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Calcul à une Boucle Avec Plusieurs Pattes Externes Dans Les Théories de Jauge : La Bibliothèque Golem95

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One-loop Multi-leg Calculation in Gauge Theories: `Golem95` Library

A ma petite "Amina"

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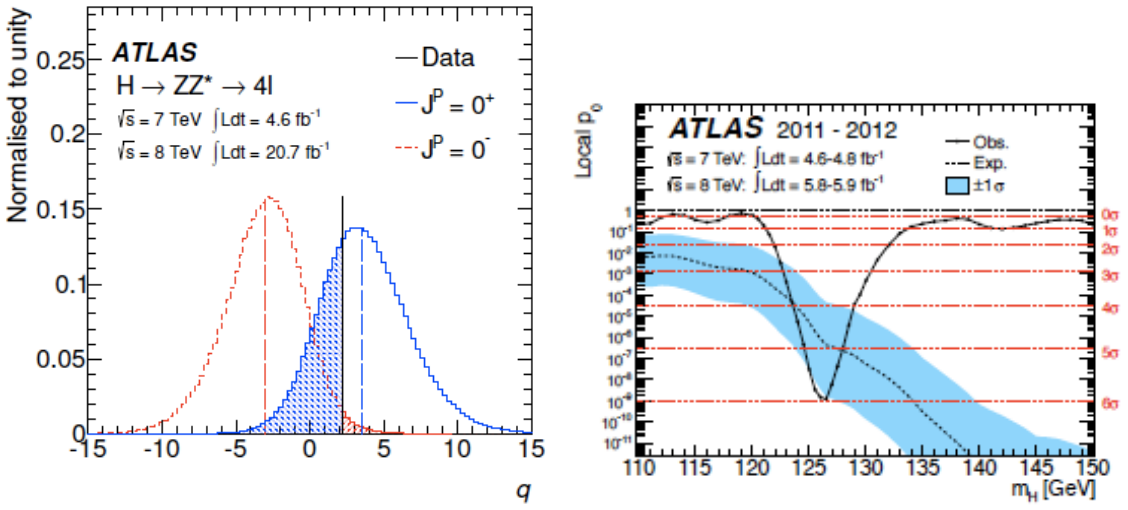
Chapter 1

Introduction

In March 14, 2013 at Moriond-QCD meeting, preliminary new results by ATLAS and CMS collaborations from CERN (*the* European Organization for Nuclear Research) made the particle physics community more certain that the new resonance in the mass range: 125-127 GeV, in the search of *the* Standard Model (SM) Higgs boson, is *a* Higgs boson!. It was confirmed that the new discovered particle is of spin-0 nature with positive parity (0^+), which are two fundamental criteria of *a* Higgs boson consistent with the SM [1, 2, 3, 4, 5]. However, it remains an open question whether this particle is *the* Higgs boson predicted by the Standard Model or the lightest boson predicted by some Beyond Standard Model (BSM) theories (such as Supersymmetric theories for example). In order to identify this particle, the other properties of *the* SM Higgs boson should be studied. For example, the decay rate to allowed particles (as WW and ZZ, bb and $\tau\tau$) must be compared with the SM predictions. Nevertheless, the collection of more data is fairly complicated, since the detection of such boson is very rare. For this reason, the LHC (Large Hadron Collider) is shut down for two years, and it will be restarted in 2015 at its nominal energy 13-14 TeV. This will confirm the Standard Model predictions or will open a new area Beyond Standard Model physics.



(a) (b)

Figure 1.1: **a)** Expected distributions of $q = \log(\mathcal{L}(J^P = 0^+)/\mathcal{L}(J^P = 0^-))$. The observed value is indicated by the vertical solid line and the expected medians by the dashed lines, see ref. [1]. **b)** Signal of Higgs boson observed at ATLAS, m_H is $126.0 \pm 0.4(\text{stat}) \pm 0.4(\text{sys})$ GeV, see ref. [2].

The Standard Model is one of the most successful gauge theories, it concerns three fundamental forces of nature except *gravity*: *electromagnetic*, *weak* and *strong nuclear interactions*. It describes the interactions between three families of quarks and leptons which are mediated by twelve gauge bosons: the eight gluons, the two $W^{\pm}$ bosons, the Z boson and the photon. The massive gauge bosons ($W^{\pm}$ and Z) acquire mass after spontaneous symmetry breaking via the Brout-Englert-Higgs mechanism [6, 7], which requires the existence of the famous Higgs boson. Each sector of the SM was developed separately in the context of quantum field theories. The first successful quantum field theory was the *Quantum ElectroDynamics* or QED, which became the first pillar of the SM and the template theory for which any theory describing a new interaction is inspired. It shows a successful marriage between the quantum theory and special relativity, and answers the main criticism that Einstein conducted over the quantum theory and its incapability to describe the matter reality in space time[8]. This theory is an abelian gauge theory, it describes how light and matter interact (the interaction of the photon with any charged particles). Its covariant formulations leads to its renormalizabilty to all orders of perturbation theory, which makes it "the jewel of physics" as Richard Feynman has called it [9]. The *Weak nuclear interaction* is the mechanism responsible for the the radiative decay and the nuclear fusion of the subatomic particles, it is the only interaction able to change the flavor of the quarks and the only interaction that violates the parity and the CP symmetry. There were many attempts to describe this interaction, a famous one was the Fermi theory. However, this interaction is well understood in the context of the Standard Model of *Electro-Weak interaction* (EW). The latter is the theory that unifies the weak and the electromagnetic interactions as two aspects of the same force, which is called the electroweak force. The strong interaction or the color force is described by the *Quantum Chromodynamics* or QCD. It describes the interactions of the quarks and the gluons inside the hadrons. It is a non-Abelian gauge theory which belongs to a large group of renormalizable gauge theories called the Yang-Mills theories [10].

To allow these theories to make accurate predictions, higher order (NLO, NNLO, ...) corrections are strongly needed especially in the new area of the LHC. However, perturbative quantum field theories lead to ultraviolet divergences in the loop diagrams. Such divergences have been discovered first in Quantum Electrodynamics, where many physicists have been ready to change some fundamental principle of physics to avoid these divergences. This problem have been solved by Bethe, Feynman, Schwinger, Tomonaga and Dyson [11, 12] and others by introducing the procedure of *renormalization*. They have shown that by the redefinition of some physical quantities as the mass, the couplings ...etc, one can get finite and sensible results to experiment. This procedure have solved the problem of the ultraviolet divergences in QED and have led to predictions agree with experiment to 8 significant digits, which is one of the most accurate calculation in all science. Despite the success of the renormalization program, many physicists viewed renormalization as an ad-hoc procedure which is justified only by its physically sensitive results. However, this idea was revolutionized by K. Wilson in the 1970's[13]. According to this new point of view, renormalization is just a simple parameterization of the sensitivity of low energy physics to high-energies physics, which means that renormalizable field theories are effective field theories, and explain why nature is described approximately! by renormalizable theories.

The renormalizabilty of the Yang-Mills theories for both broken and unbroken symmetries was proven by 'tHooft and Veltman in ref. [14, 15, 16].
The brilliant idea of renormalization is to provide a relationship between the parameters of the theory at high and low energy scales, in such way that the infinities are absorbed at high energy. The energy scale is an *unphysical scale* (it is called *renormalization scale*), since it is introduced to regularize the loop integrals by means of the *regularization methods* (Pauli-Villars, dimensional regularization ....), then the physical observable will depend on such scale if their perturbative expansion on the coupling constants is not taken into account at all orders. On top of that the coupling constant may not be small enough at given scales to provide precise results at fixed order of the expansion. Since the calculation of the physical observable at all orders is technically impossible due the dramatic increase of the Feynman diagrams and their complicated structure, we limit ourselves to some fixed orders (NLO for example) and the errors has to be included in the theoretical uncertainties. In the last decades a huge progress has been made in loop calculations (especially one-loop with multi-particles) in QCD and in EW. Nevertheless, the particle physics community wishes to enlarge the NLO and NNLO (and beyond!) calculations by including both QCD and EW radiative corrections. Here, we show a preliminary version of *les Houches wish-list* of some processes proposed in les Houches workshop of this year[1]:

Wish-list Part 1: EW gauge bosons (V=$W^{\pm}$, Z)

Process	Desired	Motivations
V	dσ(lept. V decay)@ NNNLO QCD + NLO EW MC@NNLO	Precision EW PDFs
$V+j$	dσ(lept. V decay)@ NNLO QCD + NLO EW	$Z+j$ for gluon PDF $W+c$ for strange PDF
$V+jj$	dσ(lept. V decay)@ NNLO QCD + NLO EW	Study of systematic of $H+jj$ final state
VV'	dσ(V decays)@ NNLO QCD + NLO EW	Off-shell leptonic decays TGCs (triple gauge couplings)
gg$\rightarrow$ VV	dσ(V decays)@ NLO QCD	bkg to $H \rightarrow VV$ TGCs
$V\gamma$	dσ(V decays)@ NNLO QCD + NLO EW	TGCs
$Vb\bar{b}$	dσ(lept. V decay)@NNLO QCD	bkg for $VH \rightarrow b\bar{b}$
$VV'\gamma$	dσ(V decays)@NLO QCD + NLO EW	QGCs (quartic gauge couplings)
$VV'V''$	dσ(V decays)@NLO QCD + NLO EW	QGCs, EWSB
$VV'+j$	dσ(V decays)@NLO QCD + NLO EW	bkg to H, BSM searches
$VV'+jj$	dσ(V decays)@NLO QCD + NLO EW	QGCs, EWSB

[1]The eighth les Houches workshop took place in les Houches (France), it consist of two sessions. Session I: 3-12 June 2013 with emphasis on SM-related issues. Session II: 12-21 June 2013 with emphasis on New-Physics searches, see `http://phystev.in2p3.fr/Houches2013/`.

Wish-list Part 2: Jets and heavy quarks:

Process	Desired	Motivations
$t\bar{t}$	$d\sigma$(top decays) @NNLO QCD + NLO EW	Precision top/QCD, gluon PDF, effect of extra radiation at high rapidity, top asymmetries
$t\bar{t}+j$	$d\sigma$(NWA top decays) @NLO QCD + NLO EW	Precision top/QCD, top asymmetry
dijet	$d\sigma$@ NNLO QCD+ NLO EW	Obs: inclusive jets, dijet mass PDF fits (gluon at high x) fit of the α_s x section, see [3]
$3j$	$d\sigma$@NNLO QCD + NLO EW	α_s at high p_T dom. uncertainty: scales see [4]
$\gamma+j$	$d\sigma$@NNLO QCD + NLO EW	gluon PDF, $\gamma+b$ for bottom PDF

Wish-list Part 3: Higgs (V=$W^{\pm}$, Z):

Process	Desired	Motivations
H	$d\sigma$ @NNNLO QCD + NLO EW MC@NNLO finite quark mass effects@NNLO	Higgs branching ratios and couplings
$H+j$	$d\sigma$ @NNLO QCD + NLO EW finite quark mass effects@NLO	H p_T
$H+2j$	$d\sigma$ @NNLO QCD + NLO EW	Higgs couplings
$H+V$	with H→ bb dσ@NNLO QCD + NLO EW	Higgs couplings
$t\bar{t}H$	$d\sigma$(NWA top decays) @NLO QCD + NLO EW	Top Yukawa coupling
HH	$d\sigma$@NNLO QCD finite quark mass effects dσ@NNLO QCD	Higgs self coupling

This thesis is in the stream of the `GOLEM` project or *General One Loop Evaluator of Matrix Elements*. Initially, this programs was designed for the automation of one loop QCD corrections. It is based on the Feynman diagrammatic approach, where the one-loop diagrams are reduced to basic integrals by means of a traditional reduction method, called the *Golem reduction*. It contains a library of all the building blocks of one-loop calculation, called the `Golem95` library. The calculation of one-loop amplitudes in this framework is organized as the following: 1) generate all contributing Feynman diagrams by means of `QGRAF` [17] or `FeynArts`[18], 2) separate and perform the color algebra, 3) project on helicity amplitudes, 4) reduce the one-loop diagrams to some set of basic integrals with up to 4-external legs by means of *the* Golem reduction[19], 5) evaluate these integrals with the *Golem library* (`Golem95`)[20, 21].

The Golem reduction is performed as the following: each individual Feynman graph is written as a combination of form factors times some Lorentz structures, then each form factor is reduced to a particular redundant basic integrals. Certainly, if the form factors are decomposed in term of only master integrals, i.e the scalar integral up to four-external legs in n-dimensions, inverse of Gram determinants ($\det(G)$) will appear up to certain powers in the

coefficients of the decomposition. Whereas the singularities due to the vanishing of $\det(G)$ are spurious, they might embarrass the numerical stability if the later one becomes arbitrary small, which is one of the major challenges of one-loop calculations. In Golem, this trouble is avoided by choosing a set of redundant basic integrals, which includes scalar and tensorial integrals in n or more dimensions $(n+2,\ n+4)$ instead of the *master basis of integrals*. The later one form a basis in the mathematical sense, it contains only scalar integrals in n-dimensions with up to four propagators. However, the former integrals do not form a basis in the mathematical senses, they can be expressed in term of the master integrals. The Golem choice of basic integrals guarantees that the coefficients of the expansion are free of any inverse of Gram determinant. `Golem95` set of basic integrals is made, apart some trivial one and two point functions, of $I_3^n(j_1,\cdots,j_3)$, $I_3^{n+2}(j_1)$, $I_4^{n+2}(j_1,\cdots,j_3)$ and $I_4^{n+4}(j_1)$, where the lower index indicates the number of external legs, the upper index indicates the dimension of space-time and the argument $j_1,\cdots,j_i$ means that at most i Feynman parameters appear in the numerator (i can be zero which correspond to scalar integrals). The strategy of avoiding the Gram determinant spurious singularity is the following: *in the phase space region where the* $\det(G)$ *becomes large enough, the extra elements of Golem set (the redundant integrals) are reduced to the master integrals and computed analytically in term of logarithms and dilogarithms; and in the phase space region where* $\det(G)$ *becomes arbitrary small (problematic region), the extra elements of Golem95 are used as irreducible blocks expressed as one-dimensional integral representations which are explicitly free of any inverse of Gram determinants, and leads to numerically stable results.*

On the other hand, there are other techniques of one-loop reduction, which reduce the full amplitude at once (without evaluating any Feynman diagram). They are based on the *generalized unitarity cuts* of the scattering amplitude[22, 23, 24, 25, 26], or processing the reduction at the *integrand level* [27, 28, 29]. In these approaches, the full amplitude is reduced to the set of master integrals weighted by some coefficients plus a rational term. Then, the full amplitude is calculated once these coefficients and the rational term are extracted, since the master integrals are provided by one-loop libraries as `LoopTools`[30], `OneLoop`[31], ...etc. However, the coefficients of the master integrals in this approach contain inverse of Gram determinants, which hamper the numerical stability if these determinants become arbitrary small!. In fact, the Golem library can be used as a library of master integrals as well as a library of the redundant integrals introduced above, since the master integrals correspond to some form factors which are the building blocks of this library. Then, it can be used as library for programs based on the generalized unitarity cuts or on the reduction at the integrand level. In the problematic region ($\det(G) \to 0$), the later approaches breakdown. One then can improve such methods by making use of `Golem95`. This is can be done by reconstructing the numerator of the full amplitude by means of the tensorial reconstruction at the integrand level introduced in [32], which allows to express the full amplitude as a sum of tensorial integrals up to the highest power of the loop momentum in the numerator (which cannot exceed the number of the one-loop internal propagators in renormalizable gauge theories). From there, each tensorial integral is projected into the Golem redundant basic of integrals by means of Golem reduction. Thence, the unitarity inspired approach can be improved in the problematic region. There are automated one-loop calculation programs using generalized unitarity methods on the market. Each program has a rescue system which enables to recompute in another way some phase space points which

have been marked as *bad* (because of a loss of precision). The rescue system of the `GoSam` program [33], which belongs to the list of automated loop calculation programs, relies on the ability of the `Golem95` to avoid the negative powers of the Gram determinants.

The `Golem95`, initially was designed for QCD, it did not include basic integrals with internal masses. The generalization to cases involving arbitrary internal masses -to extends its range of use- is one of the main purposes of this thesis. To handle the $\det(G)$ issues, we provide a one-dimensional integral representation rather than relying on Taylor expansions in powers of $\det(G)$. The later approach may be thought a priori better in term of CPU time and accuracy, however the order up to which the expansion shall be pushed may happen to be rather large. Originally, `Golem95` uses multi-dimensional numerical integration of the three-point and the four-point functions, or more precisely a hypercountour deformation which would be numerically more stable. Yet the computation of these multiple integrals was both slow and not very precise. The one-dimensional integral representation is more efficient in term of CPU time and accuracy. Finding such representation for `Golem95` basic integrals, especially the three-point and the four-point integrals, in the most general case, i.e. involving real and /or complex masses is the main subject of this thesis.

In chapter 2, we will give a brief introduction to gauge theories. In the first two sections we will discuss the symmetries and the conservations laws, where we will focus on the gauge symmetries. In the last two sections, we will present the QCD and the Standard Model of particle physics.
In chapter 3, we will study the main feature of perturbation field theory. In the first section, we will give a general presentation of the S-matrix theory; its definition, its properties, its analyticity and its relation to Feynman diagrams. In the second section, we will discuss the analyticity of the scattering amplitudes by giving the necessary and sufficient conditions for the occurrence of singularities of individual one-loop Feynman graphs (Landau conditions), and we will give general criteria to determine the soft and collinear divergences of these diagrams. In the third section we will present some consequences of the unitarity and the causality on the scattering amplitude computation (dispersion relation) and we conclude by giving the Cutkosky cutting rules.
In chapter 4, we will present two reduction methods based on the Feynman diagrammatic approach: the *Passarino-Veltman* reduction and the *Golem* reduction; and two reduction methods based on the inspired unitarity approaches: *Ossola-Pittau-Papadopoulos* and the *generalized unitarity*. We will close this chapter by presenting an approach uses the *tensorial reconstruction at the integrand level* (which is a unitarity inspired approach) and the *Golem reduction* (which is a Feynman diagrammatic approach) to improve the unitarity approach for vanishing Gram determinants.
In chapter 5 (which is the main part of this thesis), we will derive stable one dimensional integrals representation for each `Golem95` basic integral, where will focus on the three and four point functions in the general massive case. And in chapter 6, we will present briefly the `Golem95` program.

Introduction (Français)

A Moriond-QCD, le 14 Mars 2013, de nouveaux résultats préliminaires par les collaborations ATLAS et CMS du CERN (Organisation européenne pour la recherche nucléaire) ont contribué à convaincre la communauté des physiciens des particules que la nouvelle résonance dans la gamme de masse: 125-127 GeV, est le boson de Higgs du Modèle Standard (MS)!. Il a été confirmé que la nouvelle particule découverte est de spin-0 avec une parité positive (0^+, Fig. (1.1), qui sont deux critères fondamentaux en cohérence avec le boson de Higgs du MS [1, 2, 3, 4, 5]. Cependant, il reste une question ouverte de savoir si cette particule est vraiment *le* boson de Higgs prédit par le Modèle Standard ou peut être le plus léger boson de Higgs prédit par certaines théories au-delà du Modèle Standard (comme les théories Supersymétriques par exemple). Afin d'identifier cette particule, les autres propriétés du boson de Higgs du MS devront être étudiés. Par exemple, les taux de désintégration permis (comme WW et ZZ , bb et $\tau\tau$) doivent être comparés avec les prédictions du MS. Néanmoins, la collecte de données supplémentaires est assez compliquée car la détection d'un tel événement est très rare. Pour cette raison, le LHC (Large Hadron Collider) redémarrera en 2015 à son énergie nominale (13-14 TeV). Cela va confirmer les prédictions du Modèle Standard ou ouvrir une nouvelle zone pour la physique au-delà du Modèle Standard.

Le Modèle Standard est l'une des théories de jauge les plus réussies, il concerne trois forces fondamentales de la nature (sauf *la gravité*): *l'interaction électromagnétique*, *l'interaction faible* et *l'interaction forte*. Il décrit les interactions entre les trois familles de quarks et leptons et les douze bosons de jauge: les huit gluons, les deux bosons $W^{\pm}$, le boson Z et le photon. Les bosons de jauge massifs ($W^{\pm}$ et Z) acquièrent leur masse après la brisure spontanée de symétrie via le mécanisme de Brout-Englert-Higgs [6, 7], ce qui nécessite l'existence du fameux boson de Higgs. Chaque secteur du MS a été développé séparément dans le contexte de la théorie quantique des champs. La première théorie quantique des champs construite est l'*Electrodynamique Quantique* ou QED, qui est devenue le premier pilier du MS et la théorie modèle dont toute théorie décrivant une nouvelle interaction s'est inspirée. Elle montre un mariage réussi entre la théorie quantique et la relativité restreinte, et répond à la principale critique qu'Einstein avait émise sur la théorie quantique et de son incapacité à décrire la réalité de la matière dans l'espace-temps [8]. Cette théorie est une théorie de jauge abélienne, elle décrit comment la lumière et la matière interagissent (l'interaction des photons avec les particules chargées). Sa formulation covariante conduit à sa renormalisabiltié à tous les ordres dans la théorie des perturbations. L'*interaction nucléaire faible* est le mécanisme responsable de la désintégration radiative et à l'origine de la fusion nucléaire dans les étoiles: c'est la seule interaction capable de changer les saveurs de quarks et la seule interaction qui viole la symétrie CP. Il y a eu de nombreuses tentatives pour décrire cette interaction, une célèbre tentative a été la théorie de Fermi. Cependant, cette interaction est bien comprise dans le contexte du Modèle Standard de l'*interaction électrofaible* (EW). Cette dernière est la théorie qui unifie les interactions faibles et électromagnétiques comme deux aspects de la même force. L'interaction forte est décrite par la *Chromodynamique Quantique* ou QCD, elle décrit l'interaction des quarks et des gluons à l'intérieur des hadrons. C'est une théorie non-abélienne qui appartient au fameux groupe de théories de jauge renormalisables appelées *théories de Yang-Mills* [10].

Pour permettre à ces théories de faire des prédictions précises, des corrections d'ordres supérieurs (NLO , NNLO , ...) sont fortement nécessaires, surtout dans des calculs de précision pour le LHC. Cependant, les théories quantiques des champs perturbatives conduisent à des divergences ultraviolettes. Ces divergences ont été découvertes d'abord en QED, où de nombreux physiciens ont été prêts à changer quelques principes fondamentaux de la physique pour éviter ces dernières!. Ce problème a été résolu par Bethe, Feynman, Schwinger, Tomonoga et Dyson [11, 12] et d'autres par l'introduction de la procédure de *renormalisation*. Ils ont montré que par la redéfinition de certains quantités physiques comme la masse, les couplages ... etc, on peut obtenir des résultats finis et comparables à l'expérience. Cette procédure a permis de résoudre le problème des divergences ultraviolettes en QED et a conduit à des prédictions en accord avec 8 chiffres significatifs avec des résultats expérimentaux, ce qui est l'une des théories les mieux vérifiées dans toutes les sciences. Malgré le succès du programme de renormalisation, de nombreux physiciens ont vu la renormalisation comme une procédure ad-hoc qui ne se justifie que par ses résultats comparables à l'expérience. Toutefois, cette idée a été révolutionné par K. Wilson dans les années 1970 [13]. Selon ce nouveau point de vue, la renormalisation est un simple paramétrage! de la sensibilité de la théorie à basse énergie à la physique à haute-énergies, ce qui justifie pourquoi les théories des champs renormalisables sont efficaces, et explique pourquoi la nature est décrite par de telles théories. Le renormalisabiltié des théories de Yang-Mills spontanément brisées ou non-brisées a été prouvée par 'tHooft et Veltman dans ref. [14, 15, 16].
L'idée géniale de la renormalisation est de fournir une relation entre les paramètres de la théorie à des échelles à haute et à basse énergies, de telle façon que les infinis à haute énergie sont absorbés dans quelques paramètres. L'échelle de l'énergie est une *échelle non-physique* (elle est appelée *échelle de renormalisation*), car elle est introduite pour régulariser les intégrales de boucles au moyen des *méthodes de régularisation* (comme Pauli-Villars, la régularisation dimensionnelle .... etc). Les observables physiques ne dépendraient pas de cette échelle si leur développement perturbatif en fonction des constantes de couplage était calculé à tous les ordres. Etant donné que le calcul des observables physiques à tous les ordres est techniquement impossible, en raison de l'augmentation spectaculaire des diagrammes de Feynman et leur structure complexe, nous nous limitons à certains ordres fixés (par exemple NLO) et les erreurs doivent être prise en compte dans les incertitudes théoriques .

Dans les dernières décennies, un énorme progrès a été réalisé dans le calcul de boucles (notamment le calcul à une-boucle avec plusieurs particules externes) en QCD et en EW. Néanmoins, les physiciens des particules souhaiteraient élargir la liste des processus calculés aux ordres NLO, NNLO et au-delà! en incluant à la fois les corrections radiatives de QCD et EW. Dans les tableaux "*Wish-list Part1, Part2 et Part3*" (voir l'introduction en anglais), nous montrons une version préliminaire de *les Houches Wish list 2013*.

Cette thèse est dans le cadre du projet `GOLEM` ou *Gneral One-Loop Evaluator of Matrix Element*. Initialement, ce programme a été conçu pour l'automatisation des corrections radiatives à une boucle de la QCD. Il est basé sur l'évaluation des diagrammes de Feynman, où les diagrammes à une boucle sont réduits à des intégrales de base au moyen d'une méthode de réduction traditionnelle appelée *réduction à la Golem*. Il possède une bibliothèque qui

contient tous les *building-blocks* de n'importe quel calcul à une boucle avec jusqu'à 6-pattes externes, cette dernière est appelée la bibliothèque `Golem95`. Le calcul des amplitudes à une boucle dans ce cadre est organisé comme suit: 1) générer les diagrammes de Feynman au moyen de `QGRAF` [17] ou `FeynArts`[18], 2) séparer et effectuer l'algèbre de couleurs, 3) réduire les diagrammes à une boucle en un ensemble d'intégrales de base ayant jusqu'à 4 pattes externes en utilisant la réduction à la Golem [19], 4) évaluer ces intégrales en utilisant la bibliothèque `Golem95`[20, 21].

La réduction à la Golem est effectuée de la manière suivante: chaque diagramme de Feynman est écrit comme une combinaison de *facteurs de forme* fois des structures de Lorentz, chaque facteur de forme est exprimé en fonction des intégrales de base redondantes. Certes, si les facteurs de forme sont décomposés en terme de seulement les *master intégrales* (les intégrales scalaire à n-dimensions ayant jusqu'à quatre pattes externes, ces intégrales forment une base au sens mathématique), des puissances négatives du déterminant de Gram ($\det(G)$) apparaîtront dans les coefficients de la décomposition. Les singularités factices dues à $\det(G) \to 0$ peuvent réduire la stabilité numérique et ce problème est l'un des défis du calcul à une boucle.

Pour `Golem`, ce problème est évité en choisissant une base redondante d'intégrales de base, qui contient des intégrales scalaires et tensorielles en n-dimensions ou plus $(n{+}2, n{+}4)$, au lieu de la base des *master* intégrales. Ce choix d'intégrales de base garantit que les coefficients de la réduction sont exempts de toute puissance de l'inverse du déterminant de Gram. L'ensemble des intégrales de base de `Golem95` est donnée par: $I_3^n(j_1, \cdots, j_3)$, $I_3^{n+2}(j_1)$, $I_4^{n+2}(j_1, \cdots, j_3)$ et $I_4^{n+4}(j_1)$ et plusieurs fonctions à 2- et 1-point, où l'indice inférieur indique le nombre de pattes externes, l'indice supérieur indique la dimension de l'espace-temps et l'argument $j_1, \cdots, j_i$ signifie qu'au plus i paramètres de Feynman apparaissent dans le numérateur (i peut être égal zéro, ce qui correspond aux intégrales scalaires). La stratégie pour éviter les singularités factices induites par l'annulation de $\det(G)$ est la suivante: dans la région de l'espace de phase où le $\det(G)$ devient suffisamment grand, les intégrales redondantes sont réduites aux *master* intégrales, et calculées analytiquement en terme de logarithmes et dilogarithms. Dans la région de l'espace de phase où $\det(G)$ devient arbitrairement petit (*région problématique*), les éléments supplémentaires (les intégrales redondantes) de `Golem95` sont utilisés sous forme de blocs irréductibles exprimées en terme de représentations intégrales unidimensionnelles qui sont explicitement libres de tout inverse de $\det(G)$, ce qui conduit à des résultats numériquement stables.

D'autre part, il existe d'autres techniques de réduction d'une boucle, qui décompose l'amplitude complète (sans évaluer des diagrammes de Feynman). Elles sont basées sur les coupures d'unitarité de l'amplitude de diffusion [22, 23, 24, 25, 26], où la décomposition se fait au niveau de l'intégrant [27, 28, 29]. Dans ces approches, l'amplitude complète est réduite à l'ensemble des *master* intégrales, qui contient seulement des intégrales scalaires ayant jusqu'à quatre pattes externes en n-dimensions, multipliées par des coefficients plus un terme rationnel. Ensuite, l'amplitude est calculée une fois ces coefficients et le terme rationnel extraits, puisque les *Master* intégrales sont fournis par les bibliothèques à une boucle comme `LoopTools` [30], `OneLoop` [31], etc ... Cependant, les coefficients de ces intégrales dans ces approches sont proportionnelle à des puissance négatives de $\det(G)$, ce qui gêne la stabilité numérique si ces déterminants deviennent suffisamment petits. En fait, la bibliothèque Golem peut être utilisé comme une bibliothèque des *master* intégrales ainsi qu'une

bibliothèque d'intégrales redondantes présentées ci-dessus. Alors, elle peut être utilisée en tant que bibliothèque pour des programmes basés sur la réduction des coupures généralisées ou sur la réduction au niveau de l'intégrant. Dans la région problématique ($\det(G) \to 0$), les deux dernières approches s'effondrent! On peut alors améliorer ces méthodes en utilisant `Golem95`. Ceci peut être fait en reconstruisant le numérateur de l'amplitude complète au moyen de la *reconstruction tensorielle au niveau de l'intégrant* introduite dans [32], qui permet d'exprimer l'amplitude totale comme une somme des intégrales tensorielles avec un rang jusqu'à la plus haute puissance de l'impulsion tournant dans la boucle (qui ne peut pas dépasser le nombre de propagateurs internes dans les théories renormalisables). A partir de là, chaque intégrale tensorielle est projetée dans la base des intégrales redondantes au moyen de la réduction à la Golem. Par là, l'approche inspirée de l'unitarité est améliorée dans la région problématique. Chaque programme automatique du calcul à une boucle, qui existe sur le marché, dispose d'un système de sauvetage qui permet de recalculer d'une autre manière les points de l'espace de phase qui ont été marqués comme *mauvais* (à cause de la perte de précision). Le système de sauvetage de `GoSam` [33], qui appartient à la liste des programmes automatiques du calcul de boucles, repose sur la capacité de `Golem95` pour éviter la puissance de l'inverse des déterminants de Gram.

La bibliothèque `Golem95`, a été initialement conçue pour la QCD, elle ne comprend pas des intégrales de base avec des masses internes. La généralisation de cette bibliothèque pour des cas avec des masses arbitraires internes (les masses complexes sont incluses) est l'un des objectifs principaux de cette thèse. Pour éviter les problèmes dus à $\det(G) \to 0$, nous offrons une représentation intégrale unidimensionnelle plutôt que de s'appuyer sur des développements de Taylor en puissances de $\det(G)$. La dernière approche peut être considérée à priori comme meilleure en terme de temps CPU et de précision, cependant l'ordre jusqu'à laquelle l'expansion doit être poussée peut être assez grand. A l'origine, `Golem95` utilise l'intégration numérique multidimensionnelle des fonctions à quatre et à trois points, ou plus précisément une déformation d'hypercontour qui serait numériquement plus stable. Pourtant, le calcul de ces intégrales multiples était à la fois lent et pas très précis. La représentation intégrale unidimensionnelle est plus efficace en terme de temps CPU et de précision. Trouver une telle représentation pour toutes les intégrales de base de `Golem95`, en particulier, les intégrales à trois points et les intégrales à quatre points dans le cas le plus général, est l'objectif principal de cette thèse.

Dans le chapitre 2, nous donnerons une brève introduction aux théories de jauge. Dans les deux premières sections, nous allons discuter les symétries et les lois de conservation, où nous nous concentrerons sur les symétries de jauge. Dans les deux dernières sections, nous allons présenter la QCD et le Modèle Standard de la physique des particules.
Dans le chapitre 3, nous étudierons les principales caractéristiques de la théorie des champs perturbative. Dans la première partie, nous allons donner une présentation générale de la théorie de la matrice S, sa définition, ses propriétés, son analyticité et sa relation avec les diagrammes de Feynman. Dans la deuxième section, nous allons discuter l'analyticité des amplitudes de diffusion en donnant les conditions nécessaires et suffisantes pour l'apparition de singularités dans les différents diagrammes de Feynman à une boucle (conditions de Landau), et nous donnerons les critères généraux pour déterminer les divergences molles et

colinéaires de ces intégrales. Dans la troisième section, nous allons présenter quelques conséquences de l'unitarité et de la causalité sur le calcul de l'amplitude de diffusion (relation de dispersion) et nous conclurons par les règles de coupure de Cutkosky.
Dans le chapitre 4, nous présenterons deux méthodes de réduction basées sur l'approche des diagrammes de Feynman: la réduction de *Passarino-Veltman* et la réduction à la *Golem*; et deux méthodes de réduction basées sur l'approche de l'unitarité: *Ossola-Pittau-Papadopoulos* et la méthode des *coupures d'unitarité généralisées* . Nous terminerons ce chapitre en présentant une approche qui utilise la *reconstruction tensorielle au niveau de l'intégrant* (qui est une approche inspirée de l'unitarité) et la réduction à la *Golem* (qui est une approche basée sur les diagrammes de Feynman) pour améliorer l'approche d'unitarité dans la région de l'espace de phase problématique.
Dans le chapitre 5 (qui représente la partie principale de cette thèse), nous dériverons une représentation intégrale unidimensionnelle stable pour chaque intégrale de base de `Golem95`, où nous nous concentrerons sur les fonctions à trois et quatre point dans le cas massif le plus général. Et dans le chapitre 6, nous présenterons brièvement le programme `Golem95`.

Chapter 2

Introduction to gauge theories

Contents

Gauge theories play a crucial role in particle physics, they are the most successful theories that describes the dynamics of elementary particles. The term gauge was introduced by Herman Weyl in his attempt to unify general relativity and electromagnetism [1][34], it refers to the redundant degrees of freedom in the lagrangian of a given theory, where the transformation between possible gauges keep the lagrangian invariant, we say that the theory is invariant under gauge symmetry. The gauge transformation form a symmetry group called gauge group which forms a Lie group. To each generator of this group, it is associated a massless vector field which is responsible for the mediation of the force of interaction between the fields of the theory.
There are two type of gauge transformation: the global gauge symmetry where the transformation is identically performed at any point of space-time and the local transformation which is space-time dependent. The Lie group associated to a given theory may be Abelian (commutative) and the theory is called Abelian gauge theory, a famous example of such theories is Quantum Electrodynamics. If the gauge transformation forms a non-Abelian

[1]Nevertheless, the invariance under such transformation was known long time ago before introducing this word, where the earliest field theory having a gauge invariance was the Maxwell theory of electromagnetism[35]. Also, Hilbert have shown that the Einstein equations are invariant under coordinate transformation[36].

group (non-commutative), the theory is called non-Abelian gauge theory, a well know example is Quantum Chromodynamics which is a special case of a large category of gauge theories called the Yang-Mills theories. The later one was introduced by Chen Ning Yang and Robert Mills [10] in the context of understanding the strong interaction confining together the neutron and the proton in the atomic nuclei[2].
To each generator of the Lie group, it is associated a vector field which appears in the lagrangian of the corresponding theory as massless field to insure the gauge invariance. Such fields are associated to gauge bosons after quantization. If certain bosons are massive, then gauge invariance of the lagrangian is not satisfied and the symmetry must be broken by means of the procedure of spontaneous symmetry broken (SSB) and the bosons (massless before SSB) get masses by means of the Brout-Englert-Higgs mechanism.

In this chapter, we give an introduction to gauge theories. In the first two sections we discuss the symmetries and the conservations laws, where we focus on the gauge symmetries. In the last two sections, we present the QCD and the Standard Model of particle physics.

2.1 Symmetries and conservation laws

The fundamental object of field theory is the lagrangian density $\mathcal{L}$ which is a Lorentz invariant function of the fields $\phi_i(x)$ and their gradients $\partial_\mu \phi_i$

$$\mathcal{L}(\phi_k(x), \partial\phi_k(x)). \tag{2.1.1}$$

where the fields $\phi_k(x)$ stand for all the fields of the theory (for arbitrary spin).
Let us define the action S which is given by the four-dimensional integral over time and space

$$S = \int_{t_1}^{t_2} dt \int d^3x \, \mathcal{L}(\phi_k(x), \partial\phi_k(x)) \tag{2.1.2}$$

where the variation of these fields at the time t_1 and t_2 is chosen to be zero.
The Hamilton principle of stationary action states that the real path chosen by the particle between the position t_1 and the position t_2 is the path that makes the action S in Eq. (2.1.2) stationary, i.e. $\delta S = 0$. Then, the equations of motion of the system described by this lagrangian follow this principle which leads to the famous *Euler-Lagrange* equations of motion

$$\frac{\delta \mathcal{L}}{\delta \phi_k} = \partial_\mu \frac{\delta \mathcal{L}}{\delta \left(\partial_\mu \phi_k\right)} \tag{2.1.3}$$

These equations specify the dynamics of the system, they are Lorentz invariant which implies that the lagrangian must be Lorentz scalar. The lagrangian formalism provide an elegant and very convenient way to extract the constant of motion in classical field theory. It has been shown by Noether (Noether theorem [37]) that starting from a Lorentz invariant lagrangian density, we can prove that each continuous symmetry for which the lagrangian and the equation of motion are invariant, leads to conservation theorems and constants of

[2]The idea of Yang and Mills was the generalization of the electromagnetism Abelian gauge invariance to non-Abelian symmetry of the isospin group $SU(2)$ where the protons and the neutrons comes in a doublet.

motion. We distinguish between two kind of symmetries: *external symmetries* and *internal symmetries*. The former symmetries involve the space-time coordinate (through the fields $\phi_k(x)$). By making an infinitesimal translation $x'_\mu \to x_\mu + a_\mu$ (for a given four-vector a_μ) which let $\mathcal{L}$ invariant due to the homogeneity of space, one can prove that the four-momentum is conserved. Similarly, by making an infinitesimal rotation $x'_\mu \to x_\mu + \varepsilon_{\mu\nu}\, x_\nu$ (with $\varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu}$) leads to angular momentum conservation. The internal symmetries will be presented in the next paragraph.

2.2 Internal symetries: Gauge invariance

We call symetries which do not involve space-time coordinate, internal symmetries. As in the case of external symmetries, each of these symmetries is given by a field transformation which leaves the lagrangian density invariant. The field theories that leaves the lagrangian densities invariant under continuous groups of transformations are called gauge theories, where we distinguish between Abelian and non-Abelian gauge field theories:

2.2.1 Global gauge symmetry: Abelian case

The symmetry associated to the charge conservation is called *global gauge invariance*. It is defined by the phase transformation

$$\phi_i(x) \to \phi'_i(x) = e^{-i\, q_i\, \theta}\, \phi_i(x) \tag{2.2.4}$$

where q_i stand for the charge in e units (e stands for the charge of the positron for example) and θ is an arbitrary parameter. Since the parameter θ is independent of x, then the derivative of the field ϕ_i transforms as the field itself

$$\partial_\mu \phi_i(x) \to \partial_\mu \phi'_i(x) = e^{-i\, q_i\, \theta}\, \partial_\mu \phi_i(x) \tag{2.2.5}$$

The lagrangian density is made of product of fields ϕ_i, their hermitian conjugates $\phi_i^\dagger$ and their derivatives. Since the charge is conserved in every term of the Lagrangian, each term involving a given field must be multiplied by its hermitian conjugate. Then, the $\mathcal{L}$ is invariant under the transformation Eq. (2.2.4), or in other words it is independent of the phases of the fields ϕ_i, i.e.

$$\mathcal{L}(\phi_i, \partial_\mu \phi_i) = \mathcal{L}(\phi'_i, \partial_\mu \phi'_i) \tag{2.2.6}$$

For infinitesimal θ, the variation of the field is given by

$$\delta\phi_i(x) = \phi'_i(x) - \phi_i(x) \to -i\, \theta\, q_i\, \phi_i(x) \tag{2.2.7}$$

Under this transformation, the variation of the lagrangian must vanish and the equation of motion defined in Eq. (2.1.3) becomes

$$-i\theta\, \partial_\mu \left[\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi_i)}\, q_i \phi_i \right] = 0. \tag{2.2.8}$$

This equation shows that the current J_μ associated to this gauge transformation is conserved, i.e.

$$\partial_\mu J^\mu = 0 \tag{2.2.9}$$

$$J^\mu = -iq_i \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi_i)} \phi_i \tag{2.2.10}$$

where J^μ is called the Noether current.

The gauge transformation defined above forms a group since the elements $e^{-iq_i\,\theta}$ with the multiplication (the group law) satisfy the four requirement of the group: *1) closure 2) Associativity 3) Identity element and 4) Inverse element.* It is Abelian since these gauge transformations commute with each other. These transformations are defined by only one parameter θ, then the group is one-dimensional. This group is the $U(1)$ or the group of unitary transformation in one-dimension.

The charges q_i are the eigenvalues of an operator, called the charge operator which is defined by

$$\hat{Q} = \int d^3x\, J_0(x,t) \tag{2.2.11}$$

$$\frac{\partial}{\partial t}\hat{Q} = 0 \tag{2.2.12}$$

this operator is the only infinitesimal generator of the gauge group $U(1)$.

2.2.2 Local gauge symmetry: Abelian case

The local gauge symmetries consists of the same transformation as above, the only difference is that the parameter θ depends on the space-time coordinates. Consider the transformation

$$\phi_i(x) \to \phi_i'(x) = e^{-i\,q_i\,\theta(x)}\,\phi_i(x) \tag{2.2.13}$$

where θ is a given analytical function. For infinitesimal θ, we have

$$\delta\phi_i(x) = -iq_i\,\theta(x)\phi_i(x) \tag{2.2.14}$$

The terms of the lagrangian containing the fields and their hermitian conjugates are invariant under this transformation. However, the terms containing the derivatives are not invariant since

$$\partial_\mu\phi_i(x) \to \partial_\mu\phi_i'(x) = e^{-iq_i\,\theta(x)}\,\partial_\mu\phi_i(x) - iq_i\,(\partial_\mu\theta(x))\,e^{-iq_i\,\theta(x)}\,\phi_i(x).$$

the second term in this equation prohibits the derivative to be transformed as the field

$$\partial_\mu\phi_i(x) \not\to e^{-iq_i\,\theta(x)}\,\partial_\mu\phi_i(x) \tag{2.2.15}$$

To insure the local gauge invariance of the theory, one has to introduce a vector A_μ which must transform under the local gauge transformation as

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \frac{1}{e}\frac{\partial\theta(x)}{\partial x^\mu} \tag{2.2.16}$$

and define the quantity

$$D_\mu = \partial_\mu + ieq_i\, A_\mu \tag{2.2.17}$$

this quantity is called the covariant derivative, since it transforms as

$$D_\mu\phi_i(x) = e^{-iq_i\,\theta(x)}\, D_\mu\phi_i(x) \tag{2.2.18}$$

This procedure makes the lagrangian invariant under the local gauge invariance by means of the introduction of the vector A_μ which is interpreted as the boson field that mediates the interaction of the theory after quantization (the photon in the case of QED for example). So, we have to add a term to the $\mathcal{L}$ to describe the kinetic energy of these fields and which has to be gauge invariant. This term turns to be[3]

$$-\frac{1}{4}\, F_{\mu\nu}F^{\mu\nu} \tag{2.2.19}$$

where the strength tensor $F_{\mu\nu}$ is defined by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.2.20}$$

which is also gauge invariant.
The only mass term which can be added is of the form $-\frac{1}{2}\, m^2\, A_\mu\, A^\mu$, but it breaks the gauge invariance. Fortunately, the photon mass equals to zero and the Quantum electrodynamics is locally gauge invariant (again this transformation form a one-dimensional representation of the group $U(1)$).

2.2.3 Global gauge symmetry: non-Abelian case

The generalization of the global gauge symmetry to the non-Abelian case is quite straightforward. Let us consider the simplest non-Abelian gauge transformation which is the *isospin invariance*. The fields in this case are assumed to come in multiplets

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_n \end{pmatrix} \tag{2.2.21}$$

the gauge transformation is defined by

$$\phi \to \phi' = e^{-i\vec{L}\cdot\vec{\theta}}\phi, \tag{2.2.22}$$

[3]The kinetic term of the photon field can be derived from the Hamilton principle of stationary actions, see Bjorken [38].

where $\vec{\theta} = (\theta_1, \theta_2, \theta_3)$ the three parameters that specify the gauge transformation, $L_i (i = 1, 2, 3)$ are $n \times n$ matrices, they stand for the representation of the generators of $SU(2)$. In the case of isodoublet, say proton and neutron, $n = 2$ and the matrices L_i are equals to half of a Pauli matrices $L = \frac{1}{2}\tau$ (fundamental representation of $SU(2)$). In the case of isotriplet, say π^+, π^0 and π^- and $(L_i)_{kl} = -ic_{jkl}$ (the adjoint representation of $SU(2)$). The group $SU(2)$ has three generators T_i which satisfy the commutation relations

$$[T_j, T_k] = ic_{jkl}\, T_l \tag{2.2.23}$$

where the c_{jkl} are totally anti-symmetric.
For infinitesimal θ, we have

$$\delta\phi = -i\, \vec{L} \cdot \vec{\theta}\phi \tag{2.2.24}$$

In the case of isodoublet, this reads to

$$\delta\phi = -i\, \frac{\vec{\tau}}{2} \cdot \vec{\theta}\phi \tag{2.2.25}$$

In the case of isotriplet, this reads to

$$\delta\phi_i = c_{jkl}\theta_k\phi_l \tag{2.2.26}$$

It is quite straightforward to prove that the lagrangian is invariant under this transformation.

2.2.4 Local gauge symmetry: non-Abelian case (Yang Mills theories)

The generalization of $SU(2)$ to local gauge symmetry was first introduced by Yang and Mills in early 1954[10]. This idea was criticized by Pauli[39], since the quanta of Yang-Mills field must be massless in order to maintain the gauge invariance. This theory was neglected until when the idea that particles get masses from the spontaneous symmetry breaking was elaborated.

As we have seen above, the generators of the group obey

$$[T_j, T_k] = ic_{jkl}\, T_l \tag{2.2.27}$$

and the field

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_n \end{pmatrix} \tag{2.2.28}$$

transforms as

$$\phi \to \phi' = e^{-i\vec{L}\cdot\vec{\theta}(x)}\phi = U(\theta)\,\phi(x), \tag{2.2.29}$$

where $\theta_i(x) (i = 1, \cdots, N)$ are arbitrary functions of space-time and $L_i (I = 1, \cdots, N)$ are $n \times n$ matrices representing the generators of the group. The gradient of the field is

$$\partial_\mu\phi(x) \to U(\theta)\,\partial_\mu\phi(x) + \partial_\mu U(\theta)\,\phi(x) \tag{2.2.30}$$

As in the case QED, we want to define a covariant derivative D_μ which transforms as the field

$$D_\mu\phi(x) \to D'_\mu\phi(x)' = U(\theta)\, D_\mu\phi(x) \tag{2.2.31}$$

To do so, one has to introduce for each dimension of the group a vector field (the analogue of the photon) in order to keep the $\mathcal{L}$ invariant under the local gauge transformation. For $SU(N)$, the covariant derivative is

$$D_\mu\phi(x) = [\partial_\mu + ig\,\vec{L}\cdot\vec{W}_\mu(x)]\,\phi(x) \tag{2.2.32}$$

where g is the coupling constant and the vector field $\vec{W}_\mu$ has N components, i.e.

$$\vec{W}_\mu = (W^1_\mu(x), W^2_\mu(x), \cdots, W^N_\mu(x)) \tag{2.2.33}$$

To insure the local gauge invariance of the full lagrangian, the vector field $\vec{W}_\mu$ must transform as

$$\vec{L}\cdot\vec{W}'_\mu = U(\theta)\,[\vec{L}\cdot\vec{W}_\mu + \frac{i}{g}\,U^{-1}(\theta)\partial_\mu U(\theta)]\,U^{-1}(\theta) \tag{2.2.34}$$

one can prove that the infinitesimal transformation of the vector field is given by

$$\delta W^i_\mu(x) = \frac{1}{g}\partial_\mu\theta_i(x) + c_{jkl}\theta_k(x)W^l_\mu(x) \tag{2.2.35}$$

By analogy the kinetic energy term of the gauge boson is given by

$$\mathcal{L}_0 = -\frac{1}{4}\,G^i_{\mu\nu}G^{i,\mu\nu} \tag{2.2.36}$$

where the generalized field tensor is

$$G^j_{\mu\nu} = \partial_\mu W^j_\nu - \partial_\nu W^j_\mu + g c_{jkl}\, W^k_\mu W^l_\nu \tag{2.2.37}$$

$$\delta G^j_{\mu\nu} = c_{jkl}\theta_k\, G^l_{\mu\nu} \tag{2.2.38}$$

$\mathcal{L}_0$ is gauge invariant, $\delta\mathcal{L}_0 = 0$. As in the Abelian case, no mass term is allowed since $\vec{W}_\mu\cdot\vec{W}^\nu$ is not gauge invariant. Contrary to the abelian case, this lagrangian leads to self interaction of the vector field which is given by the term proportional c_{jkl} in the field tensor (through the term $G^i_{\mu\nu}G^{i,\mu\nu}$ in $\mathcal{L}_0$).

We notice that the labels k of W^k_μ stand for the isospin charge in the case of $SU(2)$ symmetry and the color charge in the case of $SU(3)$.

2.2.5 Spontaneous symmetry Breaking: The Higgs mechanism

Let's consider a lagrangian density for charged complex scalar field,

$$\mathcal{L} = (\partial_\mu\phi)\,(\partial^\mu\phi^*) - \mu^2\,\phi\phi^* - \lambda\,(\phi\phi^*)^2 \tag{2.2.39}$$

According to the discussion in the previous section, for this lagrangian to be invariant under the local gauge group $U(1)$. One has to replace the gradient ∂_μ by the covariant derivative

$D_\mu = \partial_\mu + ie\,A_\mu$ and add to $\mathcal{L}$ the kinetic energy term $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$. Then, this lagrangian becomes,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)\,(D^\mu\phi)^* - \mu^2\,\phi\phi^* - \lambda\,(\phi\phi^*)^2 \tag{2.2.40}$$

The local gauge transformation is given by

$$U(\theta) = e^{-i\theta(x)} \tag{2.2.41}$$

where the fields of the theory transform as

$$\begin{cases} \phi(x) \to \phi'(x) = e^{-i\theta(x)}\phi(x), \\ \phi^*(x) \to \phi^{*\prime}(x) = e^{i\theta(x)}\phi^*(x), \\ A_\mu(x) \to A'_\mu = A_\mu(x) + \frac{1}{e}\,\partial_\mu\theta(x). \end{cases} \tag{2.2.42}$$

Since the kinematic term of the lagrangian (of the field ϕ) vanishes at constant value of ϕ, the ground state of the system is obtained then by the minimum of the potential $V(\phi)$, where

$$V(\phi) = \mu^2\,\phi\phi^* + \lambda\,(\phi\phi^*)^2 \tag{2.2.43}$$

This potential has a minimum only if $\lambda > 0$. So, in the case where $\mu^2 > 0$ the minimum of the potential corresponds to $\phi = 0$. In this case, the lagrangian has a symmetric ground state. However if $\mu^2 < 0$ the minimum is given at

$$\phi\phi^* = -\frac{\mu^2}{2\lambda}, \qquad \text{or} \qquad |\phi| = \frac{v}{\sqrt{2}} = \sqrt{-\frac{\mu^2}{2\lambda}} \tag{2.2.44}$$

So, the are infinitely ground states, each of them is not symmetric since it is modified by the local gauge transformation Eq. (2.2.41). Generally, they take the form,

$$\phi_{vac} = \frac{v}{\sqrt{2}}\,e^{i\Lambda}, \qquad \text{for arbitrary real } \Lambda. \tag{2.2.45}$$

The potential for this case (for $\mu^2 < 0$) is given by the famous mexican hat. Every point of the minima is equivalent since it can be obtained from another one by the local gauge transformation (the ground state is not unique). So, we say that the symmetry of the original lagrangian is *spontaneously broken* in the case of $\mu^2 < 0$.

In the following, we will see how this phenomenon give mass to the gauge boson A_μ. Let us define the field ϕ as

$$\phi(x) = \frac{1}{\sqrt{2}}\,[v + \zeta(x) + i\,\chi(x)] \tag{2.2.46}$$

then $\mathcal{L}$ becomes

$$\begin{aligned} \mathcal{L} = &-\frac{1}{4}\,F_{\mu\nu}F^{\mu\nu} + \frac{e^2v^2}{2}\,A_\mu A^\mu + \frac{1}{2}\,(\partial_\mu\zeta)^2 + \frac{1}{2}\,(\partial_\mu\chi)^2 \\ &- \frac{1}{2}\,(2\lambda\,v^2)\,\zeta^2 - ev\,A_\mu\partial_\mu\partial^\mu\chi + \cdots \end{aligned} \tag{2.2.47}$$

The amazing surprise of the new form of $\mathcal{L}$ is that the gauge vector field involving in the theory acquire a mass (the mass term in the lagrangian is $\frac{e^2v^2}{2}A_\mu A^\mu$) and the gauge invariance still conserved since this lagrangian is completely equivalent to the previous one. This lagrangian describes the interaction between the massive vector field A_μ with two scalar field (the massive field ζ and the massless field χ). The degrees of freedom of $\mathcal{L}$ defined in Eq. (2.2.40) are four (since the massless vector field has two transverse independent modes and the complex field ϕ has two independent component). Nevertheless, the later version of the lagrangian has five degrees of freedom (three for the massive vector field and two of the two scalar fields). Actually the extra degree of freedom is superficial, since we can absorb the massless scalar field χ by a suitable gauge transformation. This can done by choosing the parameter $\theta(x)$ to be equal to the phase of the the field (see transformation above). Then the field ϕ can be chosen to be real, it can be written as

$$\phi = \frac{1}{\sqrt{2}}\left[v + \eta(x)\right]. \tag{2.2.48}$$

Inserting Eq. (2.2.48) in the original lagrangian, we get

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{e^2v^2}{2}A_\mu A^\mu + \frac{1}{2}\left(\partial_\mu\eta\right)^2 \\ &-\frac{1}{2}\left(2\lambda\, v^2\right)\eta^2 - \frac{1}{4}\lambda\,\eta^4 + \frac{1}{2}e^2\left(A_\mu\right)^2\left(2v\,\eta + \eta^2\right)\end{aligned} \tag{2.2.49}$$

So, to get this lagrangian we fixed a particular gauge called the *unitary gauge* to remove that extra degrees of freedom. The new lagrangian is no longer gauge invariant (since the gauge is fixed), and it has a four degrees of freedom (the unphysical field has been gauged). It describes the interaction between a massive vector field A_μ and a real scalar field η called the Higgs field with mass equal to $2\lambda v^2 = -2\mu^2$, for more detail see [42].

2.3 Strong interaction and QCD

Quantum Chromodynamics (QCD) is the modern theory of the strong interaction (color force). It is a non-Abelian gauge theory (the SU(3) Yang-Mills theory) which describes the interaction between the fundamental ingredients of the theory: the quarks (spin half particles) and gluons (a set of massless vector boson of spin one).

- The quarks are the matter fields of the theory, the quark field is denoted by $\psi_{q_{aj}}(x)$. They possess an internal degree of freedom called color (denoted by a) which takes the values $a = 1, 2, 3$ (very often, we refer to the colors by red, green and blue). The quarks come in six types known as flavor (denoted by j): u, s and b (up, strange and bottom) which possess a $+2/3$ fraction of electric charge and d, c and t (down , charm and top) which possess a $-1/3$ fraction of electric charge. Up and down quarks are the lowest mass quarks, they are stable and the most common in the universe. However the remaining quarks are not stable, they change to up and down quarks through the decay phenomenon and they can only be produced in high energy collision (as cosmic rays and particle accelerators).

- The gluon fields are the gauge bosons of the theory, they are massless, electrically neutral and possess a spin 1. They mediate the color force between quarks (as the photon field in the case of QED). They are denoted by G^b_μ where b labels the colors of the gluon fields ($b = 1, \cdots, 8$).

As in the other Yang-Mills theories, due to the self coupling of the gauge bosons, the theory possess remarkable properties: *i) asymptotic freedom* which signifies that at very high energy, the quarks and the gluons behave as free particle[40, 41], and *ii) color confinement* which signifies that the color charged particles (as the quarks and gluons) can not be observed directly since they cannot be isolated uniquely.

2.3.1 QCD Lagrangian

The classical lagrangian density of QCD is given by the classical lagrangian density of the Yang-Mills theory for $n = 3$,

$$\mathcal{L}_{class} = -\frac{1}{4} G^{\mu\nu}_b G^b_{\mu\nu} + \bar{\psi}_{q_{aj}} (i \not{D} - m_j) \psi_{q_{aj}}, \tag{2.3.50}$$

the field tensor is defined by

$$G^b_{\mu\nu} = \partial_\mu G^b_\nu - \partial_\nu G^b_\mu + g_s f_{abc} G^b_\mu G^c_\nu, \tag{2.3.51}$$

where f_{abc} are the structure constant of the gauge group $SU(3)$ then the group generators satisfy

$$[T^a, T^a] = i f_{abc} T^c, \tag{2.3.52}$$

$\psi_{q_{aj}}$ refers the quarks fields of the color a ($a = 1, 2, 3$) and the flavor j ($j = u, d, s, c, b, t$). The covariant derivative is defined by

$$D_\mu = \partial_\mu - i g_s \, G^a_\mu T_a, \tag{2.3.53}$$

According to the appropriate representation of the gauge group, the generators T_a are replaced by the matrices t_a for $a = 1, \cdots, 8$, where for the triplet representation of $SU(3)$ these matrices are just the half of Gell-man matrices λ_a. Then, if acting on the quark fields, the covariant derivative becomes $(D_\mu)_{ij} = \delta_{ij}\partial_\mu - i g_s (t_a)_{ij} \, G^a_\mu$ (with $t_a = \lambda_a/2$). If acting on the gluon, the generator are replaced by the structure constants and the derivative becomes: $(D_\mu)_{bc} = \delta_{bc}\partial_\mu - g_s f_{abc} \, G^a_\mu$.

If the mass of the quarks m_j are set up to zero, then the QCD lagrangian Eq. (2.3.50) is flavour and chirally symmetric, these symmetries are broken if the quarks acquire masses by the mechanism of spontaneously symmetry breaking. However, the quarks cannot exist as free particles due to the color confinement, the quarks masses may be considered as parameters in the lagrangian to be determined experimentally. Since, the perturbative calculation is valid only for high energy regime due to the asymptotic freedom (where the running coupling constant becomes small as we will see below), the masses of the quarks are not a relevant scale and it is adequate to use the massless lagrangian for the quarks u, d and s (for the heavy quarks one has to make some modifications to include quark masses t, b and perhaps c, see [42])

2.3.2 Comparison between QCD and QED

QED is the reference theory for all quantum gauge theories, QCD and the other are inspired from QED. So it will be very interesting to make a comparison between this two theories (see [42]):

- Due to the fact that the gluon carry a charge, the quark current, in contrary to the electron current, is not conserved, i.e

$$\partial_\mu J^b_\mu \not\to 0 \tag{2.3.54}$$

where $J^b_\mu = \bar{\psi}_{q_a\, i}\gamma_\nu t^a_{ij}\psi_{q_a\, j}$. There exist a conserved Noether current which does not correspond to the later one and it involves the field tensor $G^a_{\mu\nu}$. This current is given by

$$\tilde{J}^a_\mu = J^a_\mu + f_{abc}\, G^b_{\mu\nu} G^\nu_c \tag{2.3.55}$$

So, unlike QED, the gluon field G^b_μ does not couple to a conserved quark current which is one of the major difference between the two theories.

- The kinetic energy term of $\mathcal{L}_{class}$ contains the product of three and four gluon fields, this give rise to the three and four gauge bosons (gluons) self coupling which is not the case for Abelian gauge theories (the photon field).

- Both QCD current defined above (J^b_μ and $\tilde{J}^b_\mu$) are non-gauge invariant. Then,

$$\int d^3\vec{x}[\tilde{J}^a_0(\vec{x},t), J^b_\mu(y) \text{ or } \tilde{J}^b_\mu(y)] \neq 0 \tag{2.3.56}$$

2.3.3 QCD Quantization

As we mentioned above, this lagrangian describes a classical theory. The boson field G^a_μ is defined up to a given gauge transformation, to fix this freedom, one has to fix the gauge. Two choices of gauge fixing may be proposed: the covariant and the non covariant gauge fixing. Regarding the former choice, one has to add the following term to the lagrangian

$$\mathcal{L}_{Gauge} = -\frac{1}{2\alpha}\sum_{a=1}^{8}(\partial_\mu G^{\mu,a})^2, \tag{2.3.57}$$

$-\frac{1}{2\alpha}$ is lagrange multiplier. To absorb the non physical degrees of freedom, we have to add a ghost field such that

$$\mathcal{L}_{Ghost} = i(\partial\xi^a_1)D^{ab}_\mu\xi^b_2, \tag{2.3.58}$$

the ghost field ξ is a complex scalar field which follows Fermi statistics (for this reason it is not physical). The other choice of gauge fixation is the non-covariant one

$$\mathcal{L}_{Gauje} = -\frac{1}{2\lambda}(q^\mu G^a_\mu)(q^\nu G^a_\nu), \tag{2.3.59}$$

where q is an arbitrary four vector. This fixation does not require a ghost field but it leads to complicated gluon propagators.

2.3.4 The running coupling

Generally, during the perturbative expansion on $\alpha_s = g_s/(4\pi)$, we encounter ultraviolet divergences (UV divergences)[4] which must be treated by the renormalisation procedure [38]. Usually, we continue the space time to regularize these divergences by means of the of dimensional regularization technique. The later one leads to a new mass scale introduced to keep the mass dimension of the action equals to zero, this scale is, usually, denoted by μ. Then, a physical observable R (cross section, decay rate, ...) must be independent of μ, i.e the derivative of this quantity must vanish

$$\mu^2 \frac{d}{d\mu^2} R(Q^2/\mu^2, \alpha_s) = \left[\mu^2 \frac{\partial}{\partial \mu^2} + \mu^2 \frac{\partial \alpha_s}{\partial \mu^2} \frac{\partial}{\partial \alpha_s}\right] R(Q^2/\mu^2, \alpha_s) = 0, \tag{2.3.60}$$

where Q is a scale assumed to be larger than all the dimensional parameters (for example the center of mass energy $\sqrt{s}$).
The coefficient of the second term is called the β-function, where

$$\beta(\alpha_s) = \mu^2 \frac{\partial \alpha_s(\mu^2)}{\partial \mu^2}, \tag{2.3.61}$$

In the perturbative region and for a given number n_f of flavors of non massive quarks, the β-function is given by

$$\beta(\alpha_s) = -b\alpha_s\,[1 + \acute{b}\alpha_s + O(\alpha_s)], \tag{2.3.62}$$

$$b = \frac{33 - 2n_f}{12\pi}, \tag{2.3.63}$$

$$\acute{b} = \frac{153 - 19n_f}{2\pi\,(33 - 2\,n_f)}. \tag{2.3.64}$$

Neglecting b' and all higher order corrections, one can prove from Eq. (2.3.61) that the strong coupling constant at the scale Q is given by

$$\alpha_s(Q^2) = \frac{\alpha(\mu^2)}{1 + \alpha_s(\mu^2) b \ln(Q^2/\mu^2)}, \tag{2.3.65}$$

We see that if $\ln(Q^2/\mu^2)$ becomes large, the coupling $\alpha_s(Q^2)$ tends to zero, this property is the *asymptotic freedom* which is guaranteed if the theory involves 16 or fewer flavors of quarks (All the Yang-Mills theories possess this property because of the self gluon interaction.). Since the α_s becomes small for high Q, the perturbative calculation in this region is justified.

2.4 The Standard Model of Electroweak Interactions

The Standard Model (SM) is the theory that describes the interactions between the elementary particles constituting the matter. It allows to describe with an extreme precision all

[4] UV divergences occur if the energy becomes very hight (approaching infinity), i.e. the physical phenomena happens at very short distance.

the known corpuscular phenomenon and it has never been falsified. It is the gauge theory that bring together three fundamental interactions of nature except gravity. It was believed that a gauge theory able to unify the electromagnetic and the weak interaction must involve massless matter and mediator fields; however in reality this is not manifestly right. This challenge was achieved in 1964 by the pioneering idea of *spontaneous symetry breaking* introduced by Robert Brout, Francois Englert and Peter Higgs [7, 6]. Before describing this theory, it will be more convenient to talk briefly about the Fermi theory of β decay.

2.4.1 Fermi theory of β decay

The weak interaction was discovered by Bequerel in 1896. He discovered accidentally that a nucleus of an atomic number Z may decay into a different nucleus plus β rays (electrons):

$$A \to B + e^- \tag{2.4.66}$$

Theoretically, the distribution of the energy spectrum of the emitted electron should be constant

$$E_e = \frac{m_A^2 - m_B^2 + m_e^2}{2\, m_A} \tag{2.4.67}$$

where m_A and m_B are the masses of the nucleus A and B, m_e and E_e are the mass and the energy of the electron. On the other side, the experimental situation was quite confusing since it was definitely showing a continuous energy spectrum (not constant) of the electron which is in contradiction with the theoretical prediction Eq. (2.4.67) [39]. In that time, Niels Bohr was ready to abandon the principle of energy conservation. Fortunately, Pauli suggested that another neutral particle should be emitted with the electron, this particle was the *famous neutrino*. Then, the β decay was interpreted as the decay of a neutron inside the nucleus into a proton and a pair of electron-neutrino:

$$n \to p + e^- + \bar{\nu}_e \tag{2.4.68}$$

In this equation, conservation of the lepton number and the existence of various type of neutrino are assumed.

In 1934, Enrico Fermi described the β decay in term of field theory. He assumed that the production of electron-neutrino is analogue to the production of a photon in QED. Then, one can derive a model for the weak interaction by copying the main feature of QED. The QED hamiltonian density of interaction is defined by

$$\mathcal{H}_{eg}^{Int} = e \int d^3\vec{x}\, A_\mu\, \bar{\psi}\, \gamma^\mu\, \psi, \tag{2.4.69}$$

where ψ describes the charged fermion and A_μ the photon field. The Fermi ansatz for the interaction (2.4.68) is

$$\mathcal{H}_W = G_F \int d^3\vec{x}\, (\bar{\psi}_e \gamma_\mu\, \psi_\nu)\, (\bar{\psi}_p\, \gamma^\mu\, \psi_n). \tag{2.4.70}$$

More generally, this can be replaced by (see [39])

$$\mathcal{H}_W = G_F \int d^3\vec{x}\,(\bar{\psi}_e \Gamma_i\,\psi_\nu)\,(\bar{\psi}_p\,\Gamma_i\,\psi_n). \tag{2.4.71}$$

where Γ_i is 4×4 matrix, it can be of:

$\mathbf{\Gamma_i}$	**Tensor character**
$\mathbb{I}$	S (scalar)
γ^μ	V vector
$\sigma_{\mu\nu} = i\,[\gamma_\mu,\gamma_\nu]/2$	T (anti-symetric tensor)
$\gamma^\mu\,\gamma_5$	A (axial-vector)
$\gamma_5 = i\,\gamma^0\,\gamma^1\,\gamma^2\,\gamma^3$	P (pseudoscalar)

After many years of effort and after many unsuccessful attempts, it was concluded that all the weak interaction processes could be described by the local current-current interaction given by the lagrangian [5]

$$\mathcal{L} = \frac{G_F}{\sqrt{2}}\,J^\dagger_\mu(x)\,J^\mu(x). \tag{2.4.72}$$

the current J_μ is called charged current, it is formed by sum of purely hadronic and purely leptonic currents

$$J_\mu(x) = l_\mu(x) + h_\mu(x), \tag{2.4.73}$$

with

$$l_\mu(x) = \bar{\psi}_e\,\gamma_\mu\,(1-\gamma_5)\,\psi_{\nu_e} + \bar{\psi}_\mu\,\gamma_\mu\,(1-\gamma_5)\,\psi_{\nu_\mu} + ... \tag{2.4.74}$$

$$h_\mu(x) = \bar{\psi}_u\,\gamma_\mu\,(1-\gamma_5)\,\psi_d + \bar{\psi}_c\,\gamma_\mu\,(1-\gamma_5)\,\psi_s + ... \tag{2.4.75}$$

the form of this current is dictated by experimental issues, especially the angular distribution of the decay products. This has a very important consequence which is that only left handed fermions are sensitive to the weak interactions. From this lagrangian, the weak processes can be divided in three categories:

- *Leptonic process:* described by the term $l^\dagger_\mu\,l^\mu$. For example:

$$\mu^- \to e^- + \nu_e + \bar{\nu}_\mu, \tag{2.4.76}$$

$$\nu_e + e^- \to \nu_e + e^-. \tag{2.4.77}$$

the μ-decay and the ν elastic scattering, respectively.

- *Semi-leptonic process:* described by the term $l^\dagger_\mu\,h^\mu$. For example, the β-nuclear decay:

$$n \to p + e^- + \bar{\nu}_e \tag{2.4.78}$$

- *Non-leptonic process:* described by the term $h^\dagger_\mu\,h^\mu$. For example, the parity violation in nuclei:

$$n + p \to n + p. \tag{2.4.79}$$

[5] The form of the V-A interaction was developed independently by Feynman and Gell-man[45] and Sudarshan and Marshak [46] In this approach, the transition is described by a local current-current interaction given by the lagrangian Eq. (2.4.72)

Problems of Fermi theory:

The main problem of this theory is the non-renormalizability. The Fermi coupling constant has a mass dimension equals to -2, since it is multiplied by an operator of type $(\psi\,\psi)^2$ which has a 6 mass dimension. On the other side the mass dimension of the lagrangian equals to 4, then $[G_F] = 4 - 6$, which means that this theory is not renormalizable.

Also, the cross section of the process

$$\mu^- \nu_e \to e^- \nu_\mu \tag{2.4.80}$$

violated the famous Froissart-Martin unitary bound which requires that

$$\sigma \leq \ln^2 s, \qquad \text{for } s \to \infty \tag{2.4.81}$$

one can prove that this cross section for the Fermi theory is $\sigma \to G_F^2\, s$, with $s = (p_{\nu_e} + p_\mu)^2$.

To solve this problem, one can introduce a massive charged vector field that mediate the interaction between the two left handed currents (see [47]). Then, that transition matrix element of μ-decay is

$$\mathcal{M} = g_W^2 \left(\bar{\psi}_e\, \gamma_\mu \left(1 - \gamma_5\right) \psi_{\nu_e}\right) \frac{g^{\mu\nu} - \frac{q^\mu\, q^\nu}{M_W^2}}{q^2 - M_W^2} \left(\bar{\psi}_\mu\, \gamma_\mu \left(1 - \gamma_5\right) \psi_{\nu_\mu}\right)^\dagger \tag{2.4.82}$$

where M_W is the mass of the vector field, it is given by

$$M_W^2 = \sqrt{2}\, \frac{g_W^2}{G_F} \tag{2.4.83}$$

This leads to a cross section that behaves as Fermi cross section for $s << M_W^2$ and as

$$\sigma \propto \frac{g_W^4}{M_W^2}\, \frac{s}{s + M_W^2}, \qquad \text{for } s >> M_W^2 \tag{2.4.84}$$

for more detail see [43].

2.4.2 The Standard Electroweak Theory

The Standard Model is based on the gauge group $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$[6]. The $SU(3)_C$ is the QCD gauge group (it is discussed in detail above), it is non-chiral then it acts on the color indices of the left and right handed quarks. The $SU(2)_L \otimes U(1)_Y$ describes the electroweak sector, in contrast to QCD it is chiral and acts on the flavour indices of the quarks and the leptons. The $SU(2)_L$ has three gauge bosons W_i, $i = 1, 2, 3$ and a coupling constant denoted by g, it acts only on flavor indices of the left handed fermions. The Abelian group $U(1)_Y$ is chiral, it has only the gauge boson B and the coupling constant g'. It acts on flavor indices of the the left handed as well as the right handed fermions

[6]The subscripts has no mathematical significance: "C" refer to the color coupling nature of $SU(3)$, "L" refer to the left-chiral nature of the coupling of $SU(2)$ and the "Y" to the weak hyper-charge of $U(1)$.

but with different charges. The phenomenon of spontaneous symmetry breaking (SSB) breaks the group $SU(2)_L \otimes U(1)_Y$ into the unbroken $U(1)_Q$ (Q refers to the electric charge) covering the QED theory with the photon as linear combination of the bosons W^0 and B [44], and gives mass to the Z-boson and to the charged $W^{\pm}$ bosons which are responsible to neutral-current and charged current interactions, respectively.

The full SM lagrangian density can be split in four parts

$$\mathcal{L}_{SM} = \mathcal{L}_g + \mathcal{L}_f + \mathcal{L}_\Phi + \mathcal{L}_{Yuk} \tag{2.4.85}$$

The Yang-Mills lagrangian $\mathcal{L}_g$ for the gauge group $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ involves all the gauge vector fields of the theory (before SSB). It is given by

$$\mathcal{L}_g = -\frac{1}{4} G^b_{\mu\nu} G^{b\,\mu\nu} - \frac{1}{4} W^i_{\mu\nu} W^{i\,\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu}. \tag{2.4.86}$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu, \tag{2.4.87}$$

$$W^i_{\mu\nu} = \partial_\mu W^i_\nu - \partial_\nu W^i_\mu - g\,\varepsilon_{ijk}\, W^j_\mu\, W^k_\nu \qquad \text{for} \qquad i,j,k = 1,...3, \tag{2.4.88}$$

$$G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu - g_s\, f_{abc}\, G^b_\mu\, G^c_\nu \qquad \text{for} \qquad a,b,c = 1,...8. \tag{2.4.89}$$

where $B_{\mu\nu}$, $W^i_{\mu\nu}$ and $G^b_{\mu\nu}$ are the antisymmetric tensors constructed by the gauge fields B_μ associated to the gauge group $U(1)_Y$, the gauge field W^i_μ associated the three generators of the group $SU(2)_L$, and the gauge field G^b_μ associated the eight generators of the group $SU(3)_C$, respectively. ε_{ijk} and f_{abc} are the structure constants of the group $SU(2)_L$ and $SU(3)_C$ and g, g_s are the coupling constants associated to these gauge groups.

The fermion part of the SM involves 3 families of quarks and leptons. Since, the EW interaction leads to transition between fermions of different charges and since only the left handed fermions are sensitive to the EW interaction (and not the right handed) , it will be more convenient to group each family as follows:

$$\begin{array}{lll}
\text{left-doublet:} & \psi^L_{q_{am}} = \begin{pmatrix} u \\ d \end{pmatrix}_L, \begin{pmatrix} c \\ s \end{pmatrix}_L, \begin{pmatrix} t \\ b \end{pmatrix}_L, & \psi^L_{l_m} = \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L, \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L, \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_L \\
\text{right-singlet:} & \psi^R_{u_{am}} = (u,c,t)_R, & \psi^R_{\nu_m} = (\nu_e, \nu_\mu, \nu_\tau)_R \\
\text{right-singlet:} & \psi^R_{d_{am}} = (d,s,b)_R, & \psi^R_{e_m} = (e^-, \mu^-, \tau^-)_R
\end{array} \tag{2.4.90}$$

All these fields carry a weak hyper-charge Y (except the neutrinos), which is defined by

$$Y = Q - T^3_L \tag{2.4.91}$$

where Q is the electric charge operator and T^3_L is the third generator of $SU(2)$. The Y eigenvalues ($y = q - t^3_L$) of the quark fields $\psi^L_{q_{am}}$, $\psi^R_{u_{am}}$ and $\psi^R_{d_{am}}$ are given by $\frac{1}{6}, \frac{2}{3}, \text{and} -\frac{1}{3}$, respectively. And for the lepton fields $\psi^L_{l_m}$, $\psi^R_{\nu_m}$ and $\psi^R_{e_m}$ are given by $-\frac{1}{2}, 0, \text{and} -1$, respectively. We notice that the left hand and the right hand components of the fermions are defined by

$$\psi_{L,R} = [(1 \mp \gamma_5)/2]\, \psi \tag{2.4.92}$$

$$\bar{\psi}_{L,R} = [(1 \pm \gamma_5)/2]\, \bar{\psi} \tag{2.4.93}$$

where each component has different transformation properties under the SM guage group. Since the SM is chiral theory, so no fermion mass terms are allowed. Then, the fermionic lagragian $\mathcal{L}_f$ consist only of the gauge covariant kinematic energy terms

$$\mathcal{L}_f = i \sum_{m=1}^{F} \left\{ \bar{\psi}^L_{q_{am}} \not{D}\, \psi^L_{q_{bm}} + \bar{\psi}^L_{l_m} \not{D}\, \psi^L_{l_m} + \bar{\psi}^R_{q_{am}} \not{D}\, \psi^R_{q_{bm}} + \bar{\psi}^R_{l_m} \not{D}\, \psi^R_{l_m} \right\} \tag{2.4.94}$$

where F stands for the number of fermion families, and an implicit sum over the color indices (a, $b = 1, 2, 3$) is considered, the index q in $\psi^R_{q_{am}}$ ($\bar{\psi}^R_{q_{a\,m}}$) stands for u or d, and the index l in $\psi^R_{l_m}$ ($\bar{\psi}^R_{l_m}$) stands for ν or e. The general form of the covariant derivatives is given by:

$$\bar{\psi}^L_{q_{am}} \not{D}_\mu \psi^L_{q_{bm}} = \sum_{a,b=1}^{3} \bar{\psi}^L_{q_{am}} \gamma^\mu \left[\left(\partial_\mu I + \frac{ig}{2} \vec{\tau} \cdot \vec{W}_\mu + \frac{ig'}{6} I\, B_\mu \right) \delta_{ab} + \frac{ig_s}{2} \vec{\lambda}_{ab} \cdot \vec{G}_\mu I \right] \psi^L_{q_{bm}} \tag{2.4.95}$$

Acting on the remaining particles on the lagrangian $\mathcal{L}_f$, the covariant derivative is given by:

$$D_\mu \psi^L_{l_m} = \left(\partial_\mu + \frac{ig}{2} \vec{\tau} \cdot \vec{W}_\mu - \frac{ig'}{2} B_\mu \right) \psi^L_{l_m} \tag{2.4.96}$$

$$D_\mu \psi^R_{u_m} = \left(\partial_\mu + \frac{i2g'}{3} B_\mu \right) \psi^R_{u_m}, \qquad D_\mu \psi^R_{d_m} = \left(\partial_\mu - \frac{ig'}{3} B_\mu \right) \psi^R_{d_m} \tag{2.4.97}$$

$$D_\mu \psi^R_{e_m} = \left(\partial_\mu - ig'\, B_\mu \right) \psi^R_{e_m} \qquad D_\mu \psi^R_{\nu_m} = \partial_\mu \psi^R_{\nu_m} \tag{2.4.98}$$

where I is the 2×2 identity matrix of $SU(2)$. τ and λ_{ab} are the Pauli and Gell-Mann matrices.

The Higgs lagrangian part is

$$\mathcal{L}_\phi = (D^\mu \phi)^\dagger\, D_\mu \phi - V(\phi) \tag{2.4.99}$$

with

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}, \qquad \phi^\dagger = \begin{pmatrix} \phi^- \\ \phi^{0\dagger} \end{pmatrix} \tag{2.4.100}$$

$$D_\mu \phi = (\partial_\mu + \frac{i\,g}{2} \vec{\tau} \cdot \vec{W}_\mu + \frac{i\,g'}{2} B_\mu)\, \phi \tag{2.4.101}$$

V is the Higgs potential. Due to the $SU(2) \otimes U(1)$ invariance and the renormalizability restriction of the theory, the potential V must take the form

$$V(\phi) = \mu^2\, \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2 \tag{2.4.102}$$

For $\mu^2 < 0$ there will be a spontaneous symmetry breaking and the vacuum expectation value $< 0 | \phi^0 | 0 >$ (denoted by VEV) generates the masses of the Z and W bosons, for more detail se paragraph 2.2.5.

The last part of the lagrangian in Eq.(2.4.85) represents the Yukawa part which describes the Higgs doublet coupling to the fermions. This term is responsible for the generation of the fermions masses, it is given by

$$\mathcal{L}_{Yuk} = -\sum_{m,n=1}^{F} [\Gamma^u_{mn}\bar{\psi}_{q^L_{am}}\tilde{\phi}\,\psi^R_{u_{am}} + \Gamma^d_{mn}\bar{\psi}_{q^L_{am}}\phi\,\psi^R_{d_{am}} + \Gamma^e_{mn}\bar{\psi}_{l^L_m}\phi\,\psi^R_{e_m} + \Gamma^\nu_{mn}\bar{\psi}_{l^L_m}\tilde{\phi}\,\psi^R_{\nu_m}] + \text{h.c.} \tag{2.4.103}$$

$$\tilde{\phi} = i\tau^2\,\phi^\dagger = \begin{pmatrix} \phi^{0\dagger} \\ -\phi^- \end{pmatrix} \tag{2.4.104}$$

The $F \times F$ matrices $\Gamma^u, \Gamma^d, \Gamma^e$ and Γ^ν are completely arbitrary, they do not have to be real, diagonal, symmetric or hermitian (the hermitically of the lagrangian is insured by the term .h.c.). These matrices introduce the most free parameters of the SM.

