH95, propostion 4.1]) and is therefore isomorphic to $N(I,J)$. Recall that a module is elementary equivalent to its pure-injective hull. Hence $\phi(M)/\psi(M)$ is finite if and only if $\phi(N(I,J))/\psi(N(I,J))$ is finite and in this situation

$$|\phi(M)/\psi(M)| = |\phi(N(I,J))/\psi(N(I,J))|\,.$$

<u>Claim:</u> The solution set of ϕ in M is

$$\frac{[I:ag] \cap [b\mathfrak{m}:J]}{I}.$$

Take $x \in [\mathfrak{m}:J]$. Let x' be the image of x in $[\mathfrak{m}:J]/I$. For any $v \in V$, $x'v = 0$ if and only if $xv \in I$. So $x'ag = 0$ if and only if $x \in [I:ag]$. For any $v \in V$, $v|x'$ if and only if there exists $y \in [\mathfrak{m}:J]$ such that $yv - x \in I$ if and only if $x \in [\mathfrak{m}:J].v + I = [v\mathfrak{m}:J] + I$. Since $b \notin J$, $[b\mathfrak{m}:J] \supseteq V \supseteq I$, so $[b\mathfrak{m}:J] + I = [b\mathfrak{m}:J]$. So $b|x'$ if and only if $x \in [b\mathfrak{m}:J]$. Hence we have proved the claim.

<u>Claim:</u> The solution set of ψ in M is

$$\frac{[I:a] \cup [bh\mathfrak{m}:J]}{I}.$$

As in previous claim, for any $x \in [\mathfrak{m} : J]$ with image x' in $[\mathfrak{m} : J]/I$, $x'a = 0$ if and only if $x \in [I : a]$ and $bh|x'$ if and only if $x \in [bh\mathfrak{m} : J]$. Since V is commutative pp-definable subgroups are submodules. As M is uniserial, the solution set of ψ is

$$\frac{[I : a] \cup [bh\mathfrak{m} : J]}{I}.$$

Hence, since M is uniserial,

$$\left|\frac{\phi(M)}{\psi(M)}\right| = \min \left\{ \left|\frac{[I : ag]}{[I : a]}\right| , \left|\frac{[I : ag]}{[bh\mathfrak{m} : J]}\right| , \left|\frac{[b\mathfrak{m} : J]}{[bh\mathfrak{m} : J]}\right| , \left|\frac{[b\mathfrak{m} : J]}{[I : a]}\right| \right\}.$$

□

Lemma 4.4.14. *Suppose V is a valuation domain with dense value group such that the residue field of V consists of q elements. Let ϕ be the pp-1-formula $(xag = 0) \wedge (b|x)$ and let ψ be the pp-1-formula $(xa = 0) + (bh|x)$ where $a, b \in V \backslash \{0\}$ and $g, h \in \mathfrak{m}$. Then for any ideals $I, J \lhd V$ the following are equivalent:*

(i) $\left|\frac{\phi(N(I,J))}{\psi(N(I,J))}\right|$ *is finite and not equal to 1.*

(ii) $(I, J) \sim (agV, bhV)$ *or* $(I, J) \sim (a\mathfrak{m}, b\mathfrak{m})$.

Proof. (i)⇒(ii) Suppose $\left|\frac{\phi(N(I,J))}{\psi(N(I,J))}\right|$ is finite and not equal to 1. Then $N(I, J) \in \left(\frac{\phi}{\psi}\right)$ so $(I, J) \in \mathcal{W}_{a,b,g,h}$. Therefore there exists $K, L \lhd V$ such that $(K, L) \sim (I, J)$ and $a \notin K$, $b \notin L$, $ag \in K$ and $bh \in L$. Note that, since $(I, J) \sim (K, L)$, $N(I, J) = N(K, L)$. Hence, by lemma 4.4.13,

$$\left|\frac{\phi(N(I, J))}{\psi(N(I, J))}\right| = \min \left\{ \left|\frac{[K : ag]}{[K : a]}\right| , \left|\frac{[K : ag]}{[bh\mathfrak{m} : L]}\right| , \left|\frac{[b\mathfrak{m} : L]}{[bh\mathfrak{m} : L]}\right| , \left|\frac{[b\mathfrak{m} : L]}{[K : a]}\right| \right\}.$$

Thus we must consider when $\left|\frac{[K:ag]}{[K:a]}\right|$, $\left|\frac{[b\mathfrak{m}:L]}{[bh\mathfrak{m}:L]}\right|$, $\left|\frac{[K:ag]}{[bh\mathfrak{m}:L]}\right|$ and $\left|\frac{[b\mathfrak{m}:L]}{[K:a]}\right|$ are finite and not equal to 1.

By lemma 4.4.10

$$\frac{[K : ag]}{[K : a]} \cong \frac{K}{Kg}.$$

So by lemma 4.4.11 it is either infinite or 1.

By lemma 4.4.6

$$\frac{[b\mathfrak{m} : L]}{[bh\mathfrak{m} : L]} = \frac{[b\mathfrak{m} : L]}{[b\mathfrak{m} : L]h}.$$

So by lemma 4.4.11 it is either infinite or 1.

Suppose $\left|\frac{[K:ag]}{[bh\mathfrak{m}:L]}\right|$ is finite and not equal to 1. Then

$$\frac{[K:ag]}{[bh\mathfrak{m}:L]} \cong \frac{V}{\mathfrak{m}}.$$

Hence $[K:ag] = \gamma V$ for some $\gamma \in Q\backslash\{0\}$, so $K = \gamma agV$. Therefore $[bh\mathfrak{m}:L] = \gamma ag\mathfrak{m}$, so $L = 1/\gamma bhV$, by lemma 4.4.7. Hence $(I,J) \sim (K,L) \sim (agV, bhV)$.

Suppose $\left|\frac{[b\mathfrak{m}:L]}{[K:a]}\right|$ is finite and not equal to 1. Then

$$\frac{[b\mathfrak{m}:L]}{[K:a]} \cong \frac{V}{\mathfrak{m}}.$$

Hence $[b\mathfrak{m}:L] = \gamma V$ for some $\gamma \in Q\backslash\{0\}$ so $L = b/\gamma\mathfrak{m}$, by lemma 4.4.8. Therefore $[K:a] = \gamma\mathfrak{m}$, so $K = \gamma a\mathfrak{m}$. Hence $(I,J) \sim (K,L) \sim (a\mathfrak{m}, b\mathfrak{m})$.

(ii)⇒(i) We may assume $I = agV$ and $J = bhV$ or $I = a\mathfrak{m}$ and $J = b\mathfrak{m}$ since if $(I,J) \sim (K,L)$ then $N(I,J) \cong N(K,L)$.

As in lemma 4.4.13 (first paragraph), we need only consider $\frac{\phi(M)}{\psi(M)}$ for M a uniserial module realising $p(I,J)$.

Suppose $I = agV$ and $J = bhV$. Then $M = \frac{[\mathfrak{m}:J]}{I} \cong \frac{\mathfrak{m}}{abghV}$ realises $p(I,J)$.

The solution set of the formula $b|x$ in $\frac{\mathfrak{m}}{abghV}$ is $\frac{b\mathfrak{m}}{abghV}$ and the solution set of the formula $xag = 0$ in $\frac{\mathfrak{m}}{abghV}$ is $\frac{bhV}{abghV}$. So

$$\phi\left(\frac{\mathfrak{m}}{abghV}\right) = \frac{b\mathfrak{m} \cap bhV}{abghV}.$$

Since $bh \in b\mathfrak{m}$,

$$\phi\left(\frac{\mathfrak{m}}{abghV}\right) = \frac{bhV}{abghV}.$$

Similarly,

$$\psi\left(\frac{\mathfrak{m}}{abghV}\right) = \frac{bh\mathfrak{m} + bghV}{abghV}.$$

Since $bhg \in bh\mathfrak{m}$,

$$\psi\left(\frac{\mathfrak{m}}{abghV}\right) = \frac{bh\mathfrak{m}}{abghV}.$$

So

$$\frac{\phi(M)}{\psi(M)} \cong \frac{bhV}{bh\mathfrak{m}} \cong \frac{V}{\mathfrak{m}}.$$

Suppose $I = a\mathfrak{m}$ and $J = b\mathfrak{m}$. Then $M = \frac{[\mathfrak{m}:J]}{I} = \frac{1/bV}{a\mathfrak{m}} \cong \frac{V}{ab\mathfrak{m}}$ realises $p(I,J)$.

The solution set of the formula $xag = 0$ in $\frac{V}{ab\mathfrak{m}}$ is $\frac{(ab\mathfrak{m}:ag)}{ab\mathfrak{m}}$ and the solution set of the formula $b|x$ in $\frac{V}{ab\mathfrak{m}}$ is $\frac{bV}{ab\mathfrak{m}}$. So

$$\phi\left(\frac{V}{ab\mathfrak{m}}\right) = \frac{(ab\mathfrak{m} : ag) \cap bV}{ab\mathfrak{m}}.$$

As $g \in \mathfrak{m}$, $b \in (ab\mathfrak{m} : ag)$. So

$$\phi\left(\frac{V}{ab\mathfrak{m}}\right) = \frac{bV}{ab\mathfrak{m}}.$$

Similarly,

$$\psi\left(\frac{V}{ab\mathfrak{m}}\right) = \frac{b\mathfrak{m} + bhV}{ab\mathfrak{m}}.$$

Since $bh \in b\mathfrak{m}$,

$$\psi\left(\frac{V}{ab\mathfrak{m}}\right) = \frac{b\mathfrak{m}}{ab\mathfrak{m}}.$$

So

$$\frac{\phi(M)}{\psi(M)} \cong \frac{bV}{b\mathfrak{m}} \cong \frac{V}{\mathfrak{m}}.$$

□

Lemma 4.4.15. *Let V be a valuation domain with dense value group and finite residue field consisting of q elements. Let $I = tV$ and $J = sV$ for some $t, s \in \mathfrak{m}$, $N = N(I, J)$, let ϕ be the pp-1-formula $xag = 0 \wedge b|x$ and let ψ be the pp-1-formula $xa = 0 + bh|x$ where $a, b \in V\backslash\{0\}$ and $g, h \in \mathfrak{m}$. Then the following holds:*

(i) $\left|\frac{\phi(N)}{\psi(N)}\right| = 1$ *if and only if* $ab \in tsV$ *or* $abgh \notin tsV$.

(ii) $\left|\frac{\phi(N)}{\psi(N)}\right| = q$ *if and only if* $abghV = stV$.

(iii) $\left|\frac{\phi(N)}{\psi(N)}\right| = \infty$ *if and only if* $ab \notin stV$ *and* $abgh \in st\mathfrak{m}$.

In particular, if V is effectively given, then there exists an algorithm which, given any $t, s \in \mathfrak{m}$ and $\alpha, \beta, \delta, \gamma \in V$, returns the value of $\left|\frac{\phi(N)}{\psi(N)}\right|$ where ϕ is $x\alpha = 0 \wedge \beta|x$ and ψ is $x\gamma = 0 + \delta|x$ and $N = N(tV, sV)$.

Proof. (i) Since $a, b \in V\backslash\{0\}$ and $g, h \in \mathfrak{m}$, $(\frac{\phi}{\psi})$ is the basic open set $\mathcal{W}_{a,b,g,h}$. So $\left|\frac{\phi(N)}{\psi(N)}\right| = 1$ if and only if $(tV, sV) \notin \mathcal{W}_{a,b,g,h}$. The point (tV, sV) is a normal point since $\mathfrak{m}^2 = \mathfrak{m}$. So, for all $\lambda \in V\backslash\{0\}$, $(tV, sV) \in \mathcal{W}_{1,\lambda,0,0}$ if and only if $\lambda \notin tsV$.

Therefore, by proposition 3.5.6, $(tV, sV) \in \mathcal{W}_{a,b,g,h}$ if and only if $ab \notin tsV$ and $abgh \in tsV$. Thus, $\left|\frac{\phi(N)}{\psi(N)}\right| = 1$ if and only if $ab \in tsV$ or $abgh \notin tsV$.

(ii) By lemma 4.4.14 $\left|\frac{\phi(N)}{\psi(N)}\right|$ is finite and not equal to 1 if and only if $(agV, bhV) \sim (tV, sV)$ or $(a\mathfrak{m}, b\mathfrak{m}) \sim (tV, sV)$. For any pairs of ideals (I, J) and (K, L), $(I, J) \sim (K, L)$ implies $IJ = KL$. Therefore, since $\mathfrak{m}$ is not finitely generated, it is not possible that $(a\mathfrak{m}, b\mathfrak{m}) \sim (tV, sV)$. By lemma 4.4.1, $stV = abghV$ implies $(tV, sV) \sim (agV, bhV)$. Hence $\left|\frac{\phi(N)}{\psi(N)}\right| = q$ if and only if $abghV = stV$.

(iii) For any indecomposable pure-injective module N and any pair of pp-1-formulae ϕ, ψ, $\left|\frac{\phi(N)}{\psi(N)}\right|$ is either 1, q or infinite. Therefore it is enough to note that $ab \notin tsV$, $abgh \in tsV$ and $abghV \neq stV$ if and only if $ab \notin stV$ and $abgh \in st\mathfrak{m}$.

It remains to consider the final claim that if V is effectively given, then there exists an algorithm which, given any $t, s \in \mathfrak{m}$ and $\alpha, \beta, \delta, \gamma \in V$, returns the value of $\left|\frac{\phi(N)}{\psi(N)}\right|$ where ϕ is $x\alpha = 0 \wedge \beta | x$ and ψ is $x\gamma = 0 + \delta | x$ and $N = N(tV, sV)$.

Suppose V is effectively given. First note that if $\alpha \notin \gamma\mathfrak{m}$, $\delta \notin \beta\mathfrak{m}$, $\beta = 0$ or $\gamma = 0$ then $\left|\frac{\phi(M)}{\psi(M)}\right| = 1$ for all V-modules M. Since V is effectively given, we can effectively check whether $\alpha \notin \gamma\mathfrak{m}$, $\delta \notin \beta\mathfrak{m}$, $\beta = 0$ or $\gamma = 0$. Otherwise, let $a = \gamma$, $b = \beta$, $g = \alpha/\gamma$ and $h = \delta/\beta$. Hence ϕ is $xag = 0 \wedge b|x$ and ψ is $xa = 0 + bh|x$. Therefore it is enough to note that for any $r, s \in V$ we can effectively check whether $r \in sV$ and $r \in s\mathfrak{m}$. $\square$

Corollary 4.4.16. *Let ϕ, ψ be pp-1-formulae and $I = tV$, $J = sV$ for some $t, s \in \mathfrak{m}$. Then we can effectively calculate the value of $\left|\frac{\phi(N(I,J))}{\psi(N(I,J))}\right|$.*

Lemma 4.4.17. *Let V be a valuation domain with dense value group and finite residue field consisting of q elements. Let $I = t\mathfrak{m}$ and $J = s\mathfrak{m}$ for some $t, s \in V\backslash\{0\}$, $N = N(I, J)$, ϕ be the pp-1-formula $xag = 0 \wedge b|x$ and let ψ be the pp-1-formula $xa = 0 + bh|x$ where $a, b \in V\backslash\{0\}$ and $g, h \in \mathfrak{m}$. Then the following holds:*

(i) $\left|\frac{\phi(N)}{\psi(N)}\right| = 1$ *if and only if* $ab \in ts\mathfrak{m}$ *or* $abgh \notin ts\mathfrak{m}$.

(ii) $\left|\frac{\phi(N)}{\psi(N)}\right| = q$ *if and only if* $ab\mathfrak{m} = st\mathfrak{m}$.

(iii) $\left|\frac{\phi(N)}{\psi(N)}\right| = \infty$ *if and only if* $ab \notin stV$ *and* $abgh \in st\mathfrak{m}$.

In particular, if V is effectively given, then there exists an algorithm which, given any $t, s \in V\backslash\{0\}$ and $\alpha, \beta, \delta, \gamma \in V$, returns the value of $\left|\frac{\phi(N)}{\psi(N)}\right|$ where ϕ is $x\alpha = 0 \wedge \beta|x$ and ψ is $x\gamma = 0 + \delta|x$ and $N = N(t\mathfrak{m}, s\mathfrak{m})$.

Proof. (i) Since $a, b \in V\backslash\{0\}$ and $g, h \in \mathfrak{m}$, $(\frac{\phi}{\psi})$ is the basic open set $\mathcal{W}_{a,b,g,h}$. So $\left|\frac{\phi(N)}{\psi(N)}\right| = 1$ if and only if $(t\mathfrak{m}, s\mathfrak{m}) \notin \mathcal{W}_{a,b,g,h}$. The point $(t\mathfrak{m}, s\mathfrak{m})$ is a normal point since $\mathfrak{m}^2 = \mathfrak{m}$. Therefore $(t\mathfrak{m}, s\mathfrak{m}) \in \mathcal{W}_{a,b,g,h}$ if and only if $ab \notin ts\mathfrak{m}$ and $abgh \in ts\mathfrak{m}$. Thus, $\left|\frac{\phi(N)}{\psi(N)}\right| = 1$ if and only if $ab \in ts\mathfrak{m}$ or $abgh \notin ts\mathfrak{m}$.

(ii) By lemma 4.4.14 $\left|\frac{\phi(N)}{\psi(N)}\right|$ is finite and not equal to 1 if and only if $(agV, bhV) \sim (t\mathfrak{m}, s\mathfrak{m})$ or $(a\mathfrak{m}, b\mathfrak{m}) \sim (t\mathfrak{m}, s\mathfrak{m})$. For any pairs of ideals (I, J) and (K, L), $(I, J) \sim (K, L)$ implies $IJ = KL$. Therefore, since $\mathfrak{m}$ is not finitely generated, it is not possible that $(agV, bhV) \sim (t\mathfrak{m}, s\mathfrak{m})$. By lemma 4.4.2, $(t\mathfrak{m}, s\mathfrak{m}) \sim (a\mathfrak{m}, b\mathfrak{m})$ if and only if $ts\mathfrak{m} = ab\mathfrak{m}$.

(iii) For any indecomposable pure-injective module N and any pair of pp-1-formulae ϕ, ψ, $\left|\frac{\phi(N)}{\psi(N)}\right|$ is either 1, q or infinite. Therefore it is enough to note that $ab \notin ts\mathfrak{m}$, $abgh \in ts\mathfrak{m}$ and $abgh\mathfrak{m} \neq st\mathfrak{m}$ if and only if $ab \notin stV$ and $abgh \in st\mathfrak{m}$.

It remains to consider the final claim that if V is effectively given, then there exists an algorithm which, given any $t, s \in V\backslash\{0\}$ and $\alpha, \beta, \gamma, \delta \in V$, returns the value of $\left|\frac{\phi(N)}{\psi(N)}\right|$ where ϕ is $x\alpha = 0 \wedge \beta|x$ and ψ is $x\gamma = 0 + \delta|x$ and $N = N(t\mathfrak{m}, s\mathfrak{m})$.

Suppose V is effectively given. First note that if $\alpha \notin \gamma\mathfrak{m}$, $\delta \notin \beta\mathfrak{m}$, $\beta = 0$ or $\gamma = 0$ then $\left|\frac{\phi(M)}{\psi(M)}\right| = 1$ for all V-modules M. Since V is effectively given, we can effectively check whether $\alpha \notin \gamma\mathfrak{m}$, $\delta \notin \beta\mathfrak{m}$, $\beta = 0$ or $\gamma = 0$. Otherwise, let $a = \gamma$, $b = \beta$, $g = \alpha/\gamma$ and $h = \delta/\beta$. Hence ϕ is $xag = 0 \wedge b|x$ and ψ is $xa = 0 + bh|x$. Therefore it is enough to note that for any $r, s \in V$ we can effectively check whether $r \in sV$ and $r \in s\mathfrak{m}$. $\square$

Corollary 4.4.18. *Let ϕ, ψ be pp-1-formulae and $I = t\mathfrak{m}$, $J = s\mathfrak{m}$ for some $t, s \in V\backslash\{0\}$. Then we can effectively calculate the value of $\left|\frac{\phi(N(I,J))}{\psi(N(I,J))}\right|$.*

Lemma 4.4.19. *Let V be an effectively given valuation domain with dense value group and finite residue field consisting of q elements. There is an algorithm which,* *given pp-1-formulae ϕ, ψ, gives a finite list L of pairs of ideals (I, J) such that $|\phi(N)/\psi(N)| = q$ if and only if $N = N(I, J)$ for some $(I, J) \in L$.*

Proof. A priori, it is not clear that one can even explicitly write down pairs of ideals (I, J) such that $|\phi(N)/\psi(N)| = q$ where $N = N(I, J)$. Therefore first we must show, using 4.4.14, that if (I, J) is such that $|\phi(N)/\psi(N)| = q$ where $N = N(I, J)$ then $(I, J) = (aV, bV)$ for some $a, b \in \mathfrak{m}\backslash\{0\}$ or $(I, J) = (a\mathfrak{m}, b\mathfrak{m})$ for some $a, \in V\backslash\{0\}$. First, rewrite ϕ as $\sum_{i=1}^{n}(x\alpha_i = 0 \wedge \beta_i|x)$ and ψ as $\bigwedge_{j=1}^{m}(x\gamma_j = 0 + \delta_j|x)$ for some $\alpha_i, \beta_i, \gamma_i, \delta_i \in V$. Let ϕ_i be $(x\alpha_i = 0 \wedge \beta_i|x)$ and ψ_j be $(x\gamma_j = 0 + \delta_j|x)$. We can do this effectively by lemmas 4.2.1 and 4.2.2. Then for any indecomposable pure-injective N,

$$\left|\frac{\phi(N)}{\psi(N)}\right| = \max \left\{\left|\frac{\phi_i(N)}{\psi_j(N)}\right|\right\}.$$

We know that $\phi_i(N)/\psi_j(N)$ is the zero module for all indecomposable pure-injectives if $\alpha_i \notin \gamma_j\mathfrak{m}$, $\delta_j \notin \beta_i\mathfrak{m}$, $\beta_i = 0$ or $\gamma_j = 0$. If for all i, j and all indecomposable pure-injectives N $\phi_i(N)/\psi_j(N)$ is zero then there is no N indecomposable pure-injective such that $\phi(N)/\psi(N)$ is non-zero.

