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Rüdiger Murr

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"Wenn man auf etwas hindenkt, verdickt sich das, was man denkt, bis es nach und nach sichtbar wird. Wer sich was denkt, denkt das denk nur ich, kein anderer weiß es, aber dann kommt ein Tag, an dem das Gedachte als Tat oder Ding für alle Welt sichtbar wird."

Erwin Strittmatter

"Si nous nous arrêtions aux seules considérations méchaniques, il nous serait loisible d'expliquer que l'équilibre de la tête, ainsi placée sur l'empilement des vertèbres cervicales, au lieu de prolonger horizontalement la colonne vertébrale comme chez les quadrupèdes, ne peut s'obtenir que par un allègrement de la partie antérieure, c'est-à-dire la boîte crânienne et son contenu, l'encéphale."

Gabriel Camps

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Summary

This work is concerned with the characterization of certain classes of stochastic processes via duality formulae. In particular we consider reciprocal processes with jumps, a subject up to now neglected in the literature.

In the first part we introduce a new formulation of a characterization of processes with independent increments. This characterization is based on a duality formula satisfied by processes with infinitely divisible increments, in particular Lévy processes, which is well known in Malliavin calculus. We obtain two new methods to prove this duality formula, which are not based on the chaos decomposition of the space of square-integrable functionals. One of these methods uses a formula of partial integration that characterizes infinitely divisible random vectors. In this context, our characterization is a generalization of Stein's lemma for Gaussian random variables and Chen's lemma for Poisson random variables. The generality of our approach permits us to derive a characterization of infinitely divisible random measures.

The second part of this work focuses on the study of the reciprocal classes of Markov processes with and without jumps and their characterization.

We start with a resume of already existing results concerning the reciprocal classes of Brownian diffusions as solutions of duality formulae. As a new contribution, we show that the duality formula satisfied by elements of the reciprocal class of a Brownian diffusion has a physical interpretation as a stochastic Newton equation of motion. Thus we are able to connect the results of characterizations via duality formulae with the theory of stochastic mechanics by our interpretation, and to stochastic optimal control theory by the mathematical approach. As an application we are able to prove an invariance property of the reciprocal class of a Brownian diffusion under time reversal.

In the context of pure jump processes we derive the following new results. We describe the reciprocal classes of Markov counting processes, also called unit jump processes, and obtain a characterization of the associated reciprocal class via a duality formula. This formula contains as key terms a stochastic derivative, a compensated stochastic integral and an invariant of the reciprocal class. Moreover we present an interpretation of the characterization of a reciprocal class in the context of stochastic optimal control of unit jump processes. As a further application we show that the reciprocal class of a Markov counting process has an invariance property under time reversal.

Some of these results are extendable to the setting of pure jump processes, that is, we admit different jump-sizes. In particular, we show that the reciprocal classes of Markov jump processes can be compared using reciprocal invariants. A characterization of the reciprocal class of compound Poisson processes via a duality formula is possible under the assumption that the jump-sizes of the process are incommensurable.

Zusammenfassung

Diese Arbeit befasst sich mit der Charakterisierung von Klassen stochastischer Prozesse durch Dualitätsformeln. Es wird insbesondere der in der Literatur bisher unbehandelte Fall reziproker Klassen stochastischer Prozesse mit Sprüngen untersucht.

Im ersten Teil stellen wir eine neue Formulierung einer Charakterisierung von Prozessen mit unabhängigen Zuwächsen vor. Diese basiert auf der aus dem Malliavinkalkül bekannten Dualitätsformel für Prozesse mit unendlich oft teilbaren Zuwächsen. Wir präsentieren zusätzlich zwei neue Beweismethoden dieser Dualitätsformel, die nicht auf der Chaoszerlegung des Raumes quadratintegrabler Funktionale beruhen. Eine dieser Methoden basiert auf einer partiellen Integrationsformel für unendlich oft teilbare Zufallsvektoren. In diesem Rahmen ist unsere Charakterisierung eine Verallgemeinerung des Lemma für Gaußsche Zufallsvariablen von Stein und des Lemma für Zufallsvariablen mit Poissonverteilung von Chen. Die Allgemeinheit dieser Methode erlaubt uns durch einen ähnlichen Zugang die Charakterisierung unendlich oft teilbarer Zufallsmaße.

Im zweiten Teil der Arbeit konzentrieren wir uns auf die Charakterisierung reziproker Klassen ausgewählter Markovprozesse durch Dualitätsformeln.

Wir beginnen mit einer Zusammenfassung bereits existierender Ergebnisse zu den reziproken Klassen Brownscher Bewegungen mit Drift. Es ist uns möglich die Charakterisierung solcher reziproken Klassen durch eine Dualitätsformel physikalisch umzudeuten in eine Newtonsche Gleichung. Damit gelingt uns ein Brückenschlag zwischen derartigen Charakterisierungsergebnissen und der Theorie stochastischer Mechanik durch den Interpretationsansatz, sowie der Theorie stochastischer optimaler Steuerung durch den mathematischen Ansatz. Unter Verwendung der Charakterisierung reziproker Klassen durch Dualitätsformeln beweisen wir weiterhin eine Invarianzeigenschaft der reziproken Klasse Browscher Bewegungen mit Drift unter Zeitumkehrung.

Es gelingt uns weiterhin neue Resultate im Rahmen reiner Sprungprozesse zu beweisen. Wir beschreiben reziproke Klassen Markovscher Zählprozesse, d.h. Sprungprozesse mit Sprunghöhe eins, und erhalten eine Charakterisierung der reziproken Klasse vermöge einer Dualitätsformel. Diese beinhaltet als Schlüsselterme eine stochastische Ableitung nach den Sprungzeiten, ein kompensiertes stochastisches Integral und eine Invariante der reziproken Klasse. Wir präsentieren außerdem eine Interpretation der Charakterisierung einer reziproken Klasse im Rahmen der stochastischen Steuerungstheorie. Als weitere Anwendung beweisen wir eine Invarianzeigenschaft der reziproken Klasse Markovscher Zählprozesse unter Zeitumkehrung.

Einige dieser Ergebnisse werden für reine Sprungprozesse mit unterschiedlichen Sprunghöhen verallgemeinert. Insbesondere zeigen wir, dass die reziproken Klassen Markovscher Sprungprozesse vermöge reziproker Invarianten unterschieden werden können. Eine Charakterisierung der reziproken Klasse zusammengesetzter Poissonprozesse durch eine Dualitätsformel gelingt unter der Annahme inkommensurabler Sprunghöhen.

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Résumé

Ce travail est centré sur la charactérisation de certaines classes de processus aléatoires par des formules de dualité. En particulier on considérera des processus réciproques à sauts, un cas jusqu'à présent négligé dans la littérature.

Dans la première partie nous formulons de façon innovante une charactérisation des processus à accroissements indépendants. Celle-ci est basée sur une formule de dualité pour des processus infiniment divisibles, déjà connue dans le cadre du calcul de Malliavin. On va présenter deux nouvelles méthodes pour prouver cette formule, qui n'utilisent pas la décomposition en chaos de l'espace des fonctionnelles de carré intégrable. Une méthode s'appuie sur une formule d'intégration par parties satisfaite par des vecteurs aléatoires infiniment divisibles. Sous cet angle, notre charactérisation est une généralization du lemme de Stein dans le cas Gaussien et du lemme de Chen dans le cas Poissonien. La généralité de notre approche nous permet de plus, de présenter une charactérisation des mesures aléatoires infiniment divisibles.

Dans la deuxième partie de notre travail nous nous concentrons sur l'étude des classes réciproques de processus de Markov avec ou sans sauts, et sur leur charactérisation.

On commence avec un résumé des résultats déjà existants concernant les classes réciproques de diffusions browniennes comme solutions d'une formule de dualité. Nous obtenons notamment une nouvelle interprétation des classes réciproques comme les solutions d'une équation de Newton. Cela nous permet de relier nos résultats à la mécanique stochastique d'une part et à la théorie du contrôle optimale, d'autre part. La formule de dualité nous permet aussi de prouver une propriété d'invariance par retournement du temps de la classe réciproque d'une diffusion brownienne.

En outre nous obtenons une série de nouveaux résultats concernant les processus de sauts purs. Nous décrivons d'abord la classe réciproque associée à un processus markovien de comptage, c'est-à-dire un processus de sauts de taille un, puis en présentons une charactérisation par une formule de dualité. Cette formule contient une dérivée stochastique, une intégrale stochastique compensée, et une fonctionnelle qui est une grandeur invariante de la classe réciproque. De plus nous livrons une interprétation de la classe réciproque comme ensemble des solutions d'un problème de contrôle optimal. Enfin, par une utilisation appropriée de la formule de dualité, nous montrons que la classe réciproque d'un processus markovien de comptage est invariante par retournement du temps.

Quelques-uns de ces résultats restent valables pour des processus de sauts purs dont les sauts sont de taille variée. En particulier nous montrons que certaines fonctionnelles dites *invariants réciproques* permettent de distinguer différentes classes réciproques. Notre dernier résultat est la charactérisation de la classe réciproque d'un processus de Poisson composé dès lors que les (tailles des) différents sauts sont incommensurables.

Contents

Introduction

The theory of reciprocal processes basically evolved from an idea by Erwin Schrödinger. In [Sch32] he described the motion of a particle diffusing in a thermal reservoir as a stochastic boundary value problem. He proposed that the solutions of such a boundary value problem are elements of the reciprocal class associated to a Markov process. This class contains all stochastic processes that have the same bridges as that reference Markov process, where the term bridge refers to a process conditioned on deterministic initial and final states.

In [Ber32] Bernstein noted that the concept of reciprocal processes, or Markov fields indexed by time, allows to state probabilistic models based on a symmetric notion of past and future:

> *"[...] si l'on veut reconstituer cette symétrie entre le passé et le futur [...] il faut renoncer à l'emploi des chaînes du type de Markov et les remplacer par des schémas d'une nature différente."*

The properties of reciprocal processes and reciprocal classes have been examined in detail by numerous authors under various aspects.

Many important results concerning the fundamental properties of reciprocal processes where given by Jamison in a series of articles [Jam70, Jam74, Jam75]. In particular he characterizes reciprocal Gaussian processes using a differential equation satisfied by their covariance function. The theory of Gaussian reciprocal processes was amended by Chay [Cha72], Carmichael, Mass, Theodorescu [CMT82] and extended to a multivariate context by Levy [Lev97].

Important contributions to a physical interpretation and to the development of a stochastic calculus adjusted to the reciprocal class of continuous diffusions have been made by Zambrini and various co-authors in their interest of creating a "Euclidean" version of quantum mechanics. In joint work with numerous authors he develops a stochastic calculus that possesses interesting analogies to the path-integral approach to quantum mechanics as introduced by Feynman and Hibbs [FH10]. An account is given in monograph by Chung and Zambrini [CZ01], see also Zambrini's works with Cruzeiro or Thieullen [CZ91, TZ97]. Moreover Zambrini notes that reciprocal classes are an elegant way to describe the solutions of certain stochastic control problems, see [Zam86]. This interpretation was extended by Wakolbinger and Dai Pra to different cost functions in [Wak89] respectively [DP91].

Krener initiated his search for *reciprocal invariants* of the reciprocal class of continuous diffusions using short-time expansions of transition densities in [Kre88]. Clark remarked in [Cla90] that these reciprocal invariants are in fact characteristics of the reciprocal classes. They permit to identify processes belonging to the same reciprocal class. A physical interpretation of these invariants was furnished by Levy and Krener [LK93].

This thesis deals with the problem of characterizing reciprocal classes of Markov processes by duality formulae. In connection with a Brownian motion, such duality formulae first appeared as an analytical tool in Malliavin calculus: Bismut provided a probabilistic proof of Hörmander's theorem using a duality formula satisfied by the Wiener measure in [Bis81]. The term duality formula refers to a duality, or "integration by parts" relation between a stochastic derivative operator and a stochastic integral operator.

A characterization of the Poisson process as the unique process satisfying a duality formula between a difference operator and a compensated stochastic integral was first given

by Slivnjak [Sli62] and extended to Poisson measures by Mecke [Mec67]. A similar characterization of the Wiener measure was presented by Rœlly and Zessin in [RZ91]. They characterize the Brownian motion as the unique continuous process for which the Malliavin derivative and the Skorohod integral are dual operators.

The first part of this thesis deals with the characterization of processes with independent increments by a duality formula. This result serves as a basis for the study of reciprocal classes of Markov processes in the second part.

Section 1 is an introduction to the characterization of stochastic processes by duality formulae. We present the abovementioned characterizations by Slivnjak and Rœlly, Zessin and extend them to the reference-space of càdlàg processes. Our approach is based on a characterization of the Poisson, respectively the Gaussian law on $\mathbb{R}$ as unique probability distributions satisfying specific integration by parts formulae. These results are also known as Chen's Lemma, respectively Stein's Lemma, see e.g. the monograph by Stein [Ste86].

Section 2 is devoted to make explicit an integration by parts formula satisfied by infinitely divisible random vectors: *In Proposition 2.7 we show that if Z is an integrable and infinitely divisible random vector, then*

$$\mathbb{E}\,(f(Z)(Z-b)) = \mathbb{E}\,(A\nabla f(Z)) + \mathbb{E}\left(\int_{\mathbb{R}^d_*}(f(Z+q)-f(Z))qL(dq)\right). \tag{0.1}$$

Here, $f:\mathbb{R}^d\to\mathbb{R}$ is a smooth test function and $b\in\mathbb{R}^d$, $A\in\mathbb{R}^{d\times d}$ and L are the Fourier characteristics of the infinitely divisible random vector Z. In particular A is non-negative definite and L is a Lévy measure on $\mathbb{R}^d_*$. This formula is the finite dimensional version of a duality formula for càdlàg processes with independent increments, a formula which is known in the Malliavin calculus of Lévy processes with and without jumps. *Let X be a Lévy process with integrable increments. In Proposition 2.20 we prove the duality formula*

$$\begin{aligned}\mathbb{E}\left(F(X)\int_{[0,1]}u_t\cdot(dX_t-bdt)\right) &= \mathbb{E}\left(\int_{[0,1]}D_tF(X)\cdot Au_tdt\right)\\ &\quad+\mathbb{E}\left(\int_{[0,1]\times\mathbb{R}^d_*}(F(X+q\mathbb{1}_{[t,1]})-F(X))u_t\cdot qdtL(dq)\right).\end{aligned} \tag{0.2}$$

Here, $F(X)$ is a smooth functional of the Lévy process and $u:[0,1]\to\mathbb{R}^d$ is a step function. The vector b is the drift, A is the diffusion matrix and L is a Lévy measure controlling the jumps of X. The dual operator to the stochastic integral are the Malliavin derivative $D_tF(X)$ and the difference operator $F(X+q\mathbb{1}_{[t,\infty)})-F(X)$. The only known proof of this duality uses a chaos decomposition of the underlying space of square integrable functionals. We present two new and simple proofs in Propositions 2.20 and 2.38. The first one underlines the direct correspondence of infinite divisibility and this kind of duality formulae. The second one uses a new random perturbation of jumps to define the difference operator.

In **Section 3** we show that processes with independent increments are indeed the only ones that satisfy the duality formulae (0.2). Our main result is a characterization of processes with independent increments presented in Theorem 3.4. Let us state the ensuing characterization of Lévy processes: *If X is an integrable process and (b,A,L) are a tuple consisting of a vector, a non-negative definite matrix and a Lévy measure, then X is a Lévy process with characteristics (b,A,L) only if the duality formula (0.2) holds.* This is based on a characterization of infinitely divisible random vectors by (0.1) is presented in Theorem 3.1, which is a generalization of Stein's and Chen's lemma. We thus unify and extend the results of

Section 1. Our method permits us to give a new and simple proof of a characterization of infinitely divisible random measures on Polish spaces by a factorization of the Campbell measure, an old result due to Kummer and Matthes [KM70b].

The idea of using duality formulae as a tool to characterize reciprocal classes is relatively new. It has been introduced by Rœlly and Thieullen [RT02, RT05] to characterize the reciprocal classes of a Wiener measure and of Brownian diffusions.

Our main contribution in this work is the study of the reciprocal classes of pure jump processes, in particular of jump processes with unit jump size. We underline the relevance of such characterizations with applications on an optimal control problem and on time-reversal of stochastic processes.

The second part of this thesis is devoted to the study of reciprocal classes of continuous Markov processes and Markov processes with jumps. In particular our work contains the first investigation in the context of jump processes.

We define in **Section 4** the notions of reciprocal processes and of reciprocal classes associated to Markov processes. The concept of a reciprocal class is central in the second part of this thesis. *If $(X_t)_{0\leq t\leq 1}$ is an $\mathbb{R}^d$-valued Markov process, the reciprocal class of X consists of all stochastic processes $(Y_t)_{0\leq t\leq 1}$ that have the same bridges as X. The term bridge refers to the law of the process conditioned on fixed enpoints $X_0 = x$ and $X_1 = y$ for $x, y \in \mathbb{R}^d$.* We illustrate these concepts using stochastic processes with discrete time in § 4.4. This relates to the original idea of Bernstein [Ber32], who introduced reciprocal processes as a time symmetric generalization of Markov chains.

Section 5 is devoted to the study of the reciprocal classes of Brownian diffusions: *A Brownian diffusion X is the solution of the stochastic differential equation*

$$dX_t = b(t, X_t)dt + dW_t,$$

where b is a smooth function and W is a Brownian motion. As a new contribution we present the following characterization of Brownian diffusions. *In Theorem 5.14 we prove that a continuous semimartingale X with integrable increments is a Brownian diffusion if and only if*

$$\begin{aligned}\mathbb{E}\left(F(X)\int_{[0,1]} u_t dX_t\right) &= \mathbb{E}\left(\int_{[0,1]} D_t F(X)\cdot u_t dt\right)\\ &+ \mathbb{E}\left(F(X)\left[\int_{[0,1]} u_t\cdot b(t,X_t)dt + \int_{[0,1]}\sum_{i,j=1}^{d} u_{i,t}\int_{[t,1]}\partial_i b_j(s,X_s)(dX_{j,s} - b_j(s,X_s)ds)dt\right]\right)\end{aligned}$$

holds for smooth functionals $F(X)$ and $u : [0,1] \to \mathbb{R}^d$ step functions. Our assumptions on X are minimal to render the duality formula well defined, since we need to define a stochastic integral on the right side, and all terms must be integrable with respect to the law of X. Following Rœlly and Thieullen [RT02, RT05] this is extended to a characterization of the reciprocal class of a Brownian diffusion. We then present two applications. In § 5.5 we present formal analogies between the properties of processes in the reciprocal class of a Brownian diffusion and the motion of a particle in an electromagnetic field in classical mechanics. The dynamics in the examined processes similar to those of the "Bernstein processes" introduced by Zambrini [Zam85] and Lévy, Krener [LK93]. We are able to give a concise physical interpretation of the invariants associated to the reciprocal class of a Brownian diffusion Clark introduced in [Cla90]: *If $E, B : [0,1]\times\mathbb{R}^3 \to \mathbb{R}^3$ denote an electric*

respectively magnetic field, and X is a Brownian diffusion with drift b representing the motion of a particle in a thermal reservoir under the influence of this electromagnetic field, then we are able to prove that

$$\begin{aligned} E_i(t,x) &= \partial_t b_i(t,x) + \sum_{j=1}^{3} b_j(t,x)\partial_j b_i(t,x) + \frac{1}{2}\sum_{j=1}^{3} \partial_j b_i(t,x),\ i=1,2,3, \\ B(t,x) &= (\partial_2 b_3 - \partial_3 b_2, \partial_3 b_1 - \partial_1 b_3, \partial_1 b_2 - \partial_2 b_1)(t,x), \end{aligned}$$

where on the right side of these equations we encounter Clark's reciprocal invariants. Here, the motion of the particle is defined as the solution of a stochastic optimal control problem similar to the one proposed by Yasue [Yas81]. *We show that the duality formula characterizing the reciprocal class of X can then be written as*

$$\begin{aligned} &\mathbb{E}\left(\int_{[0,1]} D_t F(X)\cdot u_t dt\right) \\ =\ &\mathbb{E}\left(F(X)\left(\int_{[0,1]} u_t \circ dX_t + \int_{[0,1]} \langle u\rangle_t \times B(t,X_t)\circ dX_t + \int_{[0,1]} \langle u\rangle_t \cdot E(t,X_t)dt\right)\right), \end{aligned}$$

under the loop condition $\int_{[0,1]} u_t dt = 0$, where $\langle u\rangle_t$ denotes the primitive of u and $\circ dX_t$ is the Fisk-Stratonovic integral and "$\times$" denotes the cross product of vectors. This equation is a formal analogue to the Newton equation "directing" a particle through an electromagnetic field, see Remark 5.53. In the second application we analyze the behavior of the reciprocal class of a Brownian diffusion with respect to the time-reversal $t \mapsto 1-t$. Using the characterization of the reciprocal class by a duality formula due to Rœlly and Thieullen, we are able to identify in Proposition 5.84 the reciprocal class of reversed processes given their initial reciprocal class. A first result was presented by Thieullen [Thi93, Proposition 4.5], who identified the reciprocal invariants of a time reversed Brownian diffusion. In the context of the above mechanical interpretation we infer the following interpretation: *If X is a process in the reciprocal class of a Brownian diffusion with reciprocal invariants identical to the electromagnetic fields $E(t,x)$ and $B(t,x)$, then its reversed process is in the reciprocal class with invariants identical to $E(1-t,x)$ and $-B(1-t,x)$.*

In the study of bridges of jump processes several new problems occur. Let us briefly mention an important "algebraic" problem: If a real-valued pure jump process is conditioned to start at time $t=0$ in a point $x \in \mathbb{R}$ and to be in $y \in \mathbb{R}$ at the final time $t=1$ after $n \in \mathbb{N}$ jumps with jump-sizes $q_1, \dots, q_n \in \mathbb{R}\backslash\{0\}$, then $y = x + q_1 + \cdots + q_n$. Thus an interdependence of jump-sizes occurs for the bridges, e.g. the n'th jump depends on the size of first jump through $q_n = y - q_1 - \cdots - q_{n-1} - x$. The study of pure jump processes with unit jump size clearly permits to avoid this problem.

Section 6 is devoted to the study of the reciprocal classes of a Poisson process and of certain Markov processes with unit jumps: We define "nice unit jump processes" as jump processes with unit jumps and intensity-of-jumps functions ℓ which are bounded from below and from above. In particular a Poisson process is a nice unit jump process with intensity $\ell \equiv 1$. In Theorem 6.58 we are able introduce a new reciprocal invariant associated to the reciprocal class of Markov nice unit jump processes: *Two nice unit jump processes X and Y with intensities ℓ respectively k have the same reciprocal class, if and only if the reciprocal*

invariants

$$\Xi_\ell(t,x) = \Xi_k(t,x) \quad \textit{coincide, where} \quad \Xi_\ell(t,x) = \partial_t \log \ell(t,x) + \ell(t,x+1) - \ell(t,x).$$

Our way to characterize nice unit jump processes is based on a duality formula that was first derived by Carlen, Pardoux [CP90] and independently by Elliott, Tsoi [ET93]. In Theorem 6.69 the reciprocal invariant is shown to appear in a duality formula that characterizes the full reciprocal class: *A unit jump process X is in the reciprocal class of a nice unit jump process with intensity ℓ if and only if the duality formula*

$$\mathbb{E}\left(F(X)\int_{[0,1]} u_t dX_t\right) = \mathbb{E}\left(\mathcal{D}_u F(X)\right) - \mathbb{E}\left(F(X)\int_{[0,1]} u_t \int_{[t,1]} \Xi_\ell(s, X_{s-}) dX_s dt\right)$$

holds for smooth functionals $F(X)$ and step functions u that satisfy the loop condition $\int_{[0,1]} u_t dt = 0$. Here, $\mathcal{D}_u F(X)$ is a derivative introduced by Carlen, Pardoux and Elliott, Tsoi. *In particular a unit jump process X is a mixture of Poisson bridges if and only if the above duality formula holds with $\Xi_\ell \equiv 0$.* Moreover we show that the reciprocal class contains all solutions to an optimal control problem under the constraint of a given boundary distribution. *Given bounded cost potentials $A : \mathcal{I} \times \mathbb{R} \to (0,\infty)$ and $\Phi : \mathcal{I} \times \mathbb{R} \to \mathbb{R}$ this problem concerns the minimization of the logarithmic cost function*

$$\mathbb{E}\left(\int_{[0,1]} \left(\gamma_t \log \gamma_t - \gamma_t \log A(t, X_{t-}) + \Phi(t, X_{t-})\right) dt\right),$$

inside the class of unit jump processes X with predictable intensity function γ_t whose law is absolutely continuous with respect to the law of a Poisson process, see Proposition 6.81. In Propostion 6.90 we conclude that the minimizer of the above cost function satisfies the above duality formula with invariant

$$\Xi_\ell(t,x) = \partial_t \log A(t,x) + \Phi(t,x+1) - \Phi(t,x).$$

As a corollary we are able to identify the time-reversed reciprocal classes of nice unit jump processes. We show that *if X is a unit jump process in the reciprocal class of a nice unit jump process with invariant $\Xi_\ell(t,x)$, then the reversed process $\hat{X}_t := -X_{(1-t)-}$ is in the reciprocal class of a nice unit jump process with invariant $\Xi_\ell(1-t,-x-1)$, see Propostion 6.101.*

In **Section 7** we propose generalizations of some of our results on unit jump processes to pure jump processes with different jump-sizes. To avoid interdependence of jump-sizes of bridges, an incommensurability condition between the jumps is essential: *A finite set $Q \subset \mathbb{R}^d_*$ contains only incommensurable jump-sizes if for any two finite sums of elements of Q the equality $q_1 + \cdots + q_n = \tilde{q}_1 + \cdots + \tilde{q}_m$ implies $n = m$ and $(q_1, \ldots, q_n) = (\tilde{q}_1, \ldots, \tilde{q}_n)$ up to a permutation of the entries. Under this condition, we show that the reciprocal class of any compound Poisson process can be characterized by a duality formula, see Theorem 7.36.* We also discuss the reciprocal classes of certain Markov jump processes: *X is a nice jump process, if the exists a bounded function $\ell : [0,1] \times \mathbb{R}^d \times Q \to [\varepsilon, \infty)$ that is differentiable in time such that*

$$X_t - X_0 - \int_{[0,t]\times Q} \ell(t, X_{t-}, q) dt \Lambda(dq)$$

is a martingale, where Λ is the counting measure on Q. In Theorem 7.54 we are able to compare the reciprocal classes of nice jump processes without assuming incommensurability of jumps. *Two nice jump processes X and Y with respective intensities ℓ and k have the same reciprocal class if there exists a function $\psi : [0,1] \times \mathbb{R}^d \to \mathbb{R}$ such that $\log k(t,x,q) = \log \ell(t,x,q) +$*

$\psi(t, x+q) - \psi(t,x)$ *everywhere, and if the invariants*

$$\Xi^q_\ell(t,x) = \Xi^q_k(t,x) \text{ coincide, where } \Xi^q_\ell(t,x) = \partial_t \log \ell(t,x,q) + \int_Q (\ell(t,x+q,\bar{q}) - \ell(t,x,\bar{q}))\Lambda(dq).$$

In the **Appendix** we summarize some results from the stochastic calculus associated to pure jump semimartingales on the space of càdlàg paths. This calculus is the basis of many results in Sections 2, 6 and 7.

Notation

In this work we use the notation

$$\mathbb{N} := \{0, 1, 2, \dots\}, \quad \mathbb{N}_* := \{1, 2, \dots\},$$
$$\mathbb{R}_+ := [0, \infty), \quad \mathbb{R}_* := \mathbb{R}\backslash\{0\}, \text{ and } \mathbb{R}_*^d := \mathbb{R}^d\backslash\{0\}.$$

Any d-dimensional vector $x \in \mathbb{R}^d$ is considered as a column vector $(x_1, \dots, x_d)^t$, where ^{t} denotes the transpose. The scalar product in $\mathbb{R}^d$ is denoted by

$$x \cdot y := x_1 y_1 + \cdots + x_d y_d, \quad \forall x, y \in \mathbb{R}^d.$$

For any $x \in \mathbb{R}^d$ we define the

$$\ell_1\text{-norm (absolute value):} \quad |x|_1 := |x_1|_1 + \cdots + |x_d|_1,$$
$$\ell_2\text{-norm (Euclidean norm):} \quad |x| := \sqrt{x_1^2 + \cdots + x_d^2}.$$

Stochastic processes are indexed with time $\mathcal{I} = [0, 1]$. All finite and ordered subsets of $\mathcal{I}$ are collected in

$$\Delta_{\mathcal{I}} := \{\tau_n = \{t_1, \dots, t_n\} \in \mathcal{I}^n,\ 0 \le t_1 < t_2 < \cdots < t_n \le 1,\ n \in \mathbb{N}\}. \tag{0.3}$$

We use the shorthands $s \wedge t := \min\{s, t\}$ and $s \vee t := \max\{s, t\}$ for $s, t \in \mathcal{I}$. The space of càdlàg processes is

$$\mathbb{D}(\mathcal{I}, \mathbb{R}^d) := \{\omega : \mathcal{I} \to \mathbb{R}^d,\ \omega \text{ is càdlàg}\}. \tag{0.4}$$

Càdlàg is a shorthand for *continue à droite, limité à gauche*, which means right-continuous with left limit. We write $\mathbb{D}(\mathcal{I}) = \mathbb{D}(\mathcal{I}, \mathbb{R}^1)$ if $d = 1$. We always use the canonical setup:

- The identity $X : \mathbb{D}(\mathcal{I}, \mathbb{R}^d) \to \mathbb{D}(\mathcal{I}, \mathbb{R}^d)$ is the canonical process.
- For $t \in \mathcal{I}$ the time projection of X is $X_t(\omega) := \omega(t)$, the jump at time $t \in \mathcal{I}$ is $\Delta X_t := X_t - X_{t-}$, where we use the left limit $X_{t-} := \lim_{\varepsilon \to 0} X_{t-\varepsilon}$.
- For any $\tau \subset \mathcal{I}$ we define $\mathcal{F}_\tau := \sigma(X_t,\ t \in \tau)$, in particular $(\mathcal{F}_{[0,t]})_{t \in \mathcal{I}}$ is the canonical filtration.

We say that the process X has integrable increments with respect to some law $\mathbb{P}$ on $\mathbb{D}(\mathcal{I}, \mathbb{R}^d)$ if for any $t \in \mathcal{I}$ the random variable $|X_t - X_0|_1$ is in $\mathbb{L}^1(\mathbb{P})$. As soon as the following stochastic integrals are well defined, we denote

- the Itô-integral by $\int_{\mathcal{I}} u_t \cdot dX_t$, and $\int_{\mathcal{I}} u_t dX_t$ if $d = 1$.
- the Fisk-Stratonovich-integral by $\int_{\mathcal{I}} u_t \circ dX_t$.

Let $\mathbb{P}$ be a probability on any measurable space $(\Omega, \mathcal{F})$. If $F \in \mathbb{L}^1(\mathbb{P})$ is integrable, we denote by

$$\mathbb{E}(F) := \int_\Omega F(\omega) \mathbb{P}(d\omega)$$

the integral of F with respect to $\mathbb{P}$. Up- and subscripts on $\mathbb{P}$ are inherited in the notation of the expectation. The Dirac-measure concentrated on $\omega \in \Omega$ is denoted by $\delta_{\{\omega\}}$.

With $O(\varepsilon) : \Omega \to \mathbb{R}$ we denote a random Landau notation. In particular $(O(\varepsilon))_{\varepsilon > 0}$ is a family of random variables uniformly bounded by εK for some $K > 0$. This implies $\lim_{\varepsilon \to 0} O(\varepsilon) = 0$ in practically any sense of convergence. With $o(\varepsilon)$ we denote the small Landau notation, which signifies that $o(\varepsilon)/\varepsilon \to 0$ for $\varepsilon \to 0$.

For $d, m, n \in \mathbb{N}$ we introduce the following spaces of measurable functions:

- $C_b^n(\mathbb{R}^d, \mathbb{R}^m)$: The space of n-times differentiable functions from $\mathbb{R}^d$ into $\mathbb{R}^m$ that are bounded with bounded derivatives;

- $C_c^n(\mathbb{R}^d, \mathbb{R}^m)$: The space of n-times differentiable functions that have compact support.

Smooth functions are elements of either

$$C_b^\infty(\mathbb{R}^d, \mathbb{R}^m) := \bigcap_{n=1}^\infty C_b^n(\mathbb{R}^d, \mathbb{R}^m), \text{ or } C_c^\infty(\mathbb{R}^d, \mathbb{R}^m) := \bigcap_{n=1}^\infty C_c^n(\mathbb{R}^d, \mathbb{R}^m).$$

We write $C_b^n(\mathbb{R}^d) = C_b^n(\mathbb{R}^d, \mathbb{R})$ or $C_c^n(\mathbb{R}^d) = C_c^n(\mathbb{R}^d, \mathbb{R})$ if $m = 1$. We use similar definitions for functions in time and space, in particular $C_b^{1,2}(\mathcal{I} \times \mathbb{R}^d, \mathbb{R}^d)$ denotes the functions $f : \mathcal{I} \times \mathbb{R}^d \to \mathbb{R}^d$ that are one-time differentiable in the time direction and two-times in the space direction. The partial derivatives are noted by $\partial_t f(t,x)$ in time and by $\partial_i f(t,x)$ for the derivatives in direction x_i for $1 \le i \le d$. Let us define the supremum-norm

$$\text{for } \phi : \mathbb{R}^d \to \mathbb{R}^m, \quad \|\phi\|_\infty := \sup_{x \in \mathbb{R}^d} |\phi(x)|_1.$$

The space of cylindric, smooth and bounded functionals is

$$\mathcal{S}_d := \left\{ F : \mathbb{D}(\mathcal{I}, \mathbb{R}^d) \to \mathbb{R},\ F(\omega) = f(\omega(t_1), \dots, \omega(t_n)),\ f \in C_b^\infty(\mathbb{R}^{nd}),\ \tau_n \in \Delta_{\mathcal{I}},\ n \in \mathbb{N} \right\}, \tag{0.5}$$

simply $\mathcal{S} = \mathcal{S}_1$ if $d = 1$. The space of elementary test functions is

$$\mathcal{E}_d := \left\{ u : \mathcal{I} \to \mathbb{R}^d,\ u = \sum_{i=1}^{n-1} u_i \mathbb{1}_{(t_i, t_{i+1}]} \text{ for some } u_i \in \mathbb{R}^d,\ \tau_n \in \Delta_{\mathcal{I}},\ n \in \mathbb{N} \right\}, \tag{0.6}$$

and again $\mathcal{E} = \mathcal{E}_1$ if $d = 1$.

For any $u, v \in \mathbb{L}^2(dt)$ respectively $w \in \mathbb{L}^1(dt)$ and $t \in \mathcal{I}$ we write

$$\langle u, v \rangle_t := \int_{[0,t]} u_s v_s ds \text{ and } \langle w \rangle_t := \int_{[0,t]} w_s ds.$$

We hide the time-index if $t = 1$, e.g. $\langle w \rangle = \int_{[0,1]} w_s ds$ for $w \in \mathbb{L}^1(dt)$.

First part: Probability laws characterized by integration by parts formulae

1. Two fundamental examples

We introduce two fundamental examples of stochastic processes that can be characterized as unique processes satisfying a duality formula: The Wiener process and the Poisson process. The characterization of the Wiener process in Proposition 1.10 is due to Rœlly and Zessin [RZ91]. The characterization of the Poisson process presented in Proposition 1.21 is due to Slivnjak [Sli62]. We present a new approach for both results that is based on an integration by parts of the underlying law of the processes increments, which is the Gaussian respectively the Poisson distribution. With this technique we extend characterization results of one-dimensional random variables, as known in Stein's calculus, to the setting of càdlàg processes.

1.1. **Wiener process.**

A Wiener process has Gaussian increments. We use this, to characterize the Wiener process in § 1.1.2 based on a characterization of the Gaussian law introduced in § 1.1.1.

1.1.1. *Integration by parts of the Gaussian law.*

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be some probability space. A measurable application $Z : \Omega \to \mathbb{R}$ is called a random variable.

There are numerous ways to characterize Gaussian random variables, an account is given by Bogachev in [Bog98, Paragraph 1.9]. Let us exemplarily mention a characterization result by Darmois [Dar51]:

> *Let Z be a random variable and Z' be an independent copy of Z. Then Z is Gaussian if and only if $Z + Z'$ and $Z - Z'$ are independent.*

We present a different characterization of the standard normal distribution on $\mathbb{R}$ that is also known as Stein's Lemma. This result extends to a characterization of the Wiener process by the duality formula known from Malliavin's calculus.

If $Z \sim N(0,1)$ has standard normal distribution and $f \in C_b^\infty(\mathbb{R})$ the integration by parts formula on the real line implies

$$\mathbb{E}\,(f(Z)Z) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(z) z e^{-\frac{z^2}{2}} dz = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f'(z) e^{-\frac{z^2}{2}} dx = \mathbb{E}\,(f'(Z)). \tag{1.1}$$

We call this the integration by parts formula under the Gaussian distribution. It is essential to us that the converse conclusion also holds, the above integration by parts formula is only satisfied by the Gaussian distribution.

Proposition 1.2. *Let Z be an integrable random variable. If for every $f \in C_b^\infty(\mathbb{R})$ the integration by parts formula*

$$\mathbb{E}\,(f(Z)Z) = \mathbb{E}\,(f'(Z)) \tag{1.3}$$

holds, then $Z \sim N(0,1)$.

Proof. Take a non-negative $f \in C_c^\infty(\mathbb{R})$ and define $F(z) := \int_{(-\infty,z]} f(y)dy \in C_b^\infty(\mathbb{R})$. If $P_Z := \mathbb{P} \circ Z^{-1}$ denotes the law of Z under $\mathbb{P}$, then

$$\int_{\mathbb{R}} f(z)P_Z(dz) = \mathbb{E}\,(f(Z)) = \mathbb{E}\,(F'(Z)) = \mathbb{E}\,(F(Z)Z) \leq \|F(z)\|_\infty \mathbb{E}\,(|Z|_1) \leq c \int_{\mathbb{R}} f(z)dz,$$

for the constant $c = \mathbb{E}\,(|Z|_1)$. Therefore $P_Z(dz) \ll dz$ is absolutely continuous, say $P_Z(dz) = p(z)dz$. The weak derivative p' of the density function p is defined as the dz-a.e. unique function such that

$$\int_{\mathbb{R}} f'(z)p(z)dz = \int_{\mathbb{R}} f(z)p'(z)dz, \quad \forall f \in C_b^\infty(\mathbb{R}).$$

But then the integration by parts formula (1.3) implies that $p'(z) = p(z)z$ holds *dz-a.e.*. The unique solution of this ordinary differential equation under the condition $\int_{\mathbb{R}} p(z)dz = 1$ is the Gaussian density function, thus $Z \sim N(0,1)$. □

This characterization is due to Stein [Ste72], see also the monograph by the same author [Ste86, Lemma II.1]. The following remark hints to the fundamental idea behind Stein's calculus concerning the approximation of normal random variables.

Remark 1.4. *Loosely speaking, the distribution of a random variable V is close in distribution to $N(0,1)$ if for a class of functions $f : \mathbb{R} \to \mathbb{R}$ large enough*

$$\mathbb{E}\,(f(V)) \approx \mathbb{E}\,(f(Z)). \tag{1.5}$$

A real function g_f is called a solution to Stein's equation, if

$$g_f(x)x - g_f'(x) = f(x) - \mathbb{E}\,(f(Z)),\ \forall x \in \mathbb{R}. \tag{1.6}$$

Inserting the random variable V and taking expectations, Stein's equation then permits to change condition (1.5) into

$$\mathbb{E}\,(g_f(V)V) \approx \mathbb{E}\,(g_f'(V)).$$

Stein provided a solution of (1.6) in [Ste72]. *The fact that due to Proposition 1.2 the standard normal law is the unique probability on $\mathbb{R}$ satisfying the integration by parts formula (1.3) thus justifies Stein's method of approximation.*

Similar characterizations based on an integration by parts formula exist for different measures. Let us mention a result from the theory of differentiable measures.

Remark 1.7. *The characterization presented in Proposition 1.2 of a Gaussian probability measure is a particular case of a characterization of finite measures that are Fomin-differentiable as developed by Bogachev and Röckner* [BR95], *see also Bogachev* [Bog10, Theorem 7.6.3]. *A measure μ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is Fomin-differentiable, if for any $Q \in \mathcal{B}(\mathbb{R})$ the limit $\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(\mu(Q+\varepsilon) - \mu(Q))$ exists and is finite. The definition uses the translation $Q + \varepsilon := \{x + \varepsilon,\ x \in Q\}$. Clearly any measure that is absolutely continuous on $\mathbb{R}$ with differentiable density is Fomin-differentiable. In particular*

the result by Bogachev and Röckner includes the characterization of the Gaussian law: A normal random variable is Fomin differentiable with logarithmic gradient z, who appears on the left side of the integration by parts formula (1.3).

1.1.2. *Duality formula for the Wiener process.*

The characterization of a probability measure on the one-dimensional space $\mathbb{R}$ given in Proposition 1.2 is now lifted to an infinite-dimensional setting, we characterize a probability on the space of càdlàg processes. This probability is basically the Wiener measure, but instead of fixing the initial condition $X_0 = 0$ *a.s.* we admit arbitrary initial laws.

Definition 1.8. *A probability measure $\mathbb{P}$ on $\mathbb{D}(\mathcal{I})$ is called a Wiener measure if given X_0, for any $\{t_1, t_2, \dots, t_n\} \in \Delta_{\mathcal{I}}$ the canonical vector $(X_{t_1} - X_0, \dots, X_{t_n} - X_{t_{n-1}})^t$ has independent components with Gaussian increments $X_{t_j} - X_{t_{j-1}} \sim N(0, t_j - t_{j-1})$, $j \in \{1, \dots, n\}$, where $t_0 = 0$.*

The process X is then called a Brownian motion. In particular a Brownian motion has independent and stationary increments, but in our definition the initial law $\mathbb{P}(X_0 \in \,.\,)$ may be any probability measure on $\mathbb{R}$. The Gaussianity of increments allows a characterization of the Wiener process that is very close in spirit to Proposition 1.2.

The space of cylindric, smooth and bounded functionals $\mathcal{S}$ is going to take the place of test functions $C_b^\infty(\mathbb{R})$ used in Proposition 1.2. We introduce a derivative of functionals $F(\omega) = f(\omega(t_1), \dots, \omega(t_n)) \in \mathcal{S}$ in the direction of elementary functions $u = \sum_{i=1}^{m-1} u_i \mathbb{1}_{(s_i, s_{i+1}]} \in \mathcal{E}$ by

$$D_t F(\omega) := \sum_{i=1}^{n} \partial_i f(\omega(t_1), \dots, \omega(t_n)) \mathbb{1}_{(0,t_i]}(t), \quad \text{and } D_u F(\omega) := \int_{\mathcal{I}} D_t F(\omega) u_t dt. \tag{1.9}$$

This is a true derivative operator in the sense that a product- and a chain-formula of calculus hold. Moreover it can be interpreted as a Gâteaux-derivative, see § 2.3.1.

Proposition 1.10. *Assume that X has integrable increments with respect to $\mathbb{Q}$. Then $\mathbb{Q}$ is a Wiener measure if and only if the duality formula*

$$\mathbb{E}_{\mathbb{Q}}\left(F(X) \int_{\mathcal{I}} u_t dX_t\right) = \mathbb{E}_{\mathbb{Q}}\left(D_u F(X)\right) \tag{1.11}$$

holds for all $F \in \mathcal{S}$ and $u \in \mathcal{E}$.

Proof. Remark that formula (1.11) is well defined: On the left side we have the product of a bounded functional with a stochastic integral, that in the case $u \in \mathcal{E}$ reduces to a finite sum over the increments and is thus in $\mathbb{L}^1(\mathbb{Q})$. On the right side there is the derivative of F in direction of u which is bounded by its definition (1.9).

Fix some $\{t_1, \dots, t_n\} \in \Delta_{\mathcal{I}}$, without loss of generality we chose $u \in \mathcal{E}$ and $F \in \mathcal{S}$ with $u = \sum_{i=1}^{n-1} u_i \mathbb{1}_{(t_i, t_{i+1}]}$ and $F(\omega) = f(\omega(t_1), \dots, \omega(t_n))$. Remark that

$$\begin{aligned} f(X_{t_1}, \dots, X_{t_n}) &= f(X_0 + (X_{t_1} - X_0), X_0 + (X_{t_1} - X_0) + (X_{t_2} - X_{t_1}), \dots) \\ &=: \tilde{f}_{X_0}(X_{t_1} - X_0, \dots, X_{t_n} - X_{t_{n-1}}), \end{aligned} \tag{1.12}$$

and thus

$$D_u F(X) = \sum_{i=1}^{n} \partial_i \tilde{f}_{X_0}(X_{t_1} - X_0, \dots, X_{t_n} - X_{t_{n-1}}) u_{i-1}(t_i - t_{i-1}), \text{ where } u_0 := 0, t_0 := 0. \tag{1.13}$$

If $\mathbb{Q}$ is a Wiener measure, we condition on the initial value X_0 and get

$$\mathbb{E}_{\mathbb{Q}}\left(F(X)\int_{\mathcal{I}} u_s dX_s \,\middle|\, X_0\right) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} \tilde{f}_{X_0}(z_1,\dots,z_n)\left(\sum_{i=1}^{n-1} u_i z_{i+1}\right) e^{-\frac{z_1}{2t_1}}\cdots e^{-\frac{z_n}{2(t_n-t_{n-1})}} dz_1\cdots dz_n.$$

Integration by parts in the variables $z_2,\dots,z_n$ shows that the duality formula holds.

For the converse assume that the duality formula is satisfied under $\mathbb{Q}$. Chose $F(\omega) = f(\omega(t_1)-\omega(0),\dots,\omega(t_n)-\omega(t_{n-1})) \in \mathcal{S}$ with $f(z) = g_1(z_1)\cdots g_n(z_n)$, $g_i \in C_b^\infty(\mathbb{R})$. Fix any $j \in \{1,\dots,n\}$ and put $g_i \equiv 1$, $u_{i-1} = 0$ for $i \neq j$. The duality formula reduces to

$$\mathbb{E}_{\mathbb{Q}}(g_j(X_{t_j}-X_{t_{j-1}})u_{j-1}(X_{t_j}-X_{t_{j-1}})) = \mathbb{E}_{\mathbb{Q}}(g_j'(X_{t_j}-X_{t_{j-1}})u_{j-1}(t_j-t_{j-1})),$$

and Proposition 1.2 implies that $(X_{t_j}-X_{t_{j-1}})/(t_j-t_{j-1}) \sim N(0,1)$ under $\mathbb{Q}$. The same trick applies to any linear combinations $\alpha_1(X_{t_1}-X_0)+\cdots+\alpha_n(X_{t_n}-X_{t_{n-1}})$ with arbitrary factors $\alpha_1,\dots,\alpha_n \in \mathbb{R}$, thus $(X_{t_1}-X_0,\dots,X_{t_n}-X_{t_{n-1}})^t$ is a Gaussian vector. But then again for any $g_1,\dots,g_n \in C_b^\infty(\mathbb{R})$, we have

$$\mathbb{E}_{\mathbb{Q}}\left(g_1(X_{t_1}-X_0)\cdots g_{j-1}(X_{t_{j-1}}-X_{t_{j-2}})g_{j+1}(X_{t_{j+1}}-X_{t_j})\cdots g_n(X_{t_n}-X_{t_{n-1}})(X_{t_j}-X_{t_{j-1}})\right) = 0.$$

The random variables $X_{t_1}-X_0,\dots,X_{t_n}-X_{t_{n-1}}$ are mutually uncorrelated and therefore independent, which ends the proof. □

The duality formula as presented in Equation (1.11) is a simplified version of a well known duality from Malliavin's calculus. It can be extended to arbitrary Gaussian spaces, larger classes of differentiable functionals F and Skorohod-integrable processes u. An up to date account on Malliavin's calculus is the monograph by Nualart [Nua06].

This duality formula for a Wiener process was first introduced by Cameron as a first variation of Wiener integrals in [Cam51, Theorem II]. Bismut extended the formula and used it as an important tool in his approach on Malliavin's calculus in [Bis81]. Gaveau and Trauber furnished in [GT82] the interpretation of (1.11) as duality of operators on the Fock space isomorphic to the space of square-integrable functionals of a Wiener process.

In Proposition 1.10 we underlined the fact, that only a Wiener measure satisfies the duality relation between the stochastic integral operator and the derivative operator (1.9). This characterization was first presented by Rœlly and Zessin in [RZ91], Hsu extended it to Wiener processes on manifolds, see [Hsu05]. We presented a new simple proof that is based on the characterization of Gaussian random variables by an integration by parts formula. Our approach extends the prior characterizations to the setting of càdlàg processes.

1.2. **Poisson process.**

Analogue to the characterization of a Wiener processes presented in Proposition 1.10 we show that Poisson processes are the only processes satisfying a duality formula in Proposition 1.21. A Poisson process is defined by its independent increments which have a Poisson distribution. Analogue to the duality formula of a Wiener process, the duality formula of a Poisson process (1.22) is based on an integration by parts formula satisfied by the Poisson distribution, the law of the increments.

1.2.1. *Integration by parts of the Poisson distribution.*

Assume that the random variable $Z \sim \mathcal{P}(\lambda)$ has a Poisson distribution on $\mathbb{R}$ with mean $\lambda > 0$. Using the explicit form of the probability mass function we see that for every

bounded and measurable function $f : \mathbb{R} \to \mathbb{R}$

$$\mathbb{E}\,(f(Z)Z) = \sum_{n=0}^{\infty} f(n) n e^{-\lambda} \frac{\lambda^n}{n!} = \sum_{n=0}^{\infty} f(n+1) \lambda e^{-\lambda} \frac{\lambda^n}{n!} = \mathbb{E}\,(\lambda f(Z+1)). \tag{1.14}$$

By analogy to (1.1) we call this the integration by parts formula under the Poisson distribution. Similar to the Gaussian case, the Poisson distribution is the only one satisfying the above integration by parts formula.

Proposition 1.15. *Let Z be an integrable random variable and $\lambda > 0$. If for every $f \in C_b^\infty(\mathbb{R})$ the integration by parts formula*

$$\mathbb{E}\,(f(Z)Z) = \mathbb{E}\,(\lambda f(Z+1)), \tag{1.16}$$

holds, then $Z \sim \mathcal{P}(\lambda)$.

Proof. By dominated convergence the identity (1.16) can be extended to all bounded measurable functions on $\mathbb{R}$. Using $f = \mathbb{1}_{(-\infty,0)}$ we see that $\mathbb{P}(Z \geq 0) = 1$. Take $f = \mathbb{1}_{(0,1)}$, then

$$\int_{\mathbb{R}} \mathbb{1}_{(0,1)}(z) z P_Z(dz) = \int_{\mathbb{R}} \mathbb{1}_{(0,1)}(z+1) P_Z(dz) = \int_{\mathbb{R}} \mathbb{1}_{(-1,0)} P_Z(dz) = 0,$$

and therefore $\mathbb{P}(Z \in (0,1)) = 0$. An iteration implies $\mathbb{P}(Z \in (n, n+1)) = 0$ for all $n \in \mathbb{N}$, and thus $\mathbb{P}(Z \in \mathbb{N}) = 1$ and $\mathbb{P}(Z = n+1) = \frac{\lambda}{n+1}\mathbb{P}(Z = n)$, $\forall n \in \mathbb{N}$. Since $\mathbb{P}$ is a probability $c := \mathbb{P}(Z = 0) > 0$. Then $\mathbb{P}(Z = n) = c\frac{\lambda^n}{n!}$ and by normalization necessarily $c = e^{-\lambda}$. □

In the context of Stein's calculus the above result is known as Chen's Lemma, see [Che75] and the monograph by Stein [Ste86, Theorem VIII.1] for a proof under the additional assumption that Z is $\mathbb{N}$-valued. In our simple proof we showed that this assumption is not necessary.

Chen provided a solution to the associated Stein-Chen equation

$$g_f(x)x - \lambda g_f(x+1) = f(x) - \mathbb{E}\,(f(Z)),\ x \in \mathbb{N}, \tag{1.17}$$

which is a tool to compute bounds on the distance between certain probability distributions on $\mathbb{N}$ and the Poisson distribution, see Remark 1.4.

1.2.2. *Duality formula for the Poisson process.*

The above characterization of the Poisson distribution was first known through a similar characterization of the Poisson process by a duality formula. In our presentation we reverse this argument, and use the fact that only the Poisson distribution satisfies (1.16) to characterize a Poisson process by a duality formula.

Definition 1.18. *A càdlàg process $\mathbb{P}$ is called a Poisson process if given X_0, for any $\{t_1, \ldots, t_n\} \in \Delta_{\mathcal{I}}$ the canonical random vector $(X_{t_1} - X_0, \ldots, X_{t_n} - X_{t_{n-1}})^t$ has independent components with $X_{t_j} - X_{t_{j-1}} \sim \mathcal{P}(t_j - t_{j-1})$, $j \in \{1, \ldots, n\}$, where $t_0 = 0$.*

Note that we do not fix the initial law $\mathbb{P}(X_0 \in \,.\,)$. Next we give a complementary remark on the distribution of the jump-times of a Poisson process. This will prove to be a key result in the perturbation analysis of unit-jump processes in Section 6.

Remark 1.19. *Define the jump-times $T_0 := 0$ and $T_{i+1} = \inf\{t > T_i : X_t - X_{t-} > 0\}$ for $i \geq 0$ and the inter-jump-times $S_i := T_i - T_{i-1}$ for $i \geq 1$. If X is a Poisson process, then $(S_i)_{i \geq 1}$ is a sequence of independent random variables with exponential distribution of mean 1. If we condition*

a Poisson process on the number of jumps made between the times $t = 0$ and $t = 1$, then for any $\{t_1, \dots, t_n\} \in \Delta_I$

$$\mathbb{P}(T_1 = dt_1, \dots, T_n = dt_n | X_1 - X_0 = n) = n! \mathbb{1}_{\{t_1 < \dots < t_n\}} dt_1 \cdots dt_n. \tag{1.20}$$

For a proof of these well known results see e.g. the monograph by Brémaud [Bré99, Theorem 8.1.1].

We now lift the characterization of the Poisson distribution in Proposition 1.15 to the infinitely dimensional context of a Poisson process on càdlàg space.

Proposition 1.21. *Assume that X has integrable increments with respect to $\mathbb{Q}$. Then $\mathbb{Q}$ is a Poisson process if and only if the duality formula*

$$\mathbb{E}_{\mathbb{Q}}\left(F(X) \int_I u_t (dX_t - dt)\right) = \mathbb{E}_{\mathbb{Q}}\left(\int_I (F(X + \mathbb{1}_{[t,1]}) - F(X)) u_t dt\right) \tag{1.22}$$

holds for all $F \in \mathcal{S}$, $u \in \mathcal{E}$.

Proof. Fix $\{t_1, \dots, t_n\} \in \Delta_I$ and assume without loss of generality that $F(\omega) = f(\omega(t_1), \dots, \omega(t_n)) \in \mathcal{S}$, $u = \sum_{i=1}^{n-1} u_i \mathbb{1}_{(t_i, t_{i+1}]} \in \mathcal{E}$. Remember that as in (1.12) the functional F can be written as a cylindric functional on the initial value and the increments, analogue to (1.13) we get

$$\begin{aligned} & F(X + \mathbb{1}_{[t,1]}) - F(X) \\ = \; & \tilde{f}_{X_0}(X_{t_1} - X_0 + \mathbb{1}_{[0,t_1)}(t), \dots, X_{t_n} - X_{t_{n-1}} + \mathbb{1}_{[t_{n-1}, t_n)}(t)) - \tilde{f}_{X_0}(X_{t_1} - X_0, \dots, X_{t_n} - X_{t_{n-1}}). \end{aligned}$$

If $\mathbb{Q}$ is a Poisson process, we condition on the initial value X_0, and by the independence of increments we know the distribution of the random vector $(X_{t_1} - X_0, \dots, X_{t_n} - X_{t_{n-1}})^t$ explicitly. The derivation of equation (1.22) is the same as that of (1.16), in particular for F and u as above we get

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}}\left(F \int_I u_s dX_s \,\middle|\, X_0\right) \\ = \; & \sum_{i_1, \dots, i_n = 0}^{\infty} \tilde{f}_{X_0}(i_1, \dots, i_n) \left(\sum_{j=1}^{n-1} u_j i_{j+1}\right) e^{-t_1} \frac{t_1^{i_1}}{i_1!} e^{-(t_2 - t_1)} \frac{(t_2 - t_1)^{i_2}}{i_2!} \cdots e^{-(t_n - t_{n-1})} \frac{(t_n - t_{n-1})^{i_n}}{i_n!}, \end{aligned}$$

and the discrete analog of integration by parts, as presented above Proposition 1.15, applied to $i_2, \dots, i_n$ shows that the duality formula holds.

Now assume that the duality formula (1.22) holds. By dominated convergence it also holds for $F(X) = f(X_{t_1}, \dots, X_{t_n})$ where f is a bounded function. In particular choose $\tilde{f}_{X_0}(z) = g_1(z_1) \cdots g_n(z_n)$ with bounded functions g_i. Fix any $j \in \{1, \dots, n\}$ and put $g_i \equiv 1$, $u_{i-1} = 0$ for $i \neq j$. The duality formula reduces to

$$\mathbb{E}_{\mathbb{Q}}(g_j(X_{t_j} - X_{t_{j-1}}) u_{j-1}(X_{t_j} - X_{t_{j-1}})) = \mathbb{E}_{\mathbb{Q}}(g_j(X_{t_j} - X_{t_{j-1}} + 1) u_{j-1}(t_j - t_{j-1})),$$

which by Proposition 1.15 implies that $X_{t_j} - X_{t_{j-1}} \sim \mathcal{P}(t_j - t_{j-1})$. Now take $\tilde{f}_{X_0} = g_1(z_1) \cdots g_n(z_n)$ but $u_{i-1} = 0$ for $i \neq j$:

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}}(g_1(X_{t_1} - X_0) \cdots g_n(X_{t_n} - X_{t_{n-1}}) u_{j-1}(X_{t_j} - X_{t_{j-1}})) \\ = \; & \mathbb{E}_{\mathbb{Q}}(g_1(X_{t_1} - X_0) \cdots g_j(X_{t_j} - X_{t_{j-1}} + 1) \cdots g_n(X_{t_n} - X_{t_{n-1}}) u_{j-1}(t_j - t_{j-1})). \end{aligned}$$

Therefore $X_{t_j} - X_{t_{j-1}} \sim \mathcal{P}(t_j - t_{j-1})$ conditionally on the random variables $X_{t_1} - X_0, \dots, X_{t_{j-1}} - X_{t_{j-2}}, X_{t_{j+1}} - X_{t_j}, \dots, X_{t_n} - X_{t_{n-1}}$. This proves the independence of increments. □

Let us remark, that instead of a derivative operator as in the Wiener case (1.11) we have a difference operator on the right side of the duality formula, see § 2.3.1 for comments.

The first to present this characterization in the context of pure jump processes with unit jump-size was Slivnjak [Sli62]. A generalization to Poisson measures on Polish spaces is due to Mecke [Mec67, Theorem 3.1]. Our short and elementary proof allows to characterize a Poisson process on càdlàg space instead of restricting the setup to processes with values in $\mathbb{N}$. An interpretation of (1.22) as duality formula on the Fock space has been provided by Ito [Ito88] and also Nualart, Vives [NV90].

2. Infinitely divisible random vectors and processes with independent increments

In this section, we unify the integration by parts formulae satisfied by the Gaussian law (1.3) and the Poisson distribution (1.16) and extend them to infinitely divisible random vectors in (2.8). This new integration by parts formula is the basis of a duality formula for processes with independent increments, see (2.21). The duality formula already appeared in the context of Malliavin calculus with jumps, with a proof related to the chaos decomposition of Lévy processes as introduced by Itô [Itô56], see e.g. Løkka [Løk04]. We provide two new and elegant proofs of the duality formula satisfied by Lévy processes, see Propositions 2.20 and 2.38. As a complement we compare different definitions of the derivative and difference operator that appear in the duality formula in § 2.3.1. In particular we present a new definition of the difference operator using a random perturbation of càdlàg paths.

2.1. **Integration by parts of infinitely divisible random vectors.**

Infinitely divisible random variables and vectors were extensively investigated in the past decades. They first appeared as limit distributions of converging series of random variables, see Khintchine [Khi37]. The class of infinitely divisible random variables includes Gaussian as well as Poisson random variables. We begin this paragraph with a short introduction based on the monograph by Sato [Sat99].

A measurable function $\chi : \mathbb{R}^d \to \mathbb{R}^d$ is called a cutoff function if

$$\chi(q) = q + o(|q|^2) \text{ in a neighborhood of zero and } \chi \text{ is bounded.} \tag{2.1}$$

A Lévy measure L on $(\mathbb{R}^d_*, \mathcal{B}(\mathbb{R}^d_*))$ is a σ-finite measure such that

$$\int_{\mathbb{R}^d_*} (|q|^2 \wedge 1) L(dq) < \infty.$$

By definition every cutoff function χ is in $\mathbb{L}^2(L)$.

We are interested in the class of infinitely divisible random vectors.

Definition 2.2. *A random vector Z is called infinitely divisible if for every $k \in \mathbb{N}$ there exist independent and identically distributed random vectors $Z^{(k1)}, \dots, Z^{(kk)}$ such that*

$$Z \text{ has the same law as } Z^{(k1)} + \cdots + Z^{(kk)}.$$

The classical Lévy-Khintchine formula gives a representation for the characteristic function of Z. For any $\gamma \in \mathbb{R}^d$

$$\log \mathbb{E}\left(e^{i\gamma \cdot Z}\right) = i\gamma \cdot b - \frac{1}{2}\gamma \cdot A\gamma + \int_{\mathbb{R}^d_*} \left(e^{i\gamma \cdot q} - 1 - i\gamma \cdot \chi(q)\right) L(dq), \tag{2.3}$$

where $b \in \mathbb{R}^d$, $A \in \mathbb{R}^{d\times d}$ is a symmetric non-negative definite matrix and L a Lévy measure, [Sat99, Theorem 8.1]. The triple $(b, A, L)_\chi$ is called characteristics of Z with respect to the cutoff function χ. Only b depends on the choice of χ, [Sat99, Remark 8.4]. If χ' is another cutoff function, then Z has characteristics $(b', A, L)_{\chi'}$ with

$$b' = b + \int_{\mathbb{R}^d_*} (\chi'(q) - \chi(q)) L(dq).$$

Example 2.4. *We already introduced two important infinitely divisible distributions:*

- *A Gaussian random vector with mean b and covariance matrix A is infinitely divisible with characteristics $(b, A, 0)_\chi$ for any cutoff function χ.*

- *A Poisson random variable with mean $\lambda \geq 0$ corresponds to an infinitely divisible random variable with characteristics $(\lambda\chi(1), 0, \lambda\delta_{\{1\}})_\chi$.*

The Lévy-Khintchine formula (2.3) implies the following integrability property.

Remark 2.5. *For an infinitely divisible random vector Z we have*

$$\int_{\mathbb{R}^d_*} \left(|q|^2 \wedge |q|_1\right) L(dq) < \infty \;\Rightarrow\; Z \text{ is integrable.} \tag{2.6}$$

In this case we don't need a cutoff function in (2.3) and take $\chi(q) = q$, $\forall q \in \mathbb{R}^d_*$. The associated characteristics will be denoted simply by (b, A, L) and we call Z an integrable infinitely divisible random vector. A short proof of (2.6) is given in Remark 2.36, using the Lévy-Itô decomposition of the sample paths of a Lévy process.

The following Proposition presents an integration by parts relation satisfied by integrable infinitely divisible random vectors. It contains both integration by parts formulae (1.3) and (1.16) as special cases.

Proposition 2.7. *Let Z be an integrable infinitely divisible random vector with characteristics (b, A, L). Then the integration by parts formula*

$$\mathbb{E}\left(f(Z)\,(Z-b)\right) = \mathbb{E}\left(A\nabla f(Z)\right) + \mathbb{E}\left(\int_{\mathbb{R}^d_*} \left(f(Z+q) - f(Z)\right) qL(dq)\right) \tag{2.8}$$

holds for every $f \in C^\infty_b(\mathbb{R}^d)$.

Proof. Note that the second term on the right has a sense by (2.6) and boundedness of the test function f. We are going to prove the equality (2.8) separately for each component of the d-dimensional random vector Z. Define

$$f_\gamma(q) := e^{i\gamma\cdot q}, \quad \text{for } \gamma, q \in \mathbb{R}^d.$$

Let $1 \leq j \leq d$, we can permute differentiation and integration to obtain

$$\partial_{\gamma_j} \mathbb{E}\left(f_\gamma(Z)\right) = i\,\mathbb{E}\left(f_\gamma(Z) Z_j\right). \tag{2.9}$$

On the other hand the Lévy-Khintchine formula (2.3) implies

$$\partial_{\gamma_j} \mathbb{E}\left(f_\gamma(Z)\right) = \left(i\,b_j - (A\gamma)_j + i\int_{\mathbb{R}^d_*} q_j \left(e^{i\gamma\cdot q} - 1\right) L(dq)\right) \mathbb{E}\left(f_\gamma(Z)\right). \tag{2.10}$$

The second term on the right reduces to

$$-(A\gamma)_j \mathbb{E}\left(f_\gamma(Z)\right) = i\,\mathbb{E}\left(\left(A\nabla f_\gamma(Z)\right)_j\right).$$

The last term on the right hand side can be reformulated as

$$i\int_{\mathbb{R}^d_*} q_j \left(e^{i\gamma\cdot q} - 1\right) L(dq) \mathbb{E}\left(f_\gamma(Z)\right) = i\,\mathbb{E}\left(\int_{\mathbb{R}^d_*} \left(f_\gamma(Z+q) - f_\gamma(Z)\right) q_j L(dq)\right).$$

Comparing (2.9) and (2.10) and using the above reformulations we get (2.8) for f_γ. By linearity the equation holds for all real valued trigonometric functions.

We extend this integration by parts to smooth bounded functions by a density argument: Take any $f \in C^\infty_b(\mathbb{R}^d)$ and $\varepsilon > 0$. There exists a $k > 0$ such that $\mathbb{E}\left(\mathbb{1}_{|Z|_1 > k/2} |Z|_1\right) < \varepsilon$, $\int_{|q|_1 > k/2} (|q|^2 \wedge |q|_1) L(dq) < \varepsilon$ and *a fortiori* $\mathbb{P}(|Z|_1 > k/2) < \varepsilon/k$. By the Stone-Weierstraß theorem there exist trigonometric functions that approximate f arbitrarily well for the norm of uniform convergence on the compact set $\{q \in \mathbb{R}^d : |q|_1 \leq k\}$, and whose absolute

value on $\mathbb{R}^d$ are bounded by a multiple of $\|f\|_\infty$. Let $\phi \in C_b^\infty(\mathbb{R}^d)$ be a trigonometric function such that

$$\|(f-\phi)\mathbb{1}_{[-k,k]^d}\|_\infty < \varepsilon/k, \quad \|(\nabla f - \nabla\phi)\mathbb{1}_{[-k,k]^d}\|_\infty < \varepsilon, \text{ and } \|\phi\|_\infty \le 3\|f\|_\infty, \quad \|\nabla\phi\|_\infty \le 3\|\nabla f\|_\infty.$$

Using $1 = \mathbb{1}_{|Z|_1\le k} + \mathbb{1}_{|Z|_1>k}$ and $1 = (\mathbb{1}_{|Z|_1\le k/2} + \mathbb{1}_{|Z|_1>k/2})(\mathbb{1}_{|q|_1\le k/2} + \mathbb{1}_{|q|_1>k/2})$ this proves the four approximations

- $|\mathbb{E}(Zf(Z)) - \mathbb{E}(Z\phi(Z))|_1 \le \mathbb{E}(|Z|_1|f(Z)-\phi(Z)|_1) \le k\frac{\varepsilon}{k} + 4\|f\|_\infty\varepsilon$;
- $|\mathbb{E}(bf(Z)) - \mathbb{E}(b\phi(Z))|_1 \le \mathbb{E}(|b|_1|f(Z)-\phi(Z)|_1) \le \varepsilon/k + 4\|f\|_\infty\varepsilon$;
- $|\mathbb{E}(A\nabla f(Z)) - \mathbb{E}(A\phi(Z))|_1 \le \mathbb{E}(|A|_1|\nabla f(Z)-\nabla\phi(Z)|_1) \le |A|_1\varepsilon + |A|_1 4\|\nabla f\|_\infty\varepsilon$;
- $|\mathbb{E}(\int_{\mathbb{R}^d_*}(f(Z+q)-f(Z))qL(dq)) - \mathbb{E}(\int_{\mathbb{R}^d_*}(\phi(Z+q)-\phi(Z))qL(dq))|_1 \le 2\int_{\mathbb{R}^d_*}(|q|^2\wedge|q|_1)L(dq)\varepsilon + 8(\|f\|_\infty + \|\nabla f\|_\infty)\varepsilon + 4(\|f\|_\infty + \|\nabla f\|_\infty)\int_{\mathbb{R}^d_*}(|q|^2\wedge|q|_1)L(dq)\varepsilon$.

Define

$$\begin{aligned} K \;:=\; & \max\{1+4\|f\|_\infty, |A|_1(1+4\|\nabla f\|_\infty), \\ & 8(\|f\|_\infty + \|\nabla f\|_\infty) + 2(1+2\|f\|_\infty + 2\|\nabla f\|_\infty)\int_{\mathbb{R}^d_*}(|q|^2\wedge|q|_1)L(dq)\Big\}. \end{aligned}$$

Since (2.8) holds for ϕ we can use the above approximations to get

$$\left|\mathbb{E}(f(Z)(Z-b)) - \mathbb{E}(A\nabla f(Z)) - \mathbb{E}\left(\int_{\mathbb{R}^d_*}(f(Z+q)-f(Z))qL(dq)\right)\right|_1 \le 4K\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary we end the proof by letting ε tend to zero. □

As mentioned in the first section this integration by parts formula is already known for particular cases in Stein's calculus, see the comments following Theorem 3.1. We provided a new and simple proof that only uses the Lévy-Khintchine formula (2.3).

2.2. **Duality formula for processes with independent increments.**

Using the integration by parts (2.8) satisfied by infinitely divisible random vectors, we unify and generalize the duality formulae (1.11) of the Wiener process and (1.22) of the Poisson process: In Proposition 2.38 we prove a duality formula satisfied by d-dimensional processes with independent increments.

2.2.1. *Processes with independent increments - PII.*

We want to find a duality formula satisfied by the following class of processes.

Definition 2.11. *We say that the canonical process X is a PII under $\mathbb{P}$ if it has an arbitrary initial state, independent increments and is stochastically continuous.*

Remark that we assume that X_0 is independent of $X_t - X_0$ for any $t \in \mathcal{I}$. If X is a PII then $X_t - X_0$ is an infinitely divisible random vector for each $t \in \mathcal{I}$. This is due to the fact that $X_t - X_0$ is decomposable into a so called "null-array" of independent random variables:

$$X_t - X_0 = (X_{\frac{t}{n}} - X_0) + (X_{\frac{2t}{n}} - X_{\frac{t}{n}}) + \cdots + (X_t - X_{\frac{(n-1)t}{n}}).$$

See e.g. Sato [Sat99, Theorem 9.1] for a complete proof.

Remark 2.12. *Let us recall that for any fixed cutoff function χ the characteristics $(b_t, A_t, L_t)_\chi$ of the infinitely divisible random vector $X_t - X_0$ satisfy some regularity properties in t. Let $s \le t$ in $\mathcal{I}$. Theorem 9.8 in* [Sat99] *states, that as functions on $\mathcal{I}$*

(1) $b_t \in \mathbb{R}^d$ with $b_0 = 0$ and $t \mapsto b_t$ is continuous;

(2) $A_t \in \mathbb{R}^{d\times d}$ *is a non-negative definite matrix with* $A_0 = 0$, $q \cdot A_s q \le q \cdot A_t q$ *and* $t \mapsto q \cdot A_t q$ *is continuous for any* $q \in \mathbb{R}^d$;

(3) L_t *is a Lévy measure on* $\mathbb{R}^d_*$ *with* $L_0(\mathbb{R}^d_*) = 0$, $L_s(Q) \le L_t(Q)$ *and* $t \mapsto L_t(Q)$ *is continuous for any compact* $Q \subset \mathbb{R}^d_*$.

If the Lévy measure of X_t *is "integrable" in the sense of Remark 2.5, then* X_t *is integrable. In this case we can add the property*

(4) $\int_{\mathbb{R}^d_*} (|q|^2 \wedge |q|_1) L_t(dq) < \infty$.

If conversely there are $(b_t)_{t\in\mathcal{I}}$, $(A_t)_{t\in\mathcal{I}}$ *and* $(L_t)_{t\in\mathcal{I}}$ *such that such that conditions (1)-(3) hold, then there exists a law* $\mathbb{P}$ *on* $\mathbb{D}(\mathcal{I}, \mathbb{R}^d)$ *such that* X *is a PII and for any* $t \in \mathcal{I}$ *the triplets* $(b_t, A_t, L_t)_\chi$ *are the characteristics of* $X_t - X_0$. *Under the supplementary condition (4) we get the existence of an integrable PII.*

The law of a PII is unique up to the initial condition in the following sense: If $\mathbb{Q}$ *is another PII having the same characteristics, then* $\mathbb{P}(\,.\,|\,X_0) = \mathbb{Q}(\,.\,|\,X_0)$ *holds* $\mathbb{P}(X_0 \in\ .\,) \wedge \mathbb{Q}(X_0 \in\ .\,)$*-a.e.*

We already considered two important processes with independent increments, the Wiener process and the Poisson process. We extend these and present further fundamental examples.

Example 2.13. *The following examples shall illustrate that the law of a PII can be decomposed into an initial condition, a deterministic drift, a Gaussian part and a jump part. We will see this again in the Lévy-Itô decomposition of sample paths (2.35).*

- *Let* Z *be any* d*-dimensional random variable, then the law of the constant process* $t \mapsto Z$ *is a PII with characteristics* $(0, 0, 0)_\chi$.
- *Let* $b : \mathcal{I} \to \mathbb{R}^d$ *be continuous with* $b_0 = 0$, *then the law of the deterministic process* $t \mapsto b_t$ *is a PII with characteristics* $(b_t, 0, 0)_\chi$.
- *Let* $\sigma : \mathcal{I} \to \mathbb{R}^{d\times d}$ *be such that condition (2) of Remark 2.12 holds for* $A_s := \sigma_s^t \sigma_s$ *and let* W *be a* d*-dimensional Brownian motion with initial value* $W_0 = 0$. *Then* σW *has the law of a PII with characteristics* $(0, A_t, 0)_\chi$.
- *Let* $(Y_i)_{i\ge1}$ *be a sequence of iid* d*-dimensional random vectors such that* $Y_i \sim \rho$ *and* N *a Poisson process with initial condition* $N_0 = 0$ *a.s. that is independent of the sequence* $(Y_i)_{i\ge1}$. *Then for any function* $\ell : \mathcal{I} \to \mathbb{R}_+$ *such that* $\lambda_t := \int_{[0,t]} \ell_t dt < \infty$ *for all* $t \in \mathcal{I}$, *the process* $\sum_{i=0}^{N_{\lambda_t}} Y_i - \int_{\mathcal{I}\times\mathbb{R}^d_*} \chi(q)\ell_t dt\rho(dq)$ *has the law of a PII with characteristics* $(0, 0, \ell_t dt\rho(dq))_\chi$. *This process is a compensated compound Poisson process with time-changing intensity.*
- *Assume that the above processes are independent. Let* $\mathbb{P}$ *be the law of the* $\mathbb{D}(\mathcal{I}, \mathbb{R}^d)$*-valued process*

$$t \mapsto Z + b_t + \sigma_t W_t + \sum_{i=0}^{N_{\lambda_t}} Y_i - \int_{[0,t]\times\mathbb{R}^d_*} \chi(q)\ell_t dt\rho(dq).$$

Then X *is a PII with* $X_0 \sim Z$ *and* $X_t - X_0$ *has characteristics* $(b_t, A_t, \ell_t dt\rho(dq))_\chi$ *under* $\mathbb{P}$.

We now present a convenient form of the characteristic function of the increments of a PII. This characteristic function will be the starting point of the proof of the duality formula for PII in Proposition 2.20.

