# ALTERNATING SIGN MATRICES AND POLYTOPES

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"Everything that can be counted does not necessarily count;
everything that counts cannot necessarily be counted."

Albert Einstein

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# Summary

This thesis deals with two types of mathematical objects: alternating sign matrices and polytopes.

Alternating sign matrices were first defined in 1982 by Mills, Robbins and Rumsey. Since then, alternating sign matrices have led to some very captivating research (with multiple open problems still standing), an outline of which is presented in the opening chapter of this thesis.

Convex polytopes are extremely relevant when considering enumerations of certain classes of integer valued matrices. An overview of the relevant properties of convex polytopes is presented, before a connection is made between polytopes and alternating sign matrices: the alternating sign matrix polytope.

The vertex set of this new polytope is given, as well as a generalization of standard alternating sign matrices to give higher spin alternating sign matrices. From a result of Ehrhart a result concerning the enumeration of these matrices is obtained, namely, that for fixed size and variable line sum the enumeration is given by a particular polynomial.

In Chapter 4, we give results concerning the symmetry classes of the alternating sign matrix polytope and in Chapter 3 we study symmetry classes of the Birkhoff polytope. For this classical polytope we give some new results.

In the penultimate chapter, another polytope is defined that is a valid solution set of the transportation problem and for which a particular set of parameters gives the alternating sign matrix polytope. Importantly the transportation polytope is a subset of this new polytope.

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# Chapter 1

# Introduction

## 1.1 Alternating sign matrices

### 1.1.1 Definitions

Enumerative combinatorics studies the counting of structures obeying certain properties. Any undergraduate text on combinatorics is sure to contain some work on permutations, one class of such structures. Another class of structures of more recent interest is that of *alternating sign matrices.*

**Definition 1.1.1.** *The set of* alternating sign matrices *of size $n$, denoted $ASM(n)$, is the set of $n \times n$ matrices with the following properties:*

- *The entries are from the set {-1,0,1}.*
- *The sum of the entries in each column and in each row is 1.*
- *Disregarding the 0 entries, the 1's and -1's alternate along each row and column.*

Figure 1.1 shows the alternating sign matrices of size 3. Note that any permutation matrix is an alternating sign matrix.

How these matrices appeared and the earlier years of research that evolved around them will be the subject of our next section. Some of this material has appeared in the review papers [24, 93, 113] and the book [25].

Throughout this thesis, $\mathbb{P}$ denotes the set of positive integers, $\mathbb{N}$ denotes the set of nonnegative integers, $[m, n]$ denotes the set $\{m, m+1, \ldots, n\}$ for any $m, n \in \mathbb{Z}$, with $[m, n] = \emptyset$ for $n < m$,

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\quad\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\quad\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}\quad\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\quad\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$$

$$\begin{pmatrix}0&1&0\\1&-1&1\\0&1&0\end{pmatrix}$$

Figure 1.1: ASM(3)

and $[n]$ denotes the set $[1, n]$ for any $n \in \mathbb{Z}$. The notation $(m, n)_{\mathbb{R}}$ and $[m, n]_{\mathbb{R}}$ will be used for the open and closed intervals of real numbers between $m$ and $n$. For a finite set $T$, $|T|$ denotes the cardinality of $T$. The symbol $I_n$ denotes the identity matrix of size $n \times n$.

### 1.1.2 Historical overview

Alternating sign matrices have been appealing combinatorial objects for more than 20 years. Before explaining how they appeared we recall another mathematical object. The determinant of an $n \times n$ matrix $a$ can be defined as:

$$|a| = \sum_{\sigma \in S_n} \text{sign}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}$$

The determinant is a sum over the symmetric group $S_n$. Mills, Robbins and Rumsey worked on Dodgson condensation, an algorithm for iteratively calculating the determinant of an $n \times n$ matrix in terms of $2 \times 2$ determinants:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Changing this slightly to give:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} + \lambda a_{12}a_{21}$$

Robbins and Rumsey [95] were led to define the $\lambda$ determinant of an $n \times n$ matrix $a$, as:

$$|a|_\lambda = \sum_{\sigma \in \text{ASM}(n)} P(\sigma, \lambda) \prod_{i,j=1}^{n} a_{ij}^{\sigma_{ij}}$$

where $P$ is a certain function of $\lambda$ and $\sigma$. A simple question arose: for given $n$ how many terms are there in this sum? The first few enumerations offered were $1, 2, 7, 42, 429, 7436, \ldots$.

Not having the many references available today they asked the combinatorialist Richard Stanley if he knew of this sequence. He responded that this sequence (A005130 of [99]) also enumerates so-called *descending plane partitions* as shown by Andrews in [3]. A *plane partition* of a number $n$ is defined as an array of positive integers with non-increasing rows and columns, such that the sum of its entries is $n$. Turning their interest towards these objects, Mills, Robbins and Rumsey started working on the Macdonald conjecture [77]. Macdonald's conjecture concerned a particular type of these objects: *cyclically symmetric plane partitions.* Macdonald conjectured a form for the generating function of these elements and Stanley claimed proving this formula was "the most interesting open problem in all of enumerative combinatorics" [102]. Andrews [3] proved a particular case of this conjecture and conjectured another result. These conjectures were proved in [80] by Mills, Robbins and Rumsey. In [80] they defined alternating sign matrices and gave two conjectures concerning these new mathematical objects that they developed further in [81]. Returning to their initial query, if we let $A_n$ be the number of alternating sign matrices of size $n$, they conjectured the following formula

$$A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!} \tag{1.1}$$

It is easy to note that there can be only a single 1 in the first row of any alternating sign matrix. If we let $A_{n,k}$ be the number of alternating sign matrices with single 1 in the first row and column $k$, we have the *refined alternating sign matrix conjecture*:

$$A_{n,k} = \binom{n+k-2}{k-1} \frac{(2n-k-1)!}{(n-k)!} \prod_{j=0}^{n-2} \frac{(3j+1)!}{(n+j)!} \tag{1.2}$$

Equations (1.1)-(1.2) are the previously mentioned conjectures given in [80] by Mills, Robbins and Rumsey. In [93], Robbins offered the following opinion on a range of conjectures related to alternating sign matrices: "*These conjectures are of such compelling simplicity that it is hard to understand how any mathematician can bear the pain of living without understanding why they are true*". Some of these conjectures are still open but a review of the proof of (1.1) will follow in Section 1.1.4.

### 1.1.3 The many faces of alternating sign matrices

The title of this section is actually the title of an extremely relevant article by Propp: [89]. In [22, 89] different bijections are given between alternating sign matrices and other combinatorial objects. The 1995 paper [22] offers a nice snapshot of history since the authors were hoping that one of their bijections would offer an insight into proving the alternating sign matrix conjectures (1.1) and (1.2). However the proof of (1.1) appeared in [111] at practically the same time as did [22] without using the particular bijection mentioned.

### Edge matrix pairs

**Definition 1.1.2.** *Define the set of* edge matrix pairs *EM(n) as:*

$$EM(n) := \left\{ (h,v) = \left( \begin{pmatrix} h_{10} & \dots & h_{1n} \\ \vdots & & \vdots \\ h_{n0} & \dots & h_{nn} \end{pmatrix}, \begin{pmatrix} v_{01} & \dots & v_{0n} \\ \vdots & & \vdots \\ v_{n1} & \dots & v_{nn} \end{pmatrix} \right) \in \{0,1\}^{n\times(n+1)} \times \{0,1\}^{(n+1)\times n} \;\middle|\; \begin{array}{l} \bullet\, h_{i0} = v_{0j} = 0 \;\; \textit{for all } i,j \in [n] \\ \bullet\, h_{in} = v_{nj} = 1 \;\; \textit{for all } i,j \in [n] \\ \bullet\, h_{i,j-1} + v_{ij} = v_{i-1,j} + h_{ij} \;\; \textit{for all } i,j \in [n] \end{array} \right\} \tag{1.3}$$

We shall refer to $h$ as a *horizontal edge matrix* and $v$ as a *vertical edge matrix.* It can be checked that there is a bijection between the set ASM($n$) and EM($n$) in which the edge matrix pair $(h, v)$ which corresponds to the alternating sign matrix $a$ is given by:

$$\begin{array}{ll} h_{ij} = \sum_{j'=1}^{j} a_{i,j'} & \text{for all } \; i \in [n],\; j \in [0,n] \\ v_{ij} = \sum_{i'=1}^{i} a_{i',j} & \text{for all } \; i \in [0,n],\; j \in [n] \end{array} \tag{1.4}$$

and inversely:

$$a_{ij} = h_{ij} - h_{i,j-1} = v_{ij} - v_{i-1,j} \;\; \text{for all } i,j \in [n] \tag{1.5}$$

Therefore $h$ is the *column sum matrix* and $v$ is the *row sum matrix* of $a$. The correspondence between alternating sign matrices and edge matrix pairs was first identified in [95]. Using equation (2.4) we get for the matrices from Figure 1.1 the edge matrix pairs as shown in Figure 1.2 (the ordering of these matrices corresponds to the ordering of Figure 1.1).

### Corner sum matrices

**Definition 1.1.3.** *Define the set of* corner sum matrices *CSM(n) as:*

$$CSM(n) := \left\{ c = \begin{pmatrix} c_{0,0} & \cdots & c_{0,n} \\ \vdots & & \vdots \\ c_{n,0} & \cdots & c_{n,n} \end{pmatrix} \in [0,n]^{n+1\times n+1} \;\middle|\; \begin{array}{l} \bullet\, c_{0k} = c_{k0} = 0 \;\; \textit{for all } k \in [0,n] \\ \bullet\, c_{kn} = c_{nk} = k \;\; \textit{for all } k \in [0,n] \\ \bullet\, c_{ij} - c_{i,j-1} \in \{0,1\} \;\; \textit{for all } i,j \in [n] \\ \bullet\, c_{ij} - c_{i-1,j} \in \{0,1\} \;\; \textit{for all } i,j \in [n] \end{array} \right\} \tag{1.6}$$

It can be checked that there is a bijection between ASM($n$) and CSM($n$) in which the corner sum matrix $c$ which corresponds to the alternating sign matrix $a$ is given by:

$$c_{ij} = \sum_{i'=1}^{i} \sum_{j'=1}^{j} a_{i'j'}, \quad \text{for all } i,j \in [0,n] \tag{1.7}$$

$$\left(\begin{pmatrix}0&1&1&1\\0&0&1&1\\0&0&0&1\end{pmatrix},\begin{pmatrix}0&0&0\\1&0&0\\1&1&0\\1&1&1\end{pmatrix}\right)\quad\left(\begin{pmatrix}0&1&1&1\\0&0&0&1\\0&0&1&1\end{pmatrix},\begin{pmatrix}0&0&0\\1&0&0\\1&0&1\\1&1&1\end{pmatrix}\right)$$

$$\left(\begin{pmatrix}0&0&0&1\\0&0&1&1\\0&1&1&1\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&1\\0&1&1\\1&1&1\end{pmatrix}\right)\quad\left(\begin{pmatrix}0&0&0&1\\0&1&1&1\\0&0&1&1\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&1\\1&0&1\\1&1&1\end{pmatrix}\right)$$

$$\left(\begin{pmatrix}0&0&1&1\\0&1&1&1\\0&0&0&1\end{pmatrix},\begin{pmatrix}0&0&0\\0&1&0\\1&1&0\\1&1&1\end{pmatrix}\right)\quad\left(\begin{pmatrix}0&0&1&1\\0&0&0&1\\0&1&1&1\end{pmatrix},\begin{pmatrix}0&0&0\\0&1&0\\0&1&1\\1&1&1\end{pmatrix}\right)$$

$$\left(\begin{pmatrix}0&0&1&1\\0&1&0&1\\0&0&1&1\end{pmatrix},\begin{pmatrix}0&0&0\\0&1&0\\1&0&1\\1&1&1\end{pmatrix}\right)$$

Figure 1.2: EM(3)

and inversely,

$$a_{ij} = c_{ij} - c_{i,j-1} - c_{i-1,j} + c_{i-1,j-1}, \quad \text{for all } i,j \in [n] \tag{1.8}$$

Combining the bijections (1.4) and (1.5) between EM($n$) and ASM($n$), and (1.7) and (1.8) between ASM($n$) and CSM($n$), the corner sum matrix $c$ which corresponds to the edge matrix pair $(h, v)$ is given by:

$$c_{ij} = \sum_{i'=1}^{i} h_{i'j} = \sum_{j'=1}^{j} v_{ij'}, \quad \text{for all } i,j \in [0,n] \tag{1.9}$$

and inversely,

$$\begin{aligned} h_{ij} &= c_{ij} - c_{i-1,j}, \quad \text{for all } i \in [n], j \in [0,n] \\ v_{ij} &= c_{ij} - c_{i,j-1}, \quad \text{for all } i \in [0,n], j \in [n] \end{aligned} \tag{1.10}$$

Corner sum matrices were introduced in [95]. Figure 1.3 gives CSM(3).

$$\begin{pmatrix}0&0&0&0\\0&1&1&1\\0&1&2&2\\0&1&2&3\end{pmatrix}\quad\begin{pmatrix}0&0&0&0\\0&1&1&1\\0&1&2&2\\0&1&2&3\end{pmatrix}\quad\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&1&2\\0&1&2&3\end{pmatrix}$$

$$\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&1&1&2\\0&1&2&3\end{pmatrix}\quad\begin{pmatrix}0&0&0&0\\0&0&1&1\\0&1&2&2\\0&1&2&3\end{pmatrix}\quad\begin{pmatrix}0&0&0&0\\0&0&1&1\\0&0&1&2\\0&1&2&3\end{pmatrix}$$

$$\begin{pmatrix}0&0&0&0\\0&0&1&1\\0&1&1&2\\0&1&2&3\end{pmatrix}$$

Figure 1.3: CSM(3)

### Monotone triangles

**Definition 1.1.4.** *Define the set* monotone triangles *$MT(n)$ to be the set of all triangular arrays $t$ of the form:*

$$\begin{array}{ccccccc} & & & t_{1,1} & & & \\ & & t_{2,1} & & t_{2,2} & & \\ & \cdot^{\cdot^{\cdot}} & & & & \ddots & \\ t_{n,1} & & & \cdots & & & t_{n,n}\end{array}$$

*such that:*

- *Each entry of $t$ is in $[n]$.*
- $t_{ij} < t_{i,j+1}$ *for all $i \in [n]$, $j \in [i-1]$.*
- $t_{i+1,j} \leq t_{ij} \leq t_{i+1,j+1}$ *for all $i \in [n-1]$, $j \in [i]$.*

It follows that the last row of any monotone triangle in MT$(n)$ consists of each integer of $[n]$. It can be checked that there is a bijection between ASM$(n)$ and MT$(n)$ in which the monotone triangle $t$ which corresponds to the alternating sign matrix $a$ is obtained by first using (1.4) to find the vertical edge matrix $v$ that corresponds to $a$, and then row $i$ of $t$ corresponds to the positions of the 1's of row $i$ of $v$, with these integers being placed in increasing order along each row. Using this bijection we get Figure 1.4 for MT(3).

The set MT$(n)$ of monotone triangles was introduced in [81].

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 1 | | | | | 1 | | | | | 3 | | |
| | 2 | | 3 | | | 1 | | 3 | | | 2 | | 3 | |
| 1 | | 2 | | 3 | 1 | | 2 | | 3 | 1 | | 2 | | 3 |
| | | | | | | | | | | | | | | |
| | | 3 | | | | | 2 | | | | | 2 | | |
| | 1 | | 3 | | | 2 | | 3 | | | 1 | | 2 | |
| 1 | | 2 | | 3 | 1 | | 2 | | 3 | 1 | | 2 | | 3 |
| | | | | | | | | | | | | | | |
| | | | | | | | 1 | | | | | | | |
| | | | | | | 1 | | 3 | | | | | | |
| | | | | | 1 | | 2 | | 3 | | | | | |

Figure 1.4: MT(3)

The next form we consider is arguably one of the most important. The bijection given here made a connection between combinatorics and physics.

### Configurations of the six-vertex model with domain-wall boundary conditions

**Definition 1.1.5.** *We define the $m$ by $n$ lattice $\mathcal{L}_{m,n}$ to be the graph with vertex set:*

$$[0, m+1] \times [0, n+1] \setminus \{(0,0), (0, n+1), (m+1, 0), (m+1, n+1)\}$$

*where two points $(i,j), (i',j')$ from this set form an edge if and only if $(i,j) - (i',j') \in \{(0, \pm 1), (\pm 1, 0)\}$.*

The vertices will be depicted in a grid in which vertex $(i,j)$ appears in row $i$ from the top and column $j$ from the left as shown in Figure 1.5. If $(i,j) - (i',j') = (0, \pm 1)$ or $(i,j) - (i',j') = (\pm 1, 0)$ then the edge $\{(i,j), (i',j')\}$ is *horizontal* or *vertical* respectively.

**Definition 1.1.6.** *A configuration of the* six-vertex model *with domain-wall boundary conditions is an assignment of left or right arrows to each horizontal edge of $\mathcal{L}_{n,n}$ and an up or down arrow to each vertical edge of $\mathcal{L}_{n,n}$ such that:*

- *At each internal vertex the number of arrows directed in is the same as the number of arrows directed out.*
- *All arrows on the left/right/top/bottom boundary are oriented left/right/down/up.*

It can be shown that there is a bijection between configurations of the six-vertex model with domain-wall boundary conditions on $\mathcal{L}_{n,n}$ and ASM$(n)$. To map such a configuration to a matrix $a \in$ ASM$(n)$ the following rule is used at each vertex $(i,j)$ to give $a_{ij}$.

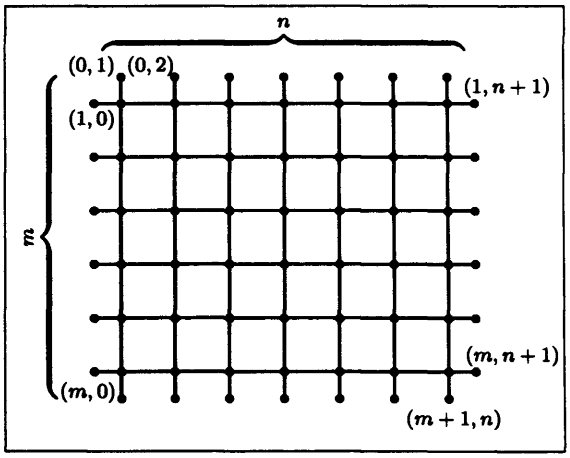
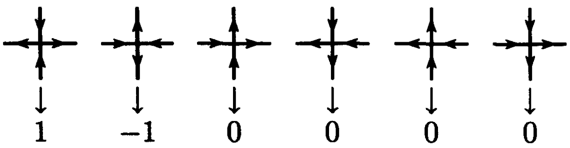



Figure 1.5: $\mathcal{L}_{m,n}$




For the inverse mapping, given an alternating sign matrix, once the arrows for the vertices corresponding to the 1's and -1's have been assigned there is a unique way in which the other arrows can be inserted into $\mathcal{L}_{n,n}$. Figure 1.6 gives the configurations corresponding to Figure 1.1. The correspondence between alternating sign matrices and configurations of the six-vertex model with domain-wall boundary conditions was first identified in [53].

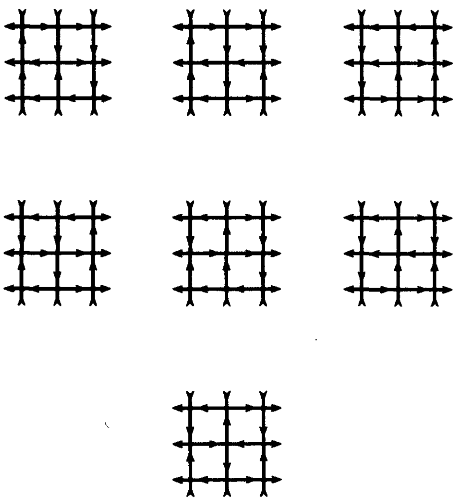


Figure 1.6: Six-vertex model configurations with domain-wall boundary conditions on $\mathcal{L}_{3,3}$

A configuration of the six-vertex model with domain wall boundary conditions can alternatively be regarded as an assignment of 0's and 1's to the edges of $\mathcal{L}_{n,n}$ such that:

- The values assigned to the four edges surrounding an internal vertex satisfy the rule given in Figure 1.8.
- All of the edges on the left and upper boundaries are assigned 0's and all of the edges on the right and lower boundaries are assigned 1's.

Thus the relationship between the two forms of configurations is simply that left and down arrows correspond to 0's and right and up arrows correspond to 1's as shown in Figure 1.7. The configurations of the six-vertex model with domain wall boundary conditions on $\mathcal{L}_{3,3}$ using 0's and 1's are shown in Figure 1.9. Also the mapping between edge matrix pairs and configurations of the six-vertex model with domain wall boundary conditions using 0's and 1's is simply that for any $(h, v) \in \mathrm{EM}(n)$, $h_{ij}$ is assigned to the horizontal edge between $(i, j)$ and $(i, j+1)$ and $v_{ij}$ is assigned to the vertical edge between $(i, j)$ and $(i+1, j)$. This is shown in Figure 1.10.

| $\rightarrow, \uparrow$ | $\leftarrow, \downarrow$ |
|---|---|
| 1 | 0 |

Figure 1.7: Orientation Convention

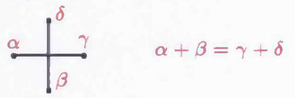



Figure 1.8: Relation between the edge matrix entries surrounding a vertex

### Lattice paths

**Definition 1.1.7.** *The set LP($n$) of* lattice paths *is the set of all sets $P$ of $n$ directed lattice paths on $\mathcal{L}_{n,n}$ such that:*

- *For each $i \in [n]$, $P$ contains a path which begins at $(n+1, i)$ and ends at $(i, n+1)$.*
- *Each step of each path of $P$ is either $(-1, 0)$ or $(0, 1)$.*
- *Different paths of $P$ do not cross or share any edge of the lattice (but may share a vertex of the lattice).*

It can be checked that there is a bijection between $\mathrm{EM}(n)$ (and hence $\mathrm{ASM}(n)$) and $\mathrm{LP}(n)$ in which the edge matrix pair $(h, v)$ which corresponds to the path set $P$ is given by:

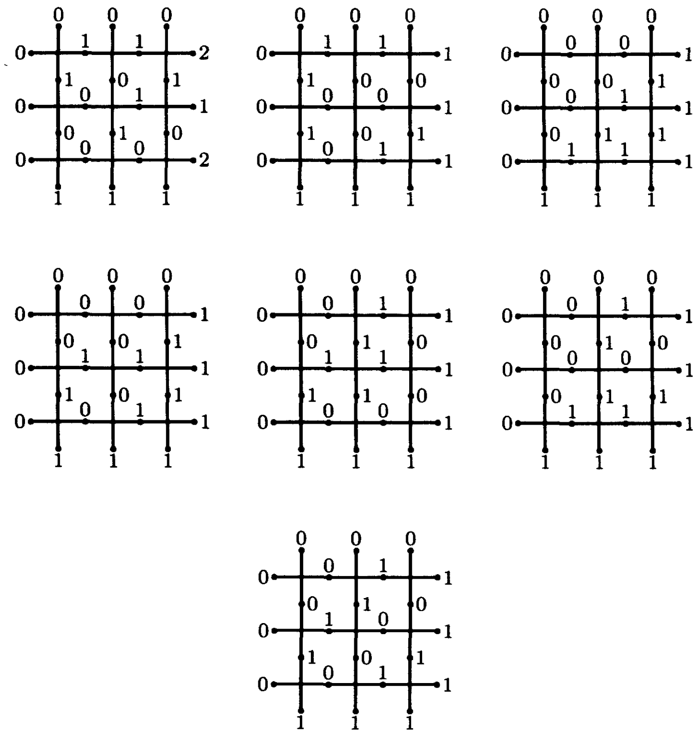


Figure 1.9: Configurations of the six-vertex model with domain wall boundary conditions on $\mathcal{L}_{3,3}$ using 0's and 1's

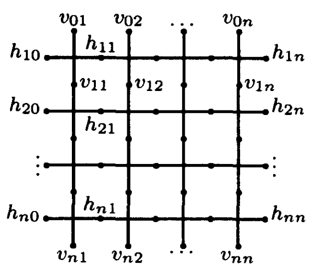



Figure 1.10: Assignment of edge matrix entries to lattice edges

- $h_{ij} = 1$ if and only if there is a path of $P$ that passes from $(i, j)$ to $(i, j+1)$.
- $v_{ij} = 1$ if and only if there is a path of $P$ that passes from $(i+1, j)$ to $(i, j)$.

The set LP($n$) is studied in [14, 22, 23, 51, 106]. Figure 1.11 gives the lattice paths corresponding to Figure 1.1.

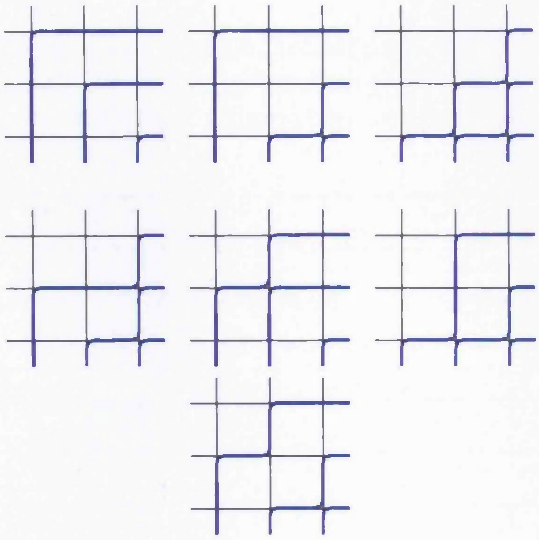


Figure 1.11: LP(3)

### Fully packed loops

**Definition 1.1.8.** *The set FPL($n$) of* fully packed loop configurations *is the set of all sets $P$ of nondirected open and closed paths on $\mathcal{L}_{n,n}$ such that:*

- *Any two edges occupied successively by a path of $P$ are different.*
- *Exactly one path of $P$ passes through each internal vertex of $\mathcal{L}_{n,n}$.*
- *Each path of $P$ does not cross itself or any other path of $P$.*
- *At each external vertex (0,2k-1) and (n+1,n-2k+2) for $k \in [\lceil \frac{n}{2} \rceil]$, and $(2k, 0)$ and $(n-2k+1, n+1)$ for $k \in [\lfloor \frac{n}{2} \rfloor]$, there is an endpoint of a path of $P$, these being the only lattice points which are path endpoints.*

It can be checked that there is a bijection between EM($n$) and FPL($n$) in which the element of FPL($n$) which corresponds to $(h, v) \in$ EM($n$) is obtained by first forming:

$$\overline{h}_{ij} = \begin{cases} h_{ij} \text{ for } i+j \text{ odd} \\ 1 - h_{ij} \text{ for } i+j \text{ even} \end{cases}$$
$$\overline{v}_{ij} = \begin{cases} 1 - v_{ij} \text{ for } i+j \text{ odd} \\ v_{ij} \text{ for } i+j \text{ even} \end{cases} \tag{1.11}$$

and then assigning a path segment to each horizontal edge of $\mathcal{L}_{n,n}$ between $(i, j)$ and $(i, j+1)$ which has $\overline{h}_{ij} = 1$ and to each vertical edge between $(i, j)$ and $(i+1, j)$ which has $\overline{v}_{ij} = 1$. Figure 1.12 gives the fully packed loop configurations on $\mathcal{L}_{3,3}$.

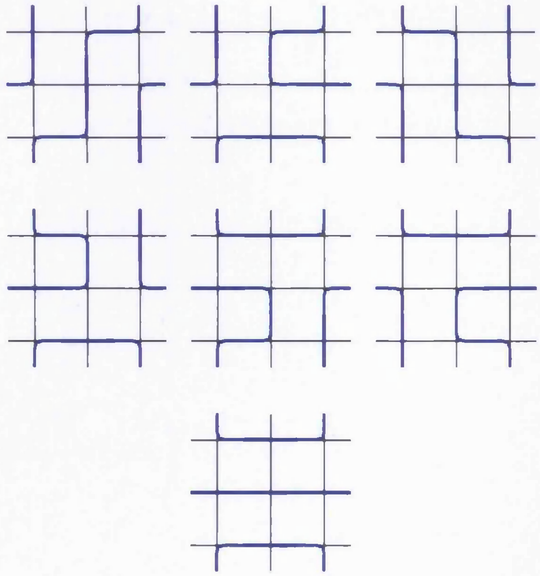


Figure 1.12: FPL(3)

Fully packed loops are interesting combinatorial objects in their own right as shown for example in [37, 38, 48, 49, 107, 109, 118].

These are most of the forms of alternating sign matrices considered by Propp in [89]. In the next section we give an overview of one of the proofs of (1.1) that uses one of these particular forms.

### 1.1.4 Proving the alternating sign matrix conjecture

The original proof of the alternating sign matrix conjecture (1.1) appeared in an article by Zeilberger [111]. He originally submitted this article as a 20 page paper in 1992. In [113] Zeilberger explains how the referee (Dave Robbins) constantly found "gaps" in his arguments. After resubmitting quite a few times, Zeilberger decided to make his paper "pre-refereed". He shared out all the lemmas in his argument to different mathematicians who checked each one individually. This led to his final 80 page paper being accepted and recognized as the original proof of the long standing conjecture. To quote Zeilberger in [113]: "this innovative *format*, and the pioneering idea of *communal checking*, are even more important than the content of my article". The proof itself involved showing that alternating sign matrices are equinumerous with totally symmetric, self-complementary plane partitions, which had already been enumerated by Andrews [4].

The second proof of conjecture (1.1) that appeared was by Kuperberg [72]. We will go over the key points in Kuperberg's proof, in very much the same way that Bressoud does in his book [25].

Kuperberg's proof relies on configurations of the six-vertex model with domain wall boundary conditions as given by Definition 1.6. Consider an internal vertex of this model using 0's and 1's: $\begin{smallmatrix} & \delta & \\ \alpha & + & \gamma \\ & \beta & \end{smallmatrix}$. A *Boltzmann weight* $W\left(\begin{smallmatrix} & \delta & \\ \alpha & + & \gamma \\ & \beta & \end{smallmatrix}, z, a\right) \in \mathbb{C}$ is defined, where $a$ and $z$ are complex variables, often called the *crossing parameter* and *spectral parameter* respectively. Defining $s(z,a) := \frac{z-\frac{1}{z}}{a^2-\frac{1}{a^2}} = \frac{a^2(z^2-1)}{z(a^4-1)}$, we use the following set of weights:

$$\begin{aligned} W\left(\begin{smallmatrix} & 0 & \\ 0 & + & 1 \\ & 1 & \end{smallmatrix}, z, a\right) &= W\left(\begin{smallmatrix} & 1 & \\ 1 & + & 0 \\ & 0 & \end{smallmatrix}, z, a\right) = 1 \\ W\left(\begin{smallmatrix} & 1 & \\ 1 & + & 1 \\ & 1 & \end{smallmatrix}, z, a\right) &= W\left(\begin{smallmatrix} & 0 & \\ 0 & + & 0 \\ & 0 & \end{smallmatrix}, z, a\right) = s(za, a) \\ W\left(\begin{smallmatrix} & 1 & \\ 0 & + & 0 \\ & 1 & \end{smallmatrix}, z, a\right) &= W\left(\begin{smallmatrix} & 0 & \\ 1 & + & 1 \\ & 0 & \end{smallmatrix}, z, a\right) = s(\tfrac{z}{a}, a) \end{aligned} \tag{1.12}$$

In general we consider $\begin{pmatrix} z_{11} & \cdots & z_{1n} \\ \vdots & & \vdots \\ z_{n1} & \cdots & z_{nn} \end{pmatrix} \in \mathbb{C}^{n\times n}$, with $z_{ij}$ being used for vertex $(i,j)$. For given $(h,v) \in \mathrm{EM}(n)$ and its corresponding six-vertex model configuration with domain wall boundary conditions using 0's and 1's we calculate the weight of this configuration by taking the product of the weights of all the vertices:

$$W(h,v) = \prod_{i,j=1}^{n} W\left(\begin{smallmatrix} & v_{i-1,j} & \\ h_{i,j-1} & + & h_{ij} \\ & v_{ij} & \end{smallmatrix}, z_{ij}, a\right)$$

In the case needed here we associate row $i$ with $x_i$ and column $j$ with $y_j$ and set $z_{ij} = \frac{x_i}{y_j}$ for some complex numbers $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$. A particular tool of statistical mechanics is the *partition function* $Z_n(x_1, \ldots x_n, y_1, \ldots y_n, a)$, which is the sum of the weights of all the possible states of a system:

$$Z_n(x_1, \ldots x_n, y_1, \ldots y_n, a) := \sum_{(h,v)\in \mathrm{EM}(n)} \prod_{i,j=1}^{n} W\left(\begin{smallmatrix} & v_{i-1,j} & \\ h_{i,j-1} & + & h_{ij} \\ & v_{ij} & \end{smallmatrix}, \frac{x_i}{y_j}, a\right)$$

This partition function had already been well studied [18, 64, 65, 70] giving the determinant formula:

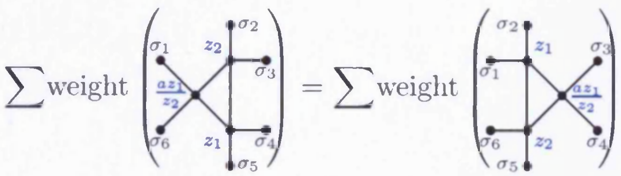
$$Z_n(x_1,\dots x_n,y_1,\dots y_n,a)=\frac{\prod_{i,j=1}^{n} s\left(\frac{x_i}{y_j a},a\right)s\left(\frac{x_i a}{y_j},a\right)}{\prod_{1\le i<j\le n} s\left(\frac{x_i}{x_j},a\right)s\left(\frac{y_j}{y_i},a\right)}\left|\frac{1}{s\left(\frac{x_i}{y_j a},a\right)s\left(\frac{x_i a}{y_j},a\right)}\right|_{i,j=1}^{n}$$
$$=\frac{\prod_{i,j=1}^{n}\left(a^2x_i^2-y_j^2\right)\left(x_i^2-a^2y_j^2\right)}{(a^4-1)^{n(n-1)}\left(\prod_{i=1}^{n}x_iy_i\right)^{n-1}\prod_{1\le i<j\le n}\left(x_i^2-x_j^2\right)\left(y_j^2-y_i^2\right)}\left|\frac{1}{\left(a^2x_i^2-y_j^2\right)\left(x_i^2-a^2y_j^2\right)}\right|_{i,j=1}^{n} \quad (1.13)$$

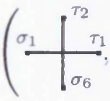
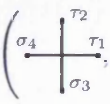
where $|a_{ij}|_{i,j=1}^{n}$ denotes the determinant of an $n\times n$ matrix $a$ with entries $a_{ij}$.

The proof of (1.13) relies on showing that the right hand side of (1.13), denoted by $F_n(x_1,\dots x_n,y_1,\dots y_n,a)$ and $Z_n(x_1,\dots x_n,y_1,\dots y_n,a)$ both satisfy the following properties:

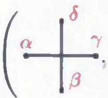
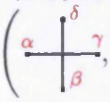
- They equal 1 for $n=1$
- They are symmetric in $x_1,\dots,x_n$ and symmetric in $y_1,\dots,y_n$ (separately).
- $\left(\prod_{i=1}^{n}x_iy_i\right)^{n-1}F_n(x_1,\dots x_n,y_1,\dots y_n,a)$ and $\left(\prod_{i=1}^{n}x_iy_i\right)^{n-1}Z_n(x_1,\dots x_n,y_1,\dots y_n,a)$ are polynomials of degree $n-1$ in each $x_i^2$ and $y_j^2$.
- $F_n(x_1,\dots x_n,y_1,\dots y_n,a)$ and $Z_n(x_1,\dots x_n,y_1,\dots y_n,a)$ obey the same recursion relation for certain values of $x_i$ and $y_j$.

The proof of symmetry is done using the Yang-Baxter equation [8]. The Yang-Baxter equation states that for a certain triangle of vertices, with each edge assigned a fixed value, if we move the left vertex across to the right, maintaining the values for the external edges but switching the top and bottom spectral parameter values, then the sum of the weights over all possible values for the internal edges stays the same. Diagrammatically this gives:

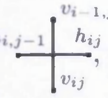



where the orientated weight of the left hand side is $W$  $\left(\ \cdot\ ,\frac{az_1}{z_2},a\right)$ and the orientated weight on the right hand side is $W$  $\left(\ \cdot\ ,\frac{az_1}{z_2},a\right)$ (with $\tau_1,\tau_2$ being summed over).

Symmetry in $x_1,\dots x_n$ follows since it is possible to introduce a new vertex on the left boundary of the lattice (thus creating a triangle) and push it across the lattice, switching the top and bottom labels at each step and keeping the same overall weight, and similarly for $y_1,\dots,y_n$.

If we can choose a particular set of parameters such that $W$  $\left(\ \cdot\ ,\frac{x_i}{y_j},a\right)=1$ for all $\alpha,\beta,\gamma,\delta$ then $Z_n(x_1,\dots x_n,y_1,\dots y_n,a)=A_n$. This is indeed what Kuperberg realized and used in [72]. Setting $x_i = q^{\frac{i}{2}} e^{\mathbb{I}\frac{\pi}{2}}, y_j = q^{\frac{1-j}{2}}, a = e^{\mathbb{I}\frac{\pi}{6}}$ (to avoid confusion with labeling of rows we use $\mathbb{I}$ to denote the imaginary constant: $\mathbb{I}^2 = -1$) and letting $q = 1$ gives $W\left(\right.$ 

$\left., \frac{x_i}{y_j}, a\right) = 1$.

Note that $\sum_{(h,v)\in \mathrm{EM}(n)} \prod_{i,j=1}^{n} W\left(\right.$ 

$\left., \frac{x_i}{y_j}, a\right)$ is well defined at $q = 1$ however the right hand side of (1.13) is not. Indeed $q = 1$ implies that $x_i = x_j$ and $y_i = y_j$ for all $i, j$ and $s(1, a) = 0$ so that the determinant and the denominator of the prefactor are both 0. We thus have $A_n = \lim_{q\to 1} Z_n(x_1, \dots x_n, y_1, \dots y_n, a)$. Kuperberg now had a valid method for the enumeration of ASM$(n)$ however to prove conjecture (1.1) there was still a fair bit of work to do. Kuperberg, using results of Cauchy, proved what is known as Kuperberg's determinant formula [25, 72]:

$$\left|\frac{1-s^{i+j-1}}{1-t^{i+j-1}}\right|_{i,j=1}^{n} = t^{\frac{n(n-1)(2n-1)}{6}} \prod_{1\le i<j\le n} (1-t^{j-i})^2 \prod_{i,j=1}^{n} \frac{1-st^{j-i}}{1-t^{i+j-1}} \tag{1.14}$$

Note:

$$\begin{aligned}
\left(ax_i^2 - y_j^2\right)\left(x_i^2 - a^2 y_j^2\right) &= q^{2-2j} e^{\mathbb{I}\frac{\pi}{3}} \left(1 + q^{i+j-1} + q^{2(i+j-1)}\right) \\
&= q^{2-2j} e^{\mathbb{I}\frac{\pi}{3}} \tfrac{1-q^{3(i+j-1)}}{1-q^{(i+j-1)}} \\
\left(x_i^2 - x_j^2\right)\left(y_j^2 - y_i^2\right) &= -q^{1+i-j}\left(1-q^{j-i}\right)^2 \\
\left(\textstyle\prod_{i=1}^{n} x_i y_i\right)^{n-1} &= (-q)^{\frac{n(n-1)}{2}} \\
\left(a^4 - 1\right)^{n(n-1)} &= \left(\sqrt{3} e^{\mathbb{I}\frac{5\pi}{6}}\right)^{n(n-1)}
\end{aligned} \tag{1.15}$$

Using this we have:

$$A_n = \lim_{q\to 1} \frac{e^{\mathbb{I}\frac{\pi(n^2-n)}{3}} q^{n(n-1)} \prod_{i,j=1}^{n} \left(q^{2-2j} \frac{1-q^{3(i+j-1)}}{1-q^{i+j-1}}\right)}{\left(\sqrt{3} e^{\mathbb{I}\frac{5\pi}{6}}\right)^{n(n-1)} (-q)^{\frac{n(n-1)}{2}} \prod_{1\le i<j\le n} \left(-q^{1+i-j}\left(1-q^{j-1}\right)^2\right)} \left|\frac{1-q^{i+j-1}}{1-q^{3(i+j-1)}}\right|_{i,j=1}^{n} \tag{1.16}$$

This reduces to:

$$A_n = (-3)^{\frac{-n(n-1)}{2}} \left(\lim_{q\to 1} \frac{q^{\frac{n(n-1)}{2}} \prod_{i,j=1}^{n} q^{2-2j}}{\prod_{1\le i<j\le n} q^{1+i-j}}\right) \left(\lim_{q\to 1} \frac{\prod_{i,j=1}^{n} \frac{1-q^{3(i+j-1)}}{1-q^{i+j-1}}}{\prod_{1\le i<j\le n} \left(1-q^{j-i}\right)^2} \left|\frac{1-q^{i+j-1}}{1-q^{3(i+j-1)}}\right|_{i,j=1}^{n}\right)$$

which gives:

$$A_n = (-3)^{\frac{-n(n-1)}{2}} \lim_{q\to 1} \frac{\prod_{i,j=1}^{n} \frac{1-q^{3(i+j-1)}}{1-q^{i+j-1}}}{\prod_{1\le i<j\le n} \left(1-q^{j-i}\right)^2} \left|\frac{1-q^{i+j-1}}{1-q^{3(i+j-1)}}\right|_{i,j=1}^{n} \tag{1.17}$$

since the term in the first limit is simply a product of $q$'s whose limit is therefore 1. Using (1.14) with $s = q$ and $t = q^3$ we have:

$$A_n = (-3)^{\frac{-n(n-1)}{2}} \lim_{q\to 1} q^{\frac{n(n-1)(2n-1)}{2}} \prod_{i,j=1}^{n} \frac{1-q^{3j-3i+1}}{1-q^{i+j-1}} \prod_{1\le i<j\le n} \left(\frac{1-q^{3(j-i)}}{1-q^{j-i}}\right)^2 \tag{1.18}$$

which gives (using the identity $1 - x^m = (1 + x + x^2 + \cdots + x^{m-1})(1-x)$)

$$A_n = (-3)^{\frac{-n(n-1)}{2}} \prod_{i,j=1}^{n} \lim_{q\to 1} \frac{1+q+q^2+\cdots+q^{3(j-i)}}{1+q+q^2+\cdots+q^{i+j-2}} \prod_{1\le i<j\le n} \lim_{q\to 1} \left(1+q^{j-i}+q^{2(j-i)}\right)^2$$

Taking the limits we have:

$$A_n = (-3)^{\frac{n(n-1)}{2}} \prod_{i,j=1}^{n} \frac{3(j-i)+1}{i+j-1}$$

Considering the numerator of $\prod_{i,j=1}^{n} \frac{3j-3i+1}{i+j-1}$ we have

$$\prod_{i,j=1}^{n} (3(j-i)+1) = \prod \begin{array}{|c|c|c|c|c|} \hline 1 & 4 & 7 & \dots & 3n-2 \\ \hline -2 & 1 & 4 & \dots & 3n-5 \\ \hline -5 & -2 & 1 & \dots & 3n-8 \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline 4-3n & 7-3n & 10-3n & \dots & 1 \\ \hline \end{array}$$

which is the product $(-1)^{\frac{n(n-1)}{2}} \prod_{k=1}^{n-1}(3k+1)^{n-k}(3k-1)^{n-k}$

Also we have the denominator as

$$\prod_{i,j=1}^{n} (i+j-1) = \prod \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & \dots & n \\ \hline 2 & 3 & 4 & \dots & n+1 \\ \hline 3 & 4 & 5 & \dots & n+2 \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline n & n+1 & n+2 & \dots & 2n-1 \\ \hline \end{array}$$

which is the product $n^n \prod_{k=1}^{n-1} k^k (n+k)^{n-k}$. Thus we have:

$$A_n = \frac{3^{\frac{n(n-1)}{2}}}{n^n} \prod_{k=1}^{n-1} \left( \frac{(3k-1)^{n-k}(3k+1)^{n-k}}{k^k(n+k)^{n-k}} \right) = \frac{1}{n^n} \prod_{k=1}^{n-1} \left( \frac{((3k-1)(3k)(3k+1))^{n-k}}{k^n(n+k)^{n-k}} \right)$$

A bit more observation gives:

$$A_n = \frac{(2\times 3\times 4)^{n-1}(5\times 6\times 7)^{n-2}\dots((3n-4)(3n-3)(3n-2))^1}{1^n 2^n 3^n \dots (n-1)^n n^n (n+1)^{n-1}(n+2)^{n-2}\dots(2n-3)^3(2n-2)^2(2n-1)} = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$$

as required.

Once Kuperberg's proof was published, Zeilberger used this same connection between alternating sign matrices and statistical mechanics to prove the refined alternating sign matrix conjecture (1.2) [112]. A good review of this proof is available in [25]. Another proof of (1.13) is given in [18]. In [40] a determinant formula is applied at the outset, however they offer a different method that gives formula (1.2) in a more natural way.

Before our next section let us consider a different set of weights to those of (1.12):

$$
\begin{aligned}
&W'\left(\overset{0}{\underset{1}{0 \!+\! 1}}, x, y\right) = 1 \quad W'\left(\overset{1}{\underset{0}{1 \!+\! 0}}, x, y\right) = x + y \\
&W'\left(\overset{1}{\underset{1}{1 \!+\! 1}}, x, y\right) = y \quad W'\left(\overset{0}{\underset{0}{0 \!+\! 0}}, x, y\right) = 1 \\
&W'\left(\overset{1}{\underset{1}{0 \!+\! 0}}, x, y\right) = x \quad W'\left(\overset{0}{\underset{0}{1 \!+\! 1}}, x, y\right) = 1
\end{aligned}
\tag{1.19}
$$

which give the following partition function:

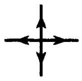
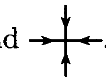
$$
\begin{aligned}
Z_n(x_1, \dots x_n, y_1, \dots y_n) &= \sum_{(h,v)\in \mathrm{EM}(n)} \prod_{i,j=1}^{n} W'\left(\overset{v_{i-1,j}}{\underset{v_{ij}}{h_{i,j-1} \!+\! h_{ij}}}, x_j, y_j\right) \\
&= \sum_{(h,v)\in \mathrm{EM}(n)} \prod_{i=1}^{n} x_i^{N_i\left(\overset{1}{\underset{1}{0 + 0}},(h,v)\right)} \, y_i^{N_i\left(\overset{1}{\underset{1}{1 + 1}},(h,v)\right)} \, (x_i + y_i)^{N_i\left(\overset{1}{\underset{0}{1 + 0}},(h,v)\right)}
\end{aligned}
$$

where $N_i\left(\overset{\delta}{\underset{\beta}{\alpha \!+\! \gamma}}, (h,v)\right)$ denotes the number of vertices of the form $\overset{\delta}{\underset{\beta}{\alpha \!+\! \gamma}}$ in column $i$ of the six vertex model configuration with domain wall boundary conditions using 0's and 1's of $(h,v)$. It has been shown bijectively in [39] that $Z_n(x_1, \dots x_n, y_1, \dots y_n) = \prod_{1 \le i < j \le n}(x_i + y_i)$. Setting $x_i = y_i = 1$ for all $i \in [n]$ we get $Z_n(x_1, \dots x_n, y_1, \dots y_n) = \prod_{1 \le i < j \le n} 2 = 2^{\frac{n(n-1)}{2}}$ and:

$$
\begin{aligned}
&\sum_{(h,v)\in \mathrm{EM}(n)} \prod_{i=1}^{n} x_i^{N_i\left(\overset{1}{\underset{1}{0 + 0}},(h,v)\right)} \, y_i^{N_i\left(\overset{1}{\underset{1}{1 + 1}},(h,v)\right)} \, (x_i + y_i)^{N_i\left(\overset{1}{\underset{0}{1 + 0}},(h,v)\right)} = \\
&\sum_{(h,v)\in \mathrm{EM}(n)} \prod_{i=1}^{n} 2^{N_i\left(\overset{1}{\underset{0}{1 + 0}},(h,v)\right)} = \sum_{(h,v)\in \mathrm{EM}(n)} 2^{\sum_{i=1}^{n} N_i\left(\overset{1}{\underset{0}{1 + 0}},(h,v)\right)}
\end{aligned}
$$

However, $\sum_{i=1}^{n} N_i\left(\overset{1}{\underset{0}{1 \!+\! 0}}, (h,v)\right)$ is the number of vertices of the form $\overset{1}{\underset{0}{1 \!+\! 0}}$ of the six vertex model configuration with domain wall boundary conditions using 0's and 1's of $(h,v)$. Recalling (1.5) we see that this corresponds to the number of $-1$'s in $a \in \mathrm{ASM}(n)$ where $a$ corresponds to $(h,v) \in \mathrm{EM}(n)$. This gives a weighted enumeration of the alternating sign matrices known as the 2-enumeration:

$$
\sum_{a \in \mathrm{ASM}(n)} 2^{\binom{\text{Number of}}{\text{-1's in } a}} = 2^{\frac{n(n-1)}{2}} \tag{1.20}
$$

This weighted enumeration is relatively simple to prove compared to the original conjectures (1.1) and (1.2). Proving these original conjectures was not however the end of research into alternating sign matrices. Indeed Kuperberg had made a connection that spurred multiple new questions that needed explanation. For example in [97] a variation of the six-vertex model is considered, namely the 8 vertex model which includes the six vertices of Definition 1.1.6 as well as the two extra vertices:  and .

Using methods similar to those of Kuperberg, Rosengren shows that the partition function of this system contains two terms: $A_n$ (as expected) and $C_n = \prod_{i=1}^{n} \frac{(3i-1)(3i-3)!}{(n+i-1)!}$ (sequence A006366 of [99]). In fact $C_n$ enumerates cyclically symmetric plane partitions (as discussed in Section 1.1.2). Surprisingly $A_nC_n$ enumerates the number of $2n \times 2n$ half turn symmetric alternating sign matrices (sequence A059475 of [99]). In 2007 yet another proof of (1.2) was offered by Fischer in [56]. This proof uses the bijection with monotone triangles together with a particular operator formula [55] for the number of certain generalized monotone triangles with specific bottom row. A lot of other work has been undertaken and this will be the subject of the next section.

### 1.1.5 Further work on alternating sign matrices

#### Alternating sign matrices invariant under symmetries of the square

Let us consider the dihedral group of symmetries of the square:

$$D_4 := \langle q, h \mid q^4 = h^2 = 1,\ hq = q^{-1}h \rangle \tag{1.21}$$

We consider $q$ to be the operation of rotating the square $90^o$ anti-clockwise about the centre, $h$ to be the operation of reflecting the square through a horizontal line through the centre. It then follows that $d = qh$ is the operation of reflecting the square through a diagonal line through the centre, $v = q^2h$ is the operation of reflecting the square through a vertical line through the centre and $a = q^3h$ is the operation of reflecting the square through an anti diagonal line through the centre. These operations are shown in Figure 1.13.

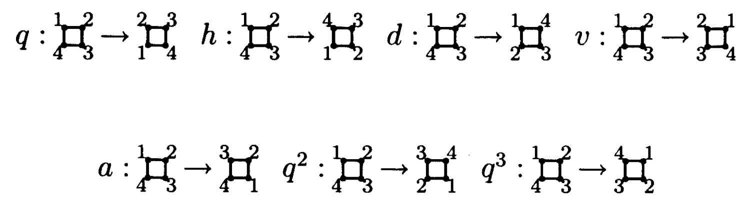


Figure 1.13: Group elements of $D_4$ as operations on the square

Figure 1.14 gives the operation table for $D_4$.

The group $D_4$ contains 10 subgroups. Ordering these by inclusion gives the Hasse diagram of Figure 1.15.

In general for any set $S$ and group $G$ such that an action of $G$ on $S$ is defined (i.e., $gs \in S$ for each $g \in G$ and $s \in S$ and $g_1(g_2s) = (g_1g_2)s$ for each $g_1, g_2 \in G$ and $s \in S$) we define the

| | $1$ | $q$ | $q^2$ | $q^3$ | $h$ | $d$ | $v$ | $a$ |
|---|---|---|---|---|---|---|---|---|
| $1$ | $1$ | $q$ | $q^2$ | $q^3$ | $h$ | $d$ | $v$ | $a$ |
| $q$ | $q$ | $q^2$ | $q^3$ | $1$ | $d$ | $v$ | $a$ | $h$ |
| $q^2$ | $q^2$ | $q^3$ | $1$ | $q$ | $v$ | $a$ | $h$ | $d$ |
| $q^3$ | $q^3$ | $1$ | $q$ | $q^2$ | $a$ | $h$ | $d$ | $v$ |
| $h$ | $h$ | $a$ | $v$ | $d$ | $1$ | $q^3$ | $q^2$ | $q$ |
| $d$ | $d$ | $h$ | $a$ | $v$ | $q$ | $1$ | $q^3$ | $q^2$ |
| $v$ | $v$ | $d$ | $h$ | $a$ | $q^2$ | $q$ | $1$ | $q^3$ |
| $a$ | $a$ | $v$ | $d$ | $h$ | $q^3$ | $q^2$ | $q$ | $1$ |

Figure 1.14: $D_4$ operation table

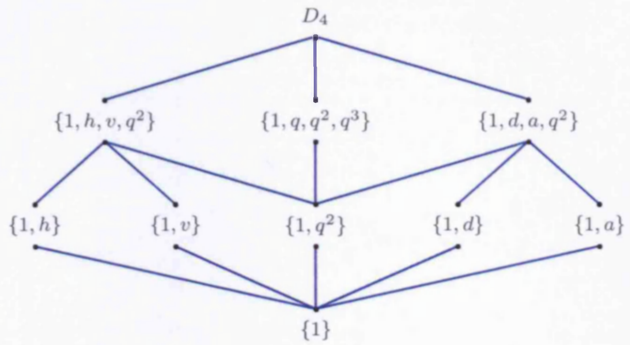



Figure 1.15: Hasse diagram for the set of subgroups of $D_4$ ordered by inclusion

subset of $S$ invariant under $G$ as:

$$S^G := \{a \in S \mid a = ga \text{ for all } g \in G\} \tag{1.22}$$

It follows that if two subgroups $H_1$ and $H_2$ of $G$ are conjugate (where $H_1$ and $H_2$ are defined to be conjugate if $H_2 = gH_1g^{-1}$ for some $g \in G$) then there is a bijection between $S^{H_1}$ and $S^{H_2}$ (since if $H_2 = gH_1g^{-1}$ then $\phi : S^{H_1} \to S^{H_2}$ given by $\phi(s) = gs$ for each $s \in S^{H_1}$ is such a bijection).

In Chapters 3 and 4, we shall consider *linear* actions of groups on subsets $S$ of $\mathbb{R}^n$, i.e. those in which the action of each element of $G$ can be regarded as a linear mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$ (which also still satisfies $g(S) \subseteq S$ for each $g \in G$, and $g_1(g_2s) = (g_1g_2)s$ for each $g_1, g_2 \in G$ and $s \in S$).

If we consider a subgroup $G$ of $D_4$, $S^G$ is the symmetry class of $S$ for some symmetry of the square. As discussed $D_4$ has 10 subgroups of which $\{1, h\}$ and $\{1, v\}$ are conjugate and $\{1, d\}$ and $\{1, a\}$ are conjugate. The elements of $D_4$ have a natural action on the set of square matrices of fixed size. For example the rotations and flips of the square shown in Figure 1.13 become rotations and flips of the square matrix entries as shown in Figure 1.16

Using the notation of (1.22), symmetry classes of alternating sign matrices can be denoted as $\mathrm{ASM}(n)^G$ where $G$ is a subgroup of $D_4$. It is of interest to study the cardinalities of $\mathrm{ASM}(n)^G$ for each subgroup $G$ of $D_4$. Due to the conjugacies of the respective subgroups, $|\mathrm{ASM}(n)^{\{1,h\}}| = |\mathrm{ASM}(n)^{\{1,v\}}|$ and $|\mathrm{ASM}(n)^{\{1,d\}}| = |\mathrm{ASM}(n)^{\{1,a\}}|$, leaving 8 cases. The conditions imposed on the matrix entries $a_{ij}$ by each of these cases are shown in Figure 1.17.

$$q:\begin{pmatrix}a_{11}&a_{12}&a_{13}&a_{14}\\a_{21}&a_{22}&a_{23}&a_{24}\\a_{31}&a_{32}&a_{33}&a_{34}\\a_{41}&a_{42}&a_{43}&a_{44}\end{pmatrix}\rightarrow\begin{pmatrix}a_{14}&a_{24}&a_{34}&a_{44}\\a_{13}&a_{23}&a_{33}&a_{43}\\a_{12}&a_{22}&a_{32}&a_{42}\\a_{11}&a_{21}&a_{31}&a_{41}\end{pmatrix}\quad h:\begin{pmatrix}a_{11}&a_{12}&a_{13}&a_{14}\\a_{21}&a_{22}&a_{23}&a_{24}\\a_{31}&a_{32}&a_{33}&a_{34}\\a_{41}&a_{42}&a_{43}&a_{44}\end{pmatrix}\rightarrow\begin{pmatrix}a_{41}&a_{42}&a_{43}&a_{44}\\a_{31}&a_{32}&a_{33}&a_{34}\\a_{21}&a_{22}&a_{23}&a_{24}\\a_{11}&a_{12}&a_{13}&a_{14}\end{pmatrix}$$

Figure 1.16: Elements of $D_4$ acting on a $4 \times 4$ matrix

| Name | Conditions | Sufficient Conditions | Subgroup | Possible Generators |
|---|---|---|---|---|
| No Symmetry | | | $\{1\}$ | |
| Horizontal Symmetry | $a_{ij} = a_{n+1-i,j}$ | $a_{ij} = a_{n+1-i,j}$ | $\{1, h\}$ | $\{h\}$ |
| Half Turn Symmetry | $a_{ij} = a_{n+1-i,n+1-j}$ | $a_{ij}=a_{n+1-i,n+1-j}$ | $\{1, q^2\}$ | $\{q^2\}$ |
| Diagonal Symmetry | $a_{ij} = a_{j,i}$ | $a_{ij} = a_{j,i}$ | $\{1, d\}$ | $\{d\}$ |
| Horizontal and Vertical Symmetry | $a_{ij}=a_{n+1-i,j}$ $=a_{i,n+1-j}$ $=a_{n+1-i,n+1-j}$ | $a_{ij}=a_{n+1-i,j}$ $=a_{i,n+1-j}$ | $\{1, h, v, q^2\}$ | $\{h, v\}$ |
| Quarter Turn Symmetry | $a_{ij}=a_{j,n+1-i}$ $=a_{n+1-i,n+1-j}$ $=a_{n+1-j,i}$ | $a_{ij}=a_{j,n+1-i}$ | $\{1, q, q^2, q^3\}$ | $\{q\}$ |
| Both Diagonal Symmetry | $a_{ij}=a_{j,i}$ $=a_{n+1-j,n+1-i}$ $=a_{n+1-i,n+1-j}$ | $a_{ij}=a_{j,i}$ $=a_{n+1-j,n+1-i}$ | $\{1, d, a, q^2\}$ | $\{d, a\}$ |
| All Symmetry | $a_{ij}=a_{n+1-i,j}$ $=a_{i,n+1-j}$ $=a_{j,n+1-i}$ $=a_{n+1-i,n+1-j}$ $=a_{n+1-j,i}$ $=a_{j,i}$ $=a_{n+1-j,n+1-i}$ | $a_{ij}=a_{n+1-i,j}$ $=a_{j,n+1-i}$ | $D_4$ | $\{q, h\}$ |

Figure 1.17: Symmetry classes of square matrices

Mills and Robbins conjectured results for the cardinalities of some of the symmetry classes of ASM$(n)$ [93, 94]. We present these conjecture in the form they were given (note the absence of conjectures for $\left|\mathrm{ASM}(n)^{\{1,d\}}\right|$, $\left|\mathrm{ASM}(2k)^{\{1,d,a,q^2\}}\right|$ and $\left|\mathrm{ASM}(n)^{D_4}\right|$):

$$\frac{|\mathrm{ASM}(2k+1)^{\{1,h\}}|}{|\mathrm{ASM}(2k-1)^{\{1,h\}}|} = \frac{\binom{6k-2}{2k}}{2\binom{4k-1}{2k}} \tag{1.23}$$

$$\frac{\left|\mathrm{ASM}(2k+1)^{\{1,q^2\}}\right|}{|\mathrm{ASM}(2k)^{\{1,q^2\}}|} = \frac{\binom{3k}{k}}{\binom{2k}{k}} \tag{1.24}$$

$$\frac{\left|\mathrm{ASM}(2k-1)^{\{1,q^2\}}\right|}{|\mathrm{ASM}(2k)^{\{1,q^2\}}|} = \frac{4\binom{3k}{k}}{3\binom{2k}{k}} \tag{1.25}$$

$$\left|\mathrm{ASM}(4k)^{\{1,q,q^2,q^3\}}\right| = \left|\mathrm{ASM}(2k)^{\{1,q^2\}}\right| |\mathrm{ASM}(k)|^2 \tag{1.26}$$

$$\left|\mathrm{ASM}(4k+1)^{\{1,q,q^2,q^3\}}\right| = \left|\mathrm{ASM}(2k+1)^{\{1,q^2\}}\right| |\mathrm{ASM}(k)|^2 \tag{1.27}$$

$$\left|\mathrm{ASM}(4k-1)^{\{1,q,q^2,q^3\}}\right| = \left|\mathrm{ASM}(2k-1)^{\{1,q^2\}}\right| |\mathrm{ASM}(k)|^2 \tag{1.28}$$

$$\frac{\left|\mathrm{ASM}(2k+1)^{\{1,d,a,q^2\}}\right|}{|\mathrm{ASM}(2k-1)^{\{1,d,a,q^2\}}|} = \frac{\binom{3k}{k}}{\binom{2k-1}{k}} \tag{1.29}$$

$$\frac{\left|\mathrm{ASM}(4k+1)^{\{1,h,v,q^2\}}\right|}{|\mathrm{ASM}(4k-1)^{\{1,h,v,q^2\}}|} = \frac{3k-1}{4k-1}\frac{\binom{6k-3}{2k-1}}{\binom{4k-2}{2k-1}} \tag{1.30}$$

$$\frac{\left|\mathrm{ASM}(4k+3)^{\{1,h,v,q^2\}}\right|}{|\mathrm{ASM}(4k+1)^{\{1,h,v,q^2\}}|} = \frac{3k+1}{4k+1}\frac{\binom{6k}{2k}}{\binom{4k}{2k}} \tag{1.31}$$

These lead to the following product formulae:

$$\left|\mathrm{ASM}(n)^{\{1,h\}}\right| = \begin{cases} 0, \ n \text{ even} \\ \prod_{i=1}^{\frac{n-1}{2}} \frac{(6i-2)!}{(n-1+2i)!}, \ n \text{ odd} \end{cases} \tag{1.32}$$

$$\left|\mathrm{ASM}(n)^{\{1,q^2\}}\right| = \prod_{i=0}^{\lceil \frac{n}{2} \rceil - 1} \frac{(3i)!}{(\lfloor \frac{n}{2} \rfloor + i)!} \prod_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} \frac{(3i+2)!}{(\lceil \frac{n}{2} \rceil + i)!} \tag{1.33}$$

$$\left|\mathrm{ASM}(n)^{\{1,q,q^2,q^3\}}\right| = \begin{cases} \prod_{i=0}^{\frac{n}{4}-1} \frac{i!(3i+1)!\left(\frac{n}{2}+i\right)!}{\left(\frac{n}{4}+i\right)!^3}, \ n = 0 \bmod 4 \\ \prod_{i=0}^{\frac{n-1}{4}} \frac{(3i)!}{\left(\frac{n-1}{4}+i\right)!} \prod_{i=0}^{\frac{n-5}{4}} \frac{(3i+1)!^2(3i+2)!}{\left(\frac{n-1}{4}+i\right)!\left(\frac{n+3}{4}+i\right)!}, \ n = 1 \bmod 4 \\ 0, \ n = 2 \bmod 4 \\ \prod_{i=0}^{\frac{n-3}{4}} \frac{(3i)!(3i+1)!^2}{\left(\frac{n-3}{4}+i\right)!\left(\frac{n+1}{4}+i\right)!} \prod_{i=0}^{\frac{n-7}{4}} \frac{(3i+2)!}{\left(\frac{n+1}{4}+i\right)!}, \ n = 3 \bmod 4 \end{cases} \tag{1.34}$$

$$\left|\mathrm{ASM}(n)^{\{1,d,a,q^2\}}\right| = \begin{cases} ? \ n \text{ even} \\ \prod_{i=0}^{\frac{n-1}{2}} \frac{(3i)!}{\left(\frac{n-1}{2}+i\right)!}, \ n \text{ odd} \end{cases} \tag{1.35}$$

$$\left|\mathrm{ASM}(n)^{\{1,h,v,q^2\}}\right| = \begin{cases} 0, \ n \text{ even} \\ \frac{\left(\lfloor \frac{3(n-3)}{4} \rfloor + 1\right)!}{3\lfloor \frac{n-3}{4} \rfloor (n-2)! \lfloor \frac{n-3}{4} \rfloor !} \prod_{i=1}^{\frac{n-3}{2}} \frac{(3i)!}{\left(\frac{n-3}{2}+i\right)!}, \ n \text{ odd} \end{cases} \tag{1.36}$$

In [73] Kuperberg uses a similar method to that of [72] to prove enumerations (1.32) and the

$n$ even cases of (1.33) and (1.34). He also proves the interesting fact that the number of off diagonally symmetric alternating sign matrices of size $2k$ (elements of $\mathrm{ASM}(2k)^{\{1,d\}}$ with all diagonal entries 0) is equal to the number of horizontally symmetric alternating sign matrices of size $2k+1$. In [85] Okada proves the enumeration formulae (1.36). The $n$ odd cases of (1.33) and (1.34) were proved by Razumov and Stroganov [91, 92]. In [22] cardinalities for the 3 remaining cases ($\mathrm{ASM}(n)^{\{1,d\}}$, $\mathrm{ASM}(n)^{\{1,d,a,q^2\}}$ and $\mathrm{ASM}(n)^{D_4}$) are given, Bousquet-Melou and Habsieger do not however prove the formula for $\mathrm{ASM}(2k+1)^{\{1,d,a,q^2\}}$ nor do they conjecture any other formulas. We believe that this is still an open problem.

### The Razumov-Stroganov conjectures

We label the alternate external vertices of $\mathcal{L}_{n,n}$ using the following rule: $(0, 2k-1)$ and $(n+1, n-2k+2)$ are labelled by $k$ and $n+k$ respectively, for each $k \in [\lceil \frac{n}{2} \rceil]$ and $(n-2k+1, n+1)$ and $(2k, 0)$ are labelled by $n+1-k$ and $2n+1-k$ respectively, for each $k \in [\lfloor \frac{n}{2} \rfloor]$ (as shown in Figure 1.18). Then any element of $\mathrm{FPL}(n)$ can be mapped to a particular non crossing pairing of these labels for which the paired labels are joined by an open path. Such a pairing is called a *link pattern* and can be represented on a disc as shown in Figure 1.19.

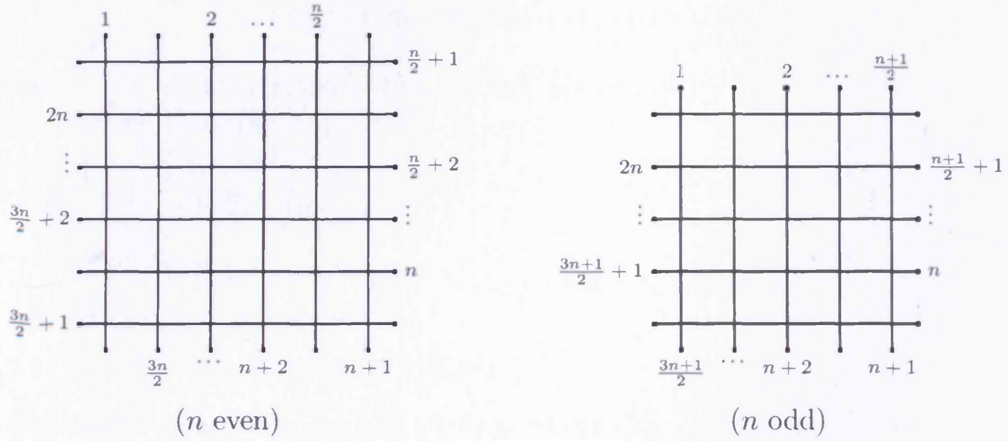



Figure 1.18: Labeling of external vertices of $\mathcal{L}_{n,n}$

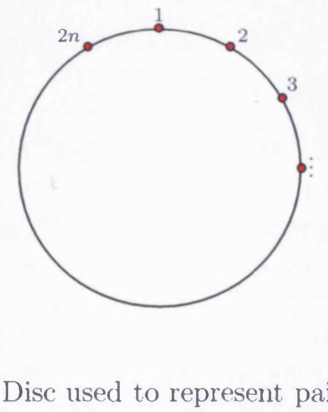



Figure 1.19: Disc used to represent pairings of $L_{2n}$

**Definition 1.1.9.** *We define $L_{2n}$ to be the set of non crossing pairings of $2n$ points (i.e. link patterns).*

It is known that $|L_{2n}| = \frac{(2n)!}{n!(n+1)!}$, i.e. that $L_{2n}$ is enumerated by the Catalan numbers (sequence A000108 of [99]). Figure 1.20 gives the set $L_6$.