After spontaneous symmetry breaking and fixing the unitary gauge, the full Higgs lagrangian becomes

$$\mathcal{L}_\phi = M_W^2 W^{\mu+} W^-_\mu \left(1 + \frac{H}{\nu}\right)^2 + \frac{1}{2} M_Z^2 Z^\mu Z_\mu \left(1 + \frac{H}{\nu}\right)^2 + \frac{1}{2}(\partial_\mu H)^2 - V(\phi) \tag{2.4.105}$$

where

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \nu + H \end{pmatrix} \tag{2.4.106}$$

the potential

$$V(\phi) = -\frac{\mu^4}{4\lambda} - \mu^2 H^2 + \lambda\nu H^3 + \frac{\lambda}{4} H^4 \tag{2.4.107}$$

The $W^\pm$ are the complex gauge bosons that mediate the charged current gauge interaction, they are defined by

$$W^\pm = (W^1 \pm i\,W^2)\sqrt{2}. \tag{2.4.108}$$

The photon field A_μ and the Z-boson field Z_μ are the mediators of the weak neutral-current interaction, they are expressed in term of the fields B_μ and W^3_μ and the weak mixing angle as the following[7]

$$A_\mu = \cos\theta_W\, B_\mu + \sin\theta_W\, W^3_\mu, \tag{2.4.109}$$

$$Z_\mu = -\sin\theta_W\, B_\mu + \cos\theta_W\, W^3_\mu. \tag{2.4.110}$$

θ_W is the weak angle, it is defined by

$$\tan\theta_W = g'/g, \quad \sin\theta_W = \frac{g'}{g_Z}, \quad \cos\theta_W = \frac{g}{g_Z}, \quad \text{with} \quad g_Z = \sqrt{g^2 + g'^2} \tag{2.4.111}$$

[7]The weak angle or the weak mixing angle it a parameter in the model of Weinberg-Salam of electroweak interaction to describe the rotation induced by spontaneous symmetry breaking to the original W^0 and B^0 vector bosons to produce the Z^0 boson and the photon

The masses of Z and W bosons are

$$M_W = \frac{g\nu}{2} = 80.398 \pm 0.025\text{GeV}, \qquad M_Z = \frac{g_Z\nu}{2} = \frac{M_W}{\cos\theta_W} = 91.1876 \pm 0.0021\text{GeV} \tag{2.4.112}$$

After spontaneous symmetry breaking, the Yukawa lagrangian become

$$\begin{aligned}\mathcal{L}_{Yuk} &= -\sum_{m,n=1}^{F} \bar{\psi}^L_{u_{a,m}} \Gamma^u_{mn} \left(\frac{\nu + H}{\sqrt{2}}\right) \psi^R_{u_{a,n}} + (d, e, \nu)\,\text{terms} + \text{h.c.} \\ &= \bar{\psi}^L_{u_a}(M^u + h^u\, H)\psi^R_{u_a} + (d, e, \nu)\text{terms} + \text{h.c.}\end{aligned} \tag{2.4.113}$$

where $\bar{\psi}^L_{u_a} = (\bar{\psi}^L_{u_{a,1}}, \cdots, \bar{\psi}^L_{u_{a,F}})$ is F-component line vector. $M^u_{mn} = \frac{\nu}{\sqrt{2}}\Gamma^u_{mn}$ is the $F \times F$ fermion mass matrix induced by spontaneous symmetry breaking. It can be diagonalized, for $F = 3$ its associated diagonal matrix is

$$M^u_D = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix} \tag{2.4.114}$$

the eigenvalue of this matrix are real, they correspond to the physical mass values of the charge $\frac{2}{3}$ quarks. Similarly we can diagonalized the Yukawa coupling matrix h^u where its eigenvalues are denoted by h_u, h_c, h_t. In a similar way, a diagonal mass matrices and Yukawa couplings are defined for the remaining particles (ψ_{q_u}, ψ_{q_e} and ψ_{q_ν}), for more detail see [44].

Chapter 3

Perturbation Theory: Analytic Properties

Contents

The procedure of quantization is based on the hamiltonian or the lagrangian formalisms which take their form from classical mechanics by means of the Bohr correspondence principle[1]. The solution of the equations of motion of a given system can be obtained from the perturbative expansion of these equations in term of the coupling constant of the interaction (electric charge for example). However two crucial difficulties are faced in perturbation theory, the infrared divergences (IR) which occur from the fact that the massless nature of gauge bosons makes the number of zero energy measurement impossible, and the ultraviolet

[1]Bohr correspondence principle states that the behavior of a system described by the quantum theory reproduces the classical theory in the limit of large quantum number.

divergences (UV) which are eliminated by the renormalization procedure. To deal with the divergences present in quantum field theory, the S-matrix approach of perturbation theory was suggested by Heisenberg [50][2], this approach is based on the idea that the S-matrix elements should be calculated directly without using the field quantities. This requires that the S-matrix should satisfy the following important properties: a) the superposition principle of quantum mechanics b) requirements of special relativity c) conservation of probability d) the short- range character of the force e) causality and existence of macroscopic time.[48, 49].
In this chapter, we study the main feature of perturbation theory. In the first section, we give a general presentation of the S-matrix theory; its definition, its properties, its analyticity and its relation to Feynman diagrams. In the second section, we discuss the analyticity of the scattering amplitude by giving the necessary and sufficient condition for the occurrence of singularities of individual one-loop Feynman graphs (Landau conditions) [52], and we give general criteria to determine the soft and collinear divergences of these diagrams. In the third section we present some consequences of the unitarity and the causality on the scattering amplitude computation (dispersion relation) and we conclude by giving the Cutkosky cutting rules [53].

3.1 S-matrix theory

3.1.1 Definition of the S-matrix

In scattering experiments, it is assumed that the force of the interaction between the particles is sufficiently weak at large distances, i.e. the incoming particles (observed long before) and the outgoing particles (observed long after) behave as free particles during the observation. Thus, in the extreme past ($t \to -\infty$) the particles can be described by state vectors denoted by $|\text{in}>$, and in the infinite future ($t \to +\infty$,) the particles can be described by state vectors denoted by $|\text{out}>$. These vector states satisfy the following properties:

-a) *The superposition principle:* if $|\psi, \bullet>$ and $|\Phi, \bullet>$ are two physically existing states,

$$\lambda\,|\psi, \bullet> + \mu\,|\Phi, \bullet>, \tag{3.1.1}$$

is also a physical state, for all complex numbers λ and μ
-b) *Orthonormality conditions*: a set of physical states is normalized, then it satisfies:

$$< m, \bullet | n, \bullet > = \delta_{mn}, \tag{3.1.2}$$

-c) *Completeness relation*: a set of physical states is complete, then it satisfies:

$$\sum_m |m, \bullet >< m, \bullet| = 1, \tag{3.1.3}$$

where n and m describe particular configurations of free particles. These last three properties are satisfied by both, incoming and outgoing states, where the "$\bullet$" stands for the labels

[2]The S-matrix approach was introduced for the first time by John Archibald Wheeler in [51].

"in" or "out".

Let us define an operator, namely S [48], by

$$S = \sum_m |m, \text{in} >< m, \text{out}| \tag{3.1.4}$$

$$S^\dagger = \sum_m |m, \text{out} >< m, \text{in}| \tag{3.1.5}$$

using the orthonormality and completeness conditions on states $|m, \text{in} >$ and $|m, \text{out} >$, one can prove that the matrix elements of the operator S satisfy

$$< \Phi, \text{in}|S|\Psi, \text{in} > = < \Phi, \text{out}|\Psi, \text{in} > = < \Phi, \text{out}|S^\dagger|\Psi, \text{out} > \tag{3.1.6}$$

Ψ and Φ refers for any physical states. In addition, the orthonormality and the completeness conditions, imply that the operator S is unitary. This operator is called the S-matrix, it plays a crucial rule in developing perturbative field theories. In the following paragraph, we will give some of its properties and some of its consequences on the perturbative calculation.

3.1.2 S-matrix properties

3.1.2.1 Unitarity

From the superposition principle, the final state can be written as $S\,|n >$ (we omit the labels "in" and "out"). In quantum mechanics, the probability that a measurement on the final state gives a result corresponding to the a given state $|m >$ is obtained from squaring the modulus of the matrix element

$$< m|S|n > \tag{3.1.7}$$

any state can be expressed by a superposition of the states $|n >$ which form a basis of a vectorial space (n, m stands for the quantum numbers specifying the state, with: $< m|n >= \delta_{nm}$ and $\sum_m |m >< m| = 1$). If $|\phi >$ is a normalized initial state in a colliding experiment, the total probability of the system must be unity. We write

$$|\Phi > = \sum_n a_n\,|n > \tag{3.1.8}$$

a_n are some complex coefficients which are characterized by

$$\sum_n |a_n|^2 = 1 \tag{3.1.9}$$

this implies

$$\begin{aligned} 1 = \sum_m |< m|S|\Phi >|^2 &= \sum_m < \Phi|S^\dagger|m >< m|S|\Phi > \\ &= < \Phi|S^\dagger\,S|\Phi > = \sum_{n,n'} a^\dagger_{n'}\,a_n\; < n'|S^\dagger\,S|n > \end{aligned} \tag{3.1.10}$$

the necessary condition for Eq. (3.1.10) to hold is

$$< n'|S^\dagger\, S|n > = \delta_{n'n}. \tag{3.1.11}$$

which implies

$$S^\dagger\, S = 1 \tag{3.1.12}$$

In the same way, we can prove

$$S\, S^\dagger = 1 \tag{3.1.13}$$

So, the condition that total probability is conserved (by unity) implies that the S-matrix must be unitary.

3.1.2.2 Relativistic invariance

If L is any Lorentz transformation and if

$$L\,|m> = |m'>, \tag{3.1.14}$$

The observable quantities must be independent of Lorentz frame, this requires that

$$|< m'|S|n' >|^2 = |< m|S|n >|^2. \tag{3.1.15}$$

Since the definition of the S-matrix given above does not specify the phase uniquely, this allows us to replace Eq.(3.1.15) by the stronger condition [48]:

$$< m'|S|n' > =< m|S|n > \tag{3.1.16}$$

The last equation has an important consequence, the matrix elements for spineless particles depend on the four-momenta only through their invariant scalar products. For example, the $2 \to 2$ spineless particle matrix element

$$< p_3, p_4|S|p_1, p_2 >, \tag{3.1.17}$$

after removing the δ-function specifying the energy-momentum conservation, this quantity can be written as a function of only these three variables

$$s = (p_1 + p_2)^2, \qquad t = (p_1 - p_4)^2, \qquad u = (p_1 - p_3)^2. \tag{3.1.18}$$

with

$$p_1 + p_2 = p_3 + p_4, \qquad s + t + u = \sum_{i=1}^{4} m_i^2, \qquad p_i^2 = m_i^2. \tag{3.1.19}$$

As a consequence of Lorentz invariance, the matrix element for the case of elastic scattering of two spineless particles is symmetric

$$< p_3, p_4|S|p_1, p_2 > =< p_1, p_2|S|p_3, p_4 >, \tag{3.1.20}$$

this can be obtained by making a simple transformation (rotation) in the center of mass frame (the Lorentz frame in which $\vec{p}_1 + \vec{p}_2 = \vec{0} = \vec{p}_3 + \vec{p}_4$), which interchanges p_1 and p_3 and interchanges p_2, p_4.

3.1.3 Consequences of unitarity

It is very useful to write the S-matrix in term of the transition matrix T:

$$S = 1 + i\,T, \tag{3.1.21}$$

The matrix T contains all the informations about the interaction, it is related the the scattering amplitude F by

$$< p_3, p_4|T|p_1, p_2 > \;\; = (2\,\pi)^4\,\delta^{(4)}(p_1 + p_2 - p_3 - p_4)\,F, \tag{3.1.22}$$

then to the experimental cross section is given by

$$\sigma = \frac{1}{(8\,\pi)^2\,|\vec{q}|\,W}\int d\Omega\,|F|^2\,\frac{|\vec{p}|}{W}, \tag{3.1.23}$$

where $\vec{q}$ and $\vec{p}$ are the center of mass momentum for particles in the initial and final states, respectively; W is the center of mass energy. Ω is the solid angle in the final state (with $d\Omega = \sin\theta\,d\theta\,d\phi$). In the following we use the unitarity property of the S-matrix to deduce some important properties of the scattering amplitude, hence of the physical observable.

3.1.3.1 Analyticity and unitarity

From the unitarity condition on the S-matrix, the T-matrix satisfies:

$$\begin{aligned} < p_3, p_4|T|p_1, p_2 > - < p_1, p_2|T|p_3, p_4 >^* \;\; &= \;\; i\;< p_3, p_4|T^\dagger\,T|p_1, p_2 > \\ &= \;\; i\;< p_1, p_2|T^\dagger\,T|p_3, p_4 >, \end{aligned} \tag{3.1.24}$$

"$\dagger$" and "$*$" stand for the transposed complex conjugate of the matrix and the complex conjugate of the matrix elements, respectively. Using the symmetry condition on S given in Eq. (3.1.20), we find that Eq. (3.1.24) is just twice the imaginary part of T-matrix element:

$$2\,i\,\mathrm{Im}\;< p_3, p_4|T|p_1, p_2 >, \tag{3.1.25}$$

By using the completeness condition, the unitarity condition becomes

$$2\,i\,\mathrm{Im}\;< p_3, p_4|T|p_1, p_2 > \;\; = \;\; \sum_n\; < p_3, p_4|T|n >< p_1, p_2|T|n >^*, \tag{3.1.26}$$

where $\sum$ denotes the sum and integration over the n-particle real intermediate states allowed by the conservation of total energy and momentum.

For sufficiently small energies of the incoming particles, no creation of new particles can occur in a collision, so each S-matrix element corresponding to a creation process must vanish. If the energies of the incoming particles are sufficiently large, a threshold will be encountered above which a particle of non-zero mass may be created (and for higher energies other particles may be created, hence other threshold may be encountered). This means that the S-matrix elements change their analytic form when crossing these thresholds [49, 48].

Then, for total energy below the inelastic energy-threshold and in term of the transition amplitude F, the unitarity condition is given by

$$\begin{aligned} 2i\,\mathrm{Im} < p_3, p_4|F|p_1, p_2 > &= (2\,\pi)^{-2} \int \frac{d^3\vec{k}_1}{2\,k_1^0}\,\frac{d^3\vec{k}_2}{2\,k_2^0}\,\delta^{(4)}(p_1+p_2-k_1-k_2) \\ &\times < p_3 p_4|F|k_1 k_2 >< p_1 p_2|F|k_1 k_2 >^* \\ &= (2\,\pi)^{-2} \int d^4k_1\,d^4k_2\,\delta^{(+)}(k_1^2-m^2)\,\delta^{(+)}(k_2^2-m^2) \\ &\times \delta^4(p_1+p_2-k_1-k_2) < p_3 p_4|F|k_1 k_2 > \\ &\times < p_1 p_2|F|k_1 k_2 >^*, \end{aligned} \tag{3.1.27}$$

where $k_i = (k_i^0, \vec{k}_i)$ for $i = 1, 2$.
For total energy above the energy-threshold for inelastic scattering, a new term must be added to the unitarity relation given by Eq. (3.1.27) to include the extra intermediate states allowed by energy conservation, which means a change in the imaginary part of the amplitude (the left hand side of this equation), and implies that S-matrix has a singularity at the corresponding energy threshold for the creation of new allowed physical processes.

The thresholds are branch points of the amplitude F [48, 49], they will be discussed in the next section. Let us consider the two-particle scattering amplitude F, this amplitude is a function of two Mandelstam variables, say the invariant energy squared s and the momentum-transfer-squared t, then

$$< p_3 p_4|F|p_1 p_2 > = F(s,t). \tag{3.1.28}$$

We keep the momentum-transfer-squared t fixed, then the branch points in term of s are $s = 4\,m^2, 9\,m^2, 16\,m^2$, these energies are called "normal thresholds", they correspond to the energies at which the production of new particles is possible (in the case of spineless and equal mass particles).

3.1.3.2 Crossing properties

Crossing properties mean that the same analytic function can be used to describe different processes. The amplitude in Eq.(3.1.28) describes the interaction of the two-two process (we assume equals masses and the only preserved conservation law is the energy momentum), say

$$A_1 + A_2 \to A_3 + A_4 \tag{3.1.29}$$

where A_1 and A_2 denote the incoming particles and A_3 and A_4 denote the outgoing particles. This can happen only if the energies of each particle p_i^0 is real positive and their three momenta are real $\vec{p}_i$. These conditions are summarized as the following

$$s \geq 4\,m^2, \qquad t \leq 0, \qquad u \leq 0. \tag{3.1.30}$$

This can be demonstrated by writing these variables in term of the three momentum $\vec{q}$ ($|\vec{q}| = |\vec{p}_1| = \vec{p}_2$) and the scattering angle θ in the center of masse frame. We get

$$\begin{cases} s &= 4\,(m^2 + |\vec{q}|^2), \\ t &= -2\,|\vec{q}|^2\,(1 - \cos\theta), \\ u &= -2\,|\vec{q}|^2\,(1 + \cos\theta). \end{cases} \tag{3.1.31}$$

These conditions define the physical region of the s-channel (see Fig.(3.1)), where the energy in the center of mass frame is $\sqrt{s}$. The function F is also analytic for the the following two conditions

$$u \geq 4\,m^2, \qquad s \leq 0, \qquad t \leq 0, \tag{3.1.32}$$

$$t \geq 4\,m^2, \qquad u \leq 0, \qquad s \leq 0. \tag{3.1.33}$$

These conditions (they are obtained from the first one by exchanging the external momenta) define the physical region of the u-channel and t-channel (see Fig.(3.1)), respectively. They correspond to the following two processes

$$A_1 + \bar{A}_3 \rightarrow \bar{A}_2 + A_4 \tag{3.1.34}$$

$$A_1 + \bar{A}_4 \rightarrow \bar{A}_2 + A_3 \tag{3.1.35}$$

where the energies in the center of mass frame are $\sqrt{t}$ and $\sqrt{u}$, respectively. $\bar{A}_i$ denotes the anti-particle of A_i.
These important properties are called "crossing properties": the same analytical function can be used to describe different physical processes for a given choice of the Mandelstam variables s, t and u [48].

3.1.4 Feynman diagrams and the S-matrix

3.1.4.1 Perturbative expansion

The equation of motion of a coupled interacting system is

$$i\,\frac{\partial|\Phi(t)>}{\partial t} = H_I(i)\,|\Phi(t)> \tag{3.1.36}$$

where $|\Phi(t)>$ describes the state of the system at time t and H_I is the interaction hamiltonian part.
Long before the interaction occurs, we assume that all the particles are far away (considered as free particles). Let us call the initial state of the system $|\Phi_i> = |\Phi(-\infty)>$. Eq. (3.1.36) tells us about the evolution of the state vector $|\Phi_i>$ in time. Hence, one can predict the final state of the system at $t = +\infty$ from its initial state and this evolution equation. In the other side, the S-matrix transforms $|\Phi(-\infty)>$ into $|\Phi(+\infty)>$, i.e

$$|\Phi(+\infty)> = S\,|\Phi(-\infty)> \tag{3.1.37}$$

So that defining this operator is equivalent to solve the differential equation of motion [54]. In addition, the integral representation of Eq. (3.1.36) is

$$|\Phi(t)> = |\Phi(-\infty)> + (-i)\int_{-\infty}^{t} dt_1\,H_I(t_1)\,|\Phi(t_1)> \tag{3.1.38}$$

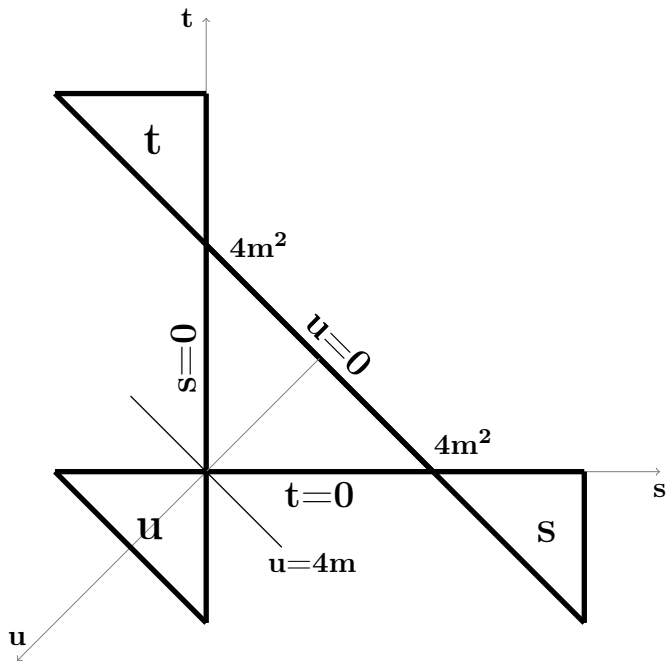


Figure 3.1: Physical region of the s, t and u channels for equal mass particles

this equation can be solved by iteration as the following

$$|\Phi(t)> = \sum_{n=0}^{\infty} (-i)^n \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 \cdots \int_{-\infty}^{t_{n-1}} dt_n \left[H_I(t_1) \cdots H_I(t_n)\right] |\Phi(-\infty)> \quad (3.1.39)$$

then

$$\begin{aligned} |\Phi(+\infty)> &= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \cdots \int_{-\infty}^{\infty} dt_n \, T\left\{H_I(t_1) \cdots H_I(t_n)\right\} |\Phi(-\infty)> \\ &= S\,\Phi(-\infty) \end{aligned} \quad (3.1.40)$$

Hence the operator S is given by

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \cdots \int_{-\infty}^{\infty} dt_n \, T\left\{H_I(t_1) \cdots H_I(t_n)\right\} \quad (3.1.41)$$

T denotes the *Dyson time-ordered product*, it is defined by

$$T\left\{\Phi(x_1)\,\Phi(x_2)\right\} = \begin{cases} \Phi(x_1)\,\Phi(x_2) & x_1 > x_2 \\ \Phi(x_2)\,\Phi(x_1) & x_2 > x_1 \end{cases} \quad (3.1.42)$$

Finally, in term of the interaction hamiltonian density $\mathcal{H}_I$, the operator S is given by

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int \cdots \int d^4x_1 \cdots d^4x_n \, T\left\{\mathcal{H}_I(x_1) \cdots \mathcal{H}_I(x_n)\right\} \quad (3.1.43)$$

where the integration is over all the space-time ($d^4x = d^3x\,dt$)

3.1.4.2 Feynman diagrams

Let us consider two examples of interactions, the quantum electromagnetic and the pseudo scalar meson theory (ps), where the two interaction are expressed in term of the normal product N respectively by

$$\mathcal{H}_I^{\rm QED} = i\,e\,N\{\bar\psi(x)\;\not{A}(x)\,\psi(x)\}, \tag{3.1.44}$$
$$\mathcal{H}_I^{\rm ps} = i\,g\,N\{\bar\psi(x)\,\gamma_5\,\phi(x)\,\psi(x)\} \tag{3.1.45}$$

The constants e and g denote the coupling, the fields in these equations are given in the interaction representation, so they satisfy the free field commutation relations. The T-matrix elements can be evaluated by rearranging the field operators in Eq. (3.1.43) in term of the normal product where the T-product is related to the N-product by

$$T\{A(x_1)\,B(x_2)\} = N\{A(x_1)\,B(x_2)\} + <0|T\{A(x_1)\,B(x_2)\}|0> \tag{3.1.46}$$

A and B stand for any field operator and $|0>$ denotes the vacuum state. The vacuum expectation value in right hand side of this equation is called the *contraction* of $A(x_1)$ and $B(x_2)$, it does not vanish only if one of the operators A and B creates particles which the other absorbs. The non-vanishing contraction for QED and ps theories are

$$<0|T\{A_\nu(x_2)\,A_\mu(x_1)\}|0> = D_F(x_2-x_1) = \frac{-i\,g_{\nu\mu}}{(2\,\pi)^4}\int\frac{d^4q\,e^{-i\,q\,x}}{q^2+i\,\lambda}, \tag{3.1.47}$$
$$<0|T\{\bar\psi_\beta(x_2)\,\psi_\alpha(x_1)\}|0> = S_{F\beta\alpha}(x_2-x_1) = \frac{i}{(2\,\pi)^4}\int\frac{d^4p\,e^{-i\,p\,x}}{\not{p}-i\,m_\psi+i\,\lambda}, \tag{3.1.48}$$
$$<0|T\{\Phi(x_2)\,\Phi(x_1)\}|0> = \Delta_{F\beta\alpha}(x_2-x_1) = \frac{i}{(2\,\pi)^4}\int\frac{d^4k\,e^{-i\,k\,x}}{k^2-m_\Phi^2+i\,\lambda}. \tag{3.1.49}$$

where D_f, S_f and Δ_F are called the Feynman propagators for the photon, the electron and the meson fields, respectively.

These relations enable us to select terms in Eq.(3.1.43) which their creation and destruction operators related to particles that we want to consider. Thus, the S-matrix can be expressed as a combination of terms involving Feynman propagators, γ-matrix ... etc. The final result will be a sum of a set of integrals over four-momentum which we call Feynman integrals. Each of these integrals can be graphically presented by a diagram called Feynman diagram, which can be converted to mathematical formula by using the Feynman rules. For example, the graph in Fig(3.2) represents a one-loop Feynman diagram, it gives a contribution to the first order of the perturbative expansion of the S-matrix. This contraction can be calculated by the following Feynman rules (we limit ourselves to QED):
- a) each *fermionic* internal line with momenta p and mass m is given by $\frac{i}{\not{p}-m+i\,\lambda}$
- b) each *photonic* internal line with momenta q is given by $\frac{g^{\mu\nu}}{q^2+i\,\lambda}$
- c) each vertex is given by $-i\,e\,\gamma^\mu$
- d) each fermion or anti-fermion incoming (outgoing) external line with momenta p and helicity l are given by the spinors $u_l(p)$ and $\bar v_l(q)$ ($\bar u_l(p)$ and $v_l(q)$), respectively.

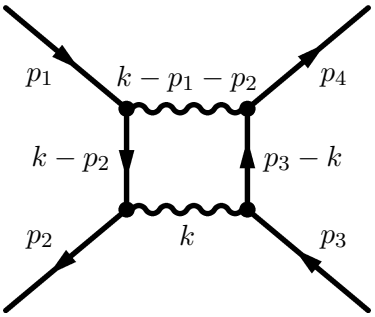


Figure 3.2: An example of one-loop Feynman diagram for QED, the straight lines denote the fermions (electron or positron) and the wavy lines denote the photon.

Then, the Feynman diagram in the Fig. (3.2) leads to the integral

$$e^4 \int \frac{d^4k}{(2\,\pi)^4} \left[\bar{v}(p_2)\,\gamma^\mu\,((\not{k} - \not{p}_2) + m_e)\,\gamma^\rho\,u(p_1)\,\bar{u}(p_3)\,\gamma_\mu\,((\not{p}_3) - \not{k} + m_\mu)\,\gamma_\rho\,v(p_4)\right] \frac{1}{(k^2 + i\,\lambda)((k-p_2)^2 - m_e^2 + i\,\lambda)((k - p_1 - p_2)^2 + i\,\lambda)((p_3 - k)^2 - m_\mu^2 + i\,\lambda)} \tag{3.1.50}$$

So, the scattering amplitude for a given process can be calculated by adding all contributing Feynman diagrams. In the next section, we study the analytic properties of such graphs.

3.2 Singularities of one-loop scattering amplitude

In the previous section, we showed how one can derive some analyticity properties of the scattering amplitude from the principle of unitarity, where we have shown that the inelastic two-body scattering amplitude $F(s,t)$ is analytic in Mandelstam variables under some suitable conditions on the value of the masses of the external particles (in the case of spineless equal mass particles).
In this section we investigate the analyticity properties of a general Feynman integral. We present a necessary and sufficient criterium to determine the singularities of a general Feynman integral on the physical sheet. We apply these equations to some Feynman graphs, and we show that one can recover the normal thresholds that we have mentioned above. On top of that, we show how we can calculate the soft and the collinear divergent contributions for an arbitrary Feynman diagram from these conditions [52, 56, 38, 57, 48, 64].

3.2.1 Singularities of an integral

A singularity of a function f is a point where f is not analytical, i.e. it is not differentiable at this point[3]. Consider the analytical function of two complex variables $g(z,w)$. For some

[3] At a singular point, either the function or its derivative or its higher derivatives become discontinued. In complex analysis, we distinguish between many type of singularities, *a) the isolated singularities* which come as, *(i) Removal singularities:* if the singular function f equals to an holomorphic function g (for

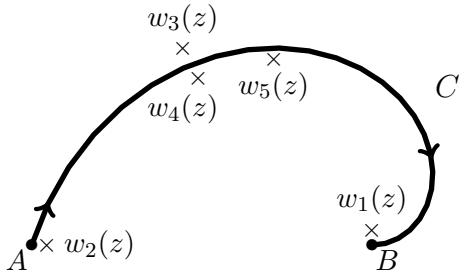


Figure 3.3: All possible singularities of the integrand $g(z, w)$: The end point of the contour of integration C are A and B, the singularities (denoted by "$\times$") are called $w_i(z)$. w_1 and w_2 are end point singularities, w_3 and w_4 are pinch singularities. No deformation of the contour C can avoid these singularities.

finite contour C of integration, we define a function f as follows

$$f(z) = \int_C g(z, w)\, dw \tag{3.2.51}$$

the singularities of the integrand in the w-complex plane are

$$w = w_r(z), \qquad\qquad r = 1, 2, \cdots \tag{3.2.52}$$

$f(z)$ is analytic as long as the integrand is analytic, i.e as long as no singularity $w_r(z)$ meets the contour of integration or this contour can be deformed in such way that these singularities are avoided. This deformation cannot avoid such singularities only in the three following cases:

- **End point singularities**: if one of the singularities w_r meets one of the end points of C (A or B, see Fig.(3.3)), thus the function $f(z)$ may be singular at the corresponding point z_1.

- **Pinch singularities:** if the contour is trapped between two (or more) singularities, i.e. these singularities approach the contour from the opposite side and coincide ($w_3(z)$ and $w_4(z)$). In this case, no deformation of the contour can avoid them.

- **Infinite deformation:** if the singularity $w_r(z)$ moves off to infinity dragging the contour when it is being deformed. Hence, f is singular at the corresponding point. This case, can be reduced to a special case of pinch singularities by making a simple change of variables, see [48].

In loop calculation, we face integrals of several variables and certainly the study of the singularities are much more complicated than in the case of integrals involving only one variable. Hence, it would be useful to generalize the previous discussion to multiple

example f(z)=$\frac{\sin(z)}{z}$, $z = 0$ is removal since $f(0) = 1$). *(ii) Pole:* the singular point z_0 is a pole if the singular function f equals to $g(z)/(z - z_0)^n$, where g is holomorphic and nonzero and n is a given natural number. *(iii) essential singularities:* if the Laurent series has infinitely many negative powers (for example $f(z) = e^{1/z}$). And *b) the branch points* which are the result of multi-valued functions as $\sqrt{z}$ and $\ln(z)$.

integrals. Let us consider the following function where the integrand is a function of several variables,

$$f(z) = \int_H \Pi_{i=1}^n \, dw_i \, g(z, w_i), \tag{3.2.53}$$

In this case, the contour of integration becomes a hypercontour (denoted by H) in w_i-space. The singularities of the integrand $g(z, w_i)$ are defined by several equations

$$\mathbb{S}_r(z, w_i) = 0, \qquad\qquad r = 1, 2, \cdots \tag{3.2.54}$$

For any value of z, $\mathbb{S}_r$ represent $2\,n - 2$-dimensional space in the $2\,n$-dimensional complex w_i-space. For example, in the case of one integration variable, the pinch singularities are given by $w_1(z_2) = w_2(z_2)$ which can be written as $\mathbb{S}_1 = w_1(z_2) - w_2(z_2) = 0$. Hence, the conditions of pinch singularities can be expressed in term of the analytic manifolds in Eq. (3.2.54).

The boundary of the hypercontour H can be described by the following analytic equations,

$$\tilde{\mathbb{S}}_r(z, w_i) = 0, \qquad\qquad r = 1, 2, \cdots \tag{3.2.55}$$

For example, in the case of one integration variable, the end point singularity is given by $w_r(z_1) = A$ which can be written as $\tilde{\mathbb{S}}_1 = w_r(z_1) - A = 0$ (A is one of the borders of the countour of integration). Hence, the conditions of the end point singularities can be expressed in term of the analytic manifolds in Eq. (3.2.55).

Singularities occur if a surface of singularities intersect with the boundary of the hyper-surface of integration (end point singularities), or if the hypercontour H is trapped by two or more surfaces $\mathbb{S}_r$, i.e. these surfaces approach H from the opposite side and the direction of their normal coincide. Hence, no deformation of H can avoid these surfaces. All possibilities that a singularity may happen are summarized by the following equations: for some complex parameters α_i and $\tilde{\alpha}_r$ not all equal to zero, we have

$$\alpha_i \, \mathbb{S}_i = 0, \qquad \text{for each } i, \qquad \text{hence } \alpha_i = 0 \text{ or } \mathbb{S}_i = 0 \tag{3.2.56}$$

$$\tilde{\alpha}_r \, \tilde{\mathbb{S}}_r = 0, \qquad \text{for each } r, \qquad \text{hence } \tilde{\alpha}_r = 0 \text{ or } \tilde{\mathbb{S}}_r = 0 \tag{3.2.57}$$

and for each integration variable w_j, we have

$$\frac{\partial}{\partial w_j} \left[\sum_i \alpha_i \, \mathbb{S}_i + \sum_r \tilde{\alpha}_r \, \tilde{\mathbb{S}}_r \right] = 0 \tag{3.2.58}$$

Eq. (3.2.58) means that the direction of the normals of the surfaces $\mathbb{S}_r$ and $\tilde{\mathbb{S}}_r$ coincide, i.e. the hyper-surfaces are tangent at the pinching point. These equations are only necessary conditions and not sufficient to such singularity to occur (Sufficient conditions will be given in the next paragraph).

I mention that a demonstration of these equations is not evident and needs the use of topology [48]. In the following, I will give only some arguments for Eqs.(3.2.56, 3.2.57, 3.2.58) in the case of one and two surfaces of singularities.

a) If two surfaces of singularities $\mathbb{S}_1$ and $\mathbb{S}_2$ approach the hypercontour H from opposite sides, and the direction of their normal coincide. Then, H may be trapped between these surfaces if:

$$\begin{cases} \mathbb{S}_1 = \mathbb{S}_2 = 0, \\ \alpha_1 \frac{\partial \mathbb{S}_1}{\partial w_i} + \alpha_2 \frac{\partial \mathbb{S}_2}{\partial w_i} = 0, \qquad i = 1, \cdots, n \end{cases} \tag{3.2.59}$$

for some non vanishing α_1, α_2.

b) Two different parts of the same singularity surface, say $\mathbb{S}_1$ may trap H. For example, if $\mathbb{S}_1$ is locally cone-like and H trapped at the vertex of the cone. The conditions for this to happen are

$$\begin{cases} \mathbb{S}_1 = 0, \\ \alpha_1 \frac{\partial \mathbb{S}_1}{\partial w_i} = 0, \qquad i = 1, \cdots, n \end{cases} \tag{3.2.60}$$

for some α_1.

In the following, we will apply the general results Eqs.(3.2.56, 3.2.57, 3.2.58) to derive the necessary and sufficient singularity conditions for the general scalar N point one-loop Feynman integral.

3.2.2 Landau equations for one-loop integrals

The general N-point one-loop scalar Feynman integral in n-dimension is given by

$$I_N^n(S) = \int \frac{d^n k}{i\,\pi^{n/2}} \frac{1}{\Pi_{i=1}^N (q_i^2 - m_i^2 + i\,\lambda)} \tag{3.2.61}$$

where S is a set containing the labels of the propagators (Fig. (3.4)), we put $S = \{1, \cdots, N\}$, the momenta through the propagators are

$$q_i = k + r_i, \qquad \text{with} \qquad r_i - r_{i-1} = p_i \tag{3.2.62}$$

p_i (with $i \in S$) are the momenta of the external legs, they are chosen ingoing for simplicity. Let us introduce the Feynman parameters z_i, with

$$\sum_i z_i = 1, \qquad \text{and} \qquad 0 \leq z_i \leq 1. \tag{3.2.63}$$

Eq. (3.2.61) becomes

$$I_N^n(S) = \Gamma(N) \int \frac{d^n k}{(2\,\pi)^{n/2}} \int_0^1 \Pi_{i=1}^N z_i \frac{\delta(1 - \sum_{i=1}^N z_i)}{[\sum_{i=1}^N z_i\,(q_i^2 - m_i^2) + i\,\lambda]^N}, \tag{3.2.64}$$

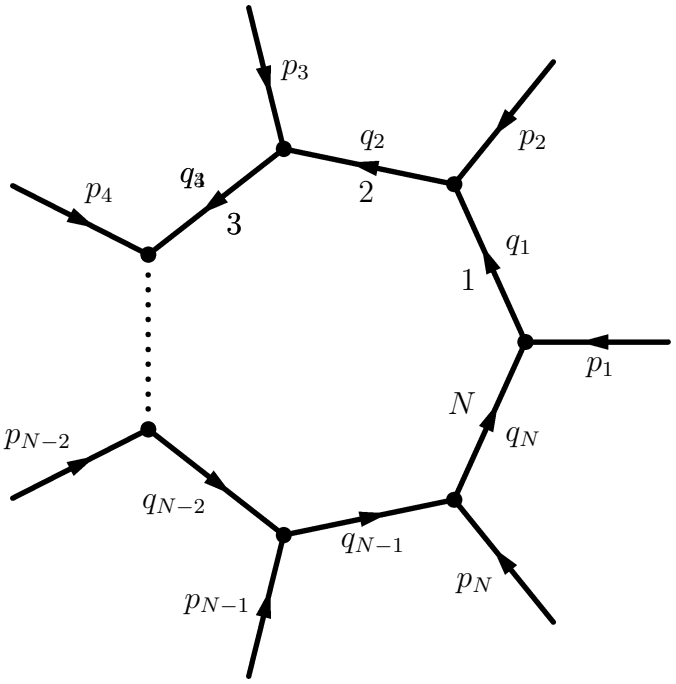


Figure 3.4: General N-point one-loop scalar integral

To perform the integration over k, we make the shift

$$l = k + \sum_{i=1}^{N} z_i \, r_i, \tag{3.2.65}$$

Hence, the denominator becomes

$$\sum_{i=1}^{N} z_i \, (q_i^2 - m_i^2) = l^2 - R^2, \tag{3.2.66}$$

with

$$R^2 = -\frac{1}{2} \sum_{i,j=1}^{N} z_i \, z_j \, \mathcal{S}_{ij} + i \, \lambda, \qquad \text{and} \qquad \mathcal{S}_{ij} = (q_i - q_j)^2 - m_i^2 - m_j^2, \tag{3.2.67}$$

After having performed the integration over l, I_N^n can be written

$$I_N^n(S) = (-1)^N \, \Gamma(N - \frac{n}{2}) \int \Pi_{i=1}^{N} \, dz_i \, \frac{\delta(1 - \sum_{i=1}^{N} z_i)}{[-\frac{1}{2} \sum_{i,j=1}^{N} \, z_i \, z_i \, \mathcal{S}_{ij} + i \, \lambda]^{N - \frac{n}{2}}}, \tag{3.2.68}$$

Eqs.(3.2.61, 3.2.64, 3.2.68) provide three representations of the same scalar integral, the first representation is in the momentum space, the second one is a mixed representation (in Feynman parameters and momentum space), the third one is in the Fyenman parameter space. In the following, we will apply the necessary conditions for the occurrence of singularities in the physical region presented in the end of the previous paragraph, to determine

the Landau equations corresponding to each representation of $I_N^n(S)$.

a) In momentum space:

The formula of the one-loop scalar integral with N external legs in n-dimension is given in Eq.(3.2.61). Since the loop momentum integration k is infinite, then the hypercontour of integration (denoted by H in the previous section) has no boundaries, i.e. $H =]-\infty, +\infty[^N$ (no boundary equations $\tilde{\mathbb{S}}_i$ has to be considered). Each singularity surface is given by

$$\mathbb{S}_i = q_i^2 - m_i^2 = 0 \tag{3.2.69}$$

we introduce the parameters α_i corresponding to each surface $\mathbb{S}_i$ and, apply Eqs.(3.2.56, 3.2.57, 3.2.58). Hence, the necessary conditions that a singularity occurs in the physical region in the representation Eq.(3.2.61) are given by

$$\begin{cases} \text{either} & q_i^2 = m_i^2, & \text{or} & \alpha_i = 0 & \forall\, i = 1, \cdots, N \\ \text{and} & \sum_{i=1}^N \alpha_i\, q_i = 0 & & & \forall\, i = 1, \cdots, N \end{cases} \tag{3.2.70}$$

The integral in Eq.(3.2.61) may have a singularity if for some non-vanishing α_i these equations have solution in the physical region. These equations are the Landau conditions corresponding to the one-loop scalar integral in the momentum space representation. The interpretation of these conditions is that, singularities may occur only when, for any internal propagator: either the four-momentum is on its mass shell or the corresponding parameter α_i vanishes.

b) Mixed representation:

The scalar integral after introducing Feynman parameters is given in Eq. (3.2.64). In this representation, the only surface of singularity of the integrand is

$$\mathbb{S} = \sum_{i=1}^{N} z_i\, (q_i^2 - m_i^2) = 0 \tag{3.2.71}$$

However, the hypercontour in the complex space (k, z_i) has several boundary equations

$$\tilde{\mathbb{S}}_i = z_i = 0 \tag{3.2.72}$$

It seems that $z_i = 1$ also form boundaries of the hypercontour, but it is not the case because of the delta function in the numerator. For example in the case of three Feynman parameters, the projection of the hypercontour (before the deformation) to the real (z_1, z_2) space (see Fig.(3.5)) has the boundaries $z_1 = 0$, $z_2 = 0$ and $z_1 + z_2 = 1$; but the last one is just $z_3 = 0$, by means of the δ-function. Then the boundaries in z_i-space are $z_i = 0$ (and not $z_i = 0$).

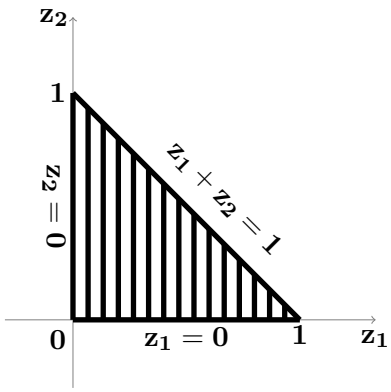


Figure 3.5: The non deformed hypercontour in z_1 and z_2 space

From Eq.(3.2.56) and Eq.(3.2.71), we get

$$\begin{cases} \text{either} & \alpha \;=\; 0, \\ \text{or} & \mathbb{S} \;=\; 0, \end{cases} \tag{3.2.73}$$

(α stands for a parameter α_i introduced above), and from Eq.(3.2.57) and Eq.(3.2.72), we get

$$\text{for each } i \begin{cases} \text{either} & \tilde{\alpha}_i \;=\; 0, \\ \text{or} & z_i \;=\; 0, \end{cases} \tag{3.2.74}$$

and from Eq.(3.2.58) and Eqs.(3.2.71, 3.2.72), we get

$$\begin{cases} \partial[\alpha\,\mathbb{S} + \sum_i \tilde{\alpha}_i\, z_i]/\partial k \;=\; 0 \\ \iff \;\; \alpha\,\partial\,\mathbb{S}/\partial k \;=\; 0 \end{cases} \quad \text{and for each } l \begin{cases} \partial[\alpha\,\mathbb{S} + \sum_i \tilde{\alpha}_i\, z_i]/\partial z_l \;=\; 0 \\ \iff \;\; \alpha\,\partial\,\mathbb{S}/\partial z_l + \tilde{\alpha}_l \;=\; 0 \end{cases} \tag{3.2.75}$$

Then, the Landau equations for this representation are given by

$$\begin{cases} \mathbb{S} \;=\; 0, \\ \partial\mathbb{S}/\partial k \;=\; 0, \\ \text{either} \quad z_l \;=\; 0, \quad \text{or} \quad \partial\mathbb{S}/\partial z_l \;=\; 0, \quad \text{for each } l \end{cases} \tag{3.2.76}$$

these conditions are completely equivalent to those corresponding to the momentum space representation.

c) In Feynman parameters space:

In this representation, the formula of the scalar integral is given in Eq.(3.2.68). The only surface of singularity is

$$\mathbb{S} = \sum_{i,j=1}^{N} z_i\, z_j\, \mathcal{S}_{ij} = 0 \tag{3.2.77}$$

and the boundary equations of the hypercontour of integration are

$$\tilde{\mathbb{S}}_i = z_i = 0 \tag{3.2.78}$$

Hence, the Landau equations of this representation are given by

$$\text{for each } i \begin{cases} & \mathbb{S} = 0, \\ & \text{and} \\ \text{either} & z_i = 0, \qquad \text{or} \quad \partial\mathbb{S}/\partial z_i = 0. \end{cases} \tag{3.2.79}$$

3.2.3 Necessary and sufficient conditions

If the loop momentum k runs over the real values, the Feynman parameters z_i are real positive and all the masses of the internal lines have a negative imaginary part. Then, the integral I_N^n evidently defines an analytical function. This is due to the fact that none of the internal propagators vanishes by means of the negative imaginary part of the masses $(-i\,\lambda)$. If this imaginary part vanishes, and the Landau conditions presented above are satisfied, then the integrand may be singular in some points of the phase space. The necessary and sufficient conditions that a singularity of the integral I_N^n occurs in the physical region are[55]

$$\begin{cases} \forall\, i\, z_i\, (q_i^2 - m_i^2) = 0, \\ \sum_{i=1}^{N}\, z_i\, q_i = 0, \\ \begin{cases} z_i \geq 0, \\ k = k^*, \end{cases} \\ \det(\mathcal{S}) = 0. \end{cases} \tag{3.2.80}$$

The two conditions in the first two lines of Eq.(3.2.80) are the necessary and not sufficient Landau conditions presented above; the two conditions in the third and the fourth line of the same equation define the physical region which is $[-\infty, +\infty]^N$ for k and $[0, +\infty]^N$ for z_i (the hyper-space $[0, 1]^N$ can be extended to $[0, +\infty]^N$ without changing the structure of singularities since $z_i = 1$ is not a boundary of the hypercontour as shown above); and the last condition $(\det(\mathcal{S}) = 0)$ defines the so called "the singular Landau curve" which together with the other conditions form the necessary and sufficient conditions for a singularity to occur in the physical region.

Proof:

We focus on the case of non vanishing z_i [55, 63], i.e all the internal lines are on the mass shell (leading singularities, see below). In the Feynman representation, the denominator is given by

$$R^2 = -\frac{1}{2}\, z^\dagger\, \mathcal{S}\, z + i\,\lambda \tag{3.2.81}$$

we call the eigenvalues of the matrix $\mathcal{S}$, $\sigma_1 \cdots \sigma_N$ (where $|\sigma_N| \ll |\sigma_i|$ for $i = 1, \cdots, N-1$) and the normalized eigenvector corresponding to σ_N is $\bar{z}$:

$$\mathcal{S}\,\bar{z} = \sigma_N\,\bar{z} \tag{3.2.82}$$

with (3.2.83)

$$\bar{z} = (\bar{z}_1 \cdots \bar{z}_N) \qquad \text{and} \qquad \sum_{i=1}^{N} \bar{z}_i = 1 \tag{3.2.84}$$

let us make the following transformation

$$\eta = z - \bar{z} \tag{3.2.85}$$

then the denominator becomes

$$R^2 = -\frac{1}{2}\,\eta^\dagger\,\mathcal{S}\,\eta - \frac{1}{2}\,\sigma_N\,|\bar{z}|^2 - \sigma_N\,\eta^\dagger\,\bar{z} - i\,\lambda \tag{3.2.86}$$

Since $\mathcal{S}$ is real and symmetric matrix, it can be diagonalized by a real orthogonal transformation such that

$$\eta_i = \sum_{j=1}^{N} A_{ij}\,\bar{\eta}_j \tag{3.2.87}$$

where the matrix A satisfies

$$\begin{cases} A & \text{orthogonal,} \\ \sum_{j=1}^{N} A_{ij} & = 1/|\bar{z}|, \\ \det(A) & = 1. \end{cases} \tag{3.2.88}$$

we notice that the normalized eigenvectors of $\mathcal{S}$ are the columns of A.
We suppose that $|\sigma_N \ll 1|$ and $\sigma_i \neq 0$, and we neglect terms that give contribution of order $\mathcal{O}(\sigma_N^2)$ to the final result. Hence, the denominator becomes

$$R^2 = -\frac{1}{2}\sum_{i=1}^{N} \sigma_i\,\bar{\eta}_i^2 - \frac{1}{2}\,\sigma_N\,|\bar{z}|^2 - i\,\lambda, \tag{3.2.89}$$

we perform the integration over $\bar{\eta}_N$ which is evident by means of the δ-function in the numerator, we get

$$I_N^n \propto \int_{-\infty}^{+\infty} d\bar{\eta}_1 \cdots d\bar{\eta}_{N-1}\,\frac{1}{[-\frac{1}{2}\sum_{i=1}^{N-1}\sigma_i\,\bar{\eta}_i^2 - \frac{1}{2}\,\sigma_N\,|\bar{z}|^2 + i\,\lambda]^{N-\frac{n}{2}}}, \tag{3.2.90}$$

where the singular contribution is not changed if the integration countour is extend to infinity provided that $N - n/2$ is sufficiently large, see [48].
If $\sigma_N \to 0$ (which means that $\det(\mathcal{S}) = \sigma_1 \cdots \sigma_N \to 0$) and $\lambda \to 0$, the zeros of the denominator of this equation correspond to $\bar{\eta}_i^{\pm} = f(\bar{\eta}_1, ..., \bar{\eta}_{i-1}, \bar{\eta}_{i+1}, ..., \bar{\eta}_{N-1}) \pm i\,\lambda$ for each $i = 1, \cdots, N-1$ since $\sigma_i \neq 0$. This means that that $\bar{\eta}_i^+$ approaches $\bar{\eta}_i^-$ from the opposite sides of the contour (due to $\pm i\,\lambda$), then the contour must be pinched between at least two singularity surfaces. Hence, the conditions given above are necessary and sufficient for the occurrence of singularities in the physical sheet. I notice that if $\mathcal{S}_{ij}$ has more than one zero eigenvalue, the demonstration becomes more complicated than this one, and the integral may have two Landau singularities one finite and the other one leads to IR divergent as in the case of double parton scattering singularity [55].

3.2.4 Nature of singularities

We call *leading singularities* of a Feynman graph, singularities which do not correspond to $z_i = 0$ while singularities corresponding to $z_i = 0$ are called *sub-leading singularities*. The later singularities are shared by the contracted graphs (For the triangle graphs obtained from the box by pinching one propagator, see Fig. (B.1)).

3.2.4.1 Leading Landau singularities

After performing all the $N-1$ integration over $\bar{\eta}$, in the approximation $|\sigma_N| << 1$, of the integral in Eq. (3.2.90), one can prove that (see [63])

$$I_N^n(S) \propto \frac{1}{\sqrt{(-1)^{N-K-1}\,\Pi_{i=1}^{N-1}\sigma_i}}\,\frac{\Gamma((N-n+1)/2)}{[\sigma_N\,\frac{-|\bar{z}|^2}{2} - i\,\varepsilon]^{(N-n+1)/2}}, \tag{3.2.91}$$

with the assumption

$$\begin{cases} \sigma_i > 0, & i = 1, \cdots, k, \\ \sigma_i < 0, & i = k+1, \cdots, N-1, \end{cases} \tag{3.2.92}$$

this result is valid only for $\sigma_i \neq 0$ and $N-n+1 > 0$ (If $N-n+1 \leq 0$, one can just expand this formula around $\varepsilon = 0$ where $n = 4-2\varepsilon$).

Let us apply this formula to some of the `Golem95` basic integrals that we will study in the next Chapters. The table (3.1) summarizes all the leading singularities that one of this integrals might have at $\sigma_N \to 0$, and the possibility if they lead to IR divergences.

N	n	Nature of singularity	IR divergences
4	$4-2\,\varepsilon$	$\sigma_4^{-1/2}$	divergent
4	$6-2\,\varepsilon$	$\sigma_4^{1/2}$	none
4	$8-2\,\varepsilon$	$\sigma_4^{3/2}$	none
4	$10-2\,\varepsilon$	$\sigma_4^{5/2}$	none
4	$12-2\,\varepsilon$	$\sigma_4^{7/2}$	none
3	$4-2\,\varepsilon$	$\ln(\sigma_3)$	divergent
3	$6-2\,\varepsilon$	σ_3	none
2	$4-2\,\varepsilon$	$\sigma_2^{1/2}$	none
1	$4-2\,\varepsilon$	σ_1	none

Table 3.1: Landau singularities of `Golem95` scalar basic integrals: IR divergences correspond to non integrable Landau singularities, this is the case for I_3^4 and I_4^4. The other function have Landau singularities, since if we derive enough number times in σ_N, the obtained derivatives will be singular if $\sigma_N \to 0$; but they do not lead to infinite results.

3.2.4.2 Sub-Leading Landau singularities

Consider the singularities corresponding to ν contraction, where ν stands for the number of propagators corresponding to $z_i = 0$, with $0 \leq \nu < N-1$. The Landau conditions for such singularities occur are given by

$$\begin{cases} D = R^2 = 0, & \\ z_i = 0, & i = 1, \cdots, \nu \\ \frac{\partial D}{\partial z_i} = 0, & i = \nu+1, \nu+2, \cdots, N \end{cases} \tag{3.2.93}$$

We perform the trivial integration over z_N (by means of the δ-function), the denominator becomes

$$\begin{aligned} D(z_1, \cdots, z_{N-1}, 1 - \sum_{i=1}^{N-1} z_i) &= D'(z_1, \cdots, z_{N-1}) \\ &= \frac{1}{2} \sum_{i,j \neq N}^{N-1} z_i \, z_j \, G_{ij}^{(N)} - \sum_{i \neq N}^{N-1} z_i \, V_i^{(N)} - \frac{1}{2} \, \mathcal{S}_{NN} - i \, \lambda, \end{aligned} \tag{3.2.94}$$

$$G_{ij}^{(N)} = -(\mathcal{S}_{ij} - \mathcal{S}_{Nj} - \mathcal{S}_{iN} + \mathcal{S}_{NN}) = 2 \, r_i \cdot r_j, \tag{3.2.95}$$

$$V_i^{(N)} = \mathcal{S}_{Ni} - \mathcal{S}_{NN}. \tag{3.2.96}$$

Hence, the Landau conditions become

$$\begin{cases} D' = 0, & \\ z_i = 0, & i = 1, \cdots, \nu \\ \frac{\partial D'}{\partial z_i} = 0, & i = \nu+1, \mu+2, \cdots, N-1 \end{cases} \tag{3.2.97}$$

for invertible Gram matrix G_{ij}, the solution of the last two equation is given by

$$\begin{gathered} \bar{z} = (0, \cdots, 0, \bar{z}_{\nu+1}, \cdots, \bar{z}_{N-1}), \\ \text{with} \\ \bar{z}_i = \frac{1}{\det(G)} \sum_{j=\nu+1}^{N-1} V_j^{(N)} \, \hat{G}_{ij}, \end{gathered} \tag{3.2.98}$$

$\det(G)$ and $\hat{G}_{ij}$ are the determinant and the element ij of the co-matrix of the Gram matrix, respectively. Inserting this result in the first Landau equations, we find

$$\begin{aligned} D'(\bar{z}_i) &= \frac{1}{2} \, \mathcal{S}_{NN} - \frac{1}{2} \sum_{i=\nu+1}^{N-1} \bar{z}_i \, V_i^{(a)} \\ &= \frac{1}{2} \, \frac{\det(\mathcal{S})}{\det(G)} \\ &= 0 \end{aligned} \tag{3.2.99}$$

where we have used

$$\det(\mathcal{S}) = (-1)^{N-1}[\mathcal{S}_{NN}\,\det(G^{(N)}) + (V^{(N)})^T \cdot \hat{G}^{(N)} \cdot V^{(N)}] \tag{3.2.100}$$

Again, this proves that $\det(\mathcal{S}) = 0$ with the other conditions form the necessary and sufficient condition for a singularity of I_N^n to occur.

To find the nature of the singularity, we expand, in the neighborhood of $\bar{z}$, the D' using Taylor expansion, then the denominator becomes

$$D'(z_i) = D'(\bar{z}_i) + \sum_{i=1}^{\nu}(z_i - \bar{z}_i)\,\frac{\partial D'}{\partial z_i}\bigg|_{z_i=\bar{z}_i} + \frac{1}{2}\sum_{j,k=\nu+1}^{N-1}(z_j - \bar{z}_j)\,(z_k - \bar{z}_k)\,\frac{\partial^2 D'}{\partial z_j\,\partial z_k}\bigg|_{z=\bar{z}}, \tag{3.2.101}$$

Since we are only concerned with some finite segments of the hypercontour near $\bar{z} = z$, the integration over z_i can be extended from $-\infty$ to $+\infty$ without changing the singular part, provided $N - n/2$ is sufficiently large. Then, one can prove that an explicit integration over the $N-1$ variables leads to [64]

$$I_N^n(S) \sim D'(\bar{z})^{-\gamma}, \tag{3.2.102}$$

this result holds provided

$$\gamma = \frac{1}{2}\,(N - \nu - n + 1) > 0, \tag{3.2.103}$$

For negative γ, one can replace Eq. (3.2.102) by

$$I_N^n(S) \sim D'(\bar{z})^{|\gamma|}\,\ln(D'(\bar{z})) \tag{3.2.104}$$

Then, the nature of singularity is of square root or logarithmic, this depends on the number of internal lines $N - \nu$ of the contracted graph. If $\nu = 0$, we see that we recover again the nature of the leading singularity discussed above in term of the eigenvalues of the matrix $\mathcal{S}$. For example:
- for N=3, $\nu = 0$ and $n = 4$ we find that $I_3^4 \sim \ln(\frac{1}{2}\,\frac{\det(S)}{\det(G)})$ which leads to IR divergences.
- for N=4, $\nu = 0$ and $n = 4$ we find that $I_4^4 \sim (\frac{1}{2}\,\frac{\det(S)}{\det(G)})^{-1/2}$ which leads to IR divergences.

3.3 Examples

3.3.1 Triangle graph

Lets consider the vertex diagram with only one off-shell external momenta and in 4-dimension [56],

$$I_3^4(S) = \int \frac{d^4k}{i\,\pi^2}\,\frac{1}{(q_1^2 - m_1^2)\,(q_2^2 - m_2^2)\,(q_3^2 - m_3^2)} \tag{3.3.105}$$

with

$$q_1 = k - p_2, \qquad q_2 = k + p_1, \qquad q_3 = k, \tag{3.3.106}$$

For the leading singularity, the Landau equations consist of

$$\text{for each } i = 1,2,3 \begin{cases} q_i^2 = m_i^2, \\ \sum_{i=1}^{3} z_i\, q_i = 0, \end{cases} \tag{3.3.107}$$

by multiplying the last line of Eq. (3.3.107) by q_j, we get the system of three simultaneous equations

$$\sum_{i=1}^{3} z_i\, q_i \cdot q_j = 0, \qquad j = 1,2,3. \tag{3.3.108}$$

which has a solution only if $\det(q_i \cdot q_j) = 0$, this can be written as

$$\frac{\det(q_i \cdot q_j)}{m_i\, m_j} = \begin{vmatrix} 1 & y_{12} & y_{13} \\ y_{12} & 1 & y_{23} \\ y_{13} & y_{23} & 1 \end{vmatrix} = 0 \tag{3.3.109}$$

with

$$\begin{aligned} y_{ij} = y_{ji} &= -\frac{1}{2}\,\frac{(q_i - q_j)^2 - m_i^2 - m_j^2}{m_i\, m_j} \\ &= -\frac{1}{2}\,\frac{\mathcal{S}_{ij}}{m_i\, m_j} \end{aligned} \tag{3.3.110}$$

Eq. (3.3.109) defines the surface on which we can find the leading Landau singularities. In a similar way, we may find the surface corresponding to the sub-leading singularities occurring at $z_i = 0$ ($z_3 = 0$ for example), the surface given by

$$\begin{vmatrix} 1 & y_{12} \\ y_{12} & 1 \end{vmatrix} = 0 \tag{3.3.111}$$

which leads to

$$p_3^2 = (m_1^2 \pm m_2^2) \tag{3.3.112}$$

Then we recover the normal threshold at $s = (m_1 + m_2)^2$ (for the case of equal masses, $s = 4\,m^2$ which we deduced by unitarity in the first part of this Chapter); therefore the singularity at $s = (m_1 - m_3)^2$ does not appear in the physical sheet. Similar singularities exist in the p_1^2 and p_2^2 channels.

3.3.2 Study of the case: $\det(\mathcal{S}) = 0$ and $\det(G) = 0$ simultaneously

The Gram matrix elements $G_{ij}^{(N)}$ in terms of the external momenta are defined by

$$G_{ij}^{(N)} = 2\,(r_i \cdot r_j) = 2 \sum_{k=1}^{i<N} \sum_{l=1}^{j<N} (p_k \cdot p_l) \tag{3.3.113}$$

the determinant of this matrix equals to the determinant of

$$\Gamma_{ij}^{(N)} = 2\,(p_i \cdot p_j), \qquad \text{for } i,j = 1,\cdots N-1. \tag{3.3.114}$$

$$\det(G^{(N)}) = \det(\Gamma^{(N)}) \tag{3.3.115}$$

where G and $\Gamma^{(N)}$ are $(N-1)\times(N-1)$ matrices.

$\det(G)$ vanishes means that at least one eigenvalue of the matrix $G^{(N)}$ (and $\Gamma^{(N)}$) vanishes. In general, to a vanishing eigenvalue of $\Gamma^{(N)}$ corresponds a family of scalars $\{x_j^{(1)}\}_{j=1,\cdots,N-1}$ not all vanishing which satisfy

$$\sum_{j=1}^{N-1} \Gamma_{ij}^{(N)}\, x_j^{(1)} = 0, \qquad i = 1,\cdots,N-1 \tag{3.3.116}$$

This shows that the vector

$$\lambda^{(1)} = \sum_{j=1}^{N-1} x_j^{(1)} p_j \tag{3.3.117}$$

is orthogonal to each of the p_i , hence it has vanishing squared pseudonorm:

$$(p_i \cdot \lambda^{(1)}) = 0, \qquad (\lambda^{(1)})^2 = 0 \tag{3.3.118}$$

which means that $\lambda^{(1)}$ is either zero i.e. $\{p_i\}_{i=1,\cdots,N-1}$ are linearly dependent or a non-zero light-like vector orthogonal to each of the 4-momentum p_i 's. Similarly, if $\Gamma^{(N)}$ has two vanishing eigenvalues, i.e. it is of rank $N-3$. There exist (at least) two linearly independent families of scalars, $\{x_j^{(1)}\}_{j=1,\cdots,N-1}$ not all vanishing, and $\{x_j^{(2)}\}_{j=1,\cdots,N-1}$ not all vanishing either.

One can prove that, if $\det(\mathcal{S}) = 0$ and $\det(G) = 0$ simultaneously, we have

$$\sum_{i=1}^{N} x_i \leq 0 \tag{3.3.119}$$

which cannot be satisfied in the physical region, the proof is given in Appendix A.5 in ref. [81]. Due this important result: *for configurations with vanishing Gram determinants, no Landau singularity can occur in the physical region.*

3.4 Infrared and collinear divergences

3.4.1 Soft divergences

Soft divergences appear if the four momentum of a massless propagator in the loop vanishes, hence Landau equations corresponding to a such situation are

$$\begin{cases} q_1^2 = m_1^2 \to 0 \\ z_1 = \mathcal{O}(1), \\ z_2, z_N = \mathcal{O}(\delta), \\ z_3,\cdots,z_{N-1} = \mathcal{O}(\delta^2) \\ q_i^2 \neq m_i^2 \end{cases} \tag{3.4.120}$$

with $\delta << 1$, it is a parameter which characterizes the values of Feynman parameters. Lets assume that all the internal and external lines are massless, then

$$R^2 = -\frac{1}{2}\sum_{i,j=1}^{N} z_i\, z_j\, \mathcal{S}_{ij} = -\sum_{i\leq j}^{N} z_i\, z_j\, \mathcal{S}_{ij}$$
$$\sim \sum_{i=3}^{N-1} z_1\, z_i\, \mathcal{S}_{1j} + z_2\, z_N\, \mathcal{S}_{2N} + \mathcal{O}(\delta^2) \tag{3.4.121}$$

By inserting Eq.(3.4.121) in $I_N^{4-2\,\varepsilon}$ and performing the integration over Feynman parameters, we prove that the soft divergent contribution of $I_N^{4-2\,\varepsilon}$ corresponding to the approximation given in Eq. (3.4.120) is given by

$$I_{N\,\mathrm{div}}^{4-2\,\varepsilon}(S) = \frac{i}{(4\,\pi)^{2-\varepsilon}}\,(-1)^{N-1}\,\frac{\Gamma(1+\varepsilon)}{\varepsilon^2}\,\frac{(-S_{2N})^{-\varepsilon}}{(-S_{2N})\,\Pi_{i=3}^{N-1}\,(-S_{1i})}. \tag{3.4.122}$$

For arbitrary dimension of space-time, i.e. $n = m + 4 - \varepsilon$, with $m = 2, 4, \cdots$, the approximated $I_3^{m+4-2\,\varepsilon}$ and $I_4^{m+4-2\,\varepsilon}$ are given by

$$I_3^{m+4-2\,\varepsilon} \sim -\frac{4\,\Gamma(1-\frac{m}{2}+\varepsilon)}{(-2\,\varepsilon+m)^2}\int_0^1 dz_2 \frac{(-S_{23}\,z_2)^{m/2-\varepsilon}}{S_{23}}. \tag{3.4.123}$$

$$I_4^{m+4-2\,\varepsilon} \sim -\frac{4\,\Gamma(2-\frac{m}{2}+\varepsilon)}{(2\,\varepsilon-m)\,(2+2\,\varepsilon-m)}\int_0^1 \frac{dz_2}{z_2}\,\frac{(-S_{13})^{-\varepsilon}(-S_{24}\,z_2)^{-\varepsilon}(-S_{13}-S_{24}\,z_2)^{-\varepsilon}}{S_{13}\,S_{24}}$$
$$\times \Big\{(-S_{13})^{m/2}\,(-S_{24}\,z2)^{\varepsilon}(-S_{13}-S_{24}\,z_2)^{\varepsilon} + (-S_{13})^{\varepsilon}(-S_{24}\,z_2)^{m/2}\,(-S_{13}-S_{24}\,z_2)^{\varepsilon}$$
$$- (-S_{13})^{\varepsilon}\,(-S_{24}\,z_2)^{\varepsilon}\,(-S_{13}-S_{24}\,z_2)^{m/2}\Big\}. \tag{3.4.124}$$

After integrating over z_2 (which can be done very easily), one can prove that Eq. (3.4.123, 3.4.124) are free of soft divergences. Then, the only soft divergent 3-point and 4-point integrals are I_3^4 and I_4^4 where the divergent parts are given by Eq. (3.4.122).

3.4.2 Collinear divergences

Collinear divergences appear when the 4-momentum of two massless propagators adjacent to an external leg becomes proportional to its 4-momentum, hence Landau equations corresponding to a such situation are

$$\begin{cases} z_1, z_N = \mathcal{O}(1), \\ z_2, \cdots, z_{N-1} = \mathcal{O}(\delta), \\ q_1^2 = m_1^2 \to 0, \\ q_N^2 = m_N^2 \to 0, \\ z_1\,q_1 + z_N\,q_N = 0. \end{cases} \tag{3.4.125}$$

let us call the momuntum of the external particle linked to the propagators number 1 and N, p_1, then

$$p_1 = q_1 - q_N, \tag{3.4.126}$$

multiplying the last equation in Eq.(3.4.125) by q_1 or q_N, we get

$$z_1\, m_1^2 + z_2\, q_1 \cdot q_N = 0, \tag{3.4.127}$$

$$z_1\; q_1 \cdot q_N + z_2\, m_N^2 = 0, \tag{3.4.128}$$

Since the masses m_1 and m_N equals to zero, then

$$p_1 \cdot q_1 = p_1 \cdot q_N = q_1 \cdot q_N = 0, \tag{3.4.129}$$

hence $q_1 \parallel q_N \parallel p_1$.
Now, let us derive a general formula of the collinear divergent contribution to $I_N^{4-2\,\varepsilon}$ in the case of massless external and internal lines. In this approximation, the denominator can be written as

$$R^2 = \sum_{i=3}^{N-2} z_j \left[z_1 \left(-S_{1j}\right) + z_N \left(-S_{jN}\right)\right] + z_2\, z_N \left(-S_{2N}\right) + z_1\, z_{N-1} \left(-S_{1N-1}\right) + \mathcal{O}(\delta) \tag{3.4.130}$$

After integrating over all Feynman parameters, we get

$$I_{N\,C}^{4-2\,\varepsilon} \sim \frac{\Gamma(1+\varepsilon)}{\varepsilon^2} \left\{ \frac{\left(-S_{2N}\right)^{-\varepsilon}}{\left(-S_{2N}\right) \Pi_{i=3}^{N-1} \left(-S_{1i}\right)} + \frac{\left(-S_{1N-1}\right)^{-\varepsilon}}{\left(-S_{1N-1}\right) \Pi_{i=2}^{N-2} \left(-S_{iN}\right)} \right.$$
$$\left. - \sum_{j=3}^{N-2} \frac{\left(-S_{ij} + S_{jN}\right)^{N-4} \left(\left(-S_{1j}\right)^{\varepsilon} - \left(-S_{jN}\right)^{\varepsilon}\right)}{\left(-S_{2N}\right) \left(-S_{1N-1}\right) \left(-S_{1j}\right) \left(-S_{jN}\right) \Pi_{i\neq j=3}^{N-2} \left(\left(-S_{1j}\right) \left(-S_{iN}\right) - \left(-S_{jN}\right) \left(-S_{1i}\right)\right)} \right\} \tag{3.4.131}$$

By following the same reasoning as in the case of the soft approximation, we can prove that scalar integrals $I_N^{m+4-2\,\varepsilon}$ are free of collinear divergences.
The total IR contribution to a given Feynman graph is given by

$$I_{N,\mathrm{div}}^{4-2\,\varepsilon} = \sum_i I_{N\,C}^{4-2\,\varepsilon} - \sum_j I_{N\,S}^{4-2\,\varepsilon} \tag{3.4.132}$$

where i and j run over all the possible collinear and soft sectors, respectively. Each sector is defined by a given Landau equations (each of Eq. (3.4.120) and Eq. (3.4.125) defines a soft and a collinear sector, respectively). For physical observable (cross section, decay rate, ...), these divergences should cancel out by means of Lee-Kinoshita-Nauenberg theorem which states that [58, 59, 61]:
In theories involving non-massive fields, the cross section is free of soft and collinear singularities by summing over all the degenerate initial and final states.

3.5 Dispersion relation

The principles of causality and the local structure of field theories equations impose certain constraints on the behavior of the scattering amplitudes. The scattering amplitude is a function of the energy and momentum transfer, if these variables are analytically continued

from the physical region to the non-physical one, many appropriate relations for computing the amplitude or expressing it in term of some measurable quantities can be derived.
In this section, we discuss one of these useful relations, the "dispersion relation and it applications in quantum field theories" [38, 56, 48].

3.5.1 Causality and Kramers-Kronig Relation

The amplitude of the forward scattered monochromatic light plan wave along the x axis is linearly related to the incident wave by

$$a_{\rm scatt}(\omega) \quad = \quad f(\omega)\, a_{\rm inc}(\omega) \tag{3.5.133}$$

ω is the frequency of the incident wave and f is an analytic function on ω.
At infinite x, it becomes

$$a_{\rm scatt}(x,t)|_{x\to\infty} \quad \longrightarrow \quad a_{\rm scatt}(\omega)\,\frac{e^{-i\,\omega\,(t-x)}}{x} \tag{3.5.134}$$

Hence, the packets formed by superposing the incident and the scattered waves are given by

$$\begin{aligned} A_{inc}(x,t) \quad &= \quad \int_{-\infty}^{+\infty} d\omega'\, a_{inc}(\omega')\, e^{-i\,\omega'\,(t-x)} \\ A_{scatt}(x,t)\Big|_{x\to\infty} \quad &= \quad \frac{1}{x}\int_{-\infty}^{+\infty} d\omega'\, f(\omega')\, a_{inc}(\omega')\, e^{-i\,\omega'\,(t-x)} \end{aligned} \tag{3.5.135}$$

The causality condition imposes that the incident packet vanishes for $x > t$ (no signal propagates faster than light), which implies that the Fourier transform of the amplitude is

$$a_{\rm inc}(\omega) \quad = \quad \frac{1}{2\,\pi}\int_{-\infty}^{0} dx\, A_{\rm inc}(x,0)\, e^{-i\,\omega\,x} \tag{3.5.136}$$

where the physical requirement of causality is

$$A_{inc}(x,t) = 0, \qquad \text{for} \qquad x > t \tag{3.5.137}$$

The amplitude in Eq. (3.5.136) can be analytically continued into the upper half of the complex ω-plan, this can be seen by replacing ω by $\omega + i\,|\gamma|$ (γ is an arbitrary real parameter) in this equation, which leads to a convergent integral in this plane. Thus, the amplitude of the scattered wave $f(\omega)\, a_{\rm inc}(\omega)$ may be also analytically continued into the upper ω complex plan; which means that the function $f(\omega)$ is also analytically continued into this complex plan.
Since f is analytically continued in the upper half of the complex ω-plane, we can apply the Cauchy's formula to this function:

$$f(z) \quad = \quad \frac{1}{2\,\pi\,i}\int_{\mathcal{C}} \frac{d\omega'\, f(\omega')}{\omega' - z} \tag{3.5.138}$$

for every z inside the closed $\mathcal{C}$.
for any $z = \omega + i\,|\gamma|$ in the upper half plan $f(\omega)$ is given by

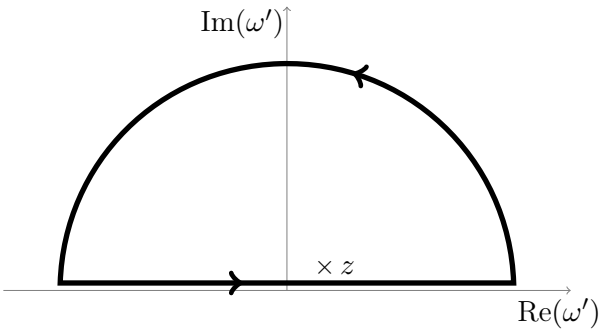


Figure 3.6: Contour in the upper half ω' plane for the equation (3.5.138)

$$\begin{aligned} f(\omega) &= \lim_{\varepsilon\to 0^+} f(\omega + i\,\varepsilon) \\ &= \frac{1}{2\,\pi\,i}\,P\int_{-\infty}^{+\infty}\frac{d\omega'\,f(\omega')}{\omega'-\omega} + \frac{1}{2}\,f(\omega) + \frac{1}{2}\,\mathcal{C}_\infty \end{aligned} \tag{3.5.139}$$

$\mathcal{C}_\infty$ stand for the circle in Fig.(3.6) (with infinite radius). The first term is the contribution of the principal value (P) of the integral across the real axis where

$$P(f(\omega)) = \lim_{\varepsilon\to 0^+}\left\{\int_{-\infty}^{-\varepsilon} f(\omega)\,d\omega + \int_{+\varepsilon}^{+\infty} f(\omega)\,d\omega\right\} \tag{3.5.140}$$

the second term is the contribution from the half circuit around the $\omega' \to \omega$ and the last term is the contribution of the infinite semicircle, with $\mathcal{C}_\infty = C_\infty^{\mathrm{Re}} + i\,C_\infty^{\mathrm{Im}}$ (C_∞^{Re} and C_∞^{Im} are real and the imaginary parts of C_∞, respectively). Hence,

$$\mathrm{Re}\,f(\omega) = \frac{1}{\pi}\,P\int_{-\infty}^{+\infty}\frac{d\omega'\,\mathrm{Im}\,f(\omega')}{\omega'-\omega} + \mathcal{C}_\infty^{(\mathrm{Re})} \tag{3.5.141}$$

$$\mathrm{Im}\,f(\omega) = -\frac{1}{\pi}\,P\int_{-\infty}^{+\infty}\frac{d\omega'\,\mathrm{Im}\,f(\omega')}{\omega'-\omega} + \mathcal{C}_\infty^{(\mathrm{Im})} \tag{3.5.142}$$

Eqs. (3.5.141) is called the dispersion relation, it gives the real part of the following equation

$$f(\omega) = \lim_{\varepsilon\to 0^+} f(\omega + i\,\varepsilon) = \lim_{\varepsilon\to 0^+}\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{d\omega'\,\mathrm{Im}\,f(\omega')}{\omega'-\omega-i\,\varepsilon} + \mathcal{C}_\infty \tag{3.5.143}$$

If $f(\omega)$ does not vanish when $\omega \to \infty$, the contribution $C_\infty \not\to 0$ does not vanish too, then we can re-derive Eq. (3.5.141) for the function $f(\omega)/\omega$ which has an extra pole at $\omega = 0$ and a better behavior at $\omega \to \infty$. Thus, the real part is given by

$$\frac{\mathrm{Re}\,f(\omega)}{\omega} = \frac{\mathrm{Re}\,f(0)}{\omega} + \frac{1}{\pi}\,P\int_{-\infty}^{+\infty}\frac{d\omega'\,\mathrm{Im}\,f(\omega')}{\omega'\,(\omega'-\omega)} \tag{3.5.144}$$

which is called dispersion relation with one subtraction.

So, the dispersion relation allows us to compute the full scattering amplitude from the knowledge of its imaginary part and its value at $\omega = 0$ if the subtraction is needed.

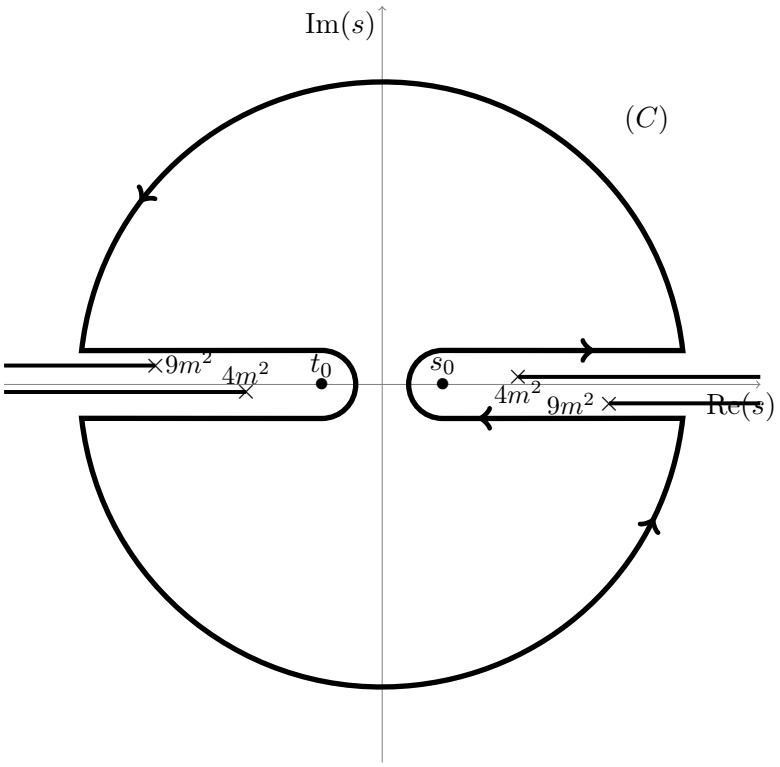


Figure 3.7: The Cauchy contour C for the amplitude F in the s-plane

3.5.2 Mandelstam dispersion relation

We consider the scattering amplitude F for equal mass spinless particles. Suppose that F has two poles at $s = s_0$ and at $t = t_0$, the variable u is fixed to the non-pole value u_0, with

$$s + t + u_0 = 4\,m^2 \tag{3.5.145}$$

The normal thresholds in the s-channel and t-channel are $s = 4\,m^2$ and $t = 4\,m^2$. Then, the amplitude F has a branch cut for

$$s \geq 4\,m^2, \qquad\qquad t \geq 4\,m^2. \tag{3.5.146}$$

From Cauchy theorem, we write

$$F(s, u_0) = \frac{1}{2\,\pi\,i} \int_C \frac{ds'\,F(s', u_0)}{s' - s} \tag{3.5.147}$$

we assume that $F(s', u_0)$ vanishes if $s' \to \infty$, i.e. the contribution along the curved part of the countour C in Fig.(3.7) vanishes by letting its radius goes to infinity. This implies

$$F(s, u_0) = P_{s_0} + P_{t_0} + \frac{1}{2\,\pi\,i} \int_{-u_0}^{-\infty} \frac{ds'\,F_t(s', u_0)}{s' - s} + \frac{1}{2\,\pi\,i} \int_{4\,m^2}^{+\infty} \frac{ds'\,F_s(s', u_0)}{s' - s} \tag{3.5.148}$$

P_{s_0} and P_{t_0} are the contributions from the two poles at $s_0 = m^2$ and at $s_0 = 3\,m^2 - u_0$, respectively. F_s and F_t are the discontinuities of F across the two branch cuts $[4\,m^2, +\infty]$ and $[4\,m^2, -\infty]$, respectively. Eq. (3.5.148) can be written in more convenient form as

$$F(s,t,u) = P_{s_0} + P_{t_0} + \frac{1}{2\,\pi\,i}\int_{4\,m^2}^{\infty} \frac{ds'\,F_s(s',t',u)}{s'-s} + \frac{1}{2\,\pi\,i}\int_{4\,m^2}^{\infty} \frac{dt'\,F_t(s',t',u)}{t'-t} \tag{3.5.149}$$

with $s' + t' + u = 4\,m^2$. Eq. (3.5.149) is the dispersion relation of the amplitude F with fixed u. Similar dispersion relations can be driven for fixed t and s, in these last two cases the discontinuity of the amplitude at the u-channel F_u will be involved. We have seen above that the discontinuity equals to twice of the imaginary part of the amplitude, .i.e

$$F_s = \text{disc}\,F = 2\,i\,\text{Im}\,F \tag{3.5.150}$$

On top of that, the discontinuity F_s for elastic scattering with

$$4\,m^2 \leq s < 9\,m^2, \qquad t \leq 0, \qquad u \leq 0. \tag{3.5.151}$$

is given by (see Eq.(3.1.27))

$$F_s(s,t) = \frac{i}{(2\,\pi)^2}\,\frac{\sqrt{s-4\,m^2}}{8\,\sqrt{s}}\int d\Omega\,F(s,t')\,F^*(s,t''), \tag{3.5.152}$$

Similar relation can be derived for the discontinuity in the t-channel (F_t) and u-channel (F_u), respectively. t' and t'' are related the square of the momentum transfer in the first and the second factor in Eq. (3.1.27).
The discontinuity relations (unitarity) in Eq. (3.5.150) (and similar equations for F_t and F_u) combined with the dispersion relations in Eq. (3.5.149) (and similar relations for the t-channel and the s- channel) are called the *dynamical equations*, since they impose many restrictions to the amplitude form [48].

To conclude this paragraph, we give the *Mandelstam dispersion relation* which involves double dispersion relation [65]:

$$\begin{aligned} F(s,t,u) = P &+ \frac{1}{\pi^2}\int\int \frac{F_{st}(s',t')}{(s'-s)\,(t'-t)}\,ds'\,dt' + \frac{1}{\pi^2}\int\int \frac{F_{tu}(t',u')}{(t'-t)\,(u'-u)}\,dt'\,du' \\ &+ \frac{1}{\pi^2}\int\int \frac{F_{us}(s',u')}{(u'-u)\,(s'-s)}\,du'\,ds' \end{aligned} \tag{3.5.153}$$

P stands for the poles contribution, $F_{s_i s_j}$ represent the double discontinuity across cuts in s_i- and s_j-channels simultaneously (where $s_1 = s$, $s_2 = t$ and $s_3 = u$). This relation is valid for scattering amplitudes of spineless particles in an equal mass theory, where the amplitude F vanishes as the variables tend to infinity in any direction in the complex plan.

