Towards Input/Output-Free Modelling of Linear Infinite-Dimensional Systems in Continuous Time

Mikael Kurula

PhD Thesis in Mathematics
Åbo Akademi University

Åbo, Finland, 2010

Towards Input/Output-Free Modelling of Linear Infinite-Dimensional Systems in Continuous Time

Mikael Kurula

Åbo Akademi University

PhD Thesis in Mathematics
Åbo Akademi University

Åbo, Finland, 2010

ISBN 978-952-12-2410-2 (Hard copy)
ISBN 978-952-12-2418-8 (Electronic)

Painosalama Oy
Åbo, Finland, 2010

Abstract

This dissertation describes a networking approach to infinite-dimensional systems theory, where there is a minimal distinction between inputs and outputs. We introduce and study two closely related classes of systems, namely the state/signal systems and the port-Hamiltonian systems, and describe how they relate to each other. Some basic theory for these two classes of systems and the interconnections of such systems is provided. The main emphasis lies on passive and conservative systems, and the theoretical concepts are illustrated using the example of a lossless transfer line. Much remains to be done in this field and we point to some directions for future studies as well.

Sammanfattning

I avhandlingen introduceras oändligtdimensionella linjära tillstånds/signalsystem i kontinuerlig tid. En av de viktigaste operationerna inom system- och reglerteorin är sammankoppling av två delsystem till ett större system. Sammankoppling i sin allmännaste form av två oändligtdimensionella system har dock visat sig vara ett utmanande problem. Tillstånds/signalsystem är väl lämpade för att kopplas samman, eftersom deras viktigaste egenskap är att systemets insignal och utsignal behandlas så lika som möjligt.

Ett klassiskt system med förbestämd ingång och utgång kan skrivas om som ett tillstånds/signalsystem genom att insignalen och utsignalen slås samman till en kombinerad yttre signal. En viktig klass av tillstånds/signalsystem är de så kallade välställda systemen, som omvänt kan tolkas som ett system med in- och utsignal genom att man spjälker den yttre signalen i en ingång och en utgång på ett lämpligt sätt. I avhandlingen redogörs för de grundläggande egenskaperna för välställda och passiva tillstånds/signalsystem.

Förutom välställda tillstånds/signalsystem presenteras också de nära besläktade porthamiltonska systemen. Dessa härstammar från modellering av konservativa, huvudsakligen ickelinjära, fysikaliska system. I motsats till tillstånds/signalsystem i kontinuerlig tid är alltså porthamiltonska system välkända sedan tidigare. Vi beskriver sammankoppling av denna typ av system och hur dessa förhåller sig till tillstånds/signalsystem. De begrepp som introduceras i avhandlingen illustreras genomgående med hjälp av transmissionslinjen.

Preface and acknowledgements

The aim of this thesis is to conclude the research that I have conducted at the mathematics department of Åbo Akademi University between spring 2004 and autumn 2009. The part which deals with Dirac structures originates from my stay at Twente University in Enschede, The Netherlands, from June to November 2005, and subsequent visits in Twente.

I wish to thank all my past and present colleagues at the mathematics department of Åbo Akademi University for the extraordinarily good atmosphere. Of course, I am in particular indebted to my supervisor, Olof Staffans, for his patience, inspiration and constant support. I am also very grateful to Arjan van der Schaft and Hans Zwart for making it possible for me to visit Twente University. It was both a very enriching experience and pleasant to be able to work among all the friendly staff and Ph.D. students at the Department of applied mathematics at Twente University.

I want to thank the referees Seppo Hassi and Joseph A. Ball for taking the time to review my thesis, including the appended articles, and the improvements that they suggested.

Financial support from the Academy of Finland, the Finnish Graduate School in Mathematical Analysis and its Applications and the Marie Curie Control Training Site is gratefully acknowledged.

I appreciate that I come from a most caring and supportive family, to whom it is always is very nice to return for a visit. Regrettably, these visits are almost without exception too short, and they take place too seldom.

I am also happy to have many good friends, many of which, but certainly not all, are associated with the Karate-Do Shotokai organisation of Sensei Mitsusuke Harada.

You have all brought me much fun, recreation and development through the years!

Åbo, 1st April 2010

Contents

Chapter 1

Introduction

We study the theory of infinite-dimensional linear systems (distributed-parameter systems) in continuous time from an input/output free perspective. We do this by combining the inputs and outputs of traditional control theory into a single external signal and take a graph approach to modelling linear systems. In this way we obtain a *state/signal system*; see Chapter 2 for more details. Considering systems theory from this point of view has many advantages, in particular when one studies interconnection of systems.

It is natural to ask the following question as a converse to the above construction: can one turn a state/signal system Σ into a sensible system with inputs and outputs by decomposing the external signal in a suitable way? One typically asks that an input signal should be unrestricted in its appropriate input space and that it, together with a given initial state, should determine both the state and the output of Σ uniquely. Another way of expressing this is to say that an input should be a *maximal free* component of the external signals.

In a general interconnection situation the characteristics of the system, which is the result of an interconnection, determine which signals can be chosen as an input. It might not be obvious how to choose this input based on the choices of inputs in the original systems, because interconnection may place additional conditions on the signals. The choice of output signals for a system is usually more straightforward, because one commonly takes the output to be the external signals which are not part of the input.

By decomposing the external signals into inputs and outputs in different ways, one can obtain different input/output behaviours, although the system itself essentially stays the same. Consider, for example, an electrical circuit welded onto a circuit board. The input can often enter the circuit in any of several places and the output can also be read off in various places. In this way several different input/output behaviours can be obtained but the electrical circuit is still the same.

It should be emphasised that the types of interconnection that we are interested in are of a quite general kind. One might for instance need to shrink the state space of the system obtained through interconnection in order to make it satisfy the common condition that the admissible initial states lie densely in the state space. This situation is not covered by the standard feedback theory.

The article [Sta06] considers state/signal systems in discrete time. There Staffans indicates how seemingly different input/output results can be seen as special cases of an input/output free state/signal result. In this way, a single unified proof can be given for the state/signal setting, rather than a separate proof for every possible input/output case.

We introduce the concepts of passivity and conservativity in Chapter 3. The intuitive interpretation of a passive system is that it has no internal energy sources. A conservative system has neither internal energy sources nor internal energy sinks. Passivity brings very useful extra structure to state/signal systems and exploiting this structure seems to be the most promising way to obtain a sensible state/signal theory.

The state/signal theory is conceptually similar to the so-called behavioural theory for finite-dimensional systems that has been developed by Willems and his co-operators. See [PW98] for a good introduction to the behavioural theory and the references in [Wil07] for more recent work in the field. The behavioural framework incorporates distributed-parameter systems by considering also the spatial variables of the system as "time variables". In this way one obtains what is called an $n-D$ (n-dimensional) system, which often has a finite-dimensional state space but multi-dimensional "time". This is one of the main differences from the state/signal approach, where one keeps time one-dimensional and makes the state space infinite-dimensional.

Willems describes the so-called "tearing, zooming and linking" approach to modelling complex systems in [Wil07]. The main idea is to model a complex system as the interconnection of simpler standard modules, whose individual behaviours are well-known. Therefore, the approach of tearing, zooming and linking is a major motivation for the interconnection theory, and by extension, for the theory presented in this thesis. The interconnection theory itself is also interesting to develop further, because control is performed by interconnection in the behavioural setting; see [MM05] or [BT02].

The idea of modular modelling is also prevalent in the theory of port-Hamiltonian systems. This theory has its roots in energy-based modelling of mainly nonlinear physical systems; see [vdS00, MvdS05]. A port-Hamiltonian system consists of two parts: the Hamiltonian, which measures the total energy of the system when it is in a given state, and the Dirac structure. The Dirac structure encodes the relations between the different variables present in the system and it also describes how the system acts under interconnection. We present Dirac structures which are defined on Hilbert spaces in Chapter 4, and we also describe how interconnection of two port-Hamiltonian systems corresponds to a so-called *composition* of their respective Dirac structures. Dirac structures and state/signal nodes are connected via an example in Chapter 5.

Systems theory in the setting of this thesis offers quite a few technical challenges. Linear operator theory provides the main tools for our study and, unlike the corresponding discrete-time analogue, many of the involved operators are unbounded. The finite-dimensional linear network theory is classical by now, see [Bel68], and thus most of the substance of the thesis lies in the technical details of the included articles.

1.1 List of included articles

The following articles are appended to this thesis:

Article I: Mikael Kurula and Olof J. Staffans, *Well-posed state/signal systems in continuous time*, To appear in Complex Analysis and Operator Theory, SpringerLink Online First version available with DOI 10.1007/s11785-009-0021-5, 2009.

Article II: Mikael Kurula, *On passive and conservative state/signal systems in continuous time*, To appear in Integral Equations and Operator Theory, SpringerLink Online First version available with DOI 10.1007/s00020-010-1787-6, 2010.

Article III: Mikael Kurula, Hans Zwart, Arjan van der Schaft, and Jussi Behrndt, *Dirac structures and their composition on Hilbert spaces*, submitted, draft available at `http://users.abo.fi/mkurula/`, 2009.

The contributions to the research field made in the above articles are listed in Section 6.4. I have also co-authored the following articles:

(i) Mikael Kurula, Hans Zwart, and Arjan van der Schaft, *Composition of infinite-dimensional linear Dirac-type structures*, Proceedings of the Mathematical Theory of Networks and Systems, 2006.

(ii) Mikael Kurula and Olof Staffans, *A complete model of a finite-dimensional impedance-passive system*, Math. Control Signals Systems **19** (2007), no. 1, 23–63.

