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Universal Properties of the Entanglement Entropy in Quantum Integrable Models

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Where is the Life we have lost in living?

Where is the wisdom we have lost in knowledge?

Where is the knowledge we have lost in information?

T. S. Eliot

Acknowledgements

Abstract

This thesis is a review of the works and ideas I have been developing in my doctoral studies, and it is mainly based on Castro-Alvaredo & Levi [2011]; Castro-Alvaredo *et al.* [2011]; Levi [2012]; Levi *et al.* [2013]. The specific aims of these works were to explore the methods developed in Calabrese & Cardy [2004]; Cardy *et al.* [2008] with the purpose of quantifying entanglement in a quantum field theory, and have a deeper understanding of their predicting power on lattice systems.

The first chapter is meant to be a review of quantum entanglement in many-body physics, and the methods we use to establish the link to QFT. In the second chapter, after a small introduction on conformal field theory, we collect the results of Calabrese & Cardy [2004], focusing in particular on the replica trick and the twist field.

The third chapter is devoted to adapting these tools to massive QFT, as performed in Cardy *et al.* [2008]. In particular we focus on the form factor program for the twist field, by means of which we are able to outline the behavior of entanglement entropy in massive theories in a non perturbative way. We expand on the results found in Castro-Alvaredo & Levi [2011], where higher particle form factors were studied for the roaming trajectory model, and the $SU(3)_2$-homogenous sine-Gordon model. We then carry out a numerical study of the Δ-function of the twist field for these two models.

In the fourth chapter we focus on the connection between the Δ-function of the twist field and Zamolodchikov c-function, as performed in Castro-Alvaredo *et al.* [2011]. In addressing this issue we perform a thorough study of the two point function of the twist field and the trace of the stress-energy tensor. This allows us to introduce a class

of composite twist fields, which were the main topic of Levi [2012].

In the fifth and last chapter we group the most common methods used to study the entanglement entropy of quantum spin chains. We start with the XY chain analysis, which is performed with a combination of analytical and numerical methods based on free fermion techniques. We then perform a numerical study of the XXZ chain by means of the density matrix renormalization group approach. Eventually we present the results obtained for these two models in Levi *et al.* [2013].

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Introduction

The first two decades of the 20th century were characterized by great intellectual ferment in the physics community. The works of J. C. Maxwell on the electromagnetic waves, summarized in his 1873 *A Treatise on Electricity and Magnetism*[1], paved the way to a technological boost which in turn gave access to a new set of experiments. Consequently new problems rose, which were not interpreted by any mathematical theory available to a scientist of that time. A first example was Maxwell's incapability of explaining the propagation of electromagnetic waves in the vacuum. The only kind of waves known then were pressure waves, which clearly needed a medium for propagation. This led to the supposition of the existence of a *luminiferous aether*, that was a fine substance which acted as a bearer of light.

It was in 1887 that an experiment led by A. Michelson and E. Morley demonstrated the nonexistence of such a substance. The problem of how electromagnetic waves could propagate in the vacuum remained then unsolved, and opened a quest that found its end with the formulation of a theory of relativity, and quantum mechanics. In particular it was A. Einstein who first introduced the concept of a light quantum (particle) in his most celebrated work on the photoelectric effect Einstein [1905].

This wave-particle duality helped solving many other difficulties that mathematical models had in describing experiments of the time. The most remarkable example was the ultraviolet catastrophe of the black-body radiation. Maxwell's

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[1]this work was originally divided into two books, and reedited Maxwell [1954a,b].

theory describes the energy of the electromagnetic field inside a cavity as an integral over the whole frequency spectrum. This leads to the divergence of the heat capacity, and then to infinite energy at any non-zero temperature. This divergence was clearly not observed in experiments. This problem was solved by M. Planck in 1900, when he assumed that the energy distribution were discrete in frequencies, finding a perfect agreement between his prediction and experimental results. The discreteness of the spectrum led again to the natural interpretation of the radiation as jets of particles, that were called *quanta*.

The scientific community started then a thorough investigation of this duality, which finally was embedded in a more comprehensive theory of quantum mechanics. Two parallel mathematical descriptions of quantum mechanics were born during those years, a *matrix mechanics* developed by W. Heisenberg, and a *wave mechanics* developed by E. Schrödinger. These two apparently distant formulations were proven equivalent, and unified later by P. M. Dirac. Such theories were based on few shared very solid principles:

1. the **wave-particle duality**. Not only radiation, but any kind of object can show particle or wave properties, depending on how it is tested.

2. The **uncertainty principle**. There are some conjugated couples of observables which cannot be measured on the same system. This can be rephrased as if the measurement of one observable would lead to errors on the second observable large enough to frustrate any prediction.

3. The **quantum essence of nature**. Performed at small scales, measurements' outcomes of any physical quantity can be counted in multiples of a *quantum*.

4. The **superposition principle**. Until a physical quantity is measured, it does not have a defined value, it is instead in a superposition of all possible outcomes of a supposed forthcoming measurement.

The last point opened a heated debate between two of the fathers of quantum mechanics, namely A. Einstein and N. Bohr. From now on, we will use the formalism of bras and kets and, unless stated differently, our dynamics will be

in the interaction picture. In this language a physical system is described by a vector $|v\rangle$ which lives in a Hilbert space $\mathscr{H}$. The superposition principle is naturally interpreted in this formalism, as the possibility of representing a state by a linear combination of vectors

$$|v\rangle = \sum_j \alpha_j |a_j\rangle, \quad \text{with} \quad \alpha_j \in \mathbb{C}, |a_j\rangle \in \mathscr{H}, \forall j. \tag{1.1}$$

The coefficients α_j are bounded by $\sum_j |\alpha_j|^2 = 1$, and they are linked to the probability that a measurement on any physical quantity contained in $|v\rangle$ gives the value contained in $|a_j\rangle$. Physical quantities are called observables, and they are represented by self-adjoint operators $\mathbf{O} = \mathbf{O}^\dagger$. The measurement process is responsible for the collapse of the state in one of the possible eigenstates of the measured observable. Calling O_k the eigenvalues of $\mathbf{O}$, and $|o_k\rangle$ its eigenvectors we define the expectation value of $\mathbf{O}$ in the physical setting described by the vector $|v\rangle$ as

$$\langle \mathbf{O} \rangle = \sum_k |\langle o_k | v \rangle|^2 O_k. \tag{1.2}$$

The interpretation of (1.2) is that a measure of $\mathbf{O}$ gives O_k as a result with probability $|\langle o_k | v \rangle|^2$. This forces the state $|v\rangle$ to "align" with the vector $|o_k\rangle$, constraining the possible outcomes of following measurements.

The probabilistic interpretation of a quantum state was unpalatable to A. Einstein, who, with B. Podolsky and N. Rosen, collected his doubts in the renowned paper Einstein *et al.* [1935]. In their opinion any theory which aims to describe the physical reality, must be complete[1]. The first objection to quantum mechanics is that if two variables do not commute (e.g. position and momentum of a wave packet) they cannot be measured at the same time, indeed a measurement on one variable which exactly determines its value would constrain the second one to have a flat statistics. Then the second quantity has no physical reality. This means either that the two quantities cannot have both physical reality, or that the theory is not complete. Their conclusion was that if both the quantities have an experimental counterpart the theory must not be complete.

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[1]the exact definition of completeness is somehow a bit elusive. In that work the gist of their interpretation was that any element of the physical reality must have a counterpart in the theory.

There is a second example in that paper, which goes a bit deeper on the matter, and opened a long history of debate. It is worth reporting it here in the fashion of the original paper. Suppose we have two quantum system, namely A and B, which interact together for a finite time $t \in [0, T]$. At time T we divide them, and we give them to two observers, Alice and Bob, who are spatially separated. The quantum state of the two systems is determined for $t < 0$, while later on it becomes the state of the composite system $A \cup B$, which has to be determined. The authors used the Schrödinger wave function formalism to describe the problem, so that the whole system is described by a wave function $\Psi(x)$. Now suppose that Alice performs a measurement of some observable $\mathbf{U}$. We call $u_1, u_2, \dots$ the eigenvalues of such operator, and $u_1(x_A), u_2(x_A), \dots$ the corresponding eigenvectors, where x_A is the set of variables associated to the system A. Then the wave function can be rewritten as

$$\Psi(x_A, x_B) = \sum_{n=1}^{\infty} \psi_n(x_B) u_n(x_A), \qquad (1.3)$$

and the measurement performed by Alice has the effect of "selecting" one of the $u_k(x_A)$ by the outcome of u_A, and then projecting Bob's system onto the state $\psi_k(x_B)$. Now suppose Alice chooses to measure another quantity $\mathbf{V}$, then we can write

$$\Psi(x_A, x_B) = \sum_{n=1}^{\infty} \phi_n(x_B) v_n(x_A). \qquad (1.4)$$

A measurement of $\mathbf{V}$ will project A onto a particular eigenstate $v_l(x_A)$, and B onto $\phi_l(x_B)$. The conclusion is that two different measures on the system A would leave the state B described by two different wave functions. On the other hand, since there is no more physical interaction between A and B, an action on one of the two sub-systems should not influence the state of the other. In their work the authors were using this feature to demonstrate the non-completeness of quantum mechanics, but this goes beyond the scope of this introduction, and we will take for granted their point of view.

These problems were left open for almost thirty years, until the work of Bell [1964]. He assumed Einstein *et al.* [1935] conclusion as his working hypothesis, and put their work in the more solid formalism of local hidden variables. He

managed to describe how the existence of local hidden variables influences the statistical correlations of a model by a set of inequalities, and finally showed that quantum mechanics violates those inequalities. By doing so he demonstrated that quantum mechanics is not a local hidden variables theory, and that its correlations cannot be reproduced by classical correlations.

We need to wait until the works of Freedman & Clauser [1972] and Aspect *et al.* [1981, 1982] to have the first convincing experimental realizations of the violation of Bell's inequalities. In the meantime the scientific community started identifying this long range effect typical of a quantum system under the name of **quantum entanglement**. In the following section we aim to formally define this phenomenon.

1.1 Quantum entanglement

Entanglement is probably the deepest (certainly the most obscure) manifestation of the quantum approach to reality. It finds its origin in the superposition principle when one is dealing with a composite system. The classical description of the dynamics of a many-body problem is performed in the phase space, whose dimension grows linearly with the number of components[1]. In a quantum mechanical description the corresponding concept is the Hilbert space $\mathcal{H}$, which is a tensor product of its components $\mathcal{H} = \bigotimes_{i=1}^{N} \mathcal{H}_i$, such that its dimension grows exponentially with the number thereof.

We will take the Hilbert spaces of each component to have the same dimension d, then a global state is represented as the vector

$$|\Psi\rangle = \sum_{\{\iota\}}^{d^N} a_\iota |\psi_\iota\rangle, \tag{1.5}$$

where ι is a N-tuple, and it is summed over all its possible configurations, that are d^N. The vector part of each term can be decomposed on the complete basis of

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[1]here we have in mind an Hamiltonian system, for which each component is represented by its position q, and its velocity p; in this way the phase space's dimension D grows linearly with the number of components N, as $D = 2N$.

the subsystems $\left|u^k\right\rangle$ in the following way

$$\left|\psi_\iota\right\rangle = \left|u^1_{i_1}\right\rangle \otimes \left|u^2_{i_2}\right\rangle \otimes ... \otimes \left|u^N_{i_N}\right\rangle, \quad \text{with} \qquad 1 \le i_k \le d, \forall i_k \in \iota. \qquad (1.6)$$

In order to ease the notation we will use the two components case in our forthcoming definitions, and will refer to it as a bipartite system. Unless stated differently all the considerations we make can be extended trivially to the generic N case. In the case of two subsystems eq. (1.5) can be written as

$$\left|\Psi\right\rangle = \sum_{i,j=1}^{d} a_{ij} \left|u^1_i\right\rangle \otimes \left|u^2_j\right\rangle. \qquad (1.7)$$

Here two cases must be distinguished:

1. the coefficients a_{ij} can be factorized into products, e.g. $a_{ij} = \alpha_i \alpha_j$. The state can be rewritten then as

$$\left|\Psi\right\rangle = \left(\sum_{i=1}^{d} \alpha_i \left|u^1_i\right\rangle\right) \otimes \left(\sum_{j=1}^{d^N} \alpha_j \left|u^2_j\right\rangle\right), \qquad (1.8)$$

   and it is called **separable**, or **factorizable**.

2. The coefficients a_{ij} cannot be factorized, and the state is called **entangled**.

The case in which the two subsystems' Hilbert spaces have different dimensions needs extra care. This case can be treated using the Schmidt decomposition Schmidt [1907]. Suppose the two systems have dimensions m and n with $m < n$. Then considering the state (1.7) the coefficients a_{ij} can be written as the entries of an $m \times n$ rectangular matrix $\mathbf{A}$. This in turn can be written as $\mathbf{A} = \mathbf{U}\lambda\mathbf{V}^\dagger$, where $\mathbf{U}$ is an $m \times m$, and $\mathbf{V}$ an $n \times n$ unitary matrix, while λ is an $m \times n$ matrix with null off-diagonal entries, and a number r of non-zero real diagonal elements $\lambda_1 \ge \lambda_2 \ge ... \ge \lambda_r$. The number of non-zero eigenvalues r is usually referred to as *rank*, and it is a property shared by the two subsystems.

This decomposition is of central importance in understanding the possible states an original pure state can be trasformed into. In fact, following Nielsen [1999], we consider two states which have been Schmidt reduced to states with the same

rank[1]

$$|\psi\rangle = \sum_{i=1}^{r} \lambda_i \left|u_i^1\right\rangle \left|u_i^2\right\rangle$$

$$|\phi\rangle = \sum_{j=1}^{r} \eta_j \left|v_j^1\right\rangle \left|v_j^2\right\rangle. \tag{1.9}$$

It can be demonstrated that $|\psi\rangle$ can be transformed into $|\phi\rangle$ by local operations and classical communication (LOCC) iff

$$\sum_{i=1}^{k} \lambda_i \le \sum_{j=1}^{k} \eta_j, \quad \forall k \in [1, r]. \tag{1.10}$$

This is called *majorization rule*.

1.2   The density matrix

States of the kind (1.6) are usually called *pure states*. There exist a second class of states which are referred to as *mixed states*, and they are described by a statistical ensemble of pure states. The statistical distribution can be for example canonical, and induced by considering a thermal state. The standard way to tackle this situations is by introducing the *density matrix*

$$\rho = \sum_{i=1} p_i \left|\psi_i\right\rangle \left\langle\psi_i\right|, \tag{1.11}$$

where p_i are the probabilities associated to the states $|\psi_i\rangle$ with the aforementioned distribution[2]. This matrix is positive definite, and of trace 1, and the expectation value of any observable $\mathbf{O}$ can be defined as $\langle\mathbf{O}\rangle = \mathrm{Tr}\left[\rho\mathbf{O}\right]$.

In parallel with the definitions in section 1.1, Werner [1989] defined as separable

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[1]from now on we will drop the tensor product symbol between Hilbert spaces for convenience, such that $|a\rangle \otimes |b\rangle$ will be abbreviated $|a\rangle|b\rangle$.

[2]in the example of a quantum state in equilibrium with a thermal bath $p_i = e^{-\frac{E_i}{kT}}$, where we are considering the case where $|\psi_i\rangle$ are eigenstates of the Hamiltonian, with eigenvalues E_i, while k is the Boltzmann constant.

those mixed states that can be written as

$$\rho = \sum_{i=1} p_i \rho_1^i \otimes \rho_2^i, \qquad (1.12)$$

and called entangled those states which cannot.

A very interesting and well understood setting in which entanglement can be studied is that of a bipartite pure system. Suppose we have a global system described by the pure state $|\psi\rangle$, which we bipartite into two subsystems A and B (a setting which is totally analogue to the one of Einstein *et al.* [1935]). The density matrix associated to this global state contains clearly just the state $|\psi\rangle$, with probability $p_\psi = 1$, that is $\rho_{AB} = |\psi\rangle\langle\psi|$. The question we want to answer is: *Given the knowledge of ρ_{AB} can we make predictions for the outcomes of a measurement on A or B?*

It is generally impossible to associate a pure state to one of the two subsystems, but if we want to study the properties of say A, we can trace out the system B. In detail this means first choosing a complete basis of B, which we call $|b_i\rangle$, and then acting on the global density matrix with the projectors on this basis in the following way

$$\rho_A = \sum_i \langle b_i | \rho_{AB} | b_i \rangle. \qquad (1.13)$$

The reader may be confused by the notation adopted in eq. (1.13). In the rhs of eq. (1.13) we are performing a partial trace on the subsystem B of the whole system $A \cup B$ on which ρ_{AB} has support. As a result the outcome of such a trace is not a scalar, but rather a matrix, which we call *reduced density matrix* of the system A. If there is entanglement between the two subsystems this will be a mixed state's density matrix. This is equivalent to admitting total ignorance about the system B, hence associating equal probability to the outcomes of any measurement of an observable with support on B. This process is usually referred to as "tracing out" the region B, such that an equivalent notation for eq. (1.13) is $\rho_A = \text{Tr}_B \rho_{AB}$.

This bipartite setting is the one that will be studied throughout this manuscript. We want to introduce here a paradigmatic setting that captures well the most important features of entanglement. The case in which the subsystems A and B

are two electrons, and their spin wave function is in the singlet state

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|\uparrow\rangle_A |\downarrow\rangle_B - |\downarrow\rangle_A |\uparrow\rangle_B),\qquad(1.14)$$

where $|\uparrow\rangle$ and $|\downarrow\rangle$ are the two spin eigenstates along the z-axis[1]. It is easy to check that the two electrons are entangled in this case, as with simple calculations one manages to express $\rho_A = \frac{1}{2}(|\uparrow\rangle_A \langle\uparrow|_A + |\downarrow\rangle_A \langle\downarrow|_A)$.

Now that we have a way to distinguish two entangled bipartitions from non-entangled ones we aim to define a way of quantifying entanglement.

1.3   Entanglement measures

In order to define an entanglement measure it is helpful to think about the concept of reduced density matrix (1.13). That matrix is used to define the expectation values of any operators acting on the subsystem A. The more these expectation values are correlated with those on B, the more the two subsystems are entangled, the more ρ_A will be mixed[2]. As a consequence the more this matrix is mixed the more we have access to information about B, and possibilities to influence the outcomes of measures of its observables[3].

Another very important aspect of entanglement is its relationship with information. Referring to the usual example of the bipartite system if the bipartitions are entangled, somehow their density matrices capture more information about the state than ρ_{AB}. This feature, even if already noticed by Schrödinger in the Thirties, was put on a solid basis and quantified in Schumacher [1995]. In this latter work a quantity called *entanglement entropy* was defined, which extends the concept of the Shannon entropy of a statistical distribution to the quantum

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[1]this example actually works with any binary systems.

[2]there is no measure of how mixed a density matrix is, more mixed means that its eigenvalues are closer to being all equal.

[3]this is the key point of many interesting applications, such as the quantum teleportation of Bennett *et al.* [1993] and Ekert [1991]'s quantum cryptography.

case, and is defined as[1]

$$S(\rho) = -\mathrm{Tr}\rho\log\rho. \tag{1.15}$$

The entanglement entropy as defined here is usually called *von Neumann entanglement entropy*, and has the following interesting properties:

- It is null for separable states and maximal in the case in which the density matrix ρ has all nonzero equal eigenvalues. In that case, due to the normalization of ρ, if the Hilbert space is d-dimensional, we will have $S(\rho) = \log d$. States of the kind defined in eq. (1.14) are the easiest examples, and they are called *maximally entangled* states.

- If ρ_{AB} is the density matrix of a pure state, $S(\rho_A) = S(\rho_B)$. This can be shown trivially by Schmidt decomposing the state, and has remarkable implications explained in section 1.7.

- It is invariant under unitary transformations of the density matrix $U^\dagger\rho U$, and then independent of the basis in which it is expressed.

- It is concave, in the sense that for a mixed state $\sum_i p_i\rho_i$ the property $\sum_i p_i S(\rho_i) \le S(\sum_i p_i\rho_i)$ holds.

- It has the remarkable property $S(\rho_{AB}) \le S(\rho_A) + S(\rho_B)$. This property does not have a counterpart in classical information theory, where the entropy of a system can never be lower than that of its components, and is called *subadditivity*. A more general version of this property can be written if one considers the possibility of an intersection between the two subsystems A and B, where the subadditivity states that $S(\rho_{A\cup B}) + S(\rho_{A\cap B}) \le S(\rho_A) + S(\rho_B)$.

The entanglement entropy (1.15) plays a central role in determining how many singlet states can be distilled from a general mixed state, outlining an operative definition of it [2]. Before walking this way though we have to consider

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[1]in this definition as in any later ones, in contrast to the quantum information notation, we are expressing the entropy in a natural basis; this in turn means that we would not be allowed to "count" the quantum information in qubits, but we should use qunats instead; this measure though is not common in literature, and with an abuse of notation we will talk about qubits referring to qunats.

[2]this concept was introduced by Bennett *et al.* [1996].

if a general state is transformable at all, and to which states. The answer comes from the majorization rule (1.10), which tells us which states are compatible, and which are not. A problem in this approach is the presence of discontinuities in the probability of transformation from one state to the other. To make this statement clearer we borrow an example from Plenio & Virmani [2007]. Let us consider the composition of two ternary states, such that the Hilbert spaces of each component are spanned by the basis $\{|0\rangle, |1\rangle, |2\rangle\}$. It is easy to check with (1.10) that the state $(|00\rangle + |11\rangle)/\sqrt{2}$ can be transformed into $0.8|00\rangle + 0.6|11\rangle$. That means there exists a local transformation which links the two states with probability equal unity. If we introduce a deformation ε leading to $(0.8|00\rangle + 0.6|11\rangle + \varepsilon|22\rangle)/\sqrt{1 + \varepsilon^2}$, we can check that this state cannot be reached for any nonzero value of ε. This can be easily understood by considering that LOCC cannot increase the rank of a state. This feature is somehow unwanted, as the expectation values of any physical quantity will depend on ε, and we do not expect discontinuities varying such a parameter.

This problem can be tackled by considering a less ideal setting. Instead of asking if two states are compatible we can ask if a set of n identical states can be transformed into a set of m states, "close enough" to a target state, that is a deformation ε of a target state which tends asymptotically to it for large n and m. Formally calling ρ the density matrix of an initial state, and σ that of a target state, we want to see if it is possible to transform n copies of ρ, $\bigotimes_{i=1}^{n} \rho_i \to \sigma_m(\varepsilon)$, where $\sigma_m(\varepsilon) \to \bigotimes_{i=1}^{m} \sigma_i$ for $m \to \infty$. An important question is then *which is the biggest rate $r = m/n$ at which we can perform this transformation?*

In particular we would like to define the *entanglement cost $E_C(\rho)$* as the rate at which we can convert a set of n maximally entangled binary states as (1.14) (whose density matrix we call ρ) into a set of m target states σ. So that calling Λ the LOCC that maps $\bigotimes_{i=1}^{n} \rho_i \to \sigma_n(\varepsilon)$

$$E_C(\sigma) = \mathrm{MIN}\left[r \,\middle|\, \lim_{n \to \infty} D\left(\Lambda\left(\bigotimes_{i=1}^{n} \rho_i\right), \bigotimes_{i=1}^{rn} \sigma_i\right) = 0\right] \qquad (1.16)$$

where $D(\rho, \sigma) = \sqrt{2(1 - F(\rho, \sigma))}$ is the *Bures distance*, $F(\rho, \sigma) = \mathrm{Tr}\sqrt{\left[\sqrt{\rho}\sigma\sqrt{\rho}\right]}$ is the *fidelity* between the two density operators. This quantity then tells us what

is the minimum number of maximally entangled states that we have to employ to "build" a set of n target states. We can invert the definition and ask ourselves how many maximally entangled states ρ we can get out of n identically prepared noisy singlets σ. This leads to the definition of the *entanglement of distillation*. This time we define the LOCC which performs the map $\Lambda(\otimes_{i=1}^{n} \sigma_i) \rightarrow \rho_n(\varepsilon)$

$$E_D(\sigma) = \mathrm{MAX} \left[r \, \middle| \quad \lim_{n \to \infty} D \left(\Lambda \left(\bigotimes_{i=1}^{n} \sigma_i \right), \bigotimes_{i=1}^{rn} \rho_i \right) = 0 \right]. \tag{1.17}$$

The entanglement cost, and entanglement of distillation are of central relevance when dealing with experimental realizations of quantum protocols. In fact in the real world preparing a binary system in a maximally entangled state is generally a difficult task. Moreover, as those protocols are based on the transmission of a quantum state, one has to deal with decoherence due to noisy channels. Quantum information is mediated by qubits, extracted from maximally entangled states, hence understanding how to optimize the conversion of a noisy state into a maximally entangled one is very important.

It was demonstrated by Bennett *et al.* [1996] that for pure states $E_C(\sigma) = E_D(\sigma)$, so that the process is asymptotically reversible. Most remarkably they are both equal to the Von Neumann entropy, which in this case is the only relevant measure of entanglement.

Other measures of entanglement are the *Rényi entropies*, introduced in their classical version by Rényi [1961]. These quantities are dependent on a real parameter $\alpha \geq 1$, and are defined as

$$S_\alpha(\rho) = \frac{1}{1-\alpha} \log \mathrm{Tr} \rho^\alpha. \tag{1.18}$$

They clearly have the Von Neumann entropy as the limit for $\alpha \to 1$. To understand their utility we have to diagonalize the density matrix ρ, such that

$$S_\alpha(\rho) = \frac{1}{1-\alpha} \log \left(\sum_{i=1}^{d} \lambda_i^\alpha \right), \tag{1.19}$$

where d is again the dimension of the Hilbert space. As $\sum_{i=1}^{d} \lambda_i = 1$, the sum in eq. (1.17) converges for any value of $\alpha \in [1, \infty)$, and $S_\alpha(\rho)$ is well defined. Moreover considering large values of α suppresses the lowest eigenvalues of ρ (lowest levels in the entanglement spectrum [1]), while small values of α take them into account. From these considerations follows that the knowledge of the Rényi entropies for general α gives access to information about the whole entanglement spectrum. In particular now we have a way to understand the majorization rule in eq. (1.10). It is telling us that LOCC cannot increase the entanglement between the two components, in the sense that $S_\alpha(\rho_\phi) \leq S_\alpha(\rho_\psi)$, for any α.

Another interesting question one can ask is: *Considering a single copy of a generic state, which is the dimension of a maximally entangled state distilled with certainty?*

The answer is given by the *single-copy entanglement* as defined by Eisert & Cramer [2005]. Suppose that the a density matrix ρ can be transformed by LOCC into $|\psi_D\rangle\langle\psi_D|$, where $|\psi_D\rangle = (|\alpha_1\rangle|\beta_1\rangle + |\alpha_2\rangle|\beta_2\rangle + ... + |\alpha_D\rangle|\beta_D\rangle)/\sqrt{D}$, then we define its single-copy entropy as $E_1(\rho) = \log D$ [2].

For this transformation to be possible condition (1.10) must be satisfied, then calling again λ_i the eigenvalues of ρ, and ordering them as decreasing with i, we have

$$\sum_{i=1}^{k} \lambda_i \leq \frac{k}{D}, \qquad \forall k \in [1, D]. \tag{1.20}$$

This naturally implies $\lambda_1 \leq 1/D$. We can then express the single-copy entanglement in terms of the original density matrix eigenvalues as

$$E_1(\rho) = \log\left(\lfloor \lambda_1^{-1} \rfloor\right), \tag{1.21}$$

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[1] we use the notion of entanglement spectrum as introduced by Li & Haldane [2008], and Calabrese & Lefevre [2008].

[2] the reader may be surprised by the fact that we are talking here about a D-dimensional maximally entangled state, in contrast to the previously defined measures, for which we considered a large number of binary maximally entangled states. It can be demonstrated though that a local unitary transformation exists which maps one into the other, so that the two approaches are equivalent.

where with $\lfloor...\rfloor$ we are taking the integer part. In particular, exploiting the fact that $\log \mathrm{Tr}\rho^\alpha \sim \alpha \log \lambda_1$ for $\alpha \to \infty$, we can link the single copy entanglement to Rényi entropies

$$E_1(\rho) = \lim_{\alpha \to \infty} S_\alpha(\rho). \tag{1.22}$$

1.4 Quantum entanglement and many-body systems

One of the most intriguing challenges in modern physics is the description of quantum many-body systems. In particular in recent years a great deal of attention has been given to the connections with quantum information theory. Indeed many open problems in many-body physics have been successfully treated with quantum information techniques. On the other hand many new protocols in quantum computing have been inspired by many-body problems, as a quantum processor is itself a many-body system.

Quantum mechanics is very effective in studying single-body problems, or systems composed of few constituents. As we increase the number of components though, systems become rapidly intractable due to the exponential growth of the Hilbert space explained in section 1.1.

A real system is seldom composed by few components, hence quantum mechanics would seem powerless in predicting any outcomes of an experiment. Fortunately in a great deal of cases interactions between components are fairly small compared to the energy scale of each constituent. Therefore the problem can be treated perturbatively using single-body techniques. Many cases though cannot be treated as such as they are characterized by strong interactions. Among them there are some very relevant problems in modern physics, such as bosonic condensates, strongly correlated electrons, studied in the quest for high-temperature superconductivity, and last but not least the physical realization of a quantum computer.

The question arises then, how can we describe strongly interacting systems?

In this context the possibility of solving numerically the equations of motion becomes crucial. Unfortunately as we are dealing with a quantum system, the

Hilbert space quickly grows too large to be implemented even on the most recent machines. Then it becomes very important to understand when a quantum system can be simulated efficiently on a classical computer[1]. The answer was given indirectly by White [1992], and it was cast in a quantum information perspective. The system's components must be weakly entangled more than weakly interacting, in order to succeed with a numerical simulation. This observation gave rise to a set of very powerful numerical techniques which go under the name of *density matrix renormalization group* (DMRG).

There are many other examples of cases in which the study of entanglement properties of a many-body system leads to new insights. One of the most remarkable is the existence of *quantum phase transitions* (QPT). The remaining part of this section is devoted to explaining qualitatively the main features of QPT. For a more detailed analysis we redirect readers to Sachdev [2007]. We focus on the zero temperature case, so that any phase transition will be driven only by quantum effects, and the system will be in a pure state rather than a thermal ensemble. We take the system to be in its ground state. To help the understanding we focus on a quantum magnet as an example. The degrees of freedom are concentrated in the magnetic cells, which, with a drastic simplification, can be represented by sites with a spin degree of freedom. The interaction among them can be represented by a link on a lattice. Here and throughout this manuscript we will consider only the case of nearest neighbours interactions. The cells can be coupled to an external magnetic field by their spin magnetic moment μ_h. Summarizing all these considerations we can write the general Hamiltonian of this system as

$$\mathcal{H}(J,h) = J \sum_{\langle i,j \rangle} \vec{S}_i \vec{S}_j - g \mu_h h \sum_i \vec{S}_i \vec{u}_i, \tag{1.23}$$

where $\vec{S} = \vec{\sigma}/2$, J sets the microscopic energy scale of the interaction between sites, g is the g-factor ($\simeq 2$ for electrons), h is the intensity of the magnetic field, and $\vec{u}$ is a versor pointing in the direction of the external field. Notice that although eq. (1.23) is independent of the lattice spacing a and the lattice shape, the physics will depend to a great extent on these quantities.

The Hamiltonian (1.23) is a representative of a family of systems which can be

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[1]this because despite our efforts we still miss a quantum computer.

written as $\mathcal{H}(h) = \mathcal{H}_0 + h\mathcal{H}_1$, h being a dimensionless parameter. For any fixed value of h we can diagonalize the Hamiltonian, and find the energy levels $E_i(h)$, and the respective eigenstates $|\psi_i(h)\rangle$. We focus on the case in which $\mathcal{H}_1$ is a conserved quantity, that is, it commutes with $\mathcal{H}_0$[1]. This means that the eigenvectors do not depend on h, as for any value of this parameter they are the eigenstates of $\mathcal{H}_0$. The energy levels on the other hand are smooth functions of h. We focus then on the ground state energy $E_0(h)$, and we want to study its behaviour. It can happen that for a given $h = h_c$, the eigenvalue of an excited state equals $E_0(h_c)$, then we have a point of non-analyticity of the ground state energy[2]. We call these points quantum critical points, and they usually separate two regions in which the system shows very different responses to an external perturbation. In particular we focus on second order QPT which are characterized by collective phenomena which in turn give rise to long-range correlations. This features can be formalized considering the model (1.23) in the case of an infinite, traslational-invariant lattice. We can define the correlation between spin components along e.g. the z-axis as

$$\langle S_i^z S_j^z \rangle = \langle \psi_0(h) | S_i^z S_j^z | \psi_0(h) \rangle - \langle \psi_0(h) | S_i^z | \psi_0(h) \rangle \langle \psi_0(h) | S_j^z | \psi_0(h) \rangle. \qquad (1.24)$$

It is generally a hard task, but it can be demonstrated that the large distance limit of this quantity is exponentially suppressed as

$$\langle S_i^z S_j^z \rangle \sim e^{-\frac{|i-j|}{\xi(h)}} (1 + \Sigma_d), \qquad (1.25)$$

where $\xi(h)$ is called *correlation length*[3]. The quantity Σ_d is a sum of infinitely many terms having a more suppressed exponential behaviour. There are some integrable theories for which it is possible to sum this series finding a power law.

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[1] notice that this is not usually the case, but we assume this in order to be able to explain the main features of a QPT in a compact and direct fashion.

[2] to be precise this feature, called *level crossing*, is possible only when $\mathcal{H}_1$ is a conserved quantity. In the most general case the two energy levels only come very close, but do not meet, and this feature is called *avoided level crossing*. In certain limits though many of these avoided level crossings become level crossings, so that our arguments are of wide applicability.

[3] in eq. (1.25) and from now on all the lattice dependent quantities will be taken to be dimensionless. We will switch to the dimensional counterpart by multiplying by the right power of the lattice spacing a, and denoting them with a hat, as e.g. $\hat{\xi} = a\xi$,

In a second order phase transition we have $\xi(h_c) = \infty$ and it is actually much easier to sum Σ_d. The correlation function will be described by a power law

$$\langle S_i^z S_j^z \rangle \sim |i - j|^{-2d} \,, \tag{1.26}$$

allowing for long-range interactions. The constant d is usually called *scaling dimension*.

The correlation length is then again an analytic function of h, except at the critical point, and it is particularly interesting to study its behaviour approaching h_c. For the model at hand, it diverges with the power law

$$\xi(h) \sim |h - h_c|^\nu \,, \tag{1.27}$$

where ν is called *critical exponent*.

Another interesting feature is the collapse of the first excited state onto the ground state. This means that if we define the *energy gap* as $\Delta(h) = E_1(h) - E_0(h)$ we have $\Delta(h_c) = 0$. The way this quantity approaches zero, and the correlation length diverges are related by

$$\Delta(h) \sim |h - h_c|^{z\nu} \,, \tag{1.28}$$

which means that $\Delta \sim \xi^{-z}$. The exponent z is usually called *dynamical exponent*. To elucidate this relation we consider the model (1.23) on a D-dimensional infinite lattice as an example. In what follows is crucial that the lattice is invariant under rotations, that is under exchange of two axes and respective coupling constants.

The Hamiltonian describes the temporal evolution of the system, and we perform a Wick rotation so that we work with imaginary time τ. Then for τ small enough we can define the transfer matrix $T = 1 - \tau \mathcal{H} \simeq e^{-\mathcal{H}\tau}$. We take a discrete time that we count in terms of the lattice spacing a, which we set to be equal one for simplicity. Then we can study this problem as a statistical model on a $D + 1$-dimensional lattice, where the wave function at two consecutive times will be connected by the transfer matrix $|\psi(\tau + 1)\rangle = T |\psi(\tau)\rangle$. Due to the invariance under rotations and translations the same holds for the wave function of any

subspace in any D dimensions, evolving in the orthogonal dimension. We want to study the same site, different times correlation function, that is $\langle S_i^z(0)S_i^z(\tau)\rangle$. We represent our operators in the Heisenberg picture, such that

$$
\begin{aligned}
\langle S_i^z(0)S_i^z(\tau)\rangle &= \langle\psi_0|S_i^z(0)e^{-H\tau}S_i^z(0)e^{H\tau}|\psi_0\rangle - \langle\psi_0|S_i^z(0)|\psi_0\rangle\langle\psi_0|S_i^z(0)|\psi_0\rangle = \\
&= \sum_{k\geq 1}\left|\langle\psi_0|S_i^z(0)|\psi_k\rangle\right|^2 e^{-(E_k-E_0)\tau},
\end{aligned}
$$
(1.29)

where we have assumed the theory has a discrete set of eigenvalues[1]. The first gap is usually much wider than the others, and we can rewrite the latter in terms of it as $E_{k+1}-E_k = \alpha_k\Delta$, where $k \geq 1$, and $0 < \alpha_k < 1$.
We can then rewrite (1.29) as

$$
\langle S_i^z(0)S_i^z(\tau)\rangle = \left|\langle\psi_0|S_i^z(0)|\psi_1\rangle\right|^2 e^{-\Delta\tau}\left(1 + \sum_{k>1}\delta_k e^{-\alpha_k\Delta\tau}\right),
$$
(1.30)

where $\delta_k = \left|\langle\psi_0|S_i^z(0)|\psi_k\rangle/\langle\psi_0|S_i^z(0)|\psi_1\rangle\right|^2$. Is now easy to compute the exponentially decaying behaviour in the limit $\tau \to \infty$ in the off-critical case, where Δ has a fixed value. The series in the rhs of eq. (1.30) is generally very hard to sum, and one has to consider the fact that the spectrum is not entirely discrete.
Rotation invariance gives us the correspondence $\xi \propto 1/\Delta$, and a direct comparison with eq. (1.26) tells us that the dynamical exponent $z = 1$, as usual for second order phase transitions. This will generally be the case for the theories studied in this manuscript.

1.5 Generalities on relativistic quantum field theory

The above lattice models are closely related to quantum field theories in their scaling limit. This connection will be explained thoroughly in the next section. Here we want to introduce some useful key aspects of $D + 1$-dimensional rela-

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[1]this is usually not the case when considering an infinite lattice, but we consider this case for simplicity.

tivistic QFT.

One of the main goals of a QFT is the study of correlation functions of local fields, as they are the only physical quantities in the theory, in the sense that they can be related to observables. First we want to settle what we mean by locality. In a relativistic theory we say that a field is local if it is causally independent of any other fields for space-like intervals. We apply this definition on a countable set of fields $\phi_a(x)$ $a \in \mathbb{N}$. This is a set of local fields if

$$\left[\phi_a(x), \phi_b(y)\right] = 0 \quad \forall a, b \in \mathbb{N} \quad \forall x, y \, \Big| \quad \left(x^0 - y^0\right)^2 < |\vec{x} - \vec{y}|^2 \qquad (1.31)$$

This definition is quite cumbersome as it is, and can be simplified further. In fact the temporal evolution of a field is governed by the Hamiltonian $\mathscr{H} = \int dx^D h(x)$, where $h(x)$ is the energy density. We can define locality by asking that $h(x)$ do not depend on any other fields for space-like intervals, that is

$$[\phi_a(x), h(y)] = 0 \quad a \in \mathbb{N} \quad \forall x, y \, \Big| \quad \left(x^0 - y^0\right)^2 < |\vec{x} - \vec{y}|^2 \qquad (1.32)$$

If we prepare a configuration of fields which satisfy eq. (1.32) we can easily see that any later configuration will automatically satisfy eq. (1.31), so that the two definitions are equivalent. This is due to the fact that due to eq. (1.32) any later couple of fields will be quantum mechanically independent, and then eq. (1.31) holds.

As we will see in the next section the real equivalence is between quantum lattice theories close to criticality, and Euclidean QFT. The next step is then to define the correlation functions in the Euclidean QFT as

$$\langle 0| \phi_{a_1}(x_1)\phi_{a_2}(x_2)...\phi_{a_n}(x_n) |0\rangle = \frac{1}{Z} \int \left[\mathscr{D}\phi\right] \phi_{a_1}(x_1)\phi_{a_2}(x_2)...\phi_{a_n}(x_n)e^{-S[\phi]}, \quad (1.33)$$

where $S[\phi]$ is the action functional, and $Z = \int \left[\mathscr{D}\phi\right] e^{-S[\phi]}$ the partition function. The connections of this expression with a lattice theory, and in particular of the integration measure, will be explained in section 1.6.

We focused on a countable set of fields because in general one could define a complete basis of operators with countable members. With this we mean that there exists a set of fields on which a general operator $\mathscr{O}$ can be decomposed

linearly as

$$\mathscr{O}(x) = \sum_{n \in \mathbb{N}} a_n \phi_n(x), \qquad (1.34)$$

where a_n are c-numbers. Here we are simplifying heavily the notation and we are not expressing explicitly any quantum number, but clearly all quantum numbers of the lhs and the rhs must match. Examples of such a set are the ordered products of powers of the field and its derivatives for a free bosonic theory. Equation (1.34) has to be taken in the weak sense, viz. is true only when the operator $\mathscr{O}$ appears into an expectation value.

Another remarkable feature of a relativistic QFT is the existence of an *operator product expansion*. Consider a physical process which is characterized by two operators $\mathscr{O}_1(x_1)$ and $\mathscr{O}_2(x_2)$ separated by a very small distance[1] compared with all the other operators $\varphi_i(y_i)$ taking part in the process. It is sensible to think that fluctuations of these two operators are not felt by φ_i, so that their product can be replaced by an effective vertex in the diagrams contributing to the process. Following this idea we can then state

$$\mathscr{O}_1(x_1)\mathscr{O}_2(x_2) = \sum_{n \in \mathbb{N}} c_{12}^n(|x_1 - x_2|)\phi_n(x_1), \qquad (1.35)$$

where c_{12}^n are c-number functions, dependent only on the nature and relative positions of the operators $\mathscr{O}_1(x_1)$ and $\mathscr{O}_2(x_2)$. Again as in eq. (1.34) the only fields allowed in the rhs are those with quantum numbers compatible with the lhs.

1.6 Quantum field theory as scaling limit

At a quantum critical point the divergence of the correlation length makes it impossible to find a length scale. One consequence is that the lattice spacing is infinitesimal compared to the range of correlations, and we can think about it as vanishing. This allows us to treat the degrees of freedom as a continuous field, and rely on quantum field theory to describe the model. In particular, as the physics is described by infinitely long fluctuations, the model is invariant under rescaling, and is well described by a conformal field theory. It is natural then to

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[1] here we are taking the Euclidean definition of distance.

ask ourselves if a quantum field theory can be used even outside criticality. The answer to this question is yes, but in a very particular case.

The idea comes from the fact that if we are "close enough" to the critical point the physics is described by low energy excitations. This can be seen with (1.28), that is in the region where ξ is very large, the gap is almost null, and energy levels are shifted towards the ground state. This means that the wave function's spectral decomposition is governed by low energy modes, whose oscillations are very big compared to the lattice spacing. Describing the low energy physics of the model means looking at large distance, and looking at larger distance is in turn equivalent to reducing the lattice spacing. The equivalence we are after is then between the low energy physics of a quantum lattice model, and a quantum field theory.

We want then to consider a vanishing lattice spacing, but the "naive" limit $a \to 0$ changes sharply the physics, as all dimensional physical quantities are measured in lattice spacings. In order to maintain the same physics we have to keep the characteristic length unmodified, and this is achieved by increasing ξ. This means taking the limit $a \to 0$, and the limit $\xi \to \infty$. Loosely speaking we are reducing the lattice spacing, but at the same time we are zooming into the system, in such a way that we observe the same characteristic distance. In a more physical fashion it means changing the coupling h in the following way

$$\lim_{h \to h_c} \lim_{a \to 0} [a\xi(h)] = \hat{\xi}, \tag{1.36}$$

such that $\hat{\xi}$ does not change. The definition of the characteristic length $\hat{\xi}$ is arbitrary, and is this definition which defines the way the two limits in the lhs of eq. (1.36) must be performed. In fact these limits are taken to keep $\hat{\xi}$ constant.

Along with the correlation length, all the lengths have to be rescaled accordingly. That is we want to keep the coordinate of the site $\hat{i} = ai$ untouched, and this means performing the limit $i \to \infty$. Performing these two limits we can observe that $\langle O_i O_j \rangle$ vanishes as $a^{2d} \to 0$. We regularize the correlator by a multiplicative renormalization. Defining the quantity $m = 1/\hat{\xi}$ we compute

$$\lim_{h \to h_c} \lim_{a \to 0} \left[(m\xi)^{2d} \langle O_i O_j \rangle \right] = \langle 0| \mathscr{O}(x)\mathscr{O}(y)|0\rangle \sim e^{-m|x-y|}. \tag{1.37}$$

Under these limits then the correlation function can be rewritten as a two point function of an Euclidean QFT where we called $\mathscr{O}$ the quantum field counterpart of the lattice operator O, and we identified the continuum coordinates $x, y = \hat{i}, \hat{j}$. This relation explains better the meaning of eq. (1.33) in light of the lattice model. In particular allows us to define the integration measure in QFT as

$$\int [\mathscr{D}\phi] = \lim_{a \to 0} \left[\sum_{\text{config}} \prod_{n \in \mathbb{N}} \prod_x d\tilde{\phi}_n(\hat{x}) \right], \qquad (1.38)$$

where $\tilde{\phi}_n$ are the lattice counterparts of fields, and we are summing over all the possible configurations. This quantity is generally divergent, but a rigorous definition would be far too detailed for our scope.

This sequence of limits is called the *scaling limit*, and the resulting QFT will be denoted as scaling theory. A first consequence of eq. (1.37) is that the mass scale of the QFT m is equivalent to the lattice quantity $1/\hat{\xi}$. In particular as it corresponds to the dimensionful gap $\hat{\Delta}$ in a relativistic theory, we can interpret this quantity as the mass of the lightest particle in the spectrum. A summary of all the relationships between QFT and lattice quantities is reported in table 1.1.

lattice theory	QFT
D-dim. lattice	$D + 1$ Euclidean manifold
rotation and translation invariance, $z = 1$	relativistic theory
dimensional position $\hat{i}$	coordinate x
$1/\hat{\xi}$	m
operator O_i	operator $\mathscr{O}(x)$ of dimension d
$\langle O_i O_j \rangle$	$\langle 0|\mathscr{O}(x)\mathscr{O}(y)|0\rangle$

Table 1.1: Summary of the map between a lattice theory and its corresponding scaling quantum field theory.

The QFT as depicted by eq. (1.37) is clearly a simplification over the corresponding lattice theory, as it captures only the low-energy features. Forgetting about the short-distance physics though means that it describes all the universal features of a lattice theory, and this is indeed the scope of a scaling limit.

Notice that the process described in this section is equivalent but opposite to the renormalization techniques developed in quantum field theory. As well ex-

plained in the introduction of Montvay & Münster [1994], in QFT we regularize by discretizing continuum theory, that is introducing a lattice spacing a. In that case there are ultra-violet divergences that come from the integration over momenta of the loop's contributions. When the QFT is discretized on a lattice, due to the periodicity $2\pi/a$ of the Fourier series on the lattice, we can integrate only over the first Brillouin zone $-\pi/a \leq p \leq \pi/a$. This clearly gives a finite result, and the next step would be sending the lattice spacing to zero, and renormalize the theory, exactly as we did in this section.

1.7   The area law

Now that we have a clearer picture on what is and how to quantify entanglement, we can ask ourselves which are the differences between a quantum system and a classical system. An attempt to answer this question is based on the comparison between the classical entropy and the von Neumann entropy.

The first difference is the very interpretation of these two quantities. Entropy enters the classical picture in thermodynamics, where it quantifies the "ignorance" of a macroscopic observer on the microscopic state of a system. The easiest example one could think about is an isolated system of fixed energy, volume, and number of constituents. Calling Ω the number of microscopic configurations compatible with the values of macroscopical quantities the entropy is $S = k \log \Omega$. The entropy then quantifies the uncertainty we have on the microscopic configuration of the system.

We can make a second more complicated example which is more directly comparable with the bipartite setting. Suppose we have a classical many-body system at fixed temperature, volume and number of constituents, and we divide it in a region A and a sorrounding environment B. We take the last to be big enough to be considered an infinite energy thermal bath. We allow the two partitions to exchange only heat, and we wait long enough for the two parts to be in thermal equilibrium. If we focus on the system A it is sensible to suppose that the probability for the system to be on an energy level is directly proportional to the number of microstates corresponding to that energy $p_i = C\Omega(E_i)$, where C is a

real number. With a bit of work one can demonstrate that $p_i = e^{-\frac{E_i}{kT}}$ holds[1], and the entropy, as defined in thermodynamics, corresponds to $S = -k\sum_i p_i \log p_i$. If we fix a temperature we can compute the energy $\langle E \rangle$, and the entropy will be $S = -k\langle \log \Omega(E) \rangle$. The same statement applies then here, the entropy is again a function of the uncertainty we have on the microscopic configurations. Clearly at zero temperature, where only the lowest energy state is accessible, the two cases are equivalent.

In the quantum case we could have a non-zero entropy even with absolute certainty on the microstate. This can be easily seen with the example of the bipartite system. If we prepare a classical system in a given microstate, and we bipartite it both the entropies of its partitions are zero. This follows from the fact that we know the global state, so that we know with certainty the microstate of any of its partitions. In the quantum case, as we have seen in section 1.2 this is not true. Even if we prepare the global state with absolute certainty its partitions can in general have a non-zero entropy. This difference is encoded in the subadditivity property of the entanglement entropy. In the classical case $S(\rho_{AB}) = S(\rho_A) + S(\rho_B)$, showing that the two partitions are uncorrelated, while in the quantum case there is an inequality, showing that there are some left-over correlations between the parties, even at zero temperature.

We want to show an example in which these correlations are well quantified in terms of quantum information. We focus on the bipartite setting, and we call d_A and d_B the dimensions of the Hilbert space of regions A and B respectively. An interesting question is the following: *if the global state were a random pure state, what would the average entropy of the region A be?*

The answer was conjectured by Page [1993], and proven by Foong & Kanno [1994] and Sen [1996] to be

$$\langle S_A \rangle \simeq \log d_A - \frac{d_A}{2d_B} \tag{1.39}$$

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[1] this case is an example of a canonical ensemble, and its probability distribution is usually normalized by a partition function Z. For simplicity of notation here we are assuming $Z = 1$, which is equivalent to dividing $\Omega(E_i)$ by the total number of possible microstates, such that $\sum_i \Omega(E_i) = 1$. The reader who wants a more complete treatment of this subject can find it on e.g. Huang [1987].

for $1 \ll d_A \le d_B$. We can see then that the mean sub-state is very close to the maximally entangled state, and the smaller the partition the closer it gets. We define the average information of a system as $I(\rho) = S_{MAX} - \langle S(\rho) \rangle$, where S_{MAX} is the entropy of the maximally entangled state in the Hilbert space, that is $S_{MAX} = \log d$, where d is the dimension of the Hilbert space. Then, continuing with the same example we have $I(\rho_{AB}) = \log d_A + \log d_B$, $I(\rho_A) = \frac{d_A}{2d_B}$, and $I(\rho_B) = \log d_B - \log d_A + \frac{d_A}{2d_B}$. The amount of information contained by the partitions and their internal correlation is then very small, leaving all the information on the global state to the correlations between the two parts.

The second, and arguably most remarkable difference between the classical case and the quantum one is the extensivity. In a classical system the uncertainty is directly proportional to the number of degrees of freedom contained in the system, and this makes the entropy an extensive quantity. This is not the case for a typical ground state of a quantum system. In the pioneering works of Bombelli *et al.* [1986], and Srednicki [1993], it was shown that in this case the entropy would rather obey an *area law*. In those works the authors where dealing with free D-dimensional massless local quantum field theories, finding that

$$S(\rho_A) = S(\rho_B) = \mathscr{A} \frac{\widehat{\partial A}}{\varepsilon^{D-1}}, \tag{1.40}$$

where $\widehat{\partial A}$ is the dimensional boundary of the region A, $\mathscr{A}$ is a dimensionless constant, and ε is an ultraviolet cut-off [1]. If we were to interpret these QFTs as scaling limit of lattice theories (or we consider their lattice regularizations), that cut-off would have a natural interpretation, and could be taken to be proportional to the lattice spacing.

This result gives us some insights on the distribution of quantum correlations in the ground state. When we are dealing with local QFT, or equivalently with lattice systems with short range interactions, quantum correlations between two regions accumulate on the boundary between them. This rather non-trivial result was demonstrated rigorously for massive (gapped) one-dimensional theories

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[1] notice that the authors were dealing with this problem in the context of black hole entropy, the two regions were then the sphere inside the event horizon, and the outer region, and the cut-off was set to be the inverse of the Plank mass.

in Hastings [2007]. Equation (1.40) would imply a constant entanglement entropy for one-dimensional critical theories. In Holzhey *et al.* [1994] it was shown that this law does not hold in this case, and entanglement entropy would rather satisfy,

$$S(\rho_A) = \mathscr{L} \log \frac{\hat{L}}{\varepsilon}, \tag{1.41}$$

where $\mathscr{L}$ is a dimensionless constant typical of the model at hand, $\hat{L}$ is the length of the chain, and ε is again a UV cut-off. We will devote Chapter 2 to proving eq. (1.41), and fixing $\mathscr{L}$.

The area law holds for gapped systems when $L \gg \xi$ as was found in Calabrese & Cardy [2004]; Cardy *et al.* [2008]. In that case one finds

$$S(\rho_A) = \mathscr{S}(\xi). \tag{1.42}$$

The constant $\mathscr{S}(\xi)$ depends both on the model at hand, and the correlation length (mass scale of the scaling QFT), and it will be studied thoroughly in Chapter 3.