3.5.3 Cutkosky rules

Cutkosky rules provide an elegant expression of the discontinuity across a branch cut starting from a singularity in the physical region defined by the Landau equations [53, 56]. Let us consider the scalar Feynamn integral I_N^n in its momentum space representation (it is defined

above), and let m be the number of internal on-shell momenta and $N-m$ is the number of the off-shell internal momenta; then the Landau conditions are given by

$$\begin{cases} q_i \;=\; m_i^2 & \text{for} \;\; i = 1, \cdots, m \\ \alpha_i \;=\; 0 & \text{for} \;\; i = m+1, \cdots, N \\ \sum_{i=1}^{N} \alpha_i \, q_i = 0 & \text{for} \;\; i = 1, \cdots, N \end{cases} \tag{3.5.154}$$

where α_i are some complex parameters.

The discontinuity in a given channel across the branch cut is given by

$$\text{disc} I_N^n(S) = (2\,\pi)^m \int \frac{d^n k}{(2\,\pi)^n} \frac{\Pi_{i=1}^m \, \theta(q_i^0) \, \delta(q_i - m_i^2)}{\Pi_{i=m+1}^N \, (q_i^2 - m_i^2 i\,\lambda)} \tag{3.5.155}$$

the role of the δ function is to put the particles corresponding to the intermediate state on their mass shell. In general, the propagator is given by the principle value P

$$\frac{1}{q_i^2 - m_i^2 + i\,\lambda} = P \, \frac{1}{q_i^2 - m_i^2} \mp i\,\pi\,\theta(q_i^0)\,\delta(q_i^2 - m_i^2) \tag{3.5.156}$$

Then, the cut propagator is obtained by removing its principal value, and replacing it by a δ function [53, 56]:

$$\frac{1}{q_i^2 - m_i^2 - i\,\lambda} \to +i\,\theta(q_i^0)\,\delta(q_i^2 - m_i^2) \tag{3.5.157}$$

$\theta(q_i^0)$ is introduced just to guarantee that the energy component of the momentum q_i along a given propagating direction is positive.

Eq. (3.5.155) is a direct consequence of the unitarity condition satisfied by the individual Feynman diagrams. It plays a crucial role in the development of powerful techniques to calculate one-loop and beyond scattering amplitudes in field theories, such as the Generalized unitarity decomposition methods that we will discuss in the next chapter.

Chapter 4

Modern Techniques of One-loop Calculation

Contents

In perturbation gauge theories, one-loop scattering amplitude are calculated by considering all contributing Feynman diagrams up to one loop order. The number of contributing Feynman diagrams grows dramatically with the number of the produced particles in the final state, and the structure of the one-loop integrals becomes much more complicated. To deal with these difficulties, many techniques of one-loop calculation are developed during the last few decades. It appears that any one-loop integral can be reduced to a combination of scalar one-loop integrals with up to four propagators weighted by some coefficients plus a rational term [67], which means that the full amplitude can be reduced to such decomposition too[27, 22]. The former approach is called the *Feynman diagrammatic approach*, where the reduction is done to each individual one-loop Feynman graphs, and the later approach is called the *unitarity inspired approach*, where the full amplitude is reduced at once. The later approach seems to be more powerful then the former one, since we can take advantage from the properties of the full amplitudes as the gauge invariance and unitarity (dispersion relations) [1].

In this chapter, we present two reduction methods based on the Feynman diagrammatic approach: the *Passarino-Veltman* reduction and the *Golem* reduction [67, 69], [19]; and two reduction methods based on the inspired unitarity approaches: *Ossola-Pittau-Papadopoulos*

[1] The full amplitude is a gauge invariant quantity, and it can be deduced from its absorptive by means of dispersion relations and unitarity. Nevertheless, individual Feynman diagrams are not gauge invariant quantities, and one has to calculate even the graphs linked to non-physical particles like the ghost for example. Then the later approach (unitarity inspired) it seems to be much more convenient than the former one.

(OPP) and the *generalized unitarity* cuts [27, 28], [22, 24, 25, 26]. We close this chapter by presenting an approach uses the *tensorial reconstruction at the integrand level* (which is a unitarity inspired approach) and the *Golem reduction* (which is a Feynman diagrammatic approach) to improve the unitarity approach for vanishing Gram determinants.

4.1 Feynman diagrammatic approache

Any one loop Feynman integral can be written as a linear combination of scalar one-loop integral with up to four external legs in n-dimensions ($n = 4 - 2\varepsilon$), and a remnant of the dimensional regularization called the rational part R. Schematically, we write

$$I_N^n \quad = \quad c_{4;j}\, I_{4;j}^n + c_{3;j}\, I_{3;j}^n + c_{2;j}\, I_{2;j}^n + c_{1;j}\, I_{1;j}^n + R + \mathcal{O}(\varepsilon) \tag{4.1.1}$$

the coefficients $c_{i;j}$ are evaluated in 4-dimensions, $I_{i;j}^n$ stands for the i-point master integrals with $i = 1, \cdots, 4$ and j specifies the combinations of the external momenta building up the momenta of the master integrals propagators.

The existence of such a reduction is one of the most crucial results of loop calculation in gauge theories. The origin of this decomposition is related to the Lorentz invariance, which allows to express tensor integrals in term of invariant form factors and to the nature of space-time, which allows to reduce a scalar integral of higher number of external legs to scalar integrals with up to four point at one loop order.

In the next two paragraphs, we present two type of reduction methods based on the Feynman diagrammatic approach. In the first paragraph, we present the *Passarino-Veltman reduction* (PV), which historically is one of the first invented reduction method of one-loop integrals. In the second paragraph, we present the *Golem reduction* method which is designed to avoid the problems induced by Gram determinant *spurious singularities*.

4.1.1 Passarino-Veltman reduction

The significance of Eq.(4.1.1) is that, any one-loop integral scalar or tensorial can be evaluated once the scalar integrals $I_{i;j}^4$, the coefficients $c_{i;j}$ in front of them and the rational part are computed. In PV framework, the basic integrals are given by

$$\begin{aligned} I_1^n(S) &= \frac{\mu^{4-n}}{i\,\pi^{\frac{n}{2}}\,r_\Gamma} \int \frac{d^n k}{D_1}, & I_2^n(S) &= \frac{\mu^{4-n}}{i\,\pi^{\frac{n}{2}}\,r_\Gamma} \int \frac{d^n k}{D_1\,D_2}, \\ I_3^n(S) &= \frac{\mu^{4-n}}{i\,\pi^{\frac{n}{2}}\,r_\Gamma} \int \frac{d^n k}{D_1\,D_2\,D_3}, & I_4^n(S) &= \frac{\mu^{4-n}}{i\,\pi^{\frac{n}{2}}\,r_\Gamma} \int \frac{d^n k}{D_1\,D_2\,D_3\,D_4}. \end{aligned} \tag{4.1.2}$$

where $D_i = q_i^2 - m_i^2 + i\,\lambda$, q_i is defined in Eq.(3.2.62), $r_\Gamma = \Gamma^2(1-\varepsilon)\,\Gamma(1+\varepsilon)/\Gamma(1-2\,\varepsilon)$ and μ is some kinematic invariant introduced to regulate potential divergences at small values of k.

The integrals I_1^n, I_2^n, I_3^n and I_4^n traditionally refer to the tadpole, bubble, triangle and box topologies in n-dimension, respectively. They form a basis in the mathematical sense, i.e. any one-loop integral can be expressed only in term of these integrals, they are called

master integrals. For real and complex internal masses, an analytical formula for each of these integrals is given in refs [66], [67], [68]. To calculate the full N-point integral, it remains to calculate the coefficients $c_{i,j}$, which is the main purpose of PV reduction method.

The calculation of individual Feynman diagrams gives rise to tensor integrals with numerators containing powers of the loop momenta. In renormalizable gauge theories, the rank of these tensors is limited by the number of the external legs. The direct computation of these tensors is not complicated but it is a bit exhausting. Passarino and Veltman introduced a new way to compute tensorial integrals in [67]. This approach allows to express any tensor integral in term of scalar integrals up to "4" external legs (master integrals). The studied tensorial integrals in the paper by Passarino and Veltman are:

$$\begin{aligned}
A_0 &= \frac{1}{i\,\pi^{\frac{n}{2}}}\int d^n k \frac{1}{D_1}, \\
B_0; B^\mu; B^{\mu\nu} &= \frac{1}{i\,\pi^{\frac{n}{2}}}\int d^n k \frac{1; k^\mu; k^\mu\,k^\nu}{D_1\,D_2}, \\
C_0; C^\mu; C^{\mu\nu}; C^{\mu\nu\alpha} &= \frac{1}{i\,\pi^{\frac{n}{2}}}\int d^n k \frac{1; k^\mu; k^\mu\,k^\nu; k^\mu\,k^\nu\,k^\alpha}{D_1\,D_2\,D_3}, \\
D_0; D^\mu; D^{\mu\nu}; D^{\mu\nu\alpha}; D^{\mu\nu\alpha\beta} &= \frac{1}{i\,\pi^{\frac{n}{2}}}\int d^n k \frac{1; k^\mu; k^\mu\,k^\nu; k^\mu\,k^\nu\,k^\alpha; k^\mu\,k^\nu\,k^\alpha\,k^\beta}{D_1\,D_2\,D_3\,D_4}.
\end{aligned} \tag{4.1.3}$$

A_0, B_0, C_0 and D_0 stand for the scalar integrals with up to four external legs in n-dimensions, respectively; $B^{\{\bullet\}}$, $C^{\{\bullet\}}$ and $D^{\{\bullet\}}$ stand for all possible tensorial 2-, 3- and 4-point integrals in n-dimensions, respectively.

In this section, we give just two simple examples of computing one-loop tensor integrals a la Passarino-Veltman: the tensorial triangles of rank "1" and "2", respectively. These two examples will be sufficient to give a full illustration of the method and show all possible problems that we can encounter. For complete description of the method, see [67, 69].

Let us consider the integrals C^μ and $C^{\mu\nu}$. As a consequence of Lorenz invariance this two quantities can be written as

$$C^\mu = p_1^\mu\,C_1 + p_2^\mu\,C_2, \tag{4.1.4}$$

$$C^{\mu\nu} = g^{\mu\nu}\,C_{00} + \sum_{i,j=1}^{2} p_i^\mu\,p_j^\nu\,C_{ij} \qquad \text{with} \qquad C_{21} = C_{12}, \tag{4.1.5}$$

where p_1 and p_2 are two linearly independent momenta, C_i, C_{00} and C_{ij} with $i,j = 1,2$ are Lorentz invariant quantities, they are called the *form factors* associated to rank one and two 3-point functions, respectively. Contracting both sides of Eq.(4.1.4) by p_1 and p_2 and using the fact that

$$k\cdot p_1 = \frac{1}{2}(f_1 + D_2 - D_1), \qquad f_1 = m_2^2 - m_1^2 - p_1^2 \tag{4.1.6}$$

$$k\cdot p_2 = \frac{1}{2}(f_2 + D_3 - D_2), \qquad f_2 = m_3^2 - m_2^2 - p_2^2 - 2p_1.p_2 \tag{4.1.7}$$

Then, we obtain the following system of equations in C_1 and C_2

$$G^{\{3\}}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} \tag{4.1.8}$$

$G^{\{3\}}$ is the Gram 2×2 matrix defined by

$$G^{\{3\}} = \begin{pmatrix} p_1^2 & p_1.p_2 \\ p_1.p_2 & p_2^2 \end{pmatrix} \tag{4.1.9}$$

and the R_i are given by

$$R_i = \int \frac{d^n k}{i\,\pi^{n/2}} \frac{k \cdot p_i}{D_1\, D_2\, D_3} \tag{4.1.10}$$

Inserting Eqs. (4.1.6, 4.1.7) in Eq. (4.1.10), the R_i can be written as

$$R_1 = \frac{1}{2}\left(f_1\, C_0(1,2,3) + B_0(1,3) - B_0(2,3)\right) \tag{4.1.11}$$

$$R_2 = \frac{1}{2}\left(f_2\, C_0(1,2,3) + B_0(1,2) - B_0(1,3)\right) \tag{4.1.12}$$

B_0 and C_0 are the scalar 1- and 2- point function, the numbers "1, 2, 3" between brackets stand for the labels of the propagators involved in these scalar functions. For example, $B(2,3)$ is defined as the integral

$$B_0(2,3) = \int \frac{d^n k}{i\,\pi^{n/2}} \frac{1}{D_2\, D_3} \tag{4.1.13}$$

Solving this system of equations for invertible Gram matrix, we find

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = (G^{\{3\}})^{-1} \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} \tag{4.1.14}$$

where the inverse of Gram matrix is given by

$$(G^{\{3\}})^{-1} = \frac{\begin{pmatrix} p_2^2 & -p_1.p_2 \\ -p_1.p_2 & p_1^2 \end{pmatrix}}{\Delta_2(p_1,p_2)}, \qquad \Delta_2(p_1,p_2) = p_1^2\, p_2^2 - (p_1 \cdot p_2)^2 \tag{4.1.15}$$

$\Delta_2(p_1,p_2)$ is the Gram determinant.

In a similar way, by contracting Eq. (4.1.5) by p_1 and independently by p_2, and using Eqs.(4.1.6, 4.1.7). We obtain the following two systems of equations

$$G^{\{3\}} \begin{pmatrix} C_{11} \\ C_{12} \end{pmatrix} = \begin{pmatrix} R_1^{(c_1)} \\ R_2^{(c_1)} \end{pmatrix}, \qquad G^{\{3\}} \begin{pmatrix} C_{12} \\ C_{22} \end{pmatrix} = \begin{pmatrix} R_1^{(c_2)} \\ R_2^{(c_2)} \end{pmatrix}. \tag{4.1.16}$$

with

$$R_1^{(c_1)} = \left(f_1\, C_1(1,2,3) + B_1(1,3) + B_0(2,3) - 2\, C_{00}(1,2,3)\right)/2 \tag{4.1.17}$$

$$R_2^{(c_1)} = \left(f_2\, C_1(1,2,3) + B_1(1,2) - B_1(1,3)\right)/2 \tag{4.1.18}$$

$$R_1^{(c_2)} = \left(f_1\, C_2(1,2,3) + B_1(1,3) - B_1(2,3)\right)/2 \tag{4.1.19}$$

$$R_2^{(c_2)} = \left(f_2\, C_2(1,2,3) - B_1(1,3) - 2\, C_{00}(1,2,3)\right)/2 \tag{4.1.20}$$

B_1 is a rank-1 two point function form factor, where $B^\mu = p_1^\mu B_1$. The term C_{00} can be expressed in term of C_i, C_0 and B_0, this can be proved by contracting $C^{\mu\nu}$ in Eq.(4.1.5) by the metric tensor. We find

$$C_{00}(1,2,3) = \frac{1}{2\,(n-2)}\left(2\,m_1^2\,C_0(1,2,3) - f_2\,C_2(1,2,3) - f_1\,C_1(1,2,3) + B_0(2,3)\right) \tag{4.1.21}$$

At this stage, we showed that the rank r triangle form factors can be expressed in term of rank $r-1$ form factors and sums of rank $r-1$ or less bubble form factors. In general, the Passarino-Veltman procedure reduce the rank r form factors of Feynman integrals with N external legs to a Feynman integrals form factor of rank $r-1$ with N and $N-1$ external legs. By repeating this procedure enough number of times, the original tensor integral will be expressed only in term of scalar integrals up to 4-point. For integrals with higher number of external legs (six and higher), this procedure can be used but it requires additional information since the external momenta are not linearly independent[2].

In conclusion, this procedure is the traditional formalism introduced to decompose tensor integrals in term of scalar integral. However, it is not the most sufficient one since it suffers from many problems: *i)* the dramatic growing number of Feynman diagrams with the number of external legs, *ii)* the fast growing number of terms in the expression of the individual integrals with the number of external legs and the rank of the integral. On top of that, *iii)* it leads to numerical instabilities due to the vanishing of the spurious Gram determinant shown above. In the next paragraph, we present another reduction method which is designed to deal with such spurious singularities and, in principle, it leads to less complicated expressions.

4.1.2 Golem algebraic reduction method

In this section, we give a short overview of the Golem reduction method, which is one of the pillars of the automatic one-loop multi-leg amplitudes calculation programs: `Golem` (and `GoSam`) [19, 71, 72, 73] (and [33]).

4.1.2.1 Overview of the method

This method is based on the Feynman diagrammatic approach, i.e. it calculates the full amplitude from calculating all contributing Feynman graphs. Of course, this increases the amount of work compared to the unitarity-inspired reduction methods (they will be shown in the next paragraph), but it enables us to avoid the problem of the spurious Gram determinants singularity in a mathematical way[3]. This is done by choosing a specific set of basic integrals and evaluate each of them numerically from their one-dimensional

[2] For example, the Gram determinant associated to these integrals vanishes in 4-dimension for $N \geq 6$, see Golem reduction in the next paragraph.

[3] Actually there are many alternatives to deal with the problems of numerical instabilities induced by the vanishing of Gram determinants, each of these approaches has its own advantages and inconveniences. *i) Interpolation:* apply some kinematic cuts to avoid unsafe regions and extrapolate the result to the problematic regions, this method can give good results in the neighborhood of $\det(G) = 0$ but not for $\det(G) = 0$. *ii) Taylor expansion:* expand the reduction coefficients ($c_{i;j}$) around $\det(G) = 0$. In this alternative, the full

integral representation in the problematic regions, i.e. regions where the Gram determinants becomes arbitrary small.

The Golem reduction library can be used as a rescue system by some automatic programs. In `GoSam` for example [33], the use of Golem library is limited only to problematic regions. This is done somehow by reconstructing the numerator of the full amplitude (or sub-amplitude) by making use of the *tensorial reconstruction at the integrand level*, and evaluating the reconstructed amplitude by means of `Golem95` library, see the last section of the current chapter.

The main feature of Golem reduction method is to reduce the scalar or tensorial integrals by adding and subtracting some terms in such way that the original integral is decomposed to a finite integral with the same number of propagators, and an infrared divergent integral with one propagator less (by pinching one propagator). This procedure is repeated several times until we end up with the wanted end-points of the reduction or what we call the Golem basic integrals, which does not form a basis in the mathematical sense but they are redundant integrals. In the Golem framework, we choose the following set:

$$\left\{ I_1(0), I_2(0;z), I_3^n(0;z_1,z_2,z_3), I_3^{n+2}(0;z), I_4^{n+2}(0;z_1,z_2,z_3), I_4^{n+4}(0;z) \right\} \tag{4.1.22}$$

Where z_i stands for Feynman parameters. This basis contains: the 4-point functions in $n+2$, which are IR and UV finite, the 4-point functions in $n+4$ dimensions which are UV divergent, the 3-point functions in $n+2$ dimensions, the 3-point functions in n dimension where all possible IR divergences are isolated, and various two and one point functions for massless configuration the one-point functions are absent). This set provides a very advantageous way to separate between IR divergent and finite contributions, since all the IR poles are contained in $I_3^4(z_i)$.

The Golem reduction formalism [19, 71, 72, 73] is designed to express any one-loop N-point Feynman diagram, with up to six external legs, as a linear combination in term of the set of the redundant basic integrals (4.1.22), and extract the reduction coefficients without facing any inverse of Gram determinant.

4.1.2.2 Form factors

Let us consider a general N-point tensor integral of rank r in n-dimensions ($n = 4 - 2\,\varepsilon$):

$$I_N^{n;\mu_1,\ldots,\mu_r}(a_1,\ldots,a_r;S) \;=\; \int \frac{d^n k}{i\,\pi^{n/2}} \frac{q_{a_1}^{\mu_1}\,\cdots\,q_{a_r}^{\mu_r}}{\Pi_{i\in S}^N\,(q_i^2 - m_i^2 + i\,\lambda)} \tag{4.1.23}$$

where $q_i = k + r_i$, k is the loop momentum, r_i is a combination of external momenta (see Eq.(3.2.62)) and m_i is the mass of the internal line "i", S is an ordered set containing the labels of the propagators. The tensor integral presented above can be expressed as a linear

calculation is done analytically (of the full amplitude or the individual Feynman diagrams) but the errors estimation is very complicated. *iii) Increasing the machine precision:* the use of high multiple-precision is not convenient in term of CPU time. *iv) Golem method:* we choose new basis of integrals other than the master integrals (which is not a basis in the mathematical sense), and we provide a stable one-dimensional integral representation for each redundant integrals.

combination of such Lorentz tensors and scalar quantities "$A^{N,r}$, $B^{N,r}$, $C^{N,r}$" called *form factors*, i.e. it can be written in the following form

$$\begin{aligned} I_N^{n;\mu_1,\ldots,\mu_r}(a_1,\ldots,a_r;S) = & \sum_{j_1\cdots j_r\in S} [\Delta^\bullet_{j_1\bullet}\cdots\Delta^\bullet_{j_r\bullet}]^{\{\mu_1\cdots\mu_r\}}_{\{a_1\cdots,a_r\}} A^{N,r}_{j_1\cdots j_r}(S) \\ & + \sum_{j_1\cdots j_{r-2}\in S} [g^{\bullet\bullet}\,\Delta_{j_1^\bullet\bullet}\cdots\Delta^\bullet_{j_{r-2}\bullet}]^{\{\mu_1\cdots\mu_r\}}_{\{a_1\cdots,a_r\}} B^{N,r}_{j_1\cdots j_{r-2}}(S) \\ & + \sum_{j_1\cdots j_{r-4}\in S} [g^{\bullet\bullet}\,g^{\bullet\bullet}\,\Delta^\bullet_{j_1\bullet}\cdots\Delta^\bullet_{j_{r-4}\bullet}]^{\{\mu_1\cdots\mu_r\}}_{\{a_1\cdots a_r\}} C^{N,r}_{j_1\cdots,j_{r-4}}(S) \end{aligned} \tag{4.1.24}$$

The shift invariant vector Δ^μ_{ij} is defined as difference of the two propagator momenta q_i and q_j ($\Delta^\mu_{ij} = r^\mu_i - r^\mu_j = q^\mu_i - q^\mu_j$). $A^{N,r}$ is the coefficient of the Lorentz structure containing only these vectors. $B^{N,r}$ and $C^{N,r}$ are the coefficients of the Lorentz tensors containing one and two metric tensor, respectively. The square brackets $[\cdots]^{\{\mu_1\cdots\mu_r\}}_{\{a_1\cdots,a_r\}}$ are interpreted as the distribution of the r Lorentz indices μ_i, and the momentum labels a_i in all distinguishable ways to the vectors $\Delta^{\mu_i}_{ja_i}$ and the metric tensors. As an example, the scalar, the tensor of rank one and the tensorial of rank two N-point integrals can be written, respectively, in the following forms

$$I_N^n(S) = A^{N,0}(S), \tag{4.1.25}$$

$$I_N^{n,\mu_1}(a_1;S) = \sum_{l\in S} \Delta^{\mu_1}_{l\,a_1}\, A^{N,1}_l(S), \tag{4.1.26}$$

$$I_N^{n,\mu_1\,\mu_2}(a_1,a_2;S) = \sum_{l_1,l_2\in S} \Delta^{\mu_1}_{l_1\,a_1}\,\Delta^{\mu_2}_{l_2\,a_2}\, A^{N,2}_{l_1 l_2}(S) + g^{\mu_1\,\mu_2}\, B^{N,2}(S), \tag{4.1.27}$$

We notice that, the form factors are independent of the vector Δ, i.e. they are shift invariant. Actually these form factors are the building blocks of the library `Golem95`. This library consist the main subject of these thesis, a detailed study will be given in the next chapters. In the following, we show how one can express these basic ingredients in terms of the Golem basic integrals presented above.

Before to close this paragraph, one has to notice that the momentum integrals, the Feynman parameters and the form factors are related by

$$\begin{aligned} I_N^{n,\mu_1\cdots\mu_r}(a_1,\cdots,a_r;S) = (-1)^r \sum_{m=0}^{[r/2]} \left(-\frac{1}{2}\right)^m & \sum_{j_1\cdots j_{r-2\,m}=1}^{N} [(g^{\cdots})^{\otimes m}\Delta^\bullet_{j_1\bullet}\cdots\Delta^\bullet_{j_r\bullet}]^{\{\mu_1\cdots\mu_r\}}_{\{a_1\cdots a_r\}} \\ & \times I_N^{n+2\,m}(j_2\cdots j_{r-2\,m};S) \end{aligned} \tag{4.1.28}$$

4.1.2.3 Reduction by subtraction

The tensorial/scalar reduction by subtraction or the so called *Golem reduction* is an algebraic reduction to the form factors introduced above. It is valid for massless as well as massive (complex masses are supported) amplitudes regularized by any scheme of dimensional regularization, provided that the external legs are living in four-dimensions (as for t'Hooft-Veltman and dimensional reduction schemes). This reduction is done by adding and

subtracting some specific terms to be fixed in such way that the original one-loop integral is separated into two integrals, a finite one with the same number of propagators and the same rank in space-time with dimensions higher by two than the original one($n+2$). And a divergent one in the same dimensions of the original space-time but with one propagator less and lesser rank. This procedure is repeated several times until when we end up by expressing all form factors in term of the Golem set of basic integrals. These method was introduced for the first time in [20], for more detail see [19].

The advantage of this reduction method is that: it leads to a very clean separation between IR divergent and finite contributions, and allows us to reduce any one loop Feynman graph or amplitude without producing any spurious Gram determinant singularity in the coefficients, and it generates less terms compared to PV reduction (introduced in the previous section). In the following, we will show how this reduction works for scalar and tensorial integrals.

4.1.2.4 Scalar reduction by subtraction

The N-point scalar one-loop integral is given by

$$I_N^n(S) = \int \frac{d^n k}{i\,\pi^{n/2}} \frac{1}{\Pi_{j\in S}\,(q_j^2 - m_j^2 + i\,\lambda)} \tag{4.1.29}$$

this integral can be split into IR and finite parts by making the ansatz

$$I_N^n(S) = \sum_{i\in S} b_i(S) \int \frac{d^n k}{i\,\pi^{n/2}} \frac{(q_i^2 - m_i^2)}{\Pi_{j\in S}\,(q_j^2 - m_j^2 + i\,\lambda)} + \int \frac{d^n k}{i\,\pi^{n/2}} \frac{1 - \sum_{i\in S} b_i(S)(q_i^2 - m_i^2)}{\Pi_{j\in S}\,(q_j^2 - m_j^2 + i\,\lambda)} \tag{4.1.30}$$

$$= I_{div}(S) + I_{fin}(S) \tag{4.1.31}$$

the b_i are fixed in such way that this integral is reduced to an IR divergent integral in n-dimension with one less propagator $I_{div} \propto \sum_i I_{N-1}^n$, and a finite integral in $n+2$-dimension with the same number of propagators $I_{fin} \propto I_N^{n+2}$. Then, the first term is a sum of all reduced integral by pinching all possible propagator "i", this term contains all possible infrared divergent terms of the original integral, since it is evaluated in n-dimensions. It can be written as

$$I_{div} \quad = \quad \sum_{i\in S} b_i(S)\, I_{N-1}^n(S \setminus \{i\}) \tag{4.1.32}$$

where $I_{N-1}^n(S \setminus \{i\})$ stands for a scalar $N-1$-point integral in n-dimensions obtained by pinching the propagator number "i" from the original scalar integral.

After introducing Feynman parameters and making the shift $k = l - \sum_{i\in S} z_i\, r_i$, the numerator of the second term in the right hand side of Eq.(4.1.30) becomes

$$1 - \sum_{i\in S} b_i(S)(q_i^2 - m_i^2) = -(l^2 + R^2) \sum_{i\in S} b_i(S) + \sum_{j\in S} z_j \left[1 - \sum_{i\in S} b_i(S)\,\{\mathcal{S}_{ij} + 2\,l\cdot\Delta_{ij}\}\right] \tag{4.1.33}$$

by imposing the condition

$$\sum_{i\in S} b_i(S)\, S_{ij} = 1, \qquad \text{for each} \qquad j = 1, \cdots, N \tag{4.1.34}$$

the term in the square brackets in Eq. (4.1.33) gives no contribution to the final result since its remaining part (after imposing the condition (4.1.34)) is linear in l, so it vanishes due the symmetric integration. Then, the integral I_{fin} becomes

$$\begin{aligned} I_{fin}(S) &= -B(S)\,\Gamma(N) \int_0^1 \Pi_{i\in S}\, dz_i\, \delta(1 - \sum_{l\in S} z_l) \int \frac{d^n l}{i\,\pi^{n/2}} \frac{l^2 + R^2}{(l^2 - R^2)^N} \\ &= -B(S)\,(N - n - 1)\, I_N^{n+2}(S), \end{aligned} \tag{4.1.35}$$

$$B(S) = \sum_{i\in S} b_i(S). \tag{4.1.36}$$

which is proportional to the scalar integral in $n + 2$-dimensions, then it is free of IR divergences. For $N \leq 6$, the matrix $\mathcal{S}$ is invertible, then the b_i are given by

$$b_i \quad = \quad \sum_{i\in S/\{j\}} \mathcal{S}^{-1}_{ki} \tag{4.1.37}$$

we notice that

$$\sum_{i\in S} b_i(S) \quad = \quad (-1)^{N+1} \det(G)/\det(\mathcal{S}) \tag{4.1.38}$$

Important Remarks

-i) For $N \geq 6$, $B = 0$ for external legs in 4-dimension which means that I_{fin} vanishes. This implies that, the integrals $I^n_{N\geq 6}$ are reduced to integrals with up to five propagators.
-ii) For $N = 5$, the finite part is $I_{fin}(S) = -2\,\varepsilon\, B(S)\, I_5^{n+2}(S) \propto \mathcal{O}(\varepsilon)$ at one-loop order. So this contribution can be dropped at one loop order, which means that I^n_5 is reduced to integrals with up to four propagators, the same thing can said for integrals with $N > 5$.
-iii) If the matrix $\mathcal{S}$ is not invertible, one has to express this matrix in term of the Gram matrix G and the vectors $V_i^{(a)}$ introduced in Eq.(3.2.96),

$$\mathcal{S}_{ij} = -G^{(a)}_{ij} + V_i^{(a)} + V_j^{(a)} \tag{4.1.39}$$

and define the generalized inverse of G by means of the Moore-Penrose generalized inverse to solve the Eq. (4.1.34) [74]. The results is given in Eqs. (43, 44) in the paper [19].

From the first two remarks, we conclude that by repeating this procedure sufficiently, we conclude that any N-point one-loop scalar integral can be reduced, ultimately, to integrals with up to four propagators at $\mathcal{O}(\varepsilon)$ order.

4.1.2.5 Tensor reduction by subtraction

The N point rank r tensor integral can be split as

$$\begin{aligned} I_N^{n,\mu_1\cdots\mu_r}(a_1,\cdots,a_r;S) &= \int \frac{d^n k}{i\,\pi^{n/2}} \frac{[q_{a_1}^{\mu_1} + \sum_{j\in S} C_{j\,a_1}^{\mu_1}\,(q_j^2 - m_i^2)]\,q_{a_2}^{\mu_2}\,\cdots q_{a_r}^{\mu_r}}{\Pi_{i\in S}\,(q_i^2 - m_i^2 + i\,\lambda)} \\ &- \sum_{j\in S} C_{j\,a_1}^{\mu_1} \int \frac{d^n k}{i\,\pi^{n/2}} \frac{(q_j^2 - m_j^2)\,q_{a_2}^{\mu_1}\,\cdots q_{a_r}^{\mu_r}}{\Pi_{i\in S}\,(q_i^2 - m_i^2 + i\,\lambda)} \end{aligned} \tag{4.1.40}$$

the last term is a $N-1$ tensor integral of rank $r-1$ containing all possible IR divergences since it is in $n = 4 - 2\,\varepsilon$ dimensions. The remaining integrals consist the finite part of this integral (it might be UV divergent), it can be written as

$$I_{fin} = \int \frac{d^n k}{i\,\pi^{n/2}} \frac{A_{a_1}^{\mu_1}\,q_{a_2}^{\mu_2}\,\cdots q_{a_r}^{\mu_r}}{\Pi_{i\in S}\,(q_i^2 - m_i^2 + i\,\lambda)}, \qquad A_{a_1}^{\mu_1} = q_{a_1}^{\mu_1} + \sum_{j\in S} C_{j\,a_1}^{\mu_1}\,(q_j^2 - m_i^2) \tag{4.1.41}$$

$A_{a_1}^{\mu_1}$ is fixed in such way that this integral is infrared safe, i.e we have to write it in $n+2$ dimensions. By introducing the Feynman parameters (z_i) and making the shift $k \to k - \sum_{i\in S} z_i\,r_i$, we can prove that this integral is IR safe only if this condition is satisfied

$$\sum_{j\in} \mathcal{S}_{kj}\,C_{jb}^{\mu} = \Delta_{kb}^{\mu} \tag{4.1.42}$$

For invertible $\mathcal{S}$, the solution of this equation is

$$C_{jb}^{\mu} = \sum_{j\in} S_{kj}^{-1}\,\Delta_{kb}^{\mu} \tag{4.1.43}$$

For not invertible case, this can happen for some exceptional kinematic or for $N \geq 7$, the solution of this equation is not unique. However, an explicit solution can be found using a similar technique pointed out in remark *iii*, see ref. [19] for more details.

4.1.2.6 Golem Form factors

In `Golem95`, one has to implement all form factors present in the reduction of 2-, 3-, 4-, 5- and 6-point integrals. These form factors are expressible in term of Golem set of basic integrals introduced above. In this description, no inverse of Gram determinant (or the inverse of its powers) are encountered, which provides a nice starting point for numerical evaluation. Such determinant appears, once the basic integrals are evaluated analytically or reduced to the set of master integrals, see below. In problematic regions, these representations give numerically unstable results. This problem is avoided by providing a one-dimensional integral representation for each redundant basic integral. In this paragraph, we give some of the needed form factors to compute one-loop integrals up-to six external legs, for more details see [19].

4.1.2.7 Form factors for 3-point integrals

The form factors for the 3-point integrals are

$$A^{3,0}(S) = I_3^n(S), \tag{4.1.44}$$
$$A_l^{3,1}(S) = -I_3^n(l;S), \tag{4.1.45}$$
$$B^{3,2}(S) = -\frac{1}{2}\, I_3^{n+2}(S) \tag{4.1.46}$$
$$A_{l_1\,l_2}^{3,2}(S) = I_3^n(l_1,l_2;S), \tag{4.1.47}$$
$$B_l^{3,3}(S) = \frac{1}{2}\, I_3^{n+2}(l;S), \tag{4.1.48}$$
$$A_{l_1\,l_2\,l_3}^{3,3}(S) = -I_3^n(l_1,l_2,l_3;S) \tag{4.1.49}$$

In term of the master integrals (scalar integral in n-dimensions), these form factors can expressed as

$$I_3^n(l_1;S) = \frac{b_{l_1}}{B}\,[I_3^n(S) - \sum_{j\in S} b_j\, I_2^n(S\setminus\{j\})] + \sum_{j\in S} \mathcal{S}^{-1}_{l_1 j}\, I_2^n(S\setminus\{j\}) \tag{4.1.50}$$

$$I_3^n(l_1,l_2;S) = -\mathcal{S}^{-1}_{l_1 l_2} I_3^{n+2}(S) + b_{l_1}\,(n-1)\, I_3^{n+2}(l_2;S) + \sum_{j\in S} \mathcal{S}^{-1}_{l_1 j}\, I_2^n(l_2;S\setminus\{j\}) \tag{4.1.51}$$

$$\begin{aligned} I_3^n(l_1,l_2,l_3;S) = &-\mathcal{S}^{-1}_{l_1 l_2}\, I_3^{n+2}(l_3;S) - \mathcal{S}^{-1}_{l_1 l_3}\, I_3^{n+2}(l_2;S) + n\, b_{l_1}\, I_3^{n+2}(l_2,l_3;S) \\ &+ \sum_{j\in S} \mathcal{S}^{-1}_{l_1 j}\, I_2^n(l_2,l_3;S\setminus\{j\}) \end{aligned} \tag{4.1.52}$$

$$I_3^{n+2}(S) = \frac{1}{B}\,\frac{1}{n-2}\,[I_3^n(S) - \sum_{l\in S} b_l\, I_2^n(S\setminus\{j\})] \tag{4.1.53}$$

$$\begin{aligned} I_3^{n+2}(l_1;S) = \frac{1}{B}\,[&b_{l_1}\, I_3^{n+2}(S) + \frac{1}{n-1}\sum_{j\in S} \mathcal{S}^{-1}_{j l_1}\, I_2^n(S\setminus\{j\}) \\ &- \frac{1}{n-1}\sum_{j\in S} b_j\, I_2^n(l_1;S\setminus\{j\})] \end{aligned} \tag{4.1.54}$$

$$\begin{aligned} I_3^{n+2}(l_1,l_2;S) = \frac{1}{n\,B}\,[&b_{l_1}\, I_3^{n+2}(l_2;S) + b_{l_2}\, I_3^{n+2}(l_1;S) + I_3^n(l_1,l_2;S) \\ &- \sum_{j\in S} b_j\, I_2^n(l_1,l_2;S\setminus\{j\})] \end{aligned} \tag{4.1.55}$$

where l_i stand for the labels of Feynman parameters, and $B = \det(G)/\det(\mathcal{S})$ ($\mathcal{S}$ and G are the kinematic and the Gram matrices associated to the 3-point functions).

By iteration, one can express these formulas in term of scalar integrals with trivial numerators (which is completely equivalent to the analytical evaluation). As a consequence of

this reduction, inverse of Gram determinants up to the third power appear in the expression of the these form factors[4]. These determinants (depend only on the external momenta) may be arbitrary small in some phase space regions, which can hamper the numerical stability. In next chapter, we will show how we can deal with this problem.

4.1.2.8 Form factors for 4-point integrals

The 4-point form factors are expressed in term the boxes (in $n+2$ and $n+4$ dimensions) and the triangles (in n and $n+2$ dimensions), with up to three Feynman parameters,

$$A^{4,0}(S) = B\, I_4^{n+2}(S) + \sum_{j\in S} b_j\, I_3^n(S\setminus\{j\}) \tag{4.1.56}$$

$$A_l^{4,1}(S) = -b_l\, I_4^{n+2}(S) - \sum_{j\in S} S_{jl}^{-1}\, I_3^n(S\setminus\{j\}) \tag{4.1.57}$$

$$B^{4,2}(S) = -\frac{1}{2}\, I_4^{n+2}(S) \tag{4.1.58}$$

$$\begin{aligned} A_{l_1l_2}^{4,2}(S) &= b_{l_1}\, I_4^{n+2}(l_2;S) + b_{l_2}\, I_4^{n+2}(l_1;S) - \mathcal{S}_{l_1l_2}^{-1}\, I_4^{n+2}(S) \\ &\quad + \frac{1}{2}\sum_{j\in S}[\mathcal{S}_{jl_2}^{-1}\, I_3^n(l_1;S\setminus\{j\}) + \mathcal{S}_{jl_1}^{-1}\, I_3^n(l_2;S\setminus\{j\})] \end{aligned} \tag{4.1.59}$$

$$B_l^{4,3} = \frac{1}{2}\, I_4^{n+2}(l;S) \tag{4.1.60}$$

$$\begin{aligned} A_{l_1l_2l_3}^{4,3}(S) &= \frac{2}{3}\,[\mathcal{S}_{l_2l_3}^{-1}\, I_4^{n+2}(l_1;S) + \mathcal{S}_{l_1l_3}^{-1}\, I_4^{n+2}(l_2;S) + \mathcal{S}_{l_1l_2}^{-1}\, I_4^{n+2}(l_3;S)] \\ &\quad - [b_{l_1}\, I_4^{n+2}(l_2,l_3;S) + b_{l_2}\, I_4^{n+2}(l_1,l_3;S) + b_{l_3}\, I_4^{n+2}(l_1,l_2;S)] \\ &\quad - \frac{1}{3}\sum_{j\in S}[\mathcal{S}_{jl_1}^{-1}\, I_3^n(l_2,l_3;S\setminus\{j\}) + \mathcal{S}_{jl_2}^{-1}\, I_3^n(l_1,l_3;S\setminus\{j\}) \\ &\quad + \mathcal{S}_{jl_3}^{-1}\, I_3^n(l_1,l_2;S\setminus\{j\})] \end{aligned} \tag{4.1.61}$$

$$C^{4,4}(S) = \frac{1}{4}\, I_4^{n+4}(S) \tag{4.1.62}$$

$$B_{l_1,l_2}^{4,4}(S) = -\frac{1}{2}\, I_4^{n+2}(l_1,l_2;S) \tag{4.1.63}$$

$$\begin{aligned} A_{l_1l_2l_3l_4}^{4,4}(S) &= f^{4,4}(l_1,l_2;l_3,l_4) + f^{4,4}(l_1,l_3;l_2,l_4) + f^{4,4}(l_1,l_4;l_3,l_2) \\ &\quad + f^{4,4}(l_2,l_3;l_1,l_4) + f^{4,4}(l_2,l_4;l_3,l_1) + f^{4,4}(l_3,l_4;l_1,l_2) \\ &\quad + g^{4,4}(l_1;l_2,l_3,l_4) + g^{4,4}(l_2;l_1,l_3,l_4) \\ &\quad + g^{4,4}(l_3;l_2,l_1,l_4) + g^{4,4}(l_4;l_2,l_3,l_1) \end{aligned} \tag{4.1.64}$$

[4] The Golem 3-point basic integrals are proportional $\det(G^{\{3\}})$ ($G^{\{3\}}$ is 3-point Gram matrix) as the following: $I_3^n(l) \propto 1/\det(G^{\{3\}})$, $I_3^n(l_1,l_2) \propto 1/\det(G^{\{3\}})^2$, $I_3^n(l_1,l_2,l_3) \propto 1/\det(G^{\{3\}})^3$, $I_3^{n+2} \propto 1/\det(G^{\{3\}})$, $I_3^{n+2}(l) \propto 1/\det(G^{\{3\}})^2$, $I_3^{n+2}(l_1,l_2) \propto 1/\det(G^{\{3\}})^3$.

$$f^{4,4}(l_1,l_2;l_3,l_4) = -\frac{1}{2}\,\mathcal{S}^{-1}_{l_1l_2}\,I_4^{n+2}(l_3,l_4;S) \tag{4.1.65}$$

$$g^{4,4}(l_1;l_2,l_3,l_4) = b_{l_1}\,I_4^{n+2}(l_2,l_3,l_4;S) + \frac{1}{4}\sum_{j\in S}\mathcal{S}^{-1}_{jl_1}\,I_3^{n}(l_2,l_3,l_4;S\setminus\{j\}) \tag{4.1.66}$$

These form factors are expressed in term of the basic integrals: the 3-point integral in n dimensions with up the three Feynman parameters in the numerator, the 3-point integral in $n+2$ dimensions with only one Feynman parameter in the numerator, the 4-point function in $n+2$ dimensions with up three Feynman parameter in the numerator, the 4-point in $n+4$ dimension with only one Feynman parameter in the numerator. As we mentioned above, these integrals with various 2- and 1-point integrals form the Golem basic integrals, i.e the endpoints of the reduction. For form factors with $N > 4$, no additional integrals are needed, see [19]. Any form factor expressed in term of this basic integrals is free of any inverse of Gram determinant. However, further reduction of these basic integrals to master integrals lead to expressions containing negative powers of the Gram determinants (up to the power 3 for Gram determinants associated to the 3-point functions, and up to the power 4 for the Gram determinant associated to the 4-point functions).
The 4-point basic integrals can be reduced to the master integrals as the following:

$$I_4^{n+2}(l;S) = \frac{1}{B}\left\{b_l\,I_4^{n+2}(S) + \frac{1}{2}\sum_{j\in S}\mathcal{S}^{-1}_{jl}\,I_3^{n}(S\setminus\{j\}) - \frac{1}{2}\sum_{j\in S}b_j\,I_3^{n}(l;S\setminus\{j\})\right\} \tag{4.1.67}$$

$$\begin{aligned} I_4^{n+2}(l_1,l_2;S) = \frac{2}{3\,B}\Big\{&b_{l_1}\,I_4^{n+2}(l_2;S) + b_{l_2}\,I_4^{n+2}(l_1;S)\\ &- \frac{1}{2}\,S^{-1}_{l_1l_2}\,I_4^{n+2}(S) + \frac{1}{4}\sum_{j\in S}\mathcal{S}^{-1}_{jl_2}\,I_3^{n}(l_1;S\setminus\{j\})\\ &+ \frac{1}{4}\sum_{j\in S}\mathcal{S}^{-1}_{jl_1}\,I_3^{n}(l_2;S\setminus\{j\}) - \frac{1}{2}\sum_{j\in S}b_j\,I_3^{n}(l_1,l_2;S\setminus\{j\})\Big\} \end{aligned} \tag{4.1.68}$$

$$\begin{aligned} I_4^{n+2}(l_1,l_2,l_3;S) = \frac{1}{2\,B}\Big\{&b_{l_3}\,I_4^{n+2}(l_1,l_2;S) + b_{l_2}\,I_4^{n+2}(l_1,l_3;S) + b_{l_1}\,I_4^{n+2}(l_2,l_3;S)\\ &- \frac{1}{3}\left(S^{-1}_{l_1l_2}\,I_4^{n+2}(l_3;S) + S^{-1}_{l_1l_3}\,I_4^{n+2}(l_2;S) + S^{-1}_{l_2l_3}\,I_4^{n+2}(l_1;S)\right)\\ &+ \frac{1}{6}\left(\sum_{i\in S}\mathcal{S}^{-1}_{il_3}\,I_3^{n}(l_1,l_2;S\setminus\{i\}) + \sum_{i\in S}\mathcal{S}^{-1}_{il_2}\,I_3^{n}(l_1,l_3;S\setminus\{i\})\right.\\ &\left.+ \sum_{i\in S}\mathcal{S}^{-1}_{il_1}\,I_3^{n}(l_2,l_3;S\setminus\{i\})\right) - \frac{1}{2}\sum_{i\in S}b_i\,I_3^{n}(l_1,l_2,l_3;S\setminus\{i\})\Big\} \end{aligned} \tag{4.1.69}$$

$$I_4^{n+4}(S) = \frac{1}{(n-1)\,B}\left\{I_4^{n+2}(S) - \sum_{j\in S}b_j\,I_3^{n+2}(S\setminus\{j\})\right\} \tag{4.1.70}$$

$$I_4^{n+4}(l;S) = \frac{1}{n\,B}\left\{b_l\, I_4^{n+4}(S) + I_4^{n+2}(l;S) - \sum_{j\in S} b_j\, I_3^{n+2}(l;S\setminus\{j\})\right\} \tag{4.1.71}$$

expressing these integrals in term of only scalar integral lead to expressions containing inverse of B up to the power 4 ($B = -\det(G)/\det(\mathcal{S})$, $\mathcal{S}$ and G are the kinematical and Gram matrices associated to the 4-point integrals) [5].

4.1.2.9 Golem basic integrals

It turns out that, the set of basic integrals that allows us to express any one-loop amplitude up to 6-external legs without producing any spurious Gram determinant singularities consists of the following redundant scalar and tensorial integrals:

$$\begin{gathered}\{I_1(S),\\ I_2^n(S), I_2^n(z_1;S), I_2^n(z_1,z_2;S),\\ I_3^n(S), I_3^n(z_1;S), I_3^n(z_1,z_2;S), I_3^{n+2}(S),\\ I_4^{n+2}(S), I_4^{n+2}(z_1;S), I_4^{n+2}(z_1,z_2;S), I_4^{n+2}(z_1,z_2,z_3;S), I_4^{n+4}(S), I_4^{n+4}(z_1;S)\}.\end{gathered} \tag{4.1.72}$$

Tensorial integral stands for integrals with Feynman parameters z_i in the numerator. This set of basic integral does not form a basis in the mathematical sense, but it is a chosen end point of the algebraic reduction. It is fixed that way to avoid the inverse of Gram determinant from the form factors. We mention that all of these integrals can be expressed in term of a complete basis called "basis of master integrals", which contain only scalar integrals in n-dimensions, and fulfills the conditions of a basis in the mathematical sense. This basis contains

$$\{I_1^n(S), I_2^n(S), I_3^n(S), I_4^n(S)\} \tag{4.1.73}$$

Expressing the form factors in term of this set of basic master integrals is equivalent to evaluate them analytically, since this procedure introduces inverse of Gram determinants in the coefficients in front of these scalar integrals, which may leads to numerical instability in problematic regions, where the associated Gram determinants become arbitrary small. However, this basis provides a fast and stable numerical evaluation in large region of phase space (non problematic region). In this configuration, these integrals are calculated analytically as we will show in the next chapter. Otherwise, when the Gram determinant is arbitrary small, one has to express the form factor in term of the basic integrals in Eq. (4.1.72) and evaluate them numerically from their one-integral representation, since their analytical formulas lead to the same problems.

Before closing this section, we mention that the tensorial 4-point integrals in Eq. (4.1.72) can be expressed in term of the scalar integrals

$$\{I_4^{n+2}(S), I_4^{n+4}(S), I_4^{n+6}(S), I_4^{n+8}(S)\} \tag{4.1.74}$$

[5] The 4-point Golem basic integrals are proportional to as the following: $I_4^{n+2} \propto 1/\det(G^{\{4\}})$, $I_4^{n+2}(l) \propto 1/\det(G^{\{4\}})^2$, $I_4^{n+2}(l_1,l_2) \propto 1/\det(G^{\{4\}})^3$, $I_4^{n+2}(l_1,l_2,l_3) \propto 1/\det(G^{\{4\}})^4$, $I_4^{n+4} \propto 1/\det(G^{\{4\}})^2$, $I_4^{n+4}(l) \propto 1/\det(G^{\{4\}})^3$.

So, instead of evaluating all the possible tensorial 4-point basic integrals directly, one has to calculate only this four scalar integrals and deduce the tensorial ones from the following relations

$$
\begin{aligned}
I_4^{n+2}(l_1,l_2,l_3;S) &= (n+1)\,(n+2)\,(n+3)\,b_{l_1}\,b_{l_2}\,b_{l_3}\,I_4^{n+8}(S)\\
&- (n+1)\left(\mathcal{S}^{-1}_{l_1\,l_2}\,b_{l_3}+\mathcal{S}^{-1}_{l_1\,l_3}\,b_{l_2}+\mathcal{S}^{-1}_{l_2\,l_3}\,b_{l_1}\right)I_4^{n+6}(S)\\
&+ \frac{(n+1)\,(n+2)}{3}\sum_{j\in S}\left(\mathcal{S}^{-1}_{j\,l_1}\,b_{l_2}\,b_{l_3}+\mathcal{S}^{-1}_{j\,l_2}\,b_{l_1}\,b_{l_3}+\mathcal{S}^{-1}_{j\,l_3}\,b_{l_1}\,b_{l_2}\right)I_4^{n+6}(S\setminus\{j\})\\
&- \frac{2}{3}\sum_{j\in S}\left(\mathcal{S}^{-1}_{l_1l_2}\,\mathcal{S}^{-1}_{jl_3}+\mathcal{S}^{-1}_{l_1l_3}\,\mathcal{S}^{-1}_{jl_2}+\mathcal{S}^{-1}_{l_2l_3}\,\mathcal{S}^{-1}_{jl_1}\right)I_3^{n+4}(S\setminus\{j\})\\
&+ \frac{n+1}{6}\sum_{j\in S}\Big[(\mathcal{S}^{-1}_{jl_2}\,b_{l_3}+\mathcal{S}^{-1}_{jl_3}\,b_{l_2})\,I_3^{n+4}(l_1;\setminus\{j\})\\
&+ (\mathcal{S}^{-1}_{jl_1}\,b_{l_3}+\mathcal{S}^{-1}_{jl_3}\,b_{l_1})\,I_3^{n+4}(l_2;\setminus\{j\})\\
&+ (\mathcal{S}^{-1}_{jl_1}\,b_{l_2}+\mathcal{S}^{-1}_{jl_2}\,b_{l_1})\,I_3^{n+4}(l_3;\setminus\{j\})\Big]\\
&\frac{1}{3}\sum_{j\in S}\Big(\mathcal{S}^{-1}_{jl_1}\,I_3^{n+2}(l_2,l_3;S\setminus\{j\})\\
&+ \mathcal{S}^{-1}_{jl_2}\,I_3^{n+2}(l_1,l_3;S\setminus\{j\})+\mathcal{S}^{-1}_{jl_3}\,I_3^{n+2}(l_1,l_2;S\setminus\{j\})\Big)
\end{aligned}
\tag{4.1.75}
$$

$$
I_4^{n+2}(l;S) = (n-1)\,b_l\,I_4^{n+4}(S)+\sum_{j\in S}\mathcal{S}^{-1}_{jl}\,I_3^{n+2}(S\setminus\{j\})
\tag{4.1.76}
$$

$$
I_4^{n+4}(l;S) = (n+1)\,b_l\,I_4^{n+6}(S)+\sum_{j\in S}\mathcal{S}^{-1}_{jl}\,I_3^{n+4}(S\setminus\{j\})
\tag{4.1.77}
$$

$$
\begin{aligned}
I_4^{n+2}(l_1,l_2;S) &= -\mathcal{S}^{-1}_{l_1\,l_2}\,I_4^{n+4}(S)+n\,(n+1)\,b_{l_1}\,b_{l_2}\,I_4^{n+6}(S)\\
&+ \frac{n}{2}\sum_{j\in S}\left(b_{l_1}\,\mathcal{S}^{-1}_{j\,l_2}+b_{l_2}\,\mathcal{S}^{-1}_{j\,l_1}\right)I_3^{n+4}(S\setminus\{j\}\\
&+ \frac{1}{2}\sum_{j\in S}\left(\mathcal{S}^{-1}_{j\,l_1}\,I_3^{n+2}(l_2;S\setminus\{j\})+\mathcal{S}^{-1}_{j\,l_2}\,I_3^{n+2}(l_1;S\setminus\{j\}\right)
\end{aligned}
\tag{4.1.78}
$$

4.2 Unitarity inspired approach

From Feynman diagrammatic approach presented in the previous sections, we have shown that any one-loop Feynman diagram can be reduced to a combination of boxes, triangles, bubbles, tadpoles and a rational terms. As a consequence, a full one-loop scattering amplitude in n dimension $\mathcal{A}_N^n$ (a sum of Feynman diagrams up to N point) can be also reduced to the same set of basis integrals,

$$
\mathcal{A}_N^n(\bar{k}) \;=\; \tilde{c}_{4;j}\,I^n_{4;j}+\tilde{c}_{3;j}\,I^n_{3;j}+\tilde{c}_{2;j}\,I^n_{2;j}+\tilde{c}_{1;j}\,I^n_{1;j}+\tilde{R}+\mathcal{O}(\varepsilon)
\tag{4.2.79}
$$

where $I^n_{i;j}$ are the same basis integrals introduced in Eq.(4.1.1), $\tilde{c}_{j;j}$ and $\tilde{R}$ are the coefficients and the rational term corresponding the full amplitude $\mathcal{A}^n_N(\bar{k})$.

In this section, we give a brief presentation of two types of unitarity based reduction method: the OPP and the Generalized unitarity cuts.

4.2.1 OPP method

The OPP (Ossola-Papadopoulos-Pittau) reduction method is a method designed to reduce any one-loop amplitude for arbitrary scattering process at the integrand level. It requires a minimum information about the form of the one-loop amplitude and enables us to extract the coefficients of the 1-, 2-, 3- and 4-point basis integrals numerically in a very efficient way with the possibility to fully construct the rational term [27, 28, 29].

4.2.1.1 Numerator parameterization

The integrand of an arbitrary N-point one-loop amplitude or a one-loop (sub) amplitude in n dimensions can be written as

$$\mathcal{A}(\bar{k}) \;=\; \frac{\mathcal{N}(k)}{\bar{D}_0\,\bar{D}_1\,...\bar{D}_{N-1}} \qquad \text{with} \quad \bar{D}_i = (\bar{k}+r_i)^2 - m_i^2 \qquad \text{and} \quad r_0 \neq 0 \tag{4.2.80}$$

with $\bar{k} = k + \tilde{k}$, where the objects $\bar{k}$, k and $\tilde{k}$ are living in n, 4 and $-2\,\varepsilon$ dimension.

Using the fact that the full amplitude can be written as a combination of at most 4-point scalar integrals implies that the numerator schematically can be expressed in the following form

$$\begin{aligned}
\mathcal{N}(k) \;=\;& \sum_{i_0<i_1<i_2<i_3}^{N-1} \left[d(i_0\,i_1\,i_2\,i_3) + \tilde{d}(k;i_0\,i_1\,i_2\,i_3)\right] \Pi^{N-1}_{i\neq i_0,i_1,i_2,i_3}\,\bar{D}_i \\
+\;& \sum_{i_0<i_1<i_2}^{N-1} \left[c(i_0\,i_1\,i_2) + \tilde{c}(k;i_0\,i_1\,i_2)\right] \Pi^{N-1}_{i\neq i_0,i_1,i_2}\,\bar{D}_i \\
+\;& \sum_{i_0<i_1}^{N-1} \left[b(i_0\,i_1) + \tilde{b}(k;i_0\,i_1)\right] \Pi^{N-1}_{i\neq i_0,i_1}\,\bar{D}_i \\
+\;& \sum_{i_0}^{N-1} \left[a(i_0) + \tilde{a}(k;i_0)\right] \Pi^{N-1}_{i\neq i_0}\,\bar{D}_i \\
+\;& \tilde{P}(k)\,\Pi^{N-1}_{i}\,\bar{D}_i
\end{aligned} \tag{4.2.81}$$

where $i_0, i_1, \cdots, i_N$ stands for the labels of the propagators.
The quantities $d(i_0\,i_1\,i_2\,i_3)$, $c(i_0\,i_1\,i_2)$, $b(i_0\,i_1)$ and $a(i_0)$ are the coefficients of all possible scalar 4-point, 3-point, 2-point and 1-point integrals, respectively. The k-dependent terms $\tilde{d}$, $\tilde{c}$, $\tilde{b}$ and $\tilde{a}$ are the so called spurious coefficients, they are defined in such way that they vanish during integration [27]. In Eq. (4.2.81), the k's momentum are living in 4-dimension, this allows to compute the 4-dimensional quantities which constitute the most difficult part. The remaining quantities living in $-2\,\varepsilon$-dimension are quite straightforward

to calculate, they will be given in the end of this paragraph. As the 4-, 3-, 2- and 1-point scalar functions are known, it remains to calculate their associated coefficients. This is done by computing the numerator several times for a well chosen set of k values and reverse the obtained system to extract these coefficients. Two problems appear during the calculation, first the explicit formula of the spurious terms is required, secondly the growing size of the system which should be feasible. For example, for one-loop integral with 6 propagators, a system of 56×56 matrix has to be inverted due to the existence of 56 independent one-loop scalar integrals. To solve this problem, one has to choose a special values for k in such way that the propagators associated to the coefficients d, c, b or a in the numerator presented above vanish, which leads to triangular system of equation as we will see later on.

4.2.1.2 Constructing of the spurious terms

To be able to construct the coefficients d, c, b and a, we should know the general explicit k-dependence of the spurious terms. To do so, we express any k in $\mathcal{N}(k)$ in term of a convenient basis of massless vectors l_i^μ

$$k^\mu = -r_0^\mu + \sum_{i=1}^{4} x_i \, l_i^\mu \qquad\qquad l_i^2 = 0 \tag{4.2.82}$$

where the 4-vectors l_1 and l_2 satisfy the relations

$$k_1 = l_1 + \alpha_1 \, l_2, \qquad k_2 = \, l_2 + \alpha_2 \, l_1 \qquad \text{with} \qquad k_i = r_i - r_0. \tag{4.2.83}$$

and in term of the spinorial notation, the 4-vectors l_3 and l_4 are given by

$$l_3^\mu = < l_1 | \gamma^\mu | l_2], \qquad l_4^\mu = < l_2 | \gamma^\mu | l_1], \qquad \text{with} \qquad (l_3 \cdot l_4) = -4 \, (l_1 \cdot l_2). \tag{4.2.84}$$

where x_i, α_i are some complex parameters.
By inserting the decomposition of the 4-dimensional vector k (Eq. (4.2.82)) in the Eq. (4.2.81), the coefficients x_i will reconstruct denominators, which give rise to coefficients d, c, b and a or they will give rise to the spurious terms $\tilde{d}$, $\tilde{c}$, $\tilde{b}$ and $\tilde{a}$, which will vanish upon integration.
For example, we insert Eq. (4.2.82) in the quantity $N^{(3)}(k)$, which is the numerator of the term containing the denominators $D_{i_0} D_{i_1} D_{i_2} D_{i_3}$ (this numerator is at most a tensor of rank 4 in renormalizable gauge theories). Ultimately, this quantity can be expressed as

$$\begin{aligned} N^{(3)}(k) &= d(i_0, i_1, i_2, i_3) + \tilde{d}(k; i_0, i_1, i_2, i_3) + \sum_{i=i_0}^{i_3} \mathcal{O}(\bar{D}_i) + \mathcal{O}(\tilde{k}^2) \\ &= d(i_0, i_1, i_2, i_3) + \tilde{d}(i_0, i_1, i_2, i_3) \, \mathrm{T}(k) + \sum_{i=i_0}^{i_3} \mathcal{O}(\bar{D}_i) + \mathcal{O}(\tilde{k}^2) \end{aligned} \tag{4.2.85}$$

Then, the explicit k-dependance of the scalar box spurious coefficients $\tilde{d}(k; i_0, i_1, i_2, i_3)$ is

$$\tilde{d}(i_0, i_1, i_2, i_3) \, \mathrm{Tr}[(\not{k} + \not{r}_0) \not{l}_1 \not{l}_2 \not{k}_3 \gamma_5] \tag{4.2.86}$$

$\tilde{d}$ is a k-independent quantity, this vanishes upon integration since

$$\int d^n q \frac{\mathrm{Tr}[(\not{k}+\not{r}_0)\ \not{l}_1\ \not{l}_2\ \not{k}_3\gamma_5]}{D_{i_0} D_{i_1} D_{i_2} D_{i_3}} = 0 \tag{4.2.87}$$

This term is the only vanishing quantity upon integration and the only k-dependent quantity associated to the box, see the demonstration in "theorem 1" page 7 in ref. [27].

Following the same reasoning as in the case of 4-point functions, one can prove that the spurious terms corresponding to the 3-point, 2-point and 1-point functions contain six, eight and four terms, respectively. For more details, see ref. [27].

At this stage, the extraction of the coefficients of the scalar integrals can be done algebraically by evaluating the full amplitude numerator $\mathcal{N}(q)$ for a well chosen set of values of the integration momentum k. This particular choice of k value, requires the vanishing of 4, 3, 2, 1 denominators to reduce the problem to the solution of a triangular system.

4.2.1.3 Extraction the coefficients

To extract the the coefficients of the scalar functions, we can work in 4-dimensions. So, we set everywhere $\tilde{k} = 0$ and $\bar{D}_i \to D_i = (k + r_i)^2 - m_i^2$. The errors generated by this approximation, is put at the rational contribution to the amplitude. This contribution will be calculated in the next paragraph.

To extract the coefficient of the 4-point function, one can select k such that (from now on we replace the labels $i_j \to j$)

$$D_0 = D_1 = D_2 = D_3 = 0 \tag{4.2.88}$$

this choice kills all the terms containing the coefficients "$c + \tilde{c}$", "$b + \tilde{b}$", "$a + \tilde{a}$" and "$\tilde{P}$" and keep only the term containing $d + \tilde{d}$ in the numerator Eq. (4.2.81). We put $k^\mu = -r_0^\mu + \sum_{i=1}^4 x_i\, l_i^\mu$, we obtain this system of equation in x_i

$$\begin{aligned}
&0 = \gamma\,(x_1\,x_2 - 4\,x_3\,x_4) - d_0, && \gamma = (k_1\cdot k_2) \pm \sqrt{\Delta},\\
&0 = d_0 - d_1 + \gamma\,(x_1\,\alpha_1 + x_2), && \Delta = (k_1\cdot k_2)^2 - k_1^2\,k_2^2,\\
&0 = d_0 - d_2 + \gamma\,(x_2\,\alpha_2 + x_1), && \alpha_i = k_i^2/\gamma,\\
&0 = d_0 - d_3 + 2\,[x_1\,(k_3\cdot l_1) + x_2\,(k_3\cdot l_2) + x_3\,(k_3\cdot l_3) + x_4\,(k_3\cdot l_4)], && d_i = m_i^2 - k_i^2,
\end{aligned} \tag{4.2.89}$$

Two possible solutions on k are found

$$(k_0^\pm)^\mu = -r_0^\mu + x_1^0\,l_1^\mu + x_2^0\,l_2^\mu + x_3^\pm\,l_3^\mu + x_4^\pm\,l_4^\mu \tag{4.2.90}$$

with

$$
\begin{aligned}
x_1^0 &= \frac{\beta}{\gamma}\,[d_2 - \alpha_2\, d_1 - d_0\,(1-\alpha_2)], \quad \beta = 1/(1-\alpha_1\,\alpha_2),\\
x_2^0 &= \frac{\beta}{\gamma}\,[d_1 - \alpha_1\, d_2 - d_0\,(1-\alpha_1)], \quad A = -k_3\cdot l_3/k_3\cdot l_4,\\
0 &= A\,x_1^{\pm 2} + B\,x_3^{\pm} - C, \qquad\qquad B = (d_3 - d_0 - 2\,x_1^0\,(k_3\cdot l_1) - 2\,x_2^0\,(k_3\cdot l_2))/(2\,k_3\cdot l_4),\\
x_4^{\pm} &= C/x_3^{\pm}, \qquad\qquad C = (x_1^0\,x_2^0 - d_0/\gamma)/4,
\end{aligned}
\tag{4.2.91}
$$

To determine d and $\tilde{d}$, the two solutions $k_0^{\pm}$ are needed, we write

$$
\mathcal{N}(k_0^{\pm}) \;=\; [d(i_0i_1i_2i_3) + \tilde{d}(i_0i_1i_2i_3)\,\mathrm{Tr}(k_0^{\pm})]\,\Pi_{i\neq 0,1,2,3}\,D_i(k_0^{\pm}) \tag{4.2.92}
$$

we have two equations with two unknowns, the solution is simply given by

$$
d(i_0i_1i_2i_3) = \frac{1}{2}\,[R(k_0^+) + R(k_0^-)], \tag{4.2.93}
$$

$$
\tilde{d}(i_0i_1i_2i_3) = \frac{1}{2}\,\frac{R(k_0^+) - R(q_0^-)}{T(k_0^+)}, \qquad \text{with} \qquad R(k^{\pm}) = \frac{N(k_0^{\pm})}{\Pi_{i\neq 0,1,2,3}\,D_i(k_0^{\pm})} \tag{4.2.94}
$$

The same strategy is adopted to extract the coefficients of the 3-, 2- and 1-point functions where k is fixed in such way that $D_0 = D_1 = D_2 = 0$, $D_0 = D_1 = 0$ and $D_0 = 0$ to determine "$c, \tilde{c}$", "$b, \tilde{b}$", and "$a, \tilde{a}$", respectively. We mention that, to extract c-coefficients, d-coefficients are needed and to extract b-coefficients the d- and c-coefficients are needed and so on [27].

4.2.1.4 Rational terms

In the previous calculation, we have assumed that $\tilde{k}^2 = 0$, which is sufficient to determine the 1-, 2-, 3- and 4-point coefficient but the rational part is missing. The rational part are computed by reintroduced $\tilde{k}^2$ in the scalar integral coefficients by making the mass shift

$$
m_i^2 \longrightarrow m_i^2 - \tilde{k}^2 \tag{4.2.95}
$$

In renormalizable gauge theories, the only possible non vanishing contribution coming from the extra integrals are

$$
\int d^n\tilde{k}\,\frac{\tilde{k}^4}{\bar{D}_i\,\bar{D}_j\,\bar{D}_k\,\bar{D}_l} \;=\; -\frac{i\,\pi}{6} + \mathcal{O}(\varepsilon) \tag{4.2.96}
$$

$$
\int d^n\tilde{k}\,\frac{\tilde{k}^2}{\bar{D}_i\,\bar{D}_j\,\bar{D}_k} \;=\; -\frac{i\,\pi^2}{2} + \mathcal{O}(\varepsilon) \tag{4.2.97}
$$

$$
\int d^n\tilde{k}\,\frac{\tilde{k}^2}{\bar{D}_i\,\bar{D}_j} \;=\; -\frac{i\,\pi^2}{2}\left[m_i^2 + m_j^2 - \frac{(p_i-p_j)^2}{3}\right] + \mathcal{O}(\varepsilon). \tag{4.2.98}
$$

From this shift, the coefficients of the master integrals get the $\tilde{k}$ dependence as

$$
d(\tilde{k}^2; ijkl) \;=\; d(ijkl) + \tilde{k}^2\,d^{(2)}(ijkl) + \tilde{k}^4\,d^{(4)}(ijkl) \tag{4.2.99}
$$

$$
c(\tilde{k}^2; ijkl) \;=\; c(ijkl) + \tilde{k}^2\,c^{(2)}(ijkl) \tag{4.2.100}
$$

$$
b(\tilde{k}^2; ijkl) \;=\; b(ijkl) + \tilde{k}^2\,b^{(2)}(ijkl) \tag{4.2.101}
$$

Then, $d^{(4)}(ijkl)$, $c^{(2)}(ijkl)$ and $b^{(2)}(ijkl)$ are the coefficients of the extra-integrals in Eqs. (4.2.99, 4.2.100, 4.2.101). These can be calculated either numerically

$$d^{(4)}(ijkl) = \lim_{\tilde{k}^2\to\infty} \frac{d(\tilde{k}^2; ijkl)}{\tilde{k}^4}, \tag{4.2.102}$$

$$c^{(2)}(ijk) = \lim_{\tilde{k}^2\to\infty} \frac{c(\tilde{k}^2; ijk)}{\tilde{k}^2}, \tag{4.2.103}$$

$$b^{(2)}(ij) = \lim_{\tilde{k}^2\to\infty} \frac{b(\tilde{k}^2; ij)}{\tilde{k}^2}. \tag{4.2.104}$$

or solving numerically the systems obtained by Eqs.(4.2.99, 4.2.100, 4.2.101) for different values of the $\tilde{k}^2$, where the result is

$$d^{(4)}(ijkl) = \frac{d(1; ijkl) + d(-1; ijkl) - 2\,d(ijkl)}{2}, \tag{4.2.105}$$

$$c^{(2)}(ijk) = c(1; ijk) - c(ijk), \tag{4.2.106}$$

$$b^{(2)}(ij) = b(1; ij) - b(ij). \tag{4.2.107}$$

where the term containing $d^{(2)}$ vanishes upon integration, so no need to calculate them. Hence, the rational terms are calculated.
This method is implemented in several reduction libraries as `SAMURAI` and `CutTools`[70, 75]

4.2.2 On-shell reduction method

To deduce some properties of the amplitude, we will use two great principles of quantum physics: the *unitarity* and the *causality*. The unitarity is a consequence of the probability conservation, it is translated by this equation

$$S\,S^+ = S^+\,S = 1 \tag{4.2.108}$$

The unitary operator S, or the diffusion matrix S, is one of the fundamental pillar of the perturbative quantum field theory. It describes the transition from an initial state $|i>$ to a final state $|f>$, where the initial and the final states are taken at time $t \to -\infty$ and time $t \to +\infty$, respectively. The probability to get the system in the state $|f>$, when the system was in the state $|i>$ is given by $|<i|S|f>|^2$, so, the unitarity nature of S is a consequence of the probability conservation, which is a constitutional demand of any consistent quantum field theory.

It is very useful to write this operator in term of the transition matrix T,

$$\begin{aligned} S_{fi} &= \delta_{fi} + i\,T_{fi} \\ &= \delta_{fi} + i\,(2\,\pi)^4\,\delta^4(p_f - p_i)\,\mathcal{A}_{fi} \qquad \text{with} \qquad T_{fi} = (2\,\pi)^4\,\delta^4(p_f - p_i)\,\mathcal{A}_{fi} \end{aligned} \tag{4.2.109}$$

$\mathcal{A}_{fi}$ is the transition amplitude from state $|i>$ to state $|f>$. Using the fact that S is unitary, one can write

$$-\,i\,(\mathcal{A} - \mathcal{A}^+) = \mathcal{A}\,\mathcal{A}^+ \tag{4.2.110}$$

which leads to this very important relation

$$2\,\mathrm{Im}(\mathcal{A}_{fi}) \;=\; \sum_j \mathcal{A}_{jf}\,\mathcal{A}^{+}_{fi}\,(2\,\pi)^4\,\delta(p_i - p_j)\,\delta(p_j - p_f) \tag{4.2.111}$$

Eq. (4.2.111) called the optical theorem. It allows us to write the imaginary part of the transition amplitude as an explicit sum of the amplitude over all intermediate physical states $|j>$ defined by $\mathcal{A}^{+}$ and $\mathcal{A}$. This sum is translated by a phase space integral over the intermediate momentum and a desecrate sum over allowed particles.
The left hand side of Eq. (4.2.111) gives the absorptive part of the amplitude which corresponds to discontinuity in the amplitude. The right hand side, can be obtained from cutting the one loop amplitude. So, this leads to the amazing results: *The imaginary part of the one-loop amplitude can be calculated by cutting two propagators of the loop*

4.2.2.1 Generalized unitarity

Let's consider, for instance, an amplitude depends only on the invariant "$s = (p_1 + p_2)^2$". The adjacent propagators of s channel are denoted by q_1 and q_2, see Fig.(4.1). s_0 is the energy threshold which correspond to a branch cut of the amplitude on s. In another way, it defines when the process

$$p_1 + p_2 \rightarrow q_1 + q_2 \tag{4.2.112}$$

can happen. If $s < s_0$, no physical processes can occur since the propagators are off the

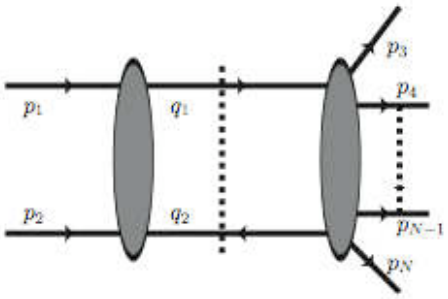


Figure 4.1: Optical theorem

mass-shell. Hence, the amplitude is analytical along the real axis. Otherwise, if $s > s_0$, the propagator can be in the mass-shell and the amplitude is not any more analytical along the real axis. The discontinuity along the real axis is (see previous chapter)[6]

$$\begin{aligned} \mathrm{Disc}_s\mathcal{A} &= \lim_{\lambda\rightarrow 0^+} i\,[\mathrm{Im}\mathcal{A}(s + i\,\lambda) - \mathrm{Im}\mathcal{A}(s - i\,\lambda)] \\ &= 2\,i\,\mathrm{Im}\mathcal{A}(s) \end{aligned} \tag{4.2.113}$$

[6] In general, we have more than one invariant. Then, the imaginary part of the amplitude is given by: $\sum_{i=1}^{m} \mathrm{Disc}_{s_i}\mathcal{A}(s_1, ..., s_m) = 2\,i\,\mathrm{Im}\mathcal{A}(s_1, ..., s_m)$, where $s_1, \cdots, s_m$ are all the invariants of the diagram, see [80].

By applying the cutkosky rules (chapter 3), the discontinuity of the amplitude along the s channel is reduced to the calculation of two tree level amplitudes as the following.

$$\text{Disc}_s = (2\,\pi)^2 \int \frac{d^n k}{(2\,\pi)^n} \frac{i}{D_1^2} \mathcal{A}_{q_1+q_2\to p_3+\ldots+p_n} \frac{i}{D_2^2} \mathcal{A}_{p_1+p_2\to q_1+q_2}\Big|_s \tag{4.2.114}$$

$$= (2\,\pi)^2 \int \frac{d^n Q}{(2\,\pi)^n} \delta^+(D_1^2)\, \mathcal{A}_{q_1+q_2\to p_3+\ldots+p_n}\, \delta^+(D_2^2)\, \mathcal{A}_{p_1+p_2\to q_1+q_2} \tag{4.2.115}$$

where $\mathcal{A}_{q_1+q_2\to p_3+\ldots+p_n}$ and $\mathcal{A}_{p_1+p_2\to q_1+q_2}$ are the tree level diagrams given by the corners of the diagram (or the amplitude) in Fig.(4.1), D_1 and D_2 are the cut propagators, they are defined as: $D_i = q_i^2 - m_{q_i}^2$ for $i = 1, 2$ (m_{q_i} is the mass of the particle carrying the momenta q_i).

From the optical theorem, this discontinuity give the absorptive part of the amplitude. The dispersive part of the amplitude can be calculated from this part (absorptive) by means of the dispersion relation. This result is one of the crucial results of loop calculation in quantum field theory.

We have seen in the beginning of this chapter that any one loop amplitude can be expressed as a combination of scalar integrals weighted by the same coefficients. Then the discontinuity of the amplitude can be expressed as combination of the discontinuity of the scalar basic integrals weighted by some coefficients, i.e.

$$\text{Disc}_{s_i}\, \mathcal{A}_N^n(\bar{k}) = \quad \tilde{c}_{4;j}\, \text{Disc}_{s_i}\, I_{4;j}^n + \tilde{c}_{3;j}\, \text{Disc}_{s_i}\, I_{3;j}^n + \tilde{c}_{2;j}\, \text{Disc}_{s_i}\, I_{2;j}^n + \tilde{c}_{1;j}\, \text{Disc}_{s_i}\, I_{1;j}^n \tag{4.2.116}$$

In the following, we apply this results to calculate the coefficients of the scalar integrals.

4.2.2.2 One loop integral coefficients

In practice, one can compute the imaginary part of the one-loop amplitude (the absorptive part of the amplitude) directly by means of the Cutkosky rules presented in the previous chapter. The real part of the amplitude (or the dispersive part) is then reconstructed from the absorptive one by means the desperation relation, thanks to unitarity and causality.

In this paragraph, we show how one can calculate the coefficients of the 4-, 3-, 2- and 1-point scalar basic integrals and the rational term of the amplitude decomposition given in Eq.(4.2.79); from the discontinuity of the amplitude in the framework of Cutkosky cutting rules in 4-dimensions, and the rational term from the on-shell recursion relations[22, 24, 25, 26]. To the decomposition coefficients, we follow the ref. [22].

- a) Quadruple cuts

To determine the coefficient of the box, one has to make quadruple cut as shown in Fig.(4.2). We mean by "Cutting a propagator": removing its principles value and replace it by a delta function $\delta^{(+)}(P^2)$, where in Minkowski space the propagator is given by

$$\frac{1}{(D_i^2 + i\,\varepsilon)} = 1/D_i^2 + \delta^{(+)}(D_i^2) \tag{4.2.117}$$

In this case, all the internal propagators are crossed by cuts, i.e. all the internal particles are on the mass shell Fig.(4.2). Hence, each cut propagator is replaced by delta function as

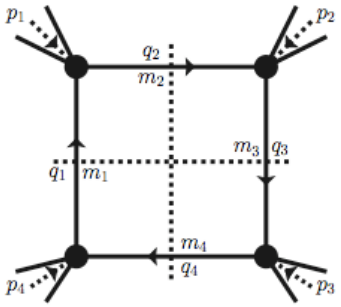


Figure 4.2: A quadruple cut

the following

$$\frac{i}{q_i^2 - m_i^2} \to (2\,\pi)\,\delta(q_i^2 - m_i^2) \tag{4.2.118}$$

The cut conditions of each internal leg implies,

$$\begin{aligned} k^2 - m_1^2 = 0, \qquad & (k + p_1)^2 - m_2^2 = 0, \\ (k + p_1 + p_2)^2 - m_3^2 = 0, \qquad & (k - p_4)^2 - m_4^2 = 0. \end{aligned} \tag{4.2.119}$$

In four dimensions, the constraints in Eq.(4.2.119) determine completely the momentum circulating in the loop (k) which, in general, has two complex conjugate solutions $k^{\pm}$. The coefficient $\tilde{c}_{4;j}$ associated to the box is obtained by pasting together the tree level diagrams obtained by cutting this box in four parts, see Fig.(4.2).

$$\begin{aligned} & \int \frac{d^4\,k}{(2\,\pi)^4}\,\delta(k^2 - m_1^2)\,A_1^{(j)}\,\delta((k + p_1)^2 - m_2^2)\,A_2^{(k)}\,\delta((k + p_1 + p_2)^2 - m_3^2) \\ & A_3^{(j)}\,\delta((k - p_4)^2 - m_4^2)\,A_4^{(j)} \\ & = \tilde{c}_{4;j}\,\mathrm{Dic}_{s_i}\,I_4^4 \\ & = \tilde{c}_{4;j} \int \frac{d^4\,k}{(2\,\pi)^4}\,\delta(k^2 - m_1^2)\,\delta((k - p_1)^2 - m_2^2)\,\delta((k + p_1 + p_2)^2 - m_3^2) \\ & \delta((k - p_4)^2 - m_4^2) \end{aligned} \tag{4.2.120}$$

Hence, the coefficient of box equals to the average of the product of the tree level amplitudes evaluated at $k^{\pm}$, we write [22],

$$\tilde{c}_{4;j} \;=\; \frac{i}{2} \sum_{\pm} A_1^{(j)}(k^{\pm})\,A_2^{(j)}(k^{\pm})\,A_3^{(j)}(k^{\pm})\,A_4^{(j)}(k^{\pm}) \tag{4.2.121}$$

where $A_n^{(k)}$, $i = 1, \cdots, 4$ are the tree level amplitudes at the corner of the box in Fig.(4.2) with the total momentum K_i. We notice that the system of equations in Eq. (4.2.119) is solve by parametrizing the cut loop momentum as the following ([22]): we introduce two massless momenta K_1^b and K_2^b in such way that the two adjacent external momenta p_1 and p_2 are given by

$$p_1 = K_1^b + \frac{S_1}{\gamma_{12}}\,K_2^b, \qquad\qquad p_2 = p_2^b + \frac{S_2}{\gamma_{12}}\,K_1^b \tag{4.2.122}$$

where

$$S_1 = p_1 \cdot p_1, \qquad S_2 = p_2 \cdot p_2, \tag{4.2.123}$$

$$\gamma_{12} = 2\, K_1^b \cdot K_2^b = p_1 \cdot p_2 \pm \sqrt{\Delta(p_1, p_2)}, \tag{4.2.124}$$

$$\Delta(p_1, p_2) = (p_1 \cdot p_2)^2 - S_1\, S_2. \tag{4.2.125}$$

For non vanishing $\Delta(p_1, p_2)$, one can introduce the massless vectors

$$a_1^\mu = K_1^{b\,\mu}, \qquad a_2^\mu = K_2^{b\,\mu}, \tag{4.2.126}$$

$$a_3^\mu =< K_1^{b-} | \gamma^\mu | K_2^{b-} >, \qquad a_4^\mu =< K_2^{b-} | \gamma^\mu | K_1^{b-} > . \tag{4.2.127}$$

and express the loop momenta as

$$k^\mu = \alpha_1\, a_1^\mu + \alpha_2\, a_2^\mu + \alpha_3\, a_3^\mu + \alpha_4\, a_4^\mu \tag{4.2.128}$$

one can prove that k has two solution $k^\pm$ where

$$\alpha_1 = \frac{S_2\,(\gamma_{12} - S_1) + (\gamma_{12} - S_2)\, m_1^2 - \gamma_{12}\, m_3^2 + S_2\, m_2^2}{\gamma_{12}^2 - S_1\, S_2}, \tag{4.2.129}$$

$$\alpha_2 = \frac{S_2\,(\gamma_{12} - S_2) + (\gamma_{12} - S_1)\, m_1^2 - \gamma_{12}\, m_2^2 + S_1\, m_3^2}{\gamma_{12}^2 - S_1\, S_2}, \tag{4.2.130}$$

$$\alpha_3 = -\frac{\beta_3 \pm \sqrt{\beta^3 - 2\,\beta_4\, Tr(\not{K}_1^b\ \not{p}_4\ \not{K}_2^b\ \not{p}_4)}}{2\ < K_1^{b-} |\ \not{p}_4 | K_2^{b-} >}, \tag{4.2.131}$$

$$\alpha_4 = \frac{\beta_4}{4\,\alpha_3}. \tag{4.2.132}$$

with

$$\beta_3 = 2\,(\alpha_1 - 1)\, K_1^b \cdot p_4 + 2\,(\alpha_2 - \frac{S_1}{\gamma_{12}})\, K_2^b \cdot p_4 - S_4 + m_4^2 - m_2^2, \tag{4.2.133}$$

$$\beta_4 = \alpha_1\, \alpha_2 - \frac{m_1^2}{\gamma_{12}}. \tag{4.2.134}$$

It seems that k has four solutions but one can prove that

$$k^\mu(\gamma_{12}^+, \alpha_3^+) = k^\mu(\gamma_{12}^-, \alpha_3^-), k^\mu(\gamma_{12}^+, \alpha_3^-) \qquad = k^\mu(\gamma_{12}^-, \alpha_3^+). \tag{4.2.135}$$

then there are only two solutions for k.