We define a natural real-valued integral of càdlàg paths over elementary functions $u = \sum_{i=1}^{n-1} u_i \mathbb{1}_{(t_i,t_{i+1}]} \in \mathcal{E}_d$ by

$$\int_{\mathcal{I}} u_s \cdot d\omega_s := \sum_{i=1}^{n-1} u_i \cdot (\omega(t_{i+1}) - \omega(t_i)) = -\sum_{i=1}^{n} (u_i - u_{i-1}) \cdot \omega(t_i), \quad \omega \in \mathbb{D}(\mathcal{I}, \mathbb{R}^d), \tag{2.14}$$

with $u_0 = u_n = 0$ as convention. Define a measure on $\mathcal{I} \times \mathbb{R}^d_*$ by

$$\bar{L}([0,t] \times Q) := L_t(Q) \text{ for } t \in \mathcal{I},\ Q \in \mathcal{B}(\mathbb{R}^d_*). \tag{2.15}$$

By independence of increments the characteristics of the infinitely divisible random vector $X_t - X_s$ for any $s \le t$ are $(b_t - b_s, A_t - A_s, \bar{L}((s,t] \times dq))_\chi$. Thus for any elementary function $u = \sum_{i=1}^{n-1} u_i \mathbb{1}_{(t_i, t_{i+1}]} \in \mathcal{E}_d$

$$\begin{aligned}\mathbb{E}\left(\exp\left(i \int_{\mathcal{I}} u_s \cdot dX_s\right)\right) = \prod_{k=1}^{n-1} \exp\Big(&iu_k \cdot (b_{t_{k+1}} - b_{t_k}) - \frac{1}{2} u_k \cdot (A_{t_{k+1}} - A_{t_k}) u_k \\ &+ \int_{\mathbb{R}^d_*} \left(e^{iu_k \cdot q} - 1 - iu_k \cdot \chi(q)\right) \bar{L}((t_k, t_{k+1}] \times dq)\Big).\end{aligned}$$

Define the integral

$$\int_{\mathcal{I}} u_s \cdot dA_s u_s := \sum_{k=1}^{n-1} u_k \cdot (A_{t_{k+1}} - A_{t_k}) u_k, \tag{2.16}$$

then we deduce the identity

$$\begin{aligned}\mathbb{E}\left(\exp\left(i \int_{\mathcal{I}} u_s \cdot dX_s\right)\right) &= \exp\Big(i \int_{\mathcal{I}} u_s \cdot db_s - \frac{1}{2} \int_{\mathcal{I}} u_s \cdot dA_s u_s \\ &\quad + \int_{\mathcal{I} \times \mathbb{R}^d_*} \left(e^{iu_s \cdot q} - 1 - iu_s \cdot \chi(q)\right) \bar{L}(dsdq)\Big).\end{aligned} \tag{2.17}$$

This is the characteristic functional of the PII, it determines law of the PII but for the initial condition.

2.2.2. *Duality formula for PII.*

Using the form (2.17) of the characteristic functional of a process with independent increments, we unify and extend the duality formulae (1.11) and (1.22) of Wiener and Poisson process to PII.

To state the duality formula we need to define d-dimensional extensions of the gradient and difference operators appearing in (1.11) and (1.22). Let $F = f(\omega(t_1), \dots, \omega(t_n))$ be a cylindric, smooth and bounded functional in $\mathcal{S}_d$. We define a

- *derivative operator*:

$$\begin{aligned} D_{s,j} F(\omega) &:= \sum_{k=0}^{n-1} \partial_{kd+j} f(\omega(t_1), \dots, \omega(t_n)) \mathbb{1}_{(0,t_k]}(s), \quad j \in \{1, \dots, d\}, \\ \text{and } D_s F(\omega) &:= (D_{s,1} F(\omega), \dots, D_{s,d} F(\omega))^t \ \text{ for } s \in \mathcal{I}; \end{aligned} \tag{2.18}$$

- *difference operator*:

$$\begin{aligned} \Psi_{s,q} F(\omega) &:= f(\omega(t_1) + q\mathbb{1}_{[0,t_1]}(s), \dots, \omega(t_n) + q\mathbb{1}_{[0,t_n]}(s)) - f(\omega(t_1), \dots, \omega(t_n)) \\ &= F(\omega + q\mathbb{1}_{[s,1]}) - F(\omega), \ \text{ for } s \in \mathcal{I},\ q \in \mathbb{R}^d_*. \end{aligned} \tag{2.19}$$

We constrain the duality formula to the classes of test functions we need for the characterization of PII in Section 3.

Proposition 2.20. *Let X be an integrable d-dimensional PII such that $X_t - X_0$ has characteristics (b_t, A_t, L_t). Then the duality formula*

(2.21)

$$\mathbb{E}\left(F(X)\left(\int_{\mathcal{I}} u_s \cdot d(X-b)_s\right)\right) = \mathbb{E}\left(\int_{\mathcal{I}} D_s F(X) \cdot dA_s u_s\right) + \mathbb{E}\left(\int_{\mathcal{I}\times\mathbb{R}^d_*} \Psi_{s,q} F(X) u_s \cdot q\bar{L}(dsdq)\right)$$

holds for all $u \in \mathcal{E}_d$, $F \in \mathcal{S}_d$.

Proof. The demonstration could be based on the result of Proposition 2.7. It will be more convenient to use the same method of proof, but with the characteristic function given in (2.17). Assume that X is an integrable PII such that $X_t - X_0$ has characteristics (b_t, A_t, L_t). Take any $u, v \in \mathcal{E}_d$ and let $F(X) = \exp\left(i\int_{\mathcal{I}} v_s \cdot dX_s\right)$ be a trigonometric path functional. Without loss of generality we assume that

$$u = \sum_{j=1}^{n-1} u_j \mathbb{1}_{(t_j, t_{j+1}]}, \text{ and } v = \sum_{j=1}^{n-1} v_j \mathbb{1}_{(t_j, t_{j+1}]}.$$

Differentiating $\mathbb{E}\left(\exp\left(i\int_{\mathcal{I}} v_s \cdot dX_s\right)\right)$ in each of the d components of v_k and using (2.17) implies

$$\begin{aligned} i\,\mathbb{E}\left(e^{i\int_{\mathcal{I}} v_s \cdot dX_s}(X_{t_{k+1}} - X_{t_k})\right) &= \mathbb{E}\left(i(b_{t_{k+1}} - b_{t_k}) - (A_{t_{k+1}} - A_{t_k})v_k \right. \\ &\qquad \left. + i\int_{\mathbb{R}^d_*} \left(e^{iv_k \cdot q} - 1\right) q\bar{L}((t_k, t_{k+1}] \times dq)\right) \mathbb{E}\left(e^{i\int_{\mathcal{I}} v_s \cdot dX_s}\right). \end{aligned}$$

Next we take the scalar product with u_k, sum over $1 \le k \le n-1$ and use the definition of the derivative (2.18) and difference operator (2.19) to get identity (2.21) for $F(X) = \exp\left(i\int_{\mathcal{I}} v_s \cdot dX_s\right)$ and $u \in \mathcal{E}_d$. The extension from trigonometric functionals to $F \in \mathcal{S}_d$ works in the same lines as in the proof of Theorem 2.7. □

We already commented in Section 1 on the first versions of the duality formulae for the Wiener process and the Poisson process. The fact that Lévy processes without Gaussian part satisfy (2.21) with $A = 0$ was first mentioned by Picard [Pic96]. More recently many authors where concerned with the duality formula on the Fock space and its interpretation in the chaos decomposition of Lévy processes based on work by Itô [Itô56]. In this context (2.21) is a duality relation between a creation operator (stochastic integral) and an annihilation operator (derivative and difference operator). We will not develop this point of view, but refer to e.g. Løkka [Løk04] or Solé, Utzet, Vives [SUV07] for the jump case. Our proof is a new and elegant way to show that the duality formula is satisfied by PII with jumps, since we do not use the chaos decomposition.

2.3. **Definition of derivative and difference operator by perturbation.**

In this paragraph we are going to make a small detour from the main subject of this thesis. We present a deterministic perturbation that leads to the definition of the derivative (2.18), and a new stochastic perturbation that leads to the difference operator (2.19). These are used to provide another proof of the duality formula (2.21). In §2.3.3 we briefly compare the derivative and difference operators defined by perturbation to corresponding definitions obtained via the chaos decomposition.

To simplify the exposition we constrain the class of reference measures on $\mathbb{D}(\mathcal{I}, \mathbb{R}^d)$ to Lévy processes. This restricts the class of perturbations needed to prove the duality formula.

Definition 2.22. *A Lévy process is a stationary PII, in particular $X_t - X_0$ has characteristics*

$$b_t = tb, \quad A_t = tA, \quad L_t(dq) = tL(dq)$$

for some $b \in \mathbb{R}^d$, $A \in \mathbb{R}^{d\times d}$ symmetric non-negative definite and L a Lévy measure on $\mathbb{R}^d_*$.

This implies $\bar{L}(dsdq) = dsL(dq)$. Since the law of the increments of a Lévy process is determined by the characteristics of $X_1 - X_0$, we just say from now on that the Lévy process X has characteristics $(b, A, L)_\chi$.

2.3.1. *Definition of derivative and difference operator.*

In this paragraph we present alternative definitions of the derivative and difference operator on $\mathcal{S}_d$ that were given in (2.18) and (2.19). These definitions apply to larger classes of functionals F.

The following interpretation of the derivative operator (2.18) on $\mathcal{S}_d$ is well known. Take some $u \in \mathcal{E}_d$, $F \in \mathcal{S}_d$, then for any symmetric non-negative definite matrix $A \in \mathbb{R}^{d\times d}$ the derivative of F in direction $\int_{[0,.]} Au_s ds$ is given by

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left(F(\omega + \varepsilon \int_{[0,.]} Au_s ds) - F(\omega)\right) = \int_{\mathcal{I}} D_s F(\omega) \cdot Au_s ds, \quad \forall \omega \in \mathbb{D}(\mathcal{I}, \mathbb{R}^d). \tag{2.23}$$

Using the Lipschitz regularity of $F \in \mathcal{S}_d$ and the dominated convergence theorem we can show that this convergence also holds in $\mathbb{L}^2(\mathbb{Q})$ for any probability $\mathbb{Q}$ on $\mathbb{D}(\mathcal{I}, \mathbb{R}^d)$: The derivative can be defined as a Gâteaux-derivative.

Denote by $\mathbb{L}^2(Adt)$ the space of functions $u : \mathcal{I} \to \mathbb{R}^d$ with $\int_{\mathcal{I}} u_s \cdot Au_s ds < \infty$. Given the non-negative definite matrix $A \in \mathbb{R}^{d\times d}$ and $u \in \mathbb{L}^2(dt)$ we define the deterministic path-perturbation

$$\theta^\varepsilon_u : \mathbb{D}(\mathcal{I}, \mathbb{R}^d) \to \mathbb{D}(\mathcal{I}, \mathbb{R}^d), \quad \omega \mapsto \theta^\varepsilon_u(\omega) = \omega + \varepsilon \int_{[0,.]} Au_s ds. \tag{2.24}$$

Definition 2.25. *Let $\mathbb{Q}$ be any probability on $\mathbb{D}(\mathcal{I}, \mathbb{R}^d)$, $F \in \mathbb{L}^2(\mathbb{Q})$. We say that F is A-differentiable if there exists a $\mathbb{R}^d$-valued process $DF \in \mathbb{L}^2(Adt \otimes \mathbb{Q})$ such that for every $u \in \mathcal{E}_d$ the equality*

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}(F \circ \theta^\varepsilon_u - F) = \int_{\mathcal{I}} D_t F \cdot Au_t dt$$

holds in $\mathbb{L}^2(\mathbb{Q})$.

If $A = \mathrm{Id}$ is the identity matrix, we will only say that the functional $F \in \mathbb{L}^2(\mathbb{Q})$ is differentiable in direction u with derivative

$$D_u F := \int_{\mathcal{I}} D_t F \cdot u_t dt. \tag{2.26}$$

Let us note, that A-differentiability depends on the matrix A and the reference measure $\mathbb{Q}$.

Remark 2.27. *Take $\mathbb{Q}$ on $\mathbb{D}(\mathcal{I}, \mathbb{R}^2)$ such that for the canonical process $X = (X_1, X_2)$ the first component X_1 is a real-valued Wiener process and $X_2 \equiv 0$ $\mathbb{Q}$-a.s. Then X is a Lévy process in $\mathbb{R}^2$ with characteristics*

$$b = 0,\ L = 0 \ \text{ and } \ A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Let us consider the simple functional $F(X) = f_1(X_{1,t}) f_2(X_{2,t})$ for some $t \in \mathcal{I}$. Then F is A-differentiable for any function f_2, but F is differentiable in direction of the identity matrix only if f_2 is differentiable in zero. In both cases f_1 has to be differentiable in the following sense: Its

weak derivative (see the proof of Theorem 1.2) is square integrable with respect to the Gaussian distribution with mean 0 *and variance* t.

The operator in Definition 2.25 is a true derivative in the following sense.

Lemma 2.28. *The standard rules of differential calculus apply:*

- *Let* F, G *and* FG *be A-differentiable in direction* u, *then the product rule holds:*

$$\int_{\mathcal{I}} D_t(GF) \cdot Au_t dt = G \int_{\mathcal{I}} D_t F \cdot Au_t dt + F \int_{\mathcal{I}} D_t G \cdot Au_t dt. \tag{2.29}$$

- *Let* F *by A-differentiable in direction* u *and* $\phi \in C_b^\infty(\mathbb{R})$, *then* $\phi(F)$ *is A-differentiable in direction* u *and the chain rule holds:*

$$\int_{\mathcal{I}} D_t \phi(F) \cdot Au_t dt = \phi'(F) \int_{\mathcal{I}} D_t F \cdot Au_t dt. \tag{2.30}$$

Proof. For the product rule, we remark that

$$\frac{1}{\varepsilon}\left((FG)\circ\theta_u^\varepsilon - FG\right) = G\frac{1}{\varepsilon}\left(F\circ\theta_u^\varepsilon - F\right) + F\frac{1}{\varepsilon}\left(G\circ\theta_u^\varepsilon - G\right) + \frac{1}{\varepsilon}\left(F\circ\theta_u^\varepsilon - F\right)\left(G\circ\theta_u^\varepsilon - G\right),$$

and in the $\mathbb{L}^2(\mathbb{Q})$-limit the last term converges to zero by the Cauchy-Schwarz inequality:

$$\mathbb{E}\left(\frac{1}{\varepsilon}\left(F\circ\theta_u^\varepsilon - F\right)\left(G\circ\theta_u^\varepsilon - G\right)\right) \le \mathbb{E}\left(\left(\frac{1}{\varepsilon}\left(F\circ\theta_u^\varepsilon - F\right)\right)^2\right)^{\frac{1}{2}} \mathbb{E}\left(\left(G\circ\theta_u^\varepsilon - G\right)^2\right)^{\frac{1}{2}},$$

where the second term tends to zero *a-fortiori*.

For the product rule use the Taylor expansion

$$\phi(F\circ\theta_u^\varepsilon) = \phi(F) + \phi'(F)\left(F\circ\theta_u^\varepsilon - F\right) + O\left(\left(F\circ\theta_u^\varepsilon - F\right)^2 \wedge \|\phi''\|_\infty\right),$$

and therefore

$$\frac{1}{\varepsilon}\left((\phi(F))\circ\theta_u^\varepsilon - \phi(F)\right) = \phi'(F)\frac{1}{\varepsilon}\left(F\circ\theta_u^\varepsilon - F\right) + \frac{1}{\varepsilon}O\left(\left(F\circ\theta_u^\varepsilon - F\right)^2 \wedge \|\phi''\|_\infty\right),$$

where the last term tends to zero by Cauchy-Schwarz again. □

Let us now introduce a definition of the difference operator (2.19) using a non-deterministic perturbation. We need the following space of elementary space-time test functions:

$$\bar{\mathcal{E}} := \left\{ \bar{u} : \mathcal{I}\times\mathbb{R}_*^d \to \mathbb{R}_+,\ \bar{u} = \sum_{j=1}^{n-1} \bar{u}_j \mathbb{1}_{(t_j,t_{j+1}]\times Q_j} \text{ for } \bar{u}_j \in \mathbb{R}_+, Q_j \subset \mathbb{R}_*^d \text{ compact},\ \tau_n \in \Delta_{\mathcal{I}}, n \in \mathbb{N} \right\}.$$

The analogue to the deterministic perturbation process $\left(\varepsilon\int_{[0,t]} Au_s ds\right)_{t\in\mathcal{I}}$ used in Definition 2.25 will be a random processes $Y^{\varepsilon\bar{u}}$ defined on some auxiliary probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ as follows.

Let $\bar{u} = \bar{u}_1 \mathbb{1}_{(s_1,s_2]\times Q} \in \bar{\mathcal{E}}$ and L any Lévy measure be fixed. We consider a family $(N^{\varepsilon\bar{u}})_{\varepsilon>0}$ of Poisson measures on $\mathcal{I}\times\mathbb{R}_*^d$ with intensity $\varepsilon\bar{u}(s,q)dsL(dq)$ living on $(\Omega', \mathcal{F}', \mathbb{P}')$. Define the $\mathbb{R}^d$-valued compound Poisson process $Y_t^{\varepsilon\bar{u}} = \int_{[0,t]\times\mathbb{R}_*^d} qN^{\varepsilon\bar{u}}(dsdq)$. Then the marginals of $Y^{\varepsilon\bar{u}}$ satisfy

$$\begin{aligned} \mathbb{P}'(Y_t^{\varepsilon\bar{u}} \in dq) &= e^{-\varepsilon((t\vee s_1)\wedge s_2)\bar{u}_1 L(Q)}\left[\delta_{\{0\}}(dq) + \varepsilon((t\vee s_1)\wedge s_2)\bar{u}_1 \mathbb{1}_Q(q)L(dq) + O(\varepsilon^2)\right] \\ &= (1 - \varepsilon((t\vee s_1)\wedge s_2)\bar{u}_1 L(Q))\delta_{\{0\}}(dq) + \varepsilon((t\vee s_1)\wedge s_2)\bar{u}_1 L(dq) + O(\varepsilon^2). \end{aligned} \tag{2.31}$$

For $F(\omega) = f(\omega(t_1), \dots, \omega(t_n)) \in \mathcal{S}_d$ the difference operator (2.19) can be derived as follows:

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\mathbb{E}'\left(F(\omega + Y^{\varepsilon\bar{u}}) - F(\omega)\right) \\
= &\sum_{j=1}^{n} \int_{\mathbb{R}^d_*} \Big(f(\omega(t_1) + q_j, \dots, \omega(t_j) + q_j, \omega(t_{j+1}), \dots, \omega(t_n)) - F(\omega)\Big) \cdot \\
&\qquad \cdot((t_j \vee s_1) \wedge s_2 - (t_{j-1} \vee s_1) \wedge s_2)\mathbb{1}_Q(q_j)\bar{u}_1 L(dq_j) \\
= &\int_{\mathcal{I}\times\mathbb{R}^d_*} \Big(F(\omega + q\mathbb{1}_{[s,1]}) - F(\omega)\Big)\bar{u}(s,q)dsL(dq) = \int_{\mathcal{I}\times\mathbb{R}^d_*} \Psi_{s,q}F(\omega)\bar{u}(s,q)dsL(dq).
\end{aligned}$$

By linearity the same limit exists for all $\bar{u} \in \bar{\mathcal{E}}$, we may define $Y^{\varepsilon\bar{u}}$ accordingly. Since for $F \in \mathcal{S}_d$ the functional $F(X)$ is bounded and Lipschitz the convergence holds in $\mathbb{L}^2(\mathbb{Q})$ for any probability $\mathbb{Q}$ on $\mathbb{D}(\mathcal{I}, \mathbb{R}^d)$. Therefore we can define the difference operator by a perturbation in $\mathbb{L}^2(\mathbb{Q})$: For $\omega' \in \Omega'$ fixed and any $\bar{u} \in \bar{\mathcal{E}}$ we define the random perturbation

$$\bar{\theta}^{\varepsilon}_{\bar{u}}(\omega', .) : \mathbb{D}(\mathcal{I}, \mathbb{R}^d) \to \mathbb{D}(\mathcal{I}, \mathbb{R}^d), \quad \omega \mapsto \bar{\theta}^{\varepsilon}_{\bar{u}}(\omega', \omega) = \omega + Y^{\varepsilon\bar{u}}(\omega'). \tag{2.32}$$

Definition 2.33. *Let $\mathbb{Q}$ be any probability on $\mathbb{D}(\mathcal{I}, \mathbb{R}^d)$, $F \in \mathbb{L}^2(\mathbb{Q})$. We say that F is L-differentiable if there exists $\Psi F \in \mathbb{L}^2(dt \otimes L \otimes \mathbb{Q})$ such that for every $\bar{u} \in \bar{\mathcal{E}}$ the equality*

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\mathbb{E}'\left(F \circ \bar{\theta}^{\varepsilon}_{\bar{u}} - F\right) = \int_{\mathcal{I}\times\mathbb{R}^d_*} \Psi_{t,q}F\, \bar{u}(s,q)dsL(dq)$$

holds in $\mathbb{L}^2(\mathbb{Q})$.

The notion of L-differentiability depends on $\mathbb{Q}$ and on the Lévy measure L in the same way as A-differentiability is dependent on A and $\mathbb{Q}$.

Remark 2.34. *The operator Ψ is not a usual derivative operator in the sense of Lemma 2.28. Take e.g. functionals $F, G \in \mathcal{S}_d$, then*

$$\begin{aligned}
\Psi_{t,q}(FG)(X) &= F(X + q\mathbb{1}_{[t,1]})G(X + q\mathbb{1}_{[t,1]}) - F(X)G(X) \\
&= G(X)\Big(F(X + q\mathbb{1}_{[t,1]}) - F(X)\Big) + F(X)\Big(G(X + q\mathbb{1}_{[t,1]}) - G(X)\Big) \\
&\quad + \Big(F(X + q\mathbb{1}_{[t,1]}) - F(X)\Big)\Big(G(X + q\mathbb{1}_{[t,1]}) - G(X)\Big) \\
&= G(X)\Psi_{t,q}F(X) + F(X)\Psi_{t,q}G(X) + \Psi_{t,q}F(X)\Psi_{t,q}G(X)
\end{aligned}$$

where the term $\Psi_{t,q}F(X)\Psi_{t,q}G(X)$ is not zero in general.

Other approaches of defining a derivative for functionals of processes with jumps exist. They use perturbations of the existing jumps of the process instead of randomly adding jumps. This leads to other operators than the difference operator of Definition 2.33. In [BJ83] Bichteler and Jacod perturb the jumps-sizes of a reference process. The drawback is that their Lévy measure has to be absolutely continuous $L(dq) \ll dq$. Carlen and Pardoux as well as Elliott and Tsoi perturbed the jump-times of a Poisson process, in particular they restricted their approach to $L = \lambda\delta_{\{1\}}$ for some $\lambda > 0$, see [CP90], [ET93] and § 6.2.1. Privault extended this approach in [Pri96] to Lévy processes with Lévy measure $L = \sum_{j=1}^{k} \lambda_j \delta_{\{q_j\}}$ with $\lambda_j > 0$, $q_j \in \mathbb{R}^d_*$ and $k \in \mathbb{N}$. Both approaches lead to a true derivative operator in the sense of Lemma 2.28. In his study of the derivation of jump process functionals [Dec98] Decreusefond unifies the approaches of perturbing the jump-sizes and jump-times.

2.3.2. *Application: An alternative proof of the duality formula for Lévy processes.*

Let X be a Lévy process with characteristics $(b, A, L)_\chi$. As an application of the perturbation analysis in § 2.3.1 we give another proof of the duality formula (2.21) for Lévy processes.

It is well known (see e.g. Sato [Sat99, Paragraph 19]) that X admits the Lévy-Itô decomposition

$$X_t - X_0 = tb + M_t^X + \int_{[0,t]\times\mathbb{R}_*^d} \chi(q)\tilde{N}^X(dsdq) + \int_{[0,t]\times\mathbb{R}_*^d} (q - \chi(q))N^X(dsdq),\ \mathbb{P}\text{-}a.s., \tag{2.35}$$

where M^X is a continuous martingale with quadratic variation process $(tA)_{t\in\mathcal{I}}$, N^X is a Poisson measure with intensity $dsL(dq)$ on $\mathcal{I}\times\mathbb{R}_*^d$ and $\tilde{N}^X$ is the compensated Poisson measure. This is an extension of the decomposition in law we introduced in Example 2.13. Before proving the duality formula we use the Lévy-Itô decomposition to derive the integrability condition presented in Remark 2.5.

Remark 2.36. *If X is a Lévy process, the canonical random vector $X_1 - X_0$ is infinitely divisible with characteristics $(b, A, L)_\chi$. Then*

$$\int_{\mathbb{R}_*^d} \left(|q|^2 \wedge |q|_1\right) L(dq) < \infty \ \Rightarrow\ X_1 - X_0 \text{ is integrable.} \tag{2.37}$$

Proof. Since L is a Lévy measure the condition on the left side of (2.37) is equivalent to $\int_{\{|q|_1>1\}} |q|_1 L(dq) < \infty$. Without loss of generality we assume that $\chi(q) = q\mathbb{1}_{|q|_1\le 1}$ in (2.35). Let us take the contra-position $X_1 - X_0 \notin \mathbb{L}^1(\mathbb{P})$. This is equivalent to the right side of (2.35) not being integrable for $t = 1$. Since b, M_1^X and $\int_{\mathcal{I}\times\{|q|_1\le 1\}} q\tilde{N}^X(dsdq)$ are integrable, this implies that $\int_{\mathcal{I}\times\{|q|_1>1\}} qN^X(dsdq)$ is not integrable. This is a contradiction to the integrability $\int_{\{|q|_1>1\}} |q|_1 L(dq) < \infty$, the mass measure of N^X. □

Let us now proceed to the proof of the duality formula for Lévy processes using Definitions 2.25 and 2.33 of the derivative and difference operator by perturbation.

Proposition 2.38. *Let X be a Lévy process with characteristics $(b, A, L)_\chi$ under $\mathbb{P}$. For every functional $F \in \mathbb{L}^2(\mathbb{P})$ that is A- and L-differentiable and every $u \in \mathbb{L}^2(Adt)$, $\bar{v} \in \mathbb{L}^2(dt \otimes L)$ the following duality formula holds*

$$\begin{aligned} &\mathbb{E}\left(F\left(\int_{\mathcal{I}} u_s \cdot dM_s^X + \int_{\mathcal{I}\times\mathbb{R}_*^d} \bar{v}(s,q)\tilde{N}^X(dsdq)\right)\right) \\ &= \mathbb{E}\left(\int_{\mathcal{I}} D_s F \cdot Au_s ds\right) + \mathbb{E}\left(\int_{\mathcal{I}\times\mathbb{R}_*^d} \Psi_{s,q}F\,\bar{v}(s,q)dsL(dq)\right). \end{aligned} \tag{2.39}$$

Proof. First let us explain the use the Girsanov theorem in the proof. Take any $u \in \mathcal{E}_d$, $\bar{v} \in \bar{\mathcal{E}}$, then for every $\varepsilon > 0$ the process

$$Y_t := \int_{[0,t]} u_s \cdot dM_s^X + \int_{[0,t]\times\mathbb{R}_*^d} \bar{v}(s,q)\tilde{N}^X(dsdq),\ t \in \mathcal{I}, \tag{2.40}$$

is a martingale. We can define its Doléans-Dade exponential as the solution of the stochastic integral equation

$$G_t^\varepsilon = 1 + \varepsilon \int_{[0,t]} G_{s-}^\varepsilon dY_s,\ t \in \mathcal{I}. \tag{2.41}$$

A solution to this equation exists in a pathwise sense and is a uniformly integrable martingale (by Theorem IV.3 of Lepingle, Mémin [LM78]). Therefore $G_1^\varepsilon \mathbb{P}$ defines a probability on $\mathbb{D}(\mathcal{I}, \mathbb{R}^d)$. By the Girsanov theorem for semimartingales (see e.g. [JS03] Theorem III.3.24) the canonical process X is a PII under $G_1^\varepsilon \mathbb{P}$ and $X_t - X_0$ has characteristics

$$tb + \varepsilon \int_{[0,t]} Au_s ds + \varepsilon \int_{[0,t]\times\mathbb{R}^d_*} \chi(q)\bar{v}(s,q) ds L(dq), \qquad tA, \qquad \left(\int_{[0,t]} (1 + \varepsilon \bar{v}(s,q)) ds\right) L(dq)$$

with respect to χ. Using Burkhölder-Davis-Gundy inequalities and Gronwall's lemma we are going to show that

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left(G_1^\varepsilon - 1\right) = \int_{\mathcal{I}} u_s \cdot dM_s^X + \int_{\mathcal{I}\times\mathbb{R}^d_*} \bar{v}(s,q)\tilde{N}^X(dsdq) = Y_1 \qquad \text{in } \mathbb{L}^2(\mathbb{P}).$$

For any function $\omega : \mathcal{I} \to \mathbb{R}$ we put

$$|\omega|_t^* := \sup_{s\le t} |\omega(s)|_1.$$

We show that $|\frac{1}{\varepsilon}(G^\varepsilon - 1) - Y|_t^*$ converges to zero in $\mathbb{L}^2(\mathbb{P})$ for $\varepsilon \to 0$. The positive constant K may change from line to line. By Burkhölder-Davis-Gundy inequalities

$$\begin{aligned}
\mathbb{E}\left(\left(\left|\frac{1}{\varepsilon}(G^\varepsilon - 1) - Y\right|_t^*\right)^2\right) &= \mathbb{E}\left(\left(\left|\int_{[0,\cdot]} (G_{s-}^\varepsilon - 1)\, dY_s\right|_t^*\right)^2\right) \\
&\le K\,\mathbb{E}\left(\int_{[0,t]} (G_{s-}^\varepsilon - 1)^2 d[Y]_s\right) \\
&\le K\,\mathbb{E}\left(\int_{[0,t]} (|G_-^\varepsilon - 1|_s^*)^2 ds\right),
\end{aligned}$$

where we used (2.41) at the first line. In the same way we can see that

$$\mathbb{E}\left(\left(|G^\varepsilon - 1|_t^*\right)^2\right) \le \varepsilon^2 K\,\mathbb{E}\left(\int_{[0,t]} \left(|G^\varepsilon|_s^*\right)^2 ds\right).$$

Using (2.41) again we get

$$\mathbb{E}\left(\left(|G^\varepsilon|_t^*\right)^2\right) \le K\left(1 + \varepsilon^2 \int_{[0,t]} \mathbb{E}\left(\left(|G_-^\varepsilon|_s^*\right)^2\right) ds\right),$$

and by Gronwall's lemma

$$\mathbb{E}\left(\left(|G^\varepsilon|_t^*\right)^2\right) \le K e^{\varepsilon^2 K t}.$$

This yields

$$\mathbb{E}\left(\left(\left|\frac{1}{\varepsilon}(G^\varepsilon - 1) - Y\right|_t^*\right)^2\right) \le \varepsilon^2 K.$$

We first prove (2.39) for $\bar{v} \equiv 0$, which is the first half of this identity. Given $\varepsilon > 0$ the process $X \circ \theta_u^\varepsilon$ clearly is a PII and $X_t \circ \theta_u^\varepsilon - X_0$ has characteristics

$$tb + \varepsilon \int_{[0,t]} Au_s ds, \qquad tA, \qquad tL(dq).$$

Now we observe by the Girsanov theorem recalled above that $X \circ \theta_u^\varepsilon$ under $\mathbb{P}$ has the same characteristics as X under $\mathbb{P}^{\varepsilon u} := G_1^\varepsilon \mathbb{P}$. By equality of the characteristic functionals and initial conditions we get the equality of laws $\mathbb{P}^{\varepsilon u} \circ X^{-1} = \mathbb{P} \circ (X \circ \theta_u^\varepsilon)^{-1}$, and therefore

$$\mathbb{E}\left(F\frac{1}{\varepsilon}\left(G_1^\varepsilon - 1\right)\right) = \frac{1}{\varepsilon}\mathbb{E}\,(F \circ \theta_u^\varepsilon - F)$$

for arbitrary A-differentiable F. We use the definition of the derivative operator to take limits, thus

$$\mathbb{E}\left(F(X)\int_{\mathcal{I}} u_t \cdot dM_t^X\right) = \mathbb{E}\left(\int_{\mathcal{I}} D_t F(X) \cdot Au_t dt\right). \tag{2.42}$$

We now prove (2.39) for $u \equiv 0$. For $\bar{v} \in \bar{\mathcal{E}}$ define the perturbation as in Section 2.3.1, then the perturbed process $X \circ \bar{\theta}^{\varepsilon}_{\bar{v}}$ is a PII under $\mathbb{P} \otimes \mathbb{P}'$ and $X_t \circ \bar{\theta}^{\varepsilon}_{\bar{v}}$ has characteristics

$$tb + \int_{[0,t]\times\mathbb{R}^d_*} \chi(q)\bar{v}(s,q)dsL(dq), \qquad tA, \qquad \left(\int_{[0,t]} (1 + \varepsilon\bar{v}(s,q))ds\right)L(dq).$$

This is an easy consequence of the fact that the sum of two independent Poisson measures is still a Poisson random measure and the intensities add up. By Girsanov theory X has the same characteristics under the measure $\mathbb{P}^{\varepsilon\bar{v}} := G_1^{\varepsilon}\mathbb{P}$, which implies $\mathbb{P}^{\varepsilon\bar{v}} \circ X^{-1} = (\mathbb{P} \otimes \mathbb{P}') \circ (X \circ \bar{\theta}^{\varepsilon}_{\bar{v}})^{-1}$. For an arbitrary L-differentiable F this means

$$\mathbb{E}\left(F\frac{1}{\varepsilon}\left(G_1^{\varepsilon} - 1\right)\right) = \frac{1}{\varepsilon}\mathbb{E}_{\mathbb{P}\otimes\mathbb{P}'}\left(F \circ \bar{\theta}^{\varepsilon}_{\bar{v}} - F\right).$$

We can apply the definition of the difference operator to get

$$\mathbb{E}\left(F\int_{\mathcal{I}\times\mathbb{R}^d_*} \bar{v}(s,q)\tilde{N}^X(dsdq)\right) = \mathbb{E}\left(\int_{\mathcal{I}\times\mathbb{R}^d_*} \Psi_{s,q}F\,\bar{v}(s,q)dsL(dq)\right). \tag{2.43}$$

Adding (2.42) and (2.43) we get (2.39) for $u \in \mathcal{E}_d$ and all $\bar{v} = \bar{v}_1 - \bar{v}_2$ with $\bar{v}_1, \bar{v}_2 \in \bar{\mathcal{E}}$. By density of these elementary functions in $\mathbb{L}^2(Adt)$ and $\mathbb{L}^2(dt \otimes L)$ and isometry of the stochastic integral with respect to the martingale M^X and the martingale measure $\tilde{N}^X$ we get the result. □

The duality formula (2.39) is extendable to predictable $u \in \mathbb{L}^2(Adt \otimes \mathbb{P})$ and $\bar{v} \in \mathbb{L}^2(dt \otimes L \otimes \mathbb{P})$.

Lemma 2.44. *Let X be as in the preceding Proposition 2.38. Then the duality formula (2.39) still holds for F that is A- and L-differentiable and predictable $u \in \mathbb{L}^2(dt \otimes \mathbb{P})$, $\bar{v} \in \mathbb{L}^2(dt \otimes L \otimes \mathbb{P})$.*

Proof. Take $F_1, F_2 \in \mathcal{S}_d$ that are $\mathcal{F}_{[0,t]}$-measurable and $u \in \mathbb{L}^2(dt)$, $\bar{v} \in \mathbb{L}^2(dt \otimes L)$ such that $u = u\mathbb{1}_{(t,1]}$ and $\bar{v} = \bar{v}\mathbb{1}_{(t,1]}$. Then the duality formula holds for all F that are A- and L-differentiable and the predictable processes F_1u and $F_2\bar{v}$ since $D_uF_1 = 0$ and $\Psi_{\bar{v}}F_2 = 0$. The rest of the proof is a monotone class argument. □

2.3.3. *Comparison of several definitions of derivative and difference operator.*

In this paragraph we investigate the connection between the definition of the derivative and difference operators by a perturbation and the annihilation operator defined via the chaos decomposition of Lévy processes. In the purely Gaussian case, a comparison of these definitions of the derivative operator can be found e.g. in the monograph by Bogachev [Bog10].

To achieve this, we need to show in Proposition 2.45 that the operators D and Ψ are closable. With the proof of closability of Ψ we extend a result by Solé, Utzet, Vives [SUV07], see also the comment after the Proposition.

Proposition 2.45. *Let X be a Lévy process with characteristics $(b, A, L)_\chi$ under $\mathbb{P}$. Define the derivative operator and the difference operator as in Definitions 2.25 and 2.33. Then*

- *the derivative operator D is closable as operator from $\mathbb{L}^2(\mathbb{P})$ into $\mathbb{L}^2(Adt \otimes \mathbb{P})$;*
- *the difference operator Ψ is closable as operator from $\mathbb{L}^2(\mathbb{P})$ into $\mathbb{L}^2(dt \otimes L \otimes \mathbb{P})$.*

Moreover for each F in the closure of the domain of Ψ the difference representation

$$\Psi_{t,q}F(X) = F(X + q\mathbb{1}_{[t,1]}) - F(X),\ dt \otimes L \otimes \mathbb{P}\text{-}a.e. \tag{2.46}$$

holds.

Proof. There is a standard proof for the closability of the derivative operator D using the duality formula (2.39) for $\bar{v} \equiv 0$ (see e.g. the monograph by Nualart [Nua06, Section 1.2]): Let $(F_j)_{j\geq 1}$ a sequence of A-differentiable functionals such that $\lim_{j\to\infty} F_j = 0$ in $\mathbb{L}^2(\mathbb{P})$. We have to show that if $\lim_{j\to\infty} DF_j = \eta$ in $\mathbb{L}^2(dt \otimes \mathbb{P})$, then $\eta = 0$. But using the product rule from Lemma 2.28 for any $G \in \mathcal{S}_d$, $u \in \mathcal{E}_d$ we can show with (2.39) that

$$\begin{aligned}\mathbb{E}\left(\int_{\mathcal{I}} \eta_s \cdot u_s ds G\right) &= \lim_{j\to\infty} \mathbb{E}\left(\int_{\mathcal{I}} D_s F_j \cdot u_s ds G\right) \\ &= \lim_{j\to\infty} \mathbb{E}\left(F_j G \int_{\mathcal{I}} u_s \cdot dM_s^X - F_j \int_{\mathcal{I}} D_s G \cdot u_s ds\right) = 0,\end{aligned}$$

which can only be the case if $\eta = 0$.

To show that Ψ is closable, we first prove the representation (2.46) for every L-differentiable functional F. We have already seen that this representation holds for $F \in \mathcal{S}_d$. Take $\bar{v} = \bar{v}_1 \mathbb{1}_{(s_1,s_2]\times Q} \in \bar{\mathcal{E}}$ and let $Y^{\varepsilon\bar{v}}$ be defined as in Section 2.3.1. Similar to (2.31), we use the density expansion

$$\begin{aligned}\mathbb{E}'\left(F(\bar{\theta}^{\varepsilon}_{\bar{v}}(\omega))\right) &= \sum_{j=0}^{\infty} \int_{(\mathcal{I}\times\mathbb{R}^d_*)^j} F(\omega + q_1\mathbb{1}_{[t_1,1]} + \cdots + q_j\mathbb{1}_{[t_j,1]}) e^{-\varepsilon\bar{v}_1(s_2-s_1)L(Q)} \\ &\quad (\varepsilon\bar{v}_1)^j \mathbb{1}_Q(q_1)L(dq_1)\cdots\mathbb{1}_Q(q_j)L(dq_j)\mathbb{1}_{\{s_1<t_1\leq t_j\leq s_2\}} dt_1\cdots dt_j.\end{aligned}$$

The fact that $\mathbb{E}'\left(\left|F(\omega\circ\bar{\theta}^{\varepsilon}_{\bar{v}}) + F(\omega)\right|_1\right) < \infty$ allows us to use the dominated convergence theorem to show that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathbb{E}'\left(F(\bar{\theta}^{\varepsilon}_{\bar{v}}(\omega)) - F(\omega)\right) = \int_{\mathcal{I}\times\mathbb{R}^d_*} \Big(F(\omega + q\mathbb{1}_{[t,1]}) - F(\omega)\Big)\bar{v}(t,q)dtL(dq).$$

By the definition of the difference operator as $\mathbb{L}^2(\mathbb{P})$-limit there exist subsequences $(\varepsilon_j)_{j\geq 1}$, $\varepsilon_j \to 0$ such that

$$\lim_{j\to\infty}\frac{1}{\varepsilon_j}\mathbb{E}'\left(F\circ\bar{\theta}^{\varepsilon}_{\bar{v}} - F\right) = \int_{\mathcal{I}\times\mathbb{R}^d_*} \Psi_{t,q}F\bar{v}(t,q)dtL(dq),\ \mathbb{P}\text{-}a.s.,$$

which implies $\Psi_{t,q}F = F(\omega + q\mathbb{1}_{[t,1]}) - F(\omega)$ holds $dt \otimes L \otimes \mathbb{P}$-*a.e.*.

The proof of closability of Ψ is oriented on a similar proof for the Poisson process by Nualart, Vives (see [NV90, Theorem 6.2]). Let $(F_j)_{j\geq 1}$ be a sequence of L-differentiable functionals such that $F_j \to 0$ in $\mathbb{L}^2(\mathbb{P})$ and $\Psi F \to \eta$ in $\mathbb{L}^2(dt \otimes L \otimes \mathbb{P})$. Ψ is closable if $\eta = 0$.

We can find a subsequence $(j_k)_{k\geq 1}$ such that $F_{j_k} \to 0\,\mathbb{P}$-*a.s.* and $\Psi F_{j_k} \to \eta$ holds $dt\otimes L\otimes\mathbb{P}$-*a.e.* By the above representation of ΨF_j we also get

$$\lim_{k\to\infty}\Psi_{t,q}F_{j_k}(\omega) = \lim_{k\to\infty}\Big(F_{j_k}(\omega + q\mathbb{1}_{[t,1]}) - F_{j_k}(\omega)\Big) = 0,\ dt \otimes L \otimes \mathbb{P}\text{-}a.e.,$$

which implies $\eta = 0$. By a similar limit argument we can prove the difference representation (2.46) on the closed domain of Ψ. □

We observe from the representation (2.46) that our definition of the closure of the operator Ψ is equivalent to

$$\Psi F(\omega) := F(\omega + q\mathbb{1}_{[t,1]}) - F(\omega) \text{ for every } F \in \mathbb{L}^2(\mathbb{P}) \text{ with } \Psi F \in \mathbb{L}^2(dt \otimes L \otimes \mathbb{P}). \tag{2.47}$$

In the literature there are at least two other approaches to define similar derivative and difference operators.

(1) Starting from definitions (2.18) and (2.19) on $\mathcal{S}_d$ one shows that the operators are closable and extends these as operators from $\mathbb{L}^2(\mathbb{P})$ into $\mathbb{L}^2(Adt\otimes\mathbb{P})$ respectively $\mathbb{L}^2(dt\otimes L\otimes\mathbb{P})$. See e.g. recent work by Geiss and Laukkarinen [GL11] for one-dimensional Lévy processes with jumps. This is a classic way to define the derivative operator for processes without jumps.

(2) Introducing a chaos decomposition of $\mathbb{L}^2(\mathbb{P})$ and defining the operators as annihilation operators on the chaos. For the jump case, see [Løk04], [SUV07] and [GL11].

In § 2.3.1 we introduced a new approach for Lévy processes with jumps:

(3) Use a deterministic respectively random perturbation to define derivative and difference operator, see Definitions 2.25 and 2.33. Show that these operators are closable (Proposition 2.45) and extend these as operators from $\mathbb{L}^2(\mathbb{P})$ into $\mathbb{L}^2(Adt \otimes \mathbb{P})$ respectively $\mathbb{L}^2(dt \otimes L \otimes \mathbb{P})$. This leads to a difference operator equivalent to the one in defined in (2.47).

Geiss and Laukkarinen prove that approaches (1) and (2) coincide for one-dimensional Lévy processes with jumps, their proof applies here as well. Solé, Utzet and Vives show that (2.47) and the definition of the difference operator on the chaos are equivalent for Lévy processes without Gaussian part. Using [SUV07, Proposition 5.5] and the closability of Ψ proved in Proposition 2.45 we deduce that the definition by perturbation we give in (3) is equivalent to the other approaches (1) and (2).

All three definitions of derivative and difference operator provide a way to prove the duality formula. We presented two proofs in Theorem 2.20 and Proposition 2.38 that are new for Lévy processes with jumps. A well known proof of a duality formula for Lévy processes with jumps is based on the chaos decomposition with annihilation and creation operators. An abstract algebraic proof of the duality formula on a Fock space isomorphic to the chaos can be found in [NV90, Proposition 4.2].

3. Characterization of infinitely divisible random vectors and PII

In this section we prove our main results of the first part of this thesis: Infinitely divisible random vectors are the unique random vectors satisfying the integration by parts formula (2.8), and PII are the unique processes satisfying the duality formula (2.21). These are new results in so far as we use the general setup of integrable random vectors respectively processes on càdlàg space and we unify the Gaussian and Poisson cases presented in Section 1. The uniqueness results imply a one-to-one connection between infinite divisibility and duality formulae including the derivative and difference operator as defined in (2.18) and (2.19). We underline this connection in § 3.3 by presenting a new proof of a characterization of infinitely divisible random measures that was first presented by Kummer and Matthes in [KM70b].

3.1. **Characterization of infinitely divisible random vectors.**

The following result is the converse of Theorem 2.7.

Theorem 3.1. *Let Z be an integrable random vector. If for every $f \in C_b^\infty(\mathbb{R}^d)$ the integration by parts formula*

$$\mathbb{E}\,(f(Z)\,(Z-b)) = \mathbb{E}\,(A\nabla f(Z)) + \mathbb{E}\left(\int_{\mathbb{R}_*^d} (f(Z+q)-f(Z))\,qL(dq)\right) \tag{3.2}$$

holds, then Z is infinitely divisible with characteristics (b, A, L).

Proof. We only need that (3.2) holds for trigonometric functions, a subset of $C_b^\infty(\mathbb{R}^d)$. For $\lambda \in \mathbb{R}$ define

$$\Phi(\lambda) = \mathbb{E}\left(e^{i\lambda\gamma\cdot Z}\right).$$

Then

$$\frac{d}{d\lambda}\Phi(\lambda) = i\,\mathbb{E}\left(e^{i\lambda\gamma\cdot Z}\gamma\cdot Z\right),$$

and since the real and complex component of $z \mapsto e^{i\lambda\gamma\cdot z}$ are in $C_b^\infty(\mathbb{R}^d)$ we can use equation (2.8) to get

$$\frac{d}{d\lambda}\Phi(\lambda) = i\left(\gamma\cdot b + i\lambda\gamma\cdot A\gamma + \int_{\mathbb{R}_*^d}\left(e^{i\lambda\gamma\cdot q}-1\right)\gamma\cdot qL(dq)\right)\Phi(\lambda).$$

This is an ordinary differential equation in λ with boundary condition $\Phi(0) = 1$ which admits the unique solution

$$\Phi(\lambda) = \exp\left(i\lambda\gamma\cdot b - \lambda^2\frac{1}{2}\gamma\cdot A\gamma + \int_{\mathbb{R}_*^d}\left(e^{i\lambda\gamma\cdot q}-1-i\lambda\gamma\cdot q\right)L(dq)\right).$$

For $\lambda = 1$ we recognize the characteristic function of an integrable infinitely divisible random vector with characteristics (b, A, L), see the Lévy-Khintchine formula (2.3). □

This characterization is a generalization of the so called Stein's Lemma for Gaussian random variables and its analogue for the Poisson distribution by Chen. We presented these results in Propositions 1.2 and 1.15. Implicitly Chen and Lou already used the above generalization in [CL87, Theorem 2.1]. Barbour, Chen and Loh considered the compound Poisson case for Lévy measures with support on $\mathbb{R}_+$ and derived a solution of the associated Stein's equation

$$g_f(x)x - \int_{(0,\infty)} g_f(x+q)qL(dq) = f(x) - \mathbb{E}\,(f(Z)),$$

see Remark 1.4 for the interpretation of such an equation. In [BCL92, Corollary 1] they conclude that the integration by parts formula for compound Poisson measures characterizes the compound Poisson distribution. Recently Gaussian random vectors have been characterized by an integration by parts formulae using the solution of the Stein's equation

$$x \cdot \nabla g_f(x) - \mathrm{Tr}(A\mathrm{Hess} g_f(x)^t) = f(x) - \mathbb{E}\,(f(Z)),$$

where $\mathrm{Hess} g_f(x)$ is the Hessian matrix of g_f and $\mathrm{Tr}(\,.\,)$ is the trace operator. Different solutions have been presented by Reinert, Röllin [RR09, Lemma 3.3], Chatterjee, Meckes [CM08, Lemma 2] and Nourdin, Peccati, Réveillac [NPR10, Lemma 3.3].

Lee and Shih obtained in [LS10, Proposition 4.5] a similar formula as (3.2) as finite dimensional projection of an (infinite dimensional) formula characterizing the associated white noise measure.

Let us give an explicit example of the characterization of an infinitely divisible random vector. This example will be discussed once more in Section 6 in the connection with Poisson processes.

Example 3.3. *Let Z have a Gamma distribution on $\mathbb{R}$ with density $\mathbb{1}_{(0,\infty)}(q)e^{-q}q^{\alpha-1}/\Gamma(\alpha)$ for some $\alpha > 0$. Then Z is infinitely divisible with characteristics given by $(\alpha, 0, \alpha\mathbb{1}_{(0,\infty)}(q)q^{-1}e^{-q}dq)$. Theorems 2.7 and 3.1 lead to the following characterizing formula:*

$$\mathbb{E}\,(f(Z)(Z-\alpha)) = \mathbb{E}\left(\int_{(0,\infty)} (f(Z+q) - f(Z))\,\alpha e^{-q}dq\right), \quad \forall f \in C_b^\infty(\mathbb{R}).$$

Diaconis and Zabell proposed a different characterization based on an integration by parts of the density function: According to [DZ91] *the random variable Z has a Gamma distribution with parameter α if and only if*

$$\mathbb{E}\,(f(Z)\,(Z-\alpha)) = \mathbb{E}\,(f'(Z)Z)\,, \quad \forall f \in C_c^\infty(\mathbb{R}).$$

3.2. **Characterization of processes with independent increments.**

The following result shows that PII are the only càdlàg processes satisfying specific duality formulae. It is the converse of the conclusion of Theorem 2.20.

Theorem 3.4. *Let $\mathbb{Q}$ be a probability on $\mathbb{D}(\mathcal{I}, \mathbb{R}^d)$ such that X has integrable increments and $(b_t)_{t\in\mathcal{I}}$, $(A_t)_{t\in\mathcal{I}}$ and $(L_t)_{t\in\mathcal{I}}$ be characteristics in the sense that (1)-(4) of Remark 2.12 hold. If for every $u \in \mathcal{E}_d$, $F \in \mathcal{S}_d$ the duality formula*

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}\left(F(X)\left(\int_{\mathcal{I}} u_s \cdot d(X-b)_s\right)\right) &= \mathbb{E}_{\mathbb{Q}}\left(\int_{\mathcal{I}} D_s F(X) \cdot dA_s u_s\right) \\ &\quad + \mathbb{E}_{\mathbb{Q}}\left(\int_{\mathcal{I}\times\mathbb{R}^d_*} \Psi_{s,q}F(X)u_s \cdot q\bar{L}(dsdq)\right) \end{aligned} \tag{3.5}$$

holds, then X is a PII with integrable increments and $X_t - X_0$ has characteristics (b_t, A_t, L_t) under $\mathbb{Q}$.

Proof. The proof is similar to the finite dimensional case presented in Theorem 3.1. Assume that X has integrable increments and satisfies (3.5). Fix any $u \in \mathcal{E}_d$ and define

$$\Phi(\lambda) := \mathbb{E}_{\mathbb{Q}}\left(\exp\left(i\lambda \int_{\mathcal{I}} u_s \cdot dX_s\right)\right) \text{ for } \lambda \in \mathbb{R}.$$

We see that

$$\begin{aligned}\frac{d}{d\lambda}\Phi(\lambda) &= i\,\mathbb{E}_{\mathbb{Q}}\left(e^{i\lambda\int_{\mathcal{I}} u_s\cdot dX_s}\int_{\mathcal{I}} u_s\cdot dX_s\right)\\ &= i\left[\int_{\mathcal{I}} u_s\cdot db_s + i\lambda\int_{\mathcal{I}} u_s\cdot dA_s u_s + \int_{\mathcal{I}\times\mathbb{R}^d_*}(e^{i\lambda u_s\cdot q}-1)u_s\cdot q\bar{L}(dsdq)\right]\Phi(\lambda),\end{aligned}$$

where we used the duality formula (3.5) to get the second equality. This is an ordinary differential equation with initial condition $\Phi(0) = 1$. It admits the unique solution

$$\Phi(\lambda) = \exp\left(i\lambda\int_{\mathcal{I}} u_s\cdot db_s - \frac{\lambda^2}{2}\int_{\mathcal{I}} u_s\cdot dA_s u_s + \int_{\mathcal{I}\times\mathbb{R}^d_*}\left(e^{i\lambda u_s\cdot q}-1-i\lambda u_s\cdot q\right)\bar{L}(dsdq)\right).$$

For $\lambda = 1$ we recognize (2.17) and identify X as PII. □

We already saw in Section 1 that these characterizations are known for the Wiener and the Poisson case. Recently Lee and Shih proved a similar infinite dimensional characterization for white noise measures on the dual of the Schwartz space, see [LS10, Theorem 3.7].

Let us present the above result in the important case of α-stable processes.

Example 3.6. *A PII is called isotropic α-stable if its characteristics for a given $t \in \mathcal{I}$ are $(tb, 0, (tC/|q|^{1+\alpha})dq)$ for some $\alpha \in (0,2)$ and any $C > 0$ and $b = \int_{\mathbb{R}^d_*} q\frac{C}{|q|^{1+\alpha}}dq$. It is well known that an α-stable process is integrable only if $\alpha \in (1,2)$. By Theorems 2.20 and 3.4 the canonical process X is α-stable under $\mathbb{Q}$ for $\alpha \in (1,2)$ if and only if X has integrable increments and*

$$\mathbb{E}_{\mathbb{Q}}\left(F(X)\int_{\mathcal{I}} u_s\cdot(dX_s - bds)\right) = \mathbb{E}_{\mathbb{Q}}\left(\int_{\mathcal{I}\times\mathbb{R}^d_*}\left(F(X+q\mathbb{1}_{[s,\infty)}) - F(X)\right)u_s\cdot q\frac{C}{|q|^{1+\alpha}}dsdq\right)$$

holds for all $F \in \mathcal{S}_d$ and $u \in \mathcal{E}_d$.

One method to extend the duality formula (3.5) to PII with non-integrable increments is to cut the large jumps of the process. For any càdlàg trajectory $\omega \in \mathbb{D}(\mathcal{I}, \mathbb{R}^d)$ the jump at time $t \in \mathcal{I}$, $\Delta\omega(t) := \omega(t) - \omega(t-)$, is well defined. Moreover for any $k > 0$, $t \in \mathcal{I}$ the sum $\sum_{s\le t}\Delta\omega(s)\mathbb{1}_{|\Delta\omega(s)|>k}$ is finite. This allows us to define the measurable application

$$\omega \mapsto \omega^k := \omega - \sum_{s\le\cdot}\Delta\omega(s)\mathbb{1}_{|\Delta\omega(s)|>k}\,.$$

If X is a PII the process $X^k(\omega) := X(\omega^k)$ is a PII with integrable increments.

Corollary 3.7. *Let $\mathbb{Q}$ be such that X^k has integrable increments for any $k > 0$. Then X is a PII with characteristics $(b, A, L)_\chi$ if and only if for every $u \in \mathcal{E}_d$, $F \in \mathcal{S}_d$ and $k > 0$*

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\left(F(X^k)\left(\int_{\mathcal{I}} u_s d(X^k - b^k)_s\right)\right) &= \mathbb{E}_{\mathbb{Q}}\left(\int_{\mathcal{I}} D_s F(X^k)\cdot dA_s u_s\right)\\ &\quad + \mathbb{E}_{\mathbb{Q}}\left(\int_{\mathcal{I}\times\mathbb{R}^d_*}\Psi_{s,q}F(X^k)u_s\cdot q\mathbb{1}_{|q|\le K}\bar{L}(dsdq)\right),\end{aligned}$$

where $b^k_t := b_t - \int_{\mathbb{R}^d_*}\left(\chi(q) - q\mathbb{1}_{|q|\le k}\right)L_t(dq) \in \mathbb{R}^d$.

Proof. Immediate from Theorems 2.20 and 3.4, since X^k has integrable increments and $X^k_t - X_0$ has characteristics $(b^k_t, A_t, \mathbb{1}_{|q|\le k}L_t(dq))$. □

This allows to treat the α-stable processes mentioned in Example 3.6 for the non-integrable case.

Example 3.8. *Take some $\alpha \in (0,2)$. By Theorems 2.20 and 3.4 the canonical process X is isotropic α-stable under $\mathbb{Q}$ if and only if each X^k has integrable increments and*

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}}\left(F(X)\int_I u_s d(X^k - b^k)_s\right) \\ = \; & \mathbb{E}_{\mathbb{Q}}\left(\int_{I\times\mathbb{R}^d_*}\left(F(X^k + q\mathbb{1}_{[t,\infty)}) - F(X^k)\right)u_t \cdot q\mathbb{1}_{|q|\le k}\frac{C}{|q|^{1+\alpha}}dtdq\right) \end{aligned}$$

holds for all $F \in \mathcal{S}_d$, $u \in \mathcal{E}_d$, $k > 0$ and some $C > 0$, where

$$b^k_t = t\int_{\mathbb{R}^d_*} q\mathbb{1}_{|q|\le k}\frac{C}{|q|^{1+\alpha}}dq.$$

3.3. **The non-negative case, characterizing infinitely divisible random measures.**

In this section we look at integration by parts formula for non-negative random vectors and a duality formula for infinitely divisible random measures on polish spaces. With our presentation we wish to point out the generality of the approaches developed in Section 2 and § 3.1, § 3.2.

3.3.1. *Non-negative infinitely divisible random vectors.*

There exists an extensive literature about the characterization of infinitely divisible random variables that are positive, see e.g. the monographs by Steutel [Ste70] and Sato [Sat99]. We present a generalization of the integration by parts formula (1.16) of the Poisson distribution that derives from the integration by parts (2.8) for infinitely divisible random vectors.

Let Z be an infinitely divisible random vector that is non-negative, $\mathbb{P}(Z \in \mathbb{R}^d_+) = 1$. It is known that the Laplace transform of Z is such that for all $\gamma \in \mathbb{R}^d_+$ we have

$$-\log \mathbb{E}\left(e^{-\gamma\cdot Z}\right) = \gamma\cdot\alpha + \int_{\mathbb{R}^d_+\setminus\{0\}}(1 - e^{-\gamma\cdot q})\,L^+(dq), \tag{3.9}$$

where $\alpha \in \mathbb{R}^d_+$ and L^+ is a Lévy measure on $\mathbb{R}^d_*$ with

$$\int_{\mathbb{R}^d_*}(|q|_1 \wedge 1)L^+(dq) < \infty, \quad L^+(\mathbb{R}^d_*\setminus\mathbb{R}^d_+) = 0.$$

The relation between the Laplace characteristics (α, L^+) and the Fourier characteristics $(b, A, L)_\chi$ of Z is given by

$$b = \alpha + \int_{\mathbb{R}^d_*}\chi(q)L^+(dq), \quad A = 0, \quad L = L^+. \tag{3.10}$$

Corollary 3.11. *Let Z be a non-negative random variable. Then Z is infinitely divisible with Laplace characteristics (α, L^+) if and only if the equation*

$$\mathbb{E}\,(f(Z)Z) = \mathbb{E}\,(f(Z)\alpha) + \mathbb{E}\left(\int_{\mathbb{R}^d_*} f(Z+q)qL^+(dq)\right). \tag{3.12}$$

holds for every $f \in C^\infty_c(\mathbb{R}^d_+)$.

Proof. The proof is the same as the proofs of Theorems 2.7 and 3.1, but for the function f_γ which is replaced by $q \mapsto e^{-\gamma\cdot q}$ and the convergence arguments can be replaced by monotone convergence. □

3.3.2. *Characterization of infinitely divisible random measures.*

We show an extension of the characterization given in Corollary 3.11 to infinitely divisible random measures. Similar to the treatment of càdlàg processes on canonical space we begin with an introduction of the canonical space of random measures over a Polish space (a Polish space is a separable, completely metrizable topological space).

Let $\mathbb{A}$ be a polish space, $\mathcal{A}_0$ the ring of bounded Borel sets and $\mathcal{A}$ the σ-field generated by $\mathcal{A}_0$. Define the space of all σ-finite measures on $(\mathbb{A}, \mathcal{A})$ by

$$\mathbb{M} := \{\mu \ \sigma\text{-finite measure on } (\mathbb{A}, \mathcal{A})\}. \tag{3.13}$$

The canonical random measure is the identity $N : \mathbb{M} \to \mathbb{M}$. We equip the space $\mathbb{M}$ with the σ-field $\mathcal{M} := \sigma(N(A),\ A \in \mathcal{A}_0)$.

Let $\mathbb{P}$ be any probability measure on $(\mathbb{M}, \mathcal{M})$. The notion of infinite divisibility is similar to the one for random vectors: The random measure N is infinitely divisible under $\mathbb{P}$ is for any $m \in \mathbb{N}$ there exists a probability $\mathbb{P}_m$ on $(\mathbb{M}, \mathcal{M})$ such that

$$\mathbb{P} = \mathbb{P}_m^{*m}, \tag{3.14}$$

where $\mathbb{P}_m^{*m} = \mathbb{P}_m * \cdots * \mathbb{P}_m$ denotes the m-times convolution product of $\mathbb{P}_m$. Define $\mathbb{M}_* := \mathbb{M}\backslash\{0\}$ and $\mathcal{M}_* = \mathcal{M} \cap \mathbb{M}_*$ (here $0 \in \mathbb{M}$ is the measure without mass $0(\mathbb{A}) = 0$). Remark that $\{0\} \in \mathcal{M}$. The Laplace transform of an infinitely divisible random measure is of the following form, see e.g. Kallenberg [Kal83, Theorem 6.1].

Proposition 3.15. *If N is infinitely divisible, there exist $\alpha \in \mathbb{M}$ and a σ-finite measure Γ over $(\mathbb{M}_*, \mathcal{M}_*)$ with $\int_{\mathbb{M}_*} (\mu(A) \wedge 1)\Gamma(d\mu) < \infty$ for every $A \in \mathcal{A}_0$ such that for all $\xi : \mathbb{A} \to \mathbb{R}_+$ we have*

$$-\log \mathbb{E}\,(e^{-\int_{\mathbb{A}} \xi(a)N(da)}) = \int_{\mathbb{A}} \xi(a)\alpha(da) + \int_{\mathbb{M}_*} (1 - e^{-\int_{\mathbb{A}} \xi(a)\mu(da)})\Gamma(d\mu), \tag{3.16}$$

where $\log(0) := -\infty$.

Proof. We refer to the proof presented by Kallenberg. The integrability condition on Γ given there is actually $\int_{\mathbb{M}_*} (1 - e^{-\mu(A)})\Gamma(d\mu) < \infty$. This is equivalent to our condition because

$$\frac{1}{2}q \le (1 - e^{-q}) \le q,\ q \in [0,1] \quad \text{and} \quad \frac{1}{2} \le (1 - e^{-q}) \le 1,\ q \in (1, \infty).$$

The existence of α and Γ can be proven by projection of the Laplace characteristics of the infinitely divisible random vectors $(N(A_1), \ldots, N(A_n))^t$ given in (3.9). □

(α, Γ) are called characteristics of the infinitely divisible random measure N. To state the characterization theorem we are going to use the following sets of test functions. The set of elementary functions is defined by

$$\mathcal{E}_{\mathbb{A}} := \left\{ \xi : \mathbb{A} \to \mathbb{R}_+,\ \xi = \sum_{i=1}^{n} \xi_i \mathbb{1}_{A_i},\ \xi_i \in \mathbb{R}_+, A_i \in \mathcal{A}_0, n \in \mathbb{N} \right\}.$$

And the set of smooth and cylindrical functionals with compact support is

$$\mathcal{S}_{\mathbb{M}} := \{F : \mathbb{M} \to \mathbb{R}_+,\ F(\mu) = f(\mu(A_1), \ldots, \mu(A_n)), f \in C_c^\infty(\mathbb{R}_+^n), A_i \in \mathcal{A}_0, n \in \mathbb{N}\}.$$

Theorem 3.17. *The random measure N is infinitely divisible with characteristics (α, Γ) under $\mathbb{Q}$ if and only if for all $F \in \mathcal{S}_{\mathbb{M}}$, $\xi \in \mathcal{E}_{\mathbb{A}}$ the duality formula*

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}\left(F(N)\int_{\mathbb{A}}\xi(a)N(da)\right) &= \mathbb{E}_{\mathbb{Q}}\left(F(N)\int_{\mathbb{A}}\xi(a)\alpha(da)\right) \\ &\quad +\mathbb{E}_{\mathbb{Q}}\left(\int_{\mathbb{M}_*}F(N+\mu)\left(\int_{\mathbb{A}}\xi(a)\mu(da)\right)\Gamma(d\mu)\right). \end{aligned} \tag{3.18}$$

holds.

Proof. Since the σ-field $\mathcal{M}$ is cylindrical, N is infinitely divisible if and only if $(N(A_1),\dots,N(A_n))^t$ is infinitely divisible as a random vector in $\mathbb{R}_+^n$ for any $A_1,\dots,A_n \in \mathcal{A}_0$ (see also Kallenberg [Kal83, Lemma 6.3]). Remark that for $\gamma \in \mathbb{R}_+^n$ we have $\gamma\cdot(N(A_1),\dots,N(A_n))^t = \int_{\mathbb{A}}\xi(a)N(da)$ if $\xi = \sum_{i=1}^n \gamma_i \mathbb{1}_{A_i} \in \mathcal{E}_{\mathbb{A}}$. By (3.16) the Lévy measure corresponding to $(N(A_1),\dots,N(A_n))^t$ is given by the image $\Gamma\circ((N(A_1),\dots,N(A_n))^t)^{-1}$. Then we conclude using Corollary 3.11 and the linearity of (3.18) with respect to ξ. □

The above theorem was first proven by Kummer and Matthes [KM70b], applying a characterization of infinitely divisible point processes proved in [KM70a]. Nehring and Zessin [NZ12] simplified the proof using a representation of infinitely divisible random measures by a Poisson measure.

A characterization of the above kind was first known for Poisson measures, a particular case of an infinitely divisible random measure, see Mecke [Mec67]. The concept of a Poisson measure generalizes the definition of a Poisson process from the index-set $\mathbb{R}_+$ to $\mathbb{A}$.

Definition 3.19. *Let Λ be a σ-finite measure without atoms on $(\mathbb{A}, \mathcal{A})$. Then N is called a Poisson measure with intensity Λ under $\mathbb{P}$ if for any $A, B \in \mathcal{A}_0$ the random variables $N(A)$ and $N(B)$ are independent with Poisson distribution $N(A) \sim \mathcal{P}(\Lambda(A))$ and $N(B) \sim \mathcal{P}(\Lambda(B))$.*

We now show that Theorem 3.17 implies a corresponding characterization of Poisson point processes on $(\mathbb{A}, \mathcal{A})$ by projection, thus we obtain Mecke's result.

Corollary 3.20. *Let Λ be some σ-finite measure on $(\mathbb{A}, \mathcal{A})$ with no atoms. The random measure N is a Poisson measure with intensity Λ under $\mathbb{Q}$ if and only if for all $F \in \mathcal{S}_{\mathbb{M}}$, $\xi \in \mathcal{E}_{\mathbb{A}}$*

$$\mathbb{E}_{\mathbb{Q}}\left(F(N)\int_{\mathbb{A}}\xi(a)N(da)\right) = \mathbb{E}_{\mathbb{Q}}\left(\int_{\mathbb{A}}F(N+\delta_{\{a\}})\xi(a)\Lambda(da)\right). \tag{3.21}$$

Proof. Suppose N is a Poisson measure with intensity Λ. Then N is infinitely divisible with independent increments. Denote by (α, Γ) its characteristics. Since N is a point process we have $\alpha \equiv 0$. By independence of increments the support of Γ is included in the set of degenerate integer-valued measures

$$\left\{\mu = n\delta_{\{a\}},\ a \in \mathbb{A},\ n \in \mathbb{N}\right\} = \bigcup_{n\geq 1}\left\{\mu = n\delta_{\{a\}},\ a \in \mathbb{A}\right\},$$

(see [Kal83], Theorem 7.2 and Lemma 7.3). Thus Γ can be projected onto some measure $\bar{\Lambda}$ on $\mathbb{N}\times\mathbb{A}$ in the sense that for every $g : \mathbb{M}\times\mathbb{A} \to \mathbb{R}_+$ we have

$$\int_{\mathbb{M}_*\times\mathbb{A}} g(\mu, a)\mu(da)\Gamma(d\mu) = \int_{\mathbb{N}\times\mathbb{A}} g(n\delta_{\{a\}}, a)\bar{\Lambda}(dnda). \tag{3.22}$$

Putting $F \equiv 1$ in (3.18) we obtain

$$\Lambda(A) = \mathbb{E}_{\mathbb{Q}}(N(A)) = \int_{\mathbb{M}_*}\mu(A)\Gamma(d\mu) = \bar{\Lambda}(\mathbb{N}\times A),\ \forall A \in \mathcal{A}_0.$$

Extending (3.18) to $F = \mathbb{1}_{\{\mu = n\delta_{\{a\}},\, a \in \mathbb{A}\}}$ and $\xi = \mathbb{1}_A$ for $A \in \mathcal{A}_0$ and applying (3.22) to $g = F\mathbb{1}_A$ we see that $\bar{\Lambda}(\mathbb{N} \times A) = \bar{\Lambda}(\{1\} \times A) = \Lambda(A)$. Therefore

$$\int_{\mathbb{M}_* \times \mathbb{A}} g(\mu, a)\mu(da)\Gamma(d\mu) = \int_{\mathbb{A}} g(\delta_{\{a\}}, a)\Lambda(da),\ g : \mathbb{M} \times \mathbb{A} \to \mathbb{R}_+ ,$$

which implies (3.21).

The sufficiency of the duality (3.21) is due to the identification of the Laplace transform of a Poisson random measure, similar to the proof of Theorem 3.17. □

Using $\xi = \mathbb{1}_A$ and $F(N) = f(N(A))$ for some $f \in C_b^\infty(\mathbb{R})$ and $A \in \mathcal{A}_0$ we see that Corollary 3.20 implies the characterization of the Poisson distribution presented in Proposition 1.15.

Second part: Characterization of Markov processes and reciprocal classes

4. The reciprocal classes of Markov processes

This section serves as an introduction to the second part of this thesis, in which we study the reciprocal classes of Markov processes. In § 4.1 and § 4.2 we state the definitions of Markov and reciprocal processes on $\mathbb{D}(\mathcal{I}, \mathbb{R}^d)$ and introduce some of their fundamental properties. In a sense reciprocal processes are Markov fields on $\mathcal{I}$, they generalize the concept of a Markov process. The reciprocal property has first been defined by Bernstein [Ber32]. He followed an idea of Schrödinger [Sch32] to introduce processes with time-symmetric dynamics. The reciprocal class, as introduced in § 4.3, contains all processes having the same bridges as a reference Markov process. Here a bridge is a process with deterministic initial and final state. We study the reciprocal classes of continuous Markov processes in Section 5 and of pure jump Markov processes in Sections 6 and 7.

As a first example we introduce in § 4.4 the concept of the reciprocal class of a Markov chain. The related concept of "reciprocal chains" is identical to the definition of reciprocal processes in a discrete time setting as originally given by Bernstein. A comparison of the bridges of homogeneous Markov chains indicates "algebraic" problems that occur when treating the reciprocal classes of càdlàg processes with jumps.

4.1. **Markov processes.**

The following is a customary definition of the Markov property.

Definition 4.1. *Let $\mathbb{P}$ be a probability on $\mathbb{D}(\mathcal{I}, \mathbb{R}^d)$. We say that $\mathbb{P}$ is Markov, if for any $t \in \mathcal{I}$ and any bounded $\mathcal{F}_{[t,1]}$-measurable functional F we have*

$$\mathbb{E}(F \,|\, \mathcal{F}_{[0,t]}) = \mathbb{E}(F \,|\, X_t), \ \mathbb{P}\text{-}a.s. \tag{4.2}$$

It is well known that a Wiener measure has the Markov property. Let us present an equivalent definition of the Markov property that appears to be time symmetric (see e.g. the introduction by Meyer [Mey67, Théorème T4]). The following property is often stated as the independence of future and past conditionally on the present.

Proposition 4.3. *The probability $\mathbb{P}$ on $\mathbb{D}(\mathcal{I}, \mathbb{R}^d)$ has the Markov property if and only if for any $t \in \mathcal{I}$ the σ-fields $\mathcal{F}_{[0,t]}$ and $\mathcal{F}_{[t,1]}$ are independent given X_t.*

Proof. Assume that $\mathbb{P}$ is Markov, let $t \in \mathcal{I}$ and F, G be bounded functionals such that F is $\mathcal{F}_{[0,t]}$-measurable and G is $\mathcal{F}_{[t,1]}$-measurable, then

$$\mathbb{E}(FG|X_t) \overset{!}{=} \mathbb{E}(\mathbb{E}(FG|\mathcal{F}_{[0,t]})|X_t) = \mathbb{E}(F\,\mathbb{E}(G|\mathcal{F}_{[0,t]})|X_t) \overset{!}{=} \mathbb{E}(F\,\mathbb{E}(G|X_t)|X_t) = \mathbb{E}(F|X_t)\mathbb{E}(G|X_t),$$

where the use of the Markov property has been marked with an exclamation.

Let on the other hand the σ-fields $\mathcal{F}_{[0,t]}$ and $\mathcal{F}_{[t,1]}$ be independent conditionally on X_t and take bounded functionals F, G as above. Then

$$\mathbb{E}(FG) = \mathbb{E}(\,\mathbb{E}(FG|X_t)) = \mathbb{E}(\,\mathbb{E}(F|X_t)\mathbb{E}(G|X_t)) = \mathbb{E}(F\,\mathbb{E}(G|X_t)),$$

where the second equality holds by assumption. □

Next we present a sufficient condition for the Markov property of càdlàg processes whose distribution is absolutely continuous with respect to a Markov process, see e.g. Léonard, Rœlly and Zambrini [LRZ12].

Lemma 4.4. *Let $\mathbb{P}$ be Markov and $\mathbb{Q}$ be the law of some càdlàg process such that $\mathbb{Q} \ll \mathbb{P}$. Then $\mathbb{Q}$ is Markov if the Radon-Nikodym density with respect to $\mathbb{P}$ factorizes as follows: For any $t \in \mathcal{I}$ there exist two random variables α_t, β_t that are $\mathcal{F}_{[0,t]}$- respectively $\mathcal{F}_{[t,1]}$-measurable such that $\mathbb{Q} = \alpha_t\beta_t\mathbb{P}$.*

Proof. Assume that $\mathbb{Q} = \alpha_t\beta_t\mathbb{P}$ for α_t, β_t as above. We prove the Markov property of $\mathbb{Q}$ using Proposition 4.3: For a bounded and $\mathcal{F}_{[0,t]}$-measurable functional F and a bounded and $\mathcal{F}_{[t,1]}$-measurable functional G we show that $\mathbb{Q}(FG|X_t) = \mathbb{Q}(F|X_t)\mathbb{Q}(G|X_t)$ holds $\mathbb{Q}$-*a.s.* First check that for any bounded $\phi : \mathbb{R} \to \mathbb{R}$ we have

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}(\phi(X_t)FG) &= \mathbb{E}(\phi(X_t)FG\alpha_t\beta_t) = \mathbb{E}(\phi(X_t)\mathbb{E}(FG\alpha_t\beta_t|X_t)) \text{ and} \\
\mathbb{E}_{\mathbb{Q}}(\phi(X_t)FG) &= \mathbb{E}_{\mathbb{Q}}(\phi(X_t)\mathbb{E}_{\mathbb{Q}}(FG|X_t)) = \mathbb{E}(\phi(X_t)\alpha_t\beta_t\mathbb{E}_{\mathbb{Q}}(FG|X_t)) \\
&= \mathbb{E}(\phi(X_t)\mathbb{E}(\alpha_t\beta_t|X_t)\mathbb{E}_{\mathbb{Q}}(FG|X_t)),
\end{aligned}$$

which implies

$$\mathbb{E}_{\mathbb{Q}}(FG|X_t) = \frac{\mathbb{E}(FG\alpha_t\beta_t|X_t)}{\mathbb{E}(\alpha_t\beta_t|X_t)} = \frac{\mathbb{E}(F\alpha_t|X_t)}{\mathbb{E}(\alpha_t|X_t)}\frac{\mathbb{E}(G\beta_t|X_t)}{\mathbb{E}(\beta_t|X_t)} \text{ holds } \mathbb{Q}\text{-}a.s.$$

since $\alpha_t\beta_t > 0$ $\mathbb{Q}$-*a.s.* and by the Markov property of $\mathbb{P}$. If we apply this with $G = 1$ and then with $F = 1$ we get

$$\mathbb{E}_{\mathbb{Q}}(F|X_t) = \frac{\mathbb{E}(F\alpha_t|X_t)}{\mathbb{E}(\alpha_t|X_t)} \text{ and } \mathbb{E}_{\mathbb{Q}}(G|X_t) = \frac{\mathbb{E}(G\beta_t|X_t)}{\mathbb{E}(\beta_t|X_t)} \text{ hold } \mathbb{Q}\text{-}a.s.,$$

which ends the proof. □

Example 4.5. *Let $\mathbb{P}$ be a Wiener measure on $\mathbb{D}(\mathcal{I})$ and $\{t_1, \dots, t_n\} \in \Delta_{\mathcal{I}}$ be arbitrary. Given the intervals $[a_1, b_1], \dots, [a_n, b_n] \subset \mathbb{R}$ we define the measure*

$$\mathbb{Q} := \frac{\mathbb{1}_{[a_1,b_1]}(X_{t_1}) \cdots \mathbb{1}_{[a_n,b_n]}(X_{t_n})}{\mathbb{P}(X_{t_1} \in [a_1, b_1], \dots, X_{t_n} \in [a_n, b_n])}\mathbb{P}.$$

Clearly $\mathbb{Q}$ is a probability on $\mathbb{D}(\mathcal{I})$ and by definition

$$\mathbb{Q}(\,.\,) = \mathbb{P}(\,.\,|X_{t_1} \in [a_1, b_1], \dots, X_{t_n} \in [a_n, b_n]).$$

Thus $\mathbb{Q}$ is a Wiener measure conditioned to pass through prescribed intervals at given times. A straightforward application of Lemma 4.4 shows that $\mathbb{Q}$ has the Markov property.