2

Entanglement Entropy in Conformal Field Theory

Conformal invariance is very constraining for a quantum field theory, nonetheless it has played a central role in the study of critical many-body systems. As we already mentioned in section 1.6, at the critical point their physics is characterized by fluctuations at all length scales, so that they are invariant under changes of scale. This feature, combined with the usual Poincaré invariance equip the theory with a symmetry under global conformal transformations. It was firstly noticed by Polyakov [1970] that local[1] theories invariant under scale transformations should be invariant under a local conformal transformation. Belavin *et al.* [1984] found that this consideration is particularly powerful in two dimensions, where the local conformal algebra becomes infinite dimensional, and constrains correlation functions enough to allow a complete description and classification of critical models.

In this chapter we review shortly their results, and we finally focus on methods for evaluating entanglement entropy in CFT.

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[1] here "local" means defined by short-range interactions; notice that this feature does not forbid long-range correlations, which are actually present in a model at its critical point.

2.1 Conformal Field Theory

What follows in these sections is a general introduction to conformal field theories, mainly based on Francesco *et al.* [1996] and Ginsparg [1988], from which we borrowed the logic path. We will start by reviewing general features of conformal invariant theories in D dimensions. A conformal transformation acts on a space equipped with a metric tensor $g_{\mu\nu}(x)$, leaving the latter invariant up to a scale

$$g_{\mu\nu}(x) \to \Lambda(x)g_{\mu\nu}(x). \tag{2.1}$$

We restrict our analysis to a flat space $g_{\mu\nu}(x) = \eta_{\mu\nu}$ of positive signature. Studying the infinitesimal transformation $x^\mu \to x^\mu + \varepsilon^\mu$ one easily arrives at

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = \frac{2}{D}\eta_{\mu\nu}\partial_\rho \varepsilon^\rho, \tag{2.2}$$

which manipulated further gives

$$(2-D)\partial_\mu\partial_\nu\partial_\rho \varepsilon^\rho = \eta_{\mu\nu}\Box\partial_\rho \varepsilon^\rho, \tag{2.3}$$

where $\Box = \partial_\mu \partial^\mu$. A rapid look to this last equation immediately suggests that the case $D = 2$ is somehow special, and we will specialize on that in the next section. For $D > 2$, eq. (2.3) forces ε to be quadratic in the coordinates, so that its most general form is

$$\varepsilon^\mu(x) = a^\mu + b^\mu_\rho x^\rho + c^\mu_{\rho\sigma} x^\rho x^\sigma, \tag{2.4}$$

and imposes $c^\mu_{\rho\sigma} = c^\mu_{\sigma\rho}$, and $b^\mu_\rho = \alpha\eta^\mu_\rho + m^\mu_\rho$, where m^μ_ρ is a skew-symmetric matrix. Analyzing separately the three powers of x in (2.4), it is possible to extract the finite conformal transformations, and their generators, which are presented in table (2.1). Clearly **translations** and **rotations** are typical of a relativistic field theory, while invariance under **dilations** and **special conformal transformations (SCT)** are the new constraints introduced by scale invariance.

It is of great importance to analyze the n-point invariants of the theory $\Gamma(n)$, as they are the quantities which will be used to build correlators in the field theory[1]. The invariance under the Poincaré group implies that $\Gamma(n)$ must depend

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[1]indeed as the n-point functions correspond to the physical quantities that we can extract

Translation	$x'^\mu = x^\mu + a^\mu$	$P_\mu = -i\partial_\mu$
Rotation	$x'^\mu = \Lambda^\mu_\nu x^\nu$	$L_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)$
Dilation	$x'^\mu = \lambda x^\mu$	$D = -ix^\rho\partial_\rho$
SCT	$x'^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2b_\rho x^\rho + b^2 x^2}$	$K_\mu = -i(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu)$

Table 2.1: Finite conformal transformations, and generators of the infinitesimal transformations in general dimensions.

on distances $|x_i - x_j|$; moreover dilations force invariants to depend on ratios $|x_i - x_j|/|x_k - x_l|$. Finally, special conformal transformations constrain all the n-points invariant with $n \leq 3$ to be constants, giving as first invariant the harmonic ratios

$$R_{ijkl} = \frac{|x_i - x_j||x_k - x_l|}{|x_i - x_k||x_j - x_l|}. \tag{2.5}$$

Another consequence of conformal invariance is the tracelessness of the energy-momentum tensor. This tensor is defined considering the variation of the action functional under the infinitesimal transformation $x^\mu \to x^\mu + \epsilon^\mu$, that is[1]

$$\delta S = \int d^D x\, T^{\mu\nu} \partial_\mu \varepsilon_\nu. \tag{2.6}$$

Now plugging (2.4) into (2.6) one can see that $T^\rho_\rho = 0$ implies invariance under dilations. So that if we want our theory to be invariant under conformal transformations we need the tracelessness of the energy-momentum tensor.

The last feature we focus on for general dimensions is the form of correlation functions for *quasi-primary operators*. These fields are defined in the classical theory and denoted by $\phi_a(x)$. They generally belong to a representation of the Lorentz group, but we focus only on spinless operators here for simplicity. They transform then under conformal maps as

$$\phi_a(x) = \left|\frac{\partial x'}{\partial x}\right|^{\frac{d_a}{D}} \phi_a(x'), \tag{2.7}$$

from the theory, they must be conformal invariant.

[1]in the following the stress-energy tensor is assumed to be symmetric. This does not pose any restriction, as in Lorentz invariant theories this tensor can always be made symmetric.

where $|\partial x'/\partial x|$ is the Jacobian of the transformation, while d_a is the scaling dimension of ϕ_a. We divided the exponent by the number of dimensions in order to have a simplification, as the dimension of the Jacobian is D. Now considering the transformation law (2.7), and assuming that the action is invariant under conformal transformations, it is clear that the n-point functions satisfy the equation

$$\langle\phi_1(x_1)\phi_2(x_2)...\phi_n(x_n)\rangle = \left|\frac{\partial x'}{\partial x}\right|^{\frac{d_1}{D}}_{x=x_1} \left|\frac{\partial x'}{\partial x}\right|^{\frac{d_2}{D}}_{x=x_2} ... \left|\frac{\partial x'}{\partial x}\right|^{\frac{d_n}{D}}_{x=x_n} \langle\phi_1(x_1')\phi_2(x_2')...\phi_n(x_n')\rangle.$$

(2.8)

Let us consider then the two point function

$$G(x_1,x_2)_{ab} = \langle\phi_a(x_1)\phi_b(x_2)\rangle.$$

(2.9)

Due to invariance under rotations and translations it depends only on the relative distance. Dilation invariance implies that (2.9) must behave as $|x_1-x_2|^{-d_1-d_2}$, and special conformal transformations lead to the condition $G(x_1,x_2)_{ab} \neq 0 \iff d_a = d_b$. It follows then, after a convenient normalization, that

$$G(x_1,x_2)_{ab} = \frac{\delta_{ab}}{|x_1-x_2|^{2d_a}}.$$

(2.10)

The same arguments can be applied quite easily to the three and four-point functions. This matter though goes beyond the scope of this manuscript, and will not be reported here. The interested reader can find them in the references reported at the beginning of this section.

2.2 CFT in 1+1 Dimensions

As we anticipated at the beginning of the chapter the case $D = 2$ is somehow special, and this section is devoted to its analysis.

In this case eq. (2.2) becomes the Cauchy-Riemann equations

$$\partial_0\varepsilon^0 = \partial_1\varepsilon^1 \quad \text{and} \quad \partial_0\varepsilon^1 = -\partial_1\varepsilon^0 \quad \text{or,}$$

(2.11)

$$\partial_0\varepsilon^0 = -\partial_1\varepsilon^1 \quad \text{and} \quad \partial_0\varepsilon^1 = \partial_1\varepsilon^0,$$

(2.12)

and its finite version can be derived by simply using eq. (2.1). Calling f the conformal map which links the initial and final coordinates we obtain

$$\left(\partial_0 f^0(x)\right)^2 + \left(\partial_1 f^1(x)\right)^2 = \left(\partial_1 f^0(x)\right)^2 + \left(\partial_0 f^1(x)\right)^2,$$
$$\partial_0 f^0(x)\partial_0 f^1(x) + \partial_1 f^0(x)\partial_1 f^1(x) = 0,$$

(2.13)

whose solutions are the finite versions of eqs. (2.11) and (2.12).

It is then natural to use complex coordinates $z = x^0 + ix^1$, $\bar{z} = x^0 - ix^1$, such that if we choose (2.11) a conformal transformation will act as a holomorphic function, while for (2.12) it will be antiholomorphic. In particular expressed in these coordinates, eq. (2.13) reads $\partial_{\bar{z}} f(z,\bar{z}) = \partial_z \bar{f}(z,\bar{z}) = 0$, that means that holomorphic transformations depend only on z, and antiholomorphic only on $\bar{z}$. We will treat z and $\bar{z}$ as two independent variables, and in particular we consider x^0 and x^1 to be complex variables too, so that f and $\bar{f}$ will be just a change of coordinates in the complex plane. This considerations open the set of coordinates to unphysical values, but we can recover the physical case by imposing $z^* = \bar{z}$.

In two dimensions the conformal transformations are embodied by the set of all the analytic functions of the complex plane. Here we have to be extremely careful when we define the conformal group. At this stage the set of analytic functions is not a group[1] on the whole complex plane. This is a direct consequence of the fact that we considered local transformations, which in general are not invertible in the whole plane. Moreover any analytic function is characterized in principle by infinitely many parameters, that are the coefficients of their Laurent series, making the group infinite dimensional. There is then a huge difference between the *local conformal group*, and the *global conformal group*. For the last we have to restrict the set to all those holomorphic functions which are invertible on the whole plane. It can be demonstrated that these transformations form the projective linear group $PGL(2,\mathbb{C})$, which is isomorphic to $SL(2,\mathbb{C})$. This group is defined by three complex parameters, which reduce to three real parameters once we set $z^* = \bar{z}$. We recover then the situation described for general dimension at the beginning of this chapter.

Far more interesting is the case in which we allow our group elements to be in-

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[1]with respect to the composition.

vertible only locally. As we have seen that the holomorphic and antiholomorphic parts are decoupled, from now on we will focus on the first case, and any definition will be naturally extended to the second case. The local group is infinite dimensional, thus we have an infinite number of generators which we aim now to calculate. We consider an infinitesimal transformation that admits Laurent expansion around $z = 0$

$$f(z) = z + \sum_{n=-\infty}^{\infty} c_n z^{n+1}, \tag{2.14}$$

where c_n are the Laurent coefficients of such a series. We can easily find the generators applying this transformation to a scalar dimensionless quasi-primary field $\phi(z)$, for which

$$\delta\phi(z) = -\left(\sum_{n=-\infty}^{\infty} c_n z^{n+1}\right) \partial_z \phi(z). \tag{2.15}$$

We can read the infinitesimal generators off this formula which we call $l_n = -z^{n+1}\partial_z$, and their antiholomorphic counterparts $\bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}}$, and finally we find their algebra

$$[l_m, l_n] = (n - m)l_{m+n}$$
$$[\bar{l}_n, \bar{l}_m] = (n - m)\bar{l}_{m+n} \tag{2.16}$$
$$[l_n, \bar{l}_m] = 0.$$

It is important now to understand which generators correspond to physical transformations. To begin with we notice that l_{-1}, l_0, l_1 form a sub-algebra, which corresponds to the global conformal transformations. In particular when $z^* = \bar{z}$ we have the identification of translations with $(l_{-1} \pm \bar{l}_{-1})$, dilations with $l_0 + \bar{l}_0$, rotations with $\pm i(l_0 - \bar{l}_0)$, and finally SCT are generated by linear combinations of l_1 and l_{-1}.

In two dimensions we can extend the definition (2.7) to fields with spin. Considering a quasi-primary field with spin s and scaling dimension d we can define the *holomorphic conformal dimension* $\Delta = (s + d)/2$, and its antiholomorphic counterpart $\bar{\Delta} = (d - s)/2$. Its transformation law under a global conformal trans-

formation $f, \bar{f}$ is then

$$\phi(z, \bar{z}) = \left(\frac{\partial f(z)}{\partial z}\right)^{\Delta} \left(\frac{\partial \bar{f}(\bar{z})}{\partial \bar{z}}\right)^{\bar{\Delta}} \phi\left(f(z), \bar{f}(\bar{z})\right). \tag{2.17}$$

There will be in general a class of fields which transform as in eq. (2.17) under any local conformal transformation, and they are called *primary fields*. For these fields eq. (2.10) can be expressed as

$$G(x_1, x_2)_{ab} = \frac{\delta_{ab}}{|z_1 - z_2|^{2\Delta_a} |\bar{z}_1 - \bar{z}_2|^{2\bar{\Delta}_a}}, \tag{2.18}$$

and the same arguments apply for three and four-point functions.

2.2.1 The Stress Energy Tensor and Conformal Ward Identity

In this section we investigate the consequences of Ward identity in two dimensional conformal field theory. First, following eq. (2.6) we introduce the conserved currents j^{μ} in terms of the stress energy tensor in euclidean coordinates as $j^{\mu} = T^{\mu}_{\nu} \varepsilon^{\nu}$, where ε^{ν} are the infinitesimal generators of the conformal algebra as defined in eq. (2.4). Switching to complex coordinates one can easily check that $T_{\bar{z}z} = T_{z\bar{z}} = \frac{1}{4} T^{\rho}_{\rho} = 0$, $T_{zz} = \frac{1}{4}(T_{00} - T_{11}) + \frac{i}{2} T_{10}$, and $T_{\bar{z}\bar{z}} = \frac{1}{4}(T_{00} - T_{11}) - \frac{i}{2} T_{10}$. The conservation law $\partial_{\mu} T^{\mu\nu} = 0$ translates into complex coordinates as $\partial_z T_{\bar{z}\bar{z}} = \partial_{\bar{z}} T_{zz} = 0$. Thus also the stress energy tensor is divided into a holomorphic and antiholomorphic part, which we will call respectively T and $\bar{T}$[1]. The Ward identity is a general result, and states that in the presence of a symmetry of the action, the variation of an operator $\mathcal{O}$ is

$$\delta\mathcal{O} = \int d^2x \partial_{\mu} \langle j^{\mu} \mathcal{O} \rangle, \tag{2.19}$$

where the integral is performed over a surface which contains the coordinates of all the fields in play. It is particularly interesting applying this law to a primary field ϕ. Equation (2.19) can be compared with the infinitesimal action of the

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[1]in particular we choose a normalization such that $T = -2\pi T_{zz}$, and $\bar{T} = -2\pi T_{\bar{z}\bar{z}}$

conformal group on ϕ, that is $\phi'(x) = \phi(x) - i\varepsilon^{\mu}(x)G_{\mu}\phi(x)$, where G_{μ} are the generators of the conformal algebra as listed in table 2.1. By doing so for translations, rotations, and dilations we find

$$\frac{\partial}{\partial x^{\mu}}\langle T_{\nu}^{\mu}(x)\phi(y)\rangle = -\delta(x-y)\frac{\partial}{\partial x^{\nu}}\langle\phi(x)\rangle, \tag{2.20}$$

$$\varepsilon_{\mu\nu}\langle T^{\mu\nu}(x)\phi(y)\rangle = -i\delta(x-y)s\langle\phi(x)\rangle, \tag{2.21}$$

$$\langle T_{\mu}^{\mu}(x)\phi(y)\rangle = -\delta(x-y)d\langle\phi(x)\rangle. \tag{2.22}$$

Now using as representation of the delta function in the complex plane $\delta(x) = \frac{1}{\pi}\partial_{\bar{z}}\frac{1}{z} = \frac{1}{\pi}\partial_{z}\frac{1}{\bar{z}}$, and with a bit of work, we can express eqs. (2.20)–(2.22) in the complex plane. The derivation is too long to be reported here, but is just a question of manipulations, and gives the definition of the OPE of a primary field with the stress energy tensor

$$T(z)\phi(w,\bar{w}) = \frac{\Delta}{(z-w)^2}\phi(w,\bar{w}) + \frac{1}{z-w}\partial_{w}\phi(w,\bar{w}) + \ldots \tag{2.23}$$

$$\bar{T}(\bar{z})\phi(w,\bar{w}) = \frac{\bar{\Delta}}{(\bar{z}-\bar{w})^2}\phi(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}}\phi(w,\bar{w}) + \ldots \tag{2.24}$$

where the dots stand for regular terms in the limit $(w,\bar{w}) \to (z,\bar{z})$. A primary field then is characterized by at most a double pole in its OPE with the stress energy tensor. Any operator with a higher pole is called *secondary field*.

2.2.2 The free Majorana Fermion

It is worth specializing these findings to the case of a free Majorana Fermion, as it will recur many times in our work. We consider this example in light of its connection with the two-dimensional Ising model found by Zuber & Itzykson [1977]. In that work the off-critical case was considered, but an adaption to the critical case is straightforward. The Ising model can be described by two Hermitian fields $\psi_1(x)$ and $\psi_2(x)$, which can be interpreted as the components of

a Majorana spinor. We introduce then

$$\Psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} \tag{2.25}$$

$$\gamma^0 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma^1 = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \gamma^5 = \gamma^0\gamma^1 = -\sigma_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.26}$$

where γ^0, γ^1 satisfy the Clifford algebra $\{\gamma^a, \gamma^b\} = 2\eta^{ab}\mathbb{1}$. Then the equations of motion will be $\gamma^\mu \partial_\mu \Psi(x) = 0$. In this representation the two components are eigenstates of γ^5, with opposite chirality, and the equation of motion can be expressed in components as the two equations $(\partial_0 - \partial_1)\psi_1(x) = (\partial_0 + \partial_1)\psi_2(x) = 0$.

To switch to complex coordinates we need first to perform a Wick rotation by defining the variable $\tau = ix^0$, and then we define the complex variables $z = \tau + ix^1$, and $\bar{z} = \tau - ix^1$. Identifying $\bar{\psi} \equiv \psi_1$ and $\psi \equiv \psi_2$ the equations of motion become $\partial \bar{\psi}(z, \bar{z}) = \bar{\partial}\psi(z, \bar{z}) = 0$. These equations are defined by the action[1]

$$S_{\text{Majorana}} = \frac{1}{2\pi} \int dz\,d\bar{z} \quad \{\psi(z,\bar{z})\bar{\partial}\psi(z,\bar{z}) + \bar{\psi}(z,\bar{z})\partial\bar{\psi}(z,\bar{z})\}. \tag{2.27}$$

First of all we want to determine the two point functions, that are the propagators of the theory. Dealing with a free theory we can achieve this by rewriting the action in the form $S = \frac{1}{2}\int d^2x\,d^2y\,\Psi(x)\mathbf{A}(x,y)\Psi(y)$, defining $\mathbf{A}(x,y) = i\delta(x-y)\gamma^0\gamma^\mu\partial_\mu$.

Now we can find the kernel, defined as $\mathbf{K}(x,y)|\ \mathbf{A}(x,y)\mathbf{K}(x,y) = \delta(x-y)\mathbf{I}$, that is

$$\frac{1}{\pi}\begin{pmatrix} \partial_{\bar{z}} & 0 \\ 0 & \partial_z \end{pmatrix}\begin{pmatrix} \langle \psi(z)\psi(w) \rangle & \langle \psi(z)\bar{\psi}(w) \rangle \\ \langle \bar{\psi}(z)\psi(w) \rangle & \langle \bar{\psi}(z)\bar{\psi}(w) \rangle \end{pmatrix} = \frac{1}{\pi}\begin{pmatrix} \partial_{\bar{z}}\frac{1}{z-w} & 0 \\ 0 & \partial_z\frac{1}{\bar{z}-\bar{w}} \end{pmatrix}. \tag{2.28}$$

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[1] in eq. (2.27) we introduce a factor $1/2\pi$ for later convenience in the expression of the two point function, this has clearly no effect on the equations of motion.

From eq. (2.28) we can easily extract

$$\langle \psi(z)\psi(w) \rangle = \frac{1}{z-w}$$
$$\langle \bar{\psi}(z)\bar{\psi}(w) \rangle = \frac{1}{\bar{z}-\bar{w}} \qquad (2.29)$$
$$\langle \psi(z)\bar{\psi}(w) \rangle = \langle \bar{\psi}(z)\psi(w) \rangle = 0.$$

Differentiating with respect to w we find

$$\langle \psi(z)\partial_w \psi(w) \rangle = \frac{1}{(z-w)^2}, \qquad (2.30)$$

and we can extend trivially this result for higher order derivatives. From these results we immediately see that $\Delta_\psi = \bar{\Delta}_{\bar{\psi}} = 1/2$, and $\Delta_{\bar{\psi}} = \bar{\Delta}_\psi = 0$. Knowing that $s_\psi = s_{\bar{\psi}} = 1/2$, we can derive $d_\psi = d_{\bar{\psi}} = 1/2$.

Now we want to test the OPE of the stress energy momentum tensor. This operator can be easily extracted from eq. (2.27), and is $T(z) = -\frac{1}{2} : \psi(z)\partial\psi(z) :$. With the colon notation we mean we take the ordered product of the operators involved, that is defined as $: a := a - \langle a \rangle$. This is needed whenever we define an operator by the product of other operators, as this will naturally diverge when we consider fields at the same point. From the definition of the stress energy tensor, eqs. (2.28) and (2.31), and applying the Wick theorem we can compute the OPEs

$$T(z)\psi(w) = \frac{1}{2}\frac{\psi(z)}{(z-w)^2} + \frac{\partial\psi(z)}{z-w} + ..., \qquad (2.31)$$

and

$$T(z)T(w) = \frac{1}{4}\frac{1}{(z-w)^4} + \frac{2T(z)}{(z-w)^2} + \frac{\partial T(z)}{z-w} + ... \qquad (2.32)$$

Equation (2.31) confirms that ψ is a primary field of conformal weight $\Delta = \frac{1}{2}$. From eq. (2.32) on the other hand we understand that the stress energy tensor is not primary. This is quite surprising at first sight as $T(z)$ is a quasi-primary in the classical theory. The fourth-order pole in the OPE of the stress energy tensor with itself has been introduced by the quantization process.

2.2.3 The central charge

The result on the two point function of the stress energy tensor in eq. (2.32) can be actually generalized, and we can state that in any theory

$$T(z)T(w) = \frac{c}{2}\frac{1}{(z-w)^4} + 2\frac{T(z)}{(z-w)^2} + \frac{\partial_z T(z)}{z-w} + ... \tag{2.33}$$

Of course we would have to report also the antiholomorphic version of eq. (2.33), through which $\bar{c}$ is defined. We consider only theories for which $c = \bar{c}$, so that such an equation is redundant. The constant c is the *central charge*, and depends on the model under consideration. The first consequence of the presence of a central charge is that the quantization of a conformal field theory changes the algebra (2.16) into the Virasoro algebra. The quantum version of the generators l_{-n} defined in eq. (2.16) are the "coefficients" of the Laurent series of the stress energy tensor

$$T(z) = \sum_{n=-\infty}^{\infty} z^{-n-2}L_n, \qquad L_n = \oint \frac{dz}{2\pi i}z^{n+1}T(z). \tag{2.34}$$

As already mentioned they satisfy the Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m}. \tag{2.35}$$

Conformal invariance in two dimensions is powerful enough to allow a division in what are usually called *conformal families*. These are defined by a primary operator, and a set of secondary operators which can be obtained from it by consecutive applications of L_n. By applying these generators we decrease the weight Δ of the resulting field, such that the primaries are called *highest weight* fields, while the secondaries are its *descendants*. A remarkable result is that the classification of the highest weight operators of a theory is enough to classify all the possible irreducible representations of the conformal group. We will not go in any detail on this matter, for which there exists a vast literature. We redirect the interested reader to one of the textbooks signaled at the beginning of this chapter.

The central charge is also an anomaly of the theory. Indeed considering the

one-loop correction to a process one has to deal with self-energy contributions $\Pi_{\mu\nu\rho\sigma}(p) = \int d^2x \ e^{ipx}\langle T_{\mu\nu}(x)T_{\rho\sigma}(0)\rangle$. In particular one has to regularize this terms as they are ultraviolet divergent, and this can be achieved e.g. by dimensional regularization. One would expect that $\Pi^{\mu}_{\mu\rho\sigma}(p) = 0$, due to the tracelessness of the stress energy tensor, but surprisingly enough this is not the case. This anomaly is proportional to the central charge, and breaks the local conformal invariance[1].

The central charge appears also in the transformation law of the stress energy tensor. With eq. (2.33) we can immediately understand that $T(z)$ is not a primary operator, and then does not transform as in eq. (2.17). Its transformation law is indeed more involved

$$T'(w) = \left(\frac{dw}{dz}\right)^{-2}\left[T(z) - \frac{c}{12}\{w,z\}\right],\tag{2.36}$$

where

$$\{w,z\} = \frac{d_z^3 w}{d_z w} - \frac{3}{2}\left(\frac{d_z^2 w}{d_z w}\right)^2,\tag{2.37}$$

with d_z the total derivative with respect to z, is the *Schwartian derivative*. This relation has a remarkable consequence. Let us consider a conformal theory on a two-dimensional manifold that can be obtained by a local conformal transformation on the complex plane. In general the trace of the stress energy tensor would not be zero. Indeed calling $\mathcal{M}$ this manifold and $w = w(z)$ its coordinates we find

$$\langle T(w)\rangle_{\mathcal{M}} = \left(\frac{dw}{dz}\right)^{-2}\left[\langle T(z)\rangle_{\mathbb{C}} - \frac{c}{12}\{w,z\}\right] = \frac{d_z^3 w}{(d_z w)^3} - \frac{3}{2}\left(\frac{d_z^2 w}{(d_z w)^2}\right)^2,\tag{2.38}$$

where in the last equality we used the fact that the vacuum expectation value of the stress energy tensor must be null in the complex plane, due to scaling invariance. We are left then with a combination of derivatives of the transformation that could be nonzero.

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[1] our presentation of the conformal anomaly ends here, but we suggest Duff [1994] for further details.

It is a general result that

$$\langle T(w) \rangle_{\mathcal{M}} = \frac{c}{24\pi} R(w), \tag{2.39}$$

where $R(w)$ is the local curvature of the manifold. This is then another example of a manifestation of an anomaly. All these considerations will become very useful in the next section, where we aim to consider a conformal field theory on a Riemann surface.

2.3   CFT on a Riemann surface

In the previous sections we have seen how conformal invariance in two dimensions constrains a QFT, and how the knowledge of all primary fields of a theory leads to a complete description of the theory. A primary field is defined by its conformal weight, so that two theories whose primary fields' dimensions match are in fact equivalent. This is the key to the success of CFT for describing critical statistical models, as it has the concept of universality classes built in. In this section we describe an expanded picture of these ideas by considering the theory on a Riemann surface.

First we would like to introduce the physical motivations for such a study. Our final goal is to evaluate the entanglement entropy of a bipartite setting in 1+1-dimensional CFT. This corresponds to evaluating the entanglement entropy of a critical one-dimensional quantum lattice system, as explained in section 1.6. To avoid confusions in notation we will study this problem directly in a quantum field theory. We consider then a critical QFT quantized on the real line, evolving in imaginary time. We take it to be in its ground state at zero temperature, and we focus on the configuration of fields at a given time, which we can take without loss of generality to be $\tau = 0$. We divide then the real line into two sub-regions which we call A, and B, as depicted in figure 2.1.

This setting is the same as the one described by eq. (1.13), and, once we have defined ρ_A, we can proceed with the evaluation of the entropy (1.15). In a quantum field theory we have an infinite number of degrees of freedom in an infinitesimal space and, even if conceptually well defined, it is technically a hard task to perform the tracing out, and then to define the reduced density matrix. In order to
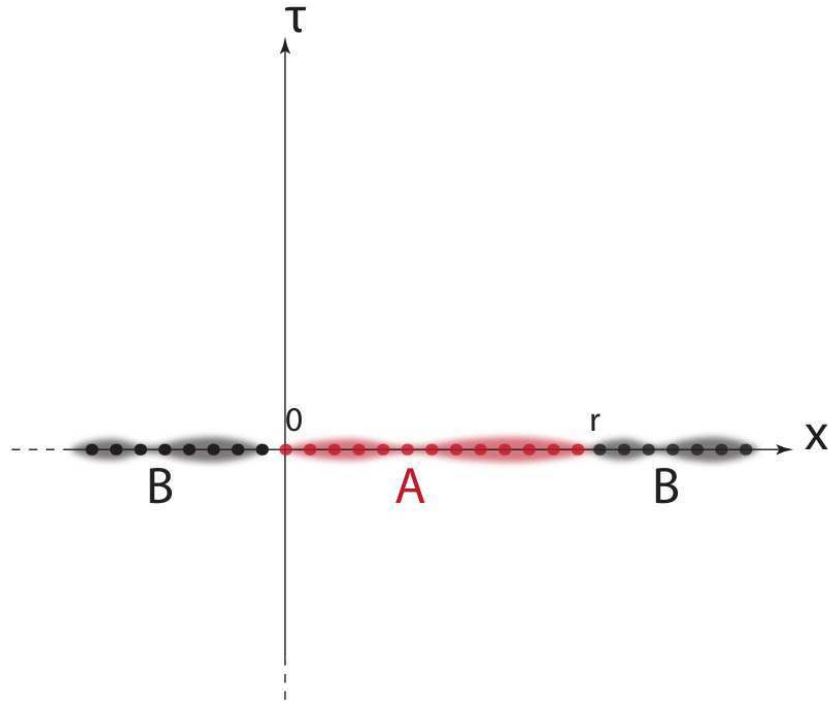
Figure 2.1: In this picture we represent a one dimensional lattice system on the real line $\mathcal{T} = 0$ which is partitioned into the two regions A and B. Due to translation invariance the region A is only defined by its length r, and we choose $A \in [0, r]$. The blurred curves around the lattice represent the field configurations of the scaling quantum field theory.

address this issue we employ a method called the *replica trick*.

2.3.1 The replica trick

This method was introduced in the context of systems with quenched disorder, as explained in Mezard *et al.* [1988]; Nishimori [2001]. Their case is physically very different from ours, but they share some technical details. They consider e.g. a classical spin system on a D-dimensional lattice with nearest-neighbour interactions, for which $\mathcal{H}(J_{ij}) = -\sum_{\langle i,j \rangle} J_{ij} S_i S_j$. The couplings J_{ij} are randomly

distributed, typically with a normal distribution $P(J_{ij})$ with mean value J_0. In particular the partition function is

$$Z(J_{ij}) = \mathrm{Tr}\, e^{-\beta \mathcal{H}(J_{ij})} \tag{2.40}$$

and they want to evaluate the mean value of the free energy

$$\langle F \rangle = -\beta^{-1} k \langle \log Z \rangle = - \int \left(\prod_{\langle i,j \rangle} dJ_{ij} \right) P(J_{ij}) \log Z(J_{ij}). \tag{2.41}$$

Generally the dependence of Z on the couplings is rather complicated, and this makes it impossible to evaluate the configurational average $\langle \log Z \rangle$ directly. One uses then the aforementioned replica trick to extract it exploiting the equality

$$\langle \log Z \rangle = \lim_{n \to 1} \frac{\langle Z^n \rangle - 1}{n}. \tag{2.42}$$

This method is clearly useful only when evaluating $\langle Z^n \rangle$ is easier than $\langle \log Z \rangle$. The process is then equivalent to creating n copies of the model, evaluating the global partition function of this collection of copies, and finally performing the limit for $n = 1$. A rather nontrivial problem is represented by the fact that the first two steps of this method require an integer n. To perform the last limit then we need to find an analytic continuation of $\langle Z^n \rangle$ for $n \in \mathbb{R}$.

Our case parallels the one just described. Instead of evaluating directly $\mathrm{Tr}_A \rho_A$ we prefer to deal with ρ_A^n. The method in eq. (2.42) was adapted to this setting by Callan & Wilczek [1994]; Holzhey *et al.* [1994], and becomes

$$S(\rho_A) = -\lim_{n \to 1} \frac{d}{dn} \mathrm{Tr}_A \rho_A^n. \tag{2.43}$$

Their results where then specialized in the context of conformal field theory by Calabrese & Cardy [2004, 2009]. The same trick of eq. (2.43) can be also employed to evaluate Rényi entropies (1.18) for integer α if we do not perform the limit.

The first step is then to find a way to implement $\mathrm{Tr}_A \rho_A^n$. In our case the degrees of freedom are encoded in the Hilbert space, which is represented by the field

configurations $\{\phi(x,0)\}$, $x \in \mathbb{R}$. We take here and henceforth the example of a theory with a single field without loss of generality. Any state of the Hilbert space $|\psi\rangle$ can be written in terms of these field configurations as

$$|\psi\rangle = \int [D\phi] \langle \phi | \psi \rangle |\phi\rangle , \qquad (2.44)$$

with coefficients which depend only on the configurations that lay in the "past", such that

$$\langle \phi | \psi \rangle = \frac{1}{\sqrt{Z}} \int_{\mathscr{C}_\downarrow} [D\varphi]_\downarrow \, e^{-S[\varphi]}. \qquad (2.45)$$

Here with $[D\phi]_\downarrow$ we mean that we integrate only in the region $\tau < 0$, and the boundary condition on the integral

$$\mathscr{C}_\downarrow: \quad \{\phi\}_\downarrow \big| \quad \varphi(x,-\infty) = \varphi_{in}(x) \quad \bigwedge \quad \varphi(x,0) = \phi(x,0) \quad \forall x \in \mathbb{R}, \qquad (2.46)$$

ensures that we are integrating the right configurations. In eq. (2.45) Z is the partition function which links in-configurations to out-configurations, and we are normalizing with $1/\sqrt{Z}$ in order to have the normalization $\langle \psi | \psi \rangle = 1$. The element $\langle \psi |$ can be then represented by

$$\langle \psi | = \int [D\phi] \langle \psi | \phi \rangle \langle \phi | , \qquad (2.47)$$

where

$$\langle \psi | \phi \rangle = \frac{1}{\sqrt{Z}} \int_{\mathscr{C}_\uparrow} [D\varphi]_\uparrow \, e^{-S[\varphi]}, \qquad (2.48)$$

and

$$\mathscr{C}_\uparrow: \quad \{\phi\}_\uparrow \big| \quad \varphi(x,\infty) = \varphi_{out}(x) \quad \bigwedge \quad \varphi(x,0) = \phi(x,0) \quad \forall x \in \mathbb{R}. \qquad (2.49)$$

Now in order to obtain ρ_A we need to trace out the field configurations of the region B from $\rho = |\psi\rangle \langle \psi|$. This practically means sewing together the conditions $\mathscr{C}_\uparrow$ and $\mathscr{C}_\downarrow$ on the line $\tau = 0$. In this way we obtain a matrix element of ρ_A

$$\langle \phi_A | \rho_A | \phi_A' \rangle = \int [D\phi]_B \langle \phi_A \phi_B | \rho_A | \phi_A' \phi_B \rangle = \frac{1}{Z} \int_{\mathscr{C}_A} [D\varphi]_{\mathbb{C} \setminus x \in A} \, e^{-S[\varphi]}, \qquad (2.50)$$

where

$$\mathscr{C}_A: \quad \{\phi\} \left| \begin{array}{l} \varphi(x, \pm\infty) = \varphi_{\substack{out \\ in}}(x) \end{array} \right. \wedge \left. \begin{array}{l} \varphi(x, \varepsilon^-) = \phi_A'(x, 0) \\ \varphi(x, \varepsilon^+) = \phi_A(x, 0) \end{array} \right. \quad \forall x \in A, \varepsilon \to 0 . \quad (2.51)$$

In the end ρ_A^n

$$\langle \phi_A | \rho_A^n | \phi_A' \rangle = \int \left(\prod_{i=1}^{n-1} D\varphi_i \right) \langle \phi_A | \rho_A | \varphi_1 \rangle \langle \varphi_1 | \rho_A | \varphi_2 \rangle \dots \langle \varphi_{n-1} | \rho_A | \phi_A' \rangle, \quad (2.52)$$

can be obtained by taking n different copies of ρ_A connected by the condition that the configuration of fields going out of the region A for positive τ of one copy must be the same configuration coming into the next copy. The final action that we want to perform is taking the trace over A, and this is simply implemented by sewing the outgoing configurations of the nth copy with the incoming configurations of the first one.

This is equivalent to plugging our CFT in the manifold $\mathscr{M}$ depicted in figure 2.2, and the evaluation of $\mathrm{Tr}_A \rho_A^n$ reduces to the evaluation of the partition function on $\mathscr{M}$. In particular if we take the region to be the segment $A = [0, r]$ we arrive at the conclusion

$$\mathrm{Tr}_A \rho_A^n = \varepsilon^{\delta_n} \frac{Z_{\mathscr{M}}(0, r)}{Z_{\mathbb{C}}^n}, \quad (2.53)$$

where we are changing the notation, and we are calling $Z_{\mathbb{C}} \equiv Z$ the partition function on the original CFT. The ratio $Z_{\mathscr{M}}(0, r)/Z_{\mathbb{C}}^n$ is known to be ultraviolet divergent[1], so that it needs the introduction of a UV cut-off ε to the right power (which we called δ_n) in order to regulate this divergence.

2.3.2 The twist field

Following the line of reasoning of the previous section in order to be able to evaluate the entanglement entropy we have to deal with

$$Z_{\mathscr{M}}(0, r) = \int [D\phi]_{\mathscr{M}} e^{-\int_{\mathscr{M}} d^2 x \mathscr{L}[\phi]}, \quad (2.54)$$

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[1]as explained in Calabrese & Cardy [2004]; Holzhey *et al.* [1994]
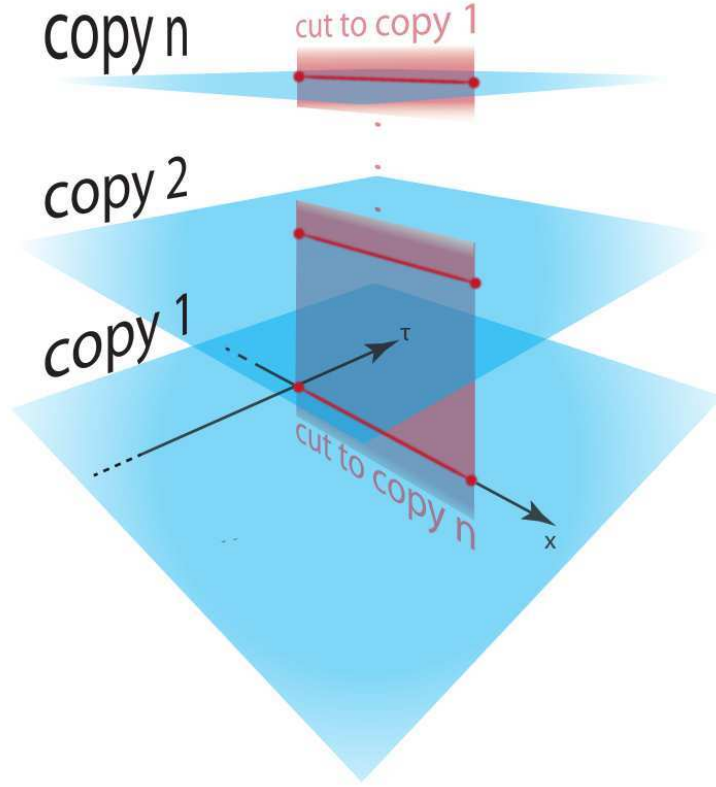
Figure 2.2: Representation of the multi-sheeted surface $\mathcal{M}$. Each blue layer represents a copy of the model at hand, and they are connected through the red branch cut.

where $\mathcal{L}$ is the Lagrangian density. Dealing directly with this quantity is rather difficult, and we prefer to reduce the problem onto a manifold that we know how to handle. The idea of considering the CFT under a conformal transformation that maps $\mathcal{M}$ into the complex plane was first introduced by Knizhnik [1986, 1987]. The reasoning goes as follows, let us consider the following single-valued covering map

$$\Xi : z \in U = \mathbb{C} \to \xi(z) \in V = \mathbb{C}, \tag{2.55}$$

and suppose it has a branch point a for which considering a point w in a neighbourhood of a

$$\chi = \Xi(w) = a + w^n. \tag{2.56}$$

The inverse map $\Xi^{-1}(\chi) = \sqrt[n]{\chi - a}$ is clearly not single-valued, but can be regularized by putting a branch cut in V, which connects n different sheets of this manifold. Along with this cut we have to define an analytic continuation of any analytic function ϕ on V, that is calling B the branch cut

$$\lim_{\varepsilon \to 0} \phi(\beta + \varepsilon) = \lim_{\varepsilon \to 0} \phi(\beta - \varepsilon) \qquad \forall \beta \in B \bigwedge \forall \varepsilon \notin B. \tag{2.57}$$

This machinery can be used to map the manifold $\mathcal{M}$ onto the complex plane, with great subsequent simplification. To achieve this goal we will use an adapted version of the inverse of the covering map in eq. (2.55), and the analytic continuation (2.57) will play the role of the sewing conditions in eq. (2.51).

The other key idea pushed forward by Knizhnik [1986, 1987] is to associate a local primary field to any branch point. A clear contextualization of this method in CFT has been pursued lately by Calabrese & Cardy [2004, 2009]. These operators were then called *twist fields* by Cardy *et al.* [2008], and in what follows we will use their formalism.

To introduce the twist field we focus on the inverse of the map in eq. (2.56) where we take $a = 0$ for simplicity. We choose to place the branch cut on the positive real axis, and we associate a twist field to the branch point in the following way. We call ϕ_i the fields on the ith sheet of V, and σ the operation of making a loop counterclockwise around $\xi = 0$. The functional integration will have to include the condition (2.57), such that in this case we have $\int_{\mathscr{C}_\sigma(0)} \prod_{i=1}^{n} [D\phi_i]$, where

$$\mathscr{C}_\sigma(0): \quad \lim_{\varepsilon \to 0} \phi_i(\beta - \varepsilon) = \lim_{\varepsilon \to 0} \phi_i(\sigma\beta + \varepsilon) = \lim_{\varepsilon \to 0} \phi_{i+1}(\beta + \varepsilon) \quad (\text{mod } n). \tag{2.58}$$

The twist field is denoted with $\mathscr{T}_\sigma(0)$ and is then formally defined by

$$\int \prod_{i=1}^{n} [D\phi_i] \, \mathscr{T}_\sigma(0)... \equiv \int_{\mathscr{C}_\sigma(0)} \prod_{i=1}^{n} [D\phi_i] ..., \tag{2.59}$$

where the dots represent the insertion of any other operators.

Notice that the locality of the twist field is ensured by the additivity property of the action. The action on the collection of the n sheets is defined as the sum of the action functionals of every sheet, such that eq. (1.32) is satisfied. eq. (2.59)

defines only one of the possible fields which implement the right branching condition. We could have considered e.g. $\tilde{\mathcal{T}}_\sigma =: \mathcal{T}_\sigma \phi :$ instead, and (2.59) would had been satisfied in any case. What uniquely fixes the twist field is the requirement that it be primary, with the lowest possible conformal weight, and invariant under all the symmetries of the theory.

Now we go back to the original manifold $\mathcal{M}$. It is clear in light of eqs. (2.56) and (2.57) it can be considered as a branched Riemann manifold with a branch cut running on the real axis from $z = 0$ to $z = r$. We need to introduce a second operator $\tilde{\mathcal{T}}_{\sigma^{-1}}(r)$, which corresponds to σ^{-1}, that is a clockwise loop, such that condition (2.57) becomes

$$\mathcal{C}_{\sigma^{-1}}(r): \quad \lim_{\varepsilon \to 0} \phi_i(\beta + \varepsilon) = \lim_{\varepsilon \to 0} \phi_i(\sigma^{-1}\beta + \varepsilon) = \lim_{\varepsilon \to 0} \phi_{i-1}(\beta - \varepsilon) \quad (\text{mod } n). \quad (2.60)$$

The two twist fields $\mathcal{T}(0)$ and $\tilde{\mathcal{T}}(r)$[1] can be thought of as if they were propagating two branch cuts which map fields respectively one copy up and one down. This affects the fields configuration in the following way. Any field crossing the negative real line does not feel any action, and will propagate on the same copy. Fields crossing the real line in the region $A \in [0, r]$ cross the branch cut propagated by $\mathcal{T}(0)$, and then will switch to the copy up. Finally those fields crossing the real line at values $x > r$ cross both the branch cuts of $\mathcal{T}(0)$, and $\tilde{\mathcal{T}}(r)$, and will propagate on the same copy. The picture as just described is represented in figure 2.3. Now putting together eqs. (2.53), (2.54) and (2.58)–(2.60) we can see that

$$\text{Tr}_A \rho_A^n = \frac{1}{Z_\mathbb{C}^n} \int \prod_{i=1}^n [D\phi_i] \, \mathcal{T}(0) \tilde{\mathcal{T}}(r) e^{-\sum_{i=1}^n S[\phi_i]} = \langle \mathcal{T}(0) \tilde{\mathcal{T}}(r) \rangle_{\mathbb{C}^n}, \quad (2.61)$$

What we achieved then is that now the entanglement entropy can be evaluated through a two-point function of primary field, a much easier object to compute than a partition function on a Riemann surface.

Next we are interested in determining the conformal weight $\Delta_{\mathcal{T}}$ of the twist field. The expectation value of an operator $\mathcal{O}$ on the ith sheet of the manifold $\mathcal{M}$ can

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[1]from now on we will abandon the subscripts σ and σ^{-1} as redundant in distinguishing $\mathcal{T}$ from $\tilde{\mathcal{T}}$.
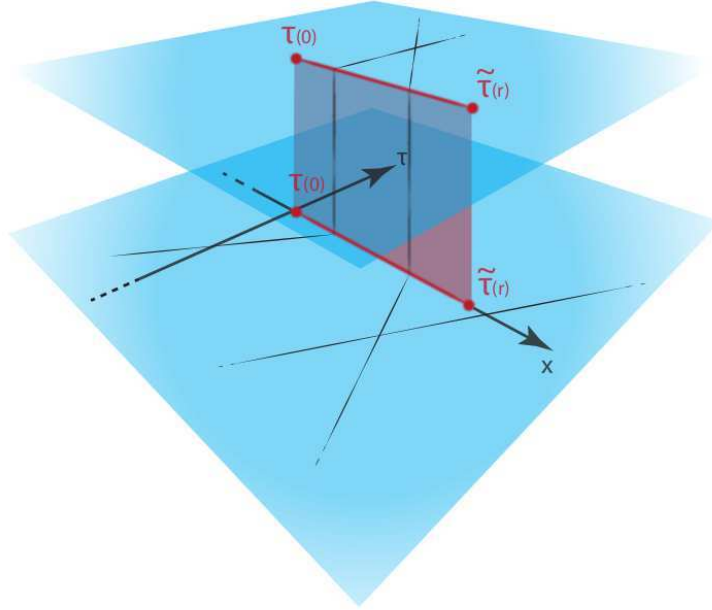
Figure 2.3: We show the effect of $\mathcal{T}$ and $\tilde{\mathcal{T}}$ on fields configurations. The black lines can be interpreted as the centre of some wave packets propagating in the incrasing τ direction.

be evaluated in terms of its correlation with the twist fields as follows

$$\langle \mathcal{O}(\xi, \text{ on } i\text{th sheet})\rangle_{\mathcal{M}} = \frac{\langle \mathcal{O}_i(\xi)\mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_{\mathbb{C}^n}}{\langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_{\mathbb{C}^n}}. \tag{2.62}$$

Then, as we are interested in the conformal weight of the twist field, we can use the OPE eq. (2.23) to extract it from its double pole. The stress energy tensor is an additive quantity, such that on the manifold it holds $T(\xi) = \sum_{i=1}^{n} T_i(\xi)$. We first focus on the inverse of the covering map (2.55) that in our case is

$$z(\xi) = \sqrt[n]{\frac{\xi}{\xi - r}}, \tag{2.63}$$

47

such that it first stretches the branch cut to infinity, and then maps the manifold to the complex plane, as illustrated in figure 2.4. As (2.63) is locally conformal
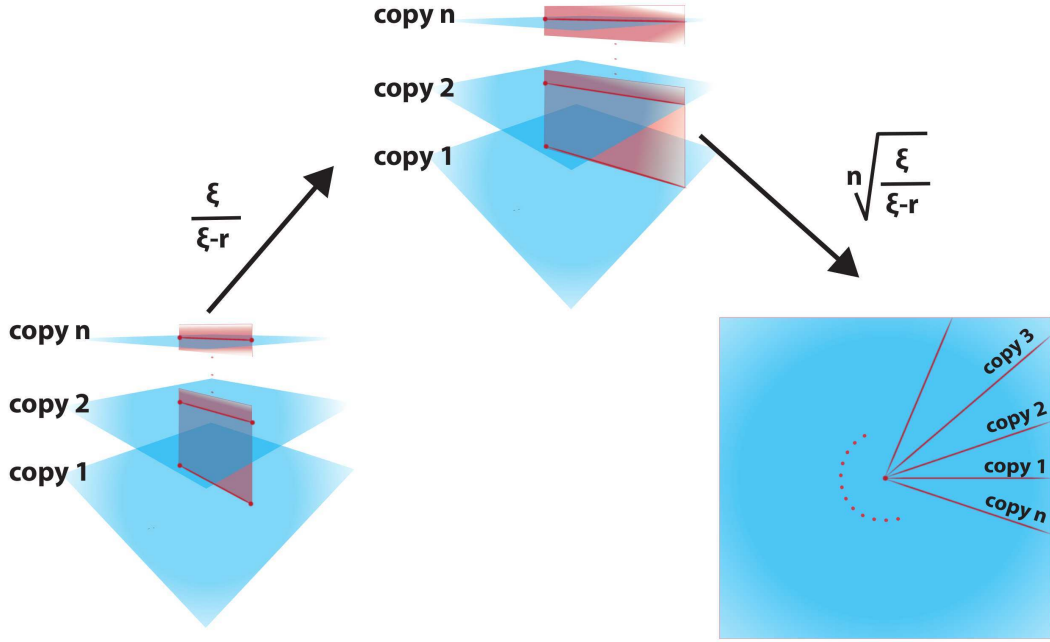
Figure 2.4: We show the sequence of transformations which unravel the manifold $\mathcal{M}$ on $\mathbb{R}^2$.

we can use eq. (2.38) in order to determine the expectation value of the stress energy tensor on the manifold, and we obtain

$$\langle T(\xi) \rangle_{\mathcal{M}} = \frac{c(n^2 - 1)}{24n} \frac{r^2}{\xi^2(\xi - r)^2}. \tag{2.64}$$

Comparing to eq. (2.39) we see we can interpret the two branch points as points of $\mathcal{M}$ with infinite curvature for any $n \neq 1$. Equation (2.62) implies that

$$\frac{\langle T(\xi)\mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_{\mathbb{C}^n}}{\langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_{\mathbb{C}^n}} = \frac{c(n^2 - 1)}{24n} \frac{r^2}{\xi^2(\xi - r)^2}. \tag{2.65}$$

Now we can use eq. (2.23) to find the following

$$\langle T(\xi)\mathscr{T}(0)\tilde{\mathscr{T}}(r)\rangle_{\mathbb{C}^n} = \left(\frac{1}{\xi-r}\frac{\partial}{\partial r}+\frac{h_{\tilde{\mathscr{T}}}}{(\xi-r)^2}+\frac{1}{\xi-\chi}\frac{\partial}{\partial\chi}+\frac{h_{\mathscr{T}}}{(\xi-\chi)}\right)\langle\mathscr{T}(\chi)\tilde{\mathscr{T}}(r)\rangle\bigg|_{\chi=0}$$

(2.66)

and using eq. (2.18) to define $\langle\mathscr{T}(\chi)\tilde{\mathscr{T}}(r)\rangle = |\chi-r|^{-4\Delta_{\mathscr{T}}}$ and $\Delta_{\mathscr{T}}=\Delta_{\tilde{\mathscr{T}}}$ we finally obtain

$$\Delta_{\mathscr{T}} = \frac{c}{24}\left(n-\frac{1}{n}\right).$$

(2.67)

2.3.3 The entanglement entropy

We are now able to evaluate the bipartite entanglement entropy for the region A. Using eqs. (2.62) and (2.67) we can finallty fix δ_n in eq. (2.53), and extract

$$\text{Tr}_A\rho_A^n = \left(\frac{r}{\varepsilon}\right)^{-\frac{c}{6}\left(n-\frac{1}{n}\right)}.$$

(2.68)

The analytic continuation needed for using the replica trick (2.43) is straightforward, such that the final result is

$$S(\rho_A) = \frac{c}{3}\log\left(\frac{r}{\varepsilon}\right).$$

(2.69)

This result allows us to fix the constant $\mathscr{L}$ in eq. (1.41), and it captures the universal features of the entanglement entropy, as it depends only on the central charge. To enjoy the full power of universality though we have to link the CFT to a critical lattice model. We have to perform a non-universal step, that is finding the relation between the lattice spacing and the UV cut-off. We can encode this non-universality with the introduction of an adimensional constant γ which links $a=\gamma\varepsilon$, such that eq. (2.69) becomes

$$S(\rho_A) = \frac{c}{3}\log\left(\frac{r}{a}\right)+\frac{c}{3}\log\gamma,$$

(2.70)

Other quantities that can be straightforwardly obtained from eq. (2.68) are the Rényi entropies (1.18), that in CFT take the form

$$S_n(\rho_A) = \frac{c}{6}\left(1+\frac{1}{n}\right)\log\left(\frac{r}{a}\right)+\frac{c}{6}\left(1+\frac{1}{n}\right)\log\gamma.$$

(2.71)

The constant corrections in eqs. (2.70) and (2.71) are non-universal, as they depend on the renormalization point, and cannot be determined by CFT arguments. They can though be found and fixed by direct calculations or numerical simulations on the lattice. This has been the case in Cardy *et al.* [2008]; Franchini *et al.* [2008]; Iglói & Juhász [2008]; Its *et al.* [2005]; Jin & Korepin [2004], where the authors were able to evaluate these constants for few integrable models.