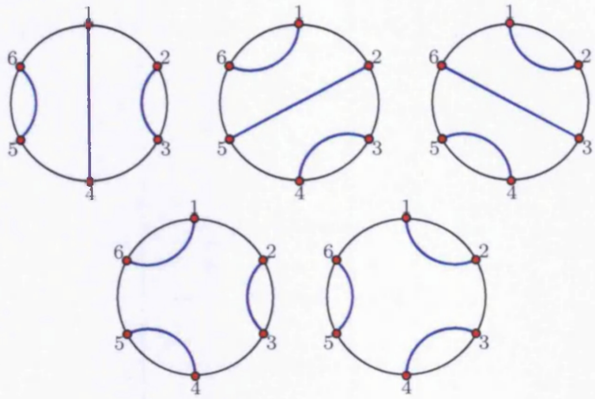


Figure 1.20: The link patterns of $L_6$

Using this we can give the classification by link pattern of FPL(3) from Figure 1.12. We define the notation $\mathrm{FPL}_\pi(n)$, by:

**Definition 1.1.10.** *For $\pi \in L_{2n}$:*

$$FPL_\pi(n) := \{a \in FPL(n) \mid a \text{ has link pattern } \pi\}$$

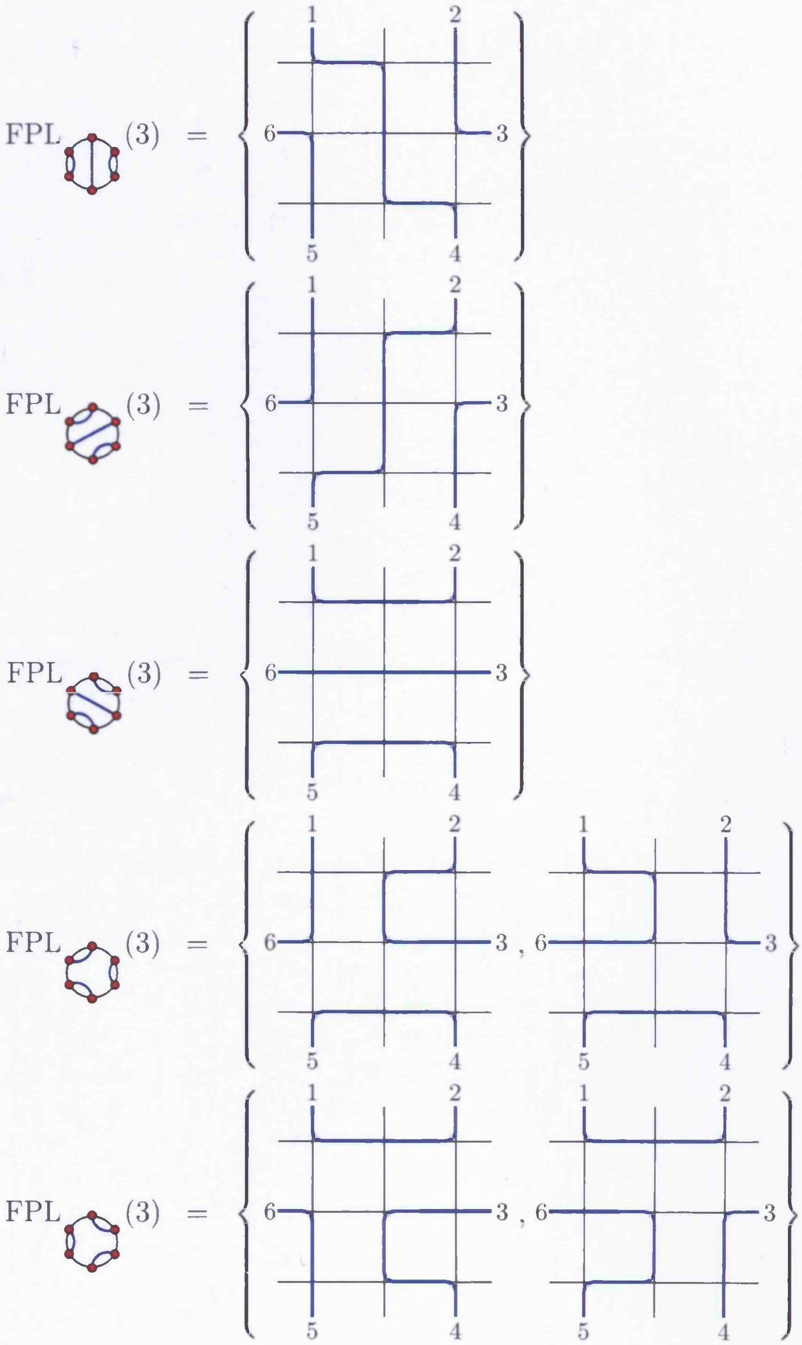
Figure 1.21 gives the classification for FPL(3).

Note the relationship between the sets of same cardinality. Indeed the link patterns corresponding to equations (1.37),(1.38),(1.39) are all rotations of each other, similarly for equations (1.40) and (1.41). This has been proved to happen for general $n$ by Wieland [109].

**Theorem 1.1.11.** *If $\pi, \pi' \in L_{2n}$ are such that $\pi$ can be obtained from $\pi'$ by rotation or reflection then:*

$$|FPL_\pi(n)| = |FPL_{\pi'}(n)|$$

This gives for the square objects of FPL($n$) a much larger symmetry group than $D_4$ namely $D_{2n}$. In this case *the square is indeed a circle.* The proof offered in [109] is bijective. The fully packed loop representation of alternating sign matrices led to the Razumov-Stroganov conjectures [90]. These conjectures relate enumerations of alternating sign matrices with elements of the wavefunctions of different integrable models. A vast amount of work has been done on the surprising connection between eigenstates of physical systems and combinatorics. See for example [7, 45, 46, 47, 117].




(1.37)

(1.38)

(1.39)

(1.40)

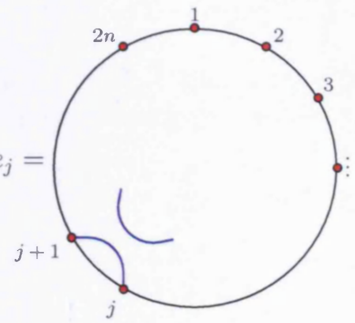
(1.41)

Figure 1.21: Classification of FPL(3)

Recalling that elements of $L_{2n}$ are non crossing pairings of $2n$ points, $\pi \in L_{2n}$ can be represented as $\pi = \{\{i_1, i_2\}, \{i_3, i_4\}, \ldots, \{i_{2n-1}, i_{2n}\}\}$, where $\{i_k, i_{k+1}\} \in \pi$ corresponds to a pairing of the points $i_k, i_{k+1}$. Let us define the operator $e_j$ for $j \in [2n]$ acting on elements $\pi \in L_{2n}$:

$$e_j\pi := \begin{cases} \pi, \ \{j, j+1 \bmod 2n\} \in \pi \\ (\pi \cup \{\{j, j+1 \bmod 2n\}, \{k, l\}\}) \setminus \{\{k, j\}, \{l, j+1 \bmod 2n\}\}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \{k, j\}, \{l, j+1 \bmod 2n\} \in \pi \end{cases}$$

i.e. we have $e_j\pi \in L_{2n}$, such that, if $\{k, j\}, \{l, j+1 \bmod 2n\}$ are pairings of $\pi$ then $\{j, j+1 \bmod 2n\}, \{k, l\}$ are pairings of $e_j\pi$ (all other pairings of $e_j\pi$ are pairings of $\pi$). If $\{j, j+1 \bmod 2n\}$ is a pairing of $\pi$ then $e_j\pi = \pi$. The operator $e_j$ can be represented diagrammatically as:

$e_j =$ 


If we order the link patterns $\pi_1, \ldots, \pi_{\frac{(2n)!}{n!(n+1)!}} \in L_{2n}$ we can construct the matrix $M_{e_j}$ corresponding to the operator $e_j$:

$$(M_{e_j})_{rs} := \begin{cases} 1, \ e_j\pi_s = \pi_r \\ 0, \ \text{otherwise} \end{cases}$$

As an example, Figure 1.22 shows the matrices corresponding to the operators $e_j$ on $L_6$ (we use the order given by Figures 1.20 and 1.21).

$$M_{e_1} = \begin{pmatrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&1&1&0 \\ 0&0&0&0&0 \\ 1&1&0&0&1 \end{pmatrix} \quad M_{e_2} = \begin{pmatrix} 1&0&0&0&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&1&1&1&0 \\ 0&0&0&0&0 \end{pmatrix} \quad M_{e_3} = \begin{pmatrix} 0&0&0&0&0 \\ 0&1&0&1&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 1&0&1&0&1 \end{pmatrix}$$

$$M_{e_4} = \begin{pmatrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&1&0&1 \\ 1&1&0&1&0 \\ 0&0&0&0&0 \end{pmatrix} \quad M_{e_5} = \begin{pmatrix} 1&0&0&1&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&1&1&0&1 \end{pmatrix} \quad M_{e_6} = \begin{pmatrix} 0&0&0&0&0 \\ 0&1&0&0&1 \\ 0&0&0&0&0 \\ 1&0&1&1&0 \\ 0&0&0&0&0 \end{pmatrix}$$

Figure 1.22: Matrices corresponding to the periodic operators defined on $L_6$

The stationary state $\Phi_{2n}$ of the periodic Hamiltonian: $H_{2n} = \sum_{j=1}^{2n} \left( I_{\frac{(2n)!}{n!(n+1)!}} - M_{e_j} \right)$, is the solution of $H_{2n}\Phi_{2n} = 0$ in which the smallest entry of $\Phi_{2n}$ is normalized to be 1 (the

Perron-Frobenius theorem and certain elementary considerations can be used to show that the solution space of $H_{2n}\Phi_{2n} = 0$ is 1 dimensional), and this brings us to our conjecture. Amazingly $(\Phi_{2n})_i = |\mathrm{FPL}_{\pi_i}(n)|$ and so $\sum_{i=1}^{\frac{(2n)!}{n!(n+1)!}} (\Phi_{2n})_i = |\mathrm{ASM}(n)|$. Therefore, $\Phi_{2n}$ is a partition of $|\mathrm{ASM}(n)|$ corresponding to the link pattern classification of $\mathrm{ASM}(n)$. The fact that $\sum_{i=1}^{\frac{(2n)!}{n!(n+1)!}} (\Phi_{2n})_i = |\mathrm{ASM}(n)|$ has been proved [47] but the fact that $(\Phi_{2n})_i = |\mathrm{FPL}_{\pi_i}(n)|$ remains a conjecture. As an example the periodic Hamiltonian corresponding to Figure 1.22 is:

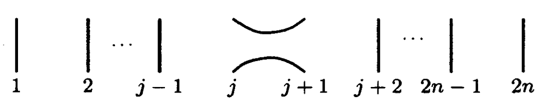
$$H_6 = \begin{pmatrix} 4 & 0 & 0 & -1 & -1 \\ 0 & 4 & 0 & -1 & -1 \\ 0 & 4 & 0 & -1 & -1 \\ -2 & -2 & -2 & 3 & 0 \\ -2 & -2 & -2 & 0 & 3 \end{pmatrix}$$

which has stationary state $\Phi_6 = (1,1,1,2,2)$. Figure 1.21 confirms this conjecture and Figure 1.23 gives $\Phi_{2n}$ for $n \in [4]$ with suitable ordering for the link patterns of $L_{2n}$.

| $n$ | $\Phi_n$ |
|---|---|
| 1 | $(1)$ |
| 2 | $(1,1)$ |
| 3 | $(1,1,1,2,2)$ |
| 4 | $(1,1,1,1,3,3,3,3,3,3,3,3,7,7)$ |

Figure 1.23: Vectors $\Phi_{2n}$ for $n \in [4]$

Until now we have represented the elements of $L_{2n}$ around a disc, we can however unfold them as shown in Figure 1.24. On these unfolded link patterns the operator $e'_j$ for $j \in [2n-1]$ is defined:

$e'_j :=$ 


Note that $e'_j = e_j$ for $j \in [2n-1]$. The difference between these two operators corresponds to different boundary conditions, in the first case we have periodic boundary conditions on our pairings but in this case we have closed boundary conditions. The closed Hamiltonian is defined as $H'_{2k} = \sum_{j=1}^{2k-1}(I - e'_j)$ and the stationary state of $H'_{2k}$, $\Phi'_{2k}$ is a partition of $|\mathrm{ASM}(2k+1)^{\{1,h\}}|$ corresponding to the link pattern classification of $\mathrm{ASM}(2k+1)^{\{1,h\}}$. Figure 1.25 gives $\Phi'_{2k}$ for $k \in [4]$ with suitable orderings for the link patterns of $L_{2k}$ [46]. Defining other boundary conditions on $L_{2n}$ gives similar conjectures.

Alternating sign matrices have been, and continue to be, a source of many unexplained conjectures. Also for those conjectures which have been confirmed (such as formula (1.1)) the proofs are unfortunately often non-combinatorial. There are connections between these simple matrices and many other fields of mathematics (including the Riemann hypothesis [51]), however, we still have no simple explanation for this. Sadly, in the next section we do not offer this explanation, but yet another connection.

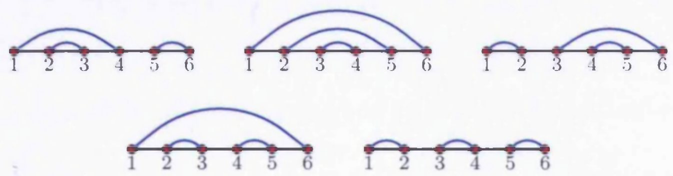



Figure 1.24: $L_6$ represented on an unfolded line segment

| $k$ | $\Phi'_{2k}$ |
|---|---|
| 1 | $(1)$ |
| 2 | $(1, 2)$ |
| 3 | $(1, 4, 5, 5, 11)$ |
| 4 | $(1, 6, 14, 14, 14, 14, 30, 50, 56, 56, 71, 75, 75, 170)$ |

Figure 1.25: Vectors $\Phi'_{2k}$ for $k \in [4]$

## 1.2 Polytopes

Another class of objects studied in combinatorics is polytopes. This section deals with these objects. Firstly, we start with a rather tedious listing of the many relevant definitions, as well as some relevant results. Most of the notions considered here can be found in [12, 60, 114].

### 1.2.1 Definitions

**Definition 1.2.1.** *A set $S \subseteq \mathbb{R}^k$ is* convex *if and only if for all $a, b \in S$, the closed line segment with endpoints $a, b$ is contained in $S$.*

The set of convex sets with straight boundaries in $\mathbb{R}^2$ is the set of convex polygons. An example of such a polygon is shown in Figure 1.26.

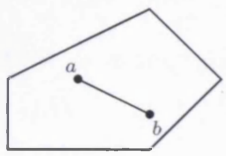



Figure 1.26: A polygon in $\mathbb{R}^2$

**Definition 1.2.2.** *For any set $S \subseteq \mathbb{R}^k$ and any $r \in \mathbb{R}$ we define the $r^{th}$ dilate of $S$:*

$$rS := \{ra \mid a \in S\}$$

In higher dimensions polygons generalize to the *convex polytopes* which will usually be referred to as simply polytopes.

**Definition 1.2.3.** *A convex polytope $\mathcal{P}$ is defined as the convex hull of a finite set of vectors:*

$$\mathcal{P} := \left\{ \lambda_1 v_1 + \cdots + \lambda_m v_m \;\middle|\; \sum_{i=1}^{m} \lambda_i = 1,\ \lambda_i \geq 0 \text{ for all } i \in [m] \right\} \tag{1.42}$$

*where $v_1, \ldots, v_m$ are fixed vectors in $\mathbb{R}^k$.*

*A convex polytope $\mathcal{P}$ has an equivalent definition as the bounded intersection of finitely-many closed half spaces (a proof of this equivalence is available in [12, 114]):*

$$\mathcal{P} := \left\{ a \in \mathbb{R}^k \mid Aa \leq b \right\} \tag{1.43}$$

*where $A \in \mathbb{R}^{l \times k}$ (with $l$ the number of half spaces used to define $\mathcal{P}$), $b \in \mathbb{R}^l$ and the inequality $Aa \leq b$ is used to denote that $(Aa)_i \leq b_i$ for all $i \in [l]$.*

Figures 1.27 and 1.28 show examples of these two equivalent definitions. Note that in the first case we write: $\mathcal{P} = \text{conv}\{v_1, \ldots, v_m\}$. It follows that for positive $r$, if $\mathcal{P}$ is given by (1.42) then its $r^{th}$ dilate is $r\mathcal{P} = \{\lambda_1 v_1 + \cdots + \lambda_m v_m \mid \sum_{i=1}^{m} \lambda_i = r,\ \lambda_i \geq 0 \text{ for all } i \in [m]\}$ and if $\mathcal{P}$ is given by (1.43) then its $r^{th}$ dilate is $r\mathcal{P} = \left\{a \in \mathbb{R}^k \mid Ax \leq rb\right\}$. It also follows that hyperplanes in $\mathbb{R}^k$ can be included together with halfspaces in the second form of the definition, since a hyperplane is simply the intersection of the two closed halfspaces which meet at the hyperplane.

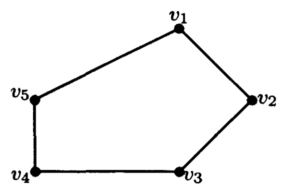



Figure 1.27: Convex hull definition of a polygon in $\mathbb{R}^2$

**Definition 1.2.4.** *The dimension* $\dim \mathcal{P}$ *of a polytope $\mathcal{P}$ is defined to be the dimension of its affine hull (where the affine hull is $Aff\mathcal{P} := \{x + \lambda(y - x) \mid x, y \in \mathcal{P},\ \lambda \in \mathbb{R})\}$).*

A polytope of dimension $d$ is called a $d$ polytope. The polygon in Figures 1.26,1.27 and 1.28 is a 2 polytope.

**Definition 1.2.5.** *A face of a polytope $\mathcal{P} \subseteq \mathbb{R}^k$ is a polytope of the form:*

$$\{a \in \mathcal{P} \mid \alpha \cdot a = \beta\}$$

*where $\alpha \cdot a \leq \beta$ is an inequality (for some $\alpha \in \mathbb{R}^k, \beta \in \mathbb{R}$) which is valid for all $a \in \mathcal{P}$.*

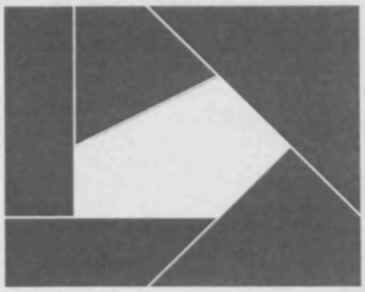


Figure 1.28: Half space definition of a polygon in $\mathbb{R}^2$

A face of dimension 0 contains a single point called a *vertex*.

**Definition 1.2.6.** *A vertex of a polytope $\mathcal{P}$ is a vector $v \in \mathcal{P}$ such that there exists a half space $H$ for which:*

$$H \cap \mathcal{P} = \{v\}$$

We denote the set of vertices of $\mathcal{P}$ as vert$\mathcal{P}$. Note that if $\mathcal{P} = \text{conv}\{v_1, \ldots, v_m\}$ then vert$\mathcal{P} \subseteq \{v_1, \ldots, v_m\}$. Also, it can be shown that $a \in \mathcal{P}$ is a vertex if and only if it does not belong to the interior of a line segment in $\mathcal{P}$, i.e. if and only if there do not exist $a_1 \neq a_2 \in \mathcal{P}$ and $0 < \lambda < 1$ such that $a = \lambda a_1 + (1 - \lambda)a_2$. This leads to the following useful lemma:

**Lemma 1.2.7.** *Let $\mathcal{P} \subseteq \mathbb{R}^k$ be a polytope, consider $a \in \mathcal{P}$. Then $a$ is not a vertex of $\mathcal{P}$ if and only if there exists a non zero point $a^* \in \mathbb{R}^k$ such that $a \pm a^* \in \mathcal{P}$.*

*Proof.* Let $a \in \mathcal{P} \setminus \text{vert}\mathcal{P}$. Then there exists $a_1 \neq a_2 \in \mathcal{P}$, $\lambda \in (0,1)_{\mathbb{R}}$ such that $a = \lambda a_1 + (1 - \lambda)a_2$. If we take $a_\pm = a \pm \min(\lambda, 1 - \lambda)(a_2 - a_1)$, then $a_\pm \in \mathcal{P}$. Indeed if $1/2 \leq \lambda < 1$, $a_\pm = a \pm (1 - \lambda)(a_2 - a_1) = (\lambda \pm (\lambda - 1))a_1 + (1 - \lambda \pm (1 - \lambda))a_2$, giving $a_- = a_1 \in \mathcal{P}$ and $a_+ = (2\lambda - 1)a_1 + (2 - 2\lambda)a_2$. However $0 \leq 2\lambda - 1 \leq 1$ so $a_+$ is contained within the line segment between $a_1$ and $a_2$ as required. We get the equivalent result for $0 < \lambda \leq 1/2$. Taking $a^* = \min(\lambda, 1 - \lambda)(a_2 - a_1)$ gives the required result. Now let there exist $a^* \neq 0$ such that $a \pm a^* \in \mathcal{P}$. Then $a = \frac{1}{2}(a - a^*) + \frac{1}{2}(a + a^*)$ and so $a$ is not a vertex of $\mathcal{P}$. □

**Lemma 1.2.8.** *If $\mathcal{K}, \mathcal{L}$ are two polytopes then $\mathcal{K} \cap \mathcal{L}$ is a polytope with $\mathcal{K} \cap \text{vert}\mathcal{L} \subseteq \text{vert}(\mathcal{K} \cap \mathcal{L})$ and $\mathcal{L} \cap \text{vert}\mathcal{K} \subseteq \text{vert}(\mathcal{K} \cap \mathcal{L})$.*

The proof of this lemma follows straightforwardly from the half space definition of a polytope (1.43), and intuitively if one considers the example given in Figure 1.29.

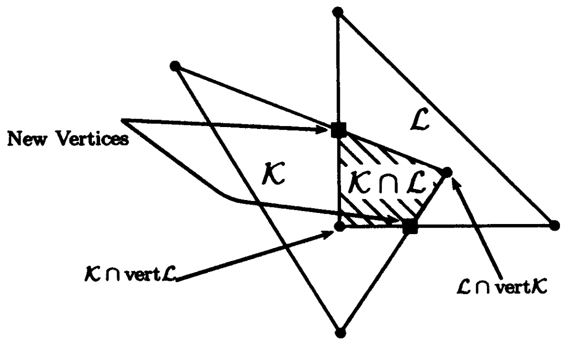



Figure 1.29: Intersection of polytopes

**Corollary 1.2.9.** *If $\mathcal{K}, \mathcal{L}$ are two polytopes and $\mathcal{K} \subseteq \mathcal{L}$ then $\mathcal{K} \cap vert\mathcal{L} \subseteq vert\mathcal{K}$.*

**Lemma 1.2.10.** *Let $\mathcal{P} \subseteq \mathbb{R}^k$ and let $\phi$ be an affine map from $\mathbb{R}^k$ to $\mathbb{R}^{k'}$. Then $\phi(\mathcal{P})$ is a polytope in $\mathbb{R}^{k'}$ and $vert\phi(\mathcal{P}) \subseteq \phi(vert\mathcal{P})$.*

This can be seen by considering the convex hull definition of $\mathcal{P}$ (1.42).

**Definition 1.2.11.** *Polytopes $\mathcal{K} \subseteq \mathbb{R}^k$ and $\mathcal{L} \subseteq \mathbb{R}^{k'}$ are defined to be affinely isomorphic if there is an affine map $\phi$ from $\mathbb{R}^k$ to $\mathbb{R}^{k'}$ which is bijective from $\mathcal{K}$ to $\mathcal{L}$.*

This leads to the following lemma:

**Lemma 1.2.12.** *If polytopes $\mathcal{K} \subseteq \mathbb{R}^k$ and $\mathcal{L} \subseteq \mathbb{R}^{k'}$ are affinely isomorphic and $\phi$ is an affine map from $\mathbb{R}^k$ to $\mathbb{R}^{k'}$ which is bijective between $\mathcal{K}$ and $\mathcal{L}$ then:*

$$\phi\,(vert\mathcal{K}) = vert\mathcal{L}$$

If all the vertices of $\mathcal{P}$ have *integer* coordinates then $\mathcal{P}$ is called an *integer* polytope. Similarly if all the vertices of $\mathcal{P}$ have *rational* coordinates then $\mathcal{P}$ is called a *rational* polytope.

**Definition 1.2.13.** *The denominator of a rational polytope $\mathcal{P}$ is defined as:*

$$D(\mathcal{P}) := min\{p \in \mathbb{P} \mid p\mathcal{P} \text{ is an integral polytope}\}$$

Note that one can also define D($\mathcal{P}$) as the least common multiple of the denominators of the coordinates of the vertices of $\mathcal{P}$ (when these coordinates are all written in lowest terms).

*Edges* are defined to be faces of dimension 1, whereas *facets* are defined to be faces of dimension $d-1$. The edge shown in Figure 1.30 is also a facet since $d-1=1$.

One last important definition before giving some results concerning polytopes is the definition of a *simplex*.

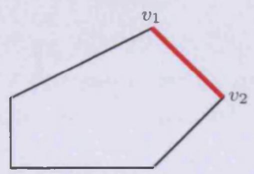



Figure 1.30: The edge between $v_1$ and $v_2$ of a polygon in $\mathbb{R}^2$

**Definition 1.2.14.** *A simplex is a d polytope with exactly $d+1$ vertices.*

Indeed it can be shown that every $d$ polytope has at least $d+1$ vertices. The polytope of Figure 1.26 is a 2 polytope with 5 vertices and thus is not a simplex. The simplices in $\mathbb{R}^2$ are triangles. It is well known that any polygon can be triangulated as shown in Figure 1.31. This leads to a generalized result (using a suitable definition of triangulation in $\mathbb{R}^d$, see for example Section 3.1 of [12]).

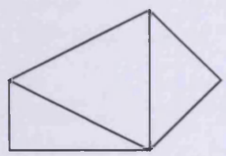


Figure 1.31: Triangulation of a polygon in $\mathbb{R}^2$

**Theorem 1.2.15.** *Any polytope $\mathcal{P}$ can be triangulated into simplices using no new vertices.*

In $\mathbb{R}^2$ triangulation can be used to prove Pick's theorem:

**Theorem 1.2.16.** *For a polygon $P$ with vertices in $\mathbb{Z}^2$ let $A$ be the area of $P$, $I$ the number of interior integer points of $P$ and $B$ the number of integer points on the boundary (where an integer point is a point in $\mathbb{Z}^2$). Then:*

$$A = I + \frac{B}{2} - 1$$

For the case in Theorem 1.2.16 the total number of integer points in $P$ is $I + B = A + \frac{B}{2} + 1$. If we dilate $P$ giving the polygon $tP$ we have new area $At^2$, and the number of boundary integer points $Bt$ so that the total number of integer points in $tP$ is

$$|tP \cap \mathbb{Z}^2| = At^2 + \frac{Bt}{2} + 1 \tag{1.44}$$

Note that this is a polynomial of degree $2 = \dim P$. Also if we only want the interior points of $tP$, denoted $tP^o$:

$$|tP^o \cap \mathbb{Z}^2| = At^2 - \frac{Bt}{2} + 1 \tag{1.45}$$

Comparing equations (1.44) and (1.45) gives $|tP^o \cap \mathbb{Z}^2| = |(-t)P \cap \mathbb{Z}^2|$. Thus evaluating the polynomial for $|tP \cap \mathbb{Z}^2|$ at negative $t$ enumerates the interior points of $tP$. Equations (1.44) and (1.45) are examples of the forthcoming Theorem 1.2.18 for the case of a polytope of dimension 2.

**Definition 1.2.17.** *A function $f : \mathbb{Z} \to \mathbb{C}$ is a quasi-polynomial if there exists an integer $N$ and polynomials $f_0, f_1, \ldots, f_{N-1}$: such that*

$$f(n) = f_i(n), \;\; n = i \; mod \; N$$

*The smallest such integer $N$ is called the* period *of $f$. The degree of $f$ is defined to be the largest degree among those of $f_0, f_1, \ldots, f_{N-1}$.*

In higher dimensions, Theorem 1.2.15 will be useful in the proof of the following important theorem based on results of Ehrhart and Macdonald:

**Theorem 1.2.18.** *If $\mathcal{P} \subset \mathbb{R}^k$ is a rational polytope of dimension $d$ then there exists a quasi-polynomial $L_{\mathcal{P}}(r)$ of degree $d$ and period which divides $D(\mathcal{P})$ such that for all $r \in \mathbb{Z}$,*

$$L_{\mathcal{P}}(r) = \begin{cases} |\mathbb{Z}^k \cap r\mathcal{P}|, \; r > 0 \\ 1, \; r = 0 \\ (-1)^d \, |\mathbb{Z}^k \cap (-r)\mathcal{P}^o|, \; r < 0 \end{cases}$$

*where $\mathcal{P}^o$ is the relative interior of $\mathcal{P}$.*

The quasi-polynomial $L_{\mathcal{P}}(r)$ is called the Ehrhart quasi-polynomial of $\mathcal{P}$. If $\mathcal{P}$ is an integral polytope then $D(\mathcal{P}) = 1$ and so $L_{\mathcal{P}}(r)$ is a polynomial, known as the Ehrhart polynomial of $\mathcal{P}$.

For an insight into Ehrhart quasi-polynomials the reader is encouraged to read [13, 75, 79, 110]. For a standard proof of Theorem 1.2.18 the reader is encouraged to read [12, 103], however here we give a review of part of the new bijective proof given in article [98]. The proof will use the following lemma from corollary 4.3.1 of [103]:

**Lemma 1.2.19.** *A function $f : \mathbb{N} \to \mathbb{C}$ is a polynomial of degree at most $d$ if and only if:*

$$\sum_{i=0}^{d+1} (-1)^{d+1-i} \binom{d+1}{i} f(r+i) = 0 \; \text{for all} \; r \in \mathbb{N}$$

Now for part of Sam's proof [98] of Theorem 1.2.18. In particular, it will be shown that for an integral simplex $\mathcal{P} \subseteq \mathbb{R}^k$ and a non negative integer $r$, $\left|r\mathcal{P} \cap \mathbb{Z}^k\right|$ is a polynomial in $r$ of degree at most $\dim \mathcal{P}$.

*Proof.* Assume $\mathcal{P}$ is an integral simplex of dimension $d$ with vertices $v_0, v_1, \ldots, v_d$ and define:

$$\mathcal{Q}_i := (r+d)\mathcal{P} + v_i \text{ for any } r \in \mathbb{N}$$

It follows that:

$$\mathcal{Q}_i := \left\{ \lambda_0 v_0 + \cdots + \lambda_d v_d \;\middle|\; \begin{array}{l} \bullet\ \lambda_0, \ldots \lambda_d \geq 0 \\ \bullet\ \lambda_i \geq 1 \\ \bullet\ \lambda_0 + \cdots + \lambda_d = r+d+1 \end{array} \right\}$$

Now consider $\mathcal{Q} := \bigcup_{i=0}^{d} \mathcal{Q}_i$. It follows that $\mathcal{Q} \subseteq (r+d+1)\mathcal{P}$. Since $r \geq 0$ any element $a \in (r+d+1)\mathcal{P}$ has a particular $\lambda_i \geq 1$ (since otherwise $\sum_{i=0}^{d} \lambda_i < d+1$) and so, $a \in \mathcal{Q}_i$ giving $(r+d+1)\mathcal{P} \subseteq \mathcal{Q}$. Thus, we have $\mathcal{Q} = (r+d+1)\mathcal{P}$. Using inclusion-exclusion we get:

$$|\mathcal{Q} \cap \mathbb{Z}^k| = \sum_{j=1}^{d+1} (-1)^{j+1} \sum_{\substack{I \subseteq [0,d] \\ |I| = j}} \left| \bigcap_{i \in I} \mathcal{Q}_i \cap \mathbb{Z}^k \right|$$

Considering $\mathcal{Q}_i \cap \mathcal{Q}_j$ for $i \neq j$:

$$\begin{aligned} \mathcal{Q}_i \cap \mathcal{Q}_j &= \left\{ \lambda_0 v_0 + \cdots + \lambda_d v_d \;\middle|\; \begin{array}{l} \bullet\ \lambda_0, \ldots \lambda_d \geq 0 \\ \bullet\ \lambda_i, \lambda_j \geq 1 \\ \bullet\ \lambda_0 + \cdots + \lambda_d = r+d+1 \end{array} \right\} \\ &= (r+d+1-2)\mathcal{P} + v_i + v_j \end{aligned}$$

It can be seen that for any $I \subseteq [0,d]$:

$$\bigcap_{i \in I} \mathcal{Q}_i = (r+d+1-|I|)\mathcal{P} + \sum_{i \in I} v_i$$

Importantly since $\mathcal{P}$ is integral, the sets $(t+d+1-|I|)\mathcal{P} + \sum_{i \in I} v_i$ and $(t+d+1-|I|)\mathcal{P}$ are simply integer translations of each other and so contain the same number of integer points. Also note that there are $\binom{d+1}{|I|}$ subsets of $[0,d]$ of size $|I|$, giving:

$$\sum_{\substack{I \subseteq [0,d] \\ |I| = j}} \left| \bigcap_{i \in I} \mathcal{Q}_i \cap \mathbb{Z}^n \right| = \binom{d+1}{j} |(r+d+1-j)\mathcal{P} \cap \mathbb{Z}^k|$$

and so:

$$\begin{aligned} |(r+d+1)\mathcal{P} \cap \mathbb{Z}^k| &= \sum_{j=1}^{d+1} (-1)^{j+1} \binom{d+1}{j} |(r+d+1-j)\mathcal{P} \cap \mathbb{Z}^k| \\ &= \sum_{j=0}^{d} (-1)^{d-j} \binom{d+1}{j} |(r+j)\mathcal{P} \cap \mathbb{Z}^k| \end{aligned}$$

Using Lemma 1.2.19 we get the result that $|r\mathcal{P} \cap \mathbb{Z}^k|$ is a polynomial in $r$ of degree at most $d$. This is not a complete proof of Theorem 1.2.18 and the reader is encouraged to read [98]. □

In the next section we consider a particular polytope: The *Birkhoff* polytope.

### 1.2.2 The Birkhoff polytope

The Birkhoff polytope has been the subject of a vast amount of research [5, 10, 11, 16, 17, 29, 30, 31, 32, 33, 52, 54, 82, 87, 100, 108]. We go over some of the main results concerning this polytope.

**Definition 1.2.20.** *The Birkhoff polytope $\mathcal{B}_n$ is defined to be the set of all doubly stochastic matrices of size n, i.e., the set of all nonnegative real entry $n \times n$ matrices in which all row and column sums are 1:*

$$\mathcal{B}_n := \left\{ a \in \mathbb{R}^{n\times n} \;\middle|\; \begin{array}{l} \bullet\ a_{ij} \geq 0 \;\; \textit{for all } (i,j) \in [n] \times [n] \\ \bullet\ \sum_{i=1}^n a_{ij} = \sum_{j=1}^n a_{ij} = 1 \;\; \textit{for all } i,j \in [n] \end{array} \right\}$$

This polytope is named after *Garrett Birkhoff* [17]. Following Definition 1.2.4 we have:

**Theorem 1.2.21.**

$$\dim \mathcal{B}_n = (n-1)^2$$

*Proof.* It follows from Definition 1.2.20 that:

$$\text{aff}\mathcal{B}_n = \left\{ a \in \mathbb{R}^{n\times n} \;\middle|\; \sum_{i=1}^{n} a_{ij} = \sum_{j=1}^{n} a_{ij} = 1 \;\text{ for all } i,j \in [n] \right\}$$

Of the $2n$ linear equations in $n^2$ variables within this set, only $2n-1$ equations are independent, so that $\dim \mathcal{B}_n = \dim(\text{aff}\mathcal{B}_n) = n^2 - (2n-1) = (n-1)^2$. □

Also of interest are the faces of this polytope. From Definition 1.2.5, it is easy to see that any face of $B_n$ is of the form:

$$\mathcal{B}_{nH} := \{ a \in \mathcal{B}_n \mid a_{ij} = 0 \;\text{ for all } (i,j) \in H \} \tag{1.46}$$

where $H \subseteq [n] \times [n]$.

This leads to the following result:

**Theorem 1.2.22.** *The Birkhoff polytope $\mathcal{B}_n$ has $n^2$ facets for $n \geq 3$.*

The proof of this follows immediately from equation (1.46) using $H = \{(i,j)\}$ for all $(i,j) \in [n] \times [n]$ and the fact that a facet is a face of dimension $\dim \mathcal{B}_n - 1$.

The $r^{th}$ dilate of $\mathcal{B}_n$ is given by:

$$r\mathcal{B}_n = \left\{ a \in \mathbb{R}^{n \times n} \;\middle|\; \begin{array}{l} \bullet\ a_{ij} \geq 0 \text{ for all } (i,j) \in [n] \times [n] \\ \bullet\ \sum_{i=1}^{n} a_{ij} = \sum_{j=1}^{n} a_{ij} = r \text{ for all } i,j \in [n] \end{array} \right\}$$

The integer points of this polytope are defined as the semi magic squares:

**Definition 1.2.23.** *The set of semi magic squares of size n and line sum r SMS(n, r) is defined as:*

$$SMS(n,r) := \left\{ a \in [0,r]^{n \times n} \;\middle|\; \sum_{i=1}^{n} a_{ij} = \sum_{j=1}^{n} a_{ij} = r \quad \text{for all } i,j \in [n] \right\}$$

Semi magic squares have been studied for a very long time, we believe that the earliest formal definition was given in [78]. For further information see for example [2, 12, 20, 52, 100, 103, 104]. Figure 1.32 gives some cardinalities of $\mathrm{SMS}(n,r)$. Note that $\mathrm{SMS}(n,0)$ contains only the $n \times n$ zero matrix, and that $\mathrm{SMS}(n,1)$ is the set of permutation matrices, isomorphic to the group of permutations of $[n]$. Also $\mathrm{SMS}(1,r)$ contains only the $1 \times 1$ matrix with entry $r$, and $\mathrm{SMS}(2,r) = \left\{ \begin{pmatrix} i & r-i \\ r-i & i \end{pmatrix} \middle| \; i \in [0,r] \right\}$.

| | $r = 0$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $n = 1$ | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 2 | 3 | 4 | 5 |
| 3 | 1 | 6 | 21 | 55 | 120 |
| 4 | 1 | 24 | 282 | 2008 | 10147 |
| 5 | 1 | 120 | 6210 | 153040 | 2224955 |
| 6 | 1 | 720 | 202410 | 20933840 | 1047649905 |

Figure 1.32: $|\mathrm{SMS}(n,r)|$ for $n \in [6]$, $r \in [0,4]$

In [17, 108] the following theorem is given. (The proof provided here is essentially that of [108].)

**Theorem 1.2.24.**

$$vert\mathcal{B}_n = SMS(n,1)$$

*Proof.* Consider $a \in \mathrm{SMS}(n,1)$ and assume $a \notin \mathrm{vert}\mathcal{B}_n$. Then by Lemma 1.2.7 there exists $a^* \neq 0$ such that $a \pm a^* \in \mathcal{B}_n$ so that $0 \leq a_{ij} \pm a^*_{ij} \leq 1$ for all $i,j \in [n]$. Choose $i,j$ such that $a^*_{ij} \neq 0$. If $a_{ij} = 0$ then $\pm a^*_{ij} \geq 0$ which gives $\pm a^*_{ij} > 0$ (since $a^*_{ij} \neq 0$) which is impossible.

If $a_{ij} = 1$ then $1 \pm a^*_{ij} \leq 1$ which leads to $\pm a^*_{ij} < 0$ (since $a_{ij} \neq 0$) which is also impossible. Therefore $\text{SMS}(n,1) \subseteq \text{vert}\mathcal{B}_n$.

Now consider $a \in \mathcal{B}_n \setminus \text{SMS}(n,1)$. This implies that we have $i_1, j_1$ such that $0 < a_{i_1,j_1} < 1$. Then we must have $i_1, j_2$ such that $0 < a_{i_1,j_2} < 1$ and $j_2 \neq j_1$ since the sum of entries in row $i_1$ of $a$ is 1. Similarly, we must have $i_2, j_2$ such that $0 < a_{i_2,j_2} < 1$ and $i_2 \neq i_1$ since the sum of entries in column $j_2$ of $a$ is 1. If we continue in this way we can create a cycle of entries $a_{i_1,j_1}, a_{i_1,j_2}, a_{i_2,j_2}, a_{i_2,j_3}, \ldots, a_{i_{s-1},j_{s-1}}, a_{i_{s-1},j_s}, a_{i_s,j_s}, a_{i_s,j_1}$ each of which is strictly between 0 and 1 where $i_k \neq i_{k+1}, i_s \neq i_1, j_k \neq j_{k+1}$ and $j_s \neq j_1$ for all $k \in [s-1]$. Taking $\epsilon$ to be the minimum of these entries, we define a matrix $a^* \in \mathbb{R}^{n \times n}$ with entries:

$$a^*_{ij} = \begin{cases} \epsilon, \ (i,j) \in \{(i_1,j_1),(i_2,j_2),\ldots,(i_s,j_s)\} \\ -\epsilon, \ (i,j) \in \{(i_1,j_2),(i_2,j_3),\ldots,(i_{s-1},j_s),(i_s,j_1)\} \\ 0, \text{ otherwise} \end{cases}$$

It then follows that $a^* \neq 0$ and $a \pm a^* \in \mathcal{B}_n$, and so $a$ is not a vertex. Thus, $\text{SMS}(n,1) \supseteq \text{vert}\mathcal{B}_n$ as required. □

Recalling Theorem 1.2.18 we define the following set:

$$\text{SMS}^\circ(n,r) := r\mathcal{B}^\circ_n \cap \mathbb{Z}^{n \times n} \tag{1.47}$$

Equivalently:

$$\text{SMS}^\circ(n,r) = \left\{ a \in [r]^{n \times n} \ \middle| \ \sum_{i=1}^{n} a_{ij} = \sum_{j=1}^{n} a_{ij} = r \ \text{ for all } i,j \in [n] \right\}$$

This leads to the following theorem:

**Theorem 1.2.25.** *For fixed $n \in \mathbb{P}$ there exists $H_n(r)$, the Ehrhart polynomial of $\mathcal{B}_n$, which satisfies:*

1. *$H_n(r)$ is a polynomial in $r$ of degree $(n-1)^2$*
2. *$|SMS(n,r)| = H_n(r)$ for all $r \in \mathbb{N}$*
3. *$|SMS^\circ(n,r)| = (-1)^{n+1}H_n(-r) = H_n(r-n)$ for all $r \in \mathbb{P}$*
4. *$H_n(-1) = H_n(-2) = \cdots = H_n(-n+1) = 0$*

5. $H_n(1) = n!$

*Proof.* Most of these results are direct implications of Theorem 1.2.18 since Theorem 1.2.24 implies that $\mathcal{B}_n$ is an integral polytope. The second equality of property (3) is obtained since we have a bijection between $\text{SMS}^o(n,r)$ and $\text{SMS}(n, r-n)$ for $r \geq n$ (in which 1 is subtracted from each entry of an element of $\text{SMS}^o(n,r)$ to give an element of $\text{SMS}(n, r-n)$). Property (4) is implied by the fact that $\text{SMS}^o(n,r) = \emptyset$ for $r < n$. □

The following enumerations illustrate this [2, 19, 84]:

$$\begin{aligned}
H_1(r) &= \binom{r}{0},\ H_2(r) = \binom{r+1}{1} \\
H_3(r) &= \binom{r+4}{4} + \binom{r+3}{4} + \binom{r+2}{4} &(1.48)\\
H_4(r) &= \binom{r+9}{9} + 14\binom{r+8}{9} + 87\binom{r+7}{9} + 148\binom{r+6}{9} + 87\binom{r+5}{9} + 14\binom{r+4}{9} \\
&\quad + \binom{r+3}{9} &(1.49)\\
H_5(r) &= \binom{r+16}{16} + 103\binom{r+15}{16} + 4306\binom{r+14}{16} + 63110\binom{r+13}{16} + 388615\binom{r+12}{16} \\
&\quad + 1115068\binom{r+11}{16} + 1575669\binom{r+10}{16} + 1115068\binom{r+9}{16} + 388615\binom{r+8}{16} \\
&\quad + 63110\binom{r+7}{16} + 4306\binom{r+6}{16} + 103\binom{r+5}{16} + \binom{r+4}{16} &(1.50)
\end{aligned}$$

Polynomials (1.48), (1.49) and (1.50) correspond to the sequences; A002817, A001496 and A003438 of [99]. Another interesting result concerns the decompositions of a semi magic square.

**Theorem 1.2.26.** *Any matrix $a \in SMS(n,r)$ can be written as the sum of $r$ permutation matrices of size $n$.*

In [20] a proof of this theorem is offered based on Hall's theorem:

**Theorem 1.2.27.** *Let $G$ be a bipartite graph with color classes $A$ and $B$, and for all $X \subseteq A$, let $N(X)$ be the set of all vertices in $B$ that have a neighbor in $X$.*

*Then $A$ has a perfect matching into $B$ (i.e. each vertex of $A$ can be matched with an adjacent vertex of $B$), if and only if $|X| \leq |N(X)|$ for all $X \subseteq A$*

Before using this theorem however we need one further definition:

**Definition 1.2.28.** *For any matrix $a \in \mathbb{R}^{m \times n}$ we define the bipartite graph of $a$ as the bipartite graph with ordered color classes $A = \{v_1, \dots, v_m\}$ and $B = \{v'_1, \dots, v'_n\}$, such that $\{v_i, v'_j\}$ is an edge if and only if $a_{ij} \neq 0$.*

Returning to the proof of Theorem 1.2.26, consider $a \in \mathrm{SMS}(n, r)$. It can then be shown that the bipartite graph of $a$ satisfies the condition in Theorem 1.2.27 and that it therefore has a perfect matching. Thus $a - p \in \mathrm{SMS}(n, r-1)$ where $p$ is the permutation matrix that corresponds to the perfect matching. Induction on $r$ then gives the required result that $a$ can be written as the sum of $r$ permutation matrices. However Theorem 1.2.26 is a special case of the following theorem concerning 0, 1 polytopes [71, 115]. A 0, 1 polytope is a polytope, for which all the coordinates of all of the vertices are either 0 or 1.

**Theorem 1.2.29.** *Let $\mathcal{P} \subseteq \mathbb{R}^k$ be a 0, 1 polytope and let $\mathcal{P}$ have the halfspace description $\mathcal{P} = \{x \in [0,1]^k \mid Ax = b\}$ for some matrix $A \in \mathbb{R}^{m \times k}$ and $b \in \mathbb{R}^m$. Then, for any $r \in \mathbb{P}$, any integer point of the $r^{th}$ dilate of $\mathcal{P}$ can be written as the sum of $r$ vertices of $\mathcal{P}$.*

*Proof.* Take $x \in \mathcal{P}$. Since $\mathcal{P}$ is a polytope then there exists a non empty set $\mathcal{V} \subseteq \mathrm{vert}\mathcal{P}$ such that $x = \sum_{v \in \mathcal{V}} \lambda_v v$, with each $\lambda_v > 0$ and $\sum_{v \in \mathcal{V}} \lambda_v = 1$. Also since $\mathcal{P}$ is 0, 1 it follows that:

- If $x_i = 0$ then $v_i = 0$ for all $v \in \mathcal{V}$.
- If $x_i = 1$ then $v_i = 1$ for all $v \in \mathcal{V}$.

Indeed if we let $\mathcal{V}_i = \{v \in \mathcal{V} \mid v_i = 1\}$ then we have $x_i = \sum_{v \in \mathcal{V}_i} \lambda_v$. So if $x_i = 0$ then $\mathcal{V}_i = \emptyset$ and if $x_i = 1$ then $\mathcal{V}_i = \mathcal{V}$.

Now consider $a \in r\mathcal{P} \cap \mathbb{Z}^k$.

Then $x = \frac{a}{r} \in \mathcal{P}$ and so there exists as before a set $\mathcal{V} \subseteq \mathrm{vert}\mathcal{P}$. Taking any element $v \in \mathcal{V}$, we have $A(a - v) = (r-1)b$. We now check that $a - v \in [0, r-1]^k$. Indeed, if $a_i = r$ then $v_i = 1$ so $(a-v)_i = r - 1$, if $a_i = 0$ then $v_i = 0$ so $(a - v)_i = 0$, and if $1 \leq a_i \leq r - 1$ then $v_i$ can be 0 or 1 which gives $0 \leq (a - v)_i \leq r - 1$.

Thus, for all $a \in r\mathcal{P} \cap \mathbb{Z}^k$ there is a $v \in \mathrm{vert}\mathcal{P}$ such that $a - v \in (r-1)\mathcal{P} \cap \mathbb{Z}^k$ and so by induction on $r$ we get the required result. □

In [19] Bona gives a combinatorial argument for the formula for $|\mathrm{SMS}(3,r)|$ as given by equation (1.48). We give an equivalent argument here. Let us label the elements of SMS(3, 1) as in Figure 1.33.

$$h_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad h_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad h_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$h_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad h_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad h_6 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Figure 1.33: SMS(3, 1)

It can be checked that we have a bijection $\phi$ between $\mathrm{SMS}(3,r)$ and the set:

$$C(r) := \left\{ (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) \in \mathbb{N}^6 \;\middle|\; \begin{array}{l} \bullet\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 = r \\ \bullet\ \lambda_4\lambda_5\lambda_6 = 0 \end{array} \right\}$$

where $\phi : C(r) \to \mathrm{SMS}(3,r)$ is defined by $\phi(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) = \sum_{i=1}^{6} \lambda_i h_i$. The condition $\lambda_4\lambda_5\lambda_6 = 0$ in $C(r)$ is related to the fact that $h_1 + h_2 + h_3 = h_4 + h_5 + h_6$, so that $\sum_{i=1}^{6} \mu_i h_i$ for arbitrary nonnegative integers $\mu_i$ can be written as $\sum_{i=1}^{6} \lambda_i h_i$ with:

$$\lambda_i = \begin{cases} \mu_i + \min(\mu_4, \mu_5, \mu_6), & i = 1, 2, 3 \\ \mu_i - \min(\mu_4, \mu_5, \mu_6), & i = 4, 5, 6 \end{cases}$$

which satisfy $\lambda_4\lambda_5\lambda_6 = 0$ and $\lambda_i \in \mathbb{N}$ for all $i \in [6]$. Note that $C(r)$ can be written as the disjoint union:

$$\begin{aligned} C(r) = &\left\{ (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) \in \mathbb{N}^6 \;\middle|\; \begin{array}{l} \bullet\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 = r \\ \bullet\ \lambda_4 = 0 \end{array} \right\} \\ \cup &\left\{ (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) \in \mathbb{N}^6 \;\middle|\; \begin{array}{l} \bullet\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 = r \\ \bullet\ \lambda_4 \geq 1, \lambda_5 = 0 \end{array} \right\} \\ \cup &\left\{ (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) \in \mathbb{N}^6 \;\middle|\; \begin{array}{l} \bullet\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 = r \\ \bullet\ \lambda_4 \geq 1, \lambda_5 \geq 1, \lambda_6 = 0 \end{array} \right\} \end{aligned}$$

Using this decomposition and the following lemma the formula (1.48) for $|\mathrm{SMS}(3,r)|$ follows.

**Lemma 1.2.30.** *The number of ways of writing an integer $s$ as a sum of $n$ integers:*

$$s = a_1 + a_2 + \cdots + a_n$$

*where $a_i \geq k_i$ for some fixed integers $k_i$ for all $i \in [n]$ (with $s + n \geq \sum_{i=1}^{n} k_i + 1$) is:*

$$\binom{s - \sum_{i=1}^{n} k_i + n - 1}{n - 1}$$

*Proof.* We let $c_i = a_i - k_i \geq 0$ and so:

$$s - \sum_{i=1}^{n} k_i = c_1 + c_2 + \cdots + c_n$$

Thus our problem is equivalent to finding the number of ways of writing the integer $s - \sum_{i=1}^{n} k_i$ as a sum of $n$ non negative integers. This is equivalent to choosing $s - \sum_{i=1}^{n} k_i$ objects from $n$ with repetition allowed which gives the required result. □

Until now we have considered enumerations of semi magic squares of fixed size $n$ and variable line sum $r$, however other problems have been considered. Indeed semi magic squares of line sum 2 and variable size $n$ (equivalent to sequence A000681 of [99]) can be enumerated using the following formulae [2]:

$$\frac{e^{\frac{x}{2}}}{\sqrt{1-x}} = \sum_{n=0}^{\infty} |\mathrm{SMS}(n,2)| \frac{x^n}{(n!)^2} \tag{1.51}$$

or

$$|\mathrm{SMS}(n,2)| = n^2 |\mathrm{SMS}(n-1,2)| - \binom{n}{2}(n-1)|\mathrm{SMS}(n-2,2)| \tag{1.52}$$

with $|\mathrm{SMS}(1,2)| = 1$ and $|\mathrm{SMS}(2,2)| = 3$ which gives:

$$|\mathrm{SMS}(n,2)| = 4^{-n} \sum_{i=0}^{n} (2(n-i))! i! \binom{n}{i}^2 2^i \tag{1.53}$$

In [68] the polytope $\mathcal{B}_n^{\{1,d\}}$ (using the convention of equation (1.22)) is considered. This is simply the polytope of size $n$ doubly stochastic matrices which are symmetric under standard transposition. It can be shown that $\mathrm{vert}\mathcal{B}_n^{\{1,d\}} \subseteq \left\{ \frac{\mu + d\mu}{2} \middle| \mu \in \mathrm{SMS}(n,1) \right\}$. However not all $\frac{\mu+d\mu}{2} = \frac{\mu+\mu^T}{2}$ give a vertex of $\mathcal{B}_n^{\{1,d\}}$. In [68] the following result is given:

**Theorem 1.2.31.** *Let $\mu \in SMS(n,1)$. Then $\frac{\mu+d\mu}{2} \in vert\mathcal{B}_n^{\{1,d\}}$ if and only if $\mu$ represents a permutation containing no cycles of even length greater than 2.*

In [101, 104] the enumerations of these vertices (corresponding to sequence A006847 of [99]) are given:

$$\begin{gathered}
\left(\tfrac{1+x}{1-x}\right)^{\frac{1}{4}} e^{\frac{x+x^2}{284}} = \textstyle\sum_{n=0}^{\infty} \left|\mathrm{vert}\mathcal{B}_n^{\{1,d\}}\right| \frac{x^n}{n!} \\
\left|\mathrm{vert}\mathcal{B}_{n+1}^{\{1,d\}}\right| = |\mathrm{vert}\mathcal{B}_n^{\{1,d\}}| + n^2 |\mathrm{vert}\mathcal{B}_{n-1}^{\{1,d\}}| - \tbinom{n}{2} |\mathrm{vert}\mathcal{B}_{n-2}^{\{1,d\}}| - n(n-1)(n-2)|\mathrm{vert}\mathcal{B}_{n-3}^{\{1,d\}}| \\
\left(\text{with } \left|\mathrm{vert}\mathcal{B}_1^{\{1,d\}}\right|, \ldots, \left|\mathrm{vert}\mathcal{B}_4^{\{1,d\}}\right| = 1,2,5,14\right)
\end{gathered} \tag{1.54}$$

For example let us consider the permutation: $\pi = (1,2,3)(4,6,5)(7)$ which gives the vertex:

$$\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

However, 3 other permutations give this same vertex:

$$\begin{gathered} (1,3,2)(4,5,6)(7) \\ (1,3,2)(4,6,5)(7) \\ (1,2,3)(4,5,6)(7) \end{gathered}$$

Applying $d$ to a permutation written in cyclic notation simply reverses the order of each disjoint cycle, however disjoint cycles of length 1 and 2 remain unchanged. Thus if a permutation (equivalent to $\frac{\mu+d\mu}{2}$) has $K$ cycles of odd length, then there are $2^K$ total permutations which give the same vertex. To enumerate the vertices we use the following lemma (a proof of which is given in [103]).

**Lemma 1.2.32.** *The number of permutations of* $[n]$, *with* $c_i$ *disjoint cycles of length* $i$ *is:*

$$\frac{n!}{\prod_{i=1}^{n} i^{c_i} c_i!}$$

If we consider a permutation with no cycle of even length greater than 2, we have:

$$c_4 = c_6 = \cdots = 0, c_1, c_2, c_3, c_5, \cdots \geq 0$$

thus using Lemma 1.2.32 the number of permutations with $c_1$ cycles of length 1, $c_2$ cycles of length 2 and $c_3, c_5, c_7, \ldots$ cycles of odd length is:

$$\frac{n!}{\prod_{i=1,2,3,5,7\ldots} i^{c_i} c_i!}$$

Our previous discussion shows that $2^{c_3+c_5+c_7\cdots}$ of these permutations give the same vertex, thus the number of vertices with disjoint cycles as above is:

$$\frac{n!}{2^{c_3+c_5+c_7\cdots} \prod_{i=1,2,3,5,7\ldots} i^{c_i} c_i!}$$

Note that $n = \sum_{i=1}^{n} ic_i$ and so summing over all possible cycles, we get the rather cumbersome enumeration:

$$|\mathrm{vert}\mathcal{B}_n^{\{1,d\}}| = \sum \frac{n!}{2^{c_3+c_5+\dots}\prod_{i=1,2,3,5,7\dots} i^{c_i}c_i!} \tag{1.55}$$

where the sum is over all non negative integers $c_1, c_2, c_3, c_5, c_7\dots$ satisfying $c_1 + 2c_2 + 3c_3 + 5c_5 + 7c_7 + \cdots = n$. Using the same argument we see that $|\mathrm{SMS}(n,1)^{\{1,d\}}| = \sum \frac{n!}{1^{c_1}c_1!2^{c_2}c_2!}$ (an element $\mu \in \mathrm{SMS}(n,1)^{\{1,d\}}$ corresponds to a permutation with disjoint cycles of length 1 or 2) where the sum is over all non negative integers $c_1$ and $c_2$ satisfying $c_1 + 2c_2 = n$. This gives:

$$|\mathrm{SMS}(n,1)^{\{1,d\}}| = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{(n-2k)!2^k k!} \tag{1.56}$$

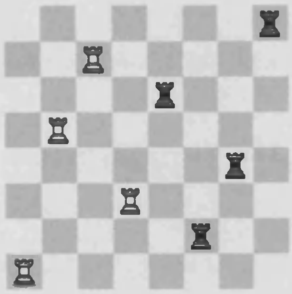
Note that the permutations here are involutions, and that $k$ counts the number of cycles of length 2 (i.e. transpositions) in each involution. This formula (1.56) could be obtained directly without using Lemma 1.2.32.

This is equivalent to sequence A000085 of [99]. The problem of enumerating the diagonal symmetry class of semi magic squares has been considered in [35, 62]. Further similar problems are considered in [1, 36, 41, 42, 43, 59, 61, 66, 68].

Symmetry classes of semi magic squares can be linked to chess problems. A chess problem of a particular type of size $n$ is the problem of finding the number of configurations of chess pieces of a particular type (Rooks, Queens, Bishops, ...), such that every square on an $n \times n$ board is attacked without any of the pieces attacking each other [9, 76, 96]. Figure 1.34 gives an example of a configuration of Queens that attack the whole board without attacking each other, this being one solution to the problem of the Queens. The added requirement that such a configuration of chess pieces be symmetric leads to a link with symmetry classes of semi magic squares. Indeed configurations of the chess problem of the Rooks correspond to permutation matrices where empty squares on the $n \times n$ chess board correspond to 0 entries in an $n \times n$ matrix, and every Rook on the board corresponds to a 1 (since for no Rooks to attack each other we must have exactly one Rook in every row and column). Figure 1.35 shows an example of a half turn symmetric solution to the problem of the Rooks of size 8 corresponding to a half turn symmetric permutation matrix of size 8.

Another set of the form $\mathcal{B}_n^G$ (but for which $G$ is not necessarily a subgroup of $D_4$) is considered

Figure 1.34: A solution to the problem of the Queens



$$\leftrightarrow \begin{pmatrix} 0&0&0&0&0&0&0&1\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 1&0&0&0&0&0&0&0 \end{pmatrix}$$

Figure 1.35: Half turn symmetric solution to the problem of the Rooks

by Brualdi in [27]. He defines the polytope $\mathcal{B}_n(P,Q)$:

$$\mathcal{B}_n(P,Q) := \{a \in \mathcal{B}_n \mid PaQ = a\}$$

where $P, Q \in \text{SMS}(n,1)$. If we consider the action $\chi = \chi(\sigma,\tau) : [n]^2 \to [n]^2$ defined by:

$$\chi(i,j) := (\sigma i, \tau j)$$

where $\sigma$ and $\tau$ are the permutations corresponding to $P$ and $Q$, then $\chi$ can be considered as a permutation on $[n]^2$. Note that any element $a$ of $\mathcal{B}_n(P,Q)$ is invariant under the action of $\chi$. Also, $\langle\chi\rangle$ is a cyclic group and therefore, $\mathcal{B}_n(P,Q) = \mathcal{B}_n^{\langle\chi\rangle}$ and so $\mathcal{B}_n(P,Q)$ is indeed of the form $\mathcal{B}_n^G$. Note also that $\chi$ can be written as a product of disjoint cycles. It can be shown that the set of elements $\omega$ (i.e. the orbit) of any cycle of $\chi$ is a subset of $\delta \times \gamma$ for some orbit $\delta$ of $\sigma$ and some orbit $\gamma$ of $\tau$. Let the orbits of $\sigma$ be $\delta_1, \ldots, \delta_p$ and the orbits of $\tau$ be $\gamma_1, \ldots, \gamma_q$. Then one of the results of Brualdi is:

$$\text{vert}\mathcal{B}_n(P,Q) = \left\{ a(e,\omega) \;\middle|\; \begin{array}{l} \bullet\ e \in \text{vert}\mathcal{T}\left((|\delta_1|,\ldots,|\delta_p|),(|\gamma_1|,\ldots,|\gamma_q|)\right) \\ \bullet\ \omega \text{ is any set } \{\omega_{rs} \mid r \in [p],\ s \in [q]\} \text{ of orbits of } \chi \\ \quad \text{with } \omega_{rs} \subseteq \delta_r \times \gamma_s \text{ for each } r \in [p],\ s \in [q] \end{array} \right\} \qquad (1.57)$$

where $a(e,\omega)$ is the $n \times n$ matrix with entries given by:

$$a(e,\omega)_{ij} = \begin{cases} \frac{e_{rs}}{\operatorname{lcm}(|\delta_r|,|\gamma_s|)}, & (i,j) \in \omega_{rs} \\ 0, & \text{otherwise} \end{cases} \quad \text{for all } (i,j) \in [n]^2$$

Also, $\mathcal{T}((|\delta_1|,\ldots,|\delta_p|),(|\gamma_1|,\ldots,|\gamma_q|))$ is a transportation polytope which will be defined in Definition 1.2.33.

In the next section we consider one more polytope, a generalization of the Birkhoff polytope with applications in operational research: the *transportation polytope.*

### 1.2.3 The transportation polytope

Let us consider the following logistical problem. A total of $m$ sources with respective supply $r_i$ for $1 \leq i \leq m$, must ensure delivery to $n$ destinations with respective demand $s_j$ for $1 \leq j \leq n$, such that the total supply is equal to the total demand i.e. $\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} s_j = \tau$. The *transportation problem* consists in finding a solution to this situation that is optimal with respect to a total cost function. This can be seen diagrammatically in Figure 1.36 where every arc contains the following information:

$$\begin{aligned} c_{ij} &: \text{ cost per unit per arc} \\ a_{ij} &: \text{ units shipped} \end{aligned}$$

or equivalently in matrix form as shown in Figure 1.37.

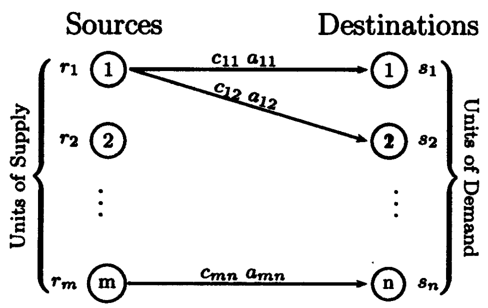



Figure 1.36: A classical transportation problem

The aim of the problem is to minimise $\rho(a) = \sum_{ij} c_{ij}a_{ij}$ over the *transportation polytope* $\mathcal{T}(r,s)$.

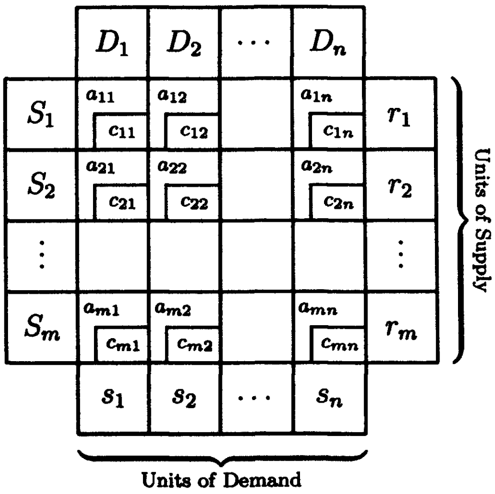



Figure 1.37: The transportation problem

**Definition 1.2.33.** *For $r \in \mathbb{R}^m, s \in \mathbb{R}^n$ such that all entries of $r$ and $s$ are non negative, and $\sum_{i=1}^m r_i = \sum_{j=1}^n s_j = \tau$, we define the* transportation polytope, *$\mathcal{T}(r,s)$:*

$$\mathcal{T}(r,s) := \left\{ a \in \mathbb{R}^{m\times n} \;\middle|\; \begin{array}{l} \bullet\ a_{ij} \geq 0 \ \ \text{for all } i \in [m], j \in [n] \\ \bullet\ \sum_{j=1}^m a_{ij} = r_i \ \ \text{for all } i \in [m] \\ \bullet\ \sum_{i=1}^n a_{ij} = s_j \ \ \text{for all } j \in [n] \end{array} \right\}$$

A class of problems similar to transportation problems was originally proposed by Kantorovich in [67] (the reference given here is the 1959 English translation of the 1939 paper originally published in Russian). However it is accepted that the standard form of the transportation problem was formulated by Hitchcock in 1941 [63]. This polytope has an obvious interest to academics in the fields of operational research but has also been examined from a purely combinatorial point of view [26, 28, 44, 69, 74, 88]. Integer points of transportation polytopes are known as contingency tables, and are investigated in, for example, [6, 50, 57]. Chapter 8 of [28] gives a great review of the results obtained for $\mathcal{T}(r,s)$. A subset of these results will follow. They are mostly generalizations of the results of Section 1.2.2.

**Theorem 1.2.34.** *$\mathcal{T}(r,s)$ is non empty and* $\dim \mathcal{T}(r,s) = (m-1)(n-1)$

The proof of the dimension is analogous to the proof of Theorem 1.2.21. Non emptiness comes from considering the matrix $a \in \mathcal{T}(r,s)$ with entries defined as follows:

$$a_{ij} = \frac{r_i s_j}{\tau} \text{ for all } i \in [m], j \in [n]$$

The vertices of $\mathcal{T}(r, s)$ are surprisingly easy to identify. One such classification is given in [69]:

**Theorem 1.2.35.** *Let $a \in \mathcal{T}(r, s)$. Then the following statements are equivalent:*

1. *$a \in vert\mathcal{T}(r, s)$.*
2. *$a$ has no non zero cycles.*
3. *The bipartite graph of $a$ has no cycle.*

Note that non zero cycle here means a cycle:

$$(i_1, j_1), (i_1, j_2), (i_2, j_2), (i_2, j_3), \dots (i_{s-1}, j_{s-1}), (i_{s-1}, j_s), (i_s, j_s), (i_s, j_1)$$

where $i_k \neq i_{k+1}, i_s \neq i_1, j_k \neq j_{k+1}, j_s \neq j_1$ for all $k \in [s-1]$, and such that each of the corresponding entries of $a$ is non zero. An example is shown in Figure 1.38. Note also that the bipartite graph of $a$ is defined in Definition 1.2.28.

*Proof.* Consider $a \in \mathcal{T}(r, s) \setminus \text{vert}\mathcal{T}(r, s)$. From Lemma 1.2.7 there exists a non zero matrix $a^*$ such that $a \pm a^* \in \mathcal{T}(r, s)$. Considering the row and column sums we have: $\sum_{i=1}^{m} a^*_{ij} = \sum_{j=1}^{n} a^*_{ij} = 0$. Thus, for each $(i, j)$ such that $a^*_{ij}$ is non zero, there must exist another $i' \neq i$ and $j' \neq j$ such that $a^*_{i'j}$ and $a^*_{ij'}$ are also non zero. Since there are only finitely many rows and columns, it follows that $a^*$ must have a cycle of non zero entries. On this cycle we have $a_{ij} \pm a^*_{ij} \geq 0$ (since each entry of an element of $\mathcal{T}(r, s)$ is non negative). If $a_{ij} = 0$ we have $\pm a^*_{ij} > 0$ (since $a^*_{ij} \neq 0$ on the cycle) which is not possible. Thus, $a_{ij} > 0$, and so $a$ must have a non zero cycle. Therefore, (2) implies (1).

Now if we consider $a \in \mathcal{T}(r, s)$ such that $a$ has a non zero cycle, we can in a similar way to the proof of Theorem 1.2.24 find $a^* \neq 0$ such that $a \pm a^* \in \mathcal{T}(r, s)$, so that $a$ is not a vertex. Thus, (1) implies (2). Statements (2) and (3) are obviously equivalent. □

If we now move onto faces of $\mathcal{T}(r, s)$, then any face of $\mathcal{T}(r, s)$ is of the form:

$$\mathcal{T}_K(r, s) := \{a \in \mathcal{T}(r, s) \mid a_{ij} = 0 \text{ if } (i, j) \in K\} \tag{1.58}$$

where $K \subseteq [m] \times [n]$.

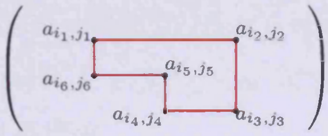



Figure 1.38: A non zero cycle

**Definition 1.2.36.** *For a polytope $\mathcal{P}$, we define $facets(\mathcal{P})$ to be the number of facets of $\mathcal{P}$ (i.e. the number of faces of $\mathcal{P}$ of dimension* $\dim \mathcal{P} - 1$*).*

The following theorem is proved in [69]:

**Theorem 1.2.37.** *For $m \leq n$:*

$$\begin{aligned} facets(\mathcal{T}(r,s)) &= 1 \text{ if } m = 1 \\ facets(\mathcal{T}(r,s)) &= 2 \text{ if } m = n = 2 \\ (m-1)n \leq facets(\mathcal{T}(r,s)) &\leq mn \text{ if } m \geq 2 \text{ and } n \geq 3 \end{aligned}$$

Before stating a result concerning the diagonal symmetry class of the transportation polytope we need the following definition:

**Definition 1.2.38.** *For any diagonally symmetric matrix $a \in \mathbb{R}^{n \times n}$ we define the* graph of *$a$ as the graph with vertex set $\{w_1, \ldots, w_n\}$, such that $\{w_i, w_j\}$ is an edge if and only if $a_{ij} \neq 0$.*

Note that $\{w_i, w_i\}$ represents a loop at vertex $w_i$, corresponding to a non zero diagonal entry $a_{ii}$ of $a$. Using Definition 1.2.38 we give the following result concerning the diagonal symmetry class of a transportation polytope $\mathcal{T}(r,r)$: $\mathcal{T}(r,r)^{\{1,d\}}$ (using the convention of equation (1.22)).