(iii) Mikael Kurula and Olof Staffans, *Well-posed state/signal systems in continuous time*, Proceedings of the Mathematical Theory of Networks and Systems, 2008.

Chapter 2

Models for linear systems

The objective of this chapter is to introduce state/signal systems. We do this by first introducing the notion of an abstract input/state/output system and then using this system to explain the ideas behind the state/signal system. We begin by discussing the lossless transfer line, which will act as the standard example in this dissertation. The various function spaces that we use are defined in Appendix B.

Example 2.1. An electrical transfer line consists of two parallel electrical wires through which an electrical current flows. Due to this current a magnetic field emerges around the wires and this results in the wires behaving like inductors. Therefore the wires have a characteristic inductance per length unit, which we for simplicity set to one. As the two cables run parallel to each other, separated by a non-conducting material, they also act as a capacitor, say with unit capacitance per length unit. By saying that the line is "lossless", we mean that the wires have no resistance and that the medium which insulates the wires from each other has zero conductance, so that no current leaks from one wire to the other. Thus the transmission line can be modelled by infinitely many small discrete inductors and capacitors as given in Figure 2.1.

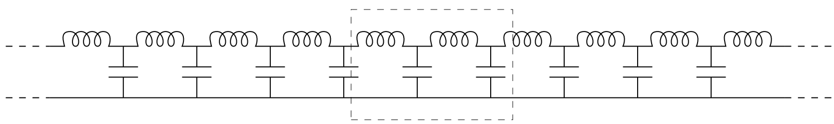

Figure 2.1: A piece of the transmission line modelled with small discrete inductors and capacitors. We proceed to study the framed part more closely.

Now consider a transmission line on the interval $[0,\infty)$. Zooming in on the framed part of Figure 2.1, we obtain Figure 2.2, where we have drawn the part of the transmission line which lies between z and $z+l$. Here $U(z,l)$ and $I(z,l)$ denote the voltage and current flowing in the left direction at the point z at time t, respectively. Since the transmission line was assumed to have unit inductivity and capacitivity, the inductance of L is l henries and the capacitance of C is l farads.

By standard knowledge of ideal inductors, the voltage over the inductor L at time t is $l\frac{\partial}{\partial t}I(z+l,t)$, where $\frac{\partial}{\partial t}I$ is the partial derivative of I with respect to the variable t. The current flowing into the capacitor C at time t is $l\frac{\partial}{\partial t}U(z,t)$. Applying Kirchhoff's

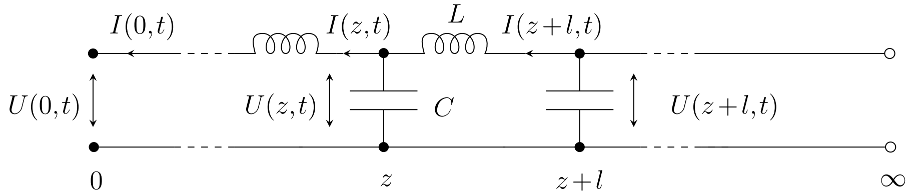

Figure 2.2: Zooming in on a part of the discretely modelled transmission line in Figure 2.1. The current and voltage at the point z at time t is denoted by $I(z,t)$ and $U(z,t)$, respectively.

laws, we obtain that for all $t \geq 0$:

$$\begin{gathered}\begin{bmatrix} I(z+l,t) \\ U(z+l,t) \end{bmatrix} = \begin{bmatrix} I(z,t) + l\frac{\partial}{\partial t}U(z,t) \\ U(z,t) + l\frac{\partial}{\partial t}I(z+l,t) \end{bmatrix} \quad \Longleftrightarrow \\ \begin{bmatrix} \frac{\partial}{\partial t}U(z,t) \\ \frac{\partial}{\partial t}I(z+l,t) \end{bmatrix} = \frac{1}{l}\begin{bmatrix} I(z+l,t) - I(z,t) \\ U(z+l,t) - U(z,t) \end{bmatrix}. \end{gathered} \tag{2.1}$$

Assume now that $\frac{\partial}{\partial z}U(z,t)$ and $\frac{\partial}{\partial z}I(z,t)$ exist and that $\frac{\partial}{\partial t}I(z,t)$ is continuous at z. Letting $l \to 0$ in the second line of (2.1), we get the "Telegrapher's equations":

$$\frac{\partial}{\partial t}\begin{bmatrix} U(z,t) \\ I(z,t) \end{bmatrix} = \begin{bmatrix} 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & 0 \end{bmatrix}\begin{bmatrix} U(z,t) \\ I(z,t) \end{bmatrix}. \tag{2.2}$$

We can specify some boundary conditions for the transmission line as well. For instance, we are free to vary the current entering the transmission line at any given time. We call this current an *input* u, so that $u(t) = -I(0,t)$ for $t \geq 0$. We can also read off a part of the system *state* $\begin{bmatrix} U(\cdot,t) \\ I(\cdot,t) \end{bmatrix}$ at time t through an *output* y. Here we choose y to be the voltage over the left end of the transfer line, so that $y(t) = U(0,t)$ for $t \geq 0$. Note that both the input and the output are obtained by evaluating the system state on the *boundary* $z = 0$ of the domain $\Omega = (0,\infty)$, on which we consider the telegrapher's equations.

Thus, for a transmission line stretching from $z = 0$ to $z = \infty$ we obtain the following initial value problem of boundary control and boundary observation type:

$$\begin{cases} \frac{\partial}{\partial t}U(z,t) = \frac{\partial}{\partial z}I(z,t) \\ \frac{\partial}{\partial t}I(z,t) = \frac{\partial}{\partial z}U(z,t) \\ I(0,t) = -u(t) \\ y(t) = U(0,t) \\ U(z,0) = U_0(z) \text{ given} \\ I(z,0) = I_0(z) \text{ given} \end{cases}, \quad t > 0,\ z > 0. \tag{2.3}$$

We ask that the solutions U and I of (2.3) have a distribution partial derivative in $\mathcal{X} := L^2(\mathbb{R}^+;\mathbb{R})$ with respect to z and that they have continuous partial derivatives with respect to t. In this case the equations (2.3) hold for all $t>0$ if and only if they hold for all $t \in \mathbb{R}^+ := [0,\infty)$. The initial state $\begin{bmatrix} U_0 \\ I_0 \end{bmatrix}$ should lie in $H^1(\mathbb{R}^+;\mathbb{R}^2)$; see Definition B.3. We finally note that we have the implicit boundary condition $U(\infty)=I(\infty)=0$ at infinity, because $U, I \in H^1(\mathbb{R}^+;\mathbb{R})$. ♦

One often sees the telegrapher's equations (2.2) written as

$$\frac{\partial}{\partial t}\begin{bmatrix} U(z,t) \\ I(z,t) \end{bmatrix} = \begin{bmatrix} 0 & -\frac{\partial}{\partial z} \\ -\frac{\partial}{\partial z} & 0 \end{bmatrix} \begin{bmatrix} U(z,t) \\ I(z,t) \end{bmatrix}.$$

The difference between the former and the latter convention is that the direction of the current is reversed. In the latter case positive current flows into the transmission line. It is also more common to consider a transfer line of finite length, but we use the infinite transfer line because we need it later, in Chapter 5.

2.1 Abstract input/state/output systems

In this section we give some definitions from the abstract input/state/output system theory. Although it is not completely obvious, Example 2.1 is a special case of this theory, as we will see in Section 2.3. Comprehensive expositions of the theory of infinite-dimensional linear input/state/output systems can be found in [CZ95] and [Sta05].

Definition 2.2. Let $\mathcal{X}$ be a Hilbert space. A family $t \to \mathfrak{A}^t$, $t \geq 0$, of bounded linear operators on $\mathcal{X}$ is a *semigroup* on $\mathcal{X}$ if $\mathfrak{A}^0 = 1$ and $\mathfrak{A}^{s+t} = \mathfrak{A}^s\mathfrak{A}^t$ for all $s,t \geq 0$.

The semigroup is *strongly continuous*, or shorter C_0, if $\lim_{t\to 0^+} \mathfrak{A}^t x_0 = x_0$ for all $x_0 \in \mathcal{X}$.

The semigroup is a *contraction semigroup* if $\|\mathfrak{A}^t\|_{\mathcal{L}(\mathcal{X})} \leq 1$ for all $t \geq 0$, where $\|\cdot\|_{\mathcal{L}(\mathcal{X})}$ denotes the operator norm.

The *generator* $A : \mathcal{X} \supset \mathrm{Dom}(A) \to \mathcal{X}$ of $\mathfrak{A}$ is the (in general unbounded) linear operator defined by

$$Ax_0 := \lim_{t\to 0^+} \frac{1}{t}(\mathfrak{A}^t x_0 - x_0), \tag{2.4}$$

with $\mathrm{Dom}(A)$ consisting of those $x_0 \in \mathcal{X}$ for which the limit (2.4) exists in $\mathcal{X}$. The domain of A is usually equipped with the inner product

$$(x^1, x^2)_{\mathrm{Dom}(A)} = (x^1, x^2)_{\mathcal{X}} + (Ax^1, Ax^2)_{\mathcal{X}}. \quad ♦ \tag{2.5}$$

The generator A of a C_0 semigroup on $\mathcal{X}$ is closed and $\mathrm{Dom}(A)$ is dense in $\mathcal{X}$; see [Paz83, Thm 1.2.7]. In particular, $\mathrm{Dom}(A)$ is then a Hilbert space with the inner product (2.5). It follows immediately from (2.5) that A is a bounded operator from $\mathrm{Dom}(A)$ to $\mathcal{X}$. Moreover, $\mathrm{Dom}(A)$ is invariant under $\mathfrak{A}$: $\mathfrak{A}^t x_0 \in \mathrm{Dom}(A)$ for all $x_0 \in \mathrm{Dom}(A)$ and $t \geq 0$; see [Sta05, Thm 3.2.1(iii)].