By 4.4.14 for any pp-1-formula ϕ of the form $xag = 0 \wedge b|x$ and ψ of the form $xa = 0 + bh|x$ where $a, b \in V\backslash\{0\}$ and $g, h \in \mathfrak{m}$, the only indecomposable pure-injective modules N such that $\left|\frac{\phi(N)}{\psi(N)}\right| = q$ are $N = N(agV, bhV)$ and $N = N(a\mathfrak{m}, b\mathfrak{m})$. If $\alpha_i \in \gamma_j\mathfrak{m}$, $\delta_j \in \beta_i\mathfrak{m}$, $\beta_i \neq 0$ and $\gamma_j \neq 0$, let $a_{i,j} = \gamma_j$, $b_{i,j} = \beta_i$, $g_{i,j} = \alpha_i/\gamma_j$ and $h_{i,j} = \delta_j/\beta_i$. So we need only consider the pure-injective modules $N(a_{i,j}\mathfrak{m}, b_{i,j}\mathfrak{m})$ and $N(a_{i,j}g_{i,j}V, b_{i,j}h_{i,j}V)$ where $a_{i,j}, b_{i,j}, g_{i,j}$ and $h_{i,j}$ are defined.

By lemmas 4.4.15 and 4.4.17, for any pp-1-formula ϕ of the form $x\alpha = 0 \wedge \beta|x$ and ψ of the form $x\gamma = 0 + \delta|x$ and any indecomposable pure-injective $N = N(a\mathfrak{m}, b\mathfrak{m})$ or $N = N(tV, sV)$ where $a, b \in V$ and $t, s \in \mathfrak{m}$ we can effectively calculate the size of $\left|\frac{\phi(N)}{\psi(N)}\right|$.

Hence we can make a finite list of indecomposable pure-injective modules N such that $\frac{\phi(N)}{\psi(N)}$ is finite and not of size 1. $\square$

Lemma 4.4.20. *Let R be a ring and χ a boolean combination of invariants sentences* *and negations of invariants sentences. If there exists an R-module M which satisfies χ then there exists a finite direct sum of pure-injective indecomposable modules satisfying χ.*

Proof. Without loss of generality we may assume that χ is the conjunction of the following invariants sentences:

$$\left|\frac{\phi_i^1}{\psi_i^1}\right| = v_i \tag{1}$$

$$\left|\frac{\phi_j^2}{\psi_j^2}\right| \geq w_j \tag{2}$$

$$\left|\frac{\phi_k^3}{\psi_k^3}\right| = 1 \tag{3}$$

where $l, m, n \in \mathbb{N}$ and for all $0 < i \leq l$, $0 < j \leq m$, $0 < k \leq n$, ϕ_i^1, ψ_i^1, ϕ_j^2, ψ_j^2, ϕ_k^3, ψ_k^3 are pp-1-formulae and $v_i, w_j \in \mathbb{N}$. This is because any boolean combination of invariants sentences and negations of invariants sentences is a disjunction of conjunctions of invariants sentences of this form.

Suppose M satisfies χ. We may assume $M = \bigoplus_{\mu \in \mathcal{M}} N_\mu$ since every module is elementary equivalent to a direct sum of pure-injective indecomposable modules. Since $M \models \chi$, for each N_μ and for all $0 < k \leq n$

$$\left|\frac{\phi_k^3(N_\mu)}{\psi_k^3(N_\mu)}\right| = 1.$$

For each $0 < i \leq l$, let Δ_i be the set of $\mu \in \mathcal{M}$ such that

$$\left|\frac{\phi_i^1(N_\mu)}{\psi_i^1(N_\mu)}\right| > 1.$$

Note that for each $0 < i \leq l$, Δ_i is a finite set, since if it had more than v_i elements then $\left|\frac{\phi_i^1(M)}{\psi_i^1(M)}\right| > 2^{v_i} > v_i$.

For each $0 < j \leq m$, let Ω_j be the set of $\mu \in \mathcal{M}$ such that

$$\left|\frac{\phi_j^2(N_\mu)}{\psi_j^2(N_\mu)}\right| > 1.$$

For each $0 < j \leq m$, if Ω_j is not finite replace it by a subset of size w_j. Then Ω_j is finite for all $0 < j \leq m$ and $\bigoplus_{\mu \in \Omega_j} N_\mu$ satisfies $\left|\frac{\phi_j^2}{\psi_j^2}\right| \geq w_j$.

Let $\Lambda = \bigcup_{i=1}^{l} \Delta_i \cup \bigcup_{j=1}^{m} \Omega_j$. Then $\bigoplus_{\mu \in \Lambda} N_\mu$ satisfies χ and Λ is a finite set. $\square$

Theorem 4.4.21. *Let V be an effectively given valuation domain with dense value group and finite residue field consisting of q elements. Then the following are equivalent:*

(i) The theory of V-modules, T_V, is decidable.

(ii) There exists an algorithm which, given $a, b \in V$, answers whether $a \in \mathrm{rad}(bV)$.

Proof. As in theorem 4.3.3 in order to show that T_V is decidable it is enough to show that there exists an algorithm which, given χ a finite conjunction of invariants sentences and negations of invariants sentences, answers whether there is a module M satisfying χ.

Suppose χ is a conjunction of the following sentences:

$$\left|\frac{\phi_i^1}{\psi_i^1}\right| = q^{v_i} \tag{1}$$

$$\left|\frac{\phi_j^2}{\psi_j^2}\right| \geq q^{w_j} \tag{2}$$

$$\left|\frac{\phi_k^3}{\psi_k^3}\right| = 1 \tag{3}$$

where $l, m, n \in \mathbb{N}$ and for all $0 < i \leq l$, $0 < j \leq m$, $0 < k \leq n$, ϕ_i^1, ψ_i^1, ϕ_j^2, ψ_j^2, ϕ_k^3, ψ_k^3 are pp-1-formulae and $v_i, w_j \in \mathbb{N}$.

It is enough to consider χ of this form since for any V-module M and any ϕ, ψ pp-1-formulae, $\left|\frac{\phi(M)}{\psi(M)}\right| = q^v$ for some $v \in \mathbb{N}$ or $\left|\frac{\phi(M)}{\psi(M)}\right|$ is infinite. See lemma 4.4.11.

If τ is a conjunction of invariants sentences like those in (1), (2) and (3) then we call $\sum_{i=1}^l v_i$ the exponent of the statement.

We proceed by induction on $\sum_{i=1}^l v_i$, the exponent of the conjunction of invariants sentences in (1).

Suppose $\sum_{i=1}^l v_i = 0$, that is (1) is empty. Suppose there exists a module M satisfying χ. By lemma 4.4.20 we may assume $M = \bigoplus_{\mu \in \mathcal{M}} N_\mu$, for some finite indexing set $\mathcal{M}$. Therefore for each $0 < j \leq m$, there is $\mu \in \mathcal{M}$ such that

$$\left|\frac{\phi_j^2(N_\mu)}{\psi_j^2(N_\mu)}\right| > 1$$

and for all $\mu \in \mathcal{M}$ and all $0 < k \leq n$,

$$\left|\frac{\phi_k^3(N_\mu)}{\psi_k^3(N_\mu)}\right| = 1.$$

Hence, for each $0 < j \leq m$, there exists N_μ such that $N_\mu \in \left(\frac{\phi_j^2}{\psi_j^2}\right)$ and $N_\mu \notin \left(\frac{\phi_k^3}{\psi_k^3}\right)$ for all $0 < k \leq n$. For each $0 < j \leq m$, let N_j be such a module. Then there exists $t \in \mathbb{N}$ such that $\left(\bigoplus_{j=1}^m N_j\right)^t$ satisfies (2) and (3).

Hence, there exists a module M satisfying (2) and (3) if and only if for all $0 < j \leq m$

$$\left(\frac{\phi_j^2}{\psi_j^2}\right) \not\subseteq \bigcup_{k=1}^n \left(\frac{\phi_k^3}{\psi_k^3}\right).$$

Theorem 4.2.15 asserts that there exists an algorithm to check this, so we are done.

Now suppose $L = \sum_{i=1}^l v_i > 0$, so (1) is not empty and that for any conjunction Θ of invariants sentences and negations of invariants sentences with exponent strictly smaller than L, there is an algorithm which answers whether there exists a module M satisfying Θ.

By lemma 4.4.12, for any indecomposable pure-injective N and any pp-1-formulae ϕ, ψ, $\left|\frac{\phi(N)}{\psi(N)}\right|$ is either 1, q or infinite.

Suppose there exists M satisfying χ. By lemma 4.4.20 we may assume $M = \bigoplus_{\mu \in \mathcal{M}} N_\mu$ where $\mathcal{M}$ is a finite indexing set and each N_μ is an indecomposable pure-injective module. Hence there exists a $\mu \in \mathcal{M}$ such that $\left|\frac{\psi_1^1(N_\mu)}{\phi_1^1(N_\mu)}\right| = q$.

By lemma 4.4.19, we can list all indecomposable pure-injective V-modules $N_1, ..., N_t$ such that

$$\left|\frac{\phi_1^1(N_s)}{\psi_1^1(N_s)}\right| = q.$$

Note that, using 4.4.14, for each module N_s, there either exists $a, b \in V\backslash\{0\}$ such that $N_s \cong N(a\mathfrak{m}, b\mathfrak{m})$ or there exists $a, b \in \mathfrak{m}\backslash\{0\}$ such that $N_s \cong N(aV, bV)$. By lemmas 4.4.16 and 4.4.18, for each N_s we can effectively calculate

$$\left|\frac{\phi_j^2(N_s)}{\psi_j^2(N_s)}\right| \text{ and } \left|\frac{\phi_k^3(N_s)}{\psi_k^3(N_s)}\right|$$

for each $0 < j \leq m$ and $0 < k \leq n$.

For each N_s, if $\left|\frac{\phi_k^3(N_s)}{\psi_k^3(N_s)}\right| \neq 1$, for any $0 < k \leq n$, remove N_s from the list. Likewise, remove N_s from the list, if $\left|\frac{\phi_i^1(N_s)}{\psi_i^1(N_s)}\right|$ is infinite.

If the list is now empty, no module M satisfying χ exists. Otherwise for each module N_s we produce new lists of sentence $(1)^s$, $(2)^s$ and $(3)^s$. For each s start with $(1)^s$ and $(2)^s$ empty, and $(3)^s$ consisting of all sentences in (3).

For each $0 < i \leq l$, if $\left|\frac{\phi_i^1(N_s)}{\psi_i^1(N_s)}\right| = q$ and $v_i > 1$, add the sentence $\left|\frac{\phi_i^1}{\psi_i^1}\right| = q^{v_i - 1}$ to $(1)^s$. If $\left|\frac{\phi_i^1(N_s)}{\psi_i^1(N_s)}\right| = q$ and $v_i \leq 1$, add the sentence $\left|\frac{\phi_i^1}{\psi_i^1}\right| = 1$ to $(3)^s$. If $\left|\frac{\phi_i^1(N_s)}{\psi_i^1(N_s)}\right| = 1$, add $\left|\frac{\phi_i^1}{\psi_i^1}\right| = q^{v_i}$ to $(1)^s$

For each $0 < j \leq m$, if $\left|\frac{\phi_j^2(N_s)}{\psi_j^2(N_s)}\right| = q$ and $w_j > 1$, add the sentence $\left|\frac{\phi_j^2}{\psi_j^2}\right| \geq q^{w_j - 1}$ to $(2)^s$. If $\left|\frac{\phi_j^2(N_s)}{\psi_j^2(N_s)}\right| = 1$, add the sentence $\left|\frac{\phi_j^2}{\psi_j^2}\right| \geq q^{w_j}$ to $(2)^s$.

For each s, if there exists a module M satisfying all sentences in $(1)^s$, $(2)^s$ and $(3)^s$ then $N_s \bigoplus M$ satisfies (1), (2) and (3) and if there exists M satisfying (1), (2) and (3) then there exists an s such that M' satisfies $(1)^s$, $(2)^s$ and $(3)^s$.

Note that for each s, the exponent of the conjunction of conditions in $(1)^s$ is strictly smaller than $\sum_{i=1}^{l} v_i$. Therefore by the induction hypothesis, for each s, there exists an algorithm which answers whether there exists a module M which satisfies $(1)^s$, $(2)^s$ and $(3)^s$.

The other direction is lemma 4.1.3. $\square$

4.5 Valuation domains with finite residue field and non-dense value group

Throughout this section let V be a valuation domain with non-dense value group and finite residue field. Recall that if V has non-dense value group then $\mathfrak{m}$ is finitely generated. Let k be a fixed generator of the maximal ideal $\mathfrak{m}$.

The main work of this section is, given ϕ, ψ pp-1-formulae and $n \in \mathbb{N}\backslash\{0\}$, to effectively determine if there exist $I, J \lhd V$ such that $\left|\frac{\phi(N(I,J))}{\psi(N(I,J))}\right| = n$ (in fact we need to determine if there exist $I, J \lhd V$ satisfying a boolean combination of such sentences).

First we show that for any pp-1-pair ϕ/ψ, if $\left|\frac{\phi(N(I,J))}{\psi(N(I,J))}\right|$ is finite and not equal to 1 then either $I^{\#} = \mathfrak{m}$ or $J^{\#} = \mathfrak{m}$ (lemma 4.5.7). Since $\mathfrak{m}$ is finitely generated, if $I \lhd V$ has attached prime $\mathfrak{m}$ then there exists $r \notin I$ such that $(I : r) = \mathfrak{m}$ (note that this means that $I = r\mathfrak{m}$, so I if finitely generated).

Therefore, if $\left|\frac{\phi(N(I,J))}{\psi(N(I,J))}\right|$ is finite and not equal to 1 for some pp-1-pair ϕ/ψ then $(I, J) \sim (\mathfrak{m}, K)$ or $(I, J) \sim (K, \mathfrak{m})$ for some $K \lhd V$.

The next step is to show that given a sentence of the form $\left|\frac{\phi}{\psi}\right| = 1$, $\left|\frac{\phi}{\psi}\right| \geq n$ or $\left|\frac{\phi}{\psi}\right| = n$ where ϕ/ψ is a pp-1-pair and $n \in \mathbb{N}$, we can effectively produce conditions on an ideal K such that if K satisfies these conditions then $N(\mathfrak{m}, K)$ satisfies the sentence and we can effectively produce conditions on an ideal L such that if L satisfies these conditions then $N(L, \mathfrak{m})$ satisfies the sentence. (See 4.5.13, 4.5.14, 4.5.15, 4.5.18, 4.5.19, 4.5.20, 4.5.24, 4.5.25 and 4.5.26).

Finally we show that given any boolean combination of conditions that we have effectively produced, we can effectively check if there exists an ideal K satisfying it. (See 4.5.27 and 4.5.28).

Unlike in the case of a valuation domain with dense value group we will not be able to make a finite list of indecomposable pure-injectives such that $\left|\frac{\phi(N)}{\psi(N)}\right| = n$, as there may not be finitely many of them. In fact there may be uncountably many of them.

Throughout this section we will tacitly use the following two lemmas.

Lemma 4.5.1. *Let V be an effectively given valuation domain with non-dense value group. For any $k \in V$ which generates $\mathfrak{m}$, the function $f : V \to V; a \mapsto ak$ is recursive.*

Proof. The function $t_k : V \to V \times V;\ a \mapsto (a, k)$ is recursive since both component maps are recursive. The map $s : V \times V \to V;\ (a, b) \mapsto a \cdot b$ is recursive since V is effectively given. Therefore $st_k : V \to V$ is recursive. □

Lemma 4.5.2. *Let V be an effectively given valuation domain. For any k which generates $\mathfrak{m}$, the function $f : \mathfrak{m} \to V;\ a \mapsto a/k$ is recursive.*

Proof. By lemma 4.1.1, the function $f : V \times V \to V \times \{0, 1\}$ defined by

$$(a, b) \mapsto \begin{cases} (0, 0) & \text{if } a \text{ does not divide } b, \\ (b/a, 1) & \text{otherwise} \end{cases}$$

is recursive. The function $g_k : V \to V \times V : a \mapsto (k, a)$ is recursive since both component maps are recursive. The function $t : V \times \{0, 1\} \to V : (a, \pi) \mapsto a$ is recursive. Therefore tfg_k is recursive and for any $a \in \mathfrak{m}$, $tfg_k(a) = a/k$ since k generates $\mathfrak{m}$.

$\square$

Lemma 4.5.3. *Let V be a valuation domain with non-dense value group and residue field consisting of q elements. Then, for all $n \in \mathbb{N}$, $\left|\frac{V}{\mathfrak{m}^n}\right| = q^n$ and all finite uniserial modules are isomorphic to $V/\mathfrak{m}^n$ for some $n \in \mathbb{N}$.*

Proof. First note that for any $n \in \mathbb{N}$, $\mathfrak{m}^n/\mathfrak{m}^{n+1} = k^nV/k^{n+1}V \cong V/\mathfrak{m}$. Hence $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ has size q. By considering the chain $V \supsetneq \mathfrak{m} \supsetneq \mathfrak{m}^2 ... \supsetneq \mathfrak{m}^n$, we see that for all $n \in \mathbb{N}$, $\left|\frac{V}{\mathfrak{m}^n}\right| = q^n$.

Note that if $I \lhd V$ and V/I is finite then $I = \mathfrak{m}^n$ for some $n \in \mathbb{N}$.

Suppose M is a finite uniserial module. Let x be an element of M with smallest annihilator. Then $xV \cong V/\mathfrak{m}^n$ for some $n \in \mathbb{N}$, since all quotients of V of finite size are of this form. Therefore $\mathrm{ann}_V x = \mathfrak{m}^n$. Suppose $y \in M$. Since M is uniserial, either $y \in xV$ or $x \in yV$. If $x \in yV$ then $x = yr$ for some $r \in V$. Therefore $\mathrm{ann}_V y = (\mathrm{ann}_V x)r \subseteq \mathrm{ann}_V x$ hence $(\mathrm{ann}_V x)r = \mathrm{ann}_V x$. So $r \notin (\mathrm{ann}_V x)^{\#} = \mathfrak{m}$. Hence r is a unit. So $y \in xV$. Therefore $M = xV$. $\square$

Corollary 4.5.4. *Let V be a valuation domain with non-dense value group and finite residue field consisting of q elements. Then, all non-zero modules of finite size are of size q^n for some $n \in \mathbb{N}$.*

Proof. Suppose M is a finite non-zero V-module. Let $0 \subsetneq M_1 \subsetneq M_2 .... \subsetneq M_l = M$ be a chain of submodules of M such that M_1 is cyclic and for each $0 < i < l$, M_{i+1}/M_i is cyclic. Since all cyclic modules over a valuation domain are uniserial, for each

$0 < i < l$, $|M_{i+1}/M_i| = q^{v_i}$ for some $v_i \in \mathbb{N}$ and $|M_1| = q^w$ for some $w \in \mathbb{N}$. Now, $|M| = |M_1| \prod_{0<i<l} |M_{i+1}/M_i|$. Hence M is of size q^n for some $n \in \mathbb{N}$. $\square$

Note that the above lemma and corollary imply that for any pp-pair ϕ/ψ and any V-module M, $\left|\frac{\phi(M)}{\psi(M)}\right| = q^n$ for some $n \in \mathbb{N}$ or $\left|\frac{\phi(M)}{\psi(M)}\right|$ is infinite.

Lemma 4.5.5. *Suppose V is a valuation domain with finite residue field and non-dense value group. Let Q be the quotient field of V and $J \subsetneq I \subseteq Q$ be V-modules. Then $|I/J|$ is finite if and only if I and J are principally generated and $Ik^n = J$ for some $n \in \mathbb{N}$ where k generates the maximal ideal.*

Proof. First note that I/J is a uniserial module because Q is uniserial. Therefore if I/J is finite and not the zero module then $I/J \cong V/\mathfrak{m}^n$ for some $n \in \mathbb{N}$ by 4.5.3. Hence I is principally generated say by $\gamma \in Q\backslash\{0\}$ and $\gamma k^n \in J$ but $\gamma k^{n-1} \notin J$ so $J = \gamma k^n V$.

For the other direction note that for any $\gamma \in Q\backslash\{0\}$ and any $n \in \mathbb{N}$, $\gamma V/k^n\gamma V \cong V/\mathfrak{m}^n$. $\square$

Lemma 4.5.6. *Suppose V is a valuation domain with non-dense value group and finite residue field consisting of q elements. For all $a, b \in V\backslash\{0\}$ with $aV \supseteq bV$ and each $v \in \mathbb{N}$, $\left|\frac{aV}{bV}\right| = q^v$ if and only if $ak^vV = bV$ and $\left|\frac{aV}{bV}\right| \geq q^v$ if and only if $ak^vV \supseteq bV$.*

Proof. For all $a, b \in V$ with $aV \supseteq bV$ there exists $c \in V$ such that $ac = b$ and $\frac{aV}{bV} \cong \frac{V}{cV}$.