**Theorem 1.2.39.** *Let $a \in \mathcal{T}(r,r)^{\{1,d\}}$. Then $a \in vert\mathcal{T}(r,r)^{\{1,d\}}$ if and only if the connected components of the graph of $a$ are either trees or odd near trees.*

A *near tree* is a connected graph that consists of a cycle with a (possibly trivial) tree rooted at each vertex. An *odd near tree* is a near tree whose cycle has odd length. Equivalently an *odd near tree* is a connected graph with no cycles of even length or odd cycles connected by a path. A proof of this result can be found in [28, 41]. Importantly it should be noted that Theorem 1.2.31 is a special case of Theorem 1.2.39.

Surprisingly it can be shown that Theorem 1.2.35 is a special case of Theorem 1.2.39. For a particular matrix $a \in \mathcal{T}(r,s)$ with $r \in \mathbb{R}^m$, $s \in \mathbb{R}^n$, we can construct a matrix $a' \in \mathcal{T}(r',r')^{\{1,d\}}$ where $r' \in \mathbb{R}^{m+n}$ has entries:

$$r'_i = \begin{cases} r_i, \; i \in [m] \\ s_{i-m}, \; i \in [m+1, m+n] \end{cases}$$

It can be checked that $a \in \text{vert}\mathcal{T}(r,s)$ if and only if $a' \in \text{vert}\mathcal{T}(r',r')$ and Theorem 1.2.39 can be used to give statement (3) of Theorem 1.2.35.

Dantzig [44] made a lot of progress on solving transportation problems efficiently. Indeed it can be shown that for a linear cost function $\rho$, the optimal solution is always a vertex. For small $m,n$ one can thus find the optima fairly simply. For larger $m,n$ this becomes a complicated problem. Dantzig developed an algorithm for solving this problem: *the simplex algorithm*. Other algorithms from linear programming can also be used.

## 1.3 Conclusion

The main aim of this thesis is to apply the methods and results described in Section 1.2 to the ever intriguing alternating sign matrices of Section 1.1. Chapter 2 is the study of a generalization of the Birkhoff polytope and semi magic squares: *the alternating sign matrix polytope* and *higher spin alternating sign matrices*. Chapter 3 is a study of $\mathcal{B}_n$ under symmetry conditions, leading to enumerations. This chapter serves as a basis to a methodology that we apply in Chapter 4 to the alternating sign matrix polytope. Finally our penultimate chapter, Chapter 5 is the logical conclusion in which we generalize our new polytope in the same way that $\mathcal{T}(r,s)$ is a generalization of $\mathcal{B}_n$.

# Chapter 2

# Higher Spin Alternating Sign Matrices

## 2.1 Introduction

This chapter is based on the article published in 2007 [15]. As described in Section 1.1.2 alternating sign matrices can be considered as generalizations of permutation matrices. Permutation matrices are semi magic squares of line sum 1 (see Definition 1.2.23). Is there an analogous set of square matrices with line sum $r$ such that setting $r = 1$ gives ASM($n$)? This leads to the definition of *higher spin alternating sign matrices.*

**Definition 2.1.1.** *We define the set ASM$(n,r)$ of* higher spin alternating sign matrices of size $n$ *and line sum* $r$*:*

$$ASM(n,r) := \left\{ a = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \in \mathbf{Z}^{n \times n} \;\middle|\; \begin{array}{l} \bullet \sum_{j'=1}^{n} a_{i,j'} = \sum_{i'=1}^{n} a_{i',j} = r \;\; \text{for all } i,j \in [n] \\ \bullet\, 0 \le \sum_{j'=1}^{j} a_{i,j'} \le r \;\; \text{for all } i \in [n],\ j \in [n-1] \\ \bullet\, 0 \le \sum_{i'=1}^{i} a_{i',j} \le r \;\; \text{for all } i \in [n-1],\ j \in [n] \end{array} \right\}$$

Note that this is equivalent to:

$$\mathrm{ASM}(n,r) = \left\{ a \in \mathbf{Z}^{n \times n} \;\middle|\; \begin{array}{l} \bullet \sum_{j'=1}^{n} a_{ij'} = \sum_{i'=1}^{n} a_{i'j} = r \;\; \text{for all } i,j \in [n] \\ \bullet \sum_{j'=1}^{j} a_{ij'} \ge 0 \;\; \text{for all } i \in [n],\ j \in [n-1] \\ \bullet \sum_{j'=j}^{n} a_{ij'} \ge 0 \;\; \text{for all } i \in [n],\ j \in [2,n] \\ \bullet \sum_{i'=1}^{i} a_{i'j} \ge 0 \;\; \text{for all } i \in [n-1],\ j \in [n] \\ \bullet \sum_{i'=i}^{n} a_{i'j} \ge 0 \;\; \text{for all } i \in [2,n],\ j \in [n] \end{array} \right\} \tag{2.1}$$

Thus ASM$(n,r)$ is the set of $n \times n$ integer-entry matrices for which all complete row and column sums are $r$ and all partial row and column sums extending from each end of the row or column are non negative. It is easy to see that setting $r = 1$ gives the set $\mathrm{ASM}(n,1) = \mathrm{ASM}(n)$ as given by Definition 1.1.1. We shall refer to the matrices of $\mathrm{ASM}(n)$ as standard alternating sign matrices whenever it is necessary to emphasize these have $r = 1$. Throughout this section we consider the following running example:

$$a = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 2 & 0 \\ 0 & 1 & 1 & -2 & 2 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix} \in \mathrm{ASM}(5,2) \tag{2.2}$$

Note that following Definition 1.2.23 we have:

$$\mathrm{SMS}(n,r) = \{a \in \mathrm{ASM}(n,r) \mid a_{ij} \geq 0 \text{ for all } i,j \in [n]\} \tag{2.3}$$

The terminology chosen, *higher spin* alternating sign matrices is derived from the statistical mechanical connection discussed in Section 1.1.4 in which there is a bijection between $\mathrm{ASM}(n,1)$ and configurations of the six-vertex model on $\mathcal{L}_{n,n}$ with domain wall boundary conditions. This model is related to the spin $\frac{1}{2}$ representation of the Lie algebra $sl(2,\mathbb{C})$. For $r > 1$ there exists a similar statistical model related to the spin $\frac{r}{2}$ representation of $sl(2,\mathbb{C})$ and it can be shown that there is a bijection between $\mathrm{ASM}(n,r)$ and configurations of this model on $\mathcal{L}_{n,n}$ with domain wall boundary conditions. See [15, 34] for details. Some cardinalities of $\mathrm{ASM}(n,r)$ are shown in Figure 2.1 (note that these cardinalities are equivalent to sequence A143670 of [99]).

| | $r=0$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $n=1$ | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 2 | 3 | 4 | 5 |
| 3 | 1 | 7 | 26 | 70 | 155 |
| 4 | 1 | 42 | 628 | 5102 | 28005 |
| 5 | 1 | 429 | 41784 | 1507128 | 28226084 |
| 6 | 1 | 7436 | 7517457 | 1749710096 | 152363972022 |

Figure 2.1: $|\mathrm{ASM}(n,r)|$ for $n \in [6]$, $r \in [0,4]$

In the next section we define multiple sets that are generalizations of the sets of Section 1.1.3.

## 2.2 The many faces of higher spin alternating sign matrices

The sets $\mathrm{EM}(n), \mathrm{CSM}(n), \mathrm{MT}(n), \mathrm{LP}(n)$ and $\mathrm{FPL}(n)$ (defined in Section 1.1.3) will now be generalized to higher spin versions $\mathrm{EM}(n,r), \mathrm{CSM}(n,r), \mathrm{MT}(n,r), \mathrm{LP}(n,r)$ and $\mathrm{FPL}(n,r)$, however, the same terminology shall be used. For all of these new sets, setting $r = 1$ gives the sets of Section 1.1.3.

### 2.2.1 Edge matrix pairs

**Definition 2.2.1.** *We define the set of edge matrix pairs, EM(n, r):*

$$EM(n,r) := \left\{ (h,v) = \left( \begin{pmatrix} h_{10} & \cdots & h_{1n} \\ \vdots & & \vdots \\ h_{n0} & \cdots & h_{nn} \end{pmatrix}, \begin{pmatrix} v_{01} & \cdots & v_{0n} \\ \vdots & & \vdots \\ v_{n1} & \cdots & v_{nn} \end{pmatrix} \right) \in [0,r]^{n\times(n+1)} \times [0,r]^{(n+1)\times n} \;\middle|\; \begin{array}{l} \bullet\ h_{i,0} = v_{0,j} = 0 \ \ \textit{for all } i,j \in [n] \\ \bullet\ h_{i,n} = v_{n,j} = r \ \ \textit{for all } i,j \in [n] \\ \bullet\ h_{i,j-1} + v_{ij} = v_{i-1,j} + h_{ij} \ \ \textit{for all } i,j \in [n] \end{array} \right\}$$

It can be checked that there is a bijection between the set $\mathrm{ASM}(n,r)$ and $\mathrm{EM}(n,r)$ in which the edge matrix pair $(h,v)$ which corresponds to the alternating sign matrix $a$ is given by:

$$\begin{array}{ll} h_{ij} = \sum_{j'=1}^{j} a_{ij'} & \text{for all } \ i \in [n],\ j \in [0,n] \\ v_{ij} = \sum_{i'=1}^{i} a_{i'j} & \text{for all } \ i \in [0,n],\ j \in [n] \end{array} \tag{2.4}$$

and inversely:

$$a_{ij} = h_{ij} - h_{i,j-1} = v_{ij} - v_{i-1,j} \ \text{ for all } \ i,j \in [n] \tag{2.5}$$

The edge matrix pair corresponding to our running example (2.2) is:

$$(h,v) = \left( \begin{pmatrix} 0 & 0 & 1 & 2 & 2 & 2 \\ 0 & 1 & 0 & 0 & 2 & 2 \\ 0 & 0 & 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 2 & 0 \\ 1 & 1 & 2 & 0 & 2 \\ 2 & 1 & 2 & 1 & 2 \\ 2 & 2 & 2 & 2 & 2 \end{pmatrix} \right) \in \mathrm{EM}(5,2) \tag{2.6}$$

We can once again represent these entries on $\mathcal{L}_{n,n}$ as in Figure 1.10. The relation given by Figure 1.8 still holds (the lattice diagram corresponding to our running example is given by Figure 2.2).

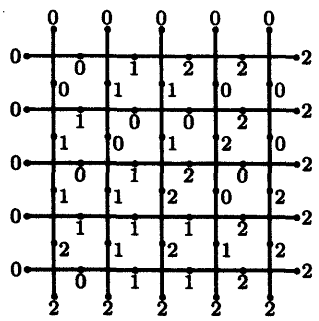


Figure 2.2: Running example represented on $\mathcal{L}_{5,5}$

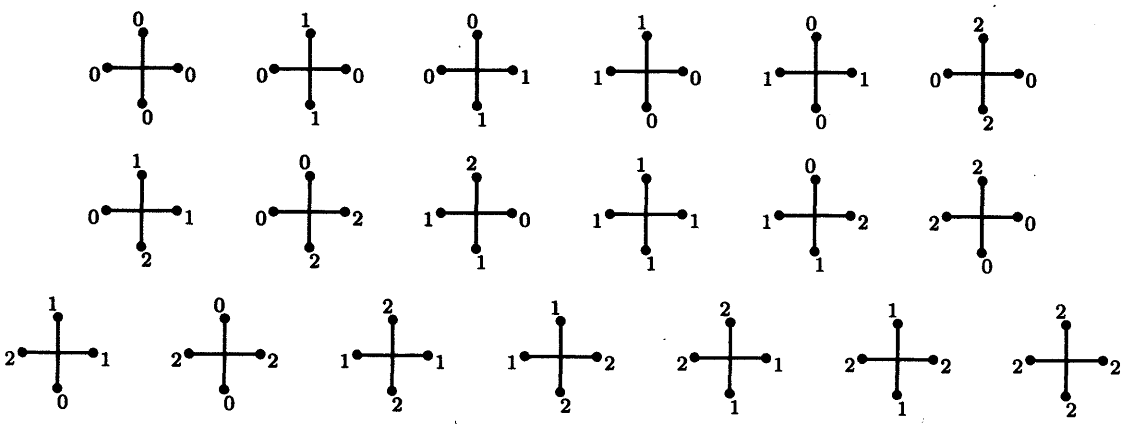
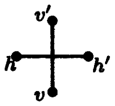


Figure 2.3: The 19 vertex types of $\mathcal{V}(2)$

**Definition 2.2.2.** *We define the set* $\mathcal{V}(r)$ *of* vertex types for a spin $\frac{r}{2}$ statistical mechanical model*:*

$$\mathcal{V}(r) := \{(h, v, h', v') \in [0, r]^4 \mid h + v = h' + v'\}$$

A vertex type $(h, v, h', v')$ is depicted as 

and the defining equality of $\mathcal{V}(r)$ is equivalent to the relation given by Figure 1.8. The 19 vertex types for $r = 2$ are given by Figure 2.3. The set $\mathcal{V}(r)$ can be expressed as the disjoint union:

$$\begin{aligned}\mathcal{V}(r)=&\{(h, v, h', h + v - h') \mid h, v, h' \in [0, r],\ h \leq h' \leq v\} \cup \\ &\{(h' + v' - v, v, h', v') \mid v, h', v' \in [0, r],\ v' \leq v < h'\} \cup \\ &\{(h, h' + v' - h, h', v') \mid h, h', v' \in [0, r],\ h' < h \leq v'\} \cup \\ &\{(h, v, h + v - v', v') \mid h, v, v' \in [0, r],\ v < v' < h\}\end{aligned} \tag{2.7}$$

Each of these sets corresponds to a weak choice of minimum value. Thus, using Lemma 1.2.30 we have:

$$|\mathcal{V}(r)| = \binom{r+3}{3} + 2\binom{r+2}{3} + \binom{r+1}{3} \tag{2.8}$$

The sequence $|\mathcal{V}(r)|$ corresponds to A005900 of [99]. Recalling equation (2.3) $\mathrm{SMS}(n, r) \subseteq \mathrm{ASM}(n, r)$. Thus there exists a particular subset $\mathrm{EM}_S(n, r)$ of $\mathrm{EM}(n, r)$ in bijection with $\mathrm{SMS}(n, r)$. The edge matrix pairs of $\mathrm{EM}_S(n, r)$ are the edge matrix pairs of $\mathrm{EM}(n, r)$ such that $h_{ij} \leq h_{i,j+1}$ (and equivalently $v_{ij} \leq v_{i+1,j}$) for all $i, j$. By the map (2.5) this implies that $a_{ij} \geq 0$. This leads to the set $\mathcal{V}_S(r) := \{(h, v, h', v') \in \mathcal{V}(r) \mid h \leq h' \text{ and } v' \leq v\}$. These requirements imposed on the previous disjoint union give:

$$\begin{aligned}\mathcal{V}_S(r)=&\{(h, v, h', h + v - h') \mid h, v, h' \in [0, r],\ h \leq h' \leq v\} \cup \\ &\{(h' + v' - v, v, h', v') \mid v, h', v' \in [0, r],\ v' \leq v < h'\}\end{aligned} \tag{2.9}$$

leading to:

$$|\mathcal{V}_S(r)| = \binom{r+2}{3} + \binom{r+3}{3}$$

as given by sequence A000330 of [99]. Recalling the Boltzmann weights (1.12), satisfying the Yang-Baxter equation, defined for the six vertex model, i.e. the vertex types of $\mathcal{V}(1)$, Boltzmann weights satisfying the Yang-Baxter equation can also be defined for the vertex types of $\mathcal{V}(r)$ for $r > 1$. Furthermore, for these weights the determinant formula (1.13) can be generalized to the higher spin vertex model with domain wall boundary conditions [34].

This is very promising, indeed if a particular choice of spectral and crossing parameters $z$ and $a$ could be found such that $W\left(\begin{smallmatrix} & v' & \\ h & + & h' \\ & v & \end{smallmatrix}, z, a\right) = 1$ for all $(h, v, h', v') \in \mathcal{V}(r)$ for $r > 1$, then a method similar to that of Section 1.1.4 could perhaps be used to obtain enumeration formulae for $|\mathrm{ASM}(n,r)|$ for fixed $r$ and variable $n$. Sadly, our investigations lead us to believe that for $r > 1$ it is not possible to find Boltzmann weights which satisfy the Yang-Baxter equation and for which there is also a choice of parameters leading to all weights being 1.

We now give a set that generalizes Definition 1.1.3.

### 2.2.2 Corner sum matrices

**Definition 2.2.3.** *We define the set CSM$(n,r)$ of* corner sum matrices*as:*

$$CSM(n,r) :=$$

$$\left\{ c = \begin{pmatrix} c_{0,0} & \cdots & c_{0,n} \\ \vdots & & \vdots \\ c_{n,0} & \cdots & c_{n,n} \end{pmatrix} \in [0,n]^{(n+1)\times(n+1)} \;\middle|\; \begin{array}{l} \bullet\ c_{0,k} = c_{k,0} = 0 \quad \textit{for all } k \in [0,n] \\ \bullet\ c_{k,n} = c_{n,k} = kr \quad \textit{for all } k \in [0,n] \\ \bullet\ c_{ij} - c_{i,j-1} \in [0,r] \quad \textit{for all } i,j \in [n] \\ \bullet\ c_{ij} - c_{i-1,j} \in [0,r] \quad \textit{for all } i,j \in [n] \end{array} \right\}$$

It can be checked that there is a bijection between $\mathrm{ASM}(n,r)$ and $\mathrm{CSM}(n,r)$ in which the corner sum matrix $c$ which corresponds to the alternating sign matrix $a$ is given by:

$$c_{ij} = \sum_{i'=1}^{i}\sum_{j'=1}^{j} a_{i',j'}, \quad \text{for all } i,j \in [0,n] \tag{2.10}$$

and inversely,

$$a_{ij} = c_{ij} - c_{i,j-1} - c_{i-1,j} + c_{i-1,j-1}, \quad \text{for all } i,j \in [n] \tag{2.11}$$

Combining the bijections (2.4) and (2.5) between $\mathrm{EM}(n,r)$ and $\mathrm{ASM}(n,r)$, and (2.10) and (2.11) between $\mathrm{ASM}(n,r)$ and $\mathrm{CSM}(n,r)$, the corner sum matrix $c$ which corresponds to the edge matrix pair $(h,v)$ is given by:

$$c_{ij} = \sum_{i'=1}^{i} h_{i'j} = \sum_{j'=1}^{j} v_{ij'}, \quad \text{for all } \; i,j \in [0,n] \tag{2.12}$$

and inversely,

$$\begin{aligned} h_{ij} &= c_{ij} - c_{i-1,j}, \quad \text{for all } i \in [n], j \in [0,n] \\ v_{ij} &= c_{ij} - c_{i,j-1}, \quad \text{for all } i \in [0,n], j \in [n] \end{aligned} \tag{2.13}$$

The corner sum matrix which corresponds to our running example (2.2) is:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 2 & 2 \\ 0 & 1 & 1 & 2 & 4 & 4 \\ 0 & 1 & 2 & 4 & 4 & 6 \\ 0 & 2 & 3 & 5 & 6 & 8 \\ 0 & 2 & 4 & 6 & 8 & 10 \end{pmatrix} \in \mathrm{CSM}(5,2) \tag{2.14}$$

Now for a generalization of Definition 1.1.4.

### 2.2.3 Monotone triangles

**Definition 2.2.4.** *We define the set MT(n, r) of* monotone triangles *to be the triangular arrays of the form:*

$$\begin{array}{ccccccc} & & & t_{1,1} \dots t_{1,r} & & & \\ & & t_{2,1} & \dots & t_{2,2r} & & \\ & \cdot^{\cdot^{\cdot}} & & & & \ddots & \\ t_{n,1} & & & \dots & & & t_{n,nr} \end{array}$$

*such that:*

- *Each entry of t is in* $[n]$.
- *In each row of t, any integer of* $[n]$ *appears at most r times.*
- $t_{ij} \le t_{i,j+1}$ *for all* $i \in [n]$, $j \in [ir-1]$.
- $t_{i+1,j} \le t_{ij} \le t_{i+1,j+r}$ *for all* $i \in [n-1]$, $j \in [ir]$.

It follows that the last row of any monotone triangle in $\mathrm{MT}(n,r)$ consists of each integer of $[n]$ repeated $r$ times. It can be checked that there is a bijection between $\mathrm{ASM}(n,r)$ and $\mathrm{MT}(n,r)$ in which the monotone triangle $t$ which corresponds to the higher spin alternating sign matrix $a$ is obtained by first using (2.4) to find the vertical edge matrix $v$ that corresponds to $a$, and then placing the integer $j$ $v_{ij}$ times in row $i$ of $t$, for each $i,j \in [n]$, with these integers being placed in weakly increasing order along each row. (Note that there is an alternative bijection in which the horizontal edge matrix $h$ which corresponds to $m$ is obtained, and the integer $i$

is then placed $h_{ij}$ times in row $j$ of $t$, for each $i, j \in [n]$). For the inverse mapping, for each $i \in [0, n]$ and $j \in [n]$, $v_{ij}$ is set to be the number of times that $j$ occurs in row $i$ of $t$, and $a$ is then obtained from $v$ using (2.5).

The monotone triangle which corresponds to our running example (2.2) is:

$$\begin{array}{c} 2\,3 \\ 1\,3\,4\,4 \\ 1\,2\,3\,3\,5\,5 \\ 1\,1\,2\,3\,3\,4\,5\,5 \\ 1\,1\,2\,2\,3\,3\,4\,4\,5\,5 \end{array} \quad \in \mathrm{MT}(5,2) \tag{2.15}$$

We now move on to a set that will be useful in generalizing a result of Section 1.2.2.

### 2.2.4 Lattice paths

**Definition 2.2.5.** *We define LP$(n, r)$ to be the set of all sets $p$ of $nr$ directed lattice paths on $\mathcal{L}_{n,n}$ such that:*

1. *For each $i \in [n]$, $p$ contains $r$ paths which begin at $(n + 1, i)$ and end at $(i, n + 1)$.*
2. *Each step of each path of $p$ is either $(-1, 0)$ or $(0, 1)$.*
3. *Different paths of $p$ do not cross.*
4. *No more than $r$ paths of $p$ pass along any edge of the lattice.*

It can be checked that there is a bijection between $\mathrm{EM}(n, r)$ (and hence $\mathrm{ASM}(n, r)$) and $\mathrm{LP}(n, r)$ in which the edge matrix pair $(h, v)$ which corresponds to the path set $p$ is given simply by

$$\begin{aligned} h_{ij} &= \text{number of paths of } p \text{ which pass from } (i, j) \\ &\quad\ \text{to } (i, j + 1), \text{ for each } i \in [n],\ j \in [0, n] \\ v_{ij} &= \text{number of paths of } p \text{ which pass from } (i + 1, j) \\ &\quad\ \text{to } (i, j), \text{ for each } i \in [0, n],\ j \in [n]. \end{aligned} \tag{2.16}$$

For the inverse mapping from $(h, v)$ to $P$, (2.16) is used to assign appropriate numbers of path segments to the horizontal and vertical edges of the lattice, and at each $(i, j) \in [n] \times [n]$, the

$h_{i,j-1} + v_{ij} = v_{i-1,j} + h_{ij}$ segments on the four neighboring edges are linked without crossing through $(i, j)$ according to the rules that

- If $h_{ij} \geq v_{ij}$ (and $h_{i,j-1} \geq v_{i-1,j}$), then $v_{i-1,j}$ paths pass from $(i, j-1)$ to $(i-1, j)$, $h_{ij} - v_{ij} = h_{i,j-1} - v_{i-1,j}$ paths pass from $(i, j-1)$ to $(i, j+1)$, and $v_{ij}$ paths pass from $(i+1, j)$ to $(i, j+1)$.
- If $v_{ij} \geq h_{ij}$ (and $v_{i-1,j} \geq h_{i,j-1}$), then $h_{i,j-1}$ paths pass from $(i, j-1)$ to $(i-1, j)$, $v_{ij} - h_{ij} = v_{i-1,j} - h_{i,j-1}$ paths pass from $(i+1, j)$ to $(i-1, j)$, and $h_{ij}$ paths pass from $(i+1, j)$ to $(i, j+1)$.

(2.17)

The two cases of (2.17) are shown diagrammatically in Figure 2.4. The lattice path vertex types corresponding to Figure 2.3 are given by Figure 2.5.

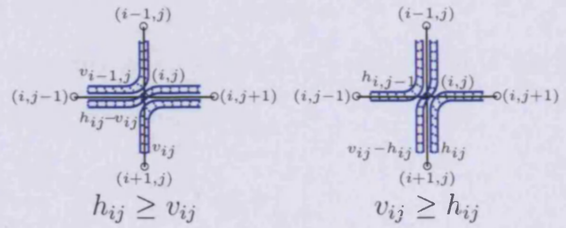



Figure 2.4: Path configurations through vertex $(i, j)$ for the cases of (2.17).

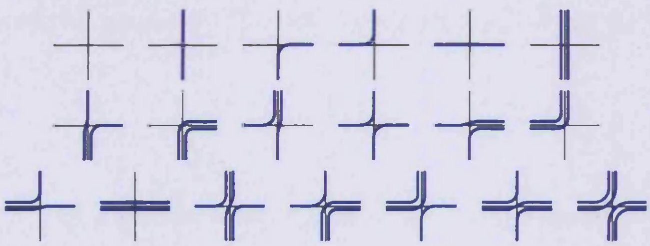
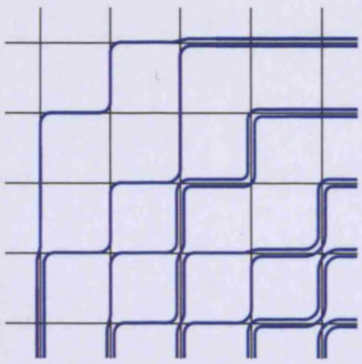


Figure 2.5: The 19 lattice path vertex types corresponding to $\mathcal{V}(2)$

The lattice path configuration corresponding to our running example (2.2) is:

 $\in \mathrm{LP}(5, 2)$ (2.18)

Note that strictly speaking the elements of $\mathrm{LP}(n,r)$ are multisets of lattice paths, since they can contain repetitions of the same paths. For example, in the case in (2.18) the paths $((6,4),(5,4),(5,5),(4,5),(4,6))$ and $((6,5),(5,5),(5,6))$ are each repeated twice. The set $\mathrm{LP}(n,r)$ leads to the following generalization of Theorem 1.2.26.

**Theorem 2.2.6.** *Any matrix $a \in ASM(n,r)$ can be written as the sum of $r$ standard alternating sign matrices of size $n$.*

*Proof.* Consider $a \in \mathrm{ASM}(n,r)$. Using bijections 2.4 and 2.16 get the corresponding $p \in \mathrm{LP}(n,r)$. Note that the $r$ nested paths which begin at $(n+1,i)$ and end at $(i,n+1)$ can be naturally ordered weakly from top to bottom at vertex $(i,n+1)$. We define $p_{ij}$ for given $i \in [n]$, $j \in [r]$ as the $j^{th}$ path which begins at $(n+1,i)$ and ends at $(i,n+1)$. Thus $p = \bigcup_{i=1}^{n} \bigcup_{j=1}^{r} p_{ij}$, swapping the ordering of this union gives $p = \bigcup_{j=1}^{r} \bigcup_{i=1}^{n} p_{ij}$. If we define $p^{(j)} := \bigcup_{i=1}^{n} p_{ij}$ for all $j \in [r]$, then we have:

$$p = \bigcup_{j=1}^{r} p^{(j)} \tag{2.19}$$

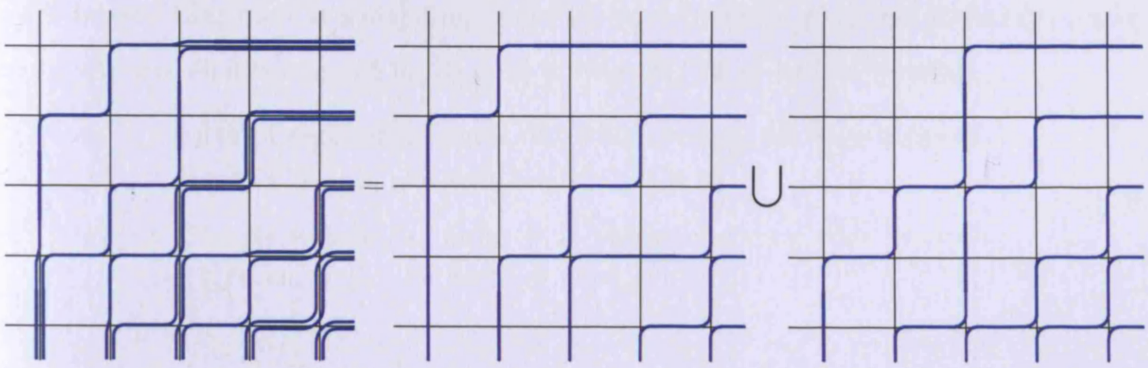
Importantly $p^{(j)} \in \mathrm{LP}(n,1)$ for all $j \in [r]$ (this follows from the ordering of our paths and the fact that no edge of $p$ is occupied by more then $r$ paths). Using the bijections (2.17) and (2.5) have:

$$a = \sum_{j=1}^{r} a^{(j)}$$

where $a^{(j)} \in \mathrm{ASM}(n,1)$ corresponds to the $p^{(j)} \in \mathrm{LP}(n,1)$ of equation (2.19).

□

This is illustrated by:



which gives the alternating sign matrix decomposition of (2.2):

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 2 & 0 \\ 0 & 1 & 1 & -2 & 2 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Note that the converse of Theorem 2.2.6, i.e. that any sum of $r$ standard alternating sign matrices is an element of $\mathrm{ASM}(n, r)$, can easily be checked to also be true. Last but not least we define another set, which is a generalization of Definition 1.1.8 however it is not in one to one correspondence with higher spin alternating sign matrices.

### 2.2.5 Fully packed loops

**Definition 2.2.7.** *We define the set $FPL(n, r)$ of* fully packed loop configurations, *to be the set of all (multi) sets $q$ of non directed open and closed lattice paths on $\mathcal{L}_{n,n}$ such that:*

- *Successive points on each path of $q$ differ by $(-1, 0), (1, 0), (0, -1)$ or $(0, 1)$.*
- *Any two edges occupied successively by a path of $q$ are different.*
- *Each path of $q$ does not cross itself or any other path of $q$.*
- *Exactly $r$ segments of paths of $q$ pass through each internal vertex of $\mathcal{L}_{n,n}$.*
- *At each (external) point $(0, 2k-1)$ and $(n+1, n-2k+2)$ for $k \in [\lfloor \frac{n+1}{2} \rfloor]$, and $(2k, 0)$ and $(n-2k+1, n+1)$ for $k \in [\lfloor \frac{n}{2} \rfloor]$, there are exactly $r$ endpoints of paths of $q$, these being the only lattice points which are path endpoints.*

It can be seen that there is a mapping from $\mathrm{FPL}(n, r)$ to $\mathrm{EM}(n, r)$ in which the fully packed loop configuration $q$ is mapped to the edge matrix pair $(h, v)$ by first forming:

$$\begin{aligned} \overline{h}_{ij} &= \text{number of segments of paths of } q \text{ which occupy the edge between} \\ &\quad (i, j) \text{ and } (i, j+1), \quad \text{for each } i \in [n],\ j \in [0, n] \\ \overline{v}_{ij} &= \text{number of segments of paths of } P \text{ which occupy the edge between} \\ &\quad (i+1, j) \text{ and } (i, j), \quad \text{for each } i \in [0, n],\ j \in [n]. \end{aligned} \tag{2.20}$$

Then similarly to (1.11), we have

$$h_{ij} = \begin{cases} \overline{h}_{ij} \text{ for } i+j \text{ odd} \\ r - \overline{h}_{ij} \text{ for } i+j \text{ odd} \end{cases}$$
$$v_{ij} = \begin{cases} r - \overline{v}_{ij} \text{ for } i+j \text{ odd} \\ \overline{v}_{ij} \text{ for } i+j \text{ odd} \end{cases} \qquad (2.21)$$

This mapping is surjective for each $r \in \mathbb{N}$ and $n \in \mathbb{P}$ and it is injective for $r \in \{0,1\}$ or $n \in \{1,2\}$. However it is not injective for $r \geq 2$ and $n \geq 3$. We define $\mathcal{W}(r)$ as the set of fully packed loop vertex types, i.e. the set of ways of linking $r$ path segments through a vertex. For example, $\mathcal{W}(2)$ is given by Figure 2.6.

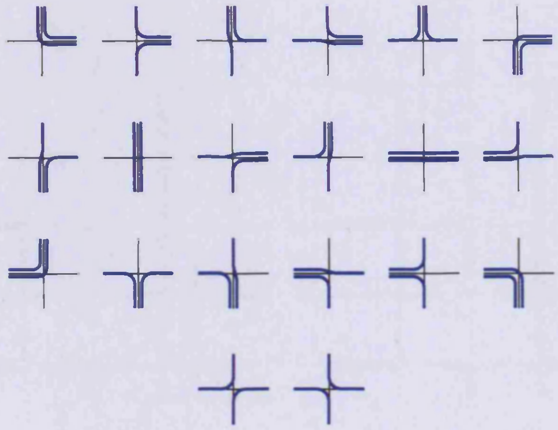
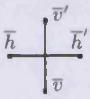


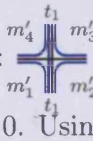
Figure 2.6: The 20 vertex types of $\mathcal{W}(2)$

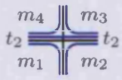
It can be shown that the number of fully packed loop vertex types with $\overline{h}, \overline{h}', \overline{v}$ and $\overline{v}'$ path segments on the surrounding edges of the vertex (i.e., 

with $\overline{h} + \overline{v} + \overline{h}' + \overline{v}' = 2r$) is $\min(\overline{h}, \overline{h}', \overline{v}, \overline{v}', r-\overline{h}, r-\overline{h}', r-\overline{v}, r-\overline{v}') + 1$. For example for $(\overline{h}, \overline{h}', \overline{v}, \overline{v}') = (1,1,1,1)$ there are two possible vertex types:  or . It is the fact that this number can be larger than 1 for $r \geq 2$ which leads to the non-injectivety of the map between FPL$(n,r)$ and EM$(n,r)$ for $r \geq 2$ and $n \geq 3$. Using this we see that $|\text{FPL}(n,r)|$ is a weighted enumeration of ASM$(n,r)$ in which each higher spin alternating sign matrix is weighted by the number of fully packed loop configurations corresponding to it. The set $\mathcal{W}(r)$ is in bijection with:

$$\mathcal{W}'(r) := \left\{ (m_1, m_2, m_3, m_4, t_1, t_2) \in \mathbb{N}^6 \;\middle|\; \begin{array}{l} \bullet\ m_1 + m_2 + m_3 + m_4 + t_1 + t_2 = r \\ \bullet\ t_1 = 0 \text{ or } (t_1 \neq 0 \text{ and } t_2 = 0) \end{array} \right\}$$

where $(m_1, m_2, m_3, m_4, t_1, t_2)$ corresponds to a vertex type with $m_1$ curved lines in quadrant 1, $m_2$ curved lines in quadrant 2, $m_3$ curved lines in quadrant 3, $m_4$ curved lines in quadrant

4, $t_1$ vertical lines and $t_2$ horizontal lines. (Note that the number of lines passing through a vertex is $r$, so that the number of endpoints of lines surrounding a vertex is $2r$). We get two possible situations: 

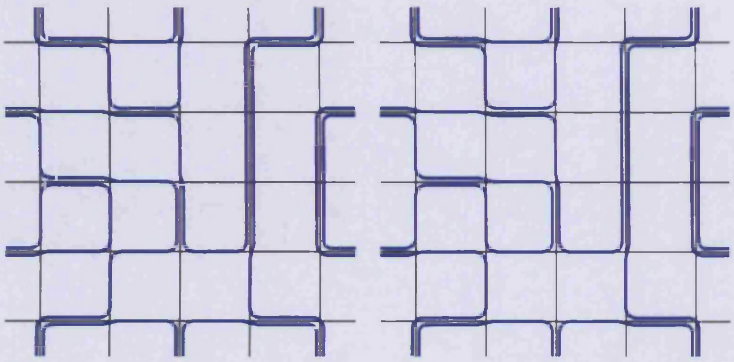
or 

corresponding to the mutually exclusive cases $t_1 = 0$ or $t_1 \neq 0$ and $t_2 = 0$. Using Lemma 1.2.30 this gives the cardinality of $\mathcal{W}(r)$ as:

$$|\mathcal{W}(r)| = \binom{r+4}{4} + \binom{r+3}{4}$$

as given by sequence A002415 of [99].

The two fully packed loop configurations of FPL(5, 2) that map to our running example (2.2) are:



(a) (b)

(2.22)

In [116] the set of link patterns $L_{2n}$ is generalized to give the definition of $L_{2n,r}$:

**Definition 2.2.8.** *$L_{2n,r}$ is the set of non crossing pairings of $2n$ points, in which $r$ arches meet at each of the points and no point is paired with itself.*

For example, $L_{6,2}$ is given by figure 2.7.

In [116], cells formed by the arches and bounding circle of a link pattern are considered, and a link pattern is defined to be admissible if and only if all of its cells are bounded by an even number of lines.

**Definition 2.2.9.** *$L'_{2n,r}$ is the set of link patterns in $L_{2n,r}$ which are admissible.*

Thus $L'_{6,2}$ is the set given by Figure 2.7 without the last 3 link patterns. In [116] it is shown that $\left|L'_{2n,r}\right| = \frac{((r+1)n)!}{(rn+1)!n!}$. This is done by showing that admissible link patterns are in bijection with Lukasiewicz words (the enumeration of which is known).

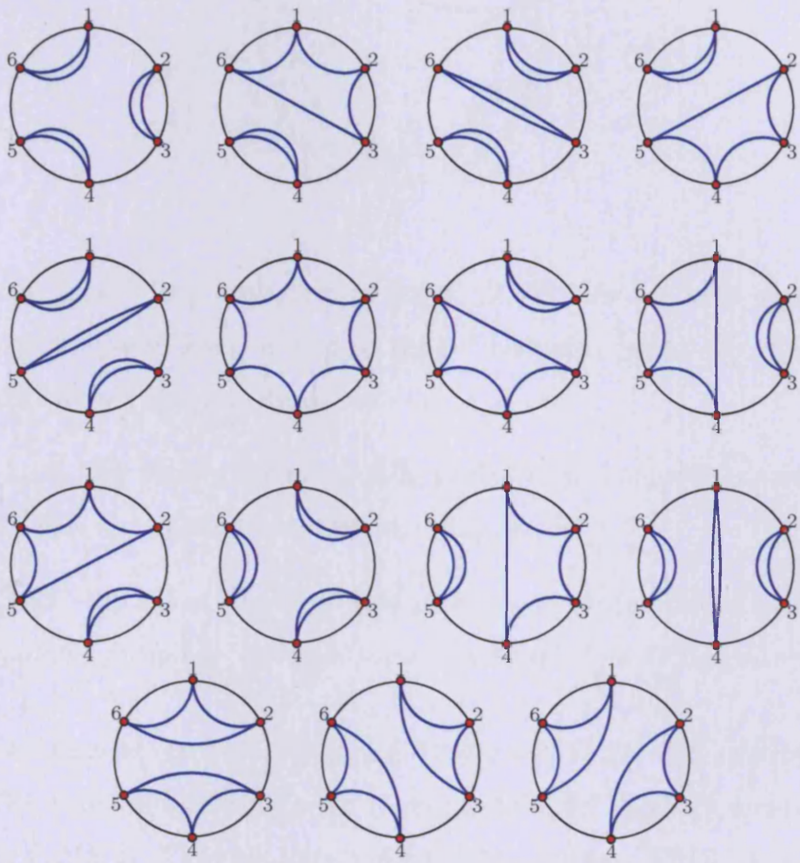
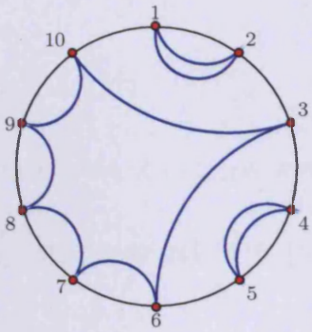



Figure 2.7: The link patterns of $L_{6,2}$

It can be seen that by labeling the external vertices of $\mathcal{L}_{n,n}$ as in Figure 1.18 a fully packed loop configuration of FPL$(n,r)$ can be naturally associated with the link pattern of $L_{2n,r}$ formed by its open paths, provided that each open path has distinct endpoints. For example, the fully packed loop configuration $(b)$ of (2.22) is associated with the link pattern:



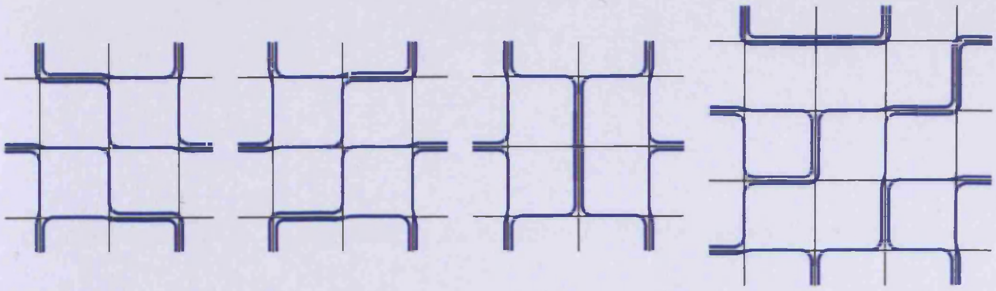

whereas the fully packed loop configuration $(a)$ of (2.22) does not have an associated link pattern as one of its paths forms a loop at the $9^{th}$ endpoint (using the labeling of Figure 1.18). This leads to the following definitions:

**Definition 2.2.10.** *We denote the set of fully packed loop configurations in FPL$(n,r)$ for which each open path has distinct endpoints as FPL$_{dis}(n,r)$.*

**Definition 2.2.11.** *We define the set of fully packed loop configurations in FPL$_{dis}(n,r)$ for which the link pattern formed by the open paths is admissible as FPL$_{adm}(n,r)$.*

Figure 2.8 shows that bijectivity of mapping (2.20) and (2.21) still fails for these newly defined sets. The configurations of $(a)$ and $(b)$ are both in FPL$_{dis}(3,2)$, however map to the same element of EM$(3,2)$. Thus our map is not injective between FPL$_{dis}(3,2)$ and EM$(3,2)$. The configuration of $(c)$ is not in FPL$_{adm}(3,2)$ ((a),(b),(c) correspond to the last three link patterns of figure 2.7) however it is the only element of FPL$(3,2)$ that maps to its image in EM$(3,2)$. This shows that our map is not surjective between FPL$_{adm}(3,2)$ and EM$(3,2)$. The configuration of $(d)$ is not in FPL$_{dis}(4,2)$ (it has a loop starting and ending at $(2,0)$), however it is the only element of FPL$(4,2)$ that maps to its image in EM$(4,2)$. Thus the map is not surjective between FPL$_{dis}(4,2)$ and EM$(4,2)$.

In the next section we make a connection between the alternating sign matrices of Section 1.1 and the convex polytopes described in Section 1.2.



(a) (b) (c) (d)

Figure 2.8: Further examples of fully packed loop configurations

## 2.3 The alternating sign matrix polytope

**Definition 2.3.1.** *We define the* alternating sign matrix polytope, $\mathcal{A}_n$ *as:*

$$\mathcal{A}_n := \left\{ a = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \in \mathbb{R}^{n\times n} \;\middle|\; \begin{array}{l} \bullet \sum_{j'=1}^{n} a_{i,j'} = \sum_{i'=1}^{n} a_{i',j} = 1 \textit{ for all } i,j \in [n] \\ \bullet\, 0 \leq \sum_{j'=1}^{j} a_{i,j'} \leq 1 \textit{ for all } i \in [n],\ j \in [n-1] \\ \bullet\, 0 \leq \sum_{i'=1}^{i} a_{i',j} \leq 1 \textit{ for all } i \in [n-1],\ j \in [n] \end{array} \right\}$$

Equivalently we have:

$$\mathcal{A}_n = \left\{ a \in \mathbb{R}^{n\times n} \;\middle|\; \begin{array}{l} \bullet \sum_{j'=1}^{n} a_{i,j'} = \sum_{i'=1}^{n} a_{i',j} = 1 \text{ for all } i,j \in [n] \\ \bullet \sum_{j'=1}^{j} a_{i,j'} \geq 0 \text{ for all } i \in [n],\ j \in [n-1] \\ \bullet \sum_{j'=j}^{n} a_{i,j'} \geq 0 \text{ for all } i \in [n],\ j \in [2,n] \\ \bullet \sum_{i'=1}^{i} a_{i',j} \geq 0 \text{ for all } i \in [n-1],\ j \in [n] \\ \bullet \sum_{i'=i}^{n} a_{i',j} \geq 0 \text{ for all } i \in [2,n],\ j \in [n] \end{array} \right\} \tag{2.23}$$

Thus $\mathcal{A}_n$ is the set of $n \times n$ real entry matrices for which all complete row and column sums are 1, and all partial row and columns sums extending from each end of the row or column are nonnegative. It follows that all entries of $\mathcal{A}_n$ are between $-1$ and 1, and along the first/last row/column all entries are non negative. Thus $\mathcal{A}_n$ is a bounded subset of $\mathbb{R}^{n\times n}$ (indeed, simply considering real entry matrices with total row and column sum 1 gives an unbounded set). Therefore, $\mathcal{A}_n$ is the bounded intersection of finitely many half spaces and is thus a convex polytope according to (1.43) in Definition 1.2.3. In [105] $\mathcal{A}_n$ is independently defined by Striker using a convex hull description (see (1.42) in Definition 1.2.3). An example of an element of $\mathcal{A}_4$ is:

$$a = \begin{pmatrix} \frac{3}{10} & 0 & \frac{3}{5} & \frac{1}{10} \\ \frac{1}{5} & \frac{1}{2} & -\frac{3}{5} & \frac{9}{10} \\ \frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{2.24}$$

Recalling Definition 1.2.20, and similarly to relation (2.3) the Birkhoff polytope is given by:

$$\mathcal{B}_n = \{a \in \mathcal{A}_n \mid a_{ij} \geq 0 \text{ for all } i,j \in [n]\} \tag{2.25}$$

Definition 1.2.23 gives $r\mathcal{B}_n \cap \mathbb{Z}^{n\times n} = \mathrm{SMS}(n,r)$. Similarly Definition 2.1.1 gives:

$$r\mathcal{A}_n \cap \mathbb{Z}^{n\times n} = \mathrm{ASM}(n,r) \tag{2.26}$$

A counterpart to Theorem 1.2.21 is:

**Theorem 2.3.2.**

$$\dim \mathcal{A}_n = (n-1)^2$$

This follows from:

$$\mathrm{aff}\mathcal{A}_n = \mathrm{aff}\mathcal{B}_n = \left\{ a \in \mathbb{R}^{n\times n} \;\middle|\; \sum_{i=1}^{n} a_{i,j} = \sum_{j=1}^{n} a_{i,j} = 1 \text{ for all } (i,j) \in [n]\times[n] \right\}$$

Before proving any further results we define another polytope.

**Definition 2.3.3.** *We define the* edge matrix polytope, *$\mathcal{E}_n$ as:*

$$\mathcal{E}_n := \left\{ (h,v) = \left( \begin{pmatrix} h_{10} & \dots & h_{1n} \\ \vdots & & \vdots \\ h_{n0} & \dots & h_{nn} \end{pmatrix}, \begin{pmatrix} v_{01} & \dots & v_{0n} \\ \vdots & & \vdots \\ v_{n1} & \dots & v_{nn} \end{pmatrix} \right) \in [0,1]_{\mathbb{R}}^{n\times(n+1)} \times [0,1]_{\mathbb{R}}^{(n+1)\times n} \;\middle|\; \begin{array}{l} \bullet\ h_{i0} = v_{0j} = 0 \ \textit{for all}\ i,j \in [n] \\ \bullet\ h_{in} = v_{nj} = 1 \ \textit{for all}\ i,j \in [n] \\ \bullet\ h_{i,j-1} + v_{ij} = v_{i-1,j} + h_{ij} \ \textit{for all}\ i,j \in [n] \end{array} \right\}$$

It is easy to see that $r\mathcal{E}_n \cap \mathbb{Z}^{2n(n+1)} = \mathrm{EM}(n,r)$, and that there are bijections between $\mathcal{E}_n$ and $\mathcal{A}_n$ equivalent to bijections (2.4) and (2.5), i.e. the $(h,v) \in \mathcal{E}_n$ which corresponds to $a \in \mathcal{A}_n$ is given by:

$$\begin{aligned} h_{ij} &= \textstyle\sum_{j'=1}^{j} a_{i,j'} \ \text{ for all } \ i \in [n],\ j \in [0,n] \\ v_{ij} &= \textstyle\sum_{i'=1}^{i} a_{i',j} \ \text{ for all } \ i \in [0,n],\ j \in [n] \end{aligned} \tag{2.27}$$

and inversely:

$$a_{ij} = h_{ij} - h_{i,j-1} \text{ or } a_{ij} = v_{ij} - v_{i-1,j} \quad \text{for all } \ i,j \in [n] \tag{2.28}$$

The equations (2.27) can be regarded as giving a linear map from $\mathbb{R}^{n\times n}$ to $\mathbb{R}^{2n(n+1)}$ and each of the equations of (2.28) can be regarded as giving a linear map from $\mathbb{R}^{2n(n+1)}$ to $\mathbb{R}^{n\times n}$. Thus $\mathcal{A}_n$ and $\mathcal{E}_n$ are affinely isomorphic as given by Definition 1.2.11. We now give an important result, which forms a counterpart to Theorem 1.2.24.

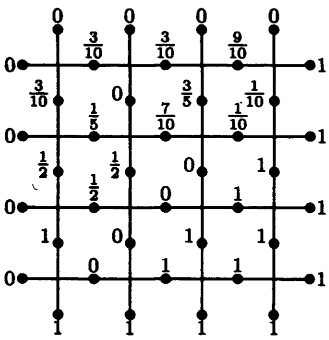
**Theorem 2.3.4.**

$$vert\mathcal{A}_n = ASM(n, 1)$$

*Proof.* The proof offered here differs slightly from the proof given in [15]. We show that $\text{vert}\mathcal{E}_n = \text{EM}(n,1)$. The result then follows from Lemma 1.2.12 since $\mathcal{A}_n$ and $\mathcal{E}_n$ are affinely isomorphic.

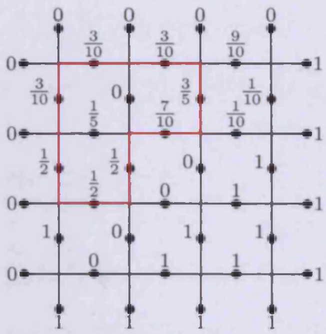
- We first show that $\text{EM}(n,1) \subseteq \text{vert}\mathcal{E}_n$. Assume $(h,v) \in \mathcal{E}_n \setminus \text{vert}\mathcal{E}_n$. Then by Lemma 1.2.7, there exists $(h^*,v^*) \neq (0,0)$ such that $(h,v)\pm(h^*,v^*) \in \mathcal{E}_n$. Since $(h^*,v^*) \neq (0,0)$ there exists $i,j \in [0,n]$ such that $h^*_{ij} \neq 0$ or $v^*_{ij} \neq 0$. Assume without loss of generality that $h^*_{ij} \neq 0$. Thus, $0 \leq h_{ij} \pm h^*_{ij} \leq 1$ which implies $0 < h_{ij} < 1$ and so $(h,v) \notin \text{EM}(n,1)$ (since $\text{EM}(n,1) = \mathcal{E}_n \cap \mathbb{Z}^{2n(n+1)}$). Thus as required $\text{EM}(n,1) \subseteq \text{vert}\mathcal{E}_n$.

- Next we show that $\text{vert}\mathcal{E}_n \subseteq \text{EM}(n,1)$. Consider $(h,v) \in \mathcal{E}_n \setminus \text{EM}(n,1)$. As in Figure 1.10 we can represent $(h,v)$ on $\mathcal{L}_{n,n}$. For example, the $(h,v)$ corresponding to the case of (2.24) on $\mathcal{L}_{4,4}$ is:




(2.29)

Since $(h,v) \in \mathcal{E}_n \setminus \text{EM}(n,1)$ this implies that there exists $i,j \in [0,n]$ such that $0 < h_{ij} < 1$ or $0 < v_{ij} < 1$. Because of the boundary conditions imposed on $\mathcal{E}_n$ all non integral entries must be associated to an internal edge. Recalling $h_{i,j-1} + v_{ij} = v_{i-1,j} + h_{ij}$ we see that we must have a cycle of non integer entries (since if any one of the four entries in this equation is non integral, then at least one of the others must be non integral). For example the following cycle of non integer entries exists for (2.29):



(2.30)

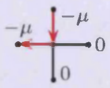
Select any such cycle on the lattice diagram of $(h, v)$, give it an orientation, say anticlockwise, and denote the sets of points $(i, j)$ for which the horizontal edge between $(i, j)$ and $(i, j+1)$ is in the cycle and directed right or left as respectively $\mathcal{H}_+$ or $\mathcal{H}_-$, and the sets of points $(i, j)$ for which the vertical edge between $(i, j)$ and $(i+1, j)$ is in the cycle and directed up or down as respectively $\mathcal{V}_+$ or $\mathcal{V}_-$. For example if we give the cycle of (2.30) an anticlockwise orientation then $\mathcal{H}_- = \{(1,1),(1,2)\}$, $\mathcal{H}_+ = \{(2,2),(3,1)\}$, $\mathcal{V}_- = \{(1,1),(2,1)\}$ and $\mathcal{V}_+ = \{(1,3),(2,2)\}$.

We now create the matrix pair $(h^*, v^*) = \left( \begin{pmatrix} h^*_{10} & \dots & h^*_{1n} \\ \vdots & & \vdots \\ h^*_{n0} & \dots & h^*_{nn} \end{pmatrix}, \begin{pmatrix} v^*_{01} & \dots & v^*_{0n} \\ \vdots & & \vdots \\ v^*_{n1} & \dots & v^*_{nn} \end{pmatrix} \right) \in \mathbb{R}^{n\times(n+1)} \times \mathbb{R}^{(n+1)\times n}$ with entries:

$$h^*_{ij} := \begin{cases} \mu \text{ if } (i,j) \in \mathcal{H}_+ \\ -\mu \text{ if } (i,j) \in \mathcal{H}_- \\ 0 \text{ otherwise} \end{cases} \quad v^*_{ij} := \begin{cases} \mu \text{ if } (i,j) \in \mathcal{V}_+ \\ -\mu \text{ if } (i,j) \in \mathcal{V}_- \\ 0 \text{ otherwise} \end{cases}$$

Note that we will have:

$$h^*_{i,j-1} + v^*_{ij} = v^*_{i-1,j} + h^*_{ij} \text{ for all } i \in [m],\ j \in [n]$$

since if the cycle does not pass through $(i, j)$ then the equation is trivial. If the cycle does pass through $(i, j)$ then because of the orientation, all appearances of $\mu$ cancel out. For example:  gives $-\mu + 0 = -\mu + 0$ or  gives $0 + \mu = \mu + 0$.

We choose:

$$\begin{aligned}\mu{:=}\min(&\{h_{ij}|(i,j)\in\mathcal{H}_+\}\cup\{\overline{h}_{ij}|(i,j)\in\mathcal{H}_+\}\cup\\&\{h_{ij}|(i,j)\in\mathcal{H}_-\}\cup\{\overline{h}_{ij}|(i,j)\in\mathcal{H}_-\}\cup\\&\{v_{ij}|(i,j)\in\mathcal{V}_+\}\cup\{\overline{v}_{ij}|(i,j)\in\mathcal{V}_+\}\cup\\&\{v_{ij}|(i,j)\in\mathcal{V}_-\}\cup\{\overline{v}_{ij}|(i,j)\in\mathcal{V}_-\})\end{aligned}$$

with $\overline{h}_{ij}=1-h_{ij}$ and $\overline{v}_{ij}=1-v_{ij}$. It can now easily be checked that: $(h^*,v^*)\neq(0,0)$ and $(h,v)\pm(h^*,v^*)\in\mathcal{E}_n$. Thus as required $\mathrm{EM}(n,1)\supseteq\mathrm{vert}\mathcal{E}_n$

We thus have $\mathrm{vert}\mathcal{E}_n=\mathrm{EM}(n,1)$ as required. □

Note that Theorem 2.3.4 implies that $\mathcal{A}_n$ is integral and that $\mathcal{E}_n$ is a 0, 1 polytope. This gives an alternate proof of Theorem 2.2.6, as an application of Theorem 1.2.29. It also follows from Theorem 2.3.4 that $\mathcal{A}_n$ is the convex hull of the standard alternating sign matrices of size $n$:

$$\mathcal{A}_n=\left\{\sum_{a\in\mathrm{ASM}(n,1)}\lambda_a a\;\middle|\;\lambda_a\in[0,1]_{\mathbb{R}}\text{ for all }a\in\mathrm{ASM}(n,1),\sum_{a\in\mathrm{ASM}(n,1)}\lambda_a=1\right\}\tag{2.31}$$

This is how Striker defines $\mathcal{A}_n$ in [105]. In this paper Striker also considers other faces of $\mathcal{A}_n$. One of the theorems given is a counterpart to Theorem 1.2.22:

**Theorem 2.3.5.** *The alternating sign matrix polytope $\mathcal{A}_n$ has $4\left[(n-2)^2+1\right]$ facets for $n\geq 3$.*

In the next section we give some structure to Figure 2.1.

## 2.4 Enumeration of higher spin alternating sign matrices of fixed size

In this section we generalize Theorem 1.2.25. Recalling Theorem 1.2.18 we need to consider $\mathcal{A}_n^o$. Similarly to $\mathcal{B}_n^o$, $\mathcal{A}_n^o$ is obtained by replacing each weak inequality in Definition 2.1.1 by a strict inequality. Recalling Relation (2.26) and Definition 1.47 we define:

$$\mathrm{ASM}^o(n,r):=r\mathcal{A}_n^o\cap\mathbb{Z}^{n\times n}$$

Equivalently:

$$\mathrm{ASM}^o(n,r) := \left\{ a \in \mathbb{Z}^{n\times n} \,\middle|\, \begin{array}{l} \bullet \sum_{j'=1}^n a_{i,j'} = \sum_{i'=1}^n a_{i',j} = r \text{ for all } i,j \in [n] \\ \bullet\, 1 \le \sum_{j'=1}^j a_{i,j'} \le r-1 \text{ for all } i \in [n],\ j \in [n-1] \\ \bullet\, 1 \le \sum_{i'=1}^i a_{i',j} \le r-1 \text{ for all } i \in [n-1],\ j \in [n] \end{array} \right\}$$

$$= \left\{ a \in \mathbb{Z}^{n\times n} \,\middle|\, \begin{array}{l} \bullet \sum_{j'=1}^n a_{i,j'} = \sum_{i'=1}^n a_{i',j} = r \text{ for all } i,j \in [n] \\ \bullet \sum_{j'=1}^j a_{i,j'} \ge 1 \text{ for all } i \in [n],\ j \in [n-1] \\ \bullet \sum_{j'=j}^n a_{i,j'} \ge 1 \text{ for all } i \in [n],\ j \in [2,n] \\ \bullet \sum_{i'=1}^i a_{i',j} \ge 1 \text{ for all } i \in [n-1],\ j \in [n] \\ \bullet \sum_{i'=i}^n a_{i',j} \ge 1 \text{ for all } i \in [2,n],\ j \in [n] \end{array} \right\}$$

Thus $\mathrm{ASM}^o(n,r) = \emptyset$ for $1 \le r < n$. Using this we have:

**Theorem 2.4.1.** *For fixed $n \in \mathbb{P}$ there exists $A_n(r)$, the Ehrhart polynomial of $\mathcal{A}_n$ which satisfies:*

1. *$A_n(r)$ is a polynomial in $r$ of degree $(n-1)^2$*
2. *$|ASM(n,r)| = A_n(r)$ for all $r \in \mathbb{N}$*
3. *$|ASM^o(n,r)| = (-1)^{n+1}A_n(-r)$ for all $r \in \mathbb{P}$*
4. *$A_n(-1) = A_n(-2) = \cdots = A_n(-n+1) = 0$*
5. *$A_n(1) = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$*

*Proof.* The proof is a direct implication of previous results. The first three properties are direct implications of Theorems 1.2.18, 2.3.2 and 2.3.4. Property (4) is obtained from the fact that $\mathrm{ASM}^o(n,r) = \emptyset$ for all $1 \le r < n$. Property (5) is implied by (1.1). □

Thus $A_n(r)$ can be interpolated by the $n+1$ values given by Theorem 2.4.1 and a further $n^2 - 3n + 1$ enumerations of $\mathrm{ASM}(n,r)$ or $\mathrm{ASM}^o(n,r)$. Similarly to equations (1.48)-(1.50) we give the polynomials $A_n(r)$ for $n \in [1,5]$:

$$
\begin{aligned}
A_1(r) &= \binom{r}{0}, \; A_2(r) = \binom{r+1}{1} \\
A_3(r) &= \binom{r+2}{4} + 2\binom{r+3}{4} + \binom{r+4}{4} && (2.32) \\
A_4(r) &= 3\binom{r+3}{9} + 80\binom{r+4}{9} + 415\binom{r+5}{9} + 592\binom{r+6}{9} + 253\binom{r+7}{9} + \\
&\quad 32\binom{r+8}{9} + \binom{r+9}{9} && (2.33) \\
A_5(r) &= 70\binom{r+4}{16} + 14468\binom{r+5}{16} + 521651\binom{r+5}{16} + 6002192\binom{r+7}{16} + \\
&\quad 28233565\binom{r+8}{16} + 61083124\binom{r+9}{16} + 64066830\binom{r+10}{16} \\
&\quad 32866092\binom{r+11}{16} + 7998192\binom{r+12}{16} + 854464\binom{r+13}{16} + \\
&\quad 34627\binom{r+14}{16} + 412\binom{r+15}{16} + \binom{r+16}{16} && (2.34)
\end{aligned}
$$

Polynomial (2.32) corresponds to sequence A006325 of [99]. Last but not least we shall pay particular attention to ASM$(n,3)$ and FPL$(n,3)$.

## 2.5 The particular case of $n = 3$

### 2.5.1 Alternating sign matrices of size 3

Recalling the argument presented in Section 1.2.2 used to prove the enumeration of SMS$(3,r)$ we here give an analogous argument for ASM$(3,r)$. We label the matrices of ASM$(3,1)$ as given in Figure 1.1.

$$h_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad h_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad h_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$h_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad h_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad h_6 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$h_7 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

It can be checked that we have a bijection $\phi$ between $\mathrm{ASM}(n,r)$ and the set:

$$C'(r) = \left\{ (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7) \in \mathbb{N}^7 \;\middle|\; \begin{array}{l} \bullet\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_7 = r \\ \bullet\ \lambda_2\lambda_3 = 0 \\ \bullet\ \lambda_5\lambda_6 = 0 \end{array} \right\}$$

where $\phi : C'(r) \to \mathrm{ASM}(3,r)$ is defined by $\phi(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7) = \sum_{i=1}^{7} \lambda_i h_i$. The conditions $\lambda_2\lambda_3 = 0$ and $\lambda_5\lambda_6 = 0$ in $C'(r)$ are related to the fact that $h_4 + h_7 = h_2 + h_3$ and $h_1 + h_7 = h_5 + h_6$. Note that the set $C'(r)$ can be written as the following disjoint union:

$$\begin{aligned} C'(r) = &\left\{ (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7) \in \mathbb{N}^7 \;\middle|\; \begin{array}{l} \bullet\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_7 = r \\ \bullet\ \lambda_3 = 0 \\ \bullet\ \lambda_6 = 0 \end{array} \right\} \\ \cup &\left\{ (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7) \in \mathbb{N}^7 \;\middle|\; \begin{array}{l} \bullet\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_7 = r \\ \bullet\ \lambda_2 = 0,\ \lambda_3 \geq 1 \\ \bullet\ \lambda_6 = 0 \end{array} \right\} \\ \cup &\left\{ (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7) \in \mathbb{N}^7 \;\middle|\; \begin{array}{l} \bullet\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_7 = r \\ \bullet\ \lambda_3 = 0 \\ \bullet\ \lambda_5 = 0,\ \lambda_6 \geq 1 \end{array} \right\} \\ \cup &\left\{ (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7) \in \mathbb{N}^7 \;\middle|\; \begin{array}{l} \bullet\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_7 = r \\ \bullet\ \lambda_2 = 0,\ \lambda_3 \geq 1 \\ \bullet\ \lambda_5 = 0,\ \lambda_6 \geq 1 \end{array} \right\} \end{aligned}$$

Using Lemma 1.2.30 we get the formula (2.32) for $|\mathrm{ASM}(3,r)|$. In fact, formula (2.32) also appears in [21] as the number of $2 \times 2$ non negative integer entry matrices with each row and column sum at most $r$. A bijection $\phi$ between $\mathrm{ASM}(3,r)$ and the set of such $2 \times 2$ matrices is given simply by:

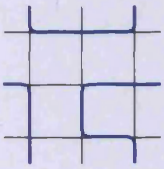
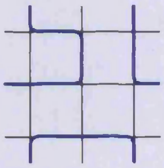
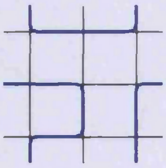
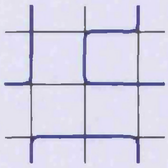
$$\phi \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{pmatrix}$$

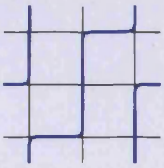
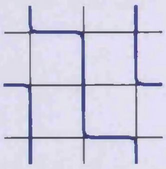
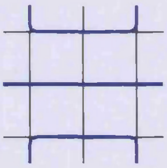
### 2.5.2 Fully packed loops of size 3

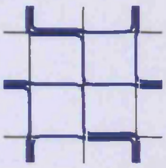
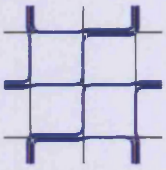
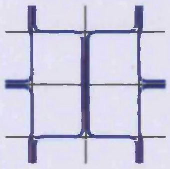
The set $\mathrm{FPL}(3,r)$ is in bijection with the following set:

$$C''(r) := \left\{ (a_1,\ldots,a_{10}) \in \mathbb{N}^{10} \;\middle|\; \begin{array}{l} \bullet\ a_1+a_2+a_3+a_4+a_5+a_6+a_7+2a_8+2a_9+2a_{10}=r \\ \bullet\ a_5a_6=a_7=a_8=a_9=0 \\ \text{ or } a_5a_7=a_6=a_8=a_{10}=0 \\ \text{ or } a_6a_7=a_5=a_9=a_{10}=0 \end{array} \right\} \tag{2.35}$$

Indeed let us consider the following fully packed loop configurations:

$B_1 =$  $B_2 =$  $B_3 =$  $B_4 =$ 

$B_5 =$  $B_6 =$  $B_7 =$ 

$B_8 =$  $B_9 =$  $B_{10} =$ 

It can be seen that $B_1,\ldots,B_7$ are the elements of $\mathrm{FPL}(3,1)$ and that $B_8$, $B_9$ and $B_{10}$ are the non admissible elements of $\mathrm{FPL}(3,2)$. It can also be seen that any element of $\mathrm{FPL}(3,r)$ can be written as:

$$\bigcup_{i=1}^{10} a_iB_i$$

where $a_iB_i$ represents $a_i$ superpositions of $B_i$. We need to make sure that when taking this union there is no crossing of paths. This corresponds to the last three equalities of $C''(r)$. The set $C''(r)$ can be written as the disjoint union:

$$\left\{(a_1,\ldots,a_4,a_5,a_6,0,0,0,a_{10})\in\mathbb{N}^{10}\;\middle|\;\begin{array}{l}\bullet\ a_1+a_2+a_3+a_4+a_5+a_6+2a_{10}=r\\ \bullet\ a_5a_6=0\end{array}\right\}$$
$$\cup\left\{(a_1,\ldots,a_4,a_5,0,a_7,0,a_9,0)\in\mathbb{N}^{10}\;\middle|\;\begin{array}{l}\bullet\ a_1+a_2+a_3+a_4+a_5+a_7+2a_9=r\\ \bullet\ a_5a_7=0\\ \bullet\ a_7+a_9\geq 1\end{array}\right\}$$
$$\cup\left\{(a_1,\ldots,a_4,0,a_6,a_7,a_8,0,0)\in\mathbb{N}^{10}\;\middle|\;\begin{array}{l}\bullet\ a_1+a_2+a_3+a_4+a_6+a_7+2a_8=r\\ \bullet\ a_6a_7=0\\ \bullet\ a_8\geq 1\end{array}\right\}$$

We now give the following lemma:

**Lemma 2.5.1.**

$$\left|\left\{(c_1,\ldots,c_{k+1})\in\mathbb{N}^{k+1}\;\middle|\;\begin{array}{l}\bullet\ c_1+\cdots+c_k+2c_{k+1}=r\\ \bullet\ c_{k-1}c_k=0\end{array}\right\}\right|=\binom{k-1+r}{k-1}$$

*Proof.* Indeed we have a bijection between the set on the left hand side and:

$$\{(a,\ldots,a_k)\in\mathbb{N}^k\mid a_1+\cdots+a_k=r\}$$

given by $\theta((a_1,\ldots,a_k))=(a_1,\ldots,a_{k-2},\max(a_{k-1}-a_k,0),\max(a_k-a_{k-1},0),\min(a_{k-1},a_k))$ and $\theta^{-1}((c_1,\ldots,c_k))=(c_1,\ldots,c_{k-2},c_{k-1}+c_{k+1},c_k+c_{k+1})$. The enumeration follows from Lemma 1.2.30. □

Using this we can enumerate the first set in the disjoint union as $\binom{r+5}{5}$. The other two sets can be enumerated using similar results, giving:

$$|\mathrm{FPL}(3,r)|=\binom{r+5}{5}+\binom{r+4}{5}+\binom{r+3}{5}$$

Recalling Definition 2.2.11 we have that any element in $\mathrm{FPL}_{\mathrm{adm}}(3,r)$ can be written as:

$$\bigcup_{i=1}^{7}a_iB_i$$

This gives $|\mathrm{FPL}_{\mathrm{adm}}(3,r)|=\binom{r+4}{4}+2\binom{r+3}{4}$, corresponding to sequence A001296 of [99]. We now prove one more result concerning these fully packed loops. Recalling Theorem 1.1.11 and Definition 2.2.8 we generalize the result for $n\in[3]$.