4.2. **Reciprocal processes.**

The following definition of the reciprocal property is stated as a conditional independence condition, close in spirit to the definition of Markov processes in Proposition 4.3.

Definition 4.6. *Let $\mathbb{P}$ be a probability on $\mathbb{D}(\mathcal{I}, \mathbb{R}^d)$. We say that $\mathbb{P}$ is reciprocal, if for any $s, t \in \mathcal{I}$, $s \le t$ the σ-fields $\mathcal{F}_{[0,s]\cup[t,1]}$ and $\mathcal{F}_{[s,t]}$ are independent given (X_s, X_t).*

Reciprocal processes have a property that looks similar to the Markov property as defined in Definition 4.1: If $\mathbb{P}$ is reciprocal, then for any $t \in \mathcal{I}$ and any bounded functionals F, G that are $\mathcal{F}_{[0,t]}$- respectively $\mathcal{F}_{[t,1]}$-measurable we have

$$\mathbb{E}(G\,|\,\mathcal{F}_{[0,t]}) = \mathbb{E}(G\,|\,X_t, X_0) \text{ and } \mathbb{E}(F\,|\,\mathcal{F}_{[t,1]}) = \mathbb{E}(F\,|\,X_t, X_1),\ \mathbb{P}\text{-}a.s. \tag{4.7}$$

This is just a consequence of the independence of the σ-fields $\mathcal{F}_{\{0\}\cup[t,1]}$ and $\mathcal{F}_{[0,t]}$ given X_0, X_t respectively $\mathcal{F}_{[0,t]\cup\{1\}}$ and $\mathcal{F}_{[t,1]}$ given X_t, X_1.

Next we recall some general properties of reciprocal processes. Most of these results are due to Jamison, who developed a general theory of reciprocal processes in a series of articles [Jam70, Jam74, Jam75].

Proposition 4.8. *Every Markov process is a reciprocal process.*

Proof. Assume that $\mathbb{P}$ has the Markov property. Let F, G and H be bounded functionals such that F is $\mathcal{F}_{[0,s]}$-measurable, G is $\mathcal{F}_{[s,t]}$-measurable and H is $\mathcal{F}_{[t,1]}$-measurable. We repeatedly use the Markov property in equations marked with an exclamation to get

$$\begin{aligned}
\mathbb{E}(FGH) &= \mathbb{E}\,(\,\mathbb{E}(FGH\,|\,\mathcal{F}_{[s,t]})) \\
&\overset{!}{=} \mathbb{E}\,(\,\mathbb{E}(F\,|X_s)\,G\,\mathbb{E}(H\,|X_t)) \\
&= \mathbb{E}\,(\,\mathbb{E}(F\,|X_s)\,\mathbb{E}(G\,|X_s, X_t)\,\mathbb{E}(H\,|X_t)) \\
&\overset{!}{=} \mathbb{E}\,(\,\mathbb{E}(F\,|\,\mathcal{F}_{[s,1]})\,\mathbb{E}(G\,|X_s, X_t)\,\mathbb{E}(H\,|X_t)) \\
&= \mathbb{E}\,(F\,\mathbb{E}(G\,|X_s, X_t)\,\mathbb{E}(H\,|X_t)) \\
&\overset{!}{=} \mathbb{E}\,(F\,\mathbb{E}(G\,|X_s, X_t)\,H\,).
\end{aligned}$$

This implies

$$\mathbb{E}(G\,|\,\mathcal{F}_{[0,s]\cup[t,1]}) = \mathbb{E}(G\,|X_s, X_t),$$

which is the reciprocal property. □

The converse is not true, we present a counterexample.

Example 4.9. *Let $\mathbb{P}$ be the law of a Wiener process with initial distribution $\mathbb{P}(X_0 \in \,.\,) = \frac{1}{2}(\delta_{\{0\}} + \delta_{\{1\}})$. Then the process $\mathbb{Q}(\,.\,) := \mathbb{P}(\,.\,|X_1 \ge X_0)$ is not Markov since for any $t \in \mathcal{I}$*

$$0 = \mathbb{Q}(X_1 < 1 | X_t \in [0,1], X_0 = 1) \neq \mathbb{Q}(X_1 < 1 | X_t \in [0,1], X_0 = 0) > 0.$$

But $\mathbb{Q}$ is a reciprocal process, since for any bounded $\mathcal{F}_{[s,t]}$-measurable functional F we have

$$\begin{aligned}
\mathbb{Q}(F\,|\,\mathcal{F}_{[0,s]\cup[t,1]}) &= \mathbb{E}\,(F\,|\,\mathcal{F}_{[0,s]\cup[t,1]}, X_1 \ge X_0) \\
&= \frac{\mathbb{1}_{\{X_1 \ge X_0\}}}{\mathbb{P}(X_1 \ge X_0)}\,\mathbb{E}\,(F\,|\,\mathcal{F}_{[0,s]\cup[t,1]}) \\
&\overset{!}{=} \frac{\mathbb{1}_{\{X_1 \ge X_0\}}}{\mathbb{P}(X_1 \ge X_0)}\,\mathbb{E}\,(F\,|X_s, X_t) \\
&= \mathbb{Q}(F\,|X_s, X_t),
\end{aligned}$$

where we marked the use of the Markov property of $\mathbb{P}$ with an exclamation sign.

Even if reciprocal processes are not generally Markov we can recover the Markov property by pinning one of the endpoints.

Proposition 4.10. *Let $\mathbb{P}$ be reciprocal. Then the pinned distribution $\mathbb{P}(\,.\,|X_0 = x)$ is $\mathbb{P}(X_0 \in\ .\,)$-a.s. well defined and Markov. Likewise $\mathbb{P}(\,.\,|X_1 = y)$ is $\mathbb{P}(X_1 \in\ .\,)$-a.s. well defined and Markov.*

Proof. We present the proof for pinning at time $t = 1$, the proof for $t = 0$ is the same. Take any $s \in [0,1]$ and bounded functionals F, G where F is $\mathcal{F}_{[0,s]}$-measurable and G is $\mathcal{F}_{[s,1]}$-measurable. For any $\phi : \mathbb{R}^d \to \mathbb{R}$ bounded we have

$$\begin{aligned}\mathbb{E}\left(\mathbb{E}\,(FG\,|\,X_s, X_1)\,\phi(X_s)\,\middle|\ X_1\right) &= \mathbb{E}\left(F\phi(X_s)G\,|\,X_1\right)\\ &= \mathbb{E}\left(\ \mathbb{E}\,(F\,|\,X_s, X_1)\,\phi(X_s)\,\mathbb{E}\,(G\,|\,X_s, X_1)\ \middle|\ X_1\right)\end{aligned}$$

which is the Markov property by Proposition 4.3. □

Example 4.11. *Let $\mathbb{Q}$ the reciprocal distribution defined in Example 4.9. By Proposition 4.10 the pinned process $\mathbb{Q}(\,.\,|X_0 = 0) = \mathbb{P}(\,.\,|X_0 = 0, X_1 \geq 0)$ has the Markov property. This can also be shown using Lemma 4.4 since $\mathbb{P}(\,.\,|X_0 = 0, X_1 \geq 0) = 2\,\mathbb{1}_{[0,\infty)}(X_1)\mathbb{P}(\,.\,|X_0 = 0)$, where $\mathbb{P}(\,.\,|X_0 = 0)$ is the law of a Brownian motion starting in zero.*

4.3. **Reciprocal class of a Markov process and harmonic transformations.**

In this paragraph we associate a class of reciprocal processes to a given distribution $\mathbb{P}$ that is Markov. The following notations will be used throughout the rest of this thesis:

- The initial law $\mathbb{P}_0(\,.\,) := \mathbb{P}(X_0 \in\ .\,)$ and the final law $\mathbb{P}_1(\,.\,) := \mathbb{P}(X_1 \in .)$ on $\mathbb{R}^d$. The joint distribution of the endpoints is $\mathbb{P}_{01}(\,.\,) := \mathbb{P}(X_0 \in .\,, X_1 \in .\,)$ on $\mathbb{R}^{2d}$.
- The fixed initial condition $\mathbb{P}^x(\,.\,) := \mathbb{P}(\,.\,|X_0 = x)$, which is $\mathbb{P}_0$-*a.s.* well defined. The bridges $\mathbb{P}^{x,y}(\,.\,) := \mathbb{P}(\,.\,|X_0 = x, X_1 = y)$, which are $\mathbb{P}_{01}$-*a.s.* well defined

For certain types of Markov processes we may assume that $\mathbb{P}^x$ is well defined for every $x \in \mathbb{R}^d$, take e.g. a Brownian motion or a Poisson process. In such a context we will say that the bridge $\mathbb{P}^{x,y}$ is well defined for every $x \in \mathbb{R}^d$ and $\mathbb{P}_1^x(dy)$-*a.s.*

Definition 4.12. *The reciprocal class $\mathcal{R}(\mathbb{P})$ associated to the Markov distribution $\mathbb{P}$ consists of all càdlàg distributions $\mathbb{Q}$ that have the disintegration*

$$\mathbb{Q}(\,.\,) = \int_{\mathbb{R}^{2d}} \mathbb{P}^{x,y}(\,.\,)\mathbb{Q}_{01}(dxdy), \tag{4.13}$$

where $\mathbb{P}^{x,y}$ has to be $\mathbb{Q}_{01}$-a.s. well defined.

The reciprocal class contains all processes that have the same bridges as a reference Markov process: From (4.13) it follows directly that

$$\mathcal{R}(\mathbb{P}) = \left\{\mathbb{Q} \text{ probability on } \mathbb{D}(\mathcal{I}, \mathbb{R}^d) \text{ such that } \mathbb{Q}^{x,y}(\,.\,) = \mathbb{P}^{x,y}(\,.\,) \text{ holds } \mathbb{Q}_{01}\text{-a.s.}\right\}. \tag{4.14}$$

This equality of bridges may be localized on subintervals of $\mathcal{I}$.

Corollary 4.15. *Let $\mathbb{Q}$ be an element of the reciprocal class $\mathcal{R}(\mathbb{P})$. Then $\mathbb{Q}$ is a reciprocal process and*

$$\mathbb{Q}(\,.\,|\mathcal{F}_{[0,s]\cup[t,1]}) = \mathbb{P}(\,.\,|X_s, X_t) \text{ on } \mathcal{F}_{[s,t]}.$$

Proof. Assume that the disintegration (4.13) holds. Let $s < t$, using the Markov property of $\mathbb{P}$ we get

$$\mathbb{Q}(\,.\,|\,\mathcal{F}_{[0,s]\cup[t,1]}) = \mathbb{Q}(\,.\,|\,\mathcal{F}_{[0,s]\cup[t,1]}, X_0, X_1) = \mathbb{P}(\,.\,|\,\mathcal{F}_{[0,s]\cup[t,1]}, X_0, X_1) \overset{!}{=} \mathbb{P}(\,.\,|X_s, X_t)$$

as measures on $\mathcal{F}_{[s,t]}$. In particular

$$\mathbb{Q}(\,.\,|X_s, X_t) = \mathbb{E}_{\mathbb{Q}}(\mathbb{Q}(\,.\,|\mathcal{F}_{[0,s]\cup[t,1]})|X_s, X_t) = \mathbb{P}(\,.\,|X_s, X_t) = \mathbb{Q}(\,.\,|\mathcal{F}_{[0,s]\cup[t,1]}),$$

we see that $\mathbb{Q}$ has indeed the reciprocal property. □

Example 4.16. *Let $\mathbb{P}$ be the law of a Brownian motion with arbitrary initial condition.*

- *Any pinned Wiener measure $\mathbb{P}^x$ is in the reciprocal class of the Wiener measure $\mathcal{R}(\mathbb{P})$. The endpoint distribution in the disintegration (4.13) is given by*

$$\mathbb{P}^x_{01}(dzdy) = \delta_{\{x\}}(dz) \otimes \frac{1}{\sqrt{2\pi}} e^{-\frac{|y-x|^2}{2}} dy.$$

- *The Brownian bridge $\mathbb{P}^{x,y}$ starting in x and ending in y is in $\mathcal{R}(\mathbb{P})$ for any $x, y \in \mathbb{R}^d$. Here the endpoint distribution is just $\mathbb{P}^{x,y}_{01} = \delta_{\{x\}} \otimes \delta_{\{y\}}$.*
- *The reciprocal process $\mathbb{Q}(\,.\,) := \mathbb{P}(\,.\,|X_1 \geq X_0)$ from Example 4.9 is in $\mathcal{R}(\mathbb{P})$ with endpoint distribution*

$$\begin{aligned}
\mathbb{Q}_{01}(dxdy) &= \mathbb{P}(X_0 \in dx, X_1 \in dy | X_1 \geq X_0) \\
&= \frac{\mathbb{P}(X_0 \in dx, X_1 \in dy, X_1 \geq X_0)}{\mathbb{P}(X_1 \geq X_0)} \\
&= 2\mathbb{1}_{\{x \geq y\}} \mathbb{P}_{01}(dxdy) \\
&= 2\mathbb{1}_{\{x \geq y\}} \mathbb{P}_0(dx) \otimes \frac{1}{\sqrt{2\pi}} e^{-\frac{|y-x|^2}{2}} dy.
\end{aligned}$$

It is easy to see, that the integrability of the increments of X is not conserved in the reciprocal class.

- *Let $x \in \mathbb{R}^d$ and μ_1 be any probability on $\mathbb{R}^d$ that does not have a finite first moment and define $\mathbb{Q}^x(\,.\,) := \int_{\mathbb{R}^d} \mathbb{P}^{x,y} \mu_1(dy) \in \mathcal{R}(\mathbb{P})$. Clearly X has integrable increments with respect to the reference Wiener measure $\mathbb{P}$. But this is not the case under $\mathbb{Q}^x$, the increment $X_1 - X_0$ is not integrable since $X_1 \sim \mu_1$.*

Let $\mathbb{P}$ be a Markov distribution. We introduce an important subclass of Markov processes in $\mathcal{R}(\mathbb{P})$: Let $h : \mathbb{R}^d \to \mathbb{R}_+$ be measurable such that $\mathbb{E}(h(X_1)) = 1$. Then $h\mathbb{P}(\,.\,) := h(X_1)\mathbb{P}(\,.\,)$ defines a probability on $\mathbb{D}(\mathcal{I}, \mathbb{R}^d)$ that has the Markov property by Lemma 4.4. Moreover $h\mathbb{P} \in \mathcal{R}(\mathbb{P})$, since for any bounded $\phi, \psi : \mathbb{R}^d \to \mathbb{R}$ and a bounded functional F we have

$$h\mathbb{E}(\phi(X_0)\psi(X_1)F) = h\mathbb{E}(\phi(X_0)\psi(X_1)h\mathbb{E}(F\,|X_0, X_1)),$$

and

$$\begin{aligned}
h\mathbb{E}(\phi(X_0)\psi(X_1)F) &= \mathbb{E}(\phi(X_0)\psi(X_1)h(X_1)F) \\
&= \mathbb{E}(\phi(X_0)\psi(X_1)h(X_1)\mathbb{E}(F\,|X_0, X_1)) \\
&= h\mathbb{E}(\phi(X_0)\psi(X_1)\mathbb{E}(F\,|X_0, X_1)).
\end{aligned}$$

Definition 4.17. *Let $\mathbb{P}$ be a Markov distribution and $h : \mathbb{R}^d \to \mathbb{R}_+$ be a measurable function such that $\mathbb{E}(h(X_1)) = 1$. Then $h\mathbb{P}(\,.\,) := h(X_1)\mathbb{P}(\,.\,)$ is called h-transform of $\mathbb{P}$.*

The notion of harmonic transformation has been introduced by Doob in [Doo57]. He used a Wiener measure as reference law and a random time instead of $t = 1$.

The notion of an h-transform in the context of reciprocal processes has been studied before by Jamison [Jam74], see also Fitzsimmons, Pitman, Yor [FPY92] and Léonard, Rœlly, Zambrini [LRZ12] for the connection between h-transforms and the bridges of a Markov process.

Let us note that a partial "converse" to the properties of h-transforms exist.

Remark 4.18. *Fix any initial state $X_0 = x$. Let $\mathbb{P}^x$ be Markov and $\mathbb{Q}^x$ an element of the reciprocal class $\mathbb{Q}^x \in \mathcal{R}(\mathbb{P})$ such that $\mathbb{Q}^x \ll \mathbb{P}^x$. Then $\mathbb{Q}^x$ is an h-transform of $\mathbb{P}^x$ with*

$$h(y) = \frac{d\mathbb{Q}_1^x}{d\mathbb{P}_1^x}(y),$$

since

$$\mathbb{Q}^x(\,.\,) = \int_{\mathbb{R}^d} \mathbb{P}^{x,y}(\,.\,)\mathbb{Q}_1^x(dy) = \int_{\mathbb{R}^d} \mathbb{P}^{x,y}(\,.\,)h(y)\mathbb{P}_1^x(dy) = h(X_1)\mathbb{P}^x(\,.\,).$$

An h-transform in the sense of Definition 4.17 may be interpreted as the prescription of a certain endpoint distribution.

Example 4.19. *Let $\mathbb{P}$ be a Wiener measure with arbitrary initial distribution μ_0. Clearly not all probabilities in $\mathcal{R}(\mathbb{P})$ are h-transforms, the bridges $\mathbb{P}^{x,y}$ already violate the necessary condition of absolute continuity with respect to $\mathbb{P}$. To abbreviate we denote the Gaussian transition kernel by*

$$\phi(x,y) := \frac{1}{(2\pi)^{\frac{d}{2}}} e^{\frac{-|x-y|^2}{2}}.$$

An h-transform changes the endpoint distribution of the reference process as follows.

- *For any probability $\mu_1(dy) = \rho_1(y)dy$ on $\mathbb{R}^d$ we define*

$$h(y) := \frac{d\mu_1}{d\mathbb{P}_1}(y) = \rho_1(y)\left(\int_{\mathbb{R}^d} \phi(x,y)\mu_0(dx)\right)^{-1}.$$

  Then $h\mathbb{P}$ is a processes in $\mathcal{R}(\mathbb{P})$ that has endpoint distribution μ_1.
- *Let h be defined as above, then the initial distribution of the h-transform changes as follows. Since $h\mathbb{P}_{01}(dxdy) = h(y)\mathbb{P}_{01}(dxdy)$ we can compute the marginal of the initial state. For any $B \in \mathcal{B}(\mathbb{R}^d)$ we have*

$$\begin{aligned} h\mathbb{P}_0(B) &= \int_{B\times\mathbb{R}^d} h(y)\mathbb{P}_{01}(dxdy) \\ &= \int_{B\times\mathbb{R}^d} \rho_1(y)\left(\int_{\mathbb{R}^d} \phi(z,y)\mu_0(dz)\right)^{-1} \phi(x,y)\mu_0(dx)dy \\ &= \int_B \left(\int_{\mathbb{R}^d} \frac{\rho_1(y)\phi(x,y)}{\int_{\mathbb{R}^d}\phi(z,y)\mu_0(dz)}dy\right)\mu_0(dx). \end{aligned}$$

  Note that for a pinned reference process the initial law does not change, we have $\mathbb{P}_0^x = \delta_{\{x\}}$ and $h\mathbb{P}_0^x = \delta_{\{x\}}$.

4.4. **The reciprocal class of a Markov chain, a basic example.**

In this paragraph we transfer the concepts of reciprocal processes and reciprocal classes to the discrete time setting for processes with finite state space. To justify this transfer we briefly explain why the dynamical properties of a Markov chain are similar to the properties of a Markov process in $\mathbb{D}(\mathcal{I}, \mathbb{R}^d)$ with deterministic jump-times that are constant between the jumps. The concept of discrete time reciprocal processes was first introduced by Bernstein [Ber32]. More recently authors have been interested in the classification of such "reciprocal chains" having Gaussian marginals, see e.g. Levy, Frezza, Krener [LFK90], Levy, Ferrante [LF02] and Carravetta [Car08].

The toy examples at the end of this paragraph then give a first impression of problems that arise, when comparing the bridges of pure jump Markov processes in continuous time.

4.4.1. *Embedding of Markov chains using deterministic jump-times.*

Let $Y = (Y_i)_{0\le i\le m}$ be an arbitrary discrete time process with finite state space $\{y_1, \dots, y_n\} \subset \mathbb{R}^d$ defined on an arbitrary probability space $(\Omega, \mathcal{A}, P)$.

Definition 4.20. *The process Y is Markov if for any $0 \le i \le m$ the σ-fields $\sigma(Y_0, \dots, Y_i)$ and $\sigma(Y_i, \dots, Y_m)$ are independent given Y_i. The process Y is reciprocal if for any $0 \le i \le j \le m$ the σ-fields $\sigma(Y_0, \dots, Y_i, Y_j, \dots Y_m)$ and $\sigma(Y_i, \dots, Y_j)$ are independent given Y_i, Y_j.*

Let us justify the above definitions by an embedding: We fix any $0 < t_1 < \cdots < t_m < 1$ throughout this paragraph. Using these as deterministic jump-times, we define the continuous time càdlàg process $Y^{(m)}$ as follows:

$$Y_t^{(m)} := \begin{cases} Y_0, \text{ for } t \in [0, t_1), \\ Y_1, \text{ for } t \in [t_1, t_2), \\ \vdots \\ Y_m, \text{ for } t \in [t_m, 1]. \end{cases} \tag{4.21}$$

The process $Y^{(m)}$ is a càdlàg process that jumps m-times at the deterministic time points $t_1, \dots, t_m$ and is constant in between the jumps. Moreover $Y^{(m)} : \Omega \to \mathbb{D}(\mathcal{I}, \mathbb{R}^d)$ is measurable. Define $\mathbb{P}_Y := P \circ (Y^{(m)})^{-1}$.

Lemma 4.22. *The process $Y = (Y_i)_{0\le i\le m}$ is Markov resp. reciprocal if and only if $\mathbb{P}_Y$ has the Markov property resp. the reciprocal property on $\mathbb{D}(\mathcal{I}, \mathbb{R}^d)$.*

Proof. Let $F(X) = f(X_{s_1}, \dots, X_{s_k}) \in \mathcal{S}_d$. By definition there exist $\{i_1, \dots, i_k\} \subset \{1, \dots, m\}$ such that $F(Y^{(m)}) = f(Y_{i_1}, \dots, Y_{i_k})$ and for arbitrary $t \in \mathcal{I}$ there exists $i \in \{1, \dots, m\}$ such that $Y_t^{(m)} = Y_i$. Then

$$\mathbb{E}_Y(F(X) \,|\, \mathcal{F}_{[0,t]}) = E(F(Y^{(m)}) | \sigma(Y_s^{(m)}, s \le t)) = E(f(Y_{i_1}, \dots, Y_{i_k}) | \sigma(Y_1, \dots, Y_i))$$

and by a similar computation

$$\mathbb{E}_Y(F(X) | X_t) = E(f(Y_{i_1}, \dots, Y_{i_k}) | Y_i).$$

The claimed equivalence concerning the Markov property is now immediate. To proof the equivalence of the reciprocal property for the discrete and continuous time models one advances in the same fashion. □

All results from § 4.1, § 4.2 and § 4.3 now apply to the setting of a discrete time process Y. In particular the definition of the reciprocal class of a Markov chain is again that of a mixture over the bridges.

Corollary 4.23. *Let Y be a Markov chain and $\tilde{Y} = (\tilde{Y}_i)_{0\le i\le m}$ another $\{y_1, \dots, y_n\}$-valued random process. Then $\mathbb{P}_{\tilde{Y}} \in \mathcal{R}(\mathbb{P}_Y)$ if and only if the following decomposition holds:*

$$P(\tilde{Y} \in \,.\,) = \sum_{i,j=1}^{n} P(Y \in \,.\, | Y_0 = y_i, Y_m = y_j) P(\tilde{Y}_0 = y_i, \tilde{Y}_m = y_j). \tag{4.24}$$

Proof. Just use the same identification of random variables as in the proof of Lemma 4.22. □

We define the reciprocal class of a given Markov chain as follows.

Definition 4.25. *Let Y be a Markov chain. Then we define the reciprocal class $\mathcal{R}(Y)$ as the collection of all $\{y_1, \dots, y_n\}$-valued processes $(\tilde{Y}_i)_{1\le i\le m}$ that have the same bridges*

$$P(\tilde{Y} \in \,.\, | \tilde{Y}_0 = y_i, \tilde{Y}_m = y_j) = P(Y \in \,.\, | Y_0 = y_i, Y_m = y_j) \text{ holds } P(\tilde{Y}_0 \in \,.\,, \tilde{Y}_m \in \,.\,)\text{-a.s.} \tag{4.26}$$

4.4.2. *The reciprocal class of a time-homogeneous Markov chain.*

Let $(Y_i)_{0\le i\le m}$ be a time-homogeneous Markov chain on the finite state space $\{y_1,\dots,y_n\}\subset\mathbb{R}^d$. Denote the initial law and transition matrix by

$$\mu_0(\{y_i\}) := P(Y_0 = y_i) \text{ for } i\in\{1,\dots,n\}, \text{ and } (p_{ij})_{1\le i,j\le n} := \Big(P(Y_1 = y_j|Y_0 = y_i)\Big)_{1\le i,j\le n}.$$

The reciprocal class of Y is defined by the identity of bridges in (4.26). Let us now study the bridges of Y for various $m,n\in\mathbb{N}$ (the number of "jumps" and the number of different states).

Example 4.27. *Let the number of states $n\in\mathbb{N}$ be arbitrary, $m=2$ and Y, $\tilde Y$ be two time-homogeneous Markov chains. We compute*

$$P(Y_1 = y_k|Y_0 = y_i, Y_2 = y_j) = \frac{p_{ik}p_{kj}}{\sum_{l=1}^n p_{il}p_{lj}}, \text{ and } P(\tilde Y_1 = y_k|\tilde Y_0 = y_i, \tilde Y_2 = y_j) = \frac{\tilde p_{ik}\tilde p_{kj}}{\sum_{l=1}^n \tilde p_{il}\tilde p_{lj}},$$

where the denominator is just the two-step probability of going from i to j. Define $p_{ij}(2) := P(Y_2 = y_j|Y_0 = y_i)$, then the equality of the bridge between y_i and y_j is given if and only if

$$\frac{p_{ik}p_{kj}}{p_{ij}(2)} = \frac{\tilde p_{ik}\tilde p_{kj}}{\tilde p_{ij}(2)}, \quad \forall k\in\{1,\dots,n\}.$$

The condition found in the above example is easily extended to arbitrary $m\in\mathbb{N}$.

Corollary 4.28. *Let $n,m\in\mathbb{N}$ and Y, $\tilde Y$ be two time-homogeneous Markov chains. Then $\tilde Y\in\mathcal{R}(Y)$ if and only if for all $1\le i,j\le n$ we have*

$$\frac{p_{ik_1}p_{k_1k_2}\cdots p_{k_{m-1}j}}{p_{ij}(m)} = \frac{\tilde p_{ik_1}\tilde p_{k_1k_2}\cdots\tilde p_{k_{m-1}j}}{\tilde p_{ij}(m)}, \quad \forall 1\le k_1,\dots,k_{m-1}\le n, \tag{4.29}$$

where $p_{ij}(m) := P(Y_m = y_j|Y_0 = y_i)$.

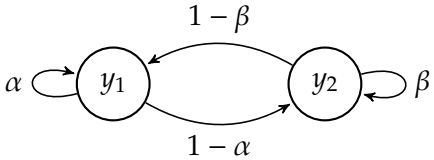
Proof. This follows directly from the definition of the reciprocal class through equality of the bridges. □

In the rest of this paragraph we compare the bridges of different time-homogeneous Markov chains Y and $\tilde Y$ in examples, using the comparison result of Corollary 4.28.

Example 4.30. *We study a simple message transmission model, a Markov chain that switches between two states $\{y_1,y_2\}$, where the transition probabilities are given by*

$$\Big(P(Y_1 = y_j\,|\,Y_0 = y_i)\Big)_{1\le i,j\le 2} = \begin{pmatrix}\alpha & 1-\alpha\\ 1-\beta & \beta\end{pmatrix}, \quad \Big(P(\tilde Y_1 = y_j\,|\,\tilde Y_0 = y_i)\Big)_{1\le i,j\le 2} = \begin{pmatrix}\tilde\alpha & 1-\tilde\alpha\\ 1-\tilde\beta & \tilde\beta\end{pmatrix},$$

where $\alpha,\beta,\tilde\alpha,\tilde\beta\in(0,1)$.



We use the localization result of Corollary 4.15 and transfer it into the discrete time setting by Lemma 4.22 to get a necessary condition for the equality of the bridges: For any lengths $m\in\mathbb{N}$ of the Markov chain, it is necessary that already the bridges for $m=2$ of Y and $\tilde Y$ are identical. Thus we have to find conditions on $\alpha,\beta,\tilde\alpha,\tilde\beta$ such that

$$P(Y_1 = y_1|Y_0 = y_1, Y_2 = y_2) = \frac{p_{11}p_{12}}{p_{11}p_{12}+p_{12}p_{22}} = \frac{\alpha(1-\alpha)}{\alpha(1-\alpha)+(1-\alpha)\beta} \overset{!}{=} \frac{\tilde\alpha(1-\tilde\alpha)}{\tilde\alpha(1-\tilde\alpha)+(1-\tilde\alpha)\tilde\beta}.$$

This can only be the case if there exists $\delta > 0$ with

$$\frac{\tilde{\alpha}}{\alpha} = \frac{\tilde{\beta}}{\beta} = \delta.$$

Another necessary condition for the equality of all bridges is

$$P(Y_1 = y_1 | Y_0 = y_1, Y_2 = y_1) = \frac{p_{11}p_{11}}{p_{11}p_{11} + p_{12}p_{21}} = \frac{\alpha^2}{\alpha^2 + (1-\alpha)(1-\beta)} \overset{!}{=} \frac{(\delta\alpha)^2}{(\delta\alpha)^2 + (1-\delta\alpha)(1-\delta\beta)},$$

which is equivalent to the quadratic equation

$$\delta^2(1-\alpha)(1-\beta) = (1-\delta\alpha)(1-\delta\beta) \Leftrightarrow \delta^2(1-\alpha-\beta) + \delta(\alpha+\beta) - 1 = 0.$$

If we assume that $\alpha, \beta \approx 1$, then the positive solutions to this quadratic equation are

$$\delta_{1,2} = \frac{\alpha+\beta \pm \sqrt{(\alpha+\beta)^2 - 4(\alpha+\beta-1)}}{2(\alpha+\beta-1)} = \frac{(\alpha+\beta) \pm (\alpha+\beta-2)}{2(\alpha+\beta-1)},$$

such that

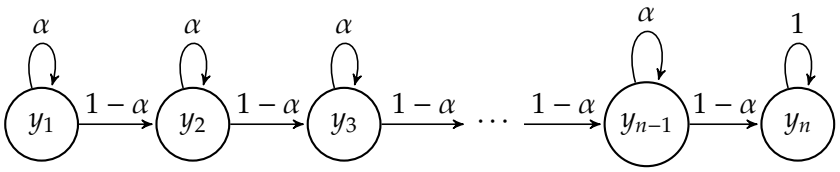
$$\delta_1 = 1, \quad \text{or} \quad \delta_2 = \frac{1}{\alpha+\beta-1}$$

are solutions. But it is easy to see, that δ_2 is too large, since in this case

$$\tilde{\alpha} = \frac{\alpha}{\alpha+\beta-1} > 1 \Leftrightarrow 1 > \beta,$$

which was an assumption. Thus the reciprocal classes of Y and $\tilde{Y}$ only coincide if $\alpha = \tilde{\alpha}$ and $\beta = \tilde{\beta}$.

Example 4.31. *Let $m, n \in \mathbb{N}$ be arbitrary and let the time-homogeneous Markov chains Y and $\tilde{Y}$ have the transition probabilities $p_{ii} = \alpha$, $p_{i,i+1} = 1-\alpha$ and $\tilde{p}_{ii} = \tilde{\alpha}$, $\tilde{p}_{i,i+1} = 1-\tilde{\alpha}$ for $1 \le i \le n-1$ and $p_{nn} = \tilde{p}_{nn} = 1$.*



We may interpret this as a comparison of birth processes with absorption in n. Let us first compare the bridges from y_i to y_j for $i \le j < n$ and $k := j - i \le m$. Condition (4.29) is

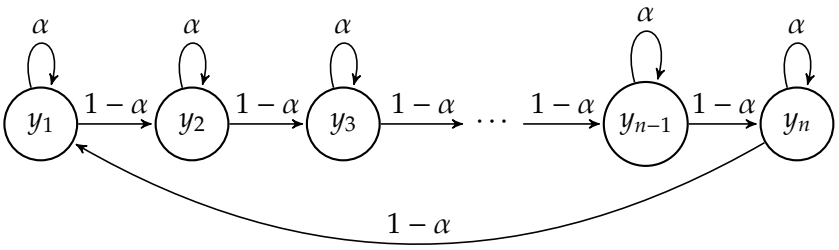
$$\frac{\alpha^{m-k}(1-\alpha)^k}{\binom{m}{k}\alpha^{m-k}(1-\alpha)^k} \overset{!}{=} \frac{\tilde{\alpha}^{m-k}(1-\tilde{\alpha})^k}{\binom{m}{k}\tilde{\alpha}^{m-k}(1-\tilde{\alpha})^k}, \tag{4.32}$$

these bridges coincide. But note that for the bridge from y_{n-1} to y_n we have

$$P(Y_1 = y_n | Y_0 = y_{n-1}, Y_m = y_n) = \frac{1-\alpha}{P(Y_m = y_n | Y_0 = y_{n-1})} \overset{!}{=} \frac{1-\tilde{\alpha}}{P(\tilde{Y}_m = y_n | \tilde{Y}_0 = y_{n-1})}.$$

The denominator is easily computed as $P(Y_m = y_n | Y_0 = y_{n-1}) = \sum_{k=0}^{m-1} \alpha^k(1-\alpha)$. Thus $\mathcal{R}(Y) = \mathcal{R}(\tilde{Y})$ if and only if $\alpha = \tilde{\alpha}$, even if most bridges coincide in this example: $\mathcal{R}(Y) \cap \mathcal{R}(\tilde{Y})$ contains all elements $\bar{Y}$ of either reciprocal class with $P(\bar{Y}_m = y_n) = 0$.

Example 4.33. *Next we consider a random walk on a circle moving in one direction: Let $p_{ii} = \alpha$, $p_{i,i+1} = 1-\alpha$ and $\tilde{p}_{ii} = \tilde{\alpha}$, $\tilde{p}_{i,i+1} = 1-\tilde{\alpha}$ for $1 \le i \le n-1$ and $p_{nn} = \alpha$, $p_{n1} = 1-\alpha$ and $\tilde{p}_{nn} = \tilde{\alpha}$, $\tilde{p}_{n1} = 1-\tilde{\alpha}$.*



Both chains are following the circle $y_1 \to y_2 \to \cdots \to y_n \to y_1 \to \ldots$. To compare the bridges we have to take into account the difference between the number of states n and the number of jumps m. If $m < n$ we can argue as in the preceding Example 4.31 that $\tilde{Y} \in \mathcal{R}(Y)$ for all $\tilde{\alpha} \in (0,1)$. But if $n \le m < 2n$, then condition (4.29) for the probability of going from y_k exactly one time around the "circle" by taking n jumps first and then staying in y_k is

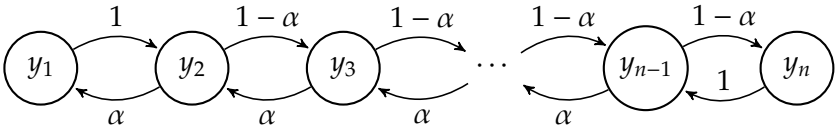
$$\begin{aligned} & P(Y_1 = y_{k+1}, \ldots, Y_{n-1} = y_{k-1}, Y_n = y_k, Y_{n+1} = y_k, \ldots, Y_m = y_k | Y_0 = y_k, Y_m = y_k) \\ = \quad & \frac{\alpha^n (1-\alpha)^{m-n}}{\frac{m!}{(n-1)!}\alpha^n(1-\alpha)^{m-n} + (1-\alpha)^m} \overset{!}{=} \frac{\tilde{\alpha}^n (1-\tilde{\alpha})^{m-n}}{\frac{m!}{(n-1)!}\tilde{\alpha}^n(1-\tilde{\alpha})^{m-n} + (1-\tilde{\alpha})^m}. \end{aligned}$$

Put $c := \frac{n!}{(m-n)!}$, then the above condition is equivalent to

$$\frac{(\frac{\alpha}{1-\alpha})^n}{c(\frac{\alpha}{1-\alpha})^n + 1} = \frac{(\frac{\tilde{\alpha}}{1-\tilde{\alpha}})^n}{c(\frac{\tilde{\alpha}}{1-\tilde{\alpha}})^n + 1} \quad \Leftrightarrow \quad \frac{1-\alpha}{\alpha} = \frac{1-\tilde{\alpha}}{\tilde{\alpha}},$$

which can only be the case if $\alpha = \tilde{\alpha}$.

Example 4.34. *Let $p_{i,i-1} = \alpha$, $p_{i,i+1} = 1-\alpha$ and $\tilde{p}_{i,i-1} = \tilde{\alpha}$, $\tilde{p}_{i,i+1} = 1-\tilde{\alpha}$ for some $\alpha, \tilde{\alpha} \in (0,1)$ with boundary reflection $p_{12} = \tilde{p}_{12} = 1$ and $p_{n,n-1} = \tilde{p}_{n,n-1} = 1$. This is a birth and death process with re-insertion of an individual if the population died out and certain death of one individual if the population has reached the number of n individuals.*



Both chains are (asymmetric) finite random walks on a domain with reflecting boundary. We can easily check that the reciprocal classes of Y and $\tilde{Y}$ do not coincide if $\alpha \neq \tilde{\alpha}$ using the localization result of Corollary 4.15: Assume that $n \ge 4$ and $m \ge 3$, then

$$P(Y_1 = y_1, Y_2 = y_2 | Y_0 = y_2, Y_3 = y_3) = \frac{\alpha(1-\alpha)}{\alpha(1-\alpha) + 2\alpha(1-\alpha)^2} = \frac{1}{3-2\alpha} \overset{!}{=} \frac{1}{3-2\tilde{\alpha}},$$

where the denominator is computed by

$$\begin{aligned} P(Y_0 = y_2, Y_3 = y_3) \quad = \quad & P(Y_0 = y_2, Y_1 = y_1, Y_2 = y_2, Y_3 = y_3) \\ & + P(Y_0 = y_2, Y_1 = y_3, Y_2 = y_4, Y_3 = y_3) \\ & + P(Y_0 = y_2, Y_1 = y_3, Y_2 = y_2, Y_3 = y_3). \end{aligned}$$

But this necessary condition is only satisfied if $\alpha = \tilde{\alpha}$.

5. The reciprocal classes of Brownian diffusions

In this section we present a theory of reciprocal classes of Brownian diffusions, which are defined in Definition 5.6. The objective is to give a synopsis of results by different authors in a uniform setting in preparation of the treatment of reciprocal classes of jump processes in Sections 6 and 7. We are able to improve some of these results, and present a new approach to *euclidean quantum mechanics* that is based on analogies to classical mechanics. Since all processes treated in this section have continuous sample paths, we present our results in the frame of the canonical setup on $C(\mathcal{I}, \mathbb{R}^d)$, the space of $\mathbb{R}^d$-valued continuous functions on $\mathcal{I}$.

In the presentation of § 5.1, § 5.2 and § 5.4 we closely follow the articles by Rœlly and Thieullen [RT02], [RT05]. They characterized the reciprocal class of a Brownian motion with drift by a duality formula. Their characterization is based on reciprocal invariants in the sense of Clark [Cla90], we present his result in § 5.3.

We present two applications. In § 5.5 we introduce an optimal control approach to determine the motion of a charged particle in an electromagnetic field. We define the dynamics of the motion of a charged particle that is immersed in a thermal reservoir and under the influence of an external electromagnetic field. The effective motion of the particle is then described by the solution of a stochastic optimal control problem under boundary constraints in the sense of Wakolbinger [Wak89] and Dai Pra [DP91]. Following their result we provide a new interpretation of the duality formula as a stochastic Newton equation. Our approach is closely related to Zambrini's euclidean quantum mechanics, see e.g. Zambrini [Zam85], but also Lévy and Krener [LK93] and Krener [Kre97].

A second application is the identification of the behavior of the reciprocal class of a Brownian diffusion under time reversal in § 5.6. This approach by duality formula is an alternative to the computations of Thieullen, who presented a similar result in [Thi93]. In combination with the results from § 5.5 we see, that the dynamics of a charged particle in a thermal reservoir are time reversible in the same sense as the deterministic dynamics of a charged particle in an electromagnetic field.

5.1. **The Brownian motion and its reciprocal class.**

Throughout this paragraph $\mathbb{P}$ denotes a d-dimensional Wiener measure with arbitrary initial state on $C(\mathcal{I}, \mathbb{R}^d)$. We show that $\mathcal{R}(\mathbb{P})$ can be characterized as the unique class of probability measures that satisfies a duality formula. By Theorem 3.4 only a Wiener measure satisfies the duality formula

$$\mathbb{E}\left(F(X)\int_{\mathcal{I}} u_s \cdot dX_s\right) = \mathbb{E}\left(\int_{\mathcal{I}} D_s F(X) \cdot u_s ds\right), \tag{5.1}$$

for all $u \in \mathcal{E}_d$, $F \in \mathcal{S}_d$. Thus this duality cannot hold for arbitrary elements of $\mathcal{R}(\mathbb{P})$.

The common feature of elements of one reciprocal class is the distribution of the bridges: Any $\mathbb{Q} \in \mathcal{R}(\mathbb{P})$ admits the disintegration (4.13) with respect to the Brownian bridges $\mathbb{P}^{x,y}$, $x, y \in \mathbb{R}^d$. Rœlly and Thieullen proved that the duality formula (5.1) holds for all Brownian bridges if the class of test functions is reduced. In the sequel we will continually use the short notation

$$\langle u\rangle_t := \int_{[0,t]} u_s ds, \text{ and } \langle u\rangle := \langle u\rangle_1, \text{ for any } u \in \mathbb{L}^1(dt).$$

Lemma 5.2. *Let $x, y \in \mathbb{R}^d$ and $\mathbb{P}^{x,y}$ be the law of the Brownian bridge from $X_0 = x$ to $X_1 = y$. Then the duality formula (5.1) holds under $\mathbb{P}^{x,y}$ for all $u \in \mathcal{E}_d$ with $\langle u\rangle = 0$ and $F \in \mathcal{S}_d$.*

Proof. Let $\phi, \psi \in C_b^\infty(\mathbb{R}^d)$, then

$$\begin{aligned}
&\mathbb{E}\left(\phi(X_0)\psi(X_1)\mathbb{E}\left(F(X)\int_{\mathcal{I}} u_s \cdot dX_s \,\middle|\, X_0, X_1\right)\right) \\
= \;& \mathbb{E}\left(\phi(X_0)\psi(X_1)F(X)\int_{\mathcal{I}} u_s \cdot dX_s\right) \\
= \;& \mathbb{E}\left(\phi(X_0)\phi(X_1)\int_{\mathcal{I}} D_s F(X)\cdot u_s ds\right) + \mathbb{E}\left(\phi(X_0)F(X)\nabla\psi(X_1)\cdot\int_{\mathcal{I}} u_s ds\right),
\end{aligned}$$

where the second term is equal to zero if $\langle u\rangle = 0$. We deduce

$$\mathbb{E}\left(\phi(X_0)\psi(X_1)\mathbb{E}\left(F(X)\int_{\mathcal{I}} u_s \cdot dX_s \,\middle|\, X_0, X_1\right)\right) = \mathbb{E}\left(\phi(X_0)\psi(X_1)\mathbb{E}\left(\int_{\mathcal{I}} D_s F(X)\cdot u_s ds \,\middle|\, X_0, X_1\right)\right).$$

□

This duality formula extends naturally to all elements of $\mathcal{R}(\mathbb{P})$ and furthermore characterizes the reciprocal class.

Theorem 5.3. *Let X have integrable increments under $\mathbb{Q}$. Then $\mathbb{Q} \in \mathcal{R}(\mathbb{P})$ if and only if the duality formula*

$$\mathbb{E}_{\mathbb{Q}}\left(F(X)\int_{\mathcal{I}} u_s \cdot dX_s\right) = \mathbb{E}_{\mathbb{Q}}\left(\int_{\mathcal{I}} D_s F(X)\cdot u_s ds\right) \tag{5.4}$$

holds for every $u \in \mathcal{E}_d$ with $\langle u\rangle = 0$ and $F \in \mathcal{S}_d$.

Proof. Let $\mathbb{Q} \in \mathcal{R}(\mathbb{P})$, $F \in \mathcal{S}_d$ and $u \in \mathcal{E}_d$ with $\langle u\rangle = 0$. By the disintegration (4.13) and Lemma 5.2 we get

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left(F(X)\int_{\mathcal{I}} u_s \cdot dX_s\right) &= \int_{\mathbb{R}^{2d}} \mathbb{E}^{x,y}\left(F(X)\int_{\mathcal{I}} u_s \cdot dX_s\right)\mathbb{Q}_{01}(dxdy) \\
&= \int_{\mathbb{R}^{2d}} \mathbb{E}^{x,y}\left(\int_{\mathcal{I}} D_s F(X)\cdot u_s ds\right)\mathbb{Q}_{01}(dxdy) \\
&= \mathbb{E}_{\mathbb{Q}}\left(\int_{\mathcal{I}} D_s F(X)\cdot u_s ds\right).
\end{aligned}$$

Conversely, let $\mathbb{Q}$ have integrable increments such that the duality formula holds. Following the proof of Lemma 5.2 we see that the duality formula still holds with respect to $\mathbb{Q}^{x,y}$ for all x, y such that the bridge is well defined. Take any $u \in \mathcal{E}_d$ and define

$$\Phi(\lambda) := \mathbb{E}_{\mathbb{Q}}^{x,y}\left(\exp\left(i\lambda\int_{\mathcal{I}} u_s \cdot dX_s\right)\right).$$

Let $\tilde{u} := u - \langle u\rangle$, then $\tilde{u} \in \mathcal{E}_d$ with $\langle\tilde{u}\rangle = 0$ and by assumption the duality formula applies. We obtain

$$\begin{aligned}
\Phi'(\lambda) &= i\,\mathbb{E}_{\mathbb{Q}}^{x,y}\left(\exp\left(i\lambda\int_{\mathcal{I}} u_s \cdot dX_s\right)\int_{\mathcal{I}} u_s \cdot dX_s\right) \\
&= i\,\mathbb{E}_{\mathbb{Q}}^{x,y}\left(\exp\left(i\lambda\int_{\mathcal{I}} u_s \cdot dX_s\right)\left(\int_{\mathcal{I}} \tilde{u}_s \cdot dX_s + (y-x)\cdot\langle u\rangle\right)\right) \\
&= -\lambda\mathbb{E}_{\mathbb{Q}}^{x,y}\left(\exp\left(i\lambda\int_{\mathcal{I}} u_s \cdot dX_s\right)\int_{\mathcal{I}} (u_s - \langle u\rangle)\cdot u_s ds\right) + i(y-x)\cdot\langle u\rangle\Phi(\lambda) \\
&= \left(i(y-x)\cdot\langle u\rangle - \lambda(\langle|u|^2\rangle - |\langle u\rangle|^2)\right)\Phi(\lambda).
\end{aligned}$$

The unique solution of this ordinary differential equation with initial condition $\Phi(0) = 1$ is

$$\Phi(\lambda) = \exp\left(i\lambda(y-x)\cdot\langle u\rangle - \frac{\lambda^2}{2}\left(\langle|u|^2\rangle - |\langle u\rangle|^2\right)\right). \tag{5.5}$$

By Lemma 5.2 the duality (5.4) holds also with respect to $\mathbb{P}^{x,y}$, in particular the above computation implies

$$\begin{aligned}\mathbb{E}^{x,y}\left(\exp\left(i\lambda\int_{\mathcal{I}} u_s\cdot dX_s\right)\right) &= \exp\left(i\lambda(y-x)\cdot\langle u\rangle - \frac{\lambda^2}{2}\left(\langle|u|^2\rangle - |\langle u\rangle|^2\right)\right)\\ &= \mathbb{E}^{x,y}_{\mathbb{Q}}\left(\exp\left(i\lambda\int_{\mathcal{I}} u_s\cdot dX_s\right)\right).\end{aligned}$$

This identifies (5.5) as the characteristic functional of a Brownian bridge: $\mathbb{P}^{x,y} = \mathbb{Q}^{x,y}$. By the disintegration formula (4.13) we deduce $\mathbb{Q} \in \mathcal{R}(\mathbb{P})$. □

In particular an arbitrary $\mathbb{Q}^{x,y}$ with fixed endpoint conditions and integrable increments is the bridge of the Wiener measure $\mathbb{P}^{x,y}$ if and only if the duality formula (5.4) holds for all $F \in \mathcal{S}_d$ and $u \in \mathcal{E}_d$ with $\langle u\rangle = 0$.

5.2. **Characterization of Brownian diffusions.**

We define the law of a Brownian motion with drift and in particular of a Brownian diffusion through its semimartingale decomposition.

Definition 5.6. *Let $b : \mathcal{I}\times C(\mathcal{I},\mathbb{R}^d) \to \mathbb{R}^d$ be a predictable process such that $\mathbb{P}_b(\int_{\mathcal{I}} |b_s|_1 ds < \infty) = 1$. Then $\mathbb{P}_b$ is the law of a Brownian motion with drift b*

$$\text{if} \quad t \mapsto X_t - \int_{[0,t]} b_s ds \quad \text{is a Brownian motion under } \mathbb{P}_b.$$

We say that $\mathbb{P}_b$ is the law of a Brownian diffusion if there exists $b \in C^{1,2}_b(\mathcal{I}\times\mathbb{R}^d,\mathbb{R}^d)$ such that $b_s = b(s, X_s)$.

Remark that analogue to Definitions 1.8, 1.18 or 2.11 a Brownian motion with drift is defined up to the arbitrary initial condition. The law of a Brownian diffusion $\mathbb{P}_b$ is a weak solution of the SDE

$$dX_t = b(t, X_t)dt + dW_t, \quad \text{with initial condition } X_0 \sim \mathbb{P}_{b,0}, \tag{5.7}$$

where W is a Brownian motion. In the sequel of the section we will use the notation

$$W_t := X_t - \int_{[0,t]} b(s, X_s)ds.$$

The terminology of a "diffusion" is not standard in probability theory, see e.g. diffusions as defined in the monograph by Protter [Pro04, Chapter V] and his comments. To justify our definition, let us mention that

- The boundedness of the drift and Novikov's condition ensure that the law of a Brownian diffusion and a Wiener measure with same initial condition are equivalent. Theorem 5.8 provides an explicit form of the density.
- Brownian diffusions are Markov processes, see Remark 5.10.
- We simplify the presentation by choosing b to be bounded with bounded derivatives, even if most results hold in a more general context.

- A Brownian diffusion has nice one-time probabilities and transition probabilities. The densities

$$p_b(s,x)dx := \mathbb{P}_b(X_s \in dx), \text{ and } p_b(s,x;t,y)dy := \mathbb{P}_b(X_t \in dy | X_s = x),$$

exist. For small $\varepsilon > 0$ we have

$$p_b(.,.) \in C_b^{1,3}([\varepsilon,1] \times \mathbb{R}^d, \mathbb{R}_+) \text{ and } p_b(.,.;t,y) \in C_b^{1,3}([0,t-\varepsilon] \times \mathbb{R}^d, \mathbb{R}_+) \text{ for all } (t,y) \in \mathcal{I} \times \mathbb{R}^d$$

The Girsanov theorem is a key tool for the generalization of results presented in § 5.1 to Brownian diffusions. The general Girsanov theorem for semimartingales may be consulted in e.g. the monograph by Jacod and Shiryaev [JS03, Chapter III].

Theorem 5.8. *Let $\mathbb{P}_b$ be the law of a Brownian diffusion, $\mathbb{P}$ a Wiener measure with the same initial law $\mathbb{P}_{b,0} = \mathbb{P}_0$. Then $\mathbb{P}_b$ is equivalent to $\mathbb{P}$ and the density process defined by $\mathbb{P}_b = G_t^b \mathbb{P}$ on $\mathcal{F}_{[0,t]}$ has the explicit form*

$$G_t^b = \exp\left(\int_{[0,t]} b(s,X_s) \cdot dX_s - \frac{1}{2}\int_{[0,t]} |b(s,X_s)|^2 ds\right). \tag{5.9}$$

In particular $G_t^b > 0$ holds $\mathbb{P}$-a.s. for all $t \in \mathcal{I}$.

Using the explicit form of the density we can make sure that Brownian diffusions are Markov processes.

Remark 5.10. *By Lemma 4.4 a Brownian diffusion is a Markov process: For any $t \in \mathcal{I}$ we can factorize*

$$\begin{aligned} \alpha_t &:= \exp\left(\int_{[0,t]} b(s,X_s) \cdot dX_s - \frac{1}{2}\int_{[0,t]} |b(s,X_s)|^2 ds\right) \\ \beta_t &:= \exp\left(\int_{[t,1]} b(s,X_s) \cdot dX_s - \frac{1}{2}\int_{[t,1]} |b(s,X_s)|^2 ds\right), \end{aligned}$$

and then $\mathbb{P}_b = \alpha_t \beta_t \mathbb{P}$, where α_t is $\mathcal{F}_{[0,t]}$-measurable and β_t is $\mathcal{F}_{[t,1]}$-measurable.

Using the Girsanov density (5.9) we derive a duality formula for Brownian diffusions.

Proposition 5.11. *Let $\mathbb{P}_b$ be the law of a Brownian diffusion with drift b. Then the duality formula*

$$\begin{aligned} \mathbb{E}_b\left(F(X)\int_{\mathcal{I}} u_s \cdot dX_s\right) &= \mathbb{E}_b\left(\int_{\mathcal{I}} D_s F(X) \cdot u_s ds\right) \\ &+ \mathbb{E}_b\left(F(X)\left[\int_{\mathcal{I}} u_t \cdot b(t,X_t)dt + \int_{\mathcal{I}} \sum_{i,j=1}^d u_{i,t} \int_{[t,1]} \partial_i b_j(s,X_s) dW_{j,s} dt\right]\right) \end{aligned} \tag{5.12}$$

holds for any $F \in \mathcal{S}_d$ and $u \in \mathcal{E}_d$, where $u = (u_1,\dots,u_d)^t$, $b = (b_1,\dots,b_d)^t$.

Proof. By Theorem 5.8 the law of a Brownian diffusion $\mathbb{P}_b$ is absolutely continuous with respect to a Wiener measure $\mathbb{P}$ with same initial condition. The Girsanov density G_1^b is differentiable in the sense of Definition 2.25: This is proven in the same lines as in Proposition 2.38 since b is bounded. Take $F \in \mathcal{S}_d$, $u \in \mathcal{E}_d$. Since F is bounded the product FG_1^b is differentiable which implies

$$\mathbb{E}\left(G_1^b F(X)\int_{\mathcal{I}} u_s \cdot dX_s\right) = \mathbb{E}\left(G_1^b \int_{\mathcal{I}} D_s F(X) \cdot u_s ds\right) + \mathbb{E}\left(G_1^b F(X)\int_{\mathcal{I}} D_s \log(G_1^b) \cdot u_s ds\right), \tag{5.13}$$

since a product formula and a chain formula hold for the Malliavin derivative, see Lemma 2.28. The last term on the right is

$$\begin{aligned}
&\int_{\mathcal{I}} D_s \log(G_1^b) \cdot u_s ds \\
= &\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left(\int_{\mathcal{I}} b(t, X_t + \varepsilon\langle u\rangle_t)\cdot(dX_t + \varepsilon u_t dt) - \int_{\mathcal{I}} b(t,X_t)\cdot dX_t\right. \\
&\left. -\frac{1}{2}\left(\int_{\mathcal{I}} b(t,X_t+\varepsilon\langle u\rangle_t)\cdot b(t,X_t+\varepsilon\langle u\rangle_t)dt - \int_{\mathcal{I}} b(t,X_t)\cdot b(t,X_t)dt\right)\right) \\
= &\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left(\int_{\mathcal{I}} (b(t,X_t+\varepsilon\langle u\rangle_t) - b(t,X_t))\cdot dX_t + \varepsilon\int_{\mathcal{I}} b(t,X_t+\varepsilon\langle u\rangle_t)\cdot u_t dt\right. \\
&\left. -\frac{1}{2}\left(\int_{\mathcal{I}} (b(t,X_t+\varepsilon\langle u\rangle_t) - b(t,X_t))\cdot b(t,X_t) + (b(t,X_t+\varepsilon\langle u\rangle_t) - b(t,X_t))\cdot b(t,X_t+\varepsilon\langle u\rangle_t)dt\right)\right),
\end{aligned}$$

and we may use a Taylor expansion of the drift b, which implies

$$\begin{aligned}
= &\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\varepsilon\int_{\mathcal{I}}\sum_{i,j=1}^d \partial_i b_j(t,X_t)\langle u_i\rangle_t dX_{j,t} + \varepsilon\int_{\mathcal{I}} b(t,X_t)\cdot u_t dt\right. \\
&\left. -\varepsilon\int_{\mathcal{I}}\sum_{i,j=1}^d \partial_i b_j(t,X_t)\langle u_i\rangle_t b_j(t,X_t)dt + \mathcal{O}(\varepsilon^2)\right) \\
= &\int_{\mathcal{I}} u_t\cdot b(t,X_t)dt + \int_{\mathcal{I}}\sum_{i,j=1}^d u_{i,t}\int_{[t,1]}\partial_i b_j(s,X_s)(dX_{j,s} - b_j(s,X_s)ds)dt,
\end{aligned}$$

in particular we see that $\log G_1^b$ is derivable. Inserting the derivative into (5.13) proves the assertion. □

In [RZ91, Lemme 3] Rœlly and Zessin presented a duality formula that holds for a Brownian motion with drift that has finite entropy with respect to a Wiener measure. They also prove a converse: A Brownian motion with drift is the only continuous process that satisfies the duality formula in the class of processes that have finite entropy with respect to a Brownian motion.

We are able to characterize the law of a Brownian diffusion $\mathbb{P}_b$ as the unique probability satisfying the duality formula (5.15) in the larger class of all continuous semimartingales with integrable increments. These are the optimal assumptions in order to render the duality formula (5.15) well defined, since it contains a stochastic integral.

Theorem 5.14. *Let X be a semimartingale with integrable increments under $\mathbb{Q}$ and $b \in C_b^{1,2}(\mathcal{I}\times\mathbb{R}^d,\mathbb{R}^d)$. If for every $u\in\mathcal{E}_d$ and $F\in\mathcal{S}_d$ the duality formula*

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\left(F(X)\int_{\mathcal{I}} u_s\cdot dX_s\right) = \mathbb{E}_{\mathbb{Q}}\left(\int_{\mathcal{I}} D_s F(X)\cdot u_s ds\right) \\
&+\mathbb{E}_{\mathbb{Q}}\left(F(X)\left[\int_{\mathcal{I}} u_t\cdot b(t,X_t)dt + \int_{\mathcal{I}}\sum_{i,j=1}^d u_{i,t}\int_{[t,1]}\partial_i b_j(s,X_{s-})(dX_{j,s} - b_j(s,X_{s-})ds)dt\right]\right) \qquad (5.15)
\end{aligned}$$

holds, then $\mathbb{Q}$ is the law of a Brownian diffusion with drift b.

Proof. First we remark that the above duality formula is indeed well defined. In particular the processes $s \to \partial_i b_j(s, X_s)$ are bounded and predictable, which renders the stochastic integral on the right integrable with respect to $\mathbb{Q}$ since X has integrable increments.

We prove that $W = X - \int_{[0,.]} b(s, X_s)ds$ is a Brownian motion under $\mathbb{E}_{\mathbb{Q}}$ using the characterization of a Wiener measure contained in Theorem 3.4. For this we extend the duality formula (5.15) to functionals of the type

$$F = f\left(X_{t_1} - \int_{[0,t_1]} b(s, X_s)ds, \dots, X_{t_n} - \int_{[0,t_n]} b(s, X_s)ds\right) = f(W_{t_1}, \dots, W_{t_n}) =: \tilde{F}(W), \tag{5.16}$$

with $f \in C_b^\infty(\mathbb{R}^{nd})$, $\{t_1, \dots, t_n\} \in \Delta_{\mathcal{I}}$ and $n \in \mathbb{N}$. The functional in (5.16) is differentiable in direction $u \in \mathcal{E}_d$, since

$$\begin{aligned} F \circ \theta_u^\varepsilon - F &= f\left(X_{t_1} + \varepsilon\langle u\rangle_{t_1} - \int_{[0,t_1]} b(s, X_s + \varepsilon\langle u\rangle_s)ds, \dots\right) - f\left(X_{t_1} - \int_{[0,t_1]} b(s, X_s)ds, \dots\right) \\ &= \sum_{i=1}^{d}\sum_{j=0}^{n-1} \partial_{i+jd} f\left(X_{t_1} - \int_{[0,t_1]} b(s, X_s)ds, \dots\right)\left(\varepsilon\langle u_i\rangle_{t_j} - \varepsilon\int_{[0,t_j]} \nabla b_i(s, X_s)\cdot\langle u\rangle_s ds\right) + \mathcal{O}(\varepsilon^2), \end{aligned}$$

where $b = (b_1, \dots, b_d)^t$ and the gradient is in direction of the space variables: $\nabla b_i(s, x) = (\partial_1 b_i(s, x), \dots, \partial_d b_i(s, x))^t$. Therefore me may take the $\mathbb{L}^2(\mathbb{Q})$-limit

$$\begin{aligned} &\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}(F \circ \theta_u^\varepsilon - F) = \int_{\mathcal{I}} D_t F \cdot u_t dt \\ &= \sum_{i=1}^{d}\sum_{j=0}^{n-1} \partial_{i+jd} f\left(X_{t_1} - \int_{[0,t_1]} b(s, X_s)ds, \dots\right)\left(\langle u_i\rangle_{t_j} - \int_{[0,t_j]} \nabla b_i(s, X_s)\cdot\langle u\rangle_s dt\right). \end{aligned} \tag{5.17}$$

Using Definition 2.18 we may write $D_u\tilde{F}(W) = \int_{\mathcal{I}} D_t\tilde{F}(W)\cdot u_t dt$ for the first term.

To show that the duality formula (5.15) still holds for the functional F as defined in (5.16) we need to approximate F by a sequence in $\mathcal{S}_d$. Let $(b^{(m)})_{m\geq 1}$ be a sequence of smooth and uniformly bounded functions $b^{(m)} \in C_b^{1,\infty}(\mathcal{I} \times \mathbb{R}^d, \mathbb{R}^d)$ such that $\|b - b^{(m)}\|_\infty \to 0$ and take $((s_0^{(m)}, s_1^{(m)}, \dots, s_m^{(m)}))_{m\geq n+1} \subset \Delta_{\mathcal{I}}$ a sequence of subdivisions of $[0, t_n]$ with $s_0 = 0$ and $s_m = t_n$ such that $\{t_1, \dots, t_n\} \subset \{s_1, \dots, s_m\}$ and $\max_{1\leq i\leq m} |s_i - s_{i-1}|_1 \to 0$ for $m \to \infty$. Define $F_m \in \mathcal{S}_d$ by

$$F_m(\omega) := f\left(\omega(t_1) - \sum_{j:s_j^{(m)}\leq t_1} b^{(m)}(s_{j-1}^{(m)}, \omega(s_{j-1}^{(m)}))(s_j^{(m)} - s_{j-1}^{(m)}), \dots\right)$$

for $m \geq n+1$. By construction $F_m(\omega) \to F(\omega)$ and $D_t F_m(\omega) \to D_t F(\omega)$ converge $\mathbb{Q}$-*a.s.* By dominated convergence (5.15) still holds for F as defined in (5.16). Inserting the derivative

(5.17) into the duality formula (5.15) we get

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\left(\tilde{F}(W)\int_{\mathcal{I}} u_t\cdot dW_t\right)-\mathbb{E}_{\mathbb{Q}}\left(\tilde{F}(W)\int_{\mathcal{I}}\sum_{i=1}^{d}u_{i,t}\int_{[t,1]}\partial_i b(s,X_s)\cdot dW_s dt\right)\\
=\;&\mathbb{E}_{\mathbb{Q}}\left(\tilde{F}(W)\int_{\mathcal{I}} u_t\cdot dW_t\right)-\mathbb{E}_{\mathbb{Q}}\left(\tilde{F}(W)\int_{\mathcal{I}}(\nabla b_1(s,X_s)\cdot\langle u\rangle_s,\dots,\nabla b_d(s,X_s)\cdot\langle u\rangle_s)^t\cdot dW_s\right)\\
=\;&\mathbb{E}_{\mathbb{Q}}\left(\int_{\mathcal{I}} D_tF\cdot u_t dt\right)\\
=\;&\mathbb{E}_{\mathbb{Q}}\left(\int_{\mathcal{I}} D_t\tilde{F}(W)\cdot u_t dt\right)-\mathbb{E}_{\mathbb{Q}}\left(\sum_{i=1}^{d}\sum_{j=0}^{n-1}\partial_{i+jd}f\,(W_{t_1},\dots)\int_{[0,t_j]}\nabla b_i(s,X_s)\cdot\langle u\rangle_s dt\right)\\
=\;&\mathbb{E}_{\mathbb{Q}}\left(\int_{\mathcal{I}} D_t\tilde{F}(W)\cdot u_t dt\right)-\mathbb{E}_{\mathbb{Q}}\left(\int_{\mathcal{I}} D_t\tilde{F}(W)\cdot(\nabla b_1(t,X_t)\cdot\langle u\rangle_t,\dots,\nabla b_d(t,X_t)\cdot\langle u\rangle_t)^t dt\right),
\end{aligned}$$

where the first two equalities are a reorganization of terms in the duality formula (5.15) and the last two equalities are a reorganization of terms using the explicit form of the derivative of F given in (5.17). Comparing the second and the last line, we recognize the duality formula of the Wiener measure with respect to the process W, the cylindric and smooth functional $\tilde{F}\in\mathcal{S}_d$ and the predictable and bounded process

$$s\mapsto(u_{1,s}-\nabla b_1(s,X_s)\cdot\langle u\rangle_s,\dots,u_{d,s}-\nabla b_d(s,X_s)\cdot\langle u\rangle_s)^t,\ u\in\mathcal{E}_d.$$

Using the same proof as for Lemma 2.44 we may extend the duality formula (5.15) from $u\in\mathcal{E}_d$ to processes $u:\mathcal{I}\times C(\mathcal{I},\mathbb{R}^d)\to\mathbb{R}$ that are predictable and bounded. Let $v\in\mathcal{E}_d$ be arbitrary, we want to find a predictable and bounded u such that the following Volterra integral equation holds:

$$u_s=v_s+(\nabla b_1(s,X_s),\dots,\nabla b_d(s,X_s))^t\cdot\int_{[0,s]}u_r dr. \tag{5.18}$$

We may find a solution by fixed-point iteration. Let the iteration start with $u^{(0)}=0$ and $u^{(1)}=v$. The iteration converges since v and ∇b_i are bounded: Indeed for any $\varepsilon>0$

$$\begin{aligned}
\|(u^{n+1}-u^n)\mathbb{1}_{[0,\varepsilon]}\|_\infty &= \|(\nabla b_1(s,X_s),\dots,\nabla b_d(s,X_s))^t\cdot(\langle u^n-u^{n-1}\rangle_\cdot\mathbb{1}_{[0,\varepsilon]})\|_\infty\\
&\le \varepsilon K\|(u^{(n)}-u^{(n-1)})\mathbb{1}_{[0,\varepsilon]}\|_\infty,
\end{aligned}$$

where K is a global bound of the derivatives of b. Moreover the solution u is the uniform limit of predictable processes and thus predictable. Inserting the solution u into (5.15), we see that the duality formula (3.5) holds for the process W under $\mathbb{Q}$. By Theorem 3.4 we conclude that W is a Brownian motion, and by Definition 5.6 we deduce that X is a Brownian diffusion with drift b. □

Note that in the proof of Theorem 5.14 the boundedness of b could be replaced by weaker assumptions, as long as the integral equation (5.18) still has a solution for every $v\in\mathcal{E}_d$.

Example 5.19. *Let $b\in C_b^\infty(\mathcal{I},\mathbb{R}^d)$, such that $\mathbb{P}_b$ is the law of a Brownian diffusion with deterministic drift b. Let X be a semimartingale with integrable increments under $\mathbb{Q}$. By the above theorem $\mathbb{Q}=\mathbb{P}_b$ if and only if $\mathbb{Q}_0=\mathbb{P}_{b,0}$ and the duality formula*

$$\mathbb{E}_{\mathbb{Q}}\left(F(X)\int_{\mathcal{I}}u_s\cdot(dX_s-b(t)dt)\right)=\mathbb{E}_{\mathbb{Q}}\left(\int_{\mathcal{I}}D_sF(X)\cdot u_s ds\right)$$

holds for every $F \in \mathcal{S}_d$ and $u \in \mathcal{E}_d$ with $\langle u \rangle = 0$. In this case the result immediately follows from Theorem 3.4 too.

5.3. **Comparison of reciprocal classes of Brownian diffusions.**

In this section we present an important result by Clark [Cla90], that permits to compare the reciprocal classes of Brownian diffusions by invariants. His result is closely connected to the following semimartingale decomposition of h-transforms of a Brownian diffusion $\mathbb{P}_b$ as computed by Jamison in [Jam75], see also Föllmer [Föl88].

Lemma 5.20. *Let $\mathbb{P}_b$ be the law of a Brownian diffusion with drift $b \in C_b^{1,2}(\mathcal{I} \times \mathbb{R}^d, \mathbb{R}^d)$. An h-transform $h\mathbb{P}_b$ is a Brownian motion with drift*

$$c(t, X_t) = b(t, X_t) + \nabla \log h(t, X_t), \text{ where } h(t, x) = \mathbb{E}_b(h(X_1)|X_t = x). \tag{5.21}$$

In particular $h \in C^{1,2}(\mathcal{I} \times \mathbb{R}^d, \mathbb{R}^d)$ and is bounded with bounded derivatives on $[0, 1 - \varepsilon] \times \mathbb{R}^d$ for arbitrary $\varepsilon > 0$.

Proof. The density process defined by $h\mathbb{P}_b = G_t \mathbb{P}_b$ on $\mathcal{F}_{[0,t]}$ is given by $G_t = \mathbb{E}_b(h(X_1)|X_t)$ since $\mathbb{P}_b$ is Markov. In particular

$$h(t, x) := \mathbb{E}_b(h(X_1)|X_t = x) = \int_{\mathbb{R}^d} h(y)\mathbb{P}_b(X_1 \in dy|X_t = x) = \int_{\mathbb{R}^d} h(y)p_b(t, x; 1, y)dy, \tag{5.22}$$

where $p_b(t, x; 1, y)$ is the transition density of $\mathbb{P}_b$. That $p_b(t, x; 1, y)$ exists, is smooth in (t, x) and locally bounded with bounded derivative on $[0, 1-\varepsilon]$, $\varepsilon > 0$, follows from Hörmander's theorem, see e.g. the famous probabilistic proof by Malliavin in [Mal78]. Thus $h(t, x)$ is smooth and $h(t, x) > 0$ everywhere. We want to write G_1 in the form of a Doléans-Dade exponential. By Itô's formula

$$h(t, X_t) = h(s, X_s) + \int_{[s,t]} \partial_t h(r, X_r)dr + \int_{[s,t]} \nabla h(r, X_r) \cdot dX_r + \frac{1}{2} \int_{[s,t]} \sum_{i=1}^{d} \partial_i^2 h(r, X_r)dr. \tag{5.23}$$

But note that $h(t, x)$ is space time harmonic and thus a solution of the Kolmogoroff backward equation

$$\mathbb{E}_b(h(t, X_t)|X_s) = h(s, X_s) \;\Rightarrow\; \partial_t h(t, x) + b(t, x) \cdot \nabla h(t, x) + \frac{1}{2} \sum_{i=1}^{d} \partial_i^2 h(t, x) = 0, \tag{5.24}$$

and if we insert this into (5.23) we derive the stochastic integral equation

$$G_t = 1 + \int_{[0,t]} G_s \nabla \log h(s, X_s) \cdot (dX_s - b(s, X_s)ds).$$

This equation has the pathwise solution

$$G_t = \exp\left(\int_{[0,t]} \nabla \log h(s, X_s) \cdot (dX_s - b(s, X_s)ds) - \frac{1}{2} \int_{[0,t]} |\nabla \log h(s, X_s)|^2 ds \right).$$

We multiply this density with G_t^b, the density of $\mathbb{P}_b$ with respect to the Wiener measure $\mathbb{P}$ given in Theorem 5.8:

$$G_t G_t^b = \exp\left(\int_{[0,t]} (b(s, X_s) + \nabla \log h(s, X_s)) \cdot dX_s - \frac{1}{2} \int_{[0,t]} |b(s, X_s) + \nabla \log h(s, X_s)|^2 ds \right).$$

Since $h\mathbb{P}_b = G_1 G_1^b \mathbb{P}$ the general Girsanov theorem implies that $h\mathbb{P}_b$ is a Brownian motion with drift $c(t, X_t)$ as specified in (5.21). □

Example 5.25. *Let $\mathbb{P}^0$ be the law of a one-dimensional Brownian motion starting in $X_0 = 0$ and $h(y) = y^2$. Then $\mathbb{E}^0(h(X_1)) = 1$, thus the conditions of Lemma 5.20 are satisfied. Moreover*

$$h(t,y) = \mathbb{E}^0(X_1^2|X_t = y) = \mathbb{E}^0((X_1 - y)^2|X_t = y) + y^2 = y^2 + (1-t).$$

Thus $h\mathbb{P}^0$ is a Brownian motion with drift $c(t,y) = (2y)/(y^2 + (1-t))$. For $t \to 1$ this drift pushes the process away from 0, which is understandable since $h\mathbb{P}^0(X_1 = 0) = 0$. Since h-transforms are in the reciprocal class, we see that the weak solution of the stochastic differential equation

$$dX_t = \frac{2X_t}{X_t^2 + (1-t)}dt + dW_t, \text{ with } X_0 = 0$$

is in the reciprocal class of a Wiener measure.

The fact that $h(t,x)$ is space-time harmonic can be further exploited to compare the reciprocal classes of different Brownian diffusions: The following characterization of reciprocal classes was first developed by Clark in a general framework of diffusions with non-unit diffusion coefficient, see [Cla90]. We present his proof in the context of Brownian diffusions.

Theorem 5.26. *Let $\mathbb{P}_b$ and $\mathbb{P}_c$ be two Brownian diffusions. Then $\mathcal{R}(\mathbb{P}_c) = \mathcal{R}(\mathbb{P}_b)$ if and only if*

(i) the "rotational" invariants coincide: $\Psi_b^{i,j}(t,y) = \Psi_c^{i,j}(t,y)$ where

$$\Psi_b^{i,j}(t,y) = \partial_i b_j(t,y) - \partial_j b_i(t,y),\ i,j \in \{1,\dots,d\}; \tag{5.27}$$

(ii) the "harmonic" invariants coincide: $\Xi_b(t,y) = \Xi_c(t,y)$ where

$$\Xi_b^i(t,y) = \partial_t b_i(t,y) + \sum_{j=1}^d b_j \partial_i b_j(t,y) + \frac{1}{2}\sum_{j=1}^d \partial_i\partial_j b_j(t,y),\ i \in \{1,\dots,d\}. \tag{5.28}$$

Proof. Assume that $\mathbb{P}_c$ is in $\mathcal{R}(\mathbb{P}_b)$. Without loss of generality we let both processes start in $x \in \mathbb{R}^d$. Since $\mathbb{P}_c^x$ is absolutely continuous with respect to $\mathbb{P}_b^x$ and has the same bridges it is an h-transform: There exists a function $h : \mathbb{R}^d \to \mathbb{R}_+$ such that $\mathbb{P}_c = h\mathbb{P}_b$. By Lemma 5.20 the drift c is of the form $c = b + \nabla \log h$, where h is a space-time harmonic function. Then *(i)* is equivalent to

$$\partial_i\partial_j \log h(t,y) - \partial_j\partial_i \log h(t,y) = 0,$$

which is known as Clairaut's theorem or Schwarz's theorem. Assertion *(ii)* stems from the fact, that h is space-time harmonic. Indeed, using the Kolmogoroff backward equation (5.24) on $h(t,y) = e^{\psi(t,y)}$ gives

$$\partial_t\psi + b\cdot\nabla\psi + \frac{1}{2}\sum_{j=1}^d \partial_j^2\psi + \frac{1}{2}\sum_{j=1}^d (\partial_j\psi)^2 = 0.$$

We take the partial derivative ∂_i and use $\partial_j\psi = c_j - b_j$ for all $1 \le j \le d$:

$$\begin{aligned} 0 \;=\; & \partial_t(c_i - b_i) + \sum_{j=1}^d (\partial_i b_j(c_j - b_j) + b_j(\partial_i c_j - \partial_i b_j)) \\ & + \frac{1}{2}\sum_{j=1}^d \partial_i\partial_j(c_j - b_j) + \sum_{j=1}^d (c_j - b_j)(\partial_i c_j - \partial_i b_j), \end{aligned}$$

which properly arranged is *(ii)*.