As was noticed by Calabrese & Cardy [2004, 2009] these results can be easily extended to the case of finite temperature or finite systems. This is because these two settings can be easily obtained from the infinite case with a conformal mapping, and the entanglement entropy is related to the two point function of twist fields, which has a simple transformation law being primary fields.
The finite temperature case is obtained by mapping the theory from $\mathbb{C}$ into a cylinder of circumference β by the conformal transformation $z(w) = \frac{\beta}{2\pi} \log w$. This corresponds to considering the manifold $\mathcal{M}$ where we fold in a cylinder the direction perpendicular to the branch-cut. The result was obtained by Calabrese & Cardy [2004]; Korepin [2004] and is[1]

$$S(\rho_A) = \frac{c}{3} \log \left(\frac{\beta}{\pi a} \sinh \frac{\pi r}{\beta} \right) + c_1' = \begin{cases} \frac{c}{3} \log ra + c_1' & r \ll \beta \\ \frac{\pi c}{3\beta} r + c_1' & r \gg \beta \end{cases}, \qquad (2.72)$$

that means that we recover the behaviour (2.69) for low temperature, while for high temperature the entropy becomes extensive and follows an area law no more. It behaves actually as a thermodynamic entropy and we have no quantum correlations left. If we were to consider finite size systems we should put the theory on a cylinder folded in the same direction of the cut. Following the same logic of (2.72) we obtain

$$S(\rho_A) = \frac{c}{3} \log \left(\frac{L}{\pi a} \sinh \frac{\pi r}{L} \right) + c_1', \qquad (2.73)$$

where L is the total length of the system. It is easy to see that eq. (2.69) is recovered by taking $r \ll L$. This is the most useful result of this section, as it

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[1]in those works the constant correction of eq. (2.70) was denoted by c_1', so that we will use their formalism in what follows, meaning that $c_1' = \frac{c}{3} \log \gamma$.

allows for a direct comparison with computer simulations of lattice models on a finite chain with periodic boundary conditions.

3

Entanglement entropy in integrable massive QFT

In this chapter we focus on the methods to evaluate entanglement entropy for $(1+1)$-dimensional relativistic QFT. As we will see these methods parallel those developed in chapter 2, with some adaptions and extensions deriving from considering a massive theory. The technology we present here was mainly introduced by Cardy *et al.* [2008]; Castro-Alvaredo & Doyon [2009b]. Although the key points in the evaluation of entanglement entropy hold for any relativistic QFT we will focus mainly on *integrable* quantum field theories. We must say here that recently it has been demonstrated by Doyon [2009] that most important results do not require integrability. But the rather special characteristics of integrable theories allow for a simplification of the problem at hand, and in many cases for a direct analytic computation of the entanglement entropy. In the next sections follows a brief introduction of common concepts of integrability, which we made extensive use of to achieve our results.

3.1   Integrability in QFT

Integrability is a vast topic, which ranges from lattice theories and QFTs to classical field theories. There is actually no universally accepted definition of integrability, and many times one has to find an "ad hoc" definition for the theory at hand. We will be rather generic, and consider integrable any theory which

has enough conserved quantities to manifest some key features in its scattering processes.

The idea that conserved charges and scattering details could be somehow linked comes from a work of Coleman & Mandula [1967]. They demonstrated that in any QFT in more than one dimension with non-trivial S-matrix the only possible symmetry Lie algebra is constituted by the Lorentz generators P_μ, and $J_{\mu\nu}$. Any other symmetry must be implemented by spinless and momentumless generators. A direct consequence of this theorem is that a QFT with a conserved charge with spin must have a trivial S-matrix $S = \mathbf{1}$. We must notice that in the derivation of this result only algebrae of commutators were considered, and that this does not hold in general for anti-commutators, super-symmetry being a remarkable exception.

This statement does not hold either for $(1+1)$-dimensional theories, where higher spin conserved charges do not imply trivial scattering. This case was studied in Iagolnitzer [1978]; Polyakov [1977]; Shankar & Witten [1978]; Zamolodchikov [1977], where the authors found that for $(1+1)$-dimensions the existence of higher-spin conserved charges is enough for the S-matrix to be factorizable, and to avoid particle production. The rest of this section is devoted to describe these two properties.

Before beginning with this matter, let us introduce some typical formalism of QFT in $(1+1)$-dimensions. In this case the on-shell condition $p_0^2 - p_1^2 = m^2$ is encoded by requiring $p_\mu = (m\cosh\theta, m\sinh\theta)$, where θ is the *rapidity*. A boost $B(\Lambda)$ acts in this representation as

$$
B(\Lambda)\begin{pmatrix} p_0 \\ p_1 \end{pmatrix} = \begin{pmatrix} \cosh\Lambda & -\sinh\Lambda \\ -\sinh\Lambda & \cosh\Lambda \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \end{pmatrix} = \begin{pmatrix} m\cosh(\theta - \Lambda) \\ m\sinh(\theta - \Lambda) \end{pmatrix}, \qquad (3.1)
$$

such that it is easy to understand that any Lorentz invariant will be built on differences of rapidities.

Second we require the existence of a set of creation/annihilation operators $A_a^\dagger(\theta)$, $A_a(\theta)$ acting on the in/out states such that

$$
A_a^\dagger(\theta)|0\rangle = |\theta\rangle_{\substack{in \\ out}}^{a}, \qquad A_a(\theta)|0\rangle = 0, \qquad A_a(\theta)|\vartheta\rangle_{\substack{in \\ out}}^{b} = \delta_{ab}\delta(\theta - \vartheta)|0\rangle, \qquad (3.2)
$$

where the subscripts represent different particle types. Thus an in state will be written as $|\theta_1 \theta_2 ... \theta_n\rangle_{in}$, with $\theta_1 > \theta_2 > ... > \theta_n$, while an out state will be written in the same fashion with $\theta_1 < \theta_2 < ... < \theta_n$.

To show what we mean by no particle production we focus on the scattering process $2 \to n$. Its properties are encoded in the S-matrix, defined as

$$|\theta_1 \theta_2\rangle_{in}^{a_1 a_2} = \sum_{n=2}^{\infty} {\sum_{\{a_1' a_2' ... a_n'\}}}' \int_{\Theta}' \prod_{i=1}^{n} d\theta_i' \ \ S_{a_1 a_2}^{a_1' a_2' ... a_n'} \left(\theta_1, \theta_2; \theta_1', \theta_2' ... \theta_n'\right) |\theta_1' \theta_2' ... \theta_n'\rangle_{out}^{a_1' a_2' ... a_n'},$$

(3.3)

where $\Theta = \theta_1' < \theta_1' < ... < \theta_n'$ is the region of integration, while with the dashed integral and sum in the rhs we mean the requirement that all the charges be conserved. In particular Lorentz invariance requires conservation of energy and momentum, from which we can define the two charges $Q_{\pm 1} = H \pm P$, whose action on eq. (3.3) gives

$$
\begin{aligned}
m_1 e^{\theta_1} + m_2 e^{\theta_2} &= \sum_{i=1}^{n} m_i' e^{\theta_i'} \\
m_1 e^{-\theta_1} + m_2 e^{-\theta_2} &= \sum_{i=1}^{n} m_i' e^{-\theta_i'}.
\end{aligned}
$$

(3.4)

We labelled these two charges with ± 1 as they transform respectively as a spin one and minus one quantity under boosts. Notice that in one dimension the Lorentz group is one-dimensional, so that we can represent it with a real number (the rapidity), and will act on a quantity Q_s with spin s as $e^{s\Lambda}$. Then Q_s is an s-order covariant tensor, and can be thought as $Q_s \propto (H + P)^s$, where the proportionality constant is a c-number. Generally in and out states are formed by well-spaced wave-packets on which these charges act locally. It is a physical requirement then that these quantities be in involution with each other, and their action on the asymptotic states can be split into the action on single particles. Then the presence of any conserved charge Q_s would require that

$$q_{a_1}^{(s)} \left(m_1 e^{\theta_1}\right)^s + q_{a_2}^{(s)} \left(m_2 e^{\theta_2}\right)^s = \sum_{i=1}^{n} q_{a_i}^{(s)} \left(m_i' e^{\theta_i'}\right)^s.$$

(3.5)

We consider theories for which it is possible to build an infinite number of higher-spin charges, and it is easy to see that the only possible solution is the trivial one, viz. when the set of rapidities and eigenvalues of all the charges of the incoming and outgoing particles match. This rules out any particle production or annihilation, and forces the scattering just to "shuffle" momenta and quantum numbers, which are strictly the same before and after the scattering.

In a later work of Parke [1980] it was shown that the same effects can be achieved with just two such conserved quantities. Moreover, assuming the conservation of only two charges, Parke demonstrated that any scattering process can be divided into a sequence of $2 \to 2$ processes, and that the S-matrix satisfies a version of the Yang-Baxter equations[1]. A very simplified version of his proof goes as follows. First we want to understand the action of a conserved charge Q_s on a localized wave-packet. We consider a Gaussian wave-packet centred on the momentum p_a and position x_a

$$\psi_\alpha(x) \sim \int dp_1 e^{-\alpha(p_1 - p_{a1})^2} e^{ip_\mu(x - x_a)^\mu}. \tag{3.6}$$

The action of a conserved quantity on this state is generated by $e^{i\sigma Q_s}$, such that

$$e^{i\sigma Q_s}\psi_\alpha(x) \sim \int dp_1 e^{-\alpha(p_1 - p_{a1})^2} e^{ip_\mu(x - x_a)^\mu + i\sigma(me^\theta)^s}. \tag{3.7}$$

Now it can be shown with a saddle point expansion that the result of acting with a charge is a shift of the centre of the wave-packet $x_a \to x_a + s\sigma (me^\theta)^{s-1}$. This shift then depends on the rapidity of the packet and the spin of the charge in general, such that if $s = 1$ the shift is independent of θ. But the existence of higher-spin charges means that the shift is different on packets with different momenta.

With this in mind it is now easy to see the factorizability property considering e.g. the $3 \to 3$ process. In $(1+1)$-dimensions this scattering can happen in three different ways with different amplitudes in principle. But if there exist higher-spin charges we can relate the three channels to one another. Then the amplitudes must factorize into products of $2 \to 2$ processes, and they satisfy the following

---

[1]Baxter [1982]

remarkable conditions

$$S_{a_1a_2a_3}^{a'_1a'_2a'_3}(\theta_1,\theta_2,\theta_3) = S_{a_1a_2}^{cb}(\theta_1,\theta_2)S_{ca_3}^{a'_1d}(\theta_1,\theta_3)S_{bd}^{a'_2a'_3}(\theta_2,\theta_3)$$

$$= S_{a_2a_3}^{bc}(\theta_2,\theta_3)S_{a_1c}^{da'_3}(\theta_1,\theta_3)S_{bd}^{a'_1a'_2}(\theta_1,\theta_2), \qquad (3.8)$$

which go under the name of Yang-Baxter equations Baxter [1972]; Yang [1967].

3.2 Conformal perturbation theory

In the previous section we have seen how the presence of an infinite number of conserved quantities implies absence of particle production, and factorization of the S-matrix. The next problem is then to understand how we can define such quantities in a QFT. There is actually no unique way of addressing this issue, but for what concerns us we will take the point of view of Zamolodchikov [1989]. We consider a CFT perturbed by a set of n primary operators

$$S = S_{CFT} + \sum_{i=1}^{n} g_i \int \phi_i(x) \ d^2x. \qquad (3.9)$$

In particular we assume them to be scalar and with scaling dimension $\Delta_i < 1$, and that there exists a finite number of them. We take our theory to be unitary[1], which in this case translates into the absence of negative conformal dimension fields. Consequently $g_i = \gamma m^{2-2\Delta_i}$, γ being an dimensionless positive constant, and m being the mass scale. The introduction of a mass scale through $\{g\}$ clearly brakes conformal invariance, and drives the theory away from the critical point[2]. Zamolodchikov demonstrated that under these conditions there exist a family of fields satisfying

$$\bar{\partial}T_{s+1} = \partial\Theta_{s-1}, \qquad (3.10)$$

---

[1]in the sense of no negative-norm states in the CFT. This implies non-negative Δ_i.

[2]acting with the renormalization group on eq. (3.9) drives the theory away from the CFT.

associated with the conserved charges

$$Q_s = \oint [T_{s+1}dz + \Theta_{s-1}d\bar{z}], \qquad (3.11)$$

where the index s, labeling various equations of motion, is a spin parameter as defined in the previous section.

In a more recent work Zamolodchikov [1991] focused on the two-point functions. The formalism is developed in the case when in eq. (3.9) we are perturbing with only one relevant field, the case of general n being extracted trivially from the considerations below. We want to study the two point function of some operator φ. If g is small enough to allow the problem to be tackled perturbatively we get

$$\langle \varphi(x)\varphi(0)\rangle_g = \frac{1}{Z_g} \sum_{n=0}^{\infty} \frac{(-g)^n}{n!} \int \left[\prod_{i=1}^{n} d^2 y_i \right] \langle \tilde{\varphi}(x)\tilde{\varphi}(0)\phi(y_1)\phi(y_2)...\phi(y_n)\rangle_{CFT}, \quad (3.12)$$

where with $\tilde{\varphi}$ we mean the CFT field corresponding to the massive φ. The partition function is defined as in (1.33), such that $Z_g = \langle e^{-g \int d^x \phi(x)}\rangle_{CFT}$. The integrals in the rhs of eq. (3.12) suffer from both UV and IR divergences. The UV divergences can be as usual regulated by standard renormalization techniques. In particular this process will lead to a redefinition of the fields, and in the marginal case $\Delta = 1$ also of the coupling constant g.

In our case we will deal only with supernormalizable theories with $\Delta < 1/2$, such that neither g nor ϕ must be renormalized.

The IR divergencies are of deeper nature and they cannot be absorbed with a renormalization.

To analyze the UV behaviour of the lhs of eq. (3.12) we make use of an OPE of the kind introduced in eq. (1.35)

$$\varphi(x)\varphi(0) = \sum_i C_{\varphi\varphi}^i(x)A_i(0), \qquad (3.13)$$

where A_i is a set of local operators of the kind (1.34). The structure constants are local quantities, and it is natural to expect that in the presence of the perturbation parameter g they be analytic. The fields A_i on the other hand are perturbations of the respective CFT field $\tilde{A}_i$, and they develop a vacuum expectation value

which is of non-local nature and therefore could be non-analytic. Calling $(\Delta_i, \bar{\Delta}_i)$ the conformal dimensions of such fields we have from dimensional arguments

$$C_{\varphi\varphi}^i(x) = x^{\Delta_i - 2\Delta_\varphi} \bar{x}^{\bar{\Delta}_i - 2\bar{\Delta}_\varphi} \sum_{n=0}^{\infty} C_{\varphi\varphi}^{i(n)} \left(gr^{2-2\Delta}\right). \tag{3.14}$$

The VEVs of fields A_i are $\langle A_i \rangle = Q_i g^{\frac{\Delta_i}{1-\Delta}}$, where Q_i are dimensionless constants. These quantities cannot be determined directly with this method. Clearly not every element of the basis will develop a non-zero VEV in the perturbed theory. In fact the perturbed theory being Lorentz invariant all those A_i with non-zero spin, or those which can be written as derivatives of other fields must have a null VEV. The same can be said for those fields generated with eq. (3.10), as they are conserved charges of the theory.

It has to be said that both the fields A_i and their structure constant have to be renormalized, and Zamolodchikov [1991] gives a prescription for that.

We study now the stress energy tensor in the perturbed theory. It is clear that, as in this theory scale invariance is broken, the stress energy tensor is traceless no more. If we remember the definitions we gave in section 2.2.1 of the stress energy tensor components in complex coordinates we can write the following conservation law

$$\partial_{\bar{z}} T + \frac{1}{4}\partial_z \Theta = 0. \tag{3.15}$$

We focus on the first order renormalization of the off-diagonal component of the stress energy tensor, that is we consider the following expectation value

$$\langle T(z)... \rangle_g = \langle \tilde{T}(z)... \rangle_{CFT} - g \int dw d\bar{w} \langle \tilde{T}(z)\tilde{\phi}(w,\bar{w})... \rangle_{CFT} + ... \tag{3.16}$$

We can use eq. (2.23) to express the expectation value under integration in the rhs of this equation up to first derivatives

$$\begin{aligned} T(z)\phi(w,\bar{w}) &= \frac{\Delta}{(z-w)^2}\phi(w,\bar{w}) + \frac{1}{z-w}\partial_w \phi(w,\bar{w}) + ... \\ &= \frac{\Delta}{(z-w)^2}\phi(z,\bar{z}) + \frac{1-\Delta}{z-w}\partial_z \phi(z,\bar{z}) + ... \end{aligned} \tag{3.17}$$

This OPE is clearly UV divergent, so that we need to introduce a small distance cut-off. This can be achieved by introducing a Heaviside step function $H\left(|z-w|^2 - a^2\right)$ under integration. Then

$$
\begin{aligned}
\partial_{\bar{z}} T &= \int dw d\bar{w} (1-\Delta) \partial_z \tilde{\phi}(z,\bar{z}) \delta\left(|z-w|^2 - a^2\right) \\
&= \pi g (1-\Delta) \partial_z \phi(z,\bar{z}) + ...,
\end{aligned}
\tag{3.18}
$$

where we ignored the most singular term in eq. (3.17) as it develops a dependence on the cut-off a, and must vanish. Then comparing eqs. (3.15) and (3.18) we deduce

$$
\Theta(z,\bar{z}) = 4\pi g (1-\Delta) \phi(z,\bar{z}) + ...
\tag{3.19}
$$

This rather important relation is a direct consequence of the braking of scale invariance by the introduction of a cut-off, and as such could have been derived only with renormalization group arguments.

3.2.1 The Ising field theory

In this section we present an example of application of the definitions formulated so far. This example is the Ising field theory, and aiming to its definition we first consider the conformal Ising model[1]. The model has $c = 1/2$ as central charge, and its operator content is reported in table 3.1.

This theory is equivalent to the Free Majorana theory of section 2.2.2, with the identification of the two spinor components with ψ and $\bar{\psi}$[2]. The expression of the *energy operator* ε in terms of the free Fermion fields is easily obtainable as $\varepsilon \propto \bar{\psi}\psi$, while σ and μ need a non-local product of Majorana fields to be expressed.

We can get a massive integrable theory out of this model by perturbing it with one of its primaries. The most studied case is the one where we perturb the action (2.27) with the energy operator, and it corresponds to the massive two-

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[1] for a proper definition of this model we should have introduced the concept of minimal conformal model. This matter as well as a complete definition would bring us too far from the main topic of this manuscript, such that we redirect interested readers to Francesco *et al.* [1996]; Ginsparg [1988].

[2] which we already denoted with the same symbol with a bit of notation freedom.

Operator	Δ	$\bar{\Delta}$
$\mathbb{1}$	0	0
ψ	$\frac{1}{2}$	0
$\bar{\psi}$	0	$\frac{1}{2}$
ε	$\frac{1}{2}$	$\frac{1}{2}$
σ	$\frac{1}{16}$	$\frac{1}{16}$
μ	$\frac{1}{16}$	$\frac{1}{16}$

Table 3.1: Operator content of the Ising model. σ and μ are usually referred to as order and disorder operators respectively in light of the correspondence with the two-dimensional Ising lattice theory.

dimensional Ising model. Adopting the same formalism of section 2.2.2, the equations of motion found by Zuber & Itzykson [1977] out of criticality are

$$\left(\gamma^\mu \partial_\mu + \gamma^5 m\right)\Psi = 0. \tag{3.20}$$

These are associated with the action

$$S_{\text{Ising}} = \int d^2 x \left[\bar{\psi}\left(\partial_x + \partial_t\right)\bar{\psi} - \psi\left(\partial_x - \partial_t\right)\psi - m\bar{\psi}\psi\right]. \tag{3.21}$$

Performing a Wick rotation and mapping onto complex variables one can see that this action corresponds to the perturbation

$$S_{\text{Ising}} = 2\pi S_{\text{Majorana}} + im \int d^2 x \bar{\psi}\psi, \tag{3.22}$$

which is indeed proportional to the energy operator. Notice that in eq. (3.22) we normalized the Majorana action by a factor 2π. The choice of this constant is not unique and we fixed it with the requirement that eq. (3.22) give (3.20) as equations of motion. Notice that this corresponds to a normalization of the fields ψ and $\bar{\psi}$, and will multiply each two point function in eq. (2.29) with a factor $1/2\pi$. We proceed with the canonical quantization of the two fields ψ and $\bar{\psi}$. This

is done by decomposing them into Fermionic modes as

$$\psi(\tau,x) = \sqrt{\frac{m}{4\pi}} \int d\theta e^{\frac{\theta}{2}} \{a(\theta)e^{m(ix\sinh\theta - \tau\cosh\theta)} + a^{\dagger}(\theta)e^{-m(ix\sinh\theta - \tau\cosh\theta)}\}$$
$$\bar{\psi}(\tau,x) = -i\sqrt{\frac{m}{4\pi}} \int d\theta e^{-\frac{\theta}{2}} \{a(\theta)e^{m(ix\sinh\theta - \tau\cosh\theta)} - a^{\dagger}(\theta)e^{-m(ix\sinh\theta - \tau\cosh\theta)}\},$$

(3.23)

where θ is the rapidity. Then we impose that the mode operators $a(\theta)$ and $a^{\dagger}(\theta)$ satisfy the canonical anti-commutation relations

$$\{a(\theta), a^{\dagger}(\theta')\} = \delta(\theta - \theta') \qquad \{a(\theta), a(\theta')\} = \{a^{\dagger}(\theta), a^{\dagger}(\theta')\} = 0. \qquad (3.24)$$

The Hilbert space of this model is the Fock space built over the vacuum $|0\rangle$ using the algebra (3.24). The action of the operators a and $a^{\dagger}$ is

$$a(\theta)|0\rangle = \langle 0|a^{\dagger}(\theta) \equiv 0 \qquad a^{\dagger}(\theta)|0\rangle \equiv |\theta\rangle \qquad \langle 0|a(\theta) \equiv \langle \theta|, \qquad (3.25)$$

and a general vector is defined by multiple applications of (3.25)

$$|\theta_1, \theta_2, ..., \theta_n\rangle = a^{\dagger}(\theta_1)a^{\dagger}(\theta_2)...a^{\dagger}(\theta_n)|0\rangle. \qquad (3.26)$$

3.3 Form Factors

The perturbative approach is characterized by slow convergence, and the need for renormalization, so that we aim for a more efficient way of computing two-point functions. A solution comes from non-perturbative approaches based on the study of the S-matrix. An exact evaluation of S-matrix elements is often accessible in the presence of integrability. Factorizability and eq. (3.8) makes the study of the two-body S-matrix of central importance, so that we will very shortly summarize its properties.

The scattering process $p_1 + p_2 \rightarrow p_3 + p_4$ can be described by the *Mandelstam variables*[1] $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$ and $u = (p_1 - p_4)^2$. In $(1+1)$-dimensions however, only two of them are independent, say s and t. Calling $\theta = \theta_1 - \theta_2$, and m_1 and m_2 the masses of the incoming particle we can express $s(\theta) = m_1^2 +$

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[1]Mandelstam [1958].

$m_2^2 + 2m_1 m_2 \cosh\theta$. The fact that the S-matrix depends on the difference of the incoming repidities is a consequence of Lorentz invariance. There are some other physical requirements that it has to fulfil, first being unitarity, that is $SS^\dagger = \mathbf{I}$. Expressed in matrix element this becomes

$$\sum_{b_1, b_2} S_{a_1 a_2}^{b_1 b_2}(\theta) \left[S_{b_1 b_2}^{a_3 a_4}(\theta) \right]^* = \delta_{a_1}^{a_3} \delta_{a_2}^{a_4}. \tag{3.27}$$

To describe all the other properties we have to study the analytic structure of the S-matrix. The threshold for particle production is $s \geq (m_1 + m_2)^2$, such that from that value on we will have a continuum of branch points of the S-matrix. The same argument holds for t, from which we can extract a continuum of poles for $s \leq (m_1 - m_2)^2$. For values of s in between we could in general have bound states, but this case will not be taken under consideration here. If we analytically continue the variable s to the complex plane, we can interpret these two sets of poles as branch cuts. Allowing for complex rapidities the two branch points $s = (m_1 \pm m_2)^2$ are mapped into $\theta = 0$ and $\theta = i\pi$. The map employed to shift between s and θ is clearly multivalued, and we choose to work in the first sheet $\text{Im}(\theta) \in (0, \pi)$, which we call *physical sheet*.

To obtain the physical values of the scattering matrix we have to "lift" it from the real axis, that is $S_{\text{phys}}(s) = \lim_{\varepsilon \to 0} S(s + i\varepsilon)$. Another property is Hermitian analyticity, which reads $\left[S_{a_1 a_2}^{a_3 a_4}(\theta) \right]^* = S_{a_4 a_3}^{a_2 a_1}(-\theta^*)$. This condition combined with parity invariance $S_{a_1 a_2}^{a_3 a_4}(\theta) = S_{a_2 a_1}^{a_4 a_3}(\theta)$ gives the real analyticity condition

$$\left[S_{a_1 a_2}^{a_3 a_4}(\theta) \right]^* = S_{a_3 a_4}^{a_1 a_2}(-\theta^*). \tag{3.28}$$

The last property we want to discuss is crossing symmetry, that is that a scattering amplitude with an in particle with given momentum must equal the amplitude with an outgoing anti-particle of opposite momentum. This translates in our case as

$$S_{a_1 a_2}^{a_3 a_4}(i\pi - \theta) = S_{\bar{a}_4 a_1}^{\bar{a}_2 a_3}(\theta). \tag{3.29}$$

This set of analytic properties can be encoded into the Zamolodchikov-Faddeev algebra[1]. For the expression of this algebra we have to postulate the existence

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[1]Faddeev [1980]; Zamolodchikov & Zamolodchikov [1979].

of a set of creation and annihilation operators such as the ones in eq. (3.2). As we have seen with that definition the nature of an in or out state is determined by the order of its rapidities. In particular we can think of a n-particle in state as the action on the vacuum of $A^{\dagger}_{a_1}(\theta_1)A^{\dagger}_{a_2}(\theta_2)...A^{\dagger}_{a_n}(\theta_n)$, with $\theta_1 \geq \theta_2 > ... \geq \theta_n$. Any other state can be obtained by exchanging rapidities and multiplying by the right scattering amplitude. These concepts are summarized in the following set of equations

$$A_{a_1}(\theta_1)A_{a_2}(\theta_2) = \sum_{a_3,a_4} S^{a_3 a_4}_{a_1 a_2}(\theta_{12}) A_{a_3}(\theta_2) A_{a_4}(\theta_1) \qquad (3.30)$$

$$A^{\dagger}_{a_1}(\theta_1)A^{\dagger}_{a_2}(\theta_2) = \sum_{a_3,a_4} S^{a_3 a_4}_{a_1 a_2}(\theta_{12}) A^{\dagger}_{a_3}(\theta_2) A^{\dagger}_{a_4}(\theta_1) \qquad (3.31)$$

$$A_{a_1}(\theta_1)A^{\dagger}_{a_2}(\theta_2) = 2\pi\delta_{a_1 a_2}\delta(\theta_{12}) + \sum_{a_3,a_4} S^{a_3 a_4}_{a_1 a_2}(\theta_{21}) A^{\dagger}_{a_3}(\theta_2) A_{a_4}(\theta_1), \qquad (3.32)$$

where we assumed the notation $\theta_{ij} = \theta_i - \theta_j$.

3.3.1   The form factor program

Form factors are expectation values of local operators between an n-particle in state and the vacuum

$$F^{\mathscr{O}|a_1 a_2...a_n}_n(\theta_1, \theta_2, ..., \theta_n) = \langle 0|\mathscr{O}(0)|\theta_1 \theta_2...\theta_n\rangle_{a_1 a_2...a_n}, \qquad (3.33)$$

where a_is label all the discrete quantum numbers of the i-th particle. Their general properties was first studied by Karowski & Weisz [1978]; Weisz [1977], and they follow directly from properties of the S-matrix.

The usual way of proceeding in finding an analytic expression of form-factors is exploiting eqs. (3.29) and (3.31) and the branched structure of the rapidities sheet to get *Watson's equations* Watson [1954]. These are

$$F^{\mathscr{O}|a_1...,a_i a_{i+1}...a_n}_n(\theta_1, ..., \theta_i, \theta_{i+1}, ..., \theta_n) = \sum_{a'_i, a'_{i+1}} S^{a'_i a'_{i+1}}_{a_i a_{i+1}}(\theta_{i i+1}) \times$$
$$F^{\mathscr{O}|a_1,...,a'_{i+1},a'_i,...,a_n}_n(\theta_1, ..., \theta_{i+1}, \theta_i, ..., \theta_n) \quad (3.34)$$

$$F^{\mathscr{O}|a_1 a_2...a_n}_n(\theta_1 + 2i\pi, \theta_2, ..., \theta_n) = F^{\mathscr{O}|a_2...a_n a_1}_n(\theta_2, ..., \theta_n, \theta_1), \qquad (3.35)$$

and have a much more compact expression in the two particle case

$$F_2^{\mathcal{O}|a_1 a_2}(\theta_{12}) = \sum_{a_i', a_{i+1}'} S_{a_i a_{i+1}}^{a_i' a_{i+1}'}(\theta_{12}) F_2^{\mathcal{O}|a_1 a_2}(-\theta_{12}) \qquad (3.36)$$

$$F_2^{\mathcal{O}|a_1 a_2}(i\pi - \theta_{12}) = F_2^{\mathcal{O}|a_1 a_2}(i\pi + \theta_{12}). \qquad (3.37)$$

For diagonal theories, that is in absence of back-scattering, the most general solution can be written as

$$F_n^{\mathcal{O}|a_1 a_2 ... a_n}(\theta_1, \theta_2, ..., \theta_n) = Q_n^{\mathcal{O}|a_1 a_2, ... a_n}(\theta_1, \theta_2, ..., \theta_n) \prod_{i<j} \frac{F_{\min}^{a_i a_j}(\theta_{ij})}{P^{a_i a_j}(\theta_{ij})}, \qquad (3.38)$$

where $Q_n^{\mathcal{O}}$ is a polynomial, characteristic of the operator under consideration, P are monomials which capture the pole structure, and $F_{\min}$ are the *minimal form factors*. These last quantities are analytic functions with no zeros in the physical strip, they are uniquely fixed by requiring that they have the weakest possible divergence for $\theta \to \pm\infty$, and they satisfy the identities

$$F_{\min}^{a_i a_j}(\theta_{ij}) = S^{a_i a_j}(\theta_{ij}) F_{\min}^{a_j a_i}(-\theta_{ij}) = F_{\min}^{a_j a_i}(2i\pi - \theta_{ij}), \qquad (3.39)$$

and they are equivalent to eqs. (3.36) and (3.37) in the diagonal case. If we are able to express the two particle S-matrix with the following integral representation

$$S^{a_i a_j}(\theta_{ij}) = e^{\int_0^\infty \frac{dt}{t} f^{a_i a_j}(t) \sinh \frac{t\theta_{ij}}{i\pi}}, \qquad (3.40)$$

for some $f^{a_i a_j}(t)$, then we can express the minimal form factor as Berg *et al.* [1979]; Watson [1954]; Weisz [1977]

$$F_{\min}^{a_i a_j}(\theta_{ij}) = \mathcal{N}_{a_i a_j} e^{\int_0^\infty \frac{dt}{t} f^{a_i a_j}(t) \frac{\sin^2\left[\frac{t(i\pi - t)}{2\pi}\right]}{\sinh t}}. \qquad (3.41)$$

From Watson's equations and eq. (3.39) it follows that $Q_n^{\mathcal{O}}$ is symmetric under the switch of two rapidities. It was found in Delfino & Mussardo [1995] that form factors depend exponentially on the difference of rapities. It is then particularly convenient to express $Q_n^{\mathcal{O}}$ in terms of $x_i = e^{\theta_i}$, so that it becomes a symmetric polynomial of these quantities. Moreover we can get some information on these

polynomials by studying the asymptotic behaviour of form factors. First they are Lorentz invariant, and we know that a boost $B(\lambda)$ of rapidity λ acts on form factors as

$$B(\lambda)F_n^{\mathcal{O}|a_1a_2...a_n}(\theta_1,\theta_2,...,\theta_n)B^{-1}(\lambda) = F_n^{\mathcal{O}|a_1a_2...a_n}(\theta_1+\lambda,\theta_2+\lambda,...,\theta_n+\lambda), \quad (3.42)$$

so that the degree of $Q_n^{\mathcal{O}}$ must be $n(n-1)/2$, in order to equal that of the denominator of eq. (3.38). Furthermore they must diverge at most as $e^{\Delta_{\mathcal{O}}\theta_i}$ for any $\theta_i \to \infty$, where $\Delta_{\mathcal{O}}$ is the weight of the CFT counterpart of the operator under consideration, Delfino & Mussardo [1995].

To have a better understanding of $Q_n^{\mathcal{O}}$ and the monomials $P^{a_ia_j}$ in eq. (3.38) we need to analyze the pole structure of n-particle form factors. As we are considering theories with no bound states the only source of poles is given by crossing conditions. These are called *kinematic poles*, and they are related to the one particle sub-channel in the three-body scattering. They lead to the residue equation

$$\text{Res}\left[F_{n+2}^{\mathcal{O}|\bar{a}aa_1,...,a_n}(\bar{\theta}_0+i\pi,\theta_0,\theta_1,...,\theta_n)\right]_{\bar{\theta}_0=\theta_0} = i\left(1 - \prod_{k=1}^{n}\sum_{a'a'_k}S_{aa_k}^{a'_0a'_k}(\theta_{0k})\right) \times$$
$$F_n^{\mathcal{O}|a_1,...,a_n}(\theta_1,...,\theta_n), \quad (3.43)$$

which connects different particle number form factors. This equation is very useful for determining higher particle form factors recursively.

Once we managed to determine the form factors of the operators we are interested in, we can finally define a non-perturbative expansion of two point functions. This is achieved by inserting the resolution of the identity

$$\mathbf{I} = \sum_{n=0}^{\infty}\sum_{a_1,...,a_n}\int_{\theta_1>\theta_2>...>\theta_n}\frac{d\theta_1...d\theta_n}{(2\pi)^n}|\theta_1,...,\theta_n\rangle_{a_1...a_n}\ _{a_1...a_n}\langle\theta_1,...,\theta_n|, \quad (3.44)$$

into the two point function $\langle \mathcal{O}(\mathscr{T}, x)\mathcal{O}(0)\rangle^{1}$, such that it can be then expressed as

$$\langle \mathcal{O}(\mathscr{T}, x)\mathcal{O}(0)\rangle = \sum_{n=0}^{\infty} \sum_{a_1,...,a_n} \int \frac{d\theta_1...d\theta_n}{n!(2\pi)^n} \langle 0| \mathcal{O}(\mathscr{T}, x)|\theta_1,...,\theta_n\rangle_{a_1...a_n} \times$$
$$\left(\langle 0| \mathcal{O}(0)|\theta_1,...,\theta_n\rangle_{a_1...a_n}\right)^{*}. \tag{3.45}$$

We consider $\mathcal{O}$ to be a spinless operator, and we act with a translation and a boost on the first operator, to rewrite eq. (3.46) as

$$\langle \mathcal{O}(\mathscr{T}, x)\mathcal{O}(0)\rangle = \sum_{n=0}^{\infty} \sum_{a_1,...,a_n} \int \frac{\prod_{i=1}^{n} d\theta_i e^{-rm_i \cosh\theta_i}}{n!(2\pi)^n} \left| F_n^{\mathcal{O}|a_1 a_2...a_n}(\theta_1, \theta_2,...,\theta_n) \right|^2,$$
$$\tag{3.46}$$

where $r = \sqrt{\mathscr{T}^2 + x^2}$. In this way a full knowledge of the form factors of an operator means having access to every order coefficient of the large-distance expansion in the lhs of eq. (3.46), and then in principle determining the two point function under examination.

3.4 Twist field form factors

We can perform the form factor program for any operator of a replica theory with few adaptions. First of all we will have an additional quantum number defining the state of particles, that is the copy they belong to. We will label by $|\theta_i\rangle_{\mu_i}$, with $\mu_i = (a_i, n_i)$ the state created by the operator $A_{a_i}^{\dagger}(\theta_i)$ acting on the the vacuum of the copy n_i[2]. The S-matrix of the n-copy model can be obtained from the single copy one with the same philosophy, and also eqs. (3.30)–(3.32). This extension is not completely trivial though, as there is no scattering between excitations on different copies, so that the S-matrix will assume the form

$$S_{\mu_1\mu_2}^{\mu_3\mu_4}(\theta_{12}) = \left[S_{a_1 a_2}^{a_3 a_4}(\theta_{12})\right]^{\delta_{n_1 n_2}} \delta_{n_3 n_1} \delta_{n_4 n_2} \tag{3.47}$$

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[1] here we are taking one of the two operators in the origin with no loss of generality, as this two point function, being invariant under translation, depends only on the distance between the two operators.

[2] here we are assuming that the global vacuum of the n-copy theory can be expressed as direct product of the vacua of single copies.

and the Zamolodchikov-Fadeev algebra must be modified to

$$A_{\mu_1}(\theta_1)A_{\mu_2}(\theta_2) = \sum_{\mu_3,\mu_4} S^{\mu_3\mu_4}_{\mu_1\mu_2}(\theta_{12})A_{\mu_3}(\theta_2)A_{\mu_4}(\theta_1) \tag{3.48}$$

$$A^\dagger_{\mu_1}(\theta_1)A^\dagger_{\mu_2}(\theta_2) = \sum_{\mu_3,\mu_4} S^{\mu_3\mu_4}_{\mu_1\mu_2}(\theta_{12})A^\dagger_{\mu_3}(\theta_2)A^\dagger_{\mu_4}(\theta_1) \tag{3.49}$$

$$A_{\mu_1}(\theta_1)A^\dagger_{\mu_2}(\theta_2) = 2\pi\delta_{\mu_1\mu_2}\delta(\theta_{12}) + \sum_{\mu_3,\mu_4} S^{\mu_3\mu_4}_{\mu_1\mu_2}(\theta_{21})A^\dagger_{\mu_3}(\theta_2)A_{\mu_4}(\theta_1). \tag{3.50}$$

We are mainly interested in form factors of the twist field, so that we want to focus onto its exchange relations with other operators of the theory. In particular we want to understand how it acts on A and $A^\dagger$. This can be achieved considering the boundary conditions which define this field in eqs. (2.58) and (2.60). We focus for simplicity on a scalar field $\phi_i(x)$, such that the only quantum number i is the number of the copy it belongs to. As the action of the twist field on creation and annihilation operators is the same we focus directly on its action on the field. In this case we find the equal time exchange relations

$$\begin{aligned}
\phi_i(y)\mathcal{T}(x) &= \mathcal{T}(x)\phi_{i+1}(y) & x^1 > y^1, \\
\phi_i(y)\mathcal{T}(x) &= \mathcal{T}(x)\phi_i(y) & x^1 < y^1,
\end{aligned} \tag{3.51}$$

and

$$\begin{aligned}
\phi_i(y)\tilde{\mathcal{T}}(x) &= \tilde{\mathcal{T}}(x)\phi_{i-1}(y) & x^1 > y^1, \\
\phi_i(y)\tilde{\mathcal{T}}(x) &= \tilde{\mathcal{T}}(x)\phi_i(y) & x^1 < y^1.
\end{aligned} \tag{3.52}$$

To allow the identification between these conditions, and eqs. (2.58) and (2.60) we take products of operators in eqs. (3.51) and (3.52) to be time-ordered. These relations define the action of the twist field at the operator level, and they allow for a comparison between matrix elements of $\mathcal{T}$ and $\tilde{\mathcal{T}}$, giving as a result the identification $\tilde{\mathcal{T}} = \mathcal{T}^\dagger$.

Taking into account eqs. (3.48)–(3.52) a modified version of eqs. (3.34), (3.35)

and (3.43) was derived in Cardy *et al.* [2008][1]

$$F_k^{\mathcal{T}|\dots\mu_i\mu_{i+1}\dots}(\dots,\theta_i,\theta_{i+1},\dots) = S_{\mu_i\mu_{i+1}}(\theta_{i\,i+1})\times$$
$$F_k^{\mathcal{T}|\dots\mu_{i+1}\mu_i\dots}(\dots,\theta_{i+1},\theta_i,\dots), \quad (3.53)$$

$$F_k^{\mathcal{T}|\mu_1\mu_2\dots\mu_k}(\theta_1+2\pi i,\dots,\theta_k) = F_k^{\mathcal{T}|\mu_2\dots\mu_n\hat{\mu}_1}(\theta_2,\dots,\theta_k,\theta_1), \quad (3.54)$$

$$\text{Res}\left[F_{k+2}^{\mathcal{T}|\bar{\mu}\mu\mu_1\dots\mu_k}(\bar{\theta}_0+i\pi,\theta_0,\theta_1\dots,\theta_k)\right]_{\bar{\theta}_0=\theta_0} = i\,F_k^{\mathcal{T}|\mu_1\dots\mu_k}(\theta_1,\dots,\theta_k), \quad (3.55)$$

$$\text{Res}\left[F_{k+2}^{\mathcal{T}|\bar{\mu}\hat{\mu}\mu_1\dots\mu_k}(\bar{\theta}_0+i\pi,\theta_0,\theta_1\dots,\theta_k)\right]_{\bar{\theta}_0=\theta_0} = -i\prod_{i=1}^{k} S_{\hat{\mu}\mu_i}(\theta_{0i})F_k^{\mathcal{T}|\mu_1\dots\mu_k}(\theta_1,\dots,\theta_k).$$
$$(3.56)$$

Equations (3.53) and (3.54) are Watson's equations for the twist field. The main difference from eqs. (3.34) and (3.35) is the introduction in the second equation of the symbol $\hat{\mu} = (a, n+1)$, by means of which we can see that a shift of $2\pi i$ in the rapidity of a particle makes it "jump" to the copy above. Another difference is that the special nature of the twist field causes the kinematic pole equation to split into eqs. (3.55) and (3.56), depending on whether $\bar{\mu}$ is in the same copy as μ, or there is a previous exchange with the twist field.

Now we focus our attention on the two-particle form factor. To ease this analysis we are going to consider a scalar single-particle theory, for which this quantity can be written as $F_2^{ij}(\theta_{ij})$, where i and j label just the copy number, as there is only one particle type. From eq. (3.54) we understood that a shift of $2\pi i$ of the rapidity of a particle shifts it to the copy above, and relativistic invariance tells us that the same shift applied to both particles leaves the form factor invariant. From these considerations we deduce that F_2^{ij} must depend only on the difference between the two copy numbers, such that

$$F_2^{\mathcal{T}|k+i\ k+j}(\theta_{ij}) = F_2^{\mathcal{T}|ij}(\theta_{ij}), \qquad \forall k \in \mathbb{Z}. \quad (3.57)$$

We will then focus on $F_2^{k1}(\theta_{k1})$, keeping in mind that we can extract the case of general copy numbers by applying eq. (3.57). Moreover acting repeatedly with

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[1]in the following equations we are simplifying the notation further writing them in the case of diagonal theories. In this case any scattering process does not change quantum numbers, so that an S-matrix element can be defined only by the set of quantum numbers of the incoming particles.

eq. (3.54) we can reduce a general two particle form factor to one with two particles on the same copy, that is

$$F_2^{\mathcal{T}|k1}(\theta_{k1}) = F_2^{\mathcal{T}|11}(\theta + 2\pi i(k-1)). \tag{3.58}$$

As a consequence we can focus on the analysis of $F_2^{\mathcal{T}|11}(\theta)$, and study its analytic structure. This is an analytic function in the region $\text{Im}(\theta) \in [0, 2\pi in)$, which we call *extended physical sheet*. The only exceptions are the single poles given by eqs. (3.55) and (3.56), which for two particles become

$$\text{Res}\left[F_2^{\mathcal{T}|\bar{1}1}(\theta + i\pi)\right]_{\theta=0} = i\langle\mathcal{T}\rangle, \tag{3.59}$$

$$\text{Res}\left[F_2^{\mathcal{T}|\bar{1}2}(\theta + i\pi)\right]_{\theta=0} = \text{Res}\left[F_2^{\mathcal{T}|\bar{1}1}(\theta + (2n-1)i\pi)\right]_{\theta=0} = -i\langle\mathcal{T}\rangle, \tag{3.60}$$

where we identified for obvious reasons $F_0^{\mathcal{T}} = \langle\mathcal{T}\rangle$. Equation (3.58) tells us that all form factors can be obtained from the same copy (and no matter which) by just shifting the rapidity, so that one can focus on $F_2^{\mathcal{T}|11}(\theta)$. In this case eqs. (3.59) and (3.60) tell us that there are two poles at $\theta = i\pi$, and $\theta = (2n-1)i\pi$, and give us their residues.

As we did for general form factors in the previous section we focus on the minimal form factor of the n-copy theory. This can be extracted from the one-copy theory considering that

$$F_{\min}^{\mathcal{T}|11}(\theta) = S(\theta)F_{\min}^{\mathcal{T}|11}(-\theta) = F_{\min}^{\mathcal{T}|11}(-\theta + 2\pi ni), \tag{3.61}$$

Using eqs. (3.40) and (3.41) in Cardy *et al.* [2008] the authors managed to extract the minimal form factor for diagonal theories

$$F_{\min}^{\mathcal{T}|11}(\theta) = \mathcal{N} e^{\int_0^\infty \frac{dt}{t} f(t) \frac{\sin^2\left[\frac{it}{2}\left(n + \frac{i\theta}{\pi}\right)\right]}{\sinh(nt)}}. \tag{3.62}$$

In the same work the form of the general copy two particle form factor was obtained for the first time with angular quantization techniques Brazhnikov &

Lukyanov [1998]; Lukyanov [1995], and is

$$F_2^{\mathscr{T}|jk}(\theta) = \frac{\langle\mathscr{T}\rangle \sin\left(\frac{\pi}{n}\right)}{2n \sinh\left(\frac{i\pi(2(j-k)-1)+\theta}{2n}\right) \sinh\left(\frac{i\pi(2(k-j)-1)-\theta}{2n}\right)} \frac{F_{\min}^{\mathscr{T}|jk}(\theta)}{F_{\min}^{\mathscr{T}|jk}(i\pi)}. \tag{3.63}$$

Before considering higher particle form factors we focus on $\tilde{\mathscr{T}}$. We understand from eqs. (3.51) and (3.52) that its form factors can be related to those of $\mathscr{T}$ by the transformation $i \to n-i$ for each particle i, that means

$$F_2^{\mathscr{T}|ij}(\theta) = F_2^{\tilde{\mathscr{T}}|(n-i)(n-j)}(\theta). \tag{3.64}$$

For higher particle form factors we can extend the considerations that brought us to eq. (3.38) to the n-copy case, and make the ansatz

$$F_k^{\mathscr{T}|11\cdots1}(\theta_1,\ldots,\theta_k) = H_k^{\mathscr{T}|1\ldots1} Q_k^{\mathscr{T}|1\ldots1}(x_1,\ldots,x_k) \prod_{i<j} \frac{F_2^{\mathscr{T}|11}(\theta_{ij})}{\langle\mathscr{T}\rangle}. \tag{3.65}$$

Here we are dimensionlessizing the two particle form factor for convenience, hence $H_k^{\mathscr{T}|1\ldots1}$ will be a dimensional constant proportional to $\langle\mathscr{T}\rangle$. For the polynomials $Q_k^{\mathscr{T}|1\ldots1}(x_1,\ldots,x_k)$ the same properties of those in eq. (3.38) hold, with the identification of $x_i = e^{\theta_i/n}$.

The general copies form factor can be extracted from eq. (3.65) by multiple applications of eq. (3.54), finding

$$F_k^{\mathscr{T}|\mu_1\cdots\mu_k}(\theta_1,\ldots,\theta_k) = F_k^{\mathscr{T}|11\cdots1}(\theta_1 + 2\pi i(\mu_1-1), \theta_2 + 2\pi i(\mu_2-1), \ldots, \theta_k + 2\pi i(\mu_k-1)), \tag{3.66}$$

with the ordering $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_k$.

We conclude this section by applying all definitions to an example, the Ising field theory defined in section 3.2.1. Being a free Fermion theory Zamolodchikov-Fadeev operators will simply be creators and annihilators of Fermion modes, and the associated S-matrix can be read directly from eq. (3.24), and is $S(\theta) = -1$. The minimal form factor can be easily obtained by plugging the S-matrix into

eq. (3.39), leading to $F_{\min}(\theta) = -i \sinh \frac{\theta}{2}$. From this it follows that

$$F_{\min}^{\mathscr{T}|11}(\theta) = -i \sinh \frac{\theta}{2n}, \tag{3.67}$$

such that

$$F_2^{\mathscr{T}|11}(\theta) = \frac{-i\langle\mathscr{T}\rangle \cos\left(\frac{\pi}{2n}\right)}{n \sinh\left(\frac{i\pi+\theta}{2n}\right)\sinh\left(\frac{i\pi-\theta}{2n}\right)} \sinh\left(\frac{\theta}{2n}\right), \tag{3.68}$$

is a nice and neat expression. Higher particle form factors were found for the first time in Castro-Alvaredo & Doyon [2009a], where they were evaluated simply applying Wick's theorem. Due to the $\mathbb{Z}_2$ symmetry of the Ising model only even particle form factors are nonzero, as the twist field is by construction invariant under any symmetry of the theory. It is easy to show that they can be expressed as Pfaffians[1]

$$F_k^{\mathscr{T}|11\ldots1}(\theta_1,\ldots,\theta_k) = \langle\mathscr{T}\rangle \mathrm{Pf}(K), \tag{3.69}$$

where K is a skew-symmetric $k \times k$ matrix with entries $K_{ij} = \frac{F_2^{\mathscr{T}|11}(\theta_{ij})}{\langle\mathscr{T}\rangle}$.

The next few sections will be devoted to two more involved models for which we carried out the form factor program for the twist field. The aim of this study is to prove the effectiveness of this program for non-free theories. We will compute higher particle form factors, so that we need a method to check the correctness of our results. As we will show in section 3.5 the Δ-sum rule Delfino *et al.* [1996] is fit to the task. This rule though gives information on the form factor expansion up to a given order, and makes it hard to check higher particle form factors. In fact usually the contribution of higher terms is so small compared to the leading term, that it can be confused with the typical errors of any numerical integration. To overcome this problem we chose to study two rather special theories, for which the contribution of higher particle form factors is very significant.

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[1]the Pfaffian of a $2n \times 2n$ skew-symmetric matrix A is defined as $\mathrm{Pf}^2(A) = \mathrm{Det}(A)$, after the observation that the determinant of a skew-symmetric matrix can always be written as the square of a polynomial of degree n of its entries.

3.4.1 The roaming trajectory model

The first theory we want to investigate is the *roaming trajectories (RT) model*, defined by Zamolodchikov [2006]. This is a model with a single particle spectrum and no bound states which is closely related to the sinh-Gordon model. The model is characterized by the two-particle S-matrix,

$$S(\theta) = \tanh \frac{1}{2}\left(\theta - \theta_0 - \frac{i\pi}{2}\right) \tanh \frac{1}{2}\left(\theta + \theta_0 - \frac{i\pi}{2}\right), \quad \theta_0 \in \mathbb{R}. \tag{3.70}$$

On the other hand, the sinh-Gordon S-matrix, as found by Arinshtein *et al.* [1979]; Mikhailov *et al.* [1981] is given by

$$S(\theta) = \frac{\tanh \frac{1}{2}\left(\theta - \frac{i\pi B}{2}\right)}{\tanh \frac{1}{2}\left(\theta + \frac{i\pi B}{2}\right)}, \quad B \in [0,2]. \tag{3.71}$$

It is easy to see that the S-matrix (3.70) can be obtained from (3.71) by the replacement

$$B \to 1 - \frac{2i\theta_0}{\pi}. \tag{3.72}$$

This relationship implies in particular that computing the form factors of the sinh-Gordon model and setting B to the value (3.72) gives the form factors of the RT-model.

The model's name emerged in the computation of the effective central charge $c_{\text{eff}}(r)$ within the thermodynamic Bethe ansatz approach Klassen & Melzer [1991]; Zamolodchikov [1990] carried out by Zamolodchikov [2006]. For massive QFTs it is expected that the function $c_{\text{eff}}(r)$ "flows" from the value zero in the infrared (large r) to a finite value in the ultraviolet (small r). For many theories, including the sinh-Gordon model, the constant value reached as $r \to 0$ is the central charge of the underlying conformal field theory associated to the model. In this case, that theory is the free massless boson, a conformal field theory with central charge $c = 1$. Therefore, in the sinh-Gordon model, the function $c_{\text{eff}}(r)$ flows from the value zero to the value 1 as r decreases.

When the same function $c_{\text{eff}}(r)$ is computed for the RT-model it shows a very different behaviour. It still flows from the value 0 to the value 1, but it does so by "visiting" infinitely many intermediate values of c giving rise to a staircase

(or roaming) pattern. The values of c that are visited correspond exactly to the central charges of the unitary minimal models of conformal field theory

$$c_p = 1 - \frac{6}{p(p+1)}, \quad \text{with} \quad p = 3, 4, 5 \ldots \tag{3.73}$$

Another observation made in Zamolodchikov [2006] is that the size of the intermediate plateaux that the function $c_{\text{eff}}(r)$ develops at the values (3.73) is determined by the value of θ_0. For $\theta_0 = 0$ there is a single plateaux at $c = 1$, thus the usual sinh-Gordon behaviour is recovered, whereas the plateaux at (3.73) become more prominent as θ_0 is increased. In the limit $\theta_0 \to \infty$ a single plateaux at $c = \frac{1}{2}$ remains which reflects the fact that the S-matrix (3.70) becomes -1 in this limit, hence the model reduces to the Ising field theory. This interesting limit behaviour was studied by Ahn et $al.$ [1993] within the form factor approach.

The two-particle minimal form factor of the sinh-Gordon model

$$F_{\min}^{\mathcal{T}|11}(\theta) = \exp\left[-2 \int_0^\infty \frac{dt \sinh \frac{tB}{4} \sinh \frac{t(2-B)}{4}}{t \sinh(nt) \cosh \frac{t}{2}} \cosh t\left(n + \frac{i\theta}{\pi}\right)\right], \tag{3.74}$$

was first obtained in Cardy et $al.$ [2008] and can be easily rewritten as an infinite product of ratios of Gamma functions. The explicit expression can be also found in Cardy et $al.$ [2008].