**Theorem 2.5.2.** *For $n \in [3]$, if $\pi, \pi' \in L_{2n,r}$ are such that $\pi$ can be obtained from $\pi'$ by rotation then $|FPL_{\pi}(n,r)| = |FPL_{\pi'}(n,r)|$.*

*Proof.* For $n \leq 2$ the result is trivial. For $n = 3$ it suffices to note that the permutation $(1,2)(3,4)(5,6,7)(8,9,10)$ is a bijection on $C''(r)$ (2.35). This particular permutation rotates the link patterns as required. □

## 2.6 Conclusion

In this chapter, firstly we offered a direct generalization of alternating sign matrices and described the many combinatorial objects which are in bijection with, or related to, these new matrices. Secondly we made the connection to polytopes by defining the alternating sign matrix polytope $\mathcal{A}_n$. The following chapters will show just how interesting an object $\mathcal{A}_n$ really is.

# Chapter 3

# Symmetry Classes of The Birkhoff Polytope

## 3.1 General results on symmetry classes

We recall Figures 1.14 - 1.17 and the set $\mathcal{P}^G$ as defined by (1.22):

$$\mathcal{P}^G := \{a \in \mathcal{P} \mid a = ga \text{ for all } g \in G\}$$

where $G$ is any group for which an action of $G$ on $\mathcal{P}$ is defined. In Section 1.2.2 we discussed the symmetry class $\mathcal{B}_n^{\{1,d\}}$ and in this chapter we consider the symmetry classes $\mathcal{B}_n^G$ for each subgroup $G$ of $D_4$ (as given by Definition 1.21), in more detail. From Chapter 1 we have a lot of results concerning $\mathcal{B}_n$ and $D_4$. The rest of this section is a list of results that will help us to deduce results concerning $\mathcal{B}_n^G$.

**Definition 3.1.1.** *For $\mathcal{P}$ a polytope and $G$ any finite group such that a linear action of $G$ on $\mathcal{P}$ is defined, we define the* group projector of $G$ on $\mathcal{P}$ *as:*

$$\Pi_G := \frac{\sum_{g \in G} g}{|G|}$$

It is easy to see that $\Pi_G$ is indeed a projector as $\Pi_G^2 = \Pi_G$. This projector is a very powerful tool (we shall be using it throughout this chapter and the next) as noted by the following lemma:

**Theorem 3.1.2.** *For $\mathcal{P}$ a polytope, $G$ any finite group such that a linear action of $G$ on $\mathcal{P}$ is defined and $\Pi_G$ as given by Definition 3.1.1,*

$$\Pi_G(\mathcal{P}) = \mathcal{P}^G$$

*Proof.* Consider any $y \in \Pi_G\mathcal{P}$. By Definition (3.1.1) $y = \frac{\sum_{g\in G} ga}{|G|}$ for some $a \in \mathcal{P}$, so $gy = \frac{\sum_{g'\in G} gg'a}{|G|} = \frac{\sum_{g\in G} ga}{|G|} = y$ for any $g \in G$. Also, $y \in \text{conv}\{ga \mid g \in G\} \subseteq \mathcal{P}$. Thus, $y \in \mathcal{P}^G$ and so $\Pi_G(\mathcal{P}) \subseteq \mathcal{P}^G$. Now consider $a \in \mathcal{P}^G$. By definition, $ga = a$ for all $g \in G$ and so $\sum_{g\in G} ga = |G|a$, giving $a = \Pi_G a \in \Pi_G(\mathcal{P})$ and so $\Pi_G(\mathcal{P}) \supseteq \mathcal{P}^G$, as required. □

These results lead to the following powerful theorem:

**Theorem 3.1.3.** *For $\mathcal{P}$ a polytope, $G$ any finite group such that a linear action of $G$ on $\mathcal{P}$ is defined and $\Pi_G$ as given by Definition 3.1.1, we have:*

$$(vert\mathcal{P})^G \subseteq vert\mathcal{P}^G \subseteq \Pi_G(vert\mathcal{P})$$

*Proof.* Since $\mathcal{P}^G \subseteq \mathcal{P}$, Lemma 1.2.8 implies that $\text{vert}\mathcal{P} \cap \mathcal{P}^G \subseteq \text{vert}\mathcal{P}^G$. Also, $\text{vert}\mathcal{P} \cap \mathcal{P}^G = (\text{vert}\mathcal{P})^G$, giving the first inclusion. Since $\Pi_G$ is affine (it is actually linear), Lemma 1.2.10 implies that $\text{vert}\Pi_G(\mathcal{P}) \subseteq \Pi_G(\text{vert}\mathcal{P})$. The second inclusion then follows from Lemma 3.1.2 □

Theorem 3.1.3 makes the study of $\mathcal{P}^G$ a lot easier as it sandwiches $\text{vert}\mathcal{P}^G$ between two sets that are easily obtained from a basic study of $\mathcal{P}$. In some cases $\text{vert}\mathcal{P}^G$ could be directly obtained (if $(\text{vert}\mathcal{P})^G = \Pi_G(\text{vert}\mathcal{P})$).

Applying Theorems 3.1.3 and 1.2.24 to $\mathcal{B}_n$ and $D_4$ we have:

$$\text{SMS}(n,1)^G \subseteq \text{vert}\mathcal{B}_n^G \subseteq \Pi_G(\text{SMS}(n,1)) \tag{3.1}$$

for each subgroup $G$ of $D_4$.

Theorem 3.1.3 leads to a convex hull description of $\mathcal{P}^G$:

**Corollary 3.1.4.** *For $\mathcal{P}$ and $G$ defined as in Theorem 3.1.3:*

$$\mathcal{P}^G = conv(\Pi_G(vert\mathcal{P}))$$

A form of this result is presented in [42]. Recalling Definition 1.2.13 the second inclusion of Theorem 3.1.3 gives:

**Corollary 3.1.5.** *For a rational polytope $\mathcal{P}$ and a finite group $G$ such that a linear action of $G$ on $\mathcal{P}$ is defined:*

$$1 \leq D(\mathcal{P}^G) \leq |G|D(\mathcal{P})$$

*Proof.* From Theorem 3.1.3 we have $\text{vert}\mathcal{P}^G \subseteq \Pi_G(\text{vert}\mathcal{P}) = \{\Pi_G v \mid v \in \text{vert}\mathcal{P}\}$, and from (3.1.1) we have $\{\Pi_G v \mid v \in \text{vert}\mathcal{P}\} = \frac{1}{|G|}\left\{\sum_{g\in G} gv \,\middle|\, v \in \text{vert}\mathcal{P}\right\}$. For all $g \in G$, $g : \mathcal{P} \to \mathcal{P}$ is an affine bijection. Thus from Lemma 1.2.12 $g(\text{vert}\mathcal{P}) = \text{vert}\mathcal{P}$. Using this we have $\frac{1}{|G|}\left\{\sum_{g\in G} gv \,\middle|\, v \in \text{vert}\mathcal{P}\right\} \subseteq \frac{1}{|G|}\left\{\sum_{i=1}^{|G|} v_i \,\middle|\, v_i \in \text{vert}\mathcal{P} \text{ for all } i \in [|G|]\right\}$. This gives, using Definition 1.2.13 for $\mathrm{D}(\mathcal{P}^G)$,

$$|G|\mathrm{D}(\mathcal{P})\text{vert}\mathcal{P}^G \subseteq \mathrm{D}(\mathcal{P})\left\{\sum_{i=1}^{|G|} v_i \,\middle|\, v_i \in \text{vert}\mathcal{P} \text{ for all } i \in [|G|]\right\} \subseteq \mathbb{Z}^d$$

Therefore, $|G|\mathrm{D}(\mathcal{P})\mathcal{P}^G$ is an integral polytope, and so using Definition 1.2.13 for $\mathrm{D}(\mathcal{P})$, $\mathrm{D}(\mathcal{P}^G) \leq |G|\mathrm{D}(\mathcal{P})$. □

Corollary 3.1.5 gives:

$$1 \leq \mathrm{D}(\mathcal{B}_n^{\{1,h\}}) \leq 2 \quad (3.2)$$
$$1 \leq \mathrm{D}(\mathcal{B}_n^{\{1,q^2\}}) \leq 2 \quad (3.3)$$
$$1 \leq \mathrm{D}(\mathcal{B}_n^{\{1,d\}}) \leq 2 \quad (3.4)$$
$$1 \leq \mathrm{D}(\mathcal{B}_n^{\{1,h,v,q^2\}}) \leq 4 \quad (3.5)$$
$$1 \leq \mathrm{D}(\mathcal{B}_n^{\{1,q,q^2,q^3\}}) \leq 4 \quad (3.6)$$
$$1 \leq \mathrm{D}(\mathcal{B}_n^{\{1,d,a,q^2\}}) \leq 4 \quad (3.7)$$
$$1 \leq \mathrm{D}(\mathcal{B}_n^{D_4}) \leq 8 \quad (3.8)$$

In this chapter we shall refine these inequalities, thus leading to enumeration theorems similar to Theorem 1.2.25. In considering the integer points of the $r^{th}$ dilate of $\mathcal{P}^G$ it is important to note the following simple fact, which follows immediately from the definitions.

**Lemma 3.1.6.** *For a polytope $\mathcal{P} \subseteq \mathbb{R}^n$, and a finite group $G$ such that a linear action of $G$ on $\mathcal{P}$ is defined:*

$$(r\mathcal{P}^G) \cap \mathbb{Z}^n = (r\mathcal{P} \cap \mathbb{Z}^n)^G$$

For an integral polytope, a further inclusion, which follows from the properties of a linear group, can be added to those of Theorem 3.1.3, giving:

**Lemma 3.1.7.** *For an integral polytope $\mathcal{P} \subseteq \mathbb{R}^n$ and a finite group $G$ such that a linear action of $G$ on $\mathcal{P}$ is defined:*

$$(vert\mathcal{P})^G \subseteq vert\mathcal{P}^G \subseteq \Pi_G(vert\mathcal{P}) \subseteq \frac{1}{|G|}(|G|\mathcal{P} \cap \mathbb{Z}^n)^G$$

Thus for $\mathcal{B}_n$ and $G \subseteq D_4$ we have:

$$\mathrm{SMS}(n,1)^G \subseteq \mathrm{vert}\mathcal{B}_n^G \subseteq \Pi_G\left(\mathrm{SMS}(n,1)\right) \subseteq \frac{1}{|G|}\left(\mathrm{SMS}(n,|G|)\right)^G \tag{3.9}$$

We will make a note of these four sets for different $G \subseteq D_4$ throughout this chapter noting the interesting relationship between them.

As well as these results we will use the idea of a *fundamental region.*

**Definition 3.1.8.** *For a polytope $\mathcal{P} \subset \mathbb{R}^m$ with coordinates of $x \in \mathbb{R}^m$ labelled $x_j$ for some index set $J$ (with $|J| = m$), a* fundamental region *is a subset $R^{\mathcal{P}}$ of $J$ for which the affine map $f_{\mathcal{P}} : \mathbb{R}^m \to \mathbb{R}^{|R^{\mathcal{P}}|}$ given by:*

$$\left(f_{\mathcal{P}}(x)\right)_j = x_j \text{ for all } j \in R^{\mathcal{P}}$$

*is injective between $\mathcal{P}$ and $f_{\mathcal{P}}(\mathcal{P})$.*

It is very important to note that $R^{\mathcal{P}}$ is not unique and that $f_{\mathcal{P}}$ depends on $R^{\mathcal{P}}$.

**Definition 3.1.9.** *For a fundamental region $R^{\mathcal{P}}$ (with $f_{\mathcal{P}}$ as in Definition 3.1.8) we define the corresponding* fundamental polytope *as $\overline{\mathcal{P}} := f_{\mathcal{P}}(\mathcal{P})$.*

For these new objects we can give the following lemmas. Since $f_{\mathcal{P}}$ is bijective between $\mathcal{P}$ and $\overline{\mathcal{P}}$ we immediately have by Lemma 1.2.12:

**Lemma 3.1.10.** *For $\mathcal{P}, \overline{\mathcal{P}}$ and $f_{\mathcal{P}}$ as given by Definitions 3.1.8 and 3.1.9:*

$$f_{\mathcal{P}}(vert\mathcal{P}) = vert\overline{\mathcal{P}}$$

By definition $f_{\mathcal{P}}$ does not change the values of the coordinates. Thus:

**Lemma 3.1.11.** *For $\mathcal{P}, \overline{\mathcal{P}}$ and $f_{\mathcal{P}}$ as given by Definitions 3.1.8 and 3.1.9:*

$$D(\mathcal{P}) = D(\overline{\mathcal{P}})$$

We have one last important result that follows straightforwardly:

**Lemma 3.1.12.** *For $\mathcal{P}, \overline{\mathcal{P}}$ and $f_{\mathcal{P}}$ as given by Definitions 3.1.8 and 3.1.9:*

$$\dim \mathcal{P} = \dim \overline{\mathcal{P}}$$

Throughout this chapter we shall be using these results to study $\mathcal{B}_n^G$ with $G$ a subgroup of $D_4$. Thus the index set $J$ of Definition 3.1.9 is $[n] \times [n]$. To lighten notation, for a given subgroup $G$ of $D_4$ we make the substitutions $R_n^G = R^{\mathcal{B}_n^G}$ and $f_G = f_{\mathcal{B}_n^G}$. In Figure 3.1 we give the fundamental regions we shall use as well as the dimensions of the corresponding polytopes.

| $G \subseteq D_4$ | $R_n^G$ | $\dim \mathcal{B}_n^G = \dim \overline{\mathcal{B}_n^G}$ |
|---|---|---|
| $\{1, h\}$ | $\begin{cases} [\frac{n}{2}] \times [n], n \text{ even} \\ [\frac{n+1}{2}] \times [n], n \text{ odd} \end{cases}$ | $\begin{cases} \frac{(n-1)(n-2)}{2}, n \text{ even} \\ \frac{(n-1)^2}{2}, n \text{ odd} \end{cases}$ |
| $\{1, h, v, q^2\}$ | $\begin{cases} [\frac{n}{2}] \times [\frac{n}{2}], n \text{ even} \\ [\frac{n+1}{2}] \times [\frac{n+1}{2}], n \text{ odd} \end{cases}$ | $\begin{cases} \frac{(n-2)^2}{4}, n \text{ even} \\ \frac{(n-1)^2}{4}, n \text{ odd} \end{cases}$ |
| $\{1, q^2\}$ | $\begin{cases} [\frac{n}{2}] \times [n], n \text{ even} \\ [\frac{n+1}{2}] \times [n], n \text{ odd} \end{cases}$ | $\begin{cases} \frac{(n-1)^2+1}{2}, n \text{ even} \\ \frac{(n-1)^2}{2}, n \text{ odd} \end{cases}$ |
| $\{1, q, q^2, q^3\}$ | $\begin{cases} [\frac{n}{2}] \times [\frac{n}{2}], n \text{ even} \\ [\frac{n+1}{2}] \times [\frac{n+1}{2}], n \text{ odd} \end{cases}$ | $\begin{cases} \frac{n(n-2)}{4}, n \text{ even} \\ \frac{(n-1)^2}{4}, n \text{ odd} \end{cases}$ |
| $\{1, d\}$ | $\{(i, j) \in [n] \times [n] \mid j \leq i\}$ | $\frac{n(n-1)}{2}$ |
| $\{1, d, a, q^2\}$ | $\{(i, j) \in [n] \times [n] \mid j \leq i \leq n + 1 - j\}$ | $\begin{cases} \frac{n^2}{4}, n \text{ even} \\ \frac{(n-1)(n+1)}{4}, n \text{ odd} \end{cases}$ |
| $D_4$ | $\begin{cases} [\frac{n}{2}] \times [\frac{n}{2}], n \text{ even} \\ [\frac{n+1}{2}] \times [\frac{n+1}{2}], n \text{ odd} \end{cases}$ | $\begin{cases} \frac{n(n-2)}{8}, n \text{ even} \\ \frac{(n-1)(n+1)}{8}, n \text{ odd} \end{cases}$ |

Figure 3.1: Fundamental regions used throughout this chapter

Figure 3.2 gives a summary of the results obtained (and the open problems) that we shall present throughout this chapter. Note that in the final column of Figure 3.2, it is assumed that $n \geq 2$ (since $\mathrm{D}(\mathcal{B}_1^G) = 1$ for all $G$).

| $G \subseteq D_4$ | $\lvert\text{SMS}(n,1)^G\rvert$ | $\dim \mathcal{B}_n^G$ | $\text{vert}\mathcal{B}_n^G$ | $\lvert\text{vert}\mathcal{B}_n^G\rvert$ | $\text{D}(\mathcal{B}_n^G)$ |
|---|---|---|---|---|---|
| $\{1,h\}$ | 0 | $\begin{cases}\frac{(n-1)(n-2)}{2}, & n \text{ even}\\ \frac{(n-1)^2}{2}, & n \text{ odd}\end{cases}$ | $\checkmark$ | $\frac{n!}{2^{\lfloor \frac{n}{2} \rfloor}}$ | 2 |
| $\{1,h,v,q^2\}$ | 0 | $\begin{cases}\frac{(n-2)^2}{4}, & n \text{ even}\\ \frac{(n-1)^2}{4}, & n \text{ odd}\end{cases}$ | $\checkmark$ | $\begin{cases}\frac{n}{2}!, & n \text{ even}\\ \sum_{i=0}^{\frac{n-1}{2}} \frac{\left(\frac{n-1}{2}!\right)^2}{i!}, & n \text{ odd}\end{cases}$ | $\begin{cases}2, & n \text{ even}\\ 4, & n \text{ odd}\end{cases}$ |
| $\{1,q^2\}$ | $2^{\lfloor \frac{n}{2} \rfloor}\lfloor \frac{n}{2} \rfloor!$ | $\begin{cases}\frac{(n-1)^2+1}{2}, & n \text{ even}\\ \frac{(n-1)^2}{2}, & n \text{ odd}\end{cases}$ | $\checkmark$ | $\begin{cases}n!!, & n \text{ even}\\ \sum_{i=0}^{\frac{n-3}{2}} \frac{2^{n-2-i}\left(\frac{n-1}{2}!\right)^2}{i!} +(n-1)!!, & n \text{ odd}\end{cases}$ | $\begin{cases}1, & n \text{ even}\\ 2, & n \text{ odd}\end{cases}$ |
| $\{1,q,q^2,q^3\}$ | $\begin{cases}\frac{(\lfloor \frac{n}{2} \rfloor)!}{(\lfloor \frac{n}{4} \rfloor)!}, & \lfloor \frac{n}{2} \rfloor \text{ even}\\ 0, & \lfloor \frac{n}{2} \rfloor \text{ odd}\end{cases}$ | $\begin{cases}\frac{n(n-2)}{4}, & n \text{ even}\\ \frac{(n-1)^2}{4}, & n \text{ odd}\end{cases}$ | $\times$ | $\times$ | $\begin{cases}2, & n \text{ even}\\ 4, & n \text{ odd}\end{cases}$ |
| $\{1,d\}$ | $\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{(n-2i)!2^i i!}$ | $\frac{n(n-1)}{2}$ | $\checkmark$ | Recursion relation and generating function. | $\begin{cases}1, & n=2\\ 2, & n \geq 3\end{cases}$ |
| $\{1,d,a,q^2\}$ | $\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} 2^{\lfloor \frac{n}{2} \rfloor - 2i} \binom{\lfloor \frac{n}{2} \rfloor}{2i} \frac{(2i)!}{i!}$ | $\begin{cases}\frac{n^2}{4}, & n \text{ even}\\ \frac{(n-1)(n+1)}{4}, & n \text{ odd}\end{cases}$ | $\times$ | $\times$ | $\begin{cases}1, & n=1,2,4\\ 2, & n=3,\ n \geq 6 \text{ even}\\ 4, & n \geq 5 \text{ odd}\end{cases}$ |
| $D_4$ | 0 | $\begin{cases}\frac{(n-2)n}{8}, & n \text{ even}\\ \frac{(n-1)(n+1)}{8}, & n \text{ odd}\end{cases}$ | $\checkmark$ | Recursion relation and generating function for $n$ even. | $\begin{cases}1, & n=1\\ 2, & n=2\\ 4, & n=3,\ n \geq 4 \text{ even}\\ 8, & n \geq 5 \text{ odd}\end{cases}$ |

Figure 3.2: Table of results for $\mathcal{B}_n^G$

## 3.2 Horizontal symmetry

Recalling the second row of Figure 1.17 the horizontal symmetry class of $\mathcal{B}_n$ is:

$$\mathcal{B}_n^{\{1,h\}} = \{a \in \mathcal{B}_n \mid a_{ij} = a_{n+1-i,j} \text{ for all } i,j \in [n]\} \tag{3.10}$$

Figure 3.3 gives the set $\text{SMS}(4,2)^{\{1,h\}}$ and some cardinalities of $\text{SMS}(n,r)^{\{1,h\}}$ are given by Figure 3.4.

$$\left\{ \begin{pmatrix} 0&0&1&1\\1&1&0&0\\1&1&0&0\\0&0&1&1 \end{pmatrix}, \begin{pmatrix} 0&1&0&1\\1&0&1&0\\1&0&1&0\\0&1&0&1 \end{pmatrix}, \begin{pmatrix} 0&1&1&0\\1&0&0&1\\1&0&0&1\\0&1&1&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&1\\0&1&1&0\\0&1&1&0\\1&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&1&0\\0&1&0&1\\0&1&0&1\\1&0&1&0 \end{pmatrix}, \begin{pmatrix} 1&1&0&0\\0&0&1&1\\0&0&1&1\\1&1&0&0 \end{pmatrix} \right\}$$

Figure 3.3: Horizontally symmetric semi magic squares of size 4 and line sum 2

Note that if $n$ is even and $r$ is odd, then $\text{SMS}(n,r)^{\{1,h\}} = \emptyset$. This follows from the fact that the column sums of a horizontally symmetric integer entry matrix of even size must be even.

| | $r=0$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $n=1$ | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 0 | 1 | 0 | 1 | 0 |
| 3 | 1 | 0 | 3 | 1 | 6 | 3 |
| 4 | 1 | 0 | 6 | 0 | 19 | 0 |
| 5 | 1 | 0 | 30 | 0 | 355 | 51 |
| 6 | 1 | 0 | 90 | 0 | 2385 | 0 |
| 7 | 1 | 0 | 630 | 0 | 94500 | 0 |

Figure 3.4: $|\mathrm{SMS}(n,r)^{\{1,h\}}|$ for $n \in [7]$, $r \in [0,5]$

Also, if $n$ and $r$ are both odd with $n > r$, then $\mathrm{SMS}(n,r)^{\{1,h\}} = \emptyset$. This occurs since if $n$ and $r$ are both odd and $\mathrm{SMS}(n,r)^{\{1,h\}}$ is non empty, then each entry in the central row of any matrix in $\mathrm{SMS}(n,r)^{\{1,h\}}$ is odd (since each column sum must be odd) which implies that $r \geq n$ (since the central row contains $n$ odd positive integers which sum to $r$).

Also note that there is a bijection between $\mathrm{SMS}(n,2)^{\{1,h\}}$ and the set of all $n$-tuples in which $1,\ldots,\lfloor\frac{n}{2}\rfloor$ each appear twice, and for $n$ odd, $\frac{n+1}{2}$ appears once. In this bijection, the tuple $(i_1,\ldots,i_n)$ corresponds to the $n \times n$ matrix $a$ whose only non zero entries are $a_{i_j j} = a_{n+1-i_j,j} = 1$ for all $j \in [n]$ with $i_j \neq \frac{n+1}{2}$, and $a_{\frac{n+1}{2},j} = 2$ for $i_j = \frac{n+1}{2}$ (and $n$ odd). For example, $(2,2,1,1)$ corresponds to $\begin{pmatrix} 0&0&1&1\\ 1&1&0&0\\ 1&1&0&0\\ 0&0&1&1 \end{pmatrix}$ and $(1,1,2)$ corresponds to $\begin{pmatrix} 1&1&0\\ 0&0&2\\ 1&1&0 \end{pmatrix}$. It follows that:

$$|\mathrm{SMS}(n,2)^{\{1,h\}}| = \binom{n}{\underbrace{2,\ldots,2}_{\lfloor\frac{n}{2}\rfloor}} = \frac{n!}{2^{\lfloor\frac{n}{2}\rfloor}} \tag{3.11}$$

which is sequence A090932 of [99].

Recalling Figure 3.1 $R_{2k}^{\{1,h\}} = [k] \times [2k]$ and $R_{2k+1}^{\{1,h\}} = [k+1] \times [2k+1]$ giving:

$$\overline{\mathcal{B}_{2k}^{\{1,h\}}} = \left\{ a \in \mathbb{R}^{k\times 2k} \;\middle|\; \begin{array}{l} \bullet\ a_{ij} \geq 0 \text{ for all } i \in [k],\ j \in [2k] \\ \bullet\ \sum_{j=1}^{2k} a_{ij} = 2\sum_{i=1}^{k} a_{ij} = 1 \text{ for all } i \in [k],\ j \in [2k] \end{array} \right\}$$

and

$$\overline{\mathcal{B}_{2k+1}^{\{1,h\}}} = \left\{ a \in \mathbb{R}^{(k+1)\times(2k+1)} \;\middle|\; \begin{array}{l} \bullet\ a_{ij} \geq 0 \text{ for all } i \in [k+1],\ j \in [2k] \\ \bullet\ \sum_{j=1}^{2k+1} a_{ij} = 2\sum_{i=1}^{k} a_{ij} + a_{k+1,j} = 1 \text{ for all } i \in [k+1], \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad j \in [2k+1] \end{array} \right\}$$

Also $f_{\{1,h\}} : \mathcal{B}_{2k}^{\{1,h\}} \to \overline{\mathcal{B}_{2k}^{\{1,h\}}}$ is given by:

$$f_{\{1,h\}}\begin{pmatrix} a_{11} & \cdots & a_{1,2k} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{k,2k} \\ a_{k1} & \cdots & a_{k,2k} \\ \vdots & & \vdots \\ a_{11} & \cdots & a_{1,2k} \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1,2k} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{k,2k} \end{pmatrix}$$

and $f_{\{1,h\}} : \mathcal{B}_{2k+1}^{\{1,h\}} \to \overline{\mathcal{B}_{2k+1}^{\{1,h\}}}$ is given by:

$$f_{\{1,h\}}\begin{pmatrix} a_{11} & \cdots & a_{1,2k+1} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{k,2k+1} \\ a_{k+1,1} & \cdots & a_{k+1,2k+1} \\ a_{k1} & \cdots & a_{k,2k+1} \\ \vdots & & \vdots \\ a_{11} & \cdots & a_{1,2k+1} \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1,2k+1} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{k,2k+1} \\ a_{k+1,1} & \cdots & a_{k+1,2k+1} \end{pmatrix}$$

**Theorem 3.2.1.**

$$vert\mathcal{B}_n^{\{1,h\}} = \Pi_{\{1,h\}}(SMS(n,1)) = \frac{1}{2}SMS(n,2)^{\{1,h\}}$$

*Proof.* We shall first prove the second equality. Then by considering different parity of $n$ we shall prove the first equality giving the overall result.

From the last inclusion of (3.9) it follows that $\Pi_{\{1,h\}}(\mathrm{SMS}(n,1)) \subseteq \frac{1}{2}\mathrm{SMS}(n,2)^{\{1,h\}}$.

Now let us consider $a \in \mathrm{SMS}(n,2)^{\{1,h\}}$. As indicated before equation (3.11), $a$ corresponds to a tuple $(i_1,\ldots,i_n)$. Now define an $n \times n$ matrix $b$ whose only non zero entries are $b_{i_j,j} = b_{n+1-i_j,j'} = 1$ for all $j,j' \in [n]$ with $i_j = i_{j'} \neq \frac{n+1}{2}$ and $j < j'$, and $b_{\frac{n+1}{2},j} = 1$ for $i_j = \frac{n+1}{2}$ (and $n$ odd). For example $a = \begin{pmatrix} 0&0&1&1\\1&1&0&0\\1&1&0&0\\0&0&1&1 \end{pmatrix}$ gives $b = \begin{pmatrix} 0&0&1&0\\1&0&0&0\\0&1&0&0\\0&0&0&1 \end{pmatrix}$ and $a = \begin{pmatrix} 1&1&0\\0&0&2\\1&1&0 \end{pmatrix}$ gives $b = \begin{pmatrix} 1&0&0\\0&0&1\\0&1&0 \end{pmatrix}$. It then follows that $b \in \mathrm{SMS}(n,1)$ and that $a = b + hb \in 2\Pi_{\{1,h\}}(\mathrm{SMS}(n,1))$, giving $\Pi_{\{1,h\}}(\mathrm{SMS}(n,1)) \supseteq \frac{1}{2}\mathrm{SMS}(n,2)^{\{1,h\}}$. We have therefore shown that $\Pi_{\{1,h\}}(\mathrm{SMS}(n,1)) = \frac{1}{2}\mathrm{SMS}(n,2)^{\{1,h\}}$.

We now move onto proving that $\text{vert}\mathcal{B}_n^{\{1,h\}} = \Pi_{\{1,h\}}(\text{SMS}(n,1))$. From the second inclusion of (3.1) it follows that $\text{vert}\mathcal{B}_n^{\{1,h\}} \subseteq \Pi_{\{1,h\}}(\text{SMS}(n,1))$. We shall now show that $\text{vert}\mathcal{B}_n^{\{1,h\}} \supseteq \Pi_{\{1,h\}}(\text{SMS}(n,1))$.

- For $n = 2k$, Definition 1.2.33 gives $\overline{\mathcal{B}_{2k}^{\{1,h\}}} = \mathcal{T}\left((\underbrace{1,\ldots,1}_{k}), (\underbrace{\frac{1}{2},\ldots,\frac{1}{2}}_{2k})\right)$. By Theorem 1.2.35 we see that any element of $\overline{\mathcal{B}_{2k}^{\{1,h\}}}$ with only non zero entries $\frac{1}{2}$ is a vertex, since such an element has only one non zero entry in each column, making it impossible for there to be a cycle of non zero entries. However, for all $a \in \Pi_{\{1,h\}}(\text{SMS}(2k,1))$ and all $(i,j) \in [k] \times [2k]$, $\left(f_{\{1,h\}}(a)\right)_{ij} \in \{0, \frac{1}{2}\}$. Thus $f_{\{1,h\}}\left(\Pi_{\{1,h\}}(\text{SMS}(2k,1))\right) \subseteq \text{vert}\overline{\mathcal{B}_{2k}^{\{1,h\}}}$. By Lemma 3.1.10 we have that $\text{vert}\overline{\mathcal{B}_{2k}^{\{1,h\}}} = f_{\{1,h\}}\left(\text{vert}\mathcal{B}_{2k}^{\{1,h\}}\right)$ and so $f_{\{1,h\}}\left(\Pi_{\{1,h\}}(\text{SMS}(2k,1))\right) \subseteq f_{\{1,h\}}\left(\text{vert}\mathcal{B}_{2k}^{\{1,h\}}\right)$. Therefore since $f_{\{1,h\}}$ is bijective on $\mathcal{B}_{2k}^{\{1,h\}}$ and $\Pi_{\{1,h\}}(\text{SMS}(2k,1))$ and $\text{vert}\mathcal{B}_{2k}^{\{1,h\}}$ are both subsets of $\mathcal{B}_{2k}^{\{1,h\}}$, $\Pi_{\{1,h\}}(\text{SMS}(2k,1)) \subseteq \text{vert}\mathcal{B}_{2k}^{\{1,h\}}$ as required.

- For $n = 2k+1$ consider the affine map $\rho : \mathbb{R}^{(k+1)\times(2k+1)} \longrightarrow \mathbb{R}^{(k+1)\times(2k+1)}$ defined by:

$$(\rho(a))_{ij} = \begin{cases} a_{ij} \text{ for all } i \in [k],\ j \in [2k+1] \\ \frac{a_{ij}}{2} \text{ for all } j \in [2k+1] \text{ and } i = k+1 \end{cases} \tag{3.12}$$

  It can be checked that this map is bijective from $\overline{\mathcal{B}_{2k+1}^{\{1,h\}}}$ to $\mathcal{T}\left((\underbrace{1,\ldots,1}_{k},\frac{1}{2}), (\underbrace{\frac{1}{2},\ldots,\frac{1}{2}}_{2k+1})\right)$ thus from Lemma 1.2.12 $\rho\left(\text{vert}\overline{\mathcal{B}_{2k+1}^{\{1,h\}}}\right) = \text{vert}\mathcal{T}\left((\underbrace{1,\ldots,1}_{k},\frac{1}{2}), (\underbrace{\frac{1}{2},\ldots,\frac{1}{2}}_{2k+1})\right)$. The fact that $\Pi_{\{1,h\}}(\text{SMS}(2k+1)) \subseteq \text{vert}\mathcal{B}_{2k+1}^{\{1,h\}}$ follows as for the $n = 2k$ case giving the required result.

□

Recalling (3.9) for $G = \{1,h\}$ we have:

$$\text{SMS}(n,1)^{\{1,h\}} = \emptyset \subsetneq \text{vert}\mathcal{B}_n^{\{1,h\}} = \Pi_{\{1,h\}}(\text{SMS}(n,1)) = \frac{1}{2}\text{SMS}(n,2)^{\{1,h\}} \tag{3.13}$$

Note that $\text{vert}\mathcal{B}_n^{\{1,h\}}$ is enumerated by (3.11). Theorem 3.2.1 immediately gives:

**Corollary 3.2.2.**

$$D(\mathcal{B}_n^{\{1,h\}}) = \begin{cases} 1, & n = 1 \\ 2, & n \geq 2 \end{cases}$$

Another result follows:

**Corollary 3.2.3.** *Any matrix $a \in SMS(2k, 2t)^{\{1,h\}}$ can be written as the sum of $t$ matrices from $SMS(2k, 2)^{\{1,h\}}$.*

*Proof.* We have:

$$\text{SMS}(2k, 2t)^{\{1,h\}} = \left(2t\mathcal{B}_{2k}^{\{1,h\}}\right) \cap \mathbb{Z}^{(2k)\times(2k)}$$

This is just the integer points of the $t^{th}$ dilate of $2\mathcal{B}_{2k}^{\{1,h\}}$. From Theorem 3.2.1 $2\mathcal{B}_{2k}^{\{1,h\}}$ is a $0, 1$ polytope (since $\text{SMS}(2k, 1)^{\{1,h\}} = \emptyset$ for all $k \in \mathbb{P}$). Thus the result follows from Theorem 1.2.29. □

We believe that other decomposition theorems can be given however they do not follow straightforwardly from Theorem 1.2.29.

The main result of this section follows from Theorems 1.2.18 and Corollary 3.2.2:

**Theorem 3.2.4.** *For fixed $n \in \mathbb{P}$ there exists $H_n^{\{1,h\}}(r)$, the Ehrhart quasi-polynomial of $\mathcal{B}_n^{\{1,h\}}$ which satisfies:*

1. *$H_n^{\{1,h\}}(r)$ is a quasi-polynomial in $r$ of degree $\dim \mathcal{B}_n^{\{1,h\}}$ and period which divides 2.*

2. *$\left|SMS(n, r)^{\{1,h\}}\right| = H_n^{\{1,h\}}(r)$ for all $r \in \mathbb{N}$*

3. *$\left|SMS^{\circ}(n, r)^{\{1,h\}}\right| = (-1)^{\dim \mathcal{B}_n^{\{1,h\}}} H_n^{\{1,h\}}(-r) = H_n^{\{1,h\}}(r - n)$ for all $r \in \mathbb{P}$*

The following enumerations illustrate this theorem:

$$H_2^{\{1,h\}}(r) = \begin{cases} 1, & r \text{ even} \\ 0, & r \text{ odd} \end{cases} \tag{3.14}$$

$$H_3^{\{1,h\}}(r) = \begin{cases} \binom{\frac{r}{2}+2}{2}, & r \text{ even} \\ \binom{\frac{r-1}{2}+1}{2}, & r \text{ odd} \end{cases} \tag{3.15}$$

$$H_4^{\{1,h\}}(r) = \begin{cases} \binom{\frac{r}{2}+1}{3} + 2\binom{\frac{r}{2}+2}{3} + \binom{\frac{r}{2}+3}{3}, & r \text{ even} \\ 0, & r \text{ odd} \end{cases} \tag{3.16}$$

$$H_5^{\{1,h\}}(r) = \begin{cases} 51\binom{\frac{r}{2}+4}{8} + 161\binom{\frac{r}{2}+5}{8} + 121\binom{\frac{r}{2}+6}{8} + 21\binom{\frac{r}{2}+7}{8} + \binom{\frac{r}{2}+8}{8}, & r \text{ even} \\ \binom{\frac{r-1}{2}+2}{8} + 21\binom{\frac{r-1}{2}+3}{8} + 121\binom{\frac{r-1}{2}+4}{8} + 161\binom{\frac{r-1}{2}+5}{8} + 51\binom{\frac{r-1}{2}+6}{8}, & r \text{ odd} \end{cases} \tag{3.17}$$

Quasi-polynomials (3.15) and the non zero values of (3.16) correspond to sequences A008795 and A005900 of [99].

Note that $\mathcal{B}_n^{\{1,h\}} = \mathcal{B}_n(hI_n, I_n)$ and so the result of Brualdi discussed in Section 1.2.2 can be used. If we consider the case $n$ odd for example we get (using the notation from the end of Section 1.2.2): $\delta_r = \begin{cases} \{r, n+1-r\}, & 1 \leq r \leq \frac{n-1}{2} \\ \{\frac{n+1}{2}\}, & r = \frac{n+1}{2} \end{cases}$ and $\gamma_s = \{s\}$ for $1 \leq s \leq n$. This gives $(|\delta_1|, \ldots, |\delta_{\frac{n+1}{2}}|) = (\underbrace{2, \ldots, 2}_{\frac{n-1}{2}}, 1)$ and $(|\gamma_1, \ldots, |\gamma_n|) = (\underbrace{1, \ldots, 1}_{n})$. Also, the orbits of $\chi$ are $\{(r,s),(n+1-r,s)\}$ for all $r \in [\frac{n-1}{2}]$, $s \in [n]$ and $\{\frac{n+1}{2}, s\}$ for all $s \in [n]$ (i.e. $\delta_r \times \delta_s$ contains only a single orbit for each $r \in [\frac{n-1}{2}]$, $s \in [n]$). Thus in Brualdi's result (1.57), $e$ is a vertex of $\mathcal{T}((\underbrace{2,\ldots,2}_{\frac{n-1}{2}},1),(\underbrace{1,\ldots,1}_{n}))$ and $\omega$ is uniquely determined by $\omega_{rs} = \delta_r \times \gamma_s$. Theorem 1.2.35 gives that vert$\mathcal{T}((\underbrace{2,\ldots,2}_{\frac{n-1}{2}},1),(\underbrace{1,\ldots,1}_{n}))$ is the set of $\frac{n+1}{2} \times n$ matrices with a single one in the last row, two ones in all the other rows, a single one in each column, and all other entries zero. Using this in (1.57) now leads to an alternate proof of Theorem 3.2.1 for $n$ odd. The $n$ even case involves $\mathcal{T}((\underbrace{2,\ldots,2}_{\frac{n}{2}}),(\underbrace{1,\ldots,1}_{n}))$ and follows similarly.

## 3.3 Horizontal and vertical symmetry

We now look at the fifth row of Figure 1.17 giving the horizontal and vertical symmetry class of $\mathcal{B}_n$:

$$\mathcal{B}_n^{\{1,h,v,q^2\}} = \{a \in \mathcal{B}_n \mid a_{ij} = a_{n+1-i,j} = a_{i,n+1-j} \text{ for all } i,j \in [n]\} \tag{3.18}$$

Figure 3.5 gives the set SMS$(4,2)^{\{1,h,v,q^2\}}$ and some cardinalities of SMS$(n,r)^{\{1,h,v,q^2\}}$ are given by Figure 3.6.

$$\left\{ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \right\}$$

Figure 3.5: Horizontally and Vertically symmetric semi magic squares of size 4 and line sum 2

| | $r=0$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $n=1$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| 3 | 1 | 0 | 1 | 1 | 2 | 1 | 2 | 2 | 3 |
| 4 | 1 | 0 | 2 | 0 | 3 | 0 | 4 | 0 | 5 |
| 5 | 1 | 0 | 2 | 0 | 11 | 3 | 16 | 4 | 49 |
| 6 | 1 | 0 | 6 | 0 | 21 | 0 | 55 | 0 | 120 |
| 7 | 1 | 0 | 6 | 0 | 120 | 0 | 370 | 55 | 2901 |
| 8 | 1 | 0 | 24 | 0 | 282 | 0 | 2008 | 0 | 10147 |

Figure 3.6: $|\mathrm{SMS}(n,r)^{\{1,h,v,q^2\}}|$ for $n \in [8]$, $r \in [0,8]$

Since $\mathrm{SMS}(n,r)^{\{1,h,v,q^2\}} \subseteq \mathrm{SMS}(n,r)^{\{1,h\}}$, it follows from the results for the horizontally symmetric case that if $n$ is even and $r$ is odd then $\mathrm{SMS}(n,r)^{\{1,h,v,q^2\}} = \emptyset$, and if $n$ and $r$ are both odd with $n > r$ then $\mathrm{SMS}(n,r)^{\{1,h,v,q^2\}} = \emptyset$. Also note that $\left|\mathrm{SMS}(2k,2t)^{\{1,h,v,q^2\}}\right| = |\mathrm{SMS}(k,t)|$ and that $\left|\mathrm{SMS}(n,2)^{\{1,h,v,q^2\}}\right| = \lfloor \frac{n}{2} \rfloor!$. These results will become clear when considering the fundamental polytope: $\overline{\mathcal{B}_n^{\{1,h,v,q^2\}}}$.

Recalling Figure 3.1 $R_{2k}^{\{1,h,v,q^2\}} = [k] \times [k]$ and $R_{2k+1}^{\{1,h,v,q^2\}} = [k+1] \times [k+1]$ giving:

$$\overline{\mathcal{B}_{2k}^{\{1,h,v,q^2\}}} = \left\{ a \in \mathbb{R}^{k \times k} \;\middle|\; \begin{array}{l} \bullet\ a_{ij} \geq 0 \text{ for all } i,j \in [k] \\ \bullet\ \sum_{j=1}^{k} a_{ij} = \sum_{i=1}^{k} a_{ij} = \frac{1}{2} \text{ for all } i,j \in [k] \end{array} \right\}$$

and

$$\overline{\mathcal{B}_{2k+1}^{\{1,h,v,q^2\}}} = \left\{ a \in \mathbb{R}^{(k+1) \times (k+1)} \;\middle|\; \begin{array}{l} \bullet\ a_{ij} \geq 0 \text{ for all } i,j \in [k+1] \\ \bullet\ 2\sum_{j=1}^{k} a_{ij} + a_{i,k+1} = 2\sum_{i=1}^{k} a_{ij} + a_{k+1,j} = 1 \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for all } i,j \in [k+1] \end{array} \right\}$$

Also $f_{\{1,h,v,q^2\}} : \mathcal{B}_{2k}^{\{1,h,v,q^2\}} \rightarrow \overline{\mathcal{B}_{2k}^{\{1,h,v,q^2\}}}$ is given by:

$$f_{\{1,h,v,q^2\}} \begin{pmatrix} a_{11} & \dots & a_{1k} & a_{1k} & \dots & a_{11} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{k1} & \dots & a_{kk} & a_{kk} & \dots & a_{k1} \\ a_{k1} & \dots & a_{kk} & a_{kk} & \dots & a_{k1} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{11} & \dots & a_{1k} & a_{1k} & \dots & a_{11} \end{pmatrix} = \begin{pmatrix} a_{11} & \dots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \dots & a_{kk} \end{pmatrix}$$

and $f_{\{1,h,v,q^2\}} : \mathcal{B}_{2k+1}^{\{1,h\}} \longrightarrow \overline{\mathcal{B}_{2k+1}^{\{1,h,v,q^2\}}}$ is given by:

$$f_{\{1,h,v,q^2\}}\begin{pmatrix} a_{11} & \dots & a_{1k} & a_{1,k+1} & a_{1k} & \dots & a_{11} \\ \vdots & & \vdots & \vdots & & & \vdots \\ a_{k1} & \dots & a_{kk} & a_{k,k+1} & a_{kk} & \dots & a_{k1} \\ a_{k+1,1} & \dots & a_{k+1,k} & a_{k+1,k+1} & a_{k+1,k} & \dots & a_{k+1,1} \\ a_{k1} & \dots & a_{kk} & a_{k,k+1} & a_{kk} & \dots & a_{k1} \\ \vdots & & \vdots & \vdots & & & \vdots \\ a_{11} & \dots & a_{1k} & a_{1,k+1} & a_{1k} & \dots & a_{11} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & \dots & a_{1k} & a_{1,k+1} \\ \vdots & & \vdots & \vdots \\ a_{k1} & \dots & a_{kk} & a_{k,k+1} \\ a_{k+1,1} & \dots & a_{k+1,k} & a_{k+1,k+1} \end{pmatrix}$$

**Theorem 3.3.1.** *For $n$ even:*

$$vert\mathcal{B}_n^{\{1,h,v,q^2\}} = \frac{1}{2}SMS(n,2)^{\{1,h,v,q^2\}}$$

*For $n$ odd:*

$$vert\mathcal{B}_n^{\{1,h,v,q^2\}} = \left\{a \in \Pi_{\{1,h,v,q^2\}}\left(SMS(n,1)\right) \,\middle|\, f_{\{1,h,v,q^2\}}(a) \text{ has no non zero cycles}\right\}$$

*Proof.*

- For $n = 2k$, $\overline{\mathcal{B}_{2k}^{\{1,h,v,q^2\}}} = \mathcal{T}\left(\big(\underbrace{\tfrac{1}{2},\dots,\tfrac{1}{2}}_{k}\big),\big(\underbrace{\tfrac{1}{2},\dots,\tfrac{1}{2}}_{k}\big)\right) = \frac{1}{2}\mathcal{B}_k$, and from Theorem 1.2.24 $\text{vert}\left(\frac{1}{2}\mathcal{B}_k\right) = \frac{1}{2}\text{SMS}(k,1)$. It can then be checked that $\text{SMS}(k,1) = f_{\{1,h,v,q^2\}}\left(\text{SMS}(2k,2)^{\{1,h,v,q^2\}}\right)$, giving the required result.

- For $n = 2k+1$, the proof follows in the same way as the proof of Theorem 3.2.1. With the affine map $\rho : \mathbb{R}^{(k+1)\times(k+1)} \longrightarrow \mathbb{R}^{(k+1)\times(k+1)}$ given by:

$$(\rho(a))_{ij} = \begin{cases} a_{ij},\ i,j \in [k] \\ \frac{a_{ij}}{2},\ i \in [k],\ j = k+1 \\ \frac{a_{ij}}{2},\ i = k+1,\ j \in [k] \\ \frac{a_{ij}}{4},\ i = j = k+1 \end{cases}$$

it can be checked that $\overline{\mathcal{B}_{2k+1}^{\{1,h,v,q^2\}}}$ is affinely isomorphic with:

$$\mathcal{T}\left(\big(\underbrace{\tfrac{1}{2},\dots,\tfrac{1}{2}}_{k},\tfrac{1}{4}\big),\big(\underbrace{\tfrac{1}{2},\dots,\tfrac{1}{2}}_{k},\tfrac{1}{4}\big)\right)$$

and the result follows.

□

**Theorem 3.3.2.**

$$\left|vert\mathcal{B}_n^{\{1,h,v,q^2\}}\right| = \begin{cases} \frac{n}{2}!, \ n \ even \\ \sum_{i=0}^{\frac{n-1}{2}} \frac{\left(\frac{n-1}{2}!\right)^2}{i!}, \ n \ odd \end{cases}$$

*Proof.* For $n$ even the proof is trivial. For $n = 2k+1$, we shall prove the result by showing that we have a bijection between $\text{vert}\mathcal{T}\left(\left(\underbrace{\frac{1}{2},\dots,\frac{1}{2}}_{k},\frac{1}{4}\right),\left(\underbrace{\frac{1}{2},\dots,\frac{1}{2}}_{k},\frac{1}{4}\right)\right)$ and the following set:

$$\bigcup_{s=0}^{k}\left\{((i_1,\dots,i_s),(j_1,\dots,j_s),a') \in [k]^s \times [k]^s \times \text{SMS}(k-s,1) \,\middle|\, \begin{array}{l} \bullet\ i_l \neq i_{l'} \text{ for all } l,l' \in [s] \\ \bullet\ j_l \neq j_{l'} \text{ for all } l,l' \in [s] \end{array}\right\}$$

Consider $a \in \text{vert}\mathcal{T}\left(\left(\underbrace{\frac{1}{2},\dots,\frac{1}{2}}_{k},\frac{1}{4}\right),\left(\underbrace{\frac{1}{2},\dots,\frac{1}{2}}_{k},\frac{1}{4}\right)\right)$. Recalling that $a$ has no non zero cycles $a_{ij} \in \left\{0,\frac{1}{2},\frac{1}{4}\right\}$ for all $i,j \in [k+1]$. In particular $a_{k+1,k+1} = \frac{1}{4}$ or $a$ must have a non zero path.

If $a_{k+1,k+1} = \frac{1}{4}$, removing row and column $k+1$ gives an element of:

$$\text{vert}\mathcal{T}\left(\left(\underbrace{\frac{1}{2},\dots,\frac{1}{2}}_{k}\right),\left(\underbrace{\frac{1}{2},\dots,\frac{1}{2}}_{k}\right)\right)$$

which corresponds to an element $a'$ of $\text{SMS}(k,1)$. Thus $a$ corresponds to an element of:

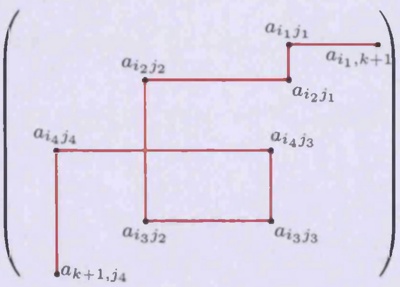
$$\bigcup_{s=0}^{k}\left\{((i_1,\dots,i_s),(j_1,\dots,j_s),a') \in [k]^s \times [k]^s \times \text{SMS}(k-s,1) \,\middle|\, \begin{array}{l} \bullet\ i_l \neq i_{l'} \text{ for all } l,l' \in [s] \\ \bullet\ j_l \neq j_{l'} \text{ for all } l,l' \in [s] \end{array}\right\}$$

with $s$ set to 0.

If $a_{k+1,k+1} \neq \frac{1}{4}$, then there exists $i_1 \in [k]$ such that $a_{i_1,k+1} = \frac{1}{4}$. Since $\sum_{j=1}^{k+1} a_{i_1 j} = \frac{1}{2}$ there exists $j_1 \in [k]$ such that $a_{i_1 j_1} = \frac{1}{4}$ (recall that the only non zero values of $a$ are $\frac{1}{4}$ or $\frac{1}{2}$). We can create a non zero path:

$$(i_1,k+1),(i_1,j_1),\dots,(i_s,j_s),(k+1,j_s)$$

with $i_l,j_l \in [k]$ and $i_l \neq i_{l'}, j_l \neq j_{l'}$ for all $l,l' \in [s]$ such that each of the corresponding values of $a$ is $\frac{1}{4}$. An example of such a path with $s=4$ is:



Removing the $s+1$ rows $i_1, i_2, \ldots, i_s, k+1$ and columns $j_1, \ldots, j_s, k+1$ gives an element of

$$\mathrm{vert}\mathcal{T}\Big(\underbrace{\Big(\frac{1}{2}, \ldots, \frac{1}{2}\Big)}_{k-s}, \underbrace{\Big(\frac{1}{2}, \ldots, \frac{1}{2}\Big)}_{k-s}\Big)$$

which corresponds to an element $a'$ of $\mathrm{SMS}(k-s, 1)$. Thus $a$ corresponds to a triplet:

$$((i_1, \ldots, i_s), (j_1, \ldots, j_s), a')$$

as required.

The enumeration result follows from the fact that for given $s$ we have $k(k-1)\ldots(k-s+1)$ choices for $(i_1, \ldots, i_s)$ and for $(j_1, \ldots, j_s)$, thus:

$$\left|\bigcup_{s=0}^{k}\left\{((i_1, \ldots, i_s), (j_1, \ldots, j_s), a') \in [k]^s \times [k]^s \times \mathrm{SMS}(k-s, 1) \,\middle|\, \begin{array}{l}\bullet\ i_l \neq i_{l'} \text{ for all } l, l' \in [s] \\ \bullet\ j_l \neq j_{l'} \text{ for all } l, l' \in [s]\end{array}\right\}\right|$$
$$= \sum_{s=0}^{k} \left(k(k-1)\ldots(k-s+1)\right)^2 (k-s)!$$

which gives the required result. □

Recalling (3.9) for $G = \{1, h, v, q^2\}$ we have:

$$\mathrm{SMS}(n,1)^{\{1,h,v,q^2\}} = \emptyset \subsetneq \mathrm{vert}\mathcal{B}_n^{\{1,h,v,q^2\}} \subsetneq \Pi_{\{1,h,v,q^2\}}\left(\mathrm{SMS}(n,1)\right) = \frac{1}{4}\mathrm{SMS}(n,4)^{\{1,h,v,q^2\}} \tag{3.19}$$

where the second inclusion is strict for $n \geq 4$. (In a similar (but far more tedious) way to the horizontal symmetry case it can be checked that $\Pi_{\{1,h,v,q^2\}}\left(\mathrm{SMS}(n,1)\right) \supseteq \frac{1}{4}\mathrm{SMS}(n,4)^{\{1,h,v,q^2\}}$.)

From Theorem 3.3.1 and the proof of Theorem 3.3.2 we have:

**Corollary 3.3.3.**

$$D(\mathcal{B}_n^{\{1,h,v,q^2\}}) = \begin{cases} 1, & n = 1 \\ 2, & n \text{ even} \\ 4, & n \geq 3 \text{ odd} \end{cases}$$

Also note that we can give a similar result to that of Corollary 3.2.3.

**Corollary 3.3.4.** *Any matrix $a \in SMS(2k, 2t)^{\{1,h,v,q^2\}}$ can be written as the sum of $t$ matrices from $SMS(2k, 2)^{\{1,h,v,q^2\}}$.*

We believe that other decomposition theorems can be given however they do not follow straightforwardly from Theorem 1.2.29.

The enumeration result follows from Theorem 1.2.18 and Corollary 3.3.3:

**Theorem 3.3.5.** *For fixed $n \in \mathbb{P}$ there exists $H_n^{\{1,h,v,q^2\}}(r)$, the Ehrhart quasi-polynomial of $\mathcal{B}_n^{\{1,h,v,q^2\}}$ which satisfies:*

1. *$H_n^{\{1,h,v,q^2\}}(r)$ is a quasi-polynomial in $r$ of degree $\dim \mathcal{B}_n^{\{1,h,v,q^2\}}$ and period which divides 2/4 for n even/odd respectively.*
2. $\left|SMS(n,r)^{\{1,h,v,q^2\}}\right| = H_n^{\{1,h,v,q^2\}}(r)$ *for all $r \in \mathbb{N}$*
3. $\left|SMS^{\circ}(n,r)^{\{1,h,v,q^2\}}\right| = (-1)^{\dim \mathcal{B}_n^{\{1,h,v,q^2\}}} H_n^{\{1,h,v,q^2\}}(-r) = H_n^{\{1,h,v,q^2\}}(r-n)$ *for all $r \in \mathbb{P}$*

This is illustrated by the following enumerations:

$$H_2^{\{1,h,v,q^2\}}(r) = \begin{cases} 1, & r \text{ even} \\ 0, & r \text{ odd} \end{cases} \tag{3.20}$$

$$H_3^{\{1,h,v,q^2\}}(r) = \begin{cases} \binom{\frac{r}{4}+1}{1}, & r = 0 \bmod 4 \\ \binom{\frac{r-1}{4}}{1} & r = 1 \bmod 4 \\ \binom{\frac{r-2}{4}+1}{1} & r = 2 \bmod 4 \\ \binom{\frac{r-3}{4}+1}{1}, & r = 3 \bmod 4 \end{cases} \tag{3.21}$$

$$H_4^{\{1,h,v,q^2\}}(r) = \begin{cases} \binom{\frac{r}{2}+1}{1}, & r \text{ even} \\ 0, & r \text{ odd} \end{cases} \tag{3.22}$$

$$H_5^{\{1,h,v,q^2\}}(r) = \begin{cases} 4\binom{\frac{r}{4}+2}{4} + 6\binom{\frac{r}{4}+3}{4} + \binom{\frac{r}{4}+4}{4}, & r = 0 \bmod 4 \\ 2\binom{\frac{r-1}{4}+1}{4} + 6\binom{\frac{r-1}{4}+2}{4} + 3\binom{\frac{r-1}{4}+3}{4}, & r = 1 \bmod 4 \\ 3\binom{\frac{r-2}{4}+2}{4} + 6\binom{\frac{r-2}{4}+3}{4} + 2\binom{\frac{r-2}{4}+4}{4}, & r = 2 \bmod 4 \\ \binom{\frac{r-3}{4}+1}{4} + 6\binom{\frac{r-3}{4}+2}{4} + 4\binom{\frac{r-3}{4}+3}{4}, & r = 3 \bmod 4 \end{cases} \tag{3.23}$$

Quasi polynomial (3.21) corresponds to sequence A008624 of [99].

## 3.4 Half turn symmetry

The third row of Figure 1.17 gives:

$$\mathcal{B}_n^{\{1,q^2\}} = \{a \in \mathcal{B}_n \mid a_{ij} = a_{n+1-i,n+1-j} \text{ for all } i,j \in [n]\} \tag{3.24}$$

Figure 3.7 gives the set $\text{SMS}(3,2)^{\{1,q^2\}}$ and Figure 3.8 gives some cardinalities of $\text{SMS}(n,r)^{\{1,q^2\}}$.

$$\left\{ \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \right\}$$

Figure 3.7: Half turn symmetric semi magic squares of size 3 and line sum 2

| | $r=0$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $n=1$ | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 2 | 3 | 4 | 5 | 6 |
| 3 | 1 | 2 | 5 | 7 | 12 | 15 |
| 4 | 1 | 8 | 34 | 104 | 259 | 560 |
| 5 | 1 | 8 | 320 | 1611 | 3987 | 13392 |
| 6 | 1 | 48 | 978 | 11264 | 87633 | 513360 |

Figure 3.8: $|SMS(n,r)^{\{1,q^2\}}|$ for $n \in [6]$, $r \in [0,5]$

As discussed in Section 1.2.2 symmetry classes of SMS$(n,r)$ are in some cases equivalent to chess problems. We recall Figure 1.35 which shows a half turn symmetric solution to the problem of the Rooks. Note also that half turn symmetric matrices are sometimes referred to as centrosymmetric matrices.

In [76, 96] the set SMS$(n,1)^{\{1,q^2\}}$ is considered and the following enumeration is given:

$$\left|\text{SMS}(n,1)^{\{1,q^2\}}\right| = (2\lfloor\frac{n}{2}\rfloor)!! = 2^{\lfloor\frac{n}{2}\rfloor}\lfloor\frac{n}{2}\rfloor! = \begin{cases} n!!,\ n \text{ even} \\ (n-1)!!,\ n \text{ odd} \end{cases} \tag{3.25}$$

which corresponds to sequence A037223 of [99]. If we consider a matrix of SMS$(n,1)^{\{1,q^2\}}$, there are $n$ choices for the location of the 1 in the first row, by symmetry this also forces the location of the 1 in the $n^{th}$ row. Recalling that we have a single 1 in every column we then have $n-2$ choices for the location of the 1 in the second row, by symmetry this also forces the location of the 1 in the $(n-1)^{th}$ row. This gives the enumeration: $\left|\text{SMS}(n,1)^{\{1,q^2\}}\right| = n!!$ for $n$ even. For $n = 2k+1$ odd we must have middle row/column:

$$\{\underbrace{0,\ldots,0}_{k},1,\underbrace{0\ldots,0}_{k}\}$$

Removing this row and column gives an element of SMS$(n-1,1)^{\{1,q^2\}}$ giving the required enumeration.

Recalling Figure 3.1: $R_{2k}^{\{1,q^2\}} = [k] \times [2k]$ and $R_{2k+1}^{\{1,q^2\}} = [k+1] \times [2k+1]$. Thus:

$$\overline{B_{2k}^{\{1,q^2\}}} = \left\{ a \in \mathbb{R}^{k\times 2k} \;\middle|\; \begin{array}{l} \bullet\ a_{ij} \geq 0 \text{ for all } i \in [k],\ j \in [2k] \\ \bullet\ \sum_{j=1}^{2k} a_{ij} = \sum_{i=1}^{k}(a_{ij} + a_{i,2k+1-j}) = 1 \text{ for all } i \in [k],\ j \in [k] \end{array} \right\}$$

and

$$\overline{\mathcal{B}_{2k+1}^{\{1,q^2\}}} = \left\{ a \in \mathbb{R}^{(k+1)\times(2k+1)} \;\middle|\; \begin{array}{l} \bullet\ a_{ij} \geq 0 \text{ for all } i \in [k+1],\ j \in [2k+1] \\ \bullet\ a_{k+1,j} = a_{k+1,2(k+1)-j} \text{ for all } j \in [k] \\ \bullet\ \sum_{j=1}^{2k+1} a_{ij} = \sum_{i=1}^{k}(a_{ij} + a_{i,2(k+1)-j}) + a_{k+1,j} = 1 \\ \qquad\qquad \text{for all } i \in [k+1],\ j \in [k+1] \end{array} \right\}$$

Also $f_{\{1,q^2\}} : \mathcal{B}_{2k}^{\{1,q^2\}} \longrightarrow \overline{\mathcal{B}_{2k}^{\{1,q^2\}}}$ is given by:

$$f_{\{1,q^2\}} \begin{pmatrix} a_{11} & \cdots & a_{1,2k} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{k,2k} \\ a_{k,2k} & \cdots & a_{k1} \\ \vdots & & \vdots \\ a_{1,2k} & \cdots & a_{11} \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1,2k} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{k,2k} \end{pmatrix}$$

and $f_{\{1,q^2\}} : \mathcal{B}_{2k+1}^{\{1,q^2\}} \longrightarrow \overline{\mathcal{B}_{2k+1}^{\{1,q^2\}}}$ is given by:

$$f_{\{1,q^2\}} \begin{pmatrix} a_{11} & \cdots & a_{1,2k+1} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{k,2k+1} \\ a_{k+1,1} & \cdots & a_{k+1,1} \\ a_{k,2k+1} & \cdots & a_{k1} \\ \vdots & & \vdots \\ a_{1,2k+1} & \cdots & a_{11} \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1,2k+1} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{k,2k+1} \\ a_{k+1,1} & \cdots & a_{k+1,1} \end{pmatrix}$$

In [43] half turn symmetric semi magic squares are considered and the following theorem is given:

**Theorem 3.4.1.** *Any matrix a in $\mathcal{B}_n^{\{1,q^2\}}$ can be written as the convex combination of elements of $SMS(n,1)^{\{1,q^2\}}$ if and only if, either, n is even, or n is odd and $a_{\frac{n+1}{2},\frac{n+1}{2}} = 1$.*

**Theorem 3.4.2.** *For n even:*

$$vert\mathcal{B}_n^{\{1,q^2\}} = SMS(n,1)^{\{1,q^2\}}$$

*For n odd:*

$$vert\mathcal{B}_n^{\{1,q^2\}} = \left\{ a \in \Pi_{\{1,q^2\}}(SMS(n,1)) \;\middle|\; \begin{array}{l} a \text{ has a unique non zero cycle} \\ \text{containing an entry in its} \\ \text{central column} \end{array} \right\} \cup SMS(n,1)^{\{1,q^2\}}$$

*Proof.* We shall prove this result by considering different parity of $n$.

- For $n = 2k$, the result is an immediate corollary of Theorem 3.4.1.

- For $n = 2k+1$, consider $a \in \overline{B_{2k+1}^{\{1,q^2\}}}$. If $a$ has a non zero cycle or an open non zero path with no entries in column $k+1$ (the central column of $a$) then $a_{k+1,k+1} = 1$ and from Theorem 3.4.1we can deduce that $a$ is not a vertex and so:

$$\text{vert}\mathcal{B}_{2k+1}^{\{1,q^2\}} \subseteq \left\{ a \in \Pi_{\{1,q^2\}}(\text{SMS}(n,1)) \;\middle|\; \begin{array}{l}\text{All the non zero cycles of } a \\ \text{contain an entry in the central} \\ \text{column of } a\end{array} \right\} \cup \text{SMS}(2k+1,1)^{\{1,q^2\}}$$

From (3.1) we have $\text{SMS}(2k+1,1)^{\{1,q^2\}} \subseteq \text{vert}\mathcal{B}_{2k+1}^{\{1,q^2\}}$. Thus we now need to show that if $a \in \Pi_{\{1,q^2\}}(\text{SMS}(n,1))$ has a single non zero path with an entry in column $k+1$ then $a$ is a vertex. To prove this we note the fact that for all $a \in \overline{\mathcal{B}_{2k+1}^{\{1,q^2\}}}$ we have:

$$a_{i,k+1} \leq \frac{1}{2} \text{ for all } i \in [k]$$

(this is because $\sum_{i=1}^{k}(a_{i,k+1} + a_{i,2(k+1)-(k+1)}) + a_{k+1,k+1} = 2\sum_{i=1}^{k} a_{i,k+1} + a_{k+1,k+1} = 1$ and so $2\sum_{i=1}^{k} a_{i,k+1} \leq 1$).

Consider $a \in \overline{\mathcal{B}_{2k+1}^{\{1,q^2\}}} \setminus \text{vert}\overline{\mathcal{B}_{2k+1}^{\{1,q^2\}}}$ with $f_{\{1,q^2\}}^{-1}(a) \in \Pi_{\{1,q^2\}}(\text{SMS}(2k+1,1))$ and assume that all the non zero cycles of $a$ contain an entry in the central column of $a$. Thus $a_{k+1,k+1} \neq 1$. Thus by Lemma 1.2.7 there exists $a^* \neq 0$ such that $a \pm a^* \in \overline{\mathcal{B}_{2k+1}^{\{1,q^2\}}}$ and there exists $i_0 \in [k]$ such that $a^*_{i_0,k+1} \neq 0$. Since $f_{\{1,q^2\}}^{-1}(a) \in \Pi_{\{1,q^2\}}(\text{SMS}(2k+1,1))$, $a_{i,k+1} \in \{0, \frac{1}{2}\}$ for all $i \in [k]$. If $a_{i_0,k+1} = 0$ then $a_{i_0,k+1} \pm a^*_{i_0,k+1} \geq 0$ gives $\pm a^*_{i_0,k+1} \geq 0$ which is impossible (since $a^*_{i_0,k+1} \neq 0$). Similarly if $a_{i_0,k+1} = \frac{1}{2}$ then $a_{i_0,k+1} \pm a^*_{i_0,k+1} \leq \frac{1}{2}$ (since as noted we have $0 \leq a_{i,k+1} \leq \frac{1}{2}$ for all $i \in [k]$) gives $\pm a^*_{i_0,k+1} \leq 0$ which is also impossible. Thus $a \in \text{vert}\overline{\mathcal{B}_{2k+1}^{\{1,q^2\}}}$ and so $f_{\{1,q^2\}}^{-1}(a) \in \text{vert}\mathcal{B}_{2k+1}^{\{1,q^2\}}$ giving:

$$\text{vert}\mathcal{B}_{2k+1}^{\{1,q^2\}} \supseteq \left\{ a \in \Pi_{\{1,q^2\}}(\text{SMS}(n,1)) \;\middle|\; \begin{array}{l}\text{All the non zero cycles of } a \\ \text{contain an entry in the central} \\ \text{column of } a\end{array} \right\} \cup \text{SMS}(2k+1,1)^{\{1,q^2\}}$$

The fact that

$$\left\{ a \in \Pi_{\{1,q^2\}}(\text{SMS}(n,1)) \;\middle|\; \begin{array}{l}a \text{ has a unique non zero cycle} \\ \text{containing an entry in its} \\ \text{central column}\end{array} \right\} = \left\{ a \in \Pi_{\{1,q^2\}}(\text{SMS}(n,1)) \;\middle|\; \begin{array}{l}\text{All the non zero cycles of } a \\ \text{contain an entry in the central} \\ \text{column of } a\end{array} \right\}$$

follows from the fact that all the values on the non zero cycle of $a$ are $\frac{1}{2}$ (since $a \in \Pi_{\{1,q^2\}}$ (SMS$(2k+1,1)$)), which gives the required result.