For the converse, assume that $\mathbb{P}_b^x$ and $\mathbb{P}_c^x$ are Brownian diffusions with drift b respectively c such that *(i)* and *(ii)* hold. Condition *(i)* implies that there exists a function $\psi \in C^1(\mathbb{R}^d)$ such

that $\nabla\psi(t,y) = c(t,y) - b(t,y)$: By the Poincaré lemma closed forms are exact on $\mathbb{R}^d$, $t \in \mathcal{I}$ being fixed. We may add to $\psi(t,y)$ any $\phi \in C^1(\mathcal{I})$ and condition *(ii)* still holds. The choice of ϕ is specific and comes from the following considerations. Define $h(t,y) := e^{\phi(t)+\psi(t,y)}$, then h is a solution of the Kolmogoroff backward equation (5.24) if

$$\partial_t\psi(t,y) + \sum_{i=1}^d b_i(t,y)\partial_i\psi(t,y) + \frac{1}{2}\sum_{i=1}^d \partial_i^2\psi(t,y) + (\partial_i\psi(t,y))^2 = -\partial_t\phi(t). \tag{5.29}$$

If we derive the left side by ∂_j for $1 \le j \le d$ we can see with $\partial_j\psi = c_j - b_j$ and condition *(ii)* that the left side indeed does not depend on $y \in \mathbb{R}^d$. We may chose ϕ such that (5.29) holds and moreover $\mathbb{E}_b^x(h(1,X_1)) = 1$. This normalization is possible, since if we insert the Kolmogoroff backward equation satisfied by $h(t,y)$ into the Itô-formula (5.23) we see that

$$\begin{aligned} h(t,X_t) &= h(0,x) + \int_{[0,t]} h(s,X_s)\,\nabla\log h(s,X_s)\cdot(dX_s - b(s,X_s)ds) \\ &= h(0,x) + \int_{[0,t]} h(s,X_s)(c(s,X_s) - b(s,X_{s-}))\cdot(dX_s - b(s,X_s)ds). \end{aligned}$$

Thus $h(t,X_t)$ is integrable and can *a-priori* be normalized. But then Lemma 5.20 implies that $h(1,X_1)\mathbb{P}_b^x$ is a diffusion with drift c, and by uniqueness of the associated martingale problem we get $\mathbb{P}_c^x = h(1,X_1)\mathbb{P}_b^x$. We have shown that the Brownian diffusion $\mathbb{P}_c^x$ is an h-transform of $\mathbb{P}_b^x$ therefore *a-fortiori* in the reciprocal class $\mathcal{R}(\mathbb{P}_b)$. □

Example 5.30. *Let $\mathbb{P}_b$ and $\mathbb{P}_c$ be two Brownian diffusions with deterministic drifts $b, c \in C_b^1(\mathcal{I},\mathbb{R}^d)$. Following the "harmonic" condition (ii) we have $\mathcal{R}(\mathbb{P}_b) = \mathcal{R}(\mathbb{P}_c)$ if and only if $t \mapsto b(t) - c(t)$ is constant.*

The invariants (5.27) and (5.28) appear in a duality formula that characterizes the reciprocal class of a Brownian diffusion.

5.4. **Characterization of the reciprocal class $\mathcal{R}(\mathbb{P}_b)$ by a duality formula.**

In this paragraph we unify the results of § 5.2 and § 5.3 to present a characterization of the reciprocal class of a Brownian diffusion by a duality formula. Let us first present a new complementary result of technical nature concerning the definition of stochastic integrals.

Proposition 5.31. *Let $\mathbb{P}_b$ be a Brownian diffusion, $\mathbb{Q} \in \mathcal{R}(\mathbb{P}_b)$ be arbitrary. Then $\mathbb{Q}$ is a semimartingale.*

Proof. Take any $x, y \in \mathbb{R}^d$, we show that $\mathbb{P}_b^{x,y}$ is a semimartingale first. It is well known, that a Brownian bridge is a Brownian motion with drift, and thus a semimartingale. Now for any bounded $\phi, \psi : \mathbb{R}^d \to \mathbb{R}$

$$\begin{aligned} \mathbb{E}_b\left(\mathbb{E}_b(\,.\,|X_0,X_1)\phi(X_0)\psi(X_1)\right) &= \mathbb{E}\left(\mathbb{E}_b(\,.\,|X_0,X_1)\mathbb{E}(G^b\,|\,X_0,X_1)\phi(X_0)\psi(X_1)\right) \\ &= \mathbb{E}\left(\mathbb{E}\,(\,.\,G^b\,|\,X_0,X_1)\phi(X_0)\psi(X_1)\right) \end{aligned}$$

and thus

$$\mathbb{E}_b^{x,y}(\,.\,) = \frac{\mathbb{E}^{x,y}(\,.\,G^b)}{\mathbb{E}^{x,y}(G^b)}.$$

Therefore $\mathbb{P}_b^{x,y} \ll \mathbb{P}^{x,y}$ and by Girsanov's theorem $\mathbb{P}_b^{x,y}$ is the law of a semimartingale too.

To show that an arbitrary mixture of bridges in the sense of the disintegration formula (4.13) is still a semimartingale, we use a definition of semimartingales as introduced by

Protter [Pro04]: We say that $H : \mathcal{I} \times C(\mathcal{I}, \mathbb{R}^d) \to \mathbb{R}^d$ is a simple integrand if it is of the form

$$H = H_0 \mathbb{1}_{\{0\}} + \sum_{i=1}^{n} H_i \mathbb{1}_{(T_i, T_{i+1}]},$$

where $0 = T_1 \leq \cdots \leq T_{n+1} < \infty$ are stopping times, and $H_i \in \mathcal{F}_{[0,T_i]}$, $\forall 1 \leq i \leq n+1$. For any simple integrand $H = H_0 \mathbb{1}_{\{0\}} + \sum_{i=1}^{n} H_i \mathbb{1}_{(T_i, T_{i+1}]}$ we may define the natural real valued integral

$$\int_{\mathcal{I}} H_t \cdot dX_t := H_0 \cdot X_0 + \sum_{i=1}^{n} H_i \cdot (X_{T_{i+1}} - X_{T_i}).$$

Let $(H^{(i)})_{i \geq 1}$ a sequence of simple integrands converging to the simple integrand H in the sense that $\sup_{(t,\omega)} |H^{(i)}(t,\omega) - H(t,\omega)|_1$ converges to zero for $i \to \infty$. Then $\mathbb{Q}$ is a semimartingale if any only if the integrals converge in probability:

$$\lim_{i \to \infty} \mathbb{Q}\left(A_i(\varepsilon)\right) = 0, \text{ for any } \varepsilon > 0, \text{ where } A_i(\varepsilon) := \left\{ \left| \int_{\mathcal{I}} H_t^{(i)} \cdot dX_s - \int_{\mathcal{I}} H_t \cdot dX_t \right|_1 > \varepsilon \right\}.$$

We use the disintegration (4.13) to show that

$$\begin{aligned} \lim_{i \to \infty} \mathbb{Q}(A_i(\varepsilon)) &= \lim_{i \to \infty} \mathbb{E}_{\mathbb{Q}} \left(\mathbb{E}_b \left(\mathbb{1}_{A_i(\varepsilon)} \,\middle|\, X_0, X_1 \right) \right) \\ &= \mathbb{E}_{\mathbb{Q}} \left(\lim_{i \to \infty} \mathbb{E}_b \left(\mathbb{1}_{A_i(\varepsilon)} \,\middle|\, X_0, X_1 \right) \right) = 0, \end{aligned}$$

where the second equation holds by bounded convergence. □

Now we prove that every process with integrable increments in the reciprocal class $\mathcal{R}(\mathbb{P}_b)$ satisfies a duality formula which is expressed in terms of the reciprocal invariants defined in Theorem 5.26. Following the proof of Lemma 5.2 and the well definedness of the stochastic integral by Proposition 5.31 we can drop a finite entropy assumption used by Rœlly and Thieullen in [RT05].

Proposition 5.32. *Let $\mathbb{Q} \in \mathcal{R}(\mathbb{P}_b)$ be such that X has integrable increments. Then the duality formula*

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}} \left(F(X) \int_{\mathcal{I}} u_t \cdot dX_t \right) &= \mathbb{E}_{\mathbb{Q}} \left(\int_{\mathcal{I}} D_t F(X) \cdot u_t dt \right) \\ &\quad - \mathbb{E}_{\mathbb{Q}} \left(F(X) \int_{\mathcal{I}} \sum_{i=1}^{d} u_{i,t} \left[\int_{[t,1]} \sum_{j=1}^{d} \Psi_b^{i,j}(s, X_s) dX_{j,s} \right.\right. \\ &\quad \left.\left. + \int_{[t,1]} \left(\Xi_b^i(s, X_s) + \frac{1}{2} \sum_{j=1}^{d} \partial_j \Psi_b^{i,j}(s, X_s) \right) ds \right] dt \right), \end{aligned} \tag{5.33}$$

holds for all $F \in \mathcal{S}_d$ and $u \in \mathcal{E}_d$ with $\langle u \rangle = 0$.

Proof. Remark, that the above formula is indeed well defined, since $\mathbb{Q}$ is a semimartingale and the characteristics are bounded integrands. We start our derivation with the duality formula (5.15) under $\mathbb{P}_b$. We exchange

$$\int_{[t,1]} \partial_i b_j(s, X_s) dX_{j,s} = \int_{[t,1]} \Psi_b^{i,j}(s, X_s) dX_{j,s} + \int_{[t,1]} \partial_j b_i(s, X_s) dX_{j,s}.$$

The last term also appears in the Itô-expansion of $b(t, X_t)$:

$$\begin{aligned} b(t,X_t) &= b(1,X_1) - \int_{[t,1]} (\partial_t b_1(s,X_s),\dots,\partial_t b_d(s,X_s))^t ds \\ &\quad - \int_{[t,1]} \sum_{j=1}^d (\partial_j b_1(s,X_s),\dots,\partial_j b_d(s,X_s))^t dX_{j,s} \\ &\quad -\frac{1}{2}\int_{[t,1]} \sum_{j=1}^d (\partial_j^2 b_1(s,X_s),\dots,\partial_j^2 b_d(s,X_s))^t ds. \end{aligned}$$

We insert this expansion into (5.15) and recognize the reciprocal invariants defined in (5.27) and (5.28):

$$\begin{aligned} &\mathbb{E}_b\left(F(X)\left[\int_{\mathcal{I}} u_t\cdot b(t,X_t)dt + \int_{\mathcal{I}}\sum_{i,j=1}^d u_{i,t}\int_{[t,1]}\partial_i b_j(s,X_s)(dX_{j,s}-b_j(s,X_s)ds)dt\right]\right) \\ = \;& \mathbb{E}_b\left(F(X)\left[\int_{\mathcal{I}}\sum_{i=1}^d u_{i,t}\left[b_i(1,X_1) - \int_{[t,1]}\sum_{j=1}^d \Psi_b^{i,j}(s,X_s)dX_{j,s}\right.\right.\right. \\ &\left.\left.\left. - \int_{[t,1]}\left(\Xi_b^i(s,X_s) + \frac{1}{2}\sum_{j=1}^d \partial_j\Psi_b^{i,j}(s,X_s)\right)ds\right]dt\right]\right). \end{aligned}$$

As in the proof of Lemma 5.2 we apply this identity to the functional $F(X)\phi(X_0)\psi(X_1)$ for $F \in \mathcal{S}_d$ and $\phi, \psi \in C_b^\infty(\mathbb{R}^d)$ and $u \in \mathcal{E}_d$ such that $\langle u\rangle = 0$. This shows that (5.33) holds for all bridges $\mathbb{P}_b^{x,y}$. Using the disintegration (4.13) one obtains that (5.33) is indeed true for all $\mathbb{Q} \in \mathcal{R}(\mathbb{P}_b)$. □

We end this paragraph with a partial converse to the above proposition as presented by Rœlly and Thieullen in [RT05]. For the sake of completeness we quote their result including all the hypotheses: They characterize the elements of $\mathcal{R}(\mathbb{P}_b)$ in a class of probabilities on $C(\mathcal{I},\mathbb{R}^d)$ that satisfy regularity assumptions. In particular, they assume that $\mathbb{Q}$ is a probability on $C(\mathcal{I},\mathbb{R}^d)$ such that

- $\mathbb{Q}$ *has finite entropy with respect to a Wiener measure.*
- $\sup_{t\in\mathcal{I}}|X_t|_1 \in \mathbb{L}^1(\mathbb{Q})$.
- *Conditional density: Regularity and domination.*
  - *(a)* $\forall 0 \le t < u < 1$, $\forall (x,y)\in\mathbb{R}^{2d}$, *there exists a function* q *such that*
    $$\mathbb{Q}(X_r \in dz | X_t = x, X_1 = y) = q(t,x;r,z;1,y)dz.$$
  - *(b)* $\forall 0 < r < 1$, $\forall (x,y)\in\mathbb{R}^{2d}$, $q(0,x;r,z;1,y) > 0$.
  - *(c)* $\forall \varepsilon > 0$, $\forall (s,x)\in[0,1-\varepsilon]\times\mathbb{R}^d$, *there exists a neighborhood* V *of* (s,x) *and a function* $\phi_V(r,z,1,y)$ *such that whenever* ∂_α *denotes* ∂_s, ∂_i *or* ∂_{ij}^2 *for* $i,j\in\{1,\dots,d\}$ *it holds:* $\sup_{(s',x')\in V}|\partial_\alpha q(s',x';r,z;1,y)| \le \phi_V(r,z,1,y)$, *and* $\int_{[0,1-\varepsilon]}\int_{\mathbb{R}^d}(1+|z|^2)\phi_V(r,z,1,y)(1+\frac{\phi_V(r,z,1,y)}{q(0,x',r,z,1,y)})dzdr < +\infty$
- *Integrability condition on the derivatives of the conditional density.*
  Let $0 \le s \le 1-\varepsilon$.
  - *(a)* $\int_{[s,1-\varepsilon]}\int_{\mathbb{R}^d}|\partial_\alpha q(s,X_s;r,z;1,X_1)|(1+|z|^2)dzdr \in \mathbb{L}^1(\mathbb{Q})$, *where* ∂_α *denotes* ∂_s *or* ∂_{ij} *for* $1\le i,j\le d$.
  - *(b)* $\int_{[s,1-\varepsilon]}\int_{\mathbb{R}^d}|\partial_i q(s,X_s;r,z;1,X_1)|(1+|z|^2)dzdr \in \mathbb{L}^2(\mathbb{Q})$.

(c) $\int_{[s,1-\varepsilon]}\int_{\mathbb{R}^d}(1+|z|^2)\frac{\partial_\alpha q(s,X_s,r,z,1,X_1)^2}{q(0,x,r,z,1,X_1)}dzdr \in \mathbb{L}^1(\mathbb{Q})$, *where* ∂_α *denotes* ∂_s, ∂_i *or* ∂_{ij} *for* $1 \le i,j \le d$.

These hypotheses exclude e.g. the bridges $\mathbb{P}_b^{x,y}$.

Theorem 5.34. *Let* $\mathbb{Q}$ *be a probability measure on* $C(\mathcal{I},\mathbb{R}^d)$ *such that the above assumptions hold. If for every* $F \in \mathcal{S}_d$, $u \in \mathcal{E}_d$ *with* $\langle u\rangle = 0$ *the duality formula (5.33) holds, then* $\mathbb{Q}$ *is in the reciprocal class* $\mathcal{R}(\mathbb{P}_b)$.

In the last two paragraphs of this section we apply and interpret this characterization result. Let us first present a continuation of Examples 5.19 and 5.30.

Example 5.35. *Let* $\mathbb{P}_b$ *be the law of a Brownian diffusion with deterministic drift* $b \in C_b^1(\mathcal{I},\mathbb{R}^d)$. *Then the above duality formula (5.33) reduces to*

$$\mathbb{E}_b\left(F(X)\int_{\mathcal{I}} u_t\cdot dX_t\right) = \mathbb{E}_b\left(\int_{\mathcal{I}} D_tF(X)\cdot u_t dt\right) - \mathbb{E}_b\left(F(X)\int_{\mathcal{I}}\sum_{i=1}^d u_{i,t}\int_{[t,1]}\partial_t b_i(s)dsdt\right).$$

Since $b(1)-b(t) = \int_{[t,1]}\partial_t b(s)ds$, *this is equivalent to the duality formula (5.4) that holds for the process* $t \mapsto X_t - \int_{[0,t]} b(s)ds$. *By Theorem 5.3 the above duality characterizes the reciprocal class* $\mathcal{R}(\mathbb{P}_b)$ *in the class of all probabilities* $\mathbb{Q}$ *on* $C(\mathcal{I},\mathbb{R}^d)$ *with integrable increments.*

5.5. **A physical interpretation of the reciprocal invariants.**

The origin, and an important inspiration for the development of the theory of reciprocal classes, was an idea of Schrödinger [Sch32]: He remarks certain analogies between the computation of transition probabilities in a reciprocal class and the probabilistic interpretation of the wave function in quantum dynamics:

> *"Il s'agit d'un problème classique: problème de probabilités dans la théorie du mouvement brownien. Mais en fin de compte, il ressortira une analogie avec la mécanique ondulatoire, qui fut si frappante pour moi lorsque je l'eus trouvée, qu'il m'est difficile de la croire purement accidentelle."*

Schrödinger introduces a time-symmetric formulation of diffusion problems that are usually posed in the form of Fokker-Planck equations:

> *"[..] on peut dire qu'aucune des deux directions du temps n'est privilégiée."*

His considerations can be summarized as follows: The motion of a particle in a thermal reservoir in the absence of external forces is described by the law of a Brownian motion $\mathbb{P}$. He interprets this well accepted model as a special solution of a diffusion problem with fixed endpoint distributions. If instead of the endpoint measure $\mathbb{P}_{01}$ we want to prescribe another boundary distribution μ_{01} on the particle, the physical meaningful motion of the particle will be described by the unique element of $\mathcal{R}(\mathbb{P})$ with boundary distribution μ_{01}. His choice of dynamics is based on a consideration on the statistics of a large number of particles, essentially a large deviation argument, an idea made rigorous by Föllmer [Föl88].

In this paragraph we want to generalize the original considerations of Schrödinger to particles moving in a thermal reservoir under the influence of an external electromagnetic field in dimension $d = 3$. Our presentation is more formal than strict, allthough we tried to present the results in a conclusive mathematical frame. We provide an interpretation of the reciprocal invariants (5.27) and (5.28) of a Brownian diffusion in terms of the electric and magnetic field in Proposition 5.71. Using this interpretation of the invariants we propose a new stochastic formulation of Newton's equation using the duality formula (5.33) in

Remark 5.72. As a continuation of this idea we can show that our stochastic dynamics are time-reversible in the same sense as the deterministic dynamics controlled by a Newton equation, see Proposition 5.84.

Of course our model is highly idealized, we do not claim that any effective physical behavior of diffusing particles is described by our results. The physical picture of our model may rather serve as an analogy to the formulation of quantum mechanics, in a sense proposed by Zambrini and other authors: The setting of a diffusing particle in an electromagnetic field has been examined among others by Zambrini, Cruzeiro [CZ91] and Lévy, Krener [LK93]. Both approaches are inspired by formal analogies to quantum mechanics, but their construction leads to the same type of processes and reciprocal classes as our construction that is purely based on analogies to classical mechanics. In particular Lévy and Krener were able to identify the reciprocal invariants of a Brownian diffusion with the electromagnetic field by a short-time expansion of reciprocal transition probabilities, see also Proposition 5.76. Let us note that the concept of stochastic mechanics as introduced by Nelson seems to lead to different kinds of processes for a similar physical problem. This is due to his alternative definition of the velocity and acceleration of a diffusing process, see e.g. Nelson's monograph [Nel85, Paragraph 14].

Our results are based on a stochastic control approach under boundary restrictions developed by Wakolbinger in [Wak89]. Let us note that a similar control approach has been studied by Guerra, Morato in [GM83] for particles under the influence of an electric field. Their results are based on different methods than Wakolbinger's and do not underline the role of the reciprocal invariants.

5.5.1. *A particle in an electromagnetic field.*

In this paragraph we describe the deterministic motion of a particle with unit mass and unit charge in an electromagnetic field. There are several ways to describe such a dynamical problem in classical mechanics. We use the formulation of Lagrangian mechanics as a variational problem, which we then reformulate as a control problem with boundary restrictions.

The trajectory of a particle is described by a function $\omega \in C^2(\mathcal{I}, \mathbb{R}^3)$, such that the

$$\text{velocity: } \dot{\omega} := \frac{d}{dt}\omega, \text{ and the acceleration: } \ddot{\omega} := \frac{d^2}{dt^2}\omega$$

are well defined. All admissible trajectories given endpoints $x, y \in \mathbb{R}^3$ are contained in

$$\Gamma(x, y) := \left\{\omega \in C^2(\mathcal{I}, \mathbb{R}^3) : \ \omega(0) = x \text{ and } \omega(1) = y\right\}.$$

The Lagrangian of a particle with unit mass and unit charge in an electromagnetic field is defined by

$$L(\omega(t), \dot{\omega}(t), t) = \frac{1}{2}|\dot{\omega}(t)|^2 + \dot{\omega}(t) \cdot A(t, \omega(t)) - \Phi(t, \omega(t)),$$

where $A \in C_b^{1,1}(\mathcal{I}\times\mathbb{R}^3, \mathbb{R}^3)$ is the vector potential of the magnetic field and $\Phi \in C_b^{1,1}(\mathcal{I}\times\mathbb{R}^3, \mathbb{R})$ is the scalar potential of the electric field. In particular we can compute the

$$\text{magnetic field:} \quad B(t, z) := \quad (\partial_2 A_3 - \partial_3 A_2, \partial_3 A_1 - \partial_1 A_3, \partial_1 A_2 - \partial_2 A_1)^t(t, z); \tag{5.36}$$

$$\text{electric field:} \quad E_i(t, z) := \quad -\partial_i \Phi(t, z) - \partial_t A_i(t, z), \quad i = 1, 2, 3. \tag{5.37}$$

The Hamiltonian action functional associated to the Lagrangian of the motion in an electromagnetic field is

$$\mathcal{J}(\omega) := \int_{\mathcal{I}} L(\omega(t), \dot{\omega}(t), t)dt = \int_{\mathcal{I}} \Big(\frac{1}{2}|\dot{\omega}(t)|^2 + \dot{\omega}(t) \cdot A(t, \omega(t)) - \Phi(t, \omega(t))\Big)dt. \tag{5.38}$$

We may now state the general Hamiltonian principle of stationary action.

Definition 5.39. *The effective trajectory of the particle moving from a point $x \in \mathbb{R}^3$ to $y \in \mathbb{R}^3$ in an electromagnetic field defined in (5.36) and (5.37) is an extremal of the action functional (5.38) in the class of all admissible trajectories $\omega \in \Gamma(x, y)$.*

To state this physical problem as a deterministic control problem with boundary restrictions let us assume that the motion of the particle is controlled: The trajectory of a particle is described by a function $\omega_c \in C^2(\mathcal{I}, \mathbb{R}^3)$ that is the solution of an ordinary differential equation of the form

$$\frac{d}{dt}\omega_c(t) = c(t, \omega_c(t)), \text{ where } c \in C_b^{1,1}(\mathcal{I} \times \mathbb{R}^3, \mathbb{R}^3) \text{ is the control vector.} \tag{5.40}$$

The control condition implies

$$\dot{\omega}_{c,i}(t) = \quad c_i(t, \omega_c(t)), \quad i = 1, 2, 3, \tag{5.41}$$

$$\ddot{\omega}_{c,i}(t) = \quad \partial_t c_i(t, \omega_c(t)) + \sum_{j=1}^{3} c_j(t, \omega_c(t))\partial_j c_i(t, \omega_c(t)), \quad i = 1, 2, 3. \tag{5.42}$$

Since we want to describe the motion of the particle with given boundary conditions, we collect all control vectors defining trajectories that have the same endpoint values $x, y \in \mathbb{R}^3$ in

$$\Gamma_c(x, y) := \Big\{c \in C_b^{1,1}(\mathcal{I} \times \mathbb{R}^d, \mathbb{R}^d) : \ \omega_c(0) = x \text{ and } \dot{\omega}_c(t) = c(t, \omega_c(t)) \ \Rightarrow \ \omega_c(1) = y\Big\}. \tag{5.43}$$

We may now state the Hamiltonian principle of stationary action as a control problem, the boundary restriction is expressed through the restriction on the set of admissible controls $\Gamma_c(x, y)$.

Definition 5.44. *A control vector $c \in \Gamma_c(x, y)$ is called an optimal control if the trajectory ω_c is an extremal of the action (5.38) in the class of all admissible trajectories $\omega \in \Gamma(x, y)$.*

We are going to see, that the effective trajectory $\omega \in C^2(\mathcal{I}, \mathbb{R}^3)$ is unique, see also e.g. the introduction by Arnold [Arn78]. The optimal control is unique in the following sense.

Remark 5.45. *Let $c \in \Gamma_c(x, y)$ be an optimal control, then it is unique in the following sense: If $\tilde{c} \in \Gamma_c(x, y)$ is another optimal control, then $\omega_c = \omega_{\tilde{c}}$ if $\omega_c(0) = \omega_{\tilde{c}}(0) = x$, since the minimizing trajectory is unique.*

Deriving a solution of the ensuing Lagrange equations of motion gives a sufficient condition on a control vector $c \in \Gamma_c(x, y)$ to be optimal. Thus the next result is a reformuation of the Euler-Lagrange equations in terms of "reciprocal invariants".

Proposition 5.46. *Let $x, y \in \mathbb{R}^3$ be arbitrary boundary values. The control vector $c \in \Gamma_c(x, y)$ is optimal if it has*

(i) the "rotational" invariant: $\Big(\Psi_c^{3,2}(s, \omega_c(s)), \Psi_c^{1,3}(s, \omega_c(s)), \Psi_c^{2,1}(s, \omega_c(s))\Big)^t = B(s, \omega_c(s))$ where

$$\Psi_c^{i,j}(s, z) := \partial_i c_j(s, z) - \partial_j c_i(s, z), \quad i, j = 1, 2, 3; \tag{5.47}$$

(ii) the "harmonic" invariant: $\left(\Xi_c^1(s,\omega_c(s)),\Xi_c^2(s,\omega_c(s)),\Xi_c^3(s,\omega_c(s))\right)^t = E(s,\omega_c(s))$*, where*

$$\Xi_c^i(s,z) := \partial_t c_i(s,z) + \sum_{j=1}^3 c_j(s,z)\partial_i c_j(s,z), \quad i=1,2,3. \tag{5.48}$$

Proof. This is just a reformulation of the Lagrange equations of motion. It is well known, that an admissible path $\omega \in \Gamma(x,y)$ is a critical value of the action functional (5.38) if the Lagrange equations of motion are satisfied:

$$\begin{aligned} 0 = &\quad \frac{d}{dt}\frac{\partial}{\partial\dot\omega_i}L(\omega,\dot\omega,t) - \frac{\partial}{\partial\omega_i}L(\omega,\dot\omega,t), \quad i=1,2,3 \\ \Leftrightarrow \quad 0 = &\quad \ddot\omega_i + \partial_t A_i + \sum_{j=1}^3 \dot\omega_j\partial_j A_i - \sum_{j=1}^3 \dot\omega_j\partial_i A_j + \partial_i\Phi, \quad i=1,2,3. \end{aligned}$$

For a controlled trajectory ω_c with $c\in\Gamma_c(x,y)$ we insert (5.41) and (5.42) to get

$$\begin{aligned} \partial_t c_i + \textstyle\sum_{j=1}^3 c_j\partial_j c_i = &\quad -\partial_t A_i - \sum_{j=1}^3 c_j\partial_j A_i + \sum_{j=1}^3 c_j\partial_i A_j - \partial_i\Phi, \quad i=1,2,3, \\ \Leftrightarrow \quad \partial_t c_i + \textstyle\sum_{j=1}^3 c_j\partial_i c_j = &\quad \sum_{j=1}^3 c_j\left(\partial_i c_j - \partial_j c_i + \partial_i A_j - \partial_j A_i\right) - \partial_t A_i - \partial_i\Phi, \quad i=1,2,3, \end{aligned}$$

where all functions are evaluated in $(s,\omega_c(s))$. A term-by-term comparison with the definition of the electromagnetic field gives the result. □

By assumptions on the electromagnetic potentials the Lagrange equations have a unique solution, and since the action functional is bounded from below we may presume that this solution is even the unique minimizer of the action functional.

In analogy to Theorem 5.26, the solutions of the deterministic optimal control problem for given electromagnetic potentials A and Φ and varying boundary conditions $x,y\in\mathbb{R}^3$ are characterized as elements of the same "reciprocal class". Condition *(i)* on the invariant in the above Proposition 5.46 states that $\Psi_c^{i,j}(s,\omega_c(s)) = \Psi_{-A}^{i,j}(s,\omega_c(s))$ for $i,j=1,2,3$. We would like to formulate condition *(ii)* in a similar way, to get an identity of the harmonic invariant Ξ_c^i to the harmonic invariant of a reference control $b\in C_b^{1,1}(\mathcal{I}\times\mathbb{R}^3,\mathbb{R}^3)$. This reference control cannot be $-A$, since generally $\Xi_{-A}^i(t,z) \neq -\partial_t A_i(t,z) - \partial_i\Phi(t,z)$ would not imply the Lagrange equation of motion for the controlled trajectory. Instead we use the following ansatz

$$b(t,z) = -A(t,z) + \nabla\psi(t,z),$$

where $\psi\in C^{1,2}(\mathcal{I}\times\mathbb{R}^3,\mathbb{R}^3)$, since in this case $\Psi_c^{i,j}(s,\omega_c(s)) = \Psi_b^{i,j}(s,\omega_c(s))$, $i,j=1,2,3$, that is, condition *(i)* still holds. The Lagrange equation of motion

$$\Xi_c^i(s,\omega_c(s)) = \sum_{j=1}^3 c_j\left(\Psi_c^{i,j}(s,\omega_c(s)) - \Psi_b^{i,j}(s,\omega_c(s))\right) - \partial_t A_i(s,\omega_c(s)) - \partial_i\Phi(s,\omega_c(s)), \quad i=1,2,3$$

holds if we find ψ such that

$$\Xi_c^i(s,\omega_c(s)) = \Xi_b^i(s,\omega_c(s)) = -\partial_t A_i(s,\omega_c(s)) - \partial_i\Phi(s,\omega_c(s)),\ i=1,2,3.$$

This is implied if ψ is the solution of the partial differential equation

$$-\partial_t A_i - \partial_i \Phi = -\partial_t A_i + \partial_t \partial_i \psi + \sum_{j=1}^{3} (-A_j + \partial_j \psi)\partial_i(-A_j + \partial_j \psi)$$

$$\Leftrightarrow \quad 0 = \partial_i \left(\partial_t \psi + \frac{1}{2} \sum_{j=1}^{3} (-A_j + \partial_j \psi)^2 + \Phi \right)$$

globally on $(s, y) \in \mathcal{I} \times \mathbb{R}^3$. Since Φ is defined up to a constant we may assume that the term in the bracket is already zero. Now we reformulate Proposition 5.46 using a reference control vector.

Proposition 5.49. *Assume that there exists a solution $\psi \in C^{1,2}(\mathcal{I} \times \mathbb{R}^3, \mathbb{R}^3)$ of the non-linear partial differential equation*

$$0 = \partial_t \psi + \frac{1}{2} \sum_{j=1}^{3} (-A_j + \partial_j \psi)^2 + \Phi, \tag{5.50}$$

such that $b(t, z) := -A(t, z) + \nabla\psi(t, z) \in C_b^{1,1}(\mathcal{I} \times \mathbb{R}^3, \mathbb{R}^3)$ is regular enough. Then $c \in \Gamma_c(x, y)$ is optimal if

(i) the "rotational" invariants coincide: $\Psi_c^{i,j}(s, \omega_c(s)) = \Psi_b^{i,j}(s, \omega_c(s))$, $i, j = 1, 2, 3$;

(ii) the "harmonic" invariants coincide: $\Xi_c^{i,j}(s, \omega_c(s)) = \Xi_b^{i,j}(s, \omega_c(s))$, $i, j = 1, 2, 3$.

Of course the conditions in Proposition 5.46 and 5.49 are only sufficient and do not guarantee the existence of a controlled trajectory ω_c that minimizes (5.38). The existence depends on the solvability of the non-linear partial differential equation (5.50) and the regularity of its solution.

Before treating the same dynamical problem for a particle in a thermal reservoir, let us make some remarks for later reference on other physical aspects of the motion of a particle in an electromagnetic field. The first remark is on the famous time-reversibility of classical mechanics.

Remark 5.51. *Let $c \in \Gamma_c(x, y)$ be an optimal control vector for the electromagnetic potentials $A \in C_b^{1,1}(\mathcal{I} \times \mathbb{R}^3, \mathbb{R}^3)$ and $\Phi \in C_b^{1,1}(\mathcal{I} \times \mathbb{R}^3, \mathbb{R})$. The time reversed trajectory is defined by*

$$\hat{\omega}_c(t) := \omega_c(1 - t).$$

The first question of time-reversibility is, wether $\hat{\omega}_c$ is a physical meaningful trajectory. The second question concerns the electromagnetic potential governing the trajectory. We can answer both questions by observing that $\frac{d}{dt}\hat{\omega}_c(t) = -c(1 - t, \hat{\omega}_c(t)) = \hat{c}(t, \hat{\omega}_c(t))$ with $\hat{c}(t, y) := -c(1 - t, y)$, since this implies

$$\begin{aligned} \left(\Psi_{\hat{c}}^{3,2}(s, \hat{\omega}_c(s)), \Psi_{\hat{c}}^{1,3}(s, \hat{\omega}_c(s)), \Psi_{\hat{c}}^{2,1}(s, \hat{\omega}_c(s))\right)^t &= -B(1 - s, \hat{\omega}_c(s)), \\ \left(\Xi_{\hat{c}}^{1}(s, \hat{\omega}_c(s)), \Xi_{\hat{c}}^{2}(s, \hat{\omega}_c(s)), \Xi_{\hat{c}}^{3}(s, \hat{\omega}_c(s))\right)^t &= E(1 - s, \hat{\omega}_c(s)). \end{aligned}$$

By Proposition 5.46 $\hat{\omega}_c$ is the trajectory of a particle in the same electromagnetic field with reversed boundary conditions.

Now we give a comment on the problem of finding an extremal contained in Definition 5.44 and an equivalent formulation used in the random setting of the next paragraph.

Remark 5.52. *The minimization of the action functional (5.38) in the class is equivalent to the minimization of the functional*

$$\begin{aligned} &\frac{1}{2}\int_{\mathcal{I}} |c(s,\omega_c(s))|^2 ds + \int_{\mathcal{I}} A(s,\omega_c(s))\cdot c(s,\omega_c(s))ds - \int_{\mathcal{I}} \Phi(s,\omega_c(s))ds \\ = &\int_{\mathcal{I}} c(s,\omega_c(s))\cdot \dot{\omega}_c(s)ds - \frac{1}{2}\int_{\mathcal{I}} |c(s,\omega_c(s))|^2 ds + \int_{\mathcal{I}} A(s,\omega_c(s))\cdot \dot{\omega}_c(s)ds - \int_{\mathcal{I}} \Phi(s,\omega_c(s))ds, \end{aligned}$$

subject to $c \in \Gamma_c(x,y)$. A similar minimization problem will appear in the stochastic setting of the next paragraph, where it has the interpretation of an entropy minimization problem.

The last remark is a reformulation of Newton's equation of motion for a particle in an electromagnetic field. This "integral" formulation provides an analogue to the duality formula of Brownian diffusions in the next paragraph.

Remark 5.53. *Given the electromagnetic field (5.37) and (5.36), Newton's equation of motion for a particle of unit mass and unit charge is an equality between the acceleration of the particle and the Lorentz force*

$$\ddot{\omega}(t) = E(t,\omega(t)) + \dot{\omega}\times B(t,\omega(t)), \quad \text{were } \omega \in \Gamma(x,y), \tag{5.54}$$

and we used the cross product of vectors $x\times y = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1)^t$ for any $x, y \in \mathbb{R}^3$. We use a simple integration by parts to give a formulation that does not need the second derivative of ω, thus admitting classes of admissible trajectories that are larger than $\Gamma(x,y)$. Take any $u \in \mathcal{E}_3$ with $\langle u\rangle = 0$, then (5.54) implies

$$\begin{aligned} &\int_{\mathcal{I}} \langle u\rangle_t \cdot \ddot{\omega}(t)dt = \int_{\mathcal{I}} \langle u\rangle_t \cdot (E(t,\omega(t) + \dot{\omega}(t)\times B(t,\omega(t)))\, dt \\ \Leftrightarrow\quad &-\int_{\mathcal{I}} u_t\dot{\omega}(t)dt = \int_{\mathcal{I}} \langle u\rangle_t \cdot E(t,\omega(t))dt + \int_{\mathcal{I}} (\langle u\rangle_t \times B(t,\omega(t)))\cdot \dot{\omega}(t)dt, \end{aligned} \tag{5.55}$$

where the equation holds due to integration by parts. Boundary terms vanish because of the loop condition on u.

5.5.2. *A particle in an electromagnetic field immersed in a thermal reservoir.*

In this paragraph we describe the motion of a particle with unit mass and unit charge immersed in a thermal reservoir and under the influence of an external electromagnetic field. We start with the description of the stochastic Lagrangian mechanics as a variational problem, then reformulate these as a stochastic control problem. A sufficient condition to be the solution of the stochastic control problem can be stated using the invariants of the reciprocal class of a Brownian diffusion.

In the deterministic case we have chosen the class of admissible trajectories such that we where able to define a velocity and an acceleration. Here we model the motion of the particle by a Brownian motion with drift, since the drift can be interpreted as velocity. Instead of endpoints $x, y \in \mathbb{R}^3$ of a deterministic trajectory we prescribe an endpoint distribution $(X_0, X_1) \sim \mu_{01}$ on $\mathbb{R}^6$. For technical reasons we will assume that the endpoint distribution and the drift are regular enough: The set of admissible distributions of the particle is

$$\Gamma(\mu_{01}) = \left\{\mathbb{P}_b \text{ such that } X \text{ is a Brownian motion with drift } b,\ \mathbb{P}_{b,01} = \mu_{01} \text{ and } \mathbb{P}_b \ll \mathbb{P}\right\}.$$

In particular we exclude bridges. The identification of the velocity of the particle with the predictable drift b under the law $\mathbb{P}_b$ allows us to introduce the canonical stochastic

Lagrangian of the particle diffusing in an electromagnetic field by

$$L(X_t, b_t, t) := \frac{1}{2}|b_t|^2 + b_t \cdot A(t, X_t) - \Phi(t, X_t).$$

We define a stochastic Hamiltonian action functional by

$$\mathcal{J}(\mathbb{P}_b) = \mathbb{E}_b\left(\int_{\mathcal{I}} L(X_t, b_t, t)dt\right) = \mathbb{E}_b\left(\int_{\mathcal{I}} \left(\frac{1}{2}|b_t|^2 + b_t \cdot A(t, X_t) - \Phi(t, X_t)\right)dt\right), \tag{5.56}$$

for any $\mathbb{P}_b \in \Gamma(\mu_{01})$. We may now state the general Hamiltonian principle of least action in the random setup. This definition is inspired by Definition 5.39, but similar minimization problems have been associated to stochastic problems of motion before, see e.g. Yasue [Yas81].

Definition 5.57. *The effective distribution of the particle moving between the endpoint distribution μ_{01} in the electromagnetic field defined in (5.36) and (5.37) minimizes the stochastic action functional (5.56) in the class of all admissible distributions $\mathbb{P}_b \in \Gamma(\mu_{01})$.*

This minimization problem does not always have a solution, since $\Gamma(\mu_{01})$ may be empty or the action may be infinite for all distributions in $\Gamma(\mu_{01})$. As for the deterministic case we will not deal with the question of the existence of a solution.

Let us state this physical problem as a stochastic control problem. Instead of introducing a control vector c we assume that the particle moves according to the law of a Brownian diffusion. As the control vector specified the trajectory by (5.40), the drift specifies the law of a Brownian diffusion $\mathbb{P}_c$ as the weak solution of the stochastic differential equation

$$dX_t = c(t, X_t)dt + dW_t, \text{ where } c \in C_b^{1,2}(\mathcal{I} \times \mathbb{R}^d, \mathbb{R}^d) \text{ is the drift.} \tag{5.58}$$

Here W denotes a Brownian motion. The drift could be called the stochastic control vector. We collect all drifts that agree with the prescribed endpoint distribution

$$\Gamma_c(\mu_{01}) := \left\{c \in C_b^{1,2}(\mathcal{I} \times \mathbb{R}^3, \mathbb{R}^3) : \text{ such that } \mathbb{P}_{c,01} = \mu_{01}\right\}. \tag{5.59}$$

Clearly $\{\mathbb{P}_c : c \in \Gamma_c(\mu_{01})\} \subset \Gamma(\mu_{01})$. The stochastic optimal control problem is defined as follows.

Definition 5.60. *The drift $c \in \Gamma_c(\mu_{01})$ is called an optimal drift under the boundary constraints $(X_0, X_1) \sim \mu_{01}$ if it minimizes the stochastic action functional (5.56) in the class of admissible distributions $\Gamma(\mu_{01})$.*

We present a solution of the minimization problem of Definition 5.57 for a particular case: Assume that the particle starts to move from a fixed point $x \in \mathbb{R}^3$. All admissible trajectories are absolutely continuous with respect to $\mathbb{P}^x$, therefore the final distribution $\mu_1(dy) = \mu_{01}(\{x\} \times dy)$ has at least to be absolutely continuous with respect to the Lebesgue-measure. Thus we look at the motion of the particle with boundary distribution $\mu_{01} = \mu_0 \otimes \mu_1$, where

$$\mu_0 = \delta_{\{x\}} \text{ and } \mu_1(dy) = \rho_1(y)dy \text{ where } \rho_1(y) > 0 \text{ for all } y \in \mathbb{R}^3. \tag{5.61}$$

The form of μ_{01} as a product measure is due to the deterministic initial condition. Let us note, that if μ_{01} is of the form (5.61), then the set of admissible distributions is non-empty since $h(X_1)\mathbb{P}^x \in \Gamma(\mu_{01})$ with $h(z) = (d\mu_1/d\mathbb{P}_1^x)(z)$, see also Example 4.19.

Using an approach by Wakolbinger [Wak89], we now compute the effective distribution of the particle. This approach is based on the identification of a relative entropy with the

stochastic action functional. Given two distributions $\mathbb{P}$ and $\mathbb{Q}$ with $\mathbb{Q} \ll \mathbb{P}$ on $C(\mathcal{I}, \mathbb{R}^3)$, the entropy of $\mathbb{Q}$ with respect to $\mathbb{P}$ is defined through

$$\mathbb{E}_{\mathbb{Q}}(\log G), \text{ where } \mathbb{Q} = G\mathbb{P}.$$

In particular if $\mathbb{P}$ is a Wiener measure, and $\mathbb{Q} = \mathbb{P}_b \in \Gamma(\mu_{01})$, then the relative entropy is given by

$$\mathbb{E}_b^x(\log G^b) = \mathbb{E}_b^x\left(\int_{\mathcal{I}} b_t \cdot dX_t - \frac{1}{2}\int_{\mathcal{I}} |b_t|^2 dt\right).$$

Proposition 5.62. *Let μ_{01} be as in (5.61). Then the effective distribution of the particle is a Brownian motion with drift $b(t, X_t) = -A(t, X_t) + \nabla \log h(t, X_t)$, where*

$$h(t, y) := \mathbb{E}_{-A}^x\left(h(X_1)\exp\left(\int_{[t,1]}\left(\Phi(s, X_s) + \frac{1}{2}|A(s, X_s)|^2\right)dt\right)\middle|\ X_t = y\right), \tag{5.63}$$

and we used

$$h(y) := \rho_1(y)(2\pi)^{\frac{3}{2}} e^{\frac{|y-x|^2}{2}}\left[\mathbb{E}^{x,y}\left(\exp\left(-\int_{\mathcal{I}} A(t, X_t)\cdot dX_t + \int_{\mathcal{I}} \Phi(t, X_t)dt\right)\right)\right]^{-1}. \tag{5.64}$$

Proof. We introduce an auxiliary measure on the canonical space: If $\mathbb{P}^x$ is the Wiener measure starting in $x \in \mathbb{R}^3$ put

$$\tilde{\mathbb{Q}}^x := G^{A,\Phi}\mathbb{P}^x \text{ with } G^{A,\Phi} := \exp\left(-\int_{\mathcal{I}} A(t, X_t)\cdot dX_t + \int_{\mathcal{I}} \Phi(t, X_t)dt\right). \tag{5.65}$$

Note that $\tilde{\mathbb{Q}}^x$ is not necessarily a probability measure. By assumption every $\mathbb{P}_b^x \in \Gamma(\mu_{01})$ is absolutely continuous with respect to $\mathbb{P}^x$ with Girsanov density

$$G^b = \exp\left(\int_{\mathcal{I}} b_t \cdot dX_t - \frac{1}{2}\int_{\mathcal{I}} |b_t|^2 dt\right). \tag{5.66}$$

Assume that there exists an element $\mathbb{P}_b \in \Gamma(\mu_{01})$ such that $\mathcal{J}(\mathbb{P}_b)$ is finite. Since

$$\mathbb{E}_b^x\left(\int_{\mathcal{I}} b_t \cdot (dX_t - b_t dt)\right) = 0 \text{ and } \mathbb{E}_b^x\left(\int_{\mathcal{I}} A(t, X_t)\cdot(dX_t - b_t dt)\right) = 0,$$

we can rewrite the action functional

$$\begin{aligned}\mathcal{J}(\mathbb{P}_b) &= \mathbb{E}_b^x\left(\int_{\mathcal{I}}\left(\frac{1}{2}|b_t|^2 + b_t\cdot A(t, X_t) - \Phi(t, X_t)\right)dt\right)\\ &= \mathbb{E}_b^x\left(\int_{\mathcal{I}} b_t\cdot dX_t - \frac{1}{2}\int_{\mathcal{I}}|b_t|^2 dt + \int_{\mathcal{I}} A(t, X_t)\cdot dX_t - \int_{\mathcal{I}}\Phi(t, X_t)dt\right).\end{aligned}$$

Note that this minimization problem is similar to the one in the deterministic setting proposed in Remark 5.52. Here we encounter the densities given in (5.65) and (5.66), which implies the entropy formulation

$$\mathcal{J}(\mathbb{P}_b) = \mathbb{E}_b^x\left(\log G^b - \log G^{A,\Phi}\right) = \mathbb{E}_b^x\left(\log\left(\frac{d\mathbb{P}_b^x}{d\mathbb{P}^x}\frac{d\mathbb{P}^x}{d\tilde{\mathbb{Q}}^x}\right)\right) = \mathbb{E}_b^x\left(\log\left(\frac{d\mathbb{P}_b^x}{d\tilde{\mathbb{Q}}^x}\right)\right).$$

We use the multiplication formula

$$\frac{d\mathbb{P}_b^x}{d\tilde{\mathbb{Q}}^x}(\omega) = \frac{d\mathbb{P}_{b,1}^x}{d\tilde{\mathbb{Q}}_1^x}(\omega(1))\frac{d\mathbb{P}_b^{x,\omega(1)}}{d\tilde{\mathbb{Q}}^{x,\omega(1)}}(\omega),$$

but the first term must be nothing else than the function $h(y)$ defined in (5.64): Indeed we know $\mathbb{P}^x_{b,1}(dy) = \rho_1(y)dy$ and the endpoint distribution of $\tilde{\mathbb{Q}}^x$ is

$$\tilde{\mathbb{Q}}^x(\mathbb{1}_Q(X_1)) = \mathbb{E}^x(\mathbb{1}_Q(X_1)G^{A,\Phi}) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3}\mathbb{1}_Q(y)e^{-\frac{|x-y|^2}{2}}\mathbb{E}^{x,y}(G^{A,\Phi})dy.$$

If we insert this into the above representation of the action functional

$$\mathcal{J}(\mathbb{P}_b) = \mathbb{E}_b\left(\log h(X_1)\right) + \mathbb{E}^{x_0}_b\left(\log\left(\frac{d\mathbb{P}^{x_0,X_1}_b}{d\tilde{\mathbb{Q}}^{x_0,X_1}}\right)\right). \tag{5.67}$$

The second term is zero if and only if $\mathbb{P}^{x,y}_b = \tilde{\mathbb{Q}}^{x,y}$ hold $\mathbb{P}^x_{b,1}$-*a.s.* Since the h-transform $h(X_1)\tilde{\mathbb{Q}}^x$ would satisfy this condition, we only have to show that $h(X_1)\tilde{\mathbb{Q}}^x$ is a Brownian motion with drift in which case $h(X_1)\tilde{\mathbb{Q}}^x \in \Gamma(\mu_{01})$ is the unique minimizer of the entropy, and thus of the action functional (5.59). We can already remark, that following Lemma 4.4 the h-transform $h(X_1)\tilde{\mathbb{Q}}^x$ has the Markov property.

Define $h(t,y)$ as in (5.64), by the Feynman-Kac formula h is a solution of the partial differential equation

$$\partial_t h + \sum_{j=1}^3 A_j\partial_j h + \frac{1}{2}\sum_{j=1}^3\partial_j^2 h + \left(\frac{1}{2}\sum_{j=1}^3 A_j^2 + \Phi\right)h = 0, \tag{5.68}$$

see e.g. the monograph by Karatzas and Shreve [KS91]. We use this to rewrite the Itô-formula

$$\log h(1,X_1) = \log h(0,X_0) + \int_{\mathcal{I}}\partial_t\log h(t,X_t)dt + \int_{\mathcal{I}}\nabla\log h(t,X_t)\cdot dX_t + \frac{1}{2}\int_{\mathcal{I}}\sum_{j=1}^3\partial_j^2\log h(t,X_t)dt,$$

which implies

$$\begin{aligned}
h(X_1)\tilde{\mathbb{Q}}^x &= \exp\left(\log h(1,X_1) - \log h(0,X_0)\right)\tilde{\mathbb{Q}}^x \\
&= \exp\left(\int_{\mathcal{I}}\nabla\log h(t,X_t)\cdot dX_t - \frac{1}{2}\int_{\mathcal{I}}|\nabla\log h(t,X_t)|^2dt\right. \\
&\qquad \left. + \int_{\mathcal{I}}(h(t,X_t))^{-1}\left(\frac{1}{2}\sum_{j=1}^3\partial_j^2 h(t,X_t) + \partial_t h(t,X_t)\right)dt\right)G^{A,\Phi}\mathbb{P}^x \\
&= \exp\left(\int_{\mathcal{I}}\left(-A + \nabla\log h\right)(t,X_t)\cdot dX_t - \frac{1}{2}\int_{\mathcal{I}}\left|\left(-A+\nabla\log h\right)(t,X_t)\right|^2dt\right)\mathbb{P}^x,
\end{aligned}$$

and we recognize $h(X_1)\tilde{\mathbb{Q}}^x$ as the law of a Brownian motion with drift $b(t,X_t) = -A(t,X_t) + \nabla\log h(t,X_t)$. □

As mentioned above the Proposition, the original idea of identifying the stochastic Hamiltonian action functional to a relative entropy is due to Wakolbinger [Wak89]. A similar optimal control problem, in the case $\Phi \equiv -\frac{1}{2}\sum_{i=1}^3 A_i$, has been solved by Dai Pra [DP91] using a logarithmic transformation approach.

In the above Proposition we showed that for given electromagnetic potentials A and Φ the effective distribution of the particle is given by $h(X_1)\tilde{\mathbb{Q}}^x$. If $h > 0$ on $\mathbb{R}^3$, then every processes $g(X_1)\tilde{\mathbb{Q}}^x$ with $\mathbb{E}^x_{\tilde{\mathbb{Q}}}(g(X_1)) = 1$ is given by $(g(X_1)/h(X_1))h(X_1)\tilde{\mathbb{Q}}^x$, and thus in the reciprocal class of the Brownian motion with drift $h(X_1)\tilde{\mathbb{Q}}^x$. Therefore the effective distribution of a particle in a thermal reservoir given electromagnetic potentials A and Φ is characterized as an element of the reciprocal class of $h(X_1)\tilde{\mathbb{Q}}^x$.

We use this fact, to combine the above result with Theorem 5.26 on the reciprocal invariants and state a necessary condition on the drift $c \in \Gamma_c(\mu_{01})$ to be optimal using the invariants of a reference drift $b \in C_b^{1,2}(\mathcal{I} \times \mathbb{R}^3, \mathbb{R}^3)$ in the reciprocal class of $h(X_1)\tilde{\mathbb{Q}}^x$.

Corollary 5.69. *Assume that there exists a solution $\psi \in C^{1,3}(\mathcal{I} \times \mathbb{R}^3, \mathbb{R}^3)$ of the non-linear partial differential equation*

$$0 = \partial_t \psi + \frac{1}{2}\left(-A_j + \partial_j \psi\right)^2 + \frac{1}{2}\sum_{j=1}^{3} \partial_j^2 \psi + \Phi, \tag{5.70}$$

with endpoint condition $\mathbb{E}^x(e^{\psi(1,X_1)}G^{A,\Phi}) = 1$ such that $b(t,y) := -A(t,y) + \nabla\psi(t,y) \in C_b^{1,2}(\mathcal{I} \times \mathbb{R}^3, \mathbb{R}^3)$ is regular enough. Then the drift $c \in \Gamma_c(\mu_{01})$ is optimal if and only if

(i) the rotational invariants coincide: $\Psi_c^{i,j}(s,y) = \Psi_b^{i,j}(s,y)$, $i = 1,2,3$;
(ii) the harmonic invariants coincide: $\Xi_c^i(s,y) = \Xi_b^i(s,y)$, $i = 1,2,3$.

Proof. Define $h(t,y) = e^{\psi(t,y)}$, then h is solution of the Feynman-Kac partial differential equation (5.68). By the proof of Proposition 5.62 we have $\mathbb{P}_b^x = h(1,X_1)\tilde{\mathbb{Q}}^x$ and $h(1,.) > 0$ identically by definition.

Let $\mathbb{P}_c^x \in \Gamma(\mu_{01})$ such that conditions *(i)* and *(ii)* hold. By Theorem 5.26 $\mathbb{P}_c^x$ is in the reciprocal class of $\mathbb{P}_b^x$ and thus of the form $\mathbb{P}_c = \tilde{h}(X_1)\mathbb{P}_b^x$. Therefore $\mathbb{P}_c$ is an h-transform of the auxiliary measure $\tilde{\mathbb{Q}}^x$ and with Proposition 5.62 we deduce that c is an optimal drift.

If on the other hand c is the optimal drift we may apply the proof of Proposition 5.62 to see that $\mathbb{P}_c$ is in the reciprocal class of $\mathbb{P}_b$ and by Theorem 5.26 the reciprocal invariants of the two Brownian diffusions coincide. □

We can reformulate the condition of equality of invariants using the electromagnetic fields B and E as defined in (5.37) and (5.36). This provides an interesting physical interpretation of the reciprocal invariants. We note also, that this development is precisely analogue to the deterministic one, see Proposition 5.46.

Proposition 5.71. *The drift $c \in \Gamma_c(\mu_{01})$ is optimal if it has*

(i) the rotational invariant: $\left(\Psi_c^{3,2}(s,y), \Psi_c^{1,3}(s,y), \Psi_c^{2,1}(s,y)\right)^t = B(s,y)$;
(ii) the harmonic invariant: $\left(\Xi_c^1(s,y), \Xi_c^2(s,y), \Xi_c^3(s,y)\right)^t = E(s,y)$.

Proof. By condition *(i)* there exists a $\psi \in C_b^{1,3}(\mathcal{I} \times \mathbb{R}^3, \mathbb{R})$ such that $c(t,y) = -A(t,y) + \nabla\psi(t,y)$ (closed forms are exact in $\mathbb{R}^3$). Then condition *(ii)* states that

$$\Xi_c^i = -\partial_t A_i + \partial_t \partial_i \psi + \sum_{j=1}^{3}(-A_j + \partial_j\psi)\partial_i(-A_j + \partial_j\psi) + \frac{1}{2}\sum_{j=1}^{3}\partial_i\partial_j(-A_j + \partial_j\psi) = -\partial_t A_i - \partial_i\Phi.$$

Since conditions *(i)* and *(ii)* are invariant under electromagnetic gauge, we may assume the Coulomb-gauge $\sum_{j=1}^{3}\partial_j A_j = 0$ holds. This implies

$$0 = \partial_i\left(\partial_t\psi + \frac{1}{2}\sum_{j=1}^{3}(A_j + \partial_j\psi)^2 + \frac{1}{2}\sum_{j=1}^{3}\partial_j^2\psi + \Phi\right).$$

We proceed as in the proof of Theorem 5.26 to find a function $\phi \in C^1(\mathcal{I})$ such that the right normalization condition $\mathbb{E}^x(e^{\psi(1,X_1)}G^{A,\Phi}) = 1$ holds. Thus $\mathbb{P}_c^x$ is an h-transform of $\tilde{\mathbb{Q}}^x$ and by Proposition 5.62 the drift c is optimal. □

Let us introduce a reformulation of the duality formula (5.33) using the above interpretation of rotational and harmonic invariants.

Remark 5.72. *Let us exchange the Itô integral in the duality formula (5.33) by a Fisk-Stratonovich integral. With respect to the law of a Brownian diffusion $\mathbb{P}_b$ that is*

$$\int_{[t,1]} \Psi_b^{i,j}(s, X_s) \circ dX_{j,r} = \int_{[t,1]} \Psi_b^{i,j}(s, X_s) dX_{j,s} + \frac{1}{2} \int_{[t,1]} \partial_j \Psi_b^{i,j}(s, X_s) ds,$$

for $1 \leq i, j \leq 3$. Assume that the drift $c \in \Gamma_c(\mu_{01})$ is optimal and let $b \in \mathcal{C}_b^{1,2}(\mathcal{I} \times \mathbb{R}^3, \mathbb{R}^3)$ be defined as in Corollary 5.69. Then the duality formula

$$\begin{aligned} \mathbb{E}_c^x \left(F(X) \int_{\mathcal{I}} u_t \circ dX_t \right) &= \mathbb{E}_c^x \left(\int_{\mathcal{I}} D_t F(X) \cdot u_t dt \right) \\ &- \mathbb{E}_c^x \left(F(X) \int_{\mathcal{I}} \sum_{i=1}^3 u_{i,t} \left[\int_{[t,1]} \sum_{j=1}^3 \Psi_b^{i,j}(s, X_s) \circ dX_{j,s} + \int_{[t,1]} \Xi_b^i(s, X_s) ds \right] dt \right), \end{aligned} \tag{5.73}$$

holds for all $F \in \mathcal{S}_3$ and $u \in \mathcal{E}_3$ with $\langle u \rangle = 0$. But identifying the characteristics and the electromagnetic field this is equivalent to

$$\begin{aligned} &\mathbb{E}_c^x \left(\int_{\mathcal{I}} D_t F(X) \cdot u_t dt \right) \\ = \; &\mathbb{E}_c^x \left(F(X) \left(\int_{\mathcal{I}} u_t \circ dX_t + \int_{\mathcal{I}} \langle u \rangle_t \times B(t, X_t) \circ dX_t + \int_{\mathcal{I}} \langle u \rangle_t \cdot E(t, X_t) dt \right) \right), \end{aligned} \tag{5.74}$$

where we used the cross product $x \times y = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1)^t$ for any $x, y \in \mathbb{R}^3$. Take any $t \in \mathcal{I}$ and chose $F \in \mathcal{S}_3$ such that F is $\mathcal{F}_{[0,t]}$ measurable and $u \in \mathcal{E}_3$ such that $u = u \mathbb{1}_{(t,1]}$. Using the computation of the proof of Lemma 2.44 for the predictable process Fu, the above equation implies

$$\mathbb{E}_c^x \left(- \int_{\mathcal{I}} u_t \circ dX_t \,\middle|\, \mathcal{F}_{[0,t]} \right) = \mathbb{E}_c^x \left(\int_{\mathcal{I}} \langle u \rangle_t \times B(t, X_t) \circ dX_t + \int_{\mathcal{I}} \langle u \rangle_t \cdot E(t, X_t) dt \,\middle|\, \mathcal{F}_{[0,t]} \right). \tag{5.75}$$

By analogy we can identify the duality formula (5.74) with the formulation of Newton's equation in Remark 5.53.

Theorem 5.34 may thus be interpreted as a criterion of uniqueness of the solution to the stochastic Newton equation (5.74).

Before considering the time-reversal of the motion of the diffusion particle, we want to point out a remarkable result by Krener [Kre97]. He showed that the invariants from the reciprocal class of a Brownian diffusion appear in a short-time expansion. By Proposition 5.71 this implies that we can locally detect the electric and magnetic field.

Proposition 5.76. *Let $\mathbb{Q}$ be any process in the reciprocal class of a Brownian diffusion $\mathbb{P}_b$. Then for small $\varepsilon > 0, \delta > 0$ the expansion*

$$\begin{aligned} &\mathbb{E}_{\mathbb{Q}} \left(X_{i,t-\varepsilon} - 2X_{i,t} + X_{i,t+\varepsilon} \,\middle|\, \frac{X_{t-\varepsilon} + X_{t+\varepsilon}}{2} = x_o, \frac{X_{t+\varepsilon} - X_{t-\varepsilon}}{2} = \delta x \right) \\ = \; &\Xi_b^i(t, x_o) \varepsilon^2 + \sum_{j=1}^3 \Psi_b^{i,j}(t, x_o) (\varepsilon \delta x_j) + O((\varepsilon \vee \delta)^3) \\ = \; &E_i(t, x_o) \varepsilon^2 + (x \times B(t, x_o))_i \, \varepsilon \delta + O((\varepsilon \vee \delta)^3) \end{aligned} \tag{5.77}$$

holds for any $t \in (0, 1)$, $x_o, x \in \mathbb{R}^d$.

We will only give the sketch of a proof using the duality formula (5.33), for the proof of this and similar results we refer to Krener [Kre97, Theorem 2.1]. Let us note, that the above short-time expansion may be interpreted as a second derivative of X_t in direction x, since the condition is equivalent to $X_{t-\varepsilon} = x_o - \delta x$ and $X_{t+\varepsilon} = x_o + \delta x$, whereas

$$X_{t-\varepsilon} - 2X_t + X_{t+\varepsilon} = (X_{t+\varepsilon} - X_t) - (X_t - X_{t-\varepsilon}).$$

Sketch of a proof. By Corollary 4.15 the identity $\mathbb{Q}(\,.\,|X_{t-\varepsilon} = x_o - \delta x, X_{t+\varepsilon} = x_o + \delta x) = \mathbb{P}_b(\,.\,|X_{t-\varepsilon} = x_o - \delta x, X_{t+\varepsilon} = x_o + \delta x)$ holds on $\mathcal{F}_{[t-\varepsilon,t+\varepsilon]}$. It is therefore sufficient to prove (5.77) for under $\mathbb{P}_b$.

Let $\phi, \psi \in C_b^\infty(\mathbb{R}^d)$. For the derivation of the conditional density on the left side of (5.77) we have to examine

$$\mathbb{E}_b\left(\phi(X_{t+\varepsilon} - X_{t-\varepsilon})\psi(X_{t-\varepsilon} + X_{t+\varepsilon})\left(\int_{[t,t+\varepsilon]} dX_{i,s} - \int_{[t-\varepsilon,t]} dX_{i,s}\right)\right). \tag{5.78}$$

Define the functionals $F(X) = \phi(X_{t+\varepsilon} - X_{t-\varepsilon})$ and $G(X) = \psi(X_{t-\varepsilon} + X_{t+\varepsilon})$. By (2.18) the derivative of FG is

$$\begin{aligned} D_{i,s}(FG)(X) &= \partial_i\phi(X_{t+\varepsilon} - X_{t-\varepsilon})\psi(X_{t-\varepsilon} + X_{t+\varepsilon})\mathbb{1}_{[t-\varepsilon,t+\varepsilon]}(s) \\ &\quad +\phi(X_{t+\varepsilon} - X_{t-\varepsilon})\partial_i\psi(X_{t-\varepsilon} + X_{t+\varepsilon})\mathbb{1}_{[0,t+\varepsilon]}(s). \end{aligned} \tag{5.79}$$

Since $\langle \mathbb{1}_{[t,t+\varepsilon]} - \mathbb{1}_{[t-\varepsilon,t]} \rangle = 0$ we can apply duality formula (5.73) to (5.78) and get

$$\begin{aligned} &\mathbb{E}_b\left(F(X)G(X)\left(\int_{[t,t+\varepsilon]} dX_{i,s} - \int_{[t-\varepsilon,t]} dX_{i,s}\right)\right) = \mathbb{E}_b\left(\int_{\mathcal{I}} D_{i,s}(FG)(X)(\mathbb{1}_{[t,t+\varepsilon]}(s) - \mathbb{1}_{[t-\varepsilon,t]}(s))ds\right) \\ &+\mathbb{E}_b\left(F(X)G(X)\int_{\mathcal{I}}(\mathbb{1}_{[t,t+\varepsilon]}(s) - \mathbb{1}_{[t-\varepsilon,t]}(s))\left[\int_{[s,1]}\sum_{j=1}^{d}\Psi_b^{i,j}(r,X_r)\circ dX_{j,r} + \int_{[s,1]}\Xi_b^i(r,X_r)dr\right]ds\right). \end{aligned}$$

The first term is zero by (5.79). In the second term we change the order of integration to get

$$\begin{aligned} &\mathbb{E}_b\left(F(X)G(X)\left(\int_{[t,t+\varepsilon]} dX_{i,s} - \int_{[t-\varepsilon,t]} dX_{i,s}\right)\right) \\ = \;&\mathbb{E}_b\left(F(X)G(X)\left(\int_{\mathcal{I}}\sum_{j=1}^{d}\Psi_b^{i,j}(s,X_s)\langle \mathbb{1}_{[t,t+\varepsilon]} - \mathbb{1}_{[t-\varepsilon,t]}\rangle_s \circ dX_{j,s}\right.\right. \\ &\left.\left. + \int_{\mathcal{I}}\Xi_b^i(s,X_s)\langle \mathbb{1}_{[t,t+\varepsilon]} - \mathbb{1}_{[t-\varepsilon,t]}\rangle_s ds\right)\right). \end{aligned}$$

The rest of the proof is a Taylor expansion around (t, x_o) of the invariants an a careful treatment of the stochastic integral. □

Using a similar short-time expansion Lévy and Krener [LK93] where able to identify the reciprocal invariants with the electromagnetic field. We have been able to give this identification in Proposition 5.71 using an optimal control approach.

5.6. **Time reversal of Brownian diffusions.**

The time-reversed canonical process is defined by

$$\hat{X}_t := X_{1-t},\ t \in \mathcal{I}. \tag{5.80}$$

The limit from the left has to appear here, since we are working on the space $C(\mathcal{I}, \mathbb{R}^d)$, even if the trajectories of X are *a.s.* continuous with respect to the reference measures we use in this paragraph. If $\mathbb{Q}$ is any probability on $C(\mathcal{I}, \mathbb{R}^d)$, we denote by $\hat{\mathbb{Q}} := \mathbb{Q} \circ \hat{X}^{-1}$ the image of $\mathbb{Q}$ under time-reversal. Clearly $\hat{\mathbb{Q}}$ is a probability on $C(\mathcal{I}, \mathbb{R}^d)$ too.

Lemma 5.81. *The Markov and reciprocal property are stable under time-reversal.*

Proof. Let $\mathbb{Q}$ be a distribution on $C(\mathcal{I},\mathbb{R}^d)$ having the Markov property. Take any $t\in\mathcal{I}$ and $F,G\in\mathcal{S}_d$ such that $F(X)=f(X_{t_1},\dots,X_{t_n})$ is $\mathcal{F}_{[0,t]}$-measurable and $G(X)=g(X_{s_1},\dots,X_{s_m})$ is $\mathcal{F}_{[t,1]}$-measurable. Remark that $\mathbb{E}_{\hat{\mathbb{Q}}}(F(X)|X_t)=\mathbb{E}_{\mathbb{Q}}(F(\hat{X})|\hat{X}_t)$ since

$$\hat{\mathbb{Q}}(\,.\,|X_t\in Q)=\frac{\hat{\mathbb{Q}}(\,.\,\cap\{X_t\in Q\})}{\hat{\mathbb{Q}}(X_t\in Q)}=\frac{\mathbb{Q}(\,.\,\circ\hat{X}^{-1}\cap\{\hat{X}_t\in Q\})}{\mathbb{Q}(\hat{X}_t\in Q)}=\mathbb{Q}(\,.\,\circ\hat{X}^{-1}|\hat{X}_t\in Q),\ \text{ for } Q\in\mathcal{B}(\mathbb{R}^d).$$

Then

$$\begin{aligned}\mathbb{E}_{\hat{\mathbb{Q}}}(F(X)G(X)|X_t) &= \mathbb{E}_{\mathbb{Q}}\left(F(\hat{X})G(\hat{X})\,\middle|\,\hat{X}_t\right)\\ &= \mathbb{E}_{\mathbb{Q}}\left(f(X_{1-t_1},\dots,f(X_{1-t_n}))g(X_{1-s_1},\dots,X_{1-s_m})\,\middle|\,X_{1-t}\right)\\ &= \mathbb{E}_{\mathbb{Q}}\left(F(\hat{X})\,\middle|\,\hat{X}_t\right)\mathbb{E}_{\mathbb{Q}}\left(G(\hat{X})\,\middle|\,\hat{X}_t\right)\\ &= \mathbb{E}_{\hat{\mathbb{Q}}}(F(X)|\,X_t)\,\mathbb{E}_{\hat{\mathbb{Q}}}(G(X)|\,X_t),\end{aligned}$$

which is the Markov property of $\hat{\mathbb{Q}}$.

Now assume that $\mathbb{Q}$ has the reciprocal property. Take any $s,t\in\mathcal{I}$, $s<t$ and $F,G\in\mathcal{S}_d$ such that $F(X)$ is $\mathcal{F}_{[0,s]\cup[t,1]}$-measurable and $G(X)$ is $\mathcal{F}_{[s,t]}$-measurable. Then the above computation for the Markov property can be applied, but for the conditioning that is on (X_s,X_t) now. □

In particular time reversed Brownian diffusions still have the Markov property. Numerous authors have been interested in the question, whether the diffusion property is preserved under time reversal too. Let us quote the following result by Haussmann, Pardoux [HP86].

Proposition 5.82. *Let $\mathbb{P}_b$ be a Brownian diffusion, then the image of $\mathbb{P}_b$ under time-reversal is a Brownian motion with drift*

$$\hat{b}(t,X_t)=-b(1-t,X_t)+\nabla\log p_b(1-t,X_t). \tag{5.83}$$

In other words: $\hat{\mathbb{P}}_b=\mathbb{P}_{\hat{b}}$, where $\mathbb{P}_{\hat{b}}$ is a Brownian motion with drift $\hat{b}$ and initial condition $\mathbb{P}_{\hat{b},0}=\mathbb{P}_{b,1}$. If a Brownian diffusion $\mathbb{P}_b^x$ with pinned initial condition is reversed, the backward process will have a very singular drift when nearing the endpoint since $\mathbb{P}_{\hat{b},1}=\delta_{\{x\}}$ in this case. Different initial conditions may lead to densities $p_b(t,x)$ that have bounded logarithmic derivatives.

> *Assume that there exists an initial condition $\mathbb{P}_{b,0}$ such that $\hat{b}\in C_b^{1,2}(\mathcal{I}\times\mathbb{R}^d,\mathbb{R}^d)$. In particular assume that $\Psi_{\hat{b}}^{i,j}$ and $\Xi_{\hat{b}}^i$ are the invariants of the reciprocal class of a Brownian diffusion.*

In [Thi93, Proposition 4.5] Thieullen identified the reciprocal class of the time-reversed diffusion $\mathbb{P}_{\hat{b}}$ using the explicit form (5.83) of the drift and the Kolmogoroff forward equation satisfied by p_b. We present a different proof, to show that this result is connected with the duality formula (5.33) that is characteristic for the reciprocal class.