Suppose $c \in V$. Then, by lemma 4.5.3, $\left|\frac{V}{cV}\right| = q^v$ if and only if $\frac{V}{cV} \cong \frac{V}{\mathfrak{m}^v}$ if and only if $cV = k^vV$.

Suppose $c \in V$. Then $\left|\frac{V}{cV}\right| \geq q^v$ if and only if $cV = k^nV$ for some $n \geq v$ or $\left|\frac{V}{cV}\right|$ is infinite. Note that $\left|\frac{V}{cV}\right|$ is infinite if and only if $c \in \cap_{n\in\mathbb{N}} k^nV$. Therefore $\left|\frac{V}{cV}\right| \geq q^v$ if and only if $k^vV \supseteq cV$.

Hence, for all $a, b \in V$ with $aV \supseteq bV$ and $v \in \mathbb{N}$, $\left|\frac{aV}{bV}\right| = q^v$ if and only if $ak^vV = bV$ and for all $a, b \in V$ with $aV \supseteq bV$ and $v \in \mathbb{N}$, $\left|\frac{aV}{bV}\right| \geq q^v$ if and only if $ak^vV \supseteq bV$.

$\square$

Lemma 4.5.7. *Let V be a valuation domain with non-dense value group and finite residue field consisting of q elements. Let ϕ, ψ be pp-1-formulae. Then, for all $I, J \lhd V$, $\left|\frac{\phi(N(I,J))}{\psi(N(I,J))}\right|$ is finite and not equal to 1 implies either $I^{\#} = \mathfrak{m}$ or $J^{\#} = \mathfrak{m}$.*

Proof. Suppose $\left|\frac{\phi(N)}{\psi(N)}\right|$ is finite and not equal to 1. Then there exists a pp-1-formula ψ' such that $\phi(N) \supsetneq \psi'(N) \supseteq \psi(N)$ and ϕ/ψ' is a minimal pair (see definition 2.2.8) in the theory of N. Since $\left|\frac{\phi(N)}{\psi(N)}\right|$ is finite and not equal to 1, $\left|\frac{\phi(N)}{\psi'(N)}\right|$ is finite and not equal to 1. Suppose N has attached prime $\mathfrak{p}$ not equal to $\mathfrak{m}$. Then, for all $r \in \mathfrak{p}$ and all non-zero $x \in N$, xr has strictly greater pp-type than x by lemma 2.3.24. Hence if $x \in \phi(N)$ then $xr \in \psi'(N)$. Therefore $\frac{\phi(N)}{\psi'(N)}$ is an $V/\mathfrak{p}$ module. All $r \notin \mathfrak{p}$ act as automorphisms on N. Hence $\frac{\phi(N)}{\psi'(N)}$ is a $V_{\mathfrak{p}}/\mathfrak{p}$-module (i.e. vector space) and therefore infinite or the zero module since $V/\mathfrak{p}$ is of infinite size.

Therefore, if $\left|\frac{\phi(N)}{\psi(N)}\right|$ is finite and not equal to 1 then $\mathrm{Att}N = \mathfrak{m}$. By lemma 3.3.6, $I^{\#} \cup J^{\#} = \mathfrak{m}$. Therefore either $I^{\#} = \mathfrak{m}$ or $J^{\#} = \mathfrak{m}$. $\square$

Recall that, since $\mathfrak{m}$ is finitely generated and so $\mathfrak{m}^2 \neq \mathfrak{m}$, if $I \lhd V$ with $I^{\#} = \mathfrak{m}$ then $I = a\mathfrak{m}$ for some $a \in V\backslash\{0\}$. See lemma 3.5.7. So by lemma 4.5.7 above, for any pair of pp-1-formulae ϕ, ψ, if $I, J \lhd V$ such that $\left|\frac{\phi(N(I,J))}{\psi(N(I,J))}\right|$ is finite and not 1 then either $(I, J) \sim (\mathfrak{m}, K)$ or $(I, J) \sim (K, \mathfrak{m})$ for some $K \lhd V$.

Lemma 4.5.8. *Let V be a valuation domain with non-dense value group and finite residue field. Then for all $b \in V\backslash\{0\}$ if $J \lhd V$ is not principal then $[b\mathfrak{m} : J]$ is not principally generated.*

Proof. Let $b \in V\backslash\{0\}$ and $J \lhd V$. Suppose $[b\mathfrak{m} : J] = \gamma V$ for some $\gamma \in Q$. Then $J\gamma \subseteq b\mathfrak{m} = bkV$ and $bkV \subsetneq J\gamma k^{-1}$, since $\gamma k^{-1} \notin [b\mathfrak{m} : J]$. Hence $bV \subseteq J\gamma k^{-1}$ so $bk \in J\gamma$. Therefore $J\gamma = bkV$, so $J = bk/\gamma V$. So J is principal. $\square$

Lemma 4.5.9. *Let V be a valuation domain with non-dense value group and finite residue field consisting of q elements. Let $v \in \mathbb{N}\backslash\{0\}$, let ϕ be the pp-formula $(xag = 0 \wedge b|x)$ and let ψ be the pp-formula $(xa = 0 + bh|x)$ where $a, b \in V\backslash\{0\}$ and $g, h \in \mathfrak{m}$.*

Suppose $J \lhd V$ is such that $J^{\#} \subsetneq \mathfrak{m}$ and $N(\mathfrak{m}, J) \in \left(\frac{\phi}{\psi}\right)$. Then $\left|\frac{\phi(N(\mathfrak{m},J))}{\psi(N(\mathfrak{m},J))}\right| = q^v$ if and only if $gV = \mathfrak{m}^v$.

Proof. Since $N(\mathfrak{m}, J) \in \left(\frac{\phi}{\psi}\right)$, there exists some $t \notin J$ such that $a \notin t\mathfrak{m}$, $ag \in t\mathfrak{m}$, $b \notin (J : t)$ and $bh \in (J : t)$. By lemma 4.4.13

$$\left|\frac{\phi(N(\mathfrak{m}, J))}{\psi(N(\mathfrak{m}, J))}\right| = \min\left\{\left|\frac{[t\mathfrak{m} : ag]}{[t\mathfrak{m} : a]}\right|, \left|\frac{[t\mathfrak{m} : ag]}{[bh\mathfrak{m} : (J : t))]}\right|, \left|\frac{[b\mathfrak{m} : (J : t)]}{[t\mathfrak{m} : a]}\right|, \left|\frac{[b\mathfrak{m} : (J : t)]}{[bh\mathfrak{m} : (J : t)]}\right|\right\}.$$

Note that since $J^{\#} \subsetneq \mathfrak{m}$, J is not principal. Hence $(J : t)$ is not principal. So by lemma 4.5.8, $[b\mathfrak{m} : (J : t)]$ and $[bh\mathfrak{m} : (J : t)]$ are not principally generated. So by lemma 4.5.5, $\left|\frac{[t\mathfrak{m}:ag]}{[bh\mathfrak{m}:(J:t))]}\right|$, $\left|\frac{[b\mathfrak{m}:(J:t)]}{[t\mathfrak{m}:a]}\right|$ and $\left|\frac{[b\mathfrak{m}:(J:t)]}{[bh\mathfrak{m}:(J:t)]}\right|$ are either 1 or infinite. But since $N(\mathfrak{m}, J) \in \left(\frac{\phi}{\psi}\right)$, $\left|\frac{\phi(N(\mathfrak{m},J))}{\psi(N(\mathfrak{m},J))}\right| > 1$. Hence $\left|\frac{[t\mathfrak{m}:ag]}{[bh\mathfrak{m}:(J:t))]}\right|$, $\left|\frac{[b\mathfrak{m}:(J:t)]}{[t\mathfrak{m}:a]}\right|$ and $\left|\frac{[b\mathfrak{m}:(J:t)]}{[bh\mathfrak{m}:(J:t)]}\right|$ must all be infinite.

Hence $\left|\frac{\phi(N(\mathfrak{m},J))}{\psi(N(\mathfrak{m},J))}\right| = q^v$ if and only if $\left|\frac{[t\mathfrak{m}:ag]}{[t\mathfrak{m}:a]}\right| = q^v$.

By lemma 4.4.10 and since $\mathfrak{m}$ is finitely generated ,

$$\frac{[t\mathfrak{m} : ag]}{[t\mathfrak{m} : a]} \cong \frac{t\mathfrak{m}}{tg\mathfrak{m}} \cong \frac{V}{gV}.$$

By lemma 4.5.6, $\left|\frac{V}{gV}\right| = q^v$ if and only if $gV = \mathfrak{m}^v$. Hence $\left|\frac{\phi(N(\mathfrak{m},J))}{\psi(N(\mathfrak{m},J))}\right| = q^v$ if and only if $gV = \mathfrak{m}^v$. □

Lemma 4.5.10. *Let V be a valuation domain with non-dense value group and finite residue field consisting of q elements. Let $v \in \mathbb{N}\backslash\{0\}$, let ϕ be the pp-formula $(xag = 0 \wedge b|x)$ and let ψ be the pp-formula $(xa = 0 + bh|x)$ where $a, b \in V\backslash\{0\}$ and $g, h \in \mathfrak{m}$. Suppose $I \lhd V$ is such that $I^{\#} \subsetneq \mathfrak{m}$ and $N(I, \mathfrak{m}) \in \left(\frac{\phi}{\psi}\right)$. Then $\left|\frac{\phi(N(I,\mathfrak{m}))}{\psi(N(I,\mathfrak{m}))}\right| = q^v$ if and only if $hV = \mathfrak{m}^v$.*

Proof. The following proof is very similar to that of 4.5.9, it is included for the convenience of the reader. Since $N(I, \mathfrak{m}) \in \left(\frac{\phi}{\psi}\right)$, there exists some $t \notin I$ such that $a \notin (I : t)$, $b \notin t\mathfrak{m}$, $ag \in (I : t)$ and $bh \in t\mathfrak{m}$. By lemma 4.4.13

$$\left|\frac{\phi(N(I, \mathfrak{m}))}{\psi(N(I, \mathfrak{m}))}\right| = \min\left\{\left|\frac{[(I : t) : ag]}{[(I : t) : a]}\right|, \left|\frac{[(I : t) : ag]}{[bh\mathfrak{m} : t\mathfrak{m}]}\right|, \left|\frac{[b\mathfrak{m} : t\mathfrak{m}]}{[(I : t) : a]}\right|, \left|\frac{[b\mathfrak{m} : t\mathfrak{m}]}{[bh\mathfrak{m} : t\mathfrak{m}]}\right|\right\}.$$

Note that since $I^\# \subsetneq \mathfrak{m}$, I is not principal. Hence $(I : t)$ is not principal. So $[(I : t) : a]$ and $[(I : t) : ag]$ are not principally generated. So by 4.5.5, $\left|\frac{[(I:t):ag]}{[(I:t):a]}\right|$, $\left|\frac{[(I:t):ag]}{[bh\mathfrak{m}:t\mathfrak{m}]}\right|$ and $\left|\frac{[b\mathfrak{m}:t\mathfrak{m}]}{[(I:t):a]}\right|$ are either 1 or infinite. But since $N(I, \mathfrak{m}) \in \left(\frac{\phi}{\psi}\right)$, $\left|\frac{\phi(N(I,\mathfrak{m}))}{\psi(N(I,\mathfrak{m}))}\right| > 1$. So $\left|\frac{[(I:t):ag]}{[(I:t):a]}\right|$, $\left|\frac{[(I:t):ag]}{[bh\mathfrak{m}:t\mathfrak{m}]}\right|$ and $\left|\frac{[b\mathfrak{m}:t\mathfrak{m}]}{[(I:t):a]}\right|$ are infinite.

Hence $\left|\frac{\phi(N(I,\mathfrak{m}))}{\psi(N(I,\mathfrak{m}))}\right| = q^v$ if and only if $\left|\frac{[b\mathfrak{m}:t\mathfrak{m}]}{[bh\mathfrak{m}:t\mathfrak{m}]}\right| = q^v$.

By lemma 4.4.10 and since $\mathfrak{m}$ is finitely generated,

$$\frac{[b\mathfrak{m} : t\mathfrak{m}]}{[bh\mathfrak{m} : t\mathfrak{m}]} \cong \frac{b\mathfrak{m}}{bh\mathfrak{m}} \cong \frac{V}{hV}.$$

By lemma 4.5.6 $\left|\frac{V}{hV}\right| = q^v$ if and only if $hV = \mathfrak{m}^v$. Hence $\left|\frac{\phi(N(I,\mathfrak{m}))}{\psi(N(I,\mathfrak{m}))}\right| = q^v$ if and only if $hV = \mathfrak{m}^v$. □

Lemma 4.5.11. *Let V be an effectively given valuation domain with non-dense value group and finite residue field consisting of q elements. There exists an algorithm which, given $v \in \mathbb{N}\backslash\{0\}$ and $\alpha, \beta, \gamma, \delta \in V$, produces Δ, a boolean combination of conditions on an ideal of the form $r \in J$, $s \in J^\#$, such that for all $J \lhd V$, J satisfies Δ if and only if $J^\# \subsetneq \mathfrak{m}$ and*

$$\left|\frac{\phi(N(\mathfrak{m}, J))}{\psi(N(\mathfrak{m}, J))}\right| = q^v$$

where ϕ is $x\alpha = 0 \wedge \beta|x$ and ψ is $x\gamma = 0 + \delta|x$.

Proof. First note that if $\alpha \notin \gamma\mathfrak{m}$, $\delta \notin \beta\mathfrak{m}$, $\gamma = 0$ or $\beta = 0$ then for all V-modules M, $\left|\frac{\phi(M)}{\psi(M)}\right| = 1$. We can effectively check if $\alpha \notin \gamma\mathfrak{m}$, $\delta \notin \beta\mathfrak{m}$, $\gamma = 0$ or $\beta = 0$. In this situation let $\Delta = FALSE$.

Otherwise let $a = \gamma$, $b = \beta$, $g = \alpha/\gamma$ and $h = \delta/\beta$.

By lemma 4.5.9, if $J^\# \subsetneq \mathfrak{m}$, the following are equivalent:

1. $\left|\frac{\phi(N(\mathfrak{m},J))}{\psi(N(\mathfrak{m},J))}\right| = q^v$.

2. $(\mathfrak{m}, J) \in \mathcal{W}_{a,b,g,h}$ and $\left|\frac{V}{gV}\right| = q^v$.

By lemma 4.5.5, $\left|\frac{V}{gV}\right| = q^v$ if and only if $k^vV = gV$. This can be checked effectively by lemmas 4.5.2 and 4.5.1. Hence, if $k^vV \neq gV$, let $\Delta = FALSE$.

If $k^v V = gV$, let $\Delta = (abgh \in J) \wedge (ab \notin J) \wedge (h \in J^{\#}) \wedge (k \notin J^{\#})$. The last conjunct is equivalent to $J^{\#} \subsetneq \mathfrak{m}$. Recall that if $J^{\#} \subsetneq \mathfrak{m}$ then $(\mathfrak{m}, J)$ is a normal point by lemma 3.5.16. Given that $J^{\#} \subsetneq \mathfrak{m}$, the first 3 conjuncts are equivalent to $(\mathfrak{m}, J) \in \mathcal{W}_{a,b,g,h}$ since $(\mathfrak{m}, J)$ is a normal point and $\mathfrak{m}J = J$. □

Corollary 4.5.12. *Let V be an effectively given valuation domain with non-dense value group and finite residue field consisting of q elements. There exists an algorithm which, given $v \in \mathbb{N}\backslash\{0\}$ and $\alpha, \beta, \gamma, \delta \in V$, produces Δ, a boolean combination of conditions on an ideal of the form $r \in J$, $s \in J^{\#}$, such that for all $J \lhd V$, J satisfies Δ if and only if $J^{\#} \subsetneq \mathfrak{m}$ and*

$$\left|\frac{\phi(N(\mathfrak{m}, J))}{\psi(N(\mathfrak{m}, J))}\right| \geq q^v$$

where ϕ is $x\alpha = 0 \wedge \beta|x$ and ψ is $x\gamma = 0 + \delta|x$.

Lemma 4.5.13. *Let V be an effectively given valuation domain with non-dense value group and finite residue field consisting of q elements. There exists an algorithm which, given $v \in \mathbb{N}\backslash\{0\}$ and ϕ, ψ pp-1-formulae, produces Δ a boolean combination of conditions on an ideal of the form $r \in J$, $s \in J^{\#}$, such that for all $J \lhd V$, J satisfies Δ if and only if $J^{\#} \subsetneq \mathfrak{m}$ and*

$$\left|\frac{\phi(N(\mathfrak{m}, J))}{\psi(N(\mathfrak{m}, J))}\right| = q^v.$$

Proof. By lemmas 4.2.1 and 4.2.2 we can effectively rewrite ϕ as $\sum_{i=1}^{n} \phi_i$ where ϕ_i is $(xa_i = 0 \wedge b_i|x)$and ψ as $\bigwedge_{j=1}^{m} \psi_j$ where ψ_j is $(xc_j = 0 + d_j|x)$. Then by lemma 3.3.8, for any pure-injective module N

$$\left|\frac{\phi(N)}{\psi(N)}\right| = \max_{i,j}\left\{\left|\frac{\phi_i(N)}{\psi_j(N)}\right|\right\}.$$

Hence a pure-injective module N satisfies $\left|\frac{\phi(N)}{\psi(N)}\right| = q^v$ if and only if there exists $0 < i \leq n$ and $0 < j \leq m$ such that $\left|\frac{\phi_i(N)}{\psi_j(N)}\right| = q^v$ and for all $0 < i \leq n$ and $0 < j \leq m$, $\left|\frac{\phi_i(N)}{\psi_j(N)}\right| \leq q^v$.

For each $0 < i \leq n$ and $0 < j \leq m$, let $\Delta_{i,j}$ be the boolean combination of conditions on an ideal J of the form $r \in J$ and $s \in J^{\#}$ such that J satisfies $\Delta_{i,j}$ if and only if $J^{\#} \subsetneq \mathfrak{m}$ and

$$\left|\frac{\phi_i(N(\mathfrak{m}, J))}{\psi_j(N(\mathfrak{m}, J))}\right| = q^v.$$

Such a condition exists and can be effectively produced by lemma 4.5.11.

For each $0 < i \leq n$ and $0 < j \leq m$, let $\Omega_{i,j}$ be the boolean combination of conditions on an ideal J of the form $r \in J$ and $s \in J^{\#}$ such that J satisfies $\Omega_{i,j}$ if and only if $J^{\#} \subsetneq \mathfrak{m}$ and

$$\left|\frac{\phi_i(N(\mathfrak{m}, J))}{\psi_j(N(\mathfrak{m}, J))}\right| \geq q^{v+1}.$$

Such a condition exists and can be effectively produced by corollary 4.5.12.

Therefore a pure injective module $N = N(\mathfrak{m}, J)$ satisfies $\left|\frac{\phi(N)}{\psi(N)}\right| = q^v$ and $J^{\#} \subsetneq \mathfrak{m}$ if and only if J satisfies

$$\left(\bigwedge_{i,j} \neg\Omega_{i,j}\right) \wedge \left(\bigvee_{i,j} \Delta_{i,j}\right).$$

$\square$

Corollary 4.5.14. *Let V be an effectively given valuation domain with non-dense value group and finite residue field consisting of q elements. There exists an algorithm which, given $v \in \mathbb{N}\backslash\{0\}$ and ϕ, ψ pp-1-formulae, produces Δ a boolean combination of conditions on an ideal of the form $r \in J$, $s \in J^{\#}$, such that for all $J \lhd V$, J satisfies Δ if and only if $J^{\#} \subsetneq \mathfrak{m}$ and*

$$\left|\frac{\phi(N(\mathfrak{m}, J))}{\psi(N(\mathfrak{m}, J))}\right| \geq q^v.$$

Lemma 4.5.15. *Let V be an effectively given valuation domain with non-dense value group and finite residue field consisting of q elements. There exists an algorithm which, given ϕ, ψ pp-1-formulae, produces Δ a boolean combination of conditions on an ideal $J \lhd V$ of the form $r \in J$, $s \in J^{\#}$ where $r, s \in V$ such that J satisfies Δ if and only if $J^{\#} \subsetneq \mathfrak{m}$ and $\left|\frac{\phi(N(\mathfrak{m},J))}{\psi(N(\mathfrak{m},J))}\right| = 1$.*

Proof. By corollary 4.2.3, there is an algorithm which, given ϕ, ψ pp-1-formulae, either returns $\emptyset$ exactly when $\left(\frac{\phi}{\psi}\right)$ is empty or produces $n \in \mathbb{N}$ and for each $0 < i \leq n$, $a_i, b_i \in V\backslash\{0\}$ and $g_i, h_i \in \mathfrak{m}$ such that $\left(\frac{\phi}{\psi}\right) = \bigcup_{i=1}^{n} \left(\frac{xa_ig_i=0 \wedge b_i|x}{xa_i=0+b_ih_i|x}\right)$.

If $\left(\frac{\phi}{\psi}\right)$ is empty then for all indecomposable pure-injective modules N, $\left|\frac{\phi(N)}{\psi(N)}\right| = 1$. So let $\Delta = TRUE$.