We now briefly return to (2.3). It is easy to see that for any two functions $f,\widetilde{f}\in C^1(\mathbb{R};\mathbb{R})$, the function

$$\begin{bmatrix} U(z,t) \\ I(z,t) \end{bmatrix} := \begin{bmatrix} f(t+z)+\widetilde{f}(z-t) \\ f(t+z)-\widetilde{f}(z-t) \end{bmatrix} \tag{2.6}$$

solves the telegrapher's equations (2.2) for $t,z>0$. This can be interpreted as $f(z)$ and $\widetilde{f}(z)$ describing two waves travelling in the left and right directions, respectively, as time t increases.

By taking the input u of (2.3) to be identically zero, we obtain the boundary condition $I(0,t)=0$ for all $t\geq 0$. This is easily seen to be equivalent to the condition $\widetilde{f}(-t)=f(t)$ for all $t\geq 0$, and (2.6) then reduces to

$$\begin{bmatrix} U(z,t) \\ I(z,t) \end{bmatrix} := \begin{bmatrix} f(t+z)+f(t-z) \\ f(t+z)-f(t-z) \end{bmatrix}, \quad t>0,\ z>0. \tag{2.7}$$

If $U_0,I_0\in C^1(\mathbb{R}^+;\mathbb{R})$ with $I_0(0)=0$ and $(\frac{\mathrm{d}}{\mathrm{d}z}U_0)(0)=0$, then the initial conditions $U(z,0)=U_0(z)$ and $I(z,0)=I_0(z)$, $z\geq 0$, determine $f\in C^1(\mathbb{R}^+;\mathbb{R})$ as

$$f(z)=\begin{cases} \dfrac{U_0(z)+I_0(z)}{2}, & z\geq 0, \\ \dfrac{U_0(-z)-I_0(-z)}{2}, & z<0. \end{cases}$$

Substituting this into (2.7), we get

$$\begin{bmatrix} U(z,t) \\ I(z,t) \end{bmatrix} = \begin{cases} \dfrac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}(U_0(t+z)+I_0(t+z))+\dfrac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}(U_0(t-z)+I_0(t-z)), & 0\leq z<t \\ \dfrac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}(U_0(t+z)+I_0(t+z))+\dfrac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}(U_0(z-t)-I_0(z-t)), & z\geq t\geq 0. \end{cases}$$

Using the operators ρ, τ and $\text{\textbf{Я}}$ given in Definition B.1, we can express this equation concisely as

$$\begin{bmatrix} U(\cdot,t) \\ I(\cdot,t) \end{bmatrix} = \mathfrak{A}^t \begin{bmatrix} U_0(\cdot) \\ I_0(\cdot) \end{bmatrix}, \tag{2.8}$$

$$\text{where}\quad \operatorname{Dom}\left(\mathfrak{A}^t\right)=\left\{\begin{bmatrix} U_0 \\ I_0 \end{bmatrix}\in C^1(\mathbb{R}^+;\mathbb{R}^2)\,\middle|\,\left(\frac{\partial}{\partial z}U_0\right)(0)=I_0(0)=0\right\}\quad\text{and}$$

$$\mathfrak{A}^t := \rho_+\tau^t\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}+\rho_{[0,t)}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\text{\textbf{Я}}\tau^t\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}+\tau^{-t}\rho_+\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad t\geq 0. \tag{2.9}$$

The map $\mathfrak{A}^t$ thus describes how a given initial state $\begin{bmatrix} U_0 \\ I_0 \end{bmatrix}\in\operatorname{Dom}(\mathfrak{A}^t)$ is mapped into the state $\begin{bmatrix} U(\cdot,t) \\ I(\cdot,t) \end{bmatrix}$ of the system (2.3) at time $t\geq 0$ in case the input u is zero. For every $t\geq 0$, the operator $\mathfrak{A}^t$ is densely defined and continuous on $\mathcal{X}:=L^2(\mathbb{R}^+;\mathbb{R}^2)$, and we may therefore extend $\mathfrak{A}^t$ uniquely to all of $\mathcal{X}$ by continuity for all $t\geq 0$. We now prove that the family $t\to\mathfrak{A}^t$, $t\geq 0$, is a C_0 semigroup on $\mathcal{X}$.

The operators in the family $t \to \mathfrak{A}^t$, $t \geq 0$ are bounded operators on $\mathcal{X}$, as we just mentioned. Moreover, it is easy to see that $\mathfrak{A}^0$ is the identity operator $1_\mathcal{X}$ on $\mathcal{X}$ and that $\lim_{t\to 0^+}(\mathfrak{A}^t - 1_\mathcal{X})\begin{bmatrix} U_0 \\ I_0 \end{bmatrix} = 0$ for all $\begin{bmatrix} U_0 \\ I_0 \end{bmatrix} \in \mathcal{X}$. A rather lengthy but straightforward computation shows that $\mathfrak{A}^s\mathfrak{A}^t = \mathfrak{A}^{s+t}$ for all $s,t \geq 0$ and thus $\mathfrak{A}$ is a C_0 semigroup on $\mathcal{X}$. Another, rather long, computation proves that the generator of $\mathfrak{A}$ is

$$A = \begin{bmatrix} 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & 0 \end{bmatrix} \quad \text{with domain} \quad \mathrm{Dom}(A) = \begin{bmatrix} H^1(\mathbb{R}^+;\mathbb{R}) \\ H_0^1(\mathbb{R}^+;\mathbb{R}) \end{bmatrix}. \tag{2.10}$$

Note that $I_0(0) = 0$ for all I_0 such that $\begin{bmatrix} U_0 \\ I_0 \end{bmatrix} \in \mathrm{Dom}(A)$ for some U_0.

The following result has been proved e.g. in [Paz83, Thm 4.1.3].

Lemma 2.3. *Let A be a densely defined operator on the Hilbert space $\mathcal{X}$ with nonempty resolvent set and denote $\dot{x} := \frac{\partial}{\partial t}x$. The homogeneous Cauchy problem*

$$\dot{x}(t) = Ax(t), \quad t > 0, \quad x(0) = x_0, \tag{2.11}$$

has a unique solution $x \in C^1(\mathbb{R}^+;\mathcal{X})$ for every initial value $x_0 \in \mathrm{Dom}(A)$ if and only if A generates a C_0 semigroup on $\mathcal{X}$.

According to [Paz83, Thm 1.2.6], a C_0 semigroup $\mathfrak{A}$ is uniquely determined by its generator A in the following way. For every $x_0 \in \mathrm{Dom}(A)$, the function $x : t \to \mathfrak{A}^t x_0$, $t \geq 0$, is the unique continuously differentiable solution of the initial value problem $\dot{x}(t) = Ax(t)$, $t \geq 0$, $x(0) = x_0$. Thus (2.8) is the unique solution in $C^1(\mathbb{R}^+;\mathcal{X})$ of the system (2.3) with $u = 0$, $U_0 \in H^1(\mathbb{R}^+;\mathbb{R})$ and $I_0 \in H_0^1(\mathbb{R}^+;\mathbb{R})$.

Let A be a closed operator on the Banach space $\mathcal{X}$. The *resolvent set* $\mathrm{Res}(A)$ of A is the set of all $\lambda \in \mathbb{C}$ such that $\lambda - A$ maps $\mathrm{Dom}(A)$ one-to-one onto $\mathcal{X}$. The complement $\mathbb{C} \setminus \mathrm{Res}(A)$ of the resolvent set is called the *spectrum* of A. From [Sta05, Thm 3.2.9(i)] we know that $\mathrm{Res}(A) \neq \emptyset$ for every C_0-semigroup generator.

Let $\alpha \in \mathrm{Res}(A)$ and assume that $\mathcal{X}_1 := \mathrm{Dom}(A)$ with the norm $\|x\|_1 := \|(\alpha - A)x\|_\mathcal{X}$ is dense in $\mathcal{X}$. Denote by $\mathcal{X}_{-1}$ the completion of $\mathcal{X}$ with respect to the norm $\|x\|_{-1} = \|(\alpha - A)^{-1}x\|_\mathcal{X}$. The operator A can also be considered as a continuous operator which maps the dense subspace $\mathcal{X}_1$ of $\mathcal{X}$ into $\mathcal{X}_{-1}$, and we denote the unique continuous extension of A to an operator $\mathcal{X} \to \mathcal{X}_{-1}$ by $A|_\mathcal{X}$.

Now take $\mathcal{X}$, $\mathcal{U}$ and $\mathcal{Y}$ to be Banach spaces and let $x \in C^1(\mathbb{R}^+;\mathcal{X})$, $u \in C(\mathbb{R}^+;\mathcal{U})$, and $y \in C(\mathbb{R}^+;\mathcal{Y})$. We turn our attention to the abstract partial differential equation

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \geq 0, \ x(0) = x_0 \text{ given}, \tag{2.12}$$

where the so-called *input/state/output system node* S is a closed operator, which maps a dense subset of $\begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}$ into $\begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$. By projecting S onto $\begin{bmatrix} \mathcal{X} \\ \{0\} \end{bmatrix}$ and $\begin{bmatrix} \{0\} \\ \mathcal{Y} \end{bmatrix}$, respectively, we can always write $S = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \supset \mathrm{Dom}(S) \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$.