Proposition 5.84. *Let $\mathbb{Q}$ be an element of $\mathcal{R}(\mathbb{P}_b)$ such that X has integrable increments. Assume that the image of $\mathbb{Q}$ under time-reversal $\hat{\mathbb{Q}}:=\mathbb{Q}\circ\hat{X}^{-1}$ satisfies the hypotheses of Theorem 5.34. Then $\hat{\mathbb{Q}}$ is an element of the reciprocal class $\mathcal{R}(\mathbb{P}_{\hat{b}})$ with invariants*

$$\Psi_{\hat{b}}^{i,j}(t,x)=-\Psi_b^{i,j}(1-t,x),\ \textit{and}\ \Xi_{\hat{b}}^i(t,x)=\Xi_b^i(1-t,x),\ \textit{for}\ (t,x)\in\mathcal{I}\times\mathbb{R}^d\ \textit{and}\ i,j\in\{1,\dots,d\}.$$

Proof. We present a proof using the characterization of the reciprocal class of a diffusion by the duality formula (5.33). Take an arbitrary $F(X) = f(X_{t_1}, \dots, X_{t_n}) \in \mathcal{S}_d$ and $u = \sum_{i=1}^m u_i \mathbb{1}_{(s_i, s_{i+1}]} \in \mathcal{E}_d$ with $\langle u \rangle = 0$. Without loss of generality we may assume that $n = m$ and $\{t_1, \dots, t_n\} = \{s_1, \dots, s_n\}$. Then

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\left(F(\hat{X})\int_{\mathcal{I}} u_t \cdot d\hat{X}_t\right) &= \mathbb{E}_{\mathbb{Q}}\left(f(X_{1-t_1}, \dots, X_{1-t_n})\sum_{i=1}^{n-1} u_i \cdot (X_{1-t_{i+1}} - X_{1-t_i})\right)\\ &= \mathbb{E}_{\mathbb{Q}}\left(\hat{F}(X)\int_{\mathcal{I}} \hat{u}_t \cdot dX_t\right),\end{aligned}$$

where we define $\hat{F}(X) = f(X_{1-t_1}, \dots, X_{1-t_n}) \in \mathcal{S}_d$ and $\hat{u} = -\sum_{i=1}^{n-1} u_i \mathbb{1}_{(1-t_{i+1}, 1-t_i]} \in \mathcal{E}_d$ with $\langle \hat{u} \rangle = 0$. Since $\mathbb{Q}$ is in the reciprocal class of $\mathbb{P}_b$ we can apply (5.33):

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\left(\hat{F}(X)\int_{\mathcal{I}} \hat{u}_t \cdot dX_t\right) &= \mathbb{E}_{\mathbb{Q}}\left(\int_{\mathcal{I}} D_t\hat{F}(X) \cdot \hat{u}_t dt\right)\\ &\quad + \mathbb{E}_{\mathbb{Q}}\left(\hat{F}(X)\int_{\mathcal{I}}\sum_{i=1}^{d}\hat{u}_{i,t}\left[\int_{[t,1]}\sum_{j=1}^{d}\Psi_b^{i,j}(s, X_s)dX_{j,s}\right.\right.\\ &\quad \left.\left. + \int_{[t,1]}\left(\Xi_b^i(s, X_s) + \frac{1}{2}\sum_{j=1}^{d}\partial_j\Psi_b^{i,j}(s, X_s)\right)ds\right]dt\right).\end{aligned}$$

The first term on the right side can be rewritten as

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\left(\int_{\mathcal{I}} D_t\hat{F}(X) \cdot \hat{u}_t dt\right) &= \mathbb{E}_{\mathbb{Q}}\left(\sum_{i=1}^{n}\int_{[0,1-t_i]}\partial_{(i-1)d+.}f(X_{1-t_1}, \dots, X_{1-t_n}) \cdot \hat{u}_s ds\right)\\ &= \mathbb{E}_{\mathbb{Q}}\left(\sum_{i=1}^{n}\partial_{(i-1)d+.}f(\hat{X}_{t_1}, \dots, \hat{X}_{t_n}) \cdot \sum_{j=1}^{i-1} u_j(t_{j+1} - t_j)\right) = \mathbb{E}_{\mathbb{Q}}\left(\int_{\mathcal{I}} D_t F(\hat{X}) \cdot u_t dt\right).\end{aligned}$$

As for the second term on the right side, we remark that $\hat{F}(X) = F(\hat{X})$ under the integration by $\mathbb{Q}$. Let us first treat the non-stochastic integrals:

$$\begin{aligned}&\mathbb{E}_{\mathbb{Q}}\left(\hat{F}(X)\int_{\mathcal{I}}\hat{u}_{i,t}\int_{[t,1]}\left(\Xi_b^i(s, X_s) + \frac{1}{2}\sum_{j=1}^{d}\partial_j\Psi_b^{i,j}(s, X_s)\right)dsdt\right)\\ =\ &\mathbb{E}_{\mathbb{Q}}\left(F(\hat{X})\int_{\mathcal{I}}\left(\Xi_b^i(s, X_s) + \frac{1}{2}\sum_{j=1}^{d}\partial_j\Psi_b^{i,j}(s, X_s)\right)\langle\hat{u}_i\rangle_s ds\right)\\ =\ &\mathbb{E}_{\mathbb{Q}}\left(F(\hat{X})\int_{\mathcal{I}}\left(\Xi_b^i(1-s, X_{1-s}) + \frac{1}{2}\sum_{j=1}^{d}\partial_j\Psi_b^{i,j}(1-s, X_{1-s})\right)\langle\hat{u}_i\rangle_{1-s} ds\right),\end{aligned}$$

and using the relation $\langle\hat{u}\rangle_{1-s} = -\int_{[s,1]} u_r dr = \langle u\rangle_s$ gives

$$\begin{aligned}&= \mathbb{E}_{\mathbb{Q}}\left(F(\hat{X})\int_{\mathcal{I}}\left(\Xi_b^i(1-s, \hat{X}_s) + \frac{1}{2}\sum_{j=1}^{d}\partial_j\Psi_b^{i,j}(1-s, \hat{X}_s)\right)\langle u_i\rangle_s ds\right)\\ &= \mathbb{E}_{\mathbb{Q}}\left(F(\hat{X})\int_{\mathcal{I}} u_{i,t}\int_{[t,1]}\left(\Xi_b^i(1-s, \hat{X}_s) + \frac{1}{2}\sum_{j=1}^{d}\partial_j\Psi_b^{i,j}(1-s, \hat{X}_s)\right)dsdt\right).\end{aligned}$$

In the last term we have to rewrite a stochastic Itô integral:

$$\mathbb{E}_{\mathbb{Q}}\left(\hat{F}(X)\int_{\mathcal{I}}\hat{u}_{i,t}\int_{[t,1]}\sum_{j=1}^{d}\Psi_b^{i,j}(s,X_s)dX_{j,s}dt\right)=\mathbb{E}_{\mathbb{Q}}\left(F(\hat{X})\int_{\mathcal{I}}\sum_{j=1}^{d}\Psi_b^{i,j}(s,X_s)\langle\hat{u}_i\rangle_s dX_{j,s}\right).$$

We have to check the behavior of the Itô-integral under time-reversal. With $\{0,1/m,\dots,(m-1)/m,1\}=:\{s_0,\dots,s_m\}\in\Delta_{\mathcal{I}}$ we define the sequence of equipartition subdivisions of $\mathcal{I}$. We expand the stochastic integral

$$\int_{\mathcal{I}}\Psi_b^{i,j}(s,X_s)\langle\hat{u}_i\rangle_s dX_{j,s}=\lim_{m\to\infty}\sum_{k=0}^{m}\Psi_b^{i,j}(s_k,X_{s_k})\langle\hat{u}_i\rangle_{s_k}(X_{j,s_{k+1}}-X_{j,s_k}),$$

by a Taylor expansion of $\Psi_b^{i,j}(t,x)$ in the j'th space coordinate

$$\begin{aligned}
&\sum_{k=0}^{m}\Psi_b^{i,j}(s_k,X_{s_k})\langle\hat{u}_i\rangle_{s_k}(X_{j,s_{k+1}}-X_{j,s_k})\\
=&\sum_{k=1}^{m+1}\Psi_b^{i,j}(1-s_k,X_{1-s_k})\langle\hat{u}_i\rangle_{1-s_k}(X_{j,1-s_{k-1}}-X_{j,1-s_k})\\
=&-\sum_{k=1}^{m+1}\Psi_b^{i,j}(1-s_k,\hat{X}_{s_k})\langle\hat{u}_i\rangle_{1-s_k}(\hat{X}_{j,s_k}-\hat{X}_{j,s_{k-1}})\\
=&-\sum_{k=0}^{m}\Psi_b^{i,j}(1-s_k,\hat{X}_{s_k})\langle\hat{u}_i\rangle_{1-s_k}(\hat{X}_{j,s_{k+1}}-\hat{X}_{j,s_k})\\
&-\sum_{k=0}^{m}\partial_j\Psi_b^{i,j}(1-s_k,\hat{X}_{s_k})\langle\hat{u}_i\rangle_{1-s_k}(\hat{X}_{j,s_{k+1}}-\hat{X}_{j,s_k})^2+R(m).
\end{aligned}$$

The last term is bounded from above by $K(\hat{X}_{s_{k+1}}-\hat{X}_{s_k})^2\max_l|\hat{X}_{s_{l+1}}-\hat{X}_{s_l}|$ for some $K>0$ and therefore converges to 0 in the $\mathbb{L}^1(\mathbb{Q})$ limit $m\to\infty$. The other terms converge to the backward Itô stochastic integral plus a correction term, thus

$$\begin{aligned}
\int_{\mathcal{I}}\Psi_b^{i,j}(s,X_s)\langle\hat{u}_i\rangle_s dX_{j,s}&=-\int_{\mathcal{I}}\Psi_b^{i,j}(1-s,\hat{X}_s)\langle\hat{u}_i\rangle_{1-s}d\hat{X}_{j,s}-\int_{\mathcal{I}}\partial_j\Psi_b^{i,j}(1-s,\hat{X}_s)\langle\hat{u}_i\rangle_{1-s}ds\\
&=-\int_{\mathcal{I}}\Psi_b^{i,j}(1-s,\hat{X}_s)\langle u_i\rangle_s d\hat{X}_{j,s}-\int_{\mathcal{I}}\partial_j\Psi_b^{i,j}(1-s,\hat{X}_s)\langle u_i\rangle_s ds.
\end{aligned}$$

This proves

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\left(\hat{F}(X)\int_{\mathcal{I}}\sum_{i=1}^{d}\hat{u}_{i,t}\int_{[t,1]}\sum_{j=1}^{d}\Psi_b^{i,j}(s,X_s)dX_{j,s}dt\right)\\
=&-\mathbb{E}_{\mathbb{Q}}\left(F(\hat{X})\int_{\mathcal{I}}u_{i,t}\left[\int_{[t,1]}\sum_{j=1}^{d}\Psi_b^{i,j}(1-s,\hat{X}_s)d\hat{X}_{j,s}+\int_{[t,1]}\sum_{j=1}^{d}\partial_j\Psi_b^{i,j}(1-s,\hat{X}_s)ds\right]dt\right).
\end{aligned}$$

Therefore we have shown, that (5.33) holds under $\hat{\mathbb{Q}}$ with respect to the claimed characteristics. An application of the characterization in Theorem 5.34 ends the proof, since the same duality holds with respect to $\mathbb{P}_{\hat{b}}$, and by assumption $\Psi_{\hat{b}}^{i,j}$ and $\Xi_{\hat{b}}^{i}$ are the characteristics of the reciprocal class of a Brownian diffusion. □

The situation is especially time-symmetric if the drift b is deterministic.

Example 5.85. *The reciprocal class of $\hat{\mathbb{P}}_b = \mathbb{P}_{\hat{b}}$ is in general different than the reciprocal class of the diffusion with drift $\tilde{b}(t,x) := -b(1-t,x)$. In particular*

$$\Psi_{\tilde{b}}(t,x) = \Psi_{\hat{b}}(t,x),$$

but

$$\begin{aligned}
\Xi^i_{\tilde{b}}(t,x) &= \partial_t b_i(1-t,x) + \sum_{j=1}^d b_j(1-t,x)\partial_i b_j(1-t,x) - \frac{1}{2}\sum_{j=1}^d \partial_i\partial_j b_j(1-t,x) \\
&\neq \partial_t b_i(1-t,x) + \sum_{j=1}^d b_j(1-t,x)\partial_i b_j(1-t,x) + \frac{1}{2}\sum_{j=1}^d \partial_i\partial_j b_j(1-t,x) \\
&= \Xi^i_{\hat{b}}(t,x).
\end{aligned}$$

If on the other hand $b \in C^1_b(\mathcal{I},\mathbb{R}^d)$ and $\mathbb{P}_b$ is the law of a Brownian diffusion with deterministic drift b, then the reciprocal invariants coincide in the sense that $\Psi^{i,j}_b(1-t) = \Psi^{i,j}_{\hat{b}}(t) = 0$ and $\Xi^i_b(1-t) = \Xi^i_{\hat{b}}(t)$.

Let us return to the physical interpretation of § 5.5.

Remark 5.86. *In dimension $d = 3$ we may use the interpretation of the reciprocal invariants as electric and magnetic field. The behavior of a diffusion particle that is immersed in a thermal reservoir and under the influence of an external electromagnetic field under time reversal may be loosely describes as follows. If $\mathbb{Q}$ describes the motion of a particle in an electromagnetic field given by $E(t,x)$ and $B(t,x)$ and boundary conditions μ_{01}, then $\hat{\mathbb{Q}}$ is the motion of a particle in an electromagnetic field given by $E(1-t,x)$ and $-B(1-t,x)$ with boundary conditions $\hat{\mathbb{Q}}_{01}(dxdy) = \mu_{01}(dydx)$. Comparing this to Remark 5.51 on the time reversal of the deterministic motion, we see that the stochastic dynamics are reversible in the same sense as the deterministic dynamics.*

6. The reciprocal classes of unit jump processes

The canonical space of unit jump processes $\mathbb{J}_1(\mathcal{I})$, as defined in (6.1), is the subspace of $\mathbb{D}(\mathcal{I})$ that consists of counting processes. A typical unit jump process is the Poisson process, whose law is supported by $\mathbb{J}_1(\mathcal{I})$.

In [CP90] Carlen and Pardoux introduced a Malliavin calculus for the Poisson process based on a derivative of its jump times. In particular they derive a duality formula for the Poisson process that contains a "true" derivative operator in the sense that a chain and a product formula are satisfied.

We derive similar duality formulae satisfied for Markov unit jump processes, which leads to our main result in Theorem 6.69: In this section we obtain a new characterization of the reciprocal classes of Markov unit jump processes as the unique class of processes satisfying a certain duality formula. This characterization is preceded by a study of the bridges of Markov unit jump processes, for which we point out the existence of reciprocal invariants presented in Theorem 6.58. We apply the characterization result to an optimal control problem and the time-reversal of unit jump processes.

The section is organized as follows. In the first paragraph we give an introduction to unit jump processes, see also the essential results of stochastic calculus of pure jump processes in the Appendix. Paragraph 6.2 is devoted to the definition of the derivation operator of Carlen and Pardoux that acts on jump-times.

In Paragraph 6.3 we study the Poisson process and its reciprocal class. We present a duality formula satisfied by the Poisson process in Proposition 6.22. Moreover we show that a Poisson process is the unique unit jump process that satisfies this duality formula. Our first new result is the extension of this characterization to the bridges of a Poisson process: By a loop-constraint on the class of test functions we prove that the reciprocal class of a Poisson process is completely determined as the set of probability measures on $\mathbb{J}_1(\mathcal{I})$ that satisfies a duality formula.

This approach is extended to Markov processes on $\mathbb{J}_1(\mathcal{I})$. In § 6.4 we introduce the setting of "nice" unit jump processes. These processes are Markov and admit a regular jump intensity given in Definition 6.46. In Theorem 6.58 we specify a characterizing reciprocal invariant for them: Two nice unit jump processes belong to the same reciprocal class if and only if they have the same invariant. Our main result is Theorem 6.69 in § 6.6. The reciprocal class of a nice unit jump process is then characterized via a duality formula that contains the reciprocal invariant.

We present two applications of this new characterization. In § 6.7 we introduce an optimal control problem for unit jump processes. The solutions of such a problem are contained in the reciprocal class of a reference nice unit jump process, thus they are characterized by a duality formula.

We study the time reversal of unit jump processes in § 6.8. This subject has been addressed before by Elliott and Tsoi [ET90]. They use the duality formula satisfied by a Poisson process to compute the intensities of time reversed processes, an approach inspired by the treatment of time-reversed Brownian motions with drift by Föllmer [Föl86]. Our approach is different and leads to different results. In particular we compute the intensity of a time-reversed nice unit jump process and characterize the behavior of the reciprocal class of a nice unit jump process under time reversal.

6.1. **Unit jump processes.**

Throughout this section we consider real valued càdlàg processes that have jumps of unit size 1 and are constant in between the jumps. Instead of working with general càdlàg functions it is natural to focus on the canonical space of unit jump paths

$$\mathbb{J}_1(\mathcal{I}) := \left\{ \omega = x + \sum_{i=1}^{m} \mathbb{1}_{[t_i,1]},\ x \in \mathbb{R},\ \{t_1,\dots,t_m\} \in \Delta_{\mathcal{I}},\ m \in \mathbb{N} \right\}. \tag{6.1}$$

In particular $\mathbb{J}_1(\mathcal{I}) \subset \mathbb{D}(\mathcal{I})$ and we use the canonical setup induced by the space of càdlàg functions:

- The canonical unit jump process $X : \mathbb{J}_1(\mathcal{I}) \to \mathbb{J}_1(\mathcal{I})$ is the identity;
- $\mathcal{F}_\tau := \sigma(X_s,\ s \in \tau)$ for every subset $\tau \subset \mathcal{I}$.

We identify $\omega \in \mathbb{J}_1(\mathcal{I})$ with the tuple containing the initial condition and the jump-times: The spaces $\mathbb{J}_1(\mathcal{I})$ and $\mathbb{R} \times \Delta_{\mathcal{I}}$ are isomorphic through the identification

$$\mathbb{J}_1(\mathcal{I}) \ni \omega = x + \sum_{i=1}^{m} \mathbb{1}_{[t_i,1]} \ \leftrightarrow\ (x, \{t_1,\dots,t_m\}) \in \mathbb{R} \times \Delta_{\mathcal{I}}.$$

The integer random variable that counts the total number of jumps is denoted by $\eta := X_1 - X_0$, in particular $\eta((x,\{t_1,\dots,t_m\})) = m$. Let $T_1, T_2, \dots$ be the consecutive jump-times of X defined by $T_0 = 0$, $T_i(\omega) = T_i((x,\{t_1,\dots,t_m\})) = t_i$ for $1 \le i \le m$ and $T_i(\omega) := \infty$ if $i > m$, where ∞ may be interpreted as abstract cemetery time. The knowledge of the initial value X_0 and the jump-times is equivalent to the information given by the canonical unit jump process.

Lemma 6.2. *Let $t \in \mathcal{I}$ be arbitrary, then*

$$\mathcal{F}_{[0,t]} = \sigma\left(X_0, T_1 \wedge t, T_2 \wedge t, \dots\right), \tag{6.3}$$

where $t \wedge \infty = t$.

Proof. Denote by $\mathcal{G}_{[0,t]}$ the σ-field on the right side of (6.3). By definition $\mathcal{F}_{[0,t]} = \sigma(X_s,\ s \le t)$. In order to prove $\mathcal{F}_{[0,t]} \subset \mathcal{G}_{[0,t]}$ we show that every X_s is $\mathcal{G}_{[0,t]}$ measurable if $s \le t$. But

$$X_s = X_0 + \sum_{i=1}^{\infty} \mathbb{1}_{\{T_i \le s\}},$$

and X_0 as well as the events $\{T_i \le s\}$ are $\mathcal{G}_{[0,t]}$-measurable for any $i \in \mathbb{N}$. For the reverse inclusion $\mathcal{G}_{[0,t]} \subset \mathcal{F}_{[0,t]}$ we have to show that X_0 as well as $(T_i \wedge t)$ for $i \in \mathbb{N}$ are $\mathcal{F}_{[0,t]}$-measurable. For X_0 this is immediate, as for the jump-times we have to use the fact that $\Delta X_s = X_s - X_{s-}$ is $\mathcal{F}_{[0,t]}$-measurable for any $s \le t$ and the jump-times may be defined as e.g. $T_1 = \inf\{s > 0 : \Delta X_s = 1\}$. □

Let us point out, that our setup of unit jump processes is more regular than similar setups of point processes on the line, see e.g. the monograph by Brémaud [Bré81].

Remark 6.4. *Usually a point process on the line is defined by a sequence $(S_i)_{i\ge1}$ of non-negative random variables, which through $T_i = S_1 + \cdots + S_i$ define the jump-times of a point process. We avoid the following irregularities:*

- *There is no immediate jump: $T_1(\omega) > 0$, $\forall \omega \in \mathbb{J}_1(\mathcal{I})$.*
- *There are no simultaneous jumps: $T_i(\omega) < T_{i+1}(\omega)$, $\forall 1 \le i \le \eta - 1$, $\forall \omega \in \mathbb{J}_1(\mathcal{I})$.*
- *There is no accumulation of jumps: $\eta(\omega) < \infty$ for every trajectory $\omega \in \mathbb{J}_1(\mathcal{I})$.*

If we would allow $T_i = T_{i+1}$, then the trajectory X could have jumps of size 2 or more. An accumulation of jumps could lead to an "explosion" of the trajectory, which in turn would no longer be càdlàg. For the sake of examining the reciprocal classes, and thus the bridges, of unit jump processes it is natural to assume that no explosion of the trajectory takes place, since we will have to condition the processes on finite endpoint values.

Following the above remark, the canonical process X is a semimartingale with respect to any probability $\mathbb{Q}$ on $\mathbb{J}_1(\mathcal{I})$. In the Appendix we give a brief introduction to the stochastic calculus of pure jump semimartingales, see also Remark 7.3. In particular there exists a predictable process $A : \mathcal{I} \times \mathbb{J}_1(\mathcal{I}) \to \mathbb{R}$ of locally bounded variation such that

$$X - X_0 - A \text{ is a local martingale with respect to } \mathbb{Q}. \tag{6.5}$$

We will call dA the intensity of the unit jump process under $\mathbb{Q}$.

Remark 6.6. *Any probability $\mathbb{Q}$ on $\mathbb{J}_1(\mathcal{I})$ can be decomposed into*

$$\mathbb{Q}(\,.\,) = \int_{\mathbb{R}} \mathbb{Q}^x(\,.\,)\mathbb{Q}_0(dx), \tag{6.7}$$

where $\mathbb{Q}^x(\,.\,) = \mathbb{Q}(\,.\,|\,X_0 = x)$ as defined in Section 4. Clearly $\mathbb{Q}^x(X_t \in\ .\,) \ll \delta_{\{x\}} * (\sum_{m=0}^{\infty} \delta_{\{m\}})$, and with the above decomposition we deduce $\mathbb{Q}(X_t \in\ .\,) \ll \mathbb{Q}_0 * (\sum_{m=0}^{\infty} \delta_{\{m\}})$ for any $t \in \mathcal{I}$. We define the density of the one-time projection by

$$\mathbb{Q}(X_t \in dy) =: q(t,y)\left(\mathbb{Q}_0 * \left(\sum_{m=0}^{\infty} \delta_{\{m\}}\right)\right)(dy).$$

In the case $\mathbb{Q}_0 = \delta_{\{x\}}$ we have $q(t,y) = \mathbb{Q}\,(X_t = y)$ whenever $y - x \in \mathbb{N}$. The transition probabilities are defined in a similar fashion by

$$q(s,x;t,y) := \mathbb{Q}\,(X_t = y | X_s = x) \text{ for } s < t,\ y - x \in \mathbb{N}.$$

Clearly $q(t,y) = \int_{\mathbb{R}} q(0,x;t,y)\mathbb{Q}_0(dx)$. Most of our results concerning the reciprocal classes of unit jump processes are first proven for $\mathbb{Q}^x$ with $x \in \mathbb{R}$ arbitrary and then extended by the decomposition (6.7).

In § 1.2 we presented a first definition of a Poisson process on the space $\mathbb{D}(\mathcal{I})$ with intensity one. The space of unit jump processes $\mathbb{J}_1(\mathcal{I})$ is large enough to admit a Poisson process with arbitrary initial condition. In this section a Poisson process will play the role of reference processes, similar to the role of a Wiener measure in Section 5.

Using Watanabe's characterization we can identify a Poisson process by its intensity. Remark that we use the term Poisson process always for a Poisson process with intensity one.

Lemma 6.8. *The probability $\mathbb{P}$ on $\mathbb{J}_1(\mathcal{I})$ is the law of a Poisson process if and only if $t \mapsto X_t - X_0 - t$ is a martingale.*

Proof. First assume that X is a Poisson process with respect to $\mathbb{P}$. By Proposition 1.21 the duality formula

$$\mathbb{E}\left(F(X)\int_{\mathcal{I}} u_s dX_s\right) = \mathbb{E}\left(\int_{\mathcal{I}} F(X + \mathbb{1}_{[s,1]})u_s ds\right)$$

holds for any $F(X) = f(X_{t_1}, \dots, X_{t_n}) \in \mathcal{S}$ and $u = \sum_{i=1}^{m-1} u_i \mathbb{1}_{(s_i, s_{i+1}]} \in \mathcal{E}$. If F is $\mathcal{F}_{[0,s]}$-measurable and $u = \mathbb{1}_{(s,t]}$ for some $s < t$ this implies

$$\mathbb{E}\,((X_t - X_s)F) = \mathbb{E}\,(F)(t-s) \quad \Rightarrow \quad \mathbb{E}\left(X_t - X_s \,|\, \mathcal{F}_{[0,s]}\right) = t - s,$$

which is the martingale property.

If on the other hand $X_t - X_0 - t$ is a martingale we may define the process

$$Y_t^\gamma := \exp\left(i\gamma(X_t - X_0) - t(e^{i\gamma} - 1)\right),\ \gamma \in \mathbb{R},$$

which is the solution of the Doléans-Dade stochastic differential equation

$$Y_t^\gamma = 1 + \int_{[0,t]} \left(e^{i\gamma} - 1\right) Y_{s-}^\gamma (dX_t - dt),$$

since X only has jumps of unit size. Thus Y^γ is a local martingale, and since Y^γ is bounded we get $\mathbb{E}(Y_t^\gamma) = 1$: The characteristic functional of X coincides with that of a Poisson process. □

In the sense of (6.5) a Poisson process has intensity dt. Throughout Section 6 we denote by $\mathbb{P}$ the law of a Poisson process.

6.2. **Derivative operator for functionals of unit jump processes.**

In § 5.1 we restricted the path perturbation $\theta_u^\varepsilon(\omega) = \omega + \varepsilon\langle u\rangle_\cdot$ to directions $u \in \mathcal{E}_d$ with $\langle u\rangle = \int_{\mathcal{I}} u_t dt = 0$. These perturbations do not influence the final state of the path $X_1 \circ \theta_u^\varepsilon = X_1$, an essential factor in the derivation of the duality formula (5.33) that characterized the reciprocal class of the Wiener measure.

In the unit jump context the perturbation $\bar\theta_{\bar v}^\varepsilon(\omega', \omega) = \omega + Y^{\varepsilon\bar v}(\omega')$ as defined in (2.32) is a random addition of jumps of size one. This addition of jumps will influence the final state $X_1 \circ \bar\theta_{\bar v}^\varepsilon = X_1 + Y_1^{\varepsilon\bar v}$, regardless of the direction of perturbation $\bar v \in \bar{\mathcal{E}}$. For the study of the reciprocal class of a Poisson process, we therefore have to introduce a different perturbation.

6.2.1. *Definition and properties of the jump-time derivative.*

We follow an idea of perturbation of the jump-times that was introduced independently by Carlen, Pardoux [CP90] and Elliott, Tsoi [ET93].

Definition 6.9. *Given a bounded, measurable $u : \mathcal{I} \to \mathbb{R}$ and small $\varepsilon \geq 0$ we define the perturbation*

$$\pi_u^\varepsilon : \mathbb{J}_1(\mathcal{I}) \to \mathbb{J}_1(\mathcal{I}), \quad (x, \{t_1, \dots, t_m\}) \mapsto (x, \{t_1 + \varepsilon\langle u\rangle_{t_1}, \dots, t_m + \varepsilon\langle u\rangle_{t_m}\} \cap \mathcal{I}).$$

Let $\mathbb{Q}$ be any probability on $\mathbb{J}_1(\mathcal{I})$. A functional $F \in \mathbb{L}^2(\mathbb{Q})$ is called differentiable in direction u if the limit

$$\mathcal{D}_u F := -\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(F \circ \pi_u^\varepsilon - F) \tag{6.10}$$

exists in $\mathbb{L}^2(\mathbb{Q})$. If F is differentiable in all directions $u \in \mathcal{E}$, we denote by $\mathcal{D}F = (\mathcal{D}_t F)_{t\in\mathcal{I}} \in \mathbb{L}^2(dt \otimes \mathbb{Q})$ the unique process such that

$$\mathcal{D}_u F = \int_{\mathcal{I}} \mathcal{D}_t F u_t dt = \langle \mathcal{D}F, u\rangle \text{ holds } \mathbb{Q}\text{-a.s. for every } u \in \mathcal{E}.$$

Note that the perturbation is well defined for ε small enough, since $-1 < \varepsilon u_t$ for all $t \in \mathcal{I}$ implies that the mapping $t \mapsto t + \varepsilon\langle u\rangle_t$ is strictly monotone. We defined a true derivative operator in the following sense.

Lemma 6.11. *The standard rules of differential calculus apply:*

- *Let F, G and FG be differentiable in direction u, then the product rule holds:*

$$\mathcal{D}_u(FG) = G\mathcal{D}_u F + F\mathcal{D}_u G.$$

- *Let F be differentiable in direction u, $\phi \in C_b^\infty(\mathbb{R})$, then $\phi(F)$ is differentiable in direction u and the chain rule holds:*

$$\mathcal{D}_u(\phi(F)) = \phi'(F)\mathcal{D}_u F.$$

Proof. Let us first proof the product rule: For F and G given in the statement we get

$$\begin{aligned} -\mathcal{D}_u(FG) &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left((F\circ\pi_u^\varepsilon)(G\circ\pi_u^\varepsilon) - FG\right) \\ &= G\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(F\circ\pi_u^\varepsilon - F\right) + F\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(G\circ\pi_u^\varepsilon - G\right) + \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(F\circ\pi_u^\varepsilon - F\right)\left(G\circ\pi_u^\varepsilon - G\right). \end{aligned}$$

The last term converges to zero by the Cauchy-Schwarz inequality:

$$\mathbb{Q}\left(\left(\frac{1}{\varepsilon}\left(F\circ\pi_u^\varepsilon - F\right)\left(G\circ\pi_u^\varepsilon - G\right)\right)^2\right) \le \mathbb{Q}\left(\left(\frac{1}{\varepsilon}(F\circ\pi_u^\varepsilon - F)\right)^2\right)^{\frac{1}{2}} \mathbb{Q}\left(\left(G\circ\pi_u^\varepsilon - G\right)^2\right)^{\frac{1}{2}}.$$

As for the proof of the chain rule, we just have to use the Taylor expansion of ϕ

$$\frac{1}{\varepsilon}\left(\phi(F\circ\pi_u^\varepsilon) - \phi(F)\right) = \phi'(F)\frac{1}{\varepsilon}(F\circ\pi_u^\varepsilon - F) + \frac{1}{\varepsilon}\left((F\circ\pi^\varepsilon - F)^2\, O(1)\right) \wedge K,$$

for some $K > 0$. Then the second term converges to zero. □

6.2.2. *Fundamental examples and counterexamples of derivability.*

By Lemma 6.2 the knowledge of the jump-times and initial condition is equivalent to the knowledge of the canonical process X. Instead of cylindric functionals $F(X) = f(X_{t_1}, \dots, X_{t_n})$ of the canonical process, it is natural in the context of unit jump processes to consider functionals of the jump-times:

$$\mathcal{S}_{\mathbb{J}_1} := \left\{F : \mathbb{J}_1(\mathcal{I}) \to \mathbb{R},\ F((x,\{t_1,\dots,t_m\})) = f(x,t_1,\dots,t_{n\wedge m},1,\dots),\ f \in C_b^\infty(\mathbb{R}\times\mathcal{I}^n),\ n\in\mathbb{N}\right\}.$$

For $F \in \mathcal{S}_{\mathbb{J}_1}$ we have $F(X) = f(X_0, T_1\wedge 1, \dots, T_n \wedge 1)$ under the convention $\infty \wedge 1 = 1$. We use the shorthand $F(X) = f_{\wedge 1}(X_0, T_1, \dots, T_n) := f(X_0, T_1\wedge 1, \dots, T_n\wedge 1) \in \mathcal{S}_{\mathbb{J}_1}$.

Proposition 6.12. *Let $\mathbb{Q}$ be an arbitrary probability on the space of unit jump paths $\mathbb{J}_1(\mathcal{I})$. Then all functionals $F(X) = f_{\wedge 1}(X_0, T_1, \dots, T_n) \in \mathcal{S}_{\mathbb{J}_1}$ are differentiable in direction of any bounded function $u : \mathcal{I} \to \mathbb{R}$. The derivative is given by $\mathcal{D}_u F(X) = \int_{\mathcal{I}} \mathcal{D}_t F(X)\, u_t dt$ with*

$$\mathcal{D}_t F(\omega) = \mathcal{D}_t F((x,\{t_1,\dots,t_m\})) := -\sum_{i=1}^{m\wedge n} \partial_{1+i} f(x,t_1,\dots,t_{n\wedge m},1,\dots)\mathbb{1}_{[0,t_i]}(t).$$

Proof. Let $F(X) = f_{\wedge 1}(X_0, T_1, \dots, T_n) \in \mathcal{S}_{\mathbb{J}_1}$ and a bounded $u : \mathcal{I} \to \mathbb{R}$ be arbitrary. By definition of the perturbation

$$F(X)\circ\pi_u^\varepsilon = f_{\wedge 1}(X_0, T_1 + \varepsilon\langle u\rangle_{T_1}, \dots, T_n + \varepsilon\langle u\rangle_{T_n}),$$

where $T_i + \varepsilon\langle u\rangle_{T_i} := \infty$ if $T_i = \infty$ already. We use the Taylor expansion to get

$$-\frac{1}{\varepsilon}\left(F(X)\circ\pi_u^\varepsilon - F(X)\right) = -\sum_{i=1}^{\eta\wedge n} \partial_{1+i} f_{\wedge 1}(X_0, T_1, \dots, T_n)\langle u\rangle_{T_i} + O(\varepsilon).$$

Then dominated convergence gives the $\mathbb{L}^2(\mathbb{Q})$-limit. □

Unfortunately it is fairly easy to find functionals that are not differentiable. The following example explains why functionals of the type $F(X) = f(X_{t_1}, \dots, X_{t_n}) \in \mathcal{S}$ are in general not differentiable.

Example 6.13. *Let $\mathbb{P}$ be the law of a Poisson process. Take $F = X_t \in \mathbb{L}^2(\mathbb{P})$ for some fixed $t \in \mathcal{I}$ and chose $u \in \mathcal{E}$ such that $u \geq 0$ on $\mathcal{I}$ and $\langle u \rangle_t > 0$. We define the time-inversion related to the perturbation π_u^ε by*

$$\int_{[0,\tau_u^\varepsilon(t)]} (1 + \varepsilon u_s) ds = t. \tag{6.14}$$

Then $\tau_u^\varepsilon : \mathcal{I} \to [0, \tau_u^\varepsilon(1)]$ is a deterministic function and $(\tau_u^\varepsilon)^{-1}(t) = t + \varepsilon \langle u \rangle_t$. The perturbed functional is

$$F \circ \pi_u^\varepsilon = X_{\tau_u^\varepsilon(t)},$$

since by $u \geq 0$ and $\langle u \rangle_t > 0$ we have $\tau_u^\varepsilon(t) < t$ and $\tau_u^\varepsilon(t) \to t$ if $\varepsilon \to 0$. Since X is stochastically continuous under $\mathbb{P}$ we get

$$-\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (X_{\tau_u^\varepsilon(t)} - X_t) = 0 \ \textit{holds } \mathbb{P}\textit{-a.s.}$$

Thus if a derivative of X_t in direction u in the sense of Definition 6.9 exists, it has to be zero: The $\mathbb{L}^2(\mathbb{P})$-convergence of the perturbed functional $X_{\tau_u^\varepsilon(t)}$ would imply the almost sure convergence of a subsequence to the $\mathbb{L}^2(\mathbb{P})$-limit. But since the almost sure limit exists and is zero, the $\mathbb{L}^2(\mathbb{P})$-limit needs to be zero too. But since $t - \tau_u^\varepsilon(t) = \varepsilon \langle u \rangle_{\tau_u^\varepsilon(t)}$ we get

$$\mathbb{E}\left(\left(\frac{1}{\varepsilon} (X_t - X_{\tau_u^\varepsilon(t)}) - 0 \right)^2 \right) = \frac{1}{\varepsilon^2} \sum_{k=0}^{\infty} k^2 e^{-\varepsilon \langle u \rangle_{\tau_u^\varepsilon(t)}} \frac{(\varepsilon \langle u \rangle_{\tau_u^\varepsilon(t)})^k}{k!} = \frac{1}{\varepsilon^2} \left(\varepsilon^2 \langle u \rangle^2_{\tau_u^\varepsilon(t)} + \varepsilon \langle u \rangle_{\tau_u^\varepsilon(t)} \right),$$

which diverges for $\varepsilon \to 0$. Thus X_t is not differentiable in direction u if $\langle u \rangle_t > 0$.

In the next example we present a class of jump-time functionals that are not generally differentiable.

Example 6.15. *Given a sequence $(f_j)_{j \geq 0}$ of functions $f_j \in C_b^\infty(\mathbb{R} \times \mathcal{I}^j)$ we define*

$$F(\omega) = F((x, \{t_1, \ldots, t_m\})) := \sum_{j=0}^{\infty} f_j(x, t_1, \ldots, t_j) \mathbb{1}_{\{j=m\}}. \tag{6.16}$$

In the canonical form such functionals are conveniently given by $F(X) = f_\eta(X_0, T_1, \ldots, T_\eta)$. Using an alternating sequence of $(f_j)_{j \geq 0}$ we present a specific example of such functionals that is not differentiable with respect to the bridge of a Poisson process. Take $f_{2j} \equiv 1$ and $f_{2j+1} \equiv 0$ for $j \in \mathbb{N}$ and $u \equiv 1 \in \mathcal{E}$, then

$$\frac{1}{\varepsilon} \left| F(X \circ \pi_u^\varepsilon) - F(X) \right| = \frac{1}{\varepsilon} (\mathbb{1}_{\eta \circ \pi_u^\varepsilon = \eta - 1} + \mathbb{1}_{\eta \circ \pi_u^\varepsilon = \eta - 3} + \ldots),$$

and with respect to a Poisson bridge $\mathbb{P}^{0,1}(\,.\,) = \mathbb{P}(\,.\,|X_0 = 0, X_1 = 1)$ the a.s. limit of the above is zero since $\mathbb{P}^{0,1}(T_1 = 1) = 0$. Remark that $\tau_u^\varepsilon(t) = t/(1 + \varepsilon)$ for $u \equiv 1$. We use $\mathbb{P}^{0,1}(\eta = 1) = 1$ to compute

$$\begin{aligned}
\mathbb{E}^{0,1}\left(\left(\frac{1}{\varepsilon} (F(X \circ \pi_u^\varepsilon) - F(X)) - 0 \right)^2 \right)^{\frac{1}{2}} &= \frac{1}{\varepsilon} \mathbb{P}^{0,1}(\eta \circ \pi_u^\varepsilon = 0)^{\frac{1}{2}} \\
&= \frac{1}{\varepsilon} \mathbb{P}^{0,1}(X_{\tau_u^\varepsilon(1)} = 0)^{\frac{1}{2}} \\
&= \frac{1}{\varepsilon} \left(\mathbb{P}(X_{\tau_u^\varepsilon(1)} = 0, X_1 = 1 | X_0 = 0) / \mathbb{P}(X_1 = 1 | X_0 = 0) \right)^{\frac{1}{2}} \\
&= \frac{1}{\varepsilon} \frac{\sqrt{\varepsilon}}{\sqrt{1 + \varepsilon}},
\end{aligned}$$

which diverges for $\varepsilon \to 0$. We conclude that functionals of the form (6.16) are not in general differentiable in the sense of Definition 6.9.

In what follows, we are in particular interested in the differentiability of functionals of the stochastic integral type. Let us remark that the stochastic integral over the canonical unit jump process is always well defined as the finite sum

$$\int_{\mathcal{I}} u_s dX_s := \sum_{i=1}^{\eta} u(T_i) \text{ for any measurable } u : \mathcal{I} \times \mathbb{J}_1(\mathcal{I}) \to \mathbb{R}.$$

We define the compensated integral

$$\delta(u) := \int_{\mathcal{I}} u_s(dX_s - ds),\ u : \mathcal{I} \times \mathbb{J}_1(\mathcal{I}) \to \mathbb{R} \text{ predictable},\ u(.,\omega) \in \mathbb{L}^1(dt),\ \forall \omega \in \mathbb{J}_1(\mathcal{I}). \tag{6.17}$$

It will be the dual operator of the derivative operator $\mathcal{D}$ with respect to a Poisson process. In Example 6.13 we have seen that $\delta(\mathbb{1}_{[0,t]}) = X_t - X_0 - t$ is not differentiable with respect to the law $\mathbb{P}$ of a Poisson process. In the last two examples we develop conditions that guarantee differentiability for the compensated integrals $\delta(u)$.

Example 6.18. *Let $\mathbb{Q}$ be an arbitrary probability on $\mathbb{J}_1(\mathcal{I})$, $v \in C_b^1(\mathcal{I})$ be deterministic and the total number of jumps $\eta \in \mathbb{L}^2(\mathbb{Q})$ be square integrable. Then the functional $\delta(v)$ is differentiable in direction of every $u \in \mathcal{E}$ with $\langle u \rangle = 0$.*

Proof. We know that

$$\delta(v) = \sum_{i=1}^{\eta} v(T_i) - \int_{\mathcal{I}} v_s ds, \text{ and } \delta(v) \circ \pi_u^\varepsilon = \sum_{i=1}^{\eta \circ \pi_u^\varepsilon} v(T_i + \varepsilon \langle u \rangle_{T_i}) - \int_{\mathcal{I}} v_s ds.$$

Under the assumption $\langle u \rangle = 0$ we have

$$t + \varepsilon \langle u \rangle_t = t - \varepsilon \int_{[t,1]} u_s ds \leq t + \varepsilon K(1-t) \leq 1,\ \forall t \in \mathcal{I},$$

if $\varepsilon K < 1$, where $K > 0$ is a global bound for the absolute value of u. Therefore $\eta \circ \pi_u^\varepsilon = \eta$ identically on $\mathbb{J}_1(\mathcal{I})$ for small $\varepsilon > 0$, and we can apply the differentiability of $v \in C_b^1(\mathcal{I})$ to see that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (\delta(v) \circ \pi_u^\varepsilon - \delta(v)) = \sum_{i=1}^{\eta} v'(T_i) \langle u \rangle_{T_i} \quad \text{holds } \mathbb{Q}\text{-}a.s. \tag{6.19}$$

Since the increment of X is square-integrable and $v'_\cdot \langle u \rangle_\cdot$ is bounded we can apply dominated convergence to get the $\mathbb{L}^2(\mathbb{Q})$-convergence. □

In particular we have seen that the derivative of $\delta(v)$ in direction $u \in \mathcal{E}$ with $\langle u \rangle = 0$ is given by (6.19). We define the process

$$\mathcal{D}\delta(v) = (\mathcal{D}_t \delta(v))_{t \in \mathcal{I}}, \text{ with } \mathcal{D}_t \delta(v) = -\sum_{i=1}^{\eta} v'(T_i) \mathbb{1}_{[0,T_i]}(t), \tag{6.20}$$

even if $\delta(v)$ is not differentiable in directions of bounded u with $\langle u \rangle \neq 0$. An additional assumption on v will allow differentiation in every direction.

Example 6.21. *Let $\mathbb{Q}$ be a probability on $\mathbb{J}_1(\mathcal{I})$ such that $\eta \in \mathbb{L}^2(\mathbb{Q})$ and $v \in C_c^1(\mathcal{I})$ in the sense that $v = 0$ in a neighborhood of $t = 0$ and $t = 1$. Then the functional $\delta(v)$ is differentiable in every direction $u \in \mathcal{E}$ with respect to $\mathbb{Q}$.*

Proof. By assumption there exists a constant $\delta > 0$ such that $v_t = 0$ for all $t \in [2\delta, 1-2\delta]^c$. Define $\eta_{1-\delta} := X_{1-\delta} - X_0$ as the number of jumps up to time $t = 1-\delta$. Then $\delta(v) = \sum_{i=1}^{\eta_{1-\delta}} v(T_i) - \langle v \rangle$, and

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\left(\left(\frac{1}{\varepsilon}\left(\sum_{i=1}^{\eta_{1-\delta}} v(T_i + \varepsilon\langle u\rangle_{T_i}) - \sum_{i=1}^{\eta_{1-\delta}} v(T_i)\right) - \sum_{i=1}^{\eta_{1-\delta}} v'(T_i)\langle u\rangle_{T_i}\right)^2\right)^{\frac{1}{2}} \\
\leq \quad &\sup_{t\in\mathcal{I}}\left\{\frac{v(t+\varepsilon\langle u\rangle_t) - v(t)}{\varepsilon} - v'(t)\langle u\rangle_t\right\}\mathbb{Q}((X_{1-\delta} - X_0)^2)^{\frac{1}{2}} \to 0, \text{ for } \varepsilon \to 0.
\end{aligned}$$

Furthermore if ε is small enough such that $\varepsilon|\langle u\rangle_{1-2\delta}| < \delta$ and $\varepsilon|\langle u\rangle_{1-\delta}| < \delta$, then

$$\sum_{i=1}^{\eta_{1-\delta}} v(T_i + \varepsilon\langle u\rangle_{T_i}) = \sum_{i=1}^{\eta_{1-\delta}\circ\pi_u^\varepsilon} v(T_i + \varepsilon\langle u\rangle_{T_i}),$$

which in combination with the above implies the result. □

We define the derivative of $\mathcal{D}\delta(v)$ of $\delta(v)$ as given in Example 6.21 by (6.20).

6.3. **The Poisson process and its reciprocal class.**

In the first part of this paragraph we prove a duality formula for a Poisson process that includes the derivative operator introduced in Definition 6.9 on the one hand, and the compensated integral (6.17) on the other hand. We can prove that Poisson processes are the only unit jump processes satisfying this formula, a characterization result that uses a different duality formula as the one given in Proposition 1.21. In § 6.3.3 we obtain a new characterization of the reciprocal class of a Poisson process. This characterization is achieved under a loop condition on the test functions in the duality formula.

6.3.1. *Characterization of the Poisson process by a duality formula.*

The integral operator (6.17) and the derivative (6.10) are dual operators in the following sense.

Proposition 6.22. *Let X be a Poisson process under $\mathbb{P}$. Then the duality formula*

$$\mathbb{E}\,(F\delta(u)) = \mathbb{E}\,(\mathcal{D}_u F) \tag{6.23}$$

holds for any $u \in \mathbb{L}^2(dt)$ and $F \in \mathbb{L}^2(\mathbb{P})$ that is differentiable.

Proof. For arbitrary $u \in \mathcal{E}$ we define the stochastic exponential process G^ε as the unique solution of the stochastic integral equation

$$G_t^\varepsilon = 1 + \varepsilon \int_{[0,t]} G_{s-}^\varepsilon u_s d(X_s - s).$$

Let us first show that $X^\varepsilon := X \circ \pi_u^\varepsilon$ is a Poisson process with respect to the probability $\mathbb{P}^\varepsilon := G_1^\varepsilon \mathbb{P}$ on $\mathbb{J}_1(\mathcal{I})$. Girsanov's theorem implies that X has intensity $(1+\varepsilon u_t)dt$ with respect to $\mathbb{P}^\varepsilon$. Take any $s < t \in \mathcal{I}$ then

$$\begin{aligned}
\mathbb{E}^\varepsilon(X_t^\varepsilon - X_s^\varepsilon | X_r^\varepsilon,\, r \leq s) &= \mathbb{E}^\varepsilon(X_{\tau_u^\varepsilon(t)} - X_{\tau_u^\varepsilon(s)} | \mathcal{F}_{[0,\tau_u^\varepsilon(s)]}) \\
&= \tau_u^\varepsilon(t) + \varepsilon\langle u\rangle_{\tau_u^\varepsilon(t)} - \tau_u^\varepsilon(s) - \varepsilon\langle u\rangle_{\tau_u^\varepsilon(s)} \\
&= t - s,
\end{aligned}$$

where τ_u^ε was defined in (6.14). Therefore $t \mapsto X_t^\varepsilon - X_0 - t$ is a martingale with respect to $\mathbb{P}^\varepsilon$ and its proper filtration. By Lemma 6.8 the law of X^ε with respect to $\mathbb{P}^\varepsilon$ is that of a Poisson process. We deduce the identity

$$\frac{1}{\varepsilon}\mathbb{E}\left(F \circ \pi_u^\varepsilon(G_1^\varepsilon - 1)\right) = -\frac{1}{\varepsilon}\mathbb{E}\left(F \circ \pi_u^\varepsilon - F\right), \tag{6.24}$$

for any differentiable $F \in \mathbb{L}^2(\mathbb{P})$ since $\mathbb{E}\left(F \circ \pi_u^\varepsilon G^\varepsilon\right) = \mathbb{E}\left(F\right)$. In the limit $\varepsilon \to 0$ this is the duality formula: The right side converges to $\mathcal{D}_u F$ by definition of the derivative. The convergence of the left side follows from Gronwall's lemma and the $\mathbb{L}^2(\mathbb{P})$ isometry of the compensated Poisson integral, which imply

$$\mathbb{E}\left((G_t^\varepsilon - 1)^2\right)^{\frac{1}{2}} \leq \varepsilon K\left(1 + \int_{[0,t]} \mathbb{E}\left((G_{s-}^\varepsilon - 1)^2\right)^{\frac{1}{2}} ds\right) \leq \varepsilon K e^{\varepsilon K t}.$$

where $\|u\|_\infty \leq K$. This permits us to compute

$$\mathbb{E}\left(\left(\frac{1}{\varepsilon}(G_1^\varepsilon - 1) - \delta(u)\right)^2\right)^{\frac{1}{2}} \leq K \int_{[0,1]} \mathbb{E}\left((G_{t-}^\varepsilon - 1)^2\right)^{\frac{1}{2}} dt \leq \varepsilon K^2 e^{\varepsilon K}.$$

The first inequality implies that

$$\frac{1}{\varepsilon}(F \circ \pi_u^\varepsilon - F)(G_1^\varepsilon - 1) \to 0 \text{ in } \mathbb{L}^1(\mathbb{P}),$$

since $F \circ \pi_u^\varepsilon \to F$ in $\mathbb{L}^2(\mathbb{P})$. The second inequality implies that $\frac{1}{\varepsilon}(G_1^\varepsilon - 1) \to \delta(u)$ in $\mathbb{L}^2(\mathbb{P})$ and since $F \in \mathbb{L}^2(\mathbb{P})$

$$\frac{1}{\varepsilon}F \circ \pi_u^\varepsilon(G_1^\varepsilon - 1) = \frac{1}{\varepsilon}(F \circ \pi_u^\varepsilon - F)(G_1^\varepsilon - 1) + \frac{1}{\varepsilon}F(G_1^\varepsilon - 1) \to F\delta(u) \text{ in } \mathbb{L}^1(\mathbb{P}).$$

The extension of the test functions to $u \in \mathbb{L}^2(dt)$ follows by a density argument. □

The duality formula (6.23) holds in particular for $F \in \mathcal{S}_{\mathbb{J}_1}$ and $u \in \mathcal{E}$. This will be enough to characterize the distribution of a Poisson process on the space of unit jump functions.

The following theorem is fundamental to us since it is a first step to the characterization of the reciprocal class of a Poisson process. We give one complete proof and outline another one. In the first proof, we show that the duality formula "contains" Watanabe's characterization of Lemma 6.8. The second proof is an outline of an iteration procedure to derive the characteristic functional of a Poisson process. This latter approach is due to Nicolas Privault, who is hereby gratefully acknowledged for sharing his ideas in a private communication during the conference *"Applications of Stochastic Processes VI"* at the *University of Potsdam*. We extend it to characterize the reciprocal class of a Poisson process.

Theorem 6.25. *Let $\mathbb{Q}$ be an arbitrary probability on $\mathbb{J}_1(\mathcal{I})$ with $\eta \in \mathbb{L}^1(\mathbb{P})$. If for all $F \in \mathcal{S}_{\mathbb{J}_1}$ and $u \in \mathcal{E}$ the duality formula*

$$\mathbb{E}_{\mathbb{Q}}\left(F(X)\delta(u)\right) = \mathbb{E}_{\mathbb{Q}}\left(\mathcal{D}_u F(X)\right) \tag{6.26}$$

holds, then X is a Poisson process under $\mathbb{Q}$.

First proof. Let $s \in \mathcal{I}$ be arbitrary, then there exists a sequence of functions $(f_i)_{i\geq 1} \subset C_b^\infty(\mathcal{I}, \mathcal{I})$ with $f_i|_{[s,1]} = s$ such that $\|f_i(\,.\,) - (\,.\,\wedge s)\|_\infty \to 0$ for $i \to \infty$. For any $F(X) = f_{\wedge 1}(X_0, T_1, \ldots, T_n) \in \mathcal{S}_{\mathbb{J}_1}$ we may define the jump-time functional $F_i(X) := f(X_0, f_i(T_1), \ldots, f_i(T_n)) \in \mathcal{S}_{\mathbb{J}_1}$. Then $F_i \to f(X_0, T_1 \wedge s, \ldots, T_n \wedge s)$ uniformly in ω. By Lemma 6.2 the functionals F_i are $\mathcal{F}_{[0,s]}$-measurable and by Proposition 6.12 we have that $\langle \mathcal{D}F_i(X), \mathbb{1}_{(s,t]}\rangle = 0$ holds $\mathbb{Q}$-*a.s.* for any $t > s$. A density argument implies $\mathbb{Q}(X_t - X_s|\mathcal{F}_{[0,s]}) = t - s$ such that $t \mapsto X_t - X_0 - t$

is a martingale under $\mathbb{Q}$. With Watanabe's characterization presented in Lemma 6.8 we conclude that X is a Poisson process. □

Second proof. In the second proof we compute the characteristic functional of $\mathbb{Q}$. We need two technical results.

Lemma 6.27. *Assume that $v \in C^1(\mathcal{I})$ with $v(1) = 0$. Then the commutation relation $\mathcal{D}_t\delta(v) = v_t + \delta(\kappa_t v)$ holds, where $\kappa_t v_s := -v_s' \mathbb{1}_{[0,s]}(t)$.*

Proof. Just use the definition

$$\delta(\kappa_t v) = \delta(-v_\cdot' \mathbb{1}_{[0,\cdot]}(t)) = -\sum_{k=1}^{\eta} v'(T_k)\mathbb{1}_{[0,T_k]}(t) + \int_{[t,1]} v_s' ds = \mathcal{D}_t\delta(v) - v_t,$$

and the last equality is implied by (6.20). □

Lemma 6.28. *For $u, v \in C^1(\mathcal{I})$ with $u(1) = 0$ we have $\langle v(\kappa u)^n, u\rangle = \frac{1}{(n+1)!}\langle v, u^{n+1}\rangle$, where*

$$\begin{aligned} v(\kappa u)_t^n &:= \int_{\mathcal{I}^n} v_{t_1}\kappa_{t_1}u_{t_2}\kappa_{t_2}u_{t_3}\cdots\kappa_{t_n}u_t dt_1\cdots dt_n \\ &= (-1)^n \int_{[0,t]}\left(\cdots\left(\int_{[0,t_2]} v_{t_1}u_{t_2}'\cdots u_{t_n}'u_t' dt_1\right)\cdots\right)dt_n. \end{aligned}$$

is bounded.

Proof. For $n = 0$ the equation holds trivially. For $n = 1$ we have

$$\begin{aligned} \int_{\mathcal{I}^2} v_{t_1}\kappa_{t_1}u_{t_2}u_{t_2}dt_1dt_2 &= -\int_{\mathcal{I}^2} v_{t_1}u_{t_2}'\mathbb{1}_{[0,t_2]}(t_1)u_{t_2}dt_1dt_2 \\ &= -\int_{\mathcal{I}} v_{t_1}\int_{[t_1,1]} u_{t_2}'u_{t_2}dt_2dt_1 \\ &= \int_{\mathcal{I}} v_{t_1}\left(u_{t_1}^2 + \int_{[t_1,1]} u_{t_2}u_{t_2}'dt_2\right)dt_1 \\ &= \langle v, u^2\rangle - \int_{\mathcal{I}^2} v_{t_1}\kappa_{t_1}u_{t_2}u_{t_2}dt_1dt_2. \end{aligned}$$

Using the same calculus for arbitrary $n \in \mathbb{N}$ we see that

$$\begin{aligned} \langle v(\kappa u)^n, u\rangle &= \int_{\mathcal{I}^{n+1}} v_{t_1}\kappa_{t_1}u_{t_2}\kappa_{t_2}u_{t_3}\cdots\kappa_{t_n}u_{t_{n+1}}u_{t_{n+1}}dt_1\cdots dt_{n+1} \\ &= \langle v(\kappa u)^{n-1}, u^2\rangle - \langle v(\kappa u)^n, u\rangle \\ &= \langle v(\kappa u)^{n-2}, u^3\rangle - 2\langle v(\kappa u)^{n-1}, u^2\rangle - \langle v(\kappa u)^n, u\rangle, \end{aligned}$$

which by iteration proves the result. □

Now we may continue with the second proof of Theorem 6.25. We will compute the Laplace transform of the random variable $\delta(u)$ for $u \in C_c^\infty(\mathcal{I})$ such that $u \geq 0$. Let $\lambda < 0$, then the duality formula holds for $F_n = \exp\left(\lambda\left(\sum_{i=1}^n u(T_i \wedge 1) - \langle u\rangle\right)\right) \in \mathcal{S}_{\mathbb{J}_1}$ and u. Considering Lemma 6.11 the functional $F = \exp(\lambda\delta(u))$ is differentiable in direction u. Dominated convergence implies that the duality formula holds for F and u, since $F_n \to F$ converges

$\mathbb{Q}$-*a.s.* An application of the duality formula and Lemma 6.27 gives

$$\begin{aligned}\partial_\lambda \mathbb{E}_\mathbb{Q}\left(e^{\lambda\delta(u)}\right) &= \mathbb{E}_\mathbb{Q}\left(\delta(u)e^{\lambda\delta(u)}\right)\\ &= \mathbb{E}_\mathbb{Q}\left(\langle u, \mathcal{D}e^{\lambda\delta(u)}\rangle\right)\\ &= \lambda\mathbb{E}_\mathbb{Q}\left(\langle u, \mathcal{D}\delta(u)\rangle e^{\lambda\delta(u)}\right)\\ &= \lambda\mathbb{E}_\mathbb{Q}\left(\langle u, u\rangle e^{\lambda\delta(u)}\right) + \lambda\mathbb{E}_\mathbb{Q}\left(\int_\mathcal{I} u_s\delta(\kappa_s u)ds e^{\lambda\delta(u)}\right).\end{aligned}$$

Since $\kappa_s u_\cdot$ is bounded we can use Lemma 6.27 and the duality formula for F and $\kappa_s u$ on the second term:

$$\begin{aligned}\partial_\lambda \mathbb{E}_\mathbb{Q}\left(e^{\lambda\delta(u)}\right) &= \lambda\mathbb{E}_\mathbb{Q}\left(\langle u, u\rangle e^{\lambda\delta(u)}\right) + \lambda\mathbb{E}_\mathbb{Q}\left(\int_{\mathcal{I}^2} u_{t_1}\kappa_{t_1}u_{t_2}\mathcal{D}_{t_2}e^{\lambda\delta(u)}dt_1dt_2\right)\\ &= \lambda\mathbb{E}_\mathbb{Q}\left(\langle u, u\rangle e^{\lambda\delta(u)}\right) + \lambda^2\mathbb{E}_\mathbb{Q}\left(\int_{\mathcal{I}^2} u_{t_1}\kappa_{t_1}u_{t_2}\mathcal{D}_{t_2}\delta(u)dt_1dt_2e^{\lambda\delta(u)}\right)\\ &= \lambda\mathbb{E}_\mathbb{Q}\left(\langle u, u\rangle e^{\lambda\delta(u)}\right) + \lambda^2\mathbb{E}_\mathbb{Q}\left(\int_{\mathcal{I}^2} u_{t_1}\kappa_{t_1}u_{t_2}u_{t_2}dt_1dt_2e^{\lambda\delta(u)}\right)\\ &\quad +\lambda^2\mathbb{E}_\mathbb{Q}\left(\int_{\mathcal{I}^2} u_{t_1}\kappa_{t_1}u_{t_2}\delta(\kappa_{t_2}u)dt_1dt_2e^{\lambda\delta(u)}\right).\end{aligned}$$

Iterating this on the third term, and then on the m'th term we get

$$\begin{aligned}\partial_\lambda \mathbb{E}_\mathbb{Q}\left(e^{\lambda\delta(u)}\right) &= \lambda\sum_{m=0}^\infty \lambda^m\mathbb{E}_\mathbb{Q}\left(\langle(\kappa u)^m u, u\rangle e^{\lambda\delta(u)}\right)\\ &= \sum_{m=0}^\infty \frac{\lambda^{m+1}}{(m+1)!}\int_\mathcal{I} u_s^{m+2}ds\,\mathbb{E}_\mathbb{Q}(e^{\lambda\delta(u)})\\ &= \mathbb{E}_\mathbb{Q}(e^{\lambda\delta(u)})\sum_{m=1}^\infty \frac{\lambda^m}{m!}\int_\mathcal{I} u_s^{m+1}ds\\ &= \mathbb{E}_\mathbb{Q}(e^{\lambda\delta(u)})\langle e^{\lambda u}-1, u\rangle,\end{aligned}$$

where we applied Lemma 6.28 to get the second equality. The unique solution of this differential equation with initial condition $\mathbb{Q}(e^0) = 1$, when evaluated in $\lambda = -1$, is

$$\mathbb{E}_\mathbb{Q}\left(e^{-\delta(u)}\right) = \exp\left(\int_\mathcal{I}(e^{-u_s}+u_s-1)ds\right),$$

for every $u \in C_c^\infty(\mathcal{I})$, $u \geq 0$. Thus by identification of the Laplace transform, the law of the increments of X on $[0+\varepsilon, 1-\varepsilon]$ for small $\varepsilon > 0$ is identical to the increments of a Poisson process. Stochastic continuity implies that $\mathbb{Q}$ is a Poisson process. □

Remark 6.29. *In Example 3.3 we mentioned a characterization of an exponential distribution introduced by Diaconis and Zabell. Following* [DZ91] *a random variable Z has exponential distribution if and only if*

$$\mathbb{E}\left(f(Z)(Z-1)\right) = \mathbb{E}\left(f'(Z)Z\right),$$

for every smooth function $f \in C_c^\infty(\mathbb{R})$. But the duality formula (6.26) applied to a function of the first jump time $f(T_1 \wedge 1)$ and the predictable process $u = \mathbb{1}_{[0,T_1]}$ is

$$\mathbb{E}\left(f(T_1\wedge 1)(1-T_1\wedge 1)\right) = -\mathbb{E}\left(f'(T_1\wedge 1)(T_1\wedge 1)\mathbb{1}_{\{T_1\leq 1\}}\right).$$

The duality formula (6.26) may thus be interpreted as a formula characterizing the (truncated) exponential distribution of the times between the jumps. In contrast, the duality formula (1.22) characterizes the Poisson distribution of the increments $X_t - X_s$ and thus contains a difference instead of a derivative operator as dual of the stochastic integral.

6.3.2. *The reciprocal class of a Poisson process.*

Following Definition 4.12 a unit jump process $\mathbb{Q}$ is in the reciprocal class $\mathcal{R}(\mathbb{P})$ of the law of a Poisson process $\mathbb{P}$ whenever the disintegration

$$\mathbb{Q}(\,.\,) = \int_{\mathbb{R}^2} \mathbb{P}^{x,y}(\,.\,)\mathbb{Q}_{01}(dxdy) \tag{6.30}$$

with respect to the Poisson bridges holds. Note that $\mathbb{P}^{x,y}$ is well defined if $y - x \in \mathbb{N}$, but since $\mathbb{Q}$ is a probability on $\mathbb{J}_1(\mathcal{I})$ we always have $\mathbb{Q}(X_1 - X_0 \in \mathbb{N}) = 1$.

In this paragraph we want to examine some examples of processes in the reciprocal class $\mathcal{R}(\mathbb{P})$. We denote by $\mathbb{P}_\alpha$ a Poisson process with intensity $\alpha > 0$, which means that under $\mathbb{P}_\alpha$ the canonical process X has stationary and independent increments with Poisson distribution $X_t - X_s \sim \mathcal{P}(\alpha(t-s))$. Surprisingly Poisson processes of any intensity are in the same reciprocal class.

Proposition 6.31. *Let $\mathbb{P}_\alpha$ be a Poisson process with intensity $\alpha > 0$. Then the reciprocal class of $\mathbb{P}_\alpha$ coincides with the reciprocal class of a Poisson process with unit intensity: $\mathcal{R}(\mathbb{P}_\alpha) = \mathcal{R}(\mathbb{P})$.*

Proof. By the disintegration formula (6.30) we only have to show that $\mathbb{P}_\alpha^{x,y} = \mathbb{P}^{x,y}$ where $y = x + m$, $m \in \mathbb{N}$. Let $0 \le k_1 \le \cdots \le k_n \le m$ and $\{t_1, \dots, t_n\} \in \Delta_{\mathcal{I}}$ for arbitrary $n \in \mathbb{N}$. Then

$$\begin{aligned}
& \mathbb{P}_\alpha^{x,y}(X_{t_1} = x + k_1, \dots, X_{t_n} = x + k_n) \\
= \;& \mathbb{P}_\alpha^{x}(X_{t_1} = x + k_1, \dots, X_{t_n} = x + k_n | X_1 = x + m) \\
= \;& \mathbb{P}_\alpha^{x}(X_{t_1} = x + k_1, \dots, X_{t_n} = x + k_n, X_1 = x + m)\,(\mathbb{P}_\alpha^{x}(X_1 = x + m))^{-1} \\
= \;& e^{-t_1\alpha}\frac{(t_1\alpha)^{k_1}}{k_1!} e^{-(t_2-t_1)\alpha}\frac{((t_2-t_1)\alpha)^{k_2-k_1}}{(k_2-k_1)!} \cdots e^{-(1-t_n)\alpha}\frac{((1-t_n)\alpha)^{m-k_n}}{(m-k_n)!} e^{\alpha}\frac{m!}{\alpha^m} \\
= \;& \frac{t_1^{k_1}}{k_1!}\frac{(t_2-t_1)^{k_2-k_1}}{(k_2-k_1)!} \cdots \frac{(1-t_n)^{m-k_n}}{(m-k_n)!} m!
\end{aligned}$$

does not depend on α. Since $\mathcal{F}_{\mathcal{I}} = \sigma(X_s,\, s \in \mathcal{I})$ is cylindric this implies $\mathbb{P}_\alpha^{x,y} = \mathbb{P}^{x,y}$ for any $\alpha > 0$. □

Clearly $\mathbb{P}_\alpha^x(X_1 \in \,.\,) \ll \mathbb{P}^x(X_1 \in \,.\,)$ for every $x \in \mathbb{R}$. Since $\mathbb{P}_\alpha^x \in \mathcal{R}(\mathbb{P})$ we can write $\mathbb{P}_\alpha^x$ as an h-transform of $\mathbb{P}^x$ with

$$h(y) = \frac{\mathbb{P}_\alpha^x(X_1 = y)}{\mathbb{P}^x(X_1 = y)} = e^{-\alpha}\frac{\alpha^{y-x}}{(y-x)!} e^1 \frac{(y-x)!}{1} = e^{(y-x)\log\alpha - (\alpha-1)}. \tag{6.32}$$

In the next example we introduce a Poisson bridge as an h-transform and compute its intensity.

Example 6.33. *Let $\mathbb{P}^{x,y}$ be the distribution of a Poisson bridge with $y - x = m \in \mathbb{N}$. For any $t \in \mathcal{I}$ and $0 \leq k \leq m$ we compute*

$$\begin{aligned}\mathbb{P}^{x,y}(X_t = x + k) &= \mathbb{P}^x(X_t = x + k | X_1 = x + m) \\ &= \mathbb{P}^x(X_t = x + k, X_1 = x + m)(\mathbb{P}^x(X_1 = x + m))^{-1} \\ &= e^{-t}\frac{t^k}{k!}e^{-(1-t)}\frac{(1-t)^{m-k}}{(m-k)!}e^1\frac{m!}{1^m} \\ &= \binom{m}{k}t^k(1-t)^{m-k},\end{aligned}$$

which means that X_t has a binomial distribution with parameter m and t. By Proposition 4.10 the Poisson bridge has the Markov property. Let $s < t$, we may use the above calculus between the points X_s at time s and y at time 1 to compute that conditionally on X_s and $X_1 = y$ the random variable $X_t - X_s$ has binomial distribution with parameter $y - X_s$ and $\frac{t-s}{1-s}$. Therefore

$$\begin{aligned}\mathbb{E}^{x,y}\left(X_t - X_s | \mathcal{F}_{[0,s]}\right) &= \mathbb{E}^{x,y}(X_t - X_s | X_s) \\ &= \frac{t-s}{1-s}(y - X_s) \\ &= \int_{[s,t]} \frac{y - X_{s-}}{1-s} dr \\ &= \mathbb{E}^{x,y}\left(\int_{[s,t]} \frac{y - X_{r-}}{1-r} dr \,\middle|\, \mathcal{F}_{[0,s]}\right),\end{aligned}$$

where the last equality follows from the fact, that

$$\mathbb{E}^{x,y}\left(\frac{y - X_r}{1-r} \,\middle|\, X_s\right) = \frac{y - X_s}{1-s}.$$

We deduce that $X - \int_{[0,\cdot]} \frac{y - X_{t-}}{1-t} dt$ is a martingale. In other words

$$X \text{ has intensity } \ell(t, X_{t-})dt := \frac{y - X_{t-}}{1-t} dt \text{ with respect to } \mathbb{P}^{x,y}.$$

Remark that the intensity explodes for $t \to 1$ if the process has not yet reached its final state $X_1 = y$. Moreover the intensity is

- *strictly positive $\ell(t, z) > 0$ for $y - x > z - x \in \mathbb{N}$ and finite for $t \in [0, 1 - \varepsilon]$, $\varepsilon > 0$;*
- *zero $\ell(., y) \equiv 0$ on $\mathcal{I}$ since $\mathbb{P}^{x,y}(X_t = y + 1) = 0$ for all $t \in \mathcal{I}$.*

Note that the bridge $\mathbb{P}^{x,y}$ is an h-transform of $\mathbb{P}^x$ with

$$h(z) = \frac{\mathbb{P}^{x,y}(X_1 = z)}{\mathbb{P}^x(X_1 = z)} = \frac{(y-x)!}{e}\mathbb{1}_{\{y\}}(z).$$

This is different from the situation in Section 5: The law of the Brownian bridge is not absolutely continuous with respect to any Wiener measure on $C(\mathcal{I}, \mathbb{R}^d)$.

Example 6.34. *Let us now construct the process in $\mathcal{R}(\mathbb{P})$ that starts from zero and has Bernoulli distribution $\mathcal{B}_p$ with parameter $p \in (0, 1)$ at the endpoint $t = 1$. As a mixture of bridges this process is given by*

$$\mathbb{Q}(.) := (1-p)\mathbb{P}^{0,0}(.) + p\mathbb{P}^{0,1}(.).$$

Equivalently $\mathbb{Q} = h\mathbb{P}^0$ with

$$h(z) = \frac{p^z(1-p)^{1-z}}{e}\mathbb{1}_{\{0,1\}}(z).$$

We want to compute the predictable compensator of X with respect to $\mathbb{Q}$. Clearly $\mathbb{E}_{\mathbb{Q}}(X_t - X_s|X_s = 1) = 0$ and conditionally on $X_s = 0$ we get

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}(X_t - X_s|X_s = 0) &= \mathbb{E}_{\mathbb{Q}}(X_t - X_s|X_s = 0, X_1 = 0)\mathbb{Q}(X_1 = 0|X_s = 0) \\ &\quad + \mathbb{E}_{\mathbb{Q}}(X_t - X_s|X_s = 0, X_1 = 1)\mathbb{Q}(X_1 = 1|X_s = 0). \end{aligned} \tag{6.35}$$

The first summand is zero, as for the second we can use the intensity of the bridges derived in Example 6.33, that is

$$\mathbb{E}_{\mathbb{Q}}(X_t - X_s|X_s = 0, X_1 = 1) = \frac{t-s}{1-s},$$

and explicitly calculate the conditional probability

$$\mathbb{Q}(X_1 = 1|X_s = 0) = \frac{\mathbb{Q}^{0,1}(X_s = 0)\mathbb{Q}^0(X_1 = 1)}{\mathbb{Q}^{0,0}(X_s = 0)\mathbb{Q}^0(X_1 = 0) + \mathbb{Q}^{0,1}(X_s = 0)\mathbb{Q}^0(X_1 = 1)} = \frac{(1-s)p}{(1-p)+(1-s)p}$$

to get

$$\mathbb{E}_{\mathbb{Q}}(X_t - X_s|X_s = 0) = \frac{t-s}{1-s}\frac{(1-s)p}{1-ps} = \int_{[s,t]} \frac{p}{1-sp} dr.$$

By a similar computations

$$\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{0\}}(X_{r-})|X_s = 0) = \mathbb{Q}(X_{r-} = 0|X_s = 0) = \frac{1-rp}{1-sp}, \text{ for } r \geq s.$$

Combining these results, we derive the intensity of X under $\mathbb{Q}$, which is

$$\mathbb{E}_{\mathbb{Q}}(X_t - X_s|X_s) = \mathbb{E}_{\mathbb{Q}}\left(\int_{[s,t]} \frac{p}{1-rp} \mathbb{1}_{\{0\}}(X_{r-}) dr \,\middle|\, X_s \right).$$

In particular the intensity is zero if $X_s = 1$, and on the set $X_s = 0$ the intensity grows but does not explode for $s \to 1$. This growth depends on the value p: A larger probability of $X_1 = 1$ under $\mathbb{Q}$ implies a larger intensity around the endpoint for $s \to 1$ if $X_s = 0$.

We extend the preceding example from a Bernoulli to a Binomial endpoint distribution.

Example 6.36. *Let $\mathbb{Q}$ be the law in $\mathcal{R}(\mathbb{P})$ with respect to which X starts from zero and has Binomial distribution $\mathcal{B}_{k,p}$ with parameters $k \in \mathbb{N}$ and $p \in (0,1)$ at the endpoint $t = 1$. This probability is defined by the mixture of bridges*

$$\mathbb{Q}(\,.\,) := \sum_{i=0}^{k} \binom{k}{i} p^i (1-p)^{k-i} \mathbb{P}^{0,i}(\,.\,).$$

The predictable compensator of X under $\mathbb{Q}$ is computed by a similar decomposition as in (6.35) and is equal to

$$\int_{[0,t]} \mathbb{1}_{\{0,\dots,k-1\}}(X_{s-}) \left[\sum_{i=x+1}^{k} \frac{p^i(1-p)^{k-i}}{(k-i)!} \frac{(1-t)^{i-X_{s-}-1}}{(i-X_{s-}-1)!} \right] / \left[\sum_{i=X_{s-}}^{k} \frac{p^i(1-p)^{k-i}}{(k-i)!} \frac{(1-t)^{i-X_{s-}}}{(i-X_{s-})!} \right] ds.$$

We see, that the predictable compensator of X for elements of the reciprocal class of a Poisson process can become arbitrarily complicated. It is therefore all the more surprising, that the reciprocal class may be characterized by a simple duality formula, as we will see in Theorem 6.39.

6.3.3. *Characterization of the reciprocal class of a Poisson process.*

The duality formula (6.23) still holds for the bridges of the Poisson process under a loop condition on the test functions.

Lemma 6.37. *Let $x, y \in \mathbb{R}$ such that $y - x \in \mathbb{N}$. Then the duality formula (6.23) holds under $\mathbb{P}^{x,y}$ for every $F \in \mathcal{S}_{\mathbb{J}_1}$ and $u \in \mathcal{E}$ with $\langle u \rangle = 0$.*

Proof. Take any $\psi \in C_b^\infty(\mathbb{R})$, $F \in \mathcal{S}_{\mathbb{J}_1}$, $u \in \mathcal{E}$ with $\langle u \rangle = 0$. Note that the loop condition on u implies $X_1 \circ \pi_u^\varepsilon = X_1$. Then

$$\begin{aligned} \mathbb{E}^x\left(\psi(X_1)\mathbb{E}^x(F(X)\delta(u)|X_1)\right) &= \mathbb{E}^x\left(\mathcal{D}_u(\psi(X_1)F(X))\right) \\ &= \mathbb{E}^x\left(\psi(X_1)\mathbb{E}^x(\mathcal{D}_u F(X)|X_1)\right), \end{aligned}$$

where the last equality holds since $\mathcal{D}_u\psi(X_1) = 0$ and the product formula of Lemma 6.11 applies. □

We are going to prove a converse of the above lemma in terms of a characterization of the reciprocal class using techniques similar to the second proof of Theorem 6.25. Therefore let us first compute the Laplace transform of $\delta(u)$ for some $u \in C^1(\mathcal{I})$ with respect to the Poisson bridge $\mathbb{P}^{x,y}$:

$$\mathbb{E}^{x,y}(e^{-\delta(u)}) = \mathbb{E}^{x,y}(e^{-\sum_{i=1}^{y-x} u(T_i) + \langle u \rangle}).$$

Denote $m = y - x$. We only have to use the conditional distribution of the jump times of a Poisson process given in Remark 1.19 to derive the Laplace transform

$$\begin{aligned} \mathbb{E}^{x,y}(e^{-\sum_{i=1}^m u(T_i)}) &= m! \int_{\mathcal{I}^m} e^{-\sum_{i=1}^m u(t_i)} \mathbb{1}_{0 \le t_1 \le \cdots \le t_m \le 1} dt_1 \cdots dt_m \\ &= \int_{\mathcal{I}^m} e^{-\sum_{i=1}^m u(t_i)} dt_1 \cdots dt_m \\ &= \langle e^{-u} \rangle^m, \end{aligned}$$

and therefore

$$\mathbb{E}^{x,y}(e^{-\delta(u)}) = \langle e^{-u} \rangle^{y-x} e^{\langle u \rangle}.$$

It follows that the function $\lambda \mapsto \mathbb{E}^{x,y}(e^{\lambda\delta(u)})$ defined for $\lambda < 0$ is solution to the ordinary differential equation

$$\partial_\lambda \mathbb{E}^{x,y}(e^{\lambda\delta(u)}) = \left[(y-x)\frac{\langle u e^{\lambda u} \rangle}{\langle e^{\lambda u} \rangle} - \langle u \rangle\right] \mathbb{E}^{x,y}(e^{\lambda\delta(u)}) \tag{6.38}$$

with initial condition $\mathbb{E}^{x,y}(e^{0\delta(u)}) = 1$. We now present our main result in this paragraph about the reciprocal class of a Poisson process.

Theorem 6.39. *Assume that $\mathbb{Q}$ is a probability on $\mathbb{J}_1(\mathcal{I})$ such that the number of jumps $\eta \in \mathbb{L}^1(\mathbb{Q})$ is integrable. Then $\mathbb{Q}$ is in the reciprocal class $\mathcal{R}(\mathbb{P})$ if and only if the duality formula*

$$\mathbb{E}_{\mathbb{Q}}\left(F(X)\delta(u)\right) = \mathbb{E}_{\mathbb{Q}}\left(\mathcal{D}_u F(X)\right) \tag{6.40}$$

holds for every $F \in \mathcal{S}_{\mathbb{J}_1}$ and $u \in \mathcal{E}$ with $\langle u \rangle = 0$.