Otherwise, note that for any $J \lhd V$, $\left|\frac{\phi(N(\mathfrak{m},J))}{\psi(N(\mathfrak{m},J))}\right| = 1$ if and only if $N(\mathfrak{m},J) \notin \left(\frac{\phi}{\psi}\right)$ if and only if $(\mathfrak{m},J) \notin \bigcup \mathcal{W}_{a_i,b_i,g_i,h_i}$ for all $0 < i \leq n$. For any $0 < i \leq n$ and $J^\# \subsetneq \mathfrak{m}$, $(\mathfrak{m},J) \notin \mathcal{W}_{a_i,b_i,g_i,h_i}$ if and only if $a_ib_i \in J$, $a_ib_ig_ih_i \notin J$ or $h_i \notin J^\#$ by lemma 3.5.16 and proposition 3.5.6. Note that $J^\# \subsetneq \mathfrak{m}$ if and only if $k \notin J^\#$. Therefore, let

$$\Delta = (k \notin J^\#) \wedge \bigwedge_{i=1}^{n} (a_ib_i \in J) \vee (a_ib_ig_ih_i \in J) \vee (h_i \notin J^\#).$$

□

Lemma 4.5.16. *Let V be an effectively given valuation domain with non-dense value group and finite residue field consisting of q elements. There exists an algorithm which, given $v \in \mathbb{N}\backslash\{0\}$ and $\alpha, \beta, \gamma, \delta \in V$, produces Δ, a boolean combination of conditions on an ideal of the form $r \in I$, $s \in I^\#$, such that for all $I \lhd V$, I satisfies Δ if and only if $I^\# \subsetneq \mathfrak{m}$ and*

$$\left|\frac{\phi(N(I,\mathfrak{m}))}{\psi(N(I,\mathfrak{m}))}\right| = q^v$$

where ϕ is $x\alpha = 0 \wedge \beta|x$ and ψ is $x\gamma = 0 + \delta|x$.

Proof. As in lemma 4.5.11, replacing 4.5.9 by 4.5.10. □

Corollary 4.5.17. *Let V be an effectively given valuation domain with non-dense value group and finite residue field consisting of q elements. There exists an algorithm which, given $v \in \mathbb{N}\backslash\{0\}$ and $\alpha, \beta, \gamma, \delta \in V$, produces Δ, a boolean combination of conditions on an ideal of the form $r \in I$, $s \in I^\#$, such that for all $I \lhd V$, I satisfies Δ if and only if $I^\# \subsetneq \mathfrak{m}$ and*

$$\left|\frac{\phi(N(I,\mathfrak{m}))}{\psi(N(I,\mathfrak{m}))}\right| \geq q^v$$

where ϕ is $x\alpha = 0 \wedge \beta|x$ and ψ is $x\gamma = 0 + \delta|x$.

Lemma 4.5.18. *Let V be an effectively given valuation domain with non-dense value group and finite residue field consisting of q elements. There exists an algorithm which, given $v \in \mathbb{N}\backslash\{0\}$ and ϕ, ψ pp-1-formulae, produces Δ a boolean combination of conditions on an ideal of the form $r \in I$, $s \in I^\#$, such that for all $I \lhd V$, I satisfies Δ if and only if $I^\# \subsetneq \mathfrak{m}$ and*

$$\left|\frac{\phi(N(I,\mathfrak{m}))}{\psi(N(I,\mathfrak{m}))}\right| = q^v.$$

Proof. Exactly as in proof of lemma 4.5.13 replacing $(\mathfrak{m}, J)$ by $(I, \mathfrak{m})$ and lemma 4.5.11 and corollary 4.5.12 by lemma 4.5.16 and corollary 4.5.17. $\square$

Corollary 4.5.19. *Let V be an effectively given valuation domain with non-dense value group and finite residue field consisting of q elements. There exists an algorithm which, given $v \in \mathbb{N}\backslash\{0\}$ and ϕ, ψ pp-1-formulae, produces Δ a boolean combination of conditions on an ideal of the form $r \in I$, $s \in I^{\#}$, such that for all $I \lhd V$, I satisfies Δ if and only if $I^{\#} \subsetneq \mathfrak{m}$ and*

$$\left|\frac{\phi(N(I,\mathfrak{m}))}{\psi(N(I,\mathfrak{m}))}\right| \geq q^v.$$

Lemma 4.5.20. *Let V be an effectively given valuation domain with non-dense value group and finite residue field consisting of q elements. There exists an algorithm which, given ϕ, ψ pp-1-formulae, produces Δ a boolean combination of conditions on an ideal $I \lhd V$ of the form $r \in I$, $s \in I^{\#}$ where $r, s \in V$ such I satisfies Δ if and only if $I^{\#} \subsetneq \mathfrak{m}$ and $\left|\frac{\phi(N(I,\mathfrak{m}))}{\psi(N(I,\mathfrak{m}))}\right| = 1$.*

Proof. As in 4.5.15. $\square$

Lemma 4.5.21. *Let V be a valuation domain with non-dense value group and finite residue field consisting of q elements. Let ϕ be the pp-1-formula $(xag = 0 \wedge b|x)$ and let ψ be the pp-1-formula $(xa = 0 + bh|x)$ where $a, b \in V\backslash\{0\}$ and $g, h \in \mathfrak{m}$. If $x \in \mathfrak{m}$ is such that $N(\mathfrak{m}, xV) \in \left(\frac{\phi}{\psi}\right)$ then*

$$\left|\frac{\phi(N(\mathfrak{m},xV))}{\psi(N(\mathfrak{m},xV))}\right| = \min \left\{\left|\frac{V}{gV}\right| , \left|\frac{V}{hV}\right| , \left|\frac{xV}{abghV}\right| , \left|\frac{abV}{xV}\right|\right\}.$$

Proof. If $N(\mathfrak{m}, xV) \in \left(\frac{\phi}{\psi}\right)$ then there exists $t \notin xV$ such that $a \notin t\mathfrak{m}$, $ag \in t\mathfrak{m}$, $b \notin (xV : t)$ and $bh \in (xV : t)$. Note that $(xV : t)$ is finitely generated.

By lemma 4.4.13,

$$\left|\frac{\phi(N(\mathfrak{m},xV))}{\psi(N(\mathfrak{m},xV))}\right| = \min\left\{\left|\frac{[t\mathfrak{m} : ag]}{[t\mathfrak{m} : a]}\right|, \left|\frac{[t\mathfrak{m} : ag]}{[bh\mathfrak{m} : (xV : t)]}\right|, \left|\frac{[b\mathfrak{m} : (xV : t)]}{[t\mathfrak{m} : a]}\right|, \left|\frac{[b\mathfrak{m} : (xV : t)]}{[bh\mathfrak{m} : (xV : t)]}\right|\right\}.$$

By lemma 4.4.10 and since $\mathfrak{m}$ is finitely generated the following equalities hold:

$$\left|\frac{[t\mathfrak{m} : ag]}{[t\mathfrak{m} : a]}\right| = \left|\frac{t\mathfrak{m}}{tg\mathfrak{m}}\right| = \left|\frac{V}{gV}\right|.$$

By lemma 4.4.10 and since $(xV : t)$ and $\mathfrak{m}$ are finitely generated the following equalities hold:

$$\left|\frac{[b\mathfrak{m} : (xV : t)]}{[bh\mathfrak{m} : (xV : t)]}\right| = \left|\frac{b\mathfrak{m}}{bh\mathfrak{m}}\right| = \left|\frac{V}{hV}\right|$$

By lemma 4.4.10 and since $(xV : t)$ and $\mathfrak{m}$ are finitely generated the following equalities hold:

$$\left|\frac{[t\mathfrak{m} : ag]}{[bh\mathfrak{m} : (xV : t)]}\right| = \left|\frac{t\mathfrak{m}(xV : t)}{abgh\mathfrak{m}}\right| = \left|\frac{x\mathfrak{m}}{abgh\mathfrak{m}}\right| = \left|\frac{xV}{abghV}\right|$$

By lemma 4.4.10 and since $(xV : t)$ and $\mathfrak{m}$ are finitely generated the following equalities hold:

$$\left|\frac{[b\mathfrak{m} : (xV : t)]}{[t\mathfrak{m} : a]}\right| = \left|\frac{ab\mathfrak{m}}{t\mathfrak{m}(xV : t)}\right| = \left|\frac{ab\mathfrak{m}}{x\mathfrak{m}}\right| = \left|\frac{abV}{xV}\right|$$

□

Lemma 4.5.22. *Let V be an effectively given valuation domain with non-dense value group and finite residue field consisting of q elements. There exists an algorithm which, given $v \in \mathbb{N}\backslash\{0\}$ and $\alpha, \beta, \gamma, \delta \in V$, produces Δ, a boolean combination of conditions on an element $x \in V$ of the form $x \in rV$ where $r \in V$ such that for all $x \in V$, x satisfies Δ if and only if $x \in \mathfrak{m}$ and*

$$\left|\frac{\phi(N(\mathfrak{m}, xV))}{\psi(N(\mathfrak{m}, xV))}\right| \geq q^v$$

where ϕ is $x\alpha = 0 \wedge \beta|x$ and ψ is $x\gamma = 0 + \delta|x$.

Proof. Note that if $\alpha \notin \gamma\mathfrak{m}$, $\delta \notin \beta\mathfrak{m}$, $\gamma = 0$ or $\beta = 0$ then for all V-modules M, $\left|\frac{\phi(M)}{\psi(M)}\right| = 1$. We can effectively check whether $\alpha \notin \gamma\mathfrak{m}$, $\delta \notin \beta\mathfrak{m}$, $\gamma = 0$ or $\beta = 0$. In this situation, let $\Delta = FALSE$.

Otherwise, let $a = \gamma$, $b = \beta$, $g = \alpha/\gamma$ and $h = \delta/\beta$. Note that we can effectively calculate the values of a, b, g and h.

Then by lemma 4.5.21, if $x \in \mathfrak{m}$ the following are equivalent:

(i) $\left|\frac{\phi(N(\mathfrak{m},xV))}{\psi(N(\mathfrak{m},xV))}\right| \geq q^v$.

(ii) $(\mathfrak{m}, xV) \in \mathcal{W}_{a,b,g,h}$ and $\min\left\{\left|\frac{V}{gV}\right| , \left|\frac{V}{hV}\right| , \left|\frac{xV}{abghV}\right| , \left|\frac{abV}{xV}\right|\right\} \geq q^v$.

Therefore $\left|\frac{\phi(N(\mathfrak{m},xV))}{\psi(N(\mathfrak{m},xV))}\right| \geq q^v$ implies $\left|\frac{V}{gV}\right| \geq q^v$ and $\left|\frac{V}{hV}\right| \geq q^v$. By lemma 4.5.6, $\left|\frac{V}{gV}\right| \geq q^v$ if and only if $g \in k^vV$ and $\left|\frac{V}{hV}\right| \geq q^v$ if and only if $h \in k^vV$. We can effective check whether $g \in k^vV$. If $g \notin k^vV$ let $\Delta = FALSE$. We can effective check whether $h \in k^vV$. If $h \notin k^vV$ let $\Delta = FALSE$.

We may now assume $g \in k^vV$ and $h \in k^vV$. Let $r = g/k^v$ and note that we can effectively calculate r.

<u>Claim</u>: $x \notin abrhkV$ and $x \in abk^vV$if and only if $x \in \mathfrak{m}$ and $\left|\frac{\phi(N(\mathfrak{m},xV))}{\psi(N(\mathfrak{m},xV))}\right| \geq q^v$.

$\Rightarrow$ Suppose $x \notin abrhkV$ and $x \in abk^vV$. Since $v > 0$, $x \in abk^vV$ implies $x \in \mathfrak{m}$. Since $x \notin abrhkV$, $x\mathfrak{m} \supseteq abrhkV$. Hence $abgh = abrhk^v \in x\mathfrak{m}$. Since $x \in abk^vV$, $x \in ab\mathfrak{m}$. Hence $ab \notin xV$. Therefore $(\mathfrak{m}, xV) \in \mathcal{W}_{a,b,g,h}$.

By lemma 4.5.6, $x \in abk^vV$ implies $\left|\frac{abV}{xV}\right| \geq q^v$. Recall that $x \notin abrhV$ implies $xkV = x\mathfrak{m} \supseteq abrhkV$. Therefore $xk^vV \supseteq abrhk^v = abghV$. Hence by lemma 4.5.6 $\left|\frac{xV}{abghV}\right| \geq q^v$. Therefore $\left|\frac{\phi(N(\mathfrak{m},xV))}{\psi(N(\mathfrak{m},xV))}\right| \geq q^v$.

$\Leftarrow$ Suppose $\left|\frac{\phi(N(\mathfrak{m},xV))}{\psi(N(\mathfrak{m},xV))}\right| \geq q^v$. Then $(\mathfrak{m}, xV) \in \mathcal{W}_{a,b,g,h}$ and

$$\min\left\{\left|\frac{V}{gV}\right| , \left|\frac{V}{hV}\right| , \left|\frac{xV}{abghV}\right| , \left|\frac{abV}{xV}\right|\right\} \geq q^v.$$

Therefore $\left|\frac{xV}{abghV}\right| \geq q^v$ and $\left|\frac{abV}{xV}\right| \geq q^v$. So $xk^vV \supseteq abghV$ and $abk^vV \supseteq xV$. Since $xk^vV \supseteq abrk^vhV$, $x\mathfrak{m} \supseteq abrkV$. Hence $x \notin abrhkV$. Since $abk^v \supseteq xV$, $x \in abk^vV$.

Therefore let $\Delta = (x \notin abrhkV) \wedge (x \in abk^vV)$.

$\square$

Corollary 4.5.23. *Let V be an effectively given valuation domain with non-dense value group and finite residue field consisting of q elements. There exists an algorithm which, given $v \in \mathbb{N}\backslash\{0\}$ and $\alpha, \beta, \gamma, \delta \in V$, produces Δ, a boolean combination of conditions on an element $x \in V$ of the form $x \in rV$ where $r \in V$ such that for all $x \in V$, x satisfies Δ if and only if $x \in \mathfrak{m}$ and*

$$\left|\frac{\phi(N(\mathfrak{m},xV))}{\psi(N(\mathfrak{m},xV))}\right| = q^v$$

where ϕ is $x\alpha = 0 \wedge \beta|x$ and ψ is $x\gamma = 0 + \delta|x$.

Lemma 4.5.24. *Let V be an effectively given valuation domain with non-dense value group and finite residue field consisting of q elements. There exists an algorithm which, given $v \in \mathbb{N}\backslash\{0\}$ and ϕ, ψ pp-1-formulae, produces Δ, a boolean combination of conditions on an element $x \in V$ of the form $x \in rV$ where $r \in V$ such that for all $x \in V$, x satisfies Δ if and only if $x \in \mathfrak{m}$ and*

$$\left|\frac{\phi(N(\mathfrak{m}, xV))}{\psi(N(\mathfrak{m}, xV))}\right| = q^v.$$

Proof. As in 4.5.13, replacing 4.5.11 by 4.5.23 and 4.5.12 by 4.5.22. □

Corollary 4.5.25. *Let V be an effectively given valuation domain with non-dense value group and finite residue field consisting of q elements. There exists an algorithm which, given $v \in \mathbb{N}\backslash\{0\}$ and ϕ, ψ pp-1-formulae, produces Δ, a boolean combination of conditions on an element $x \in V$ of the form $x \in rV$ where $r \in V$ such that for all $x \in V$, x satisfies Δ if and only if $x \in \mathfrak{m}$ and*

$$\left|\frac{\phi(N(\mathfrak{m}, xV))}{\psi(N(\mathfrak{m}, xV))}\right| \geq q^v.$$

Lemma 4.5.26. *Let V be an effectively given valuation domain with non-dense value group and finite residue field consisting of q elements. There exists an algorithm which, given ϕ, ψ pp-1-formulae, produces Δ, a boolean combination of conditions on an element $x \in V$ of the form $x \in rV$ where $r \in V$ such that for all $x \in V$, x satisfies Δ if and only if $x \in \mathfrak{m}$ and*

$$\left|\frac{\phi(N(\mathfrak{m}, xV))}{\psi(N(\mathfrak{m}, xV))}\right| = 1.$$

Proof. Note that for all $x \in \mathfrak{m}$, $\left|\frac{\phi(N(\mathfrak{m},xV))}{\psi(N(\mathfrak{m},xV))}\right| = 1$ if and only if $N(\mathfrak{m}, xV) \notin \left(\frac{\phi}{\psi}\right)$. By corollary 4.2.3, there is an algorithm which, given ϕ, ψ pp-1-formulae, either returns $\emptyset$ exactly when $\left(\frac{\phi}{\psi}\right)$ is empty or produces $n \in \mathbb{N}$ and for each $0 < i \leq n$, $a_i, b_i \in V\backslash\{0\}$ and $g_i, h_i \in \mathfrak{m}$ such that $\left(\frac{\phi}{\psi}\right) = \bigcup_{i=1}^{n} \left(\frac{xa_ig_i=0 \wedge b_i|x}{xa_i=0+b_ih_i|x}\right)$.

If $\left(\frac{\phi}{\psi}\right)$ is empty then for all indecomposable pure-injective modules N, $\left|\frac{\phi(N)}{\psi(N)}\right| = 1$. So let $\Delta = TRUE$.

Otherwise, $N(\mathfrak{m}, xV) \notin \left(\frac{\phi}{\psi}\right)$ if and only if for all $0 < i \leq n$, $(\mathfrak{m}, xV) \notin \mathcal{W}_{a_i,b_i,g_i,h_i}$. Since $\mathfrak{m}^2 \neq \mathfrak{m}$ and $g_i, h_i \in \mathfrak{m}$, $(\mathfrak{m}, xV) \notin \mathcal{W}_{a_i,b_i,g_i,h_i}$ if and only if $a_ib_i \in xV$ or

$a_ib_ig_ih_i \notin x\mathfrak{m}$. Note $a_ib_i \in xV$ if and only if $x \notin a_ib_ikV$ and $a_ib_ig_ih_i \notin x\mathfrak{m}$ if and only if $x \in a_ib_ig_ih_iV$. Finally $x \in \mathfrak{m}$ if and only if $x \in kV$. Therefore, let

$$\Delta = (x \in kV) \wedge \bigwedge_{i=1}^{n} (x \notin a_ib_ikV) \vee (x \in a_ib_ig_ih_iV).$$

□

Proposition 4.5.27. *Let V be an effectively given valuation domain with an algorithm which, given $a, b \in V$, answers whether $a \in \mathrm{rad}(bV)$. There exists an algorithm which, given a boolean combination of conditions $s \in J$ and $t \in J^{\#}$ for some $s, t \in V$, answers whether there is an ideal $J \lhd V$ satisfying these conditions.*

Proof. In order to show that we can effectively decide whether there exists an ideal $J \lhd V$ satisfying a boolean combination of conditions of the form $t \in J$ or $t \in J^{\#}$ it is enough to show that we can effectively decide whether there exists an ideal $J \lhd V$ satisfying a condition of the following form:

$$(*) \qquad \left(\bigwedge_{g=1}^{k} r_g \in J\right) \wedge \left(\bigwedge_{h=1}^{l} s_h \notin J\right) \wedge \left(\bigwedge_{i=1}^{m} t_i \in J^{\#}\right) \wedge \left(\bigwedge_{j=1}^{n} u_j \notin J^{\#}\right).$$

where $k, l, m, n \in \mathbb{N}$ and $r_g, s_h, t_i, u_j \in V$ for $0 < g \leq k$, $0 < h \leq l$, $0 < i \leq m$ and $0 < j \leq n$.

Since V is a valuation domain there exists $0 < g \leq k$ such that r_g generates the ideal $r_1V + ... + r_kV$, let $r = r_g$. There exists $0 < i \leq m$ such that t_i generates the ideal $t_1V + ... + t_mV$, let $t = t_i$. There exists $0 < h \leq l$ such that s_h generates $\cap_{h=1}^{l} s_hV$, $s = s_h$. There exists $0 < j \leq n$ such that u_j generates $\cap_{j=1}^{n} u_jV$, let $u = u_j$. It is clear that such r, s, t and u can be found effectively.

Note that $J \lhd V$ satisfies $(*)$ if and only if $r \in J$, $s \notin J$, $t \in J^{\#}$ and $u \notin J^{\#}$.

<u>Claim</u>: For any $r, s, t, u \in V$, there exists $J \lhd V$ such that $r \in J$, $s \notin J$, $t \in J^{\#}$ and $u \notin J^{\#}$ if and only if s divides r, $u \notin \mathrm{rad}(tV)$ and $u \notin \mathrm{rad}(r/sV)$.

Suppose $J \lhd V$ and $r \in J$, $s \notin J$, $t \in J^{\#}$ and $u \notin J^{\#}$. Since $J^{\#}$ is prime and $t \in J^{\#}$, $\mathrm{rad}(tV) \subseteq J^{\#}$. Therefore $u \notin \mathrm{rad}(tV)$. Clearly s divides r. Let $\gamma = r/s$. Then $s \notin J$ and $\gamma s \in J$ so $\gamma \in J^{\#}$. Therefore $\mathrm{rad}(\gamma V) \subseteq J^{\#}$ so $u \notin \mathrm{rad}(\gamma V)$.

Suppose s divides r, $u \notin \mathrm{rad}(tV)$ and $u \notin \mathrm{rad}(r/sV)$. Let $\gamma = r/s$ and $J = s(\mathrm{rad}(tV) \cup \mathrm{rad}(\gamma V))$. Then $J^{\#} = \mathrm{rad}(tV) \cup \mathrm{rad}(\gamma V)$ so $t \in J^{\#}$ and $u \notin J^{\#}$. Clearly $s \notin J$ and $\gamma \in \mathrm{rad}(\gamma V)$ so $r = s\gamma \in J$.