The domain of S is usually not of the form $\operatorname{Dom}(S)=\begin{bmatrix}\mathcal{X}_1\\ \mathcal{U}_1\end{bmatrix}$ and therefore we cannot write $S=\begin{bmatrix}A & B\\ C & D\end{bmatrix}$ as in the finite-dimensional or discrete-time cases; see (3.16). However, condition (iii) of the follwing definition says that we can always *extend* S_1 into a continuous operator $\begin{bmatrix}A|_{\mathcal{X}} & B\end{bmatrix}:\begin{bmatrix}\mathcal{X}\\ \mathcal{U}\end{bmatrix}\to\mathcal{X}_{-1}$ and thus obtain that S_1 is the restriction $\begin{bmatrix}A|_{\mathcal{X}} & B\end{bmatrix}\big|_{\operatorname{Dom}(S)}$ of $\begin{bmatrix}A|_{\mathcal{X}} & B\end{bmatrix}$ to $\operatorname{Dom}(S)$. We use the notation $S_1=A\&B$ and $S_2=C\&D$ in order to make (2.12) resemble the discrete-time and finite-dimensional cases, bearing the above relationship between A, B and $A\&B$ in mind.

Definition 2.4. By an *input/state/output-operator node (shortly operator node)* on the triple $(\mathcal{U},\mathcal{X},\mathcal{Y})$ of Banach spaces we mean a linear operator

$$\begin{bmatrix}A\&B\\ C\&D\end{bmatrix}:\begin{bmatrix}\mathcal{X}\\ \mathcal{U}\end{bmatrix}\supset\operatorname{Dom}\left(\begin{bmatrix}A\&B\\ C\&D\end{bmatrix}\right)\to\begin{bmatrix}\mathcal{X}\\ \mathcal{Y}\end{bmatrix}$$

with the following properties:

(i) The operator $\begin{bmatrix}A\&B\\ C\&D\end{bmatrix}$ is closed.

(ii) The *main operator* $A\colon \operatorname{Dom}(A)\to\mathcal{X}$, which is defined by

$$Ax=A\&B\begin{bmatrix}x\\ 0\end{bmatrix}\quad\text{on}\quad \operatorname{Dom}(A)=\left\{x\in\mathcal{X}\;\Big|\;\begin{bmatrix}x\\ 0\end{bmatrix}\in\operatorname{Dom}\left(\begin{bmatrix}A\&B\\ C\&D\end{bmatrix}\right)\right\},\tag{2.13}$$

has domain dense in $\mathcal{X}$ and nonempty resolvent set.

(iii) The operator $A\&B$ can be extended to an operator $\begin{bmatrix}A|_{\mathcal{X}} & B\end{bmatrix}$ that maps $\begin{bmatrix}\mathcal{X}\\ \mathcal{U}\end{bmatrix}$ continuously into $\mathcal{X}_{-1}$.

(iv) The domain of $\begin{bmatrix}A\&B\\ C\&D\end{bmatrix}$ satisfies the condition

$$\operatorname{Dom}\left(\begin{bmatrix}A\&B\\ C\&D\end{bmatrix}\right)=\left\{\begin{bmatrix}x\\ u\end{bmatrix}\in\begin{bmatrix}\mathcal{X}\\ \mathcal{U}\end{bmatrix}\;\Big|\;A|_{\mathcal{X}}x+Bu\in\mathcal{X}\right\}.$$

An operator node $\begin{bmatrix}A\&B\\ C\&D\end{bmatrix}$ is called an *input/state/output system node (shortly system node)* if A generates a C_0 semigroup on $\mathcal{X}$. The operator node is a *time-reflected system node* if $-A$ generates a C_0 semigroup.

The triple (u,x,y) is a *classical input/state/output trajectory* of $\begin{bmatrix}A\&B\\ C\&D\end{bmatrix}$ with initial state $x_0\in\mathcal{X}$ if $u\in C(\mathbb{R}^+;\mathcal{U})$, $x\in C^1(\mathbb{R}^+;\mathcal{X})$, $y\in C(\mathbb{R}^+;\mathcal{Y})$ and

$$\begin{bmatrix}\dot{x}(t)\\ y(t)\end{bmatrix}=\begin{bmatrix}A\&B\\ C\&D\end{bmatrix}\begin{bmatrix}x(t)\\ u(t)\end{bmatrix},\quad t>0,\quad x(0)=x_0,\tag{2.14}$$

where $\dot{x}:=\frac{\partial}{\partial t}x$. The signal u is called the *input signal*, x is the *state trajectory*, and y is the *output signal*. ♦

We give examples of system nodes in Examples 2.6 and 2.14.

Definition 2.5. The system node $\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$ on $(\mathcal{U},\mathcal{X},\mathcal{Y})$ is *(L^2-)well-posed* if there exists a $T>0$ and a constant K_T, such that all classical input/state/output trajectories (u,x,y) of $\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$ satisfy

$$\forall t\in[0,T]: \quad \|x(t)\|_{\mathcal{X}}^2+\int_0^t \|y(s)\|_{\mathcal{Y}}^2\mathrm{d}s\le K_T\left(\|x(0)\|_{\mathcal{X}}^2+\int_0^t \|u(s)\|_{\mathcal{U}}^2\mathrm{d}s\right). \quad \blacklozenge \tag{2.15}$$

Example 2.6. The operator A in (2.10) is unbounded on $\mathcal{X}=L^2(\mathbb{R}^+;\mathbb{R}^2)$ but it can be shown that it is maximally dissipative, i.e., that $\mathrm{Re}\,(x,Ax)_{\mathcal{X}}\le 0$ for all $x\in\mathrm{Dom}\,(A)$ and $\mathrm{Ran}\,(1-A)=\mathcal{X}$. By the Lumer-Phillips Theorem [Paz83, Thm 1.4.3], this implies that its semigroup $\mathfrak{A}$ in (2.9), with $\mathrm{Dom}\,(\mathfrak{A}^t)=L^2(\mathbb{R}^+;\mathbb{R}^2)$, is a contraction semigroup.

In Examples 5.3 and 5.5 of [KS09] we showed that if A is maximally dissipative but unbounded operator on $\mathcal{X}$, and $A|_{\mathcal{X}}$ is its unique extension to a continuous operator from $\mathcal{X}$ to $\mathcal{X}_{-1}$, then the linear operator

$$S:=\begin{bmatrix} A|_{\mathcal{X}} & A|_{\mathcal{X}} \\ -A|_{\mathcal{X}} & -A|_{\mathcal{X}} \end{bmatrix}\Bigg|_{\mathrm{Dom}(S)} \quad \text{with domain}$$
$$\mathrm{Dom}\,(S)=\left\{\begin{bmatrix} x \\ u \end{bmatrix}\in\begin{bmatrix} \mathcal{X} \\ \mathcal{X} \end{bmatrix}\,\middle|\,x+u\in\mathrm{Dom}\,(A)\right\}$$

is a system node on $(\mathcal{X},\mathcal{X},\mathcal{X})$ which is not L^2-well posed.

The ill-posedness was proved by noting that the transfer function of S is

$$\widehat{\mathfrak{D}}(\lambda)=-A(\lambda-A)^{-1}, \quad \lambda\in\mathbb{C}^+, \tag{2.16}$$

which tends to $-A$ as $\lambda\to\infty$ in $\mathbb{R}^+$. This transfer function is thus not bounded on any complex right-half plane, and therefore S is ill-posed, as is well-known; see e.g. [Sta05, Lem. 4.6.2]. $\blacklozenge$

See [KS09, Sect. 5] for more information on system nodes and their trajectories. Also note that the system node in Example 2.6 is symmetric with respect to $\mathcal{X}$ and $\mathcal{U}=\mathcal{Y}$. This is in general not the case, because B can in general not be recovered from the restriction of $A\&B$ to $\mathrm{Dom}\,(S)\cap\begin{bmatrix} \{0\} \\ \mathcal{U} \end{bmatrix}$ as is the case for A, cf. (2.13). Indeed, for system nodes S of boundary control type we have $\begin{bmatrix} 0 \\ u \end{bmatrix}\in\mathrm{Dom}\,(S)$ only if $u=0$; see the text before (2.24) below.

2.2 Introducing state/signal systems

The main idea of the state/signal approach is to treat the input and the output in (2.14) equally, and we formalise this idea in the following way. We first consider the input space $\mathcal{U}$ and the output space $\mathcal{Y}$ to be closed subspaces of a *combined external signal space*: $\mathcal{W}:=\mathcal{U}\dot{+}\mathcal{Y}$. We can always achieve this by setting $\mathcal{W}=\begin{bmatrix}\mathcal{Y}\\ \mathcal{U}\end{bmatrix}$ and identifying $\mathcal{Y}=\begin{bmatrix}\mathcal{Y}\\ \{0\}\end{bmatrix}$ and $\mathcal{U}=\begin{bmatrix}\{0\}\\ \mathcal{U}\end{bmatrix}$. Then we rewrite the equation (2.14) in *graph form* to get rid of the explicit input $u(t)$ and output $y(t)$:

$$\begin{bmatrix}\dot{x}(t)\\ x(t)\\ w(t)\end{bmatrix}\in V,\quad t>0,\quad x(0)=x_0,\quad \text{where} \tag{2.17}$$

$$V=\left\{\begin{bmatrix}z\\ x\\ u+y\end{bmatrix}\,\middle|\,\begin{bmatrix}z\\ y\end{bmatrix}=\begin{bmatrix}A\&B\\ C\&D\end{bmatrix}\begin{bmatrix}x\\ u\end{bmatrix}\right\}=\begin{bmatrix}A\&B\\ \begin{bmatrix}1_{\mathcal{X}} & 0\end{bmatrix}\\ C\&D+\begin{bmatrix}0 & 1_{\mathcal{U}}\end{bmatrix}\end{bmatrix}\operatorname{Dom}\left(\begin{bmatrix}A\&B\\ C\&D\end{bmatrix}\right). \tag{2.18}$$

The so-called *generating subspace* V in (2.18) is a subspace of the so-called "node space", which we now define.