- b) Triple cuts

To extract the scalar triangle coefficients, one has to use triple cuts to the one loop amplitude. In this case, we gets contributions from the triangle topologies and from the box topologies, since this later one can have three cuts. The contribution from the box is the main source of complication in extracting the triangle coefficients. So, one has to subtract this non needed contributions and impose the following three cut conditions (see [23])

$$k^2 - m_1^2 = 0, \qquad (k - p_1)^2 - m_3^2 = 0, \tag{4.2.136}$$

$$(k + p_2)^2 - m_2^2 = 0, \tag{4.2.137}$$

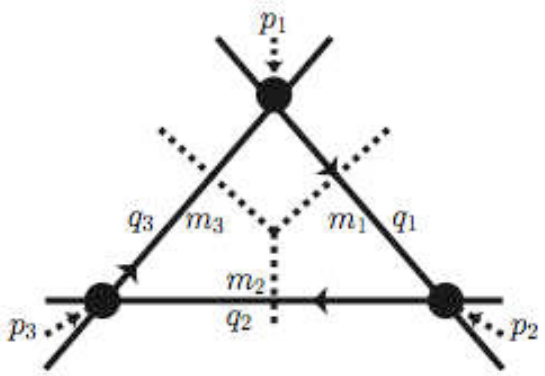


Figure 4.3: A Triple cut

these condition are not sufficient to fix the loop momentum k which must have an unconstrained degrees of freedom. k can be parametrized as

$$k^\mu = \alpha_1\, a_1^\mu + \alpha_2\, a_2^\mu + \frac{t}{2}\, a_3^\mu + \frac{\alpha_4}{2\,t}\, a_4^\mu \tag{4.2.138}$$

with

$$\alpha_1 = \frac{S_2\,(\gamma_{12} - S_1) + (\gamma_{12} - S_2)\, m_1^2 - \gamma_{12}\, m_3^2 + S_2\, m_2^2}{\gamma_{12}^2 - S_1\, S_2}, \tag{4.2.139}$$

$$\alpha_2 = \frac{S_2\,(\gamma_{12} - S_2) + (\gamma_{12} - S_1)\, m_1^2 - \gamma_{12}\, m_2^2 + S_1\, m_3^2}{\gamma_{12}^2 - S_1\, S_2}, \tag{4.2.140}$$

$$\alpha_4 = \alpha_1\, \alpha_2 - \frac{m_1^2}{\gamma_{12}}. \tag{4.2.141}$$

The cut triangle in Fig.(4.3) is given by

$$\begin{aligned}
\tilde{c}_{3;j}\, I_{3;j} &= i \int \frac{d^4k}{(2\,\pi)^4}\, \frac{A_1^{(j)}(p_1;k)\, A_2^{(j)}(p_2;k)\, A_3^{(j)}(p_3;k)}{(k^2 - m_1^2)\,((k-p_1)^2 - m_2^2)\,((k+p_2)^2 - m_3^2)} \\
&\to i\,(-2\,\pi\,i)^3 \int \frac{d^4k}{(2\,\pi)^4}\, A_1^{(j)}(K_1;k)\, A_2^{(j)}(K_2;k)\, A_3^{(j)}(K_3;k) \\
&\times \delta(k^2 - m_0^2)\, \delta((k-K-1)^2 - m_1^2)\, \delta((k+K_2)^2 - m_2^2) \\
&= i\,(-2\,\pi\,i)^3 \int \frac{dt}{(2\,\pi)^4}\, J_t\, A_1^{(j)}(t)\, A_2^{(j)}(t)\, A_3^{(j)}(t),
\end{aligned} \tag{4.2.142}$$

where J_t is the Jacobian of the transformation from the momenta k (constrained by the δ functions) to the remaining free parameter t.
In the and of the day, one can prove

$$\tilde{c}_{3;j} = i\,(-2\,\pi)^3 \int \frac{dt}{(2\,\pi)^4}\, J_t \left(\left[\mathrm{Inf}_t\, A_1^{(j)}\, A_2^{(j)}\, A_3^{(j)} \right](t) + \sum_{\{k\}} \left[\frac{\mathrm{Res}_{t=t_k}\, A_1^{(j)}\, A_2^{(j)}\, A_3^{j}}{t - t_k} \right] \right) \tag{4.2.143}$$

with

$$\lim_{t\to\infty}([\mathrm{Inf}_t\, A_1\, A_2\, A_3](t) - A_1(t)\, A_2(t)\, A_3(t)) = 0. \tag{4.2.144}$$

see section *V.* in ref. [22].

- c) Double cuts

The cuts defining the bubble topologies are

$$k^2 - m_1^2 = 0, \qquad (k+p_1)^2 - m_2^2 = 0, \tag{4.2.145}$$

These cuts are satisfied by the bubbles, triangles and boxes. The last two contribution are

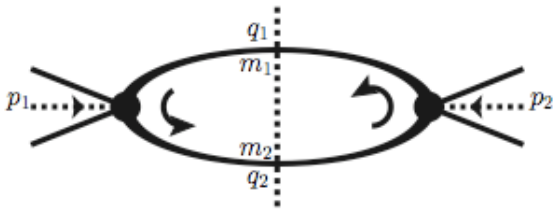


Figure 4.4: A double cut

defined above, so one can isolate them and extract the contribution from the pure bubble. The parameterization of the loop momentum is given by equations (51) and (52) in [22]. The cut babble in Fig.(4.4) is given by the integral

$$\begin{aligned}
\tilde{c}_{2;j}\, I_{2;j} &= i \int \frac{d^4k}{(2\,\pi)^4}\, \frac{A_1^{(j)}(p_1;k)\, A_2^{(j)}(p_2;k)}{(k^2-m_1^2)\,((k-p_1)^2-m_2^2)} \\
&\to i\,(-2\,\pi\,i)^2 \int \frac{d^4k}{(2\,\pi)^4}\, A_1^{(j)}(p_1;k)\, A_2^{(j)}(p_2;k) \\
&\times \delta(k^2-m_1^2)\,\delta((k-p_1)^2-m_2^2) \\
&= i\,(-2\,\pi\,i)^2 \int \frac{dt\,dy}{(2\,\pi)^4}\, J_{t,y}\, A_1^{(j)}(t,y)\, A_2^{(j)}(t,y),
\end{aligned} \tag{4.2.146}$$

where $J_{t,y}$ is the Jacobian of the transformation from the momenta k (constrained by the δ functions in Eq.(4.2.145)) to the two remaining free parameter t and y.
In the and of the day, one can prove

$$\begin{aligned}
\tilde{c}_{2;j} &= i\,(-2\,\pi)^2 \int \frac{dt\,dy}{(2\,\pi)^4}\, J_{t,y}\, A_1^{(j)}(t,y)\, A_2^{(j)}(t,y) \\
&\to i\,(-2\,\pi)^2 \int \frac{dt\,dy}{(2\,\pi)^4}\, J_{t,y} \left(\left[\mathrm{Inf}_y\,[\mathrm{Inf}_t\; A_1^{(j)}\, A_2^{(j)}]\right](t,y) + \left[\mathrm{Inf}_y \sum_{\{k\}} \left[\frac{\mathrm{Res}_{t=t_k}\, A_1^{(j)}\, A_2^{(j)}}{t-t_k}\right]\right](y) \right. \\
&\left. + \sum_{\{j\}} \left[\frac{\mathrm{Res}_{y=y_j}\,[\mathrm{Inf}_t\, A_1^{(i)}\, A_2^{(i)}](t)}{y-y_j}\right] + \sum_{\{j\}} \left[\frac{\mathrm{Res}_{y=y_j} \sum_{\{k\}} \frac{[\mathrm{Res}_{t=t_k}\, A_1^{(i)}\, A_2^{(i)}]}{t-t_k}}{y-y_j}\right] \right.
\end{aligned} \tag{4.2.147}$$

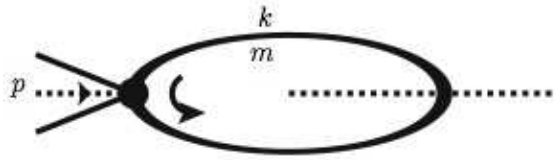


Figure 4.5: A Single cut

- c) Single cut

In this case, the only constraint on the loop momenta is

$$k^2 - m^2 = 0. \tag{4.2.148}$$

In a similar way, one finds that

$$\begin{aligned} \tilde{c}_{1;j} &= i \int \frac{d^4k}{(2\,\pi)^4} \frac{A^{(j)}(k)}{k^2 - m^2} \\ &\to (-2\,\pi\,i) \int \frac{d^4k}{(2\,\pi)^4} A^{(j)}(k)\,\delta(k^2 - m^2) \\ &= (-2\,\pi\,i) \int \frac{dt\,dy\,dw}{(2\,\pi)^4} J_{t,y,w}\, A^{(j)}(t,y,w) \end{aligned} \tag{4.2.149}$$

where $J_{t,y,w}$ is the Jacobian of the transformation from the momenta k (constrained by the δ functions in Eq.(4.2.148)) to the three remaining unconstrained parameter t, y and w. The explicit formula of $\tilde{c}_{1;j}$ is given in [22].

4.3 Improving the unitarity method with respect to $\det(G)$ problems

In the previous sections, we showed several reduction methods of one loop amplitudes, all of them are based on the Feynman diagrammatic approach or the unitarity inspired methods. Each of these methods has its own advantages and disadvantages. For example, the weakness of Golem reduction, which is based on the Feynman diagrammatic approach, is the increasing number of one-loop Feynman diagrams with the number of produced particles in the final state (to calculate the amplitude of the process $g\,g \to t\,\bar{t}\,b\,\bar{b}$ at NLO order for example, one has to calculate more than 1000 one-loop diagram); but it provides numerical stable results in problematic regions (where the Gram determinant becomes arbitrary small) in a very efficient way. On the contrary, OPP method, which is a unitarity inspired approach, decreases the amount of work since it does not require the calculation of all contributing Feynman diagrams, but it needs a very high multi-precision to avoid the numerical instability problems induced by the Gram determinants [70], which needs a longer CPU time.
In this paragraph, we present another approach of one-loop amplitude reduction obtained by reconstructing the tensorial expressing of the amplitude at the integrand level, and reducing such tensorial integrals by means of Golem reduction to avoid the problem of the spurious Gram singularities.

4.3.1 Tensor reconstruction at the integrand level

A general N-point one-loop amplitude in n-dimensions is given by

$$\mathcal{A}_N^n = \int d^n\bar{k}\,\frac{N(\bar{k},\varepsilon)}{\bar{D}_0\,\bar{D}_1\,...\bar{D}_{N-1}}, \qquad \bar{D}_i = (k+p_i)^2 - m_i^2 - \mu^2$$
$$\bar{\not{k}} = \not{k} + \not{\mu} \tag{4.3.150}$$

The *bar* stands for objects living in $n = 4 - 2\varepsilon$ dimensions. The main idea of this method is to re-write the denominator $N(\bar{k})$ as a linear combination of tensors of ranks with up to the maximum power of the integration momenta k. Hence, the full amplitude is reduced to as sum of tensorial integrals weighted by some coefficients. Once the tensor coefficients are defined, one can express each of these tensor integrals in term of Golem set of basic integrals. In the following, we show how to build numerically the tensorial representation from the numerator of the original amplitude [32].

In 4-dimensions, the numerator can be written

$$N(k) \;=\; \sum_{r=0}^{R} C_{\mu_1\cdots\mu_r}\,k_{\mu_1}\cdots k_{\mu_r} \tag{4.3.151}$$

where R is the highest power of the loop momentum in the numerator, and it stands also to the highest rank of the tensors (in renormalizable gauge theories). For each r, the coefficient $C_{\mu_1\cdots\mu_r}$ forms a contra-variant tensor. Each term of this equation can be written as

$$C_{\mu_1\cdots\mu_r}\,k_{\mu_1}\cdots k_{\mu_r} = \sum_{(i_1,i_2,i_3,i_4)\vdash r} \hat{C}^{(r)}_{i_1,i_2,i_3,i_4}\cdot(k_1)^{i_1}\,(k_2)^{i_2}\,(k_3)^{i_3}\,(k_4)^{i_4}. \tag{4.3.152}$$

where k_i are the components of k (k_4 denotes the energy component), the notation $(i_1, i_2, i_3, i_4)\vdash r$ means that the indices i_j form an integer partition of r, and each component of C contributes to one component $\hat{C}^{(r)}_{i_1,\ldots,i_4}$, where the total number of independent component (in 4-dimensions) is given by

$$n_r = \binom{4+r-1}{r} = \frac{(r+3)!}{3!\,r!} \tag{4.3.153}$$

The components of the coefficients $C_{\mu_1\cdots\mu_3}$ are calculated numerically by evaluated the numerator for an arbitrary real set of the integration momentum k. One can put $k = (x, y, z, w)$ and evaluate $N(k) = N(x, y, z, w)$ at different values of (x, y, z, w).

4.3.1.1 The coefficient calculation

The algorithm of coefficient calculation contains four levels:

- **At level-0**, we put $k = (0, 0, 0, 0)$, this trivially allow us to calculate the constant term C_0 where $C_0 = N(0, 0, 0, 0)$.

- **At level-1**, we evaluate $N^{(1)}(k) = N(k) - N^{(0)}$ for k with only one non vanishing component ($k = (x,0,0,0)$, $k = (0,y,0,0)$, $k = (0,0,z,0)$ and $k = (0,0,0,w)$). This generate a system of $4\,R$ equations and $4\,R$ unknown tensor components $C_{i_1}, \cdots, Ci_1 \cdots i_R$ which can be solved numerically.

- **At level-2**, first we have to subtract from the numerator all terms containing the calculated $4\,R+1$ coefficients from level 1 and 2. Let's call the new numerator $N^{(2)}(k)$. One has to fixe k with two non vanishing components, this lead to six possible choices of the momentum. Each choice of k leads to a system of $R\,(R-1)/2$ equation with the same number of unknown coefficients. Hence, all the $3\,R\,(R-1)$ are completely defined by solving the six systems of equations.

- **At level-3**, we call the new numerator constructed after subtracted all the known coefficients, $N^{(3)}$. The momenta k is fixed with three non vanishing components. These are four possible choices of k, each one form a system with $R\,(R-1)\,(R-2)/6$ equation and the same number of unknowns. Hence, all the $2\,R\,(R-1)\,(R-2)/3$ are completely defined by solving the four systems of equations.

- **At level-4**, k is fixed without any vanishing component. This form a system of $R\,(R-1)\,(R-2)\,(R-3)/24$ equations and unknowns.

At this stage, all the $\frac{(R+3)!}{3!\,R!}$ component of the $C_{\mu_1 \cdots \mu_r}$ are numerically computed, hence the numerator is fully reconstructed in 4-dimension without introducing any spurious source of instabilities.

For example, let's consider a numerator with two powers of k at most. Then,

$$N(k) \;=\; C_0 + C_\mu\, k_\mu + C_{\mu\,\nu}\, k_\mu\, k_\nu \tag{4.3.154}$$

The number of the independent components of the coefficients C that has to be defined is:

$$\sum_{r=0}^{2} \frac{(3+r)!}{3!\,r!} = 1 + 4 + 10 \tag{4.3.155}$$

$$= 15 \tag{4.3.156}$$

- **Level-0**: Trivially, the constant term is defined

$$N^{(0)}(0,0,0,0) \;=\; C_0 \tag{4.3.157}$$

- **Level-1** we subtract the constant term from $N(k)$, i.e. $N^{(1)}(k) = N(k) - N^{(0)}$, we get

$$N^{(1)}(x,0,0,0) \;=\; x\,C_1 + x^2\,C_{11} \tag{4.3.158}$$

$$N^{(1)}(0,y,0,0) \;=\; y\,C_2 + y^2\,C_{22} \tag{4.3.159}$$

$$N^{(1)}(0,0,z,0) \;=\; z\,C_3 + z^2\,C_{33} \tag{4.3.160}$$

$$N^{(1)}(0,0,0,w) \;=\; w\,C_4 + w^2\,C_{44} \tag{4.3.161}$$

this system defines the 8 unknown coefficients C_i and C_{ij}. It can be solved by evaluating each polynomial at two different values of k, which determine completely the 8 component C_i and C_{ii} for $i = 1, 2, 3, 4$.

- **Level-2**: we define $N^{(2)}$ such that $N^{(2)}(k) = N^{(1)}(k) - \sum_{j=1}^{4} C_j\, k_j - \sum_{j=1}^{4} C_{jj}\, k_j^2$, at the six choices of k, we get

$$\begin{aligned} N^{(2)}(x, y, 0, 0) &= x\, y\, C_{12} && (4.3.162)\\ N^{(2)}(x, 0, z, 0) &= x\, z\, C_{13} && (4.3.163)\\ N^{(2)}(x, 0, 0, w) &= x\, w\, C_{14} && (4.3.164)\\ N^{(2)}(0, y, z, 0) &= y\, z\, C_{23} && (4.3.165)\\ N^{(2)}(0, y, 0, w) &= y\, w\, C_{24} && (4.3.166)\\ N^{(2)}(0, 0, z, w) &= z\, w\, C_{34} && (4.3.167) \end{aligned}$$

which is quite straightforward to solve, then the 6 component C_{ij} for $i, j = 1, 2, 3, 4$ and $i \neq j$ are completely defined.

Hence, we have found all the numerical values of the 15 component of the tensor coefficients. And the numerator in Eq. (4.3.154) is fully reconstructed. From now on, we denote by $< N(q) >$ the reconstructed numerator in 4-dimensions. It remains to determine the contribution of μ^2-dependance, which is the subject of the next paragraph.

4.3.1.2 Reconstruction the μ^2-dependance

It remains to reconstruct the μ^2-dependance, which leads to rational part R (or the contribution to the amplitude calculated in "$-2\,\varepsilon$" dimensions). In n dimensions, the numerator is expanded in term of μ as

$$N(\bar{k}) = < N(k) > + G^{(1)}\,\mu^2 + G^{(2)}\,\mu^4 + G^{(3)}_{\alpha}\, k^{\alpha}\,\mu^2 + G^{(4)}_{\alpha\beta}\, k^{\alpha}\, k^{\beta}\,\mu^2 \quad (4.3.168)$$

where $< N(k) >$ stands for the numerator evaluated in 4-dimensions, the other numerators are the only terms leading to UV divergences in renormalizable gauge theories.
This decomposition is provided from the fact that, the rational terms are given from the combination of the $n - 4$ dimensional terms with UV divergent integrals. We notice that some term of this decomposition give no contribution to the final result (for example, the term with $G^{(2)}\,\mu^2$ is excluded by power counting from bubble and triangle diagrams) . But they may be needed to calculated other non-vanishing contribution since we are working at the integrand level (for example $G^{(1)\,\mu^2}$ in the box diagram), see [32]. Consequently, for N denominators with $N = 2, 3, 4$, the associated numerator $N_N(\bar{k})$ are

$$\begin{aligned} N_2(\bar{k}) &= < N(k) > + G^{(1)}\,\mu^2, && (4.3.169)\\ N_3(\bar{k}) &= < N(k) > + G^{(1)}\,\mu^2 + G^{(3)}_{\alpha}\, k^{\alpha}\,\mu^2, && (4.3.170)\\ N_4(\bar{k}) &= < N(k) > + G^{(1)}\,\mu^2 + G^{(2)}\,\mu^4 + G^{(3)}_{\alpha}\, k^{\alpha}\,\mu^2 + G^{(4)}_{\alpha\alpha}\, k^{\alpha}\, k^{\alpha}\,\mu^2, && (4.3.171) \end{aligned}$$

The necessary integrals to evaluate the rational parts are (see [76, 77, 78, 79])

$$\int d^n\bar{k}\frac{\mu^4}{\bar{D}_i\,\bar{D}_j\,\bar{D}_k\,\bar{D}_l} = -\frac{i\,\pi^2}{6}+\mathcal{O}(\varepsilon), \tag{4.3.172}$$

$$\int d^n\bar{k}\frac{\mu^2}{\bar{D}_i\,\bar{D}_j\,\bar{D}_k} = -\frac{i\,\pi^2}{2}+\mathcal{O}(\varepsilon), \tag{4.3.173}$$

$$\int d^n\bar{k}\frac{\mu^2}{\bar{D}_i\,\bar{D}_j} = -\frac{i\,\pi^2}{2}\left[m_i^2+m_j^2-\frac{(p_i-p_j)^2}{3}\right]+\mathcal{O}(\varepsilon), \tag{4.3.174}$$

$$\int d^n\bar{k}\frac{\mu^2\,k^\mu\,k^\nu}{\bar{D}_i\,\bar{D}_j\,\bar{D}_k\,\bar{D}_l} = -\frac{i\,\pi^2}{12}\,g^{\mu\nu}+\mathcal{O}(\varepsilon) \tag{4.3.175}$$

$$\int d^n\bar{k}\frac{\mu^2\,k^\mu}{\bar{D}_i\,\bar{D}_j\,\bar{D}_k} = \frac{i\,\pi^2}{6}\,(p_i+p_j+p_k)+\mathcal{O}(\varepsilon). \tag{4.3.176}$$

(4.3.177)

4.3.2 Projection to `Golem95` basic integrals

We have shown that the amplitude numerator can be expressed as a combination of tensors build up by products of loop momenta k weighted by tensorial coefficients $C_{\mu_1\cdots\mu_r}$, where these coefficients are kinematic dependent. Then, the procedure of *tensorial reconstruction at the integrand level* allows us to write the full scattering amplitude as a sum of one-loop tensorial integrals of rank with up to the highest power of the loop momenta in the numerator, without introducing any spurious Gram determinant singularities. These generated tensors are of the same type of the tensors encountered in the calculation of the individual Feynman diagrams. So, one can express them in term of the Golem form factors (which are a combination of the Golem basic integrals) by means of Golem reduction procedure introduced in section 4.1.2, again, this will not introduce any spurious singularities for the same reasons explained above. At this point, this procedure is a successful way to merge the reduction based on the unitarity-inspired approach and the reduction based on the Feynman diagrammatic approach (Golem reduction). This algorithm is implanted in `GoSam` package program, and it is used as a rescue system in the problematic regions where the `SaMurai` program becomes not efficient, see [33].

Chapter 5

Basic Integrals of Golem Library

Contents

The Golem project [19] initially aimed at automatically computing one loop corrections to QCD processes using Feynman diagrams techniques whereby 1) each diagram was written as form factors times Lorentz structures 2) each form factor was decomposed on a particular redundant set of basic integrals. Indeed when the form factors are reduced down to a basis of scalar integrals only, negative powers of Gram determinants, generically noted $\det(G)$ below, show up in separate coefficients of the decomposition. These $\det(G)$, albeit spurious, are sources of troublesome numerical instabilities whenever they become small. The set of basic integrals used in the Golem approach is such that all coefficients of the decomposition of any form factor on this set are free of negative powers of $\det(G)$. Let aside trivial one- and two-point functions, the Golem library of basic functions is instead made of a redundant set involving the functions $I_3^n(j_1, \cdots, j_3)$, $I_3^{n+2}(j_1)$, $I_4^{n+2}(j_1, \cdots, j_3)$ and $I_4^{n+4}(j_1)$. Here the lower indices indicate the number of external legs, the upper indices stand for the dimension of space-time, and the arguments $j_1, \cdots, j_i$ labels i Feynman parameters in the numerator of the corresponding integrand. The strategy is the following. In the phase space regions where $\det(G)$ are not troublesome, the extra elements of the Golem set are decomposed on a scalar basis and computed *analytically* in terms of logarithms and di-logarithms. In the phase space region where $\det(G)$ becomes very small, these extra Golem elements are instead used as irreducible building blocks explicitly free of Gram determinant and provided as one-dimensional integral representations computed *numerically*, see section 4.1.2 in the previous chapter and Fig. (6.2) in chapter 6.

Much faster and more efficient methods than those relying on Feynman diagrams techniques have been developed, e.g. based on unitarity cuts of transition amplitudes and not individual Feynman diagrams, and/or processing the decompositions at the level of the integrands, see previous chapter. Yet these methods still amount to a decomposition onto

a set of basic integrals. In this respect the stand-alone relevance of the Golem library of basic functions, initially developed as a part of the Golem approach, remains. Furthermore the decompositions obtained by these new methods project onto a basis of scalar integrals and thus are still submitted to numerical instabilities caused by $\det(G)$. The issue of numerical instability is then addressed in various ways ranging from smoothening numerical interpolations over the regions of instabilities to more involved rescue solutions. In [33] the solution adopted is to provide a rescue alternative relying on the Golem decomposition to compute the amplitude in the troublesome kinematic configurations, see section 4.3 in the previous chapter. The Golem library, initially designed for QCD, did not include basic functions with internal masses yet provided a convenient way of handling infrared and collinear singularities inherent in the massless case. Its completion with the cases involving internal masses, possibly complex, extends its range of use. This completion shall supply the functions $I_3^n(j_1, \cdots, j_3)$, $I_3^{n+2}(j_1)$, $I_4^{n+2}(j_1, \cdots, j_3)$ and $I_4^{n+4}(j_1)$ in the massive cases in a numerically stable with respect to $\det(G)$ issues.

To handle $\det(G)$ issues, we advocate the use of one-dimensional integral representations rather than relying on Taylor expansions in powers of $\det(G)$. The latter may be thought a priori better both in terms of CPU time and accuracy, however the order up to which the expansion shall be pushed may happen to be rather large, and its determination requires a quantitative estimate of the remainder of the truncation as a function of the order, and, as in the Taylor-Laplace expansion, the latter requires the computation of a one dimensional integral anyway. Originally, `Golem95` used the opposite option of two- or three-dimensional representations of respectively three- and four point functions [19], which were hyper-contour deformations of the definitions of these basic integrals. Yet the computation of these multiple integrals was both slow and not very precise. It is far more efficient both in terms of CPU time and accuracy to evaluate a one-dimensional integral representation, insofar as one is able to find such a representation. In the case without internal masses, we indeed found such a representation. The issue which we address in this thesis is the extension of this approach of one-dimensional integral representations for `Golem95` set of basic integrals in the most general case, i.e. with internal complex masses.

In this chapter, we will derive stable one dimensional integrals representation for each `Golem95` basic integral, where will focus on the three and four point functions in the general massive case.

5.1 Scalar three-point integrals

A generic three point function can be represented by the the diagram in Fig. (5.1). Each internal line with momentum q_i stands for the propagator of a particle of mass m_i. Then, the one-loop scalar three-point function in $n + l$ dimension associated to this triangle is defined by

$$I_3^{n+l}(S) = \int \frac{d^{n+l}k}{i\,\pi^{n/2}} \frac{1}{\Pi_{i=1}^3 \left[q_i^2 - m_i^2 + i\,\lambda\right]}, \tag{5.1.1}$$

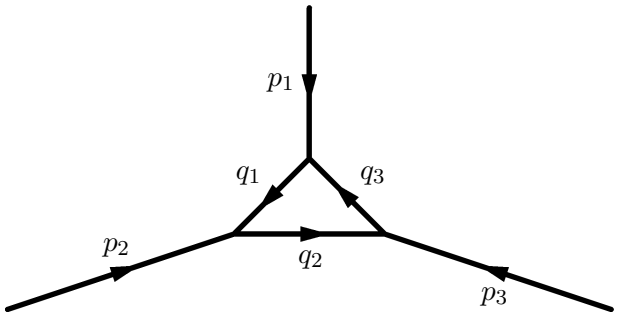


Figure 5.1: 3-point function

where $n = 4 - 2\varepsilon$ and $l = 0, 2, 4$ [1]. S is an ordered set containing the propagators labels, for the triangle in Fig.(5.1), $S = \{1, 2, 3\}$. The propagator are labeled by $q_i = k + r_i$, where k is the momentum circulating in the loop, the r_i are defined by $p_i = r_i - r_{i-1}$ for $i = 1, 2, 3$ (we specify $r_1 = 0$), and p_i are defined as incoming momenta. The momentum conservation implies

$$\sum_{i=1}^{3} p_i \quad = \quad 0 \tag{5.1.2}$$

The functions with $l = 2, 4$ are free of any infrared divergences since they are defined in a space-time with more then 4 dimension, see Table. (3.1) [2]. But they can have some ultra-violet divergences which appear as a simple $\overline{MS}$ pole ε [3]. However, the function with $l = 0$ might be infrared divergent.

[1] In this section, we limit ourself to the three scalar three point functions with $l = 0, 2, 4$. From these functions, any 3-point tensorial basic integral can be constructed.

[2] We have shown in Table. (3.1) that the power of λ_3 (λ_3 is the only vanishing eigenvalue of the matrix $\mathcal{S}$) for the scalar triangle in $n + l$ dimensions is positive for $l = 2, 4$ and negative for $l = 0$, this means that the corresponding singularity is integrable for $l = 2, 4$, then it doesn't lead to IR divergences. However, for $l = 0$ infrared divergences may occur. This was confirmed in section (1.3) of chapter 3 Eqs. (3.4.122, 3.4.131) which are obtained by the soft and collinear approximation. We have proved that the triangle with $l = 2, 3$ are free of soft and collinear divergences in the case where all the internal propagators are massless. A generalization to triangle with some massive propagator is quite straightforward!

[3] The UV divergences of these triangle are fake, since they are forced to appear by the Golem algebraic reduction. Once, these basic integral are gathered to build the form factors, these divergences will disappear.

5.1.1 Outline of the triangle calculation

Using Feynman parameterization and performing the momentum integration, $I_3^{4+l-2\,\varepsilon}(S)$ may be re-written in this form

$$I_3^{4+l-2\,\varepsilon}(S) = -\Gamma\left(\frac{2-l}{2}+\varepsilon\right)\int_0^1 \prod_{i=1}^{3} dz_i\,\delta(1-\sum_{i=1}^{3} z_i)\left(-\frac{1}{2}\,\vec{z}^{\,t}\,\mathcal{S}\,\vec{z}-i\,\lambda\right)^{\frac{l-2}{2}-\varepsilon}, \quad (5.1.3)$$

where z_i are the Feynman parameters, $i\lambda$ is the Feynman contour prescription in the propagators (it is denoted λ in order to avoid any confusion with the parameter $\varepsilon = (4-n)/2$ involved in the dimensional regularization.). We define the kinematic matrix $\mathcal{S}$, which carries all the information on the kinematics associated to this diagram, by:

$$\mathcal{S}_{i\,j} = (q_i - q_j)^2 - m_i^2 - m_j^2 \quad (5.1.4)$$

The square of the difference of two internal momenta can be written in term of the internal masses m_i and the external squared momenta $s_i = p_i^2$. Doing that, we get for $\mathcal{S}$:

$$\mathcal{S} = \begin{pmatrix} -2\,m_1^2 & s_2 - m_1^2 - m_2^2 & s_1 - m_1^2 - m_3^2 \\ s_2 - m_1^2 - m_2^2 & -2\,m_2^2 & s_3 - m_2^2 - m_3^2 \\ s_1 - m_1^2 - m_3^2 & s_3 - m_2^2 - m_3^2 & -2\,m_3^2 \end{pmatrix} \quad (5.1.5)$$

We may single out any index a in $S = \{1,2,3\}$ and write

$$z_a = 1 - \sum_{i\neq a} z_i \quad (5.1.6)$$

The quantity $\vec{z}^{\,t}\,\mathcal{S}\,\vec{z}$ becomes:

$$\begin{aligned} \vec{z}^{\,T}\,\mathcal{S}\,\vec{z} &= \sum_{i,j\neq a} z_i\,z_j\,(\mathcal{S}_{i\,j} - \mathcal{S}_{a\,j} - \mathcal{S}_{i\,a} + \mathcal{S}_{a\,a}) + 2\sum_{j\neq a} z_j\,(\mathcal{S}_{a\,j} - \mathcal{S}_{a\,a}) + \mathcal{S}_{a\,a} \\ &= -\sum_{i,j\neq a} z_i\,z_j\,G^{(a)}_{i\,j} + 2\sum_{j\neq a} z_j\,V^{(a)}_j + \mathcal{S}_{aa} \end{aligned} \quad (5.1.7)$$

with

$$G^{(a)}_{i\,j} = -(\mathcal{S}_{i\,j} - \mathcal{S}_{a\,j} - \mathcal{S}_{i\,a} + \mathcal{S}_{a\,a}), \qquad i,j \neq a \quad (5.1.8)$$

$$V^{(a)}_j = \mathcal{S}_{a\,j} - \mathcal{S}_{a\,a}, \qquad j \neq a \quad (5.1.9)$$

It is clear from its definition (5.1.8) that $G^{(a)}_{a\,j} = G^{(a)}_{i\,a} = 0$, thus the matrix $G^{(a)}$ is of rank two. This matrix is the Gram matrix built with the 4-vectors $\Delta_{i\,a} = q_i - q_a$ where $G^{(a)}_{ij} = 2\,(\Delta_{ia}\cdot\Delta_{ja})$. Its determinant does not depend on the choice of a, and it is also the determinant of the similar Gram matrix built with any subset of two external momenta (see section A.1). We note it simply $\det(G)$ without referring to a and unambiguously call it the Gram determinant associated with the kinematic matrix $\mathcal{S}$. Specifying for example $a = 3$, I_3^4 reads:

$$I_3^{4+l-2\varepsilon} = -\Gamma\left(\frac{2-l}{2}+\varepsilon\right)\int_0^1 dz_1 \int_0^{1-z_1} dz_2 \left[\frac{1}{2}\sum_{i,j=1}^{2} z_i\, z_j\, G_{ij}^{(3)} - \sum_{j=1}^{2} z_j\, V_j^{(3)} - \frac{1}{2}\mathcal{S}_{33} - i\,\lambda\right]^{\frac{l-2}{2}-\varepsilon} \tag{5.1.10}$$

With this choice, the Gram matrix $G^{(3)}$[4] and the vector $V^{(3)}$ are given by:

$$G^{(3)} = \begin{pmatrix} 2\,s_1 & s_3 - s_2 + s_1 \\ s_3 - s_2 + s_1 & 2\,s_3 \end{pmatrix} \tag{5.1.11}$$

$$V^{(3)} = \begin{pmatrix} s_1 - m_1^2 + m_3^2 \\ s_3 - m_3^2 + m_3^2 \end{pmatrix} \tag{5.1.12}$$

Then we make the following change of variables:

$$\begin{aligned} z_1 &= 1 - x \\ z_2 &= y \end{aligned}$$

we get now:

$$I_3^{4+l-2\varepsilon} = -\Gamma\left(\frac{2-l}{2}+\varepsilon\right)\int_0^1 dx \int_0^x dy\, \left[a\,x^2 + b\,y^2 + c\,x\,y + d\,x + e\,y + f - i\,\lambda\right]^{\frac{l-2}{2}-\varepsilon} \tag{5.1.13}$$

with:

$$\begin{aligned} &a = s_1 \\ &b = s_3 \\ &c = -s_3 + s_2 - s_1 \\ &d = m_3^2 - m_1^2 - s_1 \\ &e = s_1 - s_2 + m_2^2 - m_3^2 \\ &f = m_1^2 - i\lambda \end{aligned} \tag{5.1.14}$$

Eq. (5.1.13) is the starting point of the computation of the triangle in the 4-dimensions (for $l = 0$) in ref. [66] (c.f. their Eq (5.2)). We will keep the same notation for the different quantities.

First we introduce a parameter α and shift the integration variable y ($y = y' + \alpha\, x$),

$$\begin{aligned} I_3^{4+l-2\varepsilon} = -\Gamma\left(\frac{2-l}{2}+\varepsilon\right)\int_0^1 dx \int_{-\alpha\,x}^{(1-\alpha)\,x} dy' \\ \left[x^2\,(a + b\,\alpha^2 + c\,\alpha) + b\,y'^2 + x\,y'\,(2\,b\,\alpha + c)\right. \\ \left. + (d + e\,\alpha)\,x + e\,y') + f - i\,\lambda\right]^{\frac{l-2}{2}-\varepsilon} \end{aligned} \tag{5.1.15}$$

The parameter α is chosen such that

$$\alpha^2\, b + \alpha\, c + a = 0 \tag{5.1.16}$$

[4] We drop the line and the column of $G^{(3)}$ which are filled by zero.

in order that the quadratic form of x in the integrands of Eqs. (5.1.15) becomes linear in x. Note that the discriminant Δ_α of Eq. (5.1.16) is minus the Gram determinant $\det G$. For all kinematical configurations $p_1, p_2, p_3 = -p_1 - p_2$ involved in one-loop calculations of elementary processes of interest for collider physics, $\det G$ is non-positive[5]

$$\begin{aligned}\Delta_\alpha &= s_1^2 + s_2^2 + s_3^2 - 2s_1s_2 - 2s_1s_3 - 2s_2s_3 \\ &= 4\,(p_1 \cdot p_2) - 4\,p_1^2\,p_2^2.\end{aligned} \tag{5.1.17}$$

The roots $\alpha_\pm$ of the polynomial (5.1.16) are thus real in all relevant cases. So, for $l = 0, 2, 3$ Eq. (5.1.15) becomes:

$$\begin{aligned}I_3^{4-2\varepsilon}(S) = -r_\Gamma \int_0^1 dx \int_{-\alpha\,x}^{(1-\alpha)\,x} dy' \\ [\{(2\,b\,\alpha + c)\,y' + (d + e\,\alpha)\,\}x + b\,y'^2 + e\,y' + f - i\,\lambda]^{-1} + \mathcal{O}(\varepsilon) \\ = I_3^4(S) + \mathcal{O}(\varepsilon),\end{aligned} \tag{5.1.18}$$

$$\begin{aligned}I_3^{6-2\varepsilon}(S) = r_\Gamma \int_0^1 dx \int_{-\alpha\,x}^{(1-\alpha)\,x} dy' \Big\{ -\frac{1}{\varepsilon} \\ + \ln[\{(2\,b\,\alpha + c)\,y' + (d + e\,\alpha)\,\}x + b\,y'^2 + e\,y' + f - i\,\lambda] \Big\} + \mathcal{O}(\varepsilon) \\ = I_3^{6,\mathrm{div}}(S) + I_3^6(S) + \mathcal{O}(\varepsilon),\end{aligned} \tag{5.1.19}$$

$$\begin{aligned}I_3^{8-2\varepsilon}(S) = r_\Gamma \int_0^1 dx \int_{-\alpha\,x}^{(1-\alpha)\,x} dy' \Big\{ \left(\frac{1}{\varepsilon} + 1\right) \\ \times [\{(2\,b\,\alpha + c)\,y' + (d + e\,\alpha)\,\}x + b\,y'^2 + e\,y' + f] + \\ - [\{(2\,b\,\alpha + c)\,y' + (d + e\,\alpha)\,\}x + b\,y'^2 + e\,y' + f] \\ \times \ln[\{(2\,b\,\alpha + c)\,y' + (d + e\,\alpha)\,\}x + b\,y'^2 + e\,y' + f - i\,\lambda] \Big\} + \mathcal{O}(\varepsilon) \\ = I_3^{8,\mathrm{div}}(S) + I_3^8(S) + \mathcal{O}(\varepsilon).\end{aligned} \tag{5.1.20}$$

where

$$\begin{aligned}r_\Gamma &= \Gamma(1-\varepsilon)^2\Gamma(1+\varepsilon)/\Gamma(1-2\varepsilon) \\ &= 1 - \gamma_r\,\varepsilon + \mathcal{O}(\varepsilon^2)\end{aligned} \tag{5.1.21}$$

[5]The only configurations leading to a positive Gram determinant would require that all three external four-momenta $p_1, p_2, p_3 = -p_1 - p_2$ of the three point function be space-like. At the one-loop order which is our present concern, each of the three points, through which p_1, p_2 and p_3 respectively flow, shall be connected to an independent tree. In order for p_1, p_2 and p_3 to be all space-like, each of these trees should involve one leg in the initial state: this would correspond neither to a decay nor to a collision of two incoming bodies.

and γ_r is the Euler-Mascheroni Constant, it is defined by

$$\begin{aligned}\gamma_r &= \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln(n)\right)\\ &= 0.57721566490153\cdots\end{aligned} \tag{5.1.22}$$

5.1.2 The triangle in 4-dimensions

Let us consider $I_3^4(S)$,

$$\begin{aligned}I_3^4(S) = -\int_0^1 dx \int_{-\alpha\,x}^{(1-\alpha)\,x} dy'\\ [\{(2\,b\,\alpha + c)\,y' + (d + e\,\alpha)\,\}x + b\,y'^2 + e\,y' + f - i\,\lambda]^{-1}\end{aligned} \tag{5.1.23}$$

We split the integral on y', there is no problem if α is real,

$$\int_0^1 dx \int_{-\alpha\,x}^{(1-\alpha)\,x} dy' = \int_0^1 dx \int_0^{(1-\alpha)\,x} dy' - \int_0^1 dx \int_0^{-\alpha\,x} dy' \tag{5.1.24}$$

and reverse the order of integrations:

$$\int_0^1 dx \int_{-\alpha\,x}^{(1-\alpha)\,x} dy' = \int_0^{1-\alpha} dy' \int_{y'/(1-\alpha)}^1 dx - \int_0^{-\alpha} dy' \int_{y'/(-\alpha)}^1 dx \tag{5.1.25}$$

so that I_3^4 is written as (we change $y' = y$):

$$\begin{aligned}I_3^4(S) \;=\; & -\int_0^{1-\alpha} dy \int_{y/(1-\alpha)}^1 dx\, \left[\{(2\,b\,\alpha + c)\,y + (d + e\,\alpha)\,\}x + b\,y^2 + e\,y + f - i\,\lambda\right]^{-1}\\ & + \int_0^{-\alpha} dy \int_{y/(-\alpha)}^1 dx\, \left[\{(2\,b\,\alpha + c)\,y + (d + e\,\alpha)\,\}x + b\,y^2 + e\,y^{+} f - i\,\lambda\right]^{-1}\end{aligned}$$

The term in the square bracket is linear in x: the integral over this variable is easily performed and we find:

$$\begin{aligned}I_3^4(S) \;=\; & -\int_0^{1-\alpha} dy\, \frac{1}{(2\,\alpha\,b + c)\,y + d + e\,\alpha}\\ & \times \ln\left(\frac{b\,y^2 + (2\,\alpha\,b + c + e)\,y + d + e\,\alpha + f}{(y^2\,(2\,\alpha\,b + c + b\,(1-\alpha)) + y\,(d + e\,\alpha + e\,(1-\alpha)) + f\,(1-\alpha))/(1-\alpha)}\right)\\ & + \int_0^{-\alpha} dy\, \frac{1}{(2\,\alpha\,b + c)\,y + d + e\,\alpha}\\ & \times \ln\left(\frac{b\,y^2 + (2\,\alpha\,b + c + e)\,y + d + e\,\alpha + f}{(y^2\,(2\,\alpha\,b + c - b\,\alpha) + y\,(d + e\,\alpha - e\,\alpha) - f\,\alpha)/(-\alpha)}\right)\end{aligned} \tag{5.1.26}$$

The logarithms in Eq. (5.1.26) can be split into two parts without introducing η function, since the sign of their imaginary parts are not changed (they are always negatives)[6], where

[6]The imaginary part of the quadratic form on x and y in Eq. (5.1.13) is provided by: $(1-x)\mathrm{Im}(m_1^2) + \mathrm{Im}(m_2^2)y + (x-y)\mathrm{Im}(m_3^2) - \lambda$ which is negative since, $\mathrm{Im}(m_i^2) < 0$ and $x \geq y \geq 0$.

the function η is defined by

$$\begin{cases} \ln(a\,b) & = \ln a + \ln b + \eta(a,b), \\ \eta(a,b) & = 2\,\pi\,i\,\{\theta(-\mathrm{Im}a)\,\theta(-\mathrm{Im}b)\,\theta(\mathrm{Im}a\,b) - \theta(\mathrm{Im}a)\,\theta(\mathrm{Im}b)\,\theta(-\mathrm{Im}a\,b)\} \end{cases} \tag{5.1.27}$$

We split these logarithms and combine the terms coming from the upper limit in x ($x = 1$). Eq. (5.1.26) now becomes:

$$\begin{aligned} I_3^4(S) &= -\int_{-\alpha}^{1-\alpha} \frac{dy}{N}\,\ln\left(b\,y^2 + e\,y + f + N\right) \\ &\quad + \int_0^{1-\alpha} \frac{dy}{N}\,\ln\left(b\,y^2 + e\,y + f + N\,y/(1-\alpha)\right) \\ &\quad - \int_0^{-\alpha} \frac{dy}{N}\,\ln\left(b\,y^2 + e\,y + f - N\,y/\alpha\right), \end{aligned} \tag{5.1.28}$$

$$N = (2\,\alpha\,b + c)\,y + d + e\,\alpha. \tag{5.1.29}$$

One can subtract the residue for the pole $y = y_0$ ($N = 0$) with $y_0 = -(d + e\,\alpha)/(c + 2\,\alpha\,b)$ which is the same for the three integrals. Since α is real, the subtracted quantity will give no contribution to the final result:

$$\begin{aligned} I_3^4(S) &= -\int_{-\alpha}^{1-\alpha} \frac{dy}{N}\,\left[\ln\left(b\,y^2 + e\,y + f + N\right) - \ln\left(b\,y_0^2 + e\,y_0 + f\right)\right] \\ &\quad + \int_0^{1-\alpha} \frac{dy}{N}\,\left[\ln\left(b\,y^2 + e\,y + f + N\,y/(1-\alpha)\right) - \ln\left(b\,y_0^2 + e\,y_0 + f\right)\right] \\ &\quad - \int_0^{-\alpha} \frac{dy}{N}\,\left[\ln\left(b\,y^2 + e\,y + f - N\,y/\alpha\right) - \ln\left(b\,y_0^2 + e\,y_0 + f\right)\right] \end{aligned} \tag{5.1.30}$$

We make the following change of variables : $y = z - \alpha$ in the first integral of Eq. (5.1.30), $y = (1-\alpha)\,z$ in the second and $y = -\alpha\,z$ in the third. So we get for I_3^4:

$$\begin{aligned} I_3^4(S) &= -\int_0^1 dz \left\{ \frac{1}{(2\,\alpha\,b + c)\,z + (e + c)\,\alpha + d + 2\,a}\,\left[\ln\left(b\,z^2 + (c+e)\,z + a + d + f\right)\right.\right. \\ &\quad \left. - \ln\left(b\,y_0^2 + e\,y_0 + f\right)\right] \\ &\quad + \frac{-\alpha}{-\alpha\,(2\,\alpha\,b + c)\,z + d + e\,\alpha}\,\left[\ln\left(a\,z^2 + d\,z + f\right)\right. \\ &\quad \left. - \ln\left(b\,y_0^2 + e\,y_0 + f\right)\right] \\ &\quad - \frac{(1-\alpha)}{(1-\alpha)\,(2\,\alpha\,b + c)\,z + d + e\,\alpha}\,\left[\ln\left((a+b+c)\,z^2 + (d+e)\,z + f\right)\right. \\ &\quad \left.\left. - \ln\left(b\,y_0^2 + e\,y_0 + f\right)\right] \right\} \end{aligned} \tag{5.1.31}$$

Note that the subtraction terms (proportional to $\ln\left(b\,y_0^2 + e\,y_0 + f\right)$) have been added in order that the residue of each poles is zero, they sum up to zero. In term of s_i^2 and m_i^2

and for $i = 1, 2, 3$, I_3^4 can be written

$$\begin{aligned} I_3^4(S) = \frac{-1}{\varepsilon_\alpha \sqrt{-\det(G)}} \int_0^1 dz \Big\{ & \\ & \frac{1}{z - z_0^{(1)}} (\ln(s_3 z^2 + (m_2^2 - m_3^2 - s_3)z + m_3^2 - i\lambda) - \ln(\mathcal{B} - i\lambda)) \\ & + \frac{1}{z - z_0^{(2)}} (\ln(s_1 z^2 + (m_3^2 - m_1^2 - s_1)z + m_1^2 - i\lambda) - \ln(\mathcal{B} - i\lambda)) \\ & - \frac{1}{z - z_0^{(3)}} (\ln(s_2 z^2 + (m_2^2 - m_1^2 - s_2)z + m_1^2 - i\lambda) - \ln(\mathcal{B} - i\lambda)) \Big\} \end{aligned} \tag{5.1.32}$$

with

$$\mathcal{B} = -\frac{1}{2} \frac{\det(\mathcal{S})}{\det(G)} \tag{5.1.33}$$

$$z_0^{(1)} = -\frac{(m_2^2 - m_3^2 - s_3)\alpha + m_3^2 - m_1^2 + s_1}{\varepsilon_\alpha \sqrt{\det(G)}} \tag{5.1.34}$$

$$z_0^{(2)} = -\frac{(m_2^2 - m_3^2 + s_1 - s_2)\alpha + m_3^2 - m_1^2 - s_1}{-\alpha\, \varepsilon_\alpha \sqrt{\det(G)}} \tag{5.1.35}$$

$$z_0^{(3)} = -\frac{(m_2^2 - m_3^2 + s_1 - s_2)\alpha + m_3^2 - m_1^2 - s_1}{(1 - \alpha)\, \varepsilon_\alpha \sqrt{\det(G)}} \tag{5.1.36}$$

$$\alpha = \frac{s_1 + s_3 - s_2 + \varepsilon_\alpha \sqrt{-\det(G)}}{2 s_3} \tag{5.1.37}$$

and $\det(G)$ and $\det(\mathcal{S})$ are defined bellow (see Eqs. (5.1.62, 5.1.63)).

To perform the last integration analytically, one can use the formula Eq. (A.2.11) in section A.2.

If $\det(G) \to 0$, the analytical formula lead to numerically unstable results (since it is proportional to inverse of $\sqrt{|\det(G)|}$). Then, we have to switch to the numerical mode which will be the subject of the next section.

5.1.2.1 One-dimensional integral representation: $\det(G) \to 0$

To discuss the behavior of Eq. (5.1.31) when $\det(G) \to 0$, it is more enlightening to symmetrize over α. We note first that the dependence on α comes only from the coefficients of the logaritms in Eq. (5.1.31) , and not from the argument of the logarithms. Indeed, the argument of the subtracted logarithms are

$$b\, y_0^2 + e\, y_0 + f - i\lambda = -\frac{1}{2} \frac{\det(\mathcal{S})}{\det(G)} - i\lambda. \tag{5.1.38}$$

where b is real and e and f may be complex, and the sign of the imaginary part of the arguments of the quadratic logarithm must be the same for $z \in [0, 1]$[7]. So we have three

[7] The imaginary part of the argument of each quadratic logarithm is given respectively by: $\mathrm{Im}(m_2^2)z + \mathrm{Im}(m_3^2)(1 - z) - \lambda < 0$, $\mathrm{Im}(m_2^2)z + \mathrm{Im}(m_1^2)(1 - z) - \lambda < 0$, $\mathrm{Im}(m_3^2)z + \mathrm{Im}(m_1^2)(1 - z) - \lambda < 0$. They are all negative $\forall\ z \in [0, 1]$ since $\mathrm{Im}(m_i^2) \leq 0$ for $i = 1, 2, 3$.

integrals of the type:

$$I_i = \int_0^1 dy \, \frac{K_i(\alpha)}{\alpha \, A_i + B_i} \, L_i, \qquad \text{for } i = 1, 2, 3. \tag{5.1.39}$$

Since we can choose any of the two roots of equation (5.1.16) α_+ and α_-, we can symmetrize over them and write:

$$\begin{aligned} I_i &= \frac{1}{2} \int_0^1 dy \left[\frac{K(\alpha_+)}{\alpha_+ \, A_i + B_i} + \frac{K_i(\alpha_-)}{\alpha_- \, A_i + B_i} \right] L_i \\ &= \frac{1}{2} \int_0^1 dy \left[\frac{(K_i(\alpha_+) \, \alpha_- + K_i(\alpha_-) \, \alpha_+) \, A_i + (K_i(\alpha_+) + K_i(\alpha_-)) \, B_i}{\alpha_+ \, \alpha_- \, A_i^2 + A_i \, B_i \, (\alpha_- + \alpha_+) + B_i^2} \right] L_i \end{aligned} \tag{5.1.40}$$

We introduce the following quantities:

$$\begin{aligned} Q_i &= \alpha_+ \, \alpha_- \, A_i^2 + A_i \, B_i \, (\alpha_- + \alpha_+) + B_i^2 \\ &= \frac{1}{b} \, (a \, A_i^2 - c \, A_i \, B_i + b \, B_i^2) \\ N_i &= (K_i(\alpha_+) \, \alpha_- + K_i(\alpha_-) \, \alpha_+) \, A_i + (K_i(\alpha_+) + K_i(\alpha_-)) \, B_i \end{aligned}$$

We now compute Q_i and N_i for the different integrals of Eq. (5.1.31).

For the first integral in Eq. (5.1.31), $K_1(\alpha) = 1$, $A_1 = 2\,b\,z + e + c$ and $B_1 = c\,z + d + 2\,a$, so we get :

$$\begin{aligned} Q_1 &= \frac{1}{b} \left[-\Delta_\alpha \, b \, z^2 - \Delta_\alpha \, (c + e) \, z + a \, e^2 - c \, e \, d + b \, d^2 - \Delta_\alpha \, (d + a) \right] \\ &= \frac{1}{b} \left[-\Delta_\alpha \, \left(b \, z^2 + (c + e) \, z + a + d + f \right) + \det(\mathcal{S})/2 \right] \end{aligned} \tag{5.1.41}$$

$$\begin{aligned} N_1 &= \frac{1}{b} \left[2 \, b \, d - c \, e - \Delta_\alpha \right] \\ &= \frac{1}{b} \, b_1 \, \det(\mathcal{S}) \end{aligned} \tag{5.1.42}$$

For the second integral in Eq. (5.1.31), $K_2(\alpha) = -\alpha$, $A_2 = c\,z + e$ and $B_2 = 2\,a\,z + d$, so we get :

$$\begin{aligned} Q_2 &= \frac{1}{b} \left[-\Delta_\alpha \, a \, z^2 - \Delta_\alpha \, d \, z + a \, e^2 - c \, e \, d + b \, d^2 \right] \\ &= \frac{1}{b} \left[-\Delta_\alpha \, \left(a \, z^2 + d \, z + f \right) + \det(\mathcal{S})/2 \right] \end{aligned} \tag{5.1.43}$$

$$\begin{aligned} N_2 &= -\frac{1}{b} \left[2 \, a \, e - c \, d \right] \\ &= +\frac{1}{b} \, b_2 \, \det(\mathcal{S}) \end{aligned} \tag{5.1.44}$$

For the third in Eq. (5.1.31) integral, $K_3(\alpha) = -(1 - \alpha)$, $A_3 = (2\,b + c)\,z + e$ and $B_3 = (c + 2\,a)\,z + d$, so we get :

$$\begin{aligned} Q_3 &= \frac{1}{b} \left[-\Delta_\alpha \, (a + b + c) \, z^2 - \Delta_\alpha \, (e + d) \, z + a \, e^2 - c \, e \, d + b \, d^2 \right] \\ &= \frac{1}{b} \left[-\Delta_\alpha \, \left((a + b + c) \, z^2 + (e + d) \, z + f \right) + \det(\mathcal{S})/2 \right] \end{aligned} \tag{5.1.45}$$

$$\begin{aligned} N_3 &= -\frac{1}{b} \left[2 \, b \, d + c \, d - 2 \, a \, e - c \, e \right] \\ &= +\frac{1}{b} \, b_3 \, \det(\mathcal{S}) \end{aligned} \tag{5.1.46}$$

So we can write Eq. (5.1.31) as :

$$\begin{aligned} I_3^4(S) &= -\Big[b_1 \int_0^1 dz\, \frac{\ln\left(g_1(z)\right) - \ln\left(\mathcal{B} - i\lambda\right)}{2\,B\,g_1(z)+1} \\ &\quad + b_2 \int_0^1 dz\, \frac{\ln\left(g_2(z)\right) - \ln\left(\mathcal{B} - i\lambda\right)}{2\,B\,g_2(z)+1} \\ &\quad + b_3 \int_0^1 dz\, \frac{\ln\left(g_3(z)\right) - \ln\left(\mathcal{B} - i\lambda\right)}{2\,B\,g_3(z)+1}\Big] \end{aligned} \tag{5.1.47}$$

with

$$b_i = \sum_{j=1}^{4} \mathcal{S}^{-1}_{ij} \tag{5.1.48}$$

$$B = \sum_{j=1}^{4} b_j \tag{5.1.49}$$

$$= \det(G)/\det(\mathcal{S}) \tag{5.1.50}$$

and

$$g_1(z) = b\,z^2 + (c+e)\,z + a + d + f \tag{5.1.51}$$

$$g_2(z) = a\,z^2 + d\,z + f \tag{5.1.52}$$

$$g_3(z) = (a+b+c)\,z^2 + (d+e)\,z + f \tag{5.1.53}$$

The polynomials $g_j(z)$ are namely those appearing in the integral representations of the two-point functions corresponding to the three possible pinchings of a given propagator in the triangle diagram in Fig. (5.1). We parametrize the $g_j(z)$ generically as

$$g_j(z) = \gamma_j^{(2)}\, z^2 + \gamma_j^{(1)}\, z + \gamma_j^{(0)} \tag{5.1.54}$$

in order to formally handle them all at once when concerned with the zeroes of denominators of Eq. (5.1.47) $g_j(z) + 1/(2B)$. Let us note that the discriminant Δ_J of the second degree polynomial $g_j(z)$, defined by

$$\Delta_j \equiv \gamma_j^{(2)^2} - 4\,\gamma_j^{(1)}\,\gamma_j^{(0)} \tag{5.1.55}$$

turns out to be equal to minus the determinant of the reduced kinematic matrix $\mathcal{S}^{\{j\}}$. This reduced kinematic matrix corresponds to the pinching of the propagator j of the triangle in Fig. (5.1), and is obtained from the matrix $\mathcal{S}$ by suppressing line and column j. Correlatively $\gamma_j^{(2)}$ is one half of the reduced Gram determinant associated with the reduced kinematic matrix $\mathcal{S}^{\{j\}}$ (we denote it by $\det(G^{\{j\}})$, see section A.1).

Equation (5.1.47) are appealing candidates for the integral representations which we seek. Let us examine them more closely when $\det(G) \to 0$. We shall distinguish two cases: the generic case when $\det(G) \to 0$ whereas $\det(\mathcal{S})$ remains non vanishing, and the specific case $\det(G) \to 0$ and $\det(\mathcal{S}) \to 0$ simultaneously which deserves a dedicated treatment. Let us subsequently examine these two cases.

5.1.2.2 $\det(G) \to 0$ whereas $\det(\mathcal{S})$ non vanishing

Let us first consider the polynomials $g_j(z) + 1/(2B)$ appearing in the denominators of the integrals I_3^4 in Eq. (5.1.47). It is insightful to write the corresponding reduced discriminant of $g_j(z) + 1/(2B)$ as

$$\widetilde{\Delta}'_j \;=\; -\frac{\bar{b}_j^2}{\det(G)} \tag{5.1.56}$$

defining the rescaled coefficients

$$\bar{b}_j \equiv b_j \det(\mathcal{S}), \quad j = 1, 2, 3 \tag{5.1.57}$$

and using the identity

$$\bar{b}_i^2 = 2\,\gamma_i^{(2)}\,\det(\mathcal{S}) - \Delta_i\,\det(G) \tag{5.1.58}$$

where Δ_i has been defined in Eq. (5.1.55) and see ref. [81] for the proof of Eq. (5.1.58). The zeroes $z_j^{\pm}$ or $g_j(z) + 1/(2B)$ are given by

$$z_j^{\pm} \;=\; -\frac{\gamma_j^{(1)}}{2\,\gamma_j^{(2)}} \mp \frac{\bar{b}_j}{2\,\gamma_j^{(2)}\,\sqrt{-\det(G)}} \tag{5.1.59}$$

Let us remind that $\det(G) \leq 0$ as commented earlier. When $\det(G) \to 0$, both zeroes $z_j^{\pm}$ of $2Bg_j(z) + 1$ are dragged away from $[0, 1]$ towards $+\infty$ and $-\infty$ respectively, and each of the integrals

$$\mathcal{J}_j = \int_0^1 \frac{dz}{2Bg_j(z) + 1}$$

is analytically well defined and numerically safe, and furthermore the following identity holds:

$$\sum_{j=1}^{3} b_j \mathcal{J}_j = 0 \tag{5.1.60}$$

so that the contributions $\propto \ln(\mathcal{B} - i\lambda)$ sum up to zero in I_3^4. In this respect, let us stress that the contributions $\propto \ln(\mathcal{B} - i\lambda)$ are fictituous from the start. They were introduced through Eq. (5.1.30) to construct a formula with zero residues at the poles within the integration domain namely when either of $z_j^{\pm}$ is inside $[0, 1]$. When $z_j^{\pm}$ are both outside $[0, 1]$ the introduction of the $\ln(\mathcal{B}-i\lambda)$ terms is irrelevant and indeed identity (5.1.60) allows to drop them explicitly from Eq. (5.1.47). Then, I_3^4

$$I_3^4 \;=\; -\sum_{j=1}^{3} b_j \int_0^1 dz\, \frac{\ln\left(g_j(z)\right)}{2\,B\,g_j(z) + 1} \tag{5.1.61}$$

thus provide suitable integral representations in the case at hand. From a numerical point of view the explicit suppression of the $\ln(\mathcal{B} - i\lambda)$ terms from integrals (5.1.61), is preferable since $\ln(\mathcal{B} - i\lambda) \to \infty$ when $\det(G) \to 0$ thus implementing a numerical cancellation of the sum $\sum_{j=1}^{3} b_j \mathcal{J}_j \ln(-1/(2B) - i\lambda)$ after each term would have been separately calculated, may lead to numerical instabilities. Besides, if some $g_j(z)$ vanishes at some $\hat{z}_j^o$ inside $[0, 1]$,

a possible numerical improvement of the integral representation consists in deforming the integration contour in the complex z plane, to skirt the vicinity of the integrable singularity at $\hat{z}_j^o$, so as to prevent the integrand from becoming large and avoid cancelation of large contributions, according to a one-dimensional version[8] of the multidimensional deformation described in section 7 of ref. [19]. This will be discussed in detail in the case of four-point function in the second part of this chapter.

5.1.2.3 $\det(G) = 0$ and $\det(\mathcal{S}) = 0$ simultaneously

This case is more tricky and deserves further discussion. Indeed, when $\det \mathcal{S} = 0$ and $\det(G) = 0$, Eq. (5.1.48) defining the parameters b_j as $\sum_{k=1}^{3} \mathcal{S}^{-1}_{jk}$ is no longer valid as $\mathcal{S}^{-1}$ is not defined, and the parameter $B = \det \mathcal{S}/\det(G)$ is an indeterminate quantity of the type $0/0$, likewise the $z_j^{\pm}$ are indeterminate quantities not manifestly driven away from the interval $[0, 1]$.

Let us have a look to the specific kinematics which leads to such case. First, we give an expression for these two determinants in a form which is invariant under cyclic permutation of the external legs:

$$\det(G) = -\{s_1^2 + s_2^2 + s_3^2 - 2s_1s_2 - 2s_1s_3 - 2s_2s_3\} = -\lambda(s_1, s_2, s_3) \tag{5.1.62}$$

$$\begin{aligned}\det(\mathcal{S}) = 2 \Big[& s_1\, s_2\, s_3 + s_1^2\, m_2^2 + s_2^2\, m_3^2 + s_3^2\, m_1^2 \\ & - s_1\, s_2\, (m_2^2 + m_3^2) - s_1\, s_3\, (m_1^2 + m_2^2) - s_2\, s_3\, (m_1^2 + m_3^2) \\ & + s_1\, (m_1^2 - m_2^2)\, (m_3^2 - m_2^2) + s_2\, (m_2^2 - m_3^2)\, (m_1^2 - m_3^2) \\ & + s_3\, (m_3^2 - m_1^2)\, (m_2^2 - m_1^2) \Big]\end{aligned} \tag{5.1.63}$$

with $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2\,x\,y - 2\,x\,z - 2\,y\,z$.

Now, it is easy to realize that the condition under which $\det(G) = 0$ (apart the trivial condition where all the $s_i = 0$) is that one $s_i = 0$ and the other two are equal. To fix the idea, let us choose $s_2 = 0$ and $s_1 = s_3$. If we put this condition in Eq. (5.1.63), we find $\det(\mathcal{S}) = 2s_3\,(m_1^2 - m_2^2)^2$. So in order that the two determinants vanish at the same phase space point, we must have $m_1 = m_2$, so it is a necessary condition. In other words, in a triangle where all the internal masses are different, there is no phase space point where $\det(G)$ and $\det(\mathcal{S})$ vanish. To fix the idea, we will study the behavior of I_3^4 around the phase space point where :

$$m_1^2 = m_2^2 \tag{5.1.64}$$

$$s_2 = 0 \tag{5.1.65}$$

$$s_1 = s_3 \tag{5.1.66}$$

[8] In broad outline, the contour deformation is contained inside the band $0 \leq \mathcal{R}e(z) \leq 1$. It departs from the real axis at 0 with an acute angle and likewise ends at 1 in such a way that $\mathcal{I}m(g_j(z))$ is kept negative along the deformed contour so that the latter does not cross any cut of $\ln g_j(z)$. In the case at hand this type of contour never embraces any of $z_j^{\pm}$ as soon as the latter are outside $[0, 1]$, thus no subtraction of illegitimate pole residue contribution at $z_j^{\pm}$ has to be cared about.

Let's us set

$$s_- = (s_1 - s_3)/2 \tag{5.1.67}$$
$$s_+ = (s_1 + s_3)/2 \tag{5.1.68}$$
$$m_1^2 = m_2^2 = m^2 \tag{5.1.69}$$

We get:

$$\det(G) = 4\left(s_+ s_2 - s_-^2 - \frac{1}{4}s_2^2\right) \tag{5.1.70}$$
$$\det(\mathcal{S}) = 2\left(\widetilde{\lambda}\, s_2 + 4\, m^2\, s_-^2 + m_3^2\, s_2^2 - s_2\, s_-^2\right) \tag{5.1.71}$$
$$\tilde{\lambda} = \lambda(s_+^2, (m^2)^2, (m_3^2)^2) \tag{5.1.72}$$

and

$$\bar{b}_1 = m_3^2(s_2 - 2s_-) - m^2(s_2 - 2s_-) - (s_2 + 2s_-)(s_- - s_+) \tag{5.1.73}$$
$$\bar{b}_2 = m_3^2(s_2 + 2s_-) - m^2(s_2 + 2s_-) + (s_2 - 2s_-)(s_- + s_+) \tag{5.1.74}$$
$$\bar{b}_3 = -s_2(2m_3^2 - 2m^2 + s_2 - 2s_+) \tag{5.1.75}$$

Let's define the parameter $c = s_2/s_-^2$ which is of order one ($\mathcal{O}(1)$) at the limit $\det(G)$ and $\det(\mathcal{S})$ vanish simultaneously. Then,

$$\det(G) = 4\,(s_+ - 1/c)\, s_2 + \cdots \tag{5.1.76}$$
$$\det(\mathcal{S}) = 2\,(\tilde{\lambda} + 4m^2/c)\, s_2 + \cdots \tag{5.1.77}$$
$$\bar{b}_1 = \mp(m_3^2 - m^2 - s_+)\,\sqrt{s_2/c} + (m_3^2 - m^2 + s_+ - 2/c)\, s_2 + \cdots \tag{5.1.78}$$
$$\bar{b}_2 = \pm(m_3^2 - m^2 - s_+)\,\sqrt{s_2/c} + (m_3^2 - m^2 + s_+ - 2/c)\, s_2 + \cdots \tag{5.1.79}$$
$$\bar{b}_3 = 2\,(s_+ + m^2 - m_3^2)\, s_2 + \cdots \tag{5.1.80}$$

Now, let us re-write I_3^4 as :

$$I_3^4(S) = -\left[b_3\, J_3^4 + \frac{1}{2}\,(b_1 + b_2)\,\left(J_1^4 + J_2^4\right) + \frac{1}{2}\,(b_1 - b_2)\,\left(J_1^4 - J_2^4\right)\right] \tag{5.1.81}$$

where :

$$J_i^4 = \int_0^1 dz\, \frac{\ln(g_i(z)) - \ln(\mathcal{B} - i\lambda)}{2\, B\, g_i(z) + 1}$$
$$= \int_0^1 dz\, \frac{\ln(g_i(1-z)) - \ln(\mathcal{B} - i\lambda)}{2\, B\, g_i(1-z) + 1} \tag{5.1.82}$$

Now, let us study the behavior of b_3, $b_1 + b_2$, $b_1 - b_2$ and B when $s_2 \to 0$ and $s_- \to 0$,

$$b_1 + b_2 = \left.\frac{\bar{b}_1 + \bar{b}_2}{\det(\mathcal{S})}\right|_{s_2, s_- \to 0} = \frac{c\,(m_3^2 - m^2 - s_+) - 2}{4\, m^2 + c\,\tilde{\lambda}} + \cdots \to \text{finite}, \tag{5.1.83}$$

$$b_1 - b_2 = \left.\frac{\bar{b}_1 - \bar{b}_2}{\det(\mathcal{S})}\right|_{s_2, s_- \to 0} = \mp\frac{2\,(m_3^2 - m^2 - s_+)}{4\, m^2 + c\,\tilde{\lambda}}\sqrt{\frac{c}{s_2}} + \cdots \to \frac{1}{\sqrt{0}} \to \text{divergent}, \tag{5.1.84}$$

$$b_3 = \left.\frac{\bar{b}_3}{\det(\mathcal{S})}\right|_{s_2, s_- \to 0} = -\frac{c\,(m_3^2 - m^2 - s_+)}{4\, m^2 + c\,\tilde{\lambda}} + \cdots \to \text{finite} \tag{5.1.85}$$

$$B = \left.\frac{\det(G)}{\det(\mathcal{S})}\right|_{s_2, s_- \to 0} = -\frac{2\,(1 - s_+\, c)}{4\, m^2 + c\,\tilde{\lambda}} + \cdots \to \text{finite} \tag{5.1.86}$$

So, the only divergent coefficient is $b_1 - b_2$. But, in the limit $s_- \to 0$, we have $g_1(z) \to g_2(1-z)$, so the bad behavior of $b_1 - b_2$ is compensated by $J_1^4 - J_2^4$. To see that explicitly, we can write :

$$\begin{aligned} J_1^4 - J_2^4 &= \int_0^1 dz \left(\frac{\ln(g_1(z)) - \ln(\mathcal{B} - i\lambda)}{2\,B\,g_1(z)+1} - \frac{\ln(g_2(1-z)) - \ln(\mathcal{B} - i\lambda)}{2\,B\,g_2(1-z)+1} \right) \\ &= \frac{1}{2} \int_0^1 dz \left[\ln\left(\frac{g_1(z)}{g_2(1-z)}\right) \left(\frac{1}{2\,B\,g_1(z)+1} + \frac{1}{2\,B\,g_2(1-z)+1} \right) \right. \\ &\left. - 2\,B \left(\ln(g_1(z)) + \ln(g_2(1-z)) - 2\,\ln(\mathcal{B} - i\lambda) \right) \frac{g_1(z) - g_2(1-z)}{(2\,B\,g_1(z)+1)\,(2\,B\,g_2(1-z)+1)} \right] \end{aligned} \tag{5.1.87}$$

using

$$g_1(z) = g(z) + s_-\,z\,(1-z) \tag{5.1.88}$$

$$g_2(z) = g(1-z) - s_-\,z\,(1-z) \tag{5.1.89}$$

with

$$g(z) = -\,s_+\,z\,(1-z) + m^2\,z + m_3^2\,(1-z) \tag{5.1.90}$$

we prove that

$$\begin{aligned} J_1^4 - J_2^4 = s_- \int_0^1 dz &\left[\frac{\ln(g_1(z)/g_2(1-z))}{s_-} \left(\frac{1}{2\,B\,g_1(z)+1} + \frac{1}{2\,B\,g_2(1-z)+1} \right) \right. \\ &\left. - 2\,B \left(\ln(g_1(z)) + \ln(g_2(1-z)) - 2\,\ln(\mathcal{B} - i\lambda) \right) \frac{z\,(1-z)}{(2\,B\,g_1(z)+1)\,(2\,B\,g_2(1-z)+1)} \right] \end{aligned} \tag{5.1.91}$$

we explicitly factorize out a factor s_- which compensate the behavior of $b_1 - b_2$. The I_3^4 can be computed by evaluating numerically the following integral :

$$\begin{aligned} I_3^4(S) = -\Bigg[& b_3 \int_0^1 dz\, \frac{\ln(g_3(z)) - \ln\tilde{\mathcal{B}}}{2\,B\,g_3(z)+1} \\ & + \frac{1}{2}\,\frac{\bar{b}_1 + \bar{b}_2}{\det(\mathcal{S})} \int_0^1 dz \left(\frac{\ln(g_1(z)) - \ln\tilde{\mathcal{B}}}{2\,B\,g_1(z)+1} + \frac{\ln(g_2(1-z)) - \ln\tilde{\mathcal{B}}}{2\,B\,g_2(1-z)+1} \right) \\ & + \frac{1}{4}\,\frac{\bar{b}_1 - \bar{b}_2}{\det(\mathcal{S})}\,s_- \int_0^1 dz \Bigg\{ -2\,B\,(\ln(g_1(z)) + \ln(g_2(1-z)) - 2\,\ln\tilde{\mathcal{B}}) \\ & \times \frac{2\,z\,(1-z)}{(2\,B\,g_1(z)+1)\,(2\,B\,g_2(1-z)+1)} \\ & + \frac{\ln(g_1(z)/g_2(1-z))}{s_-} \left(\frac{1}{2\,B\,g_1(z)+1} + \frac{1}{2\,B\,g_2(1-z)+1} \right) \Bigg\} \Bigg] \end{aligned} \tag{5.1.92}$$

where $\tilde{\mathcal{B}} = \mathcal{B} - i\,\lambda$.

Before to close this section, let us notice that we are taking a *double limit*. Properly speaking, the limits of each of these three terms in Eq. (5.1.81) which are separately well-defined are *directional limits* $s_- \to 0$, $s_2 \to 0$ in the $\{s_-^2, s_2\}$ plane keeping the ratio $c = s_2/s_-^2$ fixed,

i.e. these directional limits as functions of c. However, the limit of the *sum* of these three terms in Eq.. (5.1.81) is indeed *independent* of c. This can be easily checked numerically, this can also be proven analytically although this is somewhat cumbersome; a proof is presented in section A.3. The ground reason why this property holds is further understood as follows. If the limit of the sum were a directional one, it would imply that the three point function would be a singular i.e. non analytical function of the kinematical invariants at such configurations. However the kinematic singularities are characterized by the so-called Landau conditions[9] [52], see chapter 3. For one loop diagrams, these conditions require not only that $\det(\mathcal{S}) = 0$, but also that the eigenvectors associated with the vanishing eigenvalue of $\mathcal{S}$ shall have only non negative components and that their sum be strictly positive. The vanishing $\det(\mathcal{S})$ in the present case is therefore *not* related to a kinematic singularity (see chapter 3 and ref. [81]): the three-point function is *regular* in the limit considered, in particular this limit shall be uniform i.e. not directional.

5.1.3 The triangle in 6-dimensions

Let's consider only the finite part of $I_3^{6-2\varepsilon}$,

$$I_3^6(S) = \int_0^1 dx \int_{-\alpha\, x}^{(1-\alpha)\, x} dy' \ln[\{(2\,b\,\alpha + c)\,y' + (d + e\,\alpha)\,\}x + b\,y'^2 + e\,y' + f - i\,\lambda] \tag{5.1.93}$$

By following the same steps as in the case of I_4^3, we find

$$\begin{aligned} I_3^6(S) = -\frac{1}{2} + \int_0^1 dz \Bigg\{ & \\ & \frac{1}{(2\,\alpha\,b + c)\,z + (e + c)\,\alpha + d + 2\,a} \left[g_1(z)\ln(g_1(z)) - \mathcal{B}\ln(\mathcal{B} - i\lambda)\right] \\ & + \frac{-\alpha}{(2\,\alpha\,b + c)\,(-\alpha)\,z + d + e\,\alpha} \left[g_2(z)\ln(g_2(z)) - \mathcal{B}\ln(\mathcal{B} - i\lambda)\right] \\ & - \frac{1-\alpha}{(2\,\alpha\,b + c)\,(1-\alpha)\,z + d + e\,\alpha} \left[g_3(z)\ln(g_3(z)) - \mathcal{B}\ln(\mathcal{B} - i\lambda)\right] \Bigg\} \end{aligned} \tag{5.1.94}$$

where the functions g_i and $\mathcal{B}$ are defined above.
If $\det(G) \to 0$ (and not $\det(\mathcal{S}) \to 0$), the substracted term $\mathcal{B}\ln(\mathcal{B} - i\lambda)$ becomes infinite. But the poles z_i^0 are infinite since their denominators are proportional to $\sqrt{-\det(G)}$. Then, the subtracted terms can be dropped (since no pole can be inside $[0, 1]$) and this formula works. However, denominators of this formula may vanish, which correspond to $z_i^0 = 0/0$. To avoid this discussion, we will peruse the symmetrization on α_+ and α_- as above (as in the case of I_3^4). The integrand of I_3^6 has the same generic form I_i defined in Eq. (5.1.39).

[9] In general the Landau conditions provide necessary conditions to face singularities, either of pinched or end-point type, in a function provided by a parametric integral. However Coleman and Norton [55] proved these conditions to be also sufficient in the case of Feynman integrals.

Then, by following the very same steps to get Eq. (5.1.47), we find

$$I_3^6(S) = -\frac{1}{2} + \sum_{i=1}^{3} b_i \int_0^z \frac{g_i(z)\ln(g_i(z))}{2\,B\,g_i(z)+1} \tag{5.1.95}$$

which is very safe if $\det(G) \to 0$.

If $\det(G) \to 0$ and $\det(\mathcal{S}) \to 0$ simultaneously, the substracted terms should not be dropped since the zeros of the denominators may be inside $[0,1]$ as shown above. For that reason, we follow the same strategy introduced for I_3^4. Let's write I_4^6 as follows:

$$I_3^6 = b_3\,J_3^6 + \frac{1}{2}\,(b_1+b_2)\,(J_1^6+J_2^6) + \frac{1}{2}\,(b_1-b_2)\,(J_1^6-J_2^6) - \frac{1}{2} \tag{5.1.96}$$

with

$$\begin{aligned} J_i^6 &= \int_0^1 dz\,\frac{g_i(z)\,\ln(g_i(z)) - \mathcal{B}\,\ln(\mathcal{B}-i\lambda)}{2\,B\,g_i(z)+1} \\ &= \int_0^1 dz\,\frac{g_i(1-z)\,\ln(g_i(1-z)) - \mathcal{B}\,\ln(\mathcal{B}-i\lambda)}{2\,B\,g_i(1-z)+1} \end{aligned} \tag{5.1.97}$$

Again, b_3, b_1+b_2 and B are finite and the divergence of b_1-b_2 is compensated by $J_1^6-J_2^6$ since $J_1^6 \to J_2^6$ in this region. Then, after some manipulation I_4^6 takes the convenient form

$$\begin{aligned} I_3^6(S) = b_3 &\int_0^1 dz\,\frac{g_3(z)\,\ln(g_3(z)) - \mathcal{B}\,\ln\tilde{\mathcal{B}}}{2\,B\,g_3(z)+1} \\ &+ \frac{1}{2}\,\frac{\bar{b}_1+\bar{b}_2}{\det(\mathcal{S})}\int_0^1 dz\left(\frac{g_1(z)\,\ln(g_1(z)) - \mathcal{B}\,\ln\tilde{\mathcal{B}}}{2\,B\,g_1(z)+1}\right. \\ &\qquad\left. + \frac{g_2(1-z)\,\ln(g_2(1-z)) - \mathcal{B}\,\ln\tilde{\mathcal{B}}}{2\,B\,g_2(1-z)+1}\right) \\ &+ \frac{1}{4}\,\frac{\bar{b}_1-\bar{b}_2}{\det(\mathcal{S})}\,s_-\int_0^1 dz\left\{\frac{\ln(g_1(z)/g_2(1-z))}{s_-}\left(\frac{g_1(z)}{2\,B\,g_1(z)+1} + \frac{g_2(1-z)}{2\,B\,g_2(1-z)+1}\right)\right. \\ &\left. + (\ln(g_1(z)) + \ln(g_2(1-z)) - 2\,\ln\tilde{\mathcal{B}})\,\frac{2\,z\,(1-z)}{(2\,B\,g_1(z)+1)\,(2\,B\,g_2(1-z)+1)}\right\} - \frac{1}{2} \end{aligned} \tag{5.1.98}$$

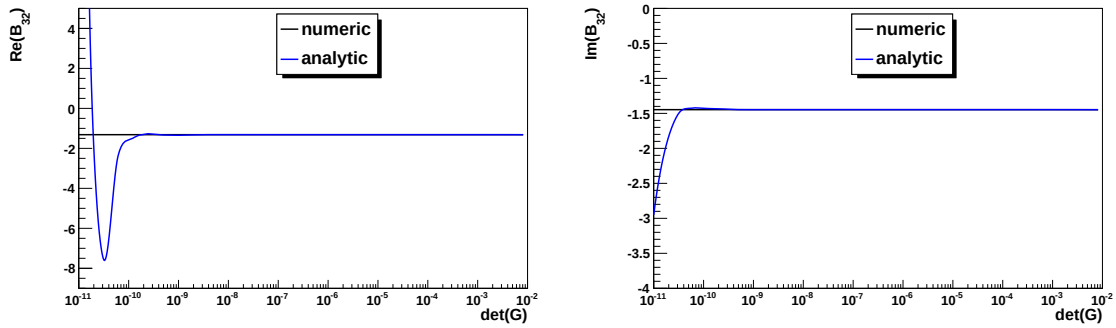
Numerical test

See Figs. (5.2, 5.3)

5.1.4 The triangle in 8-dimensions

Let's consider the finite part of $I_3^{8-2\varepsilon}$

$$\begin{aligned} I_3^8(S) = &\int_0^1 dx \int_{-\alpha\,x}^{(1-\alpha)\,x} dy'[\{(2\,b\,\alpha+c)\,y' + (d+e\,\alpha)\,\}x + b\,y'^2 + e\,y' + f] \\ &\times \ln[\{(2\,b\,\alpha+c)\,y' + (d+e\,\alpha)\,\}x + b\,y'^2 + e\,y' + f - i\,\lambda] \end{aligned} \tag{5.1.99}$$



(a) (b)

Figure 5.2: Comparison between the analytical and the numerical modes: (a) and (b) show, respectively, the real and the imaginary parts of the form factor $B^{3,2}$, where $B^{3,3} = I_3^6/2$. The analytical mod in the region where $\det(G) \to 0$ is not stable (blue line). However the numerical mode gives stable results (black line).