□

Lemma 4.5.28. *Let V be an effectively given valuation domain. There exists an algorithm which, given Δ a boolean combination of conditions on an element $x \in V$ of the form $x \in rV$ where $r \in V$, answers whether there exists $x \in V$ satisfying Δ.*

Proof. In order to show that we can effectively decide whether there exists $x \in V$ satisfying a boolean combination of conditions of the form $x \in rV$ where $r \in V$ it is enough to show that we can effectively decide whether there exists $x \in V$ satisfying a condition of the form:

$$\Delta = \bigwedge_{i=1}^{n}(x \in r_iV) \wedge \bigwedge_{j=1}^{m}(x \notin s_jV)$$

where $n, m \in \mathbb{N}$ and $r_i, s_j \in V$ for $0 < i \leq n$ and $0 < j \leq m$. Since V is a valuation domain there exists $0 < i \leq n$ such that $r_iV = \cap_{i=1}^{n} r_iV$, let $r = r_i$ and note that we can effectively find such an i. Again, since V is a valuation domain there exists $0 < j \leq m$ such that $s_jV = \cup_{j=1}^{m} s_jV$, let $s = s_j$ and note we that we can effectively find such a j.

There exists x satisfying Δ if and only if there exists $x \in V$ such that $x \in rV$ and $x \notin sV$ if and only if $sV \subsetneq rV$ if and only if $s \in r\mathfrak{m}$. Given any $r, s \in V$ we can effectively answer whether $s \in r\mathfrak{m}$. □

Theorem 4.5.29. *Let V be an effectively given valuation domain with non-dense value group and finite residue field consisting of q elements. The following are equivalent:*

(i) The theory of V-modules, T_V, is decidable.

(ii) There exists an algorithm which, given $a, b \in V$, answers whether $a \in \mathrm{rad}(bV)$.

Proof. As in theorem 4.3.3 it is enough to show that there is an algorithm which given a conjunction of invariant sentences and negations of invariants sentences χ, answers whether there exists a module M satisfying χ. Suppose χ is a conjunction of the following sentences:

$$\left|\frac{\phi_i^1}{\psi_i^1}\right| = q^{v_i} \tag{1}$$
$$\left|\frac{\phi_j^2}{\psi_j^2}\right| \geq q^{w_j} \tag{2}$$
$$\left|\frac{\phi_k^3}{\psi_k^3}\right| = 1 \tag{3}$$

where $l, m, n \in \mathbb{N}$ and for all $0 < i \leq l$, $0 < j \leq m$, $0 < k \leq n$, ϕ_i^1, ψ_i^1, ϕ_j^2, ψ_j^2, ϕ_k^3, ψ_k^3 are pp-1-formulae and $v_i, w_j \in \mathbb{N}$.

It is enough to consider sentences of this form as any finite V-module is either the zero module or has q^v elements for some strictly positive $v \in \mathbb{N}$, by corollary 4.5.4.

As in the proof of theorem 4.4.21, if τ is a conjunction of invariants sentences like those in (1), (2) and (3) then we call $\sum_{i=1}^{l} v_i$ the exponent of the statement.

We proceed by induction on $\sum_{i=1}^{l} v_i$, the exponent of the conjunction of invariants sentences in (1).

First consider the situation when $\sum_{i=1}^{l} v_i = 0$ that is (1) is empty. Exactly as in theorem 4.4.21 there exists a module M satisfying χ if and only if for all $0 < j \leq m$

$$\left(\frac{\phi_j^2}{\psi_j^2}\right) \not\subseteq \bigcup_{k=1}^{n} \left(\frac{\phi_k^3}{\psi_k^3}\right).$$

Theorem 4.2.15 asserts that there exists an algorithm to check this, so we are done.

Now suppose $L = \sum_{i=1}^{l} v_i > 0$, so (1) is not empty and that for any conjunction Θ of invariants sentences and negations of invariants sentences with exponent strictly smaller that L, there is an algorithm which answers whether there exists a module M satisfying Θ.

Suppose there exists M satisfying χ. By lemma 4.4.20 we may assume $M = \bigoplus_{\mu \in \mathcal{M}} N_\mu$ where $\mathcal{M}$ is a finite indexing set and each N_μ is an indecomposable pure-injective module. Hence there exists $\mu \in \mathcal{M}$ such that

$$q \leq \left|\frac{\phi_1^1(N_\mu)}{\psi_1^1(N_\mu)}\right| \leq q^{v_1}$$

and for all $\mu \in \mathcal{M}$, for all $0 < i \leq l$ and for all $0 < k \leq n$

$$\left|\frac{\phi_i^1(N_\mu)}{\psi_i^1(N_\mu)}\right| \leq q^{v_i} \text{ and } \left|\frac{\phi_k^3(N_\mu)}{\psi_k^3(N_\mu)}\right| = 1.$$

Let $\mathcal{U}$ be the set of functions $u : \{1, ..., l+m\} \to \mathbb{N} \cup \{\infty\}$. Let $\mathcal{U}^*$ be the subset of $\mathcal{U}$ consisting of functions $u \in \mathcal{U}$ such that $1 \leq u(1) \leq v_1$, for all $0 < i \leq l$, $0 \leq u(i) \leq v_i$ and for all $0 < j \leq m$, either $0 \leq u(l+j) < w_j$ or $u(l+j) = \infty$. Note that $\mathcal{U}^*$ is a finite set.

We now show that for each $u \in \mathcal{U}^*$ we can effectively answer whether there exists an indecomposable pure-injective V-module satisfying the following sentences:

(i) $\left|\frac{\phi_i^1}{\psi_i^1}\right| = q^{u(i)}$.

(ii) If $u(j+l) \neq \infty$, $\left|\frac{\phi_j^2}{\psi_j^2}\right| = q^{u(j+l)}$. Otherwise $\left|\frac{\phi_j^2}{\psi_j^2}\right| \geq q^{w_j}$.

(iii) $\left|\frac{\phi_k^3}{\psi_k^3}\right| = 1$.

Since $1 \leq u(1)$, by lemma 4.5.7 if $I, J \lhd V$ such that $N(I, J)$ satisfies (i), (ii) and (iii) then either $I^\# = \mathfrak{m}$ or $J^\# = \mathfrak{m}$. So, if $N(I, J)$ satisfies (i), (ii) and (iii), then we may assume either $I = \mathfrak{m}$ and $J = xV$ for some $x \in \mathfrak{m}$, $I = \mathfrak{m}$ and $J^\# \subsetneq \mathfrak{m}$ or $J = \mathfrak{m}$ and $I^\# \subsetneq \mathfrak{m}$.

Therefore it is enough to show how to answer the following 3 questions effectively:

Question 1: Does there exist $x \in \mathfrak{m}$ such that $N(\mathfrak{m}, xV)$ satisfies (i),(ii) and (iii)?

By lemma 4.5.24, given any sentence $\left|\frac{\phi}{\psi}\right| = q^v$ where ϕ, ψ are pp-1-formulae and $v \in \mathbb{N}\backslash\{0\}$ we can effectively produce Ω a boolean combination of conditions on an element $x \in V$ of the form $x \in rV$ where $r \in V$ such that x satisfies Ω if and only if $x \in \mathfrak{m}$ and $\left|\frac{\phi(N(\mathfrak{m},xV))}{\psi(N(\mathfrak{m},xV))}\right| = q^v$. By corollary 4.5.25, given any sentence $\left|\frac{\phi}{\psi}\right| \geq q^v$ where ϕ, ψ are pp-1-formulae and $v \in \mathbb{N}\backslash\{0\}$ we can effectively produce Ω a boolean combination of conditions on an element $x \in V$ of the form $x \in rV$ where $r \in V$ such that x satisfies Ω if and only if $x \in \mathfrak{m}$ and $\left|\frac{\phi(N(\mathfrak{m},xV))}{\psi(N(\mathfrak{m},xV))}\right| \geq q^v$. By lemma 4.5.26, given any sentence $\left|\frac{\phi}{\psi}\right| = 1$ where ϕ, ψ are pp-1-formulae we can effectively produce Ω a boolean combination of conditions on an element $x \in V$ of the form $x \in rV$ where $r \in V$ such that x satisfies Ω if and only if $x \in \mathfrak{m}$ and $\left|\frac{\phi(N(\mathfrak{m},xV))}{\psi(N(\mathfrak{m},xV))}\right| = 1$.

Hence we can effectively produce a boolean combination of conditions Θ on an element $x \in V$ such that x satisfies Θ if and only if $x \in \mathfrak{m}$ and $N(\mathfrak{m}, xV)$ satisfies (i), (ii) and (iii).

By lemma 4.5.28, we can effectively decide whether there exists $x \in V$ satisfying Θ.

Question 2: Does there exist $J \lhd V$ such that $J^{\#} \subsetneq \mathfrak{m}$ and $N(\mathfrak{m}, J)$ satisfies (i), (ii) and (iii)?

By lemma 4.5.13, given any sentence $\left|\frac{\phi}{\psi}\right| = q^v$ where ϕ, ψ are pp-1-formulae and $v \in \mathbb{N}\backslash\{0\}$ we can effectively produce Ω a boolean combination of conditions on an ideal $J \lhd V$ of the form $r \in J$ and $s \in J^{\#}$ where $r, s \in V$ such that J satisfies Ω if and only if $J^{\#} \subsetneq \mathfrak{m}$ and $\left|\frac{\phi(N(\mathfrak{m},J))}{\psi(N(\mathfrak{m},J))}\right| = q^v$. By corollary 4.5.14, given any sentence $\left|\frac{\phi}{\psi}\right| \geq q^v$ where ϕ, ψ are pp-1-formulae and $v \in \mathbb{N}\backslash\{0\}$ we can effectively produce Ω a boolean combination of conditions on an ideal $J \lhd V$ of the form $r \in J$ and $s \in J^{\#}$ where $r, s \in V$ such that J satisfies Ω if and only if $J^{\#} \subsetneq \mathfrak{m}$ and $\left|\frac{\phi(N(\mathfrak{m},J))}{\psi(N(\mathfrak{m},J))}\right| \geq q^v$. By lemma 4.5.15, given any sentence $\left|\frac{\phi}{\psi}\right| = 1$ where ϕ, ψ are pp-1-formulae we can effectively produce Ω a boolean combination of conditions on an ideal of the form $r \in J$, $s \in J^{\#}$ where $r, s \in V$ such that J satisfies Ω if and only if $J \subsetneq \mathfrak{m}$ and $\left|\frac{\phi(N(\mathfrak{m},J))}{\psi(N(\mathfrak{m},J))}\right| = 1$. Hence we can effectively produce Θ a boolean combination of conditions on an ideal $J \lhd V$ such that J satisfies Θ if and only if $J \subsetneq \mathfrak{m}$ and $N(\mathfrak{m}, J)$ satisfies (i), (ii) and (iii).

By lemma 4.5.27, we can effectively decide whether there exists $J \lhd V$ satisfying Θ.

Question 3: Does there exist $I \lhd V$ such that $I^{\#} \subsetneq \mathfrak{m}$ and $N(I, \mathfrak{m})$ satisfies (i), (ii) and (iii)? Same as question 2 replacing lemma 4.5.13 by 4.5.18, corollary 4.5.14 by corollary 4.5.19 and lemma 4.5.15 by lemma 4.5.20.

Let $\mathcal{U}^{**}$ be the set of $u \in \mathcal{U}^{*}$ such that an indecomposable pure-injective N exists satisfying (i),(ii) and (iii). If $\mathcal{U}^{**}$ is empty there does not exist a module M satisfying (1), (2) and (3).

For each $u \in \mathcal{U}^{**}$ we effectively produce a new list of sentences $(1)^u$, $(2)^u$ and $(3)^u$. For each u start with $(1)^u$ and $(2)^u$ empty, and $(3)^u$ containing all sentences in (3).

For each $0 < i \leq l$, if $u(i) < v_i$, add the sentence $\left|\frac{\phi_i^1}{\psi_i^1}\right| = q^{v_i - u(i)}$ to $(1)^u$. If $u(i) = v_i$, add the sentence $\left|\frac{\phi_i^1}{\psi_i^1}\right| = 1$ to $(3)^u$. For each $0 < j \leq m$, if $u(l+j) < w_j$, add the sentence $\left|\frac{\phi_j^2}{\psi_j^2}\right| \geq q^{w_j - u(l+j)}$ to $(2)^u$.

For each $u \in \mathcal{U}^{**}$ there exists a module M satisfying (1), (2) and (3) if and only if there exists a module M' satisfying $(1)^u$, $(2)^u$ and $(3)^u$.

Now there exists a module M satisfying (1), (2) and (3) if and only if there exists a module M' satisfying $(1)^u$, $(2)^u$ and $(3)^u$ for some $u \in \mathcal{U}^{**}$.

Note that for each $u \in \mathcal{U}^{**}$ the exponent of the conjunction of conditions in $(1)^u$ is strictly smaller than $L = \sum_{i=1}^{l} v_i$. Hence by the induction hypothesis, for each $u \in \mathcal{U}^{**}$ there is an algorithm which answers whether there exists a module satisfying $(1)^u$, $(2)^u$ and $(3)^u$.

The other direction is by proposition 4.1.3. $\square$

Chapter 5

The Ziegler spectrum restricted to injectives and other topologies on indecomposable injectives.

Theorem 5.0.1. *[PR10, Corollary 7.4] For any ring R the Ziegler topology restricted to the set of indecomposable injectives has a basis of open sets of the form:*

$$\left(\frac{R}{I}\right) = \{E \in \mathrm{inj}_R \mid (R/I, E) \neq 0\}$$

where I ranges over right ideals of the form $\eta({}_RR)$ where η is a pp-1-formula on left R-modules.

We call a (right) ideal I pp-definable if there is a pp-1-formula ϕ in the language of left R-modules such that $I = \phi({}_RR)$. Note that the solution set in ${}_RR$ of a pp-1-formula in the language of left R-modules will always be a right ideal of R.

We say a ring R is right coherent if every finitely generated right ideal is finitely presented, equivalently every element of R has finitely generated right annihilator and the intersection of two finitely generated right ideals is finitely generated. See [Pre09, §2.3.3].

Proposition 5.0.2. *[Rot83, Proposition 7][Zim77, 1.3a] The following are equivalent for an arbitrary ring R:*

1. *R is right coherent.*

2. *For every pp-1-formula ϕ, the right ideal $\phi({}_RR)$ is finitely generated.*

Note that, for an arbitrary ring R, any finitely generated right ideal is pp-definable in ${}_RR$, that is, every finitely generated ideal is equal to $\phi({}_RR)$ for some pp-1-formula ϕ.

Lemma 5.0.3. *Let R be a commutative ring. The map $\Gamma : \mathrm{Zg}_R|_{\mathrm{inj}} \to \mathrm{Spec}^* R$ taking indecomposable injectives to their attached prime is continuous.*

Proof. By proposition 2.3.25 the map $\Gamma : \mathrm{Zg}_R \to \mathrm{Spec}^* R$ is continuous. Therefore its restriction to the subspace of injectives is continuous. □

Combining these results we get:

Proposition 5.0.4. *[GP08a, Lemma 2.1] Let R be a commutative coherent ring. Then, after identifying topologically indistinguishable points, $\mathrm{Zg}_R|_{\mathrm{inj}}$ is homeomorphic to $\mathrm{Spec}^* R$.*

Proof. Let $\Gamma : \mathrm{Zg}_R|_{\mathrm{inj}} \to \mathrm{Spec}^* R : E \mapsto \mathrm{Att}E$.

Claim: For all finitely generated ideals I, $E \in \left(\frac{R}{I}\right)$ if and only if $\mathrm{Att}E \in \mathrm{V}(I)$.

Suppose $E \in \left(\frac{R}{I}\right)$. Then there exists a non-zero map $f : R/I \to E$. Therefore there exists $w \in E\backslash\{0\}$ such that $\mathrm{im} f \cong wR$. Hence $\mathrm{ann}_R w \supseteq I$. So $\mathrm{Att}E \supseteq I$. So $\mathrm{Att}E \in \mathrm{V}(I)$. Suppose $I = \langle r_1, ..., r_n\rangle$ and $\mathrm{Att}E \in \mathrm{V}(I)$. Then $I \subseteq \mathrm{Att}E$. So there exists $w_1, ..., w_n \in E\backslash\{0\}$ such that $r_i \in \mathrm{ann}_R w_i$ for each $0 < i \leq n$. Take $x \in \cap_{i=1}^n w_i R\backslash\{0\}$, such an x exists since E is uniform. Then $xr_i = 0$ for all $0 < i \leq n$. So $I \subseteq \mathrm{ann}_R x$. Hence, there is a non-zero map $R/I \twoheadrightarrow R/\mathrm{ann}_R x \cong xR$. Therefore $\mathrm{Hom}_R(R/I, E) \neq 0$. So $E \in \left(\frac{R}{I}\right)$.

This means that two indecomposable injectives with the same attached prime are topologically indistinguishable, so all points in a single fibre of Γ are topologically indistinguishable. For any prime ideal $\mathfrak{p}$, $\mathrm{E}(R/\mathfrak{p})$ is an indecomposable injective with $\mathrm{Att}\mathrm{E}(R/\mathfrak{p}) = \mathfrak{p}$. So Γ is surjective.

Since the sets $\mathrm{V}(I)$ where I is a finitely generated ideal are a basis of open sets for $\mathrm{Spec}^* R$, the claim implies Γ is a homeomorphism. □

Note that a valuation ring is non-coherent if and only if there exists $s \in R$ such that the ideal $\mathrm{ann}_R s$ is not finitely generated.

When R is a non-coherent commutative ring the above proposition does not necessarily hold. We give an example (Example 5.1.6) where it does not hold for a valuation ring. In the rest of this section we investigate whether $\mathrm{Zg}_R|_{\mathrm{inj}}$ is sober when R is a valuation ring, a Prüfer ring or the fibre product of two copies of the same valuation ring over the residue field. We also investigate how similar $\mathrm{Zg}_R|_{\mathrm{inj}}$ is to $\mathrm{Spec}^* R$ for these rings.

In order to show $\mathrm{Zg}_R|_{\mathrm{inj}}$ is sober when R is a valuation ring we consider a finer topology on inj_R, the ideals topology, denoted $\mathbf{ideals}_R$ and defined below. We show that for a valuation ring this topology is sober and show that for valuation rings this implies $\mathrm{Zg}_R|_{\mathrm{inj}}$ is sober.

Definition 5.0.5. *Let R be a commutative ring. We define a topology on inj_R, denoted $\boldsymbol{ideals}_R$, by declaring the set*

$$\mathcal{O}(I) = \{E \in \mathrm{inj}_R \mid Hom_R(R/I, E) \neq 0\}$$

open for each $I \lhd R$.

Working with this topology means we don't have to worry about which ideals are pp-definable. It will also be useful later. Note that, by theorem 5.0.1, the basic open sets in $\mathrm{Zg}_R|_{\mathrm{inj}}$ are exactly the open sets $\mathcal{O}(I)$ where I is pp-definable. So $\mathbf{ideals}_R$ is a refinement of $\mathrm{Zg}_R|_{\mathrm{inj}}$.

Lemma 5.0.6. *Let R be a commutative ring. The map $s : \boldsymbol{ideals}_R \to \mathrm{Spec}^* R$ taking indecomposable injectives to their attached prime is continuous.*

Proof. The $\mathbf{ideals}_R$ topology is a refinement of $\mathrm{Zg}_R|_{\mathrm{inj}}$. In lemma 5.0.3, we show that as a map from $\mathrm{Zg}_R|_{\mathrm{inj}}$ to $\mathrm{Spec}^* R$, s is continuous. Hence the map remains continuous when the $\mathbf{ideals}_R$ topology is put on inj_R. □

Remark 5.0.7. *Suppose E is an indecomposable injective R-module. Then $E \in \mathcal{O}(I)$ if and only if there exists a non-zero $w \in E$ with $\mathrm{ann}_R w \supseteq I$.*

Lemma 5.0.8. *Suppose R is a commutative ring and let $I, J \lhd R$. Then*

1. *$I \supseteq J$ implies $\mathcal{O}(I) \subseteq \mathcal{O}(J)$.*

2. *$\mathcal{O}(I) \cap \mathcal{O}(J) = \mathcal{O}(I + J)$. Therefore the open sets $\mathcal{O}(I)$ are a basis for **ideals**.*

3. *$\mathcal{O}(I) \cup \mathcal{O}(J) = \mathcal{O}(I \cap J)$.*

Proof. (1) Suppose $I \supseteq J$ and $E \in \mathcal{O}(I)$. Then there exists $w \in E \backslash \{0\}$ such that $\operatorname{ann}_R w \supseteq I$. Hence $\operatorname{ann}_R w \supseteq J$. So $E \in \mathcal{O}(J)$.

(2) Suppose $E \in \mathcal{O}(I + J)$. Then there exists $w \in E \backslash \{0\}$ such that $\operatorname{ann}_R w \supseteq I + J$. Therefore $\operatorname{ann}_R w \supseteq I$ and $\operatorname{ann}_R w \supseteq J$. So $E \in \mathcal{O}(I) \cap \mathcal{O}(J)$. Suppose $E \in \mathcal{O}(I) \cap \mathcal{O}(J)$. Then there exists $w_1, w_2 \in E \backslash \{0\}$ such that $\operatorname{ann}_R w_1 \supseteq I$ and $\operatorname{ann}_R w_2 \supseteq J$. Since E is uniform, recall that a module is uniform if the intersection of any pair of non-zero submodules is non-zero (definition 2.1.5), $w_1 R \cap w_2 R \neq 0$. Take non-zero $t \in w_1 R \cap w_2 R$. Then $\operatorname{ann}_R t \supseteq \operatorname{ann}_R w_1$ and $\operatorname{ann}_R t \supseteq \operatorname{ann}_R w_2$. Therefore $\operatorname{ann}_R t \supseteq I + J$.