Definition 2.7. Let $\mathcal{X}$ be a Hilbert space with inner product $(\cdot,\cdot)_{\mathcal{X}}$ and let $\mathcal{W}$ be a Kreĭn space with indefinite inner product $[\cdot,\cdot]_{\mathcal{W}}$; see Definition A.1. The *(continuous-time) node space* is $\mathfrak{K}:=\begin{bmatrix}\mathcal{X}\\ \mathcal{X}\\ \mathcal{W}\end{bmatrix}$ equipped with the sesquilinear *power product*

$$\left[\begin{bmatrix}z^1\\ x^1\\ w^1\end{bmatrix},\begin{bmatrix}z^2\\ x^2\\ w^2\end{bmatrix}\right]_{\mathfrak{K}}:=[w^1,w^2]_{\mathcal{W}}-(z^1,x^2)_{\mathcal{X}}-(x^1,z^2)_{\mathcal{X}}.\quad\blacklozenge \tag{2.19}$$

In Proposition A.2 we prove that the node space $\mathfrak{K}$ is a Kreĭn space. We explain why we choose $\mathcal{W}$ to be a Kreĭn space and why we call the inner product $[\cdot,\cdot]_{\mathfrak{K}}$ a "power product" at the beginning of Section 3.3.

Next we introduce a class of trajectories, which describe the dynamics induced by the generating subspace.

Definition 2.8. Let I be a subinterval of $\mathbb{R}$ with positive length, let $\mathcal{X}$ be a Hilbert space, and let $\mathcal{W}$ be a Kreĭn space. Let V be a subspace of $\begin{bmatrix}\mathcal{X}\\ \mathcal{X}\\ \mathcal{W}\end{bmatrix}$.

The space $\mathfrak{V}(I)$ of *classical trajectories* on I generated by V consists of all pairs $\begin{bmatrix}x\\ w\end{bmatrix}\in\begin{bmatrix}C^1(I;\mathcal{X})\\ C(I;\mathcal{W})\end{bmatrix}$, such that $\begin{bmatrix}\dot{x}(t)\\ x(t)\\ w(t)\end{bmatrix}\in V$ for all interior points t in I. We abbreviate $\mathfrak{V}:=\mathfrak{V}[0,\infty)$.

In preparation for the next definition, recall the following standard result. Let $\mathcal{W}=\mathcal{U}\dot{+}\mathcal{Y}$ be a direct-sum decomposition of the Kreĭn space $\mathcal{W}$. Denote the projection of $\mathcal{W}$ onto $\mathcal{U}$ along $\mathcal{Y}$ by $\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}}$ and the complementary projection by $\mathcal{P}_{\mathcal{Y}}^{\mathcal{U}}$. For any admissible norm on $\mathcal{W}$ and its restrictions to $\mathcal{U}$ and $\mathcal{Y}$, and for any $1\leq p<\infty$, the product p-norm $\left\|\begin{bmatrix} y \\ u \end{bmatrix}\right\|_{\left[\begin{smallmatrix}\mathcal{Y}\\ \mathcal{U}\end{smallmatrix}\right]} = (\|y\|_{\mathcal{Y}}^p + \|u\|_{\mathcal{U}}^p)^{1/p}$ is equivalent to the norm on $\mathcal{W}$. This means that there exists a constant $C \geq 1$, which depends on p, $\mathcal{U}$ and $\mathcal{Y}$, such that

$$\forall w \in \mathcal{W}: \quad \frac{1}{C}(\|\mathcal{P}_{\mathcal{Y}}^{\mathcal{U}} w\|^p + \|\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w\|^p)^{1/p} \leq \|w\|_{\mathcal{W}} \leq C(\|\mathcal{P}_{\mathcal{Y}}^{\mathcal{U}} w\|^p + \|\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w\|^p)^{1/p}. \tag{2.20}$$

Definition 2.9. Let $\mathcal{X}$ be a Hilbert space and $\mathcal{W}$ a Kreĭn space, and let $V \subset \mathfrak{K}$. We say that $(V;\mathcal{X},\mathcal{W})$ is a *state/signal node* if V has the following properties:

(i) The space V is closed in the norm

$$\left\|\begin{bmatrix} z \\ x \\ w \end{bmatrix}\right\| = \sqrt{\|z\|_{\mathcal{X}}^2 + \|x\|_{\mathcal{X}}^2 + \|w\|_{\mathcal{W}}^2}.$$

(ii) The space V has the property $\begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix} \in V \Longrightarrow z = 0.$

(iii) There exists some $T > 0$ such that

$$\forall \begin{bmatrix} z_0 \\ x_0 \\ w_0 \end{bmatrix} \in V \ \exists \begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{V}[0,T]: \quad \begin{bmatrix} \dot{x}(0) \\ x(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} z_0 \\ x_0 \\ w_0 \end{bmatrix}. \quad \blacklozenge$$

We are now finally able to define the notion of a state/signal system.

Definition 2.10. Let $(V;\mathcal{X},\mathcal{W})$ be a state/signal node and I a subinterval of $\mathbb{R}$ with positive length.

The space $\mathfrak{W}(I)$ of *generalised trajectories* generated by V on I is the closure of $\mathfrak{V}(I)$ in $\begin{bmatrix} C(I;\mathcal{X}) \\ L^2_{loc}(I;\mathcal{W}) \end{bmatrix}$. By this we mean that that $\begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{W}(I)$ if and only if there exists a sequence $\begin{bmatrix} x_n \\ w_n \end{bmatrix} \in \mathfrak{V}(I)$, such that $\begin{bmatrix} x_n \\ w_n \end{bmatrix} \to \begin{bmatrix} x \\ w \end{bmatrix}$ in $\begin{bmatrix} C(I;\mathcal{X}) \\ L^2_{loc}(I;\mathcal{W}) \end{bmatrix}$ as $n \to \infty$. We abbreviate $\mathfrak{W}[0,\infty)$ by $\mathfrak{W}$.

The triple $\Sigma_{s/s} = (\mathfrak{W};\mathcal{X},\mathcal{W})$ is the *state/signal system* induced by $(V;\mathcal{X},\mathcal{W})$. $\blacklozenge$

In [KS09, Def. 3.1] we define the space $\mathfrak{W}^p$ of generalised trajectories to be the closure of $\mathfrak{V}(I)$ in $\begin{bmatrix} C(I;\mathcal{X}) \\ L^p_{loc}(I;\mathcal{W}) \end{bmatrix}$ for an arbitrary finite $p \geq 1$, and we call the elements of $\mathfrak{W}^p$ "L^p trajectories". In this summary, however, we restrict us to one p for simplicity, and the natural choice for passive systems is $p = 2$.

When working with state/signal nodes and systems one needs to make some extra assumptions on the state/signal node, because Definition 2.9 imposes very little structure. We have made this choice deliberately in order to keep the state/signal node a general and flexible object. Typically one assumes the existence of an admissible or well-posed input/output decomposition of the external signal space $\mathcal{W}$, and we now proceed to describe what these assumptions mean.

Definition 2.11. Let $V\subset\mathfrak{K}$. We make the following definitions:

(i) By an *input/output pair* $(\mathcal{U},\mathcal{Y})$ of V we mean a direct-sum decomposition $\mathcal{W}=\mathcal{U}\dot{+}\mathcal{Y}$. This input/output pair is *fundamental* if the corresponding decomposition is a fundamental decomposition of the Kreĭn space $\mathcal{W}$, and $\mathcal{U}\geq 0$ and $\mathcal{Y}\leq 0$ in $\mathcal{W}$. The input/output pair is *Lagrangian* if $\mathcal{U}=\mathcal{U}^{[\perp]}$ and $\mathcal{Y}=\mathcal{Y}^{[\perp]}$ in $\mathfrak{K}$.

(ii) We say that the input/output pair $(\mathcal{U},\mathcal{Y})$ is *admissible* if there exists an operator node $\begin{bmatrix}A\&B\\C\&D\end{bmatrix}$ on $(\mathcal{U},\mathcal{X},\mathcal{Y})$, such that

$$V=\begin{bmatrix}A\&B\\ \begin{bmatrix}1_{\mathcal{X}}\ 0\end{bmatrix}\\ C\&D+\begin{bmatrix}0\ 1_{\mathcal{U}}\end{bmatrix}\end{bmatrix}\operatorname{Dom}\left(\begin{bmatrix}A\&B\\C\&D\end{bmatrix}\right). \tag{2.21}$$

Then we call $V_{op}:=\left(\begin{bmatrix}A\&B\\C\&D\end{bmatrix};\mathcal{X},\mathcal{U},\mathcal{Y}\right)$ an *operator node representation* of V.

(iii) If $(\mathcal{U},\mathcal{Y})$ is admissible for V and $\begin{bmatrix}A\&B\\C\&D\end{bmatrix}$ in (2.21) is an L^2-well-posed system node, then $(\mathcal{U},\mathcal{Y})$ is an *(L^2)-well-posed input/output pair*. We call $\mathcal{U}$ a *well-posed input space* if there exists a $\mathcal{Y}$ such that $(\mathcal{U},\mathcal{Y})$ is a well-posed input/output pair.