Proof. If $\mathbb{Q} \in \mathcal{R}(\mathbb{P})$ we just have to use Lemma 6.37 to prove that the duality formula is satisfied: Since $\mathbb{Q} \in \mathcal{R}(\mathbb{P})$ we get

$$\mathbb{E}_{\mathbb{Q}}(F(X)\delta(u)) = \int_{\mathbb{R}^2} \mathbb{E}^{xy}(F(X)\delta(u))\mathbb{Q}_{01}(dxdy) = \int_{\mathbb{R}^2} \mathbb{E}^{xy}(\mathcal{D}_u F(X))\mathbb{Q}_{01}(dxdy) = \mathbb{E}_{\mathbb{Q}}(\mathcal{D}_u F(X)).$$

Assume, on the other hand, that the duality formula holds under $\mathbb{Q}$. Then it also holds under the bridges of $\mathbb{Q}$, see the proof of Lemma 6.37. We want to compute the characteristic

functional of $\mathbb{Q}^{x,y}$ for $x, y \in \mathbb{R}$ such that the bridge is well defined. For some $u \in C_c^1(\mathcal{I})$ with $u \geq 0$ we define $\tilde{u} := u - \langle u \rangle$. We easily see that $\delta(\tilde{u}) = \delta(u) - (X_1 - X_0 - 1)\langle u \rangle$ and $\kappa\tilde{u} = \kappa u$, where the operator $\kappa_t u_s = -u_s' \mathbb{1}_{[0,s]}(t)$ was defined in Lemma 6.27. Take some $\lambda < 0$. Then we can show as in the second proof of Theorem 6.25 that the duality formula extends to $F = e^{\lambda\delta(u)}$ differentiated in direction of $\tilde{u}$. Using Examples 6.18 and 6.21 combined with the duality formula we get

$$\mathbb{E}_{\mathbb{Q}}^{x,y}(e^{\lambda\delta(u)}\delta(\tilde{u})) = \mathbb{E}_{\mathbb{Q}}^{x,y}(\mathcal{D}_{\tilde{u}}e^{\lambda\delta(u)}) = \lambda\mathbb{E}_{\mathbb{Q}}^{x,y}\left(\int_{\mathcal{I}} \tilde{u}_s(u_s + \delta(\kappa_s u))ds\, e^{\lambda\delta(u)}\right).$$

We iterate this on the derivative of the Laplace transform

$$\begin{aligned}
\partial_\lambda \mathbb{E}_{\mathbb{Q}}^{x,y}(e^{\lambda\delta(u)}) &= \mathbb{E}_{\mathbb{Q}}^{x,y}(\delta(u)e^{\lambda\delta(u)}) \\
&= \mathbb{E}_{\mathbb{Q}}^{x,y}(\delta(\tilde{u})e^{\lambda\delta(u)}) + (y - x - 1)\langle u \rangle \mathbb{E}_{\mathbb{Q}}^{x,y}(e^{\lambda\delta(u)}) \\
&= \lambda\mathbb{E}_{\mathbb{Q}}^{x,y}\left(\int_{\mathcal{I}} \tilde{u}_s(u_s + \delta(\kappa_s u))ds\, e^{\lambda\delta(u)}\right) + (y - x - 1)\langle u \rangle \mathbb{E}_{\mathbb{Q}}^{x,y}(e^{\lambda\delta(u)}) \\
&= (y - x - 1)\langle u \rangle \mathbb{E}_{\mathbb{Q}}^{x,y}(e^{\lambda\delta(u)}) + \lambda(y - x)\langle \tilde{u}, u \rangle \mathbb{E}_{\mathbb{Q}}^{x,y}(e^{\lambda\delta(u)}) \\
&\quad + \lambda\mathbb{E}_{\mathbb{Q}}^{x,y}\left(\int_{\mathcal{I}} \tilde{u}_s\delta(\kappa_s u - u_s)ds\, e^{\lambda\delta(u)}\right) \\
&= \left[-\langle u \rangle + (y - x)\langle u \rangle + (y - x)\lambda(\langle u^2 \rangle - \langle u \rangle^2)\right] \mathbb{E}_{\mathbb{Q}}^{x,y}(e^{\lambda\delta(u)}) + \lambda^2[\dots].
\end{aligned}$$

The quadratic term in λ is equal to

$$\begin{aligned}
&\lambda\mathbb{E}_{\mathbb{Q}}^{x,y}\left(\int_{\mathcal{I}} \tilde{u}_s\delta(\kappa_s u - u_s)ds\, e^{\lambda\delta(u)}\right) \\
= \;& \lambda^2\mathbb{E}_{\mathbb{Q}}^{x,y}\left(\int_{\mathcal{I}^2} \tilde{u}_{t_1}(\kappa_{t_1}u_{t_2} - u_{t_1})(u_{t_2} + \delta(\kappa_{t_2}u))dt_1dt_2e^{\lambda\delta(u)}\right) \\
= \;& \lambda^2\mathbb{E}_{\mathbb{Q}}^{x,y}\left(\int_{\mathcal{I}^2} \tilde{u}_{t_1}(\kappa_{t_1}u_{t_2} - u_{t_1})((y - x)u_{t_2} + \delta(\kappa_{t_2}u - u_{t_2}))dt_1dt_2e^{\lambda\delta(u)}\right) \\
= \;& (y - x)\lambda^2\left[\frac{1}{2}\langle u^2, \tilde{u} \rangle - \langle u, \tilde{u} \rangle\langle u \rangle\right] \mathbb{E}_{\mathbb{Q}}^{x,y}(e^{\lambda\delta(u)}) + \lambda^3[\dots] \\
= \;& (y - x)\frac{\lambda^2}{2}\left[\langle u^3 \rangle - 3\langle u^2 \rangle\langle u \rangle + 2\langle u \rangle^3\right] \mathbb{E}_{\mathbb{Q}}^{x,y}(e^{\lambda\delta(u)}) + \lambda^3[\dots].
\end{aligned}$$

The cubic term in λ is equal to

$$\begin{aligned}
&\lambda^2\mathbb{E}_{\mathbb{Q}}^{x,y}\left(\int_{\mathcal{I}^2} \tilde{u}_{t_1}(\kappa_{t_1}u_{t_2} - u_{t_1})\delta(\kappa_{t_2}u - u_{t_2})e^{\lambda\delta(u)}dt_1dt_2\right) \\
= \;& \lambda^3\mathbb{E}_{\mathbb{Q}}^{x,y}\left(\int_{\mathcal{I}^3} \tilde{u}_{t_1}(\kappa_{t_1}u_{t_2} - u_{t_1})(\kappa_{t_2}u_{t_3} - u_{t_2})((y - x)u_{t_3} + \delta(\kappa_{t_3}u - u_{t_3}))dt_1dt_2dt_3e^{\lambda\delta(u)}\right) \\
= \;& \lambda^3(y - x)\left[\frac{1}{6}\langle u^3, \tilde{u} \rangle - \frac{1}{2}\langle u^2, \tilde{u} \rangle\langle u \rangle - \frac{1}{2}\langle u, \tilde{u} \rangle\langle u^2 \rangle + \langle u, \tilde{u} \rangle\langle u \rangle^2\right] \mathbb{E}_{\mathbb{Q}}^{x,y}(e^{\lambda\delta(u)}) + \lambda^4[\dots] \\
= \;& \frac{\lambda^3}{3!}(y - x)\left[\langle u^4 \rangle - 4\langle u^3 \rangle\langle u \rangle + 12\langle u^2 \rangle\langle u \rangle^2 - 3\langle u^2 \rangle^2 - 6\langle u \rangle^4\right] \mathbb{E}_{\mathbb{Q}}^{x,y}(e^{\lambda\delta(u)}) + \lambda^4[\dots].
\end{aligned}$$

We want to show that $\lambda \mapsto \mathbb{E}_{\mathbb{Q}}^{x,y}(e^{\lambda\delta(u)})$ satisfies the same ordinary differential equation as the Laplacian of the Poisson bridge $\lambda \mapsto \mathbb{E}^{x,y}(e^{\lambda\delta(u)})$ in equation (6.38). Define the function

$$g(\lambda) := \frac{\langle ue^{\lambda u} \rangle}{\langle e^{\lambda u} \rangle} = \partial_\lambda \log\langle e^{\lambda u} \rangle.$$

Then we compute

$$\begin{aligned} g(0) &= \langle u\rangle, \\ g'(0) &= \langle u^2\rangle - \langle u\rangle^2 \\ g''(0) &= \langle u^3\rangle - 3\langle u^2\rangle\langle u\rangle + 2\langle u\rangle^3 \\ g'''(0) &= \langle u^4\rangle - 4\langle u^3\rangle\langle u\rangle + 12\langle u^2\rangle\langle u\rangle^2 - 3\langle u^2\rangle^2 - 6\langle u\rangle^4. \end{aligned}$$

For arbitrary $n \geq 0$ we have

$$g^{(n)}(0) = n! \int_{\mathcal{I}^n} \tilde{u}_{t_1}(\kappa_{t_1}u_{t_2} - u_{t_1}) \cdots (\kappa_{t_{n-1}}u_{t_n} - u_{t_{n-1}})u_{t_n} dt_1 \cdots dt_n. \tag{6.41}$$

Indeed, the above iteration in powers of λ also holds under $\mathbb{P}^{x,y}$. Therefore the uniqueness of the solution of the differential equation (6.38) implies that (6.41) has to be true. The development in λ of the derivative of the Laplace transform of $\mathbb{Q}^{x,y}$ gives

$$\begin{aligned} \partial_\lambda \mathbb{E}^{x,y}_{\mathbb{Q}}(e^{\lambda\delta(u)}) &= \left[-\langle u\rangle + (y-x)\left(g(0) + \lambda g^{(1)}(0) + \frac{\lambda^2}{2!}g^{(2)}(0) + \dots\right)\right]\mathbb{E}^{x,y}_{\mathbb{Q}}(e^{\lambda\delta(u)}) \\ &= \left[(y-x)g(\lambda) - \langle u\rangle\right]\mathbb{E}^{x,y}_{\mathbb{Q}}(e^{\lambda\delta(u)}), \end{aligned}$$

and compared with equation (6.38) we deduce that $\mathbb{E}^{x,y}_{\mathbb{Q}}(e^{\lambda\delta(u)}) = \mathbb{E}^{x,y}(e^{\lambda\delta(u)})$ holds $\mathbb{Q}_{01}(dxdy)$-*a.s.* for all $\lambda \leq 0$ and $u \in C^1_c(\mathcal{I})$ with $u \geq 0$. Since the boundary states X_0, X_1 of the bridges are deterministic, this is sufficient to prove that $\mathbb{Q}^{x,y} = \mathbb{P}^{x,y}$. By identity of its bridges to Poisson bridges, $\mathbb{Q}$ is in the reciprocal class of the Poisson process. □

An important example of the above characterization is the application to Poisson bridges.

Example 6.42. *Let $\mathbb{Q}$ be any probability on $\mathbb{J}_1(\mathcal{I})$ with $\mathbb{Q}(X_0 = x, X_1 = y) = 1$. Then $\mathbb{Q}$ is the Poisson bridge from x to y if and only if the duality formula (6.40) holds for all $F \in \mathcal{S}_{\mathbb{J}_1}$ and $u \in \mathcal{E}$ with $\langle u\rangle = 0$. Here the integrability condition $\eta \in \mathbb{L}^1(\mathbb{Q})$ is meaningless, since $\eta = y - x$ $\mathbb{Q}$-a.s.*

Let us note that the condition $\eta \in \mathbb{L}^1(\mathbb{Q})$ in Theorem 6.39 is not a real restriction in the unit jump setting, since with respect to the bridges $\mathbb{Q}^{x,y}$ the number of jumps $\eta = y - x$ is deterministic. It is thus always possible to compare the bridges of $\mathbb{Q}$ with the bridges of a Poisson process.

6.4. **Markov unit jump processes.**

In this paragraph we introduce a class of Markov unit jump processes with an especially nice intensity: A degree of regularity is required of the reference Markov processes for the definition of reciprocal invariants. We already mentioned the notion of intensity of unit jump processes in Remark 6.4. Let us now fix this idea in the following definition.

Definition 6.43. *Let $\mathbb{Q}$ be any probability distribution on $\mathbb{J}_1(\mathcal{I})$. Then there exists a $\mathbb{Q}$-a.s. unique predictable and increasing process A with $\mathbb{Q}(A_0 = 0) = 1$ such that*

$$\mathbb{E}_{\mathbb{Q}}\left(\int_{\mathcal{I}} u_s dX_s\right) = \mathbb{E}_{\mathbb{Q}}\left(\int_{\mathcal{I}} u_s dA_s\right) \tag{6.44}$$

holds for all predictable and bounded processes $u : \mathcal{I} \times \mathbb{J}_1(\mathcal{I}) \to \mathbb{R}$. The predictable measure dA is called the intensity.

We refer to Jacod [Jac75, Theorem 2.1 and Theorem 3.4] for the existence and uniqueness of the intensity. He shows that the intensity characterizes the law $\mathbb{Q}$ in the following sense: If $\mathbb{Q}'$ is another probability on $\mathbb{J}_1(\mathcal{I})$ with the same initial condition $\mathbb{Q}'_0 = \mathbb{Q}_0$ and with

intensity dA' such that $\int_I u_s dA_s = \int_I u_s dA'_s$ holds $\mathbb{Q}$-*a.s.* and $\mathbb{Q}'$-*a.s.* for any $u \in \mathcal{E}$, then $\mathbb{Q} = \mathbb{Q}'$. This is a generalization of Watanabe's characterization presented in Lemma 6.8 to all unit jump processes.

Example 6.45. *We already know several unit jump process with their respective intensities:*

- *A Poisson process $\mathbb{P}$ has intensity dt.*
- *A Poisson process $\mathbb{P}_\alpha$ with intensity $\alpha > 0$ as introduced in § 6.3.2 has intensity αdt.*
- *The Poisson bridge $\mathbb{P}^{x,y}$ has intensity $\frac{y-X_{t-}}{1-t}dt$.*

6.4.1. *Nice unit jump processes.*

We are interested in a particular class of Markov processes on $\mathbb{J}_1(\mathcal{I})$.

Definition 6.46. *Let $\varepsilon > 0$ arbitrary and ℓ a function such that*

$$\ell : \mathcal{I} \times \mathbb{R} \to [\varepsilon, \infty) \text{ is bounded, and } \ell(., x) \in \mathcal{C}^1(\mathcal{I}, \mathbb{R}_+) \text{ for all } x \in \mathbb{R}. \tag{6.47}$$

Then $\mathbb{P}_\ell$ is called the law of a "nice" unit jump process if X has intensity $\ell(t, X_{t-})dt$ under $\mathbb{P}_\ell$.

We say for short that the nice unit jump process $\mathbb{P}_\ell$ has intensity ℓ. The boundedness of the intensity $0 < \varepsilon \le \|\ell\|_\infty \le K$ assures that for every $t \in \mathcal{I}$ we have

$$\int_{[0,t]} [\ell(s, X_{s-}) \log \ell(s, X_{s-}) - \ell(s, X_{s-})]\, ds \le K(|\log \varepsilon| + 1 + K), \text{ uniformly for } \omega \in \mathbb{J}_1(\mathcal{I}).$$

Lepingle and Mémin state in [LM78, Théorème IV.3] a Novikov condition for jump processes. With the above bound their result implies that the law of a nice unit jump process $\mathbb{P}_\ell$ is equivalent to the law of a Poisson process $\mathbb{P}$ in the sense that $\mathbb{P}_\ell \ll \mathbb{P}$ and $\mathbb{P} \ll \mathbb{P}_\ell$ if $\mathbb{P}_{\ell,0} = \mathbb{P}_0$. A convenient form of the Girsanov theorem for semimartingales with jumps provides an explicit form of the density, see also the Appendix.

Proposition 6.48. *Let $\mathbb{P}_\ell$ the law of a nice unit jump process and $\mathbb{P}$ the law of a Poisson process with same initial condition $\mathbb{P}_{\ell,0} = \mathbb{P}_0$. Then $\mathbb{P}_\ell$ is equivalent to $\mathbb{P}$ and the density process defined by $\mathbb{P}_\ell = G^\ell_t \mathbb{P}$ on $\mathcal{F}_{[0,t]}$ has the explicit form*

$$G^\ell_t = \exp\left(-\int_{[0,t]} (\ell(s, X_{s-}) - 1) ds\right) \prod_{T_i \le t} \ell(T_i, X_{T_i-}). \tag{6.49}$$

*In particular $G^\ell_t > 0$ $\mathbb{P}$-*a.s.* for all $t \in \mathcal{I}$.*

Since the density G^ℓ_1 factorizes into an $\mathcal{F}_{[0,t]}$ and an $\mathcal{F}_{[t,1]}$-measurable part, $\mathbb{P}_\ell$ has indeed the Markov property, see Lemma 4.4.

Remark 6.50. *The preceding Girsanov transformation gives an explicit expression of the density of the jump-times. Assume $\mathbb{P}_{\ell,0} = \mathbb{P}_0$ as above. Then by Remark 1.19 we have*

$$\begin{aligned}
& \mathbb{P}_\ell(X_0 \in dx, T_1 \in dt_1, \dots, T_m \in dt_m, T_{m+1} = \infty) \\
= \;& G^\ell_1\, \mathbb{P}(X_0 \in dx, T_1 \in dt_1, \dots, T_m \in dt_n, T_{m+1} = \infty) \\
= \;& \mathbb{P}_0(dx) \ell(t_1, x) e^{-\int_{[0,t_1]} \ell(s,x) ds} \ell(t_2, x+1) e^{-\int_{[t_1,t_2]} \ell(s,x+1) ds} \cdots \\
& \cdots \ell(t_m, x+m-1) e^{-\int_{[t_{m-1},t_m]} \ell(s,x+m-1) ds} e^{-\int_{[t_m,1]} \ell(s,x+m) ds} \mathbb{1}_{\{0<t_1<\cdots<t_m\le 1\}} dt_1 \cdots dt_m.
\end{aligned}$$

This completely describes the law of $\mathbb{P}_\ell$.

Note that in the unit jump context the knowledge of the intensity is sufficient to identify the process. In particular we have:

Corollary 6.51. *Let $\mathbb{Q}$ be any distribution on $\mathbb{J}_1(\mathcal{I})$ such that $\eta \in \mathbb{L}^1(\mathbb{Q})$ and ℓ be as in (6.47). If for every $t \in \mathcal{I}$, $u \in \mathcal{E}$ with $u = u\mathbb{1}_{(t,1]}$ and $F \in \mathcal{S}_{\mathbb{J}_1}$ that is $\mathcal{F}_{[0,t]}$-measurable the formula*

$$\mathbb{Q}\left(F \int_{\mathcal{I}} u_t dX_t\right) = \mathbb{Q}\left(F \int_{\mathcal{I}} u_t \ell(t, X_{t-}) dt\right) \tag{6.52}$$

holds, then $\mathbb{Q}$ is a nice unit jump process with intensity ℓ.

Proof. This is just a particular case of Jacod's extension of Watanabe's characterization result quoted in the comments following Definition 6.43. □

6.5. **Comparison of reciprocal classes of nice unit jump processes.**

In Theorem 6.58 we present our main result in this paragraph. We show that the reciprocal class of any nice unit jump process is characterized by a reciprocal invariant. Let us first prove an auxiliary result: In § 4.3 we showed that h-transforms are Markov processes that preserve bridges. We are able to compute the intensity of h-tranforms of nice unit jump processes.

Proposition 6.53. *Let $\mathbb{P}_\ell$ be the law of a nice unit jump process, $h : \mathbb{R} \to \mathbb{R}_+$ be any measurable function such that $\mathbb{E}_\ell(h(X_1)) = 1$. The h-transform $h\mathbb{P}_\ell := h(X_1)\mathbb{P}_\ell$ is Markov and has intensity*

$$k(t, X_{t-})dt = \ell(t, X_{t-}) \frac{h(t, X_{t-} + 1)}{h(t, X_{t-})} dt, \quad dt \otimes h\mathbb{P}_\ell\text{-a.e.}, \tag{6.54}$$

where $h(t, x) := \mathbb{E}_\ell(h(X_1)|X_t = x)$.

Proof. We first show that $h(t, x)$ satisfies a Kolmogoroff backward equation.

Lemma 6.55. *Let $h(t, x)$ be defined as above. Then h is a solution of the Kolmogoroff backward equation*

$$\partial_t h(t, x) = -\ell(t, x)\,(h(t, x + 1) - h(t, x)) \text{ holds } \mathbb{P}_\ell(X_t \in \,.\,)\text{-a.e., for all } t \in \mathcal{I}. \tag{6.56}$$

In particular $h(., x) \in \mathcal{C}^1(\mathcal{I})$.

Proof. We express $h(t, x)$ as

$$h(t, x) = \sum_{m=0}^{\infty} h(x + m)\mathbb{P}_\ell(X_1 = x + m|X_t = x) = \sum_{m=0}^{\infty} h(x + m) p_\ell(t, x; 1, x + m).$$

Here we use Remark 6.50 and the Markov property of $\mathbb{P}_\ell$ to define the transition probability

$$\begin{aligned} p_\ell(t, x; 1, x + m) &:= \mathbb{P}_\ell(X_1 = x + m|X_t = x) \\ &= \int_{\mathcal{I}^m} \ell(t_1, x) e^{-\int_{[t,t_1]} \ell(s,x)ds} \cdots \\ &\qquad \ell(t_m, x + m - 1) e^{-\int_{[t_{m-1},t_m]} \ell(s,x+m-1)ds} e^{-\int_{[t_m,1]} \ell(s,x+m)ds} \mathbb{1}_{t \le t_1 < \cdots < t_m \le 1} dt_1 \cdots dt_m \\ &= \int_{[t,1]} \int_{[t,t_m]} \cdots \int_{[t,t_2]} f_x(t_1, \ldots, t_m, t) dt_1 \cdots dt_m, \end{aligned}$$

with

$$f_x(t_1, \ldots, t_m, t) := \ell(t_1, x) e^{-\int_{[t,t_1]} \ell(s,x)ds} \cdots \ell(t_m, x + m - 1) e^{-\int_{[t_{m-1},t_m]} \ell(s,x+m-1)ds} e^{-\int_{[t_m,1]} \ell(s,x+m)ds}.$$

Clearly $f_x(t_1,\dots,t_m,t)$ is bounded and differentiable in t, and therefore so is $p_\ell(t,x;1,x+m)$. Moreover $\partial_t f_x(t_1,\dots,t_m,t) = \ell(x,t) f_x(t_1,\dots,t_m,t)$, which implies

$$\begin{aligned}
&\frac{1}{\varepsilon}\left[p_\ell(t+\varepsilon,x;1,x+m) - p_\ell(t,x;1,x+m)\right] \\
= \;& \frac{1}{\varepsilon}\left[\int_{[t+\varepsilon,1]}\cdots\int_{[t+\varepsilon,t_2]} f_x(t_1,\dots,t_m,t+\varepsilon)dt_1\cdots dt_m - \int_{[t,1]}\cdots\int_{[t,t_2]} f_x(t_1,\dots,t_m,t)dt_1\cdots dt_m\right] \\
= \;& \frac{1}{\varepsilon}\left[\int_{[t+\varepsilon,1]}\cdots\int_{[t+\varepsilon,t_2]} \left(f_x(t_1,\dots,t_m,t+\varepsilon) - f_x(t_1,\dots,t_m,t)\right) dt_1\cdots dt_m\right. \\
& - \int_{[t,t+\varepsilon]}\int_{[t,t_m]}\cdots\int_{[t,t_2]} f_x(t_1,\dots,t_m,t)dt_1\cdots dt_m - \dots \\
& \left.\dots - \int_{[t,1]}\int_{[t,t_m]}\cdots\int_{[t,t+\varepsilon]} f_x(t_1,\dots,t_m,t)dt_1\cdots dt_m\right],
\end{aligned}$$

which by dominated convergence goes to

$$\begin{aligned}
&\overset{\varepsilon\to 0}{\to} \int_{[t,1]}\cdots\int_{[t,t_2]} \ell(t,x) f_x(t_1,\dots,t_m,t)dt_1\cdots t_m - \int_{[t,1]}\cdots\int_{[t,t_3]} f(t,t_2,\cdots,t_m,t)dt_2\cdots dt_m \\
&= \ell(t,x)p_\ell(t,x;1,x+m) - \ell(t,x)p_\ell(t,x+1;1,x+m).
\end{aligned}$$

The same computations apply to the limit of $-\frac{1}{\varepsilon}(p_\ell(t-\varepsilon,x;1,x+m) - p_\ell(t,x;1,x+m))$. The sum over the first term is

$$\sum_{m=0}^{\infty} h(y+m)\ell(t,x)p_\ell(t,x;1,x+m) = \ell(t,x)\mathbb{E}_\ell(h(X_1)|X_t = x) = \ell(t,x)h(t,x),$$

and the sum over the second term is

$$\begin{aligned}
\sum_{m=1}^{\infty} h(x+m)\ell(t,x)p_\ell(t,x+1;1,x+m) &= \sum_{m=1}^{\infty} h(x+1+m-1)\ell(t,x)p_\ell(t,x+1;1,x+m) \\
&= \ell(t,x)\mathbb{E}_\ell(h(X_1)|X_t = x+1) \\
&= \ell(t,x)h(t,x+1),
\end{aligned}$$

and we see that (6.56) holds. □

Let us resume the proof of Proposition 6.53. Since $h(t,x)$ is differentiable in the time variable we may apply the Itô-formula

$$\begin{aligned}
h(t,X_t) &= h(s,X_s) + \int_{[s,t]} \partial_t h(r,X_{r-})dr + \int_{[s,t]} (h(r,X_r) - h(r,X_{r-}))dX_r \\
&= h(s,X_s) + \int_{[s,t]} \partial_t h(r,X_{r-})dr + \int_{[s,t]} (h(r,X_{r-}+1) - h(r,X_{r-}))dX_r \\
&= h(s,X_s) + \int_{[s,t]} h(r,X_{r-})\frac{h(r,X_{r-}+1) - h(r,X_{r-})}{h(r,X_{r-})}(dX_r - \ell(r,X_{r-})dr),
\end{aligned}$$

where the last equality follows from (6.56). We recognize the Doléans-Dade differential equation, and write $h(X_1)$ in the form

$$h(X_1) = \exp\left(-\int_{\mathcal{I}}\left(\frac{h(s,X_{s-}+1)}{h(X_{s-},s)} - 1\right)\ell(s,X_{s-})ds\right)\prod_{i=1}^{\eta}\frac{h(T_i,X_0+i)}{h(T_i,X_0+i-1)}.$$

Since $h\mathbb{P}_\ell = h(X_1)G_1^\ell\mathbb{P}$ and

$$h(X_1)G_1^\ell = \exp\left(-\int_{\mathcal{I}}\left(\ell(s,X_{s-})\frac{h(s,X_{s-}+1)}{h(X_{s-},s)}-1\right)ds\right)\prod_{i=1}^{\eta}\ell(T_i,X_0+i-1)\frac{h(T_i,X_0+i)}{h(T_i,X_0+i-1)}.$$

the Girsanov theorem implies that $h\mathbb{P}_\ell$ has the intensity $k(t,X_{t-})dt$ given in (6.54). □

Using Example 6.33 we now give a qualitative statement on the intensities of the bridges of nice unit jump processes.

Remark 6.57. *Since for $y-x\in\mathbb{N}$ the bridge $\mathbb{P}^{x,y}$ of a Poisson process is an h-transform of $\mathbb{P}^x$, we deduce that $\mathbb{P}_\ell^{x,y}$ is equivalent to $\mathbb{P}^{x,y}$ as a measure on $\mathbb{J}_1(\mathcal{I})$. Therefore a relation similar to (6.54) holds between the intensity of the Poisson bridge and $\ell^{x,y}(t,X_{t-})dt$, which is the intensity of the bridge $\mathbb{P}_\ell^{x,y}$. We deduce the following qualitative statements from Example 6.33: The intensity $\ell^{x,y}(t,z)$ is*

- *strictly positive $\ell^{x,y}(t,z)>0$ for $y-x>z-x\in\mathbb{N}$ and finite for $t\in[0,1-\varepsilon]$, $\varepsilon>0$;*
- *zero $\ell(.,y)\equiv 0$ on $\mathcal{I}$ since $\mathbb{P}_\ell^{x,y}(X_t=y+1)=0$ for all $t\in\mathcal{I}$.*

In § 6.6 we prove that these qualitative properties are also implied by a duality formula.

Next we present a comparison of the reciprocal classes of two different nice unit jump processes. This is a new result, but in the same spirit as Theorem 5.26 for Brownian diffusion processes.

Theorem 6.58. *Let $\mathbb{P}_\ell$ be the law of a nice unit jump process with intensity ℓ. Then the function*

$$\Xi_\ell(t,x) := \partial_t\log\ell(t,x)+\ell(t,x+1)-\ell(t,x) \tag{6.59}$$

is a "harmonic" invariant of the reciprocal class $\mathcal{R}(\mathbb{P}_\ell)$: If $\mathbb{P}_k$ is another nice unit jump process then $\Xi_k\equiv\Xi_\ell$ if and only if $\mathcal{R}(\mathbb{P}_k)=\mathcal{R}(\mathbb{P}_\ell)$.

Proof. First assume that $\mathcal{R}(\mathbb{P}_k)=\mathcal{R}(\mathbb{P}_\ell)$. Fix $x\in\mathbb{R}$, then there exists $h:\mathbb{R}\to\mathbb{R}_+$ such that $\mathbb{P}_k^x = h(X_1)\mathbb{P}_\ell^x$ since $\mathbb{P}_\ell^x$ and $\mathbb{P}_k^x$ are equivalent measures, see Remark 4.18. We know that relation (6.54) between the intensities k and ℓ holds and that $h(t,y)=\mathbb{E}_\ell^x(h(X_1)|X_t=y)$ is space-time harmonic in the sense of (6.56). This implies

$$\begin{aligned}\frac{\partial_t h(t,y+1)}{h(t,y+1)} &= -\ell(t,y+1)\frac{h(t,y+2)-h(t,y+1)}{h(t,y+1)} \qquad \text{and}\\ \frac{\partial_t h(t,y)}{h(t,y)} &= -\ell(t,y)\frac{h(t,y+1)-h(t,y)}{h(t,y)},\end{aligned}$$

We subtract the second equation from the first

$$\partial_t\log\frac{h(t,y+1)}{h(t,y)}+\left(\frac{h(t,y+2)}{h(t,y+1)}-1\right)\ell(t,y+1)-\left(\frac{h(t,y+1)}{h(t,y)}-1\right)\ell(t,y)=0.$$

In this we can insert

$$\frac{h(t,y+1)}{h(t,y)}=\frac{k(t,y)}{\ell(t,y)},\ \text{and}\ \frac{h(t,y+2)}{h(t,y+1)}=\frac{k(t,y+1)}{\ell(t,y+1)}, \tag{6.60}$$

which leads to the equality of the invariants $\Xi_k(t,X_{t-})\equiv\Xi_\ell(t,X_{t-})$ holds $dt\otimes\mathbb{P}_\ell^x$-*a.s.* Mixing over the initial condition implies the identical equality of invariants.

Assume on the other hand, that $\mathbb{P}_\ell$ and $\mathbb{P}_k$ are nice unit jump processes such that the invariants Ξ_ℓ and Ξ_k coincide. Clearly the deterministic bridge without jumps coincides

$\mathbb{P}_\ell^{x,x} = \mathbb{P}_k^{x,x}$ for any $x \in \mathbb{R}$. Using Remark 6.50, we now prove the equality of the bridges with one jump. In particular

$$\mathbb{P}_k^{x,x+1}(T_1 \in dt) = \frac{k(t,x)e^{-\int_{[0,t]} k(s,x)ds} e^{-\int_{[t,1]} k(s,x+1)ds} dt}{\int_I k(s,x)e^{-\int_{[0,s]} k(r,x)dr} e^{-\int_{[s,1]} k(r,x+1)dr} ds},$$

and an insertion of the equality of the characteristics into the denominator gives

$$\begin{aligned} & k(t,x)e^{-\int_{[0,t]} k(s,x)ds} e^{-\int_{[t,1]} k(s,x+1)ds} \\ = \; & k(t,x)e^{-\int_{[0,t]} \partial_t \log k(s,x)+k(s,x+1)ds+\int_{[0,t]} \partial_t \log \ell(s,x)+\ell(s,x+1)-\ell(s,x)ds} e^{-\int_{[t,1]} k(s,x+1)ds} \\ = \; & \frac{k(0,x)}{\ell(0,x)} \ell(t,x) e^{-\int_{[0,t]} \ell(s,x)ds} e^{\int_{[0,t]} \ell(s,x+1)ds} e^{-\int_{[0,1]} k(s,x+1)ds}. \end{aligned}$$

The same calculus applies to the nominator which implies

$$\begin{aligned} & \mathbb{P}_k^x(T_1 \in dt | X_1 = x+1) \\ = \; & \left[\frac{k(0,x)}{\ell(0,x)} \ell(t,x) e^{-\int_{[0,t]} \ell(s,x)ds} e^{\int_{[0,t]} \ell(s,x+1)ds} e^{-\int_{[0,1]} k(s,x+1)ds} dt\right] / \left[\int_I \frac{k(0,x)}{\ell(0,x)} \ell(s,x) \cdots ds\right] \\ = \; & \left[\ell(t,x) e^{-\int_{[0,t]} \ell(s,x)ds} e^{-\int_{[t,1]} \ell(s,x+1)ds} dt\right] / \left[\int_I \ell(s,x) e^{-\int_{[0,s]} \ell(r,x)dr} e^{-\int_{[s,1]} \ell(r,x+1)dr} ds\right] \\ = \; & \mathbb{P}_\ell^x(T_1 \in dt | X_1 = x+1), \end{aligned}$$

where we used $\int_{[0,t]} \ell(s,x+1)ds = \int_{[0,1]} \ell(s,x+1)ds - \int_{[t,1]} \ell(s,x+1)ds$ to get the second equality.

The case of bridges with m jumps could be treated in a similar way. Instead, we will show that $\mathbb{P}_k^x$ is an h-transform of $\mathbb{P}_\ell^x$ for an arbitrary $x \in \mathbb{R}$. Define

$$h(t,x) := c(t), \text{ and } h(t,x+m) := \prod_{j=0}^{m-1} \frac{k(t,x+j)}{\ell(t,x+j)} c(t), \text{ with } c(t) := ce^{-\int_{[0,t]} (k(s,x)-\ell(s,x))ds}, \tag{6.61}$$

and c is a normalization constant such that $\mathbb{E}(h(1,X_1)) = 1$. Such a normalization is possible since

$$\mathbb{E}_\ell^x \left(\prod_{j=0}^{\eta-1} \frac{k(1,x+j)}{\ell(1,x+j)} \right) = \mathbb{E}^x \left(e^{-\int_I (\ell(t,X_{t-})-1)dt} \prod_{j=1}^{\eta} \frac{k(1,x+j-1)\ell(t,x+j-1)}{\ell(1,x+j-1)} \right) < \infty$$

by boundedness assumptions on ℓ and k. Put $y := x+m$ for any $m \geq 0$. The identity of the invariants implies that

$$\begin{aligned} & 0 = \partial_t \log \frac{k(t,y)}{\ell(t,y)} + k(t,y+1) - k(t,y) - \ell(t,y+1) + \ell(t,y) \\ \Leftrightarrow \quad & 0 = \partial_t \log \frac{k(t,y)}{\ell(t,y)} + \ell(t,y+1)\left(\frac{k(t,y+1)}{\ell(t,y+1)} - 1\right) - \ell(t,y)\left(\frac{k(t,y)}{\ell(t,y)} - 1\right). \end{aligned}$$

Since $\log k(t,y) - \log \ell(t,y) = \log h(t,y+1) - \log h(t,y)$ we get

$$0 = \partial_t \log \frac{h(t,y+1)}{h(t,y)} + \ell(t,y+1)\left(\frac{h(t,y+2)}{h(t,y+1)} - 1\right) - \ell(t,y)\left(\frac{h(t,y+1)}{h(t,y)} - 1\right),$$

which is equivalent to

$$\partial_t \log h(t,y+1) + \ell(t,y+1)\left(\frac{h(t,y+2)-h(t,y+1)}{h(t,y+1)}\right) = \partial_t \log h(t,y) + \ell(t,y)\left(\frac{h(t,y+1)-h(t,y)}{h(t,y)}\right).$$

For $y = x$ the right side is

$$\partial_t \log h(t,x) + \ell(t,x)\left(\frac{h(t,x+1)-h(t,x)}{h(t,x)}\right) = -(k(t,x)-\ell(t,x)) + \ell(t,x)\left(\frac{k(t,x)}{\ell(t,x)} - 1\right) = 0.$$

Therefore h is a space-time harmonic with respect to $\mathbb{P}^x_\ell$ in the sense of equation (6.56). But since $k(t,x+m) = \frac{h(t,x+m+1)}{h(t,x+m)}\ell(t,x+m)$ this implies that $\mathbb{P}^x_k = h(1,X_1)\mathbb{P}^x_\ell$ is an h-transform for any $x \in \mathbb{R}$, and in turn $\mathbb{P}^x_k \in \mathcal{R}(\mathbb{P}^x_\ell)$, the nice unit jump distributions $\mathbb{P}_k$ and $\mathbb{P}_\ell$ have the same bridges. □

Let us illustrate the above result by a comparison of the reciprocal classes of unit jump processes in two examples.

Example 6.62. *In the first example we treat counting processes with exponential decay rate. Take for example the decay of $N_0 \in \mathbb{N}$ radioactive particles. It is well known, that the mean number of particles that have not decayed until time $t \in \mathcal{I}$ is given by $N_t \approx e^{-\lambda t}N_0$, where $\lambda > 0$ is called the decay rate. This exponential decay property may be loosely stated as*

$$\frac{dN_t}{dt} = -\lambda N_t \;\Leftrightarrow\; \int_{\mathcal{I}} u_t(dN_t + \lambda N_t dt) = 0,\ \forall u \in \mathcal{E}. \tag{6.63}$$

Let us assume that the canonical unit jump process X that counts the number of decayed particles has independent increments. Then

$$\mathbb{E}^0_\ell(X_t) \approx N_0 - N_t \approx N_0(1-e^{-\lambda t}) = N_0\int_{[0,t]} \lambda e^{-\lambda s}ds,$$

thus we say that the number of decayed particles is a nice unit jump process with intensity $\ell(t)dt = N_0\lambda e^{-\lambda t}dt$. We will call this unit jump process an exponential decay process with rate λ. The associated harmonic invariant of the reciprocal class is $\Xi_\ell(t,x) = -\lambda$. Thus by Theorem 6.58 any two exponential decay processes are in the same reciprocal class if and only if the decay rates coincide.

Next we compare time-homogenous nice unit jump processes.

Example 6.64. *Assume that X is a nice unit jump process under $\mathbb{P}_\ell$ such that the intensity $\ell(X_{t-})dt$ does not depend on time. Then X is time-homogenous in the sense that*

$$\mathbb{P}_\ell(X_{t+\varepsilon} \in .\,|X_t = x) = \mathbb{P}_\ell(X_{s+\varepsilon} \in .\,|X_s = x),\ \forall \varepsilon > 0,\ s,t \in \mathcal{I}.$$

This property follows from Remark 6.50. If $\mathbb{P}_k$ is the distribution of another time-homogenous nice unit jump process with intensity $k(X_{t-})dt$, then

$$\mathcal{R}(\mathbb{P}_\ell) = \mathcal{R}(\mathbb{P}_k) \;\Leftrightarrow\; \ell(y+1) - \ell(y) = k(y+1) - k(y),$$

which is only possible if $k(y) - \ell(y)$ is constant.

6.6. Characterization of the reciprocal class $\mathcal{R}(\mathbb{P}_\ell)$ by a duality formula.

Let $\mathbb{P}_\ell$ be the law of a nice unit jump process. We first show that under any probability in $\mathcal{R}(\mathbb{P}_\ell)$ a duality formula, in which the invariant Ξ_ℓ of Theorem 6.58 appears, holds. With Theorem 6.69 we present our main result in this section: The duality formula characterizes the reciprocal class of $\mathbb{P}_\ell$ in the sense that any unit jump process that satisfies the duality formula (6.70) has the same bridges as the reference process $\mathbb{P}_\ell$. We underline the significance of our result with the introduction of two applications in § 6.7 and § 6.8.

Lemma 6.65. *Let $\mathbb{P}_\ell$ be the law of a nice unit jump process and $x, y \in \mathbb{R}$ with $y - x = m \in \mathbb{N}$. Then the duality formula*

$$\mathbb{E}_\ell^{x,y}\left(F(X)\delta(u)\right) = \mathbb{E}_\ell^{x,y}\left(\mathcal{D}_u F(X)\right) - \mathbb{E}_\ell^{x,y}\left(F(X)\int_{\mathcal{I}} u_t \int_{[t,1]} \Xi_\ell(s, X_{s-})dX_s dt\right) \tag{6.66}$$

holds for all $F \in \mathcal{S}_{\mathbb{J}_1}$ and $u \in \mathcal{E}$ with $\langle u \rangle = 0$.

Proof. Assume that the Girsanov density G_1^ℓ as defined in (6.49) is differentiable in direction of $u \in \mathcal{E}$ with $\langle u \rangle = 0$. Using the product rule and the duality formula under the Poisson measure we deduce

$$\mathbb{E}_\ell(F(X)\delta(u)) = \mathbb{E}\left(G_1^\ell F(X)\delta(u)\right) = \mathbb{E}\left(G_1^\ell \mathcal{D}_u F(X)\right) + \mathbb{E}\left(G_1^\ell F(X)\mathcal{D}_u \log G_1^\ell\right).$$

For the last term we have to differentiate the functional

$$\log G_1^\ell = \left(-\sum_{i=1}^{\eta+1}\int_{[T_{i-1},T_i]}(\ell(s,i-1)-1)ds\right) + \left(\sum_{i=1}^{\eta}\log \ell(T_i, i-1)\right),$$

with $T_0 := 0$ and $T_{\eta+1} := 1$. Using Example 6.18 we infer that $\log G_1^\ell$ is differentiable in direction u with derivative

$$\begin{aligned}\mathcal{D}_u \log G_1^\ell &= \sum_{i=1}^{\eta+1}(\ell(T_i,i-1)-1)\langle u\rangle_{T_i} - (\ell(T_{i-1},i-1)-1)\langle u\rangle_{T_{i-1}} - \sum_{i=1}^{\eta}\partial_t \log \ell(T_i,i-1)\langle u\rangle_{T_i} \\ &= -\int_{\mathcal{I}}\left[\partial_t \log \ell(s, X_{s-}) + \ell(s, X_{s-}+1) - \ell(s, X_{s-})\right]\langle u\rangle_s dX_s\end{aligned}$$

and a change of the order of integration shows that the duality formula (6.66) holds with respect to $\mathbb{P}_\ell$.

Using bounded, measurable functions $\phi, \psi : \mathbb{R} \to \mathbb{R}$ we integrate on the endpoint distributions

$$\mathbb{E}_\ell\left(\phi(X_0)\psi(X_1)F(X)\left(\delta(u) + \int_{\mathcal{I}} u_t \int_{[t,1]}\Xi_\ell(X_{s-}, s)dX_s dt\right)\right) = \mathbb{E}_\ell\left(\phi(X_0)\psi(X_1)\mathcal{D}_u F(X)\right),$$

since $\mathcal{D}_u\phi(X_0) = \mathcal{D}_u\psi(X_1) = 0$. Therefore the duality also holds for the bridges $\mathbb{P}_\ell^{x,y}$ for $x \in \mathbb{R}$ and $y = x + m$ with $m \in \mathbb{N}$. □

Example 6.67. *Let X be an exponential decay process with decay rate $\lambda > 0$, see Example 6.62. Then the duality formula (6.66) reduces to*

$$\mathbb{E}_\ell^0\left(F(X)\int_{\mathcal{I}} u_t dX_t\right) = \mathbb{E}_\ell^0\left(\mathcal{D}_u F(X)\right) + \lambda\mathbb{E}_\ell^0\left(F(X)\int_{\mathcal{I}} u_t \int_{[t,1]} dX_s dt\right)$$

for $F \in \mathcal{S}_{\mathbb{J}_1}$ and $u \in \mathcal{E}$ with $\langle u \rangle = 0$. This is equivalent to

$$\mathbb{E}_\ell^0\left(F(X)\left(\int_{\mathcal{I}} u_t(dX_t + \lambda X_{t-}dt)\right)\right) = \mathbb{E}_\ell^0\left(\mathcal{D}_u F(X)\right), \tag{6.68}$$

which may be interpreted as a probabilistic analogue of (6.63) for the reciprocal class of the exponential decay process.

The rest of this section is devoted to the proof of the converse of the above lemma in terms of a characterization of the reciprocal class.

Theorem 6.69. *Assume that $\mathbb{Q}$ is an arbitrary measure on $\mathbb{J}_1(\mathcal{I})$ such that $\eta \in \mathbb{L}^1(\mathbb{Q})$. Then $\mathbb{Q}$ is in $\mathcal{R}(\mathbb{P}_\ell)$ if and only if the duality formula*

$$\mathbb{E}_{\mathbb{Q}}\left(F(X)\delta(u)\right) = \mathbb{E}_{\mathbb{Q}}\left(\mathcal{D}_u F(X)\right) - \mathbb{E}_{\mathbb{Q}}\left(F(X)\int_{\mathcal{I}} u_t \int_{[t,1]} \Xi_\ell(s, X_{s-}) dX_s dt\right) \tag{6.70}$$

holds for every $F \in \mathcal{S}_{\mathbb{J}_1}$ and $u \in \mathcal{E}$ with $\langle u \rangle = 0$.

Proof. Note that all terms in (6.70) have a sense since Ξ_ℓ is bounded and $\eta \in \mathbb{L}^1(\mathbb{Q})$. To prove necessity we use Lemma 6.65 and the fact that elements in the reciprocal class are mixtures of bridges of $\mathbb{P}_\ell$ to extend the duality formula to any $\mathbb{Q} \in \mathcal{R}(\mathbb{P}_\ell)$ with integrable increments, see also the first part of the proof of Theorem 6.39.

For the converse it is sufficient to show that $\mathbb{Q}^x$ is in $\mathcal{R}(\mathbb{P}_\ell)$ for $\mathbb{Q}_0$-almost every $x \in \mathbb{R}$. We moreover assume that $\eta < m^*$ is bounded with respect to $\mathbb{Q}^x$ for some $m^* \in \mathbb{N}$, since the duality formula (6.70) still holds under $\mathbb{Q}^x(\,.\,|\eta \le m^*)$ for any $m^* \in \mathbb{N}$ and the bridges coincide with those of $\mathbb{Q}^x(\,.\,)$ for endpoint values smaller than $x + m^* + 1$.

Denote by dA the intensity of $\mathbb{Q}^x$. By dominated convergence the duality formula (6.70) still holds for any bounded $u : \mathcal{I} \to \mathbb{R}$ with $\langle u \rangle = 0$. Let u be the indicator function of a Lebesgue null-set in $\mathcal{I}$ and put $F \equiv 1$. We apply the duality formula (since $\langle u \rangle = 0$) and get

$$\mathbb{E}^x_{\mathbb{Q}}(\delta(u)) = \mathbb{E}^x_{\mathbb{Q}}\left(\int_{\mathcal{I}} u_s dX_s\right) = \mathbb{E}^x_{\mathbb{Q}}\left(\int_{\mathcal{I}} u_s dA_s\right) = 0.$$

But the random variable $\int_{\mathcal{I}} u_s dA_s \ge 0$ is non-negative, therefore $\int_{\mathcal{I}} u_s dA_s = 0$ $\mathbb{Q}^x$-*a.s.* which implies that $dA \ll dt$ holds $\mathbb{Q}^x$-*a.s.*

In particular there exists a predictable process $a : \mathcal{I} \times \mathbb{J}_1(\mathcal{I}) \to \mathbb{R}_+$ that is $dt \otimes \mathbb{Q}^x$-*a.e.* well defined such that $dA_t = a_t dt$. Let us now compute a nice version of a. Take any $F \in \mathcal{S}_{\mathbb{J}_1}$ that is $\mathcal{F}_{[0,s]}$-measurable and $u \in \mathcal{E}$ such that $u = u\mathbb{1}_{(s,t]}$ for $s, t \in \mathcal{I}$, $s < t$. We apply (6.70) to F and $\tilde{u} := u - \langle u \rangle (t-s)^{-1}\mathbb{1}_{(s,t]}$:

$$\begin{aligned}
\mathbb{E}^x_{\mathbb{Q}}(F(X)\int_{[s,t]} u_r a_r dr) &= \mathbb{E}^x_{\mathbb{Q}}(F(X)\int_{[s,t]} u_r dX_r) \\
&= \mathbb{E}^x_{\mathbb{Q}}(F(X)\delta(\tilde{u})) + \frac{\langle u \rangle}{t-s}\mathbb{E}^x_{\mathbb{Q}}(F(X)\int_{[s,t]} dX_r) \\
&= -\mathbb{E}^x_{\mathbb{Q}}\left(F(X)\int_{[s,t]} u_r \int_{[r,1]} \Xi_\ell(v, X_{v-}) dX_v dr\right) + \int_{[s,t]} u_r \mathbb{E}^x_{\mathbb{Q}}\left(F(X)\frac{X_t - X_s}{t-s}\right) dr \\
&\qquad + \frac{\langle u \rangle}{t-s}\mathbb{E}^x_{\mathbb{Q}}\left(F(X)\int_{[s,t]}\int_{[r,1]} \Xi_\ell(v, X_{v-}) dX_v dr\right).
\end{aligned}$$

Fubini's theorem implies that for $s \le s' \le t$

$$\begin{aligned}
\mathbb{E}^x_{\mathbb{Q}}(a(s')|\mathcal{F}_{[0,s]}) &= \mathbb{E}^x_{\mathbb{Q}}\Bigg(-\int_{[s',1]} \Xi_\ell(r, X_{r-}) dX_r \\
&\qquad + \frac{1}{t-s}\left[X_t - X_s + \int_{[s,t]}\int_{[r,1]} \Xi_\ell(v, X_{v-}) dX_v dr\right] \Bigg| \, \mathcal{F}_{[0,s]}\Bigg),
\end{aligned}$$

and taking $s' = s$ gives

$$\begin{aligned}
a(s) &= \mathbb{E}^x_{\mathbb{Q}}\left(-\int_{[s,1]} \Xi_\ell(r, X_{r-}) dX_r + \frac{1}{t-s}\left[X_t - X_s + \int_{[s,t]}\int_{[r,1]} \Xi_\ell(v, X_{v-}) dX_v dr\right] \Bigg| \, \mathcal{F}_{[0,s]}\right) \\
&= \frac{1}{t-s}\mathbb{E}^x_{\mathbb{Q}}\left(X_t - X_s - \int_{[s,t]}\int_{[s,r]} \Xi_\ell(v, X_{v-}) dX_v dr \,\Big|\, \mathcal{F}_{[0,s]}\right).
\end{aligned}$$

In particular this representation does not depend on $t \in (s,1]$. We conclude that the intensity of $\mathbb{Q}^x$ is of the form $a(s)ds$ with

$$a(s) = \frac{1}{t-s}\mathbb{E}^x_{\mathbb{Q}}\left(\int_{[s,t]}(1-(t-r)\Xi_\ell(r,X_{r-}))\,dX_r \,\middle|\, \mathcal{F}_{[0,s]}\right),\ ds\otimes\mathbb{Q}^x\text{-a.e.} \tag{6.71}$$

Let us prove that $\mathbb{Q}^x$ has the Markov property. Take some $t \in \mathcal{I}$ and define $\mathbb{Q}' := \mathbb{Q}^x(\,.\,|\mathcal{F}_{[0,t]})$ and $\mathbb{Q}'' := \mathbb{Q}^x(\,.\,|X_t)$, which we interpret as probability distributions on $\mathcal{F}_{[t,1]}$ that are $\mathbb{Q}^x$-a.s. well defined. The integration by parts formula (6.70) still holds if $u = u\mathbb{1}_{[t,1]}$ and $\langle u\rangle = 0$. Using (6.71) we see that for $s \geq t$ the point process $\mathbb{Q}'$ has intensity $a_{1,s}ds$, where

$$a_1(s) = \frac{1}{t-s}\mathbb{E}_{\mathbb{Q}'}\left(\int_{[s,t]}(1-(t-r)\Xi_\ell(r,X_{r-}))\,dX_r \,\middle|\, \mathcal{F}_{[0,s]}\right),$$

and $\mathbb{Q}''$ has intensity $a_{2,s}ds$ with

$$a_2(s) = \frac{1}{t-s}\mathbb{E}_{\mathbb{Q}''}\left(\int_{[s,t]}(1-(t-r)\Xi_\ell(r,X_{r-}))\,dX_r \,\middle|\, \mathcal{F}_{[0,s]}\right).$$

By the "tower property" of conditional expectation we deduce $a_1(s)ds = a_2(s)ds$. Thus the law of $\mathbb{Q}^x$, whether conditioned on $\mathcal{F}_{[0,t]}$ or X_t, is the same on $\mathcal{F}_{[t,1]}$, in other words $\mathcal{F}_{[0,t]}$ and $\mathcal{F}_{[t,1]}$ are independent given X_t. This is exactly the Markov property from Definition 4.1.

In particular the intensity of X under $\mathbb{Q}^x$ is given by

$$k(s,X_{s-})ds,\ \text{ with } k(s,y) := \frac{1}{t-s}\mathbb{E}^x_{\mathbb{Q}}\left(\int_{[s,t]}(1-(t-r)\Xi_\ell(r,X_{r-}))\,dX_r \,\middle|\, X_s = y\right), \tag{6.72}$$

where $k : \mathcal{I}\times\mathbb{R}\to\mathbb{R}_+$ is a measurable function that is $ds\otimes\mathbb{Q}^x(X_s\in dy)$-a.e. well defined. The measurability of k is a consequence of the measurability properties of the conditional expectation.

We now check, that the probabilities $\mathbb{Q}^x(X_s \in y)$ qualitatively behave like probabilities in the reciprocal class of $\mathbb{P}_\ell$ in the sense of Remark 6.57. This is necessary to establish a regular behaviour of $\mathbb{Q}^x$ and k in view of later computations. Fix any $y = x+m$ with $m\in\mathbb{N}$ and define the set $\mathcal{I}(y) := \{s\in\mathcal{I} : \mathbb{Q}^x(X_s = y) > 0\}$. Since unit jump functions are càdlàg we immediately deduce that this set is of the form $\mathcal{I}(y) = \cup_{i\geq 1}[s_i, s_{i+1})$ or $\mathcal{I}(y) = \cup_{i\geq 1}[s_i,s_{i+1})\cup\{1\}$ for some $s_1 < s_2 < \dots$. Using (6.72) in $t=1$ we can bound $k(s,y) \leq \frac{1}{1-s}(K + \mathbb{E}_{\mathbb{Q}}(\eta))$, such that if $s\in\mathcal{I}(y)$ we have $[s,1)\subset\mathcal{I}(y)$: A finite intensity implies a positive probability of staying in y (this is easy to see using the explicit distribution given in Remark 6.50). If we define $s_y := \inf\{s\in\mathcal{I}(y)\}$ then $\mathcal{I}(y) = [s_y,1)$ or $\mathcal{I}(y) = [s_y,1]$ and $k(s,y) < \infty$ for $s\in\mathcal{I}(y)$. Again by (6.72) the function $s\mapsto k(s,y)$ is continuous on $\mathcal{I}(y)$ since

$$k(s,y) = \frac{1}{t-s}\int_{[s,t]}\mathbb{E}^x_{\mathbb{Q}}\left((1-(t-r)\Xi_\ell(r,X_{r-}))\,k(r,X_{r-})|\,X_s = y\right)dr.$$

Clearly $\mathcal{I}(x) = [0,1)$ or $\mathcal{I}(x) = [0,1]$. Take the smallest $y = x+m$ such that $s_{y+1} > 0$ and let $\mathbb{Q}^x(X_{s_{y+1}+\varepsilon} = y+1|X_{s_{y+1}} = y) =: \delta(\varepsilon) > 0$ for any $\varepsilon > 0$ small. Assume that $\varepsilon\|\Xi_\ell\|_\infty < 1$, then using (6.72) we get the contradiction

$$0 = k(s_{y+1},y) \geq \frac{1}{\varepsilon}\mathbb{E}^x_{\mathbb{Q}}\left((1-\varepsilon\|\Xi_\ell\|_\infty)(X_{s_{y+1}+\varepsilon} - X_{s_{y+1}})|X_{s_{y+1}} = y\right) \geq (1-\varepsilon\|\Xi_\ell\|_\infty)\delta(\varepsilon) > 0.$$

Therefore $s_y = 0$ or $s_y = 1$ for all $y = x+m$, $m\in\mathbb{N}$.

Let us now compute the first time derivative of $k(s,y)$ for some $s\in(0,1)$ and y with $\mathbb{Q}^x(X_s = y) > 0$. To show that $k(s,y)$ is differentiable in s we will use (6.72) to express

$k(s+\varepsilon, y) - k(s, y)$, where $\varepsilon > 0$ with $s + \varepsilon < 1$ (the case "$\varepsilon < 0$" can be treated in a similar way). Define

$$H(s,t) = \frac{1}{t-s} \int_{[s,t]} \left(k(r, X_{r-}) - \int_{[s,r]} \Xi_\ell(v, X_{v-}) dX_v \right) dr.$$

We use the short notation $k_r = k(r, X_{r-})$ and $\Xi_{\ell,r} = \Xi_\ell(r, X_{r-})$ and the decomposition

$$\begin{aligned} \int_{[s,t]} \left(k_r - \int_{[s,r]} \Xi_{\ell,v} dX_v \right) dr &= \int_{[s+\varepsilon,t]} \left(k_r - \int_{[s+\varepsilon,r]} \Xi_{\ell,v} dX_v \right) dr \\ &\quad + \int_{[s,s+\varepsilon]} \left(k_r - \int_{[s,r]} \Xi_{\ell,v} dX_v \right) dr - \int_{[s+\varepsilon,t]} \int_{[s,s+\varepsilon]} \Xi_{\ell,v} dX_v dr \end{aligned}$$

when developing

$$\begin{aligned} & H(s,t) - H(s+\varepsilon, t) \\ = & \frac{(t-s-\varepsilon) \int_{[s,t]} \left(k_r - \int_{[s,r]} \Xi_{\ell,v} dX_v \right) dr - (t-s) \int_{[s+\varepsilon,t]} \left(k_r - \int_{[s+\varepsilon,r]} \Xi_{\ell,v} dX_v \right) dr}{(t-s)(t-s-\varepsilon)} \\ = & \frac{-\varepsilon \int_{[s+\varepsilon,t]} \left(k_r - \int_{[s+\varepsilon,r]} \Xi_{\ell,v} dX_v \right) dr + (t-s-\varepsilon) \int_{[s,s+\varepsilon]} \left(k_r - \int_{[s,r]} \Xi_{\ell,v} dX_v \right) dr}{(t-s)(t-s-\varepsilon)} \\ & + \frac{- \int_{[s+\varepsilon,t]} \int_{[s,s+\varepsilon]} \Xi_{\ell,v} dX_v dr}{(t-s)}. \end{aligned}$$

This can be rewritten into

$$H(s,t) = H(s+\varepsilon, t) - \frac{\varepsilon}{t-s} H(s+\varepsilon, t) + \frac{\varepsilon}{t-s} H(s, s+\varepsilon) - \frac{t-s-\varepsilon}{t-s} \int_{[s,s+\varepsilon]} \Xi_{\ell,r} dX_r. \tag{6.73}$$

We are going to use

$$k(s,y) = \mathbb{E}^x_{\mathbb{Q}} \left(H(s,t) | X_s = y \right) \text{ and } k(s+\varepsilon, y) = \mathbb{E}^x_{\mathbb{Q}} \left(H(s+\varepsilon, t) | X_{s+\varepsilon} = y \right).$$

But with (6.73) this implies

$$k(s,y) = \mathbb{E}_{\mathbb{Q}} \left(H(s+\varepsilon, t) - \int_{[s,s+\varepsilon]} \Xi_{\ell,r} dX_r \,\middle|\, X_s = y \right).$$

We already know that the intensity $k(s,y)$ is continuous in $s \in \mathcal{I}$ and locally bounded. Thus the following short time expansions hold:

$$\begin{aligned} \mathbb{Q}^x(X_{s+\varepsilon} = y | X_s = y) &= 1 - \varepsilon k(s,y) + O(\varepsilon^2) \\ \mathbb{Q}^x(X_{s+\varepsilon} = y+1 | X_s = y) &= \varepsilon k(s,y) + O(\varepsilon^2) \\ \mathbb{Q}^x(X_{s+\varepsilon} > y+1 | X_s = y) &= O(\varepsilon^2), \end{aligned}$$

see Remark 6.50. But then

$$\begin{aligned}
k(s,y) &= \mathbb{E}^x_{\mathbb{Q}}\left(H(s+\varepsilon,t) - \int_{[s,s+\varepsilon]} \Xi_{\ell,r} dX_r \,\middle|\, X_s = y\right) \\
&= \mathbb{E}^x_{\mathbb{Q}}\left(H(s+\varepsilon,t) - \int_{[s,s+\varepsilon]} \Xi_{\ell,r} dX_r \,\middle|\, X_{s+\varepsilon} = y, X_s = y\right)\mathbb{Q}^x(X_{s+\varepsilon} = y, X_s = y) \\
&\quad + \mathbb{E}^x_{\mathbb{Q}}\left(H(s+\varepsilon,t) - \int_{[s,s+\varepsilon]} \Xi_{\ell,r} dX_r \,\middle|\, X_{s+\varepsilon} = y+1, X_s = y\right)\mathbb{Q}^x(X_{s+\varepsilon} = y+1 | X_s = y) \\
&\quad + \mathbb{E}^x_{\mathbb{Q}}\left(H(s+\varepsilon,t) - \int_{[s,s+\varepsilon]} \Xi_{\ell,r} dX_r \,\middle|\, X_{s+\varepsilon} > y+1, X_s = y\right)\mathbb{Q}^x(X_{s+\varepsilon} > y+1 | X_s = y).
\end{aligned}$$

The first term is

$$\begin{aligned}
&\mathbb{E}^x_{\mathbb{Q}}\left(H(s+\varepsilon,t) - \int_{[s,s+\varepsilon]} \Xi_{\ell,r} dX_r \,\middle|\, X_{s+\varepsilon} = y, X_s = y\right)\mathbb{Q}^x(X_{s+\varepsilon} = y, X_s = y) \\
&= \mathbb{E}^x_{\mathbb{Q}}\left(H(s+\varepsilon,t) | X_{s+\varepsilon} = y\right)(1 - \varepsilon k(y,s) + O(\varepsilon^2)) \\
&= k(s+\varepsilon, y) - \varepsilon k(s+\varepsilon, y) k(s,y) + O(\varepsilon^2).
\end{aligned}$$

The second term is

$$\begin{aligned}
&\mathbb{E}^x_{\mathbb{Q}}\left(H(s+\varepsilon,t) - \int_{[s,s+\varepsilon]} \Xi_{\ell,r} dX_r \,\middle|\, X_{s+\varepsilon} = y+1, X_s = y\right)\mathbb{Q}^x(X_{s+\varepsilon} = y+1 | X_s = y) \\
&= (k(s+\varepsilon, y+1))(\varepsilon k(y,s) + O(\varepsilon^2)) - (\Xi_\ell(s,y) + O(\varepsilon))(\varepsilon k(y,s) + O(\varepsilon^2)) \\
&= \varepsilon k(s+\varepsilon, y+1) k(s,y) - \varepsilon \Xi_\ell(s,y) k(s,y) + O(\varepsilon^2).
\end{aligned}$$

The third term is of order $O(\varepsilon^2)$. Using these expansions we get

$$\begin{aligned}
&\frac{1}{\varepsilon}(k(s+\varepsilon, y) - k(s,y)) \\
&= k(s+\varepsilon, y) k(s,y) - k(s+\varepsilon, y+1) k(s,y) + \Xi_\ell(s,y) k(s,y) + O(\varepsilon) \\
&\to k(s,y)(k(s,y) - k(s,y+1) + \Xi_\ell(s,y)), \qquad (6.74)
\end{aligned}$$

where the last equality follows from the continuity of k. Under the condition, that $k(t,y) > 0$, this is $\Xi_k(t,y) = \Xi_\ell(t,y)$.

In particular (6.74) represents an ordinary differential equation satisfied by $k(.,y)$. We use this to check the qualitative behavior of the intensity under $\mathbb{Q}^x$ in the sense of Remark 6.57. Assume that $k(s,y) = 0$ for some $s \in \mathcal{I}$. Since the above ordinary differential equation in has a unique solution therefore $k(.,y) \equiv 0$ on $\mathcal{I}$. Thus for $y = x + m$ with $\mathbb{Q}^x(X_s = y) > 0$ we have $k(.,y) \equiv 0$ or $k(.,y) > 0$ on $\mathcal{I}$, and in the first case $\mathbb{Q}^x(X_\cdot = y+1) \equiv 0$ whereas in the second case $\mathbb{Q}^x(X_\cdot = y+1) > 0$ on $\mathcal{I}$. In particular, by our assumption that $\eta \le m^*$ is bounded $\mathbb{Q}^x$-*a.s.* we know that $k(.,x+m^*) \equiv 0$.

The rest of the argument is similar to the second part of the proof of Theorem 6.58. For $y' = x + n' \le x + m^*$ such that $\mathbb{Q}^x(X_1 = y') > 0$ we may define

$$h(t,x) := c(t), \text{ and } h(t,x+m) := 1_{\{m \le m^*\}} \prod_{j=0}^{m-1} \frac{k(t,x+j)}{\ell(t,x+j)} c(t), \text{ with } c(t) := c e^{-\int_{[0,t]} (k(s,y') - \ell(s,y')) ds},$$

since in this case $c(t)$ is strictly positive. Here c is a normalization constant such that $\mathbb{E}^x_\ell(h(1,X_1)) = 1$, which exists since we work under the assumption that $\eta \le m^*$ is bounded under $\mathbb{Q}^x$, and $k(1,y)$ is well defined and finite if $\mathbb{Q}^x(X_1 = y) > 0$. If there exists $y'' = x + n'' \ge y'$ with $k(.,y'') \equiv 0$ we put $h(.,y''+1) \equiv 0$. Then for $x \le y < y''$ the derivative (6.74)

implies $\Xi_k(t,y) = \Xi_\ell(t,y)$. As we have seen in the proof of Theorem 6.58 the function h is a space-time harmonic

$$\partial_t h(t,y) + \ell(t,y)\,(h(t,y+1) - h(t,y)) = 0.$$

By Proposition 6.53 the h-transform $h(1,X_1)\mathbb{P}_\ell^x(\,.\,)$ is a unit jump process with intensity $k(t,X_{t-})dt$. Since the intensity of a unit jump process characterizes the distribution, we identify $\mathbb{Q}^x = h\mathbb{P}_\ell^x$ as that h-transform, and in particular $\mathbb{Q}^x \in \mathcal{R}(\mathbb{P}_\ell)$. □

Example 6.75. *In Examples 6.62 and 6.67 we have treated the exponential decay process X with intensity $\ell(t)dt = \lambda e^{-\lambda t}dt$ under $\mathbb{P}_\ell$. The physical description of time-development of the number of decaying particles was given in (6.63) by*

$$\frac{dN_t}{dt} = -\lambda N_t \;\Leftrightarrow\; \int_{\mathcal{I}} u_t(dN_t + \lambda N_t dt) = 0,\ \forall u \in \mathcal{E}.$$

On the other hand, following Example 6.67 and the above characterization result, X is an exponential decay process under $\mathbb{Q}$ with $\eta \in \mathbb{L}^1(\mathbb{Q})$ if and only if

$$\mathbb{E}_{\mathbb{Q}}\left(F(X)\left(\int_{\mathcal{I}} u_t dX_t + \lambda X_{t-}dt\right)\right) = \mathbb{E}_{\mathbb{Q}}\left(\mathcal{D}_u F(X)\right),$$

for all $F \in \mathcal{S}_{\mathbb{J}_1}$ and $u \in \mathcal{E}$ with $\langle u\rangle = 0$. The duality formula may thus be interpreted as decay law. In this context Theorem 6.69 garuantees the existence and uniqueness (up to the endpoint distribution) of a solution of a decay problem under the condition, that the reciprocal invariant is the invariant of a nice unit jump process. We will generalize this interpretation of the duality formula in the next paragraph.

The last two paragraphs are devoted to applications of Theorem 6.69 to an optimal control problem and to the time reversal of unit jump processes.

6.7. **An optimal control problem associated to the reciprocal class.**

Numerous authors have been interested in optimal control problems for jump processes, see e.g. the monographs by Brémaud [Bré81], Øksendahl, Sulem [ØS07] or Fleming, Soner [FS06]. Privault and Zambrini introduced an optimal control problem whose solutions are elements of reciprocal classes of Lévy processes in [PZ04]. In this paragraph we present a similar approach to the optimal control of nice unit jump processes. Our results are based on an entropy-minimization similar to the one used for continuous processes in § 5.5.

In a few words the following control problem is the minimization of a cost function given fixed endpoint distributions. Let μ_{01} be some probability on $\mathbb{R}^2$, in the following definition $\mathbb{P}$ is the law of a Poisson process with initial condition $\mathbb{P}_0(dx) = \mu_0(dx) := \mu_{01}(dx \times \mathbb{R})$. The class of admissible unit jump processes is

$$\Gamma(\mu_{01}) := \{\mathbb{Q} :\ \text{a unit jump processes such that } \mathbb{Q}_{01} = \mu_{01} \text{ and } \mathbb{Q} \ll \mathbb{P}\}. \tag{6.76}$$

Note that this set may very well be empty. A necessary and sufficient condition for non-emptiness is $\mu_{01} \ll \mathbb{P}_{01}$, where $\mathbb{P}_0 = \mu_0$. By Girsanov's theorem the condition $\mathbb{Q} \ll \mathbb{P}$ ensures that the unit jump process $\mathbb{Q}$ has an intensity of the form $\gamma_t dt$, where $\gamma : \mathcal{I} \times \mathbb{J}_1(\mathcal{I}) \to \mathbb{R}$ is predictable. We will write $\mathbb{P}_\gamma$ instead of $\mathbb{Q}$. This notation is consistent with the notation $\mathbb{P}_\ell$ for nice unit jump processes with intensity $\ell(t,X_{t-})dt$.

Given $\varepsilon > 0$ and bounded functions $A : \mathcal{I} \times \mathbb{R} \to [\varepsilon, \infty)$, $\Phi : \mathcal{I} \times \mathbb{R} \to \mathbb{R}$ such that $A(.,x), \Phi(.,x) \in C_b^1(\mathcal{I})$ for all $x \in \mathbb{R}$, we define a Lagrangian by

$$L(X_{t-}, \gamma_t, t) := \gamma_t \log \gamma_t + \gamma_t \log A(t, X_{t-}) - \Phi(t, X_{t-}). \tag{6.77}$$

The cost function associated to the above Lagrangian is defined by

$$\mathcal{J}(\mathbb{P}_\gamma) = \mathbb{E}_\gamma\left(\int_{\mathcal{I}} L(X_{t-}, \gamma_t, t)dt\right) = \mathbb{E}_\gamma\left(\int_{\mathcal{I}} (\gamma_t \log \gamma_t + \gamma_t \log A(t, X_{t-}) - \Phi(t, X_{t-}))\, dt\right). \tag{6.78}$$

As in the diffusion case, we will make assumptions on the endpoint distribution. In particular we assume that $\mu_{01} = \mu_0 \otimes \mu_1$ with

$$\mu_0 = \delta_{\{x\}} \text{ and } \mu_1 \ll \sum_{i=0}^{\infty} \delta_{\{x+i\}} \text{ with } \mu_1(\{x+i\}) > 0, \text{ such that } \sum_{i\geq 1} i\, \mu_1(\{x+i\}) < \infty. \tag{6.79}$$

Clearly $\Gamma(\mu_{01})$ is non-empty in this case, and for every $\mathbb{Q} \in \Gamma(\mu_{01})$ we have $\eta \in \mathbb{L}^1(\mathbb{Q})$. We may now state the optimal control problem.

Definition 6.80. *The optimal control problem consists of minimizing the cost function (6.78) in the class $\Gamma(\mu_{01})$ of admissible unit jump processes.*

Under our assumptions on the endpoint distribution and the cost potential A and Φ the optimal control problem has the following solution.

Proposition 6.81. *Let μ_{01} be as in (6.79), then the unique minimizer of the cost (6.78) in the class $\Gamma(\mu_{01})$ is the Markov unit jump process $\mathbb{P}^x_k$ with intensity $k(t, X_{t-}) = (eA(t, X_{t-}))^{-1} \frac{h(t, X_{t-}+1)}{h(t, X_{t-})}$, where*

$$h(t, y) = \mathbb{E}^x_{(eA)^{-1}}\left(h(X_1) \exp\left(-\int_{[t,1]} \left(\Phi(t, X_{t-}) + (eA(t, X_{t-}))^{-1}\right) dt\right) \middle|\, X_t = y\right), \tag{6.82}$$

and we used

$$h(y) = e(y-x)!\mu_1(\{y\})\left[\mathbb{E}^{x,y}\left(\exp\left(-\int_{\mathcal{I}} (\log A(t, X_{t-}) + 1)dX_t + \int_{\mathcal{I}} (\Phi(t, X_{t-}) + 1)dt\right)\right)\right]^{-1} \tag{6.83}$$

Proof. We introduce an auxiliary measure on $\mathbb{J}_1(\mathcal{I})$ by

$$\tilde{\mathbb{Q}}^x := G^{A,\Phi} \text{ with } G^{A,\Phi} := \exp\left(-\int_{\mathcal{I}} (\log A(t, X_{t-}) + 1)dX_t + \int_{\mathcal{I}} (\Phi(t, X_{t-}) + 1)dt\right)\mathbb{P}^x. \tag{6.84}$$

Note that $\tilde{\mathbb{Q}}^x$ is not necessarily a probability measure, but is equivalent to $\mathbb{P}^x$ by boundedness assumptions on A and Φ. Every $\mathbb{P}^x_\gamma \in \Gamma(\mu_{01})$ is absolutely continuous with respect to $\mathbb{P}^x$ with Girsanov density

$$G^\gamma := \exp\left(-\int_{\mathcal{I}} (\gamma_t - 1)\, dt\right) \prod_{i=1}^{\eta} \gamma_{T_i} = \mathbb{1}_{G^\gamma > 0} \exp\left(\int_{\mathcal{I}} \log \gamma_t dX_t - \int_{\mathcal{I}} (\gamma_t - 1)\, dt\right). \tag{6.85}$$

Assume that $\mathcal{J}(\mathbb{P}^x_\gamma)$ is finite for some $\mathbb{P}^x_\gamma \in \Gamma(\mu_{01})$, then

$$\begin{aligned}
\mathcal{J}(\mathbb{P}_\gamma) &= \mathbb{E}^x_\gamma\left(\int_{\mathcal{I}} (\gamma_t \log \gamma_t + \gamma_t \log A(t, X_{t-}) - \Phi(t, X_{t-}))\, dt\right) \\
&= \mathbb{E}^x_\gamma\left(\int_{\mathcal{I}} \log \gamma_t dX_t - \int_{\mathcal{I}} (\gamma_t - 1)\, dt + \int_{\mathcal{I}} (\log A(t, X_{t-}) + 1)dX_t - \int_{\mathcal{I}} (\Phi(t, X_{t-}) + 1)dt\right),
\end{aligned}$$

and using the definitions of the densities (6.84) and (6.85) we get

$$\mathcal{J}(\mathbb{P}_\gamma) = \mathbb{E}^x_\gamma\left(\log G^\gamma - \log G^{A,\Phi}\right) = \mathbb{E}^x_\gamma\left(\log\left(\frac{d\mathbb{P}^x_\gamma}{d\mathbb{P}^x}\frac{d\mathbb{P}^x}{d\tilde{\mathbb{Q}}^x}\right)\right) = \mathbb{E}^x_\gamma\left(\log\left(\frac{d\mathbb{P}^x_\gamma}{d\tilde{\mathbb{Q}}^x}\right)\right).$$

We use the multiplication formula

$$\frac{d\mathbb{P}^x_\gamma}{d\tilde{\mathbb{Q}}^x}(\omega) = \frac{d\mathbb{P}^x_{\gamma,1}}{d\tilde{\mathbb{Q}}^x}(\omega(1))\frac{d\mathbb{P}^{x,\omega(1)}_\gamma}{d\tilde{\mathbb{Q}}^{x,\omega(1)}}(\omega),\ \forall \omega \in \mathbb{J}_1(\mathcal{I}),$$

but the first term must be nothing else than the function $h(y)$ defined in (6.83): Necessarily $\mathbb{P}^x_{\gamma,1} = \mu_1$ and the endpoint distribution of $\tilde{\mathbb{Q}}^x$ is given by

$$\tilde{\mathbb{Q}}^x_1(\{x+m\}) = \mathbb{E}^x\left(\mathbb{1}_{\{x+m\}}(X_1)G^{A,\Phi}\right) = \mathbb{E}^{x,x+m}\left(G^{A,\Phi}\right)\frac{e^{-1}}{m!}.$$

Therefore

$$\mathcal{J}(\mathbb{P}_\gamma) = \mathbb{E}^x_\gamma\left(\log h(X_1)\right) + \mathbb{E}^x_\gamma\left(\log\left(\frac{d\mathbb{P}^{x,X_1}_\gamma}{d\tilde{\mathbb{Q}}^{x,X_1}}\right)\right).$$

The first term is independent of γ by the boundary condition in $\Gamma(\mu_{01})$ and the second term is zero if and only if $\mathbb{P}^{x,y}_\gamma = \tilde{\mathbb{Q}}^{x,y}$ holds $\mathbb{P}^x_{\gamma,1}$-*a.s.* But by construction the h-transform $h\tilde{\mathbb{Q}}^x := h(X_1)\tilde{\mathbb{Q}}^x$ would satisfies this condition and moreover $h\tilde{\mathbb{Q}}^x \in \Gamma(\mu_{01})$.

Let us now identify the form of the intensity of the solution $h\tilde{\mathbb{Q}}^x$. We want to write $h(X_1)$ in the form of a Doléans-Dade exponential, therefore we need a Feynman-Kac formula for unit-jump processes.

Lemma 6.86. *Let $\tilde{A} : \mathcal{I}\times\mathbb{R} \to [\varepsilon,\infty)$, $\tilde{\Phi} : \mathcal{I}\times\mathbb{R} \to \mathbb{R}$ for some $\varepsilon > 0$ be bounded and $h : \mathbb{R} \to \mathbb{R}_+$ such that $h(X_1) \in \mathbb{L}^1(\mathbb{P}^x_{\tilde{A}})$, where $\mathbb{P}^x_{\tilde{A}}$ is a unit jump process with intensity $\tilde{A}(t, X_{t-})dt$. Then*

$$h(t,y) := \mathbb{E}^x_{\tilde{A}}\left(h(X_1)e^{-\int_{[t,1]}\tilde{\Phi}(s,X_{s-})ds}\,\middle|\, X_t = y\right)$$

is a solution of the Feynman-Kac equation

$$\partial_t h(t,y) + \tilde{A}(t,y)(h(t,y+1) - h(t,y)) - \tilde{\Phi}(t,y)h(t,y) = 0, \tag{6.87}$$

in particular $h(.,y) \in C^1(\mathcal{I})$ for all $y = x+m$, $m \in \mathbb{N}$.