Having eq. (3.65) in mind we make the following ansatz,

$$F_k^{\mathcal{T}}(x_1, \ldots, x_k) = H_k Q_k(x_1, \ldots, x_k) \prod_{i<j}^k \frac{F_{\min}^{\mathcal{T}|11}(\frac{x_i}{x_j})}{(x_i - \alpha x_j)(x_j - \alpha x_i)}, \tag{3.75}$$

where we have introduced the new variables $x_i = e^{\frac{\theta_i}{n}}$ and $\alpha = e^{\frac{i\pi}{n}}$ so that, for example

$$F_{\min}^{\mathcal{T}|11}(\theta_i - \theta_j) \equiv F_{\min}^{\mathcal{T}|11}(\frac{x_i}{x_j}), \tag{3.76}$$

A similar ansatz was already used in Niedermaier [1998] in a different context. We use the simplified notation $F_k^{\mathcal{T}}(x_1, \ldots, x_k)$ to represent the k-particle form

factor of particles all of which live in the same copy of the model. As we already introduced in the previous section $Q_k(x_1,...,x_k)$ are symmetric in all variables and have no poles on the physical sheet, while H_k are constants. We can fix the normalization by means of the pole eq. (3.59), that is $H_0 = \langle \mathcal{T} \rangle$ and $Q_0 = 1$.

Once the ansatz (3.75) has been made it remains to identify the functions $Q_k(x_1,...,x_k)$ and the constants H_k. In the sinh-Gordon model symmetry considerations imply that only even particle form factors are non-vanishing, so that our first new results would correspond to the $k = 4$ case and k will always be an even number. We therefore turn to solving equation (3.55), which we can now rewrite as

$$\lim_{\bar{\theta}_0 \to \theta_0} (\bar{\theta}_0 - \theta_0) F_{k+2}(\alpha x_0, x_0, x_1, \ldots, x_k) = i F_k(x_1, \ldots, x_k), \tag{3.77}$$

where $x_0 = e^{\frac{\theta_0}{n}}$.

In order to turn the equation (3.77) into an equation for the functions $Q_k(x_1, \ldots, x_k)$ and the constants H_k the following identity will be needed,

$$F_{\min}^{\mathcal{T}|11}(\frac{\alpha x_0}{x_i}) F_{\min}^{\mathcal{T}|11}(\frac{x_0}{x_i}) = \frac{(x_0 - x_i)(\alpha x_0 - x_i)}{(\alpha \beta^{-1} x_0 - x_i)(\beta x_0 - x_i)}, \tag{3.78}$$

where $\beta = e^{\frac{i\pi B}{2n}}$ and B is the coupling constant that appears in the sinh-Gordon S-matrix (3.71). This identity can be easily derived from the Gamma function representation of the minimal form factor of Cardy *et al.* [2008]. Plugging eq. (3.75) into eq. (3.77) and simplifying we obtain

$$H_{k+2} = \frac{2 \sin \frac{\pi}{n} \alpha^{k+2}}{n F_{\min}^{\mathcal{T}|11}(i\pi)} H_k \qquad \text{and} \qquad Q_{k+2}(\alpha x_0, x_0, x_1, \ldots, x_k) = x_0^2 P_k Q_k(x_1, \ldots, x_k). \tag{3.79}$$

where

$$
\begin{aligned}
P_k &= \prod_{a,b,c,d=1}^{k} (x_a - \alpha^2 x_0)(x_0 - \alpha x_b)(x_c - \alpha \beta^{-1} x_0)(\beta x_0 - x_d) \tag{3.80} \\
&= (-\alpha)^k \sum_{a,b,c,d=0}^{k} (-\alpha^2 x_0)^{k-a}(-\alpha^{-1} x_0)^{k-b}(-\alpha \beta^{-1} x_0)^{k-c}(-\beta x_0)^{k-d} \sigma_a^{(k)} \sigma_b^{(k)} \sigma_c^{(k)} \sigma_d^{(k)},
\end{aligned}
$$

and $\sigma_i^{(k)}$ is the i-th elementary symmetric polynomial on k variables $x_1, \ldots, x_k$,

which can be defined by means of the generating function,

$$\sum_{i=0}^{k} x^{k-i} \sigma_i^{(k)} = \prod_{i=1}^{k} (x_i + x).$$
(3.81)

The equation for H_k can be easily solved to

$$H_k = \left(\frac{2 \sin \frac{\pi}{n} \alpha^2}{n F_{\min}^{\mathcal{T}|11}(i\pi)} \right)^{\frac{k}{2}} \alpha^{\frac{k}{2}(\frac{k}{2}-1)} \langle \mathcal{T} \rangle,$$
(3.82)

whereas equations for the polynomials $Q_k(x_1, \ldots, x_k)$ will need to be solved on a case by case basis. Unfortunately the solutions get very involved very quickly. There are three main reasons for this:

- The degree of the polynomial in the denominator of (3.75) is much higher than would be the case in the standard form factor programme. Since the twist field is spinless, the degree of such polynomial must equal the degree of the polynomial $Q_k(x_1, \ldots, x_k)$ and this means that its degree will be very high for relatively small values of k. As an example, for the RT-model we will see later that the degree of $Q_2(x_1, x_2)$ is just 2, but the degrees of $Q_4(x_1, x_2, x_3, x_4)$ and $Q_6(x_1, \ldots, x_6)$ are 12 and 30 respectively.

- The polynomial P_k is a very complicated function in terms of elementary symmetric polynomials, which again complicates the solution procedure and makes it very difficult to identify any patterns as k is increased.

- The reduction properties of the elementary symmetric polynomials $\sigma_i^{(k)}$ are much more involved for the twist field than in the usual form factor programme. In general,

$$\sigma_i^{(k+2)} = \sigma_i^{(k)} + (1+\alpha)x_0 \sigma_{i-1}^{(k)} + \alpha x_0^2 \sigma_{i-2}^{(k)},$$
(3.83)

where $\sigma_i^{(k+2)}$ is an elementary symmetric polynomial on the variables $\alpha x_0, x_0, x_1, \ldots, x_k$ and $\sigma_i^{(k)}, \sigma_{i-1}^{(k)}, \sigma_{i-2}^{(k)}$ are elementary symmetric polynomials in the variables $x_1, \ldots, x_k$. We will also adopt the conventions $\sigma_i^{(k)} = 0$ for $i < 0$ and $\sigma_0^{(k)} = 1$. The usual reduction properties are recovered for $n = 1$ or $\alpha = -1$.

The polynomial $Q_2(x_1, x_2)$ can be easily obtained by setting $k = 0$ in (3.79) which gives the equation,

$$Q_2(\alpha x_0, x_0) = x_0^2, \tag{3.84}$$

There are actually two combinations of elementary symmetric polynomials of two variables $\sigma_1^{(2)}$ and $\sigma_2^{(2)}$ that correctly reduce to the identity above. The most general solution is

$$Q_2(x_1, x_2) = \alpha^{-1}\sigma_2^{(2)} + \Omega_2 K_2(x_1, x_2), \tag{3.85}$$

with Ω_2 an arbitrary constant and

$$K_2(x_1, x_2) = \alpha^{-1}\sigma_2^{(2)} - \left(\frac{\sigma_1^{(2)}}{1+\alpha}\right)^2, \tag{3.86}$$

the kernel of equation (3.84), that is the most general order 2 polynomial on the variables x_1, x_2 which solves

$$Q_{k+2}(\alpha x_0, x_0, x_1, \dots, x_k) = 0, \tag{3.87}$$

with $k = 0$. Substituting (3.85) together with H_2 in (3.75) it is easy to see that (3.63) is only recovered for $\mu_1 = \mu_2 = c_1 = c_2 = 1$ if we choose $\Omega_2 = 0$. Hence we have fixed the constant above and can now go on to compute the four particle form factor.

Solving now for $Q_4(x_1, x_2, x_3, x_4)$ we find that the most general solution to (3.79) takes the form

$$Q_4(x_1, x_2, x_3, x_4) = \sigma_4\left[\sigma_2^4 + \gamma\sigma_2(\sigma_3^2 + \sigma_1^2\sigma_4) + \delta\sigma_1\sigma_2^2\sigma_3 + \eta\sigma_1^2\sigma_3^2 + \xi\sigma_2^2\sigma_4\right.$$
$$\left. + \lambda\sigma_1\sigma_3\sigma_4 + \rho\sigma_4^2\right] + \Omega_4 K_4(x_1, x_2, x_3, x_4), \tag{3.88}$$

where we have abbreviated $\sigma_i^{(4)} \equiv \sigma_i$. The constants $\gamma, \delta, \eta, \xi, \lambda$ and ρ are fixed functions of n, whose explicit form is given in appendix A. The function $K_4(x_1, x_2, x_3, x_4)$ is the most general order 12 polynomial on the variables x_1, x_2, x_3 and x_4 that solves the equation (3.87) and Ω_4 is an arbitrary constant. The function $K_4(x_1, x_2, x_3, x_4)$

has the form

$$K_4(x_1, x_2, x_3, x_4) = A\sigma_1^2\sigma_2^2\sigma_3^2 + B(\sigma_2^3\sigma_3^2 + \sigma_1^3\sigma_3^3 + \sigma_1^2\sigma_2^3\sigma_4) + C\sigma_1\sigma_2\sigma_3(\sigma_3^2 + \sigma_1^2\sigma_4)$$
$$+ D\sigma_2^4\sigma_4 + \sigma_1^4\sigma_4^2 + \sigma_3^4 + E\sigma_1\sigma_2^2\sigma_3\sigma_4 + F\sigma_1^2\sigma_3^2\sigma_4$$
$$+ G\sigma_2\sigma_4(\sigma_3^2 + \sigma_1^2\sigma_4) + H\sigma_2^2\sigma_4^2 + I\sigma_1\sigma_3\sigma_4^2 + J\sigma_4^3, \quad\quad (3.89)$$

where the constants are given in Appendix A.

We have also computed the most general polynomial $Q_6(x_1,\ldots,x_6)$ which solves (3.79) with $k = 4$. The solution is an order 30 polynomial on the variables $x_1,\ldots,x_6$ and too cumbersome to be reported here. For $\Omega_4 = 0$ (we will see below why this choice is sensible), $Q_6(x_1,\ldots,x_6)$ depends once more on a free parameter Ω_6, which as above acts as coefficient to the function $K_6(x_1,\ldots,x_6)$ which satisfies the same equation (3.87) above.

Therefore, a structure seems to emerge where the most general $2k$-particle form factor depends on k free parameters. A similar structure was found when studying the boundary form factors of specific fields in the A_2-affine Toda field theory Castro-Alvaredo [2008]; Oota [1996], although no physical interpretation for the result was provided there. A more thorough analysis of solutions to equations of the form (3.87) was carried out in Delfino & Niccoli [2006] for the case $\alpha = -1$ and the field $T\bar{T}$.

Finally we would like to argue that choosing $\Omega_4 = 0$ in (3.88) corresponds to the specific twist field we are interested in. The general solution (3.88) is a one-parameter family of solutions characterized by the choice of the constant Ω_4. Given the usual assumption that the space of fields in a local QFT is linear, we expect that the form factor of a linear combination of fields is a linear combination of form factors, that is, in general

$$F_k^{\mathcal{O}_1 + \Omega\mathcal{O}_2}(x_1,\ldots,x_k) = F_k^{\mathcal{O}_1}(x_1,\ldots,x_k) + \Omega F_k^{\mathcal{O}_2}(x_1,\ldots,x_k) \quad\quad (3.90)$$

and therefore the solution (3.88) must describe the form factors of a linear combination of local fields (as would the solution (3.85)). Since we are interested only in one very particular field, the twist field $\mathcal{T}$, we must find a suitable mech-

anism that allows us to select the particular value of Ω_4 corresponding to the four-particle form factor of the twist field.

An interesting way of identifying the form factors of the twist field is to use the form factor cluster decomposition property, which has been studied for various models in the past Castro-Alvaredo & Fring [2001a]; Koubek & Mussardo [1993]; Smirnov [1990]; Zamolodchikov [1991] and analysed from a more general point of view in Delfino *et al.* [1996]. It is a factorization property of form factors which, for the four particle case, can be expressed as

$$\lim_{\kappa \to \infty} F_4^{\mathscr{T}}(\kappa x_1, \kappa x_2, x_3, x_4) \propto F_2^{\mathscr{T}_1}(x_1, x_2) F_2^{\mathscr{T}_2}(x_3, x_4). \qquad (3.91)$$

In general, the fields $\mathscr{T}_1$ and $\mathscr{T}_2$ on the r.h.s. may not necessarily correspond to the same field as the form factor on the l.h.s. A notable example of this is the model studied in Castro-Alvaredo & Fring [2001a] and the form factors of the field $T\bar{T}$ studied in Delfino & Niccoli [2006]. In Delfino *et al.* [1996] it was argued that for theories without internal symmetries, the cluster decomposition would be a consequence of the decoupling of right- and left-moving modes in the conformal limit and would hold for any field whose counterpart in the underlying conformal field theory is a primary field.

Given that the twist field does certainly correspond to a primary field in the underlying conformal field theory we expect a factorization of the type (3.91). Imposing (3.91) is in fact sufficient to select a single value of Ω_4 in (3.88). Indeed, if we carry out the cluster limit in (3.91) for the general expression (3.88) and we call $\sigma_i = \sigma_i(x_1, x_2)$ and $\hat{\sigma}_i = \hat{\sigma}_i(x_3, x_4)$ we find that

$$\lim_{\kappa \to \infty} F_4(\kappa x_1, \kappa x_2, x_3, x_4) \quad \sim \quad \left[\sigma_2 \hat{\sigma}_2 + \Omega_4 (A\sigma_1^2 \hat{\sigma}_1^2 + B(\sigma_2 \hat{\sigma}_1^2 + \sigma_1^2 \hat{\sigma}_2) + D\sigma_2 \hat{\sigma}_2) \right]$$
$$\times \frac{F_{\min}^{\mathscr{T}|11}(\frac{x_1}{x_2}) F_{\min}^{\mathscr{T}|11}(\frac{x_3}{x_4})}{(x_1 - \alpha x_2)(x_2 - \alpha x_1)(x_3 - \alpha x_4)(x_4 - \alpha x_3)}. \qquad (3.92)$$

Clearly, this expression factorises if and only if $\Omega_4 = 0$. In that case, we recover exactly (3.91) with $\mathscr{T}_1 = \mathscr{T}_2 = \mathscr{T}$. We will therefore choose $\Omega_4 = 0$ as our twist field solution.

If we had chosen to use the cluster decomposition property to fix the constant Ω_2

in (3.85) we would have found

$$\lim_{\kappa \to \infty} F_2^{\mathscr{T}}(\kappa x_1, x_2) \propto \Omega_2, \tag{3.93}$$

so that our choice $\Omega_2 = 0$ guarantees that $\lim_{\kappa \to \infty} F_2^{\mathscr{T}}(\kappa x_1, x_2) \propto F_1^{\mathscr{T}} F_1^{\mathscr{T}} = 0$. In general, it appears from our two-, four- and six-particle form factor solutions that for every $a \in \mathbb{Z}^+$ there exists a field $\mathscr{K}_{2a}$ whose form factors solve eqs. (3.53) and (3.56), and have the interesting property that

$$F_0^{\mathscr{K}_{2a}} = F_2^{\mathscr{K}_{2a}|\mu_1\mu_2}(\theta_1, \theta_2) = \cdots = F_{2a-2}^{\mathscr{K}_{2a}|\mu_1\ldots\mu_{2a-2}}(\theta_1,\ldots,\theta_{2a-2}) = 0, \tag{3.94}$$

consequently $F_{2a}^{\mathscr{K}_{2a}|\mu_1\ldots\mu_{2a}}(\theta_1,\ldots,\theta_{2a})$ solves (3.87) for $k = 2a$, that is, it has no kinematic poles.

3.4.2 The $SU(3)_2$-homogenous sine-Gordon model

The second model we want to study is the $SU(3)_2$-Homogeneous sine-Gordon (HSG) model. The model is just one of the simplest representatives of a large class of theories named by Fernandez-Pousa *et al.* [1997a]. Its spectrum was studied by Fernandez-Pousa & Miramontes [1998]; Fernandez-Pousa *et al.* [1997a,b], and its S-matrix by Miramontes & Fernandez-Pousa [2000]. The form factor program was carried out by Castro-Alvaredo & Fring [2001a,b]; Castro-Alvaredo *et al.* [2000a]; Castro-Alvaredo & Fring [2001c], while its thermodynamic properties were studied by Castro-Alvaredo *et al.* [2000b, 2004]; Dorey & Miramontes [2004]. The HSG-models are very interesting theories, as they include a number of distinct features rarely found for integrable models: they posses both unstable particles and bound states in their spectrum, and their S-matrices are generally non-parity invariant[1]. In particular, the $SU(3)_2$-HSG model contains two particles, which we will label as $+$ and $-$. They are self-conjugated and interact with each other by means of the following S-matrix

$$S_{\pm\pm}(\theta) = -1, \quad \text{and} \quad S_{\pm\mp}(\theta) = \pm\tanh\frac{1}{2}\left(\theta \pm \sigma - \frac{i\pi}{2}\right). \tag{3.95}$$

---

[1] $S_{ab}(\theta) \neq S_{ba}(\theta)$ for $a \neq b$.

Thus particles of the same species interact with each other as free Fermions, whereas particles of different species interact by means of parity-breaking S-matrix which depends on a free parameter σ. These S-matrix amplitudes have a pole in the unphysical sheet (that is $\Im(\theta) \in (-\pi, 0)$), with real part given by $\pm\sigma$. Such type of poles are a signature of the presence of unstable particles in the spectrum.

The scattering picture is that particles $+$ and $-$ interact with each other by creating an unstable particle, whose mass $\tilde{m}$ and decay width $\tilde{\Gamma}$ depend on the parameter σ through Breit-Wigner's formula[1]. The latter states that the S-matrix must have a pole in the Mandelstam variable s at

$$s = \left(\tilde{m} - i\frac{\tilde{\Gamma}}{2} \right)^2. \tag{3.96}$$

A comparison between eqs. (3.95) and (3.96) for $|\sigma|$ large, gives that the mass of the unstable particle can be approximated by $me^{|\sigma|/2}$, where m is the mass of the stable particles[2]. Therefore, the limit $\sigma \to \infty$ corresponds to an infinitely massive unstable particle, that is a particle that can not be formed at any finite energy scales. At the level of the S-matrix we find that $\lim_{\sigma\to\infty} S_{\pm,\mp}(\theta) = 1$, that is, the model reduces to two non-interacting copies of the Ising field theory. This property is very useful as a consistency check in form factor calculations. It implies that when $\sigma \to \infty$ the form factors of any field should reduce to those of the Ising model, which are generally known.

As for the RT-model described before, the effective central charge of the $SU(3)_2$-HSG model also exhibits a staircase pattern, albeit with only two steps (at most), as was observed by Castro-Alvaredo *et al.* [2000b]. The same structure was found for Zamolodchikov's c-function[3] and the conformal dimensions of certain local fields by Castro-Alvaredo & Fring [2001b]. In this case the appearance of steps is directly related to the presence of the unstable particle and its mass. There is only one step if $\sigma = 0$ in which case the unstable particle's mass is of the same order as that of the stable particles and a second step emerges if $\sigma \neq 0$ whose

---

[1]Breit & Wigner [1936]

[2]Castro-Alvaredo & Fring [2001b]

[3]this function was introduced by Zamolodchikov [1986], and we do not introduce it here, as we will talk about it in length in the next chapter.

onset and length are related to the precise value of σ. All these features have been analysed in detail by Castro-Alvaredo & Fring [2001b]; Castro-Alvaredo et al. [2000b]. The minimal form factors of the n-copy theory can be extracted with eqs. (3.61) and (3.62), and are

$$F_{\min}^{\pm\pm}(\theta) = -i\sinh\left(\frac{\theta}{2n}\right), \tag{3.97}$$

and

$$F_{\min}^{\pm\mp}(\theta) = A(n)e^{\pm\frac{\theta}{4n}+\frac{i\pi(1\mp1)}{4}}\exp\left(\int_{-\infty}^{\infty}\frac{dt}{t}\frac{\sinh^2\left(\frac{t}{2}\left(n+\frac{i(\theta\pm\sigma)}{\pi}\right)\right)}{\sinh(nt)\cosh(t/2)}\right), \tag{3.98}$$

with $A(n)$ given by the limit,

$$A(n) = \lim_{p\to\infty}e^{-\frac{2p+2+n}{2n}-\frac{i\pi}{4}}\sqrt{\frac{2n}{p}}\left(\frac{4p+3+2n}{4n}\right)^{\frac{4p+3+2n}{8n}}\left(\frac{4p+5+2n}{4n}\right)^{\frac{4p+5+2n}{8n}}\prod_{k=0}^{p}\frac{\Gamma\left(\frac{4k+1+2n}{4n}\right)^2}{\Gamma\left(\frac{4k+3+2n}{4n}\right)^2}. \tag{3.99}$$

The solution (3.97) is nothing but that of the Ising model, as we would expect from the first S-matrix in (3.95).

The form factors (3.98) can also be expressed in terms of an infinite product of Gamma functions

$$F_{\min}^{\pm\mp}(\theta) = A(n)e^{\pm\frac{\theta}{4n}+\frac{i\pi(1\mp1)}{4}}\prod_{k=0}^{\infty}\frac{\Gamma\left(\frac{4k+3+2n}{4n}\right)^2\Gamma\left(\frac{-2w+4k+1+2n}{4n}\right)\Gamma\left(\frac{2w+4k+1+2n}{4n}\right)}{\Gamma\left(\frac{4k+1+2n}{4n}\right)^2\Gamma\left(\frac{-2w+4k+3+2n}{4n}\right)\Gamma\left(\frac{2w+4k+3+2n}{4n}\right)}, \tag{3.100}$$

with $w = n + i(\theta\pm\sigma)/\pi$.

The function $A(n)$ defined above would seem a strange choice of normalization. The motivation for it is to ensure that the following minimal form factor relations

$$F_{\min}^{\pm\mp}(\theta)F_{\min}^{\pm\mp}(\theta+i\pi) = \pm\frac{e^{\pm\frac{\theta}{2n}\pm\frac{i\pi}{4n}}}{\sinh\left(\frac{\theta\pm\sigma}{2n}+\frac{i\pi}{4n}\right)}, \tag{3.101}$$

hold, without involving complicated constants. In particular, $A(1) = e^{-G/\pi}e^{-i\pi/4}2^{1/4}$, where G is the Catalan constant that appears in the normalization of the form factors of the one-copy model of Castro-Alvaredo et al. [2000a]; Delfino et al.

[1995]. It is worth noticing however that with respect to the latter normalization our minimal form factor at $n = 1$ is multiplied by the extra factor $e^{-i\pi/4}$.

Once the two-particle form factor and minimal form factor have been computed the basic monodromy and pole structure features of the form factors are fixed so that higher particle form factors can be constructed in terms of the solutions already found. Let us introduce the following notation:

$$F_{\ell+m}(\{x\}_\ell^+;\{x\}_m^-) := F_{\ell+m}^{\mathcal{T}|\overbrace{+\cdots+}^{\ell}\overbrace{-\cdots-}^{m}}(x_1,\ldots,x_\ell,x_{\ell+1}\ldots x_{\ell+m}), \qquad (3.102)$$

This represents the $\ell + m$-particle form factor of the twist field with ℓ particles of type $+$ and m particles of type $-$ living in one particular copy of the model. For the model under consideration, we will make the following ansatz

$$
\begin{aligned}
F_{\ell+m}(\{x\}_\ell^+;\{x\}_m^-) &= H_{\ell,m}^{+-} Q_{\ell+m}^{+-}(\{x\}_\ell^+;\{x\}_m^-) \prod_{1\leq i<j\leq \ell} \frac{F_{\min}^{\mathcal{T}|++}(\frac{x_i}{x_j})}{(x_i - \alpha x_j)(x_j - \alpha x_i)} \\
&\times \prod_{i=1}^{\ell}\prod_{j=\ell+1}^{\ell+m} F_{\min}^{\mathcal{T}|+-}(\frac{x_i}{x_j}) \prod_{\ell+1\leq i<j\leq \ell+m} \frac{F_{\min}^{\mathcal{T}|--}(\frac{x_i}{x_j})}{(x_i - \alpha x_j)(x_j - \alpha x_i)} \quad (3.103)
\end{aligned}
$$

It is easy to check that, the ansatz (3.103) automatically satisfies equations (3.53) and (3.54) provided that the functions $Q_{\ell+m}^{+-}(\{x\}_\ell^+;\{x\}_m^-)$ are separately symmetric in both sets of variables and have no poles on the physical sheet and $H_{\ell,m}^{+-}$ are constants. Notice that there are kinematic poles associated to pairs of $+$ and $-$ particles, but not to the combination $+-$, as the two particles in the model are self-conjugated (their own antiparticle). The ansatz (3.103) is reminiscent of the solution procedure used in Castro-Alvaredo & Fring [2001b]; Castro-Alvaredo *et al.* [2000a] where the form factors of local fields of the present model were also studied.

Once the ansatz (3.103) has been made it remains to identify the functions $Q_{\ell+m}^{+-}(\{x\}_\ell^+;\{x\}_m^-)$ and the constants $H_{\ell,m}^{+-}$. A useful benchmark that can be employed for this model is the fact that whenever $m = 0$ or $\ell = 0$, the resulting form factor must be the ℓ-particle or m-particle form factor of the Ising model, respectively. This relationship with the Ising model, combined with the kinematic residue equation (3.55) also implies that only form factors with both ℓ and m even will be non-vanishing.

Substituting the ansatz (3.103) into (3.55) we obtain the following recursive relations for $Q_{\ell+m}^{+-}(\{x\}_\ell^+;\{x\}_m^-)$ and the constants $H_{\ell,m}^{+-}$,

$$H_{\ell+2,m}^{+-} = \frac{\alpha^{\frac{3\ell-m}{2}+2}e^{-\frac{\sigma m}{2n}}2^{2\ell-m+1}\sin\frac{\pi}{n}}{nF_{\min}^{\mathcal{T}|++}(i\pi)}H_{\ell,m}^{+-},\qquad(3.104)$$

and

$$Q_{\ell+2+m}^{+-}(\alpha x_0,x_0,\{x\}_\ell^+;\{x\}_m^-)=P_{\ell,m}^{+-}(x_0,\{x\}_\ell^+;\{x\}_m^-)Q_{\ell+m}^{+-}(\{x\}_\ell^+;\{x\}_m^-),\qquad(3.105)$$

with

$$P_{\ell,m}^{+-}(x_0,\{x\}_\ell^+;\{x\}_m^-) = \alpha^\ell x_0^{\ell+2-m}\sigma_\ell^+ \sum_{i,j=0}^\ell \left(-\frac{x_0}{\alpha}\right)^{\ell-i}(-\alpha^2 x_0)^{\ell-j}\sigma_i^+\sigma_j^+ \sum_{k=0}^m (-\sqrt{\alpha}e^{\frac{\sigma}{n}}x_0)^{m-k}\sigma_k^-,$$
$$(3.106)$$

where σ_k^+,σ_k^- are elementary symmetric polynomials on the variables $\{x\}_\ell^+$ and $\{x\}_m^-$, respectively. To simplify notation, in (3.106) and (3.109) we have dropped the explicit variable dependence of the symmetric polynomials.

In the ansatz (3.103) we have chosen a particular ordering of the particles with type + appearing first and type - last. Of course this ordering can be changed by employing the first form factor equation (3.53). Alternatively, we could have worked with the form factor $F_{\ell+m}(\{x\}_\ell^-;\{x\}_m^+)$ where we now have ℓ particles of type - first, followed by m particles of type +. For this ordering, the recurrence equations above become instead

$$H_{\ell+2,m}^{-+} = \frac{\alpha^{\frac{3\ell}{2}+2}e^{-\frac{\sigma m}{2n}}2^{2\ell-m+1}\sin\frac{\pi}{n}}{nF_{\min}^{\mathcal{T}|--}(i\pi)}H_{\ell,m}^{-+},\qquad(3.107)$$

and

$$Q_{\ell+2+m}^{-+}(\alpha x_0,x_0,\{x\}_\ell^-;\{x\}_m^+)=P_{\ell,m}^{-+}(x_0,\{x\}_\ell^-;\{x\}_m^+)Q_{\ell+m}^{-+}(\{x\}_\ell^-;\{x\}_m^+),\qquad(3.108)$$

with

$$P_{\ell,m}^{-+}(x_0,\{x\}_\ell^-;\{x\}_m^+) = \alpha^\ell x_0^{\ell+2} \frac{\sigma_\ell^-}{\sigma_m^+} \sum_{i,j=0}^{\ell} \left(-\frac{x_0}{\alpha}\right)^{\ell-i} (-\alpha^2 x_0)^{\ell-j} \sigma_i^- \sigma_j^- \sum_{k=0}^{m} (-\sqrt{\alpha} e^{\frac{\sigma}{n}} x_0)^{m-k} \sigma_k^+.$$

(3.109)

From the definition (3.103) and equations (3.53) and (3.61) it is easy to show that

$$H_{\ell,m}^{+-} Q_{\ell,m}^{+-}(\{x\}_\ell;\{x\}_m) = H_{m,\ell}^{-+} Q_{m,\ell}^{-+}(\{x\}_m;\{x\}_\ell),$$

(3.110)

which provides a useful relationship between the solutions of (3.108) and those of (3.105).

Given the structure of the S-matrix (3.95) we know that form factors involving only particles of type + or only particles of type - should equal the form factors of the Ising model. We will consider then this case first. For $m = 0$ in (3.104)-(3.105) or equivalently $m = 0$ in (3.107)-(3.108) the equations reduce to

$$H_{\ell+2} = \frac{\alpha^{\frac{3\ell}{2}+2} 2^{2\ell+1} \sin\frac{\pi}{n}}{n F_{\min}^{\mathscr{T}|\pm\pm}(i\pi)} H_\ell,$$

(3.111)

and

$$Q_{\ell+2}(\alpha x_0, x_0, \{x\}_\ell) = P_\ell(x_0,\{x\}_\ell) Q_\ell(\{x\}_\ell),$$

(3.112)

with

$$P_\ell(x_0,\{x\}_\ell) = \alpha^\ell x_0^{\ell+2} \sigma_\ell \sum_{i,j=0}^{\ell} \left(-\frac{x_0}{\alpha}\right)^{\ell-i} (-\alpha^2 x_0)^{\ell-j} \sigma_i \sigma_j.$$

(3.113)

Interestingly even for this special case, these equations are not easy to solve and the solutions for $Q_\ell(\{x\}_\ell)$ become very cumbersome beyond $\ell = 4$. The first few

solutions are,

$$Q_2(x_1, x_2) = \alpha^{-1}\sigma_2, \tag{3.114}$$

$$Q_4(x_1, x_2, x_3, x_4) = \alpha^{-1}\sigma_4^2\left(\sigma_2^2 - \frac{p_1(\alpha)\sigma_1\sigma_3}{\alpha} + \frac{p_1(\alpha)(1+\alpha^2)^2\sigma_4}{\alpha^3}\right), \tag{3.115}$$

$$\begin{aligned}
Q_6(x_1, x_2, x_3, x_4, x_5, x_6) = \sigma_6^3\Bigg(&\sigma_2^2\sigma_4^2 + \frac{p_1(\alpha)^2\sigma_1\sigma_3^2\sigma_5}{\alpha^2} + \frac{p_2(\alpha)\sigma_1\sigma_2\sigma_4\sigma_5}{\alpha} \\
&-\frac{(1+\alpha^2)^2 p_3(\alpha)\sigma_1^2\sigma_5^2}{\alpha^4} - \frac{p_1(\alpha)\sigma_3\left(\sigma_1\sigma_4^2 + \sigma_2^2\sigma_5\right)}{\alpha} \\
&+\frac{p_2(\alpha)p_1(\alpha)^4\sigma_3^2\sigma_6}{\alpha^5} + \frac{(1+\alpha^2)^2 p_1(\alpha)\left(\sigma_4^3 + \sigma_2^3\sigma_6\right)}{\alpha^3} \\
&-\frac{p_2(\alpha)p_1(\alpha)^2 p_4(\alpha)p_3(\alpha)\sigma_1\sigma_5\sigma_6}{\alpha^7} + \frac{p_2(\alpha)^3 p_1(\alpha)^4 p_3(\alpha)\sigma_6^2}{\alpha^9} \\
&-\frac{p_1(\alpha)^2 p_3(\alpha)\sigma_3(\sigma_4\sigma_5 + \sigma_1\sigma_2\sigma_6)}{\alpha^4} - \frac{p_2(\alpha)p_1(\alpha)^2 p_5(\alpha)\sigma_2\sigma_4\sigma_6}{\alpha^5} \\
&+\frac{p_1(\alpha)p_3(\alpha)^2\left(\sigma_2\sigma_5^2 + \sigma_1^2\sigma_4\sigma_6\right)}{\alpha^5}\Bigg), \tag{3.116}
\end{aligned}$$

with

$$\begin{aligned}
p_1(\alpha) &= 1 + \alpha + \alpha^2, \\
p_2(\alpha) &= 1 - \alpha + \alpha^2 \\
p_3(\alpha) &= 1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4, \\
p_4(\alpha) &= 1 - \alpha + 3\alpha^2 - \alpha^3 + \alpha^4, \\
p_5(\alpha) &= 3 + 2\alpha + 4\alpha^2 + 2\alpha^3 + 3\alpha^4. \tag{3.117}
\end{aligned}$$

Comparing eq. (3.69) to our original ansatz we have the remarkable identity

$$Q_\ell(\{x\}_\ell) = H_\ell^{-1}\mathrm{Pf}(K^\ell)\prod_{i<j}^{\ell}(x_i - \alpha x_j)(x_j - \alpha x_i). \tag{3.118}$$

Bringing the rhs of (3.118) into the form of a combination of symmetric polynomials is highly non-trivial for $\ell > 4$. In particular, for $\ell = 6$ it yields the result (3.116).

We now focus of form factors with + and - particles. Starting with the two particle

solutions (3.114) we find the following new four particle form factor solutions

$$Q_{2+2}^{+-}(x_1, x_2; x_3, x_4) = \alpha^{-1}\sigma_2^{-}\left(\hat{\sigma}_2^{+} - \frac{\sqrt{\alpha}}{1+\alpha}\hat{\sigma}_1^{+}\sigma_1^{-} + \sigma_2^{-}\right), \qquad (3.119)$$

$$Q_{2+2}^{-+}(x_1, x_2; x_3, x_4) = \alpha^{-2}\sigma_2^{+}\left(\hat{\sigma}_2^{-} - \frac{\sqrt{\alpha}}{1+\alpha}\sigma_1^{+}\hat{\sigma}_1^{-} + \sigma_2^{+}\right), \qquad (3.120)$$

where $\hat{\sigma}_k^{\pm}$ are symmetric polynomials on the variables $\{xe^{\frac{\sigma}{n}}\}_{\ell,m}$.

Going beyond four particles is rather difficult, but because of their relationship to the form factors of the Ising model, it is possible to find closed formulae for certain types of form factors. For example, when $\ell = 2$ and m is general. In this particular case the form factor equations become simply

$$Q_{2+m}^{+-}(\alpha x_0, x_0; \{x\}_m^-) = x_0^{2-m}\sum_{k=0}^{m}(-\sqrt{\alpha}e^{\frac{\sigma}{n}}x_0)^{m-k}\sigma_k^{-}Q_m(\{x\}_m^-), \qquad (3.121)$$

and

$$Q_{2+m}^{-+}(\alpha x_0, x_0; \{x\}_m^+) = \frac{x_0^2}{\sigma_m^+}\sum_{k=0}^{m}(-\sqrt{\alpha}e^{\frac{\sigma}{n}}x_0)^{m-k}\sigma_k^{+}Q_m(\{x\}_m^+), \qquad (3.122)$$

where $Q_m(\{x\}_m)$ is the Ising model solution given by (3.118). Particular solutions to (3.121) and (3.122) take the form,

$$Q_{2+m}^{+-}(x_1, x_2; \{x\}_m^-) = \left[\alpha^{\frac{m}{2}-1}\hat{\sigma}_2^{+}\sum_{k=0}^{\frac{m}{2}}\frac{\sigma_{2k}^{-}}{(\hat{\sigma}_2^{+})^k} - \frac{\alpha^{\frac{m-1}{2}}\hat{\sigma}_1^{+}}{1+\alpha}\sum_{k=0}^{\frac{m-2}{2}}\frac{\sigma_{2k+1}^{-}}{(\hat{\sigma}_2^{+})^k}\right]Q_m(\{x\}_m^-), \quad (3.123)$$

and

$$Q_{2+m}^{-+}(x_1, x_2; \{x\}_m^+) = \frac{(\hat{\sigma}_2^{-})^{\frac{m}{2}}}{\sigma_m^+}\left[\alpha^{-1}\hat{\sigma}_2^{-}\sum_{k=0}^{\frac{m}{2}}\frac{\hat{\sigma}_{2k}^{+}}{(\hat{\sigma}_2^{-})^k} - \frac{\alpha^{-\frac{1}{2}}\hat{\sigma}_1^{-}}{1+\alpha}\sum_{k=0}^{\frac{m-2}{2}}\frac{\sigma_{2k+1}^{+}}{(\sigma_2^{-})^k}\right]Q_m(\{x\}_m^+).$$
$$(3.124)$$

They provide closed solutions to the equations (3.121)-(3.122) valid for any values of m. Unfortunately, this is not enough to conclude they are fully consistent with all form factor equations. What we mean is that the relation (3.110) must also hold, which means that for example the solution $Q_{2+4}^{-+}(x_1, x_2; \{x\}_4^+)$ constructed above, must also solve the form factor equation satisfied by $Q_{4+2}^{+-}(\{x\}_4^+; x_1, x_2)$ (up to constants). This imposes a set of further constraints on the solutions to (3.121)

and (3.122).

Let us consider an example. From (3.123) we find

$$Q_{2+4}^{-+}(x_1, x_2; \{x\}_4^-) = \alpha^{-1} \frac{\hat{\sigma}_2^-}{\sigma_4^+} \left[(\hat{\sigma}_2^-)^2 + \hat{\sigma}_2^- \sigma_2^+ + \sigma_4^+ - \frac{\sqrt{\alpha}\hat{\sigma}_1^-}{1+\alpha} \left(\hat{\sigma}_2^- \sigma_1^+ + \sigma_3^+ \right) \right] Q_4(\{x\}_4^+).$$
(3.125)

This function solves (3.121), however it does not solve the equation for $Q_{4+2}^{+-}(\{x\}_4^+; x_1, x_2)$ which can be obtained from (3.109) with $\ell = 2$, $m = 4$. If we solve that equation, we obtain a completely different solution. Therefore (3.125) is not a consistent solution to all form factor equations. As we studied in detail for the RT-model, we can generally add an extra function to any solution, as long as that function is in the kernel of the equation we are trying to solve. In our case, this means that we can always add to (3.125) any function $K_{2+4}^{-+}(x_1, x_2; \{x\}_4)$ which satisfies,

$$K_{2+4}^{-+}(\alpha x_0, x_0; \{x\}_4^+) = 0.$$
(3.126)

The most general solution to this equation, up to a multiplicative constant is,

$$K_{2+4}^{-+}(\alpha x_0, x_0; \{x\}_4^+) = \frac{\sigma_2^- \sigma_4^+ (\alpha(\sigma_1^+)^2 - (\alpha+1)^2 \sigma_2^+)}{\alpha^3(1+\alpha)^2}$$
$$\times \left(\alpha^3 \sigma_1^+ \sigma_2^+ \sigma_3^+ - \alpha(1+\alpha+\alpha^2)((\sigma_3^+)^2 + (\sigma_1^+)^2 \sigma_4^+) + (1+\alpha)^4(1+\alpha^2)\sigma_2^+ \sigma_4^+ \right)$$
(3.127)

Solving for $Q_{4+2}^{+-}(\{x\}_4^+; x_1, x_2)$ we find that

$$
\begin{aligned}
Q_{4+2}^{+-}(\{x\}_4^+; x_1, x_2) &= \alpha \frac{\hat{\sigma}_2^-}{\sigma_4^+} \left[(\hat{\sigma}_2^-)^2 + \hat{\sigma}_2^- \sigma_2^+ + \sigma_4^+ - \frac{\sqrt{\alpha}\hat{\sigma}_1^-}{1+\alpha} \left(\hat{\sigma}_2^- \sigma_1^+ + \sigma_3^+ \right) \right] Q_4(\{x\}_4^+) \\
&+ K_{2+4}^{-+}(x_1, x_2; \{x\}_4^+),
\end{aligned}
$$
(3.128)

and from equation (3.110) it follows that

$$Q_{2+4}^{-+}(x_1, x_2; \{x\}_4^-) = \alpha^{-2} Q_{4+2}^{+-}(\{x\}_4^+; x_1, x_2).$$
(3.129)

Therefore, in general, the solutions (3.123) and (3.124) need to be modified by adding some function in the kernel of (3.121) or (3.122) which is consistent with (3.110).

3.5 The Δ-sum rule

In this section we introduce the Δ-sum rule, as defined by Delfino *et al.* [1996], and eventually we use it to check the correctness of the higher particle form factors obtained in sections 3.4.1 and 3.4.2. We focus on a massive model which can be described as a perturbation of a CFT, as explained in section 3.2. We call ϕ one of its primary operators, and we study the two point function between this field and the stress energy tensor. In particular we express in complex coordinates $z, \bar{z} = re^{\pm i\alpha}$ the two following correlators

$$\langle T(z, \bar{z})\phi(0)\rangle = \frac{F(r^2)}{z^2} \tag{3.130}$$

$$\langle \Theta(z, \bar{z})\phi(0)\rangle - \langle \Theta\rangle\langle\phi\rangle = \frac{G(r^2)}{r^2}, \tag{3.131}$$

where the exact definitions of T and Θ can be found in section 2.2.1. The conservation law of the stress energy tensor was reported in eq. (3.15). Calling $t = r^2$ we can use this equation to demonstrate that for $D(t) = F(t) + \frac{1}{4}G(t)$ it holds

$$\frac{d}{d\log t}D(t) = \frac{1}{4}G(t). \tag{3.132}$$

As we have seen in section 3.2 if we consider a theory described by the action (3.9) perturbed with just one field, the trace of stress energy tensor and the perturbing field are related by eq. (3.19). Hence we can focus on $\langle \varphi(z, \bar{z})\phi(0)\rangle$ to determine G. In particular we use the short distance OPE of these two fields to express

$$G(t) = 4\pi g(1 - \Delta)C_{\varphi\phi}^0 t^{\Delta_0 - \Delta - \Delta_\phi + 1}\langle A_0\rangle + ..., \tag{3.133}$$

and distinguish two cases.

The first case is $\Delta_0 - \Delta - \Delta_\phi + 1 > 0$, which means that in the UV limit $r \to 0$ the rhs of eq. (3.133) vanishes. Hence $r = 0$ is a stationary point of $D(t)$, and $D(0) = F(0)$. We can now use eq. (2.23) to see that $D \to \Delta_\phi\langle\phi\rangle$, in the UV limit[1]. Here we have

---

[1]actually we have to say that in principle the renormalization process needed in conformal perturbation theory to cure IR divergences could mix the operators, such that this limit could be more complicated. This is not the case for any of the models considered by us.

to consider two sub-cases

- the perturbation makes the original CFT flow onto a fixed point corresponding to another CFT; then we can integrate eq. (3.132) along this massless flow obtaining

$$\Delta_\phi^{\text{UV}} - \Delta_\phi^{\text{IR}} = -\frac{1}{2\langle\phi\rangle} \int dr \;\; r\langle\Theta(r)\phi(0)\rangle_c, \qquad (3.134)$$

where with $\langle...\rangle_c$ we mean the connected correlator.

- The perturbation makes the original CFT flow onto an infinite coupling massive theory, then

$$\Delta_\phi^{\text{UV}} = -\frac{1}{2\langle\phi\rangle} \int dr \;\; r\langle\Theta(r)\phi(0)\rangle_c. \qquad (3.135)$$

The other case is when $\Delta_0 - \Delta - \Delta_\phi + 1 \le 0$. In this case $G(t)$ does not vanish for $t \to 0$, and using eq. (3.133) is useless, as the resulting integral diverges. We will not describe further this case, as it does not occur in the models we consider. Moreover both the RT and the $SU(3)_2$-homogenous sine-Gordon models have an infinite coupling theory as IR limit.

Equation (3.135) can be then employed to check the correctness of our twist field form factors, as the conformal weight of $\mathcal{T}$ is well known (2.67). This rule was already employed in Cardy *et al.* [2008] with the same scope for the Ising and sinh-Gordon models.

In what follows we will use a slightly more general version of eq. (3.134), already employed by Castro-Alvaredo & Fring [2001b]

$$\Delta_\phi(r_0) = -\frac{1}{2\langle\phi\rangle} \int_{r_0}^{\infty} dr\, r\langle\Theta(r)\phi(0)\rangle_c, \qquad (3.136)$$

where we dropped the UV superscript, so that taking $r_0 = 0$ we recover eq. (3.135), whereas for larger values of r_0 we are now able to trace changes in the value of $\Delta^{\mathcal{T}}(r_0)$ along the renormalisation group (RG) flow, that is as we move from low energies or r_0 large to high energies or $r_0 = 0$. Observing such intermediate behaviour is particularly interesting for models where the RG-flows approach the

vicinity of more than one critical point, as the RT model. It is also instructive for those models with bound states or unstable particles. For those models we expect to see a sharp change in the expectation values once we reach an energy scale comparable with the mass of these excitations. This is because of the contribution of channels involving those particles to scattering processes.

Focusing on the twist field, employing eq. (3.46) and performing the integration in r, eq. (3.136) becomes

$$\Delta_{\mathcal{T}}(r_0) = -\frac{1}{2\langle \mathcal{T} \rangle} \sum_{k=1}^{\infty} \sum_{\mu_1 \dots \mu_k} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \frac{d\theta_1 \dots d\theta_k}{k!(2\pi)^k} \frac{(1 + r_0 E) e^{-r_0 E}}{2E^2}$$
$$\times F_k^{\Theta|\mu_1 \dots \mu_k}(\theta_1, \dots, \theta_k) \left(F_k^{\mathcal{T}|\mu_1 \dots \mu_k}(\theta_1, \dots, \theta_k) \right)^* , \qquad (3.137)$$

where E stands for the sum of the on-shell energies $E = \sum_{i=1}^{k} m_{\mu_i} \cos(\theta_i)$. Unfortunately in eq. (3.137) both the number of integrals and the complexity of form factors grow very fast, making an analytic solution inaccessible. We have to rely on numerical integration methods then, and the rest of this section is devoted to this analysis for the RT and $SU(3)_2$-homogeneous sine-Gordon models.

We start by considering the roaming trajectories model. As explained in section 3.4.1, the function $c_{\text{eff}}(r)$ exhibits an infinite set of plateaux between $r = 0$ and $r \to \infty$. From Castro-Alvaredo [2011] we understand that in the RG flow of the c-function[1] each plateau is connected to the contribution of a certain order form factor. A similar type of behaviour is expected for $\Delta_{\mathcal{T}}(r_0)$ as r_0 is varied, so that we would need to perform a numerical integration of eq. (3.137) for different values of $r_0 \in [0, \infty)$ to follow the flow. This in particular means that the two particle contribution will give $c = 1/2$, the four particle $c = 7/10 - 1/2$, and so on and so forth. Thus taking $r_0 = 0$, we would expect to identify two values of $\Delta_{\mathcal{T}}$, that is the value obtained in the two-particle approximation and the value obtained in the four-particle approximation.

The two particle contribution can be expressed after a bit of manipulation by means of a single integral, which can be solved numerically with any Newton

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[1]this will be defined in the next chapter.

Cotes formulae[1]. With the same kind of manipulation we are able to express the four particle contribution by means of a triple integral, which we solved numerically with a Montecarlo integration. Since the integrand function is very peaked in a certain region we used the adaptive VEGAS algorithm of the Gnu Scientific Library (GSL)[2]. The precise height of the plateaux can be easily predicted plugging (3.73) into eq. (2.67). By doing so we obtain a value of $\Delta_{\mathscr{T}}$ for each central charge and each value of n.

The two particle contribution takes exactly the same form as for the sinh-Gordon model and was given in Cardy *et al.* [2008]. Evaluating it for $\theta_0 = 20$ we obtain the values listed in table 3.2.

n	$\frac{1}{48}\left(n - \frac{1}{n}\right)$	$\Delta_{\mathscr{T}}^{(2)}(0)$
2	0.03125	0.0312548
3	0.0555556	0.055676
4	0.078125	0.0785953
5	0.1	0.101033
6	0.121528	0.123257
7	0.142857	0.145351
8	0.164062	0.167351
9	0.185185	0.189277
10	0.20625	0.211143

Table 3.2: Two particle contribution to the conformal dimension in the RT-model. The second column shows the exact values of the conformal dimension of the twist field corresponding to central charge $c_3 = 1/2$. The third column shows the numerical values of the same quantity in the two-particle approximation for $\theta_0 = 20$.

Employing the four-particle form factors of the energy momentum tensor obtained in Fring *et al.* [1993] and Koubek & Mussardo [1993] and our solution

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[1]in particular we used the trapezoid rule.
[2]Galassi & Gough [2005]

eq. (3.88) with $\Omega_4 = 0$, the four-particle contribution is given by

$$\Delta_{\mathscr{T}}^{(4)}(0) = -\frac{\sin\left(\frac{\pi}{n}\right)\cosh(\theta_0)\left|F_{\min}^{\mathscr{T}}(i\pi)^* F_{\min}^{\Theta}(i\pi)\right|^2}{1536\pi^3}$$

$$\int_{-\infty}^{\infty} d\theta_1 d\theta_2 d\theta_3 d\theta_4 \frac{\sigma_1\sigma_2\sigma_3 Q_4(x_1,x_2,x_3,x_4)\prod\limits_{i<j} F_{\min}^{\mathscr{T}}(\theta_{ij})^* F_{\min}^{\Theta}(\theta_{ij})}{\left(\prod\limits_{i<j}\cos\left(\frac{\pi}{n}\right)-\cosh\left(\frac{\theta_{ij}}{n}\right)\right)\left(\sum\limits_{i=1}^{4}\cosh(\theta_{ij})\right)^2} \quad (3.138)$$

where σ_1, σ_2 and σ_3 above represent elementary symmetric polynomials in the variables e^{θ_i} with $i = 1,2,3,4$. The values of (3.138) for different values of n are given in table 3.3. Both tables 3.2 and 3.3 show relatively good agreement be-

n	$\frac{1}{120}\left(n-\frac{1}{n}\right)$	$\Delta_{\mathscr{T}}^{(4)}(0)$
2	0.012500	0.013086
3	0.022200	0.022169
4	0.031250	0.028611
5	0.040000	0.042555
6	0.048611	0.047566
7	0.057143	0.057996
8	0.065625	0.064736
9	0.074074	0.072281
10	0.082500	0.068762

Table 3.3: Four-particle contribution to the conformal dimension in the RT-model. The second column shows the difference between the values of the conformal dimension of the twist field corresponding to central charges $c_4 = 7/10$ and $c_2 = 1/2$. The third column shows the numerically computed four-particle contribution to the conformal dimension for $\theta_0 = 20$.

tween the values predicted by the theory and those numerically obtained. The difference between the theoretical and numerical values is considerable for some of the results in table 3.3, specially as n is increased. However it is always within the standard deviation of the computation.