(iv) If $(V;\mathcal{X},\mathcal{W})$ is a state/signal node and $V_{op}=\left(\begin{bmatrix}A\&B\\C\&D\end{bmatrix};\mathcal{X},\mathcal{U},\mathcal{Y}\right)$, then we call V_{op} an *operator node representation* of both *the state/signal node* $(V;\mathcal{X},\mathcal{W})$ and of *the state/signal system* that the node generates.

An input/output pair $(\mathcal{U},\mathcal{Y})$ is admissible (well-posed) for a *state/signal system* if it is admissible (well-posed) for at least one of its generating state/signal nodes.

(v) A *state/signal system* is *(L^2-)well-posed* if it has at least one well-posed input/output pair. ♦

We remark that the "well-posed input/output pairs" in item (iii) of the preceding definition are called "L^2-admissible input/output pairs" in [KS09]. We did a change of terminology here in order to avoid confusion with the admissible input/output pairs in item (ii).

Also note that the definition of a well-posed input/output pair in [KS09, Def. 2.7] differs from the definition in item (iii) above. However, the two definitions are equivalent. In order to see this, first note that if $(\mathcal{U},\mathcal{Y})$ is a well-posed input/output pair for V in the sense of [KS09], then by combining Theorems 4.9 and 5.8 of [KS09], there exists a well-posed system node $\begin{bmatrix}A\&B\\C\&D\end{bmatrix}$, which satisfies (2.21). Conversely, if there exists such an state/output system node, then $(\mathcal{U},\mathcal{Y})$ is a well-posed input/output pair in the sense of [KS09] for V in (2.21) by [KS09, Thm 6.4].

It follows from (2.21) that $\begin{bmatrix}x\\w\end{bmatrix}\in\mathfrak{V}$ if and only if $(\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}}w,x,\mathcal{P}_{\mathcal{Y}}^{\mathcal{U}}w)$ is a classical input/state/output trajectory of $\begin{bmatrix}A\&B\\C\&D\end{bmatrix}$, cf. (2.14). Combining Definitions 2.11(iii,v) and 2.5, we obtain that every classical trajectory $\begin{bmatrix} x \\ w \end{bmatrix}$ of a well posed state/signal system locally is a continuous linear function of the initial state $x(0)$ and some well-posed input signal $\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w$ in the following way: there exists a direct-sum decomposition $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ of the signal space, such that there for every $T > 0$ exists a constant K_T, which may depend on T, such that for all $\begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{V}$:

$$\forall t \in [0,T]: \quad \|x(t)\|_{\mathcal{X}}^2 + \int_0^t \|w(s)\|_{\mathcal{W}}^2 \mathrm{d}s \leq K_T \left(\|x(0)\|_{\mathcal{X}}^2 + \int_0^t \|\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w(s)\|_{\mathcal{U}}^2 \mathrm{d}s \right). \tag{2.22}$$

Indeed, fix $T > 0$ arbitrarily and let $(\mathcal{U}, \mathcal{Y})$ be a well-posed input/output pair for V. By (2.20) there exists a constant $C \geq 1$ such that for all $w(s) \in \mathcal{W}$ it holds that:

$$\|w(s)\|_{\mathcal{W}}^2 \leq C \|\mathcal{P}_{\mathcal{Y}}^{\mathcal{U}} w(s)\|_{\mathcal{Y}}^2 + C \|\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w(s)\|_{\mathcal{U}}^2.$$

Let $\widetilde{K}_T \geq 1$ be such that (2.15) holds with K_T replaced by $\widetilde{K}_T$. Then it also holds for all $t \in [0,T]$ that

$$\begin{aligned} \|x(t)\|_{\mathcal{X}}^2 + \int_0^t \|w(s)\|_{\mathcal{W}}^2 \mathrm{d}s &\leq \|x(t)\|_{\mathcal{X}}^2 + C \int_0^t \|\mathcal{P}_{\mathcal{Y}}^{\mathcal{U}} w(s)\|_{\mathcal{Y}}^2 \mathrm{d}s + C \int_0^t \|\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w(s)\|_{\mathcal{U}}^2 \mathrm{d}s \\ &\leq C \widetilde{K}_T \left(\|x(0)\|_{\mathcal{X}}^2 + \int_0^t \|\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w(s)\|_{\mathcal{U}}^2 \mathrm{d}s \right) + C \int_0^t \|\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w(s)\|_{\mathcal{U}}^2 \mathrm{d}s \\ &\leq 2C \widetilde{K}_T \left(\|x(0)\|_{\mathcal{X}}^2 + \int_0^t \|\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w(s)\|_{\mathcal{U}}^2 \mathrm{d}s \right), \end{aligned}$$

and thus (2.22) holds with $K_T := 2C\widetilde{K}_T$.

The inequality (2.22) holds also for generalised trajectories if $\mathcal{U}$ is a well-posed input space, since every generalised trajectory can be approximated by a sequence of classical trajectories, for which (2.22) holds.

Example 2.12. With the same set-up as in Example 2.6, let $\mathcal{W} := \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \end{bmatrix}$. In [KS09, Ex. 6.8] we proved that the subspace $V \subset \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix}$ given by

$$V := \begin{bmatrix} A|_{\mathcal{X}} & A|_{\mathcal{X}} \\ 1 & 0 \\ \begin{bmatrix} -A|_{\mathcal{X}} \\ 0 \end{bmatrix} & \begin{bmatrix} -A|_{\mathcal{X}} \\ 1 \end{bmatrix} \end{bmatrix} \mathrm{Dom}(S) \tag{2.23}$$

yields a well-posed state/signal node $(V; \mathcal{X}, \mathcal{W})$. In that example it was also proved that the input/output pair $(\mathcal{U}', \mathcal{Y}') := \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \mathcal{X}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \mathcal{X} \right)$ is well-posed and that the input/output pair $\left(\begin{bmatrix} \{0\} \\ \mathcal{X} \end{bmatrix}, \begin{bmatrix} \mathcal{X} \\ \{0\} \end{bmatrix} \right)$ is admissible but ill-posed.

The well-posedness of $(V;\mathcal{X},\mathcal{W})$ as a state/signal node was proved by replacing S with the well-posed system node S' corresponding to the pair $(\mathcal{U}',\mathcal{Y}')$, which obviously describes the same generating subspace V. Letting $\widehat{\mathfrak{D}}$ be the transfer function of S, the transfer function of S' is the map

$$\widehat{\mathfrak{D}}'(\lambda): \begin{bmatrix}1\\1\end{bmatrix}\widehat{\mu}(\lambda)\to\begin{bmatrix}-1\\1\end{bmatrix}(1-\widehat{\mathfrak{D}}(\lambda))(1+\widehat{\mathfrak{D}}(\lambda))^{-1}\widehat{\mu}(\lambda),\quad \lambda\in\mathbb{C}^+,\ \widehat{\mu}(\lambda)\in\mathcal{U},$$

which can be shown to be a contraction from $\mathcal{U}'$ to $\mathcal{Y}'$ for all $\lambda\in\mathbb{C}^+$. ♦

In the next section we proceed to explain how the transfer line example fits into the framework of abstract input/state/output systems.

2.3 A note on boundary control

We now describe how this dissertation implicitly treats boundary control; see [Fat68]. The following definition is [MS06, Def. 1.1].

Definition 2.13. A triple (L,K,G) is a *boundary colligation* on the triple $(\mathcal{U},\mathcal{X},\mathcal{Y})$ of Banach spaces if the linear operators L, K and G have the same domain $\mathcal{Z}\subset\mathcal{X}$ and take values in $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{U}$, respectively.

A boundary colligation is *strong* if both the operators $\left[\begin{smallmatrix}L\\K\\G\end{smallmatrix}\right]:\mathcal{Z}\to\left[\begin{smallmatrix}\mathcal{X}\\\mathcal{Y}\\\mathcal{U}\end{smallmatrix}\right]$ and L are closed.

A (not necessarily strong) boundary colligation is a *boundary node* if it has the following properties:

(i) The operator $\left[\begin{smallmatrix}L\\K\\G\end{smallmatrix}\right]$ is closed.

(ii) The operator G is surjective and has dense kernel.

(iii) The operator $A:=L|_{\mathcal{N}(G)}$ has a nonempty resolvent set.

If the conditions (i)–(iii) hold and A generates a C_0 semigroup on $\mathcal{X}$, then the boundary node is *internally well-posed.* ♦

As we stated in Section 2.1, if A generates a C_0 semigroup on $\mathcal{X}$, then $\mathrm{Dom}(A)=\mathcal{N}(G)$ is dense in $\mathcal{X}$ and the resolvent set is of A is nonempty. In this case (L,K,G) satisfies conditions (iii) and (iv) of Definition 2.13 if G is surjective.

If (L,K,G) is a boundary input/state/output node on $(\mathcal{U},\mathcal{X},\mathcal{Y})$ then we, according to [MS06, Thm 2.3], always obtain an operator node *of boundary control type* on $(\mathcal{U},\mathcal{X},\mathcal{Y})$ by defining $\begin{bmatrix}A\&B\\C\&D\end{bmatrix}:=\begin{bmatrix}L\\K\end{bmatrix}\begin{bmatrix}1_\mathcal{X}\\G\end{bmatrix}^{-1}$ on $\mathrm{Dom}\left(\begin{bmatrix}A\&B\\C\&D\end{bmatrix}\right)=\mathrm{Ran}\left(\begin{bmatrix}1_\mathcal{X}\\G\end{bmatrix}\right)$. In this case the operator node representation

$$V=\begin{bmatrix}A\&B\\ \begin{bmatrix}1_\mathcal{X}\ 0\end{bmatrix}\\ C\&D+\begin{bmatrix}0\ 1_\mathcal{U}\end{bmatrix}\end{bmatrix}\mathrm{Dom}(S) \tag{2.24}$$

takes the form

$$V = \begin{bmatrix} L \\ 1_{\mathcal{X}} \\ K+G \end{bmatrix} \mathrm{Dom}(L).$$

This representation is formally less dependent of the input/output pair $(\mathcal{U},\mathcal{Y})$ than (2.24), but note that conditions (iii) and (iv) in Definition 2.13 still depend on the choice of input/output pair.