□

**Theorem 3.4.3.**

$$\left|vert\mathcal{B}_n^{\{1,q^2\}}\right| = \begin{cases} n!!, & n \text{ even} \\ (n-1)!! + \sum_{i=0}^{\frac{n-3}{2}} \frac{2^{n-2-i}\left(\frac{n-1}{2}!\right)^2}{i!}, & n \text{ odd} \end{cases}$$

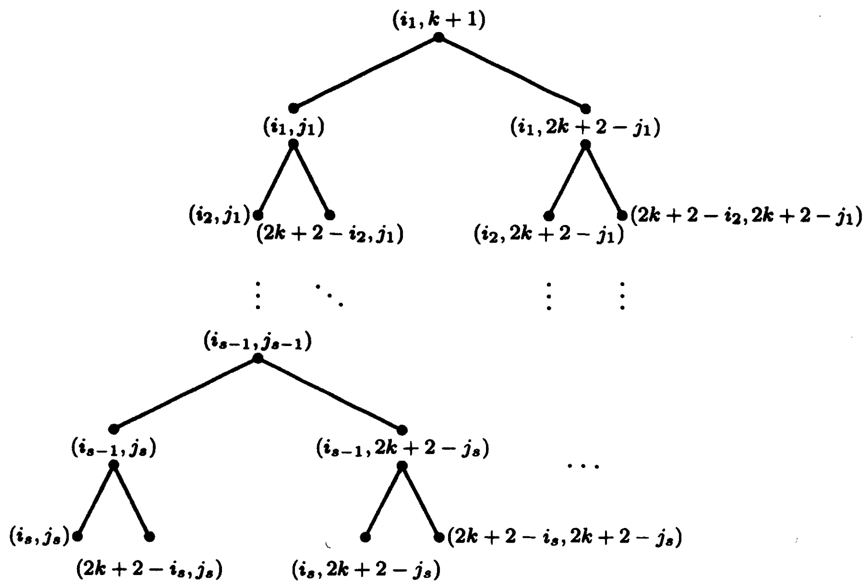
*Proof.* For $n$ even the result follows from (3.25). For $n = 2k+1$, we shall prove the result using a particular map between vert$\mathcal{B}_n^{\{1,q^2\}}$ and the set:

$$\bigcup_{s=0}^{k} \left\{ ((i_1,\dots,i_s),(j_1,\dots,j_s),a') \in [k]^s \times [k]^s \times \text{SMS}(2(k-s),1)^{\{1,q^2\}} \;\middle|\; \begin{array}{l} \bullet\ i_l \neq i_{l'} \text{ for all } l,l' \in [s] \\ \bullet\ j_l \neq j_{l'} \text{ for all } l,l' \in [s] \end{array} \right\}$$

For given $s \in [k]$ we take an element of

$$\left\{ ((i_1,\dots,i_s),(j_1,\dots,j_s),a') \in [k]^s \times [k]^s \times \text{SMS}(2(k-s),1)^{\{1,q^2\}} \;\middle|\; \begin{array}{l} \bullet\ i_l \neq i_{l'} \text{ for all } l,l' \in [s] \\ \bullet\ j_l \neq j_{l'} \text{ for all } l,l' \in [s] \end{array} \right\}$$

and create the following full binary tree:




(3.26)

Note that this full binary tree has height $2s-1$ thus we have $2^{2s-1}$ rooted paths of length $s$. We shall use this tree to create a map from vert$\mathcal{B}_{2k+1}^{\{1,q^2\}}$ and the set:

$$\bigcup_{s=0}^{k} \left\{ ((i_1,\dots,i_s),(j_1,\dots,j_s),a') \in [k]^s \times [k]^s \times \text{SMS}(2(k-s),1)^{\{1,q^2\}} \;\middle|\; \begin{array}{l} \bullet\ i_l \neq i_{l'} \text{ for all } l,l' \in [s] \\ \bullet\ j_l \neq j_{l'} \text{ for all } l,l' \in [s] \end{array} \right\}$$

Consider $a \in \text{vert}\mathcal{B}_{2k+1}^{\{1,q^2\}}$. If $a \in \text{SMS}(2k+1,1)^{\{1,q^2\}}$ then $a$ corresponds to an element $a'$ of $\text{SMS}(2k,1)^{\{1,q^2\}}$ (since $a_{k+1,k+1} = 1$) which in turn corresponds to an element of

$$\bigcup_{s=0}^{k}\left\{((i_1,\dots,i_s),(j_1,\dots,j_s),a') \in [k]^s \times [k]^s \times \text{SMS}(2(k-s),1)^{\{1,q^2\}} \,\middle|\, \begin{array}{l}\bullet\ i_l \neq i_{l'} \text{ for all } l,l' \in [s]\\ \bullet\ j_l \neq j_{l'} \text{ for all } l,l' \in [s]\end{array}\right\}$$

with $s$ set to 0 (from (3.25) we have $(2k)!!$ such vertices).

Assume that $a \notin \text{SMS}(2k+1,1)^{\{1,q^2\}}$. Thus from Theorem 3.4.2 $a_{ij} \in \left\{0,\frac{1}{2},1\right\}$ for all $i,j \in [2k+1]$ and there exists $i'_1 \in [k]$ such that $a_{i'_1,k+1} = \frac{1}{2}$. Since $a_{i'_1,k+1} = \frac{1}{2}$, by symmetry $a_{2k+2-i'_1,k+1} = \frac{1}{2}$ and so $a_{k+1,k+1} = 0$. Thus there exists $j'_s \in [k]$ such that $a_{k+1,j'_s} = \frac{1}{2}$. We can create a non zero path:

$$(i'_1,k+1),(i'_1,j'_1),\dots,(i'_s,j'_s),(k+1,j'_s)$$

with $i'_l,j'_l \in [2k+1]\setminus\{k+1\}$ and $i'_l \neq i'_{l'}$, $j'_l \neq j'_{l'}$ for all $l,l' \in [s]$, such that each of the corresponding values of $a$ is $\frac{1}{2}$. By symmetry such a path defines a non zero cycle of $a$. From Theorem 3.4.2, removing the $2s+1$ rows $i'_1,\dots,i'_s,k+1,2k+2-i'_s,\dots,2k+2-i'_1$ and columns $j'_1,\dots,j'_s,k+1,2k+2-j'_s,\dots,2k+2-j'_1$ gives an element $a'$ of $\text{SMS}(2k+1-(2s+1),1)^{\{1,q^2\}} = \text{SMS}(2(k-s),1)^{\{1,q^2\}}$. The mapping from $\text{vert}\mathcal{B}_{2k+1}^{\{1,q^2\}}$ to

$$\bigcup_{s=0}^{k}\left\{((i_1,\dots,i_s),(j_1,\dots,j_s),a') \in [k]^s \times [k]^s \times \text{SMS}(2(k-s),1)^{\{1,q^2\}} \,\middle|\, \begin{array}{l}\bullet\ i_l \neq i_{l'} \text{ for all } l,l' \in [s]\\ \bullet\ j_l \neq j_{l'} \text{ for all } l,l' \in [s]\end{array}\right\}$$

then follows since every element $a$ of $\text{vert}\mathcal{B}_{2k+1}^{\{1,q^2\}}$ corresponds to an element $a'$ of $\text{SMS}(2k+1,1)^{\{1,q^2\}}$ and a non zero path:

$$(i'_1,k+1),(i'_1,j'_1),\dots,(i'_s,j'_s),(k+1,j'_s)$$

for given $s \in [k]$. This non zero path corresponds to one of the $2^{2s-1}$ rooted paths of length $s$ of the full binary tree (3.26). Thus:

$$\left|\text{vert}\mathcal{B}_{2k+1}^{\{1,q^2\}}\right| = (2k)!!+\sum_{s=1}^{k}(k(k-1)\dots(k-s+1))^2\,2^{2s-1}2^{k-s}(k-s)! = (2k)!!+\sum_{i=0}^{k-1}\frac{2^{2k-i-1}(k!)^2}{i!}$$

as required. □

Note that Theorems 3.4.2 and 3.4.3 can be obtained using Brualdi's method [27] as given in (1.57), since $\mathcal{B}_n^{\{1,q^2\}} = \mathcal{B}_n(hI_n,hI_n)$. In this case, the transportation polytope which is used in (1.57) is $\mathcal{T}((\underbrace{2,\ldots,2}_{\frac{n}{2}}),(\underbrace{2,\ldots,2}_{\frac{n}{2}}))$ for $n$ even and $\mathcal{T}((\underbrace{2,\ldots,2}_{\frac{n-1}{2}},1),(\underbrace{2,\ldots,2}_{\frac{n-1}{2}},1))$ for $n$ odd. Theorem 3.4.3 can then be obtained using Theorems 3.3.1 and 3.3.2.

We refine (3.9) for $G=\{1,q^2\}$ to give:

- For $n=2k$:

$$\mathrm{SMS}(2k)^{\{1,q^2\}} = \mathrm{vert}\mathcal{B}_{2k}^{\{1,q^2\}} \subsetneq \Pi_{\{1,q^2\}}\left(\mathrm{SMS}(2k,1)\right) \subseteq \frac{1}{2}\mathrm{SMS}(2k,2)^{\{1,q^2\}} \tag{3.27}$$

- For $n=2k+1$:

$$\mathrm{SMS}(2k+1)^{\{1,q^2\}} \subsetneq \mathrm{vert}\mathcal{B}_{2k+1}^{\{1,q^2\}} \subsetneq \Pi_{\{1,q^2\}}\left(\mathrm{SMS}(2k+1,1)\right) \subseteq \frac{1}{2}\mathrm{SMS}(2k+1,2)^{\{1,q^2\}} \tag{3.28}$$

Theorem 3.4.2 immediately gives:

**Corollary 3.4.4.**

$$D(\mathcal{B}_n^{\{1,q^2\}}) = \begin{cases} 1, & n=1 \text{ or } n \text{ even} \\ 2, & n\geq 3 \text{ odd} \end{cases}$$

In [43] the following theorem is given:

**Theorem 3.4.5.** *For $n$ even any matrix in $SMS(n,r)^{\{1,q^2\}}$ can be written as the sum of $r$ half turn symmetric permutation matrices.*

(This is confirmed by the fact that Theorem 3.4.2 identifies $\mathcal{B}_{2k}^{\{1,q^2\}}$ as a $0,1$ polytope.)

From Theorem 1.2.18 and corollary 3.4.4 we have:

**Theorem 3.4.6.** *For fixed $n\in\mathbb{P}$ there exists $H_n^{\{1,q^2\}}(r)$, the Ehrhart quasi-polynomial of $\mathcal{B}_n^{\{1,q^2\}}$ which satisfies:*

1. *$H_n^{\{1,q^2\}}(r)$ is a quasi-polynomial in $r$ of degree $\dim\mathcal{B}_n^{\{1,q^2\}}$ and period which divides 1/2 for $n$ even/odd respectively.*

2. $\left|SMS(n,r)^{\{1,q^2\}}\right| = H_n^{\{1,q^2\}}(r)$ *for all* $r\in\mathbb{N}$

3. $\left|SMS^{o}(n,r)^{\{1,q^2\}}\right| = (-1)^{\dim\mathcal{B}_n^{\{1,q^2\}}} H_n^{\{1,q^2\}}(-r) = H_n^{\{1,q^2\}}(r-n)$ *for all* $r\in\mathbb{P}$

(Of course a quasi-polynomial with period dividing 1, is a polynomial). Here we list some quasi-polynomials illustrating this theorem:

$$H_2^{\{1,q^2\}} = \binom{r+1}{1} \tag{3.29}$$

$$H_3^{\{1,q^2\}}(r) = \begin{cases} 2\binom{\frac{r}{2}+1}{2} + \binom{\frac{r}{2}+2}{2}, & r \text{ even} \\ \binom{\frac{r-1}{2}+1}{2} + 2\binom{\frac{r-1}{2}+2}{2}, & r \text{ odd} \end{cases} \tag{3.30}$$

$$H_4^{\{1,q^2\}}(r) = \binom{r+3}{5} + 2\binom{r+4}{5} + \binom{r+5}{5} \tag{3.31}$$

Quasi-polynomials (3.30) and (3.31) correspond to sequences A001318, A033455 of [99].

## 3.5 Quarter turn symmetry

The sixth row of Figure 1.17 gives:

$$\mathcal{B}_n^{\{1,q,q^2,q^3\}} = \{a \in \mathcal{B}_n \mid a_{ij} = a_{j,n+1-i} \text{ for all } i,j \in [n]\} \tag{3.32}$$

Figure 3.9 gives the set $\mathrm{SMS}(4,2)^{\{1,q,q^2,q^3\}}$ and Figure 3.10 gives some cardinalities of $\mathrm{SMS}(n,r)^{\{1,q,q^2,q^3\}}$

$$\left\{ \begin{pmatrix} 0&0&2&0\\ 2&0&0&0\\ 0&0&0&2\\ 0&2&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&1&0\\ 1&0&0&1\\ 1&0&0&1\\ 0&1&1&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&1\\ 0&1&1&0\\ 0&1&1&0\\ 1&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&2&0&0\\ 0&0&0&2\\ 2&0&0&0\\ 0&0&2&0 \end{pmatrix} \right\}$$

Figure 3.9: Quarter turn symmetric semi magic squares of size 4 and line sum 2

| | $r=0$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $n=1$ | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 0 | 1 | 0 | 1 | 0 |
| 3 | 1 | 0 | 1 | 1 | 2 | 1 |
| 4 | 1 | 2 | 4 | 6 | 9 | 12 |
| 5 | 1 | 2 | 4 | 6 | 23 | 33 |
| 6 | 1 | 0 | 18 | 0 | 135 | 0 |

Figure 3.10: $|SMS(n,r)^{\{1,q,q^2,q^3\}}|$ for $n \in [7]$, $r \in [0,5]$

In [76, 96] the following enumeration is given:

$$\left|\mathrm{SMS}(2k,1)^{\{1,q,q^2,q^3\}}\right| = \begin{cases} \frac{(k)!}{(\frac{k}{2})!}, & k \text{ even} \\ 0, & k \text{ odd} \end{cases} \tag{3.33}$$

Note that $\frac{(k)!}{(\frac{k}{2})!} = (2k-2)(2k-6)\dots 2$. If we consider $a \in \text{SMS}(2k,1)^{\{1,q,q^2,q^3\}}$, $a$ can have no non zero entries in either diagonal (i.e. $a_{ii} = a_{i,n+1-i} = 0$ for all $i \in [2k]$). Thus there are $2k-2$ choices for the location of the 1 in the first row. By symmetry this also forces the location of the 1 in three other rows which leads to $2k-6$ choices for the location of the 1 in the next row needing a 1. This gives the required enumeration for $k$ even. For $k$ odd it can be checked that $\text{SMS}(2k,1)^{\{1,q,q^2,q^3\}} = \emptyset$ (since $a_{ii} = a_{i,n+1-i} = 0$ for all $i \in [2k]$).

We have a simple bijection between $SMS(2k,1)^{\{1,q,q^2,q^3\}}$ and $SMS(2k+1,1)^{\{1,q,q^2,q^3\}}$. Indeed if $a \in SMS(2k+1,1)^{\{1,q,q^2,q^3\}}$ then $a$ must have middle row/column:

$$\{\underbrace{0,\dots,0}_{k},1,\underbrace{0\dots,0}_{k}\}$$

Removing this row and column gives the required bijection. And so we have:

$$\left|\text{SMS}(2k,1)^{\{1,q,q^2,q^3\}}\right| = \left|\text{SMS}(2k+1,1)^{\{1,q,q^2,q^3\}}\right| = \begin{cases} \frac{(k)!}{(\frac{k}{2})!}, & k \text{ even} \\ 0, & k \text{ odd} \end{cases} \tag{3.34}$$

corresponding to sequence A122670 of [99]. Combining (3.33) and (3.34) gives:

$$\left|\text{SMS}(n,1)^{\{1,q,q^2,q^3\}}\right| = \begin{cases} \frac{\lfloor\frac{n}{2}\rfloor!}{\lfloor\frac{n}{4}\rfloor!}, & n = 0 \text{ or } 1 \bmod 4 \\ 0, & n = 2 \text{ or } 3 \bmod 4 \end{cases} \tag{3.35}$$

Recalling Figure 3.1: $R_{2k}^{\{1,q,q^2,q^3\}} = [k] \times [k]$ and $R_{2k+1}^{\{1,q,q^2,q^3\}} = [k+1] \times [k+1]$. Thus:

$$\overline{B_{2k}^{\{1,q,q^2,q^3\}}} = \left\{ a \in \mathbb{R}^{k\times k} \;\middle|\; \begin{array}{l} \bullet\ a_{ij} \geq 0 \text{ for all } i,j \in [k] \\ \bullet\ \sum_{j=1}^{k}(a_{ij}+a_{ji}) = 1 \text{ for all } i \in [k] \end{array} \right\}$$

and

$$\overline{B_{2k+1}^{\{1,q,q^2,q^3\}}} = \left\{ a \in \mathbb{R}^{(k+1)\times(k+1)} \;\middle|\; \begin{array}{l} \bullet\ a_{ij} \geq 0 \text{ for all } i,j \in [k+1] \\ \bullet\ a_{i,k+1} = a_{k+1,i} \text{ for all } i \in [k] \\ \bullet\ \sum_{j=1}^{k}(a_{ij}+a_{ji}) + a_{i,k+1} \text{ for all } i \in [k+1] \end{array} \right\}$$

Also $f_{\{1,q,q^2,q^3\}} : B_{2k}^{\{1,q,q^2,q^3\}} \longrightarrow \overline{B_{2k}^{\{1,q,q^2,q^3\}}}$ is given by:

$$f_{\{1,q,q^2,q^3\}} \begin{pmatrix} a_{11} & \dots & a_{1k} & a_{k1} & \dots & a_{11} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{k1} & \dots & a_{kk} & a_{kk} & \dots & a_{1k} \\ a_{1k} & \dots & a_{kk} & a_{kk} & \dots & a_{k1} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{11} & \dots & a_{k1} & a_{1k} & \dots & a_{11} \end{pmatrix} = \begin{pmatrix} a_{11} & \dots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \dots & a_{kk} \end{pmatrix}$$

and $f_{\{1,q,q^2,q^3\}} : \mathcal{B}_{2k+1}^{\{1,q,q^2,q^3\}} \longrightarrow \overline{\mathcal{B}_{2k+1}^{\{1,q,q^2,q^3\}}}$ is given by:

$$f_{\{1,q,q^2,q^3\}} \begin{pmatrix} a_{11} & \dots & a_{1k} & a_{1,k+1} & a_{k1} & \dots & a_{11} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{k1} & \dots & a_{kk} & a_{k,k+1} & a_{kk} & \dots & a_{1k} \\ a_{1,k+1} & \dots & a_{k,k+1} & a_{k+1,k+1} & a_{k,k+1} & \dots & a_{1,k+1} \\ a_{1k} & \dots & a_{kk} & a_{k,k+1} & a_{kk} & \dots & a_{k1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{11} & \dots & a_{k1} & a_{1,k+1} & a_{1k} & \dots & a_{11} \end{pmatrix} = \begin{pmatrix} a_{11} & \dots & a_{1k} & a_{1,k+1} \\ \vdots & & \vdots & \vdots \\ a_{k1} & \dots & a_{kk} & a_{k,k+1} \\ a_{1,k+1} & \dots & a_{k,k+1} & a_{k+1,k+1} \end{pmatrix}$$

(the fact that $a_{i,k+1} = a_{k+1,i}$ is evident from (3.32) and the definition of $\overline{\mathcal{B}_{2k+1}^{\{1,q,q^2,q^3\}}}$). Identifying the vertices for this symmetry class is omitted in this thesis. However, we give the following result:

**Theorem 3.5.1.**

$$D\left(\mathcal{B}_n^{\{1,q,q^2,q^3\}}\right) = \begin{cases} 1, & n = 1 \\ 2, & n \textit{ even} \\ 4, & n \geq 3 \textit{ odd} \end{cases}$$

*Proof.*

- For $n = 2k$, the proof involves a graphical approach and is omitted.
- For $n = 2k + 1$ it can be checked using 1.2.7 that:

$$\text{vert}\mathcal{B}_3^{\{1,q,q^2,q^3\}} = \left\{ \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \right\}$$

Taking $a^\dagger = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$ we pad $a^\dagger$ to construct $a \in \overline{\mathcal{B}_{3+2k}^{\{1,q,q^2,q^3\}}}$ with entries:

$$a_{ij} = \begin{cases} \frac{1}{2}, & i = j \in [k] \\ a^\dagger_{i-k,j-k}, & i,j \in [k+1, k+2] \\ 0, & \text{otherwise} \end{cases}$$

Thus $a$ is of the form:

$$a = \begin{pmatrix} \frac{1}{2} & \dots & 0 & \\ \vdots & \ddots & \vdots & \\ 0 & \dots & \frac{1}{2} & \\ & & & (a^\dagger) \end{pmatrix}$$

We can actually write $a$ very concisely using direct sums: $a = \left(\frac{1}{2}I_k\right) \oplus a^{\dagger}$. It is then straightforward to check that $a \in \text{vert}\overline{\mathcal{B}_{2k+3}^{\{1,q,q^2,q^3\}}}$. Thus $\text{D}\left(\mathcal{B}_{3+2k}^{\{1,q,q^2,q^3\}}\right) \geq 4$ which gives the required result.

□

We refine (3.9) for $G = \{1, q, q^2, q^3\}$ to give:

$$\text{SMS}(n,1)^{\{1,q,q^2,q^3\}} \subsetneq \text{vert}\mathcal{B}_n^{\{1,q,q^2,q^3\}} \subsetneq \Pi_{\{1,q,q^2,q^3\}}(\text{SMS}(n,1)) \subseteq \frac{1}{4}\left(\text{SMS}(n,4)\right)^{\{1,q,q^2,q^3\}} \tag{3.36}$$

From Theorems 1.2.18 and 3.5.1 we have:

**Theorem 3.5.2.** *For fixed $n \in \mathbb{P}$ there exists $H_n^{\{1,q,q^2,q^3\}}(r)$, the Ehrhart quasi-polynomial of $\mathcal{B}_n^{\{1,q,q^2,q^3\}}$ which satisfies:*

1. *$H_n^{\{1,q,q^2,q^3\}}(r)$ is a quasi-polynomial in $r$ of degree $\dim \mathcal{B}_n^{\{1,q,q^2,q^3\}}$ and period which divides 2/4 for $n$ even/odd respectively.*
2. *$\left|SMS^{\{1,q,q^2,q^3\}}(n,r)\right| = H_n^{\{1,q,q^2,q^3\}}(r)$ for all $r \in \mathbb{N}$*
3. *$\left|SMS^{o\{1,q,q^2,q^3\}}(n,r)\right| = (-1)^{\dim \mathcal{B}_n^{\{1,q,q^2,q^3\}}} H_n^{\{1,q,q^2,q^3\}}(-r) = H_n^{\{1,q,q^2,q^3\}}(r-n)$ for all $r \in \mathbb{P}$*

Here we list some quasi-polynomials illustrating this theorem.

$$H_2^{\{1,q,q^2,q^3\}}(r) = \begin{cases} 1, & r \text{ even} \\ 0, & r \text{ odd} \end{cases} \tag{3.37}$$

$$H_3^{\{1,q,q^2,q^3\}}(r) = \begin{cases} \binom{\frac{r}{4}+1}{1}, & r = 0 \bmod 4 \\ \binom{\frac{r-1}{4}}{1}, & r = 1 \bmod 4 \\ \binom{\frac{r-2}{4}+1}{1}, & r = 2 \bmod 4 \\ \binom{\frac{r-3}{4}+1}{1}, & r = 3 \bmod 4 \end{cases} \tag{3.38}$$

$$H_4^{\{1,q,q^2,q^3\}}(r) = \begin{cases} \binom{\frac{r}{2}+1}{2} + \binom{\frac{r}{2}+2}{2}, & r \text{ even} \\ 2\binom{\frac{r-1}{2}+2}{2}, & r \text{ odd} \end{cases} \tag{3.39}$$

Quasi-polynomials (3.38) and (3.39) correspond to sequences A008624 and A002620 of [99].

## 3.6 Diagonal symmetry

The fourth row of Figure 1.17 gives:

$$\mathcal{B}_n^{\{1,d\}} = \{a \in \mathcal{B}_n \mid a_{ij} = a_{ji} \text{ for all } i, j \in [n]\} \tag{3.40}$$

Recalling Theorem 1.2.31 we have already considered the diagonal symmetry class of $\mathcal{B}_n$ in Section 1.2.2. For completeness we give some cardinalities of $\mathrm{SMS}(n,r)^{\{1,d\}}$ in Figure 3.11 and a diagonally symmetric solution to the problem of the Rooks in Figure 3.12. We recall (1.56):

$$|\mathrm{SMS}(n,1)^{\{1,d\}}| = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{(n-2k)!2^k k!}$$

| | $r=0$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $n=1$ | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 2 | 3 | 4 | 5 | 6 |
| 3 | 1 | 4 | 11 | 23 | 42 | 69 |
| 4 | 1 | 10 | 56 | 214 | 641 | 1620 |
| 5 | 1 | 26 | 348 | 2698 | 14751 | 62781 |
| 6 | 1 | 76 | 2578 | 44288 | 478711 | 3710272 |

Figure 3.11: $|SMS(n,r)^{\{1,d\}}|$ for $n \in [6]$, $r \in [0,5]$

Figure 3.12: A diagonally symmetric solution to the problem of the Rooks

Recalling Figure 3.1: $R_n^{\{1,d\}} = \{(i,j) \in [n] \times [n] \mid j \leq i \text{ for all } i,j \in [n]\}$ giving:

$$\overline{\mathcal{B}_n^{\{1,d\}}} = \left\{ a = \begin{pmatrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ \vdots & & \ddots & \\ a_{n1} & \dots & \dots & a_{nn} \end{pmatrix} \in \mathbb{R}^{\frac{n(n+1)}{2}} \;\middle|\; \begin{array}{l} \bullet\ a_{ij} \geq 0 \text{ for all } (i,j) \in R_n^{\{1,d\}} \\ \bullet\ \sum_{j=1}^{i-1} a_{ij} + \sum_{j=i}^{n} a_{ji} = 1 \text{ for all } i \in [n] \end{array} \right\}$$

where $f_{\{1,d\}} : \mathcal{B}_n^{\{1,d\}} \to \overline{\mathcal{B}_n^{\{1,d\}}}$ is given by:

$$f_{\{1,d\}}\begin{pmatrix} a_{11} & a_{21} & \cdots & \cdots & a_{n1} \\ a_{21} & a_{22} & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & a_{n,n-1} \\ a_{n1} & \cdots & \cdots & a_{n,n-1} & a_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} & & & & \\ a_{21} & a_{22} & & & \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ a_{n1} & \cdots & \cdots & a_{n,n-1} & a_{nn} \end{pmatrix}$$

As stated there is no need to consider the fundamental polytope for this symmetry class as Theorem 1.2.31 identifies the set vert$\mathcal{B}_n^{\{1,d\}}$. This leads to the following corollary:

**Corollary 3.6.1.**

$$D\left(\mathcal{B}_n^{\{1,d\}}\right) = \begin{cases} 1, \; n = 1,2 \\ 2, \; n \geq 3 \end{cases}$$

Recalling Theorem 1.2.31, the enumeration of vert$\mathcal{B}_n^{\{1,d\}}$ is given by (1.55).

We refine (3.9) for $G = \{1, d\}$ to give:

$$\text{SMS}(n,1)^{\{1,d\}} \subsetneq \text{vert}\mathcal{B}_n^{\{1,d\}} \subsetneq \Pi_{\{1,d\}}(\text{SMS}(n,1)) \subsetneq \frac{1}{2}\text{SMS}(n,2)^{\{1,d\}} \tag{3.41}$$

Note that enumeration of $(\text{SMS}(n,2))^{\{1,d\}}$ is given by Gupta in [61]:

$$\begin{aligned} &(1-x)^{\frac{-1}{2}} e^{\frac{x^2}{4}+\frac{x}{2(1-x)}} = \textstyle\sum_{n=0}^{\infty} \left|\text{SMS}(n,2)^{\{1,d\}}\right| \frac{x^n}{n!} \\ &\left|\text{SMS}(n,2)^{\{1,d\}}\right| = (2n-1)\left|\text{SMS}(n-1,2)^{\{1,d\}}\right| - 2\tbinom{n-1}{2}\left|\text{SMS}(n-2,2)^{\{1,d\}}\right| \\ &\qquad\qquad -2\tbinom{n-1}{2}\left|\text{SMS}(n-3,2)^{\{1,d\}}\right| + 3\tbinom{n-1}{3}\left|\text{SMS}(n-4,2)^{\{1,d\}}\right| \\ &\qquad \left(\text{with } \left|\text{SMS}(1,2)^{\{1,d\}}\right|, \ldots, \left|\text{SMS}(4,2)^{\{1,d\}}\right| = 1, 3, 11, 56\right) \end{aligned} \tag{3.42}$$

Using a similar argument to that used to obtain (1.55) we obtain:

$$\left|\text{SMS}(n,2)^{\{1,d\}}\right| = \sum \frac{n! \prod_{i=3,4,5,\ldots} \left(\frac{i+1}{i}\right)^{c_i}}{2^{c_3+c_4+c_5+\ldots} \prod_{i=1,2,3,\ldots} c_i!} \tag{3.43}$$

where the sum is over all non negative integers $c_1, c_2, c_3, c_4, c_5 \ldots$ satisfying $c_1 + 2c_2 + 3c_3 + 4c_4 + 5c_5 + \cdots = n$.

From Theorems 1.2.18 and Corollary 3.6.1 we have:

**Theorem 3.6.2.** *For fixed $n \in \mathbb{P}$ (with $n \geq 3$) there exists $H_n^{\{1,d\}}(r)$, the Ehrhart quasi-polynomial of $\mathcal{B}_n^{\{1,d\}}$ which satisfies:*

1. *$H_n^{\{1,d\}}(r)$ is a quasi-polynomial in $r$ of degree $\frac{n(n-1)}{2}$ and period which divides 2.*

2. *$\left|SMS^{\{1,d\}}(n,r)\right| = H_n^{\{1,d\}}(r)$ for all $r \in \mathbb{N}$*

3. *$\left|SMS^{o\{1,d\}}(n,r)\right| = (-1)^{\frac{n(n-1)}{2}} H_n^{\{1,d\}}(-r) = H_n^{\{1,d\}}(r-n)$ for all $r \in \mathbb{P}$*

This is illustrated by the following enumerations:

$$H_2^{\{1,d\}}(r) = \binom{r+1}{1} \tag{3.44}$$

$$H_3^{\{1,d\}}(r) = \begin{cases} 4\binom{\frac{r}{2}+1}{3} + 7\binom{\frac{r}{2}+2}{3} + \binom{\frac{r}{2}+3}{3}, & r \text{ even} \\ \binom{\frac{r-1}{2}+1}{3} + 7\binom{\frac{r-1}{2}+2}{3} + 4\binom{\frac{r-1}{2}+3}{3}, & r \text{ odd} \end{cases} \tag{3.45}$$

$$H_4^{\{1,d\}}(r) = \begin{cases} \binom{\frac{r}{2}+1}{6} + 49\binom{\frac{r}{2}+2}{6} + 270\binom{\frac{r}{2}+3}{6} + 270\binom{\frac{r}{2}+4}{6} + 49\binom{\frac{r}{2}+5}{6} + \binom{\frac{r}{2}+6}{6}, & r \text{ even} \\ 10\binom{\frac{r-1}{2}+2}{6} + 144\binom{\frac{r-1}{2}+3}{6} + 322\binom{\frac{r-1}{2}+4}{6} + 144\binom{\frac{r-1}{2}+5}{6} + 10\binom{\frac{r-1}{2}+6}{6}, & r \text{ odd} \end{cases} \tag{3.46}$$

$$H_5^{\{1,d\}}(r) = \begin{cases} 26\binom{2+\frac{r}{2}}{10} + 2412\binom{3+\frac{r}{2}}{10} + 34533\binom{4+\frac{r}{2}}{10} + 134989\binom{5+\frac{r}{2}}{10} + 175880\binom{6+\frac{r}{2}}{10} + \\ \quad 78908\binom{7+\frac{r}{2}}{10} + 10978\binom{8+\frac{r}{2}}{10} + 337\binom{9+\frac{r}{2}}{10} + \binom{10+\frac{r}{2}}{10}, & r \text{ even} \\ \binom{2+\frac{r-1}{2}}{10} + 337\binom{3+\frac{r-1}{2}}{10} + 10978\binom{4+\frac{r-1}{2}}{10} + 78908\binom{5+\frac{r-1}{2}}{10} + 175880\binom{6+\frac{r-1}{2}}{10} + \\ \quad 134989\binom{7+\frac{r-1}{2}}{10} + 34533\binom{8+\frac{r-1}{2}}{10} + 2412\binom{9+\frac{r-1}{2}}{10} + 26\binom{10+\frac{r-1}{2}}{10}, & r \text{ odd} \end{cases} \tag{3.47}$$

Note that the quasi-polynomial for $n = 6$ is obtained in [83] however for clarity we omit it in this thesis.

The fact that $H_2^{\{1,d\}}(r)$ is a polynomial is immediate as $\text{vert}\mathcal{B}_2^{\{1,d\}} = \left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right\}$ (indeed $\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$).

The sequences given by (3.45) and (3.46) correspond to A019298, A053493 of [99].

## 3.7 Both diagonal symmetry

The seventh row of Figure 1.17 gives:

$$\mathcal{B}_n^{\{1,d,a,q^2\}} = \{a \in \mathcal{B}_n \mid a_{ij} = a_{ji} = a_{n+1-j,n+1-i} \text{ for all } i,j \in [n]\} \tag{3.48}$$

Figures 3.13 and 3.14 give the set $\text{SMS}(3,2)^{\{1,d,a,q^2\}}$ and some cardinalities of $\text{SMS}(n,r)^{\{1,d,a,q^2\}}$.

We note that $|SMS(2k,1)^{\{1,d,a,q^2\}}| = |SMS(2k+1,1)^{\{1,d,a,q^2\}}|$. Indeed if $a \in SMS(2k+1,1)^{\{1,d,a,q^2\}}$ then $q^2 a = a$ and so as before deleting the middle row and column gives the

$$\left\{ \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \right\}$$

Figure 3.13: Both diagonal symmetric semi magic squares of size 3 and line sum 2

| | $r=0$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $n=1$ | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 2 | 3 | 4 | 5 | 6 |
| 3 | 1 | 2 | 5 | 7 | 12 | 15 |
| 4 | 1 | 6 | 20 | 50 | 105 | 196 |
| 5 | 1 | 6 | 40 | 110 | 375 | 761 |
| 6 | 1 | 20 | 182 | 1040 | 4427 | |

Figure 3.14: $|\mathrm{SMS}(n,r)^{\{1,d,a,q^2\}}|$ for $n \in [6]$, $r \in [0,5]$

required bijection. Secondly $|SMS(2k,1)|^{\{1,d,a,q^2\}}$ is the sequence A000898 of [99] and is related to the rook problem. The following enumeration is given in [76, 96]:

$$|\mathrm{SMS}(2k,1)^{\{1,d,a,q^2\}}| = \sum_{i=0}^{k} 2^{k-2i} \binom{k}{2i} \frac{(2i)!}{i!} \tag{3.49}$$

Assume we have $i$ sets of 4 non zero entries that are symmetries of each other (i.e. no entries on the diagonal or anti diagonal), we obviously have $0 \le i \le \lfloor \frac{n}{4} \rfloor$. We are left with $n-4i$ other non zero entries that are pairs of either entries on the diagonal or the anti diagonal, thus we have $\frac{n}{2} - 2i$ such pairs. For given $i$ we have $2^{\frac{n}{2}-2i}\binom{\frac{n}{2}}{\frac{n}{2}-2i}$ possible configurations for fixed and anti fixed pairs. We are left with $(4i-2)(4i-6)\dots 6 \times 2 = \frac{(2i)!}{i!}$ configurations for sets of 4 non zero entries ($(4i-2)$ so that we exclude the diagonal and anti diagonal). Note we have the stronger result:

$$|\mathrm{SMS}(n,1)^{\{1,d,a,q^2\}}| = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} 2^{\lfloor \frac{n}{2} \rfloor - 2i} \binom{\lfloor \frac{n}{2} \rfloor}{2i} \frac{(2i)!}{i!}$$

equivalent to sequence A135401 of [99]. The fundamental region for this polytope is not one that we will study here, however we give it for completeness. Recalling Figure 3.1: $R_n^{\{1,d,a,q^2\}} = \{(i,j) \in [n] \times [n] \mid j \le i \text{ for all } i,j \in [n]\}$ giving:

$$\overline{\mathcal{B}_n^{\{1,d,a,q^2\}}} = \left\{ \begin{pmatrix} a_{11} & & \\ a_{21} & a_{22} & \\ \vdots & & \ddots \\ \vdots & & \cdot^{\cdot^{\cdot}} \\ a_{n-1,1} & a_{n-1,2} & \\ a_{n,1} & & \end{pmatrix} \in \mathbb{R}^{|R_n^{\{1,d,a,q^2\}}|} \;\middle|\; \begin{array}{l} \bullet\ a_{ij} \ge 0 \text{ for all } (i,j) \in R_n^{\{1,d,a,q^2\}} \\ \bullet\ \sum_{j=1}^{i-1}(a_{ij} + a_{n+1-i,j}) + \sum_{j=i}^{n+1-i} a_{ji} = 1 \\ \qquad \text{for all } i \in [n] \end{array} \right\}$$

where $f_{\{1,d,a,q^2\}} : \mathcal{B}_{2k}^{\{1,d,a,q^2\}} \longrightarrow \overline{\mathcal{B}_{2k}^{\{1,d,a,q^2\}}}$ is given by:

$$f_{\{1,d,a,q^2\}} \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{k1} & a_{k+1,1} & \cdots & a_{2k-1,1} & a_{2k,1} \\ a_{21} & \ddots & & \vdots & \vdots & & \iddots & a_{2k-1,1} \\ \vdots & & \ddots & \vdots & \vdots & \iddots & & \vdots \\ a_{k1} & \cdots & \cdots & a_{kk} & a_{k+1,k} & \cdots & \cdots & a_{k+1,1} \\ a_{k+1,1} & \cdots & \cdots & a_{k+1,k} & a_{kk} & \cdots & \cdots & a_{k1} \\ \vdots & & \iddots & \vdots & \vdots & \ddots & & \vdots \\ a_{2k-1,1} & \iddots & & \vdots & \vdots & & \ddots & a_{21} \\ a_{2k,1} & a_{2k-1,1} & \cdots & a_{k+1,1} & a_{k1} & \cdots & a_{21} & a_{11} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & & & \\ a_{21} & \ddots & & \\ \vdots & & \ddots & \\ a_{k1} & \cdots & \cdots & a_{kk} \\ a_{k+1,1} & \cdots & \cdots & a_{k+1,k} \\ \vdots & & \iddots & \\ a_{2k-1,1} & \iddots & & \\ a_{2k,1} & & & \end{pmatrix}$$

and $f_{\{1,d,a,q^2\}} : \mathcal{B}_{2k+1}^{\{1,d,a,q^2\}} \longrightarrow \overline{\mathcal{B}_{2k+1}^{\{1,d,a,q^2\}}}$ is given by:

$$f_{\{1,d,a,q^2\}} \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{k+1,1} & \cdots & a_{2k,1} & a_{2k+1,1} \\ a_{21} & \ddots & & \vdots & & \iddots & a_{2k,1} \\ \vdots & & \ddots & \vdots & \iddots & & \vdots \\ a_{k+1,1} & \cdots & \cdots & a_{k+1,k} & \cdots & \cdots & a_{k+1,1} \\ \vdots & & \iddots & \vdots & \ddots & & \vdots \\ a_{2k,1} & \iddots & & \vdots & & \ddots & a_{21} \\ a_{2k+1,1} & a_{2k,1} & \cdots & a_{k+1,1} & \cdots & a_{21} & a_{11} \end{pmatrix} = \begin{pmatrix} a_{11} & & & \\ a_{21} & \ddots & & \\ \vdots & & \ddots & \\ a_{k+1,1} & \cdots & \cdots & a_{k+1,k} \\ \vdots & & \iddots & \\ a_{2k-1,1} & \iddots & & \\ a_{2k,1} & & & \end{pmatrix}$$

It can be checked using Lemma 1.2.7 that:

$$\text{vert}\mathcal{B}_3^{\{1,d,a,q^2\}} = \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \right\}$$

We can similarly identify $\text{vert}\mathcal{B}_4^{\{1,d,a,q^2\}}$ to give $\text{D}\left(\mathcal{B}_3^{\{1,d,a,q^2\}}\right) = 2$ and $\text{D}\left(\mathcal{B}_4^{\{1,d,a,q^2\}}\right) = 1$.

However, $\mathrm{D}\left(\mathcal{B}_5^{\{1,d,a,q^2\}}\right) = 4$. Indeed using Lemma 1.2.7 we can verify that:

$$\begin{pmatrix} 0 & \frac{3}{4} & \frac{1}{4} & 0 & 0 \\ \frac{3}{4} & 0 & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & 0 & \frac{3}{4} \\ 0 & 0 & \frac{1}{4} & \frac{3}{4} & 0 \end{pmatrix} \in \mathrm{vert}\mathcal{B}_5^{\{1,d,a,q^2\}}$$

These observations lead to the following result:

**Theorem 3.7.1.**

$$D\left(\mathcal{B}_n^{\{1,d,a,q^2\}}\right) = \begin{cases} 1, & n = 1, 2 \text{ or } 4 \\ 2, & n = 3 \text{ or } n \geq 6 \text{ even} \\ 4, & n \geq 5 \text{ odd} \end{cases}$$

*Proof.* The result for $n \leq 4$ is immediate so we now consider the case $n \geq 5$ odd. From (3.7) we know that $1 \leq \mathrm{D}\left(\mathcal{B}_n^{\{1,d,a,q^2\}}\right) \leq 4$. Thus we simply need to show that $4 \leq \mathrm{D}\left(\mathcal{B}_n^{\{1,d,a,q^2\}}\right)$ for all $n \geq 5$ odd. As in Section 3.5 we pad the matrix $a^\dagger = \begin{pmatrix} 0 & \frac{3}{4} & \frac{1}{4} & 0 & 0 \\ \frac{3}{4} & 0 & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & 0 & \frac{3}{4} \\ 0 & 0 & \frac{1}{4} & \frac{3}{4} & 0 \end{pmatrix}$ to construct the matrix $a \in \mathcal{B}_{5+2k}^{\{1,d,a,q^2\}}$ with entries:

$$a_{ij} = \begin{cases} 1, & i = j \in [k] \text{ or } i = j \in [6+k, 5+2k] \\ a^\dagger_{i-k,j-k}, & (i,j) \in [1+k, 5+k] \times [1+k, 5+k] \\ 0, & \text{otherwise} \end{cases}$$

Thus $a$ is of the form:

$$a = \begin{pmatrix} 1 & \cdots & 0 & & & & \\ \vdots & \ddots & \vdots & & & & \\ 0 & \cdots & 1 & & & & \\ & & & (a^\dagger) & & & \\ & & & & 1 & \cdots & 0 \\ & & & & \vdots & \ddots & \vdots \\ & & & & 0 & \cdots & 1 \end{pmatrix}$$

Note that $a = I_k \oplus a^\dagger \oplus I_k$. Using Lemma 1.2.7 we have $a \in \mathrm{vert}\mathcal{B}_{5+2k}^{\{1,d,a,q^2\}}$, otherwise we would be able to find $a^* \neq 0$ such that $a^\dagger \pm a^* \in \mathcal{B}_5^{\{1,d,a,q^2\}}$ which is not possible as $a^\dagger \in \mathrm{vert}\mathcal{B}_5^{\{1,d,a,q^2\}}$. The proof of the result for $n \geq 6$ even is omitted in this thesis as it involves a graph theoretical approach which differs to the methodology used in this chapter and the next. □

From Theorems 1.2.18 and 3.7.1 we have:

**Theorem 3.7.2.** *For fixed $n \in \mathbb{P}$ there exists $H_n^{\{1,d,a,q^2\}}(r)$, the Ehrhart polynomial of $\mathcal{B}_n^{\{1,d,a,q^2\}}$ which satisfies:*

1. *$H_n^{\{1,d,a,q^2\}}(r)$ is a quasi polynomial in $r$ of degree $\dim \mathcal{B}_n^{\{1,d,a,q^2\}}$ and period which divides 1 for $n = 1, 2$ or 4, 2 for $n = 3$ or $n \geq 6$ even and 4 for $n \geq 5$ odd.*

2. $\left|SMS(n,r)^{\{1,d,a,q^2\}}\right| = H_n^{\{1,d,a,q^2\}}(r)$ *for all $r \in \mathbb{N}$*

3. $\left|SMS^{o}(n,r)^{\{1,d,a,q^2\}}\right| = (-1)^{\dim \mathcal{B}_n^{\{1,d,a,q^2\}}} H_n^{\{1,d,a,q^2\}}(-r) = H_n^{\{1,d,a,q^2\}}(r-n)$ *for all $r \in \mathbb{P}$*

$$H_2^{\{1,d,a,q^2\}}(r) = \binom{r+1}{1} \tag{3.50}$$

$$H_3^{\{1,d,a,q^2\}}(r) = \begin{cases} 2\binom{\frac{r}{2}+1}{2} + \binom{\frac{r}{2}+2}{2}, & r \text{ even} \\ \binom{\frac{r-1}{2}+1}{2} + 2\binom{\frac{r-1}{2}+2}{2}, & r \text{ odd} \end{cases} \tag{3.51}$$

$$H_4^{\{1,d,a,q^2\}}(r) = \binom{r+3}{4} + \binom{r+4}{4} \tag{3.52}$$

$$H_5^{\{1,d,a,q^2\}}(r) = \begin{cases} 110\binom{\frac{r}{4}+1}{6} + 2486\binom{\frac{r}{4}+2}{6} + 7256\binom{\frac{r}{4}+3}{6} + 4099\binom{\frac{r}{4}+4}{6} + 368\binom{\frac{r}{4}+5}{6} + \binom{\frac{r}{4}+6}{6}, \\ \qquad r = 0 \bmod 4 \\ 40\binom{\frac{r-1}{4}+1}{6} + 1606\binom{\frac{r-1}{4}+2}{6} + 6691\binom{\frac{r-1}{4}+3}{6} + 5258\binom{\frac{r-1}{4}+4}{6} + 719\binom{\frac{r-1}{4}+5}{6} + 6\binom{\frac{r-1}{4}+6}{6}, \\ \qquad r = 1 \bmod 4 \\ 6\binom{\frac{r-2}{4}+1}{6} + 719\binom{\frac{r-2}{4}+2}{6} + 5258\binom{\frac{r-2}{4}+3}{6} + 6691\binom{\frac{r-2}{4}+4}{6} + 1606\binom{\frac{r-2}{4}+5}{6} + 40\binom{\frac{r-2}{4}+6}{6}, \\ \qquad r = 2 \bmod 4 \\ \binom{\frac{r-3}{4}+1}{6} + 368\binom{\frac{r-3}{4}+2}{6} + 4099\binom{\frac{r-3}{4}+3}{6} + 7256\binom{\frac{r-3}{4}+4}{6} + 2486\binom{\frac{r-3}{4}+5}{6} + 110\binom{\frac{r-3}{4}+6}{6}, \\ \qquad r = 3 \bmod 4 \end{cases} \tag{3.53}$$

## 3.8 All symmetry

The last row of Figure 1.17 gives:

$$\mathcal{B}_n^{D_4} = \{a \in \mathcal{B}_n \mid a_{ij} = a_{n+1-i,j} = a_{j,n+1-i} \text{ for all} i, j \in [n]\} \tag{3.54}$$

Figure 3.15 gives the set $\text{SMS}(4,2)^{D_4}$ and Figure 3.1 gives some cardinalities of $\text{SMS}(n,r)^{D_4}$.

Recalling Figure 3.1, $R_{2k}^{D_4} = [k] \times [k]$ and $R_{2k+1}^{D_4} = [k+1] \times [k+1]$ leading to the following fundamental polytopes:

$$\overline{\mathcal{B}_{2k}^{D_4}} = \left\{ a \in \mathbb{R}^{k \times k} \;\middle|\; \begin{array}{l} \bullet\ a_{ij} \geq 0, \text{ for all } i,j \in [k] \\ \bullet\ a_{ij} = a_{ji}, \text{ for all } i,j \in [k] \\ \bullet\ \sum_{j=1}^{k} a_{ij} = \frac{1}{2} \text{ for all } i \in [k] \end{array} \right\}$$

$$\left\{ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \right\}$$

Figure 3.15: Totally symmetric semi magic squares of size 4 and line sum 2

| | $r = 0$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $n = 1$ | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 0 | 1 | 0 | 1 | 0 |
| 3 | 1 | 0 | 1 | 1 | 2 | 1 |
| 4 | 1 | 0 | 2 | 0 | 3 | 0 |
| 5 | 1 | 0 | 2 | 0 | 7 | 3 |
| 6 | 1 | 0 | 4 | 0 | 11 | 0 |
| 7 | 1 | 0 | 4 | 0 | 32 | 0 |

Table 3.1: $|\mathrm{SMS}(n,r)^{D_4}|$ for $n \in [7]$, $r \in [0,5]$

and

$$\overline{\mathcal{B}_{2k+1}^{D_4}} = \left\{ a \in \mathbb{R}^{(k+1)\times(k+1)} \;\middle|\; \begin{array}{l} \bullet\ a_{ij} \geq 0, \text{ for all } i,j \in [k+1] \\ \bullet\ a_{ij} = a_{ji}, \text{ for all } i,j \in [k+1] \\ \bullet\ 2\sum_{j=1}^{k} a_{ij} + a_{i,k+1} = 1 \text{ for all } i \in [k+1] \end{array} \right\}$$

where $f_{D_4} : \overline{\mathcal{B}_{2k}^{D_4}} \to \mathcal{B}_{2k}^{D_4}$ is given by:

$$f_{D_4}\begin{pmatrix} a_{11} & \dots & a_{1k} & a_{1k} & \dots & a_{11} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{1k} & \dots & a_{kk} & a_{kk} & \dots & a_{1k} \\ a_{1k} & \dots & a_{kk} & a_{kk} & \dots & a_{1k} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{11} & \dots & a_{1k} & a_{1k} & \dots & a_{11} \end{pmatrix} = \begin{pmatrix} a_{11} & \dots & a_{1k} \\ \vdots & & \vdots \\ a_{1k} & \dots & a_{kk} \end{pmatrix}$$

and $f_{D_4} : \overline{\mathcal{B}_{2k+1}^{D_4}} \to \mathcal{B}_{2k+1}^{D_4}$ is given by:

$$f_{D_4}\begin{pmatrix} a_{11} & \dots & a_{1k} & a_{1,k+1} & a_{1k} & \dots & a_{11} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{1k} & \dots & a_{kk} & a_{k,k+1} & a_{kk} & \dots & a_{1k} \\ a_{1,k+1} & \dots & a_{k,k+1} & a_{k+1,k+1} & a_{k,k+1} & \dots & a_{1,k+1} \\ a_{1k} & \dots & a_{kk} & a_{k,k+1} & a_{kk} & \dots & a_{1k} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{11} & \dots & a_{1k} & a_{1,k+1} & a_{1k} & \dots & a_{11} \end{pmatrix} = \begin{pmatrix} a_{11} & \dots & a_{1k} & a_{1,k+1} \\ \vdots & & \vdots & \vdots \\ a_{1k} & \dots & a_{kk} & a_{k,k+1} \\ a_{1,k+1} & \dots & a_{k,k+1} & a_{k+1,k+1} \end{pmatrix}$$

**Theorem 3.8.1.** *For $n$ even:*

$$vert\mathcal{B}_n^{D_4} = f_{D_4}^{-1}\left(\frac{1}{2} vert\mathcal{B}_{\frac{n}{2}}^{\{1,d\}}\right)$$

*For n odd:*

$$\mathit{vert}\mathcal{B}_n^{D_4} = \left\{ a \in \Pi_{D_4}(\mathit{SMS}(n,1)) \;\middle|\; \begin{array}{l} \textit{The connected components of the graph} \\ \textit{of } f_{D_4}(a) \textit{ are either trees or odd near trees} \end{array} \right\}$$

We recall Theorem 1.2.31 and Definition 1.2.38 so that this result fully identifies $\text{vert}\mathcal{B}_n^{D_4}$.

*Proof.* • For $n = 2k$, we immediately have:

$$2\overline{B_{2k}^{D_4}} = 2\mathcal{T}\Bigg(\underbrace{\Big(\frac{1}{2},\ldots,\frac{1}{2}\Big)}_{k},\underbrace{\Big(\frac{1}{2},\ldots,\frac{1}{2}\Big)}_{k}\Bigg)^{\{1,d\}} = \mathcal{B}_k^{\{1,d\}}$$

Thus, $\text{vert}2\overline{B_{2k}^{D_4}} = 2\text{vert}\overline{B_{2k}^{D_4}} = \text{vert}\mathcal{B}_k^{\{1,d\}}$. The theorem follows from Lemma 3.1.10.

• For $n = 2k+1$ using the same arguments used in the proofs of Theorem 3.2.1 and 3.3.1 we see that $\overline{B_{2k+1}^{D_4}}$ is affinely isomorphic with $\mathcal{T}\Bigg(\underbrace{\Big(\frac{1}{2},\ldots,\frac{1}{2}}_{k},\frac{1}{4}\Big),\underbrace{\Big(\frac{1}{2},\ldots,\frac{1}{2}}_{k},\frac{1}{4}\Big)\Bigg)^{\{1,d\}}$ and the result follows from Theorem 1.2.39.

□

$$\begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix} \qquad \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$


Figure 3.16: $\text{vert}\mathcal{B}_3^{D_4}$ and the corresponding graphs

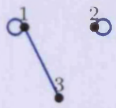
Figures 3.16 and 3.17 give the sets $\text{vert}\mathcal{B}_3^{D_4}$ and $\mathcal{B}_5^{D_4}$ and the corresponding graphs. Note that $\begin{pmatrix} \frac{3}{8} & 0 & \frac{1}{4} \\ 0 & \frac{3}{8} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & 0 \end{pmatrix} \in \Pi_{D_4}(\text{SMS}(5,1))$ however has graph 

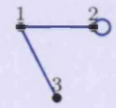
which has two connected cycles of length 1 (loops) and so is not a vertex. Indeed:

$$\begin{pmatrix} \frac{3}{8} & 0 & \frac{1}{4} \\ 0 & \frac{3}{8} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$$

Theorem 3.8.1 immediately gives $\text{D}\left(\mathcal{B}_{2k}^{D_4}\right) = 2\text{D}\left(\mathcal{B}_k^{\{1,d\}}\right)$ and so by Corollary 3.6.1 we have:

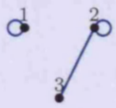
$$\begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{4} \end{pmatrix} \quad \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix}$$



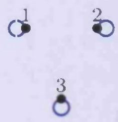





$$\begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \quad \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} \end{pmatrix} \quad \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$








$$\begin{pmatrix} 0 & \frac{3}{8} & \frac{1}{4} & \frac{3}{8} & 0 \\ \frac{3}{8} & 0 & \frac{1}{4} & 0 & \frac{3}{8} \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{3}{8} & 0 & \frac{1}{4} & 0 & \frac{3}{8} \\ 0 & \frac{3}{8} & \frac{1}{4} & \frac{3}{8} & 0 \end{pmatrix}$$

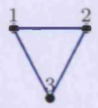



Figure 3.17: $\mathrm{vert}\mathcal{B}_5^{D_4}$ and the corresponding graphs

**Corollary 3.8.2.**

$$D(\mathcal{B}_n^{D_4}) = \begin{cases} 1, \ n = 1 \\ 2, \ n = 2 \\ 4, \ n = 3 \ \mathit{or} \ n \geq 4 \ \mathit{even} \\ 8, \ n \geq 5 \ \mathit{odd} \end{cases}$$

$\mathrm{D}(\mathcal{B}_4^{D_4}) = 2$ since $\mathrm{vert}\mathcal{B}_2^{\{1,d\}} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$. Also note that we can easily enumerate the vertices for the $n$ even case using (1.55) leading to:

$$\begin{gathered} \sum_{k\geq 0} \frac{|\mathrm{vert}\mathcal{B}_{2k}^{D_4}|}{k!} x^k = \left(\frac{1+x}{1-x}\right)^{\frac{1}{4}} e^{\frac{x+x^2}{2}} \\ |\mathrm{vert}\mathcal{B}_{2k}^{D_4}| = \sum \frac{k!}{2^{c_3+c_5+\dots} \prod_{i=1,2,3,5,7\dots} i^{c_i} c_i!} \end{gathered} \tag{3.55}$$

where the sum is over all non negative integers $c_1, c_2, c_3, c_5 \dots$ satisfying $c_1 + 2c_2 + 3c_3 + 5c_5 + \dots = k$.

Recalling (3.9) for $G = D_4$ we have:

$$\mathrm{SMS}(n,1)^{D_4} = \emptyset \subsetneq \mathrm{vert}\mathcal{B}_n^{D_4} \subsetneq \Pi_{D_4}\mathrm{SMS}(n,1) \subseteq \frac{1}{8}\mathrm{SMS}(n,8)^{D_4} \tag{3.56}$$

From Theorem 1.2.18 and Corollary 3.8.2 we have:

**Theorem 3.8.3.** *For fixed $n \in \mathbb{P}$ there exists $H_n^{D_4}(r)$, the Ehrhart quasi-polynomial of $\mathcal{B}_n^{D_4}$ which satisfies:*

1. *$H_n^{D_4}(r)$ is a quasi-polynomial in $r$ of degree $\dim \mathcal{B}_n^{D_4}$ and period which divides 4/8 for $n$ even and $n = 3$, $n \geq 5$ odd respectively.*
2. *$|SMS^{D_4}(n,r)| = H_n^{D_4}(r)$ for all $r \in \mathbb{N}$*
3. *$|SMS^{oD_4}(n,r)| = (-1)^{\dim \mathcal{B}_n^{D_4}} H_n^{D_4}(-r) = H_n^{D_4}(r-n)$ for all $r \in \mathbb{P}$*

This is illustrated by the following enumerations:

$$H_2^{D_4}(r) = \begin{cases} 1 \ r \text{ even} \\ 0 \ r \text{ odd} \end{cases} \tag{3.57}$$

$$H_3^{D_4}(r) = \begin{cases} \binom{\frac{r}{4}+1}{1}, \ r = 0 \bmod 4 \\ \binom{\frac{r-1}{4}}{1}, \ r = 1 \bmod 4 \\ \binom{\frac{r-2}{4}+1}{1}, \ r = 2 \bmod 4 \\ \binom{\frac{r-3}{4}+1}{1}, \ r = 3 \bmod 4 \end{cases} \tag{3.58}$$

$$H_4^{D_4}(r) = \begin{cases} \binom{\frac{r}{2}+1}{1} \ r \text{ even} \\ 0 \ r \text{ odd} \end{cases} \tag{3.59}$$

$$H_5^{D_4}(r) = \begin{cases} 16\binom{\frac{r}{8}+1}{3} + 19\binom{\frac{r}{8}+2}{3} + \binom{\frac{r}{8}+3}{3}, \ r = 0 \bmod 8 \\ 2\binom{\frac{r-1}{8}}{3} + 21\binom{\frac{r-1}{8}+1}{3} + 13\binom{\frac{r-1}{8}+2}{3}, \ r = 1 \bmod 8 \\ 13\binom{\frac{r-2}{8}+1}{3} + 21\binom{\frac{r-2}{8}+2}{3} + 2\binom{\frac{r-2}{8}+3}{3}, \ r = 2 \bmod 8 \\ \binom{\frac{r-3}{8}}{3} + 19\binom{\frac{r-3}{8}+1}{3} + 16\binom{\frac{r-3}{8}+2}{3}, \ r = 3 \bmod 8 \\ 4\binom{\frac{r-4}{8}+1}{3} + 25\binom{\frac{r-4}{8}+2}{3} + 7\binom{\frac{r-4}{8}+3}{3} r = 4 \bmod 8 \\ 10\binom{\frac{r-5}{8}+1}{3} + 23\binom{\frac{r-5}{8}+2}{3} + 3\binom{\frac{r-5}{8}+3}{3}, \ r = 5 \bmod 8 \\ 3\binom{\frac{r-6}{8}+1}{3} + 23\binom{\frac{r-6}{8}+2}{3} + 10\binom{\frac{r-6}{8}+3}{3}, \ r = 6 \bmod 8 \\ 7\binom{\frac{r-7}{8}+1}{3} + 25\binom{\frac{r-7}{8}+2}{3} + 4\binom{\frac{r-7}{8}+3}{3}, \ r = 7 \bmod 8 \end{cases} \tag{3.60}$$

$$H_6^{D_4}(r) = \begin{cases} 4\binom{\frac{r}{4}+1}{3} + 7\binom{\frac{r}{4}+2}{3} + \binom{\frac{r}{4}+3}{3}, \ r = 0 \bmod 4 \\ 0, \ r = 1 \bmod 4 \\ \binom{\frac{r-2}{4}+1}{3} + 7\binom{\frac{r-2}{4}+2}{3} + 4\binom{\frac{r-2}{4}+3}{3}, \ r = 2 \bmod 4 \\ 0, \ r = 3 \bmod 4 \end{cases} \tag{3.61}$$

The sequence given by (3.58) corresponds to A008624 of [99].

## 3.9 Conclusion

As this chapter contains numerous results we here give a concise listing of these results.

### 3.9.1 Conclusion for $\mathcal{B}_n^{\{1,h\}}$

- $\mathcal{B}_n^{\{1,h\}} \cap \mathbb{Z}^{n \times n} = \text{SMS}(n,1)^{\{1,h\}} = \emptyset$.
- $\dim \mathcal{B}_n^{\{1,h\}} = \begin{cases} \frac{(n-1)(n-2)}{2}, & n \text{ even} \\ \frac{(n-1)^2}{2}, & n \text{ odd} \end{cases}$.
- $\text{vert}\mathcal{B}_n^{\{1,h\}} = \frac{1}{2}\text{SMS}(n,2)^{\{1,h\}} = \Pi_{\{1,h\}}(\text{SMS}(n,1))$.
- $\left|\text{vert}\mathcal{B}_n^{\{1,h\}}\right| = \left|\text{SMS}(n,2)^{\{1,h\}}\right| = \frac{n!}{2^{\lfloor \frac{n}{2} \rfloor}}$.
- Theorem 3.2.4 gives an enumeration result for fixed $n$, quasi-polynomials of period 2 are obtained for $n \in [5]$.

### 3.9.2 Conclusion for $\mathcal{B}_n^{\{1,h,v,q^2\}}$

- $\mathcal{B}_n^{\{1,h,v,q^2\}} \cap \mathbb{Z}^{n\times n} = \mathrm{SMS}(n,1)^{\{1,h,v,q^2\}} = \emptyset$.
- $\dim \mathcal{B}_n^{\{1,h,v,q^2\}} = \begin{cases} \frac{(n-2)^2}{4}, & n \text{ even} \\ \frac{(n-1)^2}{4}, & n \text{ odd} \end{cases}$.
- – $\mathrm{vert}\mathcal{B}_{2k}^{\{1,h,v,q^2\}} = \frac{1}{2}\mathrm{SMS}(2k,2)^{\{1,h,v,q^2\}} = \Pi_{\{1,h\}}\mathrm{SMS}(2k,1)$

  – $\mathrm{vert}\mathcal{B}_{2k+1}^{\{1,h,v,q^2\}} = \left\{ a \in \Pi_{\{1,h,v,q^2\}} SMS(2k+1,1) \,\middle|\, \begin{array}{l} \text{the fundamental region of } a \\ \text{has no non zero cycles} \end{array} \right\}$
- $|\mathrm{vert}\mathcal{B}_n^{\{1,h,v,q^2\}}| = \begin{cases} \frac{n}{2}!, & n \text{ even} \\ \sum_{i=0}^{\frac{n-1}{2}} \frac{\left(\frac{n-1}{2}!\right)^2}{i!}, & n \text{ odd} \end{cases}$
- Theorem 3.3.5 gives an enumeration result for fixed $n$. Interestingly $\mathrm{D}(\mathcal{B}_{2k}^{\{1,h,v,q^2\}}) = 2$ and $\mathrm{D}(\mathcal{B}_{2k+1}^{\{1,h,v,q^2\}}) = 4$. Quasi-polynomials are obtained for $n \in [5]$.

### 3.9.3 Conclusion for $\mathcal{B}_n^{\{1,q^2\}}$

- $\left|\mathcal{B}_n^{\{1,q^2\}} \cap \mathbb{Z}^{n\times n}\right| = \left|\mathrm{SMS}(n,1)^{\{1,q^2\}}\right| = 2^{\lfloor \frac{n}{2} \rfloor} \lfloor \frac{n}{2} \rfloor !$.
- $\dim \mathcal{B}_n^{\{1,q^2\}} = \begin{cases} \frac{(n-1)^2+1}{2}, & n \text{ even} \\ \frac{(n-1)^2}{2}, & n \text{ odd} \end{cases}$.
- – $\mathrm{vert}\mathcal{B}_{2k}^{\{1,q^2\}} = \mathrm{SMS}(2k,1)^{\{1,q^2\}}$

  – $\mathrm{vert}\mathcal{B}_{2k+1}^{\{1,q^2\}} = \left\{ a \in \Pi_{\{1,q^2\}}\mathrm{SMS}(2k+1,1) \,\middle|\, \begin{array}{l} \text{All the non zero zero cycles of } a \\ \text{contain an entry in the central} \\ \text{column of } a \end{array} \right\} \cup \mathrm{SMS}(2k+1,1)^{\{1,q^2\}}$
- $\left|\mathrm{vert}\mathcal{B}_n^{\{1,q^2\}}\right| = \begin{cases} n!!, & n \text{ even} \\ \sum_{i=0}^{\frac{n-3}{2}} \frac{2^{n-2-i}\left(\frac{n-1}{2}!\right)^2}{i!} + (n-1)!!, & n \text{ odd} \end{cases}$
- Theorem 3.4.6 gives an enumeration result for fixed $n$. Interestingly $\mathrm{D}(\mathcal{B}_{2k}^{\{1,q^2\}}) = 1$ and $\mathrm{D}(\mathcal{B}_{2k+1}^{\{1,q^2\}}) = 2$. Quasi-polynomials are obtained for $n \in [4]$.

### 3.9.4 Conclusion for $\mathcal{B}_n^{\{1,q,q^2,q^3\}}$

- $\left|\mathcal{B}_n^{\{1,q,q^2,q^3\}} \cap \mathbb{Z}^{n\times n}\right| = \left|\mathrm{SMS}(n,1)^{\{1,q,q^2,q^3\}}\right| = \begin{cases} \frac{(\lfloor \frac{n}{2} \rfloor)!}{(\lfloor \frac{n}{4} \rfloor)!}, & \lfloor \frac{n}{2} \rfloor \text{ even} \\ 0, & \lfloor \frac{n}{2} \rfloor \text{ odd} \end{cases}$.

- $\dim \mathcal{B}_n^{\{1,q,q^2,q^3\}} = \begin{cases} \frac{n(n-2)}{4}, & n \text{ even} \\ \frac{(n-1)^2}{4}, & n \text{ odd} \end{cases}$.

- We have not obtained a result concerning vert$\mathcal{B}_n^{\{1,q,q^2,q^3\}}$ however a result concerning $D(\mathcal{B}_n)^{\{1,q,q^2,q^3\}}$ is obtained:

$$D(\mathcal{B}_n^{\{1,q,q^2,q^3\}}) = \begin{cases} 2, & n \text{ even} \\ 4, & n \text{ odd} \end{cases}$$

- No results concerning $\left|\text{vert}\mathcal{B}_2^{\{1,q,q^2,q^3\}}\right|$ are obtained.

- Theorem 3.5.2 gives an enumeration result for fixed $n$. Quasi-polynomials are obtained for $n \in [4]$.

### 3.9.5 Conclusion for $\mathcal{B}_n^{\{1,d\}}$

- $\left|\mathcal{B}_n^{\{1,d\}} \cap \mathbb{Z}^{n\times n}\right| = \left|\text{SMS}(n,1)^{\{1,d\}}\right| = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{(n-2k)!2^k k!}$.

- $\dim \mathcal{B}_n^{\{1,d\}} = \frac{n(n-1)}{2}$.

- vert$\mathcal{B}_n^{\{1,d\}}$ is obtained from Theorem 1.2.31.

- Recursion and generating functions are given by (3.42). A cumbersome summation formulae is given by (1.55) but no nice formula has been found.

- Theorem 3.6.2 gives an enumeration result for fixed $n$. Quasi-polynomials are obtained for $n \in [4]$.

### 3.9.6 Conclusion for $\mathcal{B}_n^{\{1,d,a,q^2\}}$

- $\left|\mathcal{B}_n^{\{1,d,a,q^2\}} \cap \mathbb{Z}^{n\times n}\right| = \left|\text{SMS}(n,1)^{\{1,d,a,q^2\}}\right| = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} 2^{\lfloor \frac{n}{2} \rfloor - 2i} \binom{\lfloor \frac{n}{2} \rfloor}{2i} \frac{(2i)!}{i!}$.

- $\dim \mathcal{B}_n^{\{1,d,a,q^2\}} = \begin{cases} \frac{n^2}{4}, & n \text{ even} \\ \frac{(n-1)(n+1)}{4}, & n \text{ odd} \end{cases}$.

- We still have not obtained a result concerning vert$\mathcal{B}_n^{\{1,d,a,q^2\}}$.

- No results concerning $\left|\text{vert}\mathcal{B}_n^{\{1,d,a,q^2\}}\right|$ are obtained.

- $\mathrm{D}\left(\mathcal{B}_n^{\{1,d,a,q^2\}}\right) = \begin{cases} 1, & n = 1, 2 \text{ or } 4 \\ 2, & n = 3 \text{ or } n \geq 6 \text{ even} \\ 4, & n \geq 5 \text{ odd} \end{cases}$

- Theorem 3.7.2 gives an enumeration result for fixed $n$. Quasi-polynomials are obtained for $n \in [5]$.

### 3.9.7 Conclusion for $\mathcal{B}_n^{D_4}$

- $\mathcal{B}_n^{D_4} \cap \mathbb{Z}^{n\times n} = \mathrm{SMS}(n,1)^{D_4} = \emptyset$.

- $\dim \mathcal{B}_n^{D_4} = \begin{cases} \frac{(n-2)n}{8}, & n \text{ even} \\ \frac{(n-1)(n+1)}{8}, & n \text{ odd} \end{cases}$.

- 
  - $\mathrm{vert}\mathcal{B}_{2k}^{D_4} = f_{D_4}^{-1}\left(\frac{1}{2}\mathrm{vert}\mathcal{B}_k^{\{1,d\}}\right)$
  - $\mathrm{vert}\mathcal{B}_{2k+1}^{D_4} = \left\{ a \in \Pi_{D_4}(\mathrm{SMS}(2k+1,1)) \;\middle|\; \begin{array}{l} \text{The connected components of} \\ \text{the graph of the fundamental} \\ \text{region of } a \text{ are either trees or} \\ \text{odd near trees} \end{array} \right\}$

- Recursion relations and generating functions are given for $\left|\mathrm{vert}\mathcal{B}_{2k}^{D_4}\right|$ by (3.55).

- Theorem 3.8.3 gives an enumeration result for fixed $n$. Quasi-polynomials are obtained for $n \in [6]$.

In the next chapter we give a similar study of the symmetry classes of the alternating sign matrix polytope $\mathcal{A}_n$ (Definition 2.3.1). Even though this polytope is "more complicated" then $\mathcal{B}_n$ in some ways the study of symmetry classes is "easier".

# Chapter 4

# Symmetry Classes of The Alternating Sign Matrix Polytope

## 4.1 Introduction

This chapter aims to give counterpart results to those obtained for $\mathcal{B}_n$ in Chapter 3. In a similar fashion to Chapter 3 we shall study the symmetry classes of $\mathcal{A}_n$ (Definition 2.3.1). Recalling Theorem 2.3.4 (the main result of Chapter 2) the proof of this theorem used bijections (2.27) and (2.28) between $\mathcal{A}_n$ and $\mathcal{E}_n$ (Definition 2.3.3). If we are to study $\mathcal{A}_n^G$ using bijections (2.27) and (2.28) we need to define an action of the elements $g$ of $D_4$ on $\mathcal{E}_n$.