(3) Suppose $E \in \mathcal{O}(I \cap J)$. The map $R/I \cap J \hookrightarrow R/I \bigoplus R/J$ which takes $1 + I \cap J$ to $(1+I, 1+J)$ is an embedding. Therefore, if there exists a non-zero map $f : R/I \cap J \to E$ then there exists a non-zero map $g : R/I \bigoplus R/J \to E$ since E is injective. Hence there either exists a non-zero map $g_1 : R/I \to E$ or there exists a non-zero map $g_2 : R/J \to E$. Therefore either $E \in \mathcal{O}(I)$ or $E \in \mathcal{O}(J)$. So $\mathcal{O}(I \cap J) \subseteq \mathcal{O}(I) \cup \mathcal{O}(J)$. By (1), $\mathcal{O}(I \cap J) \supseteq \mathcal{O}(I) \cup \mathcal{O}(J)$. $\square$

Lemma 5.0.9. *Let R be a valuation ring. The sets*

$$\mathrm{W}(I) = \{E \in \operatorname{inj}_R \mid I \supsetneq \operatorname{ann}_R w \text{ for all } w \in E \backslash \{0\}\}$$

*as I ranges over ideals (pp-definable ideals) are a basis of closed sets for **ideals**$_R$ (resp. $\mathrm{Zg}_R|_{\mathrm{inj}}$).*

Proof. Suppose $I \lhd R$. Then $E \in \mathcal{O}(I)$ if and only if there exists $w \in E \backslash \{0\}$ such that $\operatorname{ann}_R w \supseteq I$. Therefore $E \notin \mathcal{O}(I)$ if and only if for all $w \in E \backslash \{0\}$, $I \supsetneq \operatorname{ann}_R w$ if and only if $E \in \mathrm{W}(I)$. So, since the sets $\mathcal{O}(I)$ are an open basis for **ideals**$_R$, the sets $\mathrm{W}(I)$ are a closed basis for **ideals**$_R$.

The open sets $\mathcal{O}(I)$ where I ranges over pp-definable ideals are an open basis for $\mathrm{Zg}_R|_{\mathrm{inj}}$. Therefore the sets $\mathrm{W}(I)$, the complements of $\mathcal{O}(I)$, where I ranges over pp-definable ideals, are a closed basis for $\mathrm{Zg}_R|_{\mathrm{inj}}$. $\square$

Corollary 5.0.10. *Let R be a valuation ring. The closed sets in $\textbf{\textit{ideals}}_R$ ($\mathrm{Zg}_R|_{\mathrm{inj}}$) are totally ordered. In particular all closed sets in either topology are irreducible.*

Note that this means any set X irreducible closed in $\mathrm{Zg}_R|_{\mathrm{inj}}$ is closed in $\textbf{ideals}_R$ and hence irreducible. Therefore, if $\textbf{ideals}_R$ is sober then X has a generic point x in $\textbf{ideals}_R$. So $X = \mathrm{cl}_{\textbf{ideals}_R} x \subseteq \mathrm{cl}_{\mathrm{Zg}_R|_{\mathrm{inj}}} x$. Since $x \in X$, $X = \mathrm{cl}_{\mathrm{Zg}_R|_{\mathrm{inj}}} x$. Therefore, for valuation rings R, if $\textbf{ideals}_R$ is sober then $\mathrm{Zg}_R|_{\mathrm{inj}}$ is sober.

Lemma 5.0.11. *Let R be a valuation ring, $I \lhd R$ and E an indecomposable injective module. Then*

1. $\mathrm{Att}E \subsetneq I$ *implies* $E \in \mathrm{W}(I)$.
2. $E \in \mathrm{W}(I)$ *implies* $\mathrm{Att}E \subseteq I$.

Proof. (1) Suppose $\mathrm{Att}E \subsetneq I$. For all $w \in E\backslash\{0\}$, $\mathrm{ann}_R w \subseteq \mathrm{Att}E$. So, for all $w \in E\backslash\{0\}$, $\mathrm{ann}_R w \subsetneq I$. Therefore $E \in \mathrm{W}(I)$.
(2) Suppose $E \in \mathrm{W}(I)$. Then $\mathrm{ann}_R w \subsetneq I$ for all $w \in E\backslash\{0\}$. Hence $\mathrm{Att}E \subseteq I$. $\square$

We now show that basic closed sets in $\textbf{ideals}_R$ have generic points.

Lemma 5.0.12. *Let R be a valuation ring. If $I \lhd R$ is not prime then $\mathrm{W}(I)$ has a generic point. In fact there exists $\mathfrak{p} \lhd R$ prime such that $\mathrm{E}(R/\mathfrak{p})$ is generic in $\mathrm{W}(I)$.*

Proof. Let $\mathfrak{p}$ be the union of all prime ideals contained in I. Then $\mathfrak{p}$ is prime, so $\mathfrak{p} \subsetneq I$. Hence $\mathrm{E}(R/\mathfrak{p}) \in \mathrm{W}(I)$.

Suppose $E \in \mathrm{W}(I)$. Then by lemma 5.0.11, $\mathrm{Att}E \subseteq I$. Therefore $\mathrm{Att}E \subseteq \mathfrak{p}$. So if $\mathrm{E}(R/\mathfrak{p}) \in \mathrm{W}(K)$ then $\mathfrak{p} \subsetneq K$. So $\mathrm{Att}E \subsetneq K$. Hence $E \in \mathrm{W}(K)$. Therefore $E(R/\mathfrak{p})$ specialises to E. Since E was an arbitrary member of $\mathrm{W}(I)$, $\mathrm{E}(R/\mathfrak{p})$ is generic in $\mathrm{W}(I)$. $\square$

Lemma 5.0.13. *Let R be a valuation ring. Suppose $\mathfrak{p} \lhd R$ is a prime which is not the union of prime ideals strictly contained in it. Then* $\mathrm{W}(\mathfrak{p})$ *has a generic point.*

Proof. Let $\mathfrak{q}$ be the union of all prime ideals strictly contained in $\mathfrak{p}$. Note that $\mathfrak{q}$ is a prime ideal. Since $\mathrm{AttE}(R/\mathfrak{q}) = \mathfrak{q} \subsetneq \mathfrak{p}$, by lemma 5.0.11, $\mathrm{E}(R/\mathfrak{q}) \in \mathrm{W}(\mathfrak{p})$.

Suppose $E \in \mathrm{W}(\mathfrak{p})$ such that $\mathrm{E}(R/\mathfrak{q})$ does not specialise to E. We will now show that if such an E exists then E is generic in $\mathrm{W}(\mathfrak{p})$. Since $\mathrm{E}(R/\mathfrak{q})$ does not specialise to E there exists $K \lhd R$ such that $\mathrm{E}(R/\mathfrak{q}) \in \mathrm{W}(K)$ and $E \notin \mathrm{W}(K)$. Hence $\mathfrak{q} \subsetneq K$, since there exists $w \in \mathrm{E}(R/\mathfrak{q})$ with $\mathrm{ann}_R w = \mathfrak{q}$. Since $E \notin \mathrm{W}(K)$, there exists $w \in E \backslash \{0\}$ such that $K \subseteq \mathrm{ann}_R w$. Therefore $\mathrm{Att}E \supsetneq \mathfrak{q}$, by lemma 5.0.11 and $E \in \mathrm{W}(\mathfrak{p})$, so $\mathfrak{p} \supseteq \mathrm{Att}E$. Hence $\mathrm{Att}E = \mathfrak{p}$. Suppose $E' \in \mathrm{W}(\mathfrak{p})$ and $E \in \mathrm{W}(J)$ for some $J \lhd R$. Then $\mathfrak{p} = \mathrm{Att}E \subseteq J$. Since $E' \in \mathrm{W}(\mathfrak{p})$, $\mathrm{Att}E' \subseteq \mathfrak{p}$. So either $J = \mathfrak{p}$ so $E' \in \mathrm{W}(\mathfrak{p})$ or $\mathfrak{p} \subsetneq J$. If $\mathfrak{p} \subsetneq J$ then $\mathrm{Att}E' \subsetneq J$. Hence $E' \in \mathrm{W}(J)$. Therefore E is generic in $\mathrm{W}(\mathfrak{p})$ $\square$

The remaining basic open sets to consider are those of the form $\mathrm{W}(\mathfrak{p})$ where $\mathfrak{p}$ is a prime ideal which is the union of all prime ideals strictly contained in it. In order to find a generic point for such a closed set we need to find an (irreducible) ideal I which has attached prime $\mathfrak{p}$ and is such that for all $r \notin I$, $(I : r) \subsetneq \mathfrak{p}$. Below, we define an ideal which will be shown to have these properties. It is the pre-image of a finitely generated ideal in $V_{\mathfrak{p}}$.

Definition 5.0.14. *Let R be a valuation ring. Suppose $\mathfrak{p} \lhd R$ is a prime ideal which is not the nil radical, $N(R)$, and $a \in \mathfrak{p}$ which is not nilpotent. Let*

$$I_a^{\mathfrak{p}} = \{r \in R \mid rs \in aR \text{ for some } s \notin \mathfrak{p}\}.$$

Lemma 5.0.15. *Let R be a valuation ring. Suppose $\mathfrak{p} \lhd R$ is a prime ideal which is not the nil radical, $N(R)$, and $a \in \mathfrak{p}$ which is not nilpotent. Then*

1. *$I_a^{\mathfrak{p}}$ is an ideal.*

2. *$(I_a^{\mathfrak{p}})^{\#} = \mathfrak{p}$.*

3. *If $\mathfrak{p} = \mathfrak{p}^2$ and $\lambda \notin I_a^{\mathfrak{p}}$ then $(I_a^{\mathfrak{p}} : \lambda) \subsetneq \mathfrak{p}$.*

Proof. (1) Recalling 2.4.2, in order to show that $I_a^{\mathfrak{p}}$ is an ideal, we only need to show that for any $r \in R$ and $i \in I$, $ir \in I$. Suppose $v \in I_a^{\mathfrak{p}}$ and $r \in R$. Suppose, for a contradiction, that $vr \notin I_a^{\mathfrak{p}}$. Then $a = vrt$ for some $t \in \mathfrak{p}$. Since $v \in I_a^{\mathfrak{p}}$, there exists $s \notin \mathfrak{p}$ and $\mu \in R$ such that $vs = a\mu$. Hence $vs - vrt\mu = 0$, so $v(s - rt\mu) = 0$. Therefore, either $v \in N(R)$ or $s - rt\mu \in N(R)$. If $v \in N(R)$ then $vr \in N(R)$. Hence $vr \in aR \subseteq I_a^{\mathfrak{p}}$. If $s - rt\mu \in N(R)$ then $s - rt\mu \in \mathfrak{p}$. Hence $s \in \mathfrak{p}$ since $t \in \mathfrak{p}$. Either way, this contradicts our assumptions. Therefore $I_a^{\mathfrak{p}}$ is an ideal.

(2) Suppose $\gamma \in (I_a^{\mathfrak{p}})^{\#}$. Then there exists $v \notin I_a^{\mathfrak{p}}$ such that $v\gamma \in I_a^{\mathfrak{p}}$. There exists $\lambda \in \mathfrak{p}$ such that $a = v\lambda$ since $v \notin I_a^{\mathfrak{p}}$. Since $v\gamma \in I_a^{\mathfrak{p}}$ there exists $s \notin \mathfrak{p}$ and $t \in R$ such that $\gamma vs = at$. Therefore $\gamma vs = v\lambda t$, so $v(\gamma s - \lambda t) = 0$. Hence $\gamma s - \lambda t \in N(R)$ since $v \notin aR \supseteq N(R)$. Therefore $\gamma s - \lambda t \in \mathfrak{p}$. Hence $\gamma s \in \mathfrak{p}$ since $\lambda \in \mathfrak{p}$. Therefore $\gamma \in \mathfrak{p}$ since $s \notin \mathfrak{p}$. So $(I_a^{\mathfrak{p}})^{\#} \subseteq \mathfrak{p}$.

Now suppose $\gamma \in \mathfrak{p}$. Either $\gamma \in I_a^{\mathfrak{p}}$, so $\gamma \in (I_a^{\mathfrak{p}})^{\#}$ or there exists $t \in \mathfrak{p}$ such that $\gamma t = a$. In the case of the second disjunct, suppose, for a contradiction, that $t \in I_a^{\mathfrak{p}}$. Then there exists $s \notin \mathfrak{p}$ and $\mu \in R$ such that $ts = a\mu$ so $ts = \gamma t\mu$. Therefore $t(s - \gamma\mu) = 0$. Note that $t \notin N(R)$ since $a \notin N(R)$ (recalling $\gamma t = a$). Hence $s - \gamma\mu \in N(R)$. Therefore $s - \gamma\mu \in \mathfrak{p}$. So, since $s \notin \mathfrak{p}$, $\gamma\mu \notin \mathfrak{p}$. Therefore $\gamma \notin \mathfrak{p}$. A contradiction. Hence $t \notin I_a^{\mathfrak{p}}$. So $\gamma \in (I_a^{\mathfrak{p}})^{\#}$.

(3) Suppose $\lambda \notin I_a^{\mathfrak{p}}$. Then there exists $s \in \mathfrak{p}$ such that $\lambda s = a$. Since $\mathfrak{p} = \mathfrak{p}^2$ there exists $s_1, s_2 \in \mathfrak{p}$ such that $s_1 s_2 = s$. Note that $a \notin N(R)$ so $s \notin N(R)$ and $\lambda \notin N(R)$. Therefore $s_1, s_2 \notin N(R)$. Suppose, for a contradiction, that $s_1 \in (I_a^{\mathfrak{p}} : \lambda)$. Then $s_1\lambda \in I_a^{\mathfrak{p}}$, so there exists $v \notin \mathfrak{p}$ such that $s_1\lambda v \in aR$. So there exists $r \in R$ such that $s_1\lambda v = ar$ but then $s_1\lambda v = \lambda s_1 s_2 r$. So $s_1\lambda(v - s_2 r) = 0$. Since $s_1, \lambda \notin N(R)$, $v - s_2 r \in N(R)$. Hence $v - s_2 r \in \mathfrak{p}$, so $v \in \mathfrak{p}$. A contradiction. Therefore $s_1 \notin (I_a^{\mathfrak{p}} : \lambda)$ and $s_1 \in \mathfrak{p}$, so $(I_a^{\mathfrak{p}} : \lambda) \subsetneq \mathfrak{p}$. □

Recall that lemma 2.1.15 states that if R is a commutative ring and I is an irreducible ideal of R then $\mathrm{Att}(\mathrm{E}(R/I)) = I^{\#}$. Hence, by lemma 5.0.15, if a is a non-nilpotent member of $\mathfrak{p}$ then $\mathfrak{p} = \mathrm{Att}(\mathrm{E}(R/I_a^{\mathfrak{p}}))$.

Lemma 5.0.16. *Let R be a valuation ring and $\mathfrak{p}$ a prime ideal which is the union of* *all prime ideals strictly contained in it. Then* $\mathrm{W}(\mathfrak{p})$ *has a generic point.*

Proof. First note that if $\mathfrak{p}$ is the union of all prime ideals strictly contained in it then $\mathfrak{p}^2 = \mathfrak{p}$. Take $a \in \mathfrak{p}\backslash N(R)$. We will first show that $\mathrm{E}(R/I_a^{\mathfrak{p}}) \in \mathrm{W}(\mathfrak{p})$ and then show that it is a generic point of $\mathrm{W}(\mathfrak{p})$.

By 2.1.9, we know that for all $w \in E(R/I_a^{\mathfrak{p}})\backslash\{0\}$, there exists $t \notin I_a^{\mathfrak{p}}$ and $s \notin \mathrm{ann}_R w$ such that $(\mathrm{ann}_R w : s) = (I_a^{\mathfrak{p}} : t)$. Therefore, by 5.0.15 (3), $\mathrm{ann}_R w \subseteq (\mathrm{ann}_R w : s) = (I_a^{\mathfrak{p}} : t) \subsetneq \mathfrak{p}$. Hence $\mathrm{E}(R/I_a^{\mathfrak{p}}) \in \mathrm{W}(\mathfrak{p})$.

We now show that $\mathrm{E}(R/I_a^{\mathfrak{p}})$ is generic in $\mathrm{W}(\mathfrak{p})$. Suppose $E' \in \mathrm{W}(\mathfrak{p})$ and $K \lhd R$ such that $\mathrm{E}(R/I_a^{\mathfrak{p}}) \in \mathrm{W}(K)$. Then $\mathfrak{p} = \mathrm{Att}E(R/I_a^{\mathfrak{p}}) \subseteq K$ by lemma 5.0.11. If $K = \mathfrak{p}$ then $E' \in \mathrm{W}(K)$ by assumption. Otherwise $E' \in \mathrm{W}(\mathfrak{p})$ implies $\mathrm{Att}E' \subseteq \mathfrak{p}$. So if $\mathfrak{p} \subsetneq K$ then $\mathrm{Att}E' \subsetneq K$. So, by 5.0.11, $E' \in \mathrm{W}(K)$. Therefore $\mathrm{E}(R/I_a^{\mathfrak{p}})$ is generic in $\mathrm{W}(\mathfrak{p})$. □

It remains to consider non-basic closed sets.

Lemma 5.0.17. *Let R be a valuation ring. Suppose $I_i \lhd R$ is a collection of ideals indexed by $\mathcal{I}$. Suppose $W = \cap_{i\in\mathcal{I}}\mathrm{W}(I_i)$ and for all $J \lhd R$, $\mathrm{W}(J) \neq W$. Then W has a generic point. In fact it has a generic point of the form $\mathrm{E}(R/\mathfrak{p})$ for some prime ideal $\mathfrak{p}$.*

Proof. Let $\mathfrak{q}$ be the union of all primes $\mathfrak{p}$ such that $\mathrm{E}(R/\mathfrak{p}) \in W$. Hence $\mathfrak{q} \subseteq I_i$ for all $i \in \mathcal{I}$. But this means $\mathfrak{q} \subsetneq I_i$ for all $i \in \mathcal{I}$ since if $\mathfrak{q} = I_j$ then $W(I_j) = \cap_{i\in\mathcal{I}}\mathrm{W}(I_i)$. Therefore $\mathrm{E}(R/\mathfrak{q}) \in W$ and $\mathfrak{q}$ is the largest prime strictly contained in I_i for all $i \in \mathcal{I}$.

Suppose $E' \in W$. Then $\mathrm{Att}E' \subseteq I_i$ for all $i \in \mathcal{I}$. Hence $\mathrm{Att}E' \subsetneq I_i$ for all $i \in \mathcal{I}$ since if $I_i = \mathrm{Att}E'$ then $\mathrm{W}(I_i) \subseteq \cap_{i\in\mathcal{I}}\mathrm{W}(I_i)$, so $\mathrm{W}(I_i) = \cap_{i\in\mathcal{I}}\mathrm{W}(I_i)$. Therefore $\mathrm{Att}E' \subseteq \mathfrak{q}$.

Suppose $K \lhd R$ such that $E(R/\mathfrak{q}) \in \mathrm{W}(K)$. Then $\mathfrak{q} \subsetneq K$. So, if $E' \in W$ then $E' \in \mathrm{W}(K)$ since $\mathrm{Att}E' \subseteq \mathfrak{q}$ and $\mathfrak{q} \subsetneq K$. □

Theorem 5.0.18. *Let R be a valuation ring. Then* $\textbf{ideals}_R$ *and* $\mathrm{Zg}_R|_{\mathrm{inj}}$ *are sober.*

Proof. We have shown that every closed set in $\textbf{ideals}_R$ has a generic point. So $\textbf{ideals}_R$ is sober.

Suppose X is a closed subset of $\mathrm{Zg}_R|_{\mathrm{inj}}$. Then X is closed in $\mathbf{ideals}_R$. Hence X has an $\mathbf{ideals}_R$ generic point x. Since $X = \mathrm{cl}_{\mathbf{ideals}_R} x \subseteq \mathrm{cl}_{\mathrm{Zg}_R|_{\mathrm{inj}}} x$ and $x \in X$, $X = \mathrm{cl}_{\mathrm{Zg}_R|_{\mathrm{inj}}} x$. Therefore $\mathrm{Zg}_R|_{\mathrm{inj}}$ is sober. $\square$

5.1 Examples and possible behaviour

In this section we give an example of a valuation ring R with $\mathrm{Zg}_R|_{\mathrm{inj}}/\approx$ not homeomorphic to $\mathrm{Spec}^* R$. We also show that when R is a valuation ring, $\mathrm{Zg}_R|_{\mathrm{inj}}/\approx$ differs from $\mathrm{Spec}^* R$ by at most one point. That is, up to topological indistinguishability the fibres of the continuous map defined in 5.0.3 from $\mathrm{Zg}_R|_{\mathrm{inj}}$ to $\mathrm{Spec}^* R$ are all singletons except one.

In a valuation ring R the set of zero divisors union zero, ZD, is an ideal since if $x \in \mathrm{ZD}$ and $r \in R$ then $xr \in \mathrm{ZD}$.