Example 2.14. Consider the triple

$$(L,K,G) := \left(\begin{bmatrix} 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & 0 \end{bmatrix}, \begin{bmatrix} \varphi_0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -\varphi_0 \end{bmatrix} \right)$$

of operators defined on $\mathcal{Z} = H^1(\mathbb{R}^+;\mathbb{R}^2)$, where φ_a denotes evaluation at $a \in \mathbb{R}$ of a function which can be evaluated at a. Taking $x(t) = \begin{bmatrix} U(\cdot,t) \\ I(\cdot,t) \end{bmatrix}$ in $H^1(\mathbb{R}^+;\mathbb{R}^2)$, we can write the system (2.3) in the form:

$$\begin{bmatrix} \dot{x}(t) \\ x(t) \\ y(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} L \\ 1 \\ K \\ G \end{bmatrix} x(t), \quad t > 0, \quad x(0) = \begin{bmatrix} U_0 \\ I_0 \end{bmatrix} \in H^1(\mathbb{R}^+;\mathbb{R}^2).$$

Here $\frac{\partial}{\partial z} U(t)$ should be interpreted as $(\frac{\partial}{\partial z} U)(t)$.

We only need to verify that $\left[\begin{smallmatrix} L \\ K \\ G \end{smallmatrix}\right]$ is closed and G surjective in order to prove that (L,K,G) is an internally well-posed boundary node on $(\mathcal{U},\mathcal{X},\mathcal{Y}) = (\mathbb{R}, L^2(\mathbb{R}^+;\mathbb{R}^2), \mathbb{R})$, because $A = L\big|_{\mathcal{N}(G)}$ generates the C_0 semigroup (2.9) on $L^2(\mathbb{R}^+;\mathbb{R}^2)$.

The operator $\left[\begin{smallmatrix} L \\ K \\ G \end{smallmatrix}\right]$ is closed due to the fact that K, G and L are all bounded from the Hilbert space $\mathcal{Z}$ to their respective co-domains; see Definition B.3. The operator G is also surjective, because for all $a \in \mathbb{R}$ the function $f_a(z) := \begin{bmatrix} 0 \\ -a/(1+z) \end{bmatrix}$ lies in $\mathcal{Z} = H^1(\mathbb{R}^+;\mathbb{R}^2)$ and $G f_a = a$.

The system node and the generating subspace induced by (L,K,G) are

$$\begin{bmatrix} A\&B \\ C\&D \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & 0 \end{bmatrix} \\ \begin{bmatrix} \varphi_0 & 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 0 & -\varphi_0 \end{bmatrix} \end{bmatrix}^{-1} \quad \text{and} \quad V = \begin{bmatrix} \begin{bmatrix} 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & 0 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} \varphi_0 & 0 \\ 0 & -\varphi_0 \end{bmatrix} \end{bmatrix} H^1(\mathbb{R}^+;\mathbb{R}^2),$$

respectively. Note that this example is completely very from Example 2.12, although they both in a sense treat the transfer line. ♦

Chapter 3

Passive and conservative state/signal systems

We now proceed to describe passivity in the context of state/signal systems. We do this by first looking at input/state/output systems. The assumption that a system is passive essentially means that it has no internal energy sources, and thus it is not very restrictive. However, it implies a significant amount of useful extra structure. For instance, it gives us a way to construct a well-posed input/output pair, and this is essential, because there seems not to exist any general way of finding an admissible input/output pair for an arbitrary given state/signal node, even if one knows that one exists.

3.1 Passive input/state/output systems

The following definition agrees with [Sta05, Sec. 6.2]. The definition makes use of the adjoint of an unbounded operator; see Definition A.4 and the comment thereafter.

Definition 3.1. Let $\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$ be an operator node on the Hilbert-space triple $(\mathcal{U},\mathcal{X},\mathcal{Y})$. Then $\begin{bmatrix} A\&B' \\ C\&D' \end{bmatrix} := \begin{bmatrix} A\&B \\ C\&D \end{bmatrix}^*$ is called the *causal dual* of $\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$. Moreover, the operator $\begin{bmatrix} -A\&B' \\ C\&D' \end{bmatrix}$ is the *anti-causal dual* of $\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$. ♦

The causal dual of a system node is also a system node and the anti-causal dual is a time-reflected system node; see [Sta05, Lem. 6.2.14].

Definition 3.2. A system node $\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$ is *scattering passive* if it is L^2-well-posed with $K_T = 1$, i.e., for some $T > 0$ all classical input/state/output trajectories (u,x,y) of $\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$ on $[0,T]$ satisfy

$$\forall t \in [0,T]: \quad \|x(t)\|_{\mathcal{X}}^2 - \|x(0)\|_{\mathcal{X}}^2 \le \int_0^t \|u(s)\|_{\mathcal{U}}^2 \mathrm{d}s - \int_0^t \|y(s)\|_{\mathcal{Y}}^2 \mathrm{d}s. \tag{3.1}$$

The system node is *impedance passive* if there exists a unitary operator $\Psi:\mathcal{U}\to\mathcal{Y}$ and a $T>0$ such that all input/state/output trajectories on $[0,T]$ satisfy

$$\forall t\in[0,T]:\quad \|x(t)\|_{\mathcal{X}}^2-\|x(0)\|_{\mathcal{X}}^2\le 2\mathrm{Re}\int_0^t(\Psi u(s),y(s))_{\mathcal{Y}}\mathrm{d}s. \tag{3.2}$$

The system node $\begin{bmatrix}A\&B\\C\&D\end{bmatrix}$ is *scattering energy preserving* if (3.1) holds with equality instead of inequality. The system node is *scattering conservative* if both $\begin{bmatrix}A\&B\\C\&D\end{bmatrix}$ and $\begin{bmatrix}A\&B\\C\&D\end{bmatrix}^*$ are scattering energy preserving.

The system node $\begin{bmatrix}A\&B\\C\&D\end{bmatrix}$ is *impedance energy preserving* if (3.2) holds with equality instead of inequality. The system node is *impedance conservative* if both $\begin{bmatrix}A\&B\\C\&D\end{bmatrix}$ and $\begin{bmatrix}A\&B\\C\&D\end{bmatrix}^*$ are impedance energy preserving.

An internally well-posed *boundary node* (L,K,G) is scattering or impedance passive or conservative if the system node $\begin{bmatrix}A\&B\\C\&D\end{bmatrix}:=\begin{bmatrix}L\\K\end{bmatrix}\begin{bmatrix}1\\G\end{bmatrix}^{-1}$ is of the corresponding type. ♦

Note that the inequality (3.1) holds for some $T>0$ if and only if it holds for all $T>0$. The same is true for (3.2).

We now want to introduce passivity and conservativity of state/signal nodes and show how these concepts relate to passivity of input/state/output systems. In order to do this, however, we first need to study the state/signal dual.

3.2 The dual of a state/signal node

Preparing for the next definition, we recall the definition (A.1) of the orthogonal companion. Here we compute $V^{[\perp]}$ with respect to the power product (2.19).

Definition 3.3. Let $(V;\mathcal{X},\mathcal{W})$ be a state/signal node. The triple $(V^{[\perp]};\mathcal{X},\mathcal{W})$ is the *state/signal dual* of $(V;\mathcal{X},\mathcal{W})$.

For any subinterval I of $\mathbb{R}$, we denote the space of classical trajectories generated by $V^{[\perp]}$ on I by $\mathfrak{V}^d(I)$. By $\mathfrak{W}^d(I)$ we denote the space of generalised trajectories, i.e., the closure of $\mathfrak{V}^d(I)$ in $\begin{bmatrix}C(I;\mathcal{X})\\L^2_{loc}(I;\mathcal{W})\end{bmatrix}$. We shortly write $\mathfrak{V}^d:=\mathfrak{V}^d(\mathbb{R}^-)$ and $\mathfrak{W}^d:=\mathfrak{W}^d(\mathbb{R}^-)$. ♦

We identify the (continuous) dual of $\mathcal{X}$, i.e., the space of continuous linear functionals on $\mathcal{X}$, with $\mathcal{X}$ itself, as is common for Hilbert spaces, and moreover, we identify the dual of $\mathcal{W}$ with $\mathcal{W}$ itself as well. The correctness of the following argument follows from [AS07c, Sec. 2.3], where the reader can also find more details. Note, however, that the dual of $\mathcal{W}$ is identified with $-\mathcal{W}$ in [AS07c].