Proof. The proof is similar to the proof of the Kolmogoroff backward equation in Lemma 6.55. The explicit definition of $h(t,x)$ may be written as

$$\begin{aligned} h(t,y) &= \sum_{m=0}^{\infty} h(y+m)p^{\tilde{\Phi}}_{\tilde{A}}(t,y;1,y+m) \\ &= \sum_{m=0}^{\infty} h(y+m)\int_{[t,1]}\int_{[t,t_m]}\cdots\int_{[t,t_2]} f^{\tilde{\Phi}}_y(t_1,\ldots,t_m,t)dt_1\cdots dt_m, \end{aligned}$$

where

$$\begin{aligned} f^{\tilde{\Phi}}_y(t_1,\ldots,t_m,t) &:= \tilde{A}(t_1,y)e^{-\int_{[t,t_1]}(\tilde{\Phi}(s,y)+\tilde{A}(s,y))ds}\cdots \\ &\cdots\tilde{A}(t_m,y+m-1)e^{-\int_{[t_{m-1},t_m]}(\tilde{\Phi}(s,y+m-1)+\tilde{A}(s,y+m-1))ds}e^{-\int_{[t_m,1]}(\tilde{\Phi}(s,y+m)+\tilde{A}(s,y+m))ds}. \end{aligned}$$

Clearly $f_y^{\tilde\Phi}(t_1,\dots,t_m,t)$ is bounded and differentiable in t, and the derivative is given by $\partial_t f_y^{\tilde\Phi}(t_1,\dots,t_m,t) = (\tilde\Phi(t,y)+\tilde A(t,y))f_y^{\tilde\Phi}(t_1,\dots,t_m,t)$. This implies

$$\begin{aligned}
&\frac{1}{\varepsilon}\left(p_{\tilde A}^{\tilde\Phi}(t+\varepsilon,y;1,y+m)-p_{\tilde A}^{\tilde\Phi}(t,y;1,y+m)\right)\\
=\;&\frac{1}{\varepsilon}\left[\int_{[t+\varepsilon,1]}\cdots\int_{[t+\varepsilon,t_2]} f_y^{\tilde\Phi}(t_1,\dots,t_m,t+\varepsilon)dt_1\cdots dt_m-\int_{[t,1]}\cdots\int_{[t,t_2]} f_y^{\tilde\Phi}(t_1,\dots,t_m,t)dt_1\cdots dt_m\right]\\
=\;&\frac{1}{\varepsilon}\left[\int_{[t+\varepsilon,1]}\cdots\int_{[t+\varepsilon,t_2]}\left(f_y^{\tilde\Phi}(t_1,\dots,t_m,t+\varepsilon)-f_y^{\tilde\Phi}(t_1,\dots,t_m,t)\right)dt_1\cdots dt_m\right.\\
&-\int_{[t,t+\varepsilon]}\int_{[t,t_m]}\cdots\int_{[t,t_2]} f_y^{\tilde\Phi}(t_1,\dots,t_m,t)dt_1\cdots dt_m-\dots\\
&\left.\cdots-\int_{[t,1]}\int_{[t,t_m]}\cdots\int_{[t,t+\varepsilon]} f_y^{\tilde\Phi}(t_1,\dots,t_m,t)dt_1\cdots dt_m,\right.
\end{aligned}$$

which by dominated convergence goes to

$$\begin{aligned}
\overset{\varepsilon\to0}{\to}\;&\int_{[t,1]}\cdots\int_{[t,t_2]}(\tilde\Phi(t,y)+\tilde A(t,y))f_y^{\tilde\Phi}(t_1,\dots,t_m,t)dt_1\cdots t_m\\
&-\int_{[t,1]}\cdots\int_{[t,t_3]} f_y^{\tilde\Phi}(t,t_2,\cdots,t_m,t)dt_2\cdots dt_m\\
=\;&(\tilde\Phi(t,y)+\tilde A(t,y))p_{\tilde A}^{\tilde\Phi}(t,x;1,x+m)-\tilde A(t,y)p_{\tilde A}^{\tilde\Phi}(t,y+1;1,y+m).
\end{aligned}$$

The same computations apply to the limit of $-\frac{1}{\varepsilon}(p_{\tilde A}^{\tilde\Phi}(t-\varepsilon,x;1,x+m)-p_{\tilde A}^{\tilde\Phi}(t,x;1,x+m)$. The sum over the first term is

$$\begin{aligned}
\sum_{m=0}^{\infty}h(y+m)(\tilde\Phi(t,y)+\tilde A(t,y))p_{\tilde A}^{\tilde\Phi}(t,y;1,y+m) &= (\tilde\Phi(t,y)+\tilde A(t,y))\mathbb{E}_{\tilde A}^{x}(h(X_1)e^{-\int_{[t,1]}\tilde\Phi(s,X_{s-})ds}|X_t=y)\\
&= (\tilde\Phi(t,y)+\tilde A(t,y))h(t,y),
\end{aligned}$$

and the sum over the second term is

$$\begin{aligned}
\sum_{m=1}^{\infty}h(y+m)\tilde A(t,y)p_{\tilde A}^{\tilde\Phi}(t,y+1;1,y+m) &= \sum_{m=1}^{\infty}h(y+1+m-1)\tilde A(t,y)p_{\tilde A}^{\tilde\Phi}(t,y+1;1,y+m)\\
&= \tilde A(t,y)\mathbb{E}_{\tilde A}^{x}(h(X_1)|X_t=y+1)\\
&= \tilde A(t,y)h(t,y+1),
\end{aligned}$$

and we see that (6.87) holds. □

Let us resume the proof of Proposition 6.81. Since $h(.,y)$ is differentiable and $h(t,X_{t-})>0$ holds $h\tilde{\mathbb{Q}}^x$-*a.s.*, we may apply the Itô-formula

$$\log h(1,X_1)=\log h(0,x)+\int_{\mathcal{I}}\partial_t\log h(t,X_{t-})dt+\int_{\mathcal{I}}\log(h(t,X_{t-}+1)-\log h(t,X_{t-}))dX_t.$$

Using this we can rewrite

$$h(X_1)\tilde{\mathbb{Q}}^x=\mathbb{1}_{\{h(X_1)>0\}}\exp\left(\int_{\mathcal{I}}\partial_t\log h(t,X_{t-})dt+\int_{\mathcal{I}}\log\left(\frac{h(t,X_{t-}+1)}{h(t,X_{t-})}\right)dX_t\right)G^{\Phi,A}\mathbb{P}^x.$$

Now we apply the Feynman-Kac equation of Lemma 6.86 with $\tilde{A}(t,y) = (eA(t,y))^{-1}$ and $\tilde{\Phi}(t,y) = -\Phi(t,y) - (eA(t,y))^{-1}$, then

$$\begin{aligned}\partial_t \log h(t,y) &= -(eA(t,y))^{-1}\left(\frac{h(t,y+1)}{h(t,y)} - 1\right) - \Phi(t,y) - (eA(t,y))^{-1} \\ &= -(eA(t,y))^{-1}\left(\frac{h(t,y+1)}{h(t,y)}\right) - \Phi(t,y), \qquad (6.88)\end{aligned}$$

and we get

$$\begin{aligned}h(X_1)\tilde{\mathbb{Q}}^x &= \mathbb{1}_{\{h(X_1)>0\}} \exp\left(\int_{\mathcal{I}}\left(-(eA(t,X_{t-}))^{-1}\left(\frac{h(t,X_{t-}+1)}{h(t,X_{t-})}\right) - \Phi(t,X_{t-})\right)dt\right. \\ &\quad \left. + \int_{\mathcal{I}} \log\left(\frac{h(t,X_{t-}+1)}{h(t,X_{t-})}\right)dX_t - \int_{\mathcal{I}} \log(eA(t,X_{t-}))dX_t + \int_{\mathcal{I}} (\Phi(t,X_{t-})+1)\,dt\right)\mathbb{P}^x \\ &= \mathbb{1}_{\{h(X_1)>0\}} \exp\left(\int_{\mathcal{I}} \log k(t,X_{t-})dX_t - \int_{\mathcal{I}} (k(t,X_{t-})-1)\,dt\right)\mathbb{P}^x,\end{aligned}$$

where k is defined as in the statement of the proposition. By Girsanov's theorem we recognize that $\mathbb{P}^x_k = h(X_1)\tilde{\mathbb{Q}}^x$ is a unit jump process with intensity $k(t,X_{t-})dt$, moreover $\mathbb{P}^x_k$ has the Markov property since the density with respect to $\mathbb{P}^x$ factorizes in the sense of Lemma 4.4. □

Remark 6.89. *Note that the intensity of the solution given by (6.82) and (6.83) has a rather peculiar dependence on the potentials A and Φ. This is due to the fact, that we wanted to keep the form of the Lagrangian (6.77) as simple as possible, and close to the diffusion case presented in § 5.5. If e.g. we put $\tilde{A} := (eA)^{-1}$ and $\tilde{\Phi} := \Phi + 1$ we get an alternative formulation of the Lagrangian*

$$\tilde{L}(X_{t-},\gamma_t,t) = \gamma_t \log\gamma_t - \gamma_t - 1 - \gamma_t \log\tilde{A}(t,X_{t-}) - \tilde{\Phi}(t,X_{t-}),$$

and the intensity of the solution of the associated optimal control problem is then given by $k(t,y) = \tilde{A}(t,y)\frac{\tilde{h}(t,y+1)}{\tilde{h}(t,y)}$ with

$$\tilde{h}(y) = e(y-x)!\mu_1(\{y\})\left[\mathbb{E}^{x,y}\left(\int_{\mathcal{I}} \log\tilde{A}(t,X_{t-})dX_t + \int_{\mathcal{I}} \tilde{\Phi}(t,X_{t-})dt\right)\right]^{-1}$$

and

$$\tilde{h}(t,y) = \mathbb{E}^x_{\tilde{A}}\left(h(X_1)\exp\left(-\int_{[t,1]}(\tilde{\Phi}(t,X_{t-}) + \tilde{A}(t,X_{t-}) - 1)dt\right)\middle|\ X_t = y\right).$$

Using the above proposition we may now state sufficient conditions on nice unit jump processes to be the solution of the optimal control problem in terms of the reciprocal invariant.

Proposition 6.90. *Let μ_{01} be as in (6.79) and assume that there exists a bounded $\psi : \mathcal{I}\times\mathbb{R} \to \mathbb{R}$, such that $\psi(.,y) \in C^1_b(\mathcal{I})$ for every $y = x + m$ with $m \in \mathbb{N}$, that is solution of the differential equation*

$$0 = \partial_t\psi(t,y) + (eA(t,y))^{-1}e^{\psi(t,y+1)-\psi(t,y)} + \Phi(t,y), \qquad (6.91)$$

and subject to the normalization condition $\mathbb{E}^x_{\tilde{\mathbb{Q}}}(e^{\psi(1,X_1)}) = 1$, where $\tilde{\mathbb{Q}}^x$ is defined in (6.84). Define $k(t,y) = (eA(t,y))^{-1}e^{\psi(t,y+1)-\psi(t,y)}$. Then a unit jump process $\mathbb{Q}^x \in \Gamma(\mu_{01})$ is the solution of the optimal control problem if and only if the duality formula

$$\mathbb{E}^x_{\mathbb{Q}}(F(X)\delta(u)) = \mathbb{E}^x_{\mathbb{Q}}(\mathcal{D}_u F(X)) - \mathbb{E}^x_{\mathbb{Q}}\left(F(X)\int_{\mathcal{I}} u_t \int_{[t,1]} \Xi_k(s,X_{s-})dX_s dt\right) \qquad (6.92)$$

holds for every $F \in \mathcal{S}_{\mathbb{J}_1}$ and $u \in \mathcal{E}$ with $\langle u \rangle = 0$.

Proof. Put $h(t,y) := e^{\psi(t,y)}$ for $y = x + m$ and $m \in \mathbb{N}$. Then $k(t,y) = (eA(t,y))^{-1} \frac{h(t,y+1)}{h(t,y)}$, and $\mathbb{P}_k^x = h(1,X_1)\tilde{\mathbb{Q}}^x$ where $h(1,x+m) > 0$ for all $m \in \mathbb{N}$ by boundedness assumption on ψ. Proposition 6.81 implies that if a unit jump process $\mathbb{Q}^x \in \Gamma(\mu_{01})$ is the minimizer of the cost function (6.78) then there exists a function $\tilde{h} : \mathbb{R} \to \mathbb{R}_+$ such that $\mathbb{Q}^x = \tilde{h}(X_1)\tilde{\mathbb{Q}}^x$. But then $\mathbb{Q}^x = h(1,X_1)^{-1}\tilde{h}(X_1)\mathbb{P}_k^x$ is in the reciprocal class of $\mathbb{P}_k$ and thus the duality formula (6.92) holds.

If on the other hand the above duality formula holds for some $\mathbb{Q}^x \in \Gamma(\mu_{01})$, we know by Theorem 6.69 that $\mathbb{Q}^x$ is in the reciprocal class of $\mathbb{P}_k^x$ and therefore of the form $\tilde{h}(X_1)\tilde{\mathbb{Q}}^x$ for some $\tilde{h} : \mathbb{R} \to \mathbb{R}_+$. By Proposition 6.81 we deduce that $\mathbb{Q}^x$ minimizes the cost function (6.78). □

Using the Feynman-Kac formula we present a dynamical interpretation of the invariant Ξ_k in terms of the cost-potentials A and Φ. By (6.88) we have

$$\partial_t \log h(t,y) = -(eA(t,y))^{-1} \frac{h(t,y+1)}{h(t,y)} - \Phi(t,y),$$

and we insert the definition

$$k(t,y) = (eA(t,y))^{-1} \frac{h(t,y+1)}{h(t,y)}$$

to prove

$$-\Xi_k(t,y) = \partial_t \log A(t,y) + \Phi(t,y+1) - \Phi(t,y). \tag{6.93}$$

Thus we may re-write the duality formula (6.92) as

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}^x (F(X)\delta(u)) \quad &= \quad \mathbb{E}_{\mathbb{Q}}^x (\mathcal{D}_u F(X)) \\ &\quad + \mathbb{E}_{\mathbb{Q}}^x \left(F(X) \int_{\mathcal{I}} u_t \int_{[t,1]} (\partial_t \log A(s, X_{s-}) + \Phi(s, X_{s-} + 1) - \Phi(s, X_{s-}))\, dX_s dt \right) \end{aligned}$$

Let us illustrate this interpretation of the reciprocal invariant in two simple examples.

Example 6.94. *The reciprocal class of the Poisson process is characterized by the fact, that the invariant $\Xi_k = 0$ is zero. By (6.93) this is e.g. the case for a constant potential $A \in [\varepsilon, \infty)$ and $\Phi \in C_b^1(\mathcal{I})$. Thus the reciprocal class of a Poisson process contains all minimizers of the cost function*

$$\mathcal{J}(\mathbb{P}_\gamma) = \mathbb{E}_\gamma \left(\int_{\mathcal{I}} (\gamma_t \log \gamma_t - \gamma_t \log A - \Phi(t)) dt \right),$$

under the boundary condition $\mu_{01} = \delta_{\{x\}} \otimes \mu_1$ given in (6.79).

Example 6.95. *The reciprocal class of the exponential decay process X with rate $\lambda > 0$ has reciprocal invariant $\Xi_k(t,y) = -\lambda$, see Examples 6.62 and 6.67. Using (6.93) this invariant is e.g. associated to the potentials $A(t,y) = ce^{\lambda t}$ with $c > 0$ and $\Phi \in C_b^1(\mathcal{I})$. Thus the reciprocal class of the exponential decay process with rate $\lambda > 0$ contains the minimizers of the cost function*

$$\mathcal{J}(\mathbb{P}_\gamma) = \mathbb{E}_\gamma \left(\int_{\mathcal{I}} (\gamma_t \log \gamma_t - (c + \lambda t)\gamma_t - \Phi(t)) dt \right),$$

under the boundary condition $\mu_{01} = \delta_{\{x\}} \otimes \mu_1$.

6.8. **Time reversal of unit jump processes.**

On the space $\mathbb{J}_1(\mathcal{I})$ we define the time reversal as follows.

Definition 6.96. *Let $\mathbb{Q}$ be any unit jump distribution on $\mathbb{J}_1(\mathcal{I})$. The time reversed unit jump process $\hat{\mathbb{Q}}$ is defined as the image of $\mathbb{Q}$ under the transformation*

$$R : \mathbb{J}_1(\mathcal{I}) \to \mathbb{J}_1(\mathcal{I}), \quad \omega = (x, \{t_1, \dots, t_m\}) \mapsto \hat{\omega} = (-x - m, \{1 - t_m, \dots, 1 - t_1\}).$$

In comparison to the time reversal of continuous diffusions (5.80) we have to change the sign of the canonical process to get a process that still jumps up in the sense that if $\hat{X} := X \circ R$ then $\hat{X}_t = -X_{(1-t)-}$. Elliott and Tsoi have investigated time reversed unit jump processes by computing the backward intensities of the form $\mathbb{Q}(X_{t-\varepsilon} - X_t | X_t)$ for small $\varepsilon > 0$. Our concept of time-reversal is equivalent, but the results are different than those of [ET90]. We compute the intensity of a time-reversed nice unit jump process and describe the behavior of the reciprocal classes of nice unit jump processes under time-reversal.

By definition we have $\hat{\mathbb{Q}}(F) = \mathbb{Q}(F \circ R)$ for any bounded functional $F : \mathbb{J}_1(\mathcal{I}) \to \mathbb{J}_1(\mathcal{I})$. If $\mathbb{Q}$ is stochastically continuous the endpoint distribution of the reversed process is

$$\hat{\mathbb{Q}}(X_0 \in dx, X_1 \in dy) = \mathbb{Q}(X_0 \in -dy, X_1 \in -dx).$$

If $i \leq \eta = X_1 - X_0$ we also have $T_i \circ R = 1 - T_{\eta+1-i}$.

In the unit jump process context, no derivation of the intensity of a reversed nice unit jump process similar to the result for Brownian diffusions in Lemma 5.82 seems to be known yet. We compute a convenient representation of the intensity in the following proposition.

Proposition 6.97. *Let $\mathbb{P}^x_\ell$ be the law of a nice unit jump process starting in $x \in \mathbb{R}$. Then $\mathbb{Q} := \mathbb{P}^x_\ell \circ R$ is Markov and has intensity $\hat{\ell}(t, X_{t-})dt$ with*

$$\hat{\ell}(t, y) = \ell(1 - t, -y - 1) \frac{p^x_\ell(1 - t, -y - 1)}{p^x_\ell(1 - t, -y)}, \tag{6.98}$$

with $p^x_\ell(t, y) = \mathbb{P}^x_\ell(X_t = y) > 0$ for $y = x + m$, $m \in \mathbb{N}$.

Proof. Let us remark that $\mathbb{P}^x_\ell$ is the law of a unit jump process with intensity $\ell(t, X_{t-})dt$ if and only if for every $f \in C^\infty_b(\mathcal{I} \times \mathbb{R}, \mathbb{R})$ the process

$$t \mapsto f(t, X_t) - \int_{[0,t]} \left[\partial_t f(s, X_{s-}) + \ell(s, X_{s-}) \left(f(s, X_{s-} + 1) - f(s, X_{s-})\right)\right] ds$$

is a martingale with respect to $\mathbb{P}^x_\ell$ (the necessity follows easily from Itô's formula, for the sufficiency see e.g. the monograph by Jacod [Jac79]). Take any $f \in C^\infty_b(\mathcal{I} \times \mathbb{R})$ and $g \in C^\infty_b(\mathbb{R})$, then

$$\mathbb{E}^x_\ell \left(f(s, X_s) g(X_t)\right) = \mathbb{E}^x_\ell \left(f(s, X_s) g(s, X_s)\right) = \sum_{n=0}^{\infty} f(s, x + n) g(s, x + n) p^x_\ell(s, x + n),$$

where $g(s, X_s) := \mathbb{E}^x_\ell \left(g(X_t) | X_s\right)$ is a space-time harmonic function and $p^x_\ell(s, x+n) = p_\ell(0, x; s, x+n)$ is the transition probability. We use a trick to develop this term that is also used in the diffusion case (see e.g. Haussmann, Pardoux [HP86]) to get

$$\begin{aligned} &= \sum_{n=0}^{\infty} f(t, x + n) g(t, x + n) p^x_\ell(t, x + n) - \int_{[s,t]} \frac{d}{dt} \left(\sum_{n=0}^{\infty} f(r, x + n) g(r, x + n) p^x_\ell(r, x + n) \right) dr \\ &= \mathbb{E}^x_\ell \left(f(t, X_t) g(X_t)\right) - \int_{[s,t]} \frac{d}{dt} \left(\sum_{n=0}^{\infty} f(r, x + n) g(r, x + n) p^x_\ell(r, x + n) \right) dr. \end{aligned}$$

We may apply the product rule to develop the last term

$$\int_{[s,t]} \Bigg(\sum_{m=0}^{\infty} \partial_t f(r,x+m) g(r,x+m) p_\ell^x(r,x+m) \\ + f(r,x+m) \partial_t g(r,x+m) p_\ell^x(r,x+m) + f(r,x+m) g(r,x+m) \partial_t p_\ell^x(r,x+m) \Big) dr.$$

Here we may use the fact that $g(t,y)$ is solution of the Kolmogoroff backward equation and $p_\ell^x(t,y)$ of the Kolmogoroff forward equation. The backward equation has been derived in Lemma 6.55, as for the forward equation we give a similar proof in the following Lemma.

Lemma 6.99. *The counting density $p_\ell^x(r,x+m) := \mathbb{P}_\ell^x(X_r = x+m)$ is a solution of the Kolmogoroff forward equation*

$$\partial_t p_\ell^x(r,x+m) = \ell(r,x+m-1) p_\ell^x(r,x+m-1) - \ell(r,x+m) p_\ell^x(r,x+m). \tag{6.100}$$

Proof. Let $y := x+m$, by definition

$$\begin{aligned} p_\ell^x(r,y) &= \mathbb{P}_\ell^x(T_m \le r, T_{m+1} > r) \\ &= \int_{[0,r]^m} \Big[\ell(t_1,x) e^{-\int_{[0,t_1]} \ell(s,x) ds} \cdots \ell(t_m, x+m-1) e^{-\int_{[t_{m-1},t_m]} \ell(s,x+m-1) ds} \\ &\qquad e^{-\int_{[t_m,r]} \ell(s,x+m) ds} \Big] \mathbb{1}_{\{0 \le t_1 < \cdots < t_m \le r\}} dt_1 \cdots dt_m \\ &= \int_{[0,r]} \int_{[0,t_m]} \cdots \int_{[0,t_2]} f_x(t_1,\ldots,t_m,r) dt_1 \cdots dt_m, \end{aligned}$$

where $f_x(t_1,\ldots,t_m,r)$ is the defined by the term in the square brackets in the second line. Clearly $\partial_r f_x(t_1,\ldots,t_m,r) = -\ell(r,x+m) f_x(t_1,\ldots,t_m,r)$, which we can use in the expansion

$$\begin{aligned} &\frac{1}{\varepsilon}\Big(p_\ell^x(r+\varepsilon,y) - p_\ell^x(r,y) \Big) \\ = \; &\frac{1}{\varepsilon}\Bigg(\int_{[0,r+\varepsilon]} \int_{[0,t_m]} \cdots \int_{[0,t_2]} f_x(t_1,\ldots,t_m,r) dt_1 \cdots dt_m \\ &\qquad - \int_{[0,r]} \int_{[0,t_m]} \cdots \int_{[0,t_2]} f_x(t_1,\ldots,t_m,r) dt_1 \cdots dt_m \Bigg) \\ = \; &\frac{1}{\varepsilon}\Bigg(\int_{[0,r]} \int_{[0,t_m]} \cdots \int_{[0,t_2]} \left(f_x(t_1,\ldots,t_m,r+\varepsilon) - f_x(t_1,\ldots,t_m,r) \right) dt_1 \cdots dt_m \\ &\qquad + \int_{[r,r+\varepsilon]} \int_{[0,t_m]} \cdots \int_{[0,t_2]} f_x(t_1,\ldots,t_m,r+\varepsilon) dt_1 \cdots dt_m \Bigg), \end{aligned}$$

and for $\varepsilon \to 0$ this converges to

$$\begin{aligned} \frac{1}{\varepsilon}\Big(p_\ell^x(r+\varepsilon,y) - p_\ell^x(r,y) \Big) &\to -\ell(r,y) p_\ell^x(r,y) + \int_{[0,r]} \int_{[0,t_{m-1}]} \cdots \int_{[0,t_2]} f_x(t_1,\ldots,t_{m-1},r,r) dt_1 \cdots dt_m \\ &= -\ell(r,y) p_\ell^x(r,y) + \ell(r,y-1) p_\ell^x(r,y-1), \end{aligned}$$

where the exchange of integration and differentiation is justified by the boundedness of $f_x(t_1,\cdots,t_m,r)$ and its partial derivative in r. □

Let us now resume the proof of Proposition 6.97. We insert the Kolmogoroff forward equation (6.100) and the Kolmogoroff backward equation (6.56) to substitute the time

derivatives of $g(r,y)$ and $p^x_\ell(r,y)$

$$\begin{aligned}
&\mathbb{E}^x_\ell\left(\left(f(s,X_s)-f(t,X_t)\right)g(X_t)\right)\\
=\;&-\int_{[s,t]}\sum_{m=0}^{\infty}\Big(\partial_t f(r,x+m)g(r,x+m)p^x_\ell(r,x+m)\\
&\quad -f(r,x+m)\ell(r,x+m)(g(r,x+m+1)-g(r,x+m))p^x_\ell(r,x+m)\\
&\qquad +f(r,x+m)g(r,x+m)(\ell(r,x+m-1)p^x_\ell(r,x+m-1)-\ell(r,x+m)p^x_\ell(r,x+m))\Big)dr\\
=\;&-\int_{[s,t]}\sum_{m=0}^{\infty}\left(\partial_t f(r,x+m)+\left[f(r,x+m)-f(r,x+m-1)\right]\right.\\
&\quad \left.\ell(r,x+m-1)\frac{p^x_\ell(r,x+m-1)}{p^x_\ell(r,x+m)}\right)g(r,x+m)p^x_\ell(r,x+m)\\
=\;&\mathbb{E}^x_\ell\left(-g(X_t)\int_{[s,t]}\left(\partial_t f(r,X_{r-})+\left[f(r,X_{r-})-f(r,X_{r-}-1)\right]\ell(r,X_{r-}-1)\frac{p^x_\ell(r,X_{r-}-1)}{p^x_\ell(r,X_{r-})}\right)dr\right).
\end{aligned}$$

We can apply this to the time reversed nice unit jump process $\mathbb{Q}=\mathbb{P}^x_\ell\circ R$ as follows

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\left(\left(f(t,X_t)-f(s,X_s)\right)g(X_s)\right)\\
=\;&\mathbb{E}^x_\ell\left(\left(f(1-(1-t),-X_{1-t})-f(1-(1-s),-X_{1-s})\right)g(-X_{1-s})\right)\\
=\;&-\mathbb{E}^x_\ell\left(\int_{[1-t,1-s]}\left(\partial_r f(1-r,-X_{r-})+\left[f(1-r,-X_{r-})-f(1-r,-(X_{r-}-1)\right]\right.\right.\\
&\quad \left.\left.\ell(r,X_{r-}-1)\frac{p^x_\ell(r,X_{r-}-1)}{p^x_\ell(r,X_{r-})}\right)dr\right)\\
=\;&-\mathbb{E}_{\mathbb{Q}}\left(\int_{[1-t,1-s]}\left(-\partial_t f(1-r,\hat{X}_{(1-r)-})+\left[f(1-r,\hat{X}_{(1-r)-})-f(1-r,\hat{X}_{(1-r)-}+1)\right]\right.\right.\\
&\quad \left.\left.\ell(r,-\hat{X}_{(1-r)-}-1)\frac{p^x_\ell(r,-\hat{X}_{(1-r)-}-1)}{p^x_\ell(r,-\hat{X}_{(1-r)-})}\right)dr\right)\\
=\;&\mathbb{E}_{\mathbb{Q}}\left(\int_{[s,t]}\left(\partial_t f(r,\hat{X}_{r-})-\hat{\ell}(r,\hat{X}_{r-})\left(f(r,\hat{X}_{r-}+1)-f(r,\hat{X}_{r-})\right)\right)dr\right),
\end{aligned}$$

where $\hat{\ell}$ is defined as in the assertion of the proposition. Since the equation holds for all $g\in C^\infty_b(\mathbb{R})$ and we already know that $\mathbb{Q}$ has the Markov property by Lemma 5.81, the process

$$f(t,X_t)-\int_{[0,t]}\left(\partial_t f(r,\hat{X}_{r-})-\hat{\ell}(r,\hat{X}_{r-})\left(f(r,\hat{X}_{r-}+1)-f(r,\hat{X}_{r-})\right)\right)dr$$

is a martingale with respect to $\mathbb{Q}$ for all $f\in C^{1,2}_b(\mathcal{I}\times\mathbb{R})$. □

In particular the invariant $\Xi_{\hat{\ell}}(t,y)$ is well defined. To be able to apply the characterization of the reciprocal class of Theorem 6.69 we have to assume that there exists an initial condition $\mathbb{P}_{\ell,0}$ of the reference measure such that

$$\hat{\mathbb{P}}_\ell \textit{ is a nice unit jump process}.$$

This implies that $\Xi_{\hat{\ell}}$ is the invariant of the reciprocal class of a nice unit jump process. The next result shows how the reciprocal class of a nice unit jump process is transformed by time reversal.

Proposition 6.101. *Let $\mathbb{Q}$ be in the reciprocal class of $\mathbb{P}_\ell$. Then $\hat{\mathbb{Q}}$ is in the reciprocal class of $\hat{\mathbb{P}}_\ell$, and the reciprocal invariant is given by $\Xi_{\hat{\ell}}(s,y) = -\Xi_\ell(1-s,-y-1)$.*

Proof. The idea of this proof is, too identify the reciprocal class of $\hat{\mathbb{P}}_\ell$ by a duality formula using Theorem 6.69. Take any $F = f_{\wedge 1}(X_0, T_1, \dots, T_m) \in \mathcal{S}_{\mathbb{J}_1}$ and $u = \sum_{i=1}^{n-1} u_i \mathbb{1}_{(s_i, s_{i+1}]} \in \mathcal{E}$ with $\langle u \rangle = 0$. Then

$$\mathbb{E}_{\hat{\mathbb{Q}}}(F(X)\delta(u)) = \mathbb{E}_{\mathbb{Q}}\left(f_{\wedge 1}(\hat{X}_0, \hat{T}_1, \dots, \hat{T}_n)\int_{\mathcal{I}} u_s d\hat{X}_s\right) = \mathbb{E}_{\mathbb{Q}}\left(\hat{f}_{\wedge 1}(X_1, T_1, \dots, T_n)\int_{\mathcal{I}} \hat{u}_s dX_s\right),$$

with $\hat{F}(X) := \hat{f}_{\wedge 1}(X_1, T_1, \dots, T_m) = f(-X_1, 1-T_\eta, \dots, 1-T_{(\eta-m+1)\vee 1}, 1, \dots, 1)$ and $\hat{u} = \sum_{i=1}^{n-1} u_i \mathbb{1}_{(1-t_{i+1}, 1-t_i]}$ with $\langle \hat{u} \rangle = 0$. Clearly $\langle \hat{u} \rangle_{1-t} = \int_{[t,1]} u_s ds = -\langle u \rangle_t$. Since $\mathbb{Q} \in \mathcal{R}(\mathbb{P}_\ell)$ the duality formula (6.70) holds, thus

$$\begin{aligned}\mathbb{E}_{\hat{\mathbb{Q}}}(F(X)\delta(u)) &= \mathbb{E}_{\mathbb{Q}}(\hat{F}(X)\delta(\hat{u})) \\ &= \mathbb{E}_{\mathbb{Q}}(\mathcal{D}_{\hat{u}}\hat{F}(X)) - \mathbb{E}_{\mathbb{Q}}\left(\hat{F}(X)\int_{\mathcal{I}} \hat{u}_t \int_{[t,1]} \Xi_\ell(s, X_{s-}) dX_s dt\right).\end{aligned}$$

The first term in the last line is

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\left(\mathcal{D}_{\hat{u}}\hat{F}(X)\right) &= \mathbb{E}_{\mathbb{Q}}\left(\sum_{i=1}^{m\wedge\eta} \partial_{i+1}\hat{f}_{\wedge 1}(X_1, T_1, \dots, T_m)\langle \hat{u} \rangle_{T_i}\right) \\ &= \mathbb{E}_{\mathbb{Q}}\left(-\sum_{i=0}^{\eta\wedge m-1} \partial_{\eta\wedge m-i+1} f(-X_1, 1-T_\eta, \dots, 1-T_{(\eta-m+1)\vee 1}, 1, \dots, 1)\langle \hat{u} \rangle_{T_{(\eta-m+1)\vee 1-i}}\right) \\ &= \mathbb{E}_{\mathbb{Q}}\left(\sum_{i=1}^{\eta\wedge m} \partial_{i+1} f_{\wedge 1}(\hat{X}_0, \hat{T}_1, \dots, \hat{T}_m)\langle u \rangle_{\hat{T}_i}\right) \\ &= \mathbb{E}_{\hat{\mathbb{Q}}}\left(\mathcal{D}_u F(X)\right),\end{aligned}$$

which is best seen under the distinction of the cases $\eta < m$, $\eta = m$ and $\eta > m$. The second term in the duality formula can be rewritten into

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\left(\hat{F}(X)\int_{\mathcal{I}} \hat{u}_t \int_{[t,1]} \Xi_\ell(s, X_{s-}) dX_s dt\right) &= \mathbb{E}_{\mathbb{Q}}\left(\hat{F}(X)\int_{\mathcal{I}} \Xi_\ell(s, X_{s-})\langle \hat{u} \rangle_s dX_s\right) \\ &= \mathbb{E}_{\mathbb{Q}}\left(\hat{F}(X)\sum_{i=1}^{\eta} \Xi_\ell(T_i, x+i-1)\langle \hat{u} \rangle_{T_i}\right) \\ &= \mathbb{E}_{\mathbb{Q}}\left(\hat{F}(X)\sum_{i=1}^{\eta} \Xi_\ell(1-\hat{T}_{\eta-i+1}, x+i-1)\langle \hat{u} \rangle_{1-\hat{T}_{\eta-i+1}}\right) \\ &= -\mathbb{E}_{\mathbb{Q}}\left(\hat{F}(X)\sum_{i=1}^{\eta} \Xi_\ell(1-\hat{T}_{\eta-i+1}, -\hat{X}_{\hat{T}_{\eta-i+1}})\langle u \rangle_{\hat{T}_{\eta-i+1}}\right) \\ &= \mathbb{E}_{\mathbb{Q}}\left(\hat{F}(X)\sum_{i=1}^{\eta} (-\Xi_\ell(1-\hat{T}_i, -\hat{X}_{\hat{T}_i-}-1))\langle u \rangle_{\hat{T}_i}\right),\end{aligned}$$

and we may change the order of integration to get

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\left(\hat{F}(X)\int_{\mathcal{I}} \hat{u}_t \int_{[t,1]} \Xi_\ell(s, X_{s-}) dX_s dt\right) &= \mathbb{E}_{\mathbb{Q}}\left(\hat{F}(X)\int_{\mathcal{I}} u_t \int_{[t,1]} (-\Xi_\ell(1-s, -\hat{X}_{s-}-1)) d\hat{X}_s dt\right) \\ &= \mathbb{E}_{\hat{\mathbb{Q}}}\left(F(X)\int_{\mathcal{I}} u_t \int_{[t,1]} (-\Xi_\ell(1-s, -X_{s-}-1)) dX_s dt\right).\end{aligned}$$

Thus we have shown that

$$\mathbb{E}_{\hat{\mathbb{Q}}}\left(F(X)\delta(u)\right) = \mathbb{E}_{\hat{\mathbb{Q}}}\left(\mathcal{D}_u F(X)\right) - \mathbb{E}_{\hat{\mathbb{Q}}}\left(F(X)\int_{\mathcal{I}} u_t \int_{[t,1]} (-\Xi_\ell(1-s,-X_{s-}-1))dX_s dt\right),$$

and can apply the characterization of the reciprocal class in Theorem 6.69. The identity $\Xi_{\hat{\ell}}(t,y) = -\Xi_\ell(1-t,-y-1)$ is immediate from (6.98) and the Kolmogoroff forward equation satisfied by $p^x_\ell(t,y)$. □

As was the case for Brownian diffusions in § 5.6 the behavior of the reciprocal class of a nice jump process under time-reversal is symmetric.

Remark 6.102. *Let $A : \mathcal{I}\times\mathbb{R} \to [\varepsilon,\infty)$ and $\Phi : \mathcal{I}\times\mathbb{R}\to\mathbb{R}$ be the bounded potentials of the optimal control problem of Definition 6.80. Assume that $\mathbb{P}_\ell \in \Gamma(\mu_{01})$ is the law of a nice unit jump process such that the relation (6.93) holds. Then the reciprocal class of $\mathbb{P}_\ell$ contains all solutions of the optimal control problem associated to A and Φ. By Proposition 6.101 we have*

$$\Xi_{\hat{\ell}}(t,y) = -\Xi_\ell(1-t,-y-1) = \partial_t \log A(1-t,-y-1) + \Phi(1-t,-y) - \Phi(1-t,-y-1).$$

Thus the time reversed reciprocal class contains all the solutions of the optimal control problem associated to the potentials $\hat{A}(t,y) = A(1-t,-y-1)$ and $\hat{\Phi}(t,y) = \Phi(1-t,-y)$.

7. The reciprocal classes of pure jump processes

In this last section we study the reciprocal classes and bridges of Markov processes with varying jump-sizes. The only result on this subject known to the author is a work by Privault and Zambrini. In [PZ04] they derive the semimartingale decomposition of certain Markov processes in the reciprocal class of a Lévy process with jumps. Our results are different, since we are interested in the characterization of the whole reciprocal class, and in particular of the bridges with fixed deterministic boundary conditions. We study pure jump processes that only use a finite number of different jump-sizes. This setting allows us to discuss some rather geometrical aspects concerning the distribution of the bridges of such jump processes.

In the first paragraph we define the space of pure jump processes and present a natural pure jump distribution, the law of a compound Poisson process. In §7.2 we introduce the notion of incommensurable jumps, see Definition 7.23. Our main result in the second paragraph is Theorem 7.36. We are able to characterize the reciprocal class of a compound Poisson process with incommensurable jumps by a duality formula. In §7.3 we discuss possible extensions of this characterization to the reciprocal classes of Markov jump processes with regular jump-intensities, so called nice jump processes. Without assuming incommensurability of jump-sizes, we are able to compare the reciprocal classes of nice jump processes by reciprocal invariants. In addition to an "harmonic" invariant, a "gradient of a potential" condition on the intensity of the nice jump process appears that is similar to the "rotational" invariant on Brownian diffusions in Theorem 5.26.

7.1. **Pure jump processes.**

In this paragraph we define the canonical space of continuous time pure jump processes. This space is large enough to admit the distribution of compound Poisson processes. In §7.1.3 we prepare the discussion of the bridges of compound Poisson processes with some examples.

7.1.1. *Basic definitions.*

Let $Q \in \mathcal{B}(\mathbb{R}^d_*)$ be a finite set of jump-sizes. All finite and time-ordered subsets of $\mathcal{I} \times Q$ are collected in

$$\Delta_{\mathcal{I}\times Q} := \{\{(t_1, q_1), \dots, (t_m, q_m)\} \in (\mathcal{I} \times Q)^m,\ 0 \le t_1 < t_2 < \cdots < t_m \le 1,\ m \in \mathbb{N}\}.$$

Remark that the elements of Q are not numbered, a set $\{(t_1, q_1), \dots, (t_m, q_m)\}$ may contain several identical jump-sizes. The space of jump processes with jumps in Q is

$$\mathbb{J}(\mathcal{I}, Q) := \left\{\omega = x + \sum_{i=1}^{m} q_i \mathbb{1}_{[t_i,1]} :\ x \in \mathbb{R}^d,\ \{(t_1, q_1), \dots, (t_m, q_m)\} \in \Delta_{\mathcal{I}\times Q},\ m \in \mathbb{N}\right\}. \tag{7.1}$$

This setting includes as a special case $\mathbb{J}_1(\mathcal{I}) = \mathbb{J}(\mathcal{I}, \{1\})$. Since $\mathbb{J}(\mathcal{I}, Q) \subset \mathbb{D}(\mathcal{I}, \mathbb{R}^d_*)$ we use the canonical setup induced from the space of càdlàg functions:

- The canonical jump process $X : \mathbb{J}(\mathcal{I}, Q) \to \mathbb{J}(\mathcal{I}, Q)$ is the identity;
- $\mathcal{F}_\tau := \sigma(X_s,\ s \in \tau)$ for every subset $\tau \subset \mathcal{I}$.

The spaces $\mathbb{J}(\mathcal{I}, Q)$ and $\mathbb{R}^d \times \Delta_{\mathcal{I}\times Q}$ are isomorphic through the identification

$$\mathbb{J}(\mathcal{I}, Q) \ni \omega = x + \sum_{i=1}^{m} q_i \mathbb{1}_{[t_i,1]} \leftrightarrow (x, \{(t_1, q_1), \dots, (t_m, q_m)\}) \in \mathbb{R}^d \times \Delta_{\mathcal{I}\times Q}.$$

We identify $\omega \in \mathbb{J}(\mathcal{I}, Q)$ with the tuple containing the initial condition, the jumps and the jump-times. The integer random variable that counts the total number of jumps is denoted by $\eta(\omega) = \eta((x, \{(t_1, q_1), \dots, (t_m, q_m)\})) := m$. Let $T_1, T_2, \dots$ be the consecutive jump-times of X defined by $T_0 := 0$, $T_i(\omega) = T_i((x, \{(t_1, q_1), \dots, (t_m, q_m)\})) = t_i$ if $1 \le i \le m$ and $T_i(\omega) := \infty$ if $i > m$, where ∞ may be interpreted as abstract cemetery time. The consecutive jump-sizes are defined by $V_i(\omega) := V_i((x, \{(t_1, q_1), \dots, (t_m, q_m)\})) = q_i$ if $1 \le i \le m$ and $V_i(\omega) := 0$ if $i > m$. We define an integer valued random measure on $\mathcal{I} \times Q$ by

$$N^X := \sum_{i=1}^{\eta} \delta_{(T_i, V_i)}.$$

Clearly $N^X(\mathcal{I} \times Q) = \eta$. We define the number of jumps with jump-size in $B \subset Q$ up to time $t \in \mathcal{I}$ by $\eta_t^B := N^X([0, t] \times B)$. Then $(\eta_t^B)_{t \in \mathcal{I}}$ is a unit jump process with initial state $\eta_0^B = 0$.

There are several equivalent ways to define the canonical filtration.

Lemma 7.2. *The canonical filtration has the following representations*

$$\begin{aligned} \mathcal{F}_{[0,t]} &= \sigma(X_0, V_1, \dots, V_{\eta_t}, T_1 \wedge t, T_2 \wedge t, \dots) \\ &= \sigma(X_0, V_1, \dots, V_{\eta_t}, \eta_s \text{ with } 0 \le s \le t) \\ &= \sigma(X_0, \eta_s^{\{q\}} \text{ with } s \le t,\ q \in Q). \end{aligned}$$

where $t \in \mathcal{I}$ and $t \wedge \infty = t$.

Proof. Similar to the proof of Lemma 6.2 this follows from the identities

$$X_t = X_0 + \sum_{i \ge 1} V_i \mathbb{1}_{[0,t]}(T_i) = X_0 + \sum_{i=1}^{\eta_t} V_i = X_0 + \sum_{q \in Q} q \eta_t^{\{q\}},\ \forall t \in \mathcal{I}.$$

Let us also remark that $\sigma(\eta_s,\ s \le t) = \sigma(T_1 \wedge t, T_2 \wedge t, \dots)$ since $\eta_t = \sum_{i \ge 1} \mathbb{1}_{[0,t]}(T_i)$. □

In the Appendix we introduce the distributions of pure jump processes on $\mathbb{D}(\mathcal{I}, \mathbb{R}^d)$ and a stochastic calculus associated to this class of semimartingales. Let us briefly state the connection of pure jump distributions on $\mathbb{D}(\mathcal{I}, \mathbb{R}^d)$ and the space of pure jump paths $\mathbb{J}(\mathcal{I}, Q)$.

Remark 7.3. *One can easily verify that as a subset of $\mathbb{D}(\mathcal{I}, \mathbb{R}^d)$*

$$\mathbb{J}(\mathcal{I}, Q) = \left\{ \omega \in \mathbb{D}(\mathcal{I}, \mathbb{R}^d) :\ N^X(\mathcal{I} \times \mathbb{R}_*^d) = N^X(\mathcal{I} \times Q) < \infty,\ X_\cdot - X_0 = \int_{[0,\cdot] \times \mathbb{R}_*^d} q N^X(dsdq) \right\},$$

since these conditions hold for $\omega \in \mathbb{D}(\mathcal{I}, \mathbb{R}_*^d)$ if and and only if

$$\omega = x + \sum_{i=1}^{m} q_i \mathbb{1}_{[t_i, 1]},\ \text{for some } \{(t_1, q_1), \dots, (t_m, q_m)\} \in \Delta_{\mathcal{I} \times Q},\ m \in \mathbb{N}.$$

But if X is a pure jump process under the probability $\mathbb{P}$ on $\mathbb{D}(\mathcal{I}, \mathbb{R}^d)$, as introduced in Definition A.1 in the Appendix, we have

$$\mathbb{P}\left(\exists t \in \mathcal{I} :\ X_t - X_0 \ne \int_{[0,t] \times \mathbb{R}_*^d} q N^X(dsdq) \right) = 0,\ \text{and } \mathbb{P}(N^X(\mathcal{I} \times \mathbb{R}_*^d) < \infty) = 1.$$

Thus if moreover $N^X(\mathcal{I} \times \mathbb{R}_*^d) = N^X(\mathcal{I} \times Q)$ holds $\mathbb{P}$-a.s., then there exists a probability $\tilde{\mathbb{P}}$ on $\mathbb{J}(\mathcal{I}, Q)$ such that $\mathbb{P}(D') = \tilde{\mathbb{P}}(D' \cap \mathbb{J}(\mathcal{I}, Q))$ for any measurable subset $D' \subset \mathbb{D}(\mathcal{I}, \mathbb{R}^d)$. In particular for $Q = \{1\}$ we may call X a unit jump process under $\mathbb{P}$.

We can define one-time probability densities and transition densities for any probability law on $\mathbb{J}(\mathcal{I}, Q)$. As a reference measure we use

$$\Lambda := \sum_{q \in Q} \delta_{\{q\}}, \tag{7.4}$$

which will play the role of a reference measure for the distribution of the jumps, similar to the role the Lebesgue measure dt on $\mathcal{I}$ for the distribution of the jump-times. Clearly $\Lambda(Q) = |Q| \in \mathbb{N}$, since Q was assumed to contain only a finite number of arbitrary jump-sizes.

Remark 7.5. *Any probability $\mathbb{Q}$ on $\mathbb{J}(\mathcal{I}, Q)$ can be decomposed into*

$$\mathbb{Q}(\,.\,) = \int_{\mathbb{R}^d} \mathbb{Q}^x(\,.\,)\mathbb{Q}_0(dx).$$

Clearly $\mathbb{Q}^x(X_t \in \,.\,) \ll \delta_{\{x\}} * \left(\sum_{m=0}^{\infty} (\Lambda(Q))^{-m} \Lambda^{*m}\right)$, where ^{*m} denotes the m-times convolution of a measure with itself. The topological support of this reference measure is

$$Q_x := \{x + q_1 + \cdots + q_m \in \mathbb{R}^d : \, q_1, \ldots, q_m \in Q, \, m \in \mathbb{N}\}, \tag{7.6}$$

which is a countable subset of $\mathbb{R}^d$. With the above decomposition we see that $\mathbb{Q}(X_t \in \,.\,) \ll \mathbb{Q}_0 * \left(\sum_{m=0}^{\infty} (\Lambda(Q))^{-m} \Lambda^{*m}\right)$ for any $t \in \mathcal{I}$. We define the density of the one-time projection by

$$q(t, y)\left(\mathbb{Q}_0 * \left(\sum_{m=0}^{\infty} (\Lambda(Q))^{-m} \Lambda^{*m}\right)\right)(dy) := \mathbb{Q}(X_t \in dy), \; y \in Q_x. \tag{7.7}$$

The transition probabilities are defined in a similar fashion by

$$q(s, x; t, y)\delta_{\{x\}} * \left(\sum_{m=0}^{\infty} (\Lambda(Q))^{-m} \Lambda^{*m}\right)(dy) := \mathbb{Q}(X_t = y | X_s = x) \text{ for } s < t.$$

Since Q_x is at most countable, the bridge $\mathbb{Q}^{x,y}$ is well defined if and only if $y \in \mathbb{R}^d$ with $\mathbb{Q}^x(X_1 = y) > 0$. Therefore each bridge may be written in the form of an h-transform

$$\mathbb{Q}^{x,y}(\,.\,) := \frac{\mathbb{1}_{\{y\}}(X_1)}{\mathbb{Q}^x(X_1 = y)} \mathbb{Q}^x(\,.\,). \tag{7.8}$$

The difficult task when computing the distribution of the bridge is of course the computation of the normalizing factor $\mathbb{Q}^x(X_1 = y)$. An approach to facilitate these computations is the following decomposition of bridges.

Remark 7.9. *For $m \in \mathbb{N}$ define*

$$\Gamma_m(x, y) := \left\{\gamma_m = (q_1, \ldots, q_m)^t \in Q^m : \, x + q_1 + \cdots + q_m = y\right\}. \tag{7.10}$$

This set contains the possible combinations of jump-sizes that add up to $y \in \mathbb{R}^d$ starting from $x \in \mathbb{R}^d$ with $m \in \mathbb{N}$ jumps. We necessarily have $\mathbb{Q}^x(X_1 = y) = 0$ for every probability $\mathbb{Q}$ on $\mathbb{J}(\mathcal{I}, Q)$ if $\Gamma_m(x, y)$ is empty for every $m \in \mathbb{N}$. For $\gamma_m, \gamma'_m \in \Gamma_m(x, y)$ we write $\gamma_m \simeq \gamma'_m$ if γ_m and γ'_m are identical up to a permutation of their coordinates. Define $\Gamma_m^{\simeq}(x, y) := \Gamma_m(x, y)/\simeq$. Then

$$\begin{aligned}
\mathbb{Q}^{x,y}(\,.\,) &= \sum_{m=0}^{\infty} \sum_{\gamma_m \in \Gamma_m^{\simeq}(x,y)} \mathbb{Q}^{x,y}(\,.\, \cap \{(V_1, \ldots, V_m)^t \simeq \gamma_m\} \cap \{\eta = m\}) \\
&= \sum_{m=0}^{\infty} \sum_{\gamma_m \in \Gamma_m^{\simeq}(x,y)} \mathbb{Q}^x(\,.\,|\,(V_1, \ldots, V_m)^t \simeq \gamma_m, \eta = m)\mathbb{Q}^{x,y}((V_1, \ldots, V_m)^t \simeq \gamma_m, \eta = m).
\end{aligned}$$

Denote $\#_q\gamma_m := |\{i : q = q_i, \text{ where } \gamma_m = (q_1, \dots, q_m)^t\}|$ *the number of repetitions of jump-size* q *in the coordinates of* γ_m. *Since* $X_t = X_0 + \sum_{q\in Q} q\eta_t^{\{q\}}$ *we may rewrite the above conditional probability as*

$$\mathbb{Q}^x(\,.\,|\,(V_1,\dots,V_m)^t \simeq \gamma_m, \eta = m) = \mathbb{Q}^x(\,.\,|\,\eta_1^{\{q\}} = \#_q\gamma_m,\ \forall q \in Q). \tag{7.11}$$

By Lemma 7.2 the joint law of the unit jump processes $(\eta^{\{q\}},\ q \in Q)$ *determines the law of* X *up to the deterministic initial state* $X_0 = x$. *Thus for* $\gamma_m \in \Gamma_m^{\simeq}(x,y)$ *the law of the bridge* $\mathbb{Q}^{x,y}$ *is determined by the law of the bridges of unit jump processes*

$$\mathbb{Q}^x(\eta^{\{q\}} \in\ .\,|\,(V_1,\dots,V_m)^t \simeq \gamma_m, \eta = m) = \mathbb{Q}^x(\eta^{\{q\}} \in\ .\,|\,\eta_1^{\{\tilde q\}} = \#_{\tilde q}\gamma_m, \eta_0^{\{\tilde q\}} = 0,\ \forall \tilde q \in Q).$$

If the processes $(\eta^{\{q\}},\ q \in Q)$ *are independent with respect to* $\mathbb{Q}$ *we have*

$$\mathbb{Q}^x(\eta^{\{q\}} \in\ .\,|\,\eta_1^{\{\tilde q\}} = \#_{\tilde q}\gamma_m, \eta_0^{\{\tilde q\}} = 0,\ \forall \tilde q \in Q) = \mathbb{Q}^x(\eta^{\{q\}} \in\ .\,|\,\eta_1^{\{q\}} = \#_q\gamma_m, \eta_0^{\{q\}} = 0).$$

This is exactly the bridge of the unit jump process $\eta^{\{q\}}$ *from* $\eta_0^{\{q\}} = 0$ *to* $\eta_1^{\{q\}} = \#_q\gamma_m$ *under* $\mathbb{Q}^x$.

In the next paragraph we introduce an important class of processes for which the unit jump processes $(\eta^{\{q\}},\ q \in Q)$ are independent.

7.1.2. *Compound Poisson processes.*

We already mentioned compound Poisson processes in Sections 2 and 3. The space of jump processes $\mathbb{J}(\mathcal{I}, Q)$ is large enough to admit compound Poisson processes with jumps in $Q \subset \mathbb{R}^d_*$. We present one of the most common definitions.

Let us first remark that every measure L on Q is finite and $L \ll \Lambda$. In particular there exists a function $\ell : Q \to \mathbb{R}_+$ such that $L(\{q\}) = \ell(q)\Lambda(\{q\}) = \ell(q)$.

Definition 7.12. *Let* $\ell : Q \to (0,\infty)$ *by any function and define the measure* $L = \ell\Lambda$. *Let* $\mathbb{P}_\ell$ *be a probability on* $\mathbb{J}(\mathcal{I}, Q)$. *Then* X *is called a compound Poisson process with intensity* ℓ *under* $\mathbb{P}_\ell$ *if* $(\eta_t^Q)_{t\in\mathcal{I}}$ *is a Poisson process with intensity* $L(Q)$ *and is independent of the sequence* $V_1, V_2, \dots$ *of iid random vectors with* $V_i \sim (L(Q))^{-1}L(\,.\,)$.

With the assumption $\ell > 0$ we avoid a degeneracy that just reduces the number of jump-sizes Q used by the compound Poisson process $\mathbb{P}_\ell$. We will hide the subscript $\mathbb{P} = \mathbb{P}_\ell$ if $\ell \equiv 1$, that is, X is a compound Poisson process with intensity Λ under $\mathbb{P}$.

Let us state some other equivalent definitions of a compound Poisson process.

Remark 7.13. *Let* $\mathbb{P}_\ell$ *be a probability on* $\mathbb{J}(\mathcal{I}, Q)$, $\ell : Q \to (0,\infty)$ *and define the measure* $L(dq) = \ell(q)\Lambda(dq)$. *Then* X *is a compound Poisson process with intensity* ℓ *under* $\mathbb{P}_\ell$ *if and only if*

- X *is a Lévy process with characteristics* $(\int_Q \chi(q)L(dq), 0, L)_\chi$, *see Definition 2.22.*
- N^X *is a Poisson measure on* $\mathcal{I} \times Q$ *with intensity* $ds L(dq)$, *see Definition 3.19.*
- *the processes* $(\eta^{\{q\}},\ q \in Q)$ *are independent Poisson processes with respective intensities* $\ell(q)$.

Using Definition 7.12 and Remark 1.19 we provide an explicit form of the distribution of a compound Poisson process, in particular

$$\begin{aligned} &\mathbb{P}_\ell(X_0 \in dx, T_1 \in dt_1, \dots, T_m \in dt_m, V_1 = q_1, \dots, V_m = q_m, \eta = m) \\ &= \mathbb{P}_{\ell,0}(dx)\ell(q_1)\cdots\ell(q_m)e^{-L(Q)}\mathbb{1}_{0\le t_1<\cdots<t_m\le 1}dt_1\cdots dt_m. \end{aligned} \tag{7.14}$$

The distributions of the bridges of a compound Poisson process are more difficult to compute, we present some examples in § 7.1.3.

A result similar to Watanabe's characterization of a Poisson process presented in Lemma 6.8 holds. Here we characterize the compound Poisson law by the compensator of the canonical random measure.

Lemma 7.15. *Let $\ell : Q \to (0,\infty)$ and $L := \ell\Lambda$. The probability $\mathbb{P}_\ell$ on $\mathbb{J}(\mathcal{I},Q)$ is the law of a compound Poisson process with intensity ℓ if and only if $t \mapsto \int_{[0,t]\times Q} \bar{u}(s,q)(N^X(dsdq) - dsL(dq))$ is a martingale for any $\bar{u} = \sum_{i=1}^{n-1} \bar{u}_i \mathbb{1}_{(t_i,t_{i+1}]\times B_i} \in \bar{\mathcal{E}}$.*

Proof. Let X be a compound Poisson process with intensity ℓ under $\mathbb{P}_\ell$. Then the random variable $\int_{[s,t]\times Q} \bar{u}(r,q)N^X(drdq) = \sum_{q\in Q} \int_{[s,t]} \bar{u}(r,q)d\eta_r^{\{q\}}$ is independent of $\mathcal{F}_{[0,s]}$ for any $\bar{u} \in \bar{\mathcal{E}}$. Since $\eta^{\{q\}}$ is a Poisson process with intensity $\ell(q)$ we can derive the expectation

$$\mathbb{E}_\ell\left(\int_{[s,t]\times Q} \bar{u}(r,q)N^X(drdq) \,\middle|\, \mathcal{F}_{[0,s]}\right) = \int_{[s,t]\times Q} \bar{u}(r,q)drL(dq),$$

hence the martingale property.

For the converse, the martingale property of the random measure $(N^X(dtdq) - dtL(dq))$ implies that $(\eta_t^{\{q\}} - t\ell(q))_{t\in\mathcal{I}}$ is a martingale, and by Watanabe's characterization Lemma 6.8 it is a Poisson process with intensity $\ell(q)$. Using $\bar{u} = u\sum_{i=1}^n \mathbb{1}_{\{q_i\}}$ for $q_i \neq q_j$ pairwise and $q_1,\dots,q_n \in Q$, $u \in \mathcal{E}$, we compute with the same idea that any linear combination of the Poisson processes $(\eta^{\{q\}},\, q \in Q)$ is still a Poisson process and the intensities are likewise additive. Using the characterization result of Jacod [Jac75] mentioned after Definition 6.43 we deduce that $X = X_0 + \sum_q q\eta^{\{q\}}$ is distributed as the sum of independent Poisson processes $(\eta^{\{q\}},\, q \in Q)$. By Remark 7.13 we deduce that $\mathbb{P}_\ell$ is the law of a compound Poisson process with intensity ℓ. □

In Definition 6.43 we have seen that every unit jump process admits a predictable compensator that characterizes the law of the process. Since Q is finite, this idea is easily extended to the space of jump processes $\mathbb{J}(\mathcal{I},Q)$. In particular under every probability $\mathbb{Q}$ on $\mathbb{J}(\mathcal{I},Q)$ there exists a predictable random measure $\bar{A}$ on $\mathcal{I}\times Q$ such that $dN^X - d\bar{A}$ is a martingale measure in the sense that $t \mapsto \int_{[0,t]\times Q} \bar{u}(s,q)(N^X(dsdq) - \bar{A}(dsdq))$ defines a local martingale for any $\bar{u} \in \bar{\mathcal{E}}$. We will call $d\bar{A}$ the intensity of X respectively N^X. In particular a compound Poisson process has intensity $\bar{A}(dsdq) = \ell(q)ds\Lambda(dq)$ in the above sense.

7.1.3. *Examples of compound Poisson processes and their bridges.*

Let us first show that we may use the duality formula (2.39) for Lévy processes to identify the intensity of h-transforms of compound Poisson processes. This provides an important tool to compute the intensities of bridges of compound Poisson processes. Let us already note that the representation of the intensity of an h-transform presented in the lemma below holds in a far more general context: A generalization of this result to Markov jump processes is given in Proposition 7.50 by a different proof.

Lemma 7.16. *Let $\mathbb{P}_\ell$ be the law of a compound Poisson process with intensity ℓ on $\mathbb{J}(\mathcal{I},Q)$ and $h : \mathbb{R}^d \to \mathbb{R}_+$ be bounded such that $\mathbb{E}(h(X_1)) = 1$. Then the h-transform $h\mathbb{P}_\ell$ is a jump process on $\mathbb{J}(\mathcal{I},Q)$ with intensity*

$$k(t,X_{t-},q)dtL(dq) = \frac{h(t,X_{t-}+q)}{h(t,X_{t-})}dtL(dq), \tag{7.17}$$

where $h(t,y) := \mathbb{E}_\ell(h(X_1)\,|\,X_t = y)$ is $\mathbb{P}_\ell(X_t \in dy)$-a.s. well defined for every $t \in \mathcal{I}$.

Proof. Define $h(t, X_{t-}) := \mathbb{E}\,(h(X_1)\,|\,X_{t-}) = \mathbb{E}\,(h(X_1)\,|\,X_t)$, where the second equality follows from stochastic continuity of X under $\mathbb{P}_\ell$. Since X is an integrable Lévy process with characteristics $(\int_Q qL(dq), 0, L)$ Proposition 2.38 implies that for every bounded functional F that is $\mathcal{F}_{[0,t]}$ measurable and $\bar{u} \in \bar{\mathcal{E}}$ such that $\bar{u} = \bar{u}\mathbb{1}_{(t,1]\times Q}$ the duality formula

$$\mathbb{E}_\ell\left(h(X_1)F\int_{\mathcal{I}\times Q}\bar{u}(s,q)(N^X(dsdq) - dsL(dq))\right) = \mathbb{E}_\ell\left(F\int_{\mathcal{I}\times Q}(h(X_1+q)-h(X_1))\bar{u}(s,q)dsL(dq)\right)$$

holds. On the right side we may take conditional expectation and use successively a stochastic Fubini and a Fubini to get

$$\begin{aligned}
\mathbb{E}_\ell\left(h(X_1)F\int_{\mathcal{I}\times Q}\bar{u}(s,q)N^X(dsdq)\right) &= \mathbb{E}_\ell\left(F\int_{\mathcal{I}\times Q}\mathbb{E}_\ell(h(X_1+q)|X_{s-})\bar{u}(s,q)dsL(dq)\right)\\
&= \mathbb{E}_\ell\left(F\int_{\mathcal{I}\times Q}h(s,X_{s-}+q)\bar{u}(s,q)dsL(dq)\right)\\
&= \mathbb{E}_\ell\left(F\int_{\mathcal{I}\times Q}\frac{h(s,X_{s-})}{h(s,X_{s-})}h(s,X_{s-}+q)\bar{u}(s,q)dsL(dq)\right)\\
&= \mathbb{E}_\ell\left(h(X_1)F\int_{\mathcal{I}\times Q}\frac{h(s,X_{s-}+q)}{h(s,X_{s-})}\bar{u}(s,q)dsL(dq)\right),
\end{aligned}$$

where $\mathbb{E}_\ell(h(X_1+q)|X_s) = \mathbb{E}_\ell(h(X_1)|X_s+q)$ due to the space-homogeneity of the Lévy process. Since this holds for all bounded F that are $\mathcal{F}_{[0,t]}$-measurable we deduce that X has intensity (7.17) with respect to $h\mathbb{P}_\ell$. □

Let us now compare the reciprocal classes of different compound Poisson processes on $\mathbb{J}(\mathcal{I}, Q)$ in some examples. This will provide an introduction to explicit computations regarding the bridges of compound Poisson processes. The concluding statements at the end of each example will be quoted at different occasions in this paragraph.

Example 7.18. *Let $Q = \{1,2\}$ and $\mathbb{P}_\ell$ be the law of a compound Poisson process with intensity ℓ for some $\ell(1), \ell(2) > 0$. Conditioned on the initial state $X_0 = 0$ we have $\mathbb{P}^0_\ell(X_1 \in \mathbb{N}) = 1$ and $\mathbb{P}^0_\ell(X_1 = m) > 0$ for any $m \in \mathbb{N}$, as can be checked with Remark 7.13. Let us first explicitly compute the distribution of a bridge $\mathbb{P}^{0,m}_\ell(\,.\,) = \mathbb{P}_\ell(\,.\,|X_0 = 0, X_1 = m)$ for $m \in \mathbb{N}$ using the form of an h-transform given in (7.8).*

Take e.g. the bridge from $X_0 = 0$ to $X_1 = 3$ then

$$\mathbb{P}^{0,3}_\ell(\,.\,) = \frac{\mathbb{1}_{\{3\}}(X_1)}{\mathbb{P}^0_\ell(X_1=3)}\mathbb{P}^0_\ell(\,.\,). \tag{7.19}$$

The normalization factor $\mathbb{P}^0_\ell(X_1 = 3)$ is computed using

$$\{X_0 = 0,\, X_1 = 3\} = \{X_0 = 0\} \cup \{\eta_1^{\{1\}} = 3, \eta_1^{\{2\}} = 0\} \cup \{\eta_1^{\{1\}} = 1, \eta_1^{\{2\}} = 1\}, \tag{7.20}$$

and the independence of the Poisson processes $\eta^{\{1\}}$ and $\eta^{\{2\}}$:

$$\begin{aligned}
\mathbb{P}^0_\ell(X_1=3) &= \mathbb{P}^0_\ell(\eta_1^{\{1\}}=3)\mathbb{P}^0_\ell(\eta_1^{\{2\}}=0) + \mathbb{P}^0_\ell(\eta_1^{\{1\}}=1)\mathbb{P}^0_\ell(\eta_1^{\{2\}}=1)\\
&= e^{-\ell(1)}\frac{\ell(1)^3}{6}e^{-\ell(2)} + e^{-\ell(1)}\ell(1)e^{-\ell(2)}\ell(2).
\end{aligned}$$

Therefore e.g.

$$\mathbb{P}^{0,3}_\ell(T_1 \in dt_1, V_1 = 1, T_2 \in dt_2, V_2 = 2, \eta = 2) = \ell(1)\ell(2)\frac{\mathbb{1}_{\{t_1<t_2\}}dt_1dt_2}{\frac{1}{6}\ell(1)^3 + \ell(1)\ell(2)},$$

or in the case of 3 small jumps

$$\mathbb{P}^{0,3}_\ell(T_1 \in dt_1, V_1 = 1, T_2 \in dt_2, V_2 = 1, T_3 \in dt_3, V_3 = 1, \eta = 3) = \ell(1)^3 \frac{\mathbb{1}_{\{t_1<t_2<t_3\}} dt_1 dt_2 dt_3}{\frac{1}{6}\ell(1)^3 + \ell(1)\ell(2)}.$$

Let $\mathbb{P}_{\tilde\ell}$ be another compound Poisson process on $\mathbb{J}(\mathcal{I}, \{1,2\})$ with intensity $\tilde\ell$ given by $\tilde\ell(1)$, $\tilde\ell(2) > 0$. Using the above expressions of the distribution of a bridge it is quite difficult to compute conditions such that $\mathbb{P}^{0,m}_\ell = \tilde{\mathbb{P}}^{0,m}_{\tilde\ell}$ even for $m = 3$. Let us instead present a more dynamical viewpoint.

To gain insights into the dynamics of the bridge $\mathbb{P}^{0,3}_\ell$, let us compute the intensity $\ell^{0,3}(t, X_{t-}, q)dt\Lambda(dq)$ of X under $\mathbb{P}^{0,3}_\ell$ for $X_{t-} = 1$ and $q = 1$ respectively $q = 2$ and $t \in \mathcal{I}$. We apply Lemma 7.16 to h given in (7.19). To derive $h(t, y)$ for $y = 1, \dots, 3$ we see

$$\begin{aligned}
h(t,y) &= \mathbb{E}^0_\ell\left(\frac{\mathbb{1}_{\{3\}}(X_1)}{e^{-(\ell(1)+\ell(2))}(\frac{1}{6}\ell(1)^3 + \ell(1)\ell(2))} \,\middle|\, X_t = y \right) \\
&= \frac{\mathbb{P}^0_\ell(X_1 = 3 | X_t = y)}{e^{-(\ell(1)+\ell(2))}(\frac{1}{6}\ell(1)^3 + \ell(1)\ell(2))} \\
&= \frac{\mathbb{P}^0_\ell(X_1 = 3, X_t = y)}{\mathbb{P}^0_\ell(X_t = y)e^{-(\ell(1)+\ell(2))}(\frac{1}{6}\ell(1)^3 + \ell(1)\ell(2))} \\
&= \frac{\mathbb{P}^0_\ell(X_1 - X_t = 3 - y)}{e^{-(\ell(1)+\ell(2))}(\frac{1}{6}\ell(1)^3 + \ell(1)\ell(2))},
\end{aligned}$$

where the last equality holds since X has independent increments under $\mathbb{P}^0_\ell$. We use decompositions like (7.20) to compute all probabilities. Let us just state the results

$$\begin{aligned}
h(t,1) &= e^{t(\ell(1)+\ell(2))} \frac{(1-t)^2\ell(1) + (1-t)\ell(2)}{\frac{1}{6}\ell(1)^2 + \ell(2)} = e^{t(\ell(1)+\ell(2))}(1-t)\frac{(1-t)\ell(1) + \ell(2)}{\frac{1}{6}\ell(1)^2 + \ell(2)}, \\
h(t,2) &= e^{t(\ell(1)+\ell(2))} \frac{1-t}{\frac{1}{6}\ell(1)^2 + \ell(2)}, \\
h(t,3) &= e^{t(\ell(1)+\ell(2))} \frac{1}{\ell(1)(\frac{1}{6}\ell(1)^2 + \ell(2))}.
\end{aligned}$$

With (7.17) we now derive the intensity $\ell^{0,3}(t, X_{t-}, q)dt\Lambda(dq)$ of N^X with respect to $\mathbb{P}^{0,3}_\ell$. In $X_{t-} = 1$ we get for example

$$\begin{aligned}
\ell^{0,3}(t,1,1) &= \frac{h(t,2)}{h(t,1)}\ell(1) = \frac{\ell(1)}{(1-t)\ell(1) + \ell(2)}, \\
\ell^{0,3}(t,1,2) &= \frac{h(t,3)}{h(t,1)}\ell(2) = \frac{\ell(2)}{(1-t)^2\ell(1)^2 + (1-t)\ell(1)\ell(2)}.
\end{aligned}$$

The overall intensity of jumping $\ell^{0,3}(t,1,1) + \ell^{0,3}(t,1,2)$ explodes for $t \to 1$, even if $\ell^{0,3}(t,1,1) \to 1/\ell(2)$ is always finite. If $\ell(2) \approx 0$ the intensity of the jump of size one will grow faster too, but stays finite.

A necessary condition for the identity of the bridges $\mathbb{P}^{0,3}_\ell = \mathbb{P}^{0,3}_{\tilde\ell}$ is then $\ell^{0,3}(t,1,1) = \tilde\ell^{0,3}(t,1,1)$ and $\ell^{0,3}(t,1,2) = \tilde\ell^{0,3}(t,1,2)$ for all $t \in \mathcal{I}$. But

$$\ell^{0,3}(t,1,1) \equiv \tilde\ell^{0,3}(t,1,1) \Leftrightarrow (1-t) + \frac{\ell(2)}{\ell(1)} \equiv (1-t) + \frac{\tilde\ell(2)}{\tilde\ell(1)},$$

thus $\tilde{\ell}(1) = \alpha\ell(1)$ *and* $\tilde{\ell}(2) = \alpha\ell(2)$ *for some* $\alpha > 0$*. We insert this into*

$$\ell^{0,3}(t,1,2) \equiv \tilde{\ell}^{0,3}(t,1,2) \Leftrightarrow (1-t)^2\ell(1)^2 + (1-t)\ell(1)\ell(2) = \alpha\Big((1-t)^2\ell(1)^2 + (1-t)\ell(1)\ell(2)\Big),$$

to see that the intensities are only equal if $\alpha = 1$*, thus* $\ell(1) = \tilde{\ell}(1)$ *and* $\ell(2) = \tilde{\ell}(2)$*. Therefore compound Poisson processes on* $\mathbb{J}(\mathcal{I}, \{1,2\})$ *with different intensities always have different reciprocal classes.*

In the above example there are only finitely many types of paths supported by a given bridge, e.g. to go from 0 to 3 during $\mathcal{I}$, the pure jump paths which are admitted are only those with jump-size combinations $V_1 = 1, V_2 = 2$ or $V_1 = 2, V_2 = 1$ or $V_1 = 1, V_2 = 1, V_3 = 1$. In particular η only had finitely many values with respect to the bridges.

Example 7.21. *Let* $Q = \{-1, 1\}$ *and* X *be a compound Poisson process under* $\mathbb{P}_\ell$ *with intensity* ℓ *for some* $\ell(-1), \ell(1) > 0$*. Then* X *has the law of the difference of two independent Poisson process with intensities* $\ell(1)$ *respectively* $\ell(2)$*. Let us first compute the distribution of the total number of jumps* η *with respect to the bridge* $\mathbb{P}^{0,0}_\ell$*. Clearly* $\mathbb{P}^{0,0}_\ell(\eta \in \{0,2,4,\dots\}) = 1$*, the total number of jumps is a.s. pair. Moreover, there are as many jumps up, as there are jumps down*

$$\mathbb{P}^{0,0}_\ell(\eta = 2m) = \mathbb{P}^{0,0}_\ell(\eta_1^{\{1\}} = m, \eta_1^{\{-1\}} = m) = \frac{\mathbb{P}^0_\ell(\eta_1^{\{1\}} = m, \eta_1^{\{-1\}} = m)}{\sum_{i=0}^\infty \mathbb{P}^0_\ell(\eta_1^{\{1\}} = i, \eta_1^{\{-1\}} = i)}, \ \forall m \in \mathbb{N}.$$

By independence of $\eta^{\{-1\}}$ *and* $\eta^{\{1\}}$ *we get*

$$\mathbb{P}^0_\ell(\eta_1^{\{1\}} = m, \eta_1^{\{-1\}} = m) = \mathbb{P}^0_\ell(\eta_1^{\{1\}} = m)\mathbb{P}^0_\ell(\eta_1^{\{-1\}} = m) = e^{-(\ell(1)+\ell(-1))}\frac{(\ell(1)\ell(-1))^m}{(m!)^2}.$$

Let $\mathbb{P}_{\tilde{\ell}}$ *the law of a compound Poisson process on* $\mathbb{J}(\mathcal{I}, \{-1,1\})$ *with intensity* $\tilde{\ell}$*. For the equality of the bridges* $\mathbb{P}^{0,0}_\ell = \mathbb{P}^{0,0}_{\tilde{\ell}}$ *it is necessary that up to the normalizing factors* $c^{-1} = e^{\ell(1)+\ell(2)}\mathbb{P}^0_\ell(X_1 = 0)$ *and* $\tilde{c}^{-1} = e^{\tilde{\ell}(1)+\tilde{\ell}(2)}\mathbb{P}^0_{\tilde{\ell}}(X_1 = 0)$ *we have*

$$c(\ell(1)\ell(-1))^m = \tilde{c}(\tilde{\ell}(1)\tilde{\ell}(-1))^m, \ \forall m \in \mathbb{N} \Leftrightarrow \frac{\ell(1)\ell(-1)}{\tilde{\ell}(1)\tilde{\ell}(-1)} = \sqrt[m]{\frac{\tilde{c}}{c}}, \ \forall m \in \mathbb{N}.$$

This holds if and only if there exists an $\alpha > 0$ *with* $\tilde{\ell}(1) = \alpha\ell(1)$ *and* $\tilde{\ell}(-1) = \alpha^{-1}\ell(-1)$*, and in this case we also have* $c = \tilde{c}$*. Let us check, that the bridges of* $\mathbb{P}_{\tilde{\ell}}$ *and* $\mathbb{P}_\ell$ *coincide in this case for any* $\alpha > 0$*. Using Remark 7.13 it is easy to check that*

$$\mathbb{P}^0_{\tilde{\ell}}(\,.\,) = e^{-\tilde{L}(Q)+L(Q)}\prod_{i=1}^{\eta}\frac{\tilde{\ell}(V_i)}{\ell(V_i)}\mathbb{P}^0_\ell(\,.\,).$$

Since for any $n \in \mathbb{N}$ *we have*

$$\mathbb{P}^0_{\tilde{\ell}}(X_1 = n) = \sum_{i=0}^\infty \mathbb{P}^0_{\tilde{\ell}}(\eta^{\{1\}} = n+i)\mathbb{P}^0_{\tilde{\ell}}(\eta^{\{-1\}} = i) = \sum_{i=0}^\infty e^{-\tilde{\ell}(1)}\frac{\tilde{\ell}(1)^{n+i}}{(n+i)!}e^{-\tilde{\ell}(-1)}\frac{\tilde{\ell}(-1)^i}{i!}$$

we may insert $\tilde{\ell}(1) = \alpha\ell(1)$ *and* $\tilde{\ell}(-1) = \alpha^{-1}\ell(-1)$ *to compute*

$$\begin{aligned}
\mathbb{P}^{0,n}_{\tilde{\ell}}(\,.\,) &= \frac{\mathbb{1}_{\{n\}}(X_1)}{\mathbb{P}^0_{\tilde{\ell}}(X_1 = n)}\mathbb{P}^0_{\tilde{\ell}}(\,.\,) \\
&= \frac{\mathbb{1}_{\{n\}}(X_1)}{e^{-\tilde{L}(Q)}\sum_{i=0}^{\infty}\frac{(\alpha\ell(1))^{n+i}}{(n+i)!}\frac{(\alpha^{-1}\ell(-1))^i}{i!}}e^{-\tilde{L}(Q)+L(Q)}\prod_{i=1}^{\eta}\frac{\tilde{\ell}(V_i)}{\ell(V_i)}\mathbb{P}^0_{\ell}(\,.\,) \\
&= \frac{\mathbb{1}_{\{n\}}(X_1)}{e^{-\tilde{L}(Q)}\alpha^n\sum_{i=0}^{\infty}\frac{\ell(1)^{n+i}}{(n+i)!}\frac{\ell(-1)^i}{i!}}e^{-\tilde{L}(Q)+L(Q)}\alpha^n\mathbb{P}^0_{\ell}(\,.\,) \\
&= \frac{\mathbb{1}_{\{n\}}(X_1)}{\mathbb{P}^0_{\ell}(X_1 = n)}\mathbb{P}^0_{\ell}(\,.\,) \\
&= \mathbb{P}^{0,n}_{\ell}(\,.\,).
\end{aligned}$$

A similar computation shows that $\mathbb{P}^{x,y}_{\tilde{\ell}}(\,.\,) = \mathbb{P}^{x,y}_{\ell}(\,.\,)$ *for any* $x, y \in \mathbb{R}^d$ *with* $|y - x| \in \mathbb{N}$. *Thus the laws of two compound Poisson processes* $\mathbb{P}_\ell$ *and* $\mathbb{P}_{\tilde{\ell}}$ *have the same reciprocal class on* $\mathbb{J}(\mathcal{I}, \{-1, 1\})$ *if and only if* $\tilde{\ell}(1) = \alpha\ell(1)$ *and* $\tilde{\ell}(-1) = \alpha^{-1}\ell(-1)$ *for some* $\alpha > 0$.