For any $g \in D_4$ and for $(h, v) \in \mathcal{E}_n$ corresponding to $a \in \mathcal{A}_n$, we define $(gh, gv)$ to be the element of $\mathcal{E}_n$ corresponding to $ga \in \mathcal{A}_n$. Thus, $(gh)_{ij} = \sum_{j'=1}^{j}(ga)_{ij'}$ and $(gv)_{ij} = \sum_{i'=1}^{i}(ga)_{i'j}$. The conditions imposed on the matrix entries $(h_{ij}, v_{ij})$ by each of the cases of Figure 1.17 are shown in Figure 4.1.

Recalling Definition 3.1.8, as in Chapter 3 to lighten notation we make the substitutions $R_n^G = R^{\mathcal{A}_n^G}$ and $f_G = f_{\mathcal{A}_n^G}$. As described we shall use the fact that $\mathcal{A}_n$ and $\mathcal{E}_n$ are affinely isomorphic to transform the study of $\mathcal{A}_n^G$ into a study of $\mathcal{E}_n^G$, thus we make a further substitution to lighten notation: $f'_G = f_{\mathcal{E}_n^G}$ and so $f'_G(\mathcal{E}_n^G) = \overline{\mathcal{E}_n^G} \subseteq \mathbb{R}^{\left|R^{\mathcal{E}_n^G}\right|}$, where $\overline{\mathcal{E}_n^G}$ is the fundamental polytope of $\mathcal{E}_n^G$ as given by Definition 3.1.9. Also, since the mapping from $\mathcal{A}_n^G$ to $\mathcal{E}_n^G$ involves only addition of coordinates (2.27) it follows that $\mathrm{D}\left(\mathcal{A}_n^G\right) \geq \mathrm{D}\left(\mathcal{E}_n^G\right)$ and since the mapping from

| Name | Sufficient Conditions |
|---|---|
| No Symmetry | |
| Horizontal Symmetry | $h_{ij} = h_{n+1-i,j}$ <br> $v_{i,j} = 1 - v_{n-i,j}$ |
| Half Turn Symmetry | $h_{ij} = 1 - h_{n+1-i,n-j}$ <br> $v_{i,j} = 1 - v_{n-i,n+1-j}$ |
| Diagonal Symmetry | $h_{ij} = v_{ji}$ |
| Horizontal and Vertical Symmetry | $h_{ij} = h_{n+1-i,j} = 1 - h_{i,n-j}$ <br> $v_{i,j} = v_{i,n+1-j} = 1 - v_{n-i,j}$ |
| Quarter Turn Symmetry | $h_{ij} = v_{j,n+1-i} = 1 - h_{n+1-i,n-j}$ |
| Both Diagonal Symmetry | $h_{ij} = v_{ji} = 1 - h_{n+1-i,n-j}$ |
| All Symmetry | $h_{ij} = v_{ji} = h_{n+1-i,j} = 1 - h_{i,n-j}$ |

Figure 4.1: Symmetry classes of edge matrix pairs

$\mathcal{E}_n^G$ to $\mathcal{A}_n^G$ involves only subtraction of coordinates (2.28) it follows that $\mathrm{D}\left(\mathcal{A}_n^G\right) \leq \mathrm{D}\left(\mathcal{E}_n^G\right)$. Therefore:

$$\mathrm{D}\left(\mathcal{A}_n^G\right) = \mathrm{D}\left(\mathcal{E}_n^G\right) = \mathrm{D}\left(\overline{\mathcal{A}_n^G}\right) = \mathrm{D}\left(\overline{\mathcal{E}_n^G}\right) \tag{4.1}$$

In Chapter 3, for certain parity of $n$ and subgroups $G$ of $D_4$ we had $\overline{\mathcal{B}_n^G} = \mathcal{T}(r,s)$ (for some $r \in \mathbb{R}^m, s \in \mathbb{R}^n$). This motivates the definition of the following polytope:

**Definition 4.1.1.** *For*

$$H = \begin{pmatrix} H_{10} & \cdots & H_{1n} \\ \vdots & & \vdots \\ H_{m0} & \cdots & H_{mn} \end{pmatrix}, H' = \begin{pmatrix} H'_{10} & \cdots & H'_{1n} \\ \vdots & & \vdots \\ H'_{m0} & \cdots & H'_{mn} \end{pmatrix} \in \mathbb{R}^{m\times(n+1)}$$

$$V = \begin{pmatrix} V_{01} & \cdots & V_{0n} \\ \vdots & & \vdots \\ V_{m1} & \cdots & V_{mn} \end{pmatrix}, V' = \begin{pmatrix} V'_{01} & \cdots & V'_{0n} \\ \vdots & & \vdots \\ V'_{m1} & \cdots & V'_{mn} \end{pmatrix} \in \mathbb{R}^{(m+1)\times n}$$

*we define the polytope* $\Sigma(H,H',V,V')$:

$$\Sigma(H,H',V,V') := \left\{ (h,v) = \left( \begin{pmatrix} h_{10} & \cdots & h_{1n} \\ \vdots & & \vdots \\ h_{m0} & \cdots & h_{mn} \end{pmatrix}, \begin{pmatrix} v_{01} & \cdots & v_{0n} \\ \vdots & & \vdots \\ v_{m1} & \cdots & v_{mn} \end{pmatrix} \right) \in \mathbb{R}^{m\times(n+1)} \times \mathbb{R}^{(m+1)\times n} \;\middle|\; \begin{array}{l} \bullet\ H_{ij} \leq h_{ij} \leq H'_{ij} \text{ for all } i \in [m],\ j \in [0,n] \\ \bullet\ V_{ij} \leq v_{ij} \leq V'_{ij} \text{ for all } i \in [0,m],\ j \in [n] \\ \bullet\ h_{i,j-1} + v_{ij} = v_{i-1,j} + h_{ij} \text{ for all } i \in [m],\ j \in [n] \end{array} \right\}$$

Note that $\Sigma\left( \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 0 & 1 & \cdots & 1 \end{pmatrix}, \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ 1 & \cdots & 1 \end{pmatrix}, \begin{pmatrix} 0 & \cdots & 0 \\ 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix} \right) = \mathcal{E}_n$ (see Definition 2.3.3). Thus $\Sigma(H,H',V,V')$ is a generalization of $\mathcal{E}_n$. Also, elements of $\Sigma(H,H',V,V')$ have a representation on $\mathcal{L}_{m,n}$ similar to Figure 1.10 as shown in Figure 4.2.

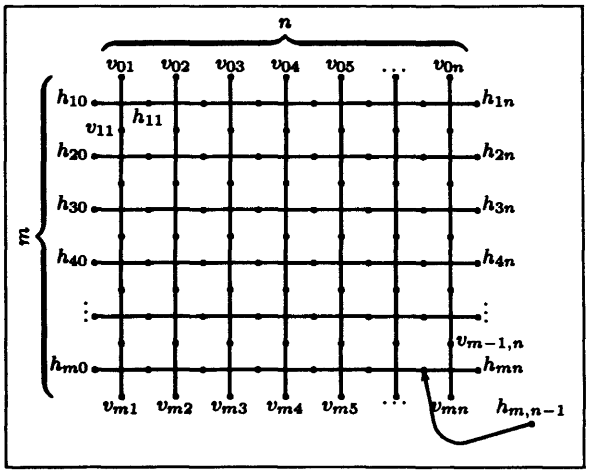



Figure 4.2: Element of $\Sigma(r, s)$ represented on $\mathcal{L}_{m,n}$

Note that if $H_{i0} = H'_{i0} = V_{0j} = V'_{0j} = 0$ for all $i \in [m]$ and $j \in [n]$, then $\Sigma(H, H', V, V')$ is in bijection (using (2.27) and (2.28)) with the following polytope:

$$\Lambda(H, H', V, V') := \left\{ a \in \mathbb{R}^{m\times n} \,\middle|\, \begin{array}{l} \bullet\ H_{ij} \le \sum_{j'=1}^{j} a_{ij'} \le H'_{ij} \text{ for all } i \in [m],\ j \in [n] \\ \bullet\ V_{ij} \le \sum_{i'=1}^{m} a_{i'j} \le V_{ij} \text{ for all } i \in [m],\ j \in [n] \end{array} \right\} \tag{4.2}$$

Recalling the proof of Theorem 2.3.4 we have:

$$\text{vert}\Sigma\left(\begin{pmatrix}0 & \dots & 0 & 1\\ \vdots & & \vdots & \vdots\\ 0 & \dots & 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1 & \dots & 1\\ \vdots & \vdots & & \vdots\\ 0 & 1 & \dots & 1\end{pmatrix}, \begin{pmatrix}0 & \dots & 0\\ \vdots & & \vdots\\ 0 & \dots & 0\\ 1 & \dots & 1\end{pmatrix}, \begin{pmatrix}0 & \dots & 0\\ 1 & \dots & 1\\ \vdots & & \vdots\\ 1 & \dots & 1\end{pmatrix}\right) =$$

$$\left\{ (h,v) \in \Sigma\left(\begin{pmatrix}0 & \dots & 0 & 1\\ \vdots & & \vdots & \vdots\\ 0 & \dots & 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1 & \dots & 1\\ \vdots & \vdots & & \vdots\\ 0 & 1 & \dots & 1\end{pmatrix}, \begin{pmatrix}0 & \dots & 0\\ \vdots & & \vdots\\ 0 & \dots & 0\\ 1 & \dots & 1\end{pmatrix}, \begin{pmatrix}0 & \dots & 0\\ 1 & \dots & 1\\ \vdots & & \vdots\\ 1 & \dots & 1\end{pmatrix}\right) \,\middle|\, (h,v) \text{ does not have a non integer cycle} \right\}$$

This leads to the following generalized result:

**Theorem 4.1.2.** *If $H_{i0} = H'_{i0}, V_{0j} = V'_{0j}, H_{in} = H'_{in}$ and $V_{mj} = V'_{mj}$ for all $i \in [m],\ j \in [n]$ then:*

$$\textit{vert}\Sigma(H, H', V, V') = \{(h, v) \in \Sigma(H, H', V, V') \mid (h, v) \textit{ does not have a non extremal cycle}\}$$

Note that we refer to entries $h_{ij}$ or $v_{ij}$ for which $H_{ij} < h_{ij} < H'_{ij}$ or $V_{ij} < v_{ij} < V_{ij}$ as non extremal. Thus, a non extremal cycle means a cycle of edges of the $m$ by $n$ lattice $\mathcal{L}_{m,n}$ such that each entry of $(h, v)$ associated with an edge of the cycle is non extremal. The proof of this theorem is similar to the proof of Theorem 2.3.4:

*Proof.* • Assume $(h, v) \in \Sigma(H, H', V, V') \setminus \mathrm{vert}\Sigma(H, H', V, V')$. Thus from Lemma 1.2.7 there exists $(h^*, v^*) = \left(\begin{pmatrix} h^*_{10} & \cdots & h^*_{1n} \\ \vdots & & \vdots \\ h^*_{m0} & \cdots & h^*_{mn} \end{pmatrix}, \begin{pmatrix} v^*_{01} & \cdots & v^*_{0n} \\ \vdots & & \vdots \\ v^*_{m1} & \cdots & v^*_{mn} \end{pmatrix}\right) \neq (0,0)$ such that $(h, v) \pm (h^*, v^*) \in \Sigma(H, H', V, V')$. Note that we will have:

$$\begin{aligned} h^*_{i0} = h^*_{in} = v^*_{0j} = v^*_{mj} = 0 \text{ for all } i \in [m],\ j \in [n] \\ h^*_{i,j-1} + v^*_{ij} = v^*_{i-1,j} + h^*_{ij} \text{ for all } i \in [m],\ j \in [n] \end{aligned}$$

The fact that $h^*_{i0} = h^*_{in} = v^*_{0j} = v^*_{mj} = 0$ for all $i \in [m]$, $j \in [n]$ follows from the requirement that $H_{i0} = H'_{i0}, V_{0j} = V'_{0j}, H_{in} = H'_{in}$ and $V_{mj} = V'_{mj}$ for all $i \in [m]$, $j \in [n]$ and so $h_{i0} = H_{i0}, h_{in} = H_{in}, v_{0j} = V_{0j}$ and $v_{mj} = V_{mj}$ for all $i \in [m]$, $j \in [n]$.

Since $(h^*, v^*) \neq (0,0)$, $(h^*, v^*)$ must have a non zero cycle. Such a cycle can be obtained by starting with a non zero entry of $(h^*, v^*)$ and repeatedly applying $h^*_{i,j-1} + v^*_{ij} = v^*_{i-1,j} + h^*_{ij}$, while noting that it is impossible for a path of non zero entries of $(h^*, v^*)$ to terminate at a boundary since the boundary entries are all zero. Recalling $(h \pm h^*, v \pm v^*) \in \Sigma(H, H', V, V')$, on the cycle we have:

$$\begin{aligned} H_{ij} \leq h_{ij} \pm h^*_{ij} \leq H'_{ij} \\ V_{ij} \leq v_{ij} \pm v^*_{ij} \leq V'_{ij} \end{aligned}$$

thus:

$$\begin{aligned} H_{ij} < h_{ij} < H'_{ij} \\ V_{ij} < v_{ij} < V'_{ij} \end{aligned}$$

and so $(h, v)$ has a non extremal cycle giving:

$$\mathrm{vert}\Sigma(H, H', V, V') \supseteq \{(h, v) \in \Sigma(r, s) \mid (h, v) \text{ does not have a non extremal cycle}\}$$

• Now assume $(h, v)$ has a non extremal cycle. Since each boundary leads only to a single boundary vertex a cycle will not include any boundary edges. Select any such cycle, give it an orientation, say anticlockwise, and denote the sets of points $(i, j)$ for which the horizontal edge between $(i, j)$ and $(i, j+1)$ is in the cycle and directed right or left as respectively $\mathcal{H}_+$ or $\mathcal{H}_-$, and the sets of points $(i, j)$ for which the vertical edge between $(i,j)$ and $(i+1,j)$ is in the cycle and directed up or down as respectively $\mathcal{V}_+$ or $\mathcal{V}_-$.

We now create the matrix pair $(h^*, v^*) = \left( \begin{pmatrix} h^*_{10} & \dots & h^*_{1n} \\ \vdots & & \vdots \\ h^*_{m0} & \dots & h^*_{mn} \end{pmatrix}, \begin{pmatrix} v^*_{01} & \dots & v^*_{0n} \\ \vdots & & \vdots \\ v^*_{m1} & \dots & v^*_{mn} \end{pmatrix} \right) \in \mathbb{R}^{m\times(n+1)} \times \mathbb{R}^{(m+1)\times n}$ with entries:

$$h^*_{ij} := \begin{cases} \mu \text{ if } (i,j) \in \mathcal{H}_+ \\ -\mu \text{ if } (i,j) \in \mathcal{H}_- \\ 0 \text{ otherwise} \end{cases} \quad v^*_{ij} := \begin{cases} \mu \text{ if } (i,j) \in \mathcal{V}_+ \\ -\mu \text{ if } (i,j) \in \mathcal{V}_- \\ 0 \text{ otherwise} \end{cases}$$

Figure 4.3 shows $(h^*, v^*)$ represented on $\mathcal{L}_{m,n}$.

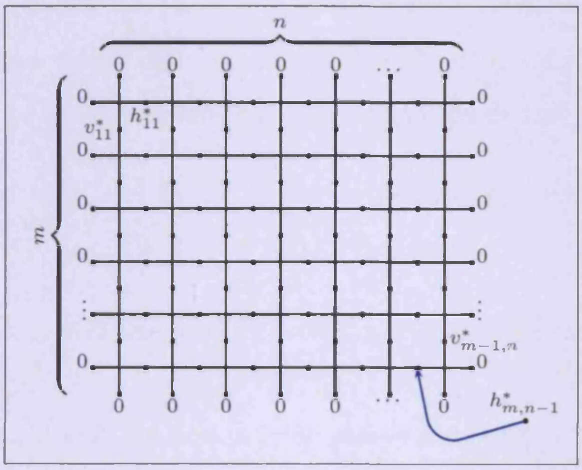
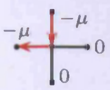
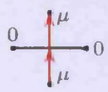



Figure 4.3: $(h^*, v^*)$ on $\mathcal{L}_{m,n}$

Note that we will have: $h^*_{i,j-1} + v^*_{ij} = v^*_{i-1,j} + v^*_{ij}$ for all $i \in [m]$, $j \in [n]$ since if the cycle does not pass through $(i,j)$ then the equation is trivial. If the cycle does pass through $(i,j)$ then because of the orientation, all appearances of $\mu$ cancel out. For example:



gives $-\mu + 0 = -\mu + 0$ or


gives $0 + \mu = \mu + 0$.

We choose:

$$\begin{aligned} \mu = \min(&\{h_{ij} - H_{ij} | (i,j) \in \mathcal{H}_+\} \cup \{H'_{ij} - h_{ij} | (i,j) \in \mathcal{H}_+\} \cup \\ &\{h_{ij} - H_{ij} | (i,j) \in \mathcal{H}_-\} \cup \{H'_{ij} - h_{ij} | (i,j) \in \mathcal{H}_-\} \cup \\ &\{v_{ij} - V_{ij} | (i,j) \in \mathcal{V}_+\} \cup \{V'_{ij} - v_{ij} | (i,j) \in \mathcal{V}_+\} \cup \\ &\{v_{ij} - V_{ij} | (i,j) \in \mathcal{V}_-\} \cup \{V'_{ij} - v_{ij} | (i,j) \in \mathcal{V}_-\}) \end{aligned}$$

Also note that:

$$h^*_{i0} = h^*_{in} = v^*_{0j} = v^*_{mj} = 0 \text{ for all } i \in [m],\ j \in [n]$$

since the cycle does not include any boundary edges. It can now easily be checked that $(h^*, v^*) \neq (0,0)$ and:

$$(h,v) \pm (h^*, v^*) \in \Sigma(r,s)$$

which gives:

$$\text{vert}\Sigma(r,s) \subseteq \{(h,v) \in \Sigma(r,s) \mid (h,v) \text{ does not have a non extremal cycle}\}$$

as required.

□

As an extension of this we define the non extremal paths of a matrix $a \in \mathbb{R}^{m\times n}$ on $\mathcal{L}_{m,n}$ as the non extremal paths of the corresponding $(h,v)$ which leads to:

**Theorem 4.1.3.** *If $H_{i0} = H'_{i0} = V_{0j} = V'_{0j} = 0$, $H_{in} = H'_{in}$ and $V_{mj} = V'_{mj}$ for all $i \in [m]$, $j \in [n]$ then:*

$$\text{vert}\Lambda(H,H',V,V') = \{a \in \text{A}(H,H',V,V') \mid a \text{ on } \mathcal{L}_{m,n} \text{ does not have a non extremal cycle}\}$$

Theorems 3.1.3, 4.1.2, 4.1.3 and the notion of fundamental regions given by Definition 3.1.8 will be the main tools used throughout this chapter. As in Chapter 3 Figure 4.4 gives a summary of the results that we obtain.

Note that $\text{aff}\mathcal{A}_n^G = \text{aff}\mathcal{B}_n^G$ and so $\dim \mathcal{A}_n^G = \dim \mathcal{B}_n^G$ for each subgroup $G$ of $D_4$.

## 4.2 Horizontal symmetry

The second row of Figures 1.17 and 4.1 give:

$$\mathcal{A}_n^{\{1,h\}} = \{a \in \mathcal{A}_n \mid a_{ij} = a_{n+1-i,j} \text{ for all } i,j \in [n]\} \quad (4.3)$$

$$\mathcal{E}_n^{\{1,h\}} = \left\{(h,v) \in \mathcal{E}_n \;\middle|\; \begin{array}{l} \bullet\ h_{ij} = h_{n+1-i,j} \text{ for all } i \in [n],\ j \in [0,n] \\ \bullet\ v_{ij} = 1 - v_{n-i,j} \text{ for all } i \in [0,n],\ j \in [n] \end{array}\right\} \quad (4.4)$$

Figure 4.5 gives the set $\text{ASM}(3,2)^{\{1,h\}}$ and some cardinalities of $\text{ASM}(n,r)^{\{1,h\}}$ are given by Figure 4.6.

| $G \subseteq D_4$ | $\lvert \mathrm{ASM}(n,1)^G \rvert$ | $\dim \mathcal{A}_n^G$ | $\mathrm{vert}\mathcal{A}_n^G$ | $\lvert \mathrm{vert}\mathcal{A}_n^G \rvert$ | $\mathrm{D}(\mathcal{A}_n^G)$ |
|---|---|---|---|---|---|
| $\{1,h\}$ | $\begin{cases} 0, & n \text{ even} \\ \prod_{i=1}^{\frac{n-1}{2}} \frac{(6i-2)!}{(n-1+2i)!}, & n \text{ odd} \end{cases}$ | $\begin{cases} \frac{(n-1)(n-2)}{2}, & n \text{ even} \\ \frac{(n-1)^2}{2}, & n \text{ odd} \end{cases}$ | √ | × | 2 |
| $\{1,h,v,q^2\}$ | $\begin{cases} 0, & n \text{ even} \\ \frac{\left(\lfloor\frac{3(n-3)}{4}\rfloor+1\right)!}{3\lfloor\frac{n-3}{4}\rfloor(n-2)!\lfloor\frac{n-3}{4}\rfloor!} \prod_{i=1}^{\frac{n-3}{2}} \frac{(3i)!}{\left(\frac{n-3}{2}+i\right)!}, & n \text{ odd} \end{cases}$ | $\begin{cases} \frac{(n-2)^2}{4}, & n \text{ even} \\ \frac{(n-1)^2}{4}, & n \text{ odd} \end{cases}$ | √ | × | 2 |
| $\{1,q^2\}$ | $\prod_{i=0}^{\lceil\frac{n}{2}\rceil-1} \frac{(3i)!}{(\lfloor\frac{n}{2}\rfloor+i)!} \prod_{i=0}^{\lfloor\frac{n}{2}\rfloor-1} \frac{(3i+2)!}{(\lceil\frac{n}{2}\rceil+i)!}$ | $\begin{cases} \frac{(n-1)^2+1}{2}, & n \text{ even} \\ \frac{(n-1)^2}{2}, & n \text{ odd} \end{cases}$ | √ | $\lvert \mathrm{ASM}(n,1)^{\{1,q^2\}} \rvert$ | 1 |
| $\{1,q,q^2,q^3\}$ | $\begin{cases} \prod_{i=0}^{\frac{n}{4}-1} \frac{i!(3i+1)!\left(\frac{n}{2}+i\right)!}{\left(\frac{n}{4}+i\right)!^3}, & n = 0 \bmod 4 \\ \prod_{i=0}^{\frac{n-1}{4}} \frac{(3i)!}{\left(\frac{n-1}{4}+i\right)!} \prod_{i=0}^{\frac{n-5}{4}} \frac{(3i+1)!^2(3i+2)!}{\left(\frac{n-1}{4}+i\right)!\left(\frac{n+3}{4}+i\right)!}, & n = 1 \bmod 4 \\ 0, & n = 2 \bmod 4 \\ \prod_{i=0}^{\frac{n-3}{4}} \frac{(3i)!(3i+1)!^2}{\left(\frac{n-3}{4}+i\right)!\left(\frac{n+1}{4}+i\right)!} \prod_{i=0}^{\frac{n-7}{4}} \frac{(3i+2)!}{\left(\frac{n+1}{4}+i\right)!}, & n = 3 \bmod 4 \end{cases}$ | $\begin{cases} \frac{n(n-2)}{4}, & n \text{ even} \\ \frac{(n-1)^2}{4}, & n \text{ odd} \end{cases}$ | √ | × | 2 |
| $\{1,d\}$ | × | $\frac{n(n-1)}{2}$ | √ | × | 1 |
| $\{1,d,a,q^2\}$ | $\begin{cases} \times, & n \text{ even} \\ ?, & n \text{ odd} \end{cases}$ | $\begin{cases} \frac{n^2}{4}, & n \text{ even} \\ \frac{(n-1)(n+1)}{4}, & n \text{ odd} \end{cases}$ | √ | $\begin{cases} \times, & n \text{ even} \\ ?, & n \text{ odd} \end{cases}$ | 1 |
| $D_4$ | × | $\begin{cases} \frac{n(n-2)}{8}, & n \text{ even} \\ \frac{(n-1)(n+1)}{8}, & n \text{ odd} \end{cases}$ | √ | × | 2 |

Figure 4.4: Table of results for $\mathcal{A}_n^G$

$$\left\{ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 0 \\ 2 & -2 & 2 \\ 0 & 2 & 0 \end{pmatrix} \right\}$$

Figure 4.5: Horizontally symmetric alternating sign matrices of size 3 and line sum 2

| | $r=0$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $n=1$ | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 0 | 1 | 0 | 1 | 0 |
| 3 | 1 | 1 | 4 | 4 | 9 | 9 |
| 4 | 1 | 0 | 8 | 0 | 27 | 0 |
| 5 | 1 | 3 | 124 | 244 | 2448 | 3960 |
| 6 | 1 | 0 | 459 | 0 | 20682 | 0 |
| 7 | 1 | 26 | 27552 | 225560 | 18687186 | 83760732 |

Figure 4.6: $\lvert \mathrm{ASM}(n,r)^{\{1,h\}} \rvert$ for $n \in [7]$, $r \in [0,5]$

Recalling (1.32) we have:

$$\left|\mathrm{ASM}(n,1)^{\{1,h\}}\right| = \begin{cases} 0, \; n \text{ even} \\ \prod_{i=1}^{\frac{n-1}{2}} \frac{(6i-2)!}{(n-1+2i)!}, \; n \text{ odd} \end{cases}$$

For $n$ odd this enumeration corresponds to sequence A005156 of [99]. Moving onto $\mathcal{A}_n^{\{1,h\}}$ we have the following fundamental polytopes:

Recalling Figure 3.1 $R_{2k}^{\{1,h\}} = [k] \times [2k]$ and $R_{2k+1}^{\{1,h\}} = [k+1] \times [2k+1]$ giving:

$$\overline{\mathcal{A}_{2k}^{\{1,h\}}} = \left\{ a \in \mathbb{R}^{k \times 2k} \;\middle|\; \begin{array}{l} \bullet\, 0 \le \sum_{j'=1}^{j} a_{ij'} \le 1 \text{ for all }, i \in [k],\; j \in [2k] \\ \bullet\, 0 \le \sum_{i'=1}^{i} a_{i'j} \le 1 \text{ for all } i \in [k],\; j \in [2k] \\ \bullet\, \sum_{j=1}^{2k} a_{ij} = 2\sum_{i=1}^{k} a_{ij} = 1 \text{ for all } i \in [k],\; j \in [2k] \end{array} \right\}$$

and

$$\overline{\mathcal{A}_{2k+1}^{\{1,h\}}} = \left\{ a \in \mathbb{R}^{(k+1) \times (2k+1)} \;\middle|\; \begin{array}{l} \bullet\, 0 \le \sum_{j'=1}^{j} a_{ij'} \le 1 \text{ for all } i \in [k+1],\; j \in [2k+1] \\ \bullet\, 0 \le \sum_{i'=1}^{i} a_{i'j} \le 1 \text{ for all } i \in [k+1],\; j \in [2k+1] \\ \bullet\, \sum_{j=1}^{2k+1} a_{ij} = 2\sum_{i=1}^{k} a_{ij} + a_{k+1,j} = 1 \text{ for all } i \in [k+1], \\ \hfill j \in [2k+1] \end{array} \right\}$$

We thus have:

$$\overline{\mathcal{E}_{2k}^{\{1,h\}}} = \left\{ (h,v) = \left( \begin{pmatrix} h_{10} & \cdots & h_{1,2k} \\ \vdots & & \vdots \\ h_{k0} & \cdots & h_{k,2k} \end{pmatrix}, \begin{pmatrix} v_{01} & \cdots & v_{0,2k} \\ \vdots & & \vdots \\ v_{k1} & \cdots & v_{k,2k} \end{pmatrix} \right) \in [0,1]_{\mathbb{R}}^{k \times (2k+1)} \times [0,1]_{\mathbb{R}}^{(k+1) \times 2k} \;\middle|\; \begin{array}{l} \bullet\, h_{i0} = v_{0j} = 0 \text{ for all } i \in [k],\; j \in [2k] \\ \bullet\, h_{i,2k} = 2v_{kj} = 1 \text{ for all } i \in [k],\; j \in [2k] \\ \bullet\, h_{i,j-1} + v_{ij} = v_{i-1,j} + h_{ij} \text{ for all } i \in [k],\; j \in [2k] \end{array} \right\}$$

and

$$\overline{\mathcal{E}_{2k+1}^{\{1,h\}}} = \left\{ (h,v) = \left( \begin{pmatrix} h_{10} & \cdots & h_{1,2k+1} \\ \vdots & & \vdots \\ h_{k+1,0} & \cdots & h_{k+1,2k+1} \end{pmatrix}, \begin{pmatrix} v_{01} & \cdots & v_{0,2k+1} \\ \vdots & & \vdots \\ v_{k+1,1} & \cdots & v_{k+1,2k+1} \end{pmatrix} \right) \in [0,1]_{\mathbb{R}}^{(k+1) \times (2k+2)} \times [0,1]_{\mathbb{R}}^{(k+2) \times (2k+1)} \;\middle|\; \begin{array}{l} \bullet\, h_{i0} = v_{0j} = 0 \text{ for all } i \in [k+1],\; j \in [2k] \\ \bullet\, h_{i2k+1} = v_{kj} + v_{k+1,j} = 1 \text{ for all } i \in [k+1],\; j \in [2k+1] \\ \bullet\, h_{i,j-1} + v_{ij} = v_{i-1,j} + h_{ij} \text{ for all } i \in [k+1],\; j \in [2k+1] \end{array} \right\}$$

where $f'_{\{1,h\}} : \mathcal{E}_{2k}^{\{1,h\}} \to \overline{\mathcal{E}_{2k}^{\{1,h\}}}$ is given by:

$$f'_{\{1,h\}}\left(\begin{pmatrix} 0 & h_{11} & \dots & h_{1,2k-1} & 1 \\ \vdots & & & & \vdots \\ 0 & h_{k1} & \dots & h_{k,2k-1} & 1 \\ 0 & h_{k1} & \dots & h_{k,2k-1} & 1 \\ \vdots & & & & \vdots \\ 0 & h_{11} & \dots & h_{1,2k-1} & 1 \end{pmatrix}, \begin{pmatrix} 0 & \dots & 0 \\ v_{11} & \dots & v_{1,2k} \\ \vdots & & \vdots \\ v_{k-1,1} & \dots & v_{k-1,2k} \\ \frac{1}{2} & \dots & \frac{1}{2} \\ 1-v_{k-1,1} & \dots & 1-v_{k-1,2k} \\ \vdots & & \vdots \\ 1-v_{11} & \dots & 1-v_{1,2k} \\ 1 & \dots & 1 \end{pmatrix}\right) =$$

$$\left(\begin{pmatrix} 0 & h_{11} & \dots & h_{1,2k-1} & 1 \\ \vdots & & & & \vdots \\ 0 & h_{k,} & \dots & h_{k,2k-1} & 1 \end{pmatrix}, \begin{pmatrix} 0 & \dots & 0 \\ v_{11} & \dots & v_{1,2k} \\ \vdots & & \vdots \\ v_{k-1,1} & \dots & v_{k-1,2k} \\ \frac{1}{2} & \dots & \frac{1}{2} \end{pmatrix}\right)$$

and $f'_{\{1,h\}} : \mathcal{E}_{2k+1}^{\{1,h\}} \to \overline{\mathcal{E}_{2k+1}^{\{1,h\}}}$ is given by:

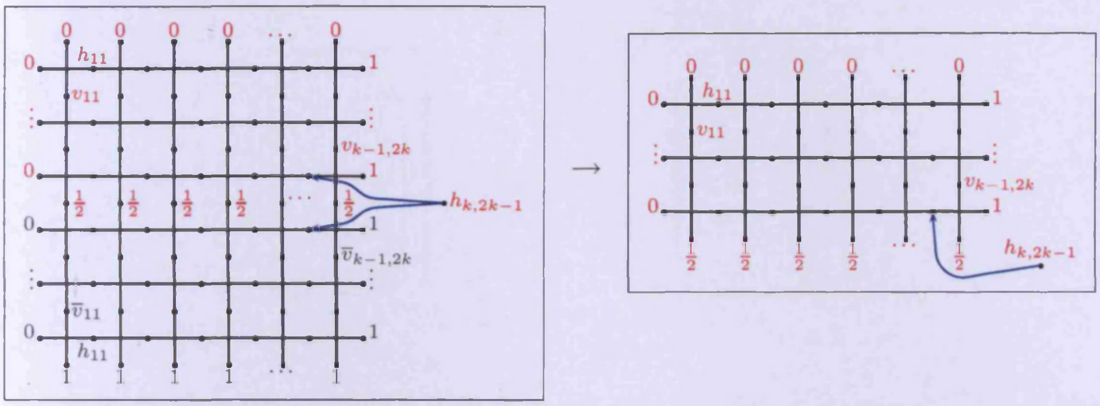
$$f'_{\{1,h\}}\left(\begin{pmatrix} 0 & h_{11} & \dots & h_{1,2k} & 1 \\ \vdots & & & & \vdots \\ 0 & h_{k1} & \dots & h_{k,2k} & 1 \\ 0 & h_{k+1,1} & \dots & h_{k+1,2k} & 1 \\ 0 & h_{k1} & \dots & h_{k,2k} & 1 \\ \vdots & & & & \vdots \\ 0 & h_{11} & \dots & h_{1,2k-1} & 1 \end{pmatrix}, \begin{pmatrix} 0 & \dots & 0 \\ v_{11} & \dots & v_{1,2k+1} \\ \vdots & & \vdots \\ v_{k1} & \dots & v_{k,2k+1} \\ 1-v_{k1} & \dots & 1-v_{k,2k+1} \\ \vdots & & \vdots \\ 1-v_{11} & \dots & 1-v_{1,2k+1} \\ 1 & \dots & 1 \end{pmatrix}\right) =$$

$$\left(\begin{pmatrix} 0 & h_{11} & \dots & h_{1,2k} & 1 \\ \vdots & & & & \vdots \\ 0 & h_{k1} & \dots & h_{k,2k} & 1 \\ 0 & h_{k+1,1} & \dots & h_{k+1,2k} & 1 \end{pmatrix}, \begin{pmatrix} 0 & \dots & 0 \\ v_{11} & \dots & v_{1,2k+1} \\ \vdots & & \vdots \\ v_{k1} & \dots & v_{k,2k+1} \\ 1-v_{k1} & \dots & 1-v_{k,2k+1} \end{pmatrix}\right)$$

These fundamental polytopes can be represented on a lattice as in Figures 4.7 and 4.8.

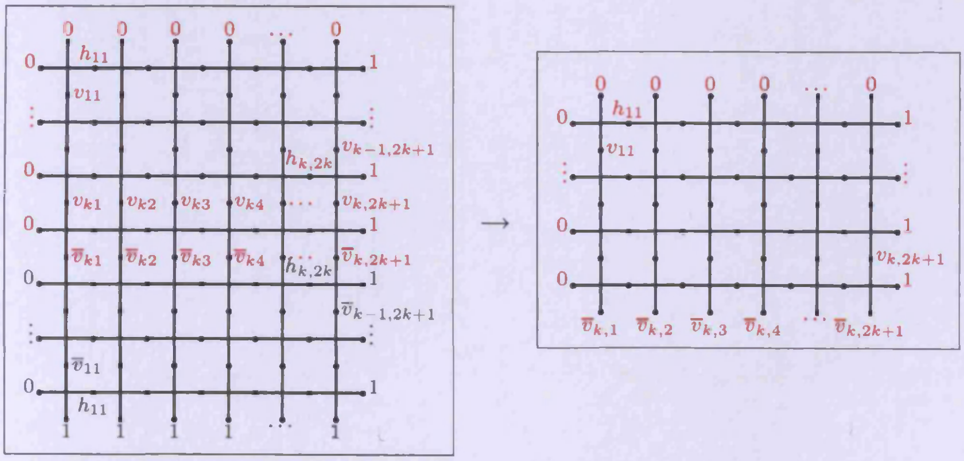
**Theorem 4.2.1.**

$$vert\mathcal{A}_n^{\{1,h\}} = \left\{ a \in \Pi_{\{1,h\}}\left(ASM(n,1)\right) \;\middle|\; \begin{array}{l} f_{\{1,h\}}(a) \textit{ on a lattice does not have a} \\ \textit{non integer cycle} \end{array} \right\}$$




where $\overline{m} = 1 - m$ for all $m \in \mathbb{R}$

Figure 4.7: Elements of $\mathcal{E}_{2k}^{\{1,h\}}$ and $\overline{\mathcal{E}_{2k}^{\{1,h\}}}$ on $\mathcal{L}_{2k,2k}$ and $\mathcal{L}_{k,2k}$




where $\overline{m} = 1 - m$ for all $m \in \mathbb{R}$

Figure 4.8: Elements of $\mathcal{E}_{2k+1}^{\{1,h\}}$ and $\overline{\mathcal{E}_{2k+1}^{\{1,h\}}}$ on $\mathcal{L}_{2k+1,2k+1}$ and $\mathcal{L}_{2k+1,k}$

*Proof.* For $n$ even recalling Definition 4.1.1 we see that $\overline{\mathcal{A}_{2k}^{\{1,h\}}} = \Lambda(H, H', V, V')$ with:

$$H = \underbrace{\begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}}_{2k+1} \Bigg\} k \quad H' = \underbrace{\begin{pmatrix} 0 & 1 & \dots & 1 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \dots & 1 & 1 \end{pmatrix}}_{2k+1} \Bigg\} k$$

$$V = \underbrace{\begin{pmatrix} 0 & \dots & 0 \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \\ \frac{1}{2} & \dots & \frac{1}{2} \end{pmatrix}}_{2k} \Bigg\} k+1 \quad V' = \underbrace{\begin{pmatrix} 0 & \dots & 0 \\ 1 & \dots & 1 \\ \vdots & & \vdots \\ 1 & \dots & 1 \\ \frac{1}{2} & \dots & \frac{1}{2} \end{pmatrix}}_{2k} \Bigg\} k+1$$

For $n$ odd consider the affine map $\rho$ defined by (3.12). The effect of $\rho$ on $\overline{\mathcal{E}_{2k+1}^{\{1,h\}}}$ is to map

$$\left( \begin{pmatrix} 0 & h_{11} & \dots & h_{1,2k} & 1 \\ \vdots & & & & \vdots \\ 0 & h_{k1} & \dots & h_{k,2k} & 1 \\ 0 & h_{k+1,1} & \dots & h_{k+1,2k} & 1 \end{pmatrix}, \begin{pmatrix} 0 & \dots & 0 \\ v_{11} & \dots & v_{1,2k+1} \\ \vdots & & \vdots \\ v_{k1} & \dots & v_{k,2k+1} \\ 1 - v_{k1} & \dots & 1 - v_{k,2k+1} \end{pmatrix} \right)$$

to

$$\left( \begin{pmatrix} 0 & h_{11} & \dots & h_{1,2k} & 1 \\ \vdots & & & & \vdots \\ 0 & h_{k1} & \dots & h_{k,2k} & 1 \\ 0 & \frac{h_{k+1,1}}{2} & \dots & \frac{h_{k+1,2k}}{2} & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 & \dots & 0 \\ v_{11} & \dots & v_{1,2k+1} \\ \vdots & & \vdots \\ v_{k1} & \dots & v_{k,2k+1} \\ \frac{1}{2} & \dots & \frac{1}{2} \end{pmatrix} \right)$$

Thus $\rho$ is bijective from $\overline{\mathcal{A}_{2k+1}^{\{1,h\}}}$ to $\Lambda(H, H', V, V')$ with:

$$H = \underbrace{\begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & \frac{1}{2} \end{pmatrix}}_{2k+2} \Bigg\} k+1 \quad H' = \underbrace{\begin{pmatrix} 0 & 1 & \dots & 1 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \dots & 1 & 1 \\ 0 & \frac{1}{2} & \dots & \frac{1}{2} & \frac{1}{2} \end{pmatrix}}_{2k+2} \Bigg\} k+1$$

$$V = \underbrace{\begin{pmatrix} 0 & \dots & 0 \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \\ \frac{1}{2} & \dots & \frac{1}{2} \end{pmatrix}}_{2k+1} \Bigg\} k+2 \quad V' = \underbrace{\begin{pmatrix} 0 & \dots & 0 \\ 1 & \dots & 1 \\ \vdots & & \vdots \\ 1 & \dots & 1 \\ \frac{1}{2} & \dots & \frac{1}{2} \end{pmatrix}}_{2k+1} \Bigg\} k+2$$

and the result follows from Theorem 4.1.3. □

$$\begin{pmatrix}0&\frac12&\frac12\\1&0&0\\0&\frac12&\frac12\end{pmatrix}\quad\begin{pmatrix}0&1&0\\1&-1&1\\0&1&0\end{pmatrix}\quad\begin{pmatrix}\frac12&0&\frac12\\0&1&0\\\frac12&0&\frac12\end{pmatrix}\quad\begin{pmatrix}\frac12&\frac12&0\\0&0&1\\\frac12&\frac12&0\end{pmatrix}$$

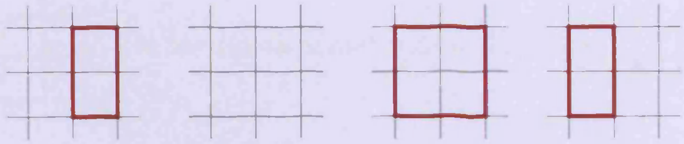
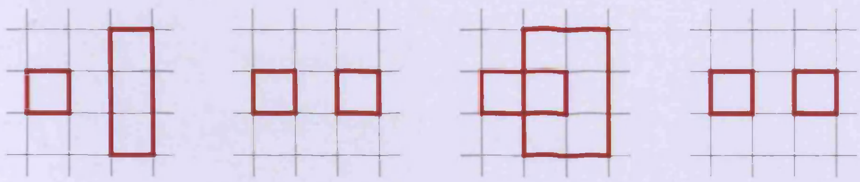


Figure 4.9: $\text{vert}\mathcal{A}_3^{\{1,h\}}$ and the corresponding non integer edges

$$\begin{pmatrix}0&0&\frac12&\frac12\\\frac12&\frac12&0&0\\\frac12&\frac12&0&0\\0&0&\frac12&\frac12\end{pmatrix}\quad\begin{pmatrix}0&0&1&0\\\frac12&\frac12&-\frac12&\frac12\\\frac12&\frac12&-\frac12&\frac12\\0&0&1&0\end{pmatrix}\quad\begin{pmatrix}0&\frac12&0&\frac12\\\frac12&0&\frac12&0\\\frac12&0&\frac12&0\\0&\frac12&0&\frac12\end{pmatrix}\quad\begin{pmatrix}0&1&0&0\\\frac12&-\frac12&\frac12&\frac12\\\frac12&-\frac12&\frac12&\frac12\\0&1&0&0\end{pmatrix}$$



$$\begin{pmatrix}\frac12&0&0&\frac12\\0&\frac12&\frac12&0\\0&\frac12&\frac12&0\\\frac12&0&0&\frac12\end{pmatrix}\quad\begin{pmatrix}\frac12&0&\frac12&0\\0&\frac12&0&\frac12\\0&\frac12&0&\frac12\\\frac12&0&\frac12&0\end{pmatrix}\quad\begin{pmatrix}\frac12&\frac12&0&0\\0&0&\frac12&\frac12\\0&0&\frac12&\frac12\\\frac12&\frac12&0&0\end{pmatrix}$$

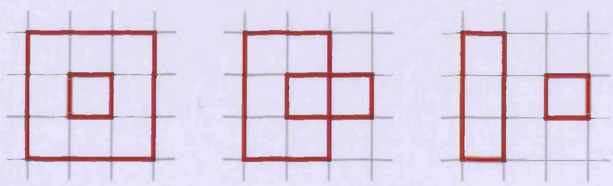
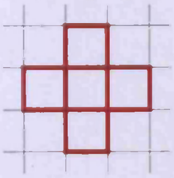
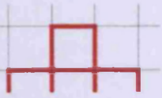


Figure 4.10: $\text{vert}\mathcal{A}_4^{\{1,h\}}$ and the corresponding non integer edges

In Figures 4.9, and 4.10 we illustrate Theorem 4.2.1 with the sets vert$\mathcal{A}_3^{\{1,h\}}$ and vert$\mathcal{A}_4^{\{1,h\}}$.

We note that $\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \in \Pi_{\{1,h\}}(\mathrm{ASM}(4,1))$ however the non integer edges for this matrix are: . In the fundamental region this gives:  which has a non integer cycle. Using this classification it is possible to identify vert$\mathcal{A}_n^{\{1,h\}}$ for given $n$. Importantly we have:

**Corollary 4.2.2.**

$$D(\mathcal{A}_n^{\{1,h\}}) = 2,\ n \geq 2$$

A consequence of Theorems 1.2.18 and Corollary 4.2.2 is:

**Theorem 4.2.3.** *For fixed $n \in \mathbb{P}$ there exists $A_n^{\{1,h\}}(r)$, the Ehrhart quasi-polynomial of $\mathcal{A}_n^{\{1,h\}}$ which satisfies:*

1. *$A_n^{\{1,h\}}(r)$ is a quasi-polynomial in $r$ of degree $\dim \mathcal{A}_n^{\{1,h\}}$ and period which divides 2.*
2. *$\left|ASM(n,r)^{\{1,h\}}\right| = A_n^{\{1,h\}}(r)$ for all $r \in \mathbb{N}$*
3. *$\left|ASM^{o}(n,r)^{\{1,h\}}\right| = (-1)^{\dim \mathcal{A}_n^{\{1,h\}}} A_n^{\{1,h\}}(-r)$ for all $r \in \mathbb{P}$*

The following enumerations illustrate this theorem.

$$A_2^{\{1,h\}}(r) = \begin{cases} 1, & r \text{ even} \\ 0, & r \text{ odd} \end{cases} \tag{4.5}$$

$$A_3^{\{1,h\}}(r) = \binom{\lfloor \frac{r}{2} \rfloor + 1}{2} + \binom{\lfloor \frac{r}{2} \rfloor + 2}{2} \tag{4.6}$$

$$A_4^{\{1,h\}}(r) = \begin{cases} \binom{\frac{r}{2}+1}{3} + 4\binom{\frac{r}{2}+2}{3} + \binom{\frac{r}{2}+3}{3}, & r \text{ even} \\ 0, & r \text{ odd} \end{cases} \tag{4.7}$$

$$A_5^{\{1,h\}}(r) = \tag{4.8}$$

$$\begin{cases} 12\binom{\frac{r}{2}+2}{8} + 459\binom{\frac{r}{2}+3}{8} + 2593\binom{\frac{r}{2}+4}{8} + 3628\binom{\frac{r}{2}+5}{8} + 1368\binom{\frac{r}{2}+6}{8} + 115\binom{\frac{r}{2}+7}{8} + \binom{\frac{r}{2}+8}{8}, & r \text{ even} \\ 4\binom{\frac{r-1}{2}+2}{8} + 261\binom{\frac{r-1}{2}+3}{8} + 2047\binom{\frac{r-1}{2}+4}{8} + 3772\binom{\frac{r-1}{2}+5}{8} + 1872\binom{\frac{r-1}{2}+6}{8} + 217\binom{\frac{r-1}{2}+7}{8} + 3\binom{\frac{r-1}{2}+8}{8}, & r \text{ odd} \end{cases}$$

Sequence (4.6) is equivalent to A008794 of [99] and coincidentally this sequence enumerates solutions of a chess problem (of the Kings).

## 4.3 Horizontal and vertical symmetry

The fifth row of Figures 1.17 and 4.1 give:

$$\mathcal{A}_n^{\{1,h,v,q^2\}} = \{a \in \mathcal{A}_n \mid a_{ij} = a_{n+1-i,j} = a_{i,n+1-j} \text{ for all } i,j \in [n]\} \tag{4.9}$$

$$\mathcal{E}_n^{\{1,h,v,q^2\}} = \left\{(h,v) \in \mathcal{E}_n \;\middle|\; \begin{array}{l} \bullet\ h_{ij} = h_{n+1-i,j} = 1 - h_{i,n-j} \text{ for all } i \in [n],\ j \in [0,n] \\ \bullet\ v_{ij} = v_{i,n+1-j} = 1 - v_{n-i,j} \text{ for all } i \in [0,n],\ j \in [n] \end{array}\right\} \tag{4.10}$$

Figure 4.11 gives the set $\mathrm{ASM}(3,5)^{\{1,h,v,q^2\}}$ and some cardinalities of $\mathrm{ASM}(n,r)^{\{1,h,v,q^2\}}$ are given by Figure 4.12.

$$\left\{ \begin{pmatrix} 0 & 5 & 0 \\ 5 & -5 & 5 \\ 0 & 5 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 3 & 1 \\ 3 & -1 & 3 \\ 1 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 2 \end{pmatrix} \right\}$$

Figure 4.11: Horizontally and vertically symmetric alternating sign matrices of size 3 and line sum 5

| | $r=0$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $n=1$ | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 0 | 1 | 0 | 1 | 0 |
| 3 | 1 | 1 | 2 | 2 | 3 | 3 |
| 4 | 1 | 0 | 2 | 0 | 3 | 0 |
| 5 | 1 | 1 | 10 | 10 | 42 | 42 |
| 6 | 1 | 0 | 11 | 0 | 48 | 0 |

Figure 4.12: $|\mathrm{ASM}(n,r)^{\{1,h,v,q^2\}}|$ for $n \in [6]$, $r \in [0,5]$

Recalling (1.36) we have: $\left|\mathrm{ASM}(n,1)^{\{1,h,v,q^2\}}\right| = \begin{cases} 0, & n \text{ even} \\ \frac{(\lfloor \frac{3(n-3)}{4} \rfloor + 1)!}{3\lfloor \frac{n-3}{4} \rfloor (n-2)! \lfloor \frac{n-3}{4} \rfloor !} \prod_{i=1}^{\frac{n-3}{2}} \frac{(3i)!}{(\frac{n-3}{2}+i)!}, & n \text{ odd} \end{cases}$

equivalent to sequence A005161 of [99].

Recalling Figure 3.1 $R_{2k}^{\{1,h,v,q^2\}} = [k] \times [k]$ and $R_{2k+1}^{\{1,h,v,q^2\}} = [k+1] \times [k+1]$. Thus:

$$\overline{\mathcal{A}_{2k}^{\{1,h,v,q^2\}}} = \left\{ a \in \mathbb{R}^{k \times k} \;\middle|\; \begin{array}{l} \bullet\ 0 \le \sum_{j'=1}^{j} a_{ij'} \le 1 \text{ for all } i,j \in [k] \\ \bullet\ 0 \le \sum_{i'=1}^{i} a_{i'j} \le 1 \text{ for all } i,j \in [k] \\ \bullet\ \sum_{j=1}^{k} a_{ij} = \sum_{i=1}^{k} a_{ij} = \frac{1}{2} \text{ for all } i \in [k],\ j \in [k] \end{array} \right\}$$

and

$$\overline{\mathcal{A}_{2k+1}^{\{1,h,v,q^2\}}} = \left\{ a \in \mathbb{R}^{(k+1) \times (k+1)} \;\middle|\; \begin{array}{l} \bullet\ 0 \le \sum_{j'=1}^{j} a_{ij'} \le 1 \text{ for all } i,j \in [k+1] \\ \bullet\ 0 \le \sum_{i'=1}^{i} a_{i'j} \le 1 \text{ for all } i,j \in [k+1] \\ \bullet\ 2\sum_{j=1}^{k} a_{ij} + a_{i,k+1} = 2\sum_{i=1}^{k} a_{ij} + a_{k+1,j} = 1 \text{ for all } \\ \qquad i,j \in [k+1] \end{array} \right\}$$

We thus have:

$$\overline{\mathcal{E}_{2k}^{\{1,h,v,q^2\}}} = \left\{ (h,v) = \left( \begin{pmatrix} h_{10} & \dots & h_{1k} \\ \vdots & & \vdots \\ h_{k0} & \dots & h_{kk} \end{pmatrix}, \begin{pmatrix} v_{01} & \dots & v_{0k} \\ \vdots & & \vdots \\ v_{k1} & \dots & v_{kk} \end{pmatrix} \right) \in [0,1]_{\mathbb{R}}^{k\times(k+1)} \times [0,1]_{\mathbb{R}}^{(k+1)\times k} \;\middle|\; \begin{array}{l} \bullet\ h_{i0} = v_{0j} = 0 \text{ for all } i \in [k],\ j \in [k] \\ \bullet\ h_{ik} = v_{kj} = \frac{1}{2} \text{ for all } i,j \in [k] \\ \bullet\ h_{i,j-1} + v_{ij} = v_{i-1,j} + h_{ij} \text{ for all } i,j \in [k] \end{array} \right\}$$

and

$$\overline{\mathcal{E}_{2k+1}^{\{1,h,v,q^2\}}} = \left\{ (h,v) = \left( \begin{pmatrix} h_{10} & \dots & h_{1,k+1} \\ \vdots & & \vdots \\ h_{k+1,0} & \dots & h_{k+1,k+1} \end{pmatrix}, \begin{pmatrix} v_{01} & \dots & v_{0,k+1} \\ \vdots & & \vdots \\ v_{k+1,1} & \dots & v_{k+1,k+1} \end{pmatrix} \right) \in [0,1]_{\mathbb{R}}^{(k+1)\times(k+2)} \times [0,1]_{\mathbb{R}}^{(k+2)\times(k+1)} \;\middle|\; \begin{array}{l} \bullet\ h_{i0} = v_{0j} = 0 \text{ for all } i,j \in [k+1] \\ \bullet\ h_{ik} + h_{i,k+1} = v_{kj} + v_{k+1,j} = 1 \text{ for all } i,j \in [k+1] \\ \bullet\ h_{i,j-1} + v_{ij} = v_{i-1,j} + h_{ij} \text{ for all } i,j \in [k+1] \end{array} \right\}$$

where $f'_{\{1,h,v,q^2\}} : \mathcal{E}_{2k}^{\{1,h,v,q^2\}} \longrightarrow \overline{\mathcal{E}_{2k}^{\{1,h,v,q^2\}}}$ is given by:

$$f'_{\{1,h,v,q^2\}} \left( \begin{pmatrix} 0 & h_{11} & \dots & h_{1,k-1} & \frac{1}{2} & 1-h_{1,k-1} & \dots & 1-h_{11} & 1 \\ \vdots & & & & \vdots & \vdots & & & \vdots \\ 0 & h_{k1} & \dots & h_{k,k-1} & \frac{1}{2} & 1-h_{k,k-1} & \dots & 1-h_{k1} & 1 \\ 0 & h_{k1} & \dots & h_{k,k-1} & \frac{1}{2} & 1-h_{k,k-1} & \dots & 1-h_{k1} & 1 \\ \vdots & & & & \vdots & \vdots & & & \vdots \\ 0 & h_{11} & \dots & h_{1,k-1} & \frac{1}{2} & 1-h_{1,k-1} & \dots & 1-h_{11} & 1 \end{pmatrix}, \begin{pmatrix} 0 & \dots & 0 & 0 & \dots & 0 \\ v_{11} & \dots & v_{1k} & v_{1k} & \dots & v_{11} \\ \vdots & & \vdots & \vdots & & \vdots \\ v_{k-1,1} & \dots & v_{k-1,k} & v_{k-1,k} & \dots & v_{k-1,1} \\ \frac{1}{2} & \dots & \frac{1}{2} & \frac{1}{2} & \dots & \frac{1}{2} \\ 1-v_{k-1,1} & \dots & 1-v_{k-1,k} & 1-v_{k-1,k} & \dots & 1-v_{k-1,1} \\ \vdots & & \vdots & \vdots & & \vdots \\ 1-v_{11} & \dots & 1-v_{1k} & 1-v_{1k} & \dots & 1-v_{11} \\ 1 & \dots & 1 & 1 & \dots & 1 \end{pmatrix} \right) =$$

$$\left( \begin{pmatrix} 0 & h_{11} & \dots & h_{1,k-1} & \frac{1}{2} \\ \vdots & & & & \vdots \\ 0 & h_{k1} & \dots & h_{k,k-1} & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 & \dots & 0 \\ v_{11} & \dots & v_{1k} \\ \vdots & & \vdots \\ v_{k-1,1} & \dots & v_{k-1,k} \\ \frac{1}{2} & \dots & \frac{1}{2} \end{pmatrix} \right)$$

and $f'_{\{1,h,v,q^2\}} : \mathcal{E}_{2k+1}^{\{1,h,v,q^2\}} \longrightarrow \overline{\mathcal{E}_{2k+1}^{\{1,h,v,q^2\}}}$ is given by:

$$f'_{\{1,h,v,q^2\}}\left(\begin{pmatrix} 0 & h_{11} & \dots & h_{1k} & 1-h_{1k} & \dots & 1-h_{11} & 1 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & h_{k1} & \dots & h_{kk} & 1-h_{kk} & \dots & 1-h_{k1} & 1 \\ 0 & h_{k+1,1} & \dots & h_{k+1,k} & 1-h_{k+1,k} & \dots & 1-h_{k+1,1} & 1 \\ 0 & h_{k1} & \dots & h_{kk} & 1-h_{kk} & \dots & h_{k1} & 1 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & h_{11} & \dots & h_{1k} & 1-h_{1k} & \dots & 1-h_{11} & 1 \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ v_{11} & \dots & v_{1k} & v_{1,k+1} & v_{1k} & \dots & v_{11} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ v_{k1} & \dots & v_{kk} & v_{k,k+1} & v_{kk} & \dots & v_{k1} \\ 1-v_{k1} & \dots & 1-v_{kk} & 1-v_{k,k+1} & 1-v_{kk} & \dots & 1-v_{k1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1-v_{11} & \dots & 1-v_{1k} & 1-v_{1,k+1} & 1-v_{1k} & \dots & 1-v_{11} \\ 1 & \dots & 1 & 1 & 1 & \dots & 1 \end{pmatrix}\right) =$$

$$\left(\begin{pmatrix} 0 & h_{11} & \dots & 1-h_{1k} \\ \vdots & \vdots & & \vdots \\ 0 & h_{k1} & \dots & 1-h_{kk} \\ 0 & h_{k+1,1} & \dots & 1-h_{k+1,k} \end{pmatrix}, \begin{pmatrix} 0 & \dots & 0 \\ v_{11} & \dots & v_{1,k+1} \\ \vdots & & \vdots \\ v_{k1} & \dots & v_{k,k+1} \\ 1-v_{k,1} & \dots & 1-v_{k,k+1} \end{pmatrix}\right)$$

These fundamental polytopes can be represented on a lattice as in Figures 4.13 and 4.14.

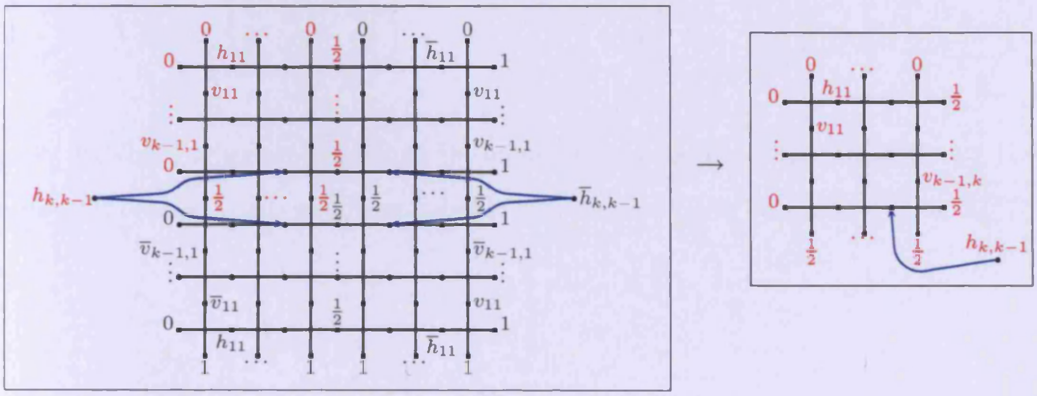



Figure 4.13: Elements of $\mathcal{E}_{2k}^{\{1,h,v,q^2\}}$ and $\overline{\mathcal{E}_{2k}^{\{1,h,v,q^2\}}}$ on a $\mathcal{L}_{2k,2k}$ and $\mathcal{L}_{k,k}$

**Theorem 4.3.1.**

$$vert\mathcal{A}_n^{\{1,h,v,q^2\}} = \left\{ a \in \Pi_{\{1,h,v,q^2\}}(ASM(n,1)) \;\middle|\; \begin{array}{l} f_{\{1,h,v,q^2\}}(a) \textit{ on a lattice does not have a} \\ \textit{non integer cycle} \end{array} \right\}$$

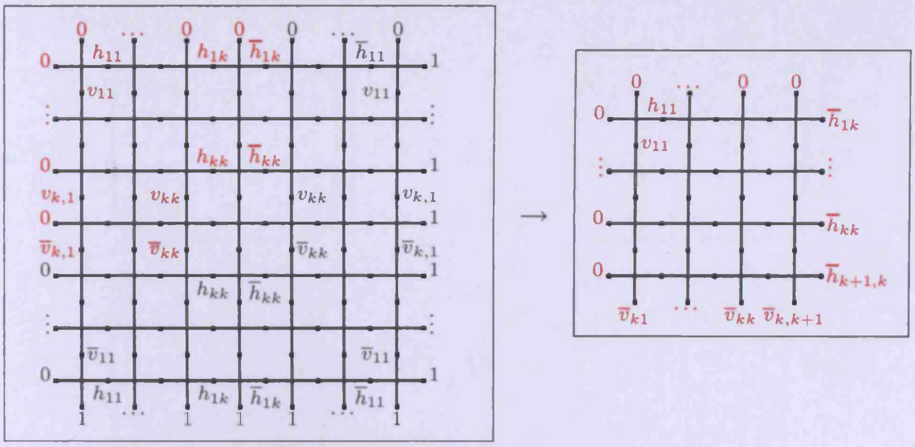



Figure 4.14: Elements of $\mathcal{E}_{2k+1}^{\{1,h,v,q^2\}}$ and $\overline{\mathcal{E}_{2k+1}^{\{1,h,v,q^2\}}}$ on $\mathcal{L}_{2k+1,2k+1}$ and $\mathcal{L}_{k+1,k+1}$

*Proof.* As before the proof of this result follows immediately for $n$ even as: $\overline{\mathcal{A}_{2k}^{\{1,h\}}} = \Lambda(H, H', V, V')$ with:

$$H = \underbrace{\begin{pmatrix} 0 & 0 & \dots & 0 & \frac{1}{2} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \frac{1}{2} \end{pmatrix}}_{k+1} \Bigg\} k \quad H' = \underbrace{\begin{pmatrix} 0 & 1 & \dots & 1 & \frac{1}{2} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \dots & 1 & \frac{1}{2} \end{pmatrix}}_{k+1} \Bigg\} k$$

$$V = \underbrace{\begin{pmatrix} 0 & \dots & 0 \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \\ \frac{1}{2} & \dots & \frac{1}{2} \end{pmatrix}}_{k} \Bigg\} k+1 \qquad V' = \underbrace{\begin{pmatrix} 0 & \dots & 0 \\ 1 & \dots & 1 \\ \vdots & & \vdots \\ 1 & \dots & 1 \\ \frac{1}{2} & \dots & \frac{1}{2} \end{pmatrix}}_{k} \Bigg\} k+1$$

The proof for the $n$ odd case follows in the same way as the proof for Theorem 4.2.1. As for the proof of Theorem 3.3.1 with a particular affine map it can be checked that $\overline{\mathcal{A}_{2k+1}^{\{1,h,v,q^2\}}}$ is affinely isomorphic to $\Lambda(H, H', V, V')$ with:

$$H = \left.\underbrace{\begin{pmatrix} 0 & 0 & \dots & 0 & \frac{1}{2} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \frac{1}{2} \\ 0 & 0 & \dots & 0 & \frac{1}{4} \end{pmatrix}}_{k+2}\right\} k+1 \quad H' = \left.\underbrace{\begin{pmatrix} 0 & 1 & \dots & 1 & \frac{1}{2} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \dots & 1 & \frac{1}{2} \\ 0 & \frac{1}{2} & \dots & \frac{1}{2} & \frac{1}{4} \end{pmatrix}}_{k+2}\right\} k+1$$

$$V = \left.\underbrace{\begin{pmatrix} 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & 0 \\ \frac{1}{2} & \dots & \frac{1}{2} & \frac{1}{4} \end{pmatrix}}_{k+1}\right\} k+2 \quad V' = \left.\underbrace{\begin{pmatrix} 0 & \dots & 0 & 0 \\ 1 & \dots & 1 & \frac{1}{2} \\ \vdots & & \vdots & \vdots \\ 1 & \dots & 1 & \frac{1}{2} \\ \frac{1}{2} & \dots & \frac{1}{2} & \frac{1}{4} \end{pmatrix}}_{k+1}\right\} k+2$$

and the result follows. □

In Figures 4.15, and 4.16 the sets $\text{vert}\mathcal{A}_3^{\{1,h,v,q^2\}}$ and $\text{vert}\mathcal{A}_4^{\{1,h,v,q^2\}}$ are given as well as the corresponding non integer edges.

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

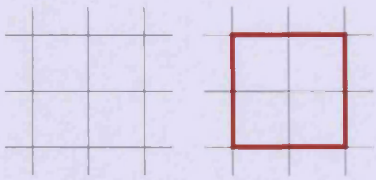


Figure 4.15: $\text{vert}\mathcal{A}_3^{\{1,h,v,q^2\}}$ and the corresponding non integer edges

$$\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \quad \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

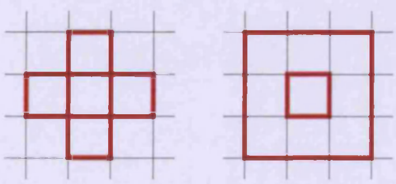
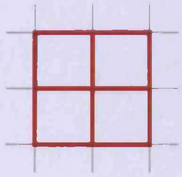
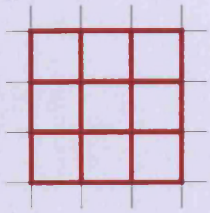
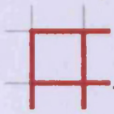


Figure 4.16: $\text{vert}\mathcal{A}_4^{\{1,h,v,q^2\}}$ and the corresponding non integer edges

We note that $\begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix} \in \Pi_{\{1,h,v,q^2\}}(\mathrm{ASM}(3,1))$ and $\begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} \in \Pi_{\{1,h,v,q^2\}}(\mathrm{ASM}(4,1))$.

However, these matrices have non-integer edges  and  both giving fundamental regions having the same non integer cycle: . Figures 4.15 and 4.16 give $D(\mathcal{A}_3^{\{1,h,v,q^2\}}) = D(\mathcal{A}_4^{\{1,h,v,q^2\}}) = 2$. We can generalize this by giving a result concerning the denominator of $\mathcal{A}_n^{\{1,h,v,q^2\}}$ similar to Corollary 4.2.2. In this section however because the result needs more work we give it in the form of a theorem:

**Theorem 4.3.2.**

$$D\left(\mathcal{A}_n^{\{1,h,v,q^2\}}\right) = 2,\ n \geq 2$$

*Proof.* For $n = 2k$, $\frac{1}{2}I_{2k} + \frac{1}{2}hI_{2k} = \begin{pmatrix} \frac{1}{2} & 0 & \dots & \dots & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \dots & \dots & \frac{1}{2} & 0 \\ \vdots & \vdots & \ddots & \cdot^{\cdot^{\cdot}} & \vdots & \vdots \\ \vdots & \vdots & \cdot^{\cdot^{\cdot}} & \ddots & \vdots & \vdots \\ 0 & \frac{1}{2} & \dots & \dots & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \dots & \dots & 0 & \frac{1}{2} \end{pmatrix} \in \mathrm{vert}\mathcal{A}_{2k}^{\{1,h,v,q^2\}}$, so $\mathrm{D}\left(\mathcal{A}_{2k}^{\{1,h,v,q^2\}}\right) \geq 2$. Now consider $(h,v) \in \overline{\mathcal{E}_{2k}^{\{1,h,v,q^2\}}}$. If there is an edge corresponding to an entry of $(h,v)$ not in $\left\{0, \frac{1}{2}, 1\right\}$ then there must be another edge connected to this edge corresponding to an entry not in $\left\{0, \frac{1}{2}, 1\right\}$. However since all boundary edges correspond to entries in $\{0, \frac{1}{2}\}$ we must have a non integer cycle and so $(h,v) \notin \mathrm{vert}\mathcal{E}_{2k}^{\{1,h,v,q^2\}}$. Therefore if $(h,v) \in \mathrm{vert}\mathcal{E}_{2k}^{\{1,h,v,q^2\}}$, then each entry of $(h,v)$ must be in $\{0, \frac{1}{2}, 1\}$ which implies $\mathrm{D}\left(\mathcal{A}_{2k}^{\{1,h,v,q^2\}}\right) \leq 2$. For $n = 2k+1$, the result follows in a very similar way (it is worth noting that $h_{k+1,k} = v_{k,k+1}$). □

As a consequence of Theorems 1.2.18 and 4.3.2 we have:

**Theorem 4.3.3.** *For fixed $n \in \mathbb{P}$ there exists $A_n^{\{1,h,v,q^2\}}(r)$, the Ehrhart quasi-polynomial of $\mathcal{A}_n^{\{1,h,v,q^2\}}$ which satisfies:*

1. $A_n^{\{1,h,v,q^2\}}(r)$ *is a quasi-polynomial in* $r$ *of degree* $\dim \mathcal{A}_n^{\{1,h,v,q^2\}}$ *and period which divides 2.*

2. $\left|ASM(n,r)^{\{1,h,v,q^2\}}\right| = A_n^{\{1,h,v,q^2\}}(r)$ *for all* $r \in \mathbb{N}$

3. $\left|ASM^{o}(n,r)^{\{1,h,v,q^2\}}\right| = (-1)^{\dim \mathcal{A}_n^{\{1,h,v,q^2\}}} A_n^{\{1,h,v,q^2\}}(-r)$ *for all* $r \in \mathbb{P}$

The following series illustrate this theorem:

$$A_2^{\{1,h,v,q^2\}}(r) = \begin{cases} 1, r \text{ even} \\ 0, r \text{ odd} \end{cases} \tag{4.11}$$

$$A_3^{\{1,h,v,q^2\}}(r) = \binom{\lfloor \frac{r}{2} \rfloor + 1}{1} \tag{4.12}$$

$$A_4^{\{1,h,v,q^2\}}(r) = \begin{cases} \binom{\frac{r}{2}+1}{1}, r \text{ even} \\ 0, r \text{ odd} \end{cases} \tag{4.13}$$

$$A_5^{\{1,h,v,q^2\}}(r) = 2\binom{\lfloor \frac{r}{2} \rfloor + 2}{4} + 5\binom{\lfloor \frac{r}{2} \rfloor + 3}{4} + \binom{\lfloor \frac{r}{2} \rfloor + 4}{4} \tag{4.14}$$

$$A_6^{\{1,h,v,q^2\}}(r) = \begin{cases} 3\binom{\frac{r}{2}+2}{4} + 6\binom{\frac{r}{2}+3}{4} + \binom{\frac{r}{2}+4}{4}, r \text{ even} \\ 0, r \text{ odd} \end{cases} \tag{4.15}$$

## 4.4 Half turn symmetry

The third row of Figures 1.17 and 4.1 give:

$$\mathcal{A}_n^{\{1,q^2\}} = \{a \in \mathcal{A}_n \mid a_{ij} = a_{n+1-i,n+1-j} \text{ for all } i,j \in [n]\} \tag{4.16}$$

$$\mathcal{E}_n^{\{1,q^2\}} = \left\{(h,v) \in \mathcal{E}_n \;\middle|\; \begin{array}{l} \bullet\ h_{ij} = 1 - h_{n+1-i,n-j} \text{ for all } i \in [n],\ j \in [0,n] \\ \bullet\ v_{ij} = 1 - v_{n-i,n+1-j} \text{ for all } i \in [0,n],\ j \in [n] \end{array}\right\} \tag{4.17}$$

Figure 4.17 gives the set $\mathrm{ASM}(3,2)^{\{1,q^2\}}$ and some cardinalities of $\mathrm{ASM}(n,r)^{\{1,q^2\}}$ are given by Figure 4.18.

$$\left\{\begin{array}{c} \begin{pmatrix}0&0&2\\0&2&0\\2&0&0\end{pmatrix}, \begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}, \begin{pmatrix}0&2&0\\2&-2&2\\0&2&0\end{pmatrix} \\ \begin{pmatrix}1&0&1\\0&2&0\\1&0&1\end{pmatrix}, \begin{pmatrix}1&1&0\\1&0&1\\0&1&1\end{pmatrix}, \begin{pmatrix}2&0&0\\0&2&0\\0&0&2\end{pmatrix} \end{array}\right\}$$

Figure 4.17: Half turn symmetric alternating sign matrices of size 3 and line sum 2

Recalling (1.33) we have:

$$\left|\mathrm{ASM}(n)^{\{1,q^2\}}\right| = \prod_{i=0}^{\lceil \frac{n}{2} \rceil - 1} \frac{(3i)!}{(\lfloor \frac{n}{2} \rfloor + i)!} \prod_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} \frac{(3i+2)!}{(\lceil \frac{n}{2} \rceil + i)!}$$

| | $r=0$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $n=1$ | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 2 | 3 | 4 | 5 | 6 |
| 3 | 1 | 3 | 6 | 10 | 15 | 21 |
| 4 | 1 | 10 | 48 | 158 | 413 | 924 |
| 5 | 1 | 25 | 256 | 1552 | 6736 | 23232 |

Figure 4.18: $|\mathrm{ASM}(n,r)^{\{1,q^2\}}|$ for $n \in [5]$, $r \in [0,5]$

$\left|\mathrm{ASM}(n,1)^{\{1,q^2\}}\right|$ corresponds to sequence A005158 of [99].