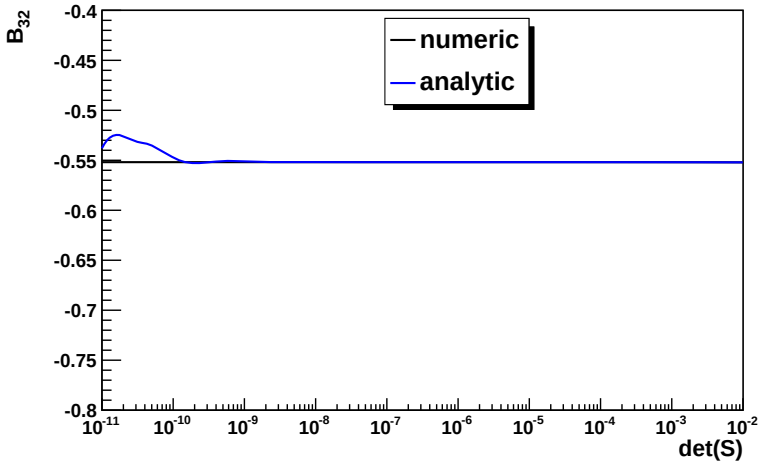


Figure 5.3: Comparison between the analytical and the numerical modes for kinematical configuration with $\det(G) \to 0$ and $\det(\mathcal{S}) \to 0$ simultaneously: the blue line stands for the analytical results (not stable) and the black line stands for the numerical results (stable).

Again we follow the same steps as above, we split the integral on y', we reverse the order of integration and introduce the parameter α. Ultimately, we find

$$\begin{aligned} I_3^8(S) = &-\frac{1}{48}(6a+2b+3c+8d+4e+12f) + \frac{1}{2}\int_0^1 dz\Big\{ \\ &\frac{1}{(2\,\alpha\,b+c)\,z+(e+c)\,\alpha+d+2\,a}\left[g_1^2(z)\ln(g_1(z)) - \mathcal{B}^2\ln(\mathcal{B}-i\lambda)\right] \\ &+\frac{-\alpha}{(2\,\alpha\,b+c)\,(-\alpha)\,z+d+e\,\alpha}\left[g_2^2(z)\ln(g_2(z)) - \mathcal{B}^2\ln(\mathcal{B}-i\lambda)\right] \\ &-\frac{1-\alpha}{(2\,\alpha\,b+c)\,(1-\alpha)\,z+d+e\,\alpha}\left[g_3^2(z)\ln(g_3(z)) - \mathcal{B}^2\ln(\mathcal{B}-i\lambda)\right]\Big\} \end{aligned} \tag{5.1.100}$$

i) For $\det(G)\to 0$ $\det(G)\not\to 0$, one can use the following formula,

$$I_3^8(S) = -\frac{1}{48}(6a+2b+3c+8d+4e+12f) + \frac{1}{2}\sum_{i=1}^{3} b_i \int_0^z \frac{g_i^2(z)\ln(g_i(z))}{2\,B\,g_i(z)+1} \tag{5.1.101}$$

this formula is obtained after symmetrization over α_+ and α_- as above.

i) For the case where $\det(G)\to 0$ *and* $\det(\mathcal{S})\to 0$ *simultaneously*, we write

$$\begin{aligned} I_3^8(S) = &\frac{1}{2}[b_3\,J_3^8 + \frac{1}{2}\,(b_1+b_2)\,(J_1^8+J_2^8) + \frac{1}{2}\,(b_1-b_2)\,(J_1^8-J_2^8)] \\ &-\frac{1}{48}(6a+2b+3c+8d+4e+12f) \end{aligned} \tag{5.1.102}$$

with

$$J_i^8 = \int_0^1 dz\,\frac{g_i^2(z)\,\ln(g_i(z)) - \mathcal{B}^2\,\ln(\mathcal{B}-i\lambda)}{2\,B\,g_i(z)+1}$$

Again, b_3, b_1+b_2 and B are finite but b_1-b_2 is divergent, its divergence is compensated by $J_1^8-J_2^8$ since $g_1(z)\to g_2(1-z)$ in this region. Then, after some manipulation I_3^8 takes the convenient form

$$\begin{aligned} I_3^8(S) = &\frac{1}{2}\int_0^1 dz\Big\{b_3\,\frac{g_3^2(z)\,\ln(g_3(z)) - \mathcal{B}^2\,\ln\tilde{\mathcal{B}}}{2\,B\,g_3(z)+1} + \frac{1}{2}\frac{\bar{b}_1+\bar{b}_2}{\det(\mathcal{S})}\Big[\\ &\frac{g_1^2(z)\,\ln(g_1(z)) - \mathcal{B}^2\,\ln\tilde{\mathcal{B}}}{2\,B\,g_1(z)+1} + \frac{g_2^2(1-z)\,\ln(g_2(1-z)) - \mathcal{B}^2\,\ln\tilde{\mathcal{B}}}{2\,B\,g_2(1-z)+1}\Big] \\ &+\frac{1}{4}\,\frac{\bar{b}_1-\bar{b}_2}{\det(\mathcal{S})}\,s_-\Big[\Big(g_2(1-z)\ln(g_1(z)) + g_1(z)\ln(g_2(1-z)) - 2\mathcal{B}\,\ln\tilde{\mathcal{B}}\Big) \\ &\qquad\times\frac{2z\,(1-z)}{(2\,B\,g_1(z)+1)\,(2\,B\,g_2(1-z)+1)} \\ &+2z(1-z)\left(\frac{g_1(z)\ln(g_1(z))}{2\,B\,g_1(z)+1} + \frac{g_2(1-z)\ln(g_2(1-z))}{2\,B\,g_2(1-z)+1}\right) \\ &+\frac{\ln(g_1(z)/g_2(1-z))}{s_-}\left(\frac{g_1^2(z)}{2\,B\,g_1(z)+1} + \frac{g_2^2(1-z)}{2\,B\,g_2(1-z)+1}\right)\Big]\Big\} \\ &-\frac{1}{48}(6a+2b+3c+8d+4e+12f) \end{aligned} \tag{5.1.103}$$

5.2 Scalar four-point functions

In this section, we calculate the general scalar 4-point functions in $n+l$ dimensions "$I_4^{n+l}(S)$" for $l = 2, 4$ (the scalar boxes for $l = 6, 8$ are given in Appendix C.). The knowledge of these scalar functions allows us to calculate all the tensorial four-point basic integrals mentioned in section 4.1.2 in the previous Chapter, i.e. the integrals $I_4^{n+2}(j_1, ..., j_3)$ and $I_4^{n+4}(j_1)$ which have the general form (after Feynman parameterization)

$$I_4^{n+2}(j_1, \cdots, j_r) = \Gamma(3 - \frac{n}{2}) \int_0^1 \Pi_{i=1}^4 dz_i \, \delta(1 - \sum_{l=1}^4 z_l) \, \frac{z_1 \cdots z_r}{(-\frac{1}{2} \, z^\dagger \cdot S \cdot z - i \, \lambda)^{3-n/2}} \quad (5.2.104)$$

$$I_4^{n+4}(j_1) = \Gamma(2 - \frac{n}{2}) \int_0^1 \Pi_{i=1}^4 dz_i \, \delta(1 - \sum_{l=1}^4 z_l) \, \frac{z_1 \cdots z_r}{(-\frac{1}{2} \, z^\dagger \cdot S \cdot z - i \, \lambda)^{2-n/2}} \quad (5.2.105)$$

For each of these integrals, we give a one-dimentional integral representation which provides numerically stable results in phase space regions where the Gram determinant becomes arbitrary small, and it is valid for arbitrary internal masses (internal complex masses are supported). In addition, analytical formulas for the integral I_4^{n+2} will be derived from the direct calculation.
We should notice that in this calculation, we limit ourself only to processes relevant for collider experiments (e.g. production of n particles in the collision of two particles: $2 \to n$) or for particle decay processes ($1 \to n$) at NLO order. The encountered one-loop Feynman diagrams in the calculation of amplitudes of such processes cannot have more than two space-like external legs, which implies that the Gram determinants associated to the triangles (obtained from pinching a given propagator of the box) are all negatives (see the proof in section B.2 in Appendix B). For more general kinematics, i.e. for kinematics with strictly positive (one or more) 3-point Gram determinants[10] (which are not interesting at NLO order! for physical processes), the validity of our formulas is not checked yet for all possible cases[11].

5.2.1 Outline of the box calculation

The one-loop four-point function in $n + l$ dimension is defined by

$$I_4^{n+l}(S) \quad = \quad \int \frac{d^{n+l}k}{i \, \pi^{n/2}} \frac{1}{\Pi_{i=1}^4 \, [q_i^2 - m_i^2 + i \, \lambda]}, \quad (5.2.106)$$

where $n = 4 - 2\,\varepsilon$ and $l = 2, 4, 6, 8$. S is an ordered set containing the propagators labels, for the box in Fig.(5.4), $S = \{1, 2, 3, 4\}$. The propagator are labeled by $q_i = k + r_i$, where k is the momenta circulating in the loop, the r_i are defined by $p_i = r_i - r_{i-1}$ for $i = 1, \cdots 4$ (we

[10] For example, boxes encountered in two or more loop diagrams, where the nature of the external legs is unknown.

[11] It works for large phase space region especially if contour deformations are not needed (e.g. if all the internal lines have different complex masses). If the contour deformations are needed, it may not work only if some parameters, say β (they will be defined later on), receive a soft imaginary part from the square root of "-" of the 3-point Gram determinants, but such configurations are not relevant for collider experiments.

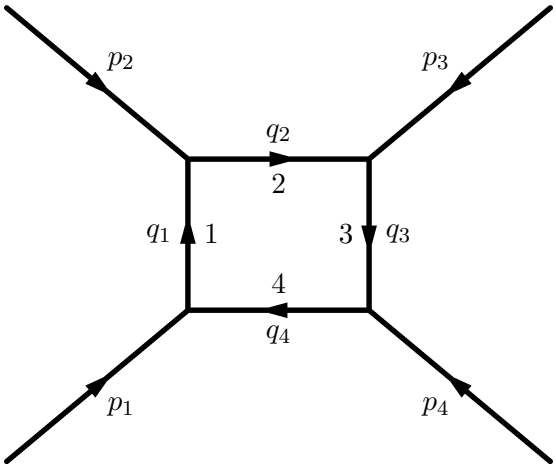


Figure 5.4: The box picturing the four-point functions

specify $r_1 = 0$), and the p_i are defined as incoming momenta. The momentum conservation implies

$$\sum_{i=1}^{4} p_i \quad = \quad 0 \tag{5.2.107}$$

These functions are free of any infrared divergences since they are defined in space-time with more than 4 dimensions, see Table.(3.1) [12]. But they can have some ultra-violet divergences which appear as simple poles in ε which are easy to handle[13]. Using Feynman parameterization and performing the momentum integration, $I_4^{4+l-2\,\varepsilon}(S)$ may be re-written in the following form

$$I_4^{4+l-2\,\varepsilon}(S) = \Gamma\left(\frac{4-l}{2}+\varepsilon\right)\int_0^1 \prod_{i=1}^{4} dz_i\,\delta(1-\sum_{i=1}^{4} z_i)\left(-\frac{1}{2}\,\vec{z}^{\,t}\,\mathcal{S}\,\vec{z}-i\,\lambda\right)^{\frac{l-4}{2}-\varepsilon}, \tag{5.2.108}$$

where z_i are the Feynman parameters, the box kinematical matrix $\mathcal{S}$ (the general form of this matrix is given in Eq, (5.1.4)) is given by

$$\mathcal{S} \quad = \quad \begin{pmatrix} -2\,m_1^2 & s_2-m_1^2-m_2^2 & t-m_1^2-m_3^2 & s_1-m_1^2-m_4^2 \\ s_2-m_1^2-m_2^2 & -2\,m_2^2 & s_3-m_2^2-m_3^2 & s-m_2^2-m_4^2 \\ t-m_1^2-m_3^2 & s_3-m_2^2-m_3^2 & -2\,m_3^2 & s_4-m_3^2-m_4^2 \\ s_1-m_1^2-m_4^2 & s-m_2^2-m_4^2 & s_4-m_3^2-m_4^2 & -2\,m_4^2 \end{pmatrix} \tag{5.2.109}$$

with

$$s_i = p_i^2, \qquad s = (p_1+p_2)^2, \qquad t = (p_2+p_3)^2. \tag{5.2.110}$$

[12] We have shown in Table.(3.1) that the powers of λ_4 (λ_4 is the only vanishing eigenvalue of the matrix $\mathcal{S}$) for the scalar boxes in $n+l$ dimensions are positive, this means that the corresponding singularities are integrable, then it doesn't lead to IR divergences. This was confirmed in section 3.2 (Chapter 3), where we have proven that the boxes are free of soft and collinear divergences in the case where all the internal propagators are massless (which is the most dangerous case!). A generalization to boxes with some massive propagator is quite straightforward!

[13] The UV divergences of these boxes are fake, since they are forced to appear by the Golem algebraic reduction. Once, these basic integral are gathered to build the form factors, these divergences will disappear.

By re-writing the integrand of Eq. (5.2.108) in term of the Gram matrix associated to the 4-point function ($G^{(a)}$) and the vectors $V_i^{(a)}$ (see Eqs.(5.1.8, 5.1.9)), we find

$$\begin{aligned} I_4^{4+l-2\,\varepsilon}(S) &= \Gamma\left(\frac{4-l}{2}+\varepsilon\right)\int_0^1 dz_1\int_0^{1-z_1} dz_2\int_0^{1-z_1-z_2} dz_3 \\ &\left(\frac{1}{2}\sum_{\substack{i,j=1\\ i,j\neq a}}^{4} z_i\,z_j\,G_{ij}^{(a)} - \sum_{\substack{j=1\\ j\neq a}}^{4} z_j\,V_i^{(a)} - \frac{1}{2}\,\mathcal{S}_{aa} - i\,\lambda\right)^{\frac{l-4}{2}-\varepsilon} \end{aligned} \tag{5.2.111}$$

For simplicity, we relabel the z_k by an appropriate permutation by fixing $a = 4$, so $i, j = 1, 2, 3$ in Eq. (5.2.111). With this choice we get

$$G^{(4)} = \begin{pmatrix} 2\,s_1 & s_1 - s_2 + s & s_1 + s_4 - t \\ s_1 - s_2 + s & 2\,s & s_4 - s_3 + s \\ s_1 + s_4 - t & s_4 - s_3 + s & 2\,s_4 \end{pmatrix}, \quad V^{(4)} = \begin{pmatrix} s_1 - m_1^2 + m_4^2 \\ s - m_2^2 + m_4^2 \\ s_4 - m_3^2 + m_4^2 \end{pmatrix}. \tag{5.2.112}$$

Since the determinant of $G^{(a)}$ (for $a = 1, 2, 3, 4$) does not depend on the choice of a (see section B.1.1), from now on we will denote it by $\det(G)$

$$\det(G) \equiv \det(G^{(1)}) = \det(G^{(2)}) = \det(G^{(3)}) = \det(G^{(4)}). \tag{5.2.113}$$

By making the following change of variables:

$$\begin{cases} z_1 = 1 - x \\ z_2 = x - y \\ z_3 = z \end{cases} \tag{5.2.114}$$

we get

$$\begin{aligned} I_4^{4+l-2\,\varepsilon}(S) = \Gamma\left(\frac{4-l}{2}+\varepsilon\right)\int_0^1 dx\int_0^x dy\int_0^y dz\,(a\,x^2 + b\,y^2 + g\,z^2 + c\,x\,y + h\,x\,z \\ + j\,y\,z + d\,x + e\,y + k\,z + f - i\,\lambda)^{\frac{l-4}{2}-\varepsilon}, \end{aligned} \tag{5.2.115}$$

with

$$\begin{aligned} a &= s_1, & f &= m_1^2, \\ b &= s_4, & g &= s_3, \\ c &= t - s_1 - s_4, & h &= s_2 + s_4 - s - t, \\ d &= m_4^2 - m_1^2 - s_1, & j &= s - s_3 - s_4, \\ e &= m_3^2 - m_4^2 + s - t, & k &= m_2^2 - m_3^2 - s_2 + t. \end{aligned} \tag{5.2.116}$$

First, we rescale the variables y and z: $y = x\,y'$ and $z = x\,z'$ respectively. Then,

$$\begin{aligned} I_4^{4+l-2\,\varepsilon}(S) = \int_0^1 dx\int_0^1 dy'\int_0^{y'} dz'\,x^2\,(x^2\,(b\,y'^2 + g\,z'^2 + j\,y'\,z') \\ + (e + c\,x)\,x\,y' + (k + h\,x)\,x\,z' + a\,x^2 + d\,x + f - i\,\lambda)^{\frac{l-4}{2}-\varepsilon} \end{aligned} \tag{5.2.117}$$

Let us now introduce a parameter α and shift z' ($z' = z'' + \alpha\, y'$):

$$I_4^{4+l-2\,\varepsilon} = \Gamma\!\left(\frac{4-l}{2}+\varepsilon\right)\int_0^1 dx\, x^2 \int_0^1 dy' \int_{-\alpha\, y'}^{(1-\alpha)\, y'} dz'' \left(x^2\,(g\,\alpha^2 + j\,\alpha + b)\, y'^2 + \mathcal{C}_1(x,z'')\, y' + \mathcal{C}_0(x,z'')\right)^{\frac{l-4}{2}-\varepsilon}, \quad (5.2.118)$$

$$\mathcal{C}_1(x,z'') = ((2\,\alpha\, g + j)\, z'' + c + \alpha\, h)\, x^2 + (e + \alpha\, k)\, x, \quad (5.2.119)$$

$$\mathcal{C}_0(x,z'') = (a + h\, z'' + g\, z''^2)\, x^2 + (d + k\, z'')\, x + f - i\,\lambda. \quad (5.2.120)$$

To linearize the integrand in Eq. (5.2.118) on y', the parameter α is chosen such that $\alpha^2\, g + \alpha\, j + b = 0$, then

$$\alpha = \frac{-j + \varepsilon_\alpha \sqrt{\Delta^{(1)}}}{2\, g}, \qquad \varepsilon_\alpha = \pm 1. \quad (5.2.121)$$

with

$$\begin{aligned}\Delta^{(1)} &= j^2 - 4b\, g \\ &= s^2 + s_3^2 + s_4^2 - 2s\, s_3 - 2s\, s_4 - 2 s_3\, s_4 \\ &= -\lambda(s, s_3, s_4) \\ &= -\det(G^{\{1\}}).\end{aligned} \quad (5.2.122)$$

The discriminant equals to "-" the Gram determinant associated to the triangle obtained by pinching the propagator number "1" of the box (see subsection B.1.2). In the physical region, i.e. for processes relevant at collider experiment or particles decays. The parameter α is real, since the discriminant $\Delta^{(1)}$ is positive.

Eq. (5.2.118) is the starting point of the box calculation in any dimension of space time. In our case, we will calculate the four scalar boxes corresponding to $l = 2, 4, 6$ and 8, respectively. Each of them is given by

$$I_4^6(S) = r_\Gamma \int_0^1 dx\, x^2 \int_0^1 dy' \int_{-\alpha\, y'}^{(1-\alpha)\, y'} dz'' \left(\mathcal{C}_1(x,z'')\, y' + \mathcal{C}_0(x,z'')\right)^{-1} \quad (5.2.123)$$

$$I_4^{8-2\,\varepsilon}(S) = r_\Gamma \int_0^1 dx\, x^2 \int_0^1 dy' \int_{-\alpha\, y'}^{(1-\alpha)\, y'} dz'' \left\{\frac{1}{\varepsilon} - \ln\!\left(\mathcal{C}_1(x,z'')\, y' + \mathcal{C}_0(x,z'')\right)\right\} \quad (5.2.124)$$

$$I_4^{10-2\,\varepsilon}(S) = r_\Gamma \int_0^1 dx\, x^2 \int_0^1 dy' \int_{-\alpha\, y'}^{(1-\alpha)\, y'} dz'' \left(\mathcal{C}_1(x,z'')\, y' + \mathcal{C}_0(x,z'')\right) \left\{-\frac{1}{\varepsilon} - 1 + \ln\!\left(\mathcal{C}_1(x,z'')\, y' + \mathcal{C}_0(x,z'')\right)\right\} \quad (5.2.125)$$

$$I_4^{12-2\,\varepsilon}(S) = r_\Gamma \int_0^1 dx\, x^2 \int_0^1 dy' \int_{-\alpha\, y'}^{(1-\alpha)\, y'} dz'' \frac{1}{2} \Big(\mathcal{C}_1(x,z'')\, y' + \mathcal{C}_0(x,z'') \Big)^2 \left\{ \frac{1}{\varepsilon} + \frac{3}{2} - \ln\Big(\mathcal{C}_1(x,z'')\, y' + \mathcal{C}_0(x,z'') \Big) \right\} \tag{5.2.126}$$

where $r_\Gamma = \Gamma(1-\varepsilon)^2\Gamma(1+\varepsilon)/\Gamma(1-2\varepsilon) = 1 - \gamma_r\varepsilon + \mathcal{O}(\varepsilon^2)$ and γ_r is defined in Eq. (5.1.22), this coefficient will be omitted in the following calculation.
From Eqs. (5.2.123), the box $I_4^6(S)$ is the only integral free of UV divergences, hence, it can be calculated in 4-dimensions (we can set $n = 4$). But, the remaining integrals are UV divergent, so one has to calculate them in $4 + l - 2\,\varepsilon$ dimensions, the UV divergent parts are quite straightforward to calculate as we will show in the next paragraphs.

In the following, we will show, in detail, how to derive the analytical formula of the scalar integral I_4^6, and the stable one-dimensional integral representation of this integral and of the scalar integral I_4^8 [14], where we will focus more in the problematic region of the phase space (the region where the Gram determinants vanish), which is the main subject of this thesis.

5.2.2 The box in 6-dimensions

We will give a detailed calculation for both the analytical and the one-dimensional representation of I_4^6. Due to the fact that α is real (see section B.2), one can inverse the order of integration in Eq. (5.2.123) as the following

$$\int_0^1 dy' \int_{-\alpha\, y'}^{(1-\alpha)\, y'} dz'' = \int_0^{1-\alpha} dz'' \int_{z''/(1-\alpha)}^1 dy' - \int_0^{-\alpha} dz'' \int_{z''/(-\alpha)}^1 dy' \tag{5.2.127}$$

so that I_4^6 is schematically written as:

$$I_4^6 = \int_0^1 dx\, x^2 \left\{ \int_0^{1-\alpha} dz'' \int_{z''/(1-\alpha)}^1 dy' - \int_0^{-\alpha} dz'' \int_{z''/(-\alpha)}^1 dy' \right\} [\cdots]^{-1}$$

We perform the integration over y' and rescale the variable $z'' = (1-\alpha)z''$ and $z'' = (-\alpha)z''$ for the first and the second term, respectively. We get

$$I_4^6(S) = \int_0^1 dx\, x \left\{ (1-\alpha) \int_0^1 dz'' \frac{1}{A_1 - i\,\lambda} \ln\left(\frac{A_1\, x + B_1 - i\,\lambda}{A_1\, z''\, x + B_1 - i\,\lambda} \right) + \alpha \int_0^1 dz'' \frac{1}{A_2 - i\,\lambda} \ln\left(\frac{A_2\, x + B_2 - i\,\lambda}{A_2\, z''\, x + B_2 - i\,\lambda} \right) \right\} \tag{5.2.128}$$

$$\begin{aligned} A_1 &= (1-\alpha)\, z''\, x\, (2\, g\, \alpha + j) + e + c\, x + \alpha\, (k + h\, x) \\ B_1 &= x^2\, (1-\alpha)^2\, z''^{\,2}\, g + (1-\alpha)\, x\, (k + h\, x)\, z'' + a\, x^2 + d\, x + f \\ A_2 &= -\alpha\, z''\, x\, (2\, g\, \alpha + j) + e + c\, x + \alpha\, (k + h\, x) \\ B_2 &= x^2\, \alpha^2\, z''^{\,2}\, g - \alpha\, x\, (k + h\, x)\, z'' + a\, x^2 + d\, x + f \end{aligned}$$

[14] The one-dimensional integral representations of I_4^{10} and I_4^{12} will be presented in the Appendix C.

where the prescription $-i\,\lambda$ in the denominators is fixed that way for some reasons that we will explain later on. However the prescription $-i\,\lambda$ in the logarithm arguments comes from the internal masses.
For instance, we consider only real internal masses, a generalization to complex internal masses will be done in the end of this section by an analytical continuation. In this prescription, the imaginary parts of the logarithms arguments in Eq. (5.2.128) is given by "$-\lambda$" (since α and the variables of integration are real). Hence, we can split them into two parts without introducing any η functions defined in (Eq. (5.1.27)). Thus

$$\begin{aligned} I_4^6(S) &= \int_0^1 dx\, x \left\{ \int_0^1 dz'' \left((1-\alpha)\, \frac{\ln(A_1\, x + B_1 - i\,\lambda)}{A_1 - i\,\lambda} + \alpha\, \frac{\ln(A_2\, x + B_2 - i\,\lambda)}{A_2 - i\,\lambda} \right) \right. \\ &\quad - \int_0^1 dz''\, (1-\alpha)\, \frac{\ln(A_1\, z''\, x + B_1 - i\,\lambda)}{A_1 - i\,\lambda} \\ &\quad \left. - \int_0^1 dz''\, \alpha\, \frac{\ln(A_2\, z''\, x + B_2 - i\,\lambda)}{A_2 - i\,\lambda} \right\} \end{aligned} \tag{5.2.129}$$

We make the change of variables $\tilde{z} = z''\, x\, (1-\alpha)$ in the first and the third integral and $\tilde{z} = -\alpha\, z''\, x$ in the second and the fourth integrals. We thus get:

$$\begin{aligned} I_4^6(S) &= \int_0^1 dx \left\{ \int_{-\alpha\, x}^{(1-\alpha)\, x} d\tilde{z}\, \frac{\ln(N + x\, M - i\,\lambda)}{M - i\,\lambda} - \int_0^{(1-\alpha)\, x} d\tilde{z}\, \frac{\ln(N + \tilde{z}/(1-\alpha)\, M - i\,\lambda)}{M - i\,\lambda} \right. \\ &\quad \left. + \int_0^{-\alpha\, x} d\tilde{z}\, \frac{\ln(N - \tilde{z}/\alpha\, M - i\,\lambda)}{M - i\,\lambda} \right\} \end{aligned} \tag{5.2.130}$$

with

$$N = \tilde{z}^2\, g + \tilde{z}\, (k + h\, x) + a\, x^2 + d\, x + f \tag{5.2.131}$$

$$M = \tilde{z}\, (2\, g\, \alpha + j) + x\, (c + \alpha\, h) + e + \alpha\, k \tag{5.2.132}$$

The $-i\,\lambda$ prescription:

In Eq. (5.2.128), the poles at $A_1 = 0$ and at $A_2 = 0$ are manifestly fake, thus the $-i\lambda$ prescription in there is irrelevant: any contour prescription, even no prescription at all yields the same result, since the residue at each pole of these terms is zero. However, if we split each of them into two parts as in (5.2.129), then one has to specify some prescription. Any prescription actually used is irrelevant but the same one has to be specified in each of the two terms resulting from the splitting. Furthermore, we combined one of the integrals of each splitting into a single integral (Eq. (5.2.130)), thus the prescription has to be the same for all four integrals; otherwise the cancellation might not be occurred. Here we fixe the prescription in such that: $-i\,\lambda$, which is the most convenient as we will show further.

Then we make the change of variables[15]. $\tilde{z} = z - \alpha\, x$ (resp. $\tilde{z} = (1-\alpha)\, z$ and $\tilde{z} = -\alpha\, z$) in the first (resp. the second and the third) integral. The three integrals take the same

[15]The notation z is used again to avoid another heavy notation.

generic form:

$$\int_0^1 dx \int_0^x dz \, \frac{K_i}{G_i\, x + H_i\, z + J_i} \, \ln\left(A_i\, x^2 + B_i\, z^2 + C_i\, x\, z + D_i\, x + E_i\, z + F_i - i\,\lambda\right)$$

Let us define as "sector I", "sector II", and "sector III" respectively these three integrals with the corresponding following sets of coefficients:

Sector I:	Sector II:	Sector III:
$A_I = a + b + c$	$A_{II} = a$	$A_{III} = a$
$B_I = g$	$B_{II} = g + j + b$	$B_{III} = b$
$C_I = h + j$	$C_{II} = c + h$	$C_{III} = c$
$D_I = d + e$	$D_{II} = d$	$D_{III} = d$
$E_I = k$	$E_{II} = e + k$	$E_{III} = e$
$F_I = f$	$F_{II} = f$	$F_{III} = f$
$G_I = c + 2\,b + h\,\alpha + j\,\alpha$	$G_{II} = c + h\,\alpha$	$G_{III} = c + h\,\alpha$
$H_I = 2\,g\,\alpha + j$	$H_{II} = (1-\alpha)\,(2\,g\,\alpha + j)$	$H_{III} = -\alpha\,(2\,g\,\alpha + j)$
$J_I = e + \alpha\,k$	$J_{II} = e + \alpha\,k$	$J_{III} = e + \alpha\,k$
$K_I = 1$	$K_{II} = -(1-\alpha)$	$K_{III} = -\alpha$

(5.2.133)

Omitting the "sector" index $i = I, II, III$ from now on, I_4^6 becomes

$$I_4^6(S) = \sum_{i=1}^{III} \int_0^1 dx \int_0^x dz \frac{K}{G\,x + H\,z + J - i\,\lambda} \ln(A\,x^2 + B\,z^2 + C\,xz + D\,x + E\,z + F - i\,\lambda) \tag{5.2.134}$$

In the following, we will adopt a strategy close to that one introduced by 'tHooft and Veltman in [66] for calculating the box in four-dimensions. First of all, let's shift $z = \bar{z} + \gamma\,x$ such that $H\,\gamma + G = 0$. The result is

$$I_4^6(S) \;=\; \sum_{i=I}^{III} \int_0^1 d\,x \int_{-\gamma\,x}^{(1-\gamma)\,x} \frac{d\,\bar{z}}{H\,\bar{z} + J} \, \ln(W_2\,x^2 + W_1\,x + W_0 - i\,\lambda) \tag{5.2.135}$$

$$W_2 \;=\; B\,\gamma^2 + C\,\gamma + A \qquad\qquad W_1 \;=\; (C + 2\,B\,\gamma)\,\bar{z} + E\,\gamma + D$$
$$W_0 \;=\; B\,\bar{z}^2 + E\,\bar{z} + F \tag{5.2.136}$$

the parameter γ is always real even if complex internal masses are involved since G and H are independent of these masses. So, the order of integration can be reversed and Eq. (5.2.135) becomes

$$I_4^6(S) \;=\; \sum_{i=I}^{III} K \left(\int_0^{1-\gamma} d\,\bar{z} \int_{\bar{z}/(1-\gamma)}^1 dx \;-\; \int_0^{-\gamma} d\,\bar{z} \int_{\bar{z}/(-\gamma)}^1 dx \right) F_0^6(x, \bar{z}) \tag{5.2.137}$$

where F_0^6 is given by:

$$F_0^6(x, \bar{z}) \;=\; \frac{1}{H\,\bar{z} + J} \, \ln(W_2\,x^2 + W_1\,x + W_0 - i\,\lambda) \tag{5.2.138}$$

we integrate F_0^6 over x, we get

$$\begin{aligned}\int dx\, F_0^6(x,z) &= \frac{1}{H\,z+J}\Big\{x\,\ln(W_2\,x^2+W_1\,x+W_0-i\,\lambda)\\ &- \int dx\,\frac{x\,(2\,W_2\,x+W_1)}{W_2\,x^2+W_1\,x+W_0-i\,\lambda}\Big\}.\end{aligned} \tag{5.2.139}$$

Let's call the contribution coming from the integrated part, in Eq. (5.2.139), $I_{4,1}^6(S)$ and from the non-integrated one $I_{4,2}^6(S)$. Then I_4^6 becomes

$$I_4^6(S) = I_{4,1}^6(S) - I_{4,2}^6(S) \tag{5.2.140}$$

By combining the terms coming from the upper limit $x=1$ and making the change of variables $\bar{z}=t-\gamma$, $\bar{z}=(1-\gamma)\,t$ and $\bar{z}=-\gamma\,t$, respectively in the terms with $\int_{-\gamma}^{1-\gamma}$, $\int_0^{1-\gamma}$ and $\int_0^{-\gamma}$ for each sector, we prove that $I_{4,1}^6$ can be written in the following form

$$I_{4,1}^6(S)=\sum_{i=I}^{III}\int_0^1 dt\,K(N_{4,1}^6(1,t-\gamma)-(1-\gamma)\,N_{4,1}^6(t,(1-\gamma)\,t)+(-\gamma)\,N_{4,1}^6(t,-\gamma\,t)) \tag{5.2.141}$$

$$N_{4,1}^6(x,z) = x\,\ln(W_2\,x^2+W_1\,x+W_0)/(H\,z+J) \tag{5.2.142}$$

$I_{4,1}^6(S)$ is a one-dimensional integral, after some manipulation it can be written as

$$\begin{aligned}I_{4,1}^6(S) &= \sum_{i=I}^{III}\frac{K}{H}\int_0^1 dt\Big\{\\ &\frac{1}{t+\frac{G+J}{H}}\,\ln(B\,t^2+(E+C)\,t+A+D+F-i\,\lambda)\\ &- \frac{1}{t+\frac{J}{G+H}}\,\ln((A+B+C)\,t^2+(E+D)\,t+F-i\,\lambda)\\ &+ \frac{1}{t+\frac{J}{G}}\,\ln(A\,t^2+D\,t+F-i\,\lambda)\Big\}\end{aligned} \tag{5.2.143}$$

By summing over all the sectors, we find that the contribution from the last two terms of this equation vanishes, this can be seen by expressing the capital letters in term of the small ones or in term of the entries of the matrix $\mathcal{S}$, see section (B.3) (the first, the second and the third quadratic form in Eq. (5.2.125) are denoted in subsection (B.3.1) by $Q_1(t)$, $Q_2(t)$ and $Q_3(t)$, respectively). Then,

$$\begin{aligned}I_{4,1}^6(S)=\sum_{i=I}^{III}\frac{K}{H}\int_0^1 dt\,\frac{1}{t+\frac{G+J}{H}}\Big\{\\ \ln(B\,t^2+(E+C)\,t+A+D+F-i\,\lambda)-\ln\Big(-\frac{1}{2}\frac{\det(\mathcal{S}^{\{1\}})}{\det(G^{\{1\}})}-i\,\lambda\Big)\Big\}\end{aligned} \tag{5.2.144}$$

where for each sector, the value of the logarithm argument taken at the pole value is given by

$$B\,t_0^2 + (E+C)\,t_0 + A + D + F - i\,\lambda|_{t_0=-\frac{G+J}{H}} = -\frac{1}{2}\frac{\det(\mathcal{S}^{\{1\}})}{\det(G^{\{1\}})} - i\,\lambda \tag{5.2.145}$$

The constant logarithm $(-\frac{1}{2}\frac{\det(\mathcal{S}^{\{1\}})}{\det(G^{\{1\}})})$ is subtracted just to build a formula with zero residue at the poles. It is the same for each sector and it gives no contribution to the final result, see demonstration bellow. $\mathcal{S}^{\{1\}}$ is a 3×3 matrix obtained from the matrix $\mathcal{S}$ by omitting the first line and the first column, $\det(G^{\{1\}})$ is the determinant of its associated Gram matrix, see section B.2.
Thus Eq. (5.2.144) can be written as

$$\begin{aligned} I^6_{4,1} =& \frac{1}{\varepsilon_\alpha\sqrt{\Delta_\alpha}}\int_0^1 dt\Bigg\{ \\ &\times\frac{1}{t-t_0^{(1)}}\left[\ln(s_3\,y^2+(m_2^2-m_3^2-s_3)\,y+m_3^2-i\lambda)-\ln(\mathcal{B}^{\{1\}}-i\lambda)\right] \\ &+\frac{1}{t-t_0^{(2)}}\left[\ln(s_4\,y^2+(m_3^2-m_4^2-s_4)\,y+m_4^2-i\lambda)-\ln(\mathcal{B}^{\{1\}}-i\lambda)\right] \\ &-\frac{1}{t-t_0^{(3)}}\left[\ln(s\,y^2+(m_2^2-m_4^2-s)\,y+m_4^2-i\lambda)-\ln(\mathcal{B}^{\{1\}}-i\lambda)\right]\Bigg\} \\ =& -I_3^4(S\setminus\{1\}) \end{aligned} \tag{5.2.146}$$

where the poles $t_0^{(i)}$ are given in subsection (B.3.1), and

$$\mathcal{B}^{\{i\}} = -\frac{1}{2}\frac{\det(\mathcal{S}^{\{i\}})}{\det(G^{\{i\}})}. \tag{5.2.147}$$

Eq. (5.2.146) represents a one-dimensional representation of *minus* the scalar 3-point function in 4-dimension obtained from the box in Fig. (5.4) by pinching the propagator number "1", see diagram (a) in Fig. (B.1) (Appendix B). One can obtain this result from Eq. (5.1.32)) by making the replacement:

$$\begin{aligned} s_1 &\longrightarrow s_4, & m_1^2 &\longrightarrow m_4^2 \\ s_2 &\longrightarrow s, & m_2^2 &\longrightarrow m_2^2 \\ s_3 &\longrightarrow s_3, & m_3^2 &\longrightarrow m_3^2 \end{aligned} \tag{5.2.148}$$

The last integration can be performed analytically by using Eq.(A.2.11) in Appendix A. We notice that $\det(G^{\{1\}})$ can be interchanged to $\det(G^{\{i\}})$ for $i=2,3,4$ by rotating the box, .i.e. by interchanging $p_i\to p_{i+1}$ and $m_i\to m_{i+1}$ for $i=1,2,3,4$. So if $\det(G^{\{1\}})$ vanishes, one has just to make this procedure to interchange it to a non vanishing reduced Gram determinant (this is completely possible since the four $\det(G^{\{i\}})$ can't vanish simultaneously). Then, the analytical formula of $I^6_{4,1}$ is stable.

The second term in the right hand side of Eq. (5.2.140) is

$$\begin{aligned} I^6_{4,2}(S) &= \left(\int_0^{1-\gamma} d\bar{z} \int_{\bar{z}/(1-\gamma)}^1 dx - \int_0^{-\gamma} d\bar{z} \int_{\bar{z}/(-\gamma)}^1 dx\right) \\ &\times \frac{K}{H\,\bar{z}+J} \frac{x\,(2\,W_2\,x+W_1)}{W_2\,x^2+W_1\,x+W_0-i\,\lambda} \end{aligned} \tag{5.2.149}$$

Since γ is real, we can reverse the order of integration again and combine the two integrals in Eq. (5.2.149), and make the change of variable $\bar{z} = z - \gamma\,x$, we get

$$I^6_{4,2}(S) = K \int_0^1 dx \int_0^x dz \frac{1}{G\,x+H\,z+J} \frac{Q^{(6)}(x,z)}{A\,x^2+B\,z^2+C\,x\,z+D\,x+E\,z+F-i\,\lambda} \tag{5.2.150}$$

The integral $I^6_{4,2}(S)$ is of the same type of I^4_4 in ref. [66], the only difference is the function $Q^{(6)}(x,z) = x\,((2\,A\,H - C\,G)\,x + (C\,H - 2\,B\,G)\,z + D\,H - E\,G)/H$ in the numerator, which does not change the nature of the logarithms and the poles from I^4_4 obtained in [66] (it has the same analytical structure of I^4_4!), but it changes only the coefficients in front off those logarithms in ref. [66]. From now on, we perform the same change of variables done by 'tHooft and Veltman. Thus we shift $z = \bar{z} + \beta\,x$[16], where β is defined as the solution of $B\,\beta^2 + C\,\beta + A = 0$, i.e.

$$\beta = \frac{-C + \varepsilon_\beta\,\sqrt{\Delta}}{2\,B}, \qquad \varepsilon_\beta = \pm 1. \tag{5.2.151}$$

Then, $I^6_{4,2}$ becomes

$$I^6_{4,2}(S) = \sum_{i=I}^{III} \int_0^1 dx \int_{-\beta\,x}^{(1-\beta)\,x} d\bar{z} \frac{K}{S\,x+T} \frac{x\,(C_1\,x+C_0)}{U\,x+V} \tag{5.2.152}$$

with

$$\begin{aligned} S &= G + H\,\beta & T &= H\,\bar{z} + J - i\,\lambda \\ U &= (C + 2\,B\,\beta)\,\bar{z} + D + \beta\,E & V &= B\,\bar{z}^2 + E\,\bar{z} + F - i\,\lambda \\ C_1 &= -\,S\,(C + 2\,B\,\beta)/H = S\tilde{C}_1 & C_0 &= ((C\,H - 2B\,G)\,\bar{z} + D\,H - E\,G)/H \end{aligned} \tag{5.2.153}$$

we put the prescription "$-i\,\lambda$" in T (At this point, the sign of λ is of no consequence if α is taken to be real). The parameter β can be real or complex, this depends on the sign of the discriminant of the quadratic form determining β, i.e on the sign of $\det(G^{(i)})$ (the Gram determinant associated to the 3-point function obtained by pinching a given propagator of the box, see section B.2). In the following, we will do the calculation for real β, a generalization to complex β will be presented in the end of this section.

Since β is real, we can exchange the order of integration

$$I^6_{4,2}(S) = K \int_0^{1-\beta} d\,\bar{z} \int_{\bar{z}/(1-\beta)}^1 dx\, F^6_1(x,\bar{z}) - \int_0^{-\beta} d\,\bar{z} \int_{\bar{z}/(-\beta)}^1 dx\, F^6_1(x,\bar{z}) \tag{5.2.154}$$

[16] we introduce z again for simplicity

F_1^6 is the integrand of Eq.(5.2.152), its primitive with respect to x is

$$\begin{aligned} N_{4,2}^6(x,z) &= \int dx\, \frac{x\,(C_1\,x+C_0)}{S\,V-T\,U}\left[\frac{S}{S\,x+T}-\frac{U}{U\,x+V}\right] \\ &= \left\{\frac{x\,\tilde{C}_1}{U}+\frac{T}{S}\,\frac{(\tilde{C}_1\,T-C_0)\,\ln(S\,x+T)}{S\,V-T\,U}\right. \\ &- \left.\frac{V}{U^2}\,\frac{(\tilde{C}_1\,S\,V-C_0\,U)\,\ln(U\,x+V)}{S\,V-T\,U}\right\} \end{aligned} \tag{5.2.155}$$

Since β is real, the sign of the imaginary parts of the logarithmic arguments is not changed during the integration over the remaining variable. Hence, there is never any problem with the logarithms. Otherwise, J may change the imaginary part of the logarithm arguments, if complex masses are required. In this case, the sign of the imaginary part of the logarithmic argument could be changed if and only if the imaginary part of J becomes positive. If this kinematics is encountered, one has to multiply Eq. (5.2.150) by a global (-1) and change G, H and J to $-G$, $-H$ and -J in the same equation, which guarantees that the branch cut of each logarithm is not crossed during the integration.

To be able to integrate this primitive analytically over the remaining integration variable, we can make a partial decomposition of the denominators $S\,V-T\,U$ and $U\,(S\,V-T\,U)$. Then, $N_{4,2}^6$ becomes

$$\begin{aligned} N_{4,2}^6(x,z) &= \frac{x\,\tilde{C}_1}{U}+\frac{2}{S}\,\ln(S\,x+T) \\ &+ \frac{1}{H}\left\{\frac{1}{2\,T_1}\sum_{\eta=\pm}\frac{H\,T_4+\eta\,T_2\,\sqrt{T_4}}{\sqrt{T_4}}\,\frac{\eta}{z-z^\eta}\,\ln\frac{S\,x+T}{U\,x+V}\right. \\ &- \frac{H\,M_0-(C+2\,B\,\beta)\,T_2}{2\,T_1}\,\frac{1}{U}\,\ln(U\,x+V) \\ &- \left.\left(\frac{B}{(C+2\,B\,\beta)}-\frac{M_1}{(C+2\,B\,\beta)}\,\frac{1}{U^2}\right)\ln(U\,x+V)\right\} \end{aligned} \tag{5.2.156}$$

with

$$T_1 = A\,H^2-C\,G\,H+B\,G^2 = -K^2\,\det(G)/2 \tag{5.2.157}$$

$$T_2 = -E\,G\,H+D\,H^2+2\,B\,G\,J-C\,H\,J \tag{5.2.158}$$

$$M_0 = C^2\,J-4\,A\,B\,J-C\,E\,G+2\,B\,D\,G-C\,D\,H+2\,A\,E\,H = K\,\det(\mathcal{S})\,b_i \tag{5.2.159}$$

$$M_1 = B\,D^2-C\,D\,E+A\,E^2-4\,A\,B\,F+C^2\,F = \det(\mathcal{S}^{\{i\}})/2 \tag{5.2.160}$$

where b_i are defined in Chapter 4, $\sqrt{T_4}$ and $z^{\eta=\pm}$ are the discriminant and the roots of the quadratic form $S\,V-T\,U$, respectively. We have

$$\begin{aligned} z^\eta &= \frac{-\alpha_2+\eta\,\sqrt{T_4}}{2\,\alpha_1} \\ &= \frac{-2\,\alpha_3}{\alpha_2+\eta\,\sqrt{T_4}}, \end{aligned} \tag{5.2.161}$$

$$\begin{aligned} T_4 &= \alpha_2^2 - 4\,\alpha_1\,\alpha_3 & \alpha_1 &= B\,G - B\,H\,\beta - C\,H, \\ &= K^2\,\det(\mathcal{S}), & \alpha_2 &= E\,G - D\,H - 2\,B\,J\,\beta - C\,J, \\ S\,V - T\,U &= \alpha_1\,z^2 + \alpha_2\,z + \alpha_3 & \alpha_3 &= F\,G - D\,J + \beta\,(F\,H - E\,J), \\ &= \alpha_1\,(z - z^+)\,(z - z^-), \end{aligned} \tag{5.2.162}$$

from now on we call α_1, "$\bar{S}$", where

$$\alpha_1 = \bar{S} = B\,(G + \bar{\beta}\,H), \qquad \bar{\beta} = -\frac{C + \varepsilon_\beta\,\sqrt{\Delta}}{2\,B}, \tag{5.2.163}$$

For more details about these definitions, see section B.3.2 in Appendox B.

At this stage, we can see the interest of choosing the prescription $-i\,\lambda$ (with $\lambda > 0$). The reason is that, the logarithms $\ln(S\,x + T)$ and $\ln(U\,x + V)$ can be combined into $\ln\frac{S\,x+T}{U\,x+V}$, as in Eq. (5.2.156), without introducing any η function, since their arguments have always negative imaginary parts for arbitrary internal masses. This will help us to construct a formula with zero-residue at the pole as we will see later on.

Inserting Eq.(5.2.156) in Eq. (5.2.154) and recombining terms coming from the upper limit $x = 1$, we get

$$\begin{aligned} I^6_{4,2}(S) &= \sum_{i=1}^{III}\sum_{\eta=\pm}\frac{K}{H}\,\frac{1}{2\,T_1}\,\frac{H\,T_4 + \eta\,T_2\,\sqrt{T_4}}{\sqrt{T_4}}\left\{\int_{-\beta}^{1-\beta} dz\frac{\eta}{z - z^\eta}\,\ln\frac{S+T}{U+V}\right. \\ &- \left.\int_0^{1-\beta} dz\frac{\eta}{z - z^\eta}\,\ln\frac{S\,z/(1-\beta)+T}{U\,z/(1-\beta)+V} + \int_0^{-\beta} dz\frac{\eta}{z - z^\eta}\,\ln\frac{S\,z/(-\beta)+T}{U\,z/(-\beta)+V}\right\} \\ &- \sum_{i=1}^{III}\frac{K}{H}\,\frac{H\,M_0 - (C + 2\,B\,\beta)T_2}{2\,T_1}\left\{\int_{-\beta}^{1-\beta}\frac{1}{U}\,\ln(U+V)\right. \\ &- \left.\int_0^{1-\beta}\frac{1}{U}\,\ln(U\,z/(1-\beta)+V) + \int_0^{-\beta}\frac{1}{U}\,\ln(U\,z/(-\beta)+V)\right\} \end{aligned} \tag{5.2.164}$$

We notice that the first, the second and the last two terms in Eq. (5.2.156) give zero-contribution to Eq. (5.2.164), this will be proved in section B.4, Appendix(B).
The Eq. (5.2.164) is proportional to inverse of T_1, i.e. to the inverse of $\det(G)$ ($\det(G)$ is Gram matrix associated to $\mathcal{S}$ matrix). If $\det(G)$ becomes arbitrary small in some phase space regions, then this equation will lead to numerical unstable results. On the other side, the primitive in Eq. (5.2.155) is not proportional explicitly to inverse of $\det(G)$ (or T_1), but it has just one term proportional to inverse of S, where

$$T_1 = S\,\bar{S} \propto \det(G) \tag{5.2.165}$$

S and $\bar{S}$ are defined in Eqs. (5.2.153, 5.2.163), respectively. So, it will be better to use the primitive Eq. (5.2.155) rather than the primitive Eq. (5.2.156) (the later one leads to Eq. (5.2.164)) to derive the one-dimensional integral representation which will be used in the in the problematic region, as we will see that in detail in section 5.2.2.2. In the following, we will use Eq. (5.2.164) to derive the analytical formula.

The poles of Eq. (5.2.164) correspond to $SV - TU = 0$ ($z = z^\eta$) and $U = 0$. In the former case, the concerning terms have the general form

$$\ln \frac{Sx+T}{Ux+V} = \ln\left(\frac{V}{U}\frac{x+T/S}{x+V/U}\right) = \ln\frac{V}{U}\Big|_{SV-TU=0} \tag{5.2.166}$$

So, at the pole value, i.e. $SV - TU = 0$ (or $T/S = V/U$), one can just subtract $\ln\frac{V}{U}$ from each of these logarithms to make the residue vanish. For real β the subtracted term give no contribution to the final result (it is true for complex β) as we will prove bellow. Similarly, we can subtract $\ln(V)|_{U=0}$ from $\ln(Ux+V)$ to make the residue at $U = 0$ vanish.
By making the following change of variables: $z_1 : z = t - \beta$, $z_2 : z = (1-\beta)\,t$ and $z_3 : z = -\beta\,t$ in the integrals $\int_{-\beta}^{1-\beta}$, $\int_0^{1-\beta}$ and $\int_0^{-\beta}$, respectively. We get

$$\begin{aligned}
I^6_{4,2}(S) = &\sum_{i=I}^{III}\sum_{\eta=\pm}\int_0^1 dt\,\frac{K}{H}\,\frac{1}{2\,T_1}\,\frac{H\,T_4+\eta\,T_2\,\sqrt{T_4}}{\sqrt{T_4}}\Bigg[\\
&\frac{\eta}{t-\beta-z^\eta}\left\{\ln\frac{H\,t+G+J-i\,\lambda}{B\,t^2+(E+C)\,z+A+D+F-i\,\lambda} - \ln(\text{"}t=\beta+z^\eta\text{"})\right\}\\
&-\frac{\eta}{t-z^\eta/(1-\beta)}\left\{\ln\frac{(H+G)\,t+J-i\,\lambda}{(A+B+C)\,t^2+(E+D)\,z+F-i\,\lambda} - \ln(\text{"}t=\frac{z^\eta}{1-\beta}\text{"})\right\}\\
&+\frac{\eta}{t-\frac{z^\eta}{-\beta}}\left\{\ln\frac{G\,t+J-i\,\lambda}{A\,t^2+D\,z+F-i\,\lambda} - \ln(\text{"}t=z^\eta/(-\beta)\text{"})\right\}\Bigg]\\
&-\sum_{i=I}^{III}\frac{K}{H}\,\frac{H\,M_0-(C+2\,B\,\beta)T_2}{2\,T_1}\,\frac{1}{C+2B\,\beta}\int_0^1 dt\,\Bigg[\\
&\frac{1}{t-\beta+R}\left\{\ln(B\,t^2+(E+C)\,z+A+D+F-i\,\lambda)-\ln\tilde{\mathcal{B}}^{\{i\}}\right\}\\
&-\frac{1}{t+R/(1-\beta)}\left\{\ln((A+B+C)\,t^2+(E+D)\,z+F-i\,\lambda)-\ln\tilde{\mathcal{B}}^{\{i\}}\right\}\\
&+\frac{1}{t+R/(-\beta)}\left\{\ln(A\,t^2+D\,z+F-i\,\lambda)-\ln\tilde{\mathcal{B}}^{\{i\}}\right\}\Bigg]
\end{aligned} \tag{5.2.167}$$

with

$$\tilde{\mathcal{B}}^{\{i\}} = \mathcal{B}^{\{i\}} - i\,\lambda \tag{5.2.168}$$

Combining this equation with $I^6_{4,1}$ and using

$$\begin{aligned}
\mathcal{J} &= \sum_{i=I}^{III}\frac{K}{H}\left\{I_4^{4,\text{rest}}(\mathcal{S}) + (C+2B\beta)I_3^4(\mathcal{S}\setminus\{i\})\right\} - I_3^4(\mathcal{S}\setminus\{1\})\\
&= 0
\end{aligned} \tag{5.2.169}$$

see subsection B.4.3 in Appendix B for the proof of this formual.

We prove that the final formula of I_4^6 is

$$
\begin{aligned}
I_4^6(S) = -\sum_{i=I}^{III}\sum_{\eta=\pm}\frac{K}{2\,T_1}\int_0^1 dt\left(\eta\sqrt{T_4}+\frac{2\,T_1+T_2}{H}\right)\Bigg[\\
\frac{1}{t-\beta-z^\eta}\left\{\ln\frac{H\,t+G+J-i\,\lambda}{B\,t^2+(E+C)\,z+A+D+F-i\,\lambda}-\ln(\text{"}t=\beta+z^\eta\text{"})\right\} \\
-\frac{1}{t-z^\eta/(1-\beta)}\left\{\ln\frac{(H+G)\,t+J-i\,\lambda}{(A+B+C)\,t^2+(E+D)\,z+F-i\,\lambda}-\ln(\text{"}t=\frac{z^\eta}{1-\beta}\text{"})\right\} \\
+\frac{1}{t-z^\eta/(-\beta)}\left\{\ln\frac{G\,t+J-i\,\lambda}{A\,t^2+D\,z+F-i\,\lambda}-\ln(\text{"}t=z^\eta/(-\beta)\text{"})\right\}\Bigg] \\
+\sum_{i=I}^{III}\frac{K}{H}\,\frac{H\,M_0-(C+2B\,\beta)\,(2\,T_1+T_2)}{2T_1}\,\frac{1}{C+2B\,\beta}\int_0^1 dt\Bigg[\\
\frac{1}{t-\beta+R}\left\{\ln(B\,t^2+(E+C)\,z+A+D+F-i\,\lambda)-\ln\tilde{\mathcal{B}}^{\{i\}}\right\} \\
-\frac{1}{t+R/(1-\beta)}\left\{\ln((A+B+C)\,t^2+(E+D)\,z+F-i\,\lambda)-\ln\tilde{\mathcal{B}}^{\{i\}}\right\} \\
+\frac{1}{t+R/(-\beta)}\left\{\ln(A\,t^2+D\,z+F-i\,\lambda)-\ln\tilde{\mathcal{B}}^{\{i\}}\right\}\Bigg]
\end{aligned}
\tag{5.2.170}
$$

where

$$R=(D+E\,\beta)/(C+2\,\beta\,B) \tag{5.2.171}$$

$$V|_{t_{0i}}=-\frac{1}{2}\frac{\det(\mathcal{S}^{\{i\}})}{\det(G^{\{i\}})}-i\,\lambda=\frac{1}{2}\mathcal{B}^{\{i\}}-i\,\lambda=\tilde{\mathcal{B}}^{\{i\}} \tag{5.2.172}$$

$$\ln(\text{"}t=t_{0i}^\eta\text{"})=\ln\left(\frac{V}{U}|_{t=t_{0i}^\eta}\right)=\ln(V|_{t=t_{0i}^\eta})-\ln(U|_{t=t_{0i}^\eta})+\eta(V|_{t=t_{0i}^\eta},1/U|_{t=t_{0i}^\eta}) \tag{5.2.173}$$

and

$$t_{01}=\beta-R \qquad t_{02}=-R/(1-\beta) \qquad t_{03}=-R/(-\beta) \tag{5.2.174}$$

$$t_{01}^\eta=\beta+z^\eta \qquad t_{02}^\eta=z^\eta/(1-\beta) \qquad t_{03}^\eta=z^\eta/(-\beta) \tag{5.2.175}$$

The subtraction logarithms gives no contribution to the final result (see next section), they are introduced to make the residues at each pole equal to zero as explained above. This formula is equivalent to the well known formula of $I_4^6(S)$ obtained from the Golem reduction,

$$I_4^6(S) \;=\; \frac{\det(\mathcal{S})}{\det(G)}\left[I_4^4(S)-\sum_{i=1}^{4}b_i\,I_3^4(S/\{i\})\right] \tag{5.2.176}$$

see the proof in subsection B.4.4 in the Appendix(B). The last integration in these two equations can be performed analytically using Eq.(A.2.11) in Appendix A.

We mention that Eq. (5.2.170) has less di-logarithmic functions compared to Eq. (5.2.176), since $I_3^4(S/\{1\})$ is omitted from the further formula by means of Eq.(5.2.169),

then it is more advantageous than the later formula.
If $\det(G)$ vanishes or becomes arbitrary small[17], these formulas (analytical mode) are not any more valid because of the problems of the numerical instability, hence we have to switch to the numerical mode by means of the one-dimensional integral representation, which will be derived in section 5.2.2.2.

5.2.2.1 Extension to complex β and/or complex internal masses

The formula of the box in 6-dimension given by Eq. (5.2.170) (or Eq. (5.2.176)) was derived for real β (the solution of $B\,\beta^2 + C\,\beta + A = 0$) and real internal masses. Actually, it is valid for any kinematical configuration in the physical region for arbitrary internal masses (complex masses are supported), and it is valid even for complex β, i.e. for configurations with at least three space-like momenta [18]..

Proof:

If β is complex [19], the problems of analyticity comes from the subtraction logarithms if their branch cuts are crossed, i.e. if their arguments become purely negatives, and in the same time, the associated poles are within the triangle $[0, -\beta, 1-\beta]$ in the z-complex plan (Eq. (5.2.164)). Otherwise, no problem of analyticity can occur since the other logarithms are independent of β and the imaginary part of their arguments are always negative as shown in section B.1.1 in Appendix(B).

Let us study the branch cuts of the subtracted logarithms and the position of the corresponding poles in Eq. (5.2.164):

- if the cut of the logarithm is not crossed and the pole is outside the triangle $[0, -\beta, 1-\beta]$: this case is safe.
- if the cut of the logarithm is crossed and the pole is outside the triangle $[0, -\beta, 1-\beta]$: also this case is safe, since the jumps across the cut are compensated in the three integrals $\int_{-\beta}^{1-\beta}$, $\int_{0}^{1-\beta}$ and $\int_{0}^{-\beta}$ in Eq.(5.2.164), i.e. For $arg_0 < 0$ (arg_0 refers to the arguments of the subtracted logarithms), $\ln(arg_0) = \ln|arg_0| \pm i\pi$, then

$$\int_{-\beta}^{1-\beta} dz \frac{\pm i\pi}{z - z_0} - \int_{0}^{1-\beta} dz \frac{\pm i\pi}{z - z_0} + \int_{0}^{-\beta} dz \frac{\pm i\pi}{z - z_0} = 0 \tag{5.2.177}$$

  where z_0 stands for the poles.
- if the cut of the logarithm is not crossed and the pole is inside the triangle $[0, -\beta, 1-\beta]$: no problem occurs, since the residue at the pole is zero by construction.

[17] We notice that T_1 may vanish for the sector II and the sector III not only for $\det(G) = 0$ but for $\alpha = 1$ or $\alpha = 0$ respectively since $T_1 \propto K^2 \det(G)$. In this case, the analytical formula of I_4^6 still valid since the K^2 coming from T_1 is compensated by the numerator.

[18] Such configuration can not be encountered at NLO order, see section B.3.1 in the Appendix B

[19] i) β^I associated to the sector I becomes complex if $s_2 < 0$, $s_3 < 0$ and $t < 0$, ii) β^{II} associated to the sector II becomes complex if $s_1 < 0$, $s_2 < 0$ and $s < 0$, iii) β^{III} associated to the sector III becomes complex if $s_1 < 0$, $s_4 < 0$ and $t < 0$, see subsection B.3.2.

- if the cut of the logarithm is crossed and the pole is inside the triangle $[0, -\beta, 1-\beta]$: problems may be produced in this region, so we have to check if this situation is encountered in our calculation or not.

Our formula have three type of poles: two simple poles corresponding to the denominator $S\,V - T\,U = 0$ and a simple pole corresponding to $U = 0$ (see the primitive in Eq.(5.2.156)). Let us calculate the argument of the subtracted logarithm associated to the pole at $U = 0$ (we call the pole z_0)

$$\begin{aligned} z_0 &= -\frac{D + E\,\beta}{C + 2\,\beta\,B} \\ &= -\frac{D + E\,(\beta_1 + \beta_2)}{i\,\varepsilon_\beta\,\sqrt{|\Delta|}} \\ &= -\frac{D + E\beta_1}{i\,\varepsilon_\beta\,\sqrt{|\Delta|}} - \frac{E}{2\,B} \end{aligned} \tag{5.2.178}$$

with

$$\beta = \beta_1 + \beta_2, \qquad \beta_1 = -\frac{C}{2\,B}, \qquad \beta_2 = \frac{i\,\varepsilon_\beta\,\sqrt{|\Delta|}}{2\,B} \tag{5.2.179}$$

is given by

$$\begin{aligned} V|_{z\to z_0} &= B\,z_0^2 + E\,z_0 + F - i\,\lambda \\ &= B\left(\frac{D + E\,\beta_1}{i\,\varepsilon_\beta\,\sqrt{|\Delta|}}\right)^2 - \frac{E^2}{4\,B} + F - i\,\lambda \\ &= \frac{\det(\mathcal{S}^{\{i\}})}{2\,|\Delta|} - i\,\lambda \end{aligned} \tag{5.2.180}$$

If the complex masses are not involved, the imaginary part of the subtraction term in Eq.(5.2.180) is given by $-i\lambda$ (it never changes the sign). Then it never crosses the negative real axis. Moreover, β complex means that the three external momenta of the corresponding triangle are space-like (otherwise the Gram determinant associated to this triangle is negative and β can not be complex, see section B.2); B is negative in the three different sectors since it is equal to p_3^2, $(p_1+p_2)^2$ and p_4^2, respectively (which are all space-like). This implies that the real part of the subtraction term is always positive in this configuration.

If the complex masses are involved and β is complex, we may encounter a problem if the pole z_0 is inside the triangle $[0, -\beta, 1-\beta]$ and the branch cut of the subtracted logarithm is crossed. Since in this case, Eq. (5.2.180) receives imaginary parts from D, E and F, and by reason of the internal complex masses have positive real part, then the real part of the argument of the subtracted logarithm is still positive. Hence, it never cross it negative real axis.

Regarding the logarithms associated to the poles $S\,V - T\,U = 0$ $(\ln((S\,x+T)/(U\,x+V)))$, we have to prove that if the poles z^η are within the triangle $[0, -\beta, 1-\beta]$, the branch cut

of these logarithms is not crossed. Let us define the z^η to be inside the triangle, then

$$z^\eta = \lambda\,(\mu - \beta), \qquad \text{with} \qquad 0 \leq \lambda, \mu \leq 1. \tag{5.2.181}$$

Inserting z^η in the arguments of these logarithms, we find

$$T + \lambda\, S = H\,\lambda\,\mu + \lambda\, G + J - i\,\lambda \tag{5.2.182}$$
$$V + \lambda\, U = B\,\lambda^2\,\mu^2 + A\,\lambda^2 + C\,\lambda^2\,\mu + E\,\lambda\,\mu + D\,\lambda + F - i\,\lambda \tag{5.2.183}$$

Eqs.(5.2.182, 5.2.183) are independent of the nature of β. If the internal masses are real, the imaginary parts of the arguments of these logarithms are given by $-\lambda$. Hence, they never cross their negative real axis. If the complex masses are involved, the imaginary part of $T + \lambda\, S$ is provided by J, so we can keep it negative by the procedure shown above. Regarding $V + \lambda\, U$, for each sector it is given by:

$$\begin{aligned} I : V + \lambda\, U = &-\lambda\,\mu\,(1-\lambda)\,p_2^2 - \lambda\,(1-\lambda)\,t + \lambda\,\mu\,(1-\lambda)\,t \\ &- \lambda^2\,\mu\,(1-\mu)\,p_3^2 + (1-\lambda)\,m_1^2 + \lambda\,\mu\,m_2^2 + \lambda\,(1-\mu)\,m_3^2 - i\,\lambda \end{aligned} \tag{5.2.184}$$
$$\begin{aligned} II : V + \lambda\, U = &\lambda\,\mu\,(1-\lambda)\,p_3^2 + \lambda\,(1-\lambda)\,p_1^2 - \lambda\,\mu\,(1-\mu)\,s \\ &- \mu\,(1-\lambda)\,p_2^2 + (1-\lambda)\,m_1^2 + \lambda\,\mu\,m_2^2 + \lambda\,(1-\mu)\,m_4^2 - i\,\lambda \end{aligned} \tag{5.2.185}$$
$$\begin{aligned} III : V + \lambda\, U = &\lambda\,\mu\,(1-\mu)\,p_1^2 - \lambda\,(1-\lambda)\,p_1^2 - \lambda\,\mu\,(1-\mu)\,p_4^2 \\ &- \mu\,(1-\lambda)\,t + (1-\lambda)\,m_1^2 + \lambda\,\mu\,m_3^2 + \lambda\,(1-\mu)\,m_4^2 - i\,\lambda \end{aligned} \tag{5.2.186}$$

where s and t are the Mandelstam variables.
From Eqs.(5.2.184, 5.2.185, 5.2.186), the imaginary parts of the logarithms argument are negatives for each sector, since they are provided by imaginary parts of the internal masses (which are negative) and $-\lambda$. Then

$$\ln \frac{T + \lambda\, S - i\,\lambda}{V + \lambda\, U - i\,\lambda} = \ln(T + \lambda\, S - i\,\lambda) - \ln(V + \lambda\, U - i\,\lambda) \tag{5.2.187}$$

Hence, if the poles z^η are inside the triangle $[0, -\beta, 1-\beta]$, the branch cuts of logarithms in Eq.(5.2.187) are never crossed.

In conclusion, Eq.(5.2.170) is valid for any configuration of the external momenta (space-like or time-like) and for arbitrary internal masses, where the complex masses are supported.

5.2.2.2 Numerical mode

In this paragraph, we show how one can avoid the inverse of the Gram determinant appearing in Eq.(5.2.170, 5.2.176) presented in the previous section. We have mentioned in the introduction that avoiding the inverse of Gram determinant (or the inverse of its powers), from the analytical formula, is possible from the mathematical point of view, since the singularity at $\det(G)$=0 is spurious (see Chapter 3). However, from the technical point of view!, as far as I know, it is not possible. But, at the level of the one-dimensional integral representation obtained from the primitive in Eq.(5.2.155), it is less complicated because of two raisons: **a)** $\det(G)$ doesn't appear explicitly in this formula but the inverse of S which appears, where $\det(G) = S\,\bar{S}$. **b)** we can always modify the primitive Eq. (5.2.155), which

provides a one-dimensional integral representation of $I^6_{4,2}$, by adding some terms without changing the final results (basically these terms are x-independent quantities and they vanish upon integration).

From the previous calculation, we have found

$$\begin{aligned}
I^6_4(S) &= I^6_{4,1}(S) - I^6_{4,2}(S),\\
I^6_{4,1}(S) &= \sum_{i=I}^{III} \int_0^1 dt\, K[N^6_{4,1}(1, t-\gamma) - (1-\gamma)\, N^6_{4,1}(t, (1-\gamma)\, t) + (-\gamma)\, N^6_{4,1}(t, -\gamma\, t)],\\
I^6_{4,2}(S) &= \sum_{i=I}^{III} \int_0^1 dt\, K[N^6_{4,2}(1, t-\beta) - (1-\beta)\, N^6_{4,2}(t, (1-\beta)\, t) + (-\beta)\, N^6_{4,2}(t, -\beta\, t)],
\end{aligned} \tag{5.2.188}$$

$N^6_{4,1}$ and $N^6_{4,2}$ are given in Eq.(5.2.142) and Eq.(5.2.155), respectively. The later one can be written as

$$\begin{aligned}
N^6_{4,2}(x,z) = \frac{\tilde{C}_1}{U^2}\left\{U\,x - V\,\ln(U\,x+V)\right\} + \frac{T}{S}\,\frac{\tilde{C}_1\,T - C_0}{S\,V - T\,U}\,\ln(S\,x+T)\\
-\frac{V}{U}\,\frac{\tilde{C}_1\,T - C_0}{S\,V - T\,U}\,\ln(U\,x+V).
\end{aligned} \tag{5.2.189}$$

Eqs. (5.2.188) provide a *one-dimensional integral representation* of $I^6_4(S)$, we have to modify $N^6_{4,2}$ in Eq. (5.2.189) to get the new representation of $I^6_4(S)$ which provides numerical stable results. From the primitive $N^6_{4,2}$, $I^6_{4,2}$ is not proportional to inverse of $\det(G)$ explicitly but it is proportional to inverse of S (the second term). So, the only remaining problem occurs when S becomes arbitrary small. $S = 0$ ($\Rightarrow \det(G) = 0$), is not a real singularity because the Landau conditions require $\det(\mathcal{S}) = 0$ which can not be satisfied simultaneously with $\det(G) = 0$ in the physical region, see Chapter 3. So it is completely possible to re-write $N^6_{4,2}$ in such way that the inverse of S is avoided. To do so, we pursue the following strategy, which will be adopted for the other scalar boxes (I^8_4, I^{10}_4 and I^{12}_4): from the term containing $\ln(S\,x+T)$ in $N^6_{4,2}$, we subtract $\ln(T)$ which give no contribution to the final result as we will show later on (the proof starts from Eq. (5.2.202)). Then

$$\begin{aligned}
\frac{T}{S}\,\frac{\tilde{C}_1\,T - C_0}{S\,V - T\,U}\left(\ln(S\,x+T) - \ln(T)\right) &= \frac{T}{S}\,\frac{\tilde{C}_1\,T - C_0}{S\,V - T\,U}\,\ln\frac{S\,x+T}{T}\\
&= -\frac{x\,(\tilde{C}_1\,T - C_0)}{S\,V - T\,U}\,\frac{\ln(1-X)}{X}
\end{aligned} \tag{5.2.190}$$

where $X = -S\,x/T$, we mention that we don't need to introduce an η function to combine the two logarithms in the left hand side of Eq.(5.2.190), since the signs of the imaginary parts of their arguments are both negative. For small X, this contribution can be written as

$$-\frac{x\,(\tilde{C}_1\,T - C_0)}{S\,V - T\,U}\,\frac{\ln(1-X)}{X} = \frac{x\,(\tilde{C}_1\,T - C_0)}{S\,V - T\,U}\,\sum_{n=0}^{\infty}\frac{X^n}{n+1} \tag{5.2.191}$$

let's introduce the function

$$q_1(X) = \begin{cases} \frac{1}{X}\ln(1-X) & \text{if} \quad X \not\to 0 \\ -\sum_{n=0}^{\infty} \frac{X^n}{n+1} & \text{if} \quad X \to 0 \end{cases} \tag{5.2.192}$$

here "$X \to 0$" means X equals to zero or becomes arbitrary small!
Each term of the expansion of q_1 around $X \to 0$ ($q_1 = -1 - \frac{1}{2}X - \frac{1}{3}X^2 - \cdots$) is free of any inverse of X (or S), which means that this expansion is finite for $S \to 0$, then for $\det(G) \to 0$. In top of that, each coefficient of the expansion is known at any order of the expansion (at the order n, the coefficient is $-1/(n+1)$)[20]. So, this function is very convenient for the numerical implementation.

Then, $N^6_{4,2}$ becomes

$$N^6_{4,2}(x,z) = \frac{\tilde{C}_1}{U^2}\left\{Ux - V\ln(Ux+V)\right\} - x\frac{\tilde{C}_1 T - C_0}{SV - TU}q_1(X) - \frac{V}{U}\frac{\tilde{C}_1 T - C_0}{SV - TU}\ln(Ux+V) \tag{5.2.193}$$

This new primitive (we keep the same name of the old one) is free of any inverse of S and it leads, in principe!, to stable results in the problematic regions. As a matter of fact, this "preliminary primitive" works very well for configurations where no pole approaches the segment $[0,1]$ (the domain of integration). For example, if the four internal masses of the box have different non negligible imaginary parts, this implies that all the poles of $N^6_{4,2}$ are far away from the integration segment by means of the imaginary parts that they receive from the complex internal masses; hence, no problem of numerical instability may occur during the numerical integration.
In general, the poles of $N^6_{4,1}$ and $N^6_{4,2}$ may approach the segment $[0,1]$ [21] (if they have sufficiently small imaginary parts) which can hamper the numerical stability. To avoid this problem, we have to perform a contour deformation to avoid the encountered poles.

5.2.2.3 Contour deformation:

We need to make a contour deformation, if one or some of the poles become very close to the segment $[0,1]$, i.e if they have a sufficiently small imaginary part which can embarrass the numerical stability. The new contours of integration should avoid these dangerous regions by avoiding the poles, and they should not cross the branch cuts of the associated logarithms, i.e. they must keep the imaginary parts of their arguments negative during the integration.

[20] The order of the expansion is fixed by comparing two result for different order of the expansion, say $n = n_1$ and $n = n_2$ ($n_1 < n_2$), with the precision of the machine. If the difference between the two results is smaller than the precision of the machine, we stop the expansion at $n = n_1$. Otherwise, we continue the expansion until when the previous condition is fulfilled.

[21] We can treat the contribution of $N^6_{4,1}$ independently as in Section 5.1, since it leads to 3-point function. Or even we can use the analytical mode, since the associated Gram determinant $\det(G^{(1)})$ can always chosen to be non-zero by rotating the box.

For terms containing quadratic logarithms, say " $\ln(a\,z^2+b\,z+c-i\,\lambda)$", we choose this contour

$$\mathcal{C}_1 \;\; : z \longrightarrow z - i\,z\,(1-z)\,[2\,a\,z+\Re(b)]\,\varepsilon_1 \tag{5.2.194}$$

For terms containing linear logarithms, say " $\ln(e\,z+f-i\,\lambda)$", we choose this contour

$$\mathcal{C}_2 \;\; : z \longrightarrow z - i\,z\,(1-z)\,\mathrm{sign}(e)\,\varepsilon_2 \tag{5.2.195}$$

and for terms without logarithms, we simply choose

$$\mathcal{C}_3 \;\; : z \longrightarrow z - i\,z\,(1-z)\,\varepsilon_3 \tag{5.2.196}$$

ε_1 and ε_2 have to be positive and their absolute values are chosen in such way that the integration contours become far enough from the dangerous region, and the sign of ε_3 is completely arbitrary!. All the possible contours and poles are given in the Fig. (5.5).

Using Cauchy's integral theorem. Schematically, I_4^6 can be written as

$$\sum_{i=1}^{3}\int_{\mathcal{C}_i}\mathcal{I}^6_{4\mathcal{C}_i}(z)\,dz = 2\,\pi\,i\sum_{i,j} I_{\mathcal{C}_{i,j}}\,\mathrm{Res}\left(\mathcal{I}^6_{4\mathcal{C}_i}(z)\right)_{z=z^0_{\mathcal{C}_{i,j}}} \tag{5.2.197}$$

$\mathcal{I}^6_{4\mathcal{C}_i}$ denotes a part of the integrand of the one-dimensional integral representation, where the contour of integration is deformed to $\mathcal{C}_i$, $z^0_{\mathcal{C}_{i,j}}$ stands for the poles of $\mathcal{I}^6_{4\mathcal{C}_i}$ where the sum runs over all the possible poles j inside the closed contour in the complex plan, $I_{\mathcal{C}_{i,j}} = \pm 1, 0$ stands for winding number of the contour $\mathcal{C}_i$ about the pole $z^0_{\mathcal{C}_{i,j}}$, it is positive if $\mathcal{C}_i$ moves in a counter clockwise manner around the pole, negative if it moves in the opposite side and 0 if it doesn't move around the pole at all.

At this stage, it remains to calculate the residues of the integrand at each pole which will be the subject of the next paragraph.

5.2.2.4 Residues calculation

The residue of a holomorphic function $f(z)$ at its pole z_0 is defined by:

$$a_{-1} \;\; = \;\; \frac{1}{(m-1)!}\,\frac{d^{m-1}}{d\,z^{m-1}}\left[(z-z_0)^m\,f(z)\right]_{z=z_0} \tag{5.2.198}$$

where m is the order of the pole z_0.

Residues of $N^6_{4,1}$:

The contribution of $N^6_{4,1}$ equals to $-I^4_3(\mathcal{S}\setminus\{1\})$ as shown in Eq. (5.2.146). Since it is free of $\det(G)$, one can use either the analytical or the numerical mode (the reduced Gram determinant $\det(G^{\{1\}})$ can be chosen to be different from zero).

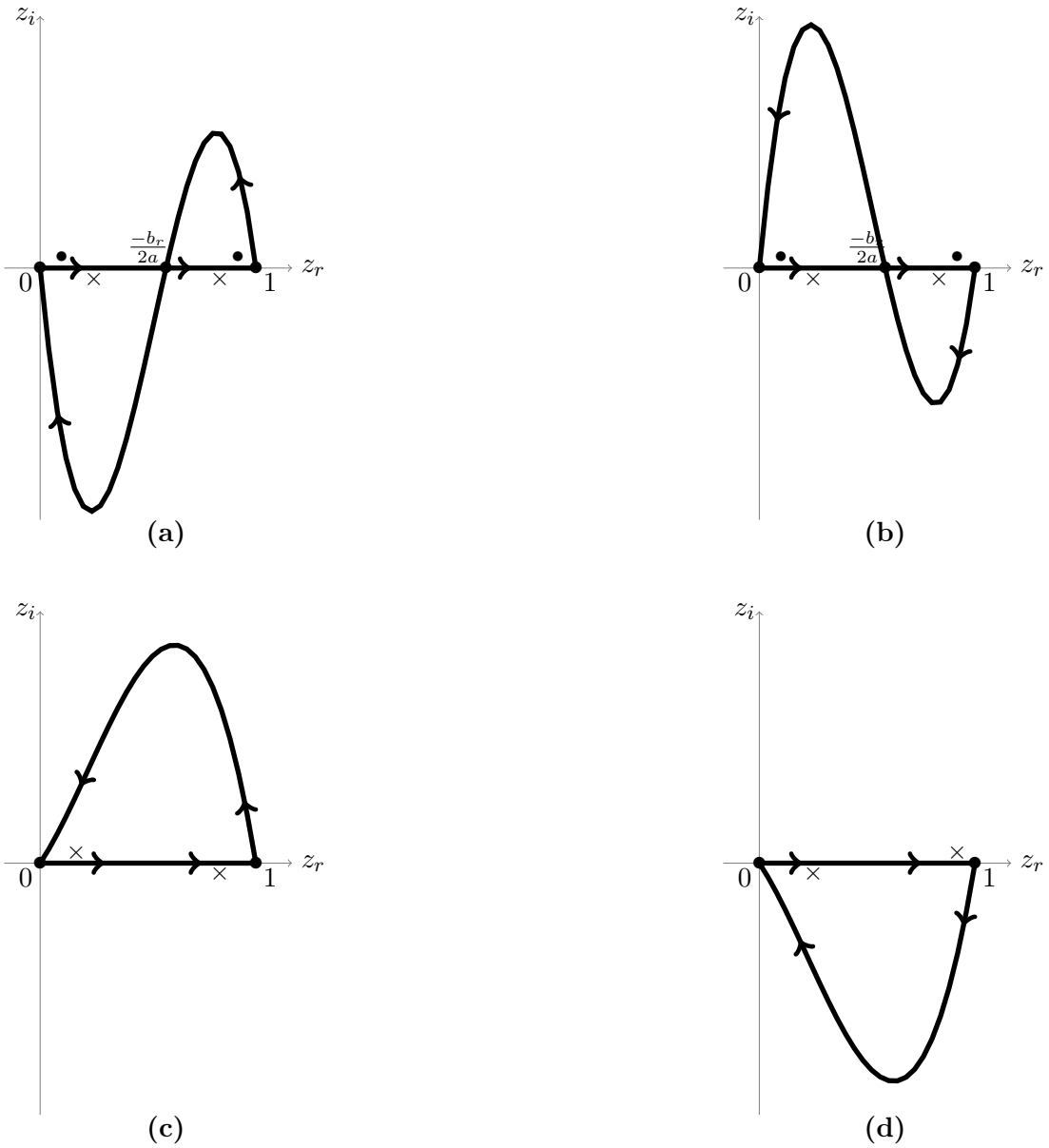


Figure 5.5: All possible contours and poles positions: **(a)** stands for the contour $\mathcal{C}_1$ with $a > 0$ and $\frac{-b}{2a} \in [0, 1]$, **(b)** stands for the contour $\mathcal{C}_1$ with $a < 0$ and $\frac{-b}{2a} \in [0, 1]$, **(c)** stands for the contour $\mathcal{C}_1$ with $a > 0$ and $\frac{-b}{2a} \notin [0, 1]$ or $\mathcal{C}_2$ with $e > 0$ or $\mathcal{C}_3$ with $\varepsilon_3 > 0$, **(d)** stands for the contour $\mathcal{C}_1$ with $a < 0$ and $\frac{-b}{2a} \notin [0, 1]$ or $\mathcal{C}_2$ with $e < 0$ or $\mathcal{C}_3$ with $\varepsilon_3 < 0$; the possible poles which can hamper the numerical stability are denotes by $\times$ or $\bullet$ (the contours (a) and (b) are not pinched between $\times$ and $\bullet$!).