In the next example the distribution of η under any bridge will be deterministic. This simplifies the comparison of the bridges of compound Poisson processes.

Example 7.22. *Let* $Q = \{e_1, e_2\}$ *with vectors* $e_1 = (1, 0)^t, e_2 = (0, 1)^t \in \mathbb{R}^2_*$ *and* X *be a compound Poisson process under* $\mathbb{P}_\ell$ *with intensity* ℓ *for some* $\ell(e_1), \ell(e_2) > 0$. *This is a two-dimensional continuous time random walk with independent motion in both directions. Starting from* $(0, 0)$ *the process can reach any point on the lattice* $\mathbb{N}^2$. *Given a fixed endpoint* $(n, m) \in \mathbb{N}^2$ *we have*

$$\mathbb{P}^{(0,0),(n,m)}_{\ell}(\eta_1^{\{e_1\}} = n, \eta_1^{\{e_2\}} = m) = 1$$

since $\{X_0 = (0, 0), X_1 = (n, m)\} = \{X_0 = (0, 0), \eta_1^{\{e_1\}} = n, \eta_1^{\{e_2\}} = m\}$. *The law of the bridge of the compound Poisson process is therefore given by the law of the bridges of the Poisson processes*

$$\begin{aligned}
&\mathbb{P}^{(0,0),(n,m)}_{\ell}(\eta^{\{e_1\}} \in \,.\,, \eta^{\{e_2\}} \in \,.\,) \\
= \;&\mathbb{P}_{\ell}(\eta^{\{e_1\}} \in \,.\,|X_0 = (0, 0), \eta^{\{e_1\}} = n)\mathbb{P}_{\ell}(\eta^{\{e_2\}} \in \,.\,|X_0 = (0, 0), \eta^{\{e_2\}} = m),
\end{aligned}$$

where $\eta^{\{e_1\}}$ *and* $\eta^{\{e_2\}}$ *are independent under* $\mathbb{P}^{(0,0),(n,m)}$ *since* $\eta = n + m$ *is deterministic and* $\Gamma^{\simeq}_{n+m}((0, 0), (n, m))$ *consists of one element. We saw in Proposition 6.31 that all Poisson processes with different intensities are in the same reciprocal class. We conclude that every distribution of a compound Poisson process* $\mathbb{P}_{\tilde{\ell}}$ *on* $\mathbb{J}(\mathcal{I}, \{e_1, e_2\})$ *with intensity* $\tilde{\ell}$ *for some* $\tilde{\ell}(e_1), \tilde{\ell}(e_2) > 0$ *has the same reciprocal class as* $\mathbb{P}_\ell$.

7.2. **Compound Poisson processes with incommensurable jumps.**

In Example 7.22 we saw that the comparison of bridges of jump processes is made easier if under any given bridge $\mathbb{Q}^{x,y}$ the number of jumps η is deterministic and $\Gamma^{\simeq}_{\eta}(x, y)$ contains at most one element. Here we generalize this idea introducing the concept of incommensurability between the jump-sizes. This will provide to be a crucial condition in the derivation of precise statements concerning the bridges of compound Poisson processes.

We first define the general property of incommensurability for jump-sizes. Then we apply this concept to identify the reciprocal class of compound Poisson processes: Our main result in this paragraph is the characterization of the reciprocal class of a compound Poisson process by a duality formula in Theorem 7.36.

7.2.1. *Incommensurability between jumps.*

Incommensurability of jump-sizes is an algebraic condition on the set Q.

Definition 7.23. *We say that the jump-sizes in Q are incommensurable if for any $q_1, \dots, q_m \in Q$ and $q'_1, \dots, q'_n \in Q$ with $m, n \in \mathbb{N}$ the equality $q_1 + \cdots + q_m = q'_1 + \cdots + q'_n$ only holds if $m = n$ and $(q_1, \dots, q_m)^t \simeq (q'_1, \dots, q'_m)^t$.*

Note that the jump-sizes $e_1 = (1, 0), e_2 = (0, 1) \in \mathbb{R}^2_*$ of Example 7.22 are incommensurable. In the following remark we show how the concept of incommensurability affects the reference measure Λ.

Remark 7.24. *Let Λ be the finite counting measure on Q introduced in (7.4). Then the jump-sizes in Q are incommensurable if and only if the measures Λ^{*m} and Λ^{*n} are singular on $\mathbb{R}^d$ for all $m, n \in \mathbb{N}$, $m \neq n$.*

Proof. Assume that the jumps in Q are incommensurable, take $n \neq m$, $n, m \in \mathbb{N}$. If there exists $q \in \mathbb{R}^d_*$ with $\Lambda^{*m}(\{q\}) > 0$ and $\Lambda^{*n}(\{q\}) > 0$, then there exist $q_1, \dots, q_m \in Q$ with $q_1 + \cdots + q_m = q$ and $q'_1, \dots, q'_n \in Q$ with $q'_1 + \cdots + q'_n = q$. But this is a direct contradiction to the incommensurability property.

Now assume that Λ^{*m} and Λ^{*n} are singular for each $n \neq m$. If Q is commensurable, there exists $m, n \in \mathbb{N}$ and $q_1, \dots, q_m, q'_1, \dots, q'_n \in Q$ with $q_1 + \cdots + q_m = q'_1 + \cdots + q'_n$. But since Λ is a discrete measure on Q we have $\Lambda^{*m}(\{q_1 + \cdots + q_m\}) > 0$ and $\Lambda^{*n}(\{q'_1 + \cdots + q'_n\}) > 0$ in contradiction to the singularity of Λ^{*m} and Λ^{*n}. □

The following Lemma will provide a decomposition of the bridges of jump processes in $\mathbb{J}(\mathcal{I}, Q)$ in the sense of Remark 7.9. It represents the probabilistic interpretation of the incommensurability of jumps and is a key result for the characterization of the reciprocal class of a compound Poisson process.

Lemma 7.25. *Let $\mathbb{Q}$ be any probability on $\mathbb{J}(\mathcal{I}, Q)$. If the jumps in Q are incommensurable then for any $x, y \in \mathbb{R}^d$ such that $\mathbb{Q}^x(X_1 = y) > 0$ there exists an $m \in \mathbb{N}$ such that $\eta = m$ $\mathbb{Q}^{x,y}$-a.s. and $\Gamma^{\simeq}_m(x, y)$ contains exactly one element.*

Proof. Let $\mathbb{Q}^x_1(y) > 0$ such that the law of η with respect to $\mathbb{Q}^{x,y}$ is not deterministic. Then there exists at least two different $m, n \in \mathbb{N}$ with $\mathbb{Q}^{x,y}(\eta = m) > 0$ and $\mathbb{Q}^{x,y}(\eta = n) > 0$. Since this implies the existence of $q_1, \dots, q_m, q'_1, \dots, q'_n \in Q_x$ with $q_1 + \cdots + q_m = y - x = q'_1 + \cdots + q'_n$ we get a contradiction to the incommensurability of Q. The fact that $\Gamma^{\simeq}_m(x, y)$ contains more than one element directly contradicts the definition of incommensurability of jump-sizes. □

Let us show that there exists sets of jumps Q that are such that every bridge of a jump process has a deterministic number of jumps $\eta = m$, but $\Gamma^{\simeq}_m(x, y)$ may contain more than one element: The condition $(q_1, \dots, q_m)^t \simeq (q'_1, \dots, q'_m)^t$ of Definition 7.23 is meaningful.

Example 7.26. *Let $\{e_1, e_2, e_3\} \subset \mathbb{R}^3$ with $e_1 = (1, 0, 0)^t$, $e_2 = (0, 1, 0)^t$ and $e_3 = (0, 0, 1)^t$ be incommensurable jumps and define $Q := \{e_1, e_2, e_1 + e_3, e_2 - e_3\} = \{e_1, e_2, (1, 0, 1)^t, (0, 1, 1)^t\}$. The jump-sizes in Q are commensurable since $e_1 + e_2 = (e_1 + e_3) + (e_2 - e_3)$. Note that the number of jumps is deterministic for any bridge of any jump process with jumps in Q. If this would not be the case, there would exist $y = x + n_1 e_1 + n_2 e_2 + n_3(e_1 + e_3) + n_4(e_2 - e_3) = x + m_1 e_1 + m_2 e_2 + m_3(e_1 + e_3) + m_4(e_2 - e_3)$ with integer $n_i, m_i \in \mathbb{N}$ $i = 1, 2, 3, 4$ and $\sum_{i=1}^4 n_i \neq \sum_{i=1}^4 m_i$. But a comparison of coefficients in front of e_1 and e_2 shows that $n_1 + n_3 = m_1 + m_3$ and $n_2 + n_4 = m_2 + m_4$ since $\{e_1, e_2, e_3\}$ are incommensurable jumps. Therefore $\sum_{i=1}^4 n_i = \sum_{i=1}^4 m_i$, in contradiction to the hypothesis that η is not generally deterministic for the bridges of any jump process.*

7.2.2. *Duality formula for compound Poisson processes.*

If $Q = \{q\}$ contains but a single jump-size, the space $\mathbb{J}(\mathcal{I}, \{q\}) \cong \mathbb{J}_1(\mathcal{I})$ is isomorphic to the space of unit jump processes. In particular X is a compound Poisson process with intensity ℓ if and only if $t \mapsto \eta_t^{\{q\}} = \eta_t$ is a Poisson process with intensity $\ell(q)$. The characterization of the Poisson process $\eta^{\{q\}}$ and its reciprocal class by a duality formula then imply similar results for the compound Poisson process and its reciprocal class. In § 7.2.3 we want to extend this principle of characterization by duality formulae to compound Poisson processes with incommensurable jumps. In turn, this paragraph is devoted to the derivation of a duality formula that holds with respect to a compound Poisson process. The basic idea is to use the additivity of the duality formula (6.26) with respect to the integrating compound Poisson distribution and the decomposition $X = \sum_{q \in Q} q\eta^{\{q\}}$. Note that the results of this paragraph are valid without the incommensurability assumption between jumps.

To present a duality formula we have to start with the introduction of a derivative operator of functionals on the space of jump processes $\mathbb{J}(\mathcal{I}, Q)$. A functional $F : \mathbb{J}(\mathcal{I}, Q) \to \mathbb{R}$ is called smooth and cylindric if there exists a bounded $f : \mathbb{R}^d \times (\mathcal{I} \times (Q \cup \{0\}))^n$ such that $f(x, (., q_1), (., q_2), \dots) \in C_b^\infty(\mathcal{I}^n)$, $\forall x \in \mathbb{R}^d$, $q_1, \dots, q_n \in Q \cup \{0\}$ and for all $\omega \in \mathbb{J}(\mathcal{I}, Q)$ we have

$$F(\omega) = F((x, \{(t_1, q_1), \dots, (t_m, q_m)\})) := f(x, (t_1, q_1), \dots, (t_{n\wedge m}, q_{n\wedge m}), (1, 0), (1, 0), \dots). \tag{7.27}$$

Define the space of smooth functionals of the jump-times and jumps by

$$\mathcal{S}_{\mathbb{J}} := \{F : \mathbb{J}(\mathcal{I}, Q) \to \mathbb{R} \text{ is smooth and cylindric as in (7.27)}\}. \tag{7.28}$$

We will also use the canonical formulation $F(X) = f(X_0, (T_1 \wedge 1, V_1), \dots, (T_n \wedge 1, V_n)) \in \mathcal{S}_{\mathbb{J}}$. Let us now present two definitions of a derivative operator that acts on functionals $F : \mathbb{J}(\mathcal{I}, Q) \to \mathbb{R}$ which are complementary. First we define a derivative of functionals $F(\omega) \in \mathcal{S}_{\mathbb{J}}$ as in (7.27), that only operates in the direction of the jump-times associated to fixed jump-sizes by

$$\begin{aligned} &\mathcal{D}_{t,q} F((x, \{(t_1, q_1), \dots, (t_m, q_m)\})) \\ &:= -\sum_{i=1}^{n\wedge m} \partial_{2i} f(x, (t_1, q_1), \dots, (t_{n\wedge m}, q_{n\wedge m}), (1, 0), \dots) \mathbb{1}_{[0,t_i]\times\{q_i\}}(t, q). \end{aligned} \tag{7.29}$$

This extends the definition of the derivative operator on smooth functionals of unit-jump processes $\mathcal{S}_{\mathbb{J}_1}$ if $Q = \{1\}$, see Proposition 6.12. Let us give another complementary definition of this derivative operator by a perturbation.

Definition 7.30. *Given a bounded, measurable $\bar{u} : \mathcal{I} \times \mathbb{R}^d \to \mathbb{R}$ and a small $\varepsilon \geq 0$ we define the perturbation $\pi_{\bar{u}}^\varepsilon : \mathbb{J}(\mathcal{I}, Q) \to \mathbb{J}(\mathcal{I}, Q)$ by*

$$(x, \{(t_1, q_1), \dots, (t_m, q_m)\}) \mapsto (x, \{(t_1 + \varepsilon\langle \bar{u}(., q_1)\rangle_{t_1}, q_1), \dots, (t_m + \varepsilon\langle \bar{u}(., q_m)\rangle_{t_m}, q_m)\} \cap \mathcal{I} \times Q).$$

Let $\mathbb{Q}$ be any probability on $\mathbb{J}(\mathcal{I}, Q)$. A functional $F \in \mathbb{L}^2(\mathbb{Q})$ is called differentiable in direction $\bar{u}$ if the limit

$$\mathcal{D}_{\bar{u}} F := -\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(F \circ \pi_{\bar{u}}^\varepsilon - F)$$

exists in $\mathbb{L}^2(\mathbb{Q})$. If F is differentiable in all directions $\bar{u} \in \bar{\mathcal{E}}$, we can denote the unique derivative by

$$\mathcal{D}F = (\mathcal{D}_{t,q} F)_{t \in \mathcal{I}, q \in Q} \in \mathbb{L}^2(dt \otimes \Lambda \otimes \mathbb{Q}), \text{ where } \mathcal{D}_{\bar{u}} F = \int_{\mathcal{I}\times Q} \mathcal{D}_{t,q} F \bar{u}(t, q) dt \Lambda(dq)$$

holds $\mathbb{Q}$-a.s. for every $\bar{u} \in \bar{\mathcal{E}}$.

The next Proposition explains why these definitions were called complementary.

Proposition 7.31. *Let $\mathbb{Q}$ be any probability on $\mathbb{J}(\mathcal{I}, Q)$. Then a functional $F(X) \in \mathcal{S}_{\mathbb{J}}$ is differentiable in the sense of Definition 7.30 in any direction $\bar{u} \in \bar{\mathcal{E}}$ with $\langle \bar{u}(., q)\rangle = 0$, $\forall q \in Q$, and the derivative coincides with (7.29).*

Proof. If $\bar{u} \in \bar{\mathcal{E}}$ with $\langle \bar{u}(., q)\rangle = 0$, $q \in Q$, then for every $\omega = (x, \{(t_1, q_1), \ldots, (t_m, q_m)\}) \in \mathbb{J}(\mathcal{I}, Q)$ we have

$$\begin{aligned} & \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(f(x, (t_1 + \varepsilon \langle \bar{u}(., q_1)\rangle_{t_1}, q_1), \ldots,) - f(x, (t_1, q_1), \ldots) \right) \\ = & \sum_{i=1}^{m \wedge n} \partial_{2i} f(x, (t_1, q_1), \ldots) \langle \bar{u}(., q_i)\rangle_{t_i} \\ = & \int_{\mathcal{I} \times Q} \sum_{i=1}^{m \wedge n} \partial_{2i} f(x, (t_1, q_1), \ldots) \mathbb{1}_{[0, t_i] \times \{q_i\}} \bar{u}(t, q) dt \Lambda(dq). \end{aligned}$$

Since $\eta \circ \pi^{\varepsilon}_{\bar{u}} = \eta$ under the condition $\langle \bar{u}(., q)\rangle = 0$ we have $V_i \circ \pi^{\varepsilon}_{\bar{u}} = V_i$ for $1 \leq i \leq \eta$ and $\varepsilon > 0$ small enough. The boundedness of f in its time derivatives implies convergence in $\mathbb{L}^2(\mathbb{Q})$ for any probability $\mathbb{Q}$ on $\mathbb{J}(\mathcal{I}, Q)$. □

Using these complementary definitions of the derivative operator we can give two complementary proofs of a duality formula that holds with respect to a compound Poisson process.

Proposition 7.32. *Let $\mathbb{P}_\ell$ be the law of a compound Poisson process with intensity ℓ on $\mathbb{J}(\mathcal{I}, Q)$. Then the duality formula*

$$\mathbb{E}_\ell \left(F \int_{\mathcal{I} \times Q} \bar{u}(t, q)(N^X(dtdq) - dtL(dq)) \right) = \mathbb{E}_\ell \left(\mathcal{D}_{\bar{u}} F \right) \tag{7.33}$$

holds for all $\bar{u} \in \bar{\mathcal{E}}$ and $F \in \mathbb{L}^2(\mathbb{P})$ that are differentiable in direction $\bar{u}$, respectively for all $F \in \mathcal{S}_{\mathbb{J}}$ if the derivative is defined by (7.29).

Proof. Let us first prove the duality formula for $\bar{u} \in \bar{\mathcal{E}}$ and $F(X) \in \mathcal{S}_{\mathbb{J}}$ using (7.29). Since $\eta^{\{q\}}$ is a Poisson process with intensity $\ell(q)$ we have

$$\begin{aligned} \mathbb{E}_\ell \left(F(X) \int_{\mathcal{I} \times Q} \bar{u}(s, q)(N^X(dsdq) - dsL(dq)) \right) &= \sum_{q \in Q} \mathbb{E}_\ell \left(F(X) \int_{\mathcal{I}} \bar{u}(s, q)(d\eta^{\{q\}}_s - \ell(q) ds) \right) \\ &= \sum_{q \in Q} \mathbb{E}_\ell \left(\int_{\mathcal{I}} \mathcal{D}_{s,q} F(X) \bar{u}(s, q) ds \right) \\ &= \mathbb{E}_\ell \left(\int_{\mathcal{I} \times Q} \mathcal{D}_{s,q} F(X) \bar{u}(s, q) ds \Lambda(dq) \right), \end{aligned}$$

where for the second equality we applied the duality formula (6.23) of Poisson processes.

The proof of (7.33) for $F \in \mathbb{L}^2(\mathbb{P}_\ell)$ that is differentiable in direction $\bar{u} \in \bar{\mathcal{E}}$ is similar to the proof of the duality formula (6.23) of the Poisson process: First we "compensate" the perturbation $\pi^{\varepsilon}_{\bar{u}}$ by a Girsanov transformation of measure. Take some $F \in \mathcal{F}_{[0,s]}$ and put $\bar{v} := F \mathbb{1}_{(s,t] \times B}$ with $s \leq t$ and $B \subset Q$. Then $\bar{v}$ is a predictable function. By definition of the perturbation we see that for any $\bar{u} \in \bar{\mathcal{E}}$ we have $N^X \circ \pi^{\varepsilon}_{\bar{u}}((s, t] \times Q)) = N^X(\omega(\bar{B}^\varepsilon))$, with $\bar{B}^\varepsilon := \{(r, q) : q \in B, \tau^{\varepsilon}_{\bar{u}}(s, q) < r \leq \tau^{\varepsilon}_{\bar{u}}(t, q)\}$ and $\tau^{\varepsilon}_{\bar{u}}(t, q)$ is defined by

$$\int_{[0, \tau^{\varepsilon}_{\bar{u}}(t,q)]} (1 + \varepsilon \bar{u}(r, q)) dr = t,$$

see also Example 6.13 for the unit jump version of this time-inversion. We define a new probability $G^\varepsilon \mathbb{P}$ on $\mathbb{J}(\mathcal{I}, Q)$ by

$$G^\varepsilon = \exp\left(\int_{\mathcal{I}} (\bar{u}(s,q) - 1) ds L(dq)\right) \prod_{i=1}^{\eta} u(T_i, V_i).$$

Then

$$\begin{aligned}
& \mathbb{E}_\ell \left(\left[F \int_{\mathcal{I}\times\mathbb{R}^d_*} \bar{v}(r,q) N^X(drdq) \right] \circ \pi^\varepsilon_{\bar{u}} G^\varepsilon \right) \\
= \; & \mathbb{E}_\ell \left(F \circ \pi^\varepsilon_{\bar{u}} N^X(\bar{B}^\varepsilon) G^\varepsilon \right) \\
= \; & \mathbb{E}_\ell \left(F \circ \pi^\varepsilon_{\bar{u}} G^\varepsilon \right) \int_{\mathcal{I}\times\mathbb{R}^d_*} \mathbb{1}_{\bar{B}^\varepsilon}(r,q)(1 + \varepsilon \bar{u}(r,q)) dr L(dq) \\
= \; & \mathbb{E}_\ell \left(F \circ \pi^\varepsilon_{\bar{u}} G^\varepsilon \right) \int_{\mathbb{R}^d_*} \left[\tau^\varepsilon_{\bar{u}}(t,q) - \tau^\varepsilon_{\bar{u}}(s,q) + \varepsilon \int_{[0,\tau^\varepsilon_{\bar{u}}(t,q)]} \bar{u}(r,q) dr - \varepsilon \int_{[0,\tau^\varepsilon_{\bar{u}}(s,q)]} \bar{u}(r,q) dr \right] \mathbb{1}_B(q) L(dq) \\
= \; & \mathbb{E}_\ell \left(F \circ \pi^\varepsilon_{\bar{u}} G^\varepsilon \right) (t-s) L(B),
\end{aligned}$$

which implies that $N^X \circ \pi^\varepsilon_{\bar{u}}$ is a Poisson measure with intensity $dtL(dq)$ on $\mathcal{I} \times Q$ under $G^\varepsilon \mathbb{P}$. We deduce the equality

$$-\mathbb{E}_\ell \left(F \circ \pi^\varepsilon_{\bar{u}} - F \right) = \mathbb{E}_\ell \left(F \circ \pi^\varepsilon_{\bar{u}} (G^\varepsilon - 1) \right)$$

for any functional F and any $\bar{u}$, since $\mathbb{E}_\ell \left(F \circ \pi^\varepsilon_{\bar{u}} G^\varepsilon \right) = \mathbb{E}_\ell(F)$. Multiplying both sides with $1/\varepsilon$ and taking the limit $\varepsilon \to 0$ proves the duality formula: The convergence arguments are similar to the ones in the proof of Proposition 6.22. □

7.2.3. *Characterization of the reciprocal class of compound Poisson processes.*

Let us first remark, that the duality formula for $F(X) \in \mathcal{S}_{\mathbb{J}}$ and $\bar{u} \in \bar{\mathcal{E}}$ in turn implies the martingale property of compensated compound Poisson processes. This result does not require incommensurability between jumps.

Proposition 7.34. *Let $\mathbb{Q}$ be an arbitrary probability on $\mathbb{J}(\mathcal{I}, Q)$ such that $\eta \in \mathbb{L}^1(\mathbb{Q})$ is integrable. If for all $F \in \mathcal{S}_{\mathbb{J}}$ and $\bar{u} \in \bar{\mathcal{E}}$ the duality formula*

$$\mathbb{E}_{\mathbb{Q}} \left(F(X) \int_{\mathcal{I}\times Q} \bar{u}(s,q) (N^X(dsdq) - ds L(dq)) \right) = \mathbb{E}_{\mathbb{Q}} \left(\int_{\mathcal{I}\times Q} \mathcal{D}_{s,q} F(X) \bar{u}(s,q) ds \Lambda(dq) \right) \tag{7.35}$$

holds, then $\mathbb{Q}$ is the law of a compound Poisson process with intensity L.

Proof. The result is due to the characterization by Watanabe presented in Lemma 7.15. In particular, if $F \in \mathcal{S}_{\mathbb{J}}$ is $\mathcal{F}_{[0,t]}$-measurable and $\bar{u} = \bar{u}\mathbb{1}_{[t,1]\times Q}$, then $\mathcal{D}_{s,q} F(X) \equiv 0$ and (7.35) is a statement of the martingale property. □

We now prove our first main result in this section: The characterization of the reciprocal class of a compound Poisson process by a duality formula. Here, the fact that the jump-sizes in Q are incommensurable is essential and necessary.

Theorem 7.36. *Let the jumps in Q be incommensurable. Assume that $\mathbb{Q}$ is a probability on $\mathbb{J}(\mathcal{I}, Q)$ such that $\eta \in \mathbb{L}^1(\mathbb{Q})$. Then $\mathbb{Q}$ is in the reciprocal class of the law of a compound Poisson process with intensity ℓ if and only if the duality formula*

$$\mathbb{E}_{\mathbb{Q}} \left(F(X) \int_{\mathcal{I}\times Q} \bar{u}(s,q) (N^X(dsdq) - ds L(dq)) \right) = \mathbb{E}_{\mathbb{Q}} \left(\int_{\mathcal{I}\times Q} \mathcal{D}_{s,q} F(X) \bar{u}(s,q) ds \Lambda(dq) \right) \tag{7.37}$$

holds for every $F \in \mathcal{S}_{\mathbb{J}}$ and $\bar{u} \in \bar{\mathcal{E}}$ with $\langle \bar{u}(.,q) \rangle = 0$ for all $q \in Q$.

Proof. Note, that the compensation of N^X by $dsL(dq)$ is not necessary under the loop condition:

$$\int_{\mathcal{I}\times Q} \bar{u}(s,q)(N^X(dsdq) - dsL(dq)) = \int_{\mathcal{I}\times Q} \bar{u}(s,q)N^X(dsdq), \text{ if } \langle \bar{u}(.,q)\rangle = 0, \forall q \in Q.$$

Let us first proof the necessity of the duality formula. In this part of the proof, the incommensurability assumption is not necessary. Under the loop condition $\langle u(.,q)\rangle = 0$ we have $\mathcal{D}_{\bar{u}}\phi(X_0) = \mathcal{D}_{\bar{u}}\psi(X_1) = 0$ for any bounded $\phi, \psi : \mathbb{R}^d \to \mathbb{R}$. This implies that the duality formula holds with respect to the bridges, which in turn implies that it holds in the reciprocal class of $\mathbb{P}_\ell$ (see also the similar proof of Lemma 6.37).

To prove the converse, let $x, y \in \mathbb{R}^d$ be such that $\mathbb{Q}^{x,y}$ is well defined. By Lemma 7.25 $\eta = m$ $\mathbb{Q}^{x,y}$-*a.s.* for some $m \in \mathbb{N}$ and there exists exactly one $\gamma_m = (q_1, \dots, q_m)^t \in \Gamma_m^{\simeq}(x,y)$. The duality formula (7.37) still holds with respect to the bridge $\mathbb{Q}^{x,y}$ by the first part of the proof. Using Remark 7.9 we decompose the expectation under the bridge

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}^{x,y}\left(F(X)\int_{\mathcal{I}\times Q}\bar{u}(s,q)N^X(dsdq)\right) &= \sum_{q\in Q}\mathbb{E}_{\mathbb{Q}}\left(F(X)\int_{\mathcal{I}}\bar{u}(s,q)d\eta_s^{\{q\}}\,\middle|\, X_0 = x, \eta_1^{\{\tilde{q}\}} = \#_{\tilde{q}}\gamma_m,\ \forall \tilde{q}\in Q\right)\\ &= \sum_{q\in Q}\mathbb{E}_{\mathbb{Q}}\left(\int_{\mathcal{I}}\mathcal{D}_{s,q}F(X)\bar{u}(s,q)ds\,\middle|\, X_0 = x, \eta_1^{\{\tilde{q}\}} = \#_{\tilde{q}}\gamma_m,\ \forall \tilde{q}\in Q\right)\end{aligned}$$

Define $F(X) \in \mathcal{S}_{\mathbb{J}}$ with $f_0 \in C_b^\infty(\mathbb{R}^d)$ and $f_q \in C_b^\infty(\mathcal{I}^n)$, $q \in Q$, as follows:

$$F((x,\{(t_1,q_1),\dots,(t_m,q_m)\})) = f_0(x)\prod_{q\in Q} f_q(t_1\mathbb{1}_{\{q=q_1\}},\dots,t_{m\wedge n}\mathbb{1}_{\{q=q_{m\wedge n}\}},1,\dots). \tag{7.38}$$

The derivative of such a functional is given by

$$\mathcal{D}_{t,q}F((x,\{(t_1,q_1),\dots,(t_m,q_m)\})) = f_0(x)\prod_{\tilde{q}\neq q} f_{\tilde{q}}(\dots)\left(\sum_{i=1}^{m\wedge n}\partial_i f(\dots)\mathbb{1}_{[0,t_i]\times\{q_i\}}(t,q)\right).$$

We apply the duality formula to this kind of functionals and conclude that $\eta^{\{q\}}$ are independent Poisson bridges conditionally on $X_0 = x, X_1 = y$: First assume that $f_{\tilde{q}} \equiv 1$ and $\bar{u}(.,\tilde{q}) \equiv 0$ for $\tilde{q} \neq q$, then the derivative operator $\mathcal{D}_{t,q}$ only acts on f_q and by Theorem 6.39 $\eta^{\{q\}}$ is a Poisson bridge from 0 to $\#_q\gamma_m$ under $\mathbb{Q}^{x,y}$. Using an arbitrary $F(X)$ as defined in (7.38) we deduce that $\eta^{\{q\}}$ still has the same law conditionally on $(\eta^{\{\tilde{q}\}},\ \tilde{q}\neq q)$. Thus $(\eta^{\{q\}},\ q\in Q)$ are independent Poisson bridges under $\mathbb{Q}^{x,y}$. But the joint law of $(\eta^{\{q\}},\ q\in Q)$ determines the distribution of X under $\mathbb{Q}^{x,y}$. Thus we conclude that $\mathbb{Q}^{x,y} = \mathbb{P}^{x,y}$, since $(\eta^{\{q\}},\ q\in Q)$ are independent Poisson bridges under $\mathbb{P}^{x,y}$, see Remark 7.9. □

If the jump-sizes in Q are commensurable, the independence of the processes $(\eta^{\{q\}},\ q\in Q)$ under the bridge $\mathbb{P}_\ell^{x,y}$ may fail. As a consequence, the above characterization result may fail as well. We illustrate this in the following example.

Example 7.39. *We resume Example 7.18: The jump-sizes $Q = \{1,2\}$ are commensurable and $\mathbb{P}_\ell$, $\mathbb{P}_{\tilde{\ell}}$ are the laws of compound Poisson process on $\mathbb{J}(\mathcal{I},Q)$ with intensities ℓ respectively $\tilde{\ell}$. We showed that if $\ell(1) \neq \tilde{\ell}(1)$ or $\ell(2) \neq \tilde{\ell}(2)$ these distributions do not have the same reciprocal class. Still, by Proposition 7.32 the duality formula (7.11) holds. Thus for all $F \in \mathcal{S}_{\mathbb{J}}$ and $\bar{u} \in \bar{\mathcal{E}}$ with $\langle \bar{u}(.,q)\rangle = 0$ we have*

$$\mathbb{E}_\ell\left(F(X)\int_{\mathcal{I}\times Q}\bar{u}(s,q)N^X(dsdq)\right) = \mathbb{E}_\ell\left(\int_{\mathcal{I}\times Q}\mathcal{D}_{t,q}F(X)\bar{u}(s,q)ds\Lambda(dq)\right).$$

In particular the same duality formula is satisfied by $\mathbb{P}_\ell$ and $\mathbb{P}_{\tilde{\ell}}$. This duality formula can not be fruitful for the characterization of the reciprocal class $\mathcal{R}(\mathbb{P}_\ell)$ on $\mathbb{J}(\mathcal{I}, \{1,2\})$.

By a similar argument as in the above example we infer the following property of reciprocal classes of compound Poisson processes with incommensurable jumps.

Corollary 7.40. *All compound Poisson processes with incommensurable jumps in Q are in the same reciprocal class.*

Proof. Under the loop condition $\langle \bar{u}(.,q) \rangle = 0$ the same duality formula holds for any compound Poisson process on $\mathbb{J}(\mathcal{I}, Q)$. With Theorem 7.36 we conclude that $\mathcal{R}(\mathbb{P}_\ell) = \mathcal{R}(\mathbb{P}_{\tilde{\ell}})$ for any two compound Poisson processes. □

7.3. **Nice jump processes.**

In this paragraph we present a first discussion of the possibility of characterizing the reciprocal classes of Markov jump processes by duality formulae. In Theorem 7.54 we present our second main result of this section: We are able to compare the reciprocal classes of Markov jump processes with regular intensities by reciprocal invariants, without relying on the incommensurability of jump-sizes.

7.3.1. *Definition and basic properties of nice jump processes.*

We are interested in a particular class of pure jump Markov processes.

Definition 7.41. *Let $\varepsilon > 0$ be arbitrary and take a function ℓ such that*

$$\ell : \mathcal{I} \times \mathbb{R}^d \times Q \to [\varepsilon, \infty) \text{ is bounded, and } \ell(.,x,q) \in C_b^1(\mathcal{I}, \mathbb{R}_+) \text{ for all } x \in \mathbb{R}^d, q \in Q. \tag{7.42}$$

Then $\mathbb{P}_\ell$ is the law of a nice jump process if X has intensity $\ell(t, X_{t-}, q)dt\Lambda(dq)$ under $\mathbb{P}_\ell$.

Clearly compound Poisson processes are nice jump processes with $\ell(t,x,q) = \ell(q)$, the notation $\mathbb{P}_\ell$ is thus consistent. In Remark 7.13 we stated that a compound Poisson process is the sum of independent Poisson process. A similar property does not hold for nice jump processes: The "sum" of nice unit jump processes $(\eta^{\{q\}}, q \in Q)$ is not necessarily a nice jump process. Let us illustrate this in the following example.

Example 7.43. *Let $Q = \{1,2\}$ as in Example 7.18 and 7.39. Since $X_t = X_0 + \eta_t^{\{1\}} + 2\eta_t^{\{2\}}$, the law of X is determined by the joint law of X_0, $\eta^{\{1\}}$ and $\eta^{\{2\}}$. We define a probability $\mathbb{Q}$ on $\mathbb{J}(\mathcal{I}, Q)$ as follows. Assume that $\mathbb{Q}_0 = \delta_{\{x\}}$, $\eta^{\{1\}}$ is a nice unit jump process with intensity $\ell^{\{1\}}(t, \eta_{t-}^{\{1\}})dt$ and $\eta^{\{2\}}$ is a nice unit jump process with intensity $\ell^{\{2\}}(t, \eta_{t-}^{\{2\}})dt$ such that $\eta^{\{1\}}$ and $\eta^{\{2\}}$ are independent.*

We may chose $\ell^{\{1\}}$ and $\ell^{\{2\}}$ such that X is not Markov: Let $t \in \mathcal{I}$ be arbitrary, $F = f(X_{t_1}, \dots, X_{t_n}) \in \mathcal{S}$ that is $\mathcal{F}_{[t,1]}$-measurable. This functional has a representation $F(X) = \tilde{F}(\eta^{\{1\}}, \eta^{\{2\}}) = f(x + \eta_{t_1}^{\{1\}} + 2\eta_{t_1}^{\{2\}}, \dots)$, and we may use the Markov property of $\eta^{\{1\}}$ and $\eta^{\{2\}}$ to deduce

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}\left(F(X) \mid \mathcal{F}_{[0,t]}\right) &= \mathbb{E}_{\mathbb{Q}}\left(\tilde{F}(\eta^{\{1\}}, \eta^{\{2\}}) \,\middle|\, \mathcal{F}_{[0,t]}\right) \\ &= \mathbb{E}_{\mathbb{Q}}\left(\tilde{F}(\eta^{\{1\}}, \eta^{\{2\}}) \,\middle|\, \eta_t^{\{1\}}, \eta_t^{\{2\}}\right) \\ &= \mathbb{E}_{\mathbb{Q}}\left(F(X) \,\middle|\, \eta_t^{\{1\}}, \eta_t^{\{2\}}\right). \end{aligned} \tag{7.44}$$

But the canonical process $X_t = x + \eta_t^{\{1\}} + 2\eta_t^{\{2\}}$ contains less information than the condition on $\eta_t^{\{1\}}$ and $\eta_t^{\{2\}}$: By bounded convergence equation (7.44) holds for all F that are $\mathcal{F}_{[t,1]}$-measurable and bounded. Take e.g. $F(X) = \mathbb{1}_{\{X_{t_1} - X_t = 1\}}$ for $t_1 > t$, then for any $n, m \in \mathbb{N}$

$$\mathbb{E}_{\mathbb{Q}}\left(F(X) \,\middle|\, \eta_t^{\{1\}} = m, \eta_t^{\{2\}} = n\right) = \mathbb{Q}\left(\eta_{t_1}^{\{1\}} - \eta_t^{\{1\}} = 1, \eta_{t_1}^{\{2\}} - \eta_t^{\{2\}} = 0 \,\middle|\, \eta_t^{\{1\}} = m, \eta_t^{\{2\}} = n\right).$$

But on the other hand

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\left(F(X) \;\middle|\; \eta_t^{\{1\}} + \eta_t^{\{2\}} = m+n\right)\\
= \;& \mathbb{Q}\left(\eta_{t_1}^{\{1\}} - \eta_t^{\{1\}} = 1, \eta_{t_1}^{\{2\}} - \eta_t^{\{2\}} = 0 \;\middle|\; \eta_t^{\{1\}} + \eta_t^{\{2\}} = m+n\right)\\
= \;& \frac{\sum_{i=0}^{n+m} \mathbb{Q}\left(\eta_{t_1}^{\{1\}} - \eta_t^{\{1\}} = 1, \eta_{t_1}^{\{2\}} - \eta_t^{\{2\}} = 0 \;\middle|\; \eta_t^{\{1\}} = i, \eta_t^{\{2\}} = n+m-i\right)\mathbb{Q}(\eta_t^{\{1\}} = i, \eta_t^{\{2\}} = n+m-i)}{\mathbb{Q}\left(\eta_t^{\{1\}} + \eta_t^{\{2\}} = m+n\right)}.
\end{aligned}$$

All probabilities can be derived using Remark 6.50. To give an explicit numeric example take $m = 1$, $n = 0$ *and* $\ell^{\{1\}}(s,0) = 1$, $\ell^{\{1\}}(s,1) = 2$, $\ell^{\{1\}}(s,2) = 3$ *and* $\ell^{\{2\}}(s,0) = 4$, $\ell^{\{2\}}(s,1) = 2$ *for any* $s \in \mathcal{I}$, *and finally put* $t = 0,5$ *and* $t_1 = 0,6$. *Then we derive*

$$\mathbb{Q}\left(\eta_{t_1}^{\{1\}} - \eta_t^{\{1\}} = 1, \eta_{t_1}^{\{2\}} - \eta_t^{\{2\}} = 0 \;\middle|\; \eta_t^{\{1\}} = 1, \eta_t^{\{2\}} = 0\right) \approx 0,11.$$

and

$$\mathbb{Q}\left(\eta_{t_1}^{\{1\}} - \eta_t^{\{1\}} = 1, \eta_{t_1}^{\{2\}} - \eta_t^{\{2\}} = 0 \;\middle|\; \eta_t^{\{1\}} + \eta_t^{\{2\}} = 1\right) \approx 0,08.$$

Thus X is not even a Markov process under $\mathbb{Q}$.

The characterization results of the reciprocal classes of nice unit jump processes by duality formulae in Section 6 thus cannot be extended directly to nice jump processes. Still, using an approach similar to the identification of the reciprocal class of a nice unit jump process by an invariant in § 6.5, we will derive a comparison of the reciprocal classes of nice jump processes in Theorem 7.54. The main technical requisite to obtain this result is the Girsanov theorem.

The law of a nice jump process $\mathbb{P}_\ell$ is equivalent to the the law of a compound Poisson process $\mathbb{P}$ with intensity $\Lambda = \sum_{q\in Q} \delta_{\{q\}}$. The Girsanov theorem for processes with jumps provides an explicit form of the density, see also Theorem A.6 in the appendix.

Proposition 7.45. *Let* $\mathbb{P}_\ell$ *be a nice jump process and* $\mathbb{P}$ *be a compound Poisson process with intensity* Λ *with same initial condition* $\mathbb{P}_{\ell,0} = \mathbb{P}_0$. *Then* $\mathbb{P}_\ell$ *is equivalent to* $\mathbb{P}$ *and the density process defined by* $\mathbb{P}_\ell = G_t^\ell \mathbb{P}$ *on* $\mathcal{F}_{[0,t]}$ *has the explicit form*

$$G_t^\ell = \exp\left(-\int_{[0,t]\times Q} (\ell(s, X_{s-}, q) - 1) ds \Lambda(dq)\right) \prod_{i=1}^{\eta_t^{\{Q\}}} \ell(T_i, X_{T_i-}, V_i). \tag{7.46}$$

Let us mention that G_1^ℓ can be written as a function of the initial state, the jump-times and jumps only:

$$G_1^\ell = \exp\left(-\sum_{i=1}^{\eta+1} \int_{[T_{i-1},T_i]\times\mathbb{R}_*^d} (\ell(s, X_0 + \sum_{j=1}^{i-1} V_j, q) - 1) ds \Lambda(dq)\right) \prod_{i=1}^{\eta} \ell(T_i, X_0 + \sum_{j=1}^{i-1} V_j, V_i), \tag{7.47}$$

where we use $T_0 = 0$ and $T_{\eta+1} := 1$ to abbreviate.

Remark 7.48. *We reformulate the Girsanov theorem to provide an explicit expression for the distribution of nice jump processes. Define the intensity of X under* $\mathbb{P}_\ell$ *to have any jump at a given time* $t \in \mathcal{I}$ *from the position* $y \in \mathbb{R}^d$ *by*

$$\bar{\ell}(t,y) := \int_Q \ell(t,y,q)\Lambda(dq) = \sum_{q\in Q} \ell(t,y,q).$$

Then the law $\mathbb{P}_\ell$ is completely described by

$$\begin{aligned}(7.49)\qquad &\mathbb{P}_\ell(X_0 \in dx, T_1 \in dt_1, V_1 = q_1, \dots, T_m \in dt_m, V_m = q_m, \eta = m)\\ &= \mathbb{P}_{\ell,0}(dx)\ell(t_1, x, q_1)e^{-\int_{[0,t_1]}\bar{\ell}(s,x)ds}\ell(t_2, x+q_1, q_2)e^{-\int_{[t_1,t_2]}\bar{\ell}(s,x+q_1)ds}\cdots\\ &\quad\cdots\ell(t_m, x+q_1+\cdots+q_{m-1}, q_m)e^{-\int_{[t_{m-1},t_m]}\bar{\ell}(s,x+q_1+\dots q_{m-1})ds}e^{-\int_{[t_m,1]}\bar{\ell}(s,x+q_1+\cdots+q_m)ds}\\ &\qquad \mathbb{1}_{\{t_1<\cdots<t_m\}}dt_1\cdots dt_m,\end{aligned}$$

for any $x \in \mathbb{R}^d$, $\{(t_1, q_1), \dots, (t_m, q_m)\} \in \Delta_{\mathcal{I}\times Q}$ and $m \in \mathbb{N}$.

7.3.2. *Comparison of the reciprocal classes of nice jump processes.*

The next proposition generalizes the result of Lemma 7.16 to h-transforms of nice jump processes.

Proposition 7.50. *Let $\mathbb{P}_\ell$ be the law of a nice jump process and $h : \mathbb{R}^d \to \mathbb{R}_+$ such that $\mathbb{E}_\ell(h(X_1)) = 1$. Then the h-transform $h\mathbb{P}_\ell$ is a Markov jump process on $\mathbb{J}(\mathcal{I}, Q)$ with intensity*

$$(7.51)\qquad k(t, X_{t-}, q)dt\Lambda(dq) = \ell(t, X_{t-}, q)\frac{h(t, X_{t-}+q)}{h(t, X_{t-})}dt\Lambda(dq),$$

where $h(t, X_t) := \mathbb{E}_\ell(h(X_1)|X_t)$.

Proof. Clearly the h-transform $h\mathbb{P}_\ell$ has the Markov property, see also Lemma 4.4. Define the space-time harmonic function $h(t, y) = \mathbb{E}_\ell(h(X_1)|X_t = y)$ and put $G_t := h(t, X_t)$. Then $h(t, y)$ is differentiable in time and solution of a Kolmogoroff backward equation.

Lemma 7.52. *Let $h(t, x)$ be defined as in the statement of the above proposition. Then h is a solution of the Kolmogoroff backward equation*

(7.53)

$$\partial_t h(t, x) = -\int_Q (h(t, x+q) - h(t, x))\,\ell(t, x, q)\Lambda(dq), \quad \text{holds } \mathbb{P}_\ell(X_t \in dx)\text{-a.s. for any } t \in \mathcal{I},$$

in particular $h(., x) \in C^1(\mathcal{I})$.

Proof. We omit the proof, since it works in the same way as the proof of Lemma 6.55 in the unit jump case. Just use Remarks 7.5 and 7.48 on the explicit distribution of $\mathbb{P}_\ell$. □

By regularity of h we can apply the Itô formula

$$h(t, X_t) = h(s, X_s) + \int_{[s,t]}\partial_t h(r, X_{r-})dr + \int_{[s,t]\times Q}[h(r, X_{r-}+q) - h(r, X_{r-})]\,N^X(drdq).$$

But since the Kolmogoroff backward equation holds, we deduce that $G_t = h(t, X_t)$ is solution of the exponential stochastic integral equation:

$$G_t = 1 + \int_{[0,t]\times Q} G_{s-}\left(\frac{h(s, X_{s-}+q)}{h(s, X_{s-})} - 1\right)(N^X(dsdq) - \ell(s, X_{s-}, q)ds\Lambda(dq)),$$

which has the explicit solution

$$G_t = \exp\left(-\int_{[0,t]\times Q}\left(\frac{h(s, X_{s-}+q)}{h(s, X_{s-})} - 1\right)\ell(s, X_{s-}, q)ds\Lambda(dq)\right)\prod_{i=1}^{\eta_t^{(Q)}}\frac{h(T_i, X_{T_i-}+V_i)}{h(T_i, X_{T_i-})}.$$

We just have to multiply the densities $G_1^k\mathbb{P} = G_1 G_1^\ell\mathbb{P}$, which gives

$$G_1 G_1^\ell = \exp\left(-\int_{\mathcal{I}\times Q}\left(\ell(s, X_{s-}, q)\frac{h(s, X_{s-}+q)}{h(s, X_{s-})} - 1\right)ds\Lambda(dq)\right)\prod_{i=1}^{\eta}\ell(T_i, X_{T_i-}, V_i)\frac{h(T_i, X_{T_i-}+V_i)}{h(T_i, X_{T_i-})},$$

and invoke the Girsanov theorem to identify

$$k(s, X_{s-}, q)ds\Lambda(dq) = \ell(s, X_{s-}, q)\frac{h(s, X_{s-} + q)}{h(s, X_{s-})}ds\Lambda(dq), \quad \mathbb{P}_\ell\text{-a.s.}$$

as intensity of $h\mathbb{P}_\ell$. □

The fact, that $h(t, x)$ is a solution of the Kolmogoroff backward equation is now used to compute reciprocal invariants for the law $\mathbb{P}_\ell$. Our main result in this paragraph is the fact, that these invariants are indeed characteristics of $\mathcal{R}(\mathbb{P}_\ell)$.

Theorem 7.54. *Let $\mathbb{P}_\ell$ and $\mathbb{P}_k$ be the laws of two nice jump processes. Then $\mathcal{R}(\mathbb{P}_k) = \mathcal{R}(\mathbb{P}_\ell)$ if and only if*

(i) $\log k - \log \ell$ *is the "gradient of a potential": For every $x \in \mathbb{R}^d$ there exists a function $\psi : \mathcal{I} \times \mathbb{R}^d \to \mathbb{R}$ such that*

$$\log k(t, y, q) - \log \ell(t, y, q) = \psi(t, y + q) - \psi(t, y)$$

holds $dt \otimes \mathbb{P}^x_\ell(X_t \in dy)$-a.e. for all $q \in Q$ and such that $e^{\psi(t,X_t)} \in \mathbb{L}^1(\mathbb{P}^x_\ell)$ for all $t \in \mathcal{I}$.

(ii) the "harmonic" invariants coincide: $\forall q \in Q$, $\Xi^q_\ell(t, y) = \Xi^q_k(t, y)$, with

$$\Xi^q_\ell(t, y) = \partial_t \log \ell(t, y, q) + \int_Q (\ell(t, y + q, \bar{q}) - \ell(t, y, \bar{q}))\, \Lambda(d\bar{q}).$$

Proof. Assume first, that $\mathcal{R}(\mathbb{P}_k) = \mathcal{R}(\mathbb{P}_\ell)$. Fix $x \in \mathbb{R}^d$. We prove *(i)* and *(ii)* under $\mathbb{P}^x_\ell$, the general result for *(ii)* follows by mixing over the initial condition. There exists $h : \mathbb{R}^d \to \mathbb{R}_+$ that is everywhere positive such that $\mathbb{P}^x_k = h(X_1)\mathbb{P}^x_\ell$ is an h-transform, see Remark 4.18. We know that relation (7.51) holds between the intensities k and ℓ with $h(t, y) = \mathbb{E}^x_\ell(h(X_1)|X_t = y)$, which by Lemma 7.52 is a solution of the Kolmogoroff backward equation. This implies condition *(i)* with potential $\log h(t, y)$. Moreover for $q, \bar{q} \in Q$ this implies

$$\begin{aligned}
\frac{\partial_t h(t, y + q)}{h(t, y + q)} &= -\int_Q \left(\frac{h(t, y + q + \bar{q}) - h(t, y + q)}{h(t, y + q)}\right) \ell(t, y + q, \bar{q})\Lambda(d\bar{q}) \text{ and} \\
\frac{\partial_t h(t, y)}{h(t, y)} &= -\int_Q \left(\frac{h(t, y + \bar{q}) - h(t, y)}{h(t, y)}\right) \ell(t, y, \bar{q})\Lambda(d\bar{q}).
\end{aligned}$$

We take the difference of these two equations and insert

$$\frac{h(t, y + \bar{q})}{h(t, y)} = \frac{k(t, y, \bar{q})}{\ell(t, y, \bar{q})}, \text{ and } \frac{h(t, y + q + \bar{q})}{h(t, y + q)} = \frac{k(t, y + q, \bar{q})}{\ell(t, y + q, \bar{q})}$$

to get the "harmonic" invariant, condition *(ii)*.

For the converse assume that *(i)* and *(ii)* hold. We will show that $\mathbb{P}^x_k$ is an h-transform of $\mathbb{P}^x_\ell$ for arbitrary $x \in \mathbb{R}^d$. Define

$$h(t, y) := e^{\psi(t,y)}c(t) \text{ with } c(t) = ce^{-\int_{[0,t]\times Q}(k(s,x,q)-\ell(s,x,q))ds\Lambda(dq)},$$

where ψ is the potential from condition *(i)* and c is a normalization constant such that $\mathbb{E}^x_\ell(h(1, X_1)) = 1$. The normalization constant exists by the boundedness of ℓ and k and the integrability condition $e^{\psi(t,X_t)} \in \mathbb{L}^1(\mathbb{P}_\ell)$ for all $t \in \mathcal{I}$. Clearly the relation (7.51) holds since $\log h$ is a potential for $\log k(t, y, q) - \log \ell(t, y, q)$ too since $c(t)$ does not depend on y. We

insert this into the equality of the harmonic invariant, condition *(ii)*, which results in

$$\begin{aligned} & \frac{\partial_t h(t,y+q)}{h(t,y+q)} + \int_Q \left(\frac{h(t,y+q+\bar{q}) - h(t,y+q)}{h(t,y+q)} \right) \ell(t,y+q,\bar{q}) \Lambda(d\bar{q}) \\ (7.55) \qquad = \quad & \frac{\partial_t h(t,y)}{h(t,y)} + \int_Q \left(\frac{h(t,y+\bar{q}) - h(t,y)}{h(t,y)} \right) \ell(t,y,\bar{q}) \Lambda(d\bar{q}) \end{aligned}$$

for all $q \in Q$ and $y \in Q_x$, the support of $\delta_{\{x\}} * \left(\sum_{m=0}^{\infty} (\Lambda(Q))^{-m} \Lambda^{*m} \right)$. In particular

$$\begin{aligned} 0 &= \partial_t \log h(t,x) + \int_Q \left(\frac{h(t,x+\bar{q})}{h(t,x)} - 1 \right) \ell(t,x,\bar{q}) \Lambda(d\bar{q}) \\ &= -\int_Q (k(s,x,\bar{q}) - \ell(s,x,\bar{q})) \Lambda(d\bar{q}) + \int_Q (k(s,x,\bar{q}) - \ell(s,x,\bar{q})) \Lambda(d\bar{q}). \end{aligned}$$

Since for every $y \in Q_x$ there exist $q_1, \dots q_m \in Q$ such that $y = x + q_1 + \cdots + q_m$ we invoke equation (7.55) and deduce that $h(t,y)$ is a solution of the Kolmogoroff backward equation with respect to ℓ. Therefore $h(t, X_{t-})$ is a martingale under $\mathbb{P}_\ell^x$ and we may apply the first part of the proof to identify $\mathbb{P}_k^x = h\mathbb{P}_\ell^x$. □

Example 7.56. *Let $Q = \{1,2\}$ and $\mathbb{P}_\ell$ the law of a compound Poisson process with intensity ℓ, where $\ell(1), \ell(2) > 0$. Then $\Xi_\ell^q(t,y) = 0$, and a nice jump process $\mathbb{P}_k$ is in the reciprocal class $\mathcal{R}(\mathbb{P}_\ell)$ if*

$$\partial_t \log k(t,y,q) + \sum_{\bar{q}=1,2} (k(t,y+q,\bar{q}) - k(t,y,\bar{q})) = 0, \text{ for } q = 1,2 \text{ and } (t,y) \in \mathcal{I} \times \mathbb{R}^d.$$

and there exists a $\psi : \mathbb{R}^d \to \mathbb{R}$ such that

$$k(t,y,q) = \ell(q) e^{\psi(t,y+q) - \psi(t,y)}, \text{ for all } t \in \mathcal{I}, q \in \{1,2\}, y \in \mathbb{R}^d.$$

If we fit the potential ψ into the harmonic invariant, we get the condition

$$\begin{aligned} & 0 = \partial_t \left(\psi(t,y+q) - \psi(t,y) \right) + \sum_{\bar{q}=1,2} \left(\ell(\bar{q}) e^{\psi(t,y+q+\bar{q}) - \psi(t,y+q)} - \ell(\bar{q}) e^{\psi(t,y+\bar{q}) - \psi(t,y)} \right) \\ \Leftrightarrow \quad & \partial_t \psi(t,y+q) + \int_Q e^{\psi(t,y+q+\bar{q}) - \psi(t,y+q)} \ell(\bar{q}) \Lambda(d\bar{q}) = \partial_t \psi(t,y) + \int_Q e^{\psi(t,y+\bar{q}) - \psi(t,y)} \ell(\bar{q}) \Lambda(d\bar{q}), \end{aligned}$$

which reduces to the statement, that $h(t,y) = \log \psi(t,y)$ is a space-time harmonic function.

A "rotational" invariant is hidden in the "gradient of a potential" condition *(i)* of the above theorem. It is easy to check that if *(i)* holds, then the "rotational invariants" $\Psi_\ell^{q,\bar{q}}(t, X_{t-}) = \Psi_k^{q,\bar{q}}(t, X_{t-})$ coincide, where

$$(7.57) \quad \Psi_\ell^{q,\bar{q}}(t, X_{t-}) := \log \ell(t, X_{t-} + q, \bar{q}) - \log \ell(t, X_{t-}, \bar{q}) - (\log \ell(t, X_{t-} + \bar{q}, q) - \log \ell(t, X_{t-}, q)).$$

This invariant is called "rotational invariant", since if $\ell(y)$ and $k(y)$ only depend on the position of the process, the condition reduces to

$$\ell(y+q) - \ell(y) - (\ell(y+\bar{q}) - \ell(y)) = k(y+q) - k(y) - (k(y+\bar{q}) - k(y)),$$

and thus resembles the rotational condition *(ii)* of the continuous case presented in Theorem 5.26: On both sides we have the difference of discrete derivative operators.

The equivalence of this identity of rotationals to the gradient of a potential condition *(i)* of the above Theorem 7.54 depends strongly on the algebraic properties of the set of jump-sizes Q. See e.g. Borwein, Lewis [BL92], who examine the gradient of a potential condition if $(Q_x, +)$ has a group structure. Both conditions are meaningless if Q only contains one

element, therefore only a "harmonic" invariant appeared in the characterization of the reciprocal class of nice unit jump processes in Theorem 6.58. Let us present a simple example that implies that the rotational invariant is weaker than the gradient of a potential condition.

Example 7.58. *Let $Q \subset \mathbb{R}^d_*$ be any finite set of jump-sizes and $\ell(t,x,q)dt\Lambda(dq)$ the intensity of a nice jump process $\mathbb{P}_\ell$. Let $k(t,x,q) := \ell(t,x,q)e^{\phi(q)-x\cdot q}$ for some function $\phi : Q \to \mathbb{R}$. It is easy to check, that the "rotational" invariants coincide $\Psi_k^{q,\bar q}(t,x) = \Psi_\ell^{q,\bar q}(t,x)$ for any $t \in \mathcal{I}$, $x \in \mathbb{R}^d$ and $q, \bar q \in Q$. But $\log k(t,x,q) - \log \ell(t,x,q) = \phi(q) - x \cdot q$ is not necessarily the gradient of a potential. Clearly if $\psi : \mathbb{R}^d \to \mathbb{R}$ with $\phi(q) - x \cdot q = \psi(x+q) - \psi(x)$ exists, we fix ψ on Q up to a constant by $\phi(q) = \psi(q) - \psi(0)$ for $x = 0$. But then for $x = q$ we get $\psi(2q) = 2\phi(q) + q \cdot q - \psi(0)$ and on the other hand $\psi(2q) = \phi(q) + \psi(0)$, which implies $\phi(q) = 2\psi(0) - q \cdot q$. This contradicts the arbitrary choice of ϕ. Chosing $Q = \{1,2\}$, $\ell : Q \to (0,\infty)$ and $k(q) = \ell(q)e^{\phi(q)}$ for some $\phi : Q \to \mathbb{R}$, we see that this result is in accordance with the statement of Example 7.18, that the reciprocal classes of the compound Poisson processes $\mathbb{P}_k$ and $\mathbb{P}_\ell$ coincide if and only if $\ell \equiv k$.*

In the next proposition we present a duality formula satisfied by any element of the reciprocal class $\mathcal{R}(\mathbb{P}_\ell)$ of a nice jump process. Note that neither the "rotational" invariant nor the "gradient of potential" condition appear in the formula.

Proposition 7.59. *Let $\mathbb{P}_\ell$ be the law of a nice jump process, $\mathbb{Q}$ be any element of the reciprocal class of $\mathbb{P}_\ell$ such that $\eta \in \mathbb{L}^1(\mathbb{Q})$. Then the duality formula*

$$\begin{aligned} &\mathbb{E}_{\mathbb{Q}}\left(F(X)\int_{\mathcal{I}\times Q}\bar u(s,q)N^X(dsdq)\right) \\ &= \mathbb{E}_{\mathbb{Q}}\left(\mathcal{D}_{\bar u}F(X)\right) - \mathbb{E}_{\mathbb{Q}}\left(F(X)\int_{\mathcal{I}\times Q}\Xi_\ell^q(s,X_{s-})\langle \bar u(.,q)\rangle_s N^X(dsdq)\right) \end{aligned} \tag{7.60}$$

holds for all $F(X) \in \mathcal{S}_{\mathbb{J}}$ and $\bar u \in \bar{\mathcal{E}}$ with $\langle \bar u(.,q)\rangle = 0$ for all $q \in Q$.

Proof. The proof is similar to the proof of Lemma 6.65 in the unit jump setting. Using the product rule satisfied by the derivative operator and the duality formula for a compound Poisson process with intensity Λ we derive

$$\mathbb{E}_\ell\left(F(X)\int_{\mathcal{I}\times Q}\bar u(s,q)N^X(dsdq)\right) = \mathbb{E}_\ell\left(\mathcal{D}_{\bar u}F(X)\right) + \mathbb{E}_\ell\left(F(X)\mathcal{D}_{\bar u}\log G_1^\ell\right),$$

if $\log G_1^\ell$ is differentiable in direction $\bar u$. By (7.47) we have to differentiate

$$\log G_1^\ell = \left(-\sum_{i=1}^{\eta+1}\int_{[T_{i-1},T_i]\times Q}(\ell(s,X_0+\sum_{j=1}^{i-1}V_j,q)-1)ds\Lambda(dq)\right) + \left(\sum_{i=1}^{\eta}\log\ell(T_i,X_0+\sum_{j=1}^{i-1}V_j,V_i)\right),$$

where $T_0 := 0$ and $T_{\eta+1} := 1$. Since $\langle \bar u(.,q)\rangle = 0$, the perturbation $\pi^\varepsilon_{\bar u}$ does not change the number of jumps, thus $\log G_1^\ell$ is differentiable and the derivative is

$$\begin{aligned} \mathcal{D}_{\bar u}\log G_1^\ell &= -\sum_{i=1}^{\eta}\int_Q\left(\ell(T_i,X_0+\sum_{j=1}^{i-1}V_j,q)\langle\bar u(.,V_i)\rangle_{T_i} - \ell(T_{i-1},X_0+\sum_{j=1}^{i-1}V_j,q)\langle\bar u(.,V_{i-1})\rangle_{T_{i-1}}\right)\Lambda(dq) \\ &\quad + \sum_{i=1}^{\eta}\partial_t\log\ell(T_i,X_0+\sum_{j=1}^{i-1}V_j,V_i)\langle\bar u(.,V_i)\rangle_{T_i} \\ &= \int_{\mathcal{I}\times Q}\Xi_\ell^q(s,X_{s-})\langle\bar u(.,q)\rangle_s N^X(dsdq). \end{aligned}$$

Thus the duality formula (7.60) holds under $\mathbb{P}_\ell$. We extend this to any $\mathbb{Q}$ in $\mathcal{R}(\mathbb{P}_\ell)$ with $\eta \in \mathbb{L}^1(\mathbb{Q})$ as usually, see also the proof of Theorem 6.69. Take bounded functions $\phi, \psi : \mathbb{R}^d \to \mathbb{R}$, then $\mathcal{D}_{\bar{u}}\phi(X_0) = 0$ and $\mathcal{D}_{\bar{u}}\psi(X_1) = 0$. The duality formula still holds for the bridges of $\mathbb{P}_\ell$ since

$$\begin{aligned} & \mathbb{E}_\ell\left(\phi(X_0)\psi(X_1)F(X)\int_{\mathcal{I}\times Q}\bar{u}(s,q)N^X(dsdq)\right) \\ = \; & \mathbb{E}_\ell\left(\phi(X_0)\psi(X_1)\mathcal{D}_{\bar{u}}F(X)\right) \\ & -\mathbb{E}_\ell\left(\phi(X_0)\psi(X_1)F(X)\int_{\mathcal{I}\times Q}\Xi^q_\ell(s,X_{s-})\langle\bar{u}(.,q)\rangle_s N^X(dsdq)\right), \end{aligned}$$

But $\mathbb{Q}$ can be decomposed into a mixture of the bridges of $\mathbb{P}_\ell$ by Definition 4.12. Thus the duality formula is also satisfied by $\mathbb{Q}$. □

The duality formula (7.35) does not contain sufficient information to characterize the reciprocal class of $\mathbb{P}_\ell$: We already remarked, that neither a gradient of a potential condition, nor a rotational invariant appear in the formula. An example to refute the idea of a characterization of the reciprocal class $\mathcal{R}(\mathbb{P}_\ell)$ by the duality formula (7.60) was given in Example 7.39, where two compound Poisson processes with different reciprocal classes where solution of the same duality formula.

Conclusion

In this thesis we studied the characterization of classes of stochastic processes by duality formulae in various settings. The first part was dedicated to the characterization of processes with independent increments by a duality formula that is well-established in Malliavin's calculus. The second part was devoted to the characterization of the reciprocal classes of various processes by duality formulae. In particular we were able to present new results concerning the reciprocal classes of pure jump processes. Let us now point to various open questions related to this work.

Our first result was a characterization of infinitely divisible random vectors by an integration by parts formula. Following the ideas of Stein's calculus, this characterization is a key to the study of approximations of the distribution of infinitely divisible random vectors. The next step in the derivation of such approximation results would be the computation of a solution to Stein's equation

$$g_f(z)(z-b) - A\nabla g_f(z) - \int_{\mathbb{R}^d_*} (g_f(z+q) - g_f(z))qL(dq) = f(z) - \mathbb{E}\,(f(Z)).$$

The distance between the distribution of a random vector V to an infinitely divisible random vector Z is then computed using

$$\begin{aligned} &\left| \mathbb{E}\left(g_f(V)(V-b) - A\nabla g_f(V) - \textstyle\int_{\mathbb{R}^d_*} (g_f(V+q) - g_f(V))qL(dq) \right) \right| < \varepsilon \\ &\qquad \Rightarrow \quad |\mathbb{E}\,(f(V)) - \mathbb{E}\,(f(Z))| < \varepsilon, \end{aligned}$$

see also Remark 1.4 and the comments after Theorem 3.1.

Next we have shown, that processes with independent increments are the unique càdlàg processes satisfying a specific duality formula. On the one hand this indicates, that approaches similar to Stein's calculus could be applied in the setting of càdlàg processes, see e.g. Chen, Xia [CX04] for results on Poisson process approximation. On the other hand our characterization result could be used to construct Gibbs-states on the configuration space $\mathbb{D}(\mathcal{I})^{\mathbb{Z}^d}$. See e.g. Dai Pra, Rœlly and Zessin [DPRZ02], who use a duality formula to show that the set of weak solutions of a class of infinite dimensional stochastic differential equations coincides with a set of space-time Gibbs fields for a certain potential.

At the end of the first part of this thesis, we give a new proof of a characterization of infinitely divisible random measures on Polish spaces. Our approach implies that similar characterizations should indeed hold for arbitrary random objects whose distribution is fixed by projections on random vectors that have infinitely divisible distributions, e.g. infinitely divisible random fields in the sense of Lee [Lee67] and Maruyama [Mar70].

The second part of this thesis begins with a recapitulation of a result by Rœlly and Thieullen concerning the characterization of the reciprocal class of Brownian diffusions by a duality formula, see Theorem 5.34. Our characterization of Brownian diffusions in the class of semimartingales with integrable increments presented in Theorem 5.14 indicates that the hypotheses connected to Rœlly, Thieullens characterization result might be relaxed.

In his original article [Cla90] Clark derived reciprocal invariants for diffusions with non-unit diffusion coefficient, that is solutions of the SDE

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t,$$

for some $b \in C_b^\infty(\mathcal{I} \times \mathbb{R}^d, \mathbb{R}^d)$ and $\sigma \in C_b^\infty(\mathcal{I} \times \mathbb{R}^d, \mathbb{R}^{d\times d})$ such that $\sigma^t\sigma - \varepsilon\mathrm{Id}$ is potive definite for some $\varepsilon > 0$. His invariants permit to distinguish reciprocal classes: We presented his result restricted to Brownian diffusions in Theorem 5.26. A corresponding characterization

of the reciprocal class of diffusions with non-unit diffusion coefficient by a duality formula does not yet exist. It would be interesting to note, if such a duality formula still exhibits the similarities to the deterministic Newton equation presented in Remark 5.72.

We characterized the reciprocal class of nice unit jump processes in the sense of a comparison of invariants in Theorem 6.58 and by a duality formula in Theorem 6.69. Both results might be extended to the reciprocal classes of Markov unit jump processes that are not nice in the sense that the reference intensity $\ell(t, X_{t-})dt$ is unbounded, ℓ has values in the whole of $[0, \infty)$. The dynamical scope of such processes is larger: A unit jump process must leave a certain state $y \in \mathbb{R}$ almost surely until a given time $t \in \mathcal{I}$, if $\varepsilon \to \int_{[0,t-\varepsilon]} \ell(r, y)dr$ explodes for $\varepsilon \to 0$. On the other hand, the unit jump process is forced stay in a certain state $y \in \mathbb{R}$ almost surely in a given time interval $[s, t] \subset \mathcal{I}$ if $\ell(r, y) = 0$, $\forall r \in [s, t]$.

In the last section of this thesis, we characterized the reciprocal class of compound Poisson processes by a duality formula in Theorem 7.36 under the assumption of incommensurable jump-sizes. We found a reciprocal invariant for nice jump processes in Theorem 7.54 without using the assumption of incommensurability. The first result indicates, that it should be possible to characterize even nice jump processes with incommensurable jumps by the duality formula (7.60). It is not verisimilar to prove such results for processes with commensurable jumps. In Example 7.39 we have given evidence, that the duality formula (7.35) containing the stochastic derivative of Definition 7.30 is not suited to characterize the reciprocal classes of jump processes with commensurable jumps. Alternative definitions of stochastic derivatives might be a key to find a duality formula that characterizes the reciprocal class of compound Poisson process with commensurable jump-sizes.

Appendix A. Stochastic calculus of pure jump processes

In this section we present a few aspects of the stochastic calculus associated to pure jump semimartingales on the space of càdlàg paths. This calculus is the basis of many results in Sections 2, 6 and 7. In particular we state the Girsanov theorem for pure jump processes, which is a key to the proof of duality formulae. Our presentation is based on the introduction to semimartingales by Jacod and Shiryaev in [JS03], the proofs are omitted.

Let X be the canonical process on $\mathbb{D}(\mathcal{I}, \mathbb{R}^d)$. The random jump measure

$$N^X := \sum_{s:\Delta X_s \neq 0} \delta_{\{(s,\Delta X_s)\}}$$

on $\mathcal{I} \times \mathbb{R}^d_*$ is well defined.

Definition A.1. *Let $\mathbb{P}$ be a probability on $\mathbb{D}(\mathcal{I}, \mathbb{R}^d)$. Then X is called a pure jump process under $\mathbb{P}$ if*

$$\mathbb{P}\left(N^X(\mathcal{I} \times \mathbb{R}^d_*) < \infty\right) = 1 \text{ and } \mathbb{P}\left(\exists t \in \mathcal{I} : X_t - X_0 \neq \int_{[0,t]\times\mathbb{R}^d_*} q N^X(dsdq)\right) = 0.$$

The last condition is an equality of processes two up to evanescence. If X is a pure jump process under $\mathbb{P}$, it is also a semimartingale since it has finite variation. The space-time version of Itô's formula has a convenient form.

Theorem A.2. *Let X be a pure jump process under $\mathbb{P}$ and $f : \mathcal{I}\times\mathbb{R}^d \to \mathbb{R}$ such that $f(.,x) \in C^1(\mathcal{I})$ for any $x \in \mathbb{R}^d$. Then the Itô-formula*

$$f(t, X_t) = f(s, X_s) + \int_{(s,t]} \partial_t f(r, X_{r-})dr + \int_{[s,t]\times\mathbb{R}^d_*} (f(r, X_{r-} + q) - f(r, X_{r-}))N^X(drdq) \quad \mathbb{P}\text{-a.s.}$$

holds for any $s < t \in \mathcal{I}$.

If f does not depend on time, the Itô-formula reduces to the canonical sum

$$f(X_t) - f(X_s) = \int_{(s,t]\times\mathbb{R}^d_*} (f(X_{r-} + q) - f(X_{r-}))\, N^X(drdq) = \sum_{s\leq r\leq t} (f(X_r) - f(X_{r-}))\,.$$

Following [JS03, Theorem II.1.8] there exists a predictable random measure $\bar{A}$ on $\mathcal{I} \times \mathbb{R}^d_*$ such that

$$\mathbb{E}\left(\int_{\mathcal{I}\times\mathbb{R}^d_*} \bar{u}(s,q)N^X(dsdq)\right) = \mathbb{E}\left(\int_{\mathcal{I}\times\mathbb{R}^d_*} \bar{u}(s,q)\bar{A}(dsdq)\right)$$

for any predictable $\bar{u} : \mathcal{I} \times \mathbb{R}^d_* \times \mathbb{D}(\mathcal{I}, \mathbb{R}^d) \to \mathbb{R}_+$. This implies that for any predictable $\bar{u} \in \mathbb{L}^1(\bar{A} \otimes \mathbb{P})$ the process

$$t \mapsto \tilde{X}^{\bar{u}}_t := \int_{[0,t]\times\mathbb{R}^d_*} \bar{u}(s,q)(N^X(dsdq) - \bar{A}(dsdq)) \tag{A.3}$$

is a martingale. It is easy to see, that the solution of the Doléans-Dade integral equation

$$Y_t = 1 + \int_{[0,t]} Y_{s-}dX_s \text{ is given by } t \mapsto Y_t = \prod_{s\leq t}(1 + \Delta X_s).$$

This is just an application of the Itô-formula with respect to the pure jump process Y. The Doléans-Dade exponential of the compensated process $\tilde{X}^{\bar{u}}$ has the following convenient form

Theorem A.4. *Let X be a pure jump process under $\mathbb{P}$. If $\tilde{X}^{\bar{u}}$ is the compensated process defined in (A.3) for some $\bar{u} \in \mathbb{L}^1(\bar{A} \otimes \mathbb{P})$, then the Doléans Dade expontial of $\tilde{X}^{\bar{u}}$ is given by*

$$Y_t^{\bar{u}} = 1 + \int_{[0,t]} Y_{s-}^{\bar{u}} d\tilde{X}_s^{\bar{u}}, \ \forall t \in \mathcal{I}$$

$$\Leftrightarrow \ Y_t^{\bar{u}} = \exp\left(-\int_{[0,t]\times\mathbb{R}_*^d} \bar{u}(s,q)\bar{A}(dsdq)\right) \prod_{s\leq t}(1+\bar{u}(s,\Delta X_s)), \ \forall t \in \mathcal{I}.$$

This follows again with Itô's formula, see [JS03, Theorem I.4.61]. The Doléans-Dade exponential is an important tool in the formulation of the Girsanov theorem for pure jump processes.

Theorem A.5. *Let X be a pure jump process under $\mathbb{P}$ with intensity $d\bar{A}$. If $\mathbb{Q} \ll \mathbb{P}$ with density G and density process $G_t = \mathbb{E}(G|\mathcal{F}_{[0,t]})$, then X is a pure jump process under $\mathbb{Q}$ with intensity $\ell d\bar{A}$, where $\ell : \mathcal{I} \times \mathbb{R}_*^d \times \mathbb{D}(\mathcal{I}, \mathbb{R}^d) \to \mathbb{R}_+$ is predictable and satisfies*

$$\mathbb{E}\left(\int_{\mathcal{I}\times\mathbb{R}_*^d} u(s,q)\ell(s,q)G_{s-}N^X(dsdq)\right) = \mathbb{E}\left(\int_{\mathcal{I}\times\mathbb{R}_*^d} u(s,q)G_sN^X(dsdq)\right)$$

for any predictable $\bar{u} : \mathcal{I} \times \mathbb{R}_*^d \times \mathbb{D}(\mathcal{I}, \mathbb{R}^d) \to \mathbb{R}_+$.

We say that X is a pure jump process without fixed jumps under $\mathbb{P}$ if the intensity $\bar{A}$ is of the form

$$\bar{A}(dsdq) = dsA_s(dq).$$

In this case the density process G_t of the preceeding theorem has the following convenient representation.

Theorem A.6. *Let X be a pure jump process without fixed jumps under $\mathbb{P}$ and with intensity $dsA_s(dq)$. Let $\mathbb{Q} \ll \mathbb{P}$ with density G and density process $G_t = \mathbb{E}(G|\mathcal{F}_{[0,t]})$ such that X has intensity $\ell(s,q)dsA_s(dq)$ for a predictable function $\ell \in \mathbb{L}^1(\bar{A} \otimes \mathbb{P})$ under $\mathbb{Q}$. Then G is the Doléans-Dade exponential of $t \mapsto \int_{[0,t]}(\ell(s,q)-1)(N^X(dsdq) - dsA_s(dq))$, that is*

$$G_t = \mathbb{1}_{\{G_t>0\}} \exp\left(-\int_{[0,t]\times\mathbb{R}_*^d} (\ell(s,q)-1)dsA_s(dq)\right) \prod_{s\leq t} \ell(s,\Delta X_s).$$

If X has fixed jumps, the form of G is somewhat more involved, although G is still a Doléans-Dade exponential. The following partial converse is one of the main tools in the derivation of duality formulae in Sections 2, 6 and 7.

Theorem A.7. *Assume that the predictable function $\ell : \mathcal{I} \times \mathbb{R}_*^d \times \mathbb{D}(\mathcal{I}, \mathbb{R}^d) \to \mathbb{R}_+$ is such that the Doléans-Dade exponential*

$$G_t = 1 + \int_{[0,t]\times\mathbb{R}_*^d} (\ell(s,q)-1)G_{s-}(N^X(dsdq) - dsA_s(dq))$$

is well defined and a martingale. Then X is a pure jump process under the probability $G_1\mathbb{P}$ with intensity $\ell(s,q)dsA_s(dq)$.

Index of notation

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