Recalling Figure 3.1: $R_{2k}^{\{1,q^2\}} = [k] \times [2k]$ and $R_{2k+1}^{\{1,q^2\}} = [k+1] \times [2k+1]$. Thus:

$$\overline{\mathcal{A}_{2k}^{\{1,q^2\}}} = \left\{ a \in \mathbb{R}^{k\times 2k} \;\middle|\; \begin{array}{l} \bullet\ 0 \le \sum_{j'=1}^{j} a_{ij'} \le 1 \text{ for all } i \in [k],\ j \in [2k] \\ \bullet\ 0 \le \sum_{i'=1}^{i} a_{i'j} \le 1 \text{ for all } i \in [k],\ j \in [2k] \\ \bullet\ \sum_{j=1}^{2k} a_{ij} = \sum_{i=1}^{k} (a_{ij} + a_{i,2k+1-j}) = 1 \text{ for all } i \in [k],\ j \in [2k] \end{array} \right\}$$

and

$$\overline{\mathcal{A}_{2k+1}^{\{1,q^2\}}} = \left\{ a \in \mathbb{R}^{(k+1)\times(2k+1)} \;\middle|\; \begin{array}{l} \bullet\ 0 \le \sum_{j'=1}^{j} a_{ij'} \le 1 \text{ for all } i \in [k+1],\ j \in [2k+1] \\ \bullet\ 0 \le \sum_{i'=1}^{i} a_{i'j} \le 1 \text{ for all } i \in [k+1],\ j \in [2k+1] \\ \bullet\ a_{k+1,j} = a_{k+1,2(k+1)-j} \text{ for all } j \in [k] \\ \bullet\ \sum_{j=1}^{2k+1} a_{ij} = \sum_{i=1}^{k} (a_{ij} + a_{i,2(k+1)-j}) + a_{k+1,j} = 1 \\ \qquad\qquad \text{for all } i \in [k+1],\ j \in [2k+1] \end{array} \right\}$$

We thus have:

$$\overline{\mathcal{E}_{2k}^{\{1,q^2\}}} = \left\{ (h,v) = \left( \begin{pmatrix} h_{10} & \cdots & h_{1,2k} \\ \vdots & & \vdots \\ h_{k0} & \cdots & h_{k,2k} \end{pmatrix}, \begin{pmatrix} v_{01} & \cdots & v_{0,2k} \\ \vdots & & \vdots \\ v_{k1} & \cdots & v_{k,2k} \end{pmatrix} \right) \in [0,1]_{\mathbb{R}}^{k\times(2k+1)} \times [0,1]_{\mathbb{R}}^{(k+1)\times 2k} \;\middle|\; \begin{array}{l} \bullet\ h_{i0} = v_{0j} = 0 \text{ for all } i \in [k],\ j \in [2k] \\ \bullet\ h_{i,2k} = v_{kj} + v_{k,2k+1-j} = 1 \text{ for all } i \in [k],\ j \in [2k] \\ \bullet\ h_{i,j-1} + v_{ij} = v_{i-1,j} + h_{ij} \text{ for all } i \in [k],\ j \in [2k] \end{array} \right\}$$

and

$$\overline{\mathcal{E}_{2k+1}^{\{1,q^2\}}} = \left\{ (h,v) = \left( \begin{pmatrix} h_{10} & \cdots & h_{1,2k+1} \\ \vdots & & \vdots \\ h_{k+1,0} & \cdots & h_{k+1,2k+1} \end{pmatrix}, \begin{pmatrix} v_{01} & \cdots & v_{0,2k+1} \\ \vdots & & \vdots \\ v_{k+1,1} & \cdots & v_{k+1,2k+1} \end{pmatrix} \right) \in [0,1]_{\mathbb{R}}^{(k+1)\times 2(k+1)} \times [0,1]_{\mathbb{R}}^{(k+2)\times(2k+1)} \;\middle|\; \begin{array}{l} \bullet\ h_{i0} = v_{0j} = 0 \text{ for all } i \in [k+1],\ j \in [2k+1] \\ \bullet\ h_{i,2k+1} = v_{k+1,j} + v_{k,2(k+1)-j} = h_{k+1,j} + h_{k+1,2k+1-j} = 1 \text{ for all } i \in [k+1], \\ \qquad\qquad j \in [2k+1] \\ \bullet\ h_{i,j-1} + v_{ij} = v_{i-1,j} + h_{ij} \text{ for all } i \in [k+1],\ j \in [2k+1] \end{array} \right\}$$

where $f'_{\{1,q^2\}} : \mathcal{E}_{2k}^{\{1,q^2\}} \to \overline{\mathcal{E}_{2k}^{\{1,q^2\}}}$ is given by:

$$f'_{\{1,q^2\}}\left(\begin{pmatrix} 0 & h_{11} & \dots & h_{1,2k-1} & 1 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & h_{k1} & \dots & h_{k,2k-1} & 1 \\ 0 & 1-h_{k,2k-1} & \dots & 1-h_{k1} & 1 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 1-h_{1,2k-1} & \dots & 1-h_{11} & 1 \end{pmatrix}, \begin{pmatrix} 0 & \dots & 0 \\ v_{11} & \dots & v_{1,2k} \\ \vdots & \dots & \vdots \\ v_{k-1,1} & \dots & v_{k-1,2k} \\ v_{k1}=1-v_{k,2k} & \dots & v_{k,2k}=1-v_{k1} \\ 1-v_{k-1,2k} & \dots & 1-v_{k-1,1} \\ \vdots & \dots & \vdots \\ 1-v_{1,2k} & \dots & 1-v_{11} \\ 1 & \dots & 1 \end{pmatrix}\right)$$

$$= \left(\begin{pmatrix} 0 & h_{11} & \dots & h_{1,2k-1} & 1 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & h_{k1} & \dots & h_{k,2k-1} & 1 \end{pmatrix}, \begin{pmatrix} 0 & \dots & 0 \\ v_{11} & \dots & v_{1,2k} \\ \vdots & \dots & \vdots \\ v_{k-1,1} & \dots & v_{k-1,2k} \\ 1-v_{k,2k}=v_{k1} & \dots & v_{k,2k}=1-v_{k,1} \end{pmatrix}\right)$$

and $f'_{\{1,q^2\}} : \mathcal{E}_{2k+1}^{\{1,q^2\}} \to \overline{\mathcal{E}_{2k+1}^{\{1,q^2\}}}$ is given by:

$$f'_{\{1,q^2\}}\left(\begin{pmatrix} 0 & h_{11} & \dots & h_{1,2k} & 1 \\ \vdots & & & & \vdots \\ 0 & h_{k1} & \dots & h_{k,2k} & 1 \\ 0 & h_{k+1,1} & \dots & h_{k+1,2k} & 1 \\ 0 & 1-h_{k,2k} & \dots & 1-h_{k,1} & 1 \\ \vdots & & & & \vdots \\ 0 & 1-h_{1,2k} & \dots & 1-h_{1,1} & 1 \end{pmatrix}, \begin{pmatrix} 0 & \dots & 0 \\ v_{11} & \dots & v_{1,2k+1} \\ \vdots & & \vdots \\ v_{k1} & \dots & v_{k,2k+1} \\ 1-v_{k,2k+1} & \dots & 1-v_{k1} \\ 1-v_{k-1,2k+1} & \dots & 1-v_{k-1,1} \\ \vdots & & \vdots \\ 1-v_{1,2k+1} & \dots & 1-v_{11} \\ 1 & \dots & 1 \end{pmatrix}\right)$$

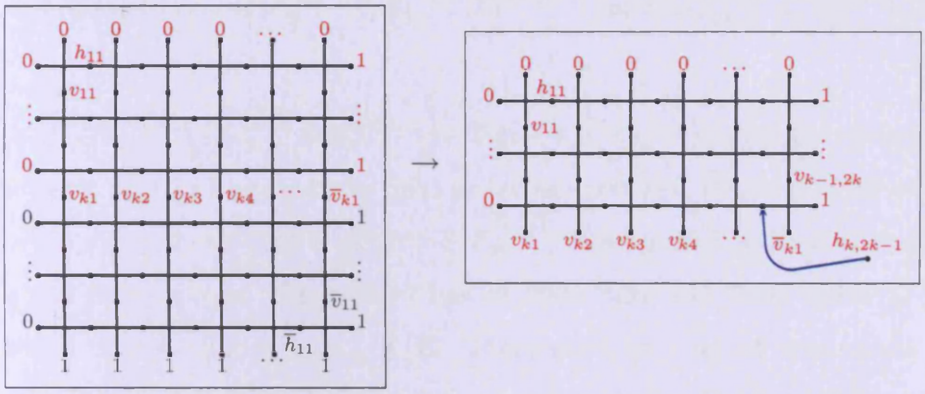
$$= \left(\begin{pmatrix} 0 & h_{11} & \dots & h_{1,2k} & 1 \\ \vdots & & & & \vdots \\ 0 & h_{k1} & \dots & h_{k,2k} & 1 \\ 0 & h_{k+1,1} & \dots & h_{k+1,2k} & 1 \end{pmatrix}, \begin{pmatrix} 0 & \dots & 0 \\ v_{11} & \dots & v_{1,2k+1} \\ \vdots & & \vdots \\ v_{k1} & \dots & v_{k,2k+1} \\ 1-v_{k,2k+1} & \dots & 1-v_{k1} \end{pmatrix}\right)$$

These fundamental polytopes can be represented on a lattice as in Figures 4.19 and 4.20.

**Theorem 4.4.1.**

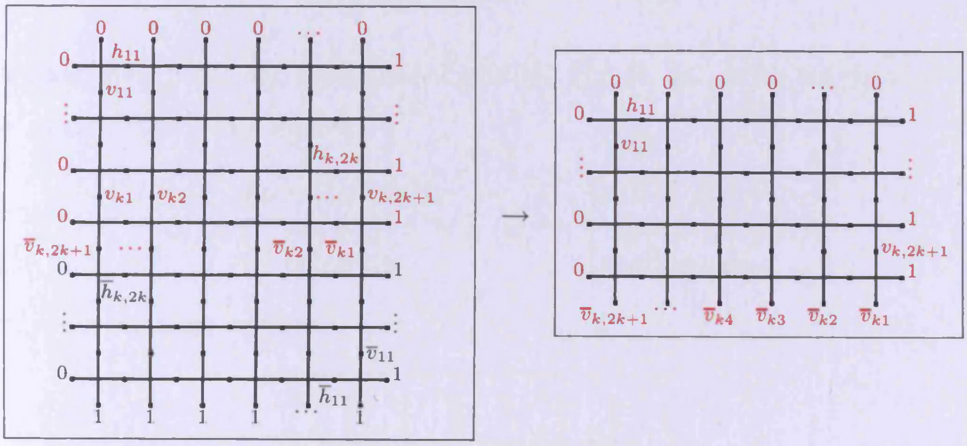
$$vert\mathcal{A}_n^{\{1,q^2\}} = ASM(n,1)^{\{1,q^2\}}$$

*Proof.* To prove this result we shall show that: $\text{vert}\overline{\mathcal{E}_{2k}^{\{1,q^2\}}} = \overline{\mathcal{E}_{2k}^{\{1,q^2\}}} \cap \mathbb{Z}^{k(4k+3)}$ and $\text{vert}\overline{\mathcal{E}_{2k+1}^{\{1,q^2\}}} = \overline{\mathcal{E}_{2k}^{\{1,q^2\}}} \cap \mathbb{Z}^{4k^2+9k+4}$. The fact that $\text{vert}\overline{\mathcal{E}_{2k}^{\{1,q^2\}}} \supseteq \overline{\mathcal{E}_{2k}^{\{1,q^2\}}} \cap \mathbb{Z}^{k(4k+3)}$ and $\text{vert}\overline{\mathcal{E}_{2k+1}^{\{1,q^2\}}} \supseteq \overline{\mathcal{E}_{2k}^{\{1,q^2\}}} \cap$




where $\overline{m} = 1 - m$ for all $m \in \mathbb{R}$

Figure 4.19: Elements of $\mathcal{E}_{2k}^{\{1,q^2\}}$ and $\overline{\mathcal{E}_{2k}^{\{1,q^2\}}}$ on a lattice diagram.



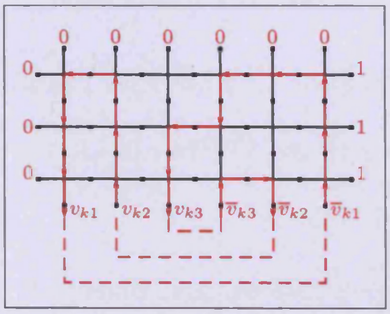

where $\overline{m} = 1 - m$ for all $m \in \mathbb{R}$

Figure 4.20: Elements of $\mathcal{E}_{2k+1}^{\{1,q^2\}}$ and $\overline{\mathcal{E}_{2k+1}^{\{1,q^2\}}}$ on a lattice diagram

$\mathbb{Z}^{4k^2+9k+4}$ follows straightforwardly from Theorem 2.3.4.

Thus we need to show that $\text{vert}\overline{\mathcal{E}_{2k}^{\{1,q^2\}}} \subseteq \overline{\mathcal{E}_{2k}^{\{1,q^2\}}} \cap \mathbb{Z}^{k(4k+3)}$ and $\text{vert}\overline{\mathcal{E}_{2k+1}^{\{1,q^2\}}} \subseteq \overline{\mathcal{E}_{2k}^{\{1,q^2\}}} \cap \mathbb{Z}^{4k^2+9k+4}$.

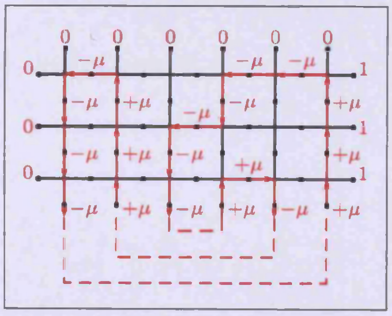
Consider $(h,v) \in \overline{\mathcal{E}_{2k}^{\{1,q^2\}}} \setminus \left(\overline{\mathcal{E}_{2k}^{\{1,q^2\}}} \cap \mathbb{Z}^{k(4k+3)}\right)$. Thus $(h,v)$ has a non integer cycle or an open non integer path. If $(h,v)$ has a cycle then as for the proof of Theorem 4.1.2 we can find $(h^*,v^*) \neq (0,0)$ such that $(h,v) \pm (h^*,v^*) \in \overline{\mathcal{E}_{2k}^{\{1,q^2\}}}$. Let us now assume that $(h,v)$ does not have a non integer cycle. Thus $(h,v)$ has an open path and there exists $j_0$ such that $v_{kj_0} \notin \mathbb{Z}$ which implies that $v_{k,2k+1-j_0} \notin \mathbb{Z}$. There must be a set of open paths (with no cycles) connecting the edges on the boundary corresponding to these coefficients (since on the boundary we have $v_{kj} = 1 - v_{k,2k+1-j}$ for all $j \in [2k]$). An example of such a set of paths is:



If we orientate these paths and create index sets $\mathcal{H}_+, \mathcal{H}_-, \mathcal{V}_+$ $\mathcal{H}_-$ as for the proof of Theorem 4.1.2, we create $(h^*,v^*)$ with entries:

$$h^*_{ij} := \begin{cases} \mu \text{ if } (i,j) \in \mathcal{H}_+ \\ -\mu \text{ if } (i,j) \in \mathcal{H}_- \\ 0 \text{ otherwise} \end{cases} \quad v^*_{ij} := \begin{cases} \mu \text{ if } (i,j) \in \mathcal{V}_+ \\ -\mu \text{ if } (i,j) \in \mathcal{V}_- \\ 0 \text{ otherwise} \end{cases}$$

To continue our example we have:



We choose:

$$\begin{aligned}\mu{:=}\min(&\{h_{ij}|(i,j)\in\mathcal{H}_+\}\cup\{\overline{h}_{ij}|(i,j)\in\mathcal{H}_+\}\cup\\&\{h_{ij}|(i,j)\in\mathcal{H}_-\}\cup\{\overline{h}_{ij}|(i,j)\in\mathcal{H}_-\}\cup\\&\{v_{ij}|(i,j)\in\mathcal{V}_+\}\cup\{\overline{v}_{ij}|(i,j)\in\mathcal{V}_+\}\cup\\&\{v_{ij}|(i,j)\in\mathcal{V}_-\}\cup\{\overline{v}_{ij}|(i,j)\in\mathcal{V}_-\})\end{aligned}$$

with $\overline{h}_{ij} = 1 - h_{ij}$ and $\overline{v}_{ij} = 1 - v_{ij}$.

Note that $v^*_{kj_0} = -v^*_{k,n+1-j_0}$ (since the corresponding edges will have opposite orientation) and so $(h,v)\pm(h^*,v^*)\in\overline{\mathcal{E}^{\{1,q^2\}}_{2k}}$ and $(h^*,v^*)\neq(0,0)$. Thus $\mathrm{vert}\overline{\mathcal{E}^{\{1,q^2\}}_{2k}}\subseteq\overline{\mathcal{E}^{\{1,q^2\}}_{2k}}\cap\mathbb{Z}^{k(4k+3)}$ as required. The proof for $\overline{\mathcal{E}^{\{1,q^2\}}_{2k+1}}$ follows in the same way. □

Theorem 4.4.1 leads to:

**Corollary 4.4.2.**

$$D(\mathcal{A}^{\{1,q^2\}}_n) = 1 \textit{ for all } n$$

As a direct implication of Theorems 1.2.29 and 4.4.1 we have:

**Corollary 4.4.3.** *Any matrix $a \in ASM(n,r)^{\{1,q^2\}}$ can be written as the sum of $r$ matrices from $ASM(n,1)^{\{1,q^2\}}$.*

As a consequence of Theorem 1.2.18 and Corollary 4.4.2 we have:

**Theorem 4.4.4.** *For fixed $n \in \mathbb{P}$ there exists $A^{\{1,q^2\}}_n(r)$, the Ehrhart polynomial of $\mathcal{A}^{\{1,q^2\}}_n$ which satisfies:*

1. *$A^{\{1,q^2\}}_n(r)$ is a polynomial in $r$ of degree $\dim\mathcal{A}^{\{1,q^2\}}_n$.*
2. *$\left|ASM(n,r)^{\{1,q^2\}}\right| = A^{\{1,q^2\}}_n(r)$ for all $r\in\mathbb{N}$*
3. *$\left|ASM^{\circ}(n,r)^{\{1,q^2\}}\right| = (-1)^{\dim\mathcal{A}^{\{1,q^2\}}_n}A^{\{1,q^2\}}_n(-r)$ for all $r\in\mathbb{P}$*

We illustrate this with the following examples:

$$A^{\{1,q^2\}}_2(r) = \binom{r+1}{1} \tag{4.18}$$

$$A^{\{1,q^2\}}_3(r) = \binom{r+2}{2} \tag{4.19}$$

$$A^{\{1,q^2\}}_4(r) = 3\binom{r+3}{5}+4\binom{r+4}{5}+\binom{r+5}{5} \tag{4.20}$$

$$A^{\{1,q^2\}}_5(r) = 10\binom{r+4}{8}+64\binom{r+5}{8}+67\binom{r+6}{8}+16\binom{r+7}{8}+\binom{r+8}{8} \tag{4.21}$$

Polynomial (4.19) corresponds to sequence A000217 of [99].

## 4.5 Quarter turn symmetry

The sixth row of Figures 1.17 and 4.1 give:

$$\mathcal{A}_n^{\{1,q,q^2,q^3\}} = \{a \in \mathcal{A}_n \mid a_{ij} = a_{j,n+1-i} \text{ for all } i,j \in [n]\} \tag{4.22}$$

$$\mathcal{E}_n^{\{1,q,q^2,q^3\}} = \{(h,v) \in \mathcal{E}_n \mid h_{ij} = v_{j,n+1-i} = 1 - h_{n+1-i,n-j} \text{ for all } i \in [n],\ j \in [0,n]\} \tag{4.23}$$

Figure 4.21 gives the set $\mathrm{ASM}(5,1)^{\{1,q,q^2,q^3\}}$ and some cardinalities of $\mathrm{ASM}(n,r)^{\{1,q,q^2,q^3\}}$ are given by Figure 4.22.

$$\left\{ \begin{pmatrix} 0&0&0&1&0\\ 1&0&0&0&0\\ 0&0&1&0&0\\ 0&0&0&0&1\\ 0&1&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&0&0\\ 0&1&-1&1&0\\ 1&-1&1&-1&1\\ 0&1&-1&1&0\\ 0&0&1&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&0&0\\ 0&0&0&0&1\\ 0&0&1&0&0\\ 1&0&0&0&0\\ 0&0&0&1&0 \end{pmatrix} \right\}$$

Figure 4.21: Quarter turn symmetric alternating sign matrices of size 5 and line sum 1

| | $r=0$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $n=1$ | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 0 | 1 | 0 | 1 | 0 |
| 3 | 1 | 1 | 2 | 2 | 3 | 3 |
| 4 | 1 | 2 | 4 | 6 | 9 | 12 |
| 5 | 1 | 3 | 12 | 24 | 56 | 92 |
| 6 | 1 | 0 | 37 | 0 | 380 | 0 |

Figure 4.22: $|ASM(n,r)^{\{1,q,q^2,q^3\}}|$ for $n \in [6]$, $r \in [0,5]$

We recall equation (1.34):

$$\left|\mathrm{ASM}(n,1)^{\{1,q,q^2,q^3\}}\right| = \begin{cases} \prod_{i=0}^{\frac{n}{4}-1} \frac{i!(3i+1)!\left(\frac{n}{2}+i\right)!}{\left(\frac{n}{4}+i\right)!^3}, & n = 0 \bmod 4 \\ \prod_{i=0}^{\frac{n-1}{4}} \frac{(3i)!}{\left(\frac{n-1}{4}+i\right)!} \prod_{i=0}^{\frac{n-5}{4}} \frac{(3i+1)!^2(3i+2)!}{\left(\frac{n-1}{4}+i\right)!\left(\frac{n+3}{4}+i\right)!}, & n = 1 \bmod 4 \\ 0, & n = 2 \bmod 4 \\ \prod_{i=0}^{\frac{n-3}{4}} \frac{(3i)!(3i+1)!^2}{\left(\frac{n-3}{4}+i\right)!\left(\frac{n+1}{4}+i\right)!} \prod_{i=0}^{\frac{n-7}{4}} \frac{(3i+2)!}{\left(\frac{n+1}{4}+i\right)!}, & n = 3 \bmod 4 \end{cases}$$

corresponding to sequence A005160 of [99].

Recalling Figure 3.1: $R_{2k}^{\{1,q,q^2,q^3\}} = [k] \times [k]$ and $R_{2k+1}^{\{1,q,q^2,q^3\}} = [k+1] \times [k+1]$. Thus:

$$\overline{\mathcal{A}_{2k}^{\{1,q,q^2,q^3\}}} := \left\{ a \in \mathbb{R}^{k\times k} \;\middle|\; \begin{array}{l} \bullet\ 0 \le \sum_{j'=1}^{j} a_{ij'} \le 1 \text{ for all } i,j \in [k] \\ \bullet\ 0 \le \sum_{i'=1}^{i} a_{i'j} \le 1 \text{ for all } i,j \in [k] \\ \bullet\ \sum_{j=1}^{k}(a_{ij} + a_{ji}) = 1 \text{ for all } i \in [k] \end{array} \right\}$$

and

$$\overline{\mathcal{A}_{2k+1}^{\{1,q,q^2,q^3\}}} := \left\{ a \in \mathbb{R}^{(k+1)\times(k+1)} \;\middle|\; \begin{array}{l} \bullet\ 0 \le \sum_{j'=1}^{j} a_{ij'} \le 1 \text{ for all } i,j \in [k+1] \\ \bullet\ 0 \le \sum_{i'=1}^{i} a_{i'j} \le 1 \text{ for all } i,j \in [k+1] \\ \bullet\ a_{i,k+1} = a_{k+1,i} \text{ for all } i \in [k] \\ \bullet\ \sum_{j=1}^{k}(a_{ij} + a_{ji}) + a_{i,k+1} = 1 \text{ for all } i \in [k+1] \end{array} \right\}$$

We thus have:

$$\overline{\mathcal{E}_{2k}^{\{1,q,q^2,q^3\}}} = \left\{ (h,v) = \left( \begin{pmatrix} h_{10} & \cdots & h_{1k} \\ \vdots & & \vdots \\ h_{k0} & \cdots & h_{kk} \end{pmatrix}, \begin{pmatrix} v_{01} & \cdots & v_{0k} \\ \vdots & & \vdots \\ v_{k1} & \cdots & v_{kk} \end{pmatrix} \right) \in [0,1]_{\mathbb{R}}^{k\times(k+1)} \times [0,1]_{\mathbb{R}}^{(k+1)\times k} \;\middle|\; \begin{array}{l} \bullet\ h_{i0} = v_{0j} = 0 \text{ for all } i \in [k],\ j \in [k] \\ \bullet\ h_{ik} + v_{ki} = 1 \text{ for all } i \in [k] \\ \bullet\ h_{i,j-1} + v_{ij} = v_{i-1,j} + h_{ij} \text{ for all } i,j \in [k] \end{array} \right\}$$

and

$$\overline{\mathcal{E}_{2k+1}^{\{1,q,q^2,q^3\}}} = \left\{ (h,v) = \left( \begin{pmatrix} h_{10} & \cdots & h_{1,k+1} \\ \vdots & & \vdots \\ h_{k+1,0} & \cdots & h_{k+1,k+1} \end{pmatrix}, \begin{pmatrix} v_{01} & \cdots & v_{0,k+1} \\ \vdots & & \vdots \\ v_{k+1,1} & \cdots & v_{k+1,k+1} \end{pmatrix} \right) \in [0,1]_{\mathbb{R}}^{(k+1)\times(k+2)} \times [0,1]_{\mathbb{R}}^{(k+2)\times(k+1)} \;\middle|\; \begin{array}{l} \bullet\ h_{i0} = v_{0j} = 0 \text{ for all } i,j \in [k+1] \\ \bullet\ h_{i,k+1} + v_{ki} = v_{k+1,j} + h_{jk} = 1 \text{ for all } i,j \in [k+1] \\ \bullet\ h_{i,j-1} + v_{ij} = v_{i-1,j} + h_{ij} \text{ for all } i,j \in [k+1] \end{array} \right\}$$

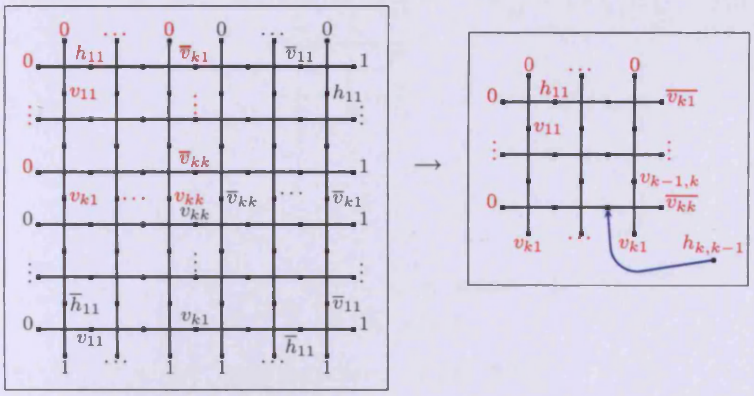
Also $f'_{\{1,q,q^2,q^3\}} : \mathcal{E}_{2k}^{\{1,q,q^2,q^3\}} \to \overline{\mathcal{E}_{2k}^{\{1,q,q^2,q^3\}}}$ is given by:

$$f'_{\{1,q,q^2,q^3\}}\left(\begin{pmatrix} 0 & h_{11} & \dots & h_{1k} & 1-v_{k-1,1} & \dots & 1-v_{11} & 1 \\ \vdots & & & \vdots & \vdots & & \vdots & \vdots \\ 0 & h_{k1} & \dots & h_{kk} & 1-v_{k-1,k} & \dots & 1-v_{1k} & 1 \\ 0 & v_{1k} & \dots & v_{kk} & 1-h_{k,k-1} & \dots & 1-h_{k1} & 1 \\ \vdots & & & \vdots & \vdots & & \vdots & \vdots \\ 0 & v_{11} & \dots & v_{k1} & 1-h_{1,k-1} & \dots & 1-h_{11} & 1 \end{pmatrix},\right.$$
$$\left.\begin{pmatrix} 0 & \dots & 0 & 0 & \dots & 0 \\ v_{11} & \dots & v_{1k} & h_{k1} & \dots & h_{11} \\ \vdots & & \vdots & \vdots & & \vdots \\ v_{k1} & \dots & v_{kk} & h_{kk} & \dots & h_{1k} \\ 1-h_{1,k-1} & \dots & 1-h_{k,k-1} & 1-v_{k-1,k} & \dots & 1-v_{k-1,1} \\ \vdots & & \vdots & \vdots & & \vdots \\ 1-h_{11} & \dots & 1-h_{k1} & 1-v_{1k} & \dots & 1-v_{11} \\ 1 & \dots & 1 & 1 & \dots & 1 \end{pmatrix}\right)$$
$$= \left(\begin{pmatrix} 0 & h_{11} & \dots & h_{1k} \\ \vdots & & & \vdots \\ 0 & h_{k1} & \dots & h_{kk} \end{pmatrix}, \begin{pmatrix} 0 & \dots & 0 \\ v_{11} & \dots & v_{1k} \\ \vdots & & \vdots \\ v_{k1} & \dots & v_{kk} \end{pmatrix}\right)$$

and $f'_{\{1,q,q^2,q^3\}} : \mathcal{E}_{2k+1}^{\{1,q,q^2,q^3\}} \to \overline{\mathcal{E}_{2k+1}^{\{1,q,q^2,q^3\}}}$ is given by:

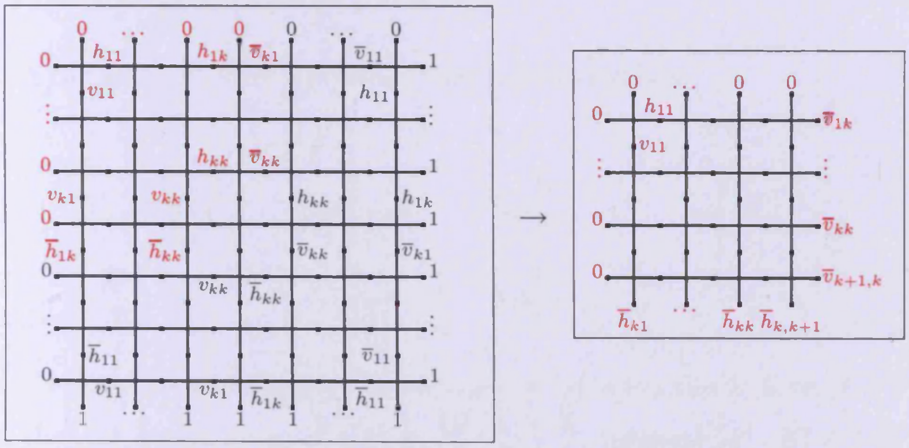
$$f'_{\{1,q,q^2,q^3\}}\left(\begin{pmatrix} 0 & h_{11} & \dots & h_{1k} & 1-v_{k1} & \dots & 1-v_{11} & 1 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & h_{k1} & \dots & h_{kk} & 1-v_{kk} & \dots & 1-v_{1k} & 1 \\ 0 & h_{k+1,1} & \dots & h_{k+1,k} & 1-v_{k,k+1} & \dots & 1-v_{1,k+1} & 1 \\ 0 & v_{1k} & \dots & v_{kk} & 1-h_{kk} & \dots & 1-h_{k1} & 1 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & v_{11} & \dots & v_{kk} & 1-h_{1k} & \dots & 1-h_{11} & 1 \end{pmatrix},\right.$$
$$\left.\begin{pmatrix} 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ v_{11} & \dots & v_{1k} & v_{1,k+1} & h_{k1} & \dots & h_{11} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ v_{k1} & \dots & v_{kk} & v_{k,k+1} & h_{kk} & \dots & h_{1k} \\ 1-h_{1k} & \dots & 1-h_{kk} & 1-h_{k+1,k} & 1-v_{kk} & \dots & 1-v_{k1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1-h_{11} & \dots & 1-h_{k1} & 1-h_{k+1,1} & 1-v_{1k} & \dots & 1-v_{11} \\ 1 & \dots & 1 & 1 & 1 & \dots & 1 \end{pmatrix}\right)$$
$$= \left(\begin{pmatrix} 0 & h_{11} & \dots & h_{1k} & 1-v_{k,1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & h_{k1} & \dots & h_{kk} & 1-v_{k,k} \\ 0 & h_{k+1,1} & \dots & h_{k+1,k} & 1-v_{k,k+1} \end{pmatrix}, \begin{pmatrix} 0 & \dots & 0 & 0 \\ v_{11} & \dots & v_{1k} & v_{1;k+1} \\ \vdots & & \vdots & \vdots \\ v_{k1} & \dots & v_{kk} & v_{k,k+1} \\ 1-h_{1k} & \dots & 1-h_{kk} & 1-h_{k+1,k} \end{pmatrix}\right)$$

These fundamental polytopes can be represented on a lattice as in Figures 4.23 and 4.24.




where $\overline{m} = 1 - m$ for all $m \in \mathbb{R}$

Figure 4.23: Elements of $\mathcal{E}_{2k}^{\{1,q,q^2,q^3\}}$ and $\overline{\mathcal{E}_{2k}^{\{1,q,q^2,q^3\}}}$ on $\mathcal{L}_{2k,2k}$ and $\mathcal{L}_{k,k}$

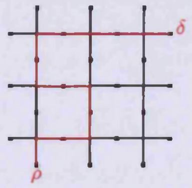



where $\overline{m} = 1 - m$ for all $m \in \mathbb{R}$

Figure 4.24: Elements of $\mathcal{E}_{2k+1}^{\{1,q,q^2,q^3\}}$ and $\overline{\mathcal{E}_{2k+1}^{\{1,q,q^2,q^3\}}}$ on $\mathcal{L}_{2k+1,2k+1}$ and $\mathcal{L}_{k+1,k+1}$

Before studying the vertices of this polytope we give the following useful lemmas:

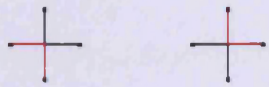
**Lemma 4.5.1.** *Consider* $(h, v) \in [0,1]_{\mathbb{R}}^{m\times(n+1)} \times [0,1]_{\mathbb{R}}^{(m+1)\times n}$ *with* $h_{i,j-1} + v_{ij} = v_{i-1,j} + h_{ij}$ *for all* $i \in [m]$, $j \in [n]$ *represented on* $\mathcal{L}_{m,n}$ *and* $\rho, \delta \in \mathbb{R} \setminus \mathbb{Z}$ *associated to two edges of* $\mathcal{L}_{m,n}$ *such that there is an isolated non integer path (i.e. at each vertex on the path only two of the adjacent edges correspond to non integer entries of* $(h, v)$*) between these two edges. The following is an example of such a configuration:*




*Then we have:*

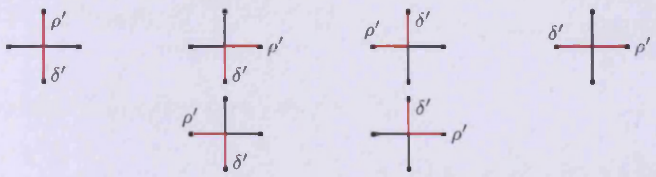
$$\rho = \begin{cases} 1 - \delta, \ K \text{ odd} \\ \delta, \ K \text{ even} \end{cases}$$

*where $K$ is the number of vertex configurations of the form:*

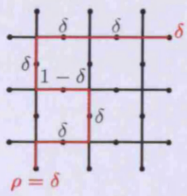


*that appear in the non integer path between the edges corresponding to $\rho$ and $\delta$.*

*Proof.* Along the path, we have 6 possible vertex configurations:




Considering the first four configurations we have an equation of the form $\rho' + \Gamma = \delta' + \Delta$ where $\Gamma, \Delta \in \{0, 1\}$, which gives $|\rho' - \delta'| = \begin{cases} 0 \ , \Gamma = \Delta \\ 1 \ , \Gamma \neq \Delta \end{cases}$. However $|\rho' - \delta'| = 1$ is impossible (since $\rho', \delta' \in (0,1)_{\mathbb{R}}$) so we have $\rho' = \delta'$. The other two configurations give an equation of the form $\rho' + \delta' = \Gamma + \Delta$ where $\Gamma, \Delta \in \{0, 1\}$ which gives $\rho' + \delta' = \begin{cases} 0 \ , \Gamma = \Delta = 0 \\ 2 \ , \Gamma = \Delta = 1 \\ 1 \ , \Gamma \neq \Delta \end{cases}$. Since $\rho', \delta' \in (0,1)_{\mathbb{R}}$ we see that $\Gamma = \Delta$ is impossible, and thus $\rho' + \delta' = 1$ as required. The result for $\rho, \delta$ then follows.




□

The proof of this lemma can easily be adapted to give the following similar lemma:

**Lemma 4.5.2.** *Consider* $(h^*, v^*) \in \mathbb{R}^{m\times(n+1)} \times \mathbb{R}^{(m+1)\times n}$ *with* $h^*_{i,j-1} + v^*_{ij} = v^*_{i-1,j} + h^*_{ij}$ *for all* $i \in [m]$, $j \in [n]$ *represented on* $\mathcal{L}_{m,n}$ *and* $\rho, \delta \in \mathbb{R} \setminus \{0\}$ *associated to two edges of* $\mathcal{L}_{m,n}$ *such that there is an isolated non zero path between these two edges. Then:*

$$\rho = \begin{cases} -\delta, & K \text{ odd} \\ \delta, & K \text{ even} \end{cases}$$

*where* $K$ *is the number of vertex configurations of the form:*

*that appear in the non zero path between the edges corresponding to* $\rho$ *and* $\delta$.

We shall use Lemma 4.5.2 to prove the following result:

**Theorem 4.5.3.**

$$vert\mathcal{A}_n^{\{1,q,q^2,q^3\}} = \left\{ a \in \Pi_{\{1,q,q^2,q^3\}}\left(ASM(n,1)\right) \;\middle|\; \begin{array}{l} \textit{All non integer cycles of } a \textit{ on } \mathcal{L}_{nn} \textit{ are} \\ \textit{invariant under quarter turn rotation} \end{array} \right\}$$

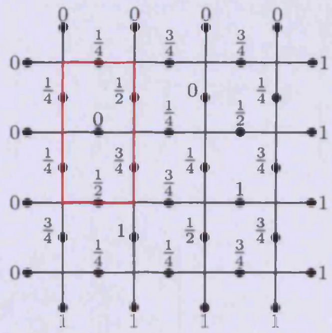
*Proof.* We shall prove the following equivalent result:

$$\text{vert}\mathcal{E}_n^{\{1,q,q^2,q^3\}} = \left\{ (h,v) \in \Pi_{\{1,q,q^2,q^3\}}\left(\text{EM}(n,1)\right) \;\middle|\; \begin{array}{l} \text{All non integer cycles of } (h,v) \text{ are} \\ \text{invariant under quarter turn rotation} \end{array} \right\}$$

- Consider

  $$(h,v) \in \mathcal{E}_n^{\{1,q,q^2,q^3\}} \setminus \left\{ (h,v) \in \Pi_{\{1,q,q^2,q^3\}}\left(\text{EM}(n,1)\right) \;\middle|\; \begin{array}{l} \text{All non integer cycles of } (h,v) \text{ are} \\ \text{invariant under quarter turn rotation} \end{array} \right\}$$

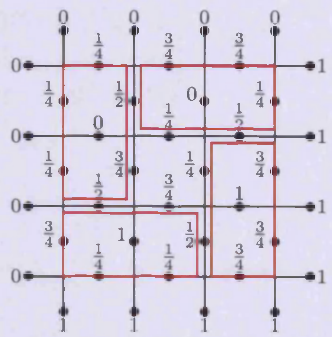
  Select any non integer cycle of $(h,v)$ that is not invariant under quarter turn rotation, give it an orientation, say anticlockwise, and denote the sets of points $(i,j)$ for which the horizontal edge between $(i,j)$ and $(i,j+1)$ is in the cycle and directed right or left as respectively $\mathcal{H}_+^{(1)}$ or $\mathcal{H}_-^{(1)}$, and the sets of points $(i,j)$ for which the vertical edge between $(i,j)$ and $(i+1,j)$ is in the cycle and directed up or down as respectively $\mathcal{V}_+^{(1)}$ or $\mathcal{V}_-^{(1)}$. For example consider $a = \begin{pmatrix} \frac14 & \frac12 & 0 & \frac14 \\ 0 & \frac14 & \frac14 & \frac12 \\ \frac12 & \frac14 & \frac14 & 0 \\ \frac14 & 0 & \frac12 & \frac14 \end{pmatrix} \in \Pi_{\{1,q,q^2,q^3\}}\left(\text{ASM}(4,1)\right)$ with the following non integer cycle on $\mathcal{L}_{4,4}$:



(4.24)

giving: $\mathcal{H}_-^{(1)} = \{(1,1)\}$, $\mathcal{H}_+^{(1)} = \{(3,1)\}$, $\mathcal{V}_-^{(1)} = \{(1,1),(2,1)\}$ and $\mathcal{V}_+^{(1)} = \{(1,2),(2,2)\}$.

Since this cycle is not invariant under quarter turn rotation, by symmetry (since $(h,v) \in \mathcal{E}_n^{\{1,q,q^2,q^3\}}$) there exists another non integer cycle that can be obtained by quarter turn rotation of the original cycle, giving another set of points $\mathcal{H}_+^{(2)}, \mathcal{H}_-^{(2)}, \mathcal{V}_+^{(2)}$ and $\mathcal{V}_-^{(2)}$. We can continue in this fashion to get two other non integer cycles giving $\mathcal{H}_+^{(3)}, \mathcal{H}_-^{(3)}, \mathcal{V}_+^{(3)}, \mathcal{V}_-^{(3)}, \mathcal{H}_+^{(4)}, \mathcal{H}_-^{(4)}, \mathcal{V}_+^{(4)}$ and $\mathcal{V}_-^{(4)}$. The other cycles of (4.24) are shown:



(4.25)

giving:

$$\begin{array}{lll} \mathcal{H}_-^{(2)} = \{(1,2),(1,3)\} & \mathcal{H}_-^{(3)} = \{(2,3)\} & \mathcal{H}_-^{(4)} = \{(3,1),(3,2)\} \\ \mathcal{H}_+^{(2)} = \{(2,2),(2,3)\} & \mathcal{H}_+^{(3)} = \{(4,3)\} & \mathcal{H}_+^{(4)} = \{(4,1),(4,2)\} \\ \mathcal{V}_-^{(2)} = \{(1,2)\} & \mathcal{V}_-^{(3)} = \{(2,3),(3,3)\} & \mathcal{V}_-^{(4)} = \{(3,1)\} \\ \mathcal{V}_+^{(2)} = \{(1,4)\} & \mathcal{V}_+^{(3)} = \{(2,4),(3,4)\} & \mathcal{V}_+^{(4)} = \{(3,3)\} \end{array} \tag{4.26}$$

We now create the matrix pairs

$$\left(h^{(k)^*}, v^{(k)^*}\right) = \left( \begin{pmatrix} h^{(k)^*}_{10} & \dots & h^{(k)^*}_{1n} \\ \vdots & & \vdots \\ h^{(k)^*}_{n0} & \dots & h^{(k)^*}_{nn} \end{pmatrix}, \begin{pmatrix} v^{(k)^*}_{01} & \dots & v^{(k)^*}_{0n} \\ \vdots & & \vdots \\ v^{(k)^*}_{n1} & \dots & v^{(k)^*}_{nn} \end{pmatrix} \right) \in \mathbb{R}^{n\times(n+1)} \times \mathbb{R}^{(n+1)\times n}$$

with entries:

$$h^{(k)^*}_{ij} := \begin{cases} (-1)^{k+1}\mu \text{ if } (i,j) \in \mathcal{H}_+^{(k)} \\ (-1)^{k}\mu \text{ if } (i,j) \in \mathcal{H}_-^{(k)} \\ 0 \text{ otherwise} \end{cases} \quad v^{(k)^*}_{ij} := \begin{cases} (-1)^{k+1}\mu \text{ if } (i,j) \in \mathcal{V}_+^{(k)} \\ (-1)^{k}\mu \text{ if } (i,j) \in \mathcal{V}_-^{(k)} \\ 0 \text{ otherwise} \end{cases}$$

The matrix pairs corresponding to (4.26) are:

$$h^{(1)*} = \begin{pmatrix} 0 & -\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad v^{(1)*} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\mu & \mu & 0 & 0 \\ -\mu & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$h^{(2)*} = \begin{pmatrix} 0 & 0 & \mu & \mu & 0 \\ 0 & 0 & -\mu & -\mu & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad v^{(2)*} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \mu & 0 & -\mu \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.27}$$

$$h^{(3)*} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mu & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 \end{pmatrix} \quad v^{(3)*} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\mu & \mu \\ 0 & 0 & -\mu & \mu \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$h^{(4)*} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \mu & \mu & 0 & 0 \\ 0 & -\mu & -\mu & 0 & 0 \end{pmatrix} \quad v^{(4)*} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mu & 0 & -\mu & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Note that we have:

$$h^{(k)*}_{i,j-1} + v^{(k)*}_{ij} = v^{(k)*}_{i-1,j} + h^{(k)*}_{ij} \text{ for all } i \in [m],\ j \in [n],\ k \in [4]$$

We choose:

$$\begin{aligned} \mu := \tfrac{1}{2}\min(&\{h_{ij} | (i,j) \in \mathcal{H}_+^{(1)}\} \cup \{\overline{h}_{ij} | (i,j) \in \mathcal{H}_+^{(1)}\} \cup \\ &\{h_{ij} | (i,j) \in \mathcal{H}_-^{(1)}\} \cup \{\overline{h}_{ij} | (i,j) \in \mathcal{H}_-^{(1)}\} \cup \\ &\{v_{ij} | (i,j) \in \mathcal{V}_+^{(1)}\} \cup \{\overline{v}_{ij} | (i,j) \in \mathcal{V}_+^{(1)}\} \cup \\ &\{v_{ij} | (i,j) \in \mathcal{V}_-^{(1)}\} \cup \{\overline{v}_{ij} | (i,j) \in \mathcal{V}_-^{(1)}\}) \end{aligned}$$

with $\overline{h}_{ij} = 1 - h_{ij}$ and $\overline{v}_{ij} = 1 - v_{ij}$ (the factor of $\frac{1}{2}$ appears here since $(h^*, v^*)$ could have some entries corresponding to edges of two cycles overlapping). Defining $(h^*, v^*) = \sum_{k=1}^{4} \left(h^{(k)*}, v^{(k)*}\right)$, it can now easily be checked that $(h, v) \pm (h^*, v^*) \in \mathcal{E}_n^{\{1,q,q^2,q^3\}}$. Thus as required

$$\text{vert}\mathcal{E}_n^{\{1,q,q^2,q^3\}} \subseteq \left\{ (h,v) \in \Pi_{\{1,q,q^2,q^3\}}(\text{EM}(n,1)) \;\middle|\; \begin{array}{l} \text{All non integer cycles of } (h,v) \text{ are} \\ \text{invariant under quarter turn rotation} \end{array} \right\}$$

The matrix pair $(h^*, v^*)$ corresponding to (4.27) is:

$$(h^*, v^*) = \left( \begin{pmatrix} 0 & -\mu & \mu & \mu & 0 \\ 0 & 0 & -\mu & -2\mu & 0 \\ 0 & 2\mu & \mu & 0 & 0 \\ 0 & -\mu & -\mu & \mu & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\mu & 2\mu & 0 & -\mu \\ -\mu & \mu & -\mu & \mu \\ \mu & 0 & -2\mu & \mu \\ 0 & 0 & 0 & 0 \end{pmatrix} \right)$$

Setting $\mu = \frac{1}{8}$ and using (2.28) we have:

$$a \pm a^* = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & \frac{1}{2} & \frac{1}{4} \end{pmatrix} \pm \begin{pmatrix} -\frac{1}{8} & \frac{1}{4} & 0 & -\frac{1}{8} \\ 0 & -\frac{1}{8} & -\frac{1}{8} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{8} & -\frac{1}{8} & 0 \\ -\frac{1}{8} & 0 & \frac{1}{4} & -\frac{1}{8} \end{pmatrix} \in \mathcal{A}_4^{\{1,q,q^2,q^3\}} \tag{4.28}$$

- Now let us consider $(h, v) \in \Pi_{\{1,q,q^2,q^3\}}(\mathrm{EM}(2k, 1))$ such that all non integer cycles of $(h, v)$ are invariant under quarter turn rotation, and assume that $(h, v) \notin \mathrm{vert}\mathcal{E}_{2k}^{\{1,q,q^2,q^3\}}$ (note that we are omitting the $n$ odd case as it follows in a similar way). Thus if we consider the fundamental region, from Lemma 1.2.7 there exists $(h^*, v^*) \neq (0, 0)$ such that $f'_{\{1,q,q^2,q^3\}}(h, v) \pm (h^*, v^*) \in \overline{\mathcal{E}_{2k}^{\{1,q,q^2,q^3\}}}$. Note that we must have: $h^*_{ik} = -v^*_{ki}$ for all $i \in [k]$.

Since all the non integer cycles of $(h, v)$ are invariant under quarter turn rotation it can be checked that the only non integer paths of $f'_{1,q,q^2,q^3}(h, v)$ are isolated non integer paths connecting the bottom and right boundaries of $\mathcal{L}_{k,k}$. Thus $(h^*, v^*)$ can only have isolated non zero paths between the right and lower boundaries of $\mathcal{L}_{k,k}$ (any other non zero paths would imply the same non integer path for $(h, v)$). These paths cannot intersect (otherwise this would create a path between two edges of the same boundary) and so there exists $i_0$ such that we have an isolated non zero path between $(k, i_0 + 1)$ and $(i_0 + 1, k)$ and so $h^*_{i_0k}, v^*_{ki_0} \neq 0$ (since these entries correspond to edges of the non zero path). Also by Lemma 4.5.2 $h^*_{i_0k} = \begin{cases} 1 - v^*_{ki_0}, & K \text{ odd} \\ v^*_{ki_0}, & K \text{ even} \end{cases}$ where $K$ is the number of vertex configurations of the form:

that appear in the isolated non zero path. Since this path connects the right and lower boundary, these vertex configurations must appear an even number of times and so $K$ is even. Thus, $h^*_{i_0k} = v^*_{ki_0}$. Recalling that we noted that $h^*_{ik} = -v^*_{ki}$ for all $i \in [k]$, this is the required contradiction and so:

$$\text{vert}\mathcal{E}_n^{\{1,q,q^2,q^3\}} \supseteq \left\{ (h,v) \in \Pi_{\{1,q,q^2,q^3\}}\left(\text{EM}(n,1)\right) \,\middle|\, \begin{array}{l}\text{All non integer cycles of } (h,v) \text{ are} \\ \text{invariant under quarter turn rotation}\end{array} \right\}$$

□

In Figures 4.25 and 4.26 we give the sets vert$\mathcal{A}_3^{\{1,q,q^2,q^3\}}$ and vert$\mathcal{A}_4^{\{1,q,q^2,q^3\}}$.

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

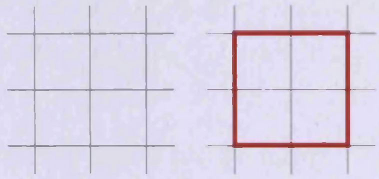


Figure 4.25: vert$\mathcal{A}_3^{\{1,q,q^2,q^3\}}$ and the corresponding non integer edges

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

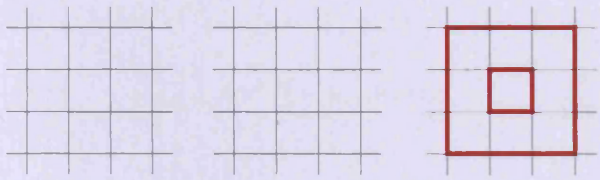


Figure 4.26: vert$\mathcal{A}_4^{\{1,q,q^2,q^3\}}$ and the corresponding non integer edges

Recalling(4.28) it can actually be checked (by setting $\mu = \frac{1}{4}$ instead of $\frac{1}{8}$) that $\begin{pmatrix} \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & \frac{1}{2} & \frac{1}{4} \end{pmatrix}$

is on the edge between $\begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

Figures 4.25 and 4.25 show that $D(\mathcal{A}_3^{\{1,q,q^2,q^3\}}) = D(\mathcal{A}_4^{\{1,q,q^2,q^3\}}) = 2$. Thus by padding we have $2 \leq D(\mathcal{A}_n^{\{1,q,q^2,q^3\}}) \leq 4$ for all $n$. However we can improve on this:

**Theorem 4.5.4.**

$$D(\mathcal{A}_n^{\{1,q,q^2,q^3\}}) = 2 \text{ for all } n \geq 2$$

*Proof.* Once again we present the proof for $n$ even as the result follows in the same way for $n$ odd. Consider $(h, v) \in \text{vert}\overline{\mathcal{E}_{2k}^{\{1,q,q^2,q^3\}}} \setminus \text{EM}(2k, 1)^{\{1,q,q^2,q^3\}}$. Thus $h_{ik} = 1 - v_{ki}$ for all $i \in [k]$ and there must exist $i_0 \in [k]$ such that $h_{i_0k} = 1 - v_{ki_0}$ is non integer. From Theorem 4.5.3 we see that the edge between $(i_0, k)$ and $(i_0, k+1)$ and the edge between $(k, i_0)$ and $(k+1, i_0)$ are connected by an isolated non integer path. By Lemma 4.5.1 we have $h_{i_0k} = \begin{cases} 1 - v_{ki_0}, & K \text{ odd} \\ v_{ki_0}, & K \text{ even} \end{cases}$ and as before $K$ must be even and so $h_{i_0k} = v_{ki_0}$. Thus (since $h_{ik} = 1 - v_{ki}$ for all $i \in [k]$), $h_{i_0k} = v_{ki_0} = \frac{1}{2}$ and all entries on the path are also $\frac{1}{2}$. Therefore all entries on each non integer cycle of an element of $\text{vert}\mathcal{A}_n^{\{1,q,q^2,q^3\}} \setminus \text{ASM}(n, 1)^{\{1,q,q^2,q^3\}}$ are $\frac{1}{2}$ and so $\text{D}(\mathcal{A}_n^{\{1,q,q^2,q^3\}}) = 2$. □

As a consequence of Theorems 1.2.18 and 4.5.4 we have:

**Theorem 4.5.5.** *For fixed $n \in \mathbb{P}$ there exists $A_n^{\{1,q,q^2,q^3\}}(r)$, the Ehrhart quasi-polynomial of $\mathcal{A}_n^{\{1,q,q^2,q^3\}}$ which satisfies:*

1. *$A_n^{\{1,q,q^2,q^3\}}(r)$ is a quasi polynomial in $r$ of degree $\dim \mathcal{A}_n^{\{1,q,q^2,q^3\}}$ and period which divides 2.*

2. *$\left|ASM(n, r)^{\{1,q,q^2,q^3\}}\right| = A_n^{\{1,q,q^2,q^3\}}(r)$ for all $r \in \mathbb{N}$*

3. *$\left|ASM^{\circ}(n, r)^{\{1,q,q^2,q^3\}}\right| = (-1)^{\dim \mathcal{A}_n^{\{1,q,q^2,q^3\}}} A_n^{\{1,q,q^2,q^3\}}(-r)$ for all $r \in \mathbb{P}$*

We illustrate this theorem with the following series:

$$A_2^{\{1,q,q^2,q^3\}}(r) = \begin{cases} 1, & r \text{ even} \\ 0, & r \text{ odd} \end{cases} \tag{4.29}$$

$$A_3^{\{1,q,q^2,q^3\}}(r) = \binom{\lfloor \frac{r}{2} \rfloor + 1}{1} \tag{4.30}$$

$$A_4^{\{1,q,q^2,q^3\}}(r) = \begin{cases} \binom{\frac{r}{2}+1}{2} + \binom{\frac{r}{2}+2}{2}, & r \text{ even} \\ 2\binom{\frac{r-1}{2}+2}{2}, & r \text{ odd} \end{cases} \tag{4.31}$$

$$A_5^{\{1,q,q^2,q^3\}}(r) = \begin{cases} 6\binom{\frac{r}{2}+2}{4} + 7\binom{\frac{r}{2}+3}{4} + \binom{\frac{r}{2}+4}{4}, & r \text{ even} \\ 2\binom{\frac{r-1}{2}+2}{4} + 9\binom{\frac{r-1}{2}+3}{4} + 3\binom{\frac{r-1}{2}+4}{4}, & r \text{ odd} \end{cases} \tag{4.32}$$

$$A_6^{\{1,q,q^2,q^3\}}(r) = \begin{cases} 9\binom{\frac{r}{2}+2}{6} + 110\binom{\frac{r}{2}+3}{6} + 142\binom{\frac{r}{2}+4}{6} + 30\binom{\frac{r}{2}+5}{6} + \binom{\frac{r}{2}+6}{6}, & r \text{ even} \\ 0, & r \text{ odd} \end{cases} \tag{4.33}$$

Quasi-polynomials (4.30) and (4.31) correspond to sequences A008619 and A002620 of [99].

## 4.6 Diagonal symmetry

The fourth row of Figures 1.17 and 4.1 give:

$$\mathcal{A}_n^{\{1,d\}} = \{a \in \mathcal{A}_n \mid a_{ij} = a_{ji} \text{ for all } i,j \in [n]\} \tag{4.34}$$

$$\mathcal{E}_n^{\{1,d\}} = \{(h,v) \in \mathcal{E}_n \mid h_{ij} = v_{ji} \text{ for all } i \in [n],\ j \in [0,n]\} \tag{4.35}$$

Figure 4.27 gives the set $\mathrm{ASM}(3,1)^{\{1,d\}}$ and some cardinalities of $\mathrm{ASM}(n,r)^{\{1,d\}}$ are given by Figure 4.28.

$$\left\{ \begin{matrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{matrix} \right\}$$

Figure 4.27: Diagonally symmetric alternating sign matrices of size 3 and line sum 1

| | $r=0$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $n=1$ | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 2 | 3 | 4 | 5 | 6 |
| 3 | 1 | 5 | 14 | 30 | 55 | 91 |
| 4 | 1 | 16 | 106 | 444 | 1407 | 3696 |
| 5 | 1 | 67 | 1332 | 13204 | 84754 | 404784 |
| 6 | 1 | 368 | 27627 | 779616 | | |

Figure 4.28: $|\mathrm{ASM}(n,r)^{\{1,d\}}|$ for $n \in [6]$, $r \in [0,5]$

No formula is known (or conjectured) for $|\mathrm{ASM}(n,1)^{\{1,d\}}|$ (sequence A005163 of [99]). Indeed in [93] Robbins states: "Apparently these numbers do not factor into small primes, so a simple product formula seems unlikely. Of course this does not rule out other very simple formulas, but these would be more difficult to discover (let alone prove)".

Recalling Figure 3.1 we have: $R_n^{\{1,d\}} = \{(i,j) \in [n] \times [n] \mid j \leq i \text{ for all } i,j \in [n]\}$ giving:

$$\overline{\mathcal{A}_n^{\{1,d\}}} = \left\{ a = \begin{pmatrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ \vdots & & \ddots & \\ a_{n1} & \cdots & \cdots & a_{nn} \end{pmatrix} \in \mathbb{R}^{\frac{n(n+1)}{2}} \;\middle|\; \begin{array}{l} \bullet\ 0 \leq \sum_{j'=1}^{j} a_{ij'} \leq 1 \text{ for all } j \in [i-1], \\ \qquad\qquad i \in [n] \\ \bullet\ 0 \leq \sum_{j'=1}^{i-1} a_{ij'} + \sum_{j'=i}^{j} a_{j'i} \leq 1 \text{ for all} \\ \qquad\qquad j \in [i,n],\ i \in [n] \\ \bullet\ \sum_{j=1}^{i-1} a_{ij} + \sum_{j=i}^{n} a_{ji} = 1 \text{ for all } i \in [n] \end{array} \right\}$$

and:

$$\overline{\mathcal{E}_n^{\{1,d\}}} = \left\{ \left( \begin{pmatrix} h_{10} & & & & \\ h_{20} & h_{21} & & & \\ h_{30} & h_{31} & h_{32} & & \\ \vdots & & & \ddots & \\ h_{n0} & h_{n1} & h_{n2} & \dots & h_{nn-1} \end{pmatrix}, \begin{pmatrix} v_{11} & & & & \\ v_{21} & v_{22} & & & \\ v_{31} & v_{32} & v_{33} & & \\ \vdots & & & \ddots & \\ v_{n1} & v_{n2} & v_{n3} & \dots & v_{nn} \end{pmatrix} \right) \in [0,1]_{\mathbb{R}}^{\frac{n(n+1)}{2}} \times [0,1]_{\mathbb{R}}^{\frac{n(n+1)}{2}} \;\middle|\; \begin{array}{l} \bullet\ h_{i0} = 0 \text{ for all } i \in [n] \\ \bullet\ v_{nj} = 1 \text{ for all } j \in [n] \\ \bullet\ h_{i,j-1} + v_{ij} = v_{i-1,j} + h_{ij} \text{ for all } j \leq i-1 \in [n-1] \end{array} \right\}$$

where $f'_{\{1,d\}} : \mathcal{E}_n^{\{1,d\}} \to \overline{\mathcal{E}_n^{\{1,d\}}}$ is given by:

$$f'_{\{1,d\}} \left( \begin{pmatrix} h_{10} & v_{11} & v_{21} & v_{31} & \dots & v_{n1} \\ h_{20} & h_{21} & v_{22} & v_{32} & \dots & v_{n2} \\ h_{30} & h_{31} & h_{32} & v_{33} & \dots & v_{n3} \\ \vdots & & & & \ddots & \vdots \\ h_{n0} & h_{n1} & h_{n2} & \dots & h_{n,n-1} & v_{nn} \end{pmatrix}, \begin{pmatrix} h_{10} & h_{20} & h_{30} & \dots & h_{n0} \\ v_{11} & h_{21} & h_{31} & \dots & h_{n1} \\ v_{21} & v_{22} & h_{32} & \dots & h_{n2} \\ v_{31} & v_{32} & v_{33} & \dots & h_{n3} \\ \vdots & & & \ddots & \dots \\ v_{n1} & v_{n2} & v_{n3} & \dots & v_{nn} \end{pmatrix} \right) =$$

$$\left( \begin{pmatrix} h_{10} & & & & \\ h_{20} & h_{21} & & & \\ h_{30} & h_{31} & h_{32} & & \\ \vdots & & & \ddots & \\ h_{n0} & h_{n1} & h_{n2} & \dots & h_{n,n-1} \end{pmatrix}, \begin{pmatrix} v_{11} & & & & \\ v_{21} & v_{22} & & & \\ v_{31} & v_{32} & v_{33} & & \\ \vdots & & & \ddots & \\ v_{n1} & v_{n2} & v_{n3} & \dots & v_{nn} \end{pmatrix} \right)$$

We can represent $\mathcal{E}_n^{\{1,d\}}$ and $\overline{\mathcal{E}_n^{\{1,d\}}}$ on $\mathcal{L}_{n,n}$ and a triangular lattice as in Figure 4.29.

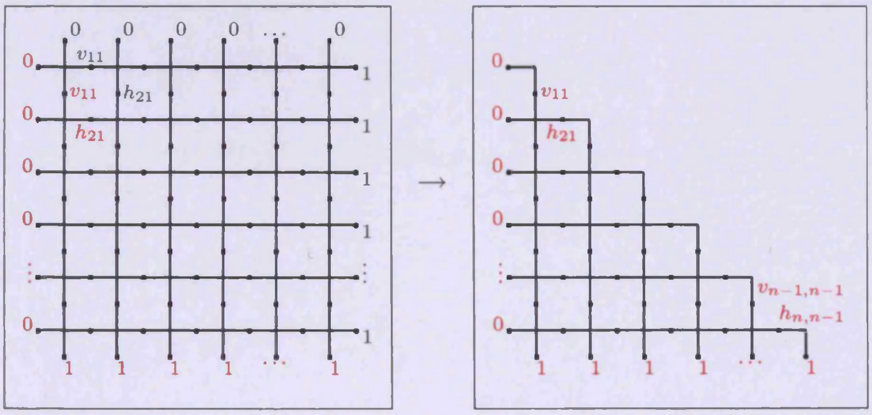



Figure 4.29: Elements of $\mathcal{E}_n^{\{1,d\}}$ and $\overline{\mathcal{E}_n^{\{1,d\}}}$ on $\mathcal{L}_{n,n}$ and a triangular lattice

**Theorem 4.6.1.**

$$vert\mathcal{A}_n^{\{1,d\}} = ASM(n,1)^{\{1,d\}}$$

*Proof.* We show that $\text{vert}\mathcal{E}_n^{\{1,d\}} = \left\{(h,v) \in \mathcal{E}_n^{\{1,d\}} \mid (h,v) \text{ does not have a non integer cycle}\right\}$. This is because, if $(h,v) \in \overline{\mathcal{E}_n^{\{1,d\}}}$ contains a non integer cycle, then as before we can find $(h^*,v^*) \neq (0,0)$ such that $(h,v) \pm (h^*,v^*) \in \overline{\mathcal{E}_n^{\{1,d\}}}$. More importantly, if $(h,v)$ contains a non integer path we can also find $(h^*,v^*) \neq (0,0)$ such that $(h,v) \pm (h^*,v^*) \in \overline{\mathcal{E}_n^{\{1,d\}}}$ without any restrictions (apart from being in $[0,1]_{\mathbb{R}}$) from the diagonal boundary, giving the required result. □

**Corollary 4.6.2.**

$$D(\mathcal{A}_n^{\{1,d\}}) = 1 \text{ for all } n$$

As a direct implication of Theorems 1.2.29 and 4.6.1 we have:

**Corollary 4.6.3.** *Any matrix from $ASM(n,r)^{\{1,d\}}$ can be written as the sum of $r$ matrices from $ASM(n,1)^{\{1,d\}}$.*

From Theorem 1.2.18 and Corollary 4.6.2 we have:

**Theorem 4.6.4.** *For fixed $n \in \mathbb{P}$ there exists $A_n^{\{1,d\}}(r)$, the Ehrhart polynomial of $\mathcal{A}_n^{\{1,d\}}$ which satisfies:*

1. *$A_n^{\{1,d\}}(r)$ is a polynomial in $r$ of degree $\frac{n(n-1)}{2}$.*
2. *$\left|ASM(n,r)^{\{1,d\}}\right| = A_n^{\{1,d\}}(r)$ for all $r \in \mathbb{N}$*
3. *$\left|ASM^{\circ}(n,r)^{\{1,d\}}\right| = (-1)^{\frac{n(n-1)}{2}} A_n^{\{1,d\}}(-r)$ for all $r \in \mathbb{P}$*

We illustrate this with the following examples:

$$A_2^{\{1,d\}}(r) = \binom{r+1}{1} \tag{4.36}$$

$$A_3^{\{1,d\}}(r) = \binom{r+2}{3} + \binom{r+3}{3} \tag{4.37}$$

$$A_4^{\{1,d\}}(r) = 3\binom{r+3}{6} + 15\binom{r+4}{6} + 9\binom{r+5}{6} + \binom{r+6}{6} \tag{4.38}$$

$$A_5^{\{1,d\}}(r) = \tag{4.39}$$

$$30\binom{r+4}{10} + 578\binom{r+5}{10} + 2045\binom{r+6}{10} + 2072\binom{r+7}{10} + 650\binom{r+8}{10} + 56\binom{r+9}{10} + \binom{r+10}{10}$$

Polynomial (4.37) corresponds to sequence A000330 of [99].