We now focus on the $SU(3)_2$-HSG model. For this model we follow the flow of $\Delta_{\mathcal{T}}(r_0)$, and this obviously increases the simulation time. We managed to reduce the running time of our program by employing a very precise, piece-wise polynomial interpolation of the functions $F_{\min}^{\mathcal{T}|\pm\mp}(\theta)F_{\min}^{\Theta|\pm\mp}(\theta)^*$. Also in this case we have carried out the integrals by means of the GSL version of VEGAS algorithm. From (3.63) and the two particle form factor of the energy-momentum tensor for the thermally perturbed Ising model[1]

$$F_2^{\Theta|\pm\pm}(\theta) = -2\pi i m^2 \sinh(\frac{\theta}{2}), \tag{3.139}$$

the two particle contribution can be easily calculated to

$$\Delta_{\mathcal{T}}^{(2)}(\tilde{r}_0) = \frac{2\cos\left(\frac{\pi}{2n}\right)}{4\pi} \int_{-\infty}^{\infty} d\theta_1 d\theta_2 \frac{(1+\tilde{r}_0\sum_{i=1}^{2}\cosh(\theta_i))e^{-\tilde{r}_0\sum_{i=1}^{2}\cosh(\theta_i)}}{(\sum_{i=1}^{2}\cosh(\theta_i))^2} \frac{\sinh\left(\frac{\theta_{12}}{2n}\right)\sinh\left(\frac{\theta_{12}}{2}\right)}{\cosh\left(\frac{\theta_{12}}{n}\right)-\cos\left(\frac{\pi}{n}\right)}, \tag{3.140}$$

where $\tilde{r}_0 = mr_0$ is a dimensionless parameter proportional to the mass scale. From a physical point of view, we expect this contribution to produce a function with a plateau at exactly $\Delta_{\mathcal{T}}^{(2)}(0) = \frac{1}{24}\left(n - \frac{1}{n}\right)$, which is the value corresponding to two copies of the Ising model or $c = 1$. The four particle contribution is also quite simple to compute, as only few form factors contribute. This is because, for each copy of the model, the only non-vanishing four particle form factors of the energy-momentum tensor are $F_{2+2}^{\Theta|+-}(\theta_1,\theta_2;\theta_3,\theta_4)$ and all other form factors that can be obtained from this one by changing the particle ordering. This form factor was given explicitly in Castro-Alvaredo *et al.* [2000a]. Together with our solution (3.120) and the ansatz (3.103) it gives the four particle contribution

$$\Delta_{\mathcal{T}}^{(4)}(\tilde{r}_0) = -\frac{\cos\left(\frac{\pi}{2n}\right)^2}{256n\pi^3 e^{\sigma/n}} \int_{-\infty}^{\infty} d\theta_1 d\theta_2 d\theta_3 d\theta_4 \frac{(1+\tilde{r}_0(\sum_{i=1}^{4}\cosh(\theta_i))e^{-\tilde{r}_0(\sum_{i=1}^{4}\cosh(\theta_i))}}{(\sum_{i=1}^{4}\cosh(\theta_i))^2} e^{(\theta_{31}+\theta_{42})/2}$$

$$\frac{(2+\sum_{i<j}^{4}\cos(\theta_{ij}))\left[\prod_{i<j}^{4}\left(F_{\min}^{\mathcal{T}|\mu_i\mu_j}(\theta_{ij})\right)^* F_{\min}^{\Theta|\mu_i\mu_j}(\theta_{ij})\right] Q_{2+2}^{+-}(x_1,x_2;x_3,x_4)e^{-(\theta_1+\theta_2+\theta_3+\theta_4)/n}}{\cosh\left(\frac{\theta_{12}}{2}\right)\cosh\left(\frac{\theta_{34}}{2}\right)\left(\cosh\left(\frac{\theta_{12}}{n}\right)-\cos\left(\frac{\pi}{n}\right)\right)\left(\cosh\left(\frac{\theta_{34}}{n}\right)-\cos\left(\frac{\pi}{n}\right)\right)}, \tag{3.141}$$

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[1]Berg *et al.* [1979]; Yurov & Zamolodchikov [1991].
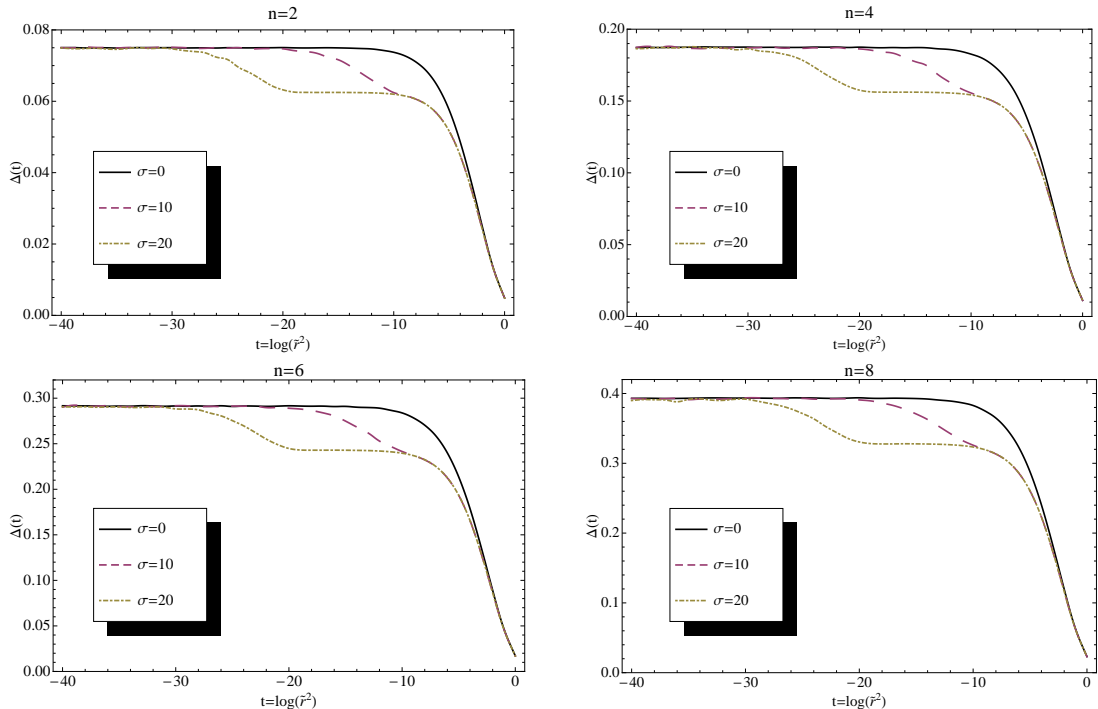
Figure 3.1: The function $\Delta(t) := \Delta_{\mathscr{T}}(t)$ with $t = 2\log(\tilde{r})$ and $\tilde{r} = mr$. In these figures we show the behaviour of $\Delta_{\mathscr{T}}(t)$ along the renormalization group flow, from the infrared to the ultraviolet fixed point, for different values of the resonance parameter σ. Our results are consistent with (2.67) and $c = 1$ for the first plateau and (2.67) with $c = 6/5$ for the second plateau.

where $\mu_1, \mu_2 = +$ and $\mu_3, \mu_4 = -$. This contribution, when added to (3.140) should bring the value of $\Delta_{\mathscr{T}}$ closer to the expected one, which is obtained by setting $c = 6/5$ in (2.67). Our numerical results shown in figure 3.1 clearly demonstrate this to be the case for various values of n. As $t \to -\infty$ the functions $\Delta(t)$ all approach the expected value (2.67) for $n = 2, 4, 6$ or 8 and $c = 6/5$ with great accuracy. This is shown in Table 3.4.

In figure 3.1 we also see that the function $\Delta(t)$ exhibits two finite plateaux along the renormalization group flow, which in numerical terms exactly correspond to the two particle and four particle contributions. The position at which the second plateau emerges changes as a function of σ, as is also illustrated in the figure. An entirely similar behaviour was found in Castro-Alvaredo & Fring

n	$\sigma = 0$	$\sigma = 10$	$\sigma = 20$	$\frac{1}{20}\left(n - \frac{1}{n}\right)$
2	0.075021	0.075041	0.074947	0.075
4	0.187194	0.187369	0.186426	0.1875
6	0.291655	0.290274	0.290568	0.291667
8	0.392992	0.392880	0.389882	0.39375

Table 3.4: In this table we display the value $\Delta_{\mathcal{T}}(t)$ for different n when we approach the UV limit $t \to -\infty$ considering different values of the resonance parameters σ. We observe a good agreement with the CFT prediction (2.67) with $c = 6/5$.

[2001b] for the c-function of the same model and in Castro-Alvaredo *et al.* [2000b] for its effective central charge. A detailed physical interpretation has been given there. Unfortunately, the errors on the 6 particles contribution were too large to give acceptable results.

3.6   The entanglement entropy

To evaluate the entanglement entropy in a massive theory we use exactly the same approach of the CFT case in section 2.3.3. Remember that the entropy we want to evaluate is that of a region $A = [0,r]$ tracing out $\backslash A$, then we can write

$$\mathrm{Tr}_A \rho_A^n = \varepsilon^{2\Delta_{\mathcal{T}}} \langle \mathcal{T}(r) \mathcal{T}(0) \rangle. \tag{3.142}$$

We already know that in the UV limit $r \ll m^{-1}$, the behaviour of entanglement entropy can be well described by eqs. (2.69) and (2.71), so that we focus on the opposite limit $r \gg m^{-1}$. In this regime the two point function in the rhs of eq. (3.142) can be expressed very efficiently with a form factor expansion. So efficiently that we expect that the two particle contribution to be enough to have a good approximation, so that, using eqs. (3.46), (3.57) and (3.58), we can write

$$\langle \mathcal{T}(r) \mathcal{T}(0) \rangle = \langle \mathcal{T} \rangle^2 \left(1 + \frac{n}{4\pi^2} \int_{-\infty}^{\infty} d\theta f(\theta,n) K_0(2mr \cosh\frac{\theta}{2}) \right), \tag{3.143}$$

where K_0 is a modified Bessel function of the second kind, and

$$f(\theta,n) = \sum_{j=1}^{n} \left| F^{\mathscr{T}|11}(-\theta + 2\pi i j) \right|^2.$$

(3.144)

In order to extract the entropy we have to use the replica trick (2.43), and this means we have to find an analytic continuation for $n \in [1,\infty)$ of eq. (3.142). Actually $f(\theta,n)$ is the only quantity to depend non-trivially on n, so that we will focus on that. The details on how to perform this analytic continuation were first reported in Cardy *et al.* [2008], where they found that for an integrable theory[1]

$$\lim_{n\to 1} \frac{\partial}{\partial n} f(\theta,n) = \frac{\pi^2}{2}\delta(\theta).$$

(3.145)

This result was checked explicitly for the Ising and sinh-Gordon models. Now putting together eqs. (2.43), (3.142), (3.143) and (3.145) we find eventually

$$S(\rho_A) = -\frac{c}{3}\log(\varepsilon m) + U - \frac{1}{8}K_0(2mr) + O(e^{-4mr}),$$

(3.146)

The dimensionless constant U is of universal nature, and is defined as

$$U = -\lim_{n\to 1} \frac{\partial}{\partial n} m^{-2\Delta_\mathscr{T}} \langle T \rangle^2,$$

(3.147)

and corrections of order $O(e^{-4mr})$ are to be expected when taking under consideration higher particle form factors.

All the considerations made so far hold for an integrable theory with just one particle in its spectrum, but in a later work it was proven by Doyon [2009] that eq. (3.145) holds also for non-integrable theories with an arbitrary number of particles. We can now regroup the results obtained in Chapters 2 and 3, and

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[1]here we are denoting with $f(\theta,n)$ also the analytic continuation when $n \in [1,\infty)$.

distinguish the following two limits for a massive QFT

$$S(\rho_A) = \frac{c}{3} \log\left(\frac{r}{\varepsilon}\right) \qquad\qquad\qquad \text{for} \quad r \ll m_1^{-1},$$

$$\text{(3.148)}$$

$$S(\rho_A) = -\frac{c}{3} \log(\varepsilon m_1) + U - \frac{1}{8} \sum_{\alpha=1}^{k} K_0(2m_\alpha r) + O(e^{-3m_1 r}) \qquad \text{for} \quad r \gg m_1^{-1},$$

$$\text{(3.149)}$$

for a theory with k particles of mass m_α in its spectrum, with $m_1 \leq m_2 \leq ... \leq m_k$. Notice that these general results confirm the behaviour of the entanglement entropy anticipated with eqs. (1.41) and (1.42).

Equation (3.149) has been obtained with a form factor expansion whose region of convergence is $m_1 r > 1$, where higher order terms are exponentially suppressed. We expect then that neglecting higher order terms gives a good approximation, which clearly becomes better for larger size of A. We have then the following qualitative picture. The entanglement entropy of the bipartition A grows logarithmically with the size of r for $r < m_1^{-1}$. When $r \simeq m_1^{-1}$, $S(\rho_A)$ comes very close to a saturation point which it reaches when $r \gg m_1^{-1}$. In the region $r > m_1^{-1}$ the difference between the entanglement entropy and its saturation value decreases exponentially with increasing r.

A remarkable feature of eqs. (3.148) and (3.149) is that, if we think of our QFT as a scaling limit of a lattice theory, then $S(\rho_A)$ can be expressed by means of universal features of the latter. Indeed its behaviour depends only on the central charge and the spectrum of the QFT. The connection between these QFT results and the lattice theory has to be thought of in the spirit of section 1.6. Equations (3.149) and (3.148) give meaningful predictions for the lattice only in its scaling limit, that is when the limits in eq. (1.37) are performed. This means that the predictive power of the QFT is restricted only to the case in which both the size of the region A and the correlation length are very large compared to the

lattice spacing. The cut-off can be related to the lattice spacing as

$$\varepsilon \propto \frac{a}{m\hat{\xi}}, \tag{3.150}$$

in a non universal way, and we can choose it in a way that it gives the same constant of eq. (2.70). Using the map reported in table 1.1 we can apply eqs. (3.148) and (3.149) to the lattice theory getting

$$S(\rho_a) = \frac{c}{3}\log(r) + c_1' \qquad\qquad \text{for} \quad r \ll \xi_1, \tag{3.151}$$

$$S(\rho_a) = \frac{c}{3}\log(\xi_1) + c_1' + U - \frac{1}{8}\sum_{\alpha=1}^{k} K_0(2\frac{r}{\xi_\alpha}) + O(e^{-4\frac{r}{\xi_1}}) \qquad \text{for} \quad r \gg \xi_1, \tag{3.152}$$

where ξ_α are the characteristic lengths of the correlations of the theory, and we are considering $\xi_1 \geq \xi_2 \geq ... \geq \xi_k$. The constant c_1' in eqs. (3.151) and (3.152) has the same origin as the one in eqs. (2.70)–(2.73), hence it is of non-universal nature, and cannot be predicted with scaling arguments. In particular, as explained in section 2.3.3 depends on the renormalization point through the parameter $\gamma = a/\varepsilon = e^{\frac{6c_1'}{c}}$. Of totally different nature is the constant U as defined in eq. (3.147), which is an off-critical universal quantity, which does not depend on the renormalization point, and can be predicted by QFT argument. Equation (3.152) is of crucial importance to have a predictive power on numerical simulations on lattice theories, and we will devote the whole Chapter 5 to this matter.

An entropic version of the c-theorem

<div style="text-align: right">4</div>

In this chapter we explore the relations between the entanglement entropy, and Zamolodchikov [1986] c-function. The connection between these two quantities has been a fertile field of research since the original formulation of the theorem, and found broad applications. In the original work Zamolodchikov [1986] defines a c-function, which counts the loss of degrees of freedom for coarse-graining under renormalization. Entanglement entropy on the other hand counts the correlation between degrees of freedom of two distinct regions, so that a connection between the two is appealing.

As we will see in section 4.1 Zamolodchikov's c-function satisfies some very peculiar properties, such that in literature any function which satisfies these properties is called a c-function.

Casini & Huerta [2004, 2007] managed to define a c-function in (1+1)-dimensional theories by means of the mutual information, demonstrating all its properties using strong subadditivity and Lorentz invariance. In an original work by Cardy [1988] the difficulties of extending this theorem to higher dimensions were collected, and a proposal for a more suitable c-function (called a-function) was given. We have to wait until Komargodski & Schwimmer [2011] though to have the first convincing proof of existence of a c-function in more than two dimensions.

We will be only concerned with theories in $(1 + 1)$ dimensions, so that our next step will be to give the definition of the original c-function of Zamolodchikov.

4.1 The c-function

This section is devoted to the proof of Zamolodchikov's c-theorem. This theory states that there exist a function $c(t)$ of the couplings[1] which

- is monotonically decreasing along the RG flow, that is $\frac{d}{d\log t}c(t) \leq 0$[2],

- is stationary only at fixed points $\left.\frac{d}{d\log t}c(t)\right|_{t=t^*} = 0$,

- at these points its value corresponds to the central charge of the corresponding CFT, $c(t^*) = c$.

To prove these propositions we define a set of functions similar to those in section 3.5. We focus on the two point functions of the three components of the stress energy tensor $T, \bar{T}, \Theta$. For dimensional and spin reasons we can express them as follows

$$\langle T(z,\bar{z})T(0)\rangle = \frac{F(t)}{z^4} \tag{4.1}$$

$$\langle \Theta(z,\bar{z})T(0)\rangle = \langle T(z,\bar{z})\Theta(0)\rangle = \frac{G(t)}{tz^2}, \tag{4.2}$$

$$\langle \Theta(z,\bar{z})\Theta(0)\rangle = \frac{H(t)}{t^2}, \tag{4.3}$$

where F, G and H are dimensionless quantities, and we are using as renormalization parameter $t = z\bar{z}$. Now we use the two point functions between the conservation law (3.15) of Θ and T to find

$$\begin{aligned}
\frac{d}{d\log t}F + \frac{1}{4}\left(\frac{d}{d\log t}G - 3G\right) &= 0 \\
\frac{d}{d\log t}G - G + \frac{1}{4}\left(\frac{d}{d\log t}H - 2H\right) &= 0.
\end{aligned} \tag{4.4}$$

Combining these two equations and defining $c = 2F - G - \frac{3}{8}H$ we find eventually

$$\frac{d}{d\log t}c(t) = -\frac{3}{4}H(t). \tag{4.5}$$

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[1]t being a renormalization parameter on which all the couplings depend.

[2]here t is a renormalization variable which has the same meaning as in eq. (3.132)

Remarkably tracelessness of the stress energy tensor implies that the critical points are stationary points for $c(t)$. Furthermore they are the only ones, because any other point could be reached by perturbing the CFT as in section 3.2, and as a consequence of eq. (3.19) the stress energy tensor would have a non-vanishing trace. For the same reason at a fixed point $c(t^*) = 2F(t^*)$, and from eq. (2.33) follows trivially that $F(t^*) = \frac{c}{2}$, hence $c(t^*) = c$. Finally to prove that $c(t)$ decreases along the RG flow we need some remarks. As Θ is a scalar operator real time Hermitian conjugation corresponds to time inversion. In the Euclidean theory clearly the concept of Hermitian conjugation does not exist, but we can extend it by considering Euclidean time reflection. As the original theory enjoys Hermitian positivity[1] we expect that

$$\langle \Theta(x, \tau) \Theta(x, -\tau) \rangle \geq 0. \tag{4.6}$$

Moreover, due to Lorentz invariance, this statement holds for two fields taken at any position $\langle \Theta(x) \Theta(y) \rangle \geq 0$. This property is called *reflection positivity*[2], and from it follows trivially $\frac{d}{d \log t} c(t) \leq 0$.

We can now integrate this function in the same way we did for eq. (3.136), and define the integral c-function

$$c(r_0) = \frac{3}{2} \int_{r_0}^{\infty} dr \ \ r^3 \langle \Theta(r) \Theta(0) \rangle. \tag{4.7}$$

4.2   Connections between $c(r)$ and $\Delta(r)$

The rather special properties of Zamolodchikov c-function suggest that it could be related to some entropic function. In chapters 2 and 3 we developed a method based on the twist field $\mathcal{T}$ to evaluate entanglement entropy. We know from eq. (2.67) that the conformal weight of the twist field is related to the central charge c, so that we suspect that this might be the bridge between a c-function and the entanglement entropy. At least at the critical point this is the case, so

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[1]this is a total general requirement of a probabilistic interpretation of quantum mechanics.
[2]Osterwalder & Schrader [1973].

that we are tempted to extend it outside criticality, that is to say

$$\Delta_{\mathcal{T}}(r) = \frac{c(r)}{24}\left(n - \frac{1}{n}\right),\qquad(4.8)$$

where $\Delta_{\mathcal{T}}(r)$ is the Δ-function (3.136) while $c(r)$ is Zamolodchikov c-function (4.7), both evaluated in the same renormalization point.

It can be checked both analytically and numerically that eq. (4.8) does not hold in general. A clear example of this is the Ising model, for which the two functions are reported in figure 4.1. We notice though from figures 3.1 and 4.1 that $\Delta_{\mathcal{T}}(r)$
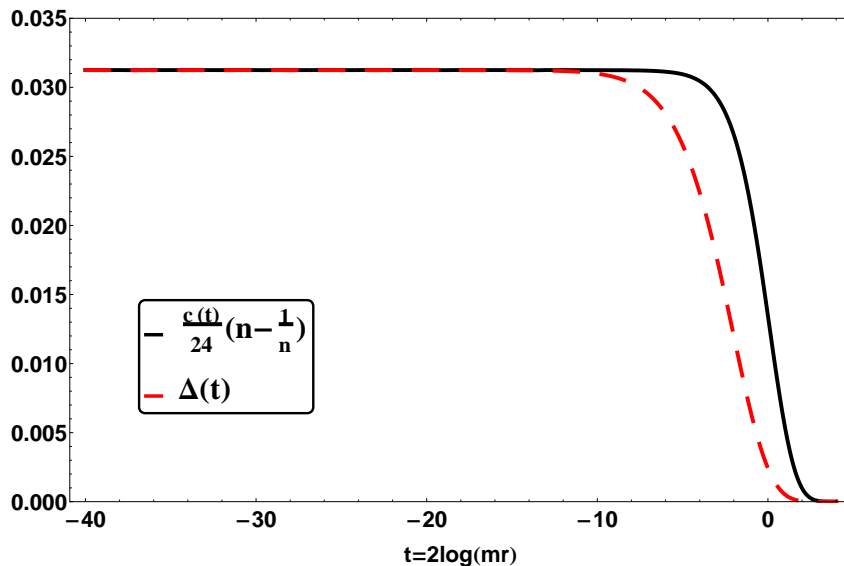
Figure 4.1: In this picture we show the difference between $\Delta(t)$, and $c(t)$ for two copies of the Ising model ($n = 2$). We carried out the same analysis until $n = 10$, finding the same qualitative behaviour. The scale is logarithmic, and we can see that, even if at the critical point $\Delta_{\mathcal{T}}(0) = \frac{c(0)}{24}\left(n - \frac{1}{n}\right)$, along the RG flow this equality does not hold anymore.

seems to have all the qualitative features shown by Zamolodchikov c-function. We want then to prove that this is actually the case, and conclude that the Δ-function of the twist field is a c-function.

The twist field exists only when we consider n copies of a model, then we can

express eq. (3.132) as

$$\frac{d\Delta_{\mathscr{T}}(r)}{dr} = \frac{r\left(\langle\Theta(r)\mathscr{T}(0)\rangle_{(\mathbb{R}^2)_n} - \langle\Theta\rangle_{(\mathbb{R}^2)_n}\langle\mathscr{T}\rangle_{(\mathbb{R}^2)_n}\right)}{2\langle\mathscr{T}\rangle_{(\mathbb{R}^2)_n}}, \tag{4.9}$$

where we denote with $(\mathbb{R}^2)_n$ the expectation values obtained considering n disconnected copies. The integral expression (3.136) becomes then

$$\Delta_{\mathscr{T}}(r) = -\frac{1}{2}\int_r^\infty ds\, s\left(\langle\Theta(s)\rangle_{\mathscr{M}_0^n} - n\langle\Theta\rangle_{\mathbb{R}^2}\right), \tag{4.10}$$

where $\mathscr{M}_0^n$ is the n sheeted Riemann manifold defined by the insertion of a branch point twist field $\mathscr{T}$ at the origin. The presence of n in front of the vev of the trace Θ in the single copy theory is a consequence of considering n copies of the model, so that in that term we have the sum of n non-interacting stress energy tensors.

If we manage to demonstrate that

$$\langle\Theta(s)\rangle_{\mathscr{M}_0^n} - n\langle\Theta\rangle_{\mathbb{R}^2} < 0 \qquad\qquad \forall r \in (0,\infty), \tag{4.11}$$

then we can prove positivity $\Delta(r) \geq 0$, and the monotonicity property $\dot\Delta(r) < 0$[1].
Notice that eq. (4.10) converges by factorization of correlation functions at large distances, and automatically implements $\Delta(\infty) = 0$. For the same reason we have that $\dot\Delta(r)$ vanishes in the IR limit. Clearly it vanishes also at the UV point due to tracelessness of the stress energy tensor, so that if we manage to prove eq. (4.11) true, we can conclude that the UV and IR points are the only stationary points, and this would prove all the wanted properties.

It is interesting to note that these properties do not necessarily hold for operators other than the branch point twist field. The results of Castro-Alvaredo & Fring [2001b] (Figs. 2 and 3) and Castro-Alvaredo & Fring [2001c] (Fig. 3) show explicit examples of fields for which the monotonicity property of $\Delta(r)$ does not hold.
Reflection positivity, used to prove the c-theorem, does not work here, so that we have to develop new arguments. We approach the problem in two different ways

- analysis of the IR and UV regions using form factor expansions and per-

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[1]this feature was also observed in Calabrese & Cardy [2004].

turbed CFT, respectively,

- general QFT-based intuitive arguments.

These two different attempts to prove eq. (4.11) will be presented respectively in sections 4.2.1 and 4.2.2. Finally we will present a thorough analysis of the Ising model in section 4.3.

4.2.1 Neighbourhood of the IR and UV fixed points

We begin by considering the large distance region. In the two-particle approximation we write

$$\langle \Theta(r) \rangle_{\mathcal{M}_0^n} - n \langle \Theta \rangle_{\mathbb{R}^2} \simeq \sum_{a,b=1}^{n} \int_{-\infty}^{\infty} \frac{d\theta_1 d\theta_2}{2(2\pi)^2} F_2^{\Theta|ab}(\theta_1,\theta_2) \left(\frac{F_2^{\mathcal{T}|ab}(\theta_1,\theta_2)}{\langle \mathcal{T} \rangle} \right)^* e^{-rm(\cosh\theta_1 + \cosh\theta_2)},$$

(4.12)

where for simplicity we assume that we have, on the right-hand side, n copies of a QFT with a single particle spectrum. The functions above are defined as $F_2^{\mathcal{O}|ab}(\theta_1,\theta_2) := \langle 0|\mathcal{O}(0)|\theta_1\theta_2\rangle_{ab}$, where $|0\rangle$ is the vacuum, $|\theta_1\theta_2\rangle_{ab}$ is a two-particle asymptotic state, and $\theta_{1,2}$ are rapidities.

For integrable models, the twist field two particle form factor is the one reported in eq. (3.63), and here we assume the form

$$F_2^{\mathcal{T}|ab}(\theta_1,\theta_2) = \frac{\frac{\langle \mathcal{T} \rangle}{2n} \sin\left(\frac{\pi}{n}\right)}{\sinh\left(\frac{i\pi(2(a-b)-1)+\theta}{2n}\right) \sinh\left(\frac{i\pi(2(b-a)-1)-\theta}{2n}\right)} \frac{F_{\min}^{ab}(\theta,n)}{F_{\min}^{ab}(i\pi,n)},$$

(4.13)

where we express explicitly the dependence on n of the minimal form factor. On the other hand the Θ form factor

$$F_2^{\Theta|ab}(\theta_1,\theta_2) = 2\pi m^2 \frac{F_{\min}^{ab}(\theta,1)}{F_{\min}^{ab}(i\pi,1)} \delta_{ab},$$

(4.14)

depends on the single copy minimal form factor. The normalization of eq. (4.14), $F_2^{\Theta|aa}(i\pi) = 2\pi m^2$, is fixed as explained in Mussardo & Simonetti [1994].

Inserting these expressions into (4.12), we can write

$$\langle \Theta(r) \rangle_{\mathcal{M}_0^n} - n \langle \Theta \rangle_{\mathbb{R}^2} = -\frac{m^2 \sin \frac{\pi}{n}}{2\pi} \int_{-\infty}^{\infty} dx \frac{K_0(2mr \cosh \frac{x}{2})}{\cosh \frac{x}{n} - \cos \frac{\pi}{n}} \frac{F_{\min}^{11}(x,1)}{F_{\min}^{11}(i\pi,1)} \frac{F_{\min}^{11}(x,n)^*}{F_{\min}^{11}(i\pi,n)^*} \quad (4.15)$$

where $K_0(t)$ is a modified Bessel function.

Clearly, the sign of (4.15) is only determined by the minimal form factor product, as all the other quantities in the integrand are positive. We can express the minimal form factor through the integral representation (eq. (3.62))

$$F_{\min}^{11}(x,n) = \exp \int_0^{\infty} \frac{dt \, f(t)}{t \sinh(nt)} \sin^2 \left[\frac{it}{2} \left(n + \frac{ix}{\pi} \right) \right], \quad (4.16)$$

where $f(t)$ can be extracted from the S-matrix with the help of eq. (3.40). The minimal form factor is in general a complex function, however the product

$$F_{\min}^{11}(x,1) F_{\min}^{11}(x,n)^* = \exp \int_0^{\infty} \frac{dt}{2t} f(t) \left(\frac{1 - \cos \frac{tx}{\pi} \cosh nt}{\sinh nt} + \frac{1 - \cos \frac{tx}{\pi} \cosh t}{\sinh t} \right), \quad (4.17)$$

is real and positive. This proves (4.11) near the infrared fixed point (mr large). In addition, the presence of the exponential (4.12) ensures that the value of the integral is larger for smaller values of mr. Note that for fields other than the branch point twist field, there is no reason to expect that the present argument, which depends on the particular form of the form factors, gives negativity.

Let us now turn to the short distance behavior of $\langle \Theta(s) \rangle_{\mathcal{M}_0^n} - n \langle \Theta \rangle_{\mathbb{R}^2}$, using the conformal perturbation formalism defined in section 3.2.

On dimensional grounds, the coupling constant g is related to a mass scale m as $g \sim m^{2-2\Delta}$, and we will take $g > 0$ and ϕ "positive" so that the spectrum of the theory is bounded from below. We can use eq. (3.19) to relate the trace of the stress energy tensor to the perturbing field ϕ, so that we can focus on the two point function between ϕ and the twist field. For $\Delta < 1$ Zamolodchikov [1989] showed that eq. (3.19) is exact in the sense that no higher order corrections in g occur. The expectation value $\langle \Theta(r) \rangle_{\mathcal{M}_0^n}$ can be evaluated through the operator

product expansion (OPE) eq. (3.13), such that

$$\frac{\langle \phi(r)\mathcal{T}(0)\rangle_{(\mathbb{R}^2)_n}}{\langle \mathcal{T}\rangle_{(\mathbb{R}^2)_n}} = \sum_{\mu=0}^{\infty} C_{\phi\mathcal{T}}^{\mu}(r)\frac{\langle A_\mu\rangle_{(\mathbb{R}^2)_n}}{\langle \mathcal{T}\rangle_{(\mathbb{R}^2)_n}}, \tag{4.18}$$

in terms of some fields A_μ of the massive QFT. Considering the zeroth order of conformal perturbation theory, we directly replace the structure functions by their CFT value. The leading term of the expansion (4.18) will involve a field A_0, written as the composite field $: \phi\mathcal{T} :$,

$$\frac{\langle \phi(r)\mathcal{T}(0)\rangle_{(\mathbb{R}^2)_n}}{\langle \mathcal{T}\rangle_{(\mathbb{R}^2)_n}} = \tilde{C}_{\phi\mathcal{T}}^{:\phi\mathcal{T}:} r^{2(\Delta_{:\phi\mathcal{T}:}-\Delta-\Delta_\mathcal{T})}\frac{\langle :\phi\mathcal{T}:\rangle_{(\mathbb{R}^2)_n}}{\langle \mathcal{T}\rangle_{(\mathbb{R}^2)_n}} + \cdots \tag{4.19}$$

It is possible to fix $\Delta_{:\phi\mathcal{T}:}$ by comparing the OPE above to the standard CFT computation of a correlation function of the form:

$$\frac{\langle \phi(z,\bar{z})\mathcal{T}(0)\mathcal{O}(x,\bar{x})\rangle_{(\mathbb{R}^2)_n}}{\langle \mathcal{T}\rangle_{(\mathbb{R}^2)_n}} = \langle \phi(z,\bar{z})\mathcal{O}(x,\bar{x})\rangle_{\mathcal{M}_0^n}$$
$$= \frac{r^{2\Delta(\frac{1}{n}-1)}}{n^{2\Delta}}\langle \phi(0)(f * \mathcal{O})(f(x),f(\bar{x}))\rangle_{\mathbb{R}^2} + ... \quad , \tag{4.20}$$

where $r = |z| = \sqrt{z\bar{z}}$ and $f(z) = z^{\frac{1}{n}}$ is the conformal transformation that unravels the Riemann sheets conformally mapping them to $\mathbb{R}^2$. Here $\mathcal{O}$ is an arbitrary product of local fields, not necessarily primary, at positions represented by the sets $x, \bar{x}$. From the comparison between (4.19) and (4.20) we can fix

$$\Delta_{:\mathcal{T}\phi:} = \frac{\Delta}{n} + \Delta_\mathcal{T} \quad \text{and} \quad \tilde{C}_{\phi\mathcal{T}}^{:\phi\mathcal{T}:} = \frac{1}{n^{2\Delta}}. \tag{4.21}$$

We can then use (4.19), and (4.21) to write

$$\langle \Theta(r)\rangle_{\mathcal{M}_0^n} - n\langle \Theta\rangle_{\mathbb{R}^2} = m^2 n\left(\frac{\alpha\beta(mr)^{2\Delta(\frac{1}{n}-1)}}{n^{2\Delta-1}} - \mu\right) + \cdots \tag{4.22}$$

where $4\pi g(1-\Delta) = \alpha m^{2-2\Delta}$, $\langle \Theta\rangle_{\mathbb{R}^2} = \mu m^2$ and

$$\frac{\langle :\phi\mathcal{T}:\rangle_{(\mathbb{R}^2)_n}}{\langle \mathcal{T}\rangle_{(\mathbb{R}^2)_n}} = \beta m^{\frac{2\Delta}{n}}, \tag{4.23}$$

and α, β and μ are all dimensionless constants. Clearly $\alpha > 0$ if $\Delta \leq 1/2$ and $g > 0$. Hence, negativity of (4.22) at short distances requires $\beta < 0$. Although expectation value $\langle \mathcal{T} \rangle_{(\mathbb{R}^2)_n}$ is positive as it represents the partition function of the theory on the manifold $\mathcal{M}_0^n$, we are not able to give a derivation of the negativity of $\langle : \phi \mathcal{T} : \rangle_{(\mathbb{R}^2)_n}$ in this context, so that we need some more general arguments to prove it.

4.2.2 General arguments

In this section we provide model-independent arguments, based on expected physical properties of unitary models, strongly suggesting that (4.11) holds for arbitrary values of mr. Here, we use in an essential way the geometric interpretation of the branch point twist field, hence these arguments do not apply to any other field.

Note that proving (4.11) (for $n > 1$) is equivalent to showing that

$$\frac{\partial}{\partial n} \left(\langle \Theta(r) \rangle_{\mathcal{M}_0^n} \right) < 0. \tag{4.24}$$

Indeed at $n = 1$ we have $\langle \Theta(s) \rangle_{\mathcal{M}_0^1} = \langle \Theta \rangle_{\mathbb{R}^2}$. Similarly, if

$$\frac{\partial}{\partial r} \left(\langle \Theta(r) \rangle_{\mathcal{M}_0^n} \right) > 0, \tag{4.25}$$

then (4.11) follows, because of factorization of correlation functions at large r. In particular, establishing (4.25) immediately shows negativity of the coefficient β in (4.23).

Our main argument uses the idea of virtual particle propagation. We re-interpret unitarity as "positivity" of the perturbing field ϕ (hence of Θ): ϕ should be an appropriate normal-ordered (i.e. renormalized) product of an operator ψ and its hermitian conjugate, $\phi = (\psi^\dagger \psi)$, in analogy with the factorization of positive-definite matrices. Then contributions to the expectation value $\langle \Theta(r) \rangle_{\mathcal{M}_0^n}$ come from virtual particles created and annihilated at the point $(r, 0)$, and propagating in $\mathcal{M}_0^n$. Every path contributes a positive amplitude proportional to the exponential of minus the single-particle Euclidean action (i.e. the Brownian motion measure of the path), with possible branching due to interactions. Some of these paths

go around the origin. As the angle around the origin $2\pi n$ increases, these self-interaction contributions become less important, because the distance traveled is greater. Whence the derivative with respect to n is negative, giving (4.24). A similar argument leads to (4.25) for $n > 1$. Indeed, as r decreases, more and more self-interaction loops must travel around the origin, hence giving lesser contributions.

We give a qualitative picture of this situation in figure 4.2.
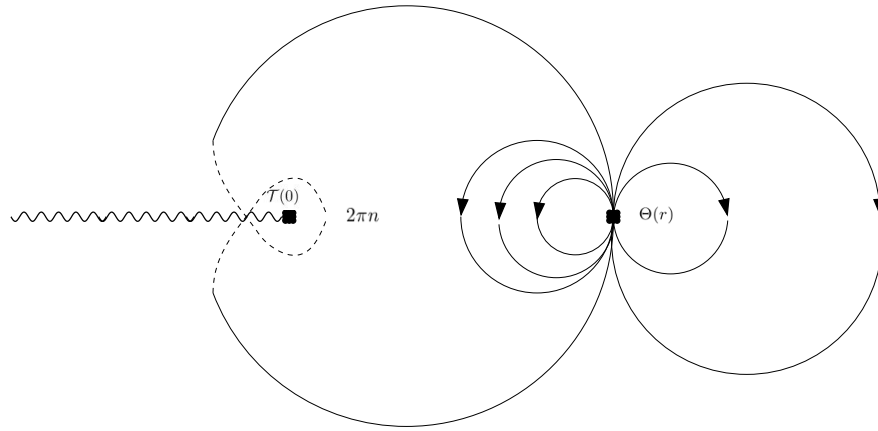
Figure 4.2: In this picture we are representing the self-interaction contributions to the stress energy tensor with an arrowed solid line. We chose to place the branch cut on the negative real axis. The dashed line represent those contributions which are forced to loop around other copies to close.

A way to study the self-interaction loops around the origin is to use angular quantization. Let us consider as an example the Klein-Gordon theory, and explicitly show (4.24) in this case. Angular quantization was developed quite generally in Brazhnikov & Lukyanov [1998]; Lukyanov [1995] in the context of form factors in integrable models; the Klein-Gordon angular quantization described in Brazhnikov & Lukyanov [1998] allows us to evaluate correlation functions. The construction of the branch point twist fields in angular quantization was described in Cardy *et al.* [2008]. Let us summarize few key ingredients. We are interested in the operator $\Theta \propto\ :\varphi^2:$ where φ is the Klein-Gordon field; the normal-ordering is a point-splitting regularization, with a subtraction proportional to the identity. We first compute the two-point function $\langle\varphi(r,0)\varphi(r',0)\rangle$, then take the limit

$r \to r'$. In the angular quantization approach, correlation functions are expressed as traces over the space of field configurations on the half-line (representation denoted by π_Z). The density matrix used in this trace is the operator performing a rotation by the angle necessary to go around the origin. The corresponding conserved charge associated to rotation, denoted by K, is the Hamiltonian of the theory. The presence of the branch-point twist field means that the angle around the origin is $2\pi n$. Hence, the density matrix is $e^{2\pi i n K}$:

$$\langle \cdots \rangle_{\mathcal{M}_0^n} = \frac{\mathrm{Tr}_{\pi_Z}\left(e^{2\pi i n K} \pi_Z(\cdots)\right)}{\mathrm{Tr}_{\pi_Z}\left(e^{2\pi i n K}\right)}. \tag{4.26}$$

For the Klein-Gordon theory, π_Z is a representation of the Heisenberg algebra with oscillators b_ν that satisfy $[b_\nu, b_{\nu'}] = 2\sinh(\pi\nu)\delta(\nu + \nu')$. Then, the following relation holds Cardy *et al.* [2008]: $\frac{\mathrm{Tr}_{\pi_Z}(e^{2\pi i n K} b_\nu b_{\nu'})}{\mathrm{Tr}_{\pi_Z}(e^{2\pi i n K})} = e^{\pi n \nu} \frac{\sinh(\pi\nu)}{\sinh(\pi n \nu)} \delta(\nu + \nu')$. Further, the bosonic field is expressed as is Brazhnikov & Lukyanov [1998]

$$\pi_Z(\varphi(r,0)) = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} d\nu\, b_\nu K_{i\nu}(mr) \quad \text{with} \quad K_{i\nu}(mr) = \frac{1}{2} \int_{-\infty}^{\infty} d\theta\, e^{-mr\cosh\theta} e^{i\nu\theta}. \tag{4.27}$$

Employing all these definitions, the two-point function can be written as

$$\langle \varphi(r,0)\varphi(r',0) \rangle_{\mathcal{M}_0^n} = \frac{4}{\pi} \int_{-\infty}^{\infty} d\nu\, e^{\pi n \nu} \frac{\sinh(\pi\nu)}{\sinh(\pi n \nu)} K_{i\nu}(mr) K_{-i\nu}(mr'). \tag{4.28}$$

Since the conformal point is a free boson, this function diverges logarithmically when $r \to r'$. However, this divergence is independent of n, whence we differentiate then take $r = r'$:

$$\frac{d\langle :\varphi(r,0)^2: \rangle_{\mathcal{M}_0^n}}{dn} = -4 \int_{-\infty}^{\infty} d\nu\, \nu \frac{\sinh(\pi\nu)}{\sinh^2(\pi n \nu)} |K_{i\nu}(mr)|^2 < 0 \quad \text{for} \quad mr \neq 0. \tag{4.29}$$

This establishes (4.24) for the Klein-Gordon theory.

For more general, unitary models, the argument goes as follows. The operator K, as it does above, should have positive imaginary eigenvalues in order for the trace (4.26) to be well-defined. Hence, let us write $iK = -J$ for a positive operator

J. Differentiating with respect to n, we find

$$\frac{d}{dn} \langle \phi(r,0) \rangle_{\mathcal{M}_0^n} = -2\pi \left(\langle J \phi(r,0) \rangle_{\mathcal{M}_0^n} - \langle J \rangle_{\mathcal{M}_0^n} \langle \phi(r,0) \rangle_{\mathcal{M}_0^n} \right). \tag{4.30}$$

Since the measure is rotation invariant, and since J is proportional to the generator of rotations, we have $\langle [J, \phi(r,0)] \rangle_{\mathcal{M}_0^n} = 0$. Hence, J and $\phi(r,0)$ can be interpreted as "classical" statistical variables, and the derivative with respect to n is the negative of their statistical correlation. We expect this statistical correlation to be positive: the statistical variable J is an "energy", composed of a kinetic energy (the conformal part) and a potential energy V (the perturbation by ϕ). Indeed, a moment's thought shows that if the average potential energy $\langle V \rangle_J$ at fixed total energy J increases with J, as should be expected, then J is positively correlated with V. Finally, let us further justify the latter angular-quantization argument through a drastic simplification. Instead of propagating a half-line around the origin for an angle of $2\pi n$, we reduce to a finite number of degrees of freedom: we consider the propagation of a quantum mechanical particle along a circle of circumference $2\pi n$. This simplification is expected to provide the right sign of the variation with respect to n, which comes from particles propagating around the origin. The operator J is replaced by the Hamiltonian H of the quantum system, and the perturbing field $\phi(r)$ is replaced by the potential energy V. The trace becomes

$$\frac{\int dx\, V(x) \langle x| e^{-2\pi n H} |x \rangle}{\int dx\, \langle x| e^{-2\pi n H} |x \rangle}. \tag{4.31}$$

Since quantum mechanics in imaginary time corresponds to a stochastic problem, we need to evaluate the average of the potential $V(x)$, with an un-normalized measure given, for any value of the position x, by the probability for a random walk in that potential to start and end at x in a time $2\pi n$. As time increases, this probability decreases for any x. However, at lower values of the potential, nearer to the absolute minimum, the additional time given to the particle is more likely to be spent near to its original position than it is at larger values, because the particle has a tendency to fall back to the minimum of the potential. Hence, as time increases, lower values of the potential get more relative weights. This implies that the average of the potential decreases as the time $2\pi n$ increases.

4.3 Perturbative renormalization analysis of the Ising model

In the previous sections we have demonstrated the validity of (4.11) for general unitary theories, giving as a specific example the Klein-Gordon case. Here we focus in much more detail on the Ising field theory, proving again (4.11) true. As a byproduct we will gain some insight on the modification of the OPE in both critical and massive theories induced by the introduction of a twist field.

The correlation function involved in (4.9) is known exactly for the Ising model. In this case, the two-particle approximation (4.12) is exact, and we can use eq. (3.139) to write

$$\langle \Theta(r) \rangle_{\mathcal{M}_0^n} - n \langle \Theta \rangle_{\mathbb{R}^2} = -\frac{m^2}{\pi} \cos \frac{\pi}{2n} \int\limits_{-\infty}^{\infty} dx \frac{K_0(2mr \cosh \frac{x}{2}) \sinh \frac{x}{2n} \sinh \frac{x}{2}}{\cosh \frac{x}{n} - \cos \frac{\pi}{n}}. \qquad (4.32)$$

This shows negativity for all $0 < mr < \infty$.

In section 3.2.1 we have seen that this model corresponds to a perturbation of the CFT Ising model by the energy operator $\varepsilon(x)$. We know moreover that this operator is related to the stress energy tensor by eq. (3.19), so that we prefer working with $\varepsilon(x)$ directly.

In the off critical model we need extra care to define ε as, due to the presence of m, there is a mixing with the identity operator. Instead of just being proportional to $\bar{\psi}\psi$ as in the critical theory here we have

$$\varepsilon = a\bar{\psi}\psi + bm\mathbb{1} \qquad \text{with} \quad a, b \in \mathbb{R} \setminus \{0\}. \qquad (4.33)$$

We fix the constants a and b in eq. (4.33) with the normalization

$$\varepsilon(x) = 2\pi \left(m\mathbb{1} + : \bar{\psi}(x)\psi(x) : \right), \qquad (4.34)$$

where $aa^\dagger =: aa^\dagger : + \mathbb{1}$ is the definition of normal ordering in this case. The two-particle form factor can be easily extracted using (3.23)

$$F_2^\varepsilon(\theta_1, \theta_2) = \langle 0 | \varepsilon(0) | \theta_1 \theta_2 \rangle = -i\frac{m}{2} \int d\phi d\eta \, e^{\frac{\phi - \eta}{2}} \langle 0 | a(\eta) a(\phi) a^\dagger(\theta_1) a^\dagger(\theta_2) | 0 \rangle =$$
$$= -im \sinh \frac{\theta_1 - \theta_2}{2}, \tag{4.35}$$

where the Wick theorem for Fermion algebrae is used to define contractions. The normalization (4.34) is chosen in light of the relation between ε and the trace of the stress energy tensor $\Theta(x) = 2\pi m \varepsilon(x)$ [1], to give $F_2^\Theta(i\pi) = 2\pi m^2$.

Clearly also for ε only the two-particle form factor is non-vanishing and it is possible to find an exact integral representation for the correlation function[2]

$$\langle \varepsilon(r) \mathscr{T}(0) \rangle = \langle \varepsilon \rangle \langle \mathscr{T} \rangle - \frac{\langle \mathscr{T} \rangle m}{2\pi^2} \cos \frac{\pi}{2n} \int\limits_{-\infty}^{\infty} dx \frac{K_0(2mr \cosh \frac{x}{2}) \sinh \frac{x}{2n} \sinh \frac{x}{2}}{\cosh \frac{x}{n} - \cos \frac{\pi}{n}}, \tag{4.36}$$

where $K_0(\phi)$ is the modified Bessel function of the second kind with argument ϕ. In the CFT, where the correlation length tends to infinity, one is allowed to use the OPE

$$\varepsilon(r) \mathscr{T}(0) = \sum_k \tilde{C}_{\varepsilon\mathscr{T}}^k r^{2(\Delta_k - \frac{1}{2} - \Delta_\mathscr{T})} A_k(0), \tag{4.37}$$

where A_k is a basis of fields, and $\tilde{C}_{\varepsilon\mathscr{T}}^k$ are the dimensionless constants of the expansion. The most relevant operator appearing in (4.37) is the composite twist field $\mathscr{O}_0 \equiv : \varepsilon\mathscr{T} :$, and it is defined implicitly as the twist operator which corresponds to the leading term. Its conformal weight and structure constant with ε and $\mathscr{T}$ can be obtained using (4.21), and they are

$$\Delta_{:\varepsilon\mathscr{T}:} = \frac{1}{2n} + \Delta_\mathscr{T} \quad \text{and} \quad \tilde{C}_{\varepsilon\mathscr{T}}^{:\varepsilon\mathscr{T}:} = \frac{1}{n}. \tag{4.38}$$

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[1]Zamolodchikov [1989]

[2]from now on the subscript labelling the manifold VEVs are evaluated on will be omitted. It is implicit that $\langle \mathscr{T} \rangle$ is always evaluated on n disconnected copies of $\mathbb{R}^2$, while other VEVs are on the real plane.

We know that, due to the arbitrariness in the choice of the argument of [1] $:\varepsilon\mathcal{T}:$, also derivatives of this field play a role in the OPE (4.37). We are able to fix the weight and structure constants for these corrections, and to identify $\mathcal{O}_\alpha \equiv\, :\partial^{2\alpha}\varepsilon\mathcal{T}:$ with $\alpha \in \mathbb{N}$

$$\Delta_{:\partial^{2\alpha}\varepsilon\mathcal{T}:} = \frac{1+2\alpha}{2n} + \Delta_\mathcal{T} \quad \text{and} \quad \tilde{C}^{:\partial^{2\alpha}\varepsilon\mathcal{T}:}_{\varepsilon\mathcal{T}} = \frac{1}{n\alpha!^2}, \tag{4.39}$$

where $\partial^{2\alpha} \equiv (\partial^2_{z\bar{z}})^\alpha$. Notice that since ε and $\mathcal{T}$ are both spinless operators only these particular derivatives can contribute to the expansion. From now on we will refer directly to the operators $:\partial^{2\alpha}\varepsilon\mathcal{T}:$, denoting $:\varepsilon\mathcal{T}:$ as the case $\alpha = 0$. We can use the perturbation process explained in section 3.2 to formulate the massive OPE, in particular with the help of eq. (3.14) we can write

$$C^k_{\varepsilon\mathcal{T}}(r) = \tilde{\mathscr{C}}^k r^{2(\Delta_k - \frac{1}{2} - \Delta_\mathcal{T})} \left[1 + \mathscr{C}^k_1 mr + \mathscr{C}^k_2 (mr)^2 + \dots \right]. \tag{4.40}$$

By means of (4.38) and (4.39) we can express this expansion for the composite twist fields as

$$C^{:\partial^{2\alpha}\varepsilon\mathcal{T}:}_{\varepsilon\mathcal{T}}(r) = \frac{r^{\frac{1+2\alpha}{n}-1}}{n(2\alpha)!} \left[1 + \mathscr{C}^{:\partial^{2\alpha}\varepsilon\mathcal{T}:}_1 mr + \mathscr{C}^{:\partial^{2\alpha}\varepsilon\mathcal{T}:}_2 (mr)^2 + \dots \right] \qquad \text{for} \quad \alpha \in \mathbb{N}_0. \tag{4.41}$$

Fixing the perturbation constants appearing in (4.41) is generally a very hard task already for the first order. However, due to the special nature of the OPE under consideration we are able to determine all $\mathscr{C}^{:\partial^{2\alpha}\varepsilon\mathcal{T}:}_j$ in a systematic way, and this will be the object of section 4.3.1. Therefore we expect the OPE in the massive theory to take the form

$$\varepsilon(r)\mathcal{T}(0) = \sum_{\alpha=0}^{\infty} \mathscr{C}^{:\partial^{2\alpha}\varepsilon\mathcal{T}:}_{\varepsilon\mathcal{T}}(r) :\partial^{2\alpha}\varepsilon\mathcal{T}:(0), \tag{4.42}$$

although we will see in eq. (3.14) that further corrections to (4.42), that are not predictable from CFT arguments, also appear.

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[1] indeed the choice to take as argument 0 is arbitrary, and one could have chosen any point in the interval $[0, r]$. The difference between these choices is represented by a Taylor expansion about 0, therefore the OPE (4.37) should include all derivatives of the field $:\varepsilon\mathcal{T}:$.