Residues of $N^6_{4,2}$:

If we calculate the residues directly from the primitive Eq. (5.2.189), we will find expressions proportional to inverse of S which might embarrass the numerical stability for small S. To avoid this trouble, one can modify the primitive Eq. (5.2.189), by adding and subtracting

some terms (x-independent terms) which give zero-contribution to the final result. The primitive becomes

$$N^6_{4,2}(x,z) = \frac{\tilde{C}_1}{U^2}\left\{U\,x + V - V_0 - V\left(\ln(U\,x+V) - \ln(V_0)\right)\right\} - x\,\frac{\tilde{C}_1\,T - C_0}{S\,V - T\,U}\,q_1(X) - \frac{V}{U}\,\frac{\tilde{C}_1\,T - C_0}{S\,V - T\,U}\left(\ln(U\,x+V) - \ln(V_0)\right) \tag{5.2.199}$$

with

$$V_0 = V|_{U=0} = -\frac{1}{2}\frac{\det(S^{\{i\}})}{\det(G^{\{i\}})} - i\,\lambda = \mathcal{B}^{\{i\}} - i\,\lambda, \tag{5.2.200}$$

The difference between this primitive and the primitive given in Eq. (5.2.193) is that: from the first line in the right hand side, we subtracted the term "$\tilde{C}_1\,(V_0 - V + V\,\ln(V_0))/U^2$" which makes the residue of this part at the double pole $U^2 = 0$ equals to zero; and from the last line, we subtracted "(...) $\ln(V_0)$" where "(...)" stands for the coefficient of the $\ln(Ux+V)$, this term makes the residue at the simple pole $U = 0$ equals to zero. We notice that the subtracted terms give no contribution to the final result since they are x-independent quantities.

Proof: In general we subtract terms as

$$\frac{f(z)}{z - z_0} \tag{5.2.201}$$

$f(z)$ is an analytical function of z ($f(z)$ is arbitrary, it can be constant as the subtraction logarithms in Eq. (5.2.170), or logarithmic as $\ln T$ in Eq. (5.2.190), or polynomials as the "V"in Eq. (5.2.199)...etc). The contribution of Eq. (5.2.202) to the final result is

$$\int_0^1 dt\,\frac{f(t-\beta)}{t-\beta-z_0} - (1-\beta)\int_0^1 dt\,\frac{f((1-\beta)\,t)}{(1-\beta)\,t - z_0} + (-\beta)\int_0^1 dt\,\frac{f((-\beta)\,t)}{(-\beta)\,t - z_0} \tag{5.2.202}$$

We make the change of variables z_i for $i = 1, 2, 3$ (see above) in these three integrals, respectively. Eq. (5.2.202) becomes

$$\int_{-\beta}^{1-\beta} dz\,\frac{f(z)}{z - z_0} - \int_0^{1-\beta} dz\,\frac{f(z)}{z - z_0} + \int_0^{-\beta} dz\,\frac{f(z)}{z - z_0} \tag{5.2.203}$$

We see that for each integral we have the same integrand (and the same pole). For arbitrary β, one can combine the last two integrals at the lower limit (at $z = 0$). Then

$$\int_{-\beta}^{1-\beta} dz\,\frac{f(z)}{z - z_0} - \int_{-\beta}^{1-\beta} dz\,\frac{f(z)}{z - z_0} \tag{5.2.204}$$

Eq. (5.2.204) vanishes if β is real or β is complex and the pole z_0 is outside the triangle $[0, \beta, 1-\beta]$. However, if β is complex and the pole z_0 is inside the triangle $[0, -\beta, 1-\beta]$,

the contribution is not any more zero but, it equals to the residue of the integrand at this pole. Generally, we write

$$\textbf{Eq.(5.2.204)} = \begin{cases} 0, & \text{if } \beta \in \mathbf{R} \quad \text{or} \quad \beta \in \mathbf{C} \text{ and } \quad z_0 \notin [0, -\beta, 1-\beta] \\ 2\,\pi\, i\, f(z_0), & \text{if } \beta \in \mathbf{C} \text{ and } \quad z_0 \in [0, -\beta, 1-\beta] \end{cases}$$

If the studied kinematical configuration corresponds to only $S \to 0$ or $\bar{S} \to 0$, then β must be real. Otherwise, the two vanish simultaneously and β may be complex, because:
let us suppose that β is complex, i.e. the discriminant Δ is negative, this implies that S and $\bar{S}$ are complex and $\bar{S}$ is proportional to the conjugate of S (for $B \neq 0$). On the other hand the configuration $S \to 0$ and $\bar{S} \to 0$ simultaneously implies that the real and the imaginary parts of S and $\bar{S}$ must $\to 0$, since each one is proportional to the complex conjugate of the other. Hence, they may become sufficiently small and have a small imaginary part, i.e $|S| = |\bar{S}| \to 0$ (in the same time). Otherwise, if one of them vanishes alone, the other one must be purely real.

From this discussion, we have to be careful in the case where $|S| = |\bar{S}| \to 0$, because if they become slightly complex and the pole is inside the triangle $[0, -\beta, 1-\beta]$, our new primitive may not be valid, we will discuss that later on[22].

Important Remarks:

-a) If the subtraction function $f(z)$ contains logarithms (for example $\ln(T)$), one has to check if the contour of integration doesn't change the sign of their argument imaginary parts, otherwise, another trouble may be generated by crossing the branch cuts of these logarithms.

-b) For example, to construct the function q_1 we have to subtracted the logarithm $\ln(T)$ from the second term of the primitive in Eq. (5.2.190); the chosen contour to integrate this part is $\mathcal{C}_2$ (defined above), this last one is chosen to keep the imaginary part of "$S\,x + T$" negative ("$S\,x + T$" is a linear polynomial of t, where t is introduced after making the change of variables from z to t (the change of variables z_i). In addition, $\mathcal{C}_2$ depends only on the signs of the coefficients of t in the argument, i.e the signs of H, $G + H$ and G for each $\ln(S\,x + T)$ in the three integrals, receptively. Thankfully, this contour keeps, also, the imaginary part of each T negative near and for $S = 0$, since the coefficients of t in T are H, $G + H - S$ and $G - S$ in the three integrals, respectively, which are approximately the same as those corresponding to $\ln(S\,x + T)$ in this region. Hence, the branch cuts of $\ln(T)$ are not crossed by $\mathcal{C}_2$ for $S \to 0$.

-c) In the case where $\bar{S} \to 0$ (and not S), the branch cuts of $\ln(T)$ may be crossed by $\mathcal{C}_2$. Fortunately, in this case we don't need to construct q_1 since $S \neq 0$, hence no need to subtract $\ln(T)$ at all from the old primitive Eq. (5.2.189).

[22] In NLO calculation for processes of interest at collider experiment, such situations are not encountered, since this occurs if the box have at lest three space-like external momenta, see section B.2.

Seeing that $\det(G) \propto S\,\bar{S}$*, we will calculate the residues for three different cases: I) only* $S \to 0$*, II) only* $\bar{S} \to 0$ *and III)* S *and* $\bar{S} \to 0$ *simultaneously, the last case implies that* $\Delta \to 0$*, i.e a reduced Gram determinant vanishes. We notice that the primitive in Eq. (5.2.199) will be slightly changed for the two later cases.*

I) The residues in the Case $S \to 0$ ($\Rightarrow \beta \to -G/H$)

I-A) Residues at $U = 0$

Let's focus on the kinematical configuration where "$S \to 0$". The primitive in Eq. (5.2.199) has at $U = 0$ a double pole, no pole and a simple pole in the first, second and the third terms, respectively. For each change of variables z_i, these poles are

$$t_{01} = \beta - R \qquad t_{02} = -R/(1-\beta) \qquad t_{03} = -R/(-\beta) \tag{5.2.205}$$

By direct application to the residue formula given in Eq. (5.2.198), one can prove that the residue of each part of $N^6_{4,2}$ is zero, thanks to the subtraction terms

$$\text{Res}\Big(N^6_{4,2}\{t\}\Big)_{t=t_{0i}} \;=\; 0 \tag{5.2.206}$$

where $N^6_{4,2}\{t\}$ stand for the primitive $N^6_{4,2}$ corresponding to each change of variable z_i.

To prove this result, let us call the first, the second and the third parts of the primitive in Eq. (5.2.199) $N^{6,(1)}_{4,2}$, $N^{6,(2)}_{4,2}$ and $N^{6,(3)}_{4,2}$, respectively. Then

$$N^{6,(1)}_{4,2}(x,z) = \frac{\tilde{C}_1}{U^2}\left\{U\,x + V - V_0 - V\left(\ln(U\,x+V) - \ln(V_0)\right)\right\} \tag{5.2.207}$$

$$N^{6,(2)}_{4,2}(x,z) = -\frac{x}{H}\,\frac{2\,\alpha_1\,z + \alpha_2}{\alpha_1\,(z-z^+)\,(z-z^-)}\,q_1(X) \tag{5.2.208}$$

$$N^{6,(3)}_{4,2}(x,z) = -\frac{1}{H}\,\frac{V}{U}\,\frac{2\,\alpha_1\,z + \alpha_2}{\alpha_1\,(z-z^+)\,(z-z^-)}\left(\ln(U\,x+V) - \ln(V_0)\right) \tag{5.2.209}$$

where

$$\tilde{C}_1\,T - C_0 = \frac{2\,\alpha_1\,z + \alpha_2}{H} \tag{5.2.210}$$

$$S\,V - T\,U = \alpha_1\,(z-z^+)\,(z-z^-) \tag{5.2.211}$$

* The residue of $N^{6,(1)}_{4,2}(x_i,z_i)$ at t_{0i} is

$$\begin{aligned}\text{Res}\Big(N^{6,(1)}_{4,2}(x_i,z_i)\Big)_{t=t_{0i}} &\propto \left[(U\,x+V)' - V'\,(\ln(U\,x+V) - \ln(V_0)) - V\,\frac{(U\,x+V)'}{U\,x+V}\right]_{t=t_{0i}} \\ &= (U\,x+V)'|_{t=t_{0i}} - V'|_{t=t_{0i}}\,(\ln(V_0) - \ln(V_0)) - \frac{V_0}{V_0}\,(U\,x+V)'|_{t=t_{0i}} \\ &= 0 \end{aligned} \tag{5.2.212}$$

$(...)'$ stands for the derivative over t, where $x_i = 1, t, t$ and $z_i = t-\beta, (1-\beta)t, -\beta t$.

* The residue of $N_{4,2}^{6,(2)}(x_i, z_i)$ at t_{0i} is

$$\begin{aligned}\text{Res}\Big(N_{4,2}^{6,(1)}(x_i, z_i)\Big)_{t=t_{0i}} &\propto \frac{b_i\ \det(\mathcal{S})}{S}\Big(\ln(U\,x+V)-\ln(V_0)\Big)_{t=t_{0i}}\\ &= \frac{b_i\ \det(\mathcal{S})}{S}\Big(\ln(V_0)-\ln(V_0)\Big)_{t=t_{0i}}\\ &= 0\end{aligned} \tag{5.2.213}$$

* The term $N_{4,2}^{6,(2)}(x_i, z_i)$ has no pole at t_{0i}, hence the result in Eq. (5.2.206).

I-B) Residues at $S\,V - T\,U = 0$

In this case, the poles (corresponding to the change of variables z_i are

$$t_{01}^{\eta} = \beta + z^{\eta} \qquad t_{02}^{\eta} = z^{\eta}/(1-\beta) \qquad t_{03}^{\eta} = z^{\eta}/(-\beta) \tag{5.2.214}$$

We notice that for $S \to 0$, $SV - TU \to -TU$. One of the poles t^{η} approaches the root of $U = 0$, i.e $U^{\eta} \to 0$, and the other one approaches the root of $T = 0$, i.e $T^{\eta} \to 0$, where $U^{\eta} = U|_{t\to t_{0i}^{\eta}}$ and $T^{\eta} = T|_{t\to t_{0i}^{\eta}}$. If the $SV - TU = 0$ has a double root, i.e $t_{0i}^{+} \to t_{0i}^{-}$, the quantities U^{η} and T^{η} vanish simultaneously which implies: $T_4 = K^2\det(\mathcal{S}) \to 0$, then $\det(\mathcal{S})$ and $\det(G)$ vanish simultaneously. *From this remark, one has to distinguish between three cases: i) $U^{\eta} \to 0$ and $T^{\eta} \not\to 0$, ii) $U^{\eta} \not\to 0$ and $T^{\eta} \to 0$ and iii) $U^{\eta} \not\to 0$ and $T^{\eta} \to 0$ simultaneously.*

* The residues of $N_{4,2}^{6,(2)}(x_i, z_i)$ at the two simple poles t_{0i}^{η} are:

-i) If $U^{\eta} \to 0$ and $T^{\eta} \not\to 0$,

$$\text{Res}\Big(N_{4,2}^{6,(2)}(x_i, z_i)\Big)_{t=t_{0i}^{\eta}} = -\frac{1}{H}\,\frac{x_i^{\eta}}{\tilde{K}_i}\,q_1(X_i^{\eta}) \tag{5.2.215}$$

$$X_i^{\eta} = -S\,x_i^{\eta}/T^{\eta} \tag{5.2.216}$$

$$\tilde{K}_1 = 1 \qquad \tilde{K}_2 = 1-\beta \qquad \tilde{K}_3 = -\beta \tag{5.2.217}$$

$$x_1 = 1, \qquad x_2 = t, \qquad x_3 = t \tag{5.2.218}$$

$$x_1^{\eta} = 1, \qquad x_2^{\eta} = \frac{-2\,\alpha_3/(1-\beta)}{\alpha_2 + \eta\,\sqrt{T_4}} \qquad x_3^{\eta} = \frac{-2\,\alpha_3/(-\beta)}{\alpha_2 + \eta\,\sqrt{T_4}} \tag{5.2.219}$$

$$z_1 = t-\beta, \qquad z_2 = (1-\beta)t, \qquad z_3 = -\beta t \tag{5.2.220}$$

Eq. (5.2.215) is valid for $S \to 0$, where the function q_1 must be expressed as a series expansion around $X_i^{\eta} = 0$.

-ii) If $U^\eta \not\to 0$ and $T^\eta \to 0$,
In this case X_i^η is not any more small, which means the series expansion of q_1 around $X_i^\eta = 0$ is not allowed. So, we have to re-write Eq. (5.2.215) as the following:

$$\mathrm{Res}\left(N_{4,2}^{6,(2)}(x_i,z_i)\right)_{t=t_{0i}^\eta} = \frac{1}{H\tilde{K}_i}\frac{V^\eta}{U^\eta}(\ln(U^\eta\, x_i^\eta + V^\eta) - \ln(V^\eta)) \tag{5.2.221}$$

-iii) If $U^\eta \to 0$ and $T^\eta \to 0$ simultaneously, we use

$$\mathrm{Res}\left(N_{4,2}^{6,(2)}(x_i,z_i)\right)_{t=t_{0i}^\eta} = \frac{1}{H\tilde{K}_i}\frac{T^\eta}{S}\ln(S/T^\eta\, x_i^\eta + 1) \tag{5.2.222}$$

where T^η/S is finite ($T^\eta/S \sim \mathcal{O}(1)$) (Eq. (5.2.222) is valid for the case -*ii*) also).

* The residues of $N_{4,2}^{6,(3)}(x_i,z_i)$ at the two simple poles t_{0i}^η are:

-i) If $U^\eta \not\to 0$ and $T^\eta \to 0$,

$$\mathrm{Res}\left(N_{4,2}^{6,(3)}(x_i,z_i)\right)_{t=t_{0i}^\eta} = -\frac{1}{H\tilde{K}_i}\frac{V^\eta}{U^\eta}\left(\ln(U^\eta\, x_i^\eta + V^\eta) - \ln(V_0)\right) \tag{5.2.223}$$

Since $S \to 0$, one of these poles, say $t_{0i}^\pm$, should approach a given pole t_{0i}; then $U^\pm \to 0$ which can hamper the numerical stability, i.e. this formula is not valid at this pole. However, it is valid for the other pole, say $t_{0i}^\mp$, since the corresponding $U^\mp \not\to 0$.

-ii) If $U^\eta \to 0$ and $T^\eta \not\to 0$,
This problem can be solved by introducing the function q_1 again. We notice that, if $t_{0i}^\eta \to t_{0i}$, we have: $V^\eta \longrightarrow V_0$ and $U^\eta \longrightarrow U_0$, and since $SV - TU = 0$, we can write

$$\frac{V^\eta}{U^\eta} = \frac{T^\eta}{S} \tag{5.2.224}$$

So, one has to express U^η in term of S, V^η and T^η in Eq. (5.2.223), we write

$$\begin{aligned}\mathrm{Res}\left(N_{4,2}^{6,(3)}(x_i,z_i)\right)_{t=t_{0i}^\eta} &= -\frac{1}{H\tilde{K}_i}\frac{T^\eta}{S}\Bigg\{ \\ &\ln\left(1+\frac{S\,x^\eta}{T^\eta}\right) + \eta\left(V^\eta, 1+\frac{S\,x^\eta}{T^\eta}\right) + \ln(V^\eta) - \ln(V_0)\Bigg\} \\ &= -\frac{1}{H\tilde{K}_i}\Bigg\{-x^\eta\; q_1(X^\eta) + \eta(V^\eta, 1-X^\eta) \\ &+ \frac{T^\eta}{S}\left(\ln\left(1-\frac{V_0-V^\eta}{V^0}\right) - \eta(V^\eta, 1/V_0)\right)\Bigg\}\end{aligned} \tag{5.2.225}$$

we have

$$\begin{aligned}Y^\eta = \frac{V_0 - V^\eta}{V_0} &= -\frac{2\,S\,(B\,(z_0+z^\eta)+E)\,(C+2\,B\,\beta)}{M_0 - \eta\,(C+2\,B\,\beta)\,\sqrt{T_4}}, \\ &= S\,\tilde{Y}^\eta. \\ \tilde{Y}^\eta &= -\frac{2(B\,(z_0+z^\eta)+E)\,(C+2\,B\,\beta)}{M_0 - \eta\,(C+2\,B\,\beta)\,\sqrt{T_4}}.\end{aligned} \tag{5.2.226}$$

then,

$$\text{Res}\left(N_{4,2}^{6,(3)}(x_i,z_i)\right)_{t=t_{0i}^{\eta}} = \frac{1}{H\tilde{K}_i}\left\{x^{\eta}\,q_1(X^{\eta}) - T^{\eta}\,\tilde{Y}^{\eta}\,q_1(Y^{\eta}) - \eta(V^{\eta},1-X^{\eta}) + \eta(V^{\eta},V_0)\right\} \tag{5.2.227}$$

The function $\eta(1/V^{\eta},V_0)$ vanishes since the imaginary parts of its arguments have the opposite sign (we have $V^{\eta}\to V_0$ in this configuration). In the case of real internal masses for example (which is the case where the contour deformation is strongly needed), the imaginary parts of V^{η} and V_0 equals to $-\lambda$ since: β is real and $\det(\mathcal{S})$ is positive ($\det(\mathcal{S}) = T_2^2/(K^2\,H^2)$), then the poles must be slightly complex for $S=0$. Regarding the function $\eta(V^{\eta},1-X^{\eta})$, the first argument and the product of the two arguments are given by V^{η} and $U^{\eta}\,x^{\eta}+V^{\eta}$, respectively. They have the same imaginary parts for $U^{\eta}\to 0$ (for arbitrary internal masses), then this function vanishes too. Thus

$$\text{Res}\left(N_{4,2}^{6,(3)}(x_i,z_i)\right)_{t=t_{0i}^{\eta}} = \frac{1}{H\tilde{K}_i}\left\{x_i^{\eta}\,q_1(X^{\eta}) - T^{\eta}\,\tilde{Y}^{\eta}\,q_1(Y^{\eta})\right\} \tag{5.2.228}$$

-iii) If $U^{\eta}\to 0$ and $T^{\eta}\to 0$ simultaneously,

$$\text{Res}\left(N_{4,2}^{6,(3)}(x_i,z_i)\right)_{t=t_{0i}^{\eta}} = -\frac{1}{H\tilde{K}_i}\left\{\frac{T^{\eta}}{S}\,\ln(S/T^{\eta}\,x_i^{\eta}+1) - T^{\eta}\,\tilde{Y}^{\eta}\,q_1(Y^{\eta})\right\} \tag{5.2.229}$$

Remark: x_i^{η} for $i=2,3$ are finite since their denominators can't vanish in this region (they are proportional to inverse of $\sqrt{-\det(G^i)}$ for $i=1,\cdots,4$). They can be indeterminate (i.e. 0/0) iff α_2 and α_3 vanish simultaneously, but this case is numerically safe.

II) The residues in the Case $\bar{S}\to 0 (\Rightarrow \beta\to G/H - C/B)$

In this case we cannot keep the second part of the primitive, given in Eq. (5.2.208), since the contour $\mathcal{C}_2$ may cross the branch cut of $\ln(T)$, as discussed above (in "Important remarks"). Fortunately, in the current case $S\not\to 0$, thus we don't need to subtract $\ln(T)$. Then $N_{4,2}^{6,(2)}$ in Eq. (5.2.208) must be replaced by

$$N_{4,2}^{6,(2)}(x,z) = \frac{1}{H}\,\frac{T}{S}\,\frac{2\,\alpha_1\,z+\alpha_2}{\alpha_1\,(z-z^{+})\,(z-z^{-})}\,\ln(Sx+T) \tag{5.2.230}$$

II-A) Residues at $U=0$

The residue of each term of $N_{4,2}^{6}$ at $U=0$ vanish as in the previous case.

II-B) Residues at $S\,V - T\,U = 0$

The two simple poles in this case are arbitrary, they may be finite as well as infinite, this depends on the parameters α_2 and α_3 ($SV-TU\to\alpha_2 z+\alpha_3$): So, we have to distinguish between four cases:
-i) If $\alpha_2\not\to 0$ and $\alpha_3\not\to 0$:
In this case, one of the simple poles t_{0i}^{η} is finite which corresponds to $\alpha_2+\eta\sqrt{T_4}\not\to 0$, and

the other one is infinite which corresponds to $\alpha_2 + \eta\sqrt{T_4} \to 0$. The later one can not be inside the closed contour of integration in the complex plan. So, one has to calculate only the residues at the finite poles which are given by

$$\text{Res}\left(N_{4,2}^{6,(2)}(x_i,z_i)\right)_{t=t_{0i}^\eta} = \frac{1}{H\tilde{K}_i}\frac{T^\eta}{S}\ln(S\,x^\eta + T^\eta) \tag{5.2.231}$$

$$\begin{aligned}\text{Res}\left(N_{4,2}^{6,(3)}(x_i,z_i)\right)_{t=t_{0i}^\eta} &= -\frac{1}{H\tilde{K}_i}\frac{V^\eta}{U^\eta}\left(\ln(U^\eta\,x^\eta + V^\eta) - \ln(V_0)\right)\\ &= -\frac{1}{H\tilde{K}_i}\frac{V^\eta}{U^\eta}\left(\ln(U^\eta/V^\eta\,x^\eta + 1) + \ln(V^\eta/V_0)\right)\end{aligned} \tag{5.2.232}$$

If U^η and V^η vanish simultaneously, the residue of $N_{4,2}^{6,(3)}$ is given in the second line in the right hand side of Eq. (5.2.232)[23].

-ii) If $\alpha_2 \to 0$ and $\alpha_3 \not\to 0$:

In this case, the two simple poles are dragged to infinity. So, no residue has to be added.

-iii) If $\alpha_2 \not\to 0$ and $\alpha_3 \to 0$:

In this case, the only pole which may be inside the closed contour is associated to a z^η of order zero ($z^\eta \sim -\alpha_3/\alpha_2 \sim \mathcal{O}(0)$) and the residue formulas in Eqs. (5.2.231, 5.2.232) are still valid.

-iv) If $\alpha_2 \to 0$ and $\alpha_3 \to 0$:

In this case, the two simple poles t_{0i}^η are finite. So, they might be inside the closed contour of integration, thus the residues are given by Eqs. (5.2.231, 5.2.232).

III) The residues in the Case $S\,,\bar{S} \to 0$ simultaneously

This configuration implies the vanishing of the following two quantities simultaneously

$$C + 2\,B\,\beta \to 0 \tag{5.2.233}$$

$$C\,H - 2B\,G \to 0 \tag{5.2.234}$$

i.e. the reduced Gram determinants $\det(G^{\{i\}})$ associated to the given sector vanishes (or becomes arbitrary small)[24]. This means that the primitives in Eq. (5.2.199) can lead to non-stable results if the zero of U (z_0) becomes infinite (since $V_0 \sim \frac{\det(\mathcal{S}^{\{i\}})}{\det(G^{\{i\}})} \to \infty$ for $\det(\mathcal{S}^{\{i\}}) \neq 0$). Let us re-write z_0 in term of $\det(\mathcal{S}^{\{i\}})$ and $\det(G^{\{i\}})$,

$$z_0 = -\frac{\bar{b}_1^{\{i\}} + \det(G^{\{i\}}) + \varepsilon_\beta E\sqrt{-\det(G^{\{i\}})}}{2B\,\varepsilon_\beta\sqrt{-\det(G^{\{i\}})}} \tag{5.2.235}$$

[23] In this case, the denominator of $N_{4,2}^{6,(3)}$ at the pole value is $SV^\eta - T^\eta U^\eta = \alpha_2 z^\eta + \alpha_3 = 0$. If $z^\eta \to z_0$, this implies that U^η and V^η vanish simultaneously (since $S \neq 0$). If z^η equal to the root of T, the V^η and T^η vanish simultaneously and the residue of $N_{4,2}^{6,(3)}$ is given by the first line in the right hand side of Eq. (5.2.232). If V^η, U^η and T^η vanish simultaneously, this formula should works.

[24] For example, Eqs. (5.2.233, 5.2.234) for the sector I are satisfied in the configuration: $s_2 = 0$, $s_3 = t$ and $s_1 = s$. If $m_1^2 = m_2^2$, these conditions will imply the vanishing of $\det(\mathcal{S}^{\{4\}})$ simultaneously with $\det(G^{\{4\}})$, $\det(G)$ and $\det(\mathcal{S})$ (and $\det(G^{\{3\}})$).

where $\bar{b}_1^{\{i\}}$ are defined in subsection B.3.2. From Eq. (5.1.58), $\bar{b}_1^{\{i\}}$ can be written as

$$\bar{b}_i^{\{j\}} = \pm\sqrt{\det(G^{\{i,j\}})\ \det(\mathcal{S}^{\{i\}}) + \det(\mathcal{S}^{\{i,j\}})\ \det(G^{\{i\}})} \tag{5.2.236}$$

where $\mathcal{S}^{\{i,j\}}$ is the matrix obtained by omitting the lines and the columns i and j, $\det(G^{\{i,j\}})$ is the Gram determinant associated this matrix. From Eq. (5.2.236), $\bar{b}_1^{\{i\}} \to 0$ if $\det(\mathcal{S}^{\{i\}}) \to 0$, which means that z_0 is finite (since $\det(\mathcal{S}^{\{i\}}) \to 0$ and $\det(G^{\{i\}}) \to 0$ vanish simultaneously) and the poles t_{0i} may be inside the integration contour. Otherwise, $\det(\mathcal{S}^{\{i\}}) \not\to 0$ which means that z_0 becomes infinite ($z_0 \sim \bar{b}_1^{\{i\}}/0$) and the poles t_{0i} are dragged to infinity. In the following, we calculate the residues for these two cases:

-a) $\det(G^{\{i\}}) \to 0$ *and* $\det(\mathcal{S}^{\{i\}}) \not\to 0$*:*
As we have mentioned above, the primitives in Eq. (5.2.199) should not be used since it leads to infinite results, it has to be replaced by,

$$\begin{aligned} N_{4,2}^{6}(x,z) = \frac{\tilde{C}_1}{U^2}\left\{U\,x - V\,\ln(U\,x+V)\right\} - x\,\frac{\tilde{C}_1\,T - C_0}{S\,V - T\,U}\,q_1(X) \\ -\frac{V}{U}\,\frac{\tilde{C}_1\,T - C_0}{S\,V - T\,U}\,\ln(U\,x+V) \end{aligned} \tag{5.2.237}$$

we call $N_{4,2}^{6,(1)}$, $N_{4,2}^{6,(2)}$ and $N_{4,2}^{6,(3)}$ the first, the second and the third parts of Eq. (5.2.237) (we keep the same notation as above).

The poles t_{0i} ($U = 0$) are infinite (they are outside the closed contour), then no residue has to be added at this poles.

The poles t_{0i}^{η} ($S\,V - T\,U = 0$) are arbitrary (they may be finite or infinite that depends on the values of α_2 and α_3):
-i) $\alpha_2 \not\to 0$ *and* $\alpha_3 \not\to 0$*:*
In this case, there is only one pole which can be inside the integration contour. This last one corresponds to $\alpha_2 + \eta\sqrt{T_4} \not\to 0$, i.e. $z^{\eta} \sim -J/H$ ($T^{\eta} \to 0$ and $U^{\eta} \not\to 0$)[25]. The residue at this pole are given by,

$$\mathrm{Res}\left(N_{4,2}^{6,(2)}(z_i)\right)_{t=t_{0i}^{\eta}} = \frac{1}{H\tilde{K}}\,\frac{T^{\eta}}{S}\,\ln(S/T^{\eta}\,x_i^{\eta}+1) \tag{5.2.238}$$

$$= \frac{1}{H\tilde{K}}\,\frac{V^{\eta}}{U^{\eta}}\,(\ln(U^{\eta}\,x_i^{\eta}+V^{\eta}) - \ln(V^{\eta})) \tag{5.2.239}$$

where Eqs. (5.2.238, 5.2.239) are completely equivalent.

$$\mathrm{Res}\left(N_{4,2}^{6,(3)}(z_i)\right)_{t=t_{0i}^{\eta}} = -\frac{1}{H\tilde{K}}\,\frac{V^{\eta}}{U^{\eta}}\,\ln(U^{\eta}\,x_i^{\eta}+V^{\eta}) \tag{5.2.240}$$

These equations lead to stable results. If $\Delta^{(1)}$ vanishes, i.e $H = 0$, one has to rotate the box by interchanging the labels of the adjacent propagators ($p_i \to p_{i+1}$ and $m_i \to m_{i+1}$

[25] The other pole is infinite, it corresponds to $\alpha_2 + \eta\sqrt{T_4} \to 0$, i.e. $z^{\eta} \sim \infty$, so no residue has to be added.

several times!), in such way that this quantity is transformed to a non vanishing $\Delta^{(i)}$. Then, z^η still finite and by consequence x_i^η are finite.
-ii) $\alpha_2 \to 0$ *and* $\alpha_3 \not\to 0$*:*
In this case, the two poles are infinite. Then, no residues are needed.
-iii) $\alpha_2 \not\to 0$ *and* $\alpha_3 \to 0$*:*
There is only one pole which may by inside the contour of integration, it corresponds to $z^\eta \sim \mathcal{O}(0)$. The residues in this case are given by Eqs. (5.2.238, 5.2.239, 5.2.240).
-iv) $\alpha_2 \to 0$ *and* $\alpha_3 \to 0$*:*
In this case, the two simple poles t_{0i}^η are finite. So, they might be inside the closed contour of integration, thus the residues are given by Eqs. (5.2.238, 5.2.239, 5.2.240).

* I notice that the β may be complex (if more than two external legs are space-like), then one has to check if the subtracted logarithm $\ln(T)$ gives no contribution to the final result since the residues are not zero by construction, otherwise our primitive given above is not valid: this can happen if $z^\eta \to -\frac{J}{H}$ is inside the triangle $[0, 1-\beta, -\beta]$. To correct this results, one has to subtract the contribution of the residue at this pole. Anyway, this configuration can not be encountered in the physical region.

-b) $\det(G^{(i)}) \to 0$ *and* $\det(\mathcal{S}^{(i)}) \to 0$:
In this case, the poles t_{0i} are finite which implies that V_0 is finite. Then, we have to use the primitive Eq. (5.2.199) to avoid the calculation of residues at the poles t_{0i} (the residues at these poles are zero by construction). The residues at t_{0i}^η are treated similarly as in paragraph **I-B)**.

Remarks:

One can construct others primitives with zero residues at all poles but we have found that they less convenient than the adopted one. These possibilities are:
1- For the first possibility, we can subtract some terms, which give no contribution, to construct a primitive with zero residue at $SV - TU = 0$ and $U = 0$. In this case, one has to use the same contour of integrations without any problem but the complication comes from introducing many new q_1 functions in the primitive.
2- The other possibly is to combine $N_{4,2}^{6,(2)}$ and $N_{4,2}^{6,(3)}$ without subtracting any term but we have to find a contour of integration which keep the imaginary part of the quadratic and the linear logarithm arguments negatives in the same time, which is not easy to find.

In conclusion, we have provided a stable one-dimensional integral representation for I_4^6, which is valid for all possible configurations in the physical region. This representation is given in Eq. (5.2.188), and the residues are provided in section 5.2.2.4. We notice that we have distinguished between three cases $S \to 0$, $\bar{S} \to 0$ and $S \to 0, \bar{S} \to 0$ simultaneously. For each of these cases, the integrand thence the residues are slightly modefied.

5.2.2.5 Numerical Test

Figs.(5.6, 5.8, 5.8). show the behavior of I_4^6 in the neighborhood of three numerical points by using the numerical and the analytical modes. These examples cover most possible cases

of the vanishing of Gram determinant, in term of S and $\bar{S}$. These points are:

Point I:

$$\begin{aligned}
&p_1^2 = -0.06636113657195236 && \det(\mathcal{S}) = 0.416 + i\,0.260\\
&p_2^2 = 0.2510540788267644 && \det(G) = 5.551\,10^{-17}\\
&p_3^2 = 0.4792850595783684 && \det(G^{\{1\}}) = -0.423\\
&p_4^2 = -0.20329972502291477 && \det(G^{\{2\}}) = -0.669\,10^{-5}\\
&s = 0.45857011915673696 && \det(G^{\{3\}}) = -0.142\\
&t = -0.5021081576535289 && \det(G^{\{4\}}) = -1.038\\
&m_1^2 = 0.022823098075160402 - i\,0.0500900358892\\
&m_2^2 = 0.045646196150320804 - i\,0.516469006659\\
&m_3^2 = 0.0684692942254812 - i\,0.700302469311\\
&m_4^2 = -1/2 - i\,0.476166102103889
\end{aligned} \tag{5.2.241}$$

Point II:

$$\begin{aligned}
&p_1^2 = 1.3757150015862212 && \det(\mathcal{S}) = 0.237 - i\,0.116\\
&p_2^2 = 1.1248280616367492 && \det(G) = 3.796\,10^{-14}\\
&p_3^2 = 0.38591856085041676 && \det(G^{\{1\}}) = -1.071\\
&p_4^2 = 0.23611461561031238 && \det(G^{\{2\}}) = -0.719\\
&s = -0.5761976737947683 && \det(G^{\{3\}}) = -3.277\\
&t = 0.19100810103734925 && \det(G^{\{4\}}) = -0.005\\
&m_1^2 = 0.39977675350726477 - i\,0.007144284265522\\
&m_2^2 = 0.3218142823922797 - i\,0.07366338099780\\
&m_3^2 = 0.37158645281693164 - i\,0.0998833365514\\
&m_4^2 = 0.09191508922087456 - i\,0.0679150240290
\end{aligned} \tag{5.2.242}$$

Point III:

$$\begin{aligned}
&p_1^2 = 1.3757150015862212 && \det(\mathcal{S}) = 0.0220 + i\,0.031\\
&p_2^2 = 0.03392179464215907 && \det(G) = 1.679\,10^{-16}\\
&p_3^2 = 0.38591856085041676 && \det(G^{\{1\}}) = -1.454\\
&p_4^2 = 0.23611461561031238 && \det(G^{\{2\}}) = -0.719\\
&s = -0.5761976737947683 && \det(G^{\{3\}}) = -0.128\\
&t = 0.19100810103734925 && \det(G^{\{4\}}) = 5.551\,10^{-17}\\
&m_1^2 = 0.39977675350726477 - i\,0.0071442842655\\
&m_2^2 = 0.3218142823922797 - i\,0.073663380997809\\
&m_3^2 = 0.37158645281693164 - i\,0.09988333655149\\
&m_4^2 = 0.09191508922087456 - i\,0.067915024029018
\end{aligned} \tag{5.2.243}$$

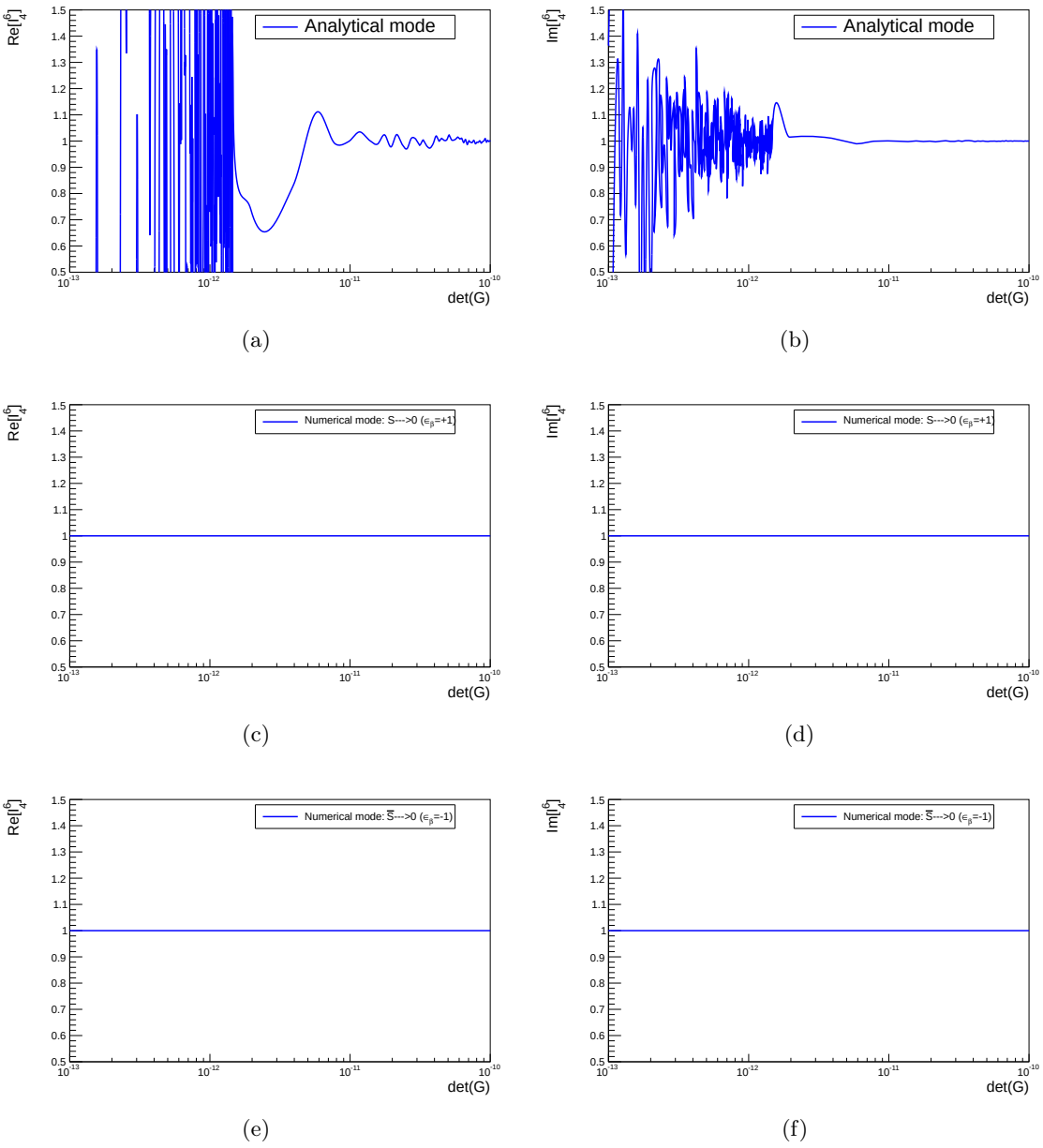


Figure 5.6: Comparison between the analytical and the numerical modes for a numerical point where all the internal masses are complex: (a) and (b) show respectively the real and the imaginary parts of $I_4^6(S)$ evaluated analytically in the region where $\det(G) \to 0$, which are not stable. However the numerical mode gives stable results, which is shown in the plots (c), (d), (e) and (f) (for the plots (c) and (d) $S \to 0$ (**for all sectors**), and for the plots (e) and (f) $\bar{S} \to 0$ (**for all sectors**), which corresponds respectively to $\varepsilon_\beta = +1$ and $\varepsilon_\beta = -1$).

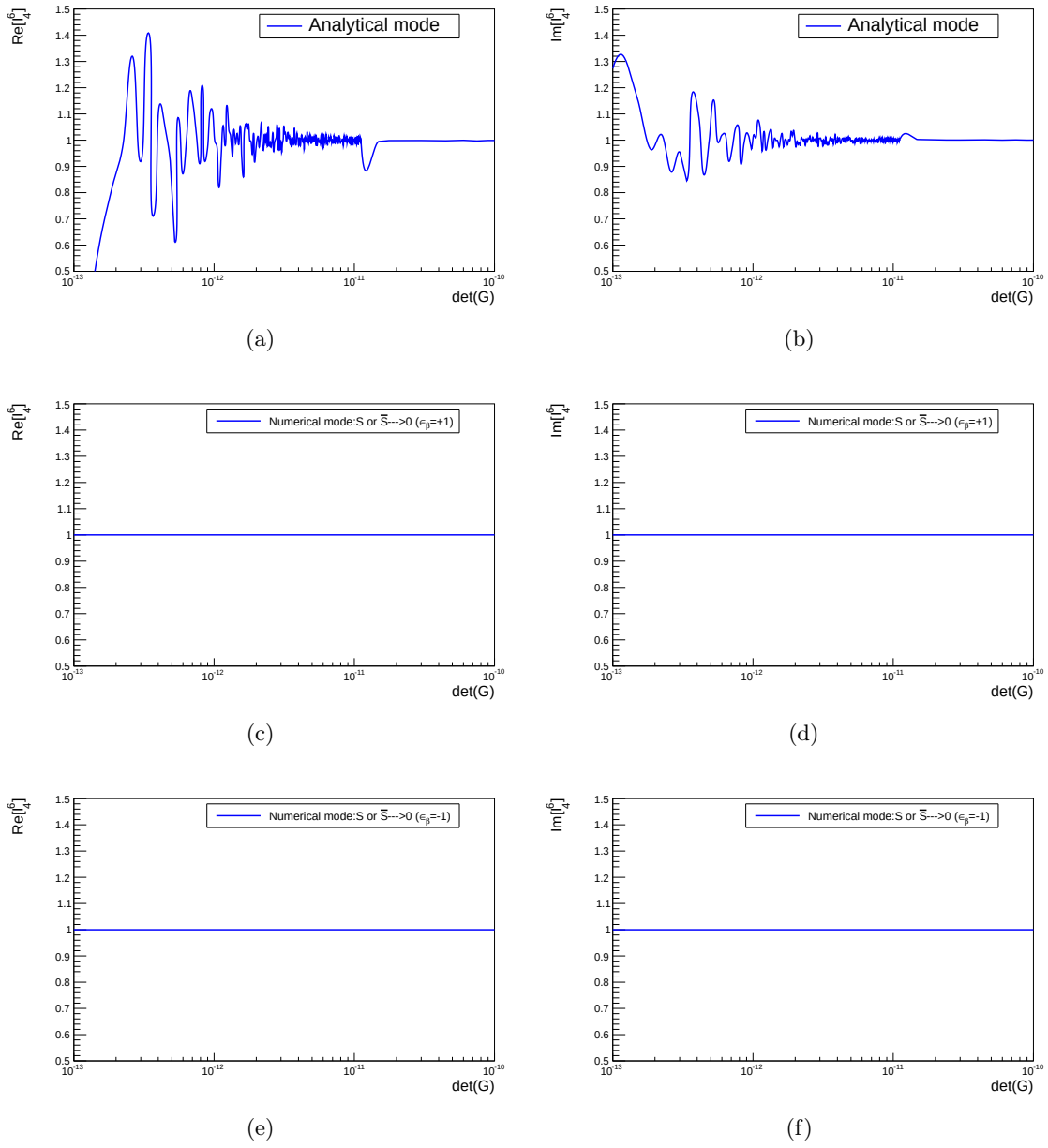


Figure 5.7: Comparison between the analytical and the numerical modes for a numerical point where all the internal masses are complex: (a) and (b) show respectively the real and the imaginary parts of $I_4^6(S)$ evaluated analytically in the region where $\det(G) \to 0$, which are not stable. However the numerical mode gives stable results, which is shown in the plots (c), (d), (e) and (f) (for the plots (c) and (d) $S \to 0$ (**for sector I**), and for the plots (e) and (f) $\bar{S} \to 0$ (**for sector II and III**), which corresponds respectively to $\varepsilon_\beta = +1$ and $\varepsilon_\beta = -1$).

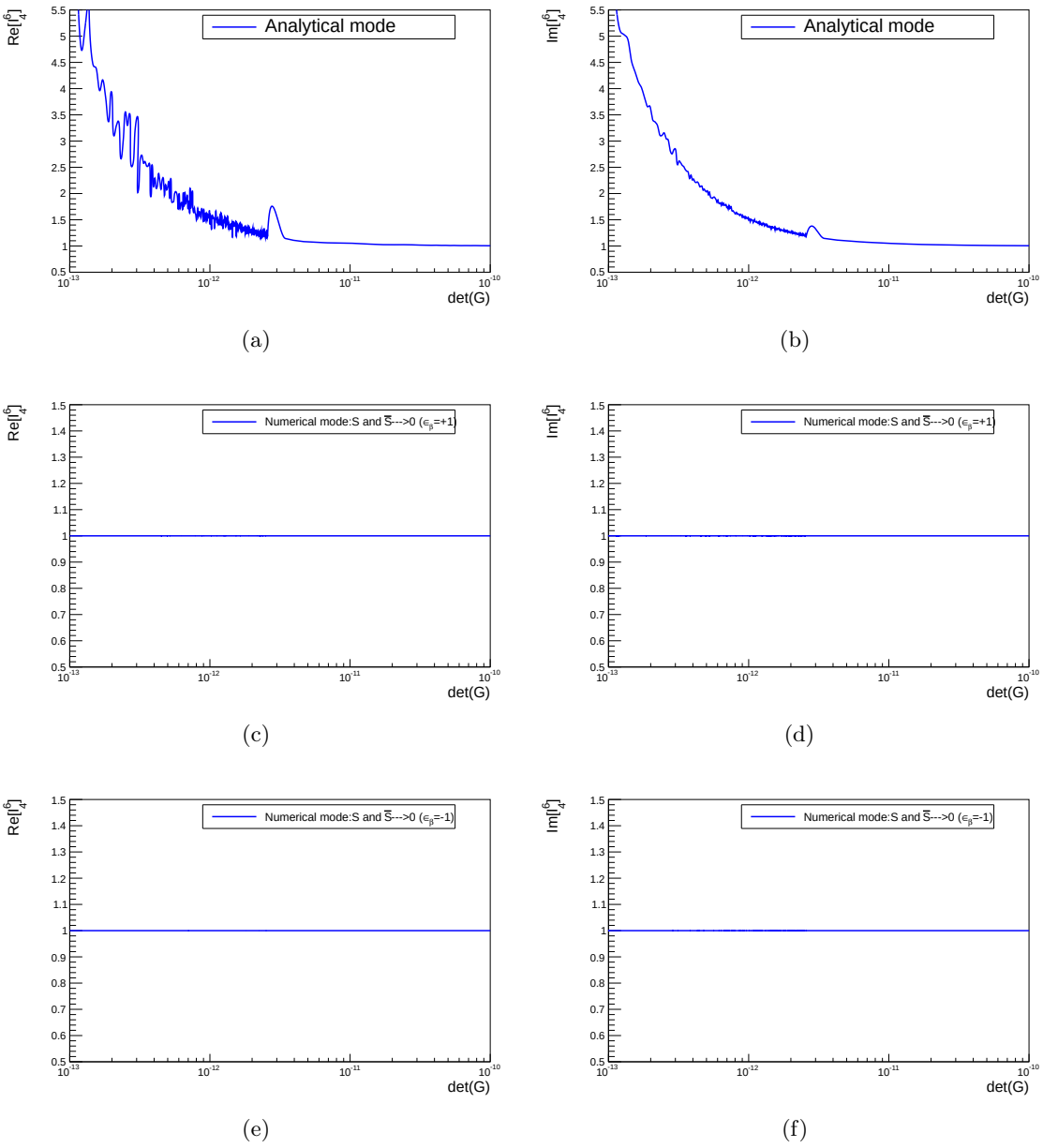


Figure 5.8: Comparison between the analytical and the numerical modes for a numerical point where all the internal masses are complex: (a) and (b) show respectively the real and the imaginary part of $I_4^6(S)$ evaluated analytically in the region where $\det(G) \to 0$, which are not stable. However the numerical mode gives stable results, which are shown in the plots (c), (d), (e) and (f): for the plots (c) and (d), $S \to 0$ and $\bar{S} \to 0$ for all sectors with $\varepsilon_\beta = +1$), and for the plots (e) and (f), $S \to 0$ and $\bar{S} \to 0$ (for all sectors with $\varepsilon_\beta = -1$).

5.2.3 The box in 8-dimensions

The ultraviolet divergent and the finite contributions of the scalar box in "$8-2\,\varepsilon$", presented in Eq. (5.2.124), are given by

$$I_4^{8,\,\mathrm{div}}(S) = \frac{1}{6}\,\frac{1}{\varepsilon} \tag{5.2.244}$$

$$I_4^{8,\,\mathrm{fin}}(S) = -\int_0^1 dx\, x^2 \int_0^1 dy' \int_{-\alpha\, y'}^{(1-\alpha)\, y'} dz''\, \ln(\mathcal{C}_1(x,z'')\, y' + \mathcal{C}_0(x,z'')) \tag{5.2.245}$$

where the functions $\mathcal{C}_1$ and $\mathcal{C}_0$ are defined in the previous section. The logarithm's argument in Eq. (5.2.245) is a linear function on y'. Since the parameter α is real, the logarithm never crosses its branch cut by varying y'. So, we can reverse the order of integration on y' and z'', and integrate I_4^8 over y'. The primitive of this function on respect to y' is

$$-y' + \frac{(\mathcal{C}_1(x,z'')\, y' + \mathcal{C}_0(x,z''))\, \ln(\mathcal{C}_1(x,z'')\, y' + \mathcal{C}_0(x,z''))}{\mathcal{C}_1(x,z'')} \tag{5.2.246}$$

We make the same changes of variables performed to calculate $I_4^6(S)$ (see section 5.2.2), we get

$$I_4^{8,fin}(S) = \mathcal{I}_4^8 - I_4^8 \tag{5.2.247}$$

$$\tag{5.2.248}$$

with

$$\mathcal{I}_4^8 = \frac{1}{6} \tag{5.2.249}$$

$$I_4^8 = \sum_{i=1}^{III} \int_0^1 dx \int_0^x dz\, \frac{K\,(A\,x^2 + B\,z^2 + C\,x\,z + D\,x + E\,z + F)}{G\,x + H\,z + J - i\,\lambda} \times \ln\left(A\,x^2 + B\,z^2 + C\,x\,z + D\,x + E\,z + F - i\,\lambda\right) \tag{5.2.250}$$

The contribution "$\mathcal{I}_4^8$" is obtained by a direct integration of "y'" in the left hand side of Eq.(5.2.246) over the two remaining variables of integration. The term I_4^8 is of type I_4^6, the only difference is the quadratic function in front of the logarithm, which does not change the analyticity of I_4^8 compared to I_4^6, since they have the same type of poles and logarithms. Hence, we can introduce safely the same parameters γ and β as before. Let us make the shift $z = z + \gamma\, x$ and chose γ in such way that $H\,\gamma + G = 0$ to eliminate the x-dependance in the denominator, the result is

$$I_4^8 = \sum_{i=1}^{III} \int_0^1 dx \int_{-\gamma\, x}^{(1-\gamma)\, x} dz\, \frac{K\,(W_2\,x^2 + W_1\,x + W_0)}{H\,z + J - i\,\lambda}\, \ln\left(W_2\,x^2 + W_1\,x + W_0 - i\,\lambda\right) \tag{5.2.251}$$

by inverting the order of integration as in Eq. (5.2.137) and integrating by part over x, one can write the primitive of this integral on respect to x in the following form

$$\begin{aligned}\int dx\, F_0^8(x,z) = \frac{K}{H\,z+J}\Bigg\{ & \\ & \left(W_2\,\frac{x^3}{3}+W_1\,\frac{x^2}{2}+W_0\,x\right)\ln\Big(W_2\,x^2+W_1\,x+W_0\Big) \\ & -\frac{2}{3}\,W_2\,x^3-\frac{1}{2}\,W_1\,x^2 \\ & +\int dx\left(\frac{2}{3}\,W_2\,x^3+\frac{1}{2}\,W_1\,x^2\right)\frac{2\,W_2\,x+W_1}{W_2\,x^2+W_1\,x+W_0}\Bigg\}\end{aligned} \tag{5.2.252}$$

where F_0^8 is the integrand of Eq. (5.2.251).
Inserting Eq. (5.2.252) in Eq. (5.2.251), we write

$$I_4^8 = I_{4,1}^8 + I_{4,2}^8 \tag{5.2.253}$$

where $I_{4,1}^8$ comes from the integrated quantity on x in Eq.(5.2.252) and $I_{4,2}^8$ from the non integrated one.

After making the three change of variables: "$z=t-\gamma$", "$(1-\gamma)\,t$" and "$-\gamma\,t$" respectively, $I_{4,1}^8$ can be writen

$$I_{4,1}^8 = \sum_{i=I}^{III}\int_0^1 dt[N_{4,1}^8(1,t-\gamma)-(1-\gamma)\,N_{4,1}^8(t,(1-\gamma)\,t)+(-\gamma)\,N_{4,1}^8(t,-\gamma\,t)] \tag{5.2.254}$$

$$\begin{aligned}N_{4,1}^8(x,z) = \frac{K}{H\,z+J}\Bigg\{ & \left(W_2\,\frac{x^3}{3}+W_1\,\frac{x^2}{2}+W_0\,x\right)\ln\Big(W_2\,x^2+W_1\,x+W_0\Big) \\ & -\frac{2}{3}\,W_2\,x^3-\frac{1}{2}\,W_1 x^2\Bigg\}\end{aligned} \tag{5.2.255}$$

$I_{4,2}^8$ is a two-dimensional integral representation, its integrand is given by the last line in the right hand side of Eq. (5.2.252). By re-introducing the x-dependance in its denominator (by making the shift $z=z+\gamma\,x$), we get

$$I_{4,2}^8 = \sum_{i=I}^{III}\int_0^1 dx\int_0^x dz\frac{K}{G\,x+H\,z+J}\frac{Q(x,z)}{A\,x^2+B\,z^2+C\,x\,z+D\,x+E\,z+F-i\,\lambda} \tag{5.2.256}$$

The function $Q(x,z)$ is a polynomial of degrees 4 in x and 2 in z. By making the shift $z=z+\beta\,x$, $I_{4,2}^8$ becomes

$$I_{4,2}^8 = \sum_{i=I}^{III} K\int_0^1 dx\int_{-\beta\,x}^{(1-\beta)\,x} d\bar{z}\,\frac{1}{S\,x+T}\,\frac{Q(x,z+\beta\,x)}{U\,x+V} \tag{5.2.257}$$

$$\begin{aligned}Q(x,z+\beta\,x) &= x^2\,[C_2\,x^2+C_1\,x+C_0] \\ &= x^2\,[C_a\,x^2+(C_c\,z+C_d)\,x+C_b\,z^2+C_e\,z+C_f]\end{aligned} \tag{5.2.258}$$

$$
\begin{aligned}
C_a &= S^2\,(2\,B\,C\,G - 8\,A\,B\,H + C^2\,H + \beta(4\,B^2\,G - 2\,B\,C\,H))/(6\,H^3)\\
C_b &= (C\,H - 2\,B\,G)^2/(2\,H^2)\\
C_c &= S\,(2\,B\,G - C\,H)\,(B\,G + 5\,B\,\beta\,H + 2\,C\,H)/(3\,H^3)\\
C_d &= S\,(B\,G + 5\,B\,\beta\,H + 2\,C\,H)(E\,G - D\,H)/(3\,H^3)\\
C_e &= (C\,H - 2\,B\,G)\,(D\,H - E\,G)/H^2\\
C_f &= (E\,G - D\,H)^2/(2\,H^2)
\end{aligned}
\tag{5.2.259}
$$

the S, V, T and U are given in the previous section.
We reverse the order of integration in the integral Eq.(5.2.257) on respect to x and z (as in Eq. (5.2.154)), and we integrate over x. The primitive is

$$
\begin{aligned}
N^8_{4,2}(x,z) &= \int dx\,\frac{Q(x,z+\beta\,x)}{S\,V - T\,U}\left(\frac{S}{S\,x+T} - \frac{U}{U\,x+V}\right)\\
&= \frac{\tilde{C}_2\,S}{U}\left(\frac{1}{3}\,x^3 - \frac{1}{2}\,\frac{V\,x^2}{U} + \frac{V^2\,x}{U^2} - \frac{V^3}{U^3}\,\ln(U\,x+V)\right)\\
&- \frac{\tilde{C}_2\,T - \tilde{C}_1}{U}\left(\frac{1}{2}\,x^2 - \frac{V\,x}{U} + \frac{V^2}{U^2}\,\ln(U\,x+V)\right)\\
&+ \frac{(C_0 - \tilde{C}_1\,T + \tilde{C}_2\,T^2)\,V}{U\,(S\,V - T\,U)}\left(x - \frac{V}{U}\,\ln(U\,x+V)\right)\\
&+ \frac{T^2}{S^2}\,\frac{(C_0 - \tilde{C}_1\,T + \tilde{C}_2\,T^2)}{S\,V - T\,U}\left(\ln(S\,x+T) - \frac{S\,x}{T}\right)
\end{aligned}
\tag{5.2.260}
$$

with $C_2 = S^2\,\tilde{C}_2$ and $C_1 = S\,\tilde{C}_1$.

In Eq. (5.2.260), only the last two terms (in the last line) in the right hand side, which are proportional to inverse of S^2 and S, respectively. The remaining terms are free of inverse of any S. In addition, this primitive is completely free of inverse of $\bar{S}$ (where $\det(G) \propto S\,\bar{S}$). At this stage, we have a one-dimentional integral representation of $I^8_{4,2}$ which has a problem when S becomes arbitrary small. This difficulty can be simply avoided by adding some non-contributing! terms to this primitive as we will show later on. But before to show that, we want to point out how the inverse of Gram determinant, technically!, appears during the calculation. To be able to integrate analytically Eq. (5.2.260), one has to arrange the formula by reducing all the complicated denominators of this function to simple elements. One gets

$$
\begin{aligned}
N^8_{4,2}(x,z) &= \frac{\theta_3}{U^3} + \frac{\theta_2}{U^2} + \frac{\theta_1}{U} + (\gamma_2\,z^2 + \gamma_1\,z + \gamma_0)\,\ln(S\,x+T)\\
&- \left(\frac{\omega_4}{U^4} + \frac{\omega_3}{U^3} + \frac{\omega_2}{U^2} + \frac{\omega_1}{U} + \zeta_2\,z^2 + \zeta_1\,z + \zeta_0\right)\ln(U\,x+V)\\
&+ \frac{1}{12}\,\frac{1}{T_1^2}\sum_{\eta=\pm}\frac{\eta}{\sqrt{T_4}}\,\frac{T_4^2 - \eta\,T_2\,(6\,T_1\,T_3 - T_2^2)\,\sqrt{T_4}/H^3}{z - z^\eta}
\end{aligned}
\tag{5.2.261}
$$

The coefficients θ_1, γ_1, γ_0, ω_1 and ω_0 are proportional to inverse of S [26]. On top of that, the

[26] We did not give the explicit form of the primitive Eq. (5.2.261) since we will not use it to derive the one-dimensional integral representation.

last term in Eq.(5.2.261) is proportional to $\det(G)^2$ ($T_1 \propto \det(G)$). These spurious singularities are very difficult to handle if we keep this primitive, which makes it not convenient candidate to derive a stable one-dimensional integral representation of $I^8_{4,2}$. So, we keep the primitive Eq. (5.2.260).
To avoid the inverse of S in the last two terms of this primitive (Eq.(5.2.260)), we re-write it as the following: from the term containing $\ln(S\,x+T)$, we subtract $\ln(T)$ which give no contribution to the final result as explained above (see section 5.2.2.2). Then, the dangerous term of this primitive can be written as

$$\frac{T^2}{S^2}\,\frac{(C_0-\tilde{C}_1\,T+\tilde{C}_2\,T^2)}{S\,V-T\,U}\left(\ln(S\,x+T)-\ln(T)-\frac{S\,x}{T}\right)=\frac{x^2\,(C_0-\tilde{C}_1\,T+\tilde{C}_2\,T^2)}{S\,V-T\,U}\,q_2(X) \tag{5.2.262}$$

$$q_2(X)=\begin{cases}\frac{1}{X^2}\,(\ln(1-X)+X) & \text{if} \quad X\not\to 0\\ -\sum_{n=0}^{\infty}\frac{X^n}{n+2} & \text{if} \quad X\to 0\end{cases} \tag{5.2.263}$$

where $X=-Sx/T$. This implies that the new primitive leads, in principle, to finite results when $\det(G)$ becomes arbitrary small, since the expansion of the dangerous term around $X\to 0$ is free of inverse of S.

$$I^8_{4,2}=\sum_{i=I}^{III}K\int_0^1 dt[N^8_{4,2}(1,t-\beta)-(1-\beta)\,N^8_{4,2}(t,(1-\beta)\,t)-\beta\,N^8_{4,2}(t,-\beta\,t)] \tag{5.2.264}$$

and

$$\begin{aligned}I^8_4=&\frac{1}{6}-\sum_{i=I}^{III}\int_0^1 dt\Big\{\\ &N^8_{4,1}(1,t-\gamma)-(1-\gamma)\,N^8_{4,1}(t,(1-\gamma)\,t)-\gamma\,N^8_{4,1}(t,-\gamma\,t)\\ &+N^8_{4,2}(1,t-\beta)-(1-\beta)\,N^8_{4,2}(t,(1-\beta)\,t)-\beta\,N^8_{4,2}(t,-\beta\,t)\Big\}\end{aligned} \tag{5.2.265}$$

In principe, Eq.(5.2.265) provides a stable one-dimensional integral representation of the function I^8_4, since the Gram determinant spurious singularities are avoided at the integrand level. However, if a pole (or more) of the integrand approaches the segment $[0,1]$, this representation may not be any more numerically stable. Then, we have to perform a contour deformation and calculate all the needed residues as in the previous section.

5.2.3.1 Residue calculation

The possible poles of I^8_4 correspond to: i) a simple pole at $T=0$ in the primitive $N^8_{4,1}$, ii) two simple poles at $S\,V-T\,U=0$ in the primitive $N^8_{4,2}$, iii) and a multiple pole at $U=0$ (of order 1, 2, 3 and 4) in the primitive $N^8_{4,2}$.
Depending on the nature of the internal masses and the kinematical configurations, these poles can have very small imaginary parts, which might hamper the numerical stability if they approach the segment $[0,1]$. To avoid this problem, we perform a contour deformation (we use the same contours introduced in section 5.2.2). Then, the analytical formulas of the

residues at each of these poles are strongly needed, which is the purpose of this paragraph. To simplify the residue calculation of $N^8_{4,2}$ and to guaranty that the residues are not proportional to inverse of S, one can add or subtract some x-independent quantities from $N^8_{4,2}$, which give no conurbation to the final result as shown above [27]. Then, from some terms of $N^8_{4,2}$, we subtract terms as $(...)\sum_{n=1}^{m}(-1)^{m-n}\frac{V^{n-m}}{n}(U^n x^n + V^n - V_0^n)$ (m is and integer which equals to the order of the multiple pole corresponding to $U=0$ mines 1, and $(...)$ stands for some coefficients) and from each $\ln(U x+V)$ we subtract $\ln(V_0)$. Hence, the new primitive takes this very convenient form

$$
\begin{aligned}
N^8_{4,2} = {} & \frac{\tilde{C}_2 S}{U^4}\left\{\sum_{n=1}^{3}(-1)^{1+n}V^{3-n}\frac{U^n x^n+(-1)^{1+n}(V-V_0)^n}{n} - V^3(\ln(Ux+V)-\ln V_0)\right\}+ \\
& \frac{\tilde{C}_2 T-\tilde{C}_1}{U^3}\left\{\sum_{n=1}^{2}(-1)^{1+n}V^{2-n}\frac{U^n x^n+(-1)^{1+n}(V-V_0)^n}{n} - V^2(\ln(Ux+V)-\ln V_0)\right\} \\
& + \frac{(C_0-\tilde{C}_1 T+\tilde{C}_2 T^2)V}{U^2(SV-TU)}\left\{Ux+V-V_0 - V(\ln(Ux+V)-\ln V_0)\right\} \\
& + \frac{x^2\,(C_0-\tilde{C}_1\,T+\tilde{C}_2\,T^2)}{S\,V-T\,U}\,q_2(X)
\end{aligned}
\tag{5.2.266}
$$

This new primitive has three principle advantages:
i) it provides a stable one-dimensional integral representation of $I^8_{4,2}$ if $S\to 0$ (then $\det(G)\to 0$), by means of the q_2
ii) the residues at the poles corresponding to $U=0$ are equal to zero by construction. This can proved by a direct application of residue formula Eq.(5.2.198)) to

$$
\begin{aligned}
& \frac{f(z)}{U^{m+1}(S\,V-T\,U)^{l=0,1}} \\
\times & \left(\sum_{n=1}^{m}(-1)^{1+n}V^{m-n}\frac{U^n x^n+(-1)^{1+n}(V-V_0)^n}{n} - V^m(\ln(Ux+V)-\ln V_0)\right)
\end{aligned}
\tag{5.2.267}
$$

where $f(z)$ is a polynomial in z.
iii) the residues at the poles corresponding to $S\,V-T\,U$ are finite when $S\to 0$, as we will see later on (if we keep the old primitive, the residues will be proportional to inverse of S). Let us call the first, the second, the third and the fourth parts of this primitive $N^{8,(i)}_{4,2}$ with

[27] This can be seen before making the shifts $t-\beta$, $(1-\beta)\,t$ and $-\beta\,t$. At this level, the added (or subtracted) quantity form the same integrand of each integral of the three terms and because β is real (or have a light imaginary part), we can combine the three integrals into a one and prove that the integrand vanishes . This trick helps us to construct some part of the primitive with zero residue at the pole or with residues free of inverse of S.

$i = 1, \cdots, 4$, respectively. Then,

$$
\begin{aligned}
N_{4,2}^{8,(1)} &= \frac{\tilde{C}_2 S}{U^4}\Bigg\{\sum_{n=1}^{3}(-1)^{1+n}V^{3-n}\frac{U^n x^n + (-1)^{1+n}(V-V_0)^n}{n} - V^3(\ln(Ux+V) - \ln V_0)\Bigg\} \\
N_{4,2}^{8,(2)} &= \frac{\tilde{C}_2 T - \tilde{C}_1}{U^3}\Bigg\{ \\
&\sum_{n=1}^{2}(-1)^{1+n}V^{2-n}\frac{U^n x^n + (-1)^{1+n}(V-V_0)^n}{n} - V^2(\ln(Ux+V) - \ln V_0)\Bigg\} \\
N_{4,2}^{8,(3)} &= \frac{(C_0 - \tilde{C}_1 T + \tilde{C}_2 T^2)V}{U^2(SV-TU)}\Bigg\{Ux + V - V_0 - V(\ln(Ux+V) - \ln V_0)\Bigg\} \\
N_{4,2}^{8,(4)} &= \frac{x^2\,(C_0 - \tilde{C}_1\,T + \tilde{C}_2\,T^2)}{SV - TU}\,q_2(X)
\end{aligned}
\tag{5.2.268}
$$

As we mentioned above, the residues at t_{0i} are equal to zero. So, it remains to calculate only the residues at t_{0i}^{η} for the cases where $S \to 0$, $\bar{S} \to 0$ and the two in the same time.

I) The residues in the case $S \to 0$:

* The residue of $N_{4,2}^{8,(3)}$ at t_{0i}^{η} is given by:

-i) If $U^\eta \not\to 0$ and $T^\eta \to 0$,

$$
\begin{aligned}
\mathrm{Res}\Big(N_{4,2}^{8,(3)}\Big)_{t=t_{0i}^{\eta}} &= \frac{1}{\tilde{K}}\,\frac{(C_0^{\eta} - \tilde{C}_1^{\eta}T^{\eta} + \tilde{C}_2 T^{\eta^2})V^{\eta}}{U^{\eta^2}\,\eta\sqrt{T_4}}\Bigg\{U^{\eta}x^{\eta} + V^{\eta} - V_0 \\
&\quad - V^{\eta}(\ln(U^{\eta}x^{\eta} + V^{\eta}) - \ln V_0)\Bigg\}
\end{aligned}
\tag{5.2.269}
$$

-ii) If $U^\eta \to 0$ and $T^\eta \not\to 0$,

$$
\mathrm{Res}\Big(N_{4,2}^{8,(3)}\Big)_{t=t_{0i}^{\eta}} = \frac{1}{\tilde{K}}\,\frac{(C_0^{\eta} - \tilde{C}_1^{\eta}\,T^{\eta} + \tilde{C}_2\,T^{\eta^2})}{\eta\,\sqrt{T_4}}\,\Big\{-x^{\eta 2}\,q_2(X^{\eta}) + T^{\eta 2}\,\tilde{Z}^{\eta 2}\,q_2(Z^{\eta})\Big\}
\tag{5.2.270}
$$

where the quantities C_0^{η} and $\tilde{C}_1^{\eta}$ correspond, respectively, to C_0 and $\tilde{C}_1$ taken at the pole value, q_2 is defined above and

$$
\begin{aligned}
Z^{\eta} &= \frac{V^{\eta} - V_0}{V^{\eta}} \\
&= S\,\tilde{Z}^{\eta} \qquad\qquad \text{with} \qquad \tilde{Z}^{\eta} = -\frac{V_0}{V^{\eta}}\,\tilde{Y}^{\eta}
\end{aligned}
\tag{5.2.271}
$$

where $\tilde{Y}^{\eta}$ is defined in Eq. (5.2.226).
-iii) If $U^\eta \to 0$ and $T^\eta \to 0$,

$$\text{Res}\left(N_{4,2}^{8,(3)}\right)_{t=t_{0i}^{\eta}} = \frac{1}{\tilde{K}} \frac{(C_0^{\eta} - \tilde{C}_1^{\eta}\, T^{\eta} + \tilde{C}_2\, T^{\eta^2})}{\eta\, \sqrt{T_4}} \left\{ -\frac{T^{\eta^2}}{S^2} (\ln(S/T^{\eta} x^{\eta} + 1) - S/T^{\eta} x^{\eta}) + T^{\eta 2}\, \tilde{Z}^{\eta 2}\, q_2(Z^{\eta}) \right\} \tag{5.2.272}$$

In this case, $T_4 \to 0$ (see section 5.2.2.4). Since C_0^{η} for $S \to 0$ is proportional to $\sqrt{T_4}$, then the $\sqrt{T_4}$ in the denominator of 5.2.272 is compensated by C_0^{η} and Eq. (5.2.272) is numerically stable.

* The residue of $\mathcal{N}_{4,2}^{8,(4)}$ at the pole t_{0i}^{η} is given by:

-i) If $U^{\eta} \to 0$ and $T^{\eta} \not\to 0$,

$$\text{Res}\left(N_{4,2}^{8,(4)}\right)_{t=t_{0i}^{\eta}} = \frac{1}{\tilde{K}} \frac{(C_0^{\eta} - \tilde{C}_1^{\eta}\, T^{\eta} + \tilde{C}_2\, T^{\eta^2})\, x^{\eta^2}}{\eta\, \sqrt{T_4}}\, q_2(X_i^{\eta}) \tag{5.2.273}$$

This formula provides stables results for $S \to 0$. If $S \to 0$ and $T^{\eta} \to 0$, then $X_i^{\eta} \not\to 0$ so q_2 has to be expressed in its logarithmic form, which correspond to the second case.

-ii) If $U^{\eta} \not\to 0$ and $T^{\eta} \to 0$,

$$\text{Res}\left(N_{4,2}^{8,(4)}\right)_{t=t_{0i}^{\eta}} = \frac{1}{\tilde{K}} \frac{(C_0^{\eta} - \tilde{C}_1^{\eta}\, T^{\eta} + \tilde{C}_2\, T^{\eta^2})}{\eta\, \sqrt{T_4}} \frac{V^{\eta}}{U^{\eta}} \left(\frac{V^{\eta}}{U^{\eta}} (\ln(U^{\eta}/V^{\eta} x^{\eta} + 1) - x^{\eta} \right) \tag{5.2.274}$$

iii) If $U^{\eta} \to 0$ and $T^{\eta} \to 0$ simultaneously

$$\text{Res}\left(N_{4,2}^{8,(4)}\right)_{t=t_{0i}^{\eta}} = \frac{1}{\tilde{K}} \frac{(C_0^{\eta} - \tilde{C}_1^{\eta}\, T^{\eta} + \tilde{C}_2\, T^{\eta^2})}{\eta\, \sqrt{T_4}} \frac{T^{\eta}}{S} \left(\frac{T^{\eta}}{S} (\ln(S/T^{\eta} x^{\eta} + 1) - x^{\eta} \right) \tag{5.2.275}$$

Again, $\sqrt{T_4}$ in the denominator is compensated by C_0^{η} in the numerator, where $C_0^{\eta} \propto \sqrt{T_4}$ for $S \to 0$.

II) The residues in the case $\bar{S} \to 0$:

In this case, we don't need to subtract $\ln(T)$ for two reasons from $\ln(Sx + T)$: a) we don't need to construct the function q_2 in the primitive, since $S \not\to 0$. b) if we keep this term, the contour of integration $\mathcal{C}_2$ might cross the branch cut of $\ln(T)$. Then, we write

$$N_{4,2}^{8,(4)} = \frac{T^2 (C_0 - \tilde{C}_1\, T + \tilde{C}_2\, T^2)}{S^2 (SV - TU)} \left(\ln(Sx + T) - \frac{Sx}{T} \right) \tag{5.2.276}$$

and we keep the other terms ($N_{4,2}^{8,(i)}$ for $i = 1, 2, 3$) as in Eq. (5.2.268).
As in the case of I_4^6, we distinguish between four cases (see above):

-i) If $\alpha_2 \not\to 0$ and $\alpha_3 \not\to 0$:
In this case, one of the simple poles t_{0i}^{η} is finite which corresponds to $\alpha_2 + \eta\sqrt{T_4} \not\to 0$, and the other one is infinite which corresponds to $\alpha_2 + \eta\sqrt{T_4} \to 0$. The later one can not be inside the closed contour of integration. So, one has to calculate only the residues at the finite poles which are given by

$$\begin{aligned}\mathrm{Res}\left(N_{4,2}^{8,(3)}\right)_{t=t_{0i}^{\eta}} &= \frac{1}{\tilde{K}}\,\frac{(C_0^{\eta} - \tilde{C}_1^{\eta}T^{\eta} + \tilde{C}_2 T^{\eta^2})V^{\eta}}{U^{\eta^2}\,\eta\sqrt{T_4}}\Big\{U^{\eta}x^{\eta} + V^{\eta} - V_0 \\ &\quad - V^{\eta}(\ln(U^{\eta}x^{\eta} + V^{\eta}) - \ln V_0)\Big\} \qquad (5.2.277)\end{aligned}$$

$$\mathrm{Res}\left(N_{4,2}^{8,(4)}(x_i, z_i)\right)_{t=t_{0i}^{\eta}} = \frac{1}{\tilde{K}}\,\frac{(C_0^{\eta} - \tilde{C}_1^{\eta}\,T^{\eta} + \tilde{C}_2\,T^{\eta^2})}{\eta\,\sqrt{T_4}}\,\frac{T^{\eta}}{S}\left(\frac{T^{\eta}}{S}\,(\ln(Sx^{\eta} + T^{\eta}) - x^{\eta}\right) \qquad (5.2.278)$$

-ii) If $\alpha_2 \to 0$ and $\alpha_3 \not\to 0$:
In this case, the two simple poles are dragged to infinity. So, no residue has to be added.
-iii) If $\alpha_2 \not\to 0$ and $\alpha_3 \to 0$:
In this case, the only pole which may be inside the closed contour is associated to a z^{η} of order zero ($z^{\eta} \sim -\alpha_3/\alpha_2 \sim \mathcal{O}(0)$) and the residue formulas in Eqs. (5.2.277, 5.2.278) are still valid.
-iv) If $\alpha_2 \to 0$ and $\alpha_3 \to 0$:
In this case, the two simple poles t_{0i}^{η} are finite. So, they might be inside the closed contour of integration, thus the residues are given by Eqs. (5.2.277, 5.2.278).

III) The residues in the Case $S\,, \bar{S} \to 0$ simultaneously

This configuration implies the vanishing of these two quantities simultaneously: $C + 2\,B\,\beta \to 0$ and $C\,H - 2B\,G \to 0$, i.e. the reduced Gram determinants $\det(G^{\{i\}}) \to 0$. This means that the primitives in Eq. (5.2.266) leads to non-stable results if the zero of U becomes infinite (since $V_0 \sim \frac{\det(\mathcal{S}^{\{i\}})}{\det(G^{\{i\}})} \to \infty$ for $\det(\mathcal{S}^{\{i\}}) \neq 0$). We distinguish between two cases as in the case of I_4^6.

-a) $\det(G^{\{i\}}) \to 0$ and $\det(\mathcal{S}^{\{i\}}) \not\to 0$:
In this case, the pole at $U = 0$ and V_0 are infinite. So the subtraction terms in Eq. (5.2.266),

which are not finites, are not needed. Thus the terms $N_{4,2}^{6,(i)}$ for $i = 1, 2, 3, 4$ becomes

$$
\begin{aligned}
N_{4,2}^{8,(1)} &= \frac{\tilde{C}_2 S}{U^4}\left\{\sum_{n=1}^{3}(-1)^{1+n}V^{3-n}\frac{U^n x^n}{n} - V^3\ln(Ux+V)\right\}\\
N_{4,2}^{8,(2)} &= \frac{\tilde{C}_2 T - \tilde{C}_1}{U^3}\left\{\sum_{n=1}^{2}(-1)^{1+n}V^{2-n}\frac{U^n x^n}{n} - V^2\ln(Ux+V)\right\}\\
N_{4,2}^{8,(3)} &= \frac{(C_0 - \tilde{C}_1 T + \tilde{C}_2 T^2)V}{U^2(SV-TU)}\left\{Ux - V\ln(Ux+V)\right\}\\
N_{4,2}^{8,(4)} &= \frac{x^2\,(C_0 - \tilde{C}_1\,T + \tilde{C}_2\,T^2)}{SV - TU}\,q_2(X)
\end{aligned}
\tag{5.2.279}
$$

The poles t_{0i} ($U = 0$) are infinite (they are outside the closed contour), then no residue has to be added at this poles.

The poles t_{0i}^{η} ($S\,V - T\,U = 0$) are arbitrary (they may be finite or infinite that depends on the values of α_2 and α_3):
-i) $\alpha_2 \not\to 0$ and $\alpha_3 \not\to 0$:
In this case, there is only one pole which can be inside the integration contour. This last one corresponds to $\alpha_2 + \eta\sqrt{T_4} \not\to 0$, i.e. $z^{\eta} \sim -J/H$ ($T^{\eta} \to 0$ and $U^{\eta} \not\to 0$)[28]. The residues at this pole are given by,

$$
\mathrm{Res}\left(N_{4,2}^{8,(3)}\right)_{t=t_{0i}^{\eta}} = \frac{1}{\tilde{K}}\,\frac{(C_0^{\eta} - \tilde{C}_1^{\eta}T^{\eta} + \tilde{C}_2 T^{\eta^2})V^{\eta}}{U^{\eta^2}\,\eta\sqrt{T_4}}\left\{U^{\eta}x^{\eta} - V^{\eta}\ln(U^{\eta}x^{\eta}+V^{\eta})\right\} \tag{5.2.280}
$$

$$
\mathrm{Res}\left(N_{4,2}^{8,(4)}(z_i)\right)_{t=t_{0i}^{\eta}} = \frac{1}{\tilde{K}}\,\frac{(C_0^{\eta} - \tilde{C}_1^{\eta}\,T^{\eta} + \tilde{C}_2\,T^{\eta^2})}{\eta\,\sqrt{T_4}}\,\frac{T^{\eta}}{S}\left(\frac{T^{\eta}}{S}\left(\ln(S/T^{\eta}x^{\eta}+1) - x^{\eta}\right)\right. \tag{5.2.281}
$$

$$
= \frac{1}{\tilde{K}}\,\frac{(C_0^{\eta} - \tilde{C}_1^{\eta}\,T^{\eta} + \tilde{C}_2\,T^{\eta^2})}{\eta\,\sqrt{T_4}}\,\frac{V^{\eta}}{U^{\eta}}\left(\frac{V^{\eta}}{U^{\eta}}\left(\ln(U^{\eta}/V^{\eta}x^{\eta}+1) - x^{\eta}\right)\right. \tag{5.2.282}
$$

where Eqs. (5.2.281, 5.2.282) are completely equivalent. *-ii) $\alpha_2 \to 0$ and $\alpha_3 \not\to 0$:*
In this case, the two poles are infinite. Then, no residues are needed.
-iii) $\alpha_2 \not\to 0$ and $\alpha_3 \to 0$:
There is only one pole which may by inside the contour of integration, it corresponds to $z^{\eta} \sim \mathcal{O}(0)$. The residues in this case are given by Eqs. (5.2.280, 5.2.281, 5.2.282).
-iv) $\alpha_2 \to 0$ and $\alpha_3 \to 0$:
In this case, the two simple poles t_{0i}^{η} are finite. So, they might be inside the closed contour of integration, thus the residues are given by Eqs. (5.2.280, 5.2.281, 5.2.282).

-b) $\det(G^{(i)}) \to 0$ and $\det(\mathcal{S}^{(i)}) \to 0$:
In this case, the poles t_{0i} are finite which implies that V_0 is finite. Then, we have to use

[28] The other pole is infinite, it corresponds to $\alpha_2 + \eta\sqrt{T_4} \to 0$, i.e. $z^{\eta} \sim \infty$, so no residue has to be added.

the primitive Eq. (5.2.266) to avoid the calculation of residues at the poles t_{0i} (the residues at these poles are zero by construction). The residues at t^{η}_{0i} are treated similarly as in paragraph **I)**.
If β is complex (which can not be encountered one-loop calculation for collider processes) and the pole at $-J/H$ is inside the triangle, the one has just to subtract the extra terms as we explained above.

5.2.3.2 Numerical tests

The results are given in Figs. (5.9, 5.10, 5.11) for the same numerical points: I, II and III given in subsection 5.2.2.5.