## 4.7 Both diagonal symmetry

The seventh row of Figure 1.17 and 4.1 give:

$$\mathcal{A}_n^{\{1,d,a,q^2\}} = \{a \in \mathcal{A}_n \mid a_{ij} = a_{ji} = a_{n+1-j,n+1-i} \text{ for all } i, j \in [n]\} \quad (4.40)$$

$$\mathcal{E}_n^{\{1,d,a,q^2\}} = \{(h, v) \in \mathcal{E}_n \mid h_{ij} = v_{ji} = 1 - h_{n+1-i,n-j} \text{ for all } i \in [n],\ j \in [0, n]\} \quad (4.41)$$

Figure 4.30 gives the set $\mathrm{ASM}(3, 2)^{\{1,d,a,q^2\}}$ and some cardinalities of $\mathrm{ASM}(n, r)^{\{1,d,a,q^2\}}$ are given by Figure 4.31.

$$\left\{ \begin{matrix} \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 0 \\ 2 & -2 & 2 \\ 0 & 2 & 0 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \end{matrix} \right\}$$

Figure 4.30: Both diagonal symmetric alternating sign matrices of size 3 and line sum 2

| | $r = 0$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $n = 1$ | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 2 | 3 | 4 | 5 | 6 |
| 3 | 1 | 3 | 6 | 10 | 15 | 21 |
| 4 | 1 | 8 | 30 | 80 | 175 | 336 |
| 5 | 1 | 15 | 94 | 378 | 1162 | 2982 |

Figure 4.31: $|\mathrm{ASM}(n, r)^{\{1,d,a,q^2\}}|$ for $n \in [6]$, $r \in [0, 5]$

Recalling equation (1.35) we have the conjectured formula:

$$\left|\mathrm{ASM}(n, 1)^{\{1,d,a,q^2\}}\right| = \begin{cases} ? & n \text{ even} \\ \prod_{i=0}^{\frac{n-1}{2}} \frac{(3i)!}{(\frac{n-1}{2}+i)!}, & n \text{ odd} \end{cases}$$

corresponding to sequence A005162 of [99]. Note that conjecturing a formula for $n$ even (represented by a question mark in Figure 4.4) and proving the formula for $n$ odd are both still open problems.

Using the same argument as for the proof of the Theorem 4.6.1 we have:

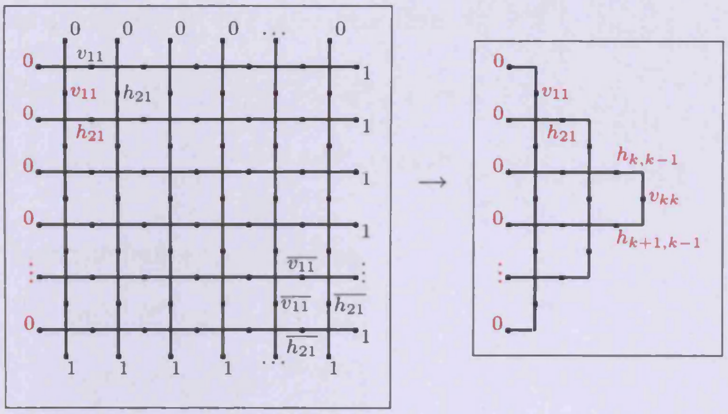
**Theorem 4.7.1.**

$$vert\mathcal{A}_n^{\{1,d,a,q^2\}} = ASM(n, 1)^{\{1,d,a,q^2\}}$$

For completeness recalling Figure 3.1 we have:

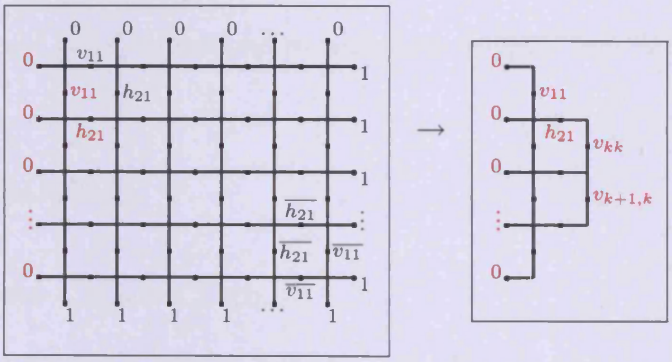
$$R_n^{\{1,d,a,q^2\}} = \{(i,j) \in [n] \times [n] \mid j \leq i \leq n+1-j \text{ for all } i,j \in [n]\}$$

Figures 4.32 and 4.33 give the fundamental polytopes on a lattice diagram.



where $\overline{m} = 1 - m$ for all $m \in \mathbb{R}$

Figure 4.32: Elements of $\mathcal{E}_{2k}^{\{1,d,a,q^2\}}$ and $\overline{\mathcal{E}_{2k}^{\{1,d,a,q^2\}}}$ on lattice diagrams



where $\overline{m} = 1 - m$ for all $m \in \mathbb{R}$

Figure 4.33: Elements of $\mathcal{E}_{2k+1}^{\{1,d,a,q^2\}}$ and $\overline{\mathcal{E}_{2k+1}^{\{1,d,a,q^2\}}}$ on lattice diagrams

Theorem 4.7.1 gives:

**Corollary 4.7.2.**

$$D(A_n^{\{1,d,a,q^2\}}) = 1 \text{ for all } n$$

As a direct implication of Theorems 1.2.29 and 4.7.1 we have:

**Corollary 4.7.3.** *Any matrix from* $ASM(n,r)^{\{1,d,a,q^2\}}$ *can be written as the sum of* $r$ *matrices from* $ASM(n,1)^{\{1,d,a,q^2\}}$.

Theorem 1.2.18 and Corollary 4.7.2 give:

**Theorem 4.7.4.** *For fixed* $n \in \mathbb{P}$ *there exists* $A_n^{\{1,d,a,q^2\}}(r)$*, the Ehrhart polynomial of* $\mathcal{A}_n^{\{1,d,a,q^2\}}$ *which satisfies:*

1. $A_n^{\{1,d,a,q^2\}}(r)$ *is a polynomial in* $r$ *of degree* $\dim \mathcal{A}_n^{\{1,d,a,q^2\}}$
2. $\left|ASM(n,r)^{\{1,d,a,q^2\}}\right| = A_n^{\{1,d,a,q^2\}}(r)$ *for all* $r \in \mathbb{N}$
3. $\left|ASM^{\circ}(n,r)^{\{1,d,a,q^2\}}\right| = (-1)^{\dim \mathcal{A}_n^{\{1,d,a,q^2\}}} A_n^{\{1,d,a,q^2\}}(-r)$ *for all* $r \in \mathbb{P}$

We illustrate this with the following examples:

$$A_2^{\{1,d,a,q^2\}}(r) = \binom{r+1}{1} \tag{4.42}$$

$$A_3^{\{1,d,a,q^2\}}(r) = \binom{r+2}{2} \tag{4.43}$$

$$A_4^{\{1,d,a,q^2\}}(r) = 3\binom{r+3}{4} + \binom{r+4}{4} \tag{4.44}$$

$$A_5^{\{1,d,a,q^2\}}(r) = 10\binom{r+4}{6} + 8\binom{r+5}{6} + \binom{r+6}{6} \tag{4.45}$$

Polynomials (4.43) and (4.44) correspond to sequences A000217 and A002417 of [99].

## 4.8 All symmetry

The last row of Figures 1.17 and 4.1 give:

$$\mathcal{A}_n^{D_4} := \{a \in \mathcal{A}_n \mid a_{ij} = a_{n+1-i,j} = a_{j,n+1-i} \text{ for all } i,j \in [n]\} \tag{4.46}$$

$$\mathcal{E}_n^{D_4} := \{(h,v) \in \mathcal{E}_n \mid h_{ij} = v_{ji} = h_{n+1-i,j} = 1 - h_{i,n-j} \text{ for all } i \in [n],\ j \in [0,n]\} \tag{4.47}$$

Figure 4.34 gives the set $\text{ASM}(5,2)^{D_4}$ and some cardinalities of $\text{ASM}(n,r)^{D_4}$ are given by Figure 4.1.

Recalling Figure 3.1 we have: $R_{2k}^{D_4} = [k] \times [k]$ and $R_{2k+1}^{D_4} = [k+1] \times [k+1]$ leading to the following fundamental polytopes:

$$\overline{\mathcal{A}_{2k}^{D_4}} := \left\{ a \in \mathbb{R}^{k \times k} \;\middle|\; \begin{array}{l} \bullet\ 0 \leq \sum_{j'=1}^{j} a_{ij'} \leq 1 \text{ for all } i,j \in [k] \\ \bullet\ \sum_{j=1}^{k} a_{ij} = \frac{1}{2} \text{ for all } i \in [k] \\ \bullet\ a_{ij} = a_{ji} \text{ for all } i,j \in [k] \end{array} \right\}$$

$$\left\{\begin{pmatrix}0&0&2&0&0\\0&1&0&1&0\\2&0&-2&0&2\\0&1&0&1&0\\0&0&2&0&0\end{pmatrix},\begin{pmatrix}0&0&2&0&0\\0&2&-2&2&0\\2&-2&2&-2&2\\0&2&-2&2&0\\0&0&2&0&0\end{pmatrix},\begin{pmatrix}0&1&0&1&0\\1&-1&2&-1&1\\0&2&-2&2&0\\1&-1&2&-1&1\\0&1&0&1&0\end{pmatrix}\right.$$
$$\left.\begin{pmatrix}0&1&0&1&0\\1&0&0&0&1\\0&0&2&0&0\\1&0&0&0&1\\0&1&0&1&0\end{pmatrix},\begin{pmatrix}1&0&0&0&1\\0&0&2&0&0\\0&2&-2&2&0\\0&0&2&0&0\\1&0&0&0&1\end{pmatrix},\begin{pmatrix}1&0&0&0&1\\0&1&0&1&0\\0&0&2&0&0\\0&1&0&1&0\\1&0&0&0&1\end{pmatrix}\right\}$$

Figure 4.34: All symmetric alternating sign matrices of size 5 and line sum 2

| | $r=0$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $n=1$ | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 0 | 1 | 0 | 1 | 0 |
| 3 | 1 | 1 | 2 | 2 | 3 | 3 |
| 4 | 1 | 0 | 2 | 0 | 3 | 0 |
| 5 | 1 | 1 | 6 | 6 | 18 | 18 |
| 6 | 1 | 0 | 7 | 0 | 22 | 0 |

Table 4.1: $|\mathrm{ASM}(n,r)^{D_4}|$ for $n \in [5]$, $r \in [0,5]$

and

$$\overline{\mathcal{A}_{2k+1}^{D_4}} := \left\{ a \in \mathbb{R}^{(k+1)\times(k+1)} \;\middle|\; \begin{array}{l} \bullet\ 0 \le \sum_{j'=1}^{j} a_{ij'} \le 1 \text{ for all } i,j \in [k+1] \\ \bullet\ 2\sum_{j=1}^{k} a_{ij} + a_{i,k+1} = 1 \text{ for all } i \in [k+1] \\ \bullet\ a_{ij} = a_{ji} \text{ for all } i,j \in [k+1] \end{array} \right\}$$

We thus have:

$$\overline{\mathcal{E}_{2k}^{D_4}} = \left\{ (h,v) = \left( \begin{pmatrix} h_{10} & \cdots & h_{1k} \\ \vdots & & \vdots \\ h_{k0} & \cdots & h_{kk} \end{pmatrix}, \begin{pmatrix} v_{01} & \cdots & v_{0k} \\ \vdots & & \vdots \\ v_{k1} & \cdots & v_{kk} \end{pmatrix} \right) \in [0,1]_{\mathbb{R}}^{k\times(k+1)} \times [0,1]_{\mathbb{R}}^{(k+1)\times k} \;\middle|\; \begin{array}{l} \bullet\ h_{i0} = v_{0j} = 0 \text{ for all } i \in [k],\ j \in [k] \\ \bullet\ h_{ik} = v_{kj} = \frac{1}{2} \text{ for all } i,j \in [k] \\ \bullet\ h_{i,j-1} + v_{ij} = v_{i-1,j} + h_{ij} \text{ for all } i,j \in [k] \\ \bullet\ h_{ij} = v_{ji} \text{ for all } i \in [k],\ j \in [0,k] \end{array} \right\}$$

and

$$\overline{\mathcal{E}_{2k+1}^{D_4}} = \left\{ (h,v) = \left( \begin{pmatrix} h_{10} & \cdots & h_{1,k+1} \\ \vdots & & \vdots \\ h_{k+1,0} & \cdots & h_{k+1,k+1} \end{pmatrix}, \begin{pmatrix} v_{01} & \cdots & v_{0,k+1} \\ \vdots & & \vdots \\ v_{k+1,1} & \cdots & v_{k+1,k+1} \end{pmatrix} \right) \right. \in [0,1]_{\mathbb{R}}^{(k+1)\times(k+2)} \times [0,1]_{\mathbb{R}}^{(k+2)\times(k+1)} \left| \begin{array}{l} \bullet\ h_{i0} = v_{0j} = 0 \text{ for all } i,j \in [k+1] \\ \bullet\ h_{ik} + h_{i,k+1} = v_{kj} + v_{k+1,j} = 1 \text{ for all } i,j \in [k+1] \\ \bullet\ h_{i,j-1} + v_{ij} = v_{i-1,j} + h_{ij} \text{ for all } i,j \in [k+1] \\ \bullet\ h_{ij} = v_{ji} \text{ for all } i \in [k+1],\ j \in [0,k+1] \end{array} \right\}$$

We note that $\overline{\mathcal{A}_n^{D_4}} = \left(\overline{\mathcal{A}_n^{\{1,h,v,q^2\}}}\right)^{\{1,d\}}$ and $\overline{\mathcal{E}_n^{D_4}} = \left(\overline{\mathcal{E}_n^{\{1,d,a,q\}}}\right)^{\{1,d\}}$ which leads to the following theorem:

**Theorem 4.8.1.**

$$vert\mathcal{A}_n^{D_4} = \left(vert\mathcal{A}_n^{\{1,h,v,q^2\}}\right)^{\{1,d\}}$$

In other words the vertices of $\mathcal{A}_n^{D_4}$ are the elements of vert$\mathcal{A}_n^{\{1,h,v,q^2\}}$ that are diagonally symmetric.

This proof follows from the proof of Theorem 4.6.1 where we see that vert $\left(\overline{\mathcal{E}_{2k}^{\{1,h,v,q^2\}}}\right)^{\{1,d\}} = \left(\text{vert}\overline{\mathcal{E}_{2k}^{\{1,h,v,q^2\}}}\right)^{\{1,d\}}$ giving the required result.

From Theorem 4.3.1 we have:

**Corollary 4.8.2.**

$$D\left(\mathcal{A}_n^{D_4}\right) = 2 \textit{ for all } n \geq 2$$

As a consequence of Theorem 1.2.18 and Corollary 4.8.2 we have:

**Theorem 4.8.3.** *For fixed $n \in \mathbb{P}$ there exists $A_n^{D_4}(r)$, the Ehrhart quasi-polynomial of $\mathcal{A}_n^{D_4}$ which satisfies:*

1. *$A_n^{D_4}(r)$ is a quasi polynomial in $r$ of degree* $\dim \mathcal{A}_n^{D_4}$ *and period which divides 2.*

2. $\left|ASM(n,r)^{D_4}\right| = A_n^{D_4}(r)$ *for all* $r \in \mathbb{N}$

3. $\left|ASM^{\circ}(n,r)^{D_4}\right| = (-1)^{\dim \mathcal{A}_n^{D_4}} A_n^{D_4}(-r)$ *for all* $r \in \mathbb{P}$

We illustrate this with the following examples

$$A_2^{D_4}(r) = \begin{cases} 1, \ r \text{ even} \\ 0, \ r \text{ odd} \end{cases} \tag{4.48}$$

$$A_3^{D_4}(r) = \binom{\lfloor \frac{r}{2} \rfloor + 1}{1} \tag{4.49}$$

$$A_4^{D_4}(r) = \begin{cases} \binom{\frac{r}{2}+1}{1}, \ r \text{ even} \\ 0, \ r \text{ odd} \end{cases} \tag{4.50}$$

$$A_5^{D_4}(r) = 2\binom{\lfloor \frac{r}{2} \rfloor + 2}{3} + \binom{\lfloor \frac{r}{2} \rfloor + 3}{3} \tag{4.51}$$

$$A_6^{D_4}(r) = \begin{cases} 3\binom{\frac{r}{2}+2}{3} + \binom{\frac{r}{2}+3}{3}, \ r \text{ even} \\ 0, \ r \text{ odd} \end{cases} \tag{4.52}$$

# 4.9 Conclusion

As this chapter contains numerous results we here give a concise listing of these results.

## 4.9.1 Conclusion for $\mathcal{A}_n^{\{1,h\}}$

- $\left|\mathcal{A}_n^{\{1,h\}} \cap \mathbb{Z}^{n\times n}\right| = \left|\mathrm{ASM}(n,1)^{\{1,h\}}\right| = \begin{cases} 0, \ n \text{ even} \\ \prod_{i=1}^{\frac{n-1}{2}} \frac{(6i-2)!}{(n-1+2i)!}, \ n \text{ odd} \end{cases}$
- $\dim \mathcal{A}_n^{\{1,h\}} = \begin{cases} \frac{(n-1)(n-2)}{2}, \ n \text{ even} \\ \frac{(n-1)^2}{2}, \ n \text{ odd} \end{cases}$
- $\mathrm{vert}\mathcal{A}_n^{\{1,h\}} = \{a \in \Pi_{\{1,h\}}(\mathrm{ASM}(n,1)) \mid f_{\{1,h\}}(a) \text{ does not have a non integer cycle}\}$
- Theorem 4.2.3 gives an enumeration result for fixed $n$. Quasi-polynomials of period 2 are obtained for $n \in [5]$

## 4.9.2 Conclusion for $\mathcal{A}_n^{\{1,h,v,q^2\}}$

- $\left|\mathcal{A}_n^{\{1,h,v,q^2\}} \cap \mathbb{Z}^{n\times n}\right| = \left|\mathrm{ASM}(n,1)^{\{1,h,v,q^2\}}\right| = \begin{cases} 0, \ n \text{ even} \\ \frac{(\lfloor \frac{3(n-3)}{4} \rfloor + 1)!}{3\lfloor \frac{n-3}{4} \rfloor (n-2)! \lfloor \frac{n-3}{4} \rfloor !} \prod_{i=1}^{\frac{n-3}{2}} \frac{(3i)!}{(\frac{n-3}{2}+i)!}, \ n \text{ odd} \end{cases}$
- $\dim \mathcal{A}_n^{\{1,h,v,q^2\}} = \begin{cases} \frac{(n-2)^2}{4}, \ n \text{ even} \\ \frac{(n-1)^2}{4}, \ n \text{ odd} \end{cases}$
- $\mathrm{vert}\mathcal{A}_n^{\{1,h,v,q^2\}} = \{a \in \Pi_{\{1,h,v,q^2\}}(\mathrm{ASM}(n,1)) \mid f_{\{1,h,v,q^2\}}(a) \text{ does not have a non integer cycle}\}$

- Theorem 4.3.3 gives an enumeration result for fixed $n$. Quasi-polynomials of period 2 are obtained for $n \in [6]$.

### 4.9.3 Conclusion for $\mathcal{A}_n^{\{1,q^2\}}$

- $\left|\mathcal{A}_n^{\{1,q^2\}} \cap \mathbb{Z}^{n\times n}\right| = \left|\mathrm{ASM}(n,1)^{\{1,q^2\}}\right| = \prod_{i=0}^{\lceil \frac{n}{2}\rceil-1} \frac{(3i)!}{(\lfloor\frac{n}{2}\rfloor+i)!} \prod_{i=0}^{\lfloor \frac{n}{2}\rfloor-1} \frac{(3i+2)!}{(\lceil\frac{n}{2}\rceil+i)!}$.
- $\dim \mathcal{A}_n^{\{1,q^2\}} = \begin{cases} \frac{(n-1)^2+1}{2}, & n \text{ even} \\ \frac{(n-1)^2}{2}, & n \text{ odd} \end{cases}$.
- $\mathrm{vert}\mathcal{A}_n^{\{1,q^2\}} = \mathrm{ASM}(n,1)^{\{1,q^2\}}$
- Theorem 4.4.4 gives an enumeration result for fixed $n$. Polynomials are obtained for $n \in [5]$.

### 4.9.4 Conclusion for $\mathcal{A}_n^{\{1,q,q^2,q^3\}}$

- $\left|\mathcal{A}_n^{\{1,q,q^2,q^3\}} \cap \mathbb{Z}^{n\times n}\right| = \left|\mathrm{ASM}(n,1)^{\{1,q,q^2,q^3\}}\right| =$
$$\begin{cases} \prod_{i=0}^{\frac{n}{4}-1} \frac{i!(3i+1)!(\frac{n}{2}+i)!}{(\frac{n}{4}+i)!^3}, & n = 0 \bmod 4 \\ \prod_{i=0}^{\frac{n-1}{4}} \frac{(3i)!}{(\frac{n-1}{4}+i)!} \prod_{i=0}^{\frac{n-5}{4}} \frac{(3i+1)!^2(3i+2)!}{(\frac{n-1}{4}+i)!(\frac{n+3}{4}+i)!}, & n = 1 \bmod 4 \\ 0, & n = 2 \bmod 4 \\ \prod_{i=0}^{\frac{n-3}{4}} \frac{(3i)!(3i+1)!^2}{(\frac{n-3}{4}+i)!(\frac{n+1}{4}+i)!} \prod_{i=0}^{\frac{n-7}{4}} \frac{(3i+2)!}{(\frac{n+1}{4}+i)!}, & n = 3 \bmod 4 \end{cases}$$
- $\dim \mathcal{A}_n^{\{1,q,q^2,q^3\}} = \begin{cases} \frac{n(n-2)}{4}, & n \text{ even} \\ \frac{(n-1)^2}{4}, & n \text{ odd} \end{cases}$
- $\mathrm{vert}\mathcal{A}_n^{\{1,q,q^2,q^3\}} = \left\{ a \in \Pi_{\{1,q,q^2,q^3\}}(\mathrm{ASM}(n,1)) \;\middle|\; \begin{array}{l}\text{All non integer cycles of } a \text{ on } \mathcal{L}_{nn} \text{ are} \\ \text{invariant under quarter turn rotation}\end{array} \right\}$
- Theorem 4.5.5 gives an enumeration result for fixed $n$. Quasi-polynomials of period 2 are obtained for $n \in [6]$.

### 4.9.5 Conclusion for $\mathcal{A}_n^{\{1,d\}}$

- There is no known enumeration for $\mathrm{ASM}(n,1)^{\{1,d\}}$
- $\dim \mathcal{A}_n^{\{1,d\}} = \frac{n(n-1)}{2}$

- $\text{vert}\mathcal{A}_n^{\{1,d\}} = \text{ASM}(n,1)^{\{1,d\}}$
- Theorem 4.6.4 gives an enumeration result for fixed $n$. Polynomials are obtained for $n \in [5]$.

### 4.9.6 Conclusion for $\mathcal{A}_n^{\{1,d,a,q^2\}}$

- $\left|\text{ASM}(n,1)^{\{1,d,a,q^2\}}\right| = \begin{cases} ?\ n \text{ even} \\ \prod_{i=0}^{\frac{n-1}{2}} \frac{(3i)!}{\left(\frac{n-1}{2}+i\right)!},\ n \text{ odd} \end{cases}$
- $\dim \mathcal{A}_n^{\{1,d,a,q^2\}} = \begin{cases} \frac{n^2}{4},\ n \text{ even} \\ \frac{(n-1)(n+1)}{4},\ n \text{ odd} \end{cases}$
- $\text{vert}\mathcal{A}_n^{\{1,d,a,q^2\}} = \text{ASM}(n,1)^{\{1,d,a,q^2\}}$
- Theorem 4.7.4 gives an enumeration result for fixed $n$. Polynomials are obtained for $n \in [5]$.

### 4.9.7 Conclusion for $\mathcal{A}_n^{D_4}$

- There is no known enumeration for $\text{ASM}(n,1)^{D_4}$.
- $\dim \mathcal{A}_n^{D_4} = \begin{cases} \frac{(n-2)n}{8},\ n \text{ even} \\ \frac{(n-1)(n+1)}{8},\ n \text{ odd} \end{cases}$
- $\text{vert}\mathcal{A}_n^{D_4} = \left(\text{vert}\mathcal{A}_n^{\{1,h,v,q^2\}}\right)^{\{1,d\}}$
- Theorem 4.8.3 gives an enumeration result for fixed $n$. Quasi-polynomials of period 2 are obtained for $n \in [6]$.

We also believe counterpart decomposition results to Theorems 3.2.3 and 3.3.4 can be obtained.

# Chapter 5

# The Alternating Transportation Polytope

## 5.1 Definition

In Section 1.2.3 the transportation problem was presented. We recall Figure 1.37, and the notation of Section 1.2.3. We here present a generalized transportation polytope based on the work done in the previous chapters. Let us allow the entries $a_{ij}$ of Figure 1.37 to be negative. A negative entry represents a delivery from a *destination* $D_j$ to a *source* $S_i$. Allowing these deliveries has been considered by Orden in [86]. He defines a new polytope: *the transshipment polytope.* In this case all nodes are considered in the same way, having both a unit of demand $r_i$ and a unit of supply $s_j$. The problem is solved by reducing it to a transportation problem and only considering polytopes whose elements have entries which are all non negative. Indeed if we remove the $a_{ij} \geq 0$ condition in Definition 1.2.33 of $\mathcal{T}(r, s)$, we are left with an unbounded space. We need to impose further conditions. Let us consider the following extra conditions to the classical transportation problem, to give a generalized transportation problem:

This is an ordered transportation problem with $m$ sources $S_1, \ldots, S_m$ and $n$ destinations $D_1, \ldots, D_n$. Thus we log the sequence of deliveries using an $m \times n$ array $T$ of numbers chosen from $[1, mn]$ without repetition, such that the delivery between $S_i$ and $D_j$ is the

$(T_{ij})^{th}$ delivery. The amount of material delivered between $S_i$ and $D_j$ is $a_{ij}$ units, where a positive/negative value of $a_{ij}$ means that material is transported from/to $S_i$ to/from $D_j$,

- $T$ must be a *standard Young tableau* [104] of rectangular shape with $m$ rows and $n$ columns , i.e. $T_{ij} < T_{i+1,j}$ for all $i \in [m-1]$, $j \in [n]$ and $T_{ij} < T_{i,j+1}$ for all $i \in [m]$, $j \in [n-1]$.

- Initially (i.e. before the first delivery) there are $r_i$ units of material at $S_i$ and finally (i.e. after the last delivery) there are 0 units of material at $S_i$ for all $i \in [m]$. This gives:
$$\sum_{j=1}^{n} a_{ij} = r_i \text{ for all } i \in [m]$$

- Initially (i.e. before the first delivery) there are 0 units of material at $D_j$ and finally (i.e. after the last delivery) there are $s_j$ units of material at $D_j$ for all $j \in [n]$. This gives:
$$\sum_{i=1}^{m} a_{ij} = s_j \text{ for all } j \in [n]$$

- At any time there are between 0 and $r_i$ units of material at $S_i$. This gives:
$$0 \leq \sum_{j'=1}^{j} a_{ij'} \leq r_i \text{ for all } i \in [m], j \in [n]$$

- At any time there are between 0 and $s_j$ units of material at $D_j$. This gives:
$$0 \leq \sum_{i'=1}^{i} a_{i'j} \leq s_j \text{ for all } i \in [m], j \in [n]$$

This leads to the following definition:

**Definition 5.1.1.** *For $r \in \mathbb{R}^m$, $s \in \mathbb{R}^n$ such that $r_i \geq 0$ and $s_j \geq 0$ for all $i \in [m]$, $j \in [n]$, and $\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} s_j$ we define $\mathcal{A}(r, s)$, the* alternating transportation polytope*:*

$$\mathcal{A}(r,s) := \left\{ a \in \mathbb{R}^{m\times n} \;\middle|\; \begin{array}{l} \bullet \sum_{j=1}^{n} a_{ij} = r_i \textit{ for all } i \in [m] \\ \bullet \sum_{i=1}^{m} a_{ij} = s_j \textit{ for all } j \in [n] \\ \bullet\, 0 \leq \sum_{j'=1}^{j} a_{ij'} \leq r_i \textit{ for all } i \in [m], j \in [n] \\ \bullet\, 0 \leq \sum_{i'=1}^{i} a_{i'j} \leq s_j \textit{ for all } i \in [m], j \in [n] \end{array} \right\}$$

It follows that:

$$\mathcal{A}(r,s) = \left\{ a \in \mathbb{R}^{m\times n} \;\middle|\; \begin{array}{l} \bullet\ \sum_{j=1}^{n} a_{ij} = r_i \ \text{ for all } i \in [m] \\ \bullet\ \sum_{i=1}^{m} a_{ij} = s_j \ \text{ for all } j \in [n] \\ \bullet\ \sum_{j'=1}^{j} a_{ij'} \geq 0 \ \text{ for all } \ i \in [m],\ j \in [n-1] \\ \bullet\ \sum_{j'=j}^{n} a_{ij'} \geq 0 \ \text{ for all } \ i \in [m],\ j \in [2,n] \\ \bullet\ \sum_{i'=1}^{i} a_{i'j} \geq 0 \ \text{ for all } \ i \in [m-1],\ j \in [n] \\ \bullet\ \sum_{i'=i}^{m} a_{i'j} \geq 0 \ \text{ for all } \ i \in [2,m],\ j \in [n] \end{array} \right\} \tag{5.1}$$

Let us assume the cost function is linear (as in Section 1.2.3) but also symmetric, i.e. we assume that a delivery from $D_j$ to $S_i$ has the same cost as a delivery from $S_i$ to $D_j$. Note that this is not always the case (for example, transportation up or down a hill could have different costs). The generalized transportation problem reduces to minimizing:

$$\rho(a) := \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} \, |a_{ij}| \tag{5.2}$$

over $\mathcal{A}(r,s)$. Note that similarly to equations 2.3 and 2.25 we have:

$$\mathcal{T}(r,s) = \{a \in \mathcal{A}(r,s) \mid a_{ij} \geq 0 \text{ for all } i \in [m],\ j \in [n]\} \tag{5.3}$$

Over $\mathcal{T}(r,s)$ the cost function (5.2) reduces to the same cost function $\rho(a) = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} a_{ij}$ considered in Section 1.2.3 and it can be seen that an extreme value of $\rho$ (over $\mathcal{T}(r,s)$) must occur at a vertex of $\mathcal{T}(r,s)$. In $\mathcal{A}(r,s)$ it is no longer true that an extreme point of $\rho(a)$ given by (5.2) must occur at a vertex of $\mathcal{A}(r,s)$. As an example consider the following transportation problem:

$$r = (1,1,1),\ s = (1,1,1),\ c = \begin{pmatrix} 18 & 14 & 20 \\ 2 & 1 & 2 \\ 20 & 14 & 18 \end{pmatrix}$$

A diagram showing the proposed cost function is given in Figure 5.1.

By Theorem 2.3.4 vert$\mathcal{A}((1,1,1),(1,1,1)) = \mathrm{ASM}(3,1)$:

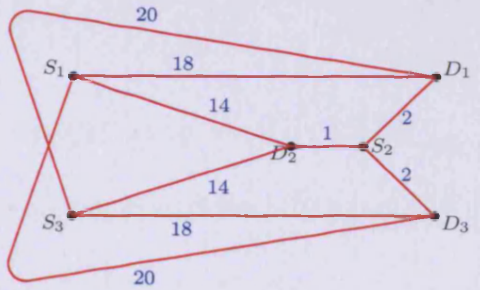



Figure 5.1: Diagram of running example

$$\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix} \quad \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix} \quad \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix} \quad \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix} \quad \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$\begin{pmatrix}0&1&0\\1&-1&1\\0&1&0\end{pmatrix}$$

Over $\mathcal{T}((1,1,1),(1,1,1))$ the minimum of $\rho$ is obtained at $\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$ or $\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$ with $\rho\left(\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\right) = \rho\left(\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\right) = 34$. However over $\mathcal{A}((1,1,1),(1,1,1))$ we have $\rho\left(\begin{pmatrix}0&1&0\\1&-1&1\\0&1&0\end{pmatrix}\right) = 33$. As shown in Figure 5.2 this avoids the longest routes.

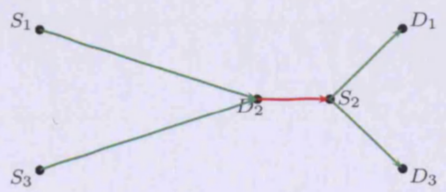



Figure 5.2: Solution of running example in $\mathcal{A}((1,1,1),(1,1,1))$

In the next section we give results analogous to those presented in Section 1.2.3.

## 5.2 Results

The first result we give is a counterpart to Theorem 1.2.34.

**Theorem 5.2.1.** *$\mathcal{A}(r,s)$ is non empty and* $\dim \mathcal{A}(r,s) = (m-1)(n-1)$.

Similarly to the edge matrix polytope given by Definition 2.3.3 we define the generalized *edge matrix pairs polytope $\mathcal{E}(r,s)$*:

$$\mathcal{E}(r,s) := \left\{ (h,v) \in \mathbb{R}^{m\times(n+1)} \times \mathbb{R}^{(m+1)\times n} \;\middle|\; \begin{array}{l} \bullet\ 0 \le h_{ij} \le r_i \text{ for all } i \in [m],\ j \in [0,n] \\ \bullet\ 0 \le v_{ij} \le s_j \text{ for all } i \in [0,m],\ j \in [n] \\ \bullet\ h_{i0} = v_{0j} = 0 \text{ for all } i \in [m], j \in [n] \\ \bullet\ h_{in} = r_i \text{ for all } i \in [m] \\ \bullet\ v_{mj} = s_j \text{ for all } j \in [n] \\ \bullet\ h_{i,j-1} + v_{ij} = v_{i-1,j} + h_{ij} \text{ for all } i \in [m],\ j \in [n] \end{array} \right\}$$

As before any element $(h,v) \in \mathcal{E}(r,s)$ can be represented on $\mathcal{L}_{m,n}$ as in Figure 5.3 where the condition given by Figure 1.8 still holds.

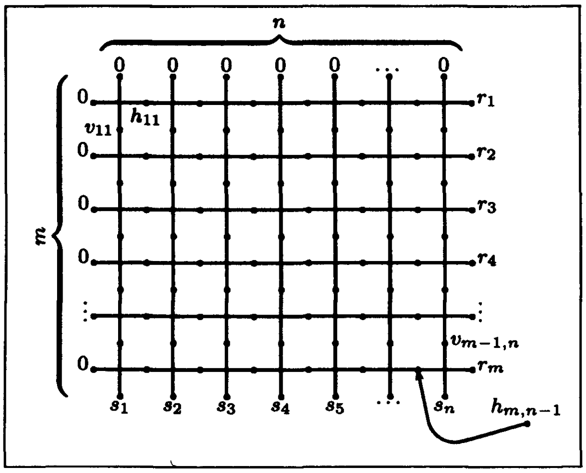



Figure 5.3: Elements of $\mathcal{E}(r,s)$ on $\mathcal{L}_{m,n}$

There is a bijection between $\mathcal{A}(r,s)$ and $\mathcal{E}(r,s)$ given by (5.4). Note that this is equivalent to bijections (2.4) and (2.5):

$$\begin{aligned} a_{ij} &= h_{ij} - h_{i,j-1} = v_{ij} - v_{i-1,j} \text{ for all } i \in [m],\ j \in [n] \\ h_{ij} &= \sum_{j'=1}^{j} a_{ij'} \text{ for all } i \in [m],\ j \in [0,n] \\ v_{ij} &= \sum_{i'=1}^{i} a_{i'j} \text{ for all } i \in [0,m],\ j \in [n] \end{aligned} \tag{5.4}$$

As for previous chapters, when referring to particular paths of $a \in \mathcal{A}(r,s)$ on $\mathcal{L}_{m,n}$, we are indeed referring to paths on the lattice diagram of the corresponding $(h,v) \in \mathcal{E}(r,s)$ as in Figure 5.3. We refer to entries $0 < h_{ij} < r_i$ or $0 < v_{ij} < s_j$ as non extremal. Thus, a non extremal path of $a \in \mathcal{A}(r,s)$ on $\mathcal{L}_{m,n}$ is a set of edges of the lattice diagram of the corresponding $(h,v)$ for which $0 < h_{ij} < r_i$ or $0 < v_{ij} < s_j$. The polytope $\mathcal{E}(r,s)$ will be used throughout the rest of this chapter, firstly to prove the following theorem (a generalization of Theorem 1.2.35):

**Theorem 5.2.2.**

$$vert\mathcal{A}(r,s) = \{a \in \mathcal{A}(r,s) \mid a \text{ on } \mathcal{L}_{m,n} \text{ has no non extremal cycles}\}$$

*Proof.* This result is actually a special case of Theorem 4.1.3 as $\mathcal{A}(r,s) = \Lambda(H,H',V,V')$ with:

$$H = \underbrace{\begin{pmatrix} 0 & 0 & \dots & 0 & r_1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 0 & r_m \end{pmatrix}}_{n+1} \Bigg\} m \quad H' = \underbrace{\begin{pmatrix} 0 & r_1 & \dots & r_1 & r_1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & r_m & \dots & r_m & r_m \end{pmatrix}}_{n+1} \Bigg\} m$$

$$V = \underbrace{\begin{pmatrix} 0 & \dots & 0 \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \\ s_1 & \dots & s_n \end{pmatrix}}_{n} \Bigg\} m+1 \quad V' = \underbrace{\begin{pmatrix} 0 & \dots & 0 \\ s_1 & \dots & s_n \\ \vdots & & \vdots \\ s_1 & \dots & s_n \\ s_1 & \dots & s_n \end{pmatrix}}_{n} \Bigg\} m+1$$

□

Note that Theorem 2.3.4 follows from Theorem 5.2.2. Also, we have the immediate corollary:

**Corollary 5.2.3.** *If* $r \in \mathbb{N}^m, s \in \mathbb{N}^n$ *then* $vert\mathcal{A}(r,s) \subseteq \mathcal{A}(r,s) \cap \mathbb{Z}^{m \times n}$

We define a generalization of the higher spin alternating sign matrices given by Definition 2.1.1.

**Definition 5.2.4.** *For $r \in \mathbb{N}^m$, $s \in \mathbb{N}^n$ such that $\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} s_j$, we define the set of alternating transportation matrices $ATM(r, s)$:*

$$ATM(r,s) := \left\{ a \in \mathbb{Z}^{m\times n} \;\middle|\; \begin{array}{l} \bullet\ \sum_{j=1}^{n} a_{ij} = r_i \text{ for all } i \in [m] \\ \bullet\ \sum_{i=1}^{m} a_{ij} = s_j \text{ for all } j \in [n] \\ \bullet\ 0 \leq \sum_{j'=1}^{j} a_{ij'} \leq r_i \text{ for all } i \in [m], j \in [n] \\ \bullet\ 0 \leq \sum_{i'=1}^{i} a_{i'j} \leq s_j \text{ for all } i \in [m], j \in [n] \end{array} \right\}$$

Note that $\mathrm{ATM}(r, s) = \mathcal{A}(r, s) \cap \mathbb{Z}^{m\times n}$.

As an example consider the set $\mathcal{A}((1,3,1),(2,1,2))$. Figure 5.4 shows the set $\mathrm{ATM}((1,3,1),(2,1,2))$ as well as their non extremal paths.

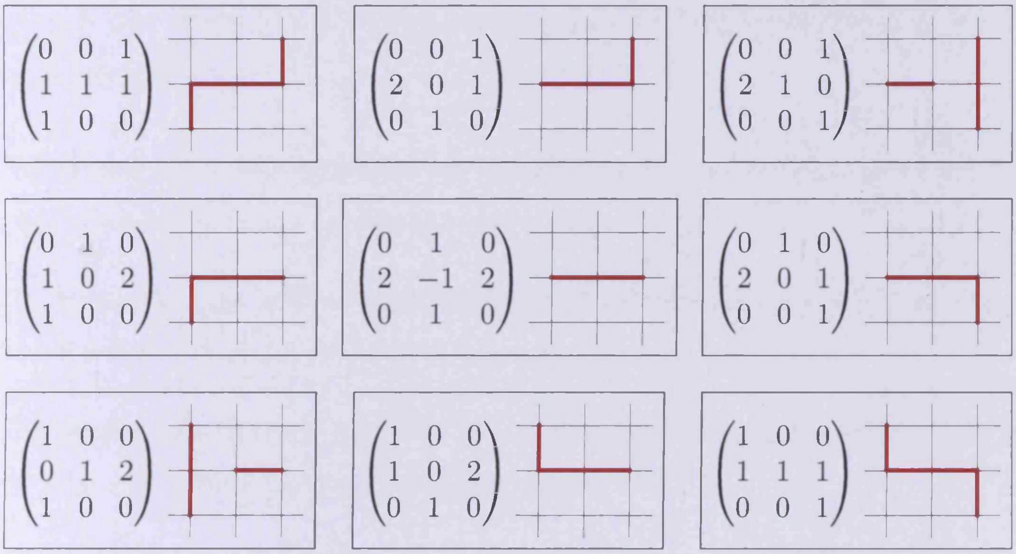



Figure 5.4: Non extremal edges of the integer elements of $\mathcal{A}((1,3,1),(2,1,2))$

First of all we note that none of these elements have a non extremal cycle and thus

$$\mathrm{vert}\mathcal{A}((1,3,1),(2,1,2)) = \mathrm{ATM}((1,3,1),(2,1,2))$$

If we now move onto faces, recalling Definition 1.2.5 we see that for $\mathcal{A}(r, s)$ the linear equalities that we need to consider correspond to the partial sum inequalities:

$$\begin{array}{l} 0 \leq \sum_{j'=1}^{j} a_{i,j'} \leq r_i \text{ for all } i \in [m], j \in [n-1] \\ 0 \leq \sum_{i'=1}^{i} a_{i'j} \leq s_i \text{ for all } i \in [m-1], j \in [n] \end{array}$$

Note that these are equivalent to:

$$\begin{array}{l} 0 \le h_{ij} \le r_i \text{ for all } i \in [m], j \in [n-1] \\ 0 \le v_{ij} \le s_i \text{ for all } i \in [m-1], j \in [n] \end{array}$$

Thus, setting one of these inequalities to be a defining equality of a face corresponds to setting one of the edges on a lattice diagram to be extremal. Consider $H \subseteq [m] \times [n-1] \times [2]$ and $V \subseteq [m-1] \times [n] \times [2]$. Then any face is of the form:

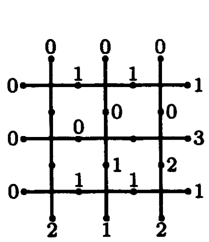
$$\mathcal{A}(r,s)_{(H,V)} := \left\{ a \in \mathcal{A}(r,s) \;\middle|\; \begin{array}{l} \bullet\ h_{ij} = \delta_{1k} r_i \text{ for all } (i,j,k) \in H \\ \bullet\ v_{ij} = \delta_{1k} s_j \text{ for all } (i,j,k) \in V \end{array} \right\} \tag{5.5}$$

where $(h, v)$ is the edge matrix pair corresponding to $a$. Note that this is similar to the notation given by (1.58) for $\mathcal{T}(r, s)$. Using this notation it is straightforward to define the following representation of faces of $\mathcal{A}(r, s)$.

**Definition 5.2.5.** *For a face $F$ of $\mathcal{A}(r, s)$, we define the lattice diagram of $F$ ld($F$) as the following labeling of $\mathcal{L}_{m,n}$:*

- *The left and upper boundary edges are labelled by zeros, and the right and lower boundary edges are labelled by the entries of $r$ and $s$ respectively.*
- *The horizontal edge between $(i, j)$ and $(i, j+1)$ is labelled $\delta_{1k} r_i$ if and only if $h_{ij} = \delta_{1k} r_i$ for all $a \in F$ (where $(h, v) \in \mathcal{E}(r, s)$ corresponds to $a$).*
- *The vertical edge between $(i, j)$ and $(i+1, j)$ is labelled $\delta_{1k} s_j$ if and only if $v_{ij} = \delta_{1k} s_j$ for all $a \in F$ (where $(h, v) \in \mathcal{E}(r, s)$ corresponds to $a$).*
- *All other edges are unlabelled.*

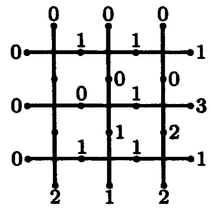
For example consider $F = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix} \right\}$. By Definition 5.2.5:

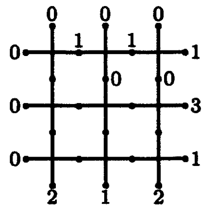
$\mathrm{ld}(F) =$ 


On the set of lattice diagrams of faces of $\mathcal{A}(r, s)$ the following operations can be defined:

**Definition 5.2.6.** *For two lattice diagrams of faces of $\mathcal{A}(r,s)$, $ld(F_1)$ and $ld(F_2)$ we define $ld(F_1) \subseteq ld(F_2)$ to mean that:*

- *If the horizontal edge between $(i,j)$ and $(i,j+1)$ of $ld(F_2)$ is labelled $\delta_{1k}r_i$ then the horizontal edge between $(i,j)$ and $(i,j+1)$ of $ld(F_1)$ is labelled $\delta_{1k}r_i$.*
- *If the vertical edge between $(i,j)$ and $(i,j+1)$ of $ld(F_2)$ is labelled $\delta_{1k}s_j$ then the vertical edge between $(i,j)$ and $(i,j+1)$ of $ld(F_1)$ is labelled $\delta_{1k}s_j$.*

For example consider $\text{ld}(F_1) =$ 

and $\text{ld}(F_2) =$ 

. Then we have $\text{ld}(F_1) \subseteq \text{ld}(F_2)$.

Using these definitions we give the following result:

**Theorem 5.2.7.** *The set of faces of $\mathcal{A}(r,s)$ ordered by inclusion is isomorphic to the set of lattice diagrams of faces ordered by inclusion.*

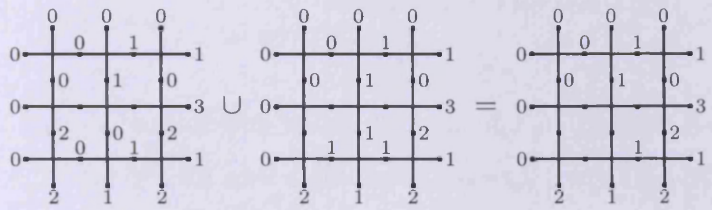
*Proof.* Let us consider two faces $F_1 = \mathcal{A}(r,s)_{(H_1,V_1)}$ and $F_2 = \mathcal{A}(r,s)_{(H_2,V_2)}$ of $\mathcal{A}(r,s)$ where $H_1, V_1, H_2$ and $V_2$ are the largest possible sets which give $F_1$ and $F_2$. Recalling (5.5), $F_1 \subseteq F_2$ if and only if $H_2 \subseteq H_1$ and $V_2 \subseteq V_1$. The result then follows from Definitions 5.2.5 and 5.2.6. □

**Definition 5.2.8.** *For two lattice diagrams of faces of $\mathcal{A}(r,s)$, $ld(F_1)$ and $ld(F_2)$ we define $ld(F_1) \cup ld(F_2)$ as the following labeling of $\mathcal{L}_{m,n}$:*

- *The horizontal edge between $(i,j)$ and $(i,j+1)$ of $ld(F_1) \cup ld(F_2)$ is labelled $\delta_{1k}r_i$ if and only if the horizontal edge between $(i,j)$ and $(i,j+1)$ of both $ld(F_1)$ and $ld(F_2)$ is labelled $\delta_{1k}r_i$.*
- *The vertical edge between $(i,j)$ and $(i+1,j)$ of $ld(F_1) \cup ld(F_2)$ is labelled $\delta_{1k}s_j$ if and only if the vertical edge between $(i,j)$ and $(i+1,j)$ of both $ld(F_1)$ and $ld(F_2)$ is labelled $\delta_{1k}s_j$.*

- *All other edges are unlabelled.*

For example we have:



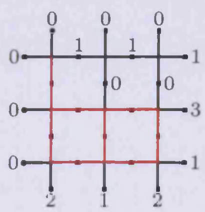
Definition 5.2.5 allows us to construct the lattice diagram ld$(F)$ from the halfspace definition of $F$, however using Definition 5.2.8 we give the following result (the proof is omitted):

**Theorem 5.2.9.** *Let $F$ be a face of $\mathcal{A}(r,s)$. Then*

$$ld(F) = \bigcup_{v \in vertF} ld(\{v\})$$

For $F = \mathcal{A}(r,s)_{(H,V)}$ a face of $\mathcal{A}(r,s)$ recalling Definition 5.2.5 we see that the entries of $h$ or $v$ on certain edges of ld$(F)$ will be determined by the boundary conditions, $H$, $V$ and the equation $h_{i,j-1} + v_{ij} = v_{i-1,j} + h_{ij}$ at each vertex. We define nec$(F)$ as the number of non extremal cycles of ld$(F)$ that do not enclose another cycle (an unlabelled edge is considered to be non extremal).

For example for $F = \mathcal{A}((1,3,1),(2,1,2))_{(\{(1,1,1)\},\emptyset)}$ we have nec$(F) = 2$, indeed:

ld$(F)$ = 

(as usual we color the non extremal edges in red). Using this we give the following result:

**Theorem 5.2.10.** *Let $F$ be a face of $\mathcal{A}(r,s)$. Then:*

$$\dim F = nec(F)$$

*Proof.* We recall that $\dim F$ is the number of values needed to uniquely determine an element of $F$. We number the nec$(F) = k$ non extremal cycles of ld$(F)$ that do not enclose another cycle and define the following sets:

$$H^l := \left\{ (i,j) \in [m] \times [n] \;\middle|\; \begin{array}{l} \text{the edge between } (i,j) \text{ and } (i,j+1) \\ \text{is in the } l^{th} \text{ cycle} \end{array} \right\} \text{ for all } 1 \leq l \leq k$$

$$V^l := \left\{ (i,j) \in [m] \times [n] \;\middle|\; \begin{array}{l} \text{the edge between } (i,j) \text{ and } (i+1,j) \\ \text{is in the } l^{th} \text{ cycle} \end{array} \right\} \text{ for all } 1 \leq l \leq k$$

Consider $(h, v) \in \mathcal{E}(r, s)$ corresponding to an element of $F$. Choosing a single value $h_{ij}$ (or $v_{ij}$) for some $(i, j) \in H^l$ (or $V^l$) for all $l \in [k]$ determines all the values of the $l^{th}$ cycle. Thus all the values of $(h, v)$ can be uniquely determined by setting $k$ values (corresponding to the $k$ non extremal cycles) giving: $\dim F \leq k$. However choosing any less than $k$ values leaves us with a cycle, and as can be seen by considering the proof of Theorem 5.2.2 this does not fully determine $(h, v)$. Thus $\dim F \geq k$ as required. □

Note that Theorem 5.2.2 is a special case of Theorem 5.2.10, since if $nec(F) = 0$ then $F$ is a vertex. Recalling Theorem 5.2.9, Theorem 5.2.10 immediately gives:

**Corollary 5.2.11.** *Taking $v_1, v_2 \in vert\mathcal{A}(r, s)$, the line segment between $v_1$ and $v_2$ is an edge if and only if:*

$$nec\left(ld(\{v_1\}) \cup ld(\{v_2\})\right) = 1$$

Figures 5.5,5.6,5.7,5.8 and 5.9 give the lattice diagrams of respectively the vertices, edges, 2 dimensional faces, facets and the whole polytope of $\mathcal{A}((1, 3, 1), (2, 1, 2))$. Note that in these diagrams, certain red edges are labelled if the corresponding values of $h$ or $v$ are fixed (but non extremal), however according to Definition 5.2.5 these edges would strictly speaking be unlabelled.

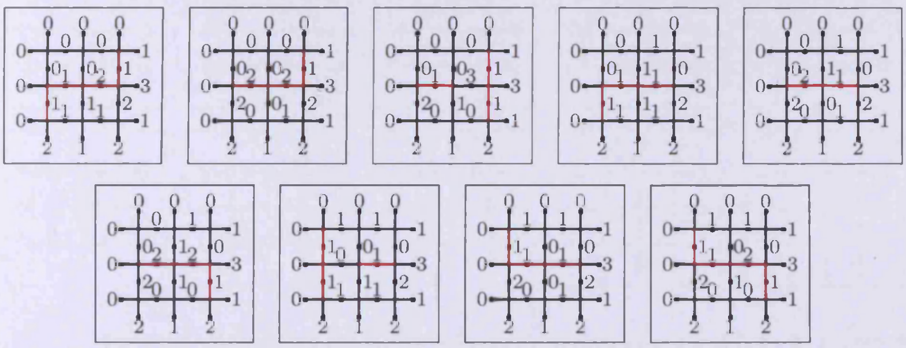


Figure 5.5: Lattice diagrams of the vertices of $\mathcal{A}\left((1, 3, 1), (2, 1, 2)\right)$

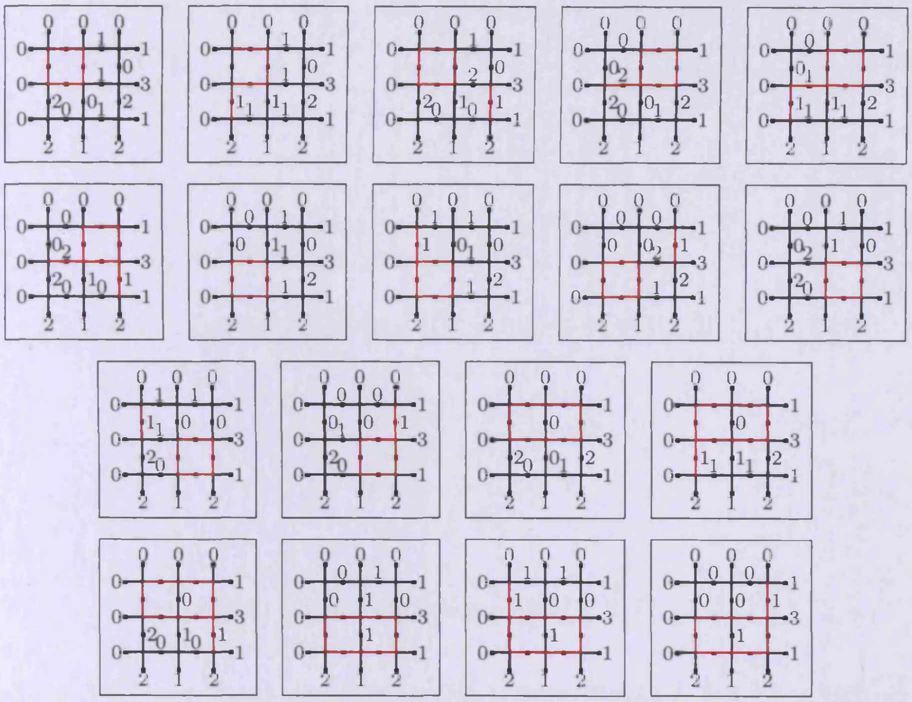


Figure 5.6: Lattice diagrams of the edges of $\mathcal{A}\left((1,3,1),(2,1,2)\right)$

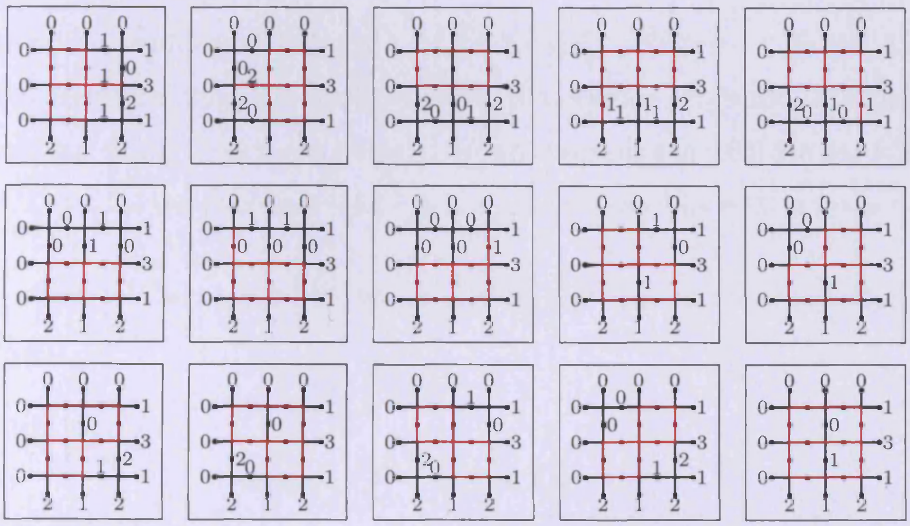


Figure 5.7: Lattice diagrams of the 2 dimensional faces of $\mathcal{A}\left((1,3,1),(2,1,2)\right)$

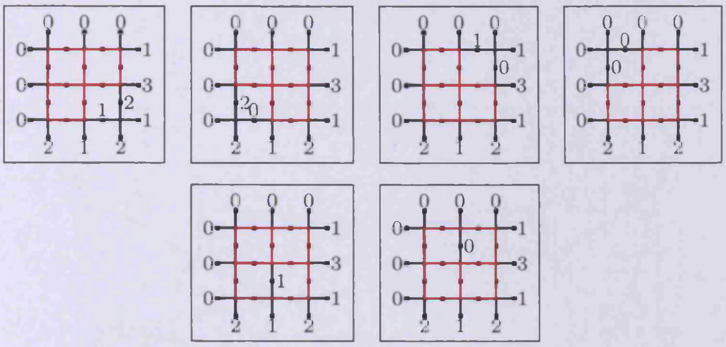


Figure 5.8: Lattice diagrams of the facets of $\mathcal{A}\left((1,3,1),(2,1,2)\right)$

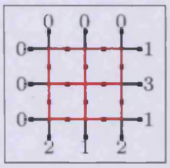



Figure 5.9: Lattice diagram of $\mathcal{A}((1,3,1),(2,1,2))$

Using Corollary 5.2.11 we are able to draw the graph of this polytope as shown in Figure 5.10. Note that the *graph* of a polytope is defined as follows: the vertices of the graph are the vertices of the polytope and two vertices are joined by an edge of the graph if and only if they are joined by an edge of the polytope. For the curious, Figure 5.11 gives the graph of the alternating sign matrix poltyope $\mathcal{A}_3$.

In [105] analogous results to Theorems 5.2.7,5.2.10 and Corollary 5.2.11 are given for $\mathcal{A}_n$. These results are given using flow girds as opposed to lattice diagrams. A simple bijection between these two sets of objects can be given and so the results of [105] are particular cases of Theorems 5.2.7,5.2.10 and Corollary 5.2.11 that can be obtained by setting $r = s = \underbrace{(1, \ldots, 1)}_{n}$.

As a counterpart to Theorem 1.2.39 we give using the convention of equation (1.22) the following result:

**Theorem 5.2.12.**

$$vert\mathcal{A}(r,r)^{\{1,d\}} = (vert\mathcal{A}(r,r))^{\{1,d\}}$$

This result follows in a similar way to the proof of Theorem 4.6.1.

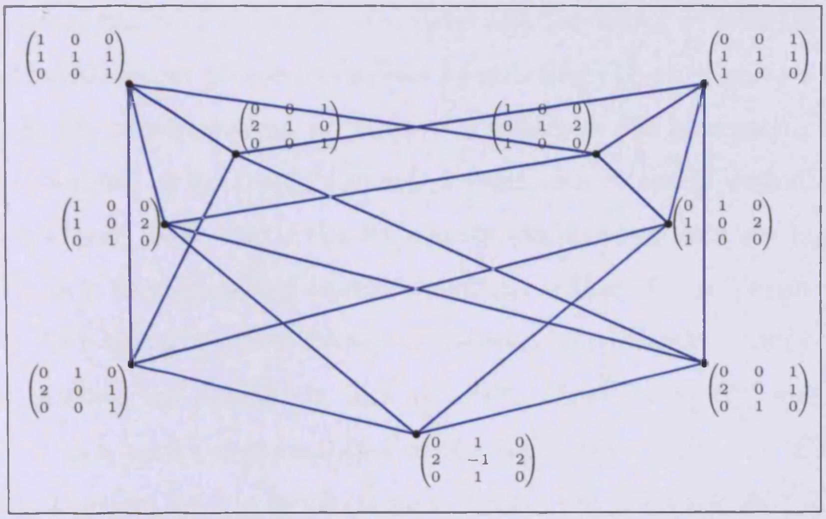



Figure 5.10: Graph of $\mathcal{A}\left((1,3,1),(2,1,2)\right)$

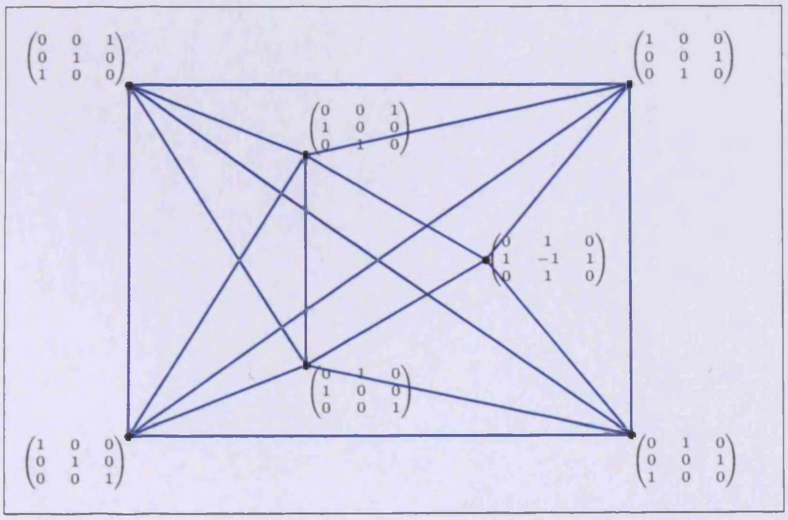



Figure 5.11: Graph of $\mathcal{A}_3$

## 5.3 Conclusion

The polytope $\mathcal{A}(r, s)$ has been shown to be a valid solution set of an ordered transportation polytope. The transshipment problem is solved by reducing the problem to a transportation problem, however the transportation polytope is a subset of the alternating transportation polytope. Thus solutions of the transshipment problem can be found within the alternating transportation polytope. Note that if the transportation problem does not have any ordered delivery restrictions it is still possible to find a solution within $\mathcal{A}(r, s)$ simply by considering permutations of the labeling of nodes.Transport between two different sources or two different destinations is however not allowed in $\mathcal{A}(r, s)$. Note that if $r \in \mathbb{Q}^m$ and $s \in \mathbb{Q}^n$ then $|\mathcal{A}(kr, ks) \cap \mathbb{Z}^{m \times n}|$ is a quasi-polynomial in $k$ of dimension $(m-1)(n-1)$. This follows from Theorem 1.2.18. Theorem 2.4.1 is a special case of this. The polytope $\mathcal{A}(r, s)$ is yet another special case of the polytope $\Lambda(H, H', V, V')$ considered in Chapter 4.

# Chapter 6

# Conclusion

## 6.1 Conclusions

This section is a summary of the work presented throughout this thesis. Every chapter has a conclusive section. Thus the summary given here will be brief.

Chapter 1 served as a review of the literature on alternating sign matrices as well as an overview of results concerning polytopes. In particular it was seen that alternating sign matrices can be considered as a generalization of permutation matrices.

In Chapter 2 we built on this, generalizing the Birkhoff polytope $\mathcal{B}_n$ to define the alternating sign matrix polytope $\mathcal{A}_n$. We introduced a new set of integer matrices and gave multiple bijections to other sets. We believe that these sets could be studied in their own right.

Chapters 3 and 4 are a study of the symmetry classes of $\mathcal{B}_n$ and $\mathcal{A}_n$. Using techniques based on the fundamental regions of these classes we were able to identify vertices of these polytopes and give results for the enumeration of symmetric semi magic squares or higher spin alternating sign matrices of fixed size and variable line sum. Once again the connection between $\mathcal{B}_n$ and $\mathcal{A}_n$ was apparent.

In Chapter 5, in a similar fashion to Chapter 2, we generalize the transportation polytope to define the alternating transportation polytope. This polytope could be the starting point for many research projects.

## 6.2 Further work

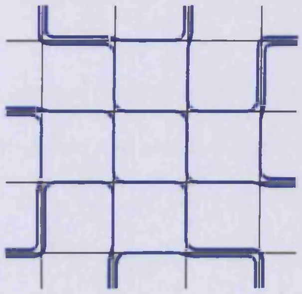
### 6.2.1 Enumeration of higher spin alternating sign matrices

Throughout this thesis we do not consider the enumeration of $\mathrm{ASM}(n,r)$ for fixed $r$ and variable $n$. Recalling (1.51),(1.52) and (1.53), enumerations of $\mathrm{SMS}(n,r)$ for $r=2$ or $r=3$ and variable $n$ are known [2, 20, 58, 104]. Obtaining similar enumerations for $\mathrm{ASM}(n,r)$ would however seemingly be a very challenging project.

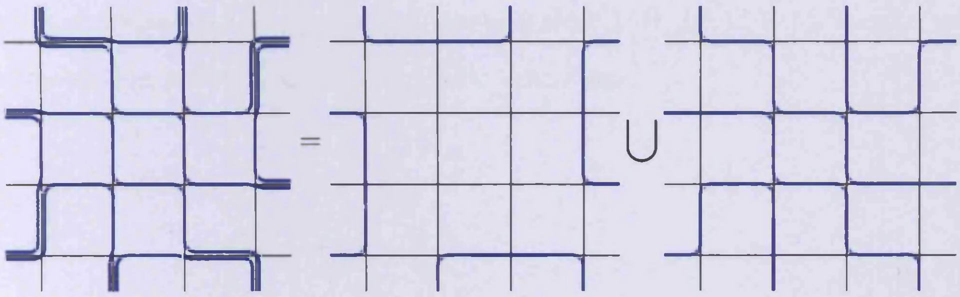
Alternative approaches to the enumeration of higher spin alternating sign matrices for fixed $n$ and variable $r$ could involve generating functions and constant term techniques. However, it would also be interesting to see whether the techniques of Section 2.5 can be generalized to give bijective derivations of enumeration formulae for $\mathrm{SMS}(n,r)$ and $\mathrm{ASM}(n,r)$ for fixed $n>3$ and variable $r$.

### 6.2.2 Generalization of the Razumov-Stroganov conjectures

In Section 2.5.2 we showed that for $n \in [3]$ and $r \in \mathbb{N}$ we have $|\mathrm{FPL}_{\pi}(n,r)| = |\mathrm{FPL}_{\pi'}(n,r)|$ where $\pi, \pi' \in L_{2n,r}$ are rotations of each other. Note that for $r=1$ this has been shown to be true for all $n \in \mathbb{P}$. Because of the decomposition of elements of $\mathrm{ASM}(n,r)$ (Theorem 2.2.6), it seems natural that this result could be generalized for all $r \in \mathbb{N}$. To generalize the bijective proof given by Wieland [109] of Theorem 1.1.11, a decomposition of elements of $\mathrm{FPL}(n,r)$ into elements of $\mathrm{FPL}(n,1)$ would be ideal. However, such a simple decomposition seems unlikely. Recalling the proof of Theorem 2.5.2 we showed that any element of $\mathrm{FPL}(3,r)$ could be decomposed using elements of $\mathrm{FPL}(3,1)$ and $\mathrm{FPL}(3,2) \setminus \mathrm{FPL}_{\mathrm{adm}}(3,2)$. Careful investigation shows that the decomposition of fully packed loop configurations for $n \geq 4$ is not as straightforward as shown by the following example:

 $\in \mathrm{FPL}_{\mathrm{adm}}(4,2)$ (6.1)

One possible decomposition of (6.1) is:

 (6.2)

however these two path configurations are not standard fully packed loop configurations.

It is reasonably straightforward to define a polytope $\mathcal{F}_n$ for which the elements of $\mathrm{FPL}(n,r)$ correspond to the integer points of $r\mathcal{F}_n$. Studying this polytope may assist with finding a method of decomposition for the elements of $\mathrm{FPL}(n,r)$.

Another project would be to generalize the Razumov-Stroganov conjectures. In [116], Zinn-Justin has defined operators on $L_{2n,r}$. Perhaps the coefficients of certain eigenvectors of these operators could be found to enumerate certain symmetry classes of $\mathrm{FPL}(n,r)$ with respect to a link pattern classification.

### 6.2.3 Considering the convex hull of symmetric vertices

In Chapters 3 and 4 we considered polytopes of the form $\mathcal{P}^G$ (using the notation of (1.22)), for $\mathcal{P}$ either $\mathcal{B}_n$ or $\mathcal{A}_n$ and $G$ a subgroup of $D_4$. Another interesting problem would be to consider the set $\mathrm{conv}\left((\mathrm{vert}\mathcal{P})^G\right)$, i.e. the convex hull of the symmetric vertices of $\mathcal{P}$. This polytope has already been studied by Cruse for the cases of $\mathcal{B}_n$ with $G=\{1,d\}$ [42] and $G=\{1,q^2\}$ [43]. Note that for some of the cases (half turn symmetry, diagonal symmetry, both diagonal symmetry) considered in Chapter 4 of $\mathcal{P}=\mathcal{A}_n$ this polytope is equal to $\mathcal{P}^G$.

### 6.2.4 Further polytopes

In Chapter 4 we defined the polytope $\Lambda(H, H', V, V')$. Studying this polytope in general would be an interesting problem, in particular when the following condition holds: $H_{i0} = H'_{i0}, V_{0j} = V'_{0j}, H_{in} = H'_{in}$ and $V_{mj} = V'_{mj}$ for all $i \in [m]$, $j \in [n]$. Theorem 4.1.2 gives the vertices of this polytope. The results concerning the faces of $\mathcal{A}(r, s)$ given in Chapter 5 should also be straightforward to generalize.

It is also worth noting that the the connection made in Chapter 5 between $\mathcal{A}(r, s)$ and the transportation problem can be generalized to show that $\Lambda(H, H', V, V')$ is a valid solution set of the transportation problem under particular conditions.

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