4.3.1 Computation of $\langle : \partial^{2\alpha} \varepsilon \mathscr{T} : \rangle$ and $\mathscr{C}_j^{:\partial^{2\alpha} \varepsilon \mathscr{T}:}$

Let us now proceed by expanding the integral in the RHS of (4.36) in a small r region, and then compare the result with the massive OPE (4.42), that is we compare terms with the same dimensions. In order to extract all the information needed it is convenient first to reexpress the integral in terms of the quantity $t = mr e^{\frac{x}{2}}$

$$-\frac{m}{\pi^2} \cos \frac{\pi}{2n} (mr)^{\frac{1}{n}-1} \int\limits_{mr}^{\infty} dt \; t^{-\frac{1}{n}} \; K_0\left(t + \frac{(mr)^2}{t}\right) \frac{\left[1 - \left(\frac{t}{mr}\right)^{-2}\right]\left[1 - \left(\frac{t}{mr}\right)^{-\frac{2}{n}}\right]}{\left(\frac{t}{mr}\right)^{-\frac{4}{n}} - 2\cos\frac{\pi}{n}\left(\frac{t}{mr}\right)^{-\frac{2}{n}} + 1}. \quad (4.43)$$

One can notice that leading contributions to this integral are given for large $t/(mr)$. This provides a natural parameter over which is possible to expand the fraction in (4.43) as

$$\frac{\left[1 - \left(\frac{t}{mr}\right)^{-2}\right]\left[1 - \left(\frac{t}{mr}\right)^{-\frac{2}{n}}\right]}{\left(\frac{t}{mr}\right)^{-\frac{4}{n}} - 2\cos\frac{\pi}{n}\left(\frac{t}{mr}\right)^{-\frac{2}{n}} + 1} = \sum_{\alpha=0}^{\infty} \Omega_\alpha(n)\left(\frac{t}{mr}\right)^{-\frac{2\alpha}{n}}, \quad (4.44)$$

where the coefficients $\Omega_\alpha(n)$ are real numbers, evaluated in Section B.1. Once (4.44) is plugged into (4.43) one gets

$$-\frac{m}{\pi^2} \cos \frac{\pi}{2n} \sum_{\alpha=0}^{\infty} \Omega_\alpha(n)(mr)^{\frac{1+2\alpha}{n}-1} \int\limits_{mr}^{\infty} dt \; t^{-\frac{1+2\alpha}{n}} K_0\left(t + \frac{(mr)^2}{t}\right). \quad (4.45)$$

The dependence on mr of the Bessel function can be extracted by expanding it for small r as

$$K_0\left(t + \frac{(mr)^2}{t}\right) = K_0(t) - \frac{K_1(t)}{t}(mr)^2 + \frac{K_0(t) + K_2(t)}{4t^2}(mr)^4 + O(mr)^6. \quad (4.46)$$

The resulting integrals can be thought of as a special case of a known integral of the Maijer G-function, as explained in Appendix B.2. Substituting (4.46) into

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(4.45) and carrying out the integrals gives

$$
\langle \varepsilon(r)\mathscr{T}(0)\rangle = -\frac{\langle\mathscr{T}\rangle m}{\pi^2}\cos\frac{\pi}{2n}\sum_{\alpha=0}^{\infty}\Omega_\alpha(n)\left[\frac{\Gamma\left(\frac{n-1-2\alpha}{2n}\right)^2}{2^{1+\frac{1+2\alpha}{n}}}(mr)^{\frac{1+2\alpha}{n}-1}+\right.
$$
$$
+\frac{n}{n+1+2\alpha}\frac{\Gamma\left(\frac{n-1-2\alpha}{2n}\right)^2}{2^{1+\frac{1+2\alpha}{n}}}(mr)^{\frac{1+2\alpha}{n}+1}+\left(\frac{n}{n+1+2\alpha}\right)^2\frac{\Gamma\left(\frac{n-1-2\alpha}{2n}\right)^2}{2^{2+\frac{2+2\alpha}{n}}}(mr)^{\frac{1+2\alpha}{n}+3}+
$$
$$
\left.-\frac{n^3}{(n+1+2\alpha)^2(3n+1+2\alpha)}\frac{\Gamma\left(\frac{n-1-2\alpha}{2n}\right)^2}{2^{2+\frac{2+2\alpha}{n}}}(mr)^{\frac{1+2\alpha}{n}+3}+\dots\right],
$$

(4.47)

where we reported only the first contributions related to the composite twist field and its descendants, as we are mainly interested in those. After a bit of manipulation we can compare term by term this expansion with (4.42), and by matching the terms with the same perturbative order we are able to extract the following VEVs

$$
\langle:\partial^{2\alpha}\varepsilon\mathscr{T}:\rangle = -\frac{\cos\frac{(1+2\alpha)\pi}{2n}(2\alpha)!n}{2^{1+\frac{1+2\alpha}{n}}\pi^2}\Gamma\left(\frac{n-1-2\alpha}{2n}\right)^2 m^{\frac{1+2\alpha}{n}}\langle\mathscr{T}\rangle. \qquad (4.48)
$$

This is the main result of this section. In the same way we can fix the constants $\mathscr{C}_j^{:\partial^{2\alpha}\varepsilon\mathscr{T}:}$ to every order. A major challenge when doing this is to be able, for terms proportional to the same power of r, to distinguish between contributions to expectation values and to structure constants. It turns out that this ambiguity can be resolved by requiring that the expectation values (4.48) are continuous functions of n for each fixed value of α. This requirement is natural because of the special relation between $\mathscr{T}$ and the entanglement entropy, in which context it is necessary to analytically continue all physical quantities to $n\in[1,\infty)$. The

first few non-vanishing coefficients are

$$\mathscr{C}_2^{:\partial^{2\alpha}\varepsilon\mathscr{T}:} = \frac{n}{n+1+2\alpha} + \frac{n^2}{(n+1+2\alpha)^2}\frac{\tan\frac{(1+2\alpha)\pi}{2n}}{\tan\frac{\pi}{2n}}$$

$$\mathscr{C}_4^{:\partial^{2\alpha}\varepsilon\mathscr{T}:} = \frac{n^2}{2(n+1+2\alpha)(3n+1+2\alpha)} + \frac{n^3(1+2\alpha+2n)}{(1+2\alpha+n)^2(1+2\alpha+3n)^2}\frac{\tan\frac{(1+2\alpha)\pi}{2n}}{\tan\frac{\pi}{2n}}$$

$$\mathscr{C}_6^{:\partial^{2\alpha}\varepsilon\mathscr{T}:} = \frac{n^3(4n+1+2\alpha)}{6(n+1+2\alpha)^2(3n+1+2\alpha)(5n+1+2\alpha)}$$
$$+ \frac{\sin\frac{(1+2\alpha)\pi}{2n}}{\sin\frac{\pi}{2n}}\frac{n^4(4n+1+2\alpha)}{2(n+1+2\alpha)^2(3n+1+2\alpha)^2(5n+1+2\alpha)}$$
$$\left(1-2n-\frac{2n^2}{1+3n+2\alpha}\right).$$

$$(4.49)$$

We find that $\mathscr{C}_{2j+1}^{:\partial^{2\alpha}\varepsilon\mathscr{T}:} = 0$, for $j \in \mathbb{N}_0$.

It is worth noticing that VEVs of these composite operators are singular whenever the argument of the Gamma-function is either zero or a negative integer. Analyzing (4.48) one can see that when $n = 1+2\alpha$ the VEV of the $2\alpha^{\text{th}}$ derivative is divergent. In other words, such singularities can only occur for odd values of n. Therefore, it follows that the analysis above is only really consistent when restricting n to be even, in which case no singularities for special values of n and α arise. In the n odd case, the singularities that occur at various orders of the expansion actually cancel each other, generating a well defined short distance expansion. This stark contrast between the n even and n odd cases is a priori rather surprising. The leading term in (4.42) is well defined for all values of $n > 1$, for both n even and odd so that our identification of the expectation value of $:\varepsilon\mathscr{T}:$, does still hold for general values of n.

In particular from eq. (4.48) we are able to prove the negativity of β in eq. (4.23), and extend the argument to higher perturbation orders. This in turn allow us to prove the validity of eq. (4.11) for the Ising field theory.

4.3.2 Logarithmic corrections to the massive OPE

Analyzing carefully the contributions reported in Appendix B.2, one can notice that further corrections involving terms of the form $(mr)^{2\alpha}$ and $(mr)^{2\alpha}\log(mr)$, with $\alpha = 0,1,\ldots$ occur (note for example the logarithmic terms in (B.10), and (B.11)), such that (4.42) is modified to

$$\varepsilon(r)\mathcal{T}(0) = \sum_{\alpha=0}^{\infty} \left[\mathcal{C}_{\varepsilon\mathcal{T}}^{:\partial^{2\alpha}\varepsilon\mathcal{T}:}(r) : \partial^{2\alpha}\varepsilon\mathcal{T} : (0) + m\mathcal{C}_{\varepsilon\mathcal{T}}^{\partial^{2\alpha}\mathcal{T}}(r)\partial^{2\alpha}\mathcal{T}(0) \right], \qquad (4.50)$$

where the new terms mentioned above are contained in the coefficients $\mathcal{C}_{\varepsilon\mathcal{T}}^{\partial^{2\alpha}\mathcal{T}}(r)$. It is clear that they are features of the massive theory as they have no counterpart in the CFT. To understand their presence one has to consider once more (4.33). The term proportional to $\bar{\psi}\psi$ is responsible for all the composite fields $: \partial^{2\alpha}\varepsilon\mathcal{T} :$ in the OPE, whereas the term proportional to the identity generates contributions proportional to $m\mathcal{T}$ and its derivatives $m\partial^{2\alpha}\mathcal{T}$. It is interesting that this paves the way for the evaluation of $\langle\partial^{2\alpha}\mathcal{T}\rangle$, although in this case, due to the structure of the expansion, this would need a resummation of infinitely many terms. The presence of logarithmic terms is imputable to the freedom in choosing b in (4.33). Indeed in the expansions carried out in Section B.2 we have general terms of the type

$$(mr)^{2\alpha} \sum_{\beta=0}^{\infty} \big(\Upsilon(\alpha,\beta,n) + \Lambda(\alpha,\beta,n)\log mr \big), \qquad (4.51)$$

where Υ and Λ are two rational functions of α, β and n. When a correction of the kind of (4.51) is plugged into (4.47) it gives for fixed α

$$-\frac{\langle\mathcal{T}\rangle m}{\pi^2} \cos\frac{\pi}{2n} (mr)^{2\alpha} \sum_{\beta=0}^{\infty} \Omega_\beta(n)\big(\Upsilon(\alpha,\beta,n) + \Lambda(\alpha,\beta,n)\log mr \big). \qquad (4.52)$$

The term containing Υ contributes to $\langle\varepsilon\rangle\langle\partial^{2\alpha}\mathcal{T}\rangle$. The presence of the logarithmic term allows us to rewrite (4.52) as

$$-\frac{\langle\mathcal{T}\rangle m}{\pi^2} \cos\frac{\pi}{2n} (mr)^{2\alpha} \sum_{\beta=0}^{\infty} \Omega_\beta(n)\Big(\Upsilon(\alpha,\beta,n) + \Lambda(\alpha,\beta,n)\log\delta + \Lambda(\alpha,\beta,n)\log\frac{mr}{\delta} \Big),$$
$$(4.53)$$

where $\delta \in \mathbb{R}^+ \setminus \{0\}$. This corresponds to a redefinition of $\langle \varepsilon \rangle$, and one can notice that a logarithmic correction is the only functional form that allows this to happen. This is not the first time a logarithmic correction to two point functions of the Ising model has been observed (see for example the spin-spin correlation function in Mikhak & Zarkesh [1994]). In both cases their presence is fully explained by the ambiguity in the definition of $\langle \varepsilon \rangle$.

4.4 Two particle form factors of composite operators

This section is dedicated to the evaluation of form factors for composite twist fields of the type introduced in Section 4.3. These operators are formally defined as the regularized limit of an operator $\mathcal{O}$ approaching the twist field in the original CFT. The regularization defines the meaning of the ordered product, which can be taken to be a point splitting

$$: \mathcal{O}\mathcal{T} : (x) \sim \lim_{\varepsilon \to 0} \mathcal{O}(x + \varepsilon)\mathcal{T}(x). \tag{4.54}$$

One can start by considering $: \psi\mathcal{T} :$ as a benchmark. Since $\mathcal{T}$ is even and ψ is odd under the $\mathbb{Z}_2$ symmetry of the Ising model, only odd particle form factors will be non-vanishing. Considering then the matrix element $\langle 0 | \psi(x)\mathcal{T}(0) | \theta \rangle$ the one-particle form factor can be extracted by looking at the leading contribution as x approaches 0. Inserting the resolution of the identity for the n-copy system

$$\mathbf{I} = \sum_{k=1}^{\infty} \sum_{\mu_1 \dots \mu_k} \int_{\theta_1 > \theta_2 > \dots > \theta_k} \frac{d\theta_1 \dots d\theta_k}{(2\pi)^k} |\theta_1, \dots, \theta_k\rangle_{\mu_1, \dots, \mu_k} {}_{\mu_k, \dots, \mu_1}\langle \theta_k, \dots, \theta_1| \tag{4.55}$$

we can write

$$\langle 0 | \psi(x)\mathcal{T}(0) | \theta \rangle = \frac{n}{2\pi} \int_{-\infty}^{\infty} d\phi \, \langle 0 | \psi(x) | \phi \rangle \, \langle \phi | \mathcal{T}(0) | \theta \rangle. \tag{4.56}$$

The matrix element $\langle 0|\psi(x)|\phi\rangle$ can be easily extracted from (3.23)

$$\langle 0|\psi(x)|\phi\rangle = \sqrt{\frac{m}{4\pi}}\, e^{\frac{\phi}{2}-ixp_\phi},\tag{4.57}$$

while $\langle\phi|\mathscr{T}(0)|\theta\rangle$ is linked to the two-particle form factor by crossing[1]

$$\langle\phi|\mathscr{T}(0)|\theta\rangle = \langle 0|\mathscr{T}(0)|\theta,\phi+i\pi-i\varepsilon^+\rangle + \langle\mathscr{T}\rangle\delta(\theta-\phi),\tag{4.58}$$

where the introduction of $i\varepsilon^+$ has to be thought of in the distributions sense, and describes how to avoid the pole at $i\pi$.

Plugging (4.57) and (4.58) into (4.56) yields

$$\langle 0|\psi(x)\mathscr{T}(0)|\theta\rangle = \langle\mathscr{T}\rangle\sqrt{\frac{m}{4\pi}}\, e^{\frac{\theta}{2}-ixp_\theta} + \frac{i}{2\pi}\sqrt{\frac{m}{4\pi}}\langle\mathscr{T}\rangle\cos\frac{\pi}{2n}$$
$$\int d\phi \quad e^{\frac{\phi}{2}-ixp_\phi}\frac{\sinh\left(\frac{\theta-\phi-i\pi}{2n}\right)}{\sinh\left(\frac{\theta-\phi+i\varepsilon^+}{2n}\right)\sinh\left(\frac{\theta-\phi-2i\pi}{2n}\right)}.\tag{4.59}$$

The leading term when x approaches 0 is given by the integral part, that is divergent, but can be made convergent by shifting the domain of integration from the real axe $\mathbb{R}$ to $\mathbb{R}+i\pi/2$. Such a change of variable does not affect the result of the integration, as the integrand has no poles in the region between the two axes[2].

The integral in (4.59) becomes then

$$e^{-i\frac{\pi}{4}}\int d\phi \quad e^{\frac{\phi}{2}-xE_\phi}\frac{\sinh\left(\frac{\theta-\phi-i\pi/2}{2n}\right)}{\sinh\left(\frac{\theta-\phi+i\pi/2}{2n}\right)\sinh\left(\frac{\theta-\phi-i3\pi/2}{2n}\right)}.\tag{4.60}$$

In this integral the main contributions come from large $|\phi|$, so that, after splitting the integration path into positive and negative regions, one can find two series expansions for the fraction in (4.60). The leading term for small x is given by the

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[1]Smirnov [1992].

[2]notice that it is the term $i\varepsilon^+$ which allows to perform this shift.

positive ψ part, that is

$$
\begin{aligned}
&-2e^{-i\frac{\pi}{4}\left(1+\frac{1}{n}\right)}e^{\frac{\theta}{2n}}\int_0^\infty d\phi \quad e^{\frac{\phi}{2}\left(1-\frac{1}{n}\right)-xE_\phi}+\ldots = \\
&-2e^{-i\frac{\pi}{4}\left(1+\frac{1}{n}\right)}e^{\frac{\theta}{2n}}\int_{-\infty}^\infty d\phi \quad e^{-mx\cosh\phi}\cosh\left[\frac{\phi}{2}\left(1-\frac{1}{n}\right)\right]+\ldots = \\
&-2e^{-i\frac{\pi}{4}\left(1+\frac{1}{n}\right)}e^{\frac{\theta}{2n}}2^{\frac{1}{2}\left(1-\frac{1}{n}\right)}\Gamma\left(\frac{n-1}{2n}\right)(mx)^{-\frac{1}{2}\left(1-\frac{1}{n}\right)}+\ldots .
\end{aligned}
\tag{4.61}
$$

The one particle form factor for $:\psi\mathscr{T}:$ can be read out from (4.59) and (4.61)

$$
F_1^{:\psi\mathscr{T}:|1}(\theta)=-\frac{ie^{-i\frac{\pi}{4}\left(1+\frac{1}{n}\right)}}{\pi}2^{\frac{1}{2}\left(1-\frac{1}{n}\right)}\Gamma\left(\frac{n-1}{2n}\right)\cos\left(\frac{\pi}{2n}\right)\langle\mathscr{T}\rangle\frac{m^{\frac{1}{2n}}}{\sqrt{4\pi}}e^{\frac{\theta}{2n}}
\tag{4.62}
$$

The procedure used to get this result was introduced for descendants of twist fields in Doyon & Silk [2011]. In that case the authors were dealing with the twist field associated to the global $U(1)$ symmetry of the Dirac Lagrangian. In this and next sections extensive use of this procedure is made, demonstrating its consistency for a different kind of twist fields.

4.4.1   Two particle form factor of $:\varepsilon\mathscr{T}:$

In this section we find the two particle form factor of the operator $:\varepsilon\mathscr{T}:$. We do so by considering the matrix element

$$
\langle 0|\varepsilon(x)\mathscr{T}(0)|\theta_1,\theta_2\rangle=\frac{1}{4\pi^2}\sum_{j=i}^n\int_{\phi_1>\phi_2}d\phi_1 d\phi_2 \quad \langle 0|\varepsilon(x)|\phi_1,\phi_2\rangle_{j,j}\,{}_{j,j}\langle\phi_2,\phi_1|\mathscr{T}(0)|\theta_1,\theta_2\rangle,
\tag{4.63}
$$

where j labels the copy number, and is repeated because ε connects only particles on the same copy.

The matrix element ${}_{j,j}\langle\phi_2,\phi_1|\mathscr{T}(0)|\theta_1,\theta_2\rangle$ is connected to the four-particle form factor of the twist field by the crossing relation (4.58), which used repeatedly

gives

$$_{j,j}\langle\phi_2\phi_1|\mathcal{T}(0)|\theta_1\theta_2\rangle = -\langle 0|\mathcal{T}(0)|\theta_1,\theta_2,\phi_1+i\pi-i\varepsilon^+,\phi_2+i\pi-i\varepsilon^+\rangle_{1,1,j,j}+$$

$$+\delta(\theta_1-\phi_1)\delta_{1,j}\langle 0|\mathcal{T}(0)|\theta_2,\phi_2+i\pi-i\varepsilon^+\rangle_{1,j}+\delta(\theta_2-\phi_2)\delta_{1,j}\langle 0|\mathcal{T}(0)|\theta_1,\phi_1-i\pi-i\varepsilon^+\rangle_{1,j}-$$

$$\delta(\theta_1-\phi_2)\delta_{1,j}\langle 0|\mathcal{T}(0)|\theta_2,\phi_1+i\pi-i\varepsilon^+\rangle_{1,j}-\delta(\theta_2-\phi_1)\delta_{1,j}\langle 0|\mathcal{T}(0)|\theta_1,\phi_2-i\pi-i\varepsilon^+\rangle_{1,j}+$$

$$+\delta(\theta_1-\phi_1)\delta(\theta_2-\phi_2)\delta_{1,j}-\delta(\theta_1-\phi_2)\delta(\theta_2-\phi_1)\delta_{1,j}.$$

$$(4.64)$$

Even if (4.64) is quite cumbersome one can notice that many terms can be extracted from the others with the exchange $\theta_1 \longleftrightarrow \theta_2$, which leaves the integration untouched. The leading contribution for small x is given by the first term (the one without deltas). This four-particle form factor considers two particles on the first copy, and two on copy j, but can be reduced to one where particles are considered on the same copy (say 1) with multiple applications of equations (3.53) and (3.54)

$$\langle 0|\mathcal{T}(0)|\theta_1,\theta_2,\phi_1+i\pi-i\varepsilon^+,\phi_2+i\pi-i\varepsilon^+\rangle_{1,1,j,j}=$$
$$F_4^{\mathcal{T}|11jj}(\theta_1,\theta_2,\phi_1+i\pi-i\varepsilon^+,\phi_2+i\pi-i\varepsilon^+)=$$
$$F_4^{\mathcal{T}|1111}(\theta_1,\theta_2,\phi_1+(2j-1)i\pi,\phi_2+(2j-1)i\pi)_+,$$

$$(4.65)$$

and this allows us to use the Pfaffian structure (3.69) to re-express it. In the last step of (4.65) the notation $(\ldots)_+$ is introduced to indicate that any pole on the real axe of what is in the brackets is avoided with the $i\varepsilon^+$ prescription. With the help of (3.69) and (4.35), one can rewrite the leading term of (4.63) as

$$\frac{im}{\langle\mathcal{T}\rangle}\frac{1}{4\pi^2}\sum_{j=1}^{n}\int_{\phi_1>\phi_2}d\phi_1 d\phi_2 \quad \sinh\left(\frac{\phi_1-\phi_2}{2}\right)e^{-ix(p_{\phi_1}+p_{\phi_2})}$$
$$[F_2^{\mathcal{T}|11}(\theta_1,\theta_2)F_2^{\mathcal{T}|11}(\phi_1,\phi_2)-F_2^{\mathcal{T}|11}(\theta_1,\phi_1+(2j-1)i\pi)F_2^{\mathcal{T}|11}(\theta_2,\phi_2+(2j-1)i\pi)+$$
$$F_2^{\mathcal{T}|11}(\theta_2,\phi_1+(2j-1)i\pi)F_2^{\mathcal{T}|11}(\theta_1,\phi_2+(2j-1)i\pi)]_+,$$

$$(4.66)$$

and proceed by evaluating the leading contribution for the three terms in (4.66). The first one is

$$-\frac{m\cos\frac{\pi}{2n}}{8\pi^2}F_2^{\mathscr{T}|11}(\theta_1,\theta_2)\int d\phi_1 d\phi_2 \quad \frac{\sinh\left(\frac{\phi_1-\phi_2}{2}\right)\sinh\left(\frac{\phi_1-\phi_2}{2n}\right)}{\sinh\left(\frac{\phi_1-\phi_2+i\pi}{2}\right)\sinh\left(\frac{\phi_1-\phi_2-i\pi}{2n}\right)}e^{-ix(p_{\phi_1}+p_{\phi_2})},$$

(4.67)

on which we can perform the change of variables $t=(\phi_1-\phi_2)/2$ and $s=(\phi_1+\phi_2)/2$, and carry out the s integration to get

$$-\frac{m\cos\frac{\pi}{2n}}{2\pi^2}F_2^{\mathscr{T}|11}(\theta_1,\theta_2)\int dt \quad \frac{\sinh t\sinh\frac{t}{n}}{\sinh\left(\frac{t}{n}+\frac{i\pi}{2n}\right)\sinh\left(\frac{t}{n}-\frac{i\pi}{2n}\right)}K_0(2mx\cosh t), \quad (4.68)$$

We notice again that the leading contribution is given for large t so that we can use the parity of the integrand to reduce the region of integration to $(0,\infty)$, and expand the fraction in (4.68) in a convergent way on this domain. The resulting integral is

$$-\frac{m\cos\frac{\pi}{2n}}{\pi^2}F_2^{\mathscr{T}|11}(\theta_1,\theta_2)\int_0^\infty dt \quad e^{t\left(1-\frac{1}{n}\right)}K_0(mxe^t)+\dots, \quad (4.69)$$

and with the change of variable $u=mxe^t$ we can extract the leading order for small x, that is

$$-\frac{m\cos\frac{\pi}{2n}}{\pi^2}F_2^{\mathscr{T}|11}(\theta_1,\theta_2)\left(\int_0^\infty dt \quad u^{-\frac{1}{n}}K_0(u)\right)(mx)^{\frac{1}{n}-1}+\dots. \quad (4.70)$$

Solving the integral we finally obtain

$$-\frac{m\cos\frac{\pi}{2n}}{2^{1+\frac{1}{n}}\pi^2}\Gamma\left(\frac{n-1}{2n}\right)^2 F_2^{\mathscr{T}|11}(\theta_1,\theta_2)(mx)^{\frac{1}{n}-1}+\dots. \quad (4.71)$$

The second and third terms in (4.66) are more involved then the first, due to their explicit dependence on the parameter j. One can start by noticing that the value of the third term can be extracted from the value of the second one by changing sign, and performing the exchange $\theta_1 \longleftrightarrow \theta_2$. Hence we focus on the first term. As, in order to have a convergent integral, the integration axe has to be risen by $i\pi/2$, we need to study the pole structure of this term. The only kinematic poles

which lie on the real axes arise from the cases $j = 1, n$ and $\phi_1 = \theta_1$ and $\phi_2 = \theta_2$; but they are avoided with the $i\varepsilon^+$ prescription. In general $F_2^{\mathscr{T}|11}(\theta, \phi + (2j-1)i\pi)$ has kinematic poles for $\phi = \theta + 2(n-j+1)i\pi$ and $\phi = \theta + 2(n-j)i\pi$. Considering that j runs from 1 to n one can see that all poles group in even multiples of $i\pi$, so that the first group above the real axe is in $\theta + 2i\pi$, and correspond to $j = n, n-1$. Hence the needed shift can be safely performed.

The integral we want to evaluate is then

$$\frac{im\langle\mathscr{T}\rangle\cos^2\frac{\pi}{2n}}{8\pi^2n^2}\sum_{j=1}^{n}\int d\phi_1 d\phi_2 \quad e^{-ix(p_{\phi_1}+p_{\phi_2})}\sinh\left(\frac{\phi_1-\phi_2}{2}\right)$$

$$\frac{\sinh\left(\frac{\theta_1-\phi_1-(2j-1)i\pi}{2n}\right)\sinh\left(\frac{\theta_2-\phi_2-(2j-1)i\pi}{2n}\right)}{\sinh\left(\frac{\theta_1-\phi_1-2(j-1)i\pi}{2n}\right)\sinh\left(\frac{\theta_1-\phi_1-2ji\pi}{2n}\right)\sinh\left(\frac{\theta_2-\phi_2-2(j-1)i\pi}{2n}\right)\sinh\left(\frac{\theta_2-\phi_2-2ji\pi}{2n}\right)}. \tag{4.72}$$

Although the fraction in (4.72) is rather complicated and mixes integration variables with parameters it can be dramatically simplified by means of the following identity

$$\sinh\left(\alpha_1 - \beta_1 \pm \gamma\right)\sinh\left(\alpha_2 - \beta_2 \pm \gamma\right) =$$
$$\frac{1}{2}\left[\cosh\left(\alpha_1 + \alpha_2 - (\beta_1 + \beta_2) \pm 2\gamma\right) - \cosh\left(\alpha_1 - \alpha_2 - (\beta_1 - \beta_2)\right)\right], \tag{4.73}$$

leading to

$$\frac{im\langle\mathscr{T}\rangle\cos^2\frac{\pi}{2n}}{4\pi^2n^2}\sum_{j=1}^{n}\int d\phi_1 d\phi_2 \quad e^{-ix(p_{\phi_1}+p_{\phi_2})}\sinh\left(\frac{\phi_1-\phi_2}{2}\right)$$

$$\frac{\cosh\left(\frac{\theta_1+\theta_2-(\phi_1+\phi_2)-2(2j-1)i\pi}{2n}\right)-\cosh\left(\frac{\theta_1-\theta_2-(\phi_1-\phi_2)}{2n}\right)}{\cosh\left(\frac{\theta_1+\theta_2-(\phi_1+\phi_2)-4(j-1)i\pi}{2n}\right)-\cosh\left(\frac{\theta_1-\theta_2-(\phi_1-\phi_2)}{2n}\right)} \times \tag{4.74}$$

$$\frac{1}{\cosh\left(\frac{\theta_1+\theta_2-(\phi_1+\phi_2)-4ji\pi}{2n}\right)-\cosh\left(\frac{\theta_1-\theta_2-(\phi_1-\phi_2)}{2n}\right)}.$$

Now performing the same change of variable as in (4.68) gives

$$\frac{im\langle\mathcal{T}\rangle\cos^2\frac{\pi}{2n}}{2\pi^2 n^2}\sum_{j=1}^{n}\int dt\,ds \quad e^{-2imx\sinh s\cosh t}\sinh t$$

$$\frac{\cosh\left(\frac{\theta_1+\theta_2}{2n}-\frac{s+(2j-1)i\pi}{n}\right)-\cosh\left(\frac{\theta_1-\theta_2}{2n}-\frac{t}{n}\right)}{\cosh\left(\frac{\theta_1+\theta_2}{2n}-\frac{s+2(j-1)i\pi}{n}\right)-\cosh\left(\frac{\theta_1-\theta_2}{2n}-\frac{t}{n}\right)}\times \tag{4.75}$$

$$\frac{1}{\cosh\left(\frac{\theta_1+\theta_2}{2n}-\frac{s-2ji\pi}{n}\right)-\cosh\left(\frac{\theta_1-\theta_2}{2n}-\frac{t}{n}\right)},$$

and to make it convergent the shift $s \longrightarrow s - i\pi/2$ has been performed.

As x approaches 0 the main contribution is given by large t, and s peaked around 0. It is natural then to expand the fraction in (4.75) in powers of t. As before there is no such expansion on the whole real axe, but splitting it into the positive and negative parts, allows us to consider two different series which converge respectively on the two regions. We can start considering $t < 0$, such that the expansion of (4.75) yields

$$e^{-\frac{\theta_1-\theta_2}{2n}}\int_{-\infty}^{0}dt \quad e^{-t\left(1-\frac{1}{n}\right)}\int_{-\infty}^{\infty}ds \quad e^{-2mxe^{-t}\cosh s}+\ldots =$$

$$2e^{-\frac{\theta_1-\theta_2}{2n}}\int_{-\infty}^{0}dt \quad e^{-t\left(1-\frac{1}{n}\right)}K_0(mxe^{-t})+\ldots. \tag{4.76}$$

Following now the same procedure which was used to obtain (4.71) out of (4.69), we arrive at the result

$$\frac{im\langle\mathcal{T}\rangle\cos^2\frac{\pi}{2n}}{2^{\frac{1}{n}+1}\pi^2 n}\Gamma\left(\frac{n-1}{2n}\right)^2 e^{-\frac{\theta_1-\theta_2}{2n}}(mx)^{\frac{1}{n}-1}+\ldots. \tag{4.77}$$

The positive part of the integral can be performed with the same logic and gives

$$-\frac{im\langle\mathcal{T}\rangle\cos^2\frac{\pi}{2n}}{2^{\frac{1}{n}+1}\pi^2 n}\Gamma\left(\frac{n-1}{2n}\right)^2 e^{\frac{\theta_1-\theta_2}{2n}}(mx)^{\frac{1}{n}-1}+\ldots, \tag{4.78}$$

so that the final result of the second part is

$$-\frac{im\langle\mathcal{T}\rangle\cos^2\frac{\pi}{2n}}{2^{\frac{1}{n}}\pi^2 n}\Gamma\left(\frac{n-1}{2n}\right)^2\sinh\frac{\theta_1-\theta_2}{2n}(mx)^{\frac{1}{n}-1}+\ldots. \tag{4.79}$$

As mentioned before, the result of of third part of the integral in (4.66) can be obtained from (4.79) by switching $\theta_1 \longleftrightarrow \theta_2$ with a minus sign in front, which doubles the result. Putting (4.71), (4.77) and (4.79) together, we finally obtain the two particle form factor for the field $:\varepsilon\mathscr{T}:$, that is

$$F_2^{:\varepsilon\mathscr{T}:|11}(\theta_1,\theta_2) = -\frac{\cos\frac{\pi}{2n}}{2^{1+\frac{1}{n}}\pi^2}\Gamma\left(\frac{n-1}{2n}\right)^2 m^{\frac{1}{n}}\left[F_2^{\mathscr{T}|11}(\theta_1,\theta_2) + \frac{4i\cos\frac{\pi}{2n}}{n}\langle\mathscr{T}\rangle\sinh\frac{\theta_1-\theta_2}{2n}\right].$$
(4.80)

This result was the aim of this section. To check its validity one can employ (3.55). Indeed, since $:\varepsilon\mathscr{T}:$ is still a twist field, the same type of residue equations as for $\mathscr{T}$ must be satisfied. Then one can easily check using (4.48) and (4.80) that

$$\lim_{\bar{\theta}\to\theta}(\bar{\theta}-\theta)F_2^{:\varepsilon\mathscr{T}:|11}(\bar{\theta}+i\pi,\theta) = i\langle:\varepsilon\mathscr{T}:\rangle, \tag{4.81}$$

which confirms the compatibility of the two results of these sections.
Notice that this result satisfies all form factors equations for the twist field, and has a structure of the type

$$F_2^{:\varepsilon\mathscr{T}:|11}(\theta_1,\theta_2) = \alpha\left[Q_2^{\mathscr{T}}(\theta_1,\theta_2) + \beta\kappa(\theta_1,\theta_2)\right]F_{\min}(\theta_1,\theta_2), \tag{4.82}$$

where α and β are two dimensional constants, $F_{\min}$ is the minimal form factor of the theory[1], and κ is a kernel solution of the form factor equations as the ones found in sections 3.4.1 and 3.4.2. Even if we could have understood that (4.80) should have had the form (4.82) to fulfill all twist properties it has to, we would have never been able to fix α and β without the methods used in this section, and section 4.3.

4.5 Higher particle form factors of composite operators

In this section we deal with the computation of higher particle form factors for $:\psi\mathscr{T}:$ and $:\varepsilon\mathscr{T}:$. They can be extracted with the same methods for both operators,

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[1]for the Ising model $F_{\min}(\theta_1,\theta_2) = -i\sinh\frac{\theta_1-\theta_2}{2n}$

employing higher particle form factors of the twist field (3.69). Indeed, focusing on $:\psi\mathcal{T}:$, when looking for the leading term for the $2k - 1$ particle form factor, one has to deal with

$$\langle 0| \psi(x)\mathcal{T}(0)|\theta_1,\theta_2,\ldots,\theta_{2k-1}\rangle \sim \frac{n}{2\pi}\int d\phi\, \langle 0|\psi(x)|\phi\rangle\,\langle 0|\mathcal{T}(0)|\theta_1,\theta_2,\ldots,\theta_{2k-1},\phi+i\pi-i\varepsilon^+\rangle. \tag{4.83}$$

One is then able to isolate term by term the higher particle part, and reduce it to the same evaluation carried out in (4.56)-(4.62), getting finally

$$F_{2k-1}^{:\psi\mathcal{T}:|11\ldots1} = \langle\mathcal{T}\rangle\mathrm{Pf}(K_{:\psi\mathcal{T}:}), \tag{4.84}$$

where $K_{:\psi\mathcal{T}:}$ is the $2k \times 2k$ matrix defined as

$$K_{:\psi\mathcal{T}:} := \begin{pmatrix} 0 & F_1^{:\psi\mathcal{T}:|1}(\theta_1) & \cdots & F_1^{:\psi\mathcal{T}:|1}(\theta_{2k-1}) \\ -F_1^{:\psi\mathcal{T}:|1}(\theta_1) & 0 & \cdots & F_2^{\mathcal{T}|11}(\theta_1,\theta_{2k-1})/\langle\mathcal{T}\rangle \\ \vdots & \vdots & \ddots & \vdots \\ -F_1^{:\psi\mathcal{T}:|1}(\theta_{2k-1}) & -F_2^{\mathcal{T}|11}(\theta_1,\theta_{2k-1})/\langle\mathcal{T}\rangle & \cdots & 0 \end{pmatrix}. \tag{4.85}$$

Higher particles form factors for $:\varepsilon\mathcal{T}:$ can be evaluated with the same logic, although they show a more complicated pattern, and can not be reduced to a Pfaffian form. This is due to the presence of a kernel part in the two particle form factor. The $2k + 2$ particle form factor is

$$F_{2k+2}^{:\varepsilon\mathcal{T}:|11\ldots1}(\theta_1,\theta_2,\ldots,\theta_{2k+2}) = \sum_{i<j}\frac{(-1)^{\sigma(i,j)}}{\langle\mathcal{T}\rangle^{2k}}F_{2k}^{\mathcal{T}|11\ldots1}(\theta_1,\ldots,\theta_{2k+2})_{ij}F_2^{:\varepsilon\mathcal{T}:|11}(\theta_i,\theta_j), \tag{4.86}$$

where $\sigma(i,j)$ is the permutation that brings θ_i and θ_j to the right of all other rapidities, while with $F(\ldots)_{ij}$ we mean a form factor of all rapidities but those two.

$$5$$

Entanglement entropy in quantum spin chains

In this chapter we present a broad review of the most commonly used methods to evaluate block entanglement entropy on quantum spin chains, and we then specialize to our results. We will use as benchmarks the XY, and XXZ models through the following sections. The first model is defined on a chain of N sites, and $N-1$ links, by the Hamiltonian

$$\mathcal{H}_{\mathrm{XY}} = \frac{J}{2} \left[\sum_{i=1}^{N-1} \left(\frac{1+\kappa}{2} \sigma_i^x \sigma_{i+1}^x + \frac{1-\kappa}{2} \sigma_i^y \sigma_{i+1}^y \right) + h \sum_{i=1}^{N} \sigma_i^z \right], \tag{5.1}$$

where $J, \kappa, h \in \mathbb{R}$,

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{5.2}$$

are Pauli matrices, and $\sigma_i^\alpha = \mathbb{I} \otimes \mathbb{I} \otimes ... \otimes \sigma^\alpha \otimes ... \otimes \mathbb{I}$ is acting on the i-th site, $\mathbb{I}$ being the identity operator on all other sites. As we are ultimately interested in the thermodynamic, and scaling limit of (5.1)[1], we consider open boundary conditions to avoid any complications. A first solution to this model for $h = 1$ was obtained by Lieb *et al.* [1961], and for $\kappa = 1$ by Pfeuty [1970]. The solution for general κ was then generalized to non-vanishing external magnetic field by Niemeijer [1967, 1968]. The correlation functions for the fundamental operators were evaluated in Barouch & McCoy [1971a,b]; Barouch *et al.* [1970]; McCoy *et al.* [1971], and

---

[1]both limits involve the consideration of an infinite chain.

were expressed in terms of Topelitz matrices. In those references also the phase diagram and critical properties of this model were studied.

The modulus of the parameter J in (5.1) sets the energy scale, while the sign sets which kind of interactions are energetically more accessible. There is a ferromagnetic regime for negative J, where the lowest-energy eigenstates of $\mathscr{H}_{\text{XY}}$ are characterized by aligned spins. Positive J on the other hand makes alternating spins preferable. We choose to work in the ferromagnetic regime, and we set $J = -1$.

The phase diagram of this model is parametrized by h and κ, the first being an external magnetic field, and the latter being the anisotropy parameter. There are two important sub-cases of this model, defined by limiting values of κ. For $\kappa = 0$ (5.1) becomes $U(1)$ invariant, and is called XX model, while for $\kappa = 1$ the Hamiltonian reduces to the Ising model

$$\mathscr{H}_{\text{Ising}} = \frac{J}{2} \left(\sum_{i=1}^{N-1} \sigma_i^x \sigma_{i+1}^x + h \sum_{i=1}^{N} \sigma_i^z \right). \tag{5.3}$$

In this limit there is Z_2 invariance. These two sub-cases belong to different universality classes, which compete in the phase diagram. The Hamiltonian (5.1) is symmetric under the change $h \to -h$, and $\kappa \to -\kappa$, so that we can focus on the sector $h, \kappa \geq 0$ with no loss of generality. In this region there are two different quantum phase transitions, one on the line $\kappa = 0 \wedge h \leq 1$, and the other one at $h = 1$. We are particularly interested in the QPT which occurs in the Ising limit, that is $\kappa = 1 \wedge h = 1$. This transition belongs to the same universality class as the two-dimensional classical Ising model. As the latter, it divides a disordered phase with Z_2 symmetry, from an ordered phase where the symmetry is spontaneously broken. These two phases in the quantum Ising model are represented by the regions $h < 1$, where the ground state is doubly degenerated and Z_2 symmetric, and $h > 1$, where it is unique. It is easy to check this feature considering the two limits $h \to 0, \infty$. In the case $h = 0$ there are two ground states corresponding to the two totally aligned states along the x-axes. Labelling by

$$|\uparrow\rangle_i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_i \qquad\qquad |\downarrow\rangle_i = \begin{pmatrix} 0 \\ 1 \end{pmatrix}_i, \tag{5.4}$$

the two eigenvectors of S_i^z, the two possible ground states are the product states $\prod_{i=1}^N (|\uparrow\rangle_i + |\downarrow\rangle_i)/\sqrt{2}$, and $\prod_{i=1}^N (|\uparrow\rangle_i - |\downarrow\rangle_i)/\sqrt{2}$. In the limit for infinite magnetic field on the other hand the ground state will be totally polarized along the z-axes $|\uparrow\uparrow \dots \uparrow\rangle$. As the energy levels $E_n(h)$ of (5.3), are smooth functions of the external magnetic field, there will be a certain value of h where some level-crossings occur. This is the $h = 1$ critical point, and as we will see it is characterized by a vanishing energy gap.

5.1 The XY chain

In this section we report a summarized version of the diagonalization procedure introduced in Lieb *et al.* [1961]. Our aim is to rewrite (5.1) as the Hamiltonian of a set of decoupled Femionic harmonic oscillators. As usual when dealing with harmonic oscillators we introduce the ladder operators $\sigma_i^\pm = \left(\sigma_i^x \pm i\sigma_i^y\right)/2$. They act on the local spins as

$$
\begin{aligned}
\sigma_i^+ |\uparrow\rangle_i &= 0 & \sigma_i^+ |\downarrow\rangle_i &= |\uparrow\rangle_i \\
\sigma_i^- |\downarrow\rangle_i &= 0 & \sigma_i^- |\uparrow\rangle_i &= |\downarrow\rangle_i,
\end{aligned}
\tag{5.5}
$$

and they satisfy the algebra $\{\sigma_i^+, \sigma_j^-\} = \mathbb{1}\delta_{ij}$. These operators are clearly not suitable for the diagonalization process, as they commute on different sites. We need to perform a Jordan-Wigner transformation to have proper Fermionic modes. This transformation relies on the basic assumption of the existence of a map between the Hilbert space of spin-1/2 chains, and the one of a system of spinless Fermions hopping from site to site. The standard way to perform it is to rewrite the Hilbert space of a single site $|s\rangle_i = \{|\uparrow\rangle_i, |\downarrow\rangle_i\}$ as $|n\rangle_i = \{|0\rangle_i, |1\rangle_i\}$, and to associate $|0\rangle_i$ with the absence of a Fermion in the site i (a hole), while $|1\rangle_i$ corresponds to an occupied site. In order to implement this transformation we notice that if we find a unitary operator $U(n)$ such that it commutes with $\sigma_n^\pm$, this would leave the same-site anti-commutation brackets invariant, so that we can define $c_n = U(n)\sigma_n^+$, which automatically satisfy $\{c_n^\dagger, c_n\} = \mathbb{1}$. In this way c_n will flip the n-th spin up, while $c_n^\dagger$ will flip it down. If we insist on this interpretation we have then the correspondence $|\uparrow\rangle_i = |0\rangle_i$, and $|\downarrow\rangle_i = |1\rangle_i$, which serves as a map between

the Hilbert spaces of the spin system, and the Fermion Hamiltonian. The next step then is to define $U(n)$ so as to have the desired anti-commutation relations for different sites. Notice that if we want to impose anti-symmetric relations, c_l and $c_l^\dagger$ must act on the Hilbert space in the following way

$$c_l^\dagger |n_1 n_2 ... n_l ... n_N\rangle = (-1)^{\delta_l}(1-n_l)|n_1 n_2 ... n_l + 1 ... n_N\rangle,$$
$$c_l |n_1 n_2 ... n_l ... n_N\rangle = (-1)^{\delta_l} n_l |n_1 n_2 ... n_l - 1 ... n_N\rangle, \tag{5.6}$$

where $\delta_l = \sum_{i<l} n_i$. Then the problem is reduced to finding an operator representation of $(-1)^{\delta_l}$. This is achieved noticing that $\sigma_i^z = 1 - 2c_i^\dagger c_i$ has all the right properties to be the building block of $U(n)$, so that

$$c_n = \left(\prod_{l<n} \sigma_l^z\right)\sigma_n^+ \qquad \sigma_n^+ = \prod_{l<n}\left(1 - 2c_j^\dagger c_j\right)c_n$$
$$c_n^\dagger = \left(\prod_{l<n} \sigma_l^z\right)\sigma_n^- \qquad \sigma_n^- = \prod_{l<n}\left(1 - 2c_j^\dagger c_j\right)c_n^\dagger \tag{5.7}$$

The highly non-local nature of the above map is forced by the constraints (5.6), which are necessary to have $\{c_i^\dagger, c_j\} = \delta_{ij}\mathbb{1}$, and $\{c_i, c_j\} = \{c_i^\dagger, c_j^\dagger\} = 0$. As a result c_i and $c_i^\dagger$ are sensitive to the number of particles on the left of the considered site[1], besides annihilating and creating one on it, respectively.

Substituting (5.7) into (5.1), we find the quadratic Fermionic Hamiltonian

$$\mathcal{H}_{extXY} = -\frac{1}{2}\sum_{i=1}^{N-1}\left[c_{i+1}^\dagger c_i + c_i^\dagger c_{i+1} + \kappa\left(c_{i+1}c_i + c_i^\dagger c_{i+1}^\dagger\right)\right] - \frac{hN}{2} + h\sum_{i=1}^{N}c_i^\dagger c_i. \tag{5.8}$$

Notice that for $\kappa \neq 0$ the number of Fermions is not conserved, that is the total magnetization in the z-direction is not conserved by (5.1). What is conserved is the oddness/evenness of the number of Fermions, that is to say that the original Hamiltonian can be split into two integer/non-integer magnetization sectors. To diagonalize (5.8) we have to find a basis of conserved Fermion operators.

---

[1]to be precise their sign changes if the total number of particles on the left of the site under consideration is even or odd. This observation becomes of central importance when considering periodic boundary conditions, for which one has to distinguish chains with even/odd number of sites.

As a first step we can take advantage of the translation invariance of the scaling Hamiltonian Fourier transforming the Fermionic modes. Clearly any such transformation would not have as a target space the momentum space, as it is impossible to define a momentum, due to the boundary conditions under consideration. We can tackle this issue though if we remember that we are interested in the limit $N \to \infty$. Having that in mind we can manipulate (5.8) and write the following

$$
\begin{aligned}
\mathcal{H}_{\mathrm{XY}} = \quad & - \frac{1}{2} \sum_{i=1}^{N} \left[c_{i+1}^{\dagger} c_i + c_i^{\dagger} c_{i+1} + \kappa \left(c_{i+1} c_i + c_i^{\dagger} c_{i+1}^{\dagger} \right) - 2h c_i^{\dagger} c_i \right] \\
& - \frac{hN}{2} - c_1^{\dagger} c_N + c_N^{\dagger} c_1 + \kappa \left(c_1 c_N + c_N^{\dagger} c_1^{\dagger} \right),
\end{aligned}
\tag{5.9}
$$

where we identified $c_{N+1} = c_1$. This Hamiltonian is not that of a periodic chain, due to the presence of bond operators outside the sum, and yet we cannot define a proper momentum. In the thermodynamic limit though, we can ignore those terms, as they are $O(1/N)$, and overcome this problem.

It is convenient firstly to center the sum in (5.8) by considering $-(N-1)/2 \leq i \leq (N-1)/2$. Then we can define the Fourier modes as

$$
d_k = \frac{1}{\sqrt{N}} \sum_{j=-\frac{N-1}{2}}^{\frac{N-1}{2}} c_j e^{-\frac{2\pi i}{N} kj},
\tag{5.10}
$$

which clearly inherit the anti-commutation properties $\{d_l^{\dagger}, d_m\} = \delta_{ml} \mathbb{1}$, and $\{d_l, d_m\} = \{d_l^{\dagger}, d_m^{\dagger}\} = 0$. The sums in (5.8), and (5.10) have been shifted in order to have a well defined, centered delta function

$$
\delta_{lm} = \frac{1}{N} \sum_{j=-\frac{N-1}{2}}^{\frac{N-1}{2}} e^{-\frac{2\pi i}{N}(l-m)j}.
\tag{5.11}
$$

We can now rewrite $\mathcal{H}_{\mathrm{XY}}$ in terms of these new modes, and we find

$$
\mathcal{H}_{\mathrm{XY}} = \sum_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}} \left[\left(h - \cos \frac{2\pi k}{N} \right) d_k^{\dagger} d_k - \frac{i\kappa}{2} \sin \frac{2\pi k}{N} \left(d_k d_{-k} + d_k^{\dagger} d_{-k}^{\dagger} \right) \right] - \frac{hN}{2}.
\tag{5.12}
$$

Our final aim is the removal from the Hamiltonian of the terms of the kind $d_k^\dagger d_{-k}^\dagger$, as they do not conserve the Fermion number. This can be achieved by a transformation such as the one introduced in Bogoliubov [1947]. This is basically a rotation on the Fermion modes, and is based on the observation that if one has a set of Fermionic oscillators a_k, and defines $b_k = e^{i\phi_u} u_k a_k + e^{i\phi_v} v_k a_k^\dagger$ where $|u_k|^2 + |v_k|^2 = 1$, then also b_k satisfy Fermionic anticommutation relations. In our case the situation is slightly complicated by the fact that we are considering also negative modes in (5.10). We define then the mixed Bogoliubov transformation $\gamma_k = e^{i\phi_u} u_k d_k + e^{i\phi_v} v_k d_{-k}^\dagger$, and notice that Fermionic relations are automatically satisfied under the condition $u_k = u_{-k}$, and $v_k = -v_{-k}$. We assume that u_k and v_k are real, and parametrize them as $u_k = \cos\theta_k$ and $v_k = \sin\theta_k$. Finally we can set $\phi_u = 0$ with no loss of generality, and choose for convenience $\phi_v = -\pi/2$, so that $d_k = \cos\theta_k \gamma_k + i \sin\theta_k \gamma_{-k}^\dagger$.

We now write (5.12) in this new Fermion basis, and notice that by the choice

$$\tan 2\theta_k = \frac{\kappa \sin\frac{2\pi k}{N}}{h - \cos\frac{2\pi k}{N}},\tag{5.13}$$

we are able to cancel the unwanted terms. The Hamiltonian can be then rewritten as the one of decoupled Fermionic oscillators

$$\mathcal{H}_{\mathrm{XY}} = \sum_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}} \varepsilon(k)\left(\gamma_k^\dagger \gamma_k - \frac{1}{2}\right),\tag{5.14}$$

with dispersion relation

$$\varepsilon(k) = \sqrt{\left(h - \cos\frac{2\pi k}{N}\right)^2 + \kappa^2 \sin^2\frac{2\pi k}{N}}.\tag{5.15}$$

Now that we have an analytic expression of the energy levels in terms of h and κ we can perform the thermodynamic limit by considering the limits $\kappa \to 1$, and $N \to \infty$ in (5.14). This is easily performed changing variable to $\phi = 2\pi k/N$ and

replacing the sum

$$\frac{1}{N}\sum_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}} \longrightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi}d\phi, \tag{5.16}$$

so that the Hamiltonian can be written as

$$\lim_{N\to\infty}\frac{\mathscr{H}_{XY}}{N} = \frac{1}{2\pi}\int_{-\pi}^{\pi}d\phi\,\varepsilon(\phi)\left(\gamma_\phi^\dagger\gamma_\phi - \frac{1}{2}\right), \tag{5.17}$$

where $\varepsilon(\phi) = \sqrt{(h-\cos\phi)^2 + \kappa^2\sin^2\phi}$. We have now a clearer picture of the phase diagram, as drawn in Figure 5.1. The Hamiltonian becomes gapless, on the lines $\kappa = 0 \wedge h < 1$, and for $h = 1$. The first critical line clearly falls into the XX universality class, and is described by a CFT with $c = 1$. The second line belongs to the Ising universality class, and is characterized by a central charge $c = 1/2$. We will focus on this second kind of phase transition henceforth, restricting ourselves in particular to the Ising model (5.3), that is we approach the critical line $h = 1$ along the green line in Figure 5.1.
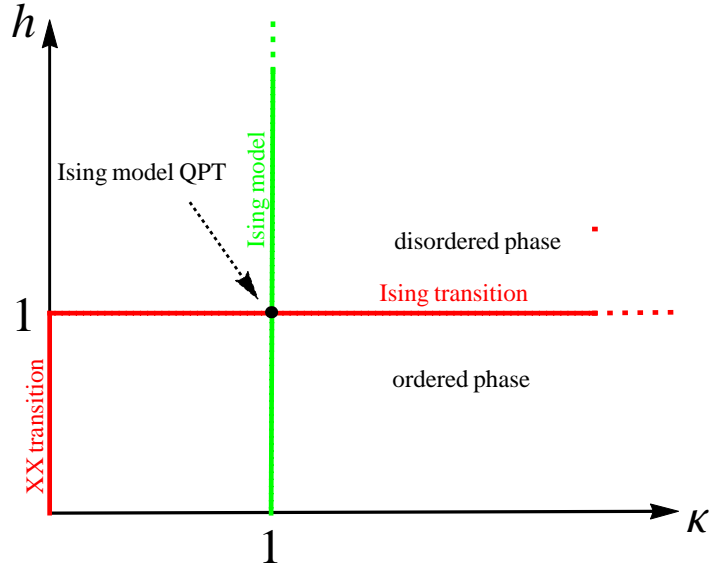

Figure 5.1: phase diagram of the XY spin chain.

5.1.1 Correlation functions

In this section we summarize the derivation of the fundamental correlation functions at zero temperature as presented by Barouch & McCoy [1971b]. We have seen in the previous section that the quantity $\varepsilon(k)$, as defined in (5.15), can be interpreted as a single particle energy. As it is positive defined, we can interpret the vacuum state as the zero particle state, that is $\gamma_\phi |0\rangle = 0$. Hence the Hilbert space is described by the number of occupied states, so that $|\phi_1\phi_2,...,\phi_k\rangle = \gamma^\dagger_{\phi_1}\gamma^\dagger_{\phi_2},...,\gamma^\dagger_{\phi_k}|0\rangle$. In this formalism it is particularly easy to express the two-point correlators, that are $\langle 0|\gamma_\phi\gamma^\dagger_\varphi|0\rangle = \delta(\phi-\varphi)$, and $\langle 0|\gamma_\phi\gamma_\varphi|0\rangle = \langle 0|\gamma^\dagger_\phi\gamma^\dagger_\varphi|0\rangle = \langle 0|\gamma^\dagger_\phi\gamma_\varphi|0\rangle = 0$. We can now proceed backwards in order to find the correlation functions of the original spin operators. We invert the Bogoliubov transformation defined in the previous section. We find then that the correlators of the set of Fermions d_k are

$$\langle 0|d_k d^\dagger_l|0\rangle = \frac{1+\cos 2\theta_k}{2}\delta_{kl} \qquad\qquad \langle 0|d_k d_l|0\rangle = -\frac{i}{2}\sin 2\theta_k\delta_{-kl}. \qquad (5.18)$$

In the thermodynamic limit we can Fourier expand these two quantities, and define the following

$$
\begin{aligned}
E_{jk} &= \langle 0|c_j c_k|0\rangle = -\frac{i}{2\pi}\int_{-\pi}^{\pi} d\phi \frac{\sin 2\theta_\phi}{2} e^{i\phi(j-k)}\\
F_{jk} &= \langle 0|c_j c^\dagger_k|0\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi \frac{1+\cos 2\theta_\phi}{2} e^{i\phi(j-k)}.
\end{aligned} \qquad (5.19)
$$

We now want to invert the Jordan-Wigner transformation (5.7) in order to express all correlation functions in terms of the original spin operators, so that we call $\rho^\alpha_{jk} = \langle 0|\sigma^\alpha_j\sigma^\alpha_k|0\rangle$, with $\alpha = x,y,z$, and $j \leq k$. By means of simple calculations we can write all the fundamental correlators in terms of the quantities $A_i = c^\dagger_i + c_i$, and $B_i = c^\dagger_i - c_i$, finding

$$
\begin{aligned}
\rho^x_{jk} &= \langle 0|B_j A_{j+1} B_{j+1}...A_{k-1}B_{k-1}A_k|0\rangle\\
\rho^y_{jk} &= (-1)^{k-j}\langle 0|A_j B_{j+1} A_{j+1}...B_{k-1}A_{k-1}B_k|0\rangle\\
\rho^z_{jk} &= \langle 0|A_j B_j A_k B_k|0\rangle.
\end{aligned} \qquad (5.20)
$$

By noticing that A_i and B_i satisfy the following anti-commutation relations $\{A_i,A_j\} = \{B_i,B_j\} = 0$ and $\{A_i,B_j\} = 2\delta_{ij}$, we can reduce the last equality in (5.20)

to $\rho_{jk}^z = \langle 0|A_j B_j|0\rangle \langle 0|A_k B_k|0\rangle - \langle 0|A_j B_k|0\rangle \langle 0|A_k B_j|0\rangle$.