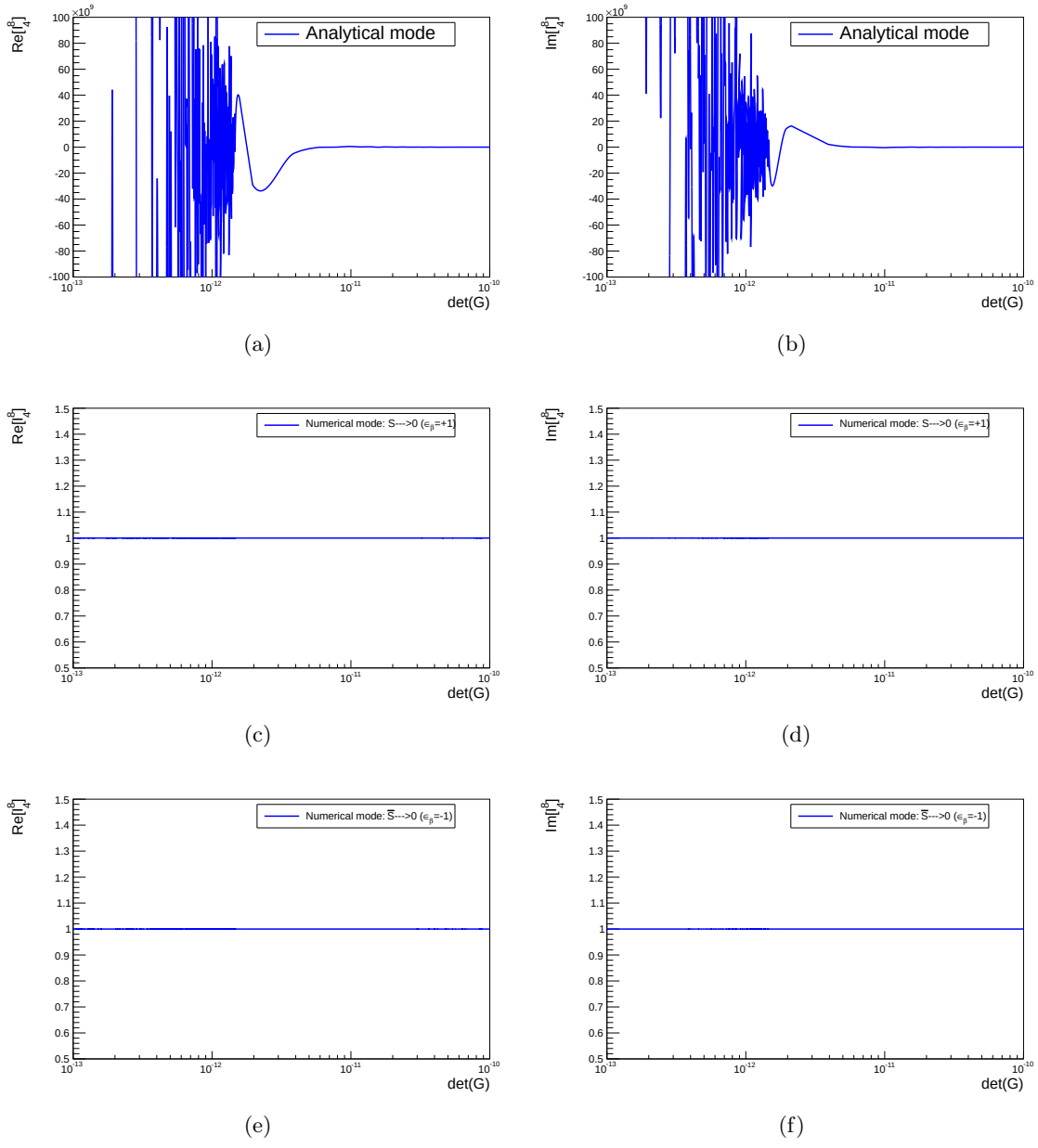


Figure 5.9: Comparison between analytical and numerical modes where all the internal masses are complex : (a) and (b) show respectively the real and the imaginary part of $I_4^8(S)$ evaluated analytically in the region where $\det(G) \to 0$, which are not stable. However the numerical mode gives stable results, which are show in the plots in (c), (d), (e) and (f) (for the plots (c) and (d) $S \to 0$ (for all sectors), and for the plots (e) and (f) $\bar{S} \to 0$ (for all sectors), which corresponds respectively to $\varepsilon_\beta = +1$ and $\varepsilon_\beta = -1$ for this numerical point).

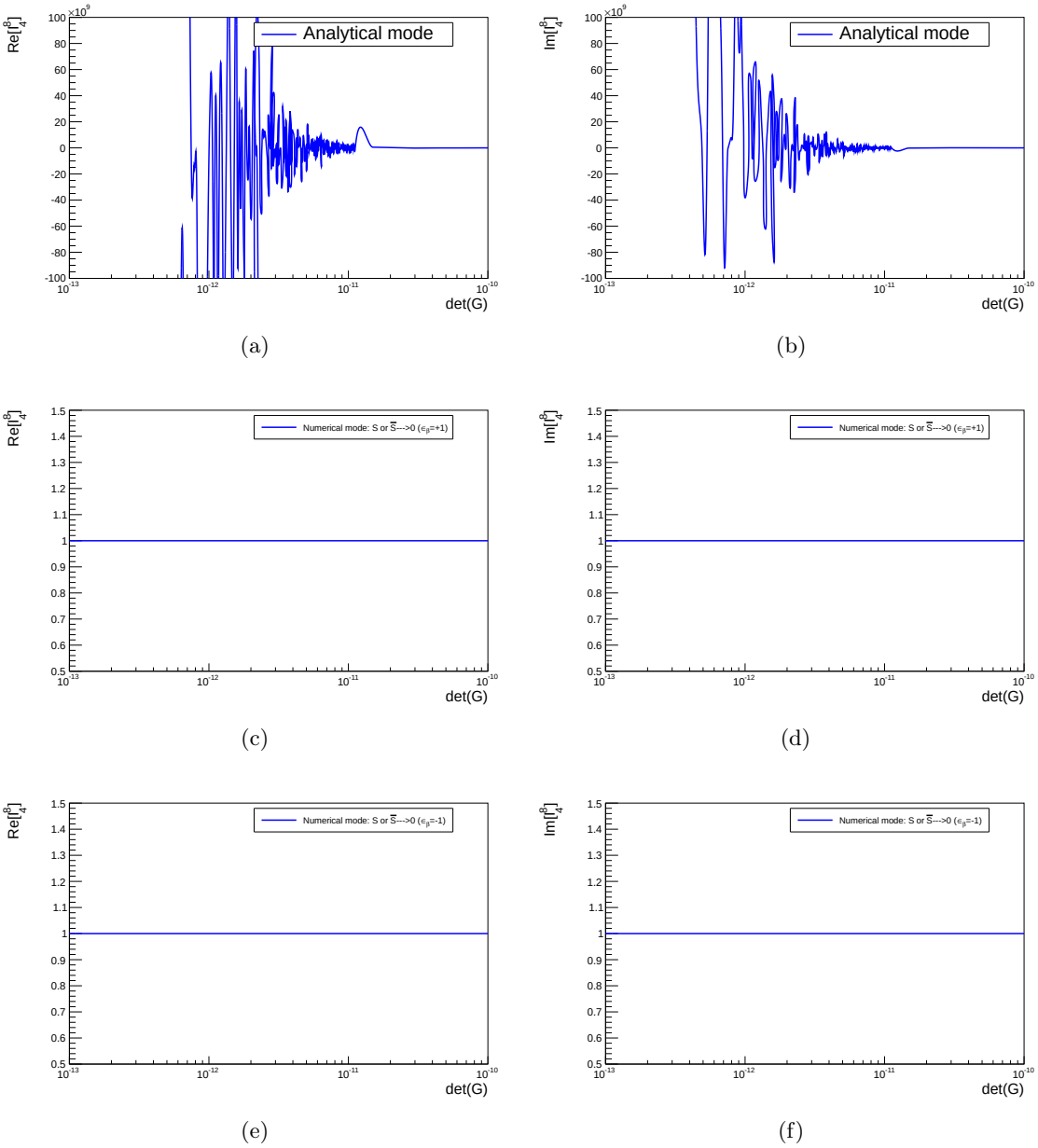


Figure 5.10: Comparison between analytical and numerical modes for numerical point where all the internal masses are complex: (a) and (b) show respectively the real and the imaginary part of $I_4^8(S)$ evaluated analytically in the region where $\det(G) \to 0$, which are not stable. However the numerical mode gives stable results, which are shown the plots in (c), (d), (e) and (f) (for the plots (c) and (d) $S \to 0$ (for sector I), and for the plots (e) and (f) $\bar{S} \to 0$ (for sectors I and II), which corresponds respectively to $\varepsilon_\beta = +1$ and $\varepsilon_\beta = -1$ for this numerical point).

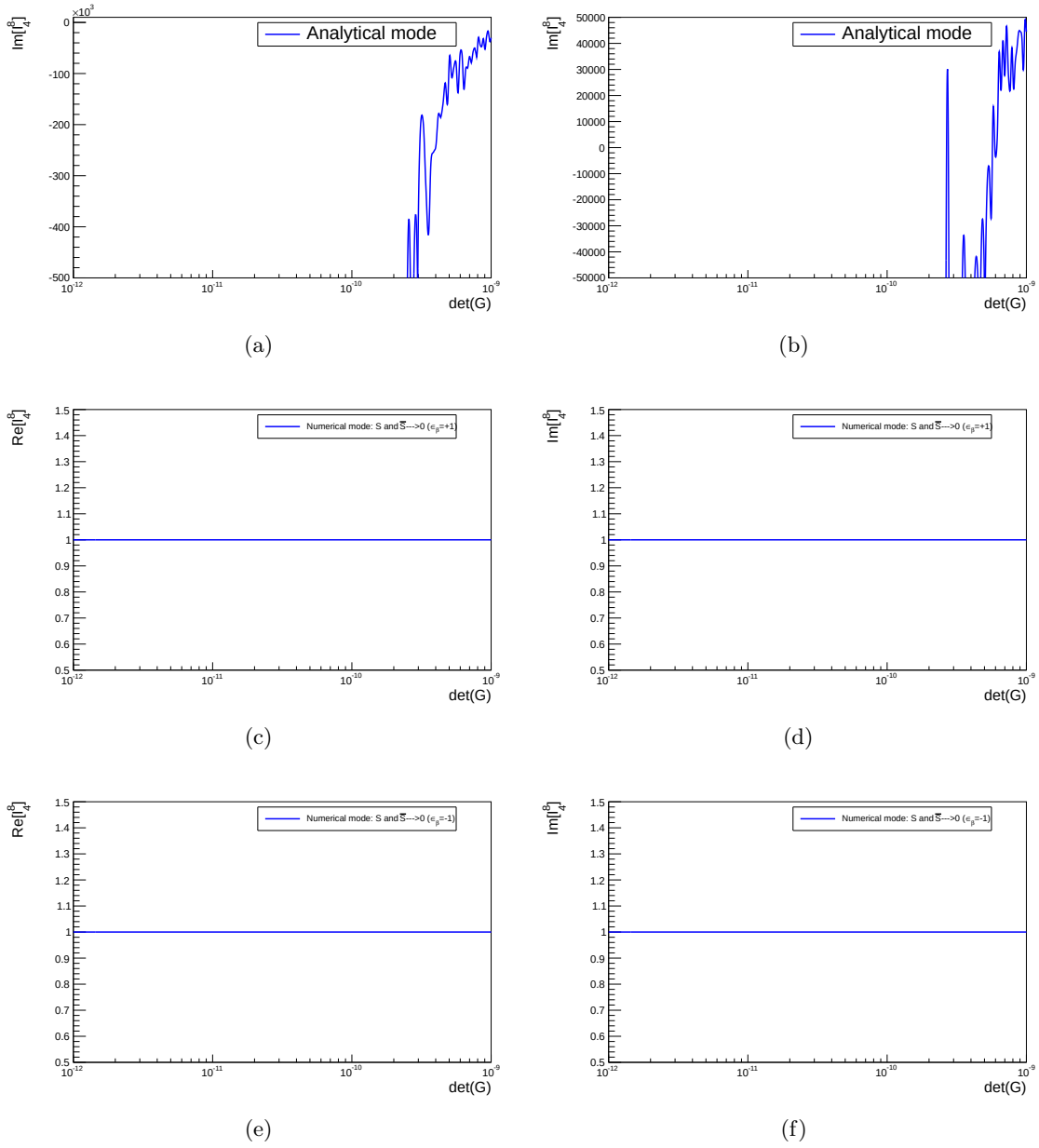


Figure 5.11: Comparison between analytical and numerical modes for numerical point where all the internal masses are complex: (a) and (b) show respectively the real and the imaginary part of $I_4^8(S)$ evaluated analytically in the region where $\det(G) \to 0$, which are not stable. However the numerical mode gives stable results, which show the plots in (c), (d), (e) and (f) (for the plots (c) and (d) $S \to 0$ and $\bar{S} \to 0$ (for all sectors with $\varepsilon_\beta = +1$), and for the plots (e) and (f) $S \to 0$ and $\bar{S} \to 0$ (for all sectors with $\varepsilon_\beta = -1$).

Chapter 6

Implementations in Golem95

Contents

In the last few years, an important progress in the automation of multi-leg one-loop scattering amplitude calculations has been made. Various programs and libraries have been developed for this task, many of them are publicly available. For example, the libraries: "Golem95 [21], LoopTools [30], OneLoop [82], ... etc, and the package programs SAMURAI [75], Golem [21], GoSam [83], BlackHat [84]" ... etc. This thesis is focused on one of these tools, that is the Golem library or Golem95. It is a program written in FORTRAN 95, it contains all the building blocks for one-loop calculation. This library is designed for the numerical evaluation of forms factors involved in the calculation of one-loop scattering amplitudes with up to six external legs for any gauge theory. It is based on the *Golem reduction* method presented in the Chapter 4; this reduction formalism enables us to avoid the spurious singularities generated by the Gram determinants encountered during the tensorial reduction which can disturb the numerical stability, the calculation of the stable one-dimensional integral representation was presented in the previous chapter. Golem95 is valid for the evaluation of amplitudes with massless as well as massive internal particles in a fast and a very efficient numerical way (complex masses are supported). This library can be used to calculate not only the form factors of the tensor reduction, but it can be used to calculate also the master basic integrals needed by some tools, hence Golem95 is a library of master integrals. In addition, this library can be used to evaluate amplitudes in the framework of the unitarity inspired reconstruction at the integrand level. Consequently, it can be interfaced with other automatic program based on this approach. So far, Golem95 is used by GoSam, which is based one the OPP reduction, as a rescue system in the problematic regions.
I notice that the basic integrals presented in the previous chapter are not all implemented in Golem95, the only implemented integrals for the case $\det(G) \to 0$ are the 3-point functions (all), and for the case $\det(G) \to 0$ and $\det(\mathcal{S})$ simultaneously I_3^4 and I_3^6 are the only implemented. So, in this chapter we give a brief overview of the library, and we show some examples involving the implemented ingredients.

6.1 Dealing with the Spurious singularities

A further reduction of the tensorial elements of the Golem redundant basic integrals to scalar integrals leads to expressions containing inverses of B (see section 4.1.2), where

$$B = (-1)^{N+1} \det(G)/\det(S) \tag{6.1.1}$$

Hence, these expressions will lead to numerical unstable results in the regions where the Gram determinant becomes arbitrary small. The philosophy of Golem95 to avoid such spurious singularities is the following:

We provide for each basic integral, an analytical formula and a stable one-dimensional integral representation. The former representation is obtained by performing all the integrations analytically; and the later one is obtained by performing the first integrations analytically and keeping the last one, which will be performed numerically after modifying the integrand such that the inverse of Gram determinants are avoided, see the previous Chapter. Then,
- i) *if $\hat{B} > \hat{B}^{cut}$, the basic integrals are evaluated analytically, which provide fast and efficient numerical evaluation in large phase space region.*
- ii) *if $\hat{B} < \hat{B}^{cut}$, we switch to the numerical mode by integrating the one-dimensional representation of each basic integral numerically.*
with

$$\hat{B} = B \times max(\mathcal{S}_{ij}) \tag{6.1.2}$$

where $\hat{B}^{cut}$ is certain cut, it is fixed to 0.005 by default.
Schematically the philosophy of the *Golem method* to avoid the Gram determinant spurious singularities is presented in Fig. (6.1).

We notice that Golem95 can be used as a library of massless as well as massive master integrals (complex masses are supported). These master integrals are the scalar integral that constitute the end point of Passarino-Veltman, OPP and Generalized unitarity cuts reductions presented in chapter 4. This basis contains all scalar boxes, triangles, babuls and tadpoles in n-dimension. The general scalar box in n-dimension is calculated by calling the appropriate 4-point form factor $A^{4,0}$. The scalar triangle in n-dimension can be calculated by calling the appropriate 3-point form factor $A^{3,0}$ and finally the scalar bubbles is calculated by calling the 2-point form factor. Depending on the kinematical configuration, Golem95 will call the appropriate form factor corresponding to the desired master integral.

6.2 Description of the Golem95 software

Golem95 contains the following four main directories:

- 1) **src:** is the source files of the program.
- 2) **demos:** contains some programs for demonstration.
- 3) **doc:** contains some documentations.
- 4) **test:** contains some tests.

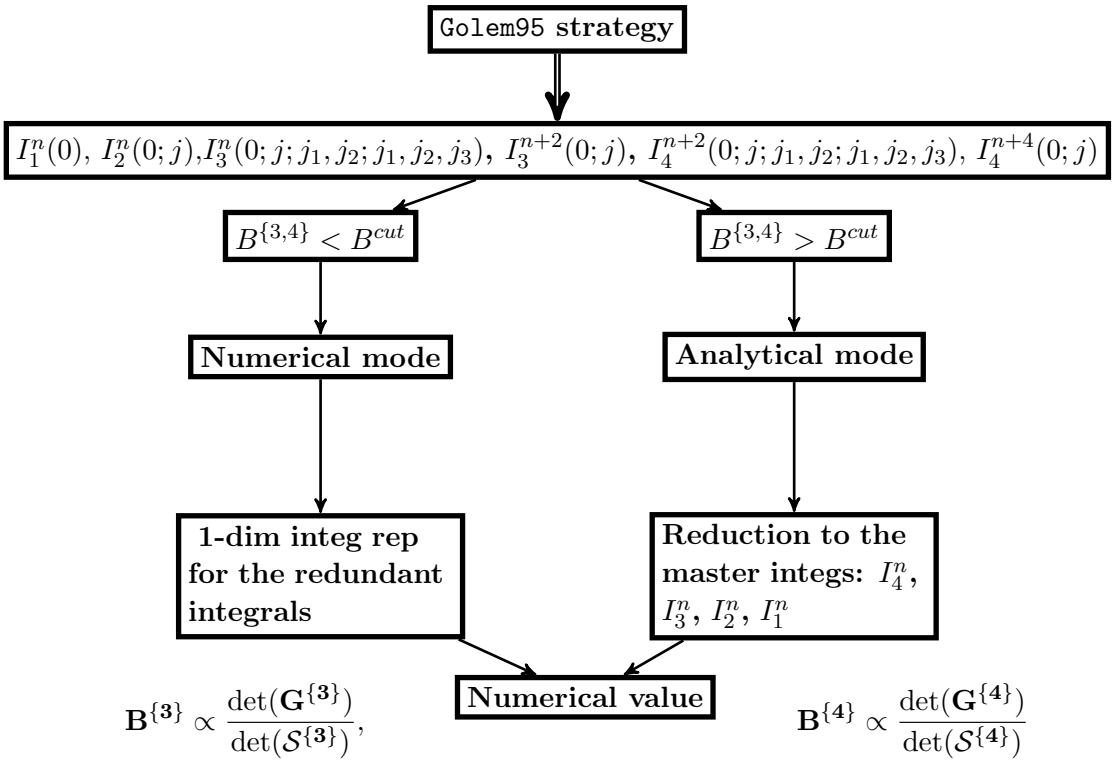


Figure 6.1: The Golem strategy to avoid the spurious singularities induced by Gram determinant. The stable one-dimensional integral representations for each of the Golem redundant basic integral are presented in the previous chapter.

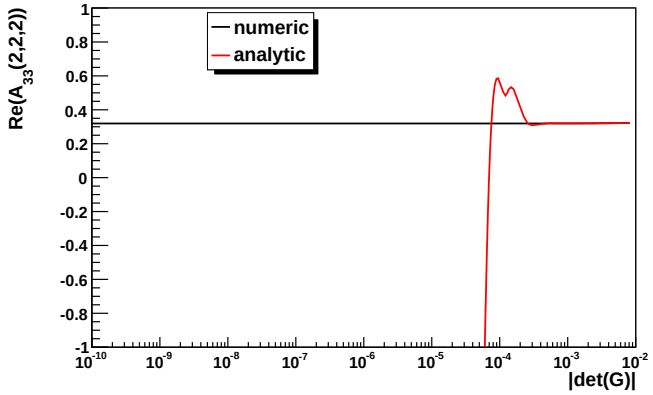


Figure 6.2: Comparison between the numerical and the analytical modes for the evaluation of the form factor $A^{3,3}_{l_1 l_2 l_3}$: The two plots represent the real part of $A^{3,3}_{l_1 l_2 l_3}$ evaluated analytically (red) and numerically (black) versus the absolute value of $\det(G)$. The former one shows fluctuations of this function in the region $\det(G) \to 0$, which means that the result is not any more stable. However, the later one shows a smooth function near and for $\det(G) = 0$, i.e. it is numerically stable.

The src directory:

src contains the following sub-directories

- a) **form_factor:** all form factors from 1-point to 6-point functions are implemented in the following five modules: `form_factor_1p.f90`, `form_factor_2p.f90`, `form_factor_3p.f90`, `form_factor_4p.f90`, `form_factor_5p.f90` and `form_factor_6p.f90`, respectively.

- b) **integrals:** this sub-directory contains

  - **four_point:** Computes the four point functions for all possible types of external momenta (on the mass shell or off the mass shell). The calculation, from one external off-shell with up to four, is organized in the five following modules: `function_4p1m.f90`, `function_4p2m_opp.f90`, `function_4p2m_adj.f90`, `function_4p3m.f90`, `function_4p4m.f90`, respectively. The `generic_function_4p.f90` contains the generic routines to compute the four point functions in n+2 and n+4 dimensions. It can compute the zero mass, one mass, two mass adjacent, two mass opposite, three mass. The `function_4p_qln.f90` for $n = 6, \cdots 16$ computes the n-dimensional four point function corresponding to QCDLoop box number n.

  - **three_point**: Computes the three point functions with 3-, 2-, 1- or 0-external off-shell legs with 2-, 1- or 0-internal mass. The calculation is organized in the modules: `function_3p0m_1mi.f90`, `function_3p1m.f90`, `function_3p2m.f90`, `function_3p3-m.f90`, `function_3p1m_1mi.f90`, `function_3p1m_2mi.f90`, `function_3p2m_1mi.f90`. The `generic_function_3p.f90` contains the generic routines to compute the three point functions in n and $n + 2$ dimensions, ...

  - **two_point:** Contains the modules: `function_2p0m_1mi.f90` (computes the two-point function with zero momentum and two equal masses $I_2^n(0, m^2, m^2)$), `function_2p_m1m2.f90` (computes $I_2^n(s, m_1^2, m_2^2)$ with/without Feynman parameters), `generic_function_2p.f90` (contains the generic routines to compute the two point functions in n and $n + 2$ dimensions).

  - **one_point:** Contains the module `generic_function_1p.f90` to compute one point functions in n dimensions.

- c) **kinematic:** Contains the modules `matrice_s.f90`, `inverse_matrice.f90` to compute the matrix $\mathcal{S}$, its inverse and the reduction coefficients b_i in the following two modules.

- d) **module:** this sub-directory contains the following functions and subroutines:

  - `parameter.f90:` The parameters $\hat{B}$ is fixed in this file to switch from the analytical mode to the numerical mode, the default value of it is 0.005. Also the parameters concerning the rational part (to be included or not) should be fixed in this file.

  - Special functions: the di-logarithmic functions and other special functions are defined in the following files: `z_log.f90`, `zdilog.f90`, `kronecker.f90` and `constante.f90`.

- spinor.f90: computes the scalar products, spinorial products and the antisymmetric tensor.
- precision_golem.f90: This module defines the parameter ki which gives the representation of the real and complex numbers in Golem95.
- cache.f90: This module is used to reserve some memory to store already computed four/three point functions.
- Other modules: tri.f90, translate.f90, sortie_erreur.f90, array.f90, equal .f90, form_factor_type.f90.

- e) **numerical:** Contains the following two modules: mod_adapt_gauss.f90 and mod_numeric.f90. These two modules are designed for the one-dimensional numerical integration. This numerical integration is based on Gauss-K adaptive integration discussed before. They allow the user to try many modes of numerical integration.
- f) **interface:** Contains the modules which allow us to re-construct the amplitude at the integrand level by means of the*tensorial reconstruction at the integrand level*, and to interface it with Golem95 or LoopTools.

- tens_rec.f90: This module offers the possibility of reconstructing the tensor coefficients that have to be contracted with tensor integrals in order to reproduce a diagram, which has been specified by a set of denominators and a numerator $N(k, \mu^2)$. This module is typically used in connection with the module tens_comb.
- tens_comb.f90: This module contains the routines necessary for the contraction of the tensor coefficients as reconstructed by the module tens_rec with the according tensor integrals.
- tensor_integrals.f90: This module provides an interface which allows to compute tensor integrals rather than form factors.
- tool_lt_to_golem.f90: This module contains one function to build the interface between LoopTools and Golem95.
- Modules containing LoopTools functions: gb0.f90, gc0.f90, gd0.f90, ge0.f90 and gf0.f90 contain B_{0i}, C_{0i}, D_{0i}, E_{0i} and F_{0i}, respectively.

6.3 Examples

Example1: rank 5 5-point form factor (version golem95-1.2.1)

In this example, we calculate only the form factor: $A^{5,5}_{j_1,j_2,j_3,j_4,j_5}$, with $j_i = 1, 2, 3, 4$. The S matrix defining the numerical point that we want to calculate is

$$S = \begin{pmatrix} 0 & p_2^2 & s_{23} & s_{51} & p_1^2 \\ p_2^2 & 0 & p_3^2 & s_{34} & s_{12} \\ s_{23} & p_3^2 & 0 & p_4^2 & s_{45} \\ s_{51} & s_{34} & p_4^2 & 0 & p_5^2 \\ p_1^2 & s_{12} & s_{45} & p_5^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -3 & -4 & 0 \\ 0 & 0 & 0 & 6 & 15 \\ -3 & 0 & 0 & 0 & 2 \\ -4 & 6 & 0 & 0 & 0 \\ 0 & 15 & 2 & 0 & 0 \end{pmatrix} \tag{6.3.3}$$

- In the directory **demos**, type 'make' if you want to compile all demos or add the name of the demo file if you want to run only one demo file, e.g. `make demo_5point` (for our example).

- By running `./demo_5point`, we produce the following output on the shell:
  `Choose what the program should compute:`
  `0) form factor for five-point function, rank 0`
  `1) form factor for five-point function, rank 3 (z1*z2*z4)`
  `2) form factor for five-point function, rank 5 (z1*z2*z3*z4*z5)`
  `3) form factor for diagram with propagator 3 pinched, rank 0`
  `4) form factor for diagram with propagators 1 and 4 pinched, rank 0`

- By choosing option "2", it will produce:
  `calculating form factor A_12345 for 5-point function rank 5`
  `The result has been written to the file test5point.txt`

- The file test5point.txt contains:
  `The kinematics is:`

  `p1+p2+p3+p4+p5 = 0`

  `S(1,3)=` $(p2+p3)^2$`= (-3.,0.)`
  `S(2,4) =` $(p3+p4)^2$`= (6.,0.)`
  `S(2,5) =` $(p1+p2)^2$`= (15.,0.)`
  `S(3,5) =` $(p4+p5)^2$`= (2.,0.)`
  `S(1,4) =` $(p1+p5)^2$`= (-4.,0.)`
  `S(1,2) =` $p2^2$`= (0.,0.)`
  `S(2,3) =` $p3^2$`= (0.,0.)`
  `S(3,4) =` $p4^2$`= (0.,0.)`
  `S(4,5) =` $p5^2$`= (0.,0.)`
  `S(1,5) =` $p1^2$`= (0.,0.)`
  $(\mu)^2$ `= 1.0`

  `normalization:`
  `defining` $I_N^n = \mu^{(4-n)} \int d^n k/(i * \pi^{(n/2)})\, func(k, p_i)$
  `=` r_Γ `(P2/`ε^2`+P1/`ε`+P0),`
  `n = 4-2`ε`,`
  $r_\Gamma = \Gamma(1+\varepsilon)\Gamma(1-\varepsilon)^2/\Gamma(1-2\varepsilon)$
  `the program gives numbers for P2,P1,P0`
  `1/`ε^2 `(0.0000000000E+00 +` i `0.0000000000E+00)`
  `+ 1/`ε `(0.0000000000E+00 +` i `0.0000000000E+00)`
  `+ 1 (-.8615520644E-04 +` i `0.1230709464E-03)`
  `CPU time= 0.003999`

Example2: form factor for rank 2 6-point functions (version `golem95-1.2.1`)

In the last example, all the input (the numerical point and the numerator) are fixed by default in the directory **demos**. If the user want to chose the input, the files `param.input` and `momunta.dat` in the directory **test** should be modified. The later one contains the programs `mask_3point`, `mask_4point`, `mask_5point` and `mask_6point` to calculates the 3-, 4-, 5- and 6-point Feynman integrals, respectively. To calculate the form factor of the rank 2 six-point functions for any possible numerator $z_j = 1, ...6$ and for a given numerical point, in the directory **test**, we follow the following steps:
-1) Fixe the six four-vectors in the file `momuea.dat`, each vector should be defined in the same line. Let's choose the following kinematical configuration[1]:

$$\begin{array}{l} p_1 = (0.500, 0.000, 0.000, 0.500) \\ p_2 = (-0.500, 0.000, 0.000, 0.500) \\ p_3 = (0.045, -0.043, 0.011, -0.186) \\ p_4 = (-0.085, -0.091, 0.245, -0.326) \\ p_5 = (-0.117, 0.192, -0.065, -0.233) \\ p_6 = (0.158, -0.057, -0.191, -0.254) \end{array} \qquad \text{with} \qquad p_i(x_i, y_i, z_i, E_i)$$

- 2) Edit the file `param.input` and choose:

- Number of legs: only 3,4,5,6 are possible: `choose 6`
- Rank: the rank is always $\leq$ number of legs (renormalizable gauge theories): `choose` 2
- Type of form factor: A, B or C (note: type B exists only for rank ≥ 2, type C for rank ≥ 4): `choose A`
- Labels of Feynman parameters in the numerators: "0" for scalar, "all" for all possible numerators, or for specific choices example: put 2,2,3 for a rank 3 integral with $z2^2z3$ in the numerator : `choose all`
- Specify the name of the file containing momenta: : `type momenta.dat`
- Choose the label to distinguish between different numerical points: `choose 1`

- 3) type the command `perl maktest.pl` to run the example.
- 4) the generated files are called N[number of legs][rank][pt].out, pt stand for the order of the chosen numerical point: In this case, the result will be written in `N6rank1zi-pt1.out` for `i` $= 1, .., 6$, `N6rank1zizj-pt1.out` for `i`, `j` $= 1, ..., 6$ and files called `N6rank1-pt1.numbers.out`, `N6ran2-pt1.numbers`.

Example 3: General massive case in the problematic region (version not public yet)

In this paragraph, the form factors of the 3-point functions are calculated for two problematic numerical points, one correspond to the vanishing of the Gram determinant alone and

[1] We notice that random momenta can be generated using `mom-rambo.f`

the other one corresponds to the vanishing of the Gram determinant and the determinant of the kinematical matrix simultaneously. We notice that the implementation of the 4-point functions are not completed yet.

I) $\det(G) \to 0$ **and not** $\det(\mathcal{S})$**:**

Let us choose this numerical point:

$$\begin{aligned} m_1^2 &= 0.022823098075160402 \\ m_2^2 &= 7.6751408576965332 - i\,0.14250832796096802 \\ m_3^2 &= 0.0684692942254812 \\ s_1 &= s_3 = 4.3248424530029297 \\ s_2 &= 0.0 \end{aligned}$$

where

$$\begin{aligned} \det(G) &= 0.0 \\ \det(\mathcal{S}) &= 505.5241686796991 - i\,18.86529162284873 \end{aligned}$$

In the directory **demos** type:
- cmake . (to generate the `Makefile`)
- make (to generate the binary files `demo_Npoint` for $N = 3, 4, 5, 6$)
- ./ `demo_3point` (to run the 3-point functions) - Choose the option 9, which corresponds to our numerical point
- The program will ask:

```
Choose what the program should compute:
0) scalar three-point function in n dimensions
1) three-point function in n dimensions with one Feynman parameter
2) three-point function in n dimensions with two Feynman parameters
3) three-point function in n dimensions with three Feynman parameters
4) scalar three-point function in n+2 dimensions
5) three-point function in n+2 dimensions with one Feynman parameter
6) test of the mu independence
```

- The results for each choice are given in the following table:

Choice	**Form Factors**	$\mathbf{1/\varepsilon^2}$	$\mathbf{1/\varepsilon}$	**Finite**
0	$A^{3,0}(\mu^2 = 1.0)$	0.0	0.0	-0.1759780388 - i 0.4017808420
1	$-A_l^{3,1}(\mu^2 = 1.0)$	0.0	0.0	-0.07731895433 - i 0.03763325320
2	$A_{l_1 l_2}^{3,2}$ $(\mu^2 = 1.0)$	0.0	0.0	-0.02255678625 -i 0.01469989033
3	$-A_{l_1 l_2 l_3}^{3,3}(\mu^2 = 1.0)$	0.0	0.0	-0.01338493750 -i 0.7647310145 $\times 10^{-3}$
4	$-2B^{3,2}(\mu^2 = 1.0)$	0.0	0.0	0.06763621562 -i 0-.2885457940
5	$2B_l^{3,3}$ $(\mu^2 = 1.0)$	0.0	0.0	0.1480748335 -i 0.01913342362
6	$A^{3,0}$ $(\mu^2 = 34.0)$	0.0	0.0	-0.1759780388 i 0-.40178084200

II) $\det(G) \to 0$ **and** $\det(S) \to 0$ **simultaneously:**

Let us choose this numerical point:

$$\begin{aligned} m_1^2 &= m_2^2 = 7.6751408576965332 - i\,0.14250832796096802 \\ m_3^2 &= 0.0684692942254812 \\ s_1 &= s_3 = 4.3248424530029297 \\ s_2 &= 0.0 \end{aligned}$$

where

$$\begin{aligned} \det(G) &= 0.0 \\ \det(\mathcal{S}) &= 0.0 \end{aligned}$$

we get

Choice	**Form Factors**	$\mathbf{1/\varepsilon^2}$	$\mathbf{1/\varepsilon}$	**Finite**
0	$A^{3,0}(\mu^2 = 1.0)$	0.0	0.0	-0.1772018325 -i 0.004642477410
1	$-A_l^{3,1}(\mu^2 = 1.0)$	–	–	not yet implemented
2	$A_{l_1 l_2}^{3,2}\ (\mu^2 = 1.0)$	–	–	not yet implemented
3	$-A_{l_1 l_2 l_3}^{3,3}(\mu^2 = 1.0)$	–	–	not yet implemented
4	$-2B^{3,2}(\mu^2 = 1.0)$	0.0	0.0	0.6646684749 -i 0.01158277702
5	$2B_l^{3,3}\ (\mu^2 = 1.0)$	–	–	not yet implemented
6	$A^{3,0}\ (\mu^2 = 34.0)$	0.0	0.0	-0.1772018325 -i 0.004642477410

For this problematic region, the implementation of the tensorial 3-point functions is in progress.

Chapter 7

Conclusion

This thesis is focused on developing and generalizing the Golem library (`Golem95`) to be used in all possible kinematical configurations, which might be encountered in the calculation of amplitudes at NLO order for process of interest at collider experiments or particle decay, and including radiative corrections of any perturbative gauge theory (QCD, SM and BSM).

`Golem95` is based on the *Golem reduction* method, presented in chapter 4, which reduces the form factors of a given Feynman integral into a combination of a certain set of redundant basic integrals with up to four-external legs, called *Golem basic integrals*, weighted by some coefficients. This reduction formalism is able to hide the negative powers of the Gram determinants from the expansion coefficients and avoid the problems of the numerical instability induced by spurious singularities due to the vanishing of these Gram determinants, thanks to the choice of the redundant basic integrals.

In renormalizable gauge theories and for one-loop Feynman integrals with up to six external legs, the needed basic integrals in this formalism are: $I_3^n(j_1, \cdots, j_3)$, $I_3^{n+2}(j_1)$, $I_4^{n+2}(j_1, \cdots, j_3)$ and $I_4^{n+4}(j_1)$ and various two- and one-point functions. Where, these functions may be scalars, i.e. without Feynman parameters in the numerators or tensorials, i.e. with Feynman parameter in the numerator (j_i stands for the labels of the Feynman parameters).

Nevertheless, new negative powers of the Gram determinant appear if these basic integrals are evaluated analytically, .i.e all the integrations over the Feynman parameters are performed analytically for each basic integral; or reduced to the set of the *master integrals*, i.e. to only scalar integrals in n-dimensions with up to four-external legs. These tow approaches are completely equivalent, so in the safe region it is recommended to use the later approach since the analytical formulas of the *master integrals* are well-known.

To avoid the problems induced by $\det(G) \to 0$, a stable one-dimensional integral representations for each redundant integral is provided, where the inverse of Gram determinants are hidden at the integrand level of each representation, i.e. no further reduction of the redundant basic integrals is performed. The last integration is performed numerically and leads to numerical stable results in the problematic region (where the Gram determinant becomes arbitrary small).

Instead to derive directly a one-dimensonal integral representation for each tensorial

redundant basic integrals, we have provided, in Chapter 5, stable one-dimensional integral representation for each scalar three-point integral up to eight dimensions ($n = 4, 6, 8$) and for each scalar four-point integral up to twelve dimensions ($n = 6, 8, 10, 12$) in the general massive case, where the complex internal masses are supported. These seven scalar integrals (I_3^{4+l} for $l = 0, 2, 4$ and I_4^{4+l} for $l = 2, 4, 6, 8$) allow us to reconstruct any tensorial basic integrals presented earlier, without introducing any new negative power of the Gram determinants (so instead to calculate ten integrals, we have calculated only seven!).

To summarize, the Golem strategy to avoid the Gram determinant problems is based on the choice of the set of the redundant basic integrals and the implementation of the stable one-dimensional integral representation for each element of this set. In practice, it works as the following:

- If the Gram determinant is large enough (safe region), `Golem95` uses the *analytical mode*, i.e. it calls the analytical formula of each basic integral.
- If the Gram determinant is arbitrary small (problematic region), `Golem95` switches to the numerical mode, i.e. the stable one-dimensional integral reperesentation of each integral are integrated numerically.

`Golem95` can be used as a library of master integrals as the scalar integrals are related directly to the form factors: the scalar four-point functions in n-dimensions is given by the form factor $A^{4,0}$, the scalar three-point functions in n-dimensions is given by the form factor $A^{3,0}$ and so on.

One of the powerful feature of `Golem95` is that it can be used to improve *inspired-unitarity reduction methods* in the problematic region. This is done by reconstructing the numerator of the full amplitude using the *tensorial reconstruction at the integrand level* and projecting the new amplitude decomposition to `Golem95` basic integrals without introducing any new inverse powers of the Gram determinants. Then, it can be used as a rescue system for automatic programs using reduction methods based on unitarity approaches (`GoSam` for example, uses `Golem95` as rescue system in the problematic region).

Conclusion (Français)

Cette thèse se concentre sur le développement et la généralisation de la bibliothèque Golem (`Golem95`) pour qu'elle soit utilisée dans toutes les configurations cinématique possibles, qui peuvent être rencontrées dans le calcul des amplitudes à l'ordre NLO pour des processus d'intérêt dans les collisionneurs ou dans les désintégrations de particules, et y compris les corrections radiatives de toutes les théories de jauge perturbatives (QCD, SM et BSM).

`Golem95` est basée sur la *réduction à la Golem*, présentée dans le chapitre 4, ce qui réduit les facteurs de forme d'une intégrale de Feynman à une boucle à une combinaison d'un certain ensemble d'intégrales de base redondantes avec jusqu'à quatre pattes externes, appelée *intégrales de base de Golem*, pondérées par des coefficients. Ce formalisme de réduction,

grâce au choix d'intégrales de base redondantes, est en mesure de cacher les puissances négatives du déterminant de Gram dans ces coefficients, et d'éviter les problèmes d'instabilité numérique induite par les singularités factices dues à l'annulation de ces déterminants de Gram.

Dans les théories de jauge renormalisables et pour une intégrale de Feynman à une boucle avec jusqu'à six pattes externes, les intégrales de base nécessaires à ce formalisme sont: $I_3^n(j_1, \cdots, j_3)$, $I_3^{n+2}(j_1)$, $I_4^{n+2}(j_1, \cdots, j_3)$ et $I_4^{n+4}(j_1)$ et plusieurs fonctions à deux et à un point. Ces fonctions peuvent être des scalaires, c.-à-d. sans paramètres Feynman dans le numérateur ou tensorielles, c.-à-d. avec des paramètres de Feynman dans le numérateur, (j_i représente les labels des paramètres de Feynman).

Néanmoins, de nouvelles puissances négatives du déterminant Gram apparaissent si ces intégrales de base sont calculées analytiquement, c.-à-d. si toutes les intégrations sur les paramètres de Feynman sont effectuées analytiquement; ou si elles sont réduites à l'ensemble des *master intégrales*, c.-à-d. si elles ne sont exprimées qu'en fonction des intégrales scalaires à n-dimensions avec jusqu'à quatre pattes externes. Ces deux approches sont parfaitement équivalentes.

Pour éviter les problèmes induits par l'annihilation de $\det(G)$, une représentation intégrale unidimensionnelles pour chaque intégrale de base redondante est fournie, où l'inverse des déterminants de Gram sont cachés au niveau de l'intégrant dans chaque représentation, c.-à-d. aucune réduction supplémentaire des intégrales de base redondante est effectuée. La dernière intégration est effectuée numériquement et conduit à des résultats stables numériquement dans la région problématique de l'espace de phase (où le déterminant de Gram devient arbitrairement petit).

Grâce au choix d'intégrales de base, le formalisme de réduction à la Golem permet d'éviter les problèmes induits par des singularités factices engendrées par des puissances négatives du déterminant de Gram dans les coefficients de l'expansion. Au lieu de dériver directement une représentation intégrale unidimensionnelle pour chaque intégrale de base redondantes, on a fourni dans le chapitre 5 (et l'appendice C.), une représentation intégrale unidimensionnelle stable pour chaque intégrale scalaire à trois points avec jusqu'à huit dimensions ($n = 4, 6, 8$) et pour chaque intégrale scalaire à quatre points avec jusqu'à douze dimensions ($n = 6, 8, 10, 12$) dans le cas massif le plus général (y compris les masses complexe). Ces sept intégrales scalaires (I_3^{4+l} pour $l = 0, 2, 4$ et I_4^{4+l} pour $l = 2, 4, 6, 8$) permettent de reconstruire les intégrales de base tensorielles présentées ci-dessus, sans introduire de nouvelles puissances négatives du déterminant de Gram (Ainsi, au lieu de calculer dix intégrales, on a calculé que sept!).

Pour résumer, la stratégie de Golem pour éviter les problèmes dus à l'annihilation du déterminant de Gram est basée sur le choix de l'ensemble des intégrales de base redondantes et la mise en oeuvre de la représentation intégrale unidimensionnelles de chaque élément de cet ensemble. Dans la pratique, cela fonctionne comme suit:

- Si le déterminant de Gram est assez grand, `Golem95` utilise le *mode analytique*, c.-à-d. on appelle la formule analytique de chaque intégrale de base.
- Si le déterminant de Gram est arbitrairement petit, `Golem95` passe au mode numérique, en intégrant numériquement la représentation intégrale unidimensionnelle de chaque intégrale de base redondante.

`Golem95` peut être utilisé comme une bibliothèque de *master intégrales*, car ces intégrales sont liés directement aux facteurs de forme (les facteurs de forme sont les *building-blocks* de `Golem95`): la fonction scalaire à quatre points en n-dimensions est donnée par le facteur de forme $A^{4,0}$, la fonction scalaire à trois points en n-dimensions est donnée par le facteur de forme $A^{3,0}$ et ainsi de suite.

Une des caractéristiques les plus importantes de `Golem95` est qu'elle peut être utilisée pour améliorer les *méthodes de réduction inspiré de l'unitarité* dans la région problématique de l'espace de phase. Cela se fait en reconstruisant le numérateur de l'amplitude complète en utilisant la *reconstruction tensorielle au niveau de l'intégrant* et en projetant la nouvelle décomposition dans la base de `Golem95` sans introduire de nouveaux inverses du déterminant de Gram. Ainsi, cette bibliothèque peut être utilisée comme un système de sauvetage pour des programmes automatiques basés sur les méthodes de réduction inspirées de l'unitarité (`GoSam` par exemple, utilise `Golem95` comme un système de secours dans le région problématique).

Appendix A

Three-point functions

A.1 Matrices and determinants involved in section 5.1

$$G^{(a=1,2,3)} \text{ associated to } \mathcal{S} \text{ in } Eq.(5.1.5): \begin{cases} G^{(1)} = \begin{pmatrix} 2s_2 & s_1+s_2-s_3 \\ s_1+s_2-s_3 & 2s_1 \end{pmatrix} \\ G^{(2)} = \begin{pmatrix} 2s_2 & -s_1+s_2+s_3 \\ -s_1+s_2+s_3 & 2s_3 \end{pmatrix} \\ G^{(3)} = \begin{pmatrix} 2s_1 & s_1-s_2+s_3 \\ s_1-s_2+s_3 & 2s_3 \end{pmatrix} \end{cases} \tag{A.1.1}$$

The matrices $G^{(1)}$, $G^{(2)}$ and $G^{(3)}$ have the same determinant

$$\begin{aligned} \det(G) &= \det(G^{(3)}) = \det(G^{(3)}) = \det(G^{(3)}) \\ &= -s_1^2 + 2s_1s_2 - s_2^2 + 2s_1s_3 + 2s_2s_3 - s_3^2 \\ &= -\lambda(s_1, s_2, s_3) \end{aligned} \tag{A.1.2}$$

which we called in section 5. 1, the Gram determinant associated with the kinematical matrix $\mathcal{S}$, where $\lambda(x,y,z) = x^2+y^2+z^2-2xy-2xz-2yz$.

The parameters $\gamma_i^{(2)}$, $\gamma_i^{(1)}$ and $\gamma_i^{(0)}$ and the discriminant Δ_i (Eq. (5.1.55)) for $i = 1, 2, 3$:

(i=1)	(i=2)	(i=3)
$\gamma_1^{(2)} = s_3$	$\gamma_2^{(2)} = s_1$	$\gamma_3^{(2)} = s_2$
$\gamma_1^{(1)} = m_2^2 - m_3^2 - s_3$	$\gamma_2^{(1)} = m_3^2 - m_1^2 - s_1$	$\gamma_3^{(1)} = m_2^2 - m_1^2 - s_2$
$\gamma_1^{(0)} = m_3^2 - i\lambda$	$\gamma_2^{(0)} = m_1^2 - i\lambda$	$\gamma_3^{(0)} = m_1^2 - i\lambda$

(A.1.3)

$$\Delta_1 = m_2^4 - 2m_2^2m_3^2 + m_3^4 - 2m_2^2s_3 - 2m_3^2s_3 + s_3^2 \tag{A.1.4}$$

$$\Delta_2 = m_1^4 - 2m_1^2m_3^2 + m_3^4 - 2m_1^2s_1 - 2m_3^2s_1 + s_1^2 \tag{A.1.5}$$

$$\Delta_3 = m_1^4 - 2m_1^2m_2^2 + m_2^4 - 2m_1^2s_2 - 2m_2^2s_2 + s_2^2 \tag{A.1.6}$$

The reduced matrices of $\mathcal{S}$ (obtained by omitting the line and the column i) are:

$$\mathcal{S}^{\{1\}} = \begin{pmatrix} -2m_2^2 & -m_2^2 - m_3^2 + s_3 \\ -m_2^2 - m_3^2 + s_3 & -2m_3^2 \end{pmatrix} \quad \Rightarrow \det(\mathcal{S}^{\{1\}}) = -\Delta_1 \tag{A.1.7}$$

$$\mathcal{S}^{\{2\}} = \begin{pmatrix} -2m_1^2 & -m_1^2 - m_3^2 + s_1 \\ -m_1^2 - m_3^2 + s_1 & -2m_3^2 \end{pmatrix} \quad \Rightarrow \det(\mathcal{S}^{\{2\}}) = -\Delta_2 \tag{A.1.8}$$

$$\mathcal{S}^{\{3\}} = \begin{pmatrix} -2m_12 & -m_1^2 - m_2^2 + s_2 \\ -m_1^2 - m_2^2 + s_2 & -2m_2^2 \end{pmatrix} \quad \Rightarrow \det(\mathcal{S}^{\{3\}}) = -\Delta_3 \tag{A.1.9}$$

The Gram determinant associated to each of these matrices (which is the Gram matrix itself, since it is given by a single element) are:

$$\det(G^{\{1\}}) = 2s_3 = 2\gamma_1^{(2)}, \quad \det(G^{\{1\}}) = 2s_1 = 2\gamma_2^{(2)}, \quad \det(G^{\{1\}}) = 2s_2 = 2\gamma_3^{(2)}. \quad \text{(A.1.10)}$$

A.2 The analytical integration

The proof of the following formula is given in ref. [66]

$$\begin{aligned} I_Q &= \int_0^1 dy\, \frac{1}{y-y_0} \left\{\ln(a\,y^2+b\,y+c) - \ln(a\,y_0^2+b\,y_0+c)\right\} \\ &= \left[\eta(-y_1,-y_2) - \eta(y_0-y_1,y_0-y_2) - \eta\left(a - i\,\varepsilon\,\frac{a}{a-i\,\delta}\right)\right] \ln\frac{y_0-1}{y_0} \\ &+ \mathrm{Li}_2\left(\frac{y_0}{y_0-y_1}\right) - \mathrm{Li}_2\left(\frac{y_0-1}{y_0-y_1}\right) + \eta\left(-y_1, \frac{1}{y_0-y_1}\right) \ln\frac{y_0}{y_0-y_1} \\ &- \eta\left(1-y_1, \frac{1}{y_0-y_1}\right) \ln\frac{y_0-1}{y_0-y_1} \\ &+ \mathrm{Li}_2\left(\frac{y_0}{y_0-y_2}\right) - \mathrm{Li}_2\left(\frac{y_0-1}{y_0-y_2}\right) + \eta\left(-y_2, \frac{1}{y_0-y_2}\right) \ln\frac{y_0}{y_0-y_2} \\ &- \eta\left(1-y_2, \frac{1}{y_0-y_2}\right) \ln\frac{y_0-1}{y_0-y_2} \end{aligned} \quad \text{(A.2.11)}$$

where ε and δ are two infinitesimal quantities having the opposite sign of the imaginary parts of the logarithms arguments in the first line in the right hand side of this equation, y_1 and y_2 are the roots of $a\,y^2+b\,y+c=0$ a must be real but b and c are arbitrary (may be complex).

A.3 The c-independence of I_3^{4+l}

This appendix presents an analytical proof that, whereas each of the three terms involved in Eq. (5.1.81) are separately functions of c in the directional limit $s_- \to 0$, $s_2 \to 0$ with $c = s_2/s_-^2$ (fixed), the limit of their sum is actually independent of c. For this purpose we compute the c-derivative of this sum in this limit and prove it to vanish identically in t. We provide an explicit proof for I_3^4; the I_3^6 and I_3^8 cases, albeit more difficult, can be handled in

a completely similar way. In the limit $s_- \to 0$, $s_2 \to 0$, c is fixed, then Eq. (5.1.81) becomes

$$\begin{aligned} I_3^4(S) &= -\Bigg[\frac{c\,(s_+ + (m^2 - m_3^2))}{(c\,\tilde{\lambda} + 4m^2)} \int_0^1 dz\, \frac{\ln(m^2) - \ln(\mathcal{B} - i\lambda)}{2\,B\,m^2 + 1} \\ &+ \frac{c\,(s_+ + (m_3^2 - m^2)) - 2}{(c\,\tilde{\lambda} + 4m^2)} \int_0^1 dz\, \frac{\ln(g(z)) - \ln(\mathcal{B} - i\lambda)}{2\,B\,g(z) + 1} \\ &+ \frac{-s_+ + (m_3^2 - m^2)}{(c\,\tilde{\lambda} + 4m^2)} \int_0^1 dz \Bigg\{ 4\,B\,z\,(z-1)\, \frac{\ln(g(z)) - \ln(\mathcal{B} - i\lambda)}{(2\,B\,g(z) + 1)^2} \\ &- \frac{z\,(1-z)}{g(z)}\, \frac{2}{2\,B\,g(z) + 1} \Bigg\} \Bigg] \end{aligned} \tag{A.3.12}$$

where

$$B = -\frac{2(1 - 2\,t\,s_+)}{(c\,\tilde{\lambda} + 4m^2)}, \qquad \mathcal{B} = -\frac{1}{2B} \tag{A.3.13}$$

$$g(z) = s_+\,z^2 + \left(-s_+ + m^2 - m_3^2\right)\, z + m_3^2 \tag{A.3.14}$$

$$\tilde{\lambda} = \left(s_+ - (m^2 + m_3^2)\right)^2 - 4\,m^2\,m_3^2 \tag{A.3.15}$$

To have more compact notation, let us introduce the following quantities : $\Delta_m = m_3^2 - m^2$, $D(z,c) = -4(1 - 2\,c\,s_+)\,g(z) + c\,\tilde{\lambda} + 4m^2$, $H(z,c) = \ln(g(z)) - \ln(\mathcal{B} - i\lambda)$, $T_1 = 1 - c\,s_+$, $T_2 = \Delta_m - s_+$ and $T_3 = \Delta_m + s_+$. Differentiating I_3^4 with respect to c. A long but straightforward computation leads to the following result:

$$\frac{d}{dc}\, I_3^4(S) = P_1 + P_2 + P_3 + P_4 + P_5 \tag{A.3.16}$$

with

$$\begin{aligned} P_1 &= \frac{1}{4}\,\frac{T_2}{T_1^2\,\mathcal{B}} \\ P_2 &= (1 + T_1 - c\,\Delta_m) \int_0^1 dz \frac{(g'(z))^2\,H(z,c)}{D(z,c)^2} \\ P_3 &= \frac{1}{4\,\mathcal{B}\,T_1^2} \int_0^1 dz\, \frac{T_2^2\,(1 + T_1 - c\,\Delta_m) + 4\,H(z,c)\,T_3 \mathcal{B}\,T_1^2}{D(z,c)} \\ P_4 &= \frac{2T_2}{T_1\,\mathcal{B}} \int_0^1 dz\, \frac{z\,(1-z)\,(g(z)\,T_2^2 + 4\,s_+\,\mathcal{B}\,T_1\,H(z,c) + (g'(z))^2\,\mathcal{B}T_1)}{g(z)\,D(z,c)^2} \\ P_5 &= 16\,T_1\,T_2 \int_0^1 dz\, \frac{z\,(1-z)\,(g'(z))^2\,H(z,c)}{D(z,c)^3} \end{aligned}$$

where $g'(z) = dg(z)/dz$. To derive these equations, we have used

$$\frac{\partial D(z,c)}{\partial c} = (g'(z))^2 \tag{A.3.17}$$

Instead of computing the integrals over z, we will perform integration by part to reduce the terms having the highest power at the denominator. Indeed, we can note that the derivative with respect to z of $D(z,c)$ is proportional to $g'(z)$:

$$\frac{\partial D(z,c)}{\partial z} = -4T_1\,g'(z) \tag{A.3.18}$$

Let us start by the last term of eq. (A.3.16) P_5. Using integration by part, and noticing that surface term vanish due to the $z\,(z-1)$ factor, we get :

$$\begin{aligned} P_5 &= 2\,T_2 \int_0^1 dz \Big\{ H(z,c)\,g'(z)\,(2\,z-1) - 2z\,(1-z)\,s_+ H(z,c) \\ &\quad - \frac{z\,(1-z)\,(g'(z))^2}{g(z)} \Big\} \frac{1}{D(z,c)^2} \end{aligned} \tag{A.3.19}$$

Let us collect all the terms with $D(z,c)^2$

$$P_2 + P_4 + P_5 = \int_0^1 dz\, \frac{2z\,(1-z)\,T_2^3 - \mathcal{B}\,T_1\,T_3\,D(z,c)\,H(z,c)}{T_1\,\mathcal{B}\,D(z,c)^2} \tag{A.3.20}$$

By comparing eq. (A.3.20) and the equation which gives P_3, we can see that the part proportional to $H(z,c)$ drops out. So the sum of the P_i $(i = 2..5)$ is equal to:

$$\sum_{i=2}^{5} P_i = \frac{T_2^2}{4\,\mathcal{B}\,T_1^2} \int_0^1 dz\, \frac{8T_1\,T_2\,z\,(1-z) + D(z,c)\,(1+T_1-c\,\Delta_m)}{D(z,c)^2} \tag{A.3.21}$$

Wee can notice that :

$$\begin{aligned} z\,(1-z) &= -\frac{1}{4s_+^2\,T_2} \Big(T_3\,s_+\,D(z,c) + (2\,\Delta_m - \Delta_m\,c\,s_+ - c\,s_+^2)\,(g'(z))^2 \\ &\quad + 2\Delta_m\,T_2\,g'(z) \Big) \end{aligned} \tag{A.3.22}$$

Inserting eq. (A.3.22) in eq. (A.3.21), we get:

$$\sum_{i=2}^{5} P_i = Q_1 + Q_2 + Q_3 \tag{A.3.23}$$

with

$$\begin{aligned} Q_1 &= \frac{T_2^2\,(s_+^2\,c + \Delta_m\,c\,s_+ - 2\,\Delta_m)}{2\,T_1\,\mathcal{B}\,s_+^2} \int_0^1 dz\, \frac{(g'(z))^2}{D(z,c)^2} \\ Q_2 &= -\frac{T_2^3\,\Delta_m}{T_1\,\mathcal{B}\,s_+^2} \int_0^1 dz\, \frac{g'(z)}{D(z,c)^2} \\ Q_3 &= -\frac{T_2^2}{4\,T_1^2\,\mathcal{B}\,s_+} \int_0^1 dz\, \frac{2\,T_1\,T_3 - s_+\,(1-T_1-\Delta_m\,c)}{D(z,c)} \end{aligned}$$

Again, an integration by part for Q_1 and Q_2, using eq. (A.3.18), gives :

$$\begin{aligned} Q_1 &= \frac{T_2^2\,(s_+^2\,c + \Delta_m\,c\,s_+ - 2\,\Delta_m)}{8\,T_1^2\,\mathcal{B}\,s_+^2} \Big(\frac{g'(1)}{D(1,c)} - \frac{g'(0)}{D(0,c)} - \int_0^1 dz\, \frac{2s_+}{D(z,c)} \Big) \\ Q_2 &= -\frac{T_2^3\,\Delta_m}{4\,T_1^2\mathcal{B}\,s_+^2} \Big(\frac{1}{D(1,c)} - \frac{1}{D(0,c)} \Big) \end{aligned}$$

It is easy to see then that the terms proportional to $1/D(z,t)$ in Q_1 and Q_3 cancel out. The definition of $D(z,t)$ and $g'(z)$ leads to

$$\begin{aligned} g'(1) &= -T_2 && \text{(A.3.24)} \\ g'(0) &= -T_3 && \text{(A.3.25)} \\ D(1,c) &= c\,T_2^2 && \text{(A.3.26)} \\ D(0,c) &= c\,T_3^2 - 4\,\Delta_m && \text{(A.3.27)} \end{aligned}$$

Using those results, we find that:

$$\sum_{i=2}^{5} P_i = -\frac{T_2}{4\,T_1^2\,\mathcal{B}} = -P_1 \qquad \text{(A.3.28)}$$

Hence

$$\frac{d}{dc}\, I_3^4(S) = 0 \qquad \text{(A.3.29)}$$

q.e.d

In similar way we can prove that

$$\frac{d}{dc}\, I_3^{4+l}(S) = 0, \qquad \text{for} \qquad l = 2, 4. \qquad \text{(A.3.30)}$$

Appendix B

Four-point functions

B.1 Matrices and determinants involved in section 5.2

B.1.1 The different Gram matrices associated with $\mathcal{S}$

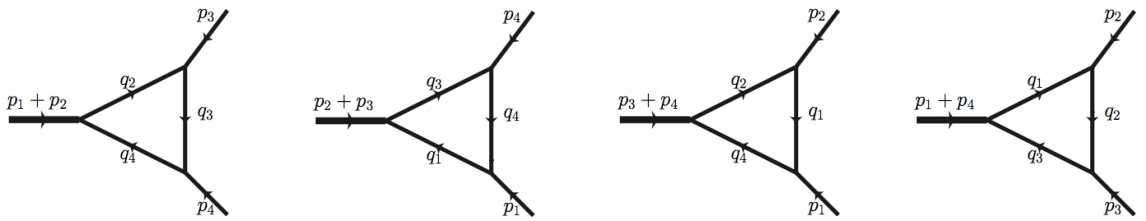
The different Gram matrices $G^{(a)}$ (for $a = 1, 2, 3, 4$) associated to $\mathcal{S}$ in Eq. (5.2.109)

$$G^{(1)} = \begin{pmatrix} 2s_2 & s_2 - s_3 + t & -s + s_1 + s_2 \\ s_2 - s_3 + t & 2t & s_1 - s_4 + t \\ -s + s_1 + s_2 & s_1 - s_4 + t & 2s_1 \end{pmatrix} \quad G^{(2)} = \begin{pmatrix} 2s_2 & s_2 + s_3 - t & s - s_1 + s_2 \\ s_2 + s_3 - t & 2s_3 & s + s_3 - s_4 \\ s - s_1 + s_2 & s + s_3 - s_4 & 2s \end{pmatrix}$$
$$G^{(3)} = \begin{pmatrix} 2t & -s_2 + s_3 + t & -s_1 + s_4 + t \\ -s_2 + s_3 + t & 2s_3 & -s + s_3 + s_4 \\ -s_1 + s_4 + t & -s + s_3 + s_4 & 2s_4 \end{pmatrix} \quad G^{(4)} = \begin{pmatrix} 2s_1 & s + s_1 - s_2 & s_1 + s_4 - t \\ s + s_1 - s_2 & 2s & s - s_3 + s_4 \\ s_1 + s_4 - t & s - s_3 + s_4 & 2s_4 \end{pmatrix} \tag{B.1.1}$$

and

$$\det(G) = \det(G^{(1)}) = \det(G^{(2)}) = \det(G^{(3)}) = \det(G^{(4)}) =$$
$$-2ss_1s_2 + 2ss_1s_3 - 2s_1^2s_3 + 2s_1s_2s_3 - 2s_1s_3^2 + 2ss_2s_4 + 2s_1s_2s_4 - 2s_2^2s_4 - 2ss_3s_4 + 2s_1s_3s_4 + 2s_2s_3s_4 - 2s_2s_4^2 - 2s^2t + 2ss_1t + 2ss_2t + 2ss_3t + 2s_1s_3t - 2s_2s_3t + 2ss_4t - 2s_1s_4t + 2s_2s_4t - 2st^2$$

B.1.2 The reduced Gram determinant of the box



(a) (b) (c) (d)

Figure B.1: The triangles in subfigure (a), (b), (c) and (d) are obtained from the box in Fig. (5.4) by pinching the propagator number 1, 2, 3 and 4, respectively.

Pinching the propagator number "1":

$$\mathcal{S}^{\{1\}} = \begin{pmatrix} -2m_2^2 & -m_2^2 - m_3^2 + s_3 & -m_2^2 - m_4^2 + s \\ -m_2^2 - m_3^2 + s_3 & -2m_3^2 & -m_3^2 - m_4^2 + s_4 \\ -m_2^2 - m_4^2 + s & -m_3^2 - m_4^2 + s_4 & -2m_4 2 \end{pmatrix} \tag{B.1.2}$$

$$\begin{aligned}\det(\mathcal{S}^{\{1\}}) = 2(&-m_2^2 m_3^2 s + m_3^4 s + m_2^2 m_4^2 s - m_3^2 m_4^2 s + m_3^2 s^2 + m_2^2 m_3^2 s_3 - m_2^2 m_4^2 s_3 - m_3^2 m_4^2 s_3 + m_4^4 s_3 - m_3^2 s s_3 \\ &- m_4^2 s s_3 + m_4^2 s_3^2 + m_2^4 s_4 - m_2^2 m_3^2 s_4 - m_2^2 m_4^2 s_4 + m_3^2 m_4^2 s_4 - m_2^2 s s_4 - m_3^2 s s_4 - m_2^2 s_3 s_4 - m_4^2 s_3 s_4 \\ &+ s s_3 s_4 + m_2^2 s_4^2)\end{aligned} \tag{B.1.3}$$

$$\det(G^{\{1\}}) = -s^2 + 2ss_3 - s_3^2 + 2ss_4 + 2s_3 s_4 - s_4^2 \tag{B.1.4}$$

Pinching the propagator number "2":

$$\mathcal{S}^{\{2\}} = \begin{pmatrix} -2m_1^2 & -m_1^2 - m_3^2 + t & -m_1^2 - m_4^2 + s_1 \\ -m_1^2 - m_3^2 + t & -2m_3^2 & -m_3^2 - m_4^2 + s_4 \\ -m_1^2 - m_4^2 + s_1 & -m_3^2 - m_4^2 + s_4 & -2m_4^2 \end{pmatrix} \tag{B.1.5}$$

$$\begin{aligned}\det(\mathcal{S}^{\{2\}}) = 2(&-m_1^2 m_3^2 s_1 + m_3^4 s_1 + m_1^2 m_4^2 s_1 - m_3^2 m_4^2 s_1 + m_3^2 s_1^2 + m_1^4 s_4 - m_1^2 m_3^2 s_4 - m_1^2 m_4^2 s_4 + m_3^2 m_4^2 s_4 \\ &- m_1^2 s_1 s_4 - m_3^2 s_1 s_4 + m_1^2 s_4^2 + m_1^2 m_3^2 t - m_1^2 m_4^2 t - m_3^2 m_4^2 t + m_4^4 t - m_3^2 s_1 t - m_4^2 s_1 t - m_1^2 s_4 t \\ &- m_4^2 s_4 t + s_1 s_4 t + m_4^2 t^2)\end{aligned} \tag{B.1.6}$$

$$\det(G^{\{2\}}) = -s_1^2 + 2s_1 s_4 - s_4^2 + 2s_1 t + 2s_4 t - t^2 \tag{B.1.7}$$

Pinching the propagator number "3":

$$\mathcal{S}^{\{3\}} = \begin{pmatrix} -2m_1^2 & -m_1^2 - m_2^2 + s_2 & -m_1^2 - m_4^2 + s_1 \\ -m_1^2 - m_2^2 + s_2 & -2m_2^2 & -m_2^2 - m_4^2 + s \\ -m_1^2 - m_4^2 + s_1 & -m_2^2 - m_4^2 + s & -2m_4^2 \end{pmatrix} \tag{B.1.8}$$

$$\begin{aligned}\det(\mathcal{S}^{\{3\}}) = 2(&m_1^4 s - m_1^2 m_2^2 s - m_1^2 m_4^2 s + m_2^2 m_4^2 s + m_1^2 s^2 - m_1^2 m_2^2 s_1 + m_2^4 s_1 + m_1^2 m_4^2 s_1 - m_2^2 m_4^2 s_1 - m_1^2 s s_1 \\ &- m_2^2 s s_1 + m_2^2 s_1^2 + m_1^2 m_2^2 s_2 - m_1^2 m_4^2 s_2 - m_2^2 m_4^2 s_2 + m_4^4 s_2 - m_1^2 s s_2 - m_4^2 s s_2 - m_2^2 s_1 s_2 - m_4^2 s_1 s_2 \\ &+ s s_1 s_2 + m_4^2 s_2^2)\end{aligned} \tag{B.1.9}$$

$$\det(G^{\{3\}}) = -s^2 + 2ss_1 - s_1^2 + 2ss_2 + 2s_1 s_2 - s_2^2 \tag{B.1.10}$$

Pinching the propagator number "4":

$$\mathcal{S}^{\{4\}} = \begin{pmatrix} -2m_1^2 & -m_1^2 - m_2^2 + s_2 & -m_1^2 - m_3^2 + t \\ -m_1^2 - m_2^2 + s_2 & -2m_2^2 & -m_2^2 - m_3^2 + s_3 \\ -m_1^2 - m_3^2 + t & -m_2^2 - m_3^2 + s_3 & -2m_3^2 \end{pmatrix} \tag{B.1.11}$$

$$\begin{aligned}\det(\mathcal{S}^{\{4\}}) = 2(&m_1^2 m_2^2 s_2 - m_1^2 m_3^2 s_2 - m_2^2 m_3^2 s_2 + m_3^4 s_2 + m_3^2 s_2^2 + m_1^4 s_3 - m_1^2 m_2^2 s_3 - m_1^2 m_3^2 s_3 + m_2^2 m_3^2 s_3 \\ &- m_1^2 s_2 s_3 - m_3^2 s_2 s_3 + m_1^2 s_3^2 - m_1^2 m_2^2 t + m_2^4 t + m_1^2 m_3^2 t - m_2^2 m_3^2 t - m_2^2 s_2 t - m_3^2 s_2 t - m_1^2 s_3 t \\ &- m_2^2 s_3 t + s_2 s_3 t + m_2^2 t^2)\end{aligned} \tag{B.1.12}$$

$$\det(G^{\{4\}}) = -s_2^2 + 2s_2 s_3 - s_3^2 + 2s_2 t + 2s_3 t - t^2 \tag{B.1.13}$$

B.2 Characterization of the studied kinematics

In this section, we want to show the physical configurations that one can encounter in the calculation of amplitudes at NLO order. It concerns the 4-point functions as well as the 3-point functions.

The encountered Feynman diagrams at next-to-leading order calculation of scattering amplitudes for processes of interest in the collision of two particles, say $2 \longrightarrow n$, or in particle decay, say $1 \longrightarrow n$ (where n is the number of particles in the final state) can not have more than two space-like external legs; since, in such process, a space-like momenta can be obtained only from the combination of incoming momenta (1 or 2 legs at most) with outgoing momenta. All possible 4-point functions and 3-point functions which may be encountered in one-loop calculation are shown in Fig. (B.2),

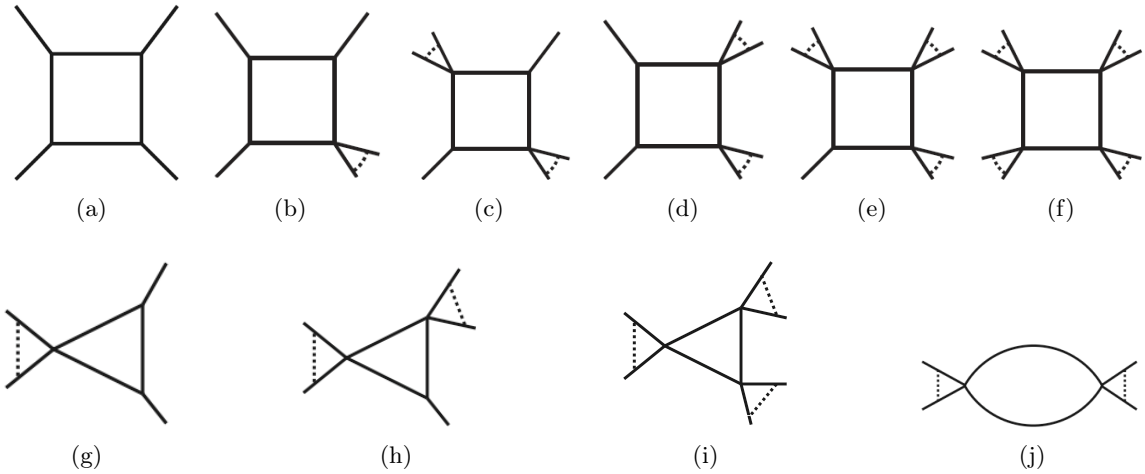


Figure B.2: These diagrams are obtained by contracting one or more (or non) propagators. Since, the one-loop Feynman diagrams involved in the amplitude calculation of the process $1 \to n$ and $2 \to n$ have at most two incoming external legs, then the momenta of the corners of each of these graphs is a combination of: **a)** only incoming momentum which implies that the resulting momenta is time-like, **b)** only outgoing momentum which implies that the resulting momenta is time-like, **c)** one incoming and one or more outgoing momentum which implies that the resulting momenta may be space-like, **d)** two incoming and one or more outgoing momentum which implies that the resulting momenta may be space-like.

For these configurations, the 3-point function Gram determinants are negative as we will prove in the next paragraph (the 3-point function Gram determinants stand for $\det(G)$ defined in Eq. (5.1.62) or for the box reduced Gram determinants $\det(G^{\{i\}})$ for $i = 1, 2, 3, 4$ defined in the previous section).

We notice that for Feynman diagrams with more than one-loop (encountered in next-to-next-to leading calculation for example), this argumentation may not be correct since the external legs can depend on the integration momenta of other loops, which means that the nature of this momenta is unknown. For example, the box in Fig.(B.3) has four external legs of unknown nature, since they depend on the momenta circulating in the two triangles. Then, the contacted boxes may have some negative 3-point Gram determinant as we show later on.

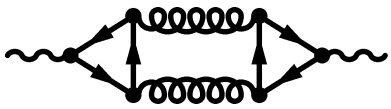


Figure B.3: Example of a box with unknown nature of external momenta

The sign of the reduced Gram determinants:

The reduced Gram determinants $\det(G^{\{i\}})$ (where $i = 1, \cdots, 4$, which stands for the label of the pinched propagator of the box in Fig. (5.4) to obtain the 3-point functions $I_3^{n+l}(\mathcal{S}\setminus\{i\})$, $\mathcal{S}\setminus\{i\}$ is 3×3 matrix obtained by omitting the line and the column i, it is denoted $\mathcal{S}^{\{i\}}$ in subsection B.1.2) is given by

$$\det(G^{\{i\}}) = 4p_a^2\, p_b^2 - 4\,(p_a\cdot p_b)^2 \tag{B.2.14}$$

$$= -\lambda(p_a^2, p_b^2, p_c^2), \tag{B.2.15}$$

where p_a, p_b and p_c are the momenta of the external legs of the contracted box, .i.e. the triangles in Fig. (B.1), with $p_c^2 = (p_a + p_b)^2$. Let us define the quantities

$$\begin{aligned}
\Delta^{(1)} &= -\lambda(s_3, s_4, s) = -\det(G^{\{1\}}) \\
\Delta^{(2)} &= -\lambda(s_1, s_4, t) = -\det(G^{\{2\}}) \\
\Delta^{(3)} &= -\lambda(s_1, s_2, s) = -\det(G^{\{3\}}) \\
\Delta^{(4)} &= -\lambda(s_2, s_3, t) = -\det(G^{\{4\}})
\end{aligned} \tag{B.2.16}$$

$\Delta^{(1)}$ is the discriminant of the equation defining α Eq. (5.2.121), and $\Delta^{(i)}$ (with $i = 2, 3, 4$) stand for the discriminants Δ of the equation defining β Eq. (5.2.151) for each sector, where

$$\Delta^{I} = \Delta^{(4)}, \qquad \Delta^{II} = \Delta^{(3)}, \qquad \Delta^{III} = \Delta^{(2)}. \tag{B.2.17}$$

As we mentioned above, the contracted box (triangle) can have two time-like, one time-like and non time-like momenta. In the following, we will study the signs of $\det(G^{(i)})$ in these possible configurations. Let us assume that the momentum of the triangle are all incoming, then

$$p_a + p_b + p_c = 0 \tag{B.2.18}$$

i) $p_a^2 > 0$ *and* p_b^2, p_c^2 *are arbitrary:* in the center of mass frame of p_a $(p_a^0, \vec{0})$ (since $p_a^2 \geq 0$ (time-like), we can find such Lorentz transformation), the scalar product of p_a and p_b is given by

$$p_a \cdot p_b = p_a^0\, p_b^0 \tag{B.2.19}$$

where $p_b = (p_b^0, \vec{p}_b)$. Then,

$$\begin{aligned}
\det(G^{(i)}) &\propto p_a^2\, p_b^2 - (p_a.p_b)^2 \\
&= (p_a^0)^2\,((p_b^0)^2 - |\vec{p}_b|^2) - (p_a^0)^2\,(p_b^0)^2 \\
&= -(p_a^0)^2\,|\vec{p}_b|^2 \leq 0.
\end{aligned} \tag{B.2.20}$$

it vanishes if $\vec{p}_b = \vec{0}$ which means that $\vec{p}_a$ and $\vec{p}_b$ are collinear.

ii) *if $p_a^2 = 0$ and p_b^2, p_c^2 are arbitrary:*

$$\det(G^{(i)}) \propto -(p_a.p_b)^2 \leq 0 \tag{B.2.21}$$

It vanishes if:
a) p_b light-like and proportional to p_a
b) p_b is space-like and orthogonal to p_a

iii) $p_a^2 < 0$ and $p_b^2 < 0$ and $p_c^2 < 0$: $\det(G^{(i)})$ can be positive. Fortunately, we can not encounter one-loop Feynman diagrams (or pinched Feynman diagrams) with more than two external legs in configurations of a decay $(1 \to n)$ or collision $(2 \to n)$.
Then, the Gram determinants associated to the 3-point functions (contracted 4-point function) can not be positives for physical configurations at NLO order. So, the parameters α and β are real.

B.3 More in I_4^{4+l}

B.3.1 The logarithms arguments of I_4^{4+l}

sector I		
Q1(y)+$i\lambda$	$a+b+c+d+e+f+(h+j+k)y+gy^2$	$m_3^2+(m_2^2-m_3^2-s_3)y+s_3y^2$
Q2(y)+$i\lambda$	$f+(d+e+k)y+(a+b+c+g+h+j)y^2$	$m_1^2+(-m_1^2+m_2^2-s_2)y+s_2y^2$
Q3(y)+$i\lambda$	$f+(d+e)y+(a+b+c)y^2$	$m_1^2+(-m_1^2+m_3^2-t)y+ty^2$
L1(y) +$i\lambda$	$2b+c+e+\alpha(h+j+k)+(2\alpha g+j)y$	$m_3^2-m_4^2+\alpha(m_2^2-m_3^2-s_3)$ $+s_4+(s-s_3+2\alpha s_3-s_4)y$
L2(y) +$i\lambda$	$e+\alpha k+(2b+c+j+\alpha(2g+h+j))y$	$m_3^2-m_4^2+s_1-t+\alpha(m_2^2-m_3^2-s_2+t)$ $+(-s_1+s_12+t-s_3+\alpha(s_2-t+s_3))y$
L3(y)+$i\lambda$	$e+\alpha k+(2b+c+\alpha(h+j))y$	$m_3^2-m_4^2+s_1-t+\alpha(m_2^2-m_3^2-s_2+t)$ $+(-s_1+t+\alpha(s_2-t-s_3)+s_4)y$
sector II		
Q1(y) +$i\lambda$	$a+d+f+(c+e+h+k)y+(b+g+j)y^2$	$m_4^2+(m_2^2-m_4^2-s)y+sy^2$
Q2(y)+$i\lambda$	$f+(d+e+k)y+(a+b+c+g+h+j)y^2$	$m_1^2+(-m_1^2+m_2^2-s_2)y+s_2y^2$
Q3(y)+$i\lambda$	$f+dy+ay^2$	$m_1^2+(-m_1^2+m_4^2-s_1)y+s_1y^2$
L1(y)+$i\lambda$	$c+e+\alpha(h+k)+(2b+j+\alpha(2g+j))y$	$m_3^2-m_4^2-s_4+\alpha(m_2^2-m_3^2-s_12+s_4)$ $+(s_12-s_3+\alpha(s_12+s_3-s_4)+s_4)y$
L2(y)+$i\lambda$	$e+\alpha k+(2b+c+j+\alpha(2g+h+j))y$	$m_3^2-m_4^2+s_1-t+\alpha(m_2^2-m_3^2-s_2+t)$ $+(-s_1+s_12+t-s_3+\alpha(s_2-t+s_3))y$
L3(y)+$i\lambda$	$e+\alpha k+(c+\alpha h)y$	$m_3^2-m_4^2+s_1-t+\alpha(m_2^2-m_3^2-s_2+t)$ $+(-s_1+t-s_4+\alpha(-s_12+s_2-t+s_4))y$
sector III		
Q1(y)+$i\lambda$	$a+d+f+(c+e)y+by^2$	$m_4^2+(m_3^2-m_4^2-s_4)y+s_4y^2$
Q2(y)+$i\lambda$	$f+(d+e)y+(a+b+c)y^2$	$m_1^2+(-m_1^2+m_3^2-t)y+ty^2$
Q3(y)+$i\lambda$	$f+dy+ay^2$	$m_1^2+(-m_1^2+m_4^2-s_1)y+s_1y^2$
L1(y)+$i\lambda$	$c+e+\alpha(h+k)+(2b+\alpha j)y$	$m_3^2-m_4^2-s_4+\alpha(m_2^2-m_3^2-s_12+s_4)$ $+(\alpha(s_12-s_3-s_4)+2s_4)y$
L2(y)+$i\lambda$	$e+\alpha k+(2b+c+\alpha(h+j))y$	$m_3^2-m_4^2+s_1-t+\alpha(m_2^2-m_3^2-s_2+t)$ $+(-s_1+t+\alpha(s_2-t-s_3)+s_4)y$
L3(y)+$i\lambda$	$e+\alpha k+(c+\alpha h)y$	$m_3^2-m_4^2+s_1-t+\alpha(m_2^2-m_3^2-s_2+t)$ $+(-s_1+t-s_4+\alpha(-s_12+s_2-t+s_4))y$

Where

$$Q_1(y) = By^2 + (E+C)y + A + D + F - i\lambda \tag{B.3.22}$$
$$Q_2(y) = (A+B+C)y^2 + (E+D)y + F - i\lambda \tag{B.3.23}$$
$$Q_3(y) = Ay^2 + Dy + F - i\lambda \tag{B.3.24}$$

and

$$L_1(y) = Hy + G + J - i\lambda \tag{B.3.25}$$
$$L_2(y) = (G+H)y + J - i\lambda \tag{B.3.26}$$
$$L_3(y) = Gy + J - i\lambda \tag{B.3.27}$$

are the quadratic (Q_i) and the linear (L_i) logarithm arguments involved in I_4^{4+l} ($l = 2, 4, 6, 8$), receptively. The imaginary parts of Q_i are always negative even complex masses are involved. For example, the imaginary part of Q_1 for the sector I is

$$\mathrm{Im}(Q_1) = \mathrm{Im}(m_3^2)(1-y) + \mathrm{Im}(m_2^2)y - \lambda < 0, \qquad \text{since} \qquad \mathrm{Im}(m_i^2) \leq 0 \tag{B.3.28}$$

The imaginary parts of L_i can, always, chosen to be negative as explained in chapter 5.