Lemma 5.1.1. *If R is a valuation ring then all pp-definable ideals are either principal or the annihilator of some $s \in R$.*

Proof. Over valuation rings (as in the special case of valuation domains, see 3.2.2) all pp-1-formulae are equivalent to a finite conjunction of formulae of the form $s|xr$ for some $s, r \in R$. The solution set of $s|xr$ in R is $(sR : r)$ (see [EH95]). The ideals of a valuation ring are totally ordered, so all pp-definable ideals are of the form $(sR : r)$ for some $s, r \in R$. If $r \in sR$ then $(sR : r) = R$. Therefore we may assume $s = rt$ for some $t \in R$.

<u>Claim</u>: $(rtR : r) = tR + \mathrm{ann}_R r$.

Suppose $\mu \in (rtR : r)$. Then $\mu r \in rtR$. Hence there exists $\lambda \in R$ such that $(\mu - t\lambda)r = 0$. Therefore $\mu \in tR + \mathrm{ann}_R r$. So $(rtR : r) \subseteq tR + \mathrm{ann}_R r$. It is clear that $\mathrm{ann}_R r \subseteq (rtR : r)$ and $tR \subseteq (rtR : r)$. Therefore $(rtR : r) = tR + \mathrm{ann}_R r$.

So, since the ideals in R are totally ordered, $(rtR : r) = tR$ or $(rtR : r) = \mathrm{ann}_R r$. So all pp-definable ideals are either principal or the annihilator of some element in R. $\square$

Lemma 5.1.2. *Let R be a valuation ring and $s \in R$. If $\mathrm{ann}_R s$ is a prime ideal then $\mathrm{ann}_R s = \mathrm{ZD}$.*

Proof. Suppose $s \in R$ and $\mathrm{ann}_R s$ is a prime ideal. Clearly $\mathrm{ann}_R s \subseteq \mathrm{ZD}$. Suppose $\mu \in \mathrm{ZD}$. Then there exists $t \in R \backslash \{0\}$ such that $\mu t = 0$. If $t|s$ then $\mu \in \mathrm{ann}_R s$. Otherwise $t = sr$ for some $r \in R \backslash \mathrm{ann}_R s$. Therefore $\mu sr = 0$ so $\mu r \in \mathrm{ann}_R s$. Hence, since $\mathrm{ann}_R s$ is prime, $\mu \in \mathrm{ann}_R s$. $\square$

Lemma 5.1.3. *Let R be a valuation ring. Suppose $r \in R$ and rR is a prime ideal. Then $\mathrm{W}(rR)$ has a generic point $\mathrm{E}(R/\mathfrak{p})$ for some prime ideal $\mathfrak{p}$.*

Proof. Suppose $\mathfrak{p}$ is the largest prime ideal not containing r. Then for all $K \lhd R$, $\mathrm{E}(R/\mathfrak{p}) \in \mathrm{W}(K)$ implies $\mathfrak{p} \subsetneq K$. Suppose $E' \in \mathrm{W}(rR)$. Then for all $w \in E' \backslash \{0\}$, $\mathrm{ann}_R w \subsetneq rR$. Therefore $r \notin \mathrm{Att}E'$. Hence $\mathrm{Att}E' \subseteq \mathfrak{p}$. So if $\mathrm{E}(R/\mathfrak{p}) \in \mathrm{W}(K)$ for some $K \lhd R$ then $\mathrm{Att}E' \subseteq \mathfrak{p} \subsetneq K$. Hence $E' \in \mathrm{W}(K)$. Therefore $\mathrm{E}(R/\mathfrak{p})$ specialises to E'. Hence $\mathrm{E}(R/\mathfrak{p})$ is a generic point of $\mathrm{W}(rR)$. $\square$

Remark 5.1.4. *Let R be a valuation ring. Suppose E is an indecomposable injective R-module with $\mathrm{Att}E = \mathfrak{p}$ but E not isomorphic to $\mathrm{E}(R/\mathfrak{p})$. Then, for all $w \in E \backslash \{0\}$, $\mathrm{ann}_R w \subsetneq \mathrm{Att}E$ since if $\mathrm{ann}_R w = \mathfrak{p}$ then $E \cong \mathrm{E}(R/\mathfrak{p})$. Therefore $E \in \mathrm{W}(K)$ if and only if $\mathfrak{p} \subseteq K$. Hence if E, F are indecomposable injective R-modules both with attached prime $\mathfrak{p}$ but neither is isomorphic to $\mathrm{E}(R/\mathfrak{p})$ then E and F are topologically indistinguishable in* ***ideals***$_R$ *and therefore in $\mathrm{Zg}_R|_{\mathrm{inj}}$.*

The following proposition shows that $\mathrm{Zg}_R|_{\mathrm{inj}}$ is very similar to $\mathrm{Spec}^* R$ when R is a valuation ring.

Proposition 5.1.5. *Let R be a valuation ring. Suppose $\mathfrak{p}$ and $\mathfrak{q}$ are non-equal prime ideals and E is an indecomposable injective R-module with attached prime $\mathfrak{p}$ and topologically distinguishable in $\mathrm{Zg}_R|_{\mathrm{inj}}$ from $\mathrm{E}(R/\mathfrak{p})$. Then if F is an indecomposable injective R-module with attached prime $\mathfrak{q}$ then F is topologically indistinguishable from $\mathrm{E}(R/\mathfrak{q})$.*

Proof. By lemma 5.0.12, 5.0.17, 5.1.3, the only closed sets which may not have a generic point isomorphic to $\mathrm{E}(R/\mathfrak{p})$ for some prime ideal $\mathfrak{p}$ are the basic open sets

W($\mathfrak{q}$) where $\mathfrak{q}$ is prime. By 5.1.1, if K is a pp-definable ideal then K is either principal or the annihilator of some element $s \in R$. Therefore, if W(K) has a generic point not isomorphic to E($R/\mathfrak{p}$) for some prime $\mathfrak{p}$ then $K = \mathrm{ann}_R s$ for some $s \in R$ and K is prime. So, by lemma 5.1.2, $\mathrm{ann}_R s = \mathrm{ZD}$. Hence there is only one closed set in $\mathrm{Zg}_R|_{\mathrm{inj}}$ with no generic point isomorphic to E($R/\mathfrak{p}$) for some prime $\mathfrak{p}$. □

Note that the above proposition shows that at most one closed set can have a generic point topologically distinguishable from an indecomposable injective of the form E($R/\mathfrak{p}$) where $\mathfrak{p}$ is a prime ideal of R. Therefore, up to topological indistinguishability, there is only one point topologically distinguishable from a point of the form E($R/\mathfrak{p}$) where $\mathfrak{p}$ is a prime ideal of R. So the fibres of the continuous map defined in 5.0.3 from $\mathrm{Zg}_R|_{\mathrm{inj}}$ to $\mathrm{Spec}^* R$ are all singletons except at most one.

We now give an example of a valuation ring for which $\mathrm{Zg}_R|_{\mathrm{inj}}/\approx$ is not homeomorphic to $\mathrm{Spec}^* R$. In order to do this we recall theorem 2.4.5 ([Kru32]) which is incredibly useful for constructing examples.

Example 5.1.6. *We want to find an example of a valuation ring R such that there exists a prime ideal $\mathfrak{p}$ and infinitely many indecomposable injectives E_i with* $\mathrm{Att}E_i = \mathfrak{p}$ *and each E_i is topologically distinguishable from* E($R/\mathfrak{p}$) *in the Ziegler topology.*

Let S be a valuation domain with value group $\mathbb{Q}$, such a ring exists by 2.4.5. Let $C \subseteq (1,2) \subseteq \mathbb{R}$ be an uncountable set of irrational numbers such that if $c, d \in C$ then $c - d \notin \mathbb{Q}$ (such a set exists since $\mathbb{R}/\mathbb{Q}$ is uncountable).

Let $J = \{s \in S \mid v(s) > 2\}$ and $R = S/J$. Then R is a valuation ring. Let $s \in S$ be such that $v(s) = 2$. Then $s \notin J$. Suppose r is in the maximal ideal of S. Then $v(r) > 0$. Hence $v(rs) = v(r) + v(s) > 2$, so $rs \in J$. Therefore $(J : s)$ contains the maximal ideal of S. So since $s \notin J$, $(J : s)$ is the maximal ideal of S. Therefore $\mathrm{ann}_{S/J}(s + J)$ *is the maximal ideal in S/J and is pp-definable.*

Let $I_c = \{r \in S \mid v(r) > c\}$ for each $c \in C$. Note that for all $c \in C$, $I_c \supseteq J$ since $c < 2$. So S/I_c is an S/J-module. We will show that for any non-equal $c, d \in C$ the S/J-injective hulls of S/I_c and S/I_d are not isomorphic. If the injective hulls of S/I_d and S/I_c were isomorphic then lemma 2.1.9 would imply that there exists $\mu \notin I_c$

and $\lambda \notin I_d$ such that $(I_c/J : \mu + J) = (I_d/J : \lambda + J)$. Suppose, for a contradiction, that there exists $\mu \notin I_c$ and $\lambda \notin I_d$ such that $(I_c/J : \mu + J) = (I_d/J : \lambda + J)$. Since $\mu \notin I_c$, $c - v(\mu) \geq 0$ and since $\lambda \notin I_d$, $d - v(\lambda) \geq 0$. Note that $c - v(\mu) \neq d - v(\lambda)$ since $v(\mu), v(\lambda) \in \mathbb{Q}$. We may assume $c - v(\mu) > d - v(\lambda)$. Take $q \in \mathbb{Q}$ such that $c - v(\mu) > q > d - v(\lambda)$ and $s \in S$ such that $v(s) = q$. Then $c > v(s) + v(\mu) = v(s\mu)$, so $s\mu \notin I_c$. Hence $s \notin (I_c : \mu)$, so $s + J \notin (I_c/J : \mu + J)$. As $v(s\lambda) = v(s) + v(\lambda) > d$, $s\lambda \in I_d$. Therefore $s \in (I_d : \lambda)$ so $s + J \in (I_d/J : \lambda + J)$. Contradicting $(I_c/J : \mu + J) = (I_d/J : \lambda + J)$. Therefore, for any non-equal $c, d \in C$, the S/J-injective hulls of S/I_c and S/I_d are not isomorphic.

For each $c \in C$, let E_c be the S/J-injective hull of S/I_c. It remains to show that for each $c \in C$, E_c is topologically distinguishable from E, the R-injective hull of $R/\mathfrak{m}$ where $\mathfrak{m}$ is the maximal ideal of R. We have shown that $\mathfrak{m}$ is pp-definable. So $\mathcal{O}(\mathfrak{m})$ is an open set in $\mathrm{Zg}_R|_{\mathrm{inj}}$ and $E \in \mathcal{O}(\mathfrak{m})$. In order to show that for all $c \in C$, $E_c \notin \mathcal{O}(\mathfrak{m})$, it is enough to show that for all $w \in E_c \backslash \{0\}$, $\mathrm{ann}_{S/J} w \subsetneq \mathfrak{m}$. Equivalently, for all $\lambda \notin I_c$, $(I_c/J : \lambda + J) \subsetneq \mathfrak{m}$. Note that if $\lambda \notin I_c$ then $c > v(\lambda)$ since $c \geq v(\lambda)$, $v(\lambda) \in \mathbb{Q}$ and $c \notin \mathbb{Q}$. Take $q \in \mathbb{Q}$ such that $c - v(\lambda) > q > 0$ and $r \in R$ such that $v(r) = q$. Note that since $q > 0$, r is not a unit. Since $v(r\lambda) = v(r) + v(\lambda) < c$, $r\lambda \notin I_c$. Hence $r\lambda + J \notin I_c/J$. So $r + J \notin (I_c/J : \lambda + J)$ and $r + J \in \mathfrak{m}$. Therefore $(I_c/J : \lambda + J) \subsetneq \mathfrak{m}$.

5.2 Prüfer rings

A Prüfer ring is a commutative ring R such that for all $\mathfrak{p} \lhd R$ prime, $R_\mathfrak{p}$ is a valuation ring.

Proposition 5.2.1. *[Pop73] The following properties of a ring morphism $u : A \to B$ are equivalent:*

(i) u is an epimorphism;

(ii) The canonical map $m : B \bigotimes_A B \to B$, with $m(b \otimes b') = bb'$ is bijective;

(iii) The functor, induced by restriction of scalars, $u_* : \mathrm{Mod}-B \to \mathrm{Mod}-A$ is full.

Lemma 5.2.2. *Let R be a commutative ring, $\mathfrak{p}$ a prime ideal and $\eta : R \to R_{\mathfrak{p}}$ the localisation map. Then any indecomposable injective $R_{\mathfrak{p}}$-module is an indecomposable injective R-module via η. Thus we have a map $t : \mathrm{inj}_{R_{\mathfrak{p}}} \to \mathrm{inj}_R$. The map t is an embedding with image $\{E \in \mathrm{inj}_R \mid \mathrm{Att}E \subseteq \mathfrak{p}\}$.*

Proof. Since $\eta : R \to R_{\mathfrak{p}}$ is an epimorphism, $\mathrm{Mod}-R_{\mathfrak{p}}$ embeds as a full subcategory of $\mathrm{Mod}-R$ by η_*. Therefore, following [Pre09], if $\mathrm{End}_R(M)$ contains a non-zero, non-identity idempotent, then so does $\mathrm{End}_{R_{\mathfrak{p}}}M$. Therefore if M is an indecomposable $R_{\mathfrak{p}}$-module then M is an indecomposable R-module.

Since $\eta : R \to R_{\mathfrak{p}}$ is flat, if E is an injective $R_{\mathfrak{p}}$-module then E is an injective R-module (via η), by [Lam99, 3.6A].

It remains to show that the image of t is $\{E \in \mathrm{inj}_R \mid \mathrm{Att}E \subseteq \mathfrak{p}\}$. If E is an indecomposable injective $R_{\mathfrak{p}}$-module then all $r \notin \mathfrak{p}$ act on E by multiplication as an automorphism. So the image of t is contained in $\{E \in \mathrm{inj}_R \mid \mathrm{Att}E \subseteq \mathfrak{p}\}$. Suppose $E \in \{E \in \mathrm{inj}_R \mid \mathrm{Att}E \subseteq \mathfrak{p}\}$. Then for all $r \notin \mathfrak{p}$, the action of r by multiplication on E is invertible. So $E \otimes_R R_{\mathfrak{p}}|_R \cong E$. Therefore the image of t is $\{E \in \mathrm{inj}_R \mid \mathrm{Att}E \subseteq \mathfrak{p}\}$. □

Lemma 5.2.3. *Let R be a commutative ring. Then* ***ideals***$_{R_{\mathfrak{p}}}$ *is homeomorphic to* ***ideals***$_R \cap \{E \in \mathrm{inj}_R \mid \mathrm{Att}E \subseteq \mathfrak{p}\}$.

Proof. Let $t : \mathrm{inj}_{R_{\mathfrak{p}}} \to \mathrm{inj}_R$ be as in 5.2.2. Then the inverse of t takes $E \in \mathrm{inj}_R \cap \{E \in \mathrm{inj}_R \mid \mathrm{Att}E \subseteq \mathfrak{p}\}$ to $E \otimes_R R_{\mathfrak{p}}$.

We now show that for any $I \lhd R$

$$t^{-1}(\mathcal{O}_R(I) \cap \{E \in \mathrm{inj}_R \mid \mathrm{Att}E \subseteq \mathfrak{p}\}) = \mathcal{O}_{R_{\mathfrak{p}}}(IR_{\mathfrak{p}}).$$

<u>Claim</u>: For any $E \in \mathrm{inj}_{R_{\mathfrak{p}}}$, $\mathrm{Hom}_R(R/I, E|_R) \cong \mathrm{Hom}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/IR_{\mathfrak{p}}, E_{R_{\mathfrak{p}}})$.

As $R_{\mathfrak{p}}$-modules, $E_{R_{\mathfrak{p}}} \cong \mathrm{Hom}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}, E_{R_{\mathfrak{p}}})$, so as R-modules $E|_R \cong \mathrm{Hom}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}, E_{R_{\mathfrak{p}}})|_R$. Therefore $\mathrm{Hom}_R(R/I, E|_R) \cong \mathrm{Hom}_R(R/I, \mathrm{Hom}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}, E_{R_{\mathfrak{p}}}))$. So by the hom-tensor adjunction $\mathrm{Hom}_R(R/I, \mathrm{Hom}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}, E_{R_{\mathfrak{p}}})) \cong \mathrm{Hom}_{R_{\mathfrak{p}}}(R/I \otimes R_{\mathfrak{p}}, E_{R_{\mathfrak{p}}})$. For any $I \lhd R$, $R/I \otimes R_{\mathfrak{p}} \cong R_{\mathfrak{p}}/IR_{\mathfrak{p}}$ by [Osb00, Proposition 2.2]. Therefore $\mathrm{Hom}_R(R/I, E|_R) \cong \mathrm{Hom}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/IR_{\mathfrak{p}}, E_{R_{\mathfrak{p}}})$.

Suppose $E \in t^{-1}(\mathcal{O}_R(I) \cap \{E \in \mathrm{inj}_R \mid \mathrm{Att}E \subseteq \mathfrak{p}\})$. Then $t(E) = E|_R \in \mathcal{O}(I)$, so $\mathrm{Hom}_R(R/I, E) \neq 0$. By the above claim this is if and only if $\mathrm{Hom}_{R_\mathfrak{p}}(R_\mathfrak{p}/IR_\mathfrak{p}, E_{R_\mathfrak{p}}) \neq 0$. So $t^{-1}(\mathcal{O}_R(I) \cap \{E \in \mathrm{inj}_R \mid \mathrm{Att}E \subseteq \mathfrak{p}\}) = \mathcal{O}_{R_\mathfrak{p}}(IR_\mathfrak{p})$.

It remains to show that for any $J \lhd R_\mathfrak{p}$, $t(\mathcal{O}_{R_\mathfrak{p}}(J))$ is open. But, since any ideal $J \lhd R_\mathfrak{p}$ is equal to $(J \cap R)R_\mathfrak{p}$ by [Mat89, Ch2 Theorem 4.1] and $J \cap R$ is ideal in R, $t(\mathcal{O}_{R_\mathfrak{p}}(J)) = \mathcal{O}_R(J \cap R)$.

Therefore t is a homeomorphism. □

Lemma 5.2.4. *Let R be a commutative ring. Then*

$$\mathrm{Zg}_{R_\mathfrak{p}}|_{\mathrm{inj}} \textit{ and } \mathrm{Zg}_R|_{\mathrm{inj}} \cap \{E \in \mathrm{inj}_R \mid \mathrm{Att}E \subseteq \mathfrak{p}\}$$

are homeomorphic.

Proof. The epimorphism $f : R \to R_\mathfrak{p}$ induces a continuous embedding g from $\mathrm{Zg}_{R_\mathfrak{p}}$ to Zg_R, (see 2.3.26). The image of $\mathrm{inj}_{R_\mathfrak{p}}$ under g is $\mathrm{inj}_R \cap \{E \in \mathrm{inj}_R \mid \mathrm{Att}E \subseteq \mathfrak{p}\}$ by lemma 5.2.2.

Therefore

$$\mathrm{Zg}_{R_\mathfrak{p}}|_{\mathrm{inj}} \text{ and } \mathrm{Zg}_R|_{\mathrm{inj}} \cap \{E \in \mathrm{inj}_R \mid \mathrm{Att}E \subseteq \mathfrak{p}\}$$

are homeomorphic. □

Proposition 5.2.5. *Let R be a commutative ring. Then*

1. *The space $\mathrm{Zg}_R|_{\mathrm{inj}}$ is sober if and only if $\mathrm{Zg}_{R_\mathfrak{p}}|_{\mathrm{inj}}$ is sober for all primes $\mathfrak{p} \lhd R$.*

2. *The space $\mathbf{ideals}_R$ is sober if and only if $\mathbf{ideals}_{R_\mathfrak{p}}$ is sober for all primes $\mathfrak{p} \lhd R$.*

Proof. First recall, proposition 2.3.20, that all irreducible closed sets in $\mathrm{Spec}^* R$ are of the form

$$\{\mathfrak{q} \in \mathrm{Spec}^* R \mid \mathfrak{q} \subseteq \mathfrak{p}\}$$

where $\mathfrak{p} \lhd R$ is prime.

Note that for any $\mathfrak{p} \lhd R$, $\{E \in \mathrm{inj}_R \mid \mathrm{Att}E \subseteq \mathfrak{p}\}$ is a closed subset of both $\mathbf{ideals}_R$ and $\mathrm{Zg}_R|_{\mathrm{inj}}$ since it is the pre-image of $\{\mathfrak{q} \in \mathrm{Spec}^* R \mid \mathfrak{q} \subseteq \mathfrak{p}\}$ which is closed in $\mathrm{Spec}^* R$.

Since for any prime ideal $\mathfrak{p}$, $\mathbf{ideals}_{R_\mathfrak{p}}$ is homeomorphic to the $\mathbf{ideals}_R$ closed set $\mathbf{ideals}_R \cap \{E \in \text{inj} \mid \text{Att}E \subseteq \mathfrak{p}\}$, if $\mathbf{ideals}_R$ is sober then $\mathbf{ideals}_{R_\mathfrak{p}}$ is sober. By the same argument, if $\text{Zg}_R|_{\text{inj}}$ is sober then for any prime ideal $\mathfrak{p}$, $\text{Zg}_{R_\mathfrak{p}}|_{\text{inj}}$ is sober.