Using the definitions of A and B, and (5.19), we can rewrite

$$\rho_{jk}^z = G^2(0) - G(j-k)G(k-j), \tag{5.21}$$

in terms of

$$
\begin{aligned}
G(j-k) &= \langle 0|B_j A_k|0\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi\, e^{2i\theta_\phi - i\phi(j-k)} \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi\, e^{-i\phi(j-k)} \frac{h - \cos\phi + i\kappa\sin\phi}{\sqrt{(h-\cos\phi)^2 + \kappa^2\sin^2\phi}}.
\end{aligned}
\tag{5.22}
$$

The other fundamental correlators are more involved to express in terms of Gs. Using the Wick theorem to reduce ρ^x and ρ^y in (5.20) one can show that

$$
\begin{aligned}
\rho_{jk}^x &= \det\begin{vmatrix} G(-1) & G(-2) & \dots & G(-l) \\ G(0) & G(-1) & \dots & G(-l+1) \\ \vdots & \vdots & \ddots & \vdots \\ G(l-2) & G(l-3) & \dots & G(-1) \end{vmatrix} \\
\rho_{jk}^y &= \det\begin{vmatrix} G(1) & G(0) & \dots & G(-l+2) \\ G(2) & G(1) & \dots & G(-l+3) \\ \vdots & \vdots & \ddots & \vdots \\ G(l) & G(l-1) & \dots & G(1) \end{vmatrix},
\end{aligned}
\tag{5.23}
$$

where $l = |j-k|$.

Matrices of the kind (5.23) are known in the literature as Topelitz matrices, and were studied in Wu [1966], Szegö [1952], Kac [1954] and Hartwig & Fisher [1969]. Their properties were exploited by Barouch & McCoy [1971b] to study the physical behaviour of the correlation functions at zero and finite temperature, in order to have a complete physical description of the XY chain. This description goes far beyond the scope of this manuscript, and we refer the interested reader to the aforementioned literature. We have though a clearer interpretation of the Ising critical line in figure 5.1. With the examination of (5.21) and (5.23) for large separations ($l \to \infty$) we find the asymptotic behavior reported in

section 5.1.1, where we identify $\lambda = \left(h + \sqrt{\kappa^2 + h^2 - 1}\right)/(1+\kappa)$.

$h < 1$	$\rho^x(l) \simeq M_x^2$	$\rho^y(l) \simeq \frac{\lambda^{2l}}{l^3}$	$\rho^z(l) - M_z^2 \simeq \frac{\lambda^{2l}}{l^2}$
$h = 1$	$\rho^x(l) \simeq l^{-\frac{1}{4}}$	$\rho^y(l) \simeq l^{-\frac{9}{4}}$	$\rho^z(l) - M_z^2 \simeq l^{-2}$
$h > 1$	$\rho^x(l) \simeq \frac{\lambda^{-l}}{l^{\frac{1}{2}}}$	$\rho^y(l) \simeq \frac{\lambda^{-l}}{l^{\frac{3}{2}}}$	$\rho^z(l) - M_z^2 \simeq \frac{1}{4} - M_z^2$

Table 5.1: Asymptotic behavior for large distance of the correlation functions of the XY chain near the Ising critical line $h = 1$.

The critical line $h = 1$ is described by a CFT, and all correlation function are characterized by a large-distance power law. This line divides two non-critical regions which can be identified by means of the order parameter[1] $M^x = \langle 0 | \sigma_j^x | 0 \rangle$. The region $h > 1$ is usually referred to as *disordered phase*, as there is no magnetization along the x-axis. The correlation functions decay exponentially, and comparing to eq. (1.25) we can extract the correlation length

$$\xi_{\text{XY}} = \frac{1}{\log \lambda}. \tag{5.24}$$

In the region $h < 1$ the local magnetization $M_x \neq 0$, and we call it ordered phase. This phase is characterized by long-range order along the x-axis, as can be seen from the large distance behavior of the correlation function, which saturates to a fixed value.

5.1.2 The Ising chain and its scaling limit

Our final goal is the study of entanglement entropy of the Ising model, it is then worth reporting a restricted version of the formulae we found for this sub-case. The Ising case can be extracted in general by setting $\kappa = 1$ in all the quantities we derived.

The Hamiltonian can then be expressed in terms of Fermion harmonic oscillators as (5.17), with the identification $\varepsilon(\phi) = \sqrt{1 + h^2 - 2h \cos \phi}$, and a qualita-

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[1] the most natural choice would be the total magnetization along this axis, but as we consider the thermodynamic limit, for which this quantity diverges, we prefer the local magnetization. Notice that due to translation invariance its definition is independent of the position.
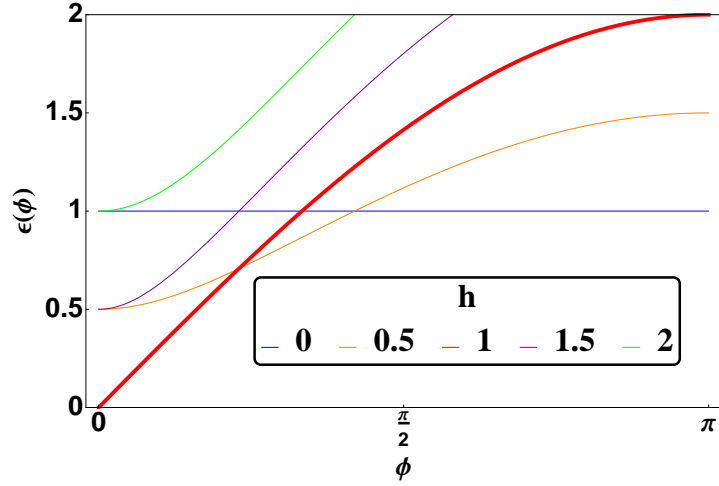
Figure 5.2: Energy levels of the scaling Ising chain.

tive description of the excited states is reported in figure 5.2. There we plot the spectrum of excited states for different values of the external magnetic field. Different values of ϕ correspond to different excitations $\gamma_\phi^\dagger |0\rangle$ of energy $E_0 + \varepsilon(\phi)$, the lowest corresponding to $\phi = 0$. The red solid line corresponds to the critical case, and we can see how the gap between the ground state and the first excited state vanishes. Also (5.22) is fairly simplified by restricting it on the $\kappa = 1$ line. In the noncritical case it can be expressed as

$$G(l) = hF(l) - F(l+1), \qquad \text{with} \qquad F(l) = \frac{1}{\pi} \int_0^\pi d\phi\, \varepsilon^{-1}(\phi) \cos(\phi l). \quad (5.25)$$

For $h = 0, 1$, expression (5.22) can be worked out exactly

$$
\begin{aligned}
G(l) &= \frac{2}{\pi} \frac{1}{2l+1}, & h &= 1 \\
G(l) &= -\delta_{-l1}, & h &= 0,
\end{aligned}
\qquad (5.26)
$$

and because of the simplicity of these expressions, the correlations (5.21) and (5.23) can be studied in great detail. This goes beyond the scope of this thesis, but has been studied in Pfeuty [1970], along with the characterization of the magnetization.

In order to perform the scaling limit on the Ising spin chain we first use the Jordan-Wigner transformation defined in eq. (5.7). The Hamiltonian (5.3) can be then rewritten as

$$\mathcal{H}_{\text{Ising}} = \frac{J}{2} \sum_i \left[\left(c_i^\dagger - c_i \right) \left(c_{i+1}^\dagger + c_{i+1} \right) - h \left(c_i^\dagger - c_i \right) \left(c_i^\dagger + c_i \right) \right], \qquad (5.27)$$

where we do not take into account boundary conditions, as we are considering an infinite chain. We have written eq. (5.27) in a factorized form to underline the connection with a free Majorana Hamiltonian. Indeed we can define the two components of a Majorana spinor as

$$\Psi(n) = \frac{c_n^\dagger + c_n}{\sqrt{2}}, \qquad \text{and} \qquad \bar{\Psi}(n) = \frac{c_n^\dagger - c_n}{\sqrt{2}i}, \qquad (5.28)$$

by means of which we can rewrite eq. (5.27) as

$$\mathcal{H}_{\text{Ising}} = iJ \sum_n \{ \bar{\Psi}(n)[\Psi(n+1) - \Psi(n)] - (h-1)\bar{\Psi}(n)\Psi(n) \}. \qquad (5.29)$$

Again checking that they satisfy the right anticommutation conditions is a simple exercise.

We now perform the scaling limit described in section 1.6. We want to consider the limit $a \to 0$, going towards the critical point $h = 1$ at the same time. If we just perform these two limits though we can see from Figure 5.2 that the energy gap would collapse. We have a gapped scaling theory if we take also the limit $J \to \infty$. Considering all the limits together we can define the Fermi velocity $c = Ja^1$, and the mass $m = J(h-1)$. We perform the aforementioned limits so as to keep c and m constant. Defining the set of coordinates $x = na$, under the scaling limit $\sum_n \to \int dx$, and $\Psi(n), \bar{\Psi}(n) \to \Psi(x), \bar{\Psi}(x)$, such that we find

$$\mathcal{H}_{\text{Ising}} = i \int dx \left(c\bar{\Psi}(x)\partial_x \Psi(x) - m\bar{\Psi}(x)\Psi(x) \right). \qquad (5.30)$$

This Hamiltonian gives the equations of motion of (3.20), hence the Ising field theory is indeed the scaling limit of the Ising model. To see explicitly the connec-

---

[1] this quantity is the speed of light of the relativistic scaling theory.

tion we represent eq. (5.29) with the new set of Majorana components

$$\psi(x) = \frac{\Psi(x) + \bar{\Psi}(x)}{\sqrt{2}}, \qquad \text{and} \qquad \bar{\psi}(x) = \frac{\Psi(x) + \bar{\Psi}(x)}{\sqrt{2}}. \tag{5.31}$$

This change of variables in turn allows us to write[1]

$$\mathcal{H}_{\text{Ising}} = i \int dx \{ \frac{c}{2} \left[\psi(x)\partial_x \psi(x) - \bar{\psi}(x)\partial_x \bar{\psi}(x) \right] + m\bar{\psi}(x)\psi(x) \}, \tag{5.32}$$

and by means of a Legendre transformation, and taking $c = 1$ we can easily see that this Hamiltonian corresponds to the action (3.21).

5.1.3 Entanglement entropy of the XY chain

In this section we use results from previous sections to compute the bi-partite entanglement entropy (1.15). The region A corresponds to a block of L contiguous spins, B being its complement. This setting is shown in Figure 5.3. We aim then first to define the reduced density matrix ρ_A, and then evaluate S_A, as defined in (1.13), by diagonalizing this matrix. The position of the block A is of no importance since we are considering a translationally-invariant infinite chain, such that we can consider spins $|s\rangle_i$ from position $i = 1$ to $i = L$, and label the reduced density matrix by ρ_L. This is a hermitian matrix, and its action on the generic spin i can be written as $\alpha_\mu \sigma_i^\mu$, where $\alpha_\mu = \left(\alpha^0, \alpha^1, \alpha^2, \alpha^3 \right)$ is real, while $\sigma_i^\mu = (\mathbb{1}, \sigma_i^x, \sigma_i^y, \sigma_i^z)$. Then the density matrix takes the form

$$\rho_L = \frac{1}{2^L} \sum_{\mu_1,\ldots,\mu_L=0,1,2,3} \rho_{\mu_1,\ldots,\mu_L} \sigma_1^{\mu_1} \otimes \ldots \otimes \sigma_L^{\mu_L}, \tag{5.33}$$

where $\rho_{\mu_1,\ldots,\mu_L} = \langle 0 | \sigma_1^{\mu_1} \otimes \ldots \otimes \sigma_L^{\mu_L} | 0 \rangle$. This is a $2^L \times 2^L$-matrix, then its dimension grows exponentially with the length of the region L. This makes the direct evaluation of its eigenvalues impossible but for very short intervals. A way to tackle this problem comes from the Bogoliubov transformation defined in section 5.1, as it reduces the XY Hamiltonian, to one of uncorrelated Fermionic harmonic

---

[1]to perform this step we are assuming that ψ and $\bar{\psi}$ be some well localized wave functions for which $\int dx\partial_x \left[\psi\bar{\psi} \right] = 0$ holds.

oscillators.

In this section we choose a different map than (5.7), and we want to work with
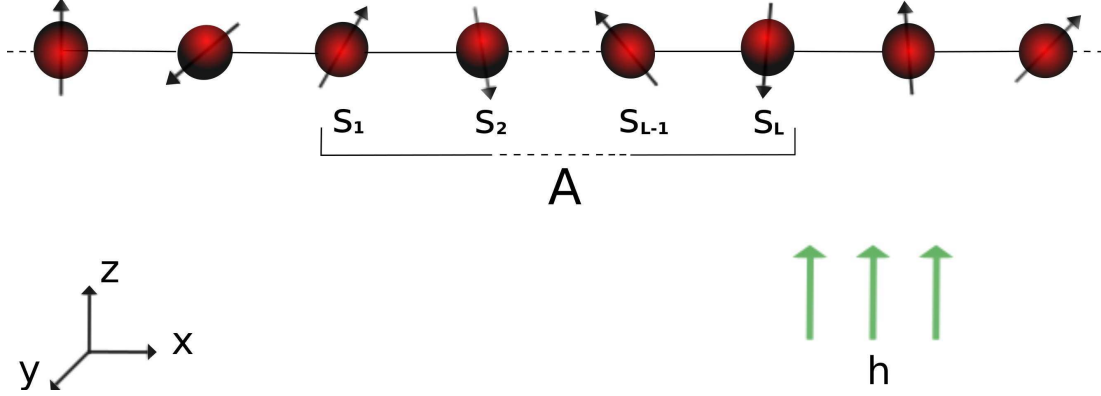
Figure 5.3: Representation of the bipatition in the two regions A, and its complement. The external magnetic field is aligned along the z-axis, while the black links between spins represent the bond interaction.

Majorana instead of Dirac Fermions[1], as it is a drastic simplification. This formalism was first introduced by Bravyi & Kitaev [2000], and it was thought as a way to implement a physical realisation of a quantum computation with local Fermion modes. It has then been employed by Vidal *et al.* [2003] to express the density matrix of the XY-chain in terms of these modes. In the following we will proceed as in section 5.1 defining a Jordan-Wigner map, and then a Bogoliubov transformation in order to find the Majorana-Fermion basis in which (5.1) is diagonal.

The central idea is to identify the two Dirac complex operators of creation and annihilation on a certain site, as real Majorana operators acting on two different sites. So that instead of (5.7) we define

$$a_{2n-1} = \left(\prod_{l<n} \sigma_l^z \right) \sigma_n^x \qquad\qquad a_{2n} = \left(\prod_{l<n} \sigma_l^z \right) \sigma_n^y, \qquad (5.34)$$

and they clearly are Hermitian Fermionic operators, as they satisfy $a_k^\dagger = a_k$, and $\{a_k, a_l\} = \delta_{kl}$. Using this map it is easy to check that $\sigma_n^z = -i a_{2n-1} a_{2n}$,

---

[1]here the words Majorana and Dirac are not to be understood in the QFT sense, as we are not talking about spinor representations; instead they mean respectively a real or complex Fermion.

$\sigma_n^x \sigma_{n+1}^x = -i a_{2n} a_{2n+1}$, and $\sigma_n^y \sigma_{n+1}^y = i a_{2n-1} a_{2n+2}$. The XY Hamiltonian can then be formulated in terms of these operators as

$$\mathcal{H}_{\mathrm{XY}} = \frac{i}{2} \sum_{n=-\frac{N-1}{2}}^{\frac{N-1}{2}} \left[\frac{1+\kappa}{2} a_{2n} a_{2n+1} - \frac{1-\kappa}{2} a_{2n-1} a_{2n+2} + h a_{2n-1} a_{2n} \right], \qquad (5.35)$$

and rewritten in the much more convenient form $H_{\mathrm{XY}} = \frac{i}{4} \vec{a}^T \mathbf{A} \vec{a}$, where $\vec{a} = (a_{-N}, a_{1-N}, ..., a_{N+1})$, while

$$\mathbf{A} = \begin{pmatrix} A_0 & A_1 & & & \\ -A_1^T & A_0 & A_1 & & \\ & & \ddots & & \\ & & -A_1^T & A_0 & A_1 \\ & & & -A_1^T & A_0 \end{pmatrix}, \qquad (5.36)$$

with

$$A_0 = \begin{pmatrix} 0 & 2h \\ -2h & 0 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 0 & \kappa - 1 \\ \kappa + 1 & 0 \end{pmatrix}. \qquad (5.37)$$

In this way the problem reduces to finding a transformation matrix $\mathbf{V} \in SO(2N)$ that makes $\mathbf{A}$ diagonal. This transformation corresponds to a combination of the Fourier expansion and Bogoliubov transformation of section 5.1. First we Fourier-expand the Majorana operators in the following way

$$a_{2n} = \sqrt{\frac{2}{N}} \sum_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}} \left[\sin\left(\frac{2\pi nk}{N}\right) e_{2k} + \cos\left(\frac{2\pi nk}{N}\right) d_{2k} \right],$$

$$a_{2n-1} = \sqrt{\frac{2}{N}} \sum_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}} \left[\sin\left(\frac{2\pi nk}{N}\right) e_{2k-1} + \cos\left(\frac{2\pi nk}{N}\right) d_{2k-1} \right], \qquad (5.38)$$

which allows to write the Hamiltonian as

$$H_{\mathrm{XY}} = \frac{i}{4} \sum_{k=0}^{\frac{N}{2}} \varepsilon(k) \begin{pmatrix} d_{2k-1} \\ e_{2k-1} \\ d_{2k} \\ e_{2k} \end{pmatrix}^T \begin{pmatrix} 0 & 0 & c_k & -s_k \\ 0 & 0 & s_k & c_k \\ -c_k & -s_k & 0 & 0 \\ s_k & -c_k & 0 & 0 \end{pmatrix} \begin{pmatrix} d_{2k-1} \\ e_{2k-1} \\ d_{2k} \\ e_{2k} \end{pmatrix}, \qquad (5.39)$$

where $c_k = \cos 2\theta_k$, $s_k = \sin 2\theta_k$, θ_k being the angle defined in eq. (5.13), while $\varepsilon(k)$ are the energy levels defined in eq. (5.15). Finally we act with the Bogoliubov transformation

$$
\begin{pmatrix} b_{-2k-1} \\ b_{-2k} \\ b_{2k-1} \\ b_{2k} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} u_k & v_k & u_k & -v_k \\ u_k & v_k & -u_k & v_k \\ v_k & -u_k & v_k & u_k \\ -v_k & u_k & v_k & u_k \end{pmatrix} \begin{pmatrix} d_{2k-1} \\ e_{2k-1} \\ d_{2k} \\ e_{2k} \end{pmatrix}. \tag{5.40}
$$

It can be readily verified that the operators b_k satisfy all the right properties to be Majorana Fermions, and they diagonalize the Hamiltonian, which finally takes the form

$$
H_{\mathrm{XY}} = \frac{i}{2} \sum_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}} \varepsilon(k) b_{2k-1} b_{2k}. \tag{5.41}
$$

The matrix $\mathbf{V}$ is then defined by the composition of the transformations (5.38), and (5.40), and connects the two sets of Fermions as $\vec{b} = \mathbf{V}\vec{a}$. Notice that this last formulation is totally equivalent to the one used to diagonalize the Hamiltonian in section 5.1. We can see that by comparing the definitions of the Dirac operators γ_k, and the real ones b_k. It takes a bit of work, but one can extract the relation between those two set of Fermions, that is $\gamma_k = (b_{2k-1} + i b_{2k})/2$. This can be readily used to prove the equivalence of (5.14) and (5.41).

We have seen in section 5.1.1 the properties of the ground state of the set of operators γ_k. Summarising here, we have $\langle \gamma_k^\dagger \gamma_l \rangle = \delta_{kl}$, and $\langle \gamma_k^\dagger \gamma_l^\dagger \rangle = \langle \gamma_k \gamma_l \rangle = 0$. These are the only important expectation values of the theory, as with Wick's theorem one can reduce any n-correlation function to products of two-point functions. Together with requiring that the ground state be the lowest energy eigenvector of the Hamiltonian, the set of correlation functions gives a full characterisation of $|0\rangle$. We can do the same for the operators b_k, and use the map between them and γ_k to obtain the only non-trivial two-point correlation $\langle b_k b_l \rangle = \delta_{kl} + i\Gamma_{kl}^b$, where

Γ_{kl}^b are the entries of the $2N \times 2N$ matrix

$$\Gamma^b = \begin{pmatrix} 0 & 1 & & & & \\ -1 & 0 & & & & \\ & & \ddots & & & \\ & & & & 0 & 1 \\ & & & & -1 & 0 \end{pmatrix}. \tag{5.42}$$

We recall here that our final goal is to extract the eigenvalues of (5.34) by diagonalizing a smaller matrix, exploiting the uncorrelation of Fermion modes. Then the next step is to express the two point function $\langle b_k b_l \rangle$ by means of the Fermion set a_n, as with (5.34) they can be easily related to the Pauli matrices appearing in eq. (5.33). This can be achieved by defining $\Gamma^a = \mathbf{V}^T \Gamma^b \mathbf{V}$, and noticing that $\langle a_i a_j \rangle = \delta_{ij} + i\Gamma_{ij}^a$. This yields

$$\Gamma^a = \begin{pmatrix} \mathbf{G}_0 & \mathbf{G}_1 & \dots & \mathbf{G}_{N-1} \\ -\mathbf{G}_1 & \mathbf{G}_0 & \dots & \mathbf{G}_{N-2} \\ \vdots & & \ddots & \vdots \\ -\mathbf{G}_{N-1} & -\mathbf{G}_{N-2} & \dots & \mathbf{G}_0 \end{pmatrix}, \quad \text{with} \quad \mathbf{G}_n = \begin{pmatrix} 0 & G(n) \\ -G(-n) & 0 \end{pmatrix}, \tag{5.43}$$

where $G(n)$ are the quantities defined in eq. (5.22) for the thermodynamic limit. Now we aim to express the coefficients $\rho_{\mu_1,\dots,\mu_L}$ of eq. (5.33) by means of the matrix elements $\langle a_i a_j \rangle$. First we notice that many $\rho_{\mu_1,\dots,\mu_L}$ are null. In fact the Hamiltonian (5.1) conserves the number of spins up/down[1]. This in turn implies that $\prod_{i=1}^{L} \sigma_i^z \rho_{\mu_1,\dots,\mu_L} \prod_{i=1}^{L} \sigma_i^z = \rho_{\mu_1,\dots,\mu_L}$, that is any $\rho_{\mu_1,\dots,\mu_L}$ with an odd number of σ^x and σ^y must be zero. Hence any non-vanishing element will have an even number of σ^x and σ^y, and correspond to matrix elements with an even number of a_n. We will then be always able to reduce such matrix elements to products of two point correlators $\langle a_i a_j \rangle$, where $i, j \in [1, 2L]^2$. Now we have all the means to express any density matrix element in terms of two-point functions of the Majorana set a_n, and compute them by dealing with $2L \times 2L$ matrices,

---

[1]that is the oddness/evenness of Fermions in eq. (5.8).

[2]recall that we are dealing only with a block of L spins.

instead of $2^L \times 2^L$. Unfortunately, a direct evaluation at this stage is difficult, due to the high degree of correlation among a_n modes[1], so that we need further simplifications. Notice that (5.43) is a real skew symmetric Topelitz matrix, and as such its eigenvalues are complex conjugated pairs, which we label $\pm i v_j$, with $j \in [1,L]$. We first reduce Γ^a in its block diagonal form by the transformation $\Gamma^f = \mathbf{U}^T \Gamma^a \mathbf{U}$, where $\mathbf{U} \in SU(2L)$. Then we end up with the matrix

$$
\Gamma^f = \begin{pmatrix}
0 & v_1 & & & & \\
-v_1 & 0 & v_2 & & & \\
& -v_2 & 0 & & & \\
& & & \ddots & & \\
& & & & 0 & v_L \\
& & & & -v_L & 0
\end{pmatrix},
\tag{5.44}
$$

and the set of Majorana operators $\vec{f} = \mathbf{U}\vec{a}$. This is a fair simplification, because as can be read from eq. (5.44), the only correlated modes are f_{2j-1} with f_{2j}, $\forall j \in [1,L]$. To conclude we define the Fermionic modes $\psi_n = (f_{2j-1} + i f_{2j})/2$, such that

$$
\langle \psi_l \psi_m^\dagger \rangle = \delta_{lm} \frac{1 + v_l}{2}.
\tag{5.45}
$$

We have finally reduced the density matrix ρ in eq. (5.33), to the density matrix of a set of uncorrelated, unentangled Fermionic operators, that is $\rho = \varrho_1 \otimes \varrho_2 \otimes ... \otimes \varrho_L$. The single mode density matrix can be reduced to the diagonal form $\varrho_m = \mathrm{diag}[(1 + v_m)/2, (1 - v_m)/2]$ which means that we have a clear expression of the 2^L eigenvalues of ρ, in terms of products of the $2L$ eigenvalues of the matrices ϱ_l. The map between eigenvalues of ρ (that we will call λs) and ϱ_l can be implemented by a string of L classical bits $\mathbb{P} = p_1 p_2 ... p_L$, where $p_i = 0,1$, such that an eigenvalue will be labelled by such a string as

$$
\lambda_{\mathbb{P}} = \prod_{j=1}^{L} \frac{1 + (-1)^{p_j} v_j}{2}.
\tag{5.46}
$$

---

[1] this is directly linked to the low "sparseness" of the matrix elements in Γ^a; this matrix is generally easy to diagonalize, but a map between its eigenvalues, and those of the density matrix would result rather complicated in general.

This map is directly related to the uncorrelation of the modes ψ_j, and allows us to extract the entanglement entropy by the sum

$$S(\lambda_{\mathbb{P}}) = \sum_{j=1}^{L} H_2(\frac{1+\nu_j}{2}),\qquad(5.47)$$

where $H_2(x) = -x\log x - (1-x)\log(1-x)$ is the binary entropy.

This is the main result of this section, and it is telling us that any simulation which aims to evaluate the entanglement of the XY-chain will be extremely fast and reliable. Indeed we were able to reduce the system to a set of Fermions which are only classically correlated, and this means that the system can be very efficiently simulated on a classical computer. Another way to see this is that to evaluate the entanglement entropy we do not need the knowledge of the $\mathbb{P}$s. So that if we define a gain as the amount of classical information that we need to extract the λs out of the νs, that is $\Delta I = L\log 2$.

5.2   Numerical results for the Ising chain

In the previous section we have defined a method to evaluate the block entanglement entropy which is suitable for numerical implementation. The main steps of a computer simulation will be to integrate (5.22) numerically, then build and diagonalize (5.43), and finally evaluate the entanglement entropy as in eq. (5.47). The program we used to obtain the results listed in this section is reported in Appendix C.

Firstly we want to check the qualitative behavior of the entanglement entropy as a function of the external magnetic field. From the scaling considerations carried out in Chapters 2 and 3 we expect that it diverge logarithmically with L at the critical point $h = 1$, and that it saturate for any other values of the magnetic field. In Figure 5.4 we can see how the critical point is manifested for finite increasing blocks, confirming the behaviour that we expect. The analytical counterpart of this plot can be found in Franchini *et al.* [2007]. In that case the authors were considering a different setting, evaluating the entropy of half an infinite chain
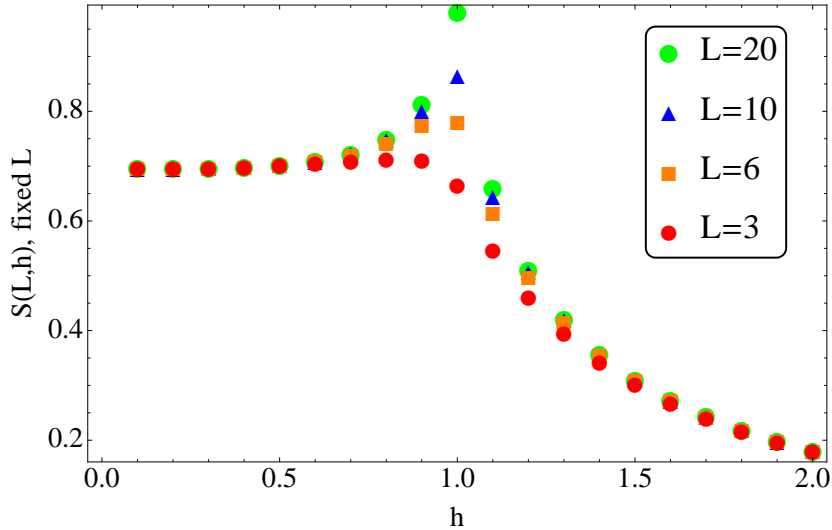
Figure 5.4: Behavior of the entanglement entropy for fixed block length as a function of h.

against the other half. Even considering two different settings Figure 5.4 is qualitatively comparable with Figure 5 in Franchini *et al.* [2007], the only difference being the slope approaching the critical point. We can also extract some information on the ground state. In the limit $h \to \infty$ the entanglement entropy vanishes for any L. that is the ground state is a separable state in that limit. On the other hand at $h = 0$ the entanglement entropy is $\log 2$, and again does not depend on the length of the block anymore. This is due to the degeneracy of the ground state explained in the comments below eq. (5.3). The ground state is composed by two separable states, and it has to be treated as a mixed state. This means that if we call the two equivalent ground states $|\phi_0\rangle$ and $|\varphi_0\rangle$, the density matrix will be

$$\rho = \frac{1}{2} \left[|\phi_0\rangle \langle \phi_0| + |\varphi_0\rangle \langle \varphi_0| \right], \quad \text{then} \quad S(L) = \log 2 + \frac{1}{2} S_{\phi_0}(L) + \frac{1}{2} S_{\varphi_0}(L) = \log 2.$$
(5.48)

Another interesting qualitative study is the behavior of the entropy for growing block lengths, at fixed h. We know from eqs. (3.151) and (3.152) that in the gapped regime, where there is a finite correlation length ξ, we have a logarithmic growth for short intervals $L << \xi$, while the entropy saturates for $L >> \xi$. At the critical point on the other hand we expect that the chain be described by a CFT,
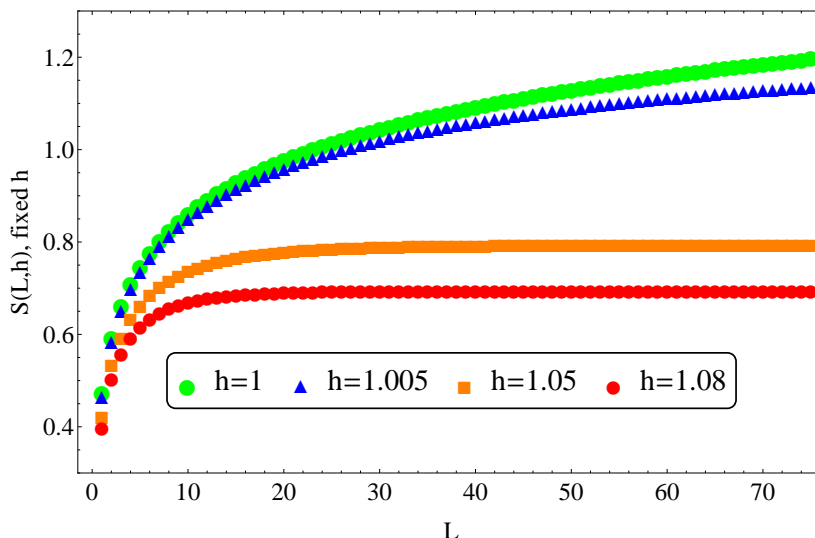
Figure 5.5: Behavior of the entanglement entropy for fixed h as a function of the block length.

so that the entropy grows logarithmically unbounded. These features can be observed in Figure 5.5. The logarithmic growth ceases sooner and sooner for increasing h, and the entropy saturates to a constant value, except for the case $h = 1$.

We need a more detailed study in order to be able to compare our results with the QFT predictions. The Ising spin chain and QFT are connected by the scaling limit reported in section 5.1.2, which allows us to use numerical results on the spin chain to confirm the QFT analytical predictions. Using the map reported in Table 1.1 and the explicit form of the correlation length (5.24) we are able to write eqs. (3.151) and (3.152) specifically for the Ising model. These are

$$S(L,h) = \frac{1}{6}\log(L) + c_1' \qquad\qquad \text{for} \quad L \ll \frac{1}{\log h}, \quad (5.49)$$

$$S(L,h) = -\frac{1}{6}\log(\log h) + c_1' + U - \frac{1}{8}K_0(2L\log h) + ... \qquad \text{for} \quad L \gg \frac{1}{\log h}. \quad (5.50)$$

The first quantity we focus on is the constant U. The value of this constant was computed analytically in Cardy *et al.* [2008], and takes the value $U = -0.131984....$ In order to obtain U from the lattice model we have to eradicate the non-universal behaviour from eqs. (5.49) and (5.50). This can be achieved by considering the

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following limits

$$U = \lim_{h \to 1} S(\infty, h) - \lim_{L \to \infty} S(L, 1). \tag{5.51}$$

In terms of numerical simulations the most accessible way to obtain U is to subtract the constant c_1' obtained by fitting the critical results with (5.49) from the constant obtained by fitting the saturation value of the off-critical data with (5.50).

We start then considering the critical case $h = 1$, and we display the results in Figure 5.6, which shows remarkable agreement between the numerical data and the interpolating function eq. (5.49). In fact extrapolating the central charge and
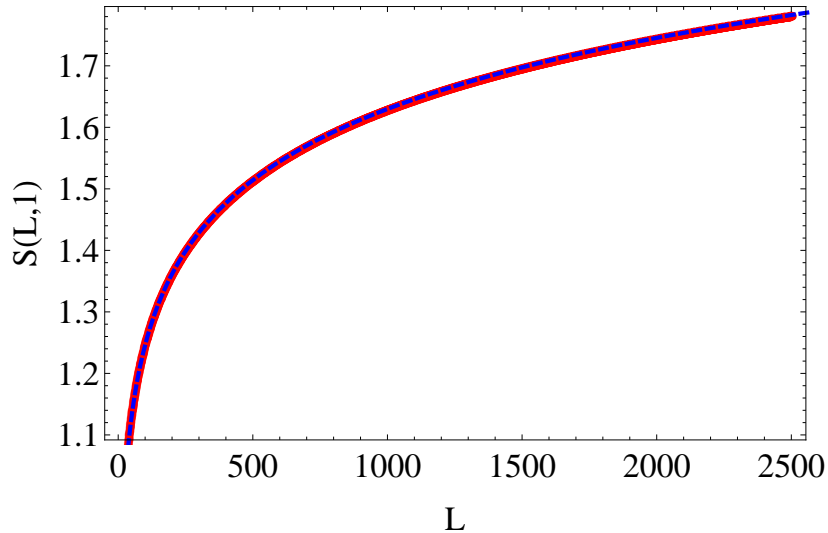
Figure 5.6: Plot of the critical entanglement entropy $S(L, 1)$. The numerical data are shown in red, along with the fitting logarithmic behaviour (5.49) as a blue dashed line.

the non-universal constant from a best fit with $\frac{c}{3} \log L + c_1'$, we find $c = 0.500003$, and $c_1' = 0.478551$. We can already appreciate the precision of our simulation comparing the numerical estimate of the central charge with the theoretical value $c = 1/2$.

While the small distance limit (5.49) is described by a unique conformal theory, the large distance limit depends explicitly on the external magnetic field, and we expect the QFT prediction (5.50) to hold only in the scaling region, that is in an infinitesimal region around the critical point. We would expect though that the

scaling behaviour deform smoothly outside this region, hence that it depend on some analytic functions of h. It is convenient to parametrize eq. (5.50) in terms of the correlation length ξ. In particular we make the ansatz that

$$S(L,\xi) = \frac{1}{6}\log(\xi) + c_1' + U(\xi) - \frac{1}{\alpha(\xi)}K_0(2\frac{L}{\xi}) + ... \qquad \text{for} \quad L \gg \xi. \qquad (5.52)$$

This ansatz is supported specially by the study of the Bessel function term. In fact we observed that an interpolation of the numerical behaviour with $K_0(2L/\xi)$ for different value of ξ fits remarkably, with some slight modification of the 1/8 constant in front. In Figure 5.7 we summarize these considerations.

We want to study then how the dimensionless constants $U(\xi)$ and $\alpha(\xi)$ run
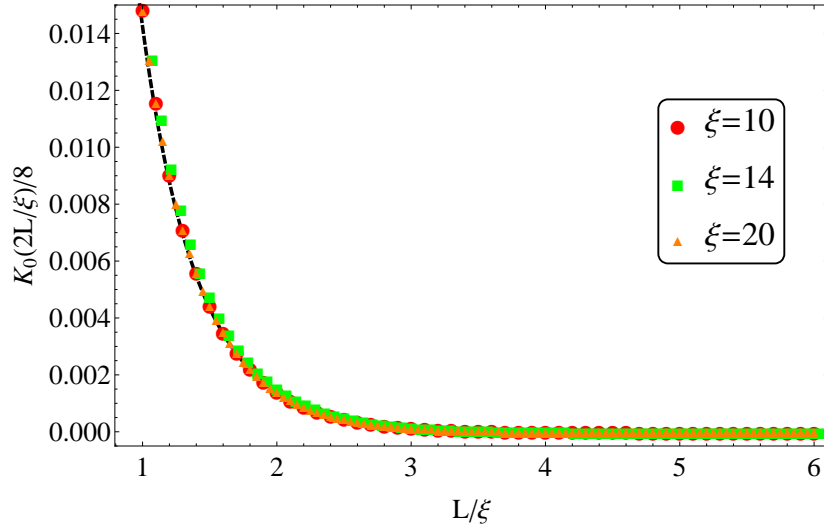
Figure 5.7: In this figure we plot the numerical data obtained for the Bessel-like term in eq. (5.52) for different values of the correlation length ξ. We can observe how even for relatively small ξ the scaling behaviour $K_0(2\frac{L}{\xi})/8$ (black dashed line) fits our numerical data.

outside the scaling region, and we want to see how they approach the critical point in order to compute their scaling value. The region we consider is $1.001 \le h \le 1.1$, that is $600 \le \xi \le 10$, and we collect data for $1 \le L/\xi \le 6$. We focus first on $U(\xi)$. As we already found c_1' we can obtain this quantity for different values of ξ using

$$U(\xi) = S(\infty,\xi) - \frac{1}{6}\log(\xi) - c_1'. \qquad (5.53)$$

In Calabrese *et al.* [2010]; Peschel [2004] the corrections to entanglement entropy in the gapped regime of the Ising chain were studied analytically, so that we can use their results to check the accuracy of our simulations. In that work the authors were considering a different setting from ours, and they were focusing on the bipartite entanglement between two infinite half chains. As explained in section 1.7 the area law prescribes that in one dimensional theories the entanglement entropy be proportional to the number of shared boundaries between the two regions. We expect then to find double the contributions of the infinite half chain case. We can extrapolate from eq. (27) in Calabrese *et al.* [2010] a theoretical prediction for $S(\infty, \xi)$, that is

$$S(\infty, \xi) - \frac{1}{6}\log(\xi) = 4 \sum_{k=1}^{\infty} \left[\frac{(-1)^k}{4k \sinh(k \log \xi)} (k \coth(k \log \xi) \log \xi - 1) \right], \qquad (5.54)$$

and exploiting eq. (5.53) we can compare the r.h.s. of (5.54) with our results for $U + c_1'$. This comparison is displayed in Figure 5.8 which shows remarkable
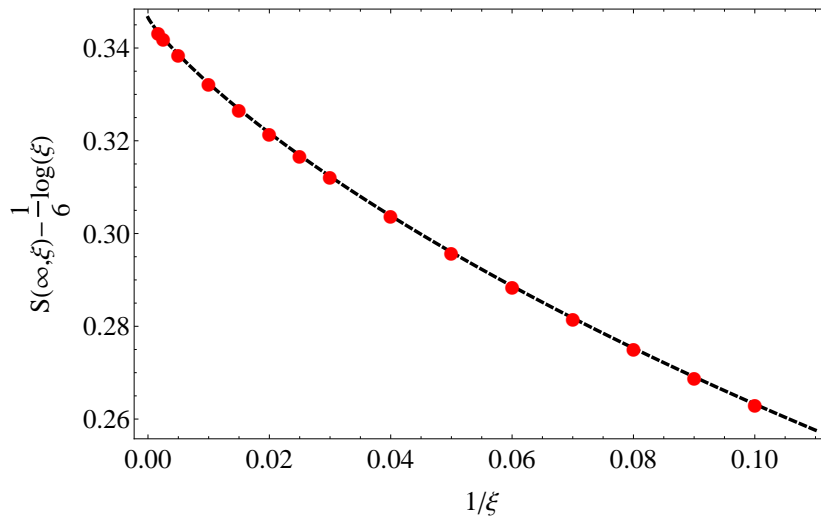
Figure 5.8: The numerical values obtained for $U + c_1'$ are presented and compared against the analytical prediction in Calabrese *et al.* [2010] for $S(\infty, \xi) - \frac{1}{6}\log \xi$, pictured as a dashed black line. The agreement leaves no doubt about the precision of our numerics.

agreement, underlying the reliability of our results. Eventually we can extract from the plot in Figure 5.8 the scaling value of $\lim_{\xi \to \infty} U(\xi) + c_1'$, and subtracting

the value of c_1' obtained with the critical analysis in Figure 5.6, we finally obtain $U(\infty) = -0.131984$, which is exactly equal to the theoretical predictions up to the precision we considered.

Now we focus on the constant $\alpha(\xi)$ of eq. (5.52). From a careful look to Figure 5.7 we can see that the agreement between numerical data and the theoretical expectation $K_0(2L/\xi)/8$ becomes better for increasing ratios L/ξ. This is explained by the fact that our theoretical prediction has been made on the basis of a form factor expansion of which we considered only the two particle contribution. As this series converges in the region $mr = L/\xi > 1$ it is totally sensible to expect that for $L/\xi \gtrsim 1$ higher particle corrections still play a big role, and truncating to the two particle contribution is not a good approximation. In order to tackle this issue we perform our analysis for $L \geq x\xi$, for $x = 1.5, 2, 2.5, 3, 3.5, 4$ and 4.5. For each value of x we give a numerical estimation on $\alpha_x(\xi)$ for different ξ fitting our data with $\frac{1}{\alpha_x(\xi)}K_0(2\frac{L}{\xi})$. The data we
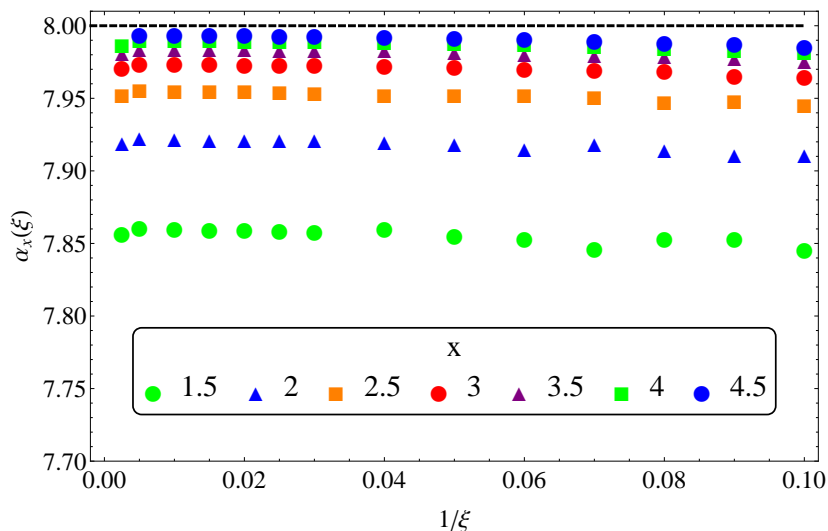
Figure 5.9: The running of the constant $\alpha_x(\xi)$ outside the scaling region is shown for different values of x.

obtained are shown in Figure 5.9, where we can see that for increasing x the value of $\alpha_x(\xi)$ approaches the scaling theoretical value $\alpha = 8$. A very interesting and unexpected feature is that $\alpha_x(\xi)$ seems to remain constant in the region $1 \leq h \leq 1.1$. Linear fittings of the seven sets of data provide the asymptotic values $\alpha_{1.5}(\infty) = 7.85581$, $\alpha_2(\infty) = 7.9186$, $\alpha_{2.5}(\infty) = 7.95252$, $\alpha_3(\infty) = 7.97121$,

$\alpha_{3.5}(\infty) = 7.9818$, $\alpha_4(\infty) = 7.98783$ and $\alpha_{4.5}(\infty) = 7.99146$. These data are dis-
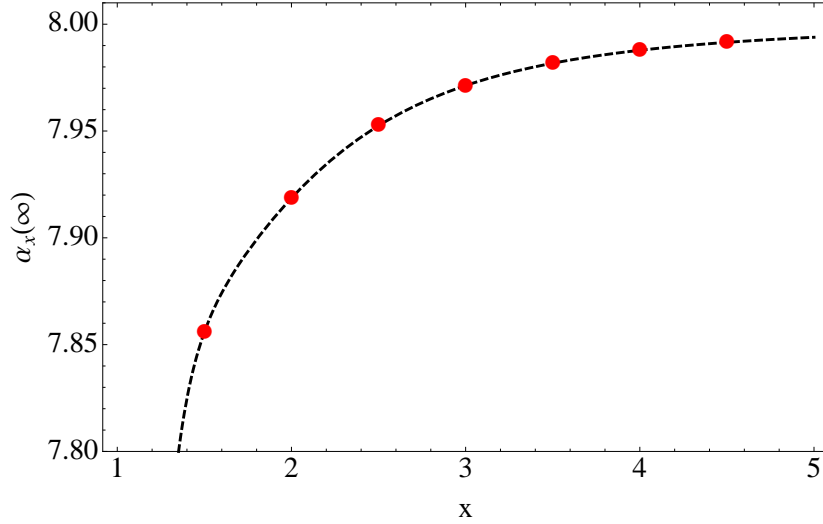
Figure 5.10: We show data obtained for $\alpha_x(\infty)$ in the region $1.5 \leq x \leq 4,5$ with a numerical fit represented by the black dashed line.

played in Figure 5.10, and we performed a numerical best-fit which is shown in the graph alongside. From this fit we extapolate $\lim_{x \to \infty} \alpha_x(\infty) = 8.00084$, which is remarkably close to the theoretical value $\alpha = 8$.

5.3 The XXZ Heisenberg chain

The XXZ Hamiltonian is defined as

$$\mathcal{H}_{\mathrm{XXZ}} = J \sum_{i=1}^{N} \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z \right), \tag{5.55}$$

where $S_i^\alpha = \sigma_i^\alpha/2$ are spin operators, and Δ plays the role of an anisotropy parameter. The parameter J sets the energy scale, and only its sign is important in the description of the phase diagram. A negative value of J makes the planar behaviour in the xy-plane antiferromagnetic, while with a positive value a ferromagnetic alignment is energetically less expensive. The axial behaviour on the z-axis in contrast is described by the sign of the product $J\Delta$. For positive $J\Delta$ a Néel state is more accessible, whereas $J\Delta < 0$ makes aligned states prefer-

able. We will consider positive J throughout this section. In the limit of large anisotropy parameter the XXZ chain becomes a Ising spin chain with zero external magnetic field. On the other hand in the opposite limit, i.e. for small values of Δ the model behaves as the isotropic XY model. The ground state in these two point is highly different, so that we expect that the model undergo a QPT somewhere in the middle. This is indeed the case. In fact this model has a critical line for $\Delta \in (-1, 1]$. The modulus of Δ describes the competition between interactions in the z direction with the ones on the xy-plane. Consequently the region $|\Delta| > 1$ is usually referred to as the axial regime, while for $|\Delta| < 1$ we talk about planar regime.

The free Fermion methods which worked so well for the Ising model do not apply to the XXZ spin chain, and we can easily understand why considering the Jordan-Wigner transformation described in eq. (5.7). For $\Delta \neq 0$ we have a term of the kind

$$\sigma_i^z \sigma_{i+1}^z = 1 - 2\left(c_i^\dagger c_i + c_{i+1}^\dagger c_{i+1}\right) + 4 c_i^\dagger c_i c_{i+1}^\dagger c_{i+1}. \tag{5.56}$$

The last term in eq. (5.56) is responsible for an interaction between Fermions. This feature makes the approach based on the correlation matrix inapplicable for the XXZ model, hence we need a more sophisticated numerical method to find the ground state properties. A particularly suitable method is the density matrix renormalization group, which we will describe in section 5.3.1.

Performing the transformation (5.7) on eq. (5.55) lets us express the XXZ Hamiltonian as that of a Fermion model closely connected to the Hubbard model Essler [2005]. Precisely we can write

$$\mathcal{H}_{\text{XXZ}} = t \sum_{i=1}^{N} \left[c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i \right] + V \sum_{i=1}^{N} \left(\hat{n}_i - \frac{1}{2} \right)\left(\hat{n}_{i+1} - \frac{1}{2} \right). \tag{5.57}$$

The operator $\hat{n}_i = c_i^\dagger c_i$ is the number operator, $t = J/2$ is usually called the hopping integral, and $V = J\Delta$ is the nearest-neighbour Coulomb repulsion.

The XXZ spectrum has been computed with the Bethe ansatz approach in Cloizeaux & Gaudin [1966]; des Cloizeaux & Pearson [1962]; Gaudin [1971]. The ground state energy was found to be $E_0 = J\Delta N/4$. The correlation length can also be evaluated with the same methods. This was done in Gu *et al.* [2002], and taking

$J = 1$ it is

$$\frac{1}{\xi(\Delta)} = \gamma(\Delta) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \tanh\left[2n\gamma(\Delta)\right], \quad \text{with} \quad \gamma(\Delta) = \cosh^{-1}\Delta. \quad (5.58)$$

5.3.1 The density matrix renormalization group (DMRG)

The density matrix renormalization group is a renormalization technique which works extremely well for describing the ground state of strongly correlated systems, specially in low dimension. The main difference between this method, and other renormalization schemes is the decimation procedure. A step of the momentum space renormalization group is performed with a truncation of high energy modes, while the real space renormalization group prescription is to approximate the short distance correlations with their mean value. The DMRG on the other hand, as introduced in White [1992, 1993], performs a truncation in the Hilbert space, keeping the most important states in the description of the ground state. The criteria used to understand which states are important, and which must be discarded allow for a quantum information interpretation of the DMRG. In what follows we describe a DMRG step.

As we applied this method to the XXZ model, we will present directly the action of the DMRG on a XXZ spin chain. We start with a block of l spins B. The quantum state which describes the block lives in a 2^l-dimensional Hilbert space H_B, and any operator with support on this block such as the Hamiltonian $\mathcal{H}$ can be expressed by $2^l \times 2^l$ matrices with matrix elements $\langle B|\mathcal{H}|B'\rangle$, where $|B\rangle, |B'\rangle \in H_B$. Now we add a site to the right of B, defining what is usually called the system $B\bullet$. The system is described by a vector of a 2^{l+1} dimensional Hilbert space $H_{B\bullet}$, and all the operators can be easily written in the product basis $|Bs\rangle = |B\rangle \otimes |s\rangle$, where $|s\rangle$ represents the single site basis. Taking the XXZ Hamiltonian[1] as an example this can be written as

$$\mathcal{H}_{B\bullet} = \mathcal{H}_B \otimes \mathbb{1}_2 + \sigma_R^x \otimes \sigma_s^x + \sigma_R^y \otimes \sigma_s^y + \Delta\sigma_R^z \otimes \sigma_s^z, \quad (5.59)$$

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[1]we consider as usual the ferromagnetic regime, and from now on we will take $J = 1$ with no loss of generality.

where σ_s^α with $\alpha = (x, y, z)$ act on the state $|s\rangle$, and $\sigma_R^\alpha = \mathbb{1}_{2^{l-1}} \otimes \sigma^\alpha$ are spin operators acting on the rightmost site of B.

Next we introduce an environment which has the aim of mimicking the thermodynamic limit. There are several ways to implement this feature. The setting we chose is the original one, introduced by White. We define the environment as $\bullet B$, that is by reflection of the system, so that we obtain the superblock $B \bullet \bullet B$. The superblock Hamiltonian is again easily defined in the basis $|B1s1s2B2\rangle = |B1\rangle \otimes |s1\rangle \otimes |s2\rangle \otimes |B2\rangle$ as

$$\mathcal{H}_{B\bullet\bullet B} = \mathcal{H}_{B\bullet} \otimes \mathbb{1}_{2^{l+1}} + \mathbb{1}_{2^{l+1}} \otimes \mathcal{H}_{\bullet B} + \mathbb{1}_{2^l} \otimes \left(\sigma_{s1}^x \otimes \sigma_{s2}^x + \sigma_{s1}^y \otimes \sigma_{s2}^y + \Delta \sigma_{s1}^z \otimes \sigma_{s2}^z \right) \otimes \mathbb{1}_{2^l}$$
$$+ \sigma_{L1}^x \otimes \mathbb{1}_4 \otimes \sigma_{R2}^x + \sigma_{L1}^y \otimes \mathbb{1}_4 \otimes \sigma_{R2}^y + \Delta \sigma_{L1}^z \otimes \mathbb{1}_4 \otimes \sigma_{R2}^z,$$

$$(5.60)$$

where we denoted with $s1$ and $s2$ respectively the left and right additional sites, with $B1$ and $B2$ the block's and environment's states, and with $L1$ the leftmost spin of the system's block, and with $R2$ the rightmost spin of the environment's block. The terms $\mathcal{H}_{B\bullet}$ and $\mathcal{H}_{\bullet B}$ are defined with the logic of eq. (5.59). With the connection terms between the two blocks in eq. (5.60) we implement periodic boundary conditions.