* The parameter α in term of the entries of $\mathcal{S}$ is given by:

$$\alpha = \frac{-(s - s_3 - s_4) + \varepsilon_\alpha \sqrt{s^2 + s_3^2 + s_4^2 - 2ss_3 - 2ss_4 - 2s_3s_4}}{2s_3}, \qquad \varepsilon_\alpha = \pm \tag{B.3.29}$$
$$= \frac{-\sqrt{\Delta_\alpha + 4\, s_3 s_4} + \varepsilon_\alpha \sqrt{\Delta_\alpha}}{2s_3} \tag{B.3.30}$$

$$\Delta_\alpha = s^2 + s_3^2 + s_4^2 - 2ss_3 - 2ss_4 - 2s_3s_4 \tag{B.3.31}$$

From Eq. (B.3.30) the parameter α vanishes (i.e. $K = 0$ for the sector III), if:
- $\varepsilon_\alpha = +1$ and $s_4 = 0$
- $\varepsilon_\alpha = \pm$ and $s_4 = 0$ and $s = s_3 \neq 0$ (implies that $\Delta_\alpha = 0$)

and it equals to 1 (i.e. $K = 0$ for the sector II), if:
- $\varepsilon_\alpha = +1$ and $s_3 = s = 0$ and $s_4 < 0$
- $\varepsilon_\alpha = -1$ and $s_4 = s = 0$ and $s_3 < 0$
- $\varepsilon_\alpha = \pm$ and $s = 0$ and $s_3 = s_4 < 0$ (implies that $\Delta_\alpha = 0$).

* The poles of $I_{4,1}^6$ in Eq. (5.2.146) are:

$$t_0^{(1)} = -\frac{(m_2^2 - m_3^2 - s_3)\alpha + m_3^2 - m_4^2 + s_4}{\varepsilon_\alpha \sqrt{\det(G^{\{1\}})}} \tag{B.3.32}$$
$$t_0^{(2)} = -\frac{(m_2^2 - m_3^2 + s_4 - s)\alpha + m_3^2 - m_4^2 - s_4}{-\alpha\, \varepsilon_\alpha \sqrt{\det(G^{\{1\}}}} \tag{B.3.33}$$
$$t_0^{(3)} = -\frac{(m_2^2 - m_3^2 + s_4 - s)\alpha + m_3^2 - m_4^2 - s_4}{(1-\alpha)\, \varepsilon_\alpha \sqrt{\det(G^{\{1\}}}} \tag{B.3.34}$$

Here α is not the same α of the tree point function given in Eq. (5.1.37), one can obtain the former one from the later one by making the replacement Eq. (5.2.148).

B.3.2 In term of the Golem reduction notation!

$$M_0^I = \det(\mathcal{S})\, b_4 = \bar{b}_4 \tag{B.3.35}$$
$$M_0^{II} = -(1-\alpha)\det(\mathcal{S})\, b_3 = -(1-\alpha)\bar{b}_3 \tag{B.3.36}$$
$$M_0^{III} = -\alpha \det(\mathcal{S})\, b_2 = -\alpha \bar{b}_2 \tag{B.3.37}$$

$$M_1^I = \det(\mathcal{S}^{\{4\}})/2 \tag{B.3.38}$$
$$M_1^{II} = \det(\mathcal{S}^{\{3\}})/2 \tag{B.3.39}$$
$$M_1^{III} = \det(\mathcal{S}^{\{2\}})/2 \tag{B.3.40}$$

$$\det(\mathcal{S}) =$$
$$m_2^4 s_1^2 - 2m_2^2 m_3^2 s_1^2 + m_3^4 s_1^2 - 2m_1^2 m_2^2 s_1 s + 2m_1^2 m_3^2 s_1 s + 2m_2^2 m_3^2 s_1 s - 2m_3^4 s_1 s + m_1^4 s^2 - 2m_1^2 m_3^2 s^2 + m_3^4 s^2 + 2m_2^2 m_3^2 s_1 s_2 - 2m_3^4 s_1 s_2 - 2m_2^2 m_4^2 s_1 s_2 + 2m_3^2 m_4^2 s_1 s_2 + 2m_1^2 m_3^2 s s_2 - 2m_3^4 s s_2 - 2m_1^2 m_4^2 s s_2 + 2m_3^2 m_4^2 s s_2 - 4m_3^2 s_1 s s_2 + m_3^4 s_2^2 - 2m_3^2 m_4^2 s_2^2 + m_4^4 s_2^2 - 2m_2^4 s_1 t + 2m_2^2 m_3^2 s_1 t + 2m_2^2 m_4^2 s_1 t - 2m_3^2 m_4^2 s_1 t + 2m_1^2 m_2^2 s t - 4m_1^2 m_3^2 s t + 2m_2^2 m_3^2 s t + 2m_1^2 m_4^2 s t - 4m_2^2 m_4^2 s t + 2m_3^2 m_4^2 s t + 2m_2^2 s_1 s t + 2m_3^2 s_1 s t - 2m_1^2 s^2 t - 2m_3^2 s^2 t - 2m_2^2 m_3^2 s_2 t + 2m_2^2 m_4^2 s_2 t + 2m_3^2 m_4^2 s_2 t - 2m_4^4 s_2 t + 2m_3^2 s s_2 t + 2m_4^2 s s_2 t + m_2^4 t^2 - 2m_2^2 m_4^2 t^2 + m_4^4 t^2 - 2m_2^2 s t^2 - 2m_4^2 s t^2 + s^2 t^2 + 2m_1^2 m_2^2 s_1 s_3 + 2m_1^2 m_3^2 s_1 s_3 - 4m_2^2 m_3^2 s_1 s_3 - 4m_1^2 m_4^2 s_1 s_3 + 2m_2^2 m_4^2 s_1 s_3 + 2m_3^2 m_4^2 s_1 s_3 - 2m_2^2 s_1^2 s_3 - 2m_3^2 s_1^2 s_3 - 2m_1^4 s s_3 + 2m_1^2 m_3^2 s s_3 + 2m_1^2 m_4^2 s s_3 - 2m_3^2 m_4^2 s s_3 + 2m_1^2 s_1 s s_3 + 2m_3^2 s_1 s s_3 - 2m_1^2 m_3^2 s_2 s_3 + 2m_1^2 m_4^2 s_2 s_3 + 2m_3^2 m_4^2 s_2 s_3 - 2m_4^4 s_2 s_3 + 2m_3^2 s_1 s_2 s_3 + 2m_4^2 s_1 s_2 s_3 - 2m_1^2 m_2^2 t s_3 + 2m_1^2 m_4^2 t s_3 + 2m_2^2 m_4^2 t s_3 - 2m_4^4 t s_3 + 2m_2^2 s_1 t s_3 + 2m_4^2 s_1 t s_3 + 2m_1^2 s t s_3 + 2m_4^2 s t s_3 - 2 s_1 s t s_3 - 4m_4^2 s_2 t s_3 + m_1^4 s_3^2 - 2m_1^2 m_4^2 s_3^2 + m_4^4 s_3^2 - 2m_1^2 s_1 s_3^2 - 2m_4^2 s_1 s_3^2 + s_1^2 s_3^2 + 2m_1^2 m_2^2 s_1 s_4 - 2m_2^4 s_1 s_4 - 2m_1^2 m_3^2 s_1 s_4 + 2m_2^2 m_3^2 s_1 s_4 - 2m_1^4 s s_4 + 2m_1^2 m_2^2 s s_4 + 2m_1^2 m_3^2 s s_4 - 2m_2^2 m_3^2 s s_4 - 4m_1^2 m_2^2 s_2 s_4 + 2m_1^2 m_3^2 s_2 s_4 + 2m_2^2 m_3^2 s_2 s_4 + 2m_1^2 m_4^2 s_2 s_4 + 2m_2^2 m_4^2 s_2 s_4 - 4m_3^2 m_4^2 s_2 s_4 + 2m_2^2 s_1 s_2 s_4 + 2m_3^2 s_1 s_2 s_4 + 2m_1^2 s s_2 s_4 + 2m_3^2 s s_2 s_4 - 2m_3^2 s_2^2 s_4 - 2m_4^2 s_2^2 s_4 + 2m_1^2 m_2^2 t s_4 - 2m_2^4 t s_4 - 2m_1^2 m_4^2 t s_4 + 2m_2^2 m_4^2 t s_4 - 4m_2^2 s_1 t s_4 + 2m_1^2 s t s_4 + 2m_2^2 s t s_4 + 2m_2^2 s_2 t s_4 + 2m_4^2 s_2 t s_4 - 2 s s_2 t s_4 - 2m_1^4 s_3 s_4 + 2m_1^2 m_2^2 s_3 s_4 + 2m_1^2 m_4^2 s_3 s_4 - 2m_2^2 m_4^2 s_3 s_4 + 2m_1^2 s_1 s_3 s_4 + 2m_2^2 s_1 s_3 s_4 - 4m_1^2 s s_3 s_4 + 2m_1^2 s_2 s_3 s_4 + 2m_4^2 s_2 s_3 s_4 - 2 s_1 s_2 s_3 s_4 + m_1^4 s_4^2 - 2m_1^2 m_2^2 s_4^2 + m_2^4 s_4^2 - 2m_1^2 s_2 s_4^2 - 2m_2^2 s_2 s_4^2 + s_2^2 s_4^2$$

$$\det(\mathbf{G}) =$$
$$-2(s_1 s s_2 - s_1 s t + s^2 t - s s_2 t + s t^2 + s_1^2 s_3 - s_1 s s_3 - s_1 s_2 s_3 - s_1 t s_3 - s t s_3 + s_2 t s_3 + s_1 s_3^2 - s_1 s_2 s_4 - s s_2 s_4 + s_2^2 s_4 + s_1 t s_4 - s t s_4 - s_2 t s_4 - s_1 s_3 s_4 + s s_3 s_4 - s_2 s_3 s_4 + s_2 s_4^2)$$

$$\bar{\mathbf{b}}_\mathbf{1} =$$
$$m_2^2 s_1 s - m_3^2 s_1 s - m_1^2 s^2 + m_3^2 s^2 - m_3^2 s s_2 + m_4^2 s s_2 - m_2^2 s t + 2m_3^2 s t - m_4^2 s t + s^2 t - m_2^2 s_1 s_3 - m_3^2 s_1 s_3 + 2m_4^2 s_1 s_3 + 2m_1^2 s s_3 - m_3^2 s s_3 - m_4^2 s s_3 - s_1 s s_3 + m_3^2 s_2 s_3 - m_4^2 s_2 s_3 + m_2^2 t s_3 - m_4^2 t s_3 - s t s_3 - m_1^2 s_3^2 + m_4^2 s_3^2 + s_1 s_3^2 - m_2^2 s_1 s_4 + m_3^2 s_1 s_4 + 2m_1^2 s s_4 - m_2^2 s s_4 - m_3^2 s s_4 + 2m_2^2 s_2 s_4 - m_3^2 s_2 s_4 - m_4^2 s_2 s_4 - s s_2 s_4 - m_2^2 t s_4 + m_4^2 t s_4 - s t s_4 + 2m_1^2 s_3 s_4 - m_2^2 s_3 s_4 - m_4^2 s_3 s_4 - s_1 s_3 s_4 + 2 s s_3 s_4 - s_2 s_3 s_4 - m_1^2 s_4^2 + m_2^2 s_4^2 + s_2 s_4^2$$

$$\bar{\mathbf{b}}_\mathbf{2} =$$
$$-(m_2^2 s_1^2) + m_3^2 s_1^2 + m_1^2 s_1 s - m_3^2 s_1 s - m_3^2 s_1 s_2 + m_4^2 s_1 s_2 + 2m_2^2 s_1 t - m_3^2 s_1 t - m_4^2 s_1 t - m_1^2 s t - m_3^2 s t + 2m_4^2 s t - s_1 s t + m_3^2 s_2 t - m_4^2 s_2 t - m_2^2 t^2 + m_4^2 t^2 + s t^2 - m_1^2 s_1 s_3 + 2m_3^2 s_1 s_3 - m_4^2 s_1 s_3 + s_1^2 s_3 + m_1^2 t s_3 - m_4^2 t s_3 - s_1 t s_3 - m_1^2 s_1 s_4 + 2m_2^2 s_1 s_4 - m_3^2 s_1 s_4 - m_1^2 s s_4 + m_3^2 s s_4 + 2m_1^2 s_2 s_4 - m_3^2 s_2 s_4 - m_4^2 s_2 s_4 - s_1 s_2 s_4 - m_1^2 t s_4 + 2m_2^2 t s_4 - m_4^2 t s_4 + 2 s_1 t s_4 - s t s_4 - s_2 t s_4 - m_1^2 s_3 s_4 + m_4^2 s_3 s_4 - s_1 s_3 s_4 + m_1^2 s_4^2 - m_2^2 s_4^2 + s_2 s_4^2$$

$$\bar{\mathbf{b}}_\mathbf{3} =$$
$$m_2^2 s_1^2 - m_3^2 s_1^2 - m_1^2 s_1 s - m_2^2 s_1 s + 2m_3^2 s_1 s + m_1^2 s^2 - m_3^2 s^2 - m_2^2 s_1 s_2 + 2m_3^2 s_1 s_2 - m_4^2 s_1 s_2 - m_1^2 s s_2 + 2m_3^2 s s_2 - m_4^2 s s_2 + 2 s_1 s s_2 - m_3^2 s_2^2 + m_4^2 s_2^2 - m_2^2 s_1 t + m_4^2 s_1 t + 2m_1^2 s t - m_2^2 s t - m_4^2 s t - s_1 s t + s^2 t + m_2^2 s_2 t - m_4^2 s_2 t - s s_2 t - m_1^2 s_1 s_3 + 2m_2^2 s_1 s_3 - m_4^2 s_1 s_3 + s_1^2 s_3 - m_1^2 s s_3 + m_4^2 s s_3 - s_1 s s_3 + m_1^2 s_2 s_3 - m_4^2 s_2 s_3 - s_1 s_2 s_3 + m_1^2 s_1 s_4 - m_2^2 s_1 s_4 - m_1^2 s s_4 + m_2^2 s s_4 - m_1^2 s_2 s_4 - m_2^2 s_2 s_4 + 2m_4^2 s_2 s_4 - s_1 s_2 s_4 - s s_2 s_4 + s_2^2 s_4$$

$$\bar{\mathbf{b}}_\mathbf{4} =$$
$$m_2^2 s_1 s_2 - m_3^2 s_1 s_2 + m_1^2 s s_2 - m_3^2 s s_2 + m_3^2 s_2^2 - m_4^2 s_2^2 - m_2^2 s_1 t + m_3^2 s_1 t - m_1^2 s t + 2m_2^2 s t - m_3^2 s t - m_2^2 s_2 t - m_3^2 s_2 t + 2m_4^2 s_2 t - s s_2 t + m_2^2 t^2 - m_4^2 t^2 + s t^2 + 2m_1^2 s_1 s_3 - m_2^2 s_1 s_3 - m_3^2 s_1 s_3 - m_1^2 s s_3 + m_3^2 s s_3 - m_1^2 s_2 s_3 - m_3^2 s_2 s_3 + 2m_4^2 s_2 s_3 - s_1 s_2 s_3 - m_1^2 t s_3 - m_2^2 t s_3 + 2m_4^2 t s_3 - s_1 t s_3 - s t s_3 + 2 s_2 t s_3 + m_1^2 s_3^2 - m_4^2 s_3^2 + s_1 s_3^2 - m_1^2 s_2 s_4 - m_2^2 s_2 s_4 + 2m_3^2 s_2 s_4 + s_2^2 s_4 + m_1^2 t s_4 - m_2^2 t s_4 - s_2 t s_4 - m_1^2 s_3 s_4 + m_2^2 s_3 s_4 - s_2 s_3 s_4$$

$$\beta^I = \frac{-s_2 + s_3 + t + \varepsilon_\beta \sqrt{-\det(G^{\{4\}})}}{2s_3} \tag{B.3.41}$$

$$\beta^{II} = \frac{s + s_1 - s_2 + \varepsilon_\beta \sqrt{-\det(G^{\{3\}})}}{2s} \tag{B.3.42}$$

$$\beta^{III} = \frac{s_1 + s_4 - t + \varepsilon_\beta \sqrt{-\det(G^{\{2\}})}}{2s_4} \tag{B.3.43}$$

* The $b_i^{\{j\}}$ associated to the three point functions $I_3^4(S \setminus \{j\})$:

Sector	$\bar{b}_j^{\{1\}}$	$(2B(G+J)-(C+E)H)/K$
I	$\bar{b}_1^{\{1\}}$	$-m_2^2 s + m_3^2 s + m_2^2 s_3 + m_3^2 s_3 - 2m_4^2 s_3 + s s_3 - s_3^2 + m_2^2 s_4 - m_3^2 s_4 + s_3 s_4$
II	$\bar{b}_2^{\{1\}}$	$m_2^2 s - 2m_3^2 s + m_4^2 s - s^2 - m_2^2 s_3 + m_4^2 s_3 + s s_3 + m_2^2 s_4 - m_4^2 s_4 + s s_4$
III	$\bar{b}_3^{\{1\}}$	$m_3^2 s - m_4^2 s - m_3^2 s_3 + m_4^2 s_3 - 2m_2^2 s_4 + m_3^2 s_4 + m_4^2 s_4 + s s_4 + s_3 s_4 - s_4^2$

Sector	$\bar{b}_1^{\{i\}}$	$-C^2 + 4BA + 2BD - EC$
I	$\bar{b}_1^{\{4\}}$	$-m_2^2 s_2 + m_3^2 s_2 - 2m_1^2 s_3 + m_2^2 s_3 + m_3^2 s_3 + s_2 s_3 - s_3^2 + m_2^2 t - m_3^2 t + s_3 t$
II	$\bar{b}_1^{\{3\}}$	$-2m_1^2 s + m_2^2 s + m_4^2 s - s^2 + m_2^2 s_1 - m_4^2 s_1 + s s_1 - m_2^2 s_2 + m_4^2 s_2 + s s_2$
III	$\bar{b}_1^{\{2\}}$	$m_3^2 s_1 - m_4^2 s_1 - 2m_1^2 s_4 + m_3^2 s_4 + m_4^2 s_4 + s_1 s_4 - s_4^2 - m_3^2 t + m_4^2 t + s_4 t$

Sector	$\bar{b}_2^{\{i\}}$	$CD - 2AE$
I	$\bar{b}_2^{\{4\}}$	$-m_1^2 s_2 + m_3^2 s_2 + m_1^2 s_3 - m_3^2 s_3 + m_1^2 t - 2m_2^2 t + m_3^2 t + s_2 t + s_3 t - t^2$
II	$\bar{b}_2^{\{3\}}$	$m_1^2 s - m_4^2 s + m_1^2 s_1 - 2m_2^2 s_1 + m_4^2 s_1 + s s_1 - s_1^2 - m_1^2 s_2 + m_4^2 s_2 + s_1 s_2$
III	$\bar{b}_2^{\{2\}}$	$m_1^2 s_1 - 2m_3^2 s_1 + m_4^2 s_1 - s_1^2 + m_1^2 s_4 - m_4^2 s_4 + s_1 s_4 - m_1^2 t + m_4^2 t + s_1 t$

Sector	$\bar{b}_3^{\{i\}}$	$-2BD + 2AE + C(E-D)$
I	$\bar{b}_3^{\{4\}}$	$m_1^2 s_2 + m_2^2 s_2 - 2m_3^2 s_2 - s_2^2 + m_1^2 s_3 - m_2^2 s_3 + s_2 s_3 - m_1^2 t + m_2^2 t + s_2 t$
II	$\bar{b}_3^{\{3\}}$	$m_1^2 s - m_2^2 s - m_1^2 s_1 + m_2^2 s_1 + m_1^2 s_2 + m_2^2 s_2 - 2m_4^2 s_2 + s s_2 + s_1 s_2 - s_2^2$
III	$\bar{b}_3^{\{2\}}$	$-m_1^2 s_1 + m_3^2 s_1 + m_1^2 s_4 - m_3^2 s_4 + m_1^2 t + m_3^2 t - 2m_4^2 t + s_1 t + s_4 t - t^2$

where $b_i^{\{j\}}$ stands for the Golem reduction coefficients b_i (see chapter 4) associated to the 3-point functions $I_3^4(S \setminus \{j\})$, they are defined by:

$$\sum_{j=1}^{3} b_i^{\{j\}} (\mathcal{S}_{ki}^{\{j\}})^{-1} = 1, \qquad \sum_{i=1}^{3} \bar{b}_i^{\{j\}} = \det(G^{\{j\}}), \qquad \bar{b}_i^{\{j\}} = \det(\mathcal{S}^{\{j\}})\, b_i^{\{j\}}. \tag{B.3.44}$$

B.4 The vanishing contributions in section 5.2.2

B.4.1 Vanishing terms of Eq. (5.2.156): *i)* $K \ln(Sx+T)/S$

Let's call the contribution of $\frac{K}{S}\ln(Sx+T)$ to I_4^6, J_1. Then,

$$\begin{aligned} J_1 &= \sum_{i=I}^{III} \left\{ \int_0^{1-\beta} dz\, \frac{K}{S} \ln(Sx+T) \Big|_{z/(1-\beta)}^{1} - \int_0^{-\beta} dz\, \frac{K}{S} \ln(Sx+T) \Big|_{z/(-\beta)}^{1} \right\} \\ &= \sum_{i=I}^{III} \int_0^1 dx \int_{-\beta x}^{(1-\beta)x} dz \frac{K}{Sx+T} \end{aligned} \tag{B.4.45}$$

By making the shift $z \to z - \beta$, J_1 become

$$J_1 = \sum_{i=I}^{III} \int_0^1 dx \int_0^x dz \frac{K}{Gx + Hz + J - i\lambda} \tag{B.4.46}$$

Moreover, we make the shift $z \to t - \gamma$ (with $\gamma = -G/H$), then J_1 becomes

$$J_1 = \sum_{i=I}^{III} \int_0^1 dx \int_{-\gamma x}^{(1-\gamma)x} dt \frac{K}{Ht + J - i\lambda} \tag{B.4.47}$$

We split the integral on t, the we exchange the oder of integration and finally integrate over x, we find

$$J_1 = \sum_{i=I}^{III} K \int_0^1 dt \left\{ \frac{1}{Ht + G + Ji\lambda} - \frac{t}{(G+H)t + Ji\lambda} + \frac{t}{Gt + J - i\lambda} \right\} \tag{B.4.48}$$

By expressing the denominators in term of the small letters (see table above) and summing over the three sectors, we find that the second term and the third term of the integrand of this equation sum up to zero. After integrating the remaining term on t, we find

$$J_1 = \sum_{i=I}^{III} \frac{K}{H} \ln\left(\frac{G + H + J - i\lambda}{G + J - i\lambda} \right) \tag{B.4.49}$$

which sum up to zero over the three sectors. Then,

$$J_1 = 0 \tag{B.4.50}$$

B.4.2 Vanishing terms of Eq. (5.2.156) *ii)*: sum of the first and the last two terms

Let's call the contribution of the of and the last two terms in Eq. (5.2.156) to I_4^6, J_2. Then,

$$\begin{aligned} J_2 = -\sum_{i=I}^{III} \frac{K}{H} \Bigg\{ & \int_0^{1-\beta} dz \left[\frac{x(C+2B\beta)}{U} + \left(\frac{B}{C+2B\beta} - \frac{M_1}{C+2B\beta} \frac{1}{U^2} \right) \ln(Ux+V) \right] \Bigg|_{z/(1-\beta)}^{1} \\ & - \int_0^{-\beta} dz \left[\frac{x(C+2B\beta)}{U} + \left(\frac{B}{C+2B\beta} - \frac{M_1}{C+2B\beta} \frac{1}{U^2} \right) \ln(Ux+V) \right] \Bigg|_{z/(-\beta)}^{1} \Bigg\} \end{aligned} \tag{B.4.51}$$

We combine the terms coming from the upper limit on x ($x = 1$) and make the shifts: $z = t - \beta$, $z = (1-\beta)t$ and $z = -\beta z$ in the parts with $\int_{-\beta} 1 - \beta$, $\int_0^{1-\beta}$ and $\int_0^{-\beta}$, respectively. We find,

$$\begin{aligned} J_2 = -\sum_{i=I}^{III} \frac{K}{H} \int_0^1 dt \Bigg\{ & \\ & \frac{C+2B\beta}{t - t_{01}} + \left(\frac{B}{C+2B\beta} - \frac{M_1}{(C+2B\beta)^3} \frac{1}{(t-t_{01})^2} \right) \ln(Q_1(t)) \\ & - \frac{(C+2B\beta)t}{t - t_{02}} - \left(\frac{B(1-\beta)}{C+2B\beta} - \frac{M_1/(1-\beta)}{(C+2B\beta)^3} \frac{1}{(t-t_{02})^2} \right) \ln(Q_2(t)) \\ & + \frac{(C+2B\beta)t}{t - t_{03}} + \left(\frac{B(-\beta)}{C+2B\beta} - \frac{M_1/(-\beta)}{(C+2B\beta)^3} \frac{1}{(t-t_{03})^2} \right) \ln(Q_3(t)) \Bigg\} \end{aligned} \tag{B.4.52}$$

where t_{0i} and R are defined in chapter. 5. To simplify this formula, we integrate by part the denominators $(t-t_{0i})^2$. We have,

$$\begin{aligned}J &= \int_0^1 dt \frac{\ln(a_2t^2+a_1t+a_0)}{b_1t+b_0} \\ &= \frac{1}{\omega}\Bigg\{\frac{b_1a_0\ln(a_0)}{b_0} - \frac{b_1(a_2+a_1+a_0)\ln(a_2+a_1+a_0)}{b_1+b_0} + 2a_2 + (a_1b_1-2a_2b_0)\int_0^1 \frac{dt}{b_1t+b_0} \\ &+ a_2\int_0^1 dt\ln(a_2t^2+a_1t+a_0)\Bigg\} \end{aligned} \tag{B.4.53}$$

with

$$\omega = b_1^2a_0 + b_0^2a_2 - b_1b_0a_1 \tag{B.4.54}$$

Applying this result to each term, we find

$$\begin{aligned}\int_0^1 dt\frac{\ln(Q_1(t))}{(t-t_{01})^2} &= \frac{\Delta}{M_1}\Bigg\{2B + \frac{(C+2B\beta)(A+D+F)\ln(A+D+F-i\lambda)}{-\beta(C+2B\beta)+D+E\beta} \\ &- \frac{(C+2B\beta)(A+B+C+D+E+F)\ln(A+B+C+D+E+F-i\lambda)}{(1-\beta)(C+2B\beta)+D+E\beta} \\ &- \frac{4AB-C^2+2BD-CE}{C+2B\beta}\int_0^1 dt\frac{1}{t-t_{01}} + B\int_0^1 dt\ln(Q_1(t))\Bigg\}\end{aligned} \tag{B.4.55}$$

$$\begin{aligned}\int_0^1 dt\frac{\ln(Q_2(t))}{(t-t_{02})^2} &= \frac{\Delta}{M_1}\Bigg\{2(A+B+C) + \frac{(1-\beta)(C+2B\beta)F\ln(F-i\lambda)}{D+E\beta} \\ &- \frac{(1-\beta)(C+2B\beta)(A+B+C+D+E+F)\ln(A+B+C+D+E+F-i\lambda)}{(1-\beta)(C+2B\beta)+D+E\beta} \\ &- \frac{2BD+CD-2AE-CE}{C+2B\beta}\int_0^1 dt\frac{1}{t-t_{02}} + (A+B+C)\int_0^1 dt\ln(Q_2(t))\Bigg\}\end{aligned} \tag{B.4.56}$$

$$\begin{aligned}\int_0^1 dt\frac{\ln(Q_3(t))}{(t-t_{03})^2} &= \frac{\Delta}{M_1}\Bigg\{2A + \frac{-\beta(C+2B\beta)F\ln(F-i\lambda)}{D+E\beta} \\ &- \frac{-\beta(C+2B\beta)(A+D+F)\ln(A+D+F-i\lambda)}{-\beta(C+2B\beta)+D+E\beta} \\ &- \frac{CD-2AE}{C+2B\beta}\int_0^1 dt\frac{1}{t-t_{03}} + A\int_0^1 dt\ln(Q_3(t))\Bigg\}\end{aligned} \tag{B.4.57}$$

To prove that J_2 vanishes, we insert Eqs. (B.4.55, B.4.56, B.4.57) in Eq. (B.4.52). The new formula of J_2 is long, so we will not express it here but we show the main contributions that cancel together. Let's call each of these contributions $J^{(i)}$ for $i = 1, \cdots !$. Then,

$$J_2 = \sum_{i=I}^{III}\sum_{j=1}^{...} J^{(j)}. \tag{B.4.58}$$

So,

$$\begin{aligned} J^{(1)} &= \frac{2B}{C+2B\beta} - \frac{2(A+B+C)}{(1-\beta)(C+2B\beta)} + \frac{2A}{(-\beta)(C+2B\beta)} \\ &= \frac{B\beta^2+C\beta+A}{-\beta(1-\beta)H(C+2B\beta)} \\ &= 0 \end{aligned} \tag{B.4.59}$$

$$\begin{aligned} J^{(2)} &= -\frac{K}{H}\int_0^1 dt\Bigg\{\frac{1}{t-t_{01}} + \frac{t}{t-t_{02}} + \frac{t}{t-t_{03}} + \frac{4AB-C^2+2BD-CE}{(C+2B\beta)^2}\frac{1}{t-t_{01}} \\ &- \frac{2BD+CD-2AE-CE}{(1-\beta)(C+2B\beta)^2}\frac{1}{t-t_{02}} + \frac{CD-2AE}{-\beta(C+2B\beta)^2}\frac{1}{t-t_{03}}\Bigg\} \\ &= \frac{k}{H}\frac{2BD-CE}{(C+2B\beta)^2}\Bigg\{\frac{1}{t-t_{01}} + \frac{t}{t-t_{02}} + \frac{t}{t-t_{03}}\Bigg\} \\ &= \frac{k}{H}\frac{2BD-CE}{(C+2B\beta)^2}\Bigg\{\ln\left(\frac{1-\beta+R}{-\beta+R}\right) - \ln\left(\frac{1-\beta+R}{R}\right) + \ln\left(\frac{-\beta+R}{R}\right)\Bigg\} \end{aligned} \tag{B.4.60}$$

To prove that $J^{(2)}$ equals to zero, we split the logarithms. In the case of real β, which is the case relevant for physical configurations at NLO order as explained above, no η functions has to be introduced since the imaginary part of the split logarithms arguments are provided by the imaginary part of R. This later one receives its imaginary part from the complex masses. In the case of real internal masses this imaginary part is provided by $-i\lambda$, the R can be written as

$$R = \frac{D+E\beta+i\lambda}{C+2B\beta} \tag{B.4.61}$$

where $i\lambda$ in R is of no consequence, it chosen to $+i\lambda$ (and not $-i\lambda$) since the denominator U in the primitive Eq. (5.2.156) comes from integrating $\propto \frac{1}{Ux+V}$ over x. The denominator of this expression has $-i\lambda$ as imaginary part (in the case of real internal masses) and x vary between zero and one, see chapter 5. Then, it can be written as $(U+i\lambda)x+V-i\lambda$. Hence, Eq, (B.4.60) vanishes. The generalization to the case with complex β is straightforward.

$$\begin{aligned} J^{(3)} = \frac{K}{H}\int_0^1 dt\Bigg\{ &\frac{B}{C+2B\beta}\Big[\ln(Q_1(t)) - (1-\beta)\ln(Q_2(t)) + (-\beta)\ln(Q_3(t))\Big] \\ &- \frac{B}{C+2B\beta}\ln(Q_1(t)) + \frac{A+B+C}{(1-\beta)(C+2B\beta)\ln(Q_2(t))} - \frac{A}{-\beta(C+2B\beta)}\ln(Q_3(t))\Bigg\} \end{aligned} \tag{B.4.62}$$

we combine this logarithms, one gets

$$J^{(3)} = -\frac{K}{H}\int_0^1 dt\Big\{\ln(Q_2(t)) - \ln(Q_3(t))\Big\} \tag{B.4.63}$$

We sum over all the sector (by expressing Q_2 and Q_3 in term of the entries of $\mathcal{S}$, see table above), we prove that

$$J^{(3)} = 0 \tag{B.4.64}$$

B.4.3 Vanishing terms of Eq. (5.2.170): "$\mathcal{J}$"

Let's prove that

$$\begin{aligned}\mathcal{J} &= \sum_{i=I}^{III} \frac{K}{H}\Big\{ I_4^{4,\text{rest}}(\mathcal{S}) + (C+2B\beta) I_3^4(\mathcal{S}\setminus\{i\})\Big\} - I_3^4(\mathcal{S}\setminus\{1\}) \\ &= 0 \end{aligned} \tag{B.4.65}$$

with

$$\begin{aligned} I_4^{4,\text{rest}} = \int_0^1 dt \sum_{\eta=\pm}\Big\{ & \\ & \frac{1}{t-\beta-z^\eta}\left(\ln\frac{H\,t+G+J-i\,\lambda}{B\,t^2+(E+C)\,z+A+D+F-i\,\lambda} - \ln(\text{"}t=\beta+z^\eta\text{"})\right) \\ & -\frac{1}{t-\frac{z^\eta}{1-\beta}}\left(\ln\frac{(H+G)\,t+J-i\,\lambda}{(A+B+C)\,t^2+(E+D)\,z+F-i\,\lambda} - \ln(\text{"}t=z^\eta/(1-\beta)\text{"})\right) \\ & +\frac{1}{t+\frac{z^\eta}{\beta}}\left(\ln\frac{G\,t+J-i\,\lambda}{A\,t^2+D\,z+F-i\,\lambda} - \ln(\text{"}t=z^\eta/(-\beta)\text{"})\right)\Big\} \end{aligned} \tag{B.4.66}$$

$$\begin{aligned} I_3^4(\mathcal{S}\setminus\{i\}) = \frac{1}{C+2B\beta}\int_0^1 dt\Big\{ & \\ & \frac{1}{t-\beta+R}\left(\ln(B\,t^2+(E+C)\,z+A+D+F-i\,\lambda) - \ln\tilde{\mathcal{B}}^{(i)}\right) \\ & -\frac{1}{t+\frac{R}{1-\beta}}\left(\ln((A+B+C)\,t^2+(E+D)\,z+F-i\,\lambda) - \ln\tilde{\mathcal{B}}^{(i)}\right) \\ & +\frac{1}{t+\frac{R}{-\beta}}\left(\ln(A\,t^2+D\,z+F-i\,\lambda) - \ln\tilde{\mathcal{B}}^{(i)}\right)\Big\} \end{aligned} \tag{B.4.67}$$

$$I_3^4(\mathcal{S}\setminus\{1\}) = -\sum_{i=I}^{III}\int_0^1 dt\frac{K}{Ht+G+J}\left(\ln(B\,t^2+(E+C)\,z+A+D+F-i\,\lambda) - \ln\tilde{\mathcal{B}}^{(1)}\right) \tag{B.4.68}$$

We will show that the sum of the first two parts $\mathcal{J}$ equals to $I_3^4(\mathcal{S}\setminus\{1\})$. Let's make the following change of variable to each term of these parts respectively

$$z = t-\beta, \qquad z=(1-\beta)t, \qquad z=-\beta t. \tag{B.4.69}$$

we get

$$\begin{aligned} I_4^{4,\text{rest}}(\mathcal{S}) = \sum_{\eta=\pm}\Big\{ & \int_0^{1-\beta}\frac{dz}{z-z^\eta}\left[\ln\left(\frac{T+S}{V+U}\right) - \ln\left(\frac{T+Sz/(1-\beta)}{V+Uz/(1-\beta)}\right)\right] \\ & -\int_0^{-\beta}\frac{dz}{z-z^\eta}\left[\ln\left(\frac{T+S}{V+U}\right) - \ln\left(\frac{T-Sz/\beta_i}{V-Uz/\beta_i}\right)\right]\Big\} \end{aligned} \tag{B.4.70}$$

Now, we go one step backward by introducing again the variable x. $I_4^{4,\text{rest}}$ becomes

$$\begin{aligned} I_4^{4,\text{rest}}(\mathcal{S}) &= \sum_{\eta=\pm} \int_0^1 dx \int_{-\beta\, x}^{(1-\beta)\, x} \frac{dz}{z - z^\eta} \left[\frac{S}{S\, x + T} - \frac{U}{U\, x + V} \right] \\ &= \sum_{\eta=\pm} \int_0^1 dx \int_{-\beta\, x}^{(1-\beta)\, x} dz \frac{2\alpha_1 z + \alpha_2}{(Sx+T)(Ux+V)} \end{aligned} \tag{B.4.71}$$

Then, we shift the variable z ($z \to z - \beta x$)

$$I_4^{4,\text{rest}}(\mathcal{S}) = \sum_{\eta=\pm} \int_0^1 dx \int_0^x dz \frac{1}{Gx + Hz + J} \frac{2\alpha_1 z - 2\alpha_1 \beta x + \alpha_2}{Ax^2 + Bz^2 + Cxz + Dx + Ez + F - i\lambda} \tag{B.4.72}$$

In a similar way, we can prove that

$$I_3^4(\mathcal{S} \setminus \{i\}) = \int_0^1 dx \int_0^x dz \frac{1}{Ux + V} \tag{B.4.73}$$

Now, let's insert $I_4^{4,\text{rest}}$ and $I_3^4(\mathcal{S} \setminus \{i\})$ in the two first parts of $\mathcal{J}$. After reducing to the same denominator, we find

$$\begin{aligned} \frac{K}{H}(I_4^{4,\text{rest}} + (C + 2B\beta) I_3^4(\mathcal{S} \setminus \{i\})) &= \frac{K}{H} \int_0^1 dx \int_0^x dz \frac{1}{Gx + Hz + J} \\ &\quad \times \frac{(2\, B\, G - C\, H)\, z + (C\, G - 2\, A\, H)\, x + E\, G - D\, H}{A\, x^2 + B\, z^2 + C\, x\, z + D\, x + E\, z + F - i\, \lambda} \end{aligned} \tag{B.4.74}$$

$$= \frac{K}{H} \int_0^1 dx \int_{-\gamma x}^{(1-\gamma)x} dz \frac{-H}{Hz + J} \frac{2W_2 x + W_1}{W_2 x^2 + W_1 x + W_0} \tag{B.4.75}$$

Eq. (B.4.75) is obtained from Eq. (B.4.74) by making the shift $z \to z + \gamma$ (with $\gamma = -G/H$). It can be integrated over x by exchanging the order of integration between x and z. Then, the primitive on x is

$$-\frac{K}{Hz + J} \ln(W_2 x^2 + W_1 x + W_0) = -N_{4,1}^6(x, z) \tag{B.4.76}$$

which equals to $-$ the primitive of $I_{4,1}^6$ which gives $-I_3^4(\mathcal{S} \setminus \{1\})$, see Eq. (5.2.142). Hence

$$\begin{aligned} \mathcal{J} &= I_3^4(\mathcal{S} \setminus \{1\}) - I_3^4(\mathcal{S} \setminus \{1\}) \\ &= 0 \end{aligned} \tag{B.4.77}$$

B.4.4 Prof of I_4^6 Golem formula Eq. (5.2.176)

Eq. (5.2.170) can be organized as the following

$$
\begin{aligned}
I_4^6(S) = -\sum_{i=1}^{3}\sum_{\eta=\pm}\frac{T_4}{2T_1}\frac{K}{\sqrt{T_4}}\int_0^1 dt\Bigg[& \\
& \frac{\eta}{t-\beta-z^\eta}\left\{\ln\frac{H\,t+G+J-i\,\lambda}{B\,t^2+(E+C)\,z+A+D+F-i\,\lambda}-\ln(\text{"}t=\beta+z^\eta\text{"})\right\} \\
& -\frac{\eta}{t-z^\eta/(1-\beta)}\left\{\ln\frac{(H+G)\,t+J-i\,\lambda}{(A+B+C)\,t^2+(E+D)\,z+F-i\,\lambda}-\ln(\text{"}t=z^\eta/(1-\beta)\text{"})\right\} \\
& +\frac{\eta}{t-z^\eta/(-\beta)}\left\{\ln\frac{G\,t+J-i\,\lambda}{A\,t^2+D\,z+F-i\,\lambda}-\ln(\text{"}t=z^\eta/(-\beta)\text{"})\right\}\Bigg] \\
& -\sum_{i=I}^{III}\frac{K}{H}\left[\frac{2T_1+T_2}{2T_1}(I_4^{4,\text{rest}}+(C+2B\,\beta)I_3^4(\mathcal{S}\setminus\{i\}))-\frac{H\,M_0}{2T_1}\,I_3^4(\mathcal{S}\setminus\{i\})\right]
\end{aligned}
\tag{B.4.78}
$$

we have

$$
\frac{T_4}{2\,T_1}=-\frac{\det(\mathcal{S})}{\det(G)} \tag{B.4.79}
$$

So, the first term between square brackets is just

$$
\frac{\det(\mathcal{S})}{\det(G)}I_4^4(\mathcal{S}) \tag{B.4.80}
$$

where

$$
\begin{aligned}
I_4^4(\mathcal{S}) = -\sum_{i=1}^{3}\sum_{\eta=\pm}\frac{K}{\sqrt{T_4}}\int_0^1 dt\Bigg[& \\
& \frac{\eta}{t-\beta-z^\eta}\left\{\ln\frac{H\,t+G+J-i\,\lambda}{B\,t^2+(E+C)\,z+A+D+F-i\,\lambda}-\ln(\text{"}t=\beta+z^\eta\text{"})\right\} \\
& -\frac{\eta}{t-z^\eta/(1-\beta)}\left\{\ln\frac{(H+G)\,t+J-i\,\lambda}{(A+B+C)\,t^2+(E+D)\,z+F-i\,\lambda}-\ln(\text{"}t=z^\eta/(1-\beta)\text{"})\right\} \\
& +\frac{\eta}{t-z^\eta/(-\beta)}\left\{\ln\frac{G\,t+J-i\,\lambda}{A\,t^2+D\,z+F-i\,\lambda}-\ln(\text{"}t=z^\eta/(-\beta)\text{"})\right\}\Bigg]
\end{aligned}
\tag{B.4.81}
$$

see Eq. (6.26) of ref. [66][1] .
We have

$$
\frac{2T_1+T_2}{2T1}=-\frac{\det(\mathcal{S})}{\det(G)}\,b_1 \tag{B.4.82}
$$

then the first term in the last line of Eq. (5.2.170) becomes

$$
-\frac{\det(\mathcal{S})}{\det(G)}\,b_1 I_3^4(\mathcal{S}\setminus\{1\}) \tag{B.4.83}
$$

[1] we notice that K in this formula equals to $-K$ in Eq. (6.26) of ref. [66]

where we used Eq. (B.4.65)

In addition, we have

$$\frac{KM_0}{2T_1} = -\frac{\det(\mathcal{S})}{\det(G)}\, b_i, \qquad i = 2,3,4 \tag{B.4.84}$$

then the last term in the last line of Eq. (B.4.78) becomes

$$-\frac{\det(\mathcal{S})}{\det(G)}\, b_i I_3^4(\mathcal{S} \setminus \{i\}) \tag{B.4.85}$$

From these equations, I_4^6 can be written as

$$I_4^6(\mathcal{S}) = \frac{\det(\mathcal{S})}{\det(G)}\{I_4^4(\mathcal{S}) - \sum_{i=1}^{4} b_i I_3^4(\mathcal{S} \setminus \{i\})\} \tag{B.4.86}$$

which is exactly the formula obtained by Golem reduction[19].
q.e.d.

Appendix C

The four-point functions: I_4^{10} and I_4^{12}

C.1 The scalar box in 10-dimensions

The UV divergent and the finite contributions of the box in $10-2\,\varepsilon$ dimensions, presented in Eq. (5.2.125) are

$$I_4^{10,\,\mathrm{div}}(S) = -r_\Gamma \mathcal{I}_4^{(10)}/\varepsilon \tag{C.1.1}$$

$$I_4^{10,\mathrm{fin}}(S) = r_\Gamma (I_4^{10,a}(S) + I_4^{10,b}(S)) \tag{C.1.2}$$

with

$$I_4^{10,a}(S) = -\mathcal{I}_4^{(10)}$$

$$I_4^{10,b}(S) = \int_0^1 dx\, x^2 \int_0^1 dy' \int_{-\alpha\, y'}^{(1-\alpha)\, y'} dz'' \left(C_1(x,z'')\, y' + C_0(x,z'') \right) \times \ln\left(C_1(x,z'')\, y' + C_0(x,z'') \right)$$

$$= I_4^{10}(S)$$

and

$$\mathcal{I}_4^{(10)} = \frac{12\,a + 6\,b + 8\,c + 15\,d + 10\,e + 20\,f + 2\,g + 4\,h + 3\,j + 5\,k}{120} \tag{C.1.3}$$

As we mentioned before, since α is real, we can easily revert the order of integration as in Eq. (5.2.127) and integrate I_4^{10} over y'. The primitive is

$$\mathcal{J}_4^{10}(S) \;=\; \mathcal{J}_{4,1}^{10}(S) + \mathcal{J}_{4,2}^{10}(S) \tag{C.1.4}$$

with

$$\mathcal{J}_{4,1}^{10}(S) \;=\; -\frac{x^2}{4}\left(C_1(x,z)\, y' + 2\, C_0(x,z) \right) y' \tag{C.1.5}$$

$$\mathcal{J}_{4,2}^{10}(S) \;=\; \frac{x^2}{2}\, \frac{(C_1(x,z)\, y' + C_0(x,z))^2\, \ln(C_1(x,z)\, y' + C_0(x,z))}{C_1(x,z)} \tag{C.1.6}$$

So, I_4^{10} can be written

$$I_4^{10}(S) \;=\; J_{4,1}^{10}(S) + J_{4,2}^{10}(S) \tag{C.1.7}$$

where $J_{4,1}^{10}$ and $J_{4,2}^{10}$ are two-dimensional integral representations corresponding to the primitives $\mathcal{J}_{4,1}^{10}$ and $\mathcal{J}_{4,2}^{10}$, receptively. The first one (without logarithm) can be easily integrated

over the remaining variables of integration, it gives

$$J_{4,1}^{10}(S) = -\mathcal{I}_4^{(10)}/2 \tag{C.1.8}$$

The second integral is of the same nature as I_4^8 (Eq. (5.2.250) chapter 5), the only difference is the power 2 of the quadratic function in front of the logarithm. By following the same procedure, we get

$$\begin{aligned} J_{4,2}^{10}(S) &= \frac{1}{2}\sum_{i=1}^{III}\int_0^1 dx \int_0^x dz \frac{K}{G\,x + H\,z + J - i\,\lambda}(A\,x^2 + B\,z^2 + C\,x\,z + D\,x + E\,z + F)^2 \\ &\times \ln\left(A\,x^2 + B\,z^2 + C\,x\,z + D\,x + E\,z + F - i\,\lambda\right) \end{aligned} \tag{C.1.9}$$

The quantities A, B, ... are defined in Eq.(5.2.133). We can proceed to the same strategy adopted to calculate the other scalar box functions since $J_{4,2}^{10}$ has the same logarithms and the same poles as Eq. (5.2.250), hence the same analyticity. We make the shift $z = z + \gamma\,x$ in such way that $H\,\gamma + G = 0$, the result is

$$\begin{aligned} J_{4,2}^{10}(S) = \frac{1}{2}\sum_{i=1}^{III}\int_0^1 dx \int_{-\gamma\,x}^{(1-\gamma)\,x} dz \frac{K}{H\,z + J - i\,\lambda}(W_2\,x^2 + W_1\,x + W_0)^2 \\ \times \ln\left(W_2\,x^2 + W_1\,x + W_0 - i\,\lambda\right) \end{aligned} \tag{C.1.10}$$

Inverting the order of integration and integrating over x, the primitive is

$$\begin{aligned} \int dx F_{4,2}^{10} = \\ &\frac{1}{H\,z + J}\Big\{x\,(W_0^2 + W_0\,W_1\,x + (W_1^2 + 2\,W_0\,W_2)\,x^2/3 + W_1\,W_2\,x^3/2 + W_2^2\,x^4/5) \\ &\ln\Big(W_2\,x^2 + W_1\,x + W_0\Big) \\ &- \int dx\,x\,(W_0^2 + W_0\,W_1\,x + (W_1^2 + 2\,W_0\,W_2)\,x^2/3 + W_1\,W_2\,x^3/2 + W_2^2\,x^4/5) \\ &\frac{2\,W_2\,x + W_1}{W_2\,x^2 + W_1\,x + W_0}\Big\} \end{aligned} \tag{C.1.11}$$

where $F_{4,2}^{10}$ is double of the integrand of Eq. (C.1.10). So, $J_{4,2}^{10}$ can be written as

$$J_{4,2}^{10}(S) = \frac{1}{2}\sum_{i=I}^{III}[I_{4,1}^{10}(S) - I_{4,2}^{10}(S)] \tag{C.1.12}$$

$I^{10}_{4,1}(S)$ comes from the integrated quantity in Eq. (C.1.11). After some manipulation (see chapter 5), it can be written as

$$\begin{aligned} I^{10}_{4,1}(S) &= \sum_{i=I}^{III} K \int_0^1 dt (N^{10}_{4,1}(1, t-\gamma) - (1-\gamma)\, N^{10}_{4,1}(t, (1-\gamma)\, t) - \gamma\, N^{10}_{4,1}(t, -\gamma\, t)) \\ N^{10}_{4,1}(x, z) &= \frac{1}{H\, z + J}\, x\, (W_0^2 + W_0\, W_1\, x + (W_1^2 + 2\, W_0\, W_2)\, x^2/3 + W_1\, W_2\, x^3/2 + W_2^2\, x^4/5) \\ &\quad \ln\Big(W_2\, x^2 + W_1\, x + W_0\Big) \end{aligned} \tag{C.1.13}$$

Eq.(C.1.13) provides a numerical stable one-dimensional integral representation, since it is free of any inverse of Gram determinant.

$I^{10}_{4,2}(S)$ comes from the remaining part of Eq. (C.1.11), it can be written

$$\begin{aligned} I^{10}_{4,2}(S) &= \sum_{i=I}^{III} \int_0^1 dx \int_0^x dz\, \frac{K}{G\, x + H\, z + J}\, \frac{Q^{10}(x, z)}{A\, x^2 + B\, z^2 + C\, x\, z + D\, x + E\, z + F - i\, \lambda} \\ &= \sum_{i=I}^{III} K \int_0^1 dx \int_{-\beta\, x}^{(1-\beta)\, x} d\bar{z}\, \frac{1}{S\, x + T}\, \frac{Q^{10}(x, z + \beta\, x)}{U\, x + V} \end{aligned} \tag{C.1.14}$$

where Q^{10} is a polynomial of degrees 6 in x and 5 in z. After making the shift $z = z + \beta\, x$, it can be written as

$$Q(x, z + \beta\, x) = x \left(C_5^{(10)}\, x^5 + C_4^{(10)}\, x^4 + C_3^{(10)}\, x^3 + C_2^{(10)}\, x^2 + C_1^{(10)}\, x + C_0^{(10)}\right) \tag{C.1.15}$$

with

$$C_5^{(10)} = S^3\, C_5^{(10,0)} \tag{C.1.16}$$
$$C_4^{(10)} = S^2\, (C_4^{(10,1)}\, z + C_4^{(10,0)}) \tag{C.1.17}$$
$$C_3^{(10)} = S\, (C_3^{(10,2)}\, z^2 + C_3^{(10,1)}\, z + C_3^{(10,0)}) \tag{C.1.18}$$
$$C_2^{(10)} = C_2^{(10,3)}\, z^3 + C_2^{(10,2)}\, z^2 + C_2^{(10,1)}\, z + C_2^{(10,0)}\, z \tag{C.1.19}$$
$$C_1^{(10)} = C_1^{(10,4)}\, z^4 + C_1^{(10,3)}\, z^3 + C_1^{(10,2)}\, z^2 + C_1^{(10,1)}\, z + C_1^{(10,0)} \tag{C.1.20}$$
$$C_0^{(10)} = C_0^{(10,5)}\, z^5 + C_0^{(10,4)}\, z^4 + C_0^{(10,3)}\, z^3 + C_0^{(10,2)}\, z^2 + C_0^{(10,1)}\, z + C_0^{(10,0)} \tag{C.1.21}$$

$$C_5^{(10)} = S^3\, \tilde{C}_5 \qquad C_4^{(10)} = S^2\, \tilde{C}_4 \qquad C_3^{(10)} = S\, \tilde{C}_3 \tag{C.1.22}$$

We notice that the $C_i^{(10,n)}$ (for $i, n = 1, \cdots 5$) are free of inverse of $\det(G)$ or S.

We revert the order of integration on x and z in Eq. (C.1.14), we write

$$I_{4,2}^{10}(S) = \sum_{i=I}^{III} K \left[\int_0^{1-\beta} d\,z \int_{z/(1-\beta)}^{1} dx\, F_{4,2}^{10}(x,z) - \int_0^{-\beta} d\,\bar{z} \int_{z/(-\beta)}^{1} dx\, F_{4,2}^{10}(x,z) \right] \tag{C.1.23}$$

where $F_{4,2}^{10}$ is the integrand of Eq. (C.1.14).

$$\begin{aligned}
N_{4,2}^{10}(x,z) &= \int dx\, \frac{Q^{(10)}(x, z+\beta\, x)}{S\,V - T\,U} \left(\frac{S}{S\,x+T} - \frac{U}{U\,x+V} \right) \\
&= \frac{1}{SV - TU} \Bigg\{ \\
&- \frac{1}{3}\, T\, (\tilde{C}_3 - \tilde{C}_4 T + \tilde{C}_5 T^2)\, x^3 + \frac{1}{4}\, ST\, (\tilde{C}_5 T - \tilde{C}_4)\, x^4 - \frac{1}{5} \tilde{C}_5\, S^2 T\, x^5 \\
&+ \frac{\tilde{C}_5\, V\, S^3}{U} \left(\frac{x^5}{5} - \frac{V\, x^4}{4\,U} + \frac{V^2\, x^3}{3\,U^2} - \frac{V^3\, x^2}{2\,U^3} + \frac{V^4\, x}{U^4} - \frac{V^5}{U^5}\, \ln(U\,x+V) \right) \\
&+ \frac{\tilde{C}_4\, V\, S^2}{U} \left(\frac{x^4}{4} - \frac{V\, x^3}{3\,U} + \frac{V^2\, x^2}{2\,U^2} - \frac{V^3\, x}{U^3} + \frac{V^4}{U^4}\, \ln(U\,x+V) \right) \\
&+ \frac{\tilde{C}_3\, V\, S}{U} \left(\frac{x^3}{3} - \frac{V\, x^2}{2\,U} + \frac{V^2\, x}{U^2} - \frac{V^3}{U^3}\, \ln(U\,x+V) \right) \\
&+ \frac{C_2\, V}{U} \left(\frac{x^2}{2} - \frac{V\, x}{U} + \frac{V^2}{U^2}\, \ln(U\,x+V) \right) + \frac{C_1\, V}{U} \left(x - \frac{V}{U}\, \ln(U\,x+V) \right) \\
&+ \frac{C_0\, V}{U}\, \ln(U\,x+V) - \frac{T\, C_0}{S}\, \ln(S\,x+T) + \frac{T\, C_1}{S} \left(\frac{T}{S}\, \ln(S\,x+T) - x \right) \\
&- \frac{T(C_2 - \tilde{C}_3\, T + \tilde{C}_4\, T^2 - \tilde{C}_5\, T^3)}{S} \left(\frac{T^2}{S^2}\, \ln(S\,x+T) - \frac{T\,x}{S} + \frac{x^2}{2} \right) \Bigg\}
\end{aligned} \tag{C.1.24}$$

In Eq.(C.1.24), only the last contributions which are proportional to inverse of S with up to the power 3 (the last three terms). The remaining terms are free of any inverse of S or $\bar{S}$ (with $\det(G) \propto S\,\bar{S}$), the last one ($\bar{S}$) appears if we reduce the denominators $(S\,V - T\,U)\,U^n$ (with $n = 0, \cdots$) to simple elements as we explained in the case of I_4^8. Because of that, we keep this primitive which provides a stable one-dimensional integral representation after modifying it as the following: From each term containing $\ln(S\,x+T)$, we subtract $\ln(T)$ (which give no contribution to the final result as we explained above), and re-write them in term the functions q_3, q_2 and q_1. We have

$$\begin{aligned}
\frac{T}{S} \left(\frac{T^2}{S^2}\, (\ln(S\,x+T) - \ln(T)) - \frac{T\,x}{S} + \frac{x^3}{2} \right) &= -\frac{x^3}{X^3}\, (\ln(1-X) + X + X^2/2) \\
&= -x^3\, q_3(X) \qquad \text{(C.1.25)} \\
\frac{T}{S} \left(\frac{T}{S}\, (\ln(S\,x+T) - \ln(T)) - x \right) &= \frac{x^2}{X^2}\, (\ln(1-X) + X) \\
&= x^2\, q_2(X) \qquad \text{(C.1.26)} \\
\frac{T}{S}\, (\ln(S\,x+T) - \ln(T)) &= -\frac{x}{X}\, \ln(1-X) \\
&= -x\, q_1(X) \qquad \text{(C.1.27)}
\end{aligned}$$

where q_1 and q_2 are defined above and q_3 is given by

$$q_3(X) = \begin{cases} \frac{1}{X^3}\left(\ln(1-X) + X + X^2/2\right) & \text{if} \quad X \not\to 0 \\ -\sum_{n=0}^{\infty} \frac{X^n}{n+3} & \text{if} \quad X \to 0 \end{cases} \tag{C.1.28}$$

Contour deformation

As in the previous cases, we have to perform a contour deformation to avoid the poles when they become close to the segment $[0,1]$, such contours are presented in paragraph 5.2.2.3. It will be very gainful if we can find a new primitive (which give the same result as the previous one) with zero residue at the poles corresponding to $U=0$, since these poles appear as multi-poles up to 6th order, which make the work extremely complicated. Such primitive can be constructed by adding to each $U^n x^n$ in the numerator the term $(-1)^{1+n}(V-V_0)^n$, and from $\ln(Ux+V)$ we subtract the $\ln(V_0)$ [1]. The new primitive is given by

$$\begin{aligned} N^{10}_{4,2} = \frac{1}{SV-TU}\Bigg\{ & \\ & -\frac{1}{3}T\left(\tilde{C}_3 - \tilde{C}_4 T + \tilde{C}_5 T^2\right)x^3 + \frac{1}{4}ST\left(\tilde{C}_5 T - \tilde{C}_4\right)x^4 - \frac{1}{5}\tilde{C}_5\, S^2 T\, x^5 \\ & + \frac{\tilde{C}_5 S^3}{U^6}\left[\sum_{n=1}^{5}\frac{(-1)^{1+n}\left(U^n x^n + (-1)^{1+n}(V-V_0)^n\right)V^{6-n}}{n} - V^6\left(\ln(Ux+V) - \ln(V_0)\right)\right] \\ & - \frac{\tilde{C}_4\, S^2}{U^5}\left[\sum_{n=1}^{4}\frac{(-1)^{1+n}\left(U^n x^n + (-1)^{1+n}(V-V_0)^n\right)V^{5-n}}{n} - V^5\left(\ln(Ux+V) - \ln(V_0)\right)\right] \\ & + \frac{\tilde{C}_3 S}{U^4}\left[\sum_{n=1}^{3}\frac{(-1)^{1+n}(U^n x^n + (-1)^{1+n}(V-V_0)^n)V^{4-n}}{n} - V^4\left(\ln(Ux+V) - \ln(V_0)\right)\right] \\ & - \frac{C_2}{U^3}\left[\sum_{n=1}^{2}\frac{(-1)^{1+n}(U^n x^n + (-1)^{1+n}(V-V_0)^n)V^{(3-n)}}{n} - V^3\left(\ln(Ux+V) - \ln(V_0)\right)\right] \\ & + \frac{C_1 V}{U^2}\left[Ux + V - V_0 - V\left(\ln(U\,x+V) - \ln(V_0)\right)\right] + \frac{C_0 V}{U}\left(\ln(U\,x+V) - \ln(V_0)\right) \\ & + x\, q_1(X) + x^2\, C_1\, q_2(X) + x^3\left(C_2 - \tilde{C}_3\, T + \tilde{C}_4\, T^2 - \tilde{C}_5 T^3\right) q_3(X)\Bigg\} \end{aligned} \tag{C.1.29}$$

This new primitive has three advantages, *i)* it provides a stable one-dimensional integral representation for $I^{10}_{4,2}$ when $S \to 0$ (then $\det(G) \to 0$), *ii)* the residues at the poles t_{0i} (or $U=0$) vanish (one can prove this by direct application to residue formula Eq. (5.2.198)), *iii)* the residues at the poles t^{η}_{0i} (or $S\,V - T\,U = 0$) are finite when $S \to 0$. The calculation of these residues, albeit more difficult than in the case of I^6_4 and I^8_4, can be handled in completely similar way. In this case the function $q_3(X^\eta)$ and $q_3(Z^\eta)$ are used, where X^η and Z^η are defined in chapter 5.

[1] Regarding the poles $SV - TU = 0$, we can construct a primitive with zero residues at these poles but we found that the contours may cross branch cuts of the subtracted terms (they are logarithmic). So, we decide to don't use this primitive and simply calculate the residues at each of these poles.

Then, $I_4^{10}(S)$ is given by

$$\begin{aligned} I_4^{10}(S) &= -\frac{3}{2}\mathcal{I}_4^{10} + \frac{1}{2}\sum_{i=I}^{III}\int_0^1 K\,dt\Big\{ \\ &+ N_{4,1}^{10}(1,t-\gamma) - (1-\gamma)\,N_{4,1}^{10}(t,(1-\gamma)\,t) - \gamma\,N_{4,1}^{10}(t,-\gamma\,t) \\ &- [N_{4,2}^{10}(1,t-\beta) - (1-\beta)\,N_{4,2}^{10}(t,(1-\beta)\,t) - \beta\,N_{4,2}^{10}(t,-\beta\,t)]\Big\} \\ &+ \text{residues} \end{aligned} \tag{C.1.30}$$

which provides a numerical stable results for all the configuration discussed in paragraph above.

C.2 The scalar box in 12-dimensions

The UV divergent and the finite contributions of the box in $12-2\,\varepsilon$ dimensions, defines in Eq. (5.2.126) are

$$I_4^{12,\,\mathrm{div}}(S) = \frac{\mathcal{I}_4^{(12)}}{2\varepsilon} \tag{C.2.31}$$

$$I_4^{12,\mathrm{fin}}(S) = I_4^{12,a}(S) + I_4^{12,b}(S) \tag{C.2.32}$$

with

$$\begin{aligned} I_4^{12,a}(S) &= \frac{3}{4}\,\mathcal{I}_4^{(12)} \\ I_4^{12,b}(S) &= -\frac{1}{2}\int_0^1 dx\,x^2\int_0^1 dy'\int_{-\alpha\,y'}^{(1-\alpha)\,y'} dz''\left(C_1(x,z'')\,y' + C_0(x,z'')\right)^2 \\ &\qquad\qquad \times\,\ln\left(C_1(x,z'')\,y' + C_0(x,z'')\right) \\ &= I_4^{12}(S) \end{aligned} \tag{C.2.33}$$

and

$$\begin{aligned} \mathcal{I}_4^{(12)} &= \frac{\mathrm{a}^2}{14} + \frac{\mathrm{b}^2}{42} + \frac{\mathrm{c}^2}{28} + \frac{\mathrm{d}^2}{10} + \frac{\mathrm{e}^2}{20} + \frac{\mathrm{f}^2}{6} + \frac{\mathrm{g}^2}{210} + \frac{\mathrm{h}^2}{84} + \frac{\mathrm{j}^2}{126} + \frac{\mathrm{k}^2}{60} + \frac{\mathrm{j}\,\mathrm{k}}{45} \\ &+ \mathrm{h}\left(\frac{2\,\mathrm{j}}{105} + \frac{\mathrm{k}}{36}\right) + \mathrm{b}\left(\frac{2\,\mathrm{c}}{35} + \frac{\mathrm{d}}{12} + \frac{\mathrm{e}}{15} + \frac{\mathrm{f}}{10} + \frac{\mathrm{g}}{63} + \frac{\mathrm{h}}{35} + \frac{\mathrm{j}}{42} + \frac{\mathrm{k}}{30}\right) \\ &+ \mathrm{c}\left(\frac{\mathrm{d}}{9} + \frac{\mathrm{e}}{12} + \frac{2\,\mathrm{f}}{15} + \frac{2\,\mathrm{g}}{105} + \frac{\mathrm{h}}{28} + \frac{\mathrm{j}}{35} + \frac{\mathrm{k}}{24}\right) + \mathrm{e}\left(\frac{\mathrm{f}}{6} + \frac{\mathrm{g}}{45} + \frac{\mathrm{h}}{24} + \frac{\mathrm{j}}{30} + \frac{\mathrm{k}}{20}\right) \\ &+ \mathrm{a}\left(\frac{\mathrm{b}}{14} + \frac{2\,\mathrm{c}}{21} + \frac{\mathrm{d}}{6} + \frac{\mathrm{e}}{9} + \frac{\mathrm{f}}{5} + \frac{\mathrm{g}}{42} + \frac{\mathrm{h}}{21} + \frac{\mathrm{j}}{28} + \frac{\mathrm{k}}{18}\right) + \mathrm{g}\left(\frac{\mathrm{h}}{70} + \frac{\mathrm{j}}{84} + \frac{\mathrm{k}}{60}\right) \\ &+ \mathrm{d}\left(\frac{2\,\mathrm{e}}{15} + \frac{\mathrm{f}}{4} + \frac{\mathrm{g}}{36} + \frac{\mathrm{h}}{18} + \frac{\mathrm{j}}{24} + \frac{\mathrm{k}}{15}\right) + \mathrm{f}\left(\frac{\mathrm{g}}{30} + \frac{\mathrm{h}}{15} + \frac{\mathrm{j}}{20} + \frac{\mathrm{k}}{12}\right) \end{aligned} \tag{C.2.34}$$

We revert the order of integration and integrate the finite part over y'. The primitive of the integrand in Eq. (C.2.33) is

$$\mathcal{J}_4^{12}(S) = \mathcal{J}_{4,1}^{12}(S) + \mathcal{J}_{4,2}^{12}(S) \tag{C.2.35}$$

(C.2.36)

with

$$\mathcal{J}_{4,1}^{12}(S) = -\frac{x^2}{9}\left(C_1(x,z)\,y + C_0(x,z)\right)^3 \tag{C.2.37}$$

$$\mathcal{J}_{4,2}^{12}(S) = \frac{x^2}{3}\,\frac{(C_1(x,z)\,y + C_0(x,z))^3\,\ln(C_1(x,z)\,y + C_0(x,z))}{C_0(x,z)} \tag{C.2.38}$$

then

$$I_4^{12}(S) = -\frac{1}{2}\,(J_{4,1}^{12}(S) + J_{4,2}^{12}(S)) \tag{C.2.39}$$

where $J_{4,1}^{12}$ and $J_{4,2}^{12}$ are two-dimensional integral representations associated to the primitives $\mathcal{J}_{4,1}^{12}$ and $\mathcal{J}_{4,2}^{12}$, receptively. The first one (without logarithm) can be easy integrated over the remaining variables of integration, we get

$$J_{4,1}^{12}(S) = -\frac{1}{3}\,\mathcal{I}_4^{(12)} \tag{C.2.40}$$

The second integral is of the same type as $J_{4,2}^{10}$, the only difference is the power 3 of the quadratic function in front of the logarithm. By following the same procedure, it can be written as

$$J_{4,2}^{12}(S) = \frac{1}{3}\sum_{i=1}^{III}\int_0^1 dx \int_0^x dz\,\frac{K}{G\,x + H\,z + J - i\,\lambda}(A\,x^2 + B\,z^2 + C\,x\,z + D\,x + E\,z + F)^3 \ln\left(A\,x^2 + B\,z^2 + C\,x\,z + D\,x + E\,z + F - i\,\lambda\right) \tag{C.2.41}$$

One more time, we proceed to the same strategy adopted to calculate the other scalar box functions (chapter 5). We make the shift $z = z + \gamma\,x$ in such way that $H\,\gamma + G = 0$, the result is

$$J_{4,2}^{12}(S) = \frac{1}{3}\sum_{i=1}^{III}\int_0^1 dx \int_{-\gamma\,x}^{(1-\gamma)\,x} dz\,\frac{K}{H\,z + J - i\,\lambda}(W_2\,x^2 + W_1\,x + W_0)^3 \ln\left(W_2\,x^2 + W_1\,x + W_0 - i\,\lambda\right) \tag{C.2.42}$$

Inverting the order of integration and integrating over x, the primitive is

$$\begin{aligned}\int dx\, F_4^{12} = \frac{K}{H\,z+J}\Big\{ & x\,(W_0^3 + 3/2\,W_0^2\,W_1\,x + W_0\,(W_1^2 + W_0\,W_2)\,x^2 \\ & + 1/4\,W_1\,(W_1^2 + 6\,W_0\,W_2)\,x^3 + 3/5\,W_2\,(W_1^2 + W_0\,W_2)\,x^4 + 1/2\,W_1\,W_2^2\,x^5 \\ & + W_2^3\,x^6/7)\,\ln\Big(W_2\,x^2 + W_1\,x + W_0\Big) \\ & - \int dx\,x\,(W_0^3 + 3/2\,W_0^2\,W_1\,x + W_0\,(W_1^2 + W_0\,W_2)\,x^2 \\ & + 1/4\,W_1\,(W_1^2 + 6\,W_0\,W_2)\,x^3 + 3/5\,W_2\,(W_1^2 + W_0\,W_2)\,x^4 + 1/2\,W_1\,W_2^2\,x^5 \\ & + W_2^3\,x^6/7)\,\frac{2\,W_2\,x + W_1}{W_2\,x^2 + W_1\,x + W_0}\Big\} \end{aligned} \tag{C.2.43}$$

F_4^{12} is the integrand of Eq. (C.2.42) (without dividing by 3!). So, $J_{4,2}^{12}$ can be written as

$$J_{4,2}^{12}(S) \;=\; \frac{1}{3}\sum_{i=I}^{III}\,[I_{4,1}^{12}(S) - I_{4,2}^{12}(S)] \tag{C.2.44}$$

$I_{4,1}^{12}(S)$ comes from the integrated quantity (first part of in Eq. (C.2.43)). After some manipulation, its final one-dimensional integral representation is given by

$$I_{4,1}^{12}(S) = \sum_{i=I}^{III}\int_0^1 dt K[N_{4,1}^{12}(1,t-\gamma) - (1-\gamma)\,N_{4,1}^{12}(t,(1-\gamma)\,t) - \gamma\,N_{4,1}^{12}(t,-\gamma\,t)] \tag{C.2.45}$$

$$\begin{aligned} N_{4,1}^{12}(x,z) \;=\; & \frac{1}{H\,z+J}\,x\,(W_0^3 + 3/2\,W_0^2\,W_1\,x + W_0\,(W_1^2 + W_0\,W_2)\,x^2 \\ + \; & 1/4\,W_1\,(W_1^2 + 6\,W_0\,W_2)\,x^3 + 3/5\,W_2\,(W_1^2 + W_0\,W_2)\,x^4 + 1/2\,W_1\,W_2^2\,x^5 \\ + \; & W_2^3\,x^6/7)\,\ln\Big(W_2\,x^2 + W_1\,x + W_0\Big) \end{aligned} \tag{C.2.46}$$

$I_{4,2}^{12}(S)$ comes from the remaining part in Eq. (C.2.43), it can be written as

$$\begin{aligned} I_{4,2}^{12}(S) \;&=\; \sum_{i=I}^{III}\int_0^1 dx\int_0^x dz\,\frac{K}{G\,x+H\,z+J}\,\frac{Q^{12}(x,z)}{A\,x^2+B\,z^2+C\,x\,z+D\,x+E\,z+F-i\,\lambda} \\ &=\; \sum_{i=I}^{III} K\int_0^1 dx\int_{-\beta\,x}^{(1-\beta)\,x} d\bar{z}\,\frac{1}{S\,x+T}\,\frac{Q^{12}(x,z+\beta\,x)}{U\,x+V} \end{aligned} \tag{C.2.47}$$

Q^{12} is a polynomial of degrees 8 in x and 7 in z. After making the shift $z = z + \beta\,x$, it can be written as

$$Q(x,z+\beta\,x) = x\,\Big(C_7\,x^7 + C_6\,x^6 + C_5\,x^5 + C_4\,x^4 + C_3\,x^3 + C_2\,x^2 + C_1\,x + C_0\Big) \tag{C.2.48}$$

the coefficients C_i are defined below.
We revert the order of integration in x, we write

$$I^{12}_{4,2}(S) \;=\; \sum_{i=I}^{III}\Big[\, K\int_0^{1-\beta} d\,\bar z \int_{z/(1-\beta)}^{1} dx\, F^{12}_{4,2}(x,z) - K\int_0^{-\beta} d\,z \int_{z/(-\beta)}^{1} dx\, F^{12}_{4,2}(x,z)\Big] \tag{C.2.49}$$

where $F^{12}_{4,2}$ is the integrand of Eq. (C.2.47).
This primitive of $F^{12}_{4,2}$ on x can be written as

$$\begin{aligned}
N^{12}_{4,2}(x,z) =& \frac{1}{SV-TU}\Big\{ \\
&-\frac{1}{7}\tilde C_7 S^3 T x^7 - \frac{1}{6}S^2T(\tilde C_6 - \tilde C_7 T)x^6 - \frac{1}{5}ST(\tilde C_5 - \tilde C_6 T + \tilde C_7 T^2)x^5 \\
&\qquad + \frac{1}{4}T(-\tilde C_4 + \tilde C_5 T - \tilde C_6 T^2 + \tilde C_7 T^3)x^4 \\
&\frac{\tilde C_7 S^4}{U^8}\Big[\sum_{n=1}^{7}\frac{(-1)^{1+n}\,(U^n x^n + (-1)^{1+n}\,(V-V_0)^n)V^{8-n}}{n} - V^8(\ln(Ux+V) - \ln V_0)\Big] \\
&-\frac{\tilde C_6 S^3}{U^7}\Big[\sum_{n=1}^{6}\frac{(-1)^{1+n}\,(U^n x^n + (-1)^{1+n}\,(V-V_0)^n)V^{7-n}}{n} - V^7(\ln(Ux+V) - \ln V_0)\Big] \\
&+\frac{\tilde C_5 S^2}{U^6}\Big[\sum_{n=1}^{5}\frac{(-1)^{1+n}\,(U^n x^n + (-1)^{1+n}\,(V-V_0)^n)V^{6-n}}{n} - V^6(\ln(Ux+V) - \ln V_0)\Big] \\
&-\frac{\tilde C_4 S}{U^5}\Big[\sum_{n=1}^{4}\frac{(-1)^{1+n}\,(U^n x^n + (-1)^{1+n}\,(V-V_0)^n)V^{5-n}}{n} - V^5(\ln(Ux+V) - \ln V_0)\Big] \\
&+\frac{C_3}{U^4}\Big[\sum_{n=1}^{3}\frac{(-1)^{1+n}\,(U^n x^n + (-1)^{1+n}\,(V-V_0)^n)V^{4-n}}{n} - V^4(\ln(Ux+V) - \ln V_0)\Big] \\
&-\frac{C_2}{U^3}\Big[\sum_{n=1}^{2}\frac{(-1)^{1+n}\,(U^n x^n + (-1)^{1+n}\,(V-V_0)^n)V^{3-n}}{n} - V^3(\ln(Ux+V) - \ln V_0)\Big] \\
&+\frac{C_1}{U^2}\Big[(Ux+V-V_0)V - V^2(\ln(Ux+V) - \ln V_0)\Big] + \frac{C_0 V}{U}\Big[\ln(Ux+V) - \ln V_0\Big] \\
&\qquad + C_0\, x\, q_1(X) + C_1\, x^2\, q_2(X) + C_2\, x^3\, q_3(X) \\
&\qquad + (C_3 - \tilde C_4 T + \tilde C_5 T T^2 - \tilde C_5 T^3 + \tilde C_7 T T^4)\, x^4\, q_4(X)\Big\}
\end{aligned} \tag{C.2.50}$$

In this primitive, the singular parts on S are expressed in term of the functions q_1, q_2, q_3 and q_4, where

$$q_4(X) = \begin{cases} \frac{1}{X^4}\,(\ln(1-X) + X + X^2/2 + X^3/3) & \text{if} \quad X \not\to 0 \\ -\sum_{n=0}^{\infty}\frac{X^n}{n+4} & \text{if} \quad X \quad \to 0 \end{cases} \tag{C.2.51}$$

and residues at $U=0$ of each term of equals to zero, by construction (we added to each $U^n x^n$ the term $(V-V_0)^n$ and subtracted $\ln V_0$ from $\ln(Ux+V)$ to make these residues at t_{0i} vanish).

The residues of $N_{4,2}^{12}(x,z)$ at the poles t_{0i}^{η}, albeit more cumbersome, they can be treated similarly as in the case of I_4^6 and I_4^8. In this case the function $q_4(X^\eta)$ and $q_4(Z^\eta)$ are used, where X^η and Z^η are defined in chapter 5.

$$I_{4,2}^{12}(S) = \sum_{i=I}^{III} \int_0^1 dt K[N_{4,2}^{12}(1,t-\beta) - (1-\beta)\, N_{4,2}^{12}(t,(1-\beta)\,t) - \beta\, N_{4,2}^{12}(t,-\beta\,t)] \tag{C.2.52}$$

and

$$I_4^{12,\text{fin}}(S) = I_4^{12,a} + \frac{1}{6}[\mathcal{I}_4^{12}(S) - (I_{4,1}^{12}(S) - I_{4,2}^{12}(S))] + \text{residues} \tag{C.2.53}$$

where $I_4^{12,a}$, $\mathcal{I}_4^{12}$ and $I_{4,1}^{12}$ are defined above.

with

$$C_7 = S^4\, C_7^{(0)} \tag{C.2.54}$$

$$C_6 = S^3\, (C_6^{(1)}\, z + C_6^{(0)}) \tag{C.2.55}$$

$$C_5 = S^2\, (C_5^{(2)}\, z^2 + C_5^{(1)}\, z + C_5^{(0)}) \tag{C.2.56}$$

$$C_4 = S\, C_4^{(3)}\, z^3 + C_4^{(2)}\, z^2 + C_4^{(1)}\, z + C_4^{(0)} \tag{C.2.57}$$

$$C_3 = C_3^{(4)}\, z^4 + C_3^{(3)}\, z^3 + C_3^{(2)}\, z^2 + C_3^{(1)}\, z + C_3^{(0)} \tag{C.2.58}$$

$$C_2 = C_2^{(5)}\, z^5 + C_2^{(4)}\, z^4 + C_2^{(3)}\, z^3 + C_2^{(2)}\, z^2 + C_2^{(1)}\, z + C_2^{(0)} \tag{C.2.59}$$

$$C_1 = C_1^{(6)}\, z^6 + C_1^{(5)}\, z^5 + C_1^{(4)}\, z^4 + C_1^{(3)}\, z^3 + C_1^{(2)}\, z^2 + C_1^{(1)}\, z + C_1^{(0)} \tag{C.2.60}$$

$$C_0 = C_0^{(7)}\, z^7 + C_0^{(6)}\, z^6 + C_0^{(5)}\, z^5 + C_0^{(4)}\, z^4 + C_0^{(3)}\, z^3 + C_0^{(2)}\, z^2 + C_0^{(1)}\, z + C_0^{(0)} \tag{C.2.61}$$

$$C_7 = S^4 \tilde{C}_7, \qquad C_6 = S^3 \tilde{C}_6, \qquad C_5 = S^2 \tilde{C}_5, \qquad C_4 = S \tilde{C}_4 \tag{C.2.62}$$

We notice that the $C_i^{(j)}$ for $i,j = 0,\cdots,7$ are free of inverse of $\det(G)$ and inverse of S.

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Abstract: Higher order corrections in gauge theories play a crucial role in studying physics within the standard model and beyond at TeV colliders, like LHC, TeVatron and ILC. Therefore, it is of extreme importance to provide tools for next-to-leading order amplitude computation which are fast, stable, efficient and highly automatized. This thesis aims at developing the library of integrals `Golem95`. This library is a program written in `Fortran95`, it contains all the necessary ingredients to calculate any one-loop scalar or tensorial integral with up to six external legs. `Golem95` uses the traditional reduction method (*Golem* reduction) to reduce the form factors into redundant basic integrals, which can be scalar (without Feynman parameters in the numerator) or tensorial (with Feynman parameter in the numerator); this formalism allows us to avoid the problems of numerical instabilities generated by the spurious singularities induced by the vanishing of the Gram determinants. In addition, this library can be interfaced with automatic programs of NLO calculation based on the unitarity inspired reduction methods as `GoSam` for example. Earlier versions of `Golem95` were designed for the calculation of amplitudes without internal masses. The purpose of this thesis is to extend this library for more general configurations (complex masses are supported); and to provide numerically stable calculation in the problematic regions ($\det(G) \to 0$), by providing a stable one-dimensional integral representation for each `Golem95` basic integral.
Key words: `Golem95`, NLO computations, One-loop Feynman integrals, Complex masses, Gram determinant, Gauge theories.

Résumé: Les calculs de précision dans les théories de jauge jouent un rôle très important pour l'étude de la physique du Modèle Standard et au-delà dans les super-collisionneurs de particules comme le LHC, TeVatron et ILC. Par conséquent, il est extrêmement important de fournir des outils du calcul d'amplitudes à une boucle stables, rapides, efficaces et hautement automatisés. Cette thèse a pour but de développer la bibliothèque d'intégrales `Golem95`. Cette bibliothèque est un programme écrit en `Fortran95`, qui contient tous les ingrédients nécessaires pour calculer une intégrale scalaire ou tensorielle à une boucle avec jusqu'à six pattes externes. `Golem95` utilise une méthode traditionnelle de réduction (*réduction à la Golem*) qui réduit les facteurs de forme en des intégrales de base redondantes qui peuvent être scalaires (sans paramètres de Feynman au numérateur) ou tensorielles (avec des paramètres de Feynman au numérateur); ce formalisme permet d'éviter les problèmes de l'instabilité numérique engendrés par des singularités factices dues à l'annulation des déterminants de Gram. En plus, cette bibliothèque peut être interfacée avec des programmes du calcul automatique basés sur les méthodes d'unitarité comme `GoSam` par exemple. Les versions antérieures de `Golem95` ont été conçues pour le calcul des amplitudes sans masses internes. Le but de ce travail de thèse est de généraliser cette bibliothèque pour les configurations les plus générales (les masses complexes sont incluses), et de fournir un calcul numériquement stable dans les régions problématique en donnant une représentation intégrale unidimensionnelle stable pour chaque intégrale de base de `Golem95`.
Mots clé: `Golem95`, Calcul NLO, Intégrales de Feynman à une boucle, Masses complexes, Déterminant de Gram, Théories de jauge.