Suppose that for all prime $\mathfrak{p} \lhd R$, $\mathbf{ideals}_{R_\mathfrak{p}}$ is sober. Suppose X is an irreducible closed subset of $\mathbf{ideals}_R$. Then the image of X under the map $s : \mathbf{ideals}_R \to \text{Spec}^* R$ given in 5.0.6 is irreducible, so the closure of $s(X)$ is irreducible. Hence there exists a prime $\mathfrak{p} \lhd R$ such that the closure of $s(X)$ in $\text{Spec}^* R$ is $\{\mathfrak{q} \in \text{Spec}^* \mid \mathfrak{q} \subseteq \mathfrak{p}\}$ by proposition 2.3.20. Therefore $X \subseteq \{E \in \text{inj}_R \mid \text{Att}E \subseteq \mathfrak{p}\}$. Since $\{E \in \text{inj}_R \mid \text{Att}E \subseteq \mathfrak{p}\}$ is homeomorphic to $\mathbf{ideals}_{R_\mathfrak{p}}$ and $\mathbf{ideals}_{R_\mathfrak{p}}$ is sober, X has a generic point. Therefore $\mathbf{ideals}_R$ is sober.

By the same argument, using 5.0.3 in place of 5.0.6, if for all primes $\mathfrak{p} \lhd R$, $\text{Zg}_{R_\mathfrak{p}}|_{\text{inj}}$ is sober then $\text{Zg}_R|_{\text{inj}}$ is sober.

□

Proposition 5.2.6. *Let R be a Prüfer ring. Then $\boldsymbol{ideals}_R$ and $\text{Zg}_R|_{\text{inj}}$ are sober.*

Proof. For any prime $\mathfrak{p} \lhd R$, $R_\mathfrak{p}$ is a valuation ring. Therefore, by 5.2.5 and 5.0.18, $\mathbf{ideals}_R$ and $\text{Zg}_R|_{\text{inj}}$ are sober. □

5.3 Fibre products

Throughout this section fix V a valuation ring, let $\mathfrak{m}$ be the maximal ideal and $k = V/\mathfrak{m}$ the residue field. Let $V \underset{k}{\times} V$ denote the fibre product of two copies of the valuation ring V over its residue field k. Note that $V \underset{k}{\times} V$ is the sub-ring of $V \times V$ consisting of elements $(x, y) \in V \times V$ with $x - y \in \mathfrak{m}$ and $V \underset{k}{\times} V$ is a local ring with maximal ideal $\mathfrak{m} \times \mathfrak{m}$.

We will show that both $\mathbf{ideals}_{V \underset{k}{\times} V}$ and $\text{Zg}_{V \underset{k}{\times} V}|_{\text{inj}}$ is sober.

Lemma 5.3.1. *Let R be a commutative ring and $I \lhd R$. Then $\mathcal{O}_R(I)$ and $\boldsymbol{ideals}_{R/I}$ are homeomorphic.*

Proof. Let $f : R \to R/I$ be the quotient map. Then any R/I-module is an R-module via f. If E is a uniform R/I-module then E is a uniform R-module. Let

$t : \mathbf{ideals}_{R/I} \to \mathbf{ideals}_R : F_{R/I} \mapsto \mathrm{E}_R(F)$. This map is well-defined since the injective hull of a uniform module is indecomposable. Suppose F_1, F_2 are indecomposable injective R/I-modules and $t(F_1) \cong t(F_2)$. Then there exists $w_1 \in F_1 \backslash \{0\}$ and $w_2 \in F_2 \backslash \{0\}$ such that $\mathrm{ann}_R w_1 = \mathrm{ann}_R w_2$. Therefore $\mathrm{ann}_{R/I} w_1 = \mathrm{ann}_{R/I} w_2$ so $F_1 \cong F_2$. Hence t is an injective map. It remains to show that the image of t is $\mathcal{O}(I)$ and t is a homeomorphism.

Suppose $E_R \in \mathcal{O}(I)$. Then there exists $w \in E \backslash \{0\}$ such that $\mathrm{ann}_R w \supseteq I$. Note that $R/\mathrm{ann}_R w$ is a uniform R/I-module. So $\mathrm{im} t \subseteq \mathcal{O}(I)$. If $K \lhd R$ is irreducible and $K \supseteq I$ then R/K is a uniform R/I-module. Hence its R/I injective hull E is indecomposable and $t(E)$ is the R injective hull of R/K. Therefore $\mathrm{im} t \supseteq \mathcal{O}(I)$.

We now show t is a homeomorphism onto $\mathcal{O}_R(I)$.

The open sets $\mathcal{O}_R(J)$ where $J \lhd R$ and $I \subseteq J$ are a basis for $\mathcal{O}_R(I)$ since $\mathcal{O}_R(I) \cap \mathcal{O}_R(K) = \mathcal{O}_R(I + K)$ for any $K \lhd R$.

Claim: $t(\mathcal{O}_{R/I}(J/I)) = \mathcal{O}_R(J)$ for all $J \lhd R$ with $J \supseteq I$.

Suppose $E \in t(\mathcal{O}_{R/I}(J/I))$. Then there exists F an indecomposable injective R/I-modules such that $\mathrm{E}_R(F) = E$ and $w \in F \backslash \{0\}$ such that $\mathrm{ann}_{R/I} w \supseteq J/I$. Therefore $\mathrm{ann}_R w \supseteq J$. Hence $E \in \mathcal{O}_R(J)$.

Suppose $E \in \mathcal{O}(J)$. Then $E \in \mathcal{O}(I)$ so there exists F an indecomposable injective R/I-module with $\mathrm{E}_R(F) = E$ and $w \in E \backslash \{0\}$ with $\mathrm{ann}_R w \supseteq J$. Since E is uniform, $wR \cap F \neq 0$. Take non-zero $u \in wR \cap F$. Then $\mathrm{ann}_R u \supseteq \mathrm{ann}_R w \supseteq J$. Hence $u \in F$ and $\mathrm{ann}_{R/I} u \supseteq J/I$. Therefore $F \in \mathcal{O}_{R/I}(J/I)$. Hence $E \in \mathcal{O}_{R/I}(J/I)$.

It follows from the claim that t is a homeomorphism onto $\mathcal{O}_R(I)$ $\square$

Lemma 5.3.2. *Let V be a valuation ring with residue field k. Then $\frac{V \underset{k}{\times} V}{\mathfrak{m} \times 0}$ and $\frac{V \underset{k}{\times} V}{0 \times \mathfrak{m}}$ are isomorphic to V.*

Proof. The map $f : V \underset{k}{\times} V \to V; (a, b) \mapsto a$ is a homomorphism and $f(a, b) = 0$ if and only if $a = 0$. For any $b \in V$, $(0, b) \in V \underset{k}{\times} V$ if and only if $0 - b = -b \in \mathfrak{m}$. Therefore the kernel of f is $0 \times \mathfrak{m}$. Since for any $a \in V$, $(a, a) \in V \underset{k}{\times} V$, f is surjective. Therefore $\frac{V \underset{k}{\times} V}{0 \times \mathfrak{m}}$ is isomorphic to V. $\square$

Lemma 5.3.3. *Let V be a valuation ring with maximal ideal $\mathfrak{m}$ and residue field k. If $X \subseteq \textbf{\textit{ideals}}_{V \underset{k}{\times} V}$ is closed in $\textbf{\textit{ideals}}_{V \underset{k}{\times} V}$ then*

(i) there exist $x, y \in \textbf{\textit{ideals}}_{V \underset{k}{\times} V}$ such that $X = \mathrm{cl}_{\textbf{\textit{ideals}}}(x) \cup \mathrm{cl}_{\textbf{\textit{ideals}}}(y)$.

(ii) if X is irreducible then X has a generic point.

Hence $\textbf{\textit{ideals}}_{V \underset{k}{\times} V}$ is sober.

Proof. (i)First note that $\mathcal{O}(\mathfrak{m}\times 0)\cup\mathcal{O}(0\times\mathfrak{m}) = \mathcal{O}(\mathfrak{m}\times 0\cap 0\times\mathfrak{m}) = \mathcal{O}(0) = \textbf{ideals}_{V \underset{k}{\times} V}$. By lemma 5.3.2, $\frac{V \underset{k}{\times} V}{\mathfrak{m}\times 0} \cong V$ and $\frac{V \underset{k}{\times} V}{0\times\mathfrak{m}} \cong V$. Therefore, by lemma 5.3.1, $\mathcal{O}(\mathfrak{m} \times 0)$ is homeomorphic to $\textbf{ideals}_V$ and $\mathcal{O}(0 \times \mathfrak{m})$ is homeomorphic to $\textbf{ideals}_V$.

Since V is a valuation ring, every closed set in $\textbf{ideals}_V$ is irreducible. Let X be a closed set in $\textbf{ideals}_{V \underset{k}{\times} V}$. Then $X \cap \mathcal{O}(\mathfrak{m} \times 0)$ is irreducible in the subspace topology and has a generic point x. Hence $\mathrm{cl}_{\textbf{ideals}_{V \underset{k}{\times} V}}(x) \supseteq X \cap \mathcal{O}(\mathfrak{m} \times 0)$. Also, $X \cap \mathcal{O}(0\times\mathfrak{m})$ is irreducible in the subspace topology and has a generic point y. Hence $\mathrm{cl}_{\textbf{ideals}_{V \underset{k}{\times} V}}(y) \supseteq X \cap \mathcal{O}(0 \times \mathfrak{m})$. Therefore $X \subseteq \mathrm{cl}_{\textbf{ideals}_{V \underset{k}{\times} V}}(x) \cup \mathrm{cl}_{\textbf{ideals}_{V \underset{k}{\times} V}}(y)$. Since $x, y \in X$, $X = \mathrm{cl}_{\textbf{ideals}_{V \underset{k}{\times} V}}(x) \cup \mathrm{cl}_{\textbf{ideals}_{V \underset{k}{\times} V}}(y)$.
(ii) Now, let X be an irreducible closed set in $\textbf{ideals}_{V \underset{k}{\times} V}$. By part (i), there exists $x, y \in \textbf{ideals}_{V \underset{k}{\times} V}$ such that $X = \mathrm{cl}_{\textbf{ideals}_{V \underset{k}{\times} V}}(x) \cup \mathrm{cl}_{\textbf{ideals}_{V \underset{k}{\times} V}}(y)$. Since X is irreducible, either $X = \mathrm{cl}_{\textbf{ideals}_{V \underset{k}{\times} V}}(x)$ or $X = \mathrm{cl}_{\textbf{ideals}_{V \underset{k}{\times} V}}(y)$. So either x is a generic point of X or y is a generic point of X. □

Lemma 5.3.4. *Let V be a valuation ring with maximal ideal $\mathfrak{m}$ and residue field k. Then $\mathrm{Zg}_{V \underset{k}{\times} V}|_{\mathrm{inj}}$ is sober.*

Proof. Suppose $X \subseteq \mathrm{Zg}_{V \underset{k}{\times} V}|_{\mathrm{inj}}$ is an irreducible closed set. Then X is closed in $\textbf{ideals}_{V \underset{k}{\times} V}$. So by lemma 5.3.3, there exists $x, y \in X$ such that $X = \mathrm{cl}_{\textbf{ideals}}(x) \cup \mathrm{cl}_{\textbf{ideals}}(y)$. Now $\mathrm{cl}_{\textbf{ideals}}(x) \subseteq \mathrm{cl}_{\mathrm{Zg}_{V \underset{k}{\times} V}|_{\mathrm{inj}}}(x)$ and $\mathrm{cl}_{\textbf{ideals}}(y) \subseteq \mathrm{cl}_{\mathrm{Zg}_{V \underset{k}{\times} V}|_{\mathrm{inj}}}(y)$. Hence $X = \mathrm{cl}_{\mathrm{Zg}_{V \underset{k}{\times} V}|_{\mathrm{inj}}}(x) \cup \mathrm{cl}_{\mathrm{Zg}_{V \underset{k}{\times} V}|_{\mathrm{inj}}}(y)$. Therefore, since X is irreducible, $X = \mathrm{cl}_{\mathrm{Zg}_{V \underset{k}{\times} V}|_{\mathrm{inj}}}(x)$ or $X = \mathrm{cl}_{\mathrm{Zg}_{V \underset{k}{\times} V}|_{\mathrm{inj}}}(y)$. So $\mathrm{Zg}_{V \underset{k}{\times} V}|_{\mathrm{inj}}$ is sober. □

5.3.1 Examples

In this section we will give an example of a ring R with 3 indecomposable injectives with the same attached prime which are pairwise topologically distinguishable in $\mathrm{Zg}_R|_{\mathrm{inj}}$.

Lemma 5.3.5. *Let $R = V \underset{k}{\times} V$ where V is a valuation ring with maximal ideal $\mathfrak{m}$. Then, for all $(u, v) \in R$, (u, v) is a unit if and only if u and v are units in V.*

Proof. Suppose $(u, v) \in R$ and (u, v) is a unit. Then there exists $(k, l) \in R$ such that $(u, v)(k, l) = (1, 1)$. Hence $uk = 1$ and $vl = 1$. So u and v are both units in V. Suppose $(u, v) \in R$ and u, v are units in V. Then there exists $k, l \in V$ such that $uk = 1$ and $vl = 1$. Since $(u, v) \in R$, $u - v \in \mathfrak{m}$. Therefore $uv(k - l) = v - u \in \mathfrak{m}$. Since $u, v \notin \mathfrak{m}$, $k - l \in \mathfrak{m}$. Hence $(k, l) \in R$ and $(u, v).(k, l) = (1, 1)$. So (u, v) is a unit in R. □

Lemma 5.3.6. *Let $R = V \underset{k}{\times} V$ where V is a valuation ring with maximal ideal $\mathfrak{m}$. Then, for all $I \lhd V$, $I \times \mathfrak{m}$ and $\mathfrak{m} \times I$ are irreducible ideals in R.*

Proof. Suppose $K, L \lhd R$, $I \times \mathfrak{m} \subsetneq K$ and $I \times \mathfrak{m} \subsetneq L$. Take $(k_1, k_2) \in K \backslash I \times \mathfrak{m}$ and $(l_1, l_2) \in L \backslash I \times \mathfrak{m}$. Then $k_1, k_2, l_1, l_2 \in \mathfrak{m}$, so $(k_1, 0) = (k_1, k_2) - (0, k_2) \in K$ and $(l_1, 0) = (l_1, l_2) - (0, l_2) \in L$ since $(0, k_2) \in I \times \mathfrak{m}$ and $(0, l_1) \in I \times \mathfrak{m}$. Since V is a valuation ring, either $k_1 \in l_1 V$ or $l_1 \in k_1 V$. Without loss of generality we may assume $k_1 = l_1 r$ for some $r \in V$. Hence $(k_1, 0) = (l_1 r, 0) = (l_1, 0) \cdot (r, r) \in L$. Hence $(k_1, k_2) \in L$ since $(0, k_2) \in I \times \mathfrak{m}$. Therefore $I \times \mathfrak{m} \subsetneq K \cap L$. So $I \times \mathfrak{m}$ is irreducible. By symmetry, $\mathfrak{m} \times I$ is irreducible. □

Lemma 5.3.7. *Let $R = V \underset{k}{\times} V$ where V is a valuation ring with maximal ideal $\mathfrak{m}$. Then, for all $I \lhd V$, $(I \times \mathfrak{m})^{\#} = I^{\#} \times \mathfrak{m}$ and $(\mathfrak{m} \times I)^{\#} = \mathfrak{m} \times I^{\#}$.*

Proof. Suppose $c \in I^{\#}$ and $m \in \mathfrak{m}$. Then there exists $\gamma \notin I$ such that $\gamma c \in I$. Hence $(c, m) \cdot (\gamma, \gamma) = (c\gamma, m\gamma) \in I \times \mathfrak{m}$ and $(\gamma, \gamma) \notin I \times \mathfrak{m}$. So $(c, m) \in (I \times \mathfrak{m})^{\#}$.

Suppose $(a, b) \in (I \times \mathfrak{m})^{\#}$. Then, there exists $(c, d) \notin I \times \mathfrak{m}$ such that $(a, b) \cdot (c, d) = (ac, bd) \in I \times \mathfrak{m}$. Since $(c, d) \notin I \times \mathfrak{m}$ either $c \notin I$ or $d \notin \mathfrak{m}$. But, if $d \notin \mathfrak{m}$ then, since $c - d \in \mathfrak{m}$, $c \notin \mathfrak{m}$. Hence $c \notin I$. Therefore $ac \in I$ and $c \notin I$. So $a \in I^{\#}$. Hence $(a, b) \in I^{\#} \times \mathfrak{m}$. $\square$

Example 5.3.8. *Let V be a valuation domain with value group $\mathbb{R} \bigoplus \mathbb{R}$ with the lexicographic order. Then V has two non-zero prime ideals, $\mathfrak{m}$ maximal and $\mathfrak{p}$ non-maximal. Note that since the value group is dense, $\mathfrak{m}$ is not finitely generated. Take non-zero $s \in \mathfrak{p}$ and let R be V/s^2V. Let s' denote the image of s in R and note that $\mathrm{ann}_R s' = s'R$. Now R is a valuation ring with two prime ideals $\mathfrak{m}'$ maximal and $\mathfrak{p}'$ non-maximal. We now consider $R \underset{k}{\times} R$ where k is the residue field of R.*

We now show that the $R \underset{k}{\times} R$ ideals $\mathfrak{m}' \times s'R$ and $s'R \times \mathfrak{m}'$ are pp-definable. Suppose $(x, y) \in R \underset{k}{\times} R$ and $(x, y)(s', 0) = 0$. Then $xs' = 0$, so $x \in s'R = \mathrm{ann}_R s'$. Hence $x \in \mathfrak{m}'$. Since $(x, y) \in R \underset{k}{\times} R$, $x - y \in \mathfrak{m}'$. Hence $y \in \mathfrak{m}'$. Therefore $(x, y) \in s'R \times \mathfrak{m}'$. It is clear that $s'R \times \mathfrak{m}' \subseteq \mathrm{ann}_{R \underset{k}{\times} R}(s', 0)$. So $s'R \times \mathfrak{m}' = \mathrm{ann}_{R \underset{k}{\times} R}(s', 0)$. Hence $s'R \times \mathfrak{m}'$ is pp-definable and similarly, $\mathfrak{m}' \times s'R$ is pp-definable.

Take a non-unit $t \in R \backslash \{s'R\}$. By lemma 5.3.6, $I_1 = tR \times \mathfrak{m}'$ and $I_2 = \mathfrak{m}' \times tR$ are irreducible ideals. By lemma 5.3.7, $I_1^{\#} = (tR)^{\#} \times \mathfrak{m}'$ and $I_2^{\#} = \mathfrak{m}' \times (tR)^{\#}$. Therefore $I_1^{\#} = \mathfrak{m}' \times \mathfrak{m}'$ and $I_2^{\#} = \mathfrak{m}' \times \mathfrak{m}'$ since $(tR)^{\#} = \mathfrak{m}'$.

Therefore the injective hulls of $R \underset{k}{\times} R/I_1$ and $R \underset{k}{\times} R/I_2$ both have attached prime $\mathfrak{m}' \times \mathfrak{m}'$.

Let E be the injective hull of $R \underset{k}{\times} R/\mathfrak{m}' \underset{k}{\times} \mathfrak{m}'$, F_1 the injective hull of $R \underset{k}{\times} R/I_1$ and F_2 the injective hull of $R \underset{k}{\times} R/I_2$.

It remains to show that E, F_1 and F_2 are topologically distinguishable in $\mathrm{Zg}_{R \underset{k}{\times} R}|_{\mathrm{inj}}$. Clearly $F_1 \in \mathcal{O}(s'R \times \mathfrak{m}')$ since $t \notin s'R$, so $tR \times \mathfrak{m}' \supseteq s'R \times \mathfrak{m}'$. Similarly $F_2 \in \mathcal{O}(\mathfrak{m}' \times s'R)$.

Suppose, for a contradiction, $F_1 \in \mathcal{O}(\mathfrak{m}' \times s'R)$. Then there exists $w \in F_1 \backslash \{0\}$ such that $\mathrm{ann}_{R \underset{k}{\times} R}(w) \supseteq \mathfrak{m}' \times s'R$. Hence there exists $(a, b) \notin I_1$ such that $(I_1 : (a, b)) \supseteq \mathfrak{m}' \times s'R$. So there exists $a \notin tR$ such that $(tR : a) = \mathfrak{m}'$. Since $a \notin tR$, there exists $r \in R$ such that $ar = t$. Hence $(tR : a) = (arR : a) = \mathrm{ann}_R a + rR$. Since $r \notin \mathrm{ann}_R a$, $(tR : a) = rR$. But the maximal ideal is not finitely generated, since $\mathbb{R} \bigoplus \mathbb{R}$ is dense. Therefore $F_1 \notin \mathcal{O}(\mathfrak{m}' \times s'R)$. Similarly $F_2 \notin \mathcal{O}(s'R \times \mathfrak{m}')$. Finally

$E \in \mathcal{O}(\mathfrak{m}' \times s'R) \cap \mathcal{O}(s'R \times \mathfrak{m}')$ since $\mathfrak{m}' \times \mathfrak{m}' \supseteq \mathfrak{m}' \times s'R$ and $\mathfrak{m}' \times \mathfrak{m}' \supseteq s'R \times \mathfrak{m}'$.

Therefore E, F_1 and F_2 are topologically distinguishable in $\mathrm{Zg}_{V \underset{k}{\times} V}|_{\mathrm{inj}}$ and $\mathrm{Att}E = \mathrm{Att}F_1 = \mathrm{Att}F_2$.

Hence E, F_1 and F_2 are all in a single fibre, $\Gamma^{-1}(\mathrm{Att}E)$, of the continuous map defined in 5.0.3 but they are pairwise topologically distinguishable. Hence $\Gamma^{-1}(\mathrm{Att}E)$ contains at least three topologically distinguishable points.

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