The aim of our algorithm is to obtain the best approximation of the ground state, so that we diagonalize the superblock Hamiltonian. This is the most time demanding step in the DMRG, requiring a fair amount of computational power and memory, already for relatively small blocks[1]. This task can be accomplished using any diagonalization algorithm. As $\mathcal{H}_{B\bullet\bullet B}$ is a sparse symmetric real matrix we used the "dsyev" routine of the Lapack Anderson *et al.* [1999]. This is a standard realization of the Lanczos algorithm Lanczos [1950], by which we obtain an approximation to the superblock ground state. The computational time of this step can be reduced sharply by using conserved quantum numbers. The Hamiltonian (5.55) conserves e.g. the global magnetization along the z-axis, $m_z = \langle S_{tot}^z \rangle = \frac{1}{2} \langle \sum_i \sigma_i^z \rangle$. This means that $\left[S_{tot}^z, \mathcal{H}_{XXZ} \right] = 0$, and $\mathcal{H}_{XXZ}$ connects

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[1]if the length of the block is e.g. 10 sites finding the ground state of eq. (5.60) would require the numerical diagonalization of a $2^{22} \times 2^{22}$ matrix.

only states with equal definite global magnetization. We can then build the basis of eigenstates of S_{tot}^z. The Hamiltonian $\mathscr{H}_{\mathrm{XXZ}}$ is block diagonal expressed in this basis, each block corresponding to a different value of m_z. We can focus on each block separately, reducing the computational effort which is required by a full diagonalization, and making this step parallelizable. The global magnetization can assume any integer value between $-l-1 \le m_z \le l+1$, and the largest block is the $m_z = 0$ sector, which is a square matrix of dimension[1]

$$\binom{2l+2}{l+1} = \frac{(2l+2)!}{(l+1)!(l+1)!} \sim \frac{2^{2l}}{\sqrt{l}}, \qquad \text{when} \quad l \to \infty. \qquad (5.61)$$

Our algorithm outputs the ground state as a 2^{2l+2} vector in the basis

$$\left|\psi\right\rangle = \sum_{\beta_1=1}^{2^l} \sum_{\sigma_1=1,2} \sum_{\sigma_2=1,2} \sum_{\beta_2=1}^{2^l} \psi_{\beta_1\sigma_1\sigma_2\beta_2} \left|B1_{\beta_1}s1_{\sigma_1}s2_{\sigma_2}B2_{\beta_2}\right\rangle. \qquad (5.62)$$

It is convenient to change basis, and rewrite $\left|\psi\right\rangle$ in the factor basis $\left|B1_{\beta_1}s1_{\sigma_1}\right\rangle \otimes \left|s2_{\sigma_2}B2_{\beta_2}\right\rangle$. We simplify the notation calling $\left|i\right\rangle$ the system's basis, and $\left|j\right\rangle$ the environment basis, such that eq. (5.62) takes the form

$$\left|\psi\right\rangle = \sum_{i,j=1}^{2^{l+1}} \psi_{ij} \left|i\right\rangle \left|j\right\rangle. \qquad (5.63)$$

The coefficients ψ_{ij} in eq. (5.63) can be conveniently stored in a $2^{l+1} \times 2^{l+1}$ matrix.

Now comes the central part of the DMRG, the truncation protocol. The original idea in White [1992, 1993] is that, as $\left|\psi\right\rangle$ is a pure state, the physical properties of the system are encoded in the reduced density matrix $\rho_{B\bullet} = \mathrm{Tr}_{\bullet B} \left|\psi\right\rangle\left\langle\psi\right|$. If we stored the ground state as suggested in eq. (5.63) then the trace can be performed with just one matrix multiplication, and the reduced density matrix

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[1]to compute this dimension we have to think that we are looking for all the states with an equal number of up and down spins. These states can be counted considering just half of the spins, as the other half is fixed once chosen the first half. This means that we want to count the number of combinations of half of the spins, such that eq. (5.61) follows.

elements can be computed as

$$\langle i | \rho_{B_\bullet} | i' \rangle = \sum_j \psi_{ij} \psi_{ji'}^*. \tag{5.64}$$

As explained in section 1.2 this density matrix has positive eigenvalues λ_α which sum to one. We choose a number m, which we call truncation number. We diagonalize $\rho_{B_\bullet}$, and we keep the largest m eigenvalues $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_m$, and the respective eigenvectors $|v_1\rangle, |v_2\rangle, ..., |v_\alpha\rangle$ stored as columns of the $2^{l+1} \times m$ truncation matrix V. We define the truncation error as

$$\varepsilon = 1 - \sum_{\alpha=1}^{m} \lambda_\alpha, \tag{5.65}$$

which as we will see is a good control parameter of the precision of our approximation. Now we want to understand how this truncation affects the representation of operators, and the ground state. Consider a bounded operator, with support on the system, e.g. the Hamiltonian $\mathcal{H}_{B_\bullet}$. As explained in section 1.2 we can evaluate the exact value of the ground state energy of the system as

$$E_0 = \text{Tr}\left[\rho_{B_\bullet} \mathcal{H}_{B_\bullet}\right]. \tag{5.66}$$

We define the approximated version $\tilde{E}_0$ as

$$\tilde{E}_0 = \frac{\sum_{\alpha=1}^{m} \lambda_\alpha \langle v_\alpha | \mathcal{H}_{B_\bullet} | v_\alpha \rangle}{\sum_{\alpha=1}^{m} \lambda_\alpha}, \tag{5.67}$$

and the error on E_0 introduced by the truncation is easily obtained as

$$\begin{aligned}
|E_0 - \tilde{E}_0| &= \left| \sum_{\alpha=1}^{2^{l+1}} \lambda_\alpha \langle v_\alpha | \mathcal{H}_{B_\bullet} | v_\alpha \rangle - \frac{\sum_{\alpha=1}^{m} \lambda_\alpha \langle v_\alpha | \mathcal{H}_{B_\bullet} | v_\alpha \rangle}{1-\varepsilon} \right| \\
&\leq \varepsilon \left| \sum_{\alpha=m+1}^{2^{l+1}} \langle v_\alpha | \mathcal{H}_{B_\bullet} | v_\alpha \rangle \right| + O(\varepsilon^2).
\end{aligned} \tag{5.68}$$

The error on the estimate of an observable is then of the order of ε, then we want the truncation error to be as small as possible. Now we turn our attention to the ground state, and we want to demonstrate that the prescription of keeping

the largest m eigenstates of $\rho_{B_\bullet}$ gives the most faithful approximation. After the truncation procedure the superblock state (5.63) is left in the form

$$\left|\tilde{\psi}\right\rangle = \sum_{\alpha,j} a_{\alpha j} \left|v_\alpha\right\rangle \left|j\right\rangle. \tag{5.69}$$

The quantities $a_{\alpha j}$ are elements of a $m \times 2^{l+1}$ matrix $\tilde{A} = V\Psi$, where Ψ is the matrix of the coefficients ψ_{ij} in eq. (5.63), and V is the truncation matrix. The matrix $\tilde{A}$ can be brought into a more convenient form if we Schmidt decompose it, as explained in section 1.1. This means we bring it to the form $\mathscr{A} = VAU^\dagger$, where $\mathscr{A}$ is a diagonal $m \times m$ matrix with diagonal elements $a_1 \geq a_2 \geq ... \geq a_m$, and U is a $m \times 2^{l+1}$ matrix of matrix elements $u_{\alpha j}$. This in turn means we are rewriting (5.63) as

$$\left|\tilde{\psi}\right\rangle = \sum_\alpha a_\alpha \left|v_\alpha\right\rangle \left|u_\alpha\right\rangle. \tag{5.70}$$

We have to be careful here as the Schmidt decomposition does not preserve the norm. That is, considering $u_{\alpha j} = \langle u_\alpha | j \rangle$, in general $\sum_j |u_{\alpha j}|^2 \neq 1$, leading to the necessity of a normalization of the basis $|u_\alpha\rangle$.

To find the best approximation we want to minimize the quadratic norm

$$\mathscr{D} = \left|\left|\psi\right\rangle - \left|\tilde{\psi}\right\rangle\right|^2, \tag{5.71}$$

and in components

$$\mathscr{D} = \mathrm{Tr}\left|\psi_{ij} - \sum_{\alpha=1}^m a_\alpha v_{\alpha i}^* u_{\alpha j}\right|^2, \tag{5.72}$$

is given in terms of the Hilbert-Schmidt distance. The Schmidt decomposition is a sub-case of the more general singular value decomposition, and we can apply the Eckart—Young theorem Eckart & Young [1936]. This theorem states that the optimal way of approximating a matrix M of rank r with a matrix $\tilde{M}$ of lower rank $\tilde{r} < r$ is keeping the largest $\tilde{r}$ singular values and singular vectors. This is exactly what we are doing in eq. (5.72). Notice moreover that (5.64) can be diagonalized by means of U, and its eigenvalues are $\lambda_\alpha = |a_\alpha|^2$. We have demonstrated then that the DMRG prescription is the optimal way to approximate the ground state with a lower rank state. Moreover, with a bit of manipulation on

eq. (5.72), one is able to demonstrate that $\mathscr{D} = \varepsilon$, giving to the truncation error a new interpretation. A connection with quantum information theory can be found if we rewrite eq. (5.70) as

$$\left| \tilde{\psi} \right\rangle = \sum_{\alpha} \sqrt{\lambda_\alpha} \left| v_\alpha \right\rangle \left| u_\alpha \right\rangle. \tag{5.73}$$

Then $\rho_{B\bullet} = \sum_{\alpha=1}^{m} \lambda_\alpha \left| u_\alpha \right\rangle \left\langle u_\alpha \right|$, and $\rho_{\bullet B} = \sum_{\alpha=1}^{m} \lambda_\alpha \left| v_\alpha \right\rangle \left\langle v_\alpha \right|$, such that the block entropy

$$S_{B\bullet} = S_{\bullet B} = - \sum_{\alpha=1}^{m} \lambda_\alpha \log \lambda_\alpha. \tag{5.74}$$

The DMRG truncation procedure is then that which maximizes the block entropy between the system and the environment.

We are now ready to summarize the key steps of the infinite algorithm

1. Import as input parameter the chosen truncation number m, and the Lanczos precision at will. Consider a block of small size l if it is the first loop, or consider the output block of step 4 if the program has already looped. Consider or import also all the operators of interest. For simplicity we will assume it is the first loop in what follows. As we want to form the superblock in the next step in particular we have to define the rightmost and leftmost spin operators $\sigma_R^\alpha = \mathbb{1}_{2^{l-1}} \otimes \sigma^\alpha$, and $\sigma_L^\alpha = \sigma^\alpha \otimes \mathbb{1}_{2^{l-1}}$.

2. Form the superblock Hamiltonian (5.60), and find the ground state with the Lanczos algorithm at the chosen precision. The output is a 2^{2l+2} vector.

3. Store the ground state as in eq. (5.63), and build the system reduced density matrix as described in eq. (5.64). Now we need to fully diagonalize $\rho_{B\bullet}$, store the first m eigenvalues, and arrange the corresponding eigenvectors as columns of the truncation matrix V. The columns of V must be normalized correctly to achieve the right result. Use the found eigenvalues λ_α to define ε as in eq. (5.65), and $S_{B\bullet}$ as in eq. (5.74), these will be used as control parameters.

4. Change basis to the block Hamiltonian doing $\tilde{\mathscr{H}}_{B\bullet} = V^\dagger \mathscr{H}_{B\bullet} V$, which is stored as a $m \times m$ matrix. This process is called update. Do the same

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with all operators of interest after giving them support on the system. In particular the update σ_R^α and σ_L^α is done as $\tilde{\sigma}_L^\alpha = V^\dagger(\sigma^\alpha \otimes \mathbb{1}_{2l})V$, and $\tilde{\sigma}_R^\alpha = V^\dagger(\mathbb{1}_{2l} \otimes \sigma^\alpha)V$. Use the updated matrices to define the new superblock Hamiltonian in steps 1–2.

Once we set the truncation number m, as long as $m \leq 2^{l+1}$ the infinite algorithm results only in a change of basis. The truncation begins the first step in which $m > 2^{l+1}$, and the superblock Hamiltonian will be a $2m \times 2m$ matrix in all subsequent loops.

We implemented this algorithm in C++ with the use of Lapack package, and Boost C++ libraries Demming & Duffy [2012, 2010]. The typical truncation number our realization is able to keep is $m < 64$ without considering conserved quantum numbers. With the consideration of quantum numbers the truncation number grows to $m < 200$. The most precise and reliable results though were obtained with the help of the super optimized ALPS project realization of the DMRG Albuquerque *et al.* [2007]; Bauer *et al.* [2011], which allowed us to consider up to $m = 1000$ states.

The second part of our DMRG simulation is the finite algorithm. This is designed with the aim of gaining precision in reproducing the ground state of finite chains. Its mechanism is similar to the infinite algorithm, but the dimension of the super block is kept fixed, and the system grows at the expense of the environment, and vice versa. If we want to extract the ground state properties of a chain of length $2L$, we start performing the finite algorithm until the left and right blocks are both $L-1$-sites long. At this stage we begin what is called the first sweep. We grow the left block to L sites, and decrease the right block to $L-2$ sites. The system growth is performed as in the infinite algorithm, and we use the updated operators of the $L-2$ step to describe the environment[1]. We keep on with this process until we grew the left block to $L-3$ sites. Now we invert the process, and we make the right block grow at the expense of the left one. When the left block is just one site long we consider the sweep ended. This part of the program introduces a kind of thermalization of the chain. In fact in

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[1]here we are assuming that we started with a block of just one spin, as is usually the case.

the infinite algorithm the changing length from step to step can introduce some unwanted features. Specially in those electronic systems with definite particle density the number of particles grows at each infinite step, and the system cannot thermalize[1]. Other cases in which the finite systems does not perform well are in presence of impurities or randomness in the Hamiltonian, and in general whenever the Hamiltonian changes from step to step.

5.4 The scaling limit of the XXZ chain

Performing the scaling limit for the XXZ chain is a rather hard task compared to the Ising case. In this case the scaling theory would be the $SU(2)$-Thirring (sine-Gordon) model, which has a much richer structure than the Ising field theory. The sine-Gordon model presents in its spectrum a massive soliton and antisoliton (the Fermion and antiFermion of the Thirring model), and a class of bound states called breathers. The action can be written as

$$S_{\text{sG}} = \int d^2 x \left[\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \lambda \cos(\beta \phi) \right],$$ (5.75)

in terms of the pseudoscalar field ϕ, the mass scale λ, and the parameter β.

The connection between the XXZ chain and the sine-Gordon model has been studied in Ercolessi *et al.* [2010]; Klassen & Melzer [1993]; Pallua & Prester [1999], using two different approaches. In Pallua & Prester [1999] they considered the ferromagnetic chain, i.e. $J = -1$ in eq. (5.55), plugged in an external magnetic field h along the x-axis. They were studying then the limits $\Delta \to -1^+$, and $h \to 0$, and they managed to find a precise relation between the XXZ and sine-Gordon Hamiltonians. The introduction of a magnetic field in that case had the effect of "lifting" the theory off the critical line, so as to end up with a massive scaling theory.

The authors of Ercolessi *et al.* [2010] were considering a different setting. They were dealing with the XYZ chain, that is the case where the coupling constants of the bound terms along the three axes are all different. In that work the entanglement entropy for both the XXZ chain and the sine-Gordon model was evaluated

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[1] this feature is well explained e.g. in Schollwöck [2005]

analytically. Yet a different kind of setting was considered. In fact the authors exploited the connection between XXZ spin chain and the six-vertex model Baxter [1982], and used the transfer matrix approach to evaluate the entropy. The same study was carried out, along with the Ising and XY chains, in Calabrese *et al.* [2010]; Peschel [2004]. This method by construction allows to compute the bipartite entanglement entropy of a semi-infinite chain, and the modifications explained in section 5.2 are needed to compare these results to ours.

5.5 Numerical results on the XXZ chain

In section 5.2 we have seen that the ansatz (5.52) seems to describe very well the entanglement entropy in the gapped region, and it reduces to the right scaling behaviour when we approach the critical point. We want to perform the same study for the XXZ model in order to have a nontrivial confirmation of the validity of our hypothesis. In fact, as explained in section 5.4 this model contains two lightest particle, so that eq. (3.152) reduces to

$$S(L,\xi) = \frac{1}{6}\log\xi + U - \frac{1}{4}K_0(2\frac{L}{\xi}) + ...,\qquad(5.76)$$

as both soliton and anti-soliton give the same contribution to the Bessel-like term.

In order to obtain meaningful numerical data the program we use can be divided into the following two steps

- We start with the infinite algorithm until we reach the desired length. We are interested in the scaling properties of the block entropy when the length of the block is from comparable to much bigger than the correlation length. We need then to grow our block far beyond the correlation length.

- We perform then the finite algorithm. We take a number of sweeps and states which ensures convergence, and we use the entropy of the final sweep for increasing (e.g. left) blocks to extract our results. The setting we want to reproduce is the one of a finite block, plugged into an infinite chain, so that we are forced to consider periodic boundary conditions.

Δ	1.92833	1.85021	1.79041	1.74286	1.70394	1.67137
L_s	60	70	80	90	100	110
m_i	600	600	800	800	800	1000
m_f	600	600	600	800	800	1000
n	60	60	10	80	80	100
$\epsilon(10^{-11})$	1.3	3.4	6.5	3.7	6.2	3.06
$|\langle\sigma_i^z\rangle|$	0.0089	0.0068	0.0011	0.01	0.01	0.009

Table 5.2: In this table the details of our DMRG simulations are reported. L_s is the length of the chain, m_i the number of states kept in the infinite phase, m_f the number of states kept in the finite phase, and finally ϵ is the truncation error.

> DMRG's precision relies heavily on the decaying of the entanglement spectrum, which is much slower for periodic boundaries than open ones. This forces us to keep more states, and sweep more before observing a good convergence. Good convergence is characterized by a low truncation error, and a symmetric behaviour of the entanglement entropy for growing left/right blocks.

We used conserved quantum numbers to speed up the computation. In fact we know that the XXZ Hamiltonian conserves the global magnetization along the anisotropy axis. Moreover we know that the ground state is in the $\langle S_z \rangle = 0$ sector of the Hamiltonian. As in section 5.2 we extend our study outside the scaling region, making the ansatz that the behaviour be described by smooth functions in the anisotropy parameter Δ.

The DMRG, in contrast with free Fermion methods, gives access only to finite chain results. The chain must grow much longer than whichever L we consider, and, as we want to study the region $L \gg \xi$, this feature forces us to perform our study quite far from the critical point $\Delta = 1$. We study the cases $\Delta = 1.67137, 1.70394, 1.74286, 1.79041, 1.85021$ and 1.92833, which correspond respectively to $\xi = 12, 14, 16, 18, 20$ and 22. We find as optimal length of the chain five times the correlation length. This is long enough to allow us to perform our scaling study, but small enough to ensure contained running times. The values of the input parameters alongside with some properties of the ground state are reported in Table 5.2. A major problem that arose in performing DMRG simu-

lations on long chains in this regime is the presence of oscillations in the local magnetization and the entropy. As found in Baxter [1982]; Izergin *et al.* [1999], in the thermodynamic limit the ground state is two-fold degenerate for $\Delta > 1$, as there are two lowest energy states, with different momenta 0, and π, which we call respectively $|\psi_0\rangle$, and $|\psi_\pi\rangle$. This degeneracy is removed in the case of finite periodic chains, where the ground state has zero momentum, while the lowest energy state of the momentum π sector is an excited state. We will refer to these two states again as $|\psi_0\rangle$, and $|\psi_\pi\rangle$ with an abuse of notation. The gap between these two states though becomes very narrow for chains of the length we are considering. Our DMRG implementation struggles to distinguish these two states at the precision we are working . A source of error could be the employ-ment of the Lanczos algorithm for sparse diagonalization, which has difficulties in resolving degeneracies. As a result we observe a small staggered behaviour for the local magnetization, and a small oscillation in the block entropy between even/odd lengths of the block. Even if $\langle\psi_0|\sigma_i^z|\psi_0\rangle = \langle\psi_\pi|\sigma_i^z|\psi_\pi\rangle = 0$, a confu-sion between these two states can end up giving as target state a superposition $|\Psi\rangle = \alpha|\psi_0\rangle + \beta|\psi_\pi\rangle$, and as a result $\langle\Psi|\sigma_i^z|\Psi\rangle \neq 0$ in general. We use as signal of a good convergence $|\langle\sigma_i^z\rangle|$, which we want to be as low as possible. We used differ-ent implementations of the DMRG, but the most precise and reliable results were obtained with ALPS Albuquerque *et al.* [2007]; Bauer *et al.* [2011]. Throughout all our simulations we managed to keep $|\langle\sigma_i^z\rangle| \leq 0.01$, and the truncation error on the order of 10^{-11}. This is a quite big value of ε, but is totally acceptable as we are considering periodic boundary conditions.

In Figure 5.11 we show how well numerical data for the Bessel-like function in eq. (5.76) are fitted by the analytical prediction. For this plot the same qual-itative considerations valid for Figure 5.7 hold, and data are fitted better for increasing L/ξ. Moreover, isolating any value of ξ in the plot, we can appreciate the oscillation introduced by the aforementioned degeneracy.

We consider now the ansatz (5.52) and we focus on $U(\xi)$. We the same logic used for eq. (5.54) we can extrapolate from (14) of Ercolessi *et al.* [2010]

$$S(\infty, \Delta) - \frac{1}{3}\log\xi(\Delta) = \log 2 + 2\left[\varepsilon(\Delta)\sum_{j=0}^{\infty}\frac{j}{1 + e^{j\varepsilon(\Delta)}} + \sum_{j=0}^{\infty}\log\left(1 + e^{j\varepsilon(\Delta)}\right)\right], \quad (5.77)$$
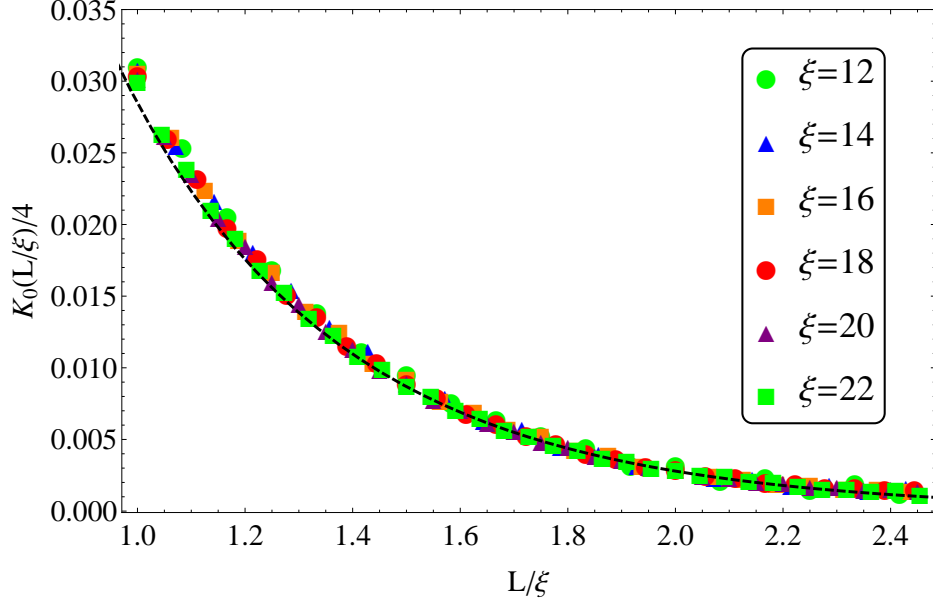
Figure 5.11: Numerical data for the Bessel-like term in eq. (5.76) are compared with the analytical behaviour represented by the dashed black line.

where $\varepsilon(\Delta) = \cosh^{-1}\Delta$. Our results are shown, and compared with eq. (5.77) in Figure 5.12. We can observe in Figure 5.12 that this case is not particularly affected by oscillation between odd and even blocks. The appearance of the term $\log 2$ in eq. (5.77) is related to the fact that the ground state, that is the state with zero momentum $|\psi_0\rangle$, is itself a composition of two orthogonal Néel-states. From eq. (5.77) we can compute $U(\infty) + c'_1 = 2\log 2/3$.

We performed a numerical study of the critical $\Delta = 1$ case, from which we can extract c'_1. This case is really well fitted by the behaviour $c/3 \log L + c'_1$, with $c = 1.00024$, and $c'_1 = 0.73375$. The numerical value of the central charge is very close to the analytical value $c = 1$, sine-Gordon being a deformation of a free boson. We use the value of c'_1 to give a numerical prediction of the constant $U(\infty) = -0.27166$ for the $SU(2)$-Thirring model.

Finally we focus on the behaviour of $\alpha(\xi)$ in the region considered by us. As usual we consider each value of the correlation length separately, and we estimate $\alpha(\xi)$ by a numerical fit of our data with $K_0(2L/\xi)/\alpha(\xi)$. We observe sharper oscillations between even and odd blocks for the extracted value of $\alpha(\xi)$ than for $U(\xi)$, and we report in Figure 5.13 our data. Taking the mean value of results on odd
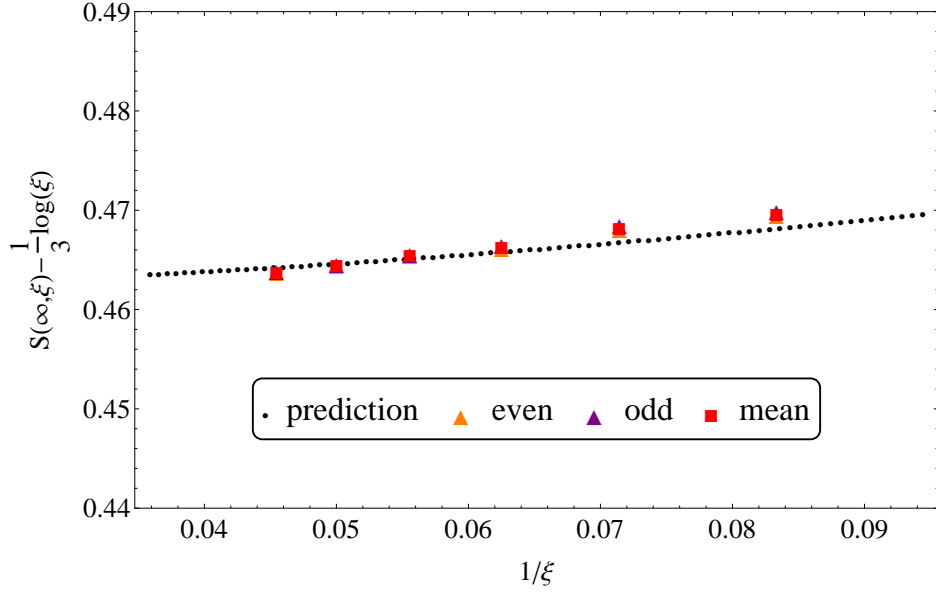
Figure 5.12: The numerical results obtained for $U + c'_1$ are displayed, and compared with the analytic prediction in Ercolessi *et al.* [2010]. The accordance is lower than for the Ising case, but good enough to confirm confidently the analytic behaviour.

and even block lengths we find the best fit is of linear form, from which we can easily extrapolate $\alpha(\infty) = 4.0039$, which is remarkably close to the scaling QFT prediction $\alpha = 4$. Contrary to the Ising model in the XXZ case $\alpha(\xi)$ has a non trivial running, which seems to be linear in the region that we considered.

Figure 5.13: Numerical estimate of $\alpha(\xi)$ obtained with a best fit of the Bessel-like term in eq. (5.52). The black dashed line is the best linear fit $\alpha(\xi) = -4.3261\xi^{-1} + 4.0039$.

6

Conclusions

In the first part of this thesis we have reviewed the methods commonly used to study entanglement in QFT, stressing particularly their connections to the scaling regime of lattice theories. We then achieved a series of results which we are going to list in this section.

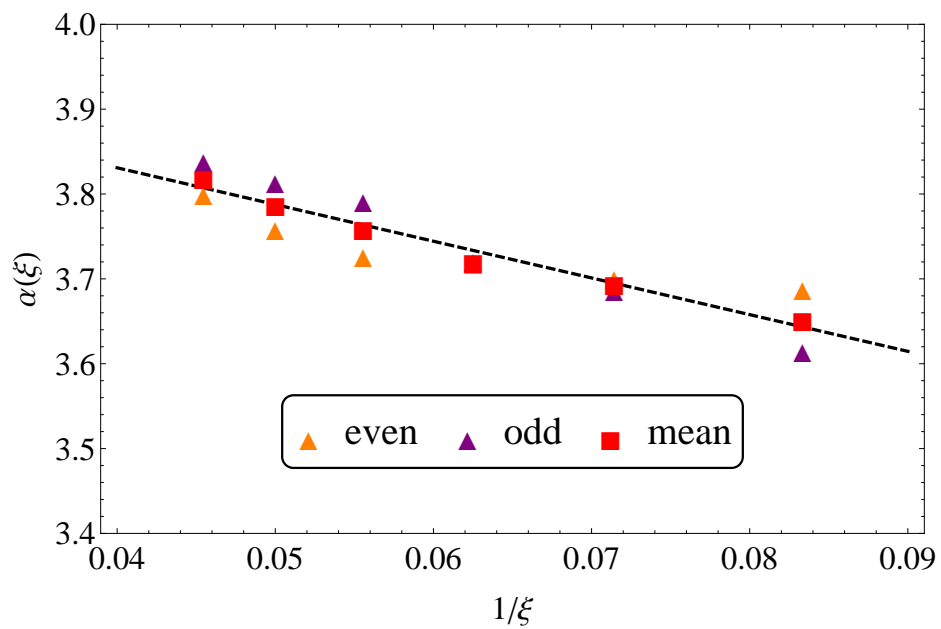In Chapter 3 we focused on the replica trick in the context of massive relativistic QFT, and in particular on the twist field approach. There we have solved eqs. (3.53)-(3.56) in order to compute higher particle form factors of the twist field. We considered two different models, namely the roaming trajectory model, and the $SU(3)_2$-homogeneous sine-Gordon model. This study have been motivated by the desire to find solutions to the form factor equations for non-trivial, non-free models. The choice of models have been performed in light of an eventual test by the Δ-sum rule. Our computations have revealed a number of interesting features: first, although the solution procedure and equations have many similarities with those for other local fields, it is considerably harder to find higher particle solutions for the twist field. This is mainly due to the increased number of poles the form factors have within the extended physical sheet. As a consequence, even for simple models it does not seem possible to find the nice closed determinant formulae found for example in Castro-Alvaredo & Fring [2001a]; Castro-Alvaredo *et al.* [2000a]; Fring *et al.* [1993]; Koubek & Mussardo [1993] for free Fermion models.

For the RT-model we have noted that solutions to the form factors equations for

branch point twist fields are generally not unique. This lack of uniqueness is not unexpected. This is because this geometric picture of the twist field as an object that connects the various sheets in a Riemann surface is not the only feature that characterizes the twist field. As its name indicates it is mainly characterized by a branch cut. One may also change the features of the twist field by putting other fields at the corresponding branch point as we did in Chapter 4. The expectation is that in this way we find form factors of descendants of the twist field. This has been one of the main inspirations for Castro-Alvaredo *et al.* [2011], and the study performed in Chapter 4. Our analysis of the RT-model, including the investigation of the cluster decomposition property of form factors, confirms that some of these other twist fields correspond to non primary fields at conformal level and are likely to be related to composite fields involving the entropy-related twist field and other fields of the theory. We have found that for the RT-model and generally any model with a single particle spectrum, the most general solution for the $2k$-particle form factor of the twist field depends on k free parameters.

Concerning the numerical computations performed in this chapter, our aim has been to test the few form factor solutions obtained for two theories: the roaming trajectories model and the $SU(3)_2$-homogeneous sine-Gordon model. Both share the appearance of staircase patterns for the associated effective central charges Castro-Alvaredo *et al.* [2000b]; Zamolodchikov [2006]. For the HSG-model the same pattern has been reproduced for Zamolodchikov's c-function Castro-Alvaredo & Fring [2001b]; Zamolodchikov [1986] and for the conformal dimensions of certain local fields Castro-Alvaredo & Fring [2001b]. Our numerics demonstrate that such pattern is again reproduced for the conformal dimension of the twist field which exhibits two plateaux at $\Delta_{\mathcal{T}} = \frac{1}{24}\left(n - \frac{1}{n}\right)$ and $\Delta_{\mathcal{T}} = \frac{1}{20}\left(n - \frac{1}{n}\right)$. For the RT-model we focused on the first and second steps in the staircase pattern only, corresponding to $\Delta_{\mathcal{T}} = \frac{1}{48}\left(n - \frac{1}{n}\right)$ and $\Delta_{\mathcal{T}} = \frac{7}{240}\left(n - \frac{1}{n}\right)$, respectively.

An interesting conclusion that can be drawn from this study, specially for the $SU(3)_2$-HSG model, is that the function $\Delta_{\mathcal{T}}(r_0)$ given by (3.136) seems to behave

exactly as

$$\Delta_{\mathcal{T}}(r_0) = \frac{c(r_0)}{24}\left(n - \frac{1}{n}\right),$$ (6.1)

where $c(r_0)$ is Zamolodchikov's c-function. The study of this relation is the main subject of Chapter 4.

In Chapter 4 we have first provided evidence that

$$\Delta_{\mathcal{T}}(r_0) \neq \frac{c(r_0)}{24}\left(n - \frac{1}{n}\right),$$ (6.2)

but that the function

$$\tilde{c}(r) = \frac{24n\Delta_{\mathcal{T}}(r)}{n^2 - 1},$$ (6.3)

satisfies Zamolodchikov's c-theorem. A study of the Δ-function of the twist field, due to its definition (4.10), relies on the full knowledge of the two point function $\langle\Theta(r)\mathcal{T}(0)\rangle$, so that we focused on this last quantity. First, we used a form factor expansion of $\langle\Theta(r)\mathcal{T}(0)\rangle$ to show that $\Delta(r)$ is monotonically decreasing for large distances. Second we investigated the short distance behavior of $\Delta(r)$ by using conformal perturbation theory considerations.

We exploited the relation (3.19) between Θ and the perturbing field ϕ, and then used the OPE of operators ϕ and $\mathcal{T}$ to attempt to prove that $\Delta(r)$ is a c-function in this region. This proof relies on the negativity of the composed twist field $\langle : \phi\mathcal{T} : \rangle$, which we did not manage to ensure for general theories. We managed though to compute $\langle\varepsilon(r)\mathcal{T}(0)\rangle$, where ε is the energy operator of the Ising model, and we showed negativity in this case.

This in particular led us to the identification of the vacuum expectation values of a new class of twist fields, including

$$: \varepsilon\mathcal{T} : (x) \sim \lim_{\delta \to 0} \varepsilon(x + \delta)\mathcal{T}(x),$$ (6.4)

and its derivatives $: \partial^{2\alpha}\varepsilon\mathcal{T} :$. In the process of showing the negativity of $\langle\varepsilon(r)\mathcal{T}(0)\rangle$ we managed to compute massive corrections to the structure constants up to $(mr)^6$ for both $: \varepsilon\mathcal{T} :$, and $: \partial^{2\alpha}\varepsilon\mathcal{T} :$. Moreover we computed the exact expression

of $\langle : \varepsilon \mathscr{T} : \rangle$ and $: \langle \partial^{2\alpha} \varepsilon \mathscr{T} : \rangle$, showing the negativity of all of them. The very fact that we are able to compute the expectation values of such a large class of operators is remarkable, as this is in general a rather hard task, and there is no general prescription to tackle it.

For the Klein-Gordon model we proved that $\Delta(r)$ is also monotonically decreasing for all r using angular quantization. Finally provided general arguments, based on various physical considerations, we showed that this holds for general unitary models. We eventually conclude that the function $\tilde{c}(r)$, as Zamolodchikov $c(r)$, measures the loss of degrees of freedom in a renormalization process. This important and somehow unexpected result could help to establish a connection between the loss of degrees of freedom, and the increasing of the entanglement entropy along the renormalization group flow.

In Chapter 5 we focus on giving numerical evidence for the behaviour in eqs. (3.148) and (3.149) of the entanglement entropy in QFT. We used the scaling map to translate QFT quantities in the context of gapped spin chains, on which we performed our numerics. We considered the Ising and the XXZ chains, which have very different characteristics, as the first corresponds to a free Fermion problem, while the second does not. We employed different numerical methods, taking advantage of the free Fermion map of the Ising model, and performing a DMRG study of the XXZ chain.

We managed to confirm QFT predictions with an amazing precision up to the first exponentially decaying correction, and we observed that they can be extended off the scaling region with few small changes. By means of this study we show that computing the entanglement entropy is a good numerical way to determine the number of lightest particles in the QFT spectrum. Finally we gave a numerical prediction of the constant U in eq. (3.149) in the $SU(2)$-Thirring model.

Explicit formulae for Q_4 and K_4

The constants in (3.88) are given by

$$\gamma = 2\left(1+2\cos(\frac{\pi}{n})\right)\sec(\frac{\pi}{2n})\sin(\frac{(B-4)\pi}{4n})\sin(\frac{(2+B)\pi}{4n}), \tag{A.1}$$

$$\delta = -\left(4\cos(\frac{\pi}{2n})-\cos(\frac{3\pi}{2n})-\cos(\frac{(B-1)\pi}{2n})\right)\sec(\frac{\pi}{2n}), \tag{A.2}$$

$$\eta = 3+2\cos(\frac{\pi}{n})+4\cos(\frac{\pi}{2n})\cos(\frac{(B-1)\pi}{2n}), \tag{A.3}$$

$$\xi = 2\left(1+3\cos(\frac{\pi}{n})-\cos(\frac{2\pi}{n})+8\cos(\frac{\pi}{2n})^3\cos(\frac{(B-1)\pi}{2n})\right.$$
$$\left.+\cos(\frac{(B-1)\pi}{n})\right), \tag{A.4}$$

$$\lambda = -2\left(6+6\cos(\frac{\pi}{n})+4\cos(\frac{2\pi}{n})+\cos(\frac{3\pi}{n})+\left(1+2\cos(\frac{\pi}{n})\right)\cos(\frac{(B-1)\pi}{n})\right)$$
$$-4\left(5\cos(\frac{\pi}{2n})+2\cos(\frac{3\pi}{2n})+\cos(\frac{5\pi}{2n})\right)\cos(\frac{(B-1)\pi}{2n}), \tag{A.5}$$

$$\rho = 8\cos(\frac{\pi}{n})^2\left(3+3\cos(\frac{\pi}{n})+\cos(\frac{2\pi}{n})+8\cos(\frac{\pi}{2n})^3\cos(\frac{(B-1)\pi}{2n})\right)$$
$$+8\cos(\frac{\pi}{n})^2\left(1+2\cos(\frac{\pi}{n})\right)\cos(\frac{(B-1)\pi}{n}). \tag{A.6}$$

All the constants above are real for B real and they remain real when $B = 1-\frac{2i\theta_0}{\pi}$, as one would expect.

The constants in (3.89) are given by

$$A = -\frac{1}{\left(1 + 2\cos(\frac{\pi}{n})\right)^3}, \tag{A.7}$$

$$B = \frac{2\left(1 + \cos(\frac{\pi}{n})\right)}{\left(1 + 2\cos(\frac{\pi}{n})\right)^3}, \tag{A.8}$$

$$C = -\frac{2\left(2 + \cos(\frac{\pi}{n})\right)}{\left(1 + 2\cos(\frac{\pi}{n})\right)^2}, \tag{A.9}$$

$$D = -\frac{16\cos^4(\frac{\pi}{2n})}{\left(1 + 2\cos(\frac{\pi}{n})\right)^3}, \tag{A.10}$$

$$E = \frac{8\cos(\frac{\pi}{2n})^2\left(3 + 6\cos(\frac{\pi}{n}) + \cos(\frac{2\pi}{n})\right)}{\left(1 + 2\cos(\frac{\pi}{n})\right)^3}, \tag{A.11}$$

$$F = \frac{2\left(2\cos(\frac{\pi}{n}) + \cos(\frac{2\pi}{n})\right)}{\left(1 + 2\cos(\frac{\pi}{n})\right)^3}, \tag{A.12}$$

$$G = -\frac{16\cos(\frac{\pi}{2n})^2\cos(\frac{\pi}{n})\left(2 + \cos(\frac{\pi}{n})\right)}{\left(1 + 2\cos(\frac{\pi}{n})\right)^2}, \tag{A.13}$$

$$H = \frac{128\cos(\frac{\pi}{2n})^4\cos(\frac{\pi}{n})^2}{\left(1 + 2\cos(\frac{\pi}{n})\right)^3}, \tag{A.14}$$

$$I = \frac{8\left(\cos(\frac{\pi}{2n}) + \cos(\frac{3\pi}{2n})\right)^2\left(3 + 2\cos(\frac{\pi}{n}) + \cos(\frac{2\pi}{n})\right)}{\left(1 + 2\cos(\frac{\pi}{n})\right)^3}, \tag{A.15}$$

$$J = -\frac{256\cos(\frac{\pi}{2n})^4\cos(\frac{\pi}{n})^4}{\left(1 + 2\cos(\frac{\pi}{n})\right)^3}. \tag{A.16}$$

Coefficients $\Omega_\alpha(n)$ and Maijer's G-function

B.1 Coefficients $\Omega_\alpha(n)$

This section is devoted to the computation of coefficients in (4.44). As explained in Section 4.3.1 the aim is to expand the fraction on the LHS of that equation for large t/mr. First of all, let us introduce the more convenient variable $u = (t/mr)^{-2/n}$. The denominator can be then treated, as long as $t > mr$ as the generating function of the Chebishev polynomials of the second kind, that is

$$\frac{1}{u^2 - 2xu + 1} = \sum_{\alpha=0}^{\infty} U_\alpha(x)u^\alpha, \tag{B.1}$$

for $-1 < x < 1$, and $|u| < 1$. The first condition is satisfied for every $n \geq 2$ as $x = \cos(\pi/n)$, while the second is satisfied in the whole integration domain of (4.43) except for the lower limit $t = mr$. This divergence is "cured" by integrating over the domain $[mr + \epsilon, \infty)$, where ϵ is a small parameter. Once this expansion is plugged into (4.43) the sum and the integration can be safely exchanged. After the integration is performed one has then to be sure that the result does not depend on ϵ, and finally set it to zero. We have performed these steps showing that indeed (B.1) in this case can be taken as valid also at the point $t = mr$. The details are technical and cumbersome, and they will not be reported here. From now on, and throughout the calculation in Section 4.3.1 we will take (B.1) as series representation on the whole integration path.

The polynomials $U_\alpha(x)$ in our case are formally defined as follows

$$U_\alpha\left(\cos\frac{\pi}{n}\right) = \frac{\sin\frac{(1+\alpha)\pi}{n}}{\sin\frac{\pi}{n}}, \tag{B.2}$$

The LHS in (4.44) can then be expanded as shown in the RHS with

$$\Omega_\alpha(n) = \begin{cases} \frac{\cos\frac{(1+2\alpha)\pi}{2n}}{\cos\frac{\pi}{2n}} & \text{if } \alpha < n \\ \frac{\cos\frac{(1+2\alpha)\pi}{2n}}{\cos\frac{\pi}{2n}} + \frac{\sin\frac{(1+2\alpha)\pi}{2n}}{\sin\frac{\pi}{2n}} & \text{if } \alpha \geq n \end{cases}. \tag{B.3}$$

B.2 Definite integrals of Bessel functions and powers

In this appendix we present a solution to integrals of the kind

$$\int_{mr}^{\infty} dt \quad t^{-\mu} K_\nu(t), \tag{B.4}$$

where both μ and mr are positive real numbers. In Section 4.3.1, in particular, an expansion for small values of mr was needed, so that this will be the aim of this appendix. First of all let us introduce the function

$$G_{p,q}^{m,n}\left(t \begin{vmatrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{vmatrix}\right) = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^m \Gamma(b_j - s) \prod_{j=1}^n \Gamma(1 - a_j + s)}{\prod_{j=m+1}^q \Gamma(1 - b_j + s) \prod_{j=n+1}^p \Gamma(a_j - s)} t^s \, ds. \tag{B.5}$$

This is a representation of the Meijer G-function, in the formalism adopted by Gradshteyn & Ryzhik [1980], and the details and properties about this function will not be reported here. A useful identity is

$$K_\nu(t) = \frac{1}{2} G_{0,2}^{2,0}\left(\frac{t^2}{4} \bigg| \frac{\nu}{2}, \frac{-\nu}{2}\right), \tag{B.6}$$

which holds for $|\arg(t)| \leq \pi/2$, and the empty sets of gamma functions' poles are omitted. In the light of (B.6) the integral in (B.4) can be rewritten as follows

$$\frac{(mr)^{1-\mu}}{4} \int_1^\infty dt \ \ t^{-\frac{\mu+1}{2}} G_{0,2}^{2,0}\left(\frac{m^2}{4}t \Big| \frac{\nu}{2}, -\frac{\nu}{2}\right). \tag{B.7}$$

This is a special case of a known integral of the G-function, which in the most general form is

$$\int_1^\infty dt \ \ t^{-\rho}(t-1)^{\sigma-1} G_{p,q}^{m,n}\left(\alpha t \left| \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\right.\right) = \Gamma(\sigma) G_{p+1,q+1}^{m+1,n}\left(\alpha \left| \begin{matrix} a_1,\ldots,a_p,\rho \\ \rho-\sigma,b_1,\ldots,b_q \end{matrix}\right.\right), \tag{B.8}$$

which holds for real $|\arg(t)| \leq (m+n-p/2-q/2)\pi$, $p+q \leq 2(m+n)$, $\Re(\sigma) > 0$ and $\Re(\rho-\sigma-a_j) > -1 \ \forall j \in [1,n]$. These conditions are all clearly satisfied by (B.7), so that the result is

$$\int_{mr}^\infty dt \ \ t^{-\mu} K_\nu(t) = \frac{(mr)^{1-\mu}}{4} G_{1,3}^{3,0}\left(\frac{m^2}{4} \left| \begin{matrix} \frac{\mu+1}{2} \\ \frac{\mu-1}{2}, \frac{\nu}{2}, -\frac{\nu}{2} \end{matrix}\right.\right) \tag{B.9}$$

Now this result has to be restricted to the cases (4.46) to be useful for the OPE that was considered in Section 4.3.1. The calculations are tedious and the results cumbersome, hence only the first few terms of the expansion of the first two contributions are given

$$\int_{mr}^\infty dt \ \ t^{-\frac{2\alpha+1}{n}} K_0(t) = 2^{-1-\frac{1+2\alpha}{n}} \Gamma\left(\frac{n-1-2\alpha}{2n}\right)^2 +$$
$$+ \frac{n[(n-1-2\alpha)(\gamma-\log 2)-n]}{(n-1-2\alpha)^2}(mr)^{1-\frac{1+2\alpha}{n}} + \frac{n}{(n-1-2\alpha)}(mr)^{1-\frac{1+2\alpha}{n}} \log(mr) +$$
$$+ \frac{n[(3n-1-2\alpha)(\gamma-1-\log 2)-n]}{4(3n-1-2\alpha)^2}(mr)^{3-\frac{1+2\alpha}{n}} +$$
$$+ \frac{n}{4(3n-1-2\alpha)}(mr)^{3-\frac{1+2\alpha}{n}} \log(mr) + O(mr)^{5-\frac{1+2\alpha}{n}}$$

$$\tag{B.10}$$

$$\int_{mr}^{\infty} dt \quad t^{-1-\frac{2\alpha+1}{n}} K_1(t) =$$

$$\frac{(mr)^{-1-\frac{1+2\alpha}{n}} n}{1+2\alpha+n} + 2^{-2-\frac{1+2\alpha}{n}} \Gamma\left(\frac{n-1-2\alpha}{2n}\right) \Gamma\left(-\frac{1+2\alpha+n}{2n}\right) +$$

$$+ \frac{n[(n-1-2\alpha)(1+2\log 2 - 2\gamma) - 2n]}{4(n-1-2\alpha)^2} (mr)^{1-\frac{1+2\alpha}{n}} +$$

$$- \frac{n}{2(n-1-2\alpha)} (mr)^{1-\frac{1+2\alpha}{n}} \log(mr) +$$

$$+ \frac{n[(3n-1-2\alpha)(5+4\log 2 - 4\gamma) + 4n]}{64(3n-1-2\alpha)^2} (mr)^{3-\frac{1+2\alpha}{n}} +$$

$$+ \frac{n}{16(3n-1-2\alpha)} (mr)^{3-\frac{1+2\alpha}{n}} \log(mr) + O(mr)^{5-\frac{1+2\alpha}{n}},$$

$$(B.11)$$

where $\gamma = 0.577216$ is the Euler-Mascheroni constant.

Mathematica code for the Ising model

In this appendix we report the details of the program used to extract the block entanglement entropy (5.47) of the Ising spin chain (5.3) numerically. The program is implemented in Mathematica. Our program uses the Stream objects present in Mathematica's libraries to write the outputs in external files. Then the first step is to give the command

```
1  ee = OpenWrite["ee.nb"];
2
3  ...
4
5  Close[ee];
```

Listing C.1: Stream commands

in lines 1 of listing C.1. With this we open the file ee.nb on which we write the results. The stream will be closed after the program has ended with the commands in lines 5 of listing C.1.

The second step is the evaluation of $\mathbf{G}_n$ in eq. (5.43) by means of the function GG of listing C.2. This is achieved first by defining $G(n)$s of eq. (5.25) and (5.26) as done in lines 2–9, and then by setting them as entries of $\mathbf{G}_n$ as done in lines 10–13.

```
 1 GG[L_, h_] := Block[{gg},
 2   gg[x_, a_] := KroneckerDelta[-x, 1] /; h == 0;
 3   gg[x_, a_] := -1./(\[Pi] (x + 1/2)) /; h == 1;
 4   gg[x_, a_] :=
 5    NIntegrate[
 6      Re[(Cos[t] - a + I*Sin[t])/Sqrt[(Cos[t] - a)^2 + (Sin[t])^2] E^(
 7        I*t*x)/(2 \[Pi])], {t, 0, 2 \[Pi]},
 8      Method -> {GlobalAdaptive, MaxErrorIncreases -> 10000},
 9      PrecisionGoal -> 12, MaxRecursion -> 20] /; h != 1 && h != 0;
10   Do[
11    G[l] = {{{0, gg[l, h]}, {-gg[-l, h], 0}}};
12    Print[l, "ok!"];
13    , {l, 0, L - 1}];
14   ];
```

Listing C.2: Definition of the function GG

The third and last step is to implement the definition of the entanglement entropy from $\mathbf{G}_n$s. This is achieved with the function CorrToEntropy defined in listing C.3. In lines 3–11 we define the matrix Γ^a of eq. (5.43). We diagonalize it completely and select its positive eigenvalues in lines 12–16. In lines 17–22 we use those eigenvalues to extract the entanglement entropy (EE). We finally print on ee.nb, and sce.nb respectively the two measures of entanglement.

```
 1 CorrToEntropy[L_] :=
 2   Block[{MM, MMM, MMMM, l1, l2, gg, EE, ss, SCE, EigenC, H},
 3    MM = Table[0, {i, 1, 2*L}, {j, 1, 2*L}];
 4    Do[MMM[j] =
 5      ArrayFlatten[Transpose[Table[G[i], {i, 0, L - j}]]], {j, 1, L}];
 6    Do[Do[MM[[i, j]] =
 7      MMM[Ceiling[i/2]][[Mod[i + 1, 2] + 1,
```

```
8      j - 2 Ceiling[i/2] + 2]], {j, 2 Ceiling[i/2] - 1, 2*L}}, {i, 1,
9        2*L}];
10   Do[Do[MM[[j, i]] = -MM[[i, j]], {i, 1, 2*Ceiling[j/2] - 2}], {j, 3,
11       2*L}];
12   EigenC = Eigenvalues[MM];
13   (*Selects only positive eigenvalues and build (1+l[i])/
14   2 upper eigenvalues of the density matrices of each fermion mode*)
15
16       l1 = Select[Im[EigenC], # > 0 &];
17   l2 = Table[(l1[[i]] + 1)/2, {i, 1, Length[l1]}];
18   (*Von Neumann entanglement entropy as binary entropy*)
19   H[x_] :=
20     If[Abs[x] > 10^-22 &&
21       Abs[1 - x] > 10^-22, -x*Log[x] - (1 - x) Log[1 - x], 0];
22   EE = Chop[Re[Sum[H[l2[[i]]], {i, 1, Length[l2]}]]];
23   WriteString[ee, "{", L, ",", SetPrecision[EE, 20], "},\n"];
24   Return[Re[EE]];
25   ];
```

Listing C.3: Definition of the function CorrToEntropy

The time taken by this program to run is independent on the value of the external magnetic field, apart from the cases $h = 0, 1$, for which the definitions of gg in lines 2 and 3 of listing C.1 boost the simulations, as no integration has to be performed. On the other hand it is highly dependent on the length L of the interval A considered, as one would expect. The most time consuming step is by far the exact diagonalization of the $2L \times 2L$ matrix MM, as this is in general not very sparse. We run our simulations with Mathematica 9.0, an a 3.06 GHz Intel Core 2 Duo CPU, with 4 GB 1067 MHz DDR3 memory. The longest simulation we run was for $h = 1.0025$, on a block growing from $L = 1$ to $L = 1770$. Its running time has been roughly of six hours.

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