By [AS07c, Lemma 2.3], if $\mathcal{W}=\mathcal{U}\dotplus\mathcal{Y}$ then also $\mathcal{W}=\mathcal{Y}^{[\perp]}\dotplus\mathcal{U}^{[\perp]}$. According to the discussion after that lemma, we can identify the continuous duals $\mathcal{U}'$ and $\mathcal{Y}'$ of $\mathcal{U}$ and $\mathcal{Y}$ as

$$\mathcal{U}'=\mathcal{Y}^{[\perp]} \quad \text{and} \quad \mathcal{Y}'=\mathcal{U}^{[\perp]}$$

using the duality pairings

$$\begin{aligned}\langle u,u'\rangle_{\langle\mathcal{U},\mathcal{Y}^{[\perp]}\rangle}&=[u,u']_{\mathcal{W}}, \quad u\in\mathcal{U},\ u'\in\mathcal{Y}^{[\perp]} \quad \text{and}\\ \langle y,y'\rangle_{\langle\mathcal{Y},\mathcal{U}^{[\perp]}\rangle}&=[y,y']_{\mathcal{W}}, \quad y\in\mathcal{Y},\ y'\in\mathcal{U}^{[\perp]}.\end{aligned} \tag{3.3}$$

We thus obtain the following pairings between $\begin{bmatrix}\mathcal{X}\\ \mathcal{U}\end{bmatrix}$ and $\begin{bmatrix}\mathcal{X}\\ \mathcal{Y}\end{bmatrix}$ and their respective duals:

$$\begin{aligned}\left\langle \begin{bmatrix}x\\ u\end{bmatrix},\begin{bmatrix}z'\\ u'\end{bmatrix}\right\rangle_{\left\langle\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right],\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}^{[\perp]}\end{smallmatrix}\right]\right\rangle}&=(x,z')_{\mathcal{X}}+\langle u,u'\rangle_{\langle\mathcal{U},\mathcal{Y}^{[\perp]}\rangle} \quad \text{and}\\ \left\langle \begin{bmatrix}z\\ y\end{bmatrix},\begin{bmatrix}x'\\ y'\end{bmatrix}\right\rangle_{\left\langle\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right],\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}^{[\perp]}\end{smallmatrix}\right]\right\rangle}&=(z,x')_{\mathcal{X}}+\langle y,y'\rangle_{\langle\mathcal{Y},\mathcal{U}^{[\perp]}\rangle}.\end{aligned} \tag{3.4}$$

Adjoint operators computed with respect to these duality pairings are denoted by the dagger †; see also Definition A.4. For instance, if $S:\begin{bmatrix}\mathcal{X}\\ \mathcal{U}\end{bmatrix}\supset\mathrm{Dom}(S)\to\begin{bmatrix}\mathcal{X}\\ \mathcal{Y}\end{bmatrix}$ is densely defined, then $S^\dagger:\begin{bmatrix}\mathcal{X}\\ \mathcal{U}^{[\perp]}\end{bmatrix}\supset\mathrm{Dom}\left(S^\dagger\right)\to\begin{bmatrix}\mathcal{X}\\ \mathcal{Y}^{[\perp]}\end{bmatrix}$ is the maximally defined operator such that for all $\begin{bmatrix}x\\ u\end{bmatrix}\in\mathrm{Dom}(S)$ and $\begin{bmatrix}x'\\ y'\end{bmatrix}\in\mathrm{Dom}\left(S^\dagger\right)$:

$$\left\langle S\begin{bmatrix}x\\ u\end{bmatrix},\begin{bmatrix}x'\\ y'\end{bmatrix}\right\rangle_{\left\langle\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right],\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}^{[\perp]}\end{smallmatrix}\right]\right\rangle}=\left\langle \begin{bmatrix}x\\ u\end{bmatrix},S^\dagger\begin{bmatrix}x'\\ y'\end{bmatrix}\right\rangle_{\left\langle\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right],\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}^{[\perp]}\end{smallmatrix}\right]\right\rangle}. \tag{3.5}$$

In the following theorem we construct the operator $\begin{bmatrix}A\&B^d\\ C\&D^d\end{bmatrix}$ from $\begin{bmatrix}A\&B\\ C\&D\end{bmatrix}$. Here $A\&B^d$ is not determined by $A\&B$ alone – one should rather understand $A\&B^d$ as the projection of $\begin{bmatrix}A\&B^d\\ C\&D^d\end{bmatrix}$, which is a certain type of system dual of $\begin{bmatrix}A\&B\\ C\&D\end{bmatrix}$, onto $\begin{bmatrix}\mathcal{X}\\ \{0\}\end{bmatrix}$, cf. the explanation before Definition 2.4.

Theorem 3.4. *Let $V\subset\mathfrak{K}$ and $\mathcal{W}=\mathcal{U}\dotplus\mathcal{Y}$. Assume that there exists a densely defined operator $S=\begin{bmatrix}A\&B\\ C\&D\end{bmatrix}:\begin{bmatrix}\mathcal{X}\\ \mathcal{U}\end{bmatrix}\supset\mathrm{Dom}(S)\to\begin{bmatrix}\mathcal{X}\\ \mathcal{Y}\end{bmatrix}$, such that V has the graph representation*

$$V=\begin{bmatrix}A\&B\\ \begin{bmatrix}1 & 0\end{bmatrix}\\ C\&D+\begin{bmatrix}0 & 1\end{bmatrix}\end{bmatrix}\mathrm{Dom}(S). \tag{3.6}$$

Let $S^\dagger$ be the adjoint of S, as given in (3.5), *and define*

$$\begin{aligned} S^d := \begin{bmatrix} A\&B^d \\ C\&D^d \end{bmatrix} := \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} S^\dagger \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{on} \\ \operatorname{Dom}\left(S^d\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \operatorname{Dom}\left(S^\dagger\right) \subset \begin{bmatrix} \mathcal{X} \\ \mathcal{U}^{[\perp]} \end{bmatrix}. \end{aligned} \tag{3.7}$$

Then $V^{[\perp]}$ is given by

$$V^{[\perp]} = \begin{bmatrix} A\&B^d \\ \begin{bmatrix} 1 & 0 \end{bmatrix} \\ C\&D^d + \begin{bmatrix} 0 & 1 \end{bmatrix} \end{bmatrix} \operatorname{Dom}\left(S^d\right). \tag{3.8}$$

If S is an operator node, then so are S^d and $S^\dagger$. In this case, the main operator of S^d is $-A^*$, where A^* is the adjoint of A as an unbounded operator on the Hilbert space $\mathcal{X}$. If S is an ordinary system node, then so is $S^\dagger$ and in this case S^d is a time-reflected system node; see Definition 2.4.

We have the following important corollary to Theorem 3.4.

Corollary 3.5. *A given input/output pair $(\mathcal{U},\mathcal{Y})$ is admissible for the state/signal node $(V;\mathcal{X},\mathcal{W})$ if and only if the "dual input/output pair" $(\mathcal{U}^{[\perp]},\mathcal{Y}^{[\perp]})$ is admissible for the state/signal dual $(V^{[\perp]};\mathcal{X},\mathcal{W})$.*

Usually $(V^{[\perp]};\mathcal{X},\mathcal{W})$ is not a state/signal node even if $(V;\mathcal{X},\mathcal{W})$ is, as can be seen by taking $V=\{0\}$ and $\mathcal{X}\neq\{0\}$. Then $\{0\}\neq \begin{bmatrix} \mathcal{X} \\ \{0\} \\ \{0\} \end{bmatrix} \subset V^{[\perp]}$ violates condition (ii) of Definition 2.9 with V replaced by $V^{[\perp]}$. However, if $V_{op}=(S;\mathcal{X},\mathcal{U},\mathcal{Y})$ is an operator node representation where S is a system node, then $(V^{[\perp]};\mathcal{X},\mathcal{W})$ is a *time-reflected state/signal node*. Time-reflected state/signal nodes are characterised by conditions (i) and (ii) of Definition 2.9 together with the condition that there exists some $T<0$ such that

$$\forall \begin{bmatrix} z_0 \\ x_0 \\ w_0 \end{bmatrix} \in V \ \exists \begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{V}[T,0]: \quad \begin{bmatrix} \dot{x}(0) \\ x(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} z_0 \\ x_0 \\ w_0 \end{bmatrix}. \tag{3.9}$$

Compare (3.9) to condition (iii) of Definition 2.9.

We are now able to proceed to the main topic of this chapter.

3.3 Passive state/signal systems

The power product on $\mathfrak{K}$ which is given in (2.19) can be interpreted in the following way. Let $\|x(t)\|_{\mathcal{X}}^2$ represent the amount of energy stored in state $x(t)\in\mathcal{X}$ at time t and let $[w(t),w(t)]_{\mathcal{W}}$ be the energy flowing into the system per time unit through the external signal when it takes the value $w(t)$.

If all energy flowing in through the external signals is stored in the state then

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|_{\mathcal{X}}^2 = (\dot{x}(t),x(t))_{\mathcal{X}} + (x(t),\dot{x}(t))_{\mathcal{X}} = [w(t),w(t)]_{\mathcal{W}}.$$

Thus, if $\begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{V}$ is a classical trajectory of some state/signal node and $t \geq 0$, then

$$p(t) := \left[\begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix}, \begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix}\right]_{\mathfrak{K}} = [w(t),w(t)]_{\mathcal{W}} - (\dot{x}(t),x(t))_{\mathcal{X}} - (x(t),\dot{x}(t))_{\mathcal{X}}$$

describes the energy absorbed through the external signals, which is not stored in the state, per time unit.

If $p(t) > 0$, then the trajectory $\begin{bmatrix} x \\ w \end{bmatrix}$ *dissipates energy* at a rate of $p(t)$ per time unit at time t. If $p(t) < 0$, then $\begin{bmatrix} x \\ w \end{bmatrix}$ *accumulates energy* at a rate of $|p(t)|$ per time unit and if $p(t) = 0$ then $\begin{bmatrix} x \\ w \end{bmatrix}$ *preserves energy* at time t. This motivates the following definition.

Definition 3.6. An *ordinary state/signal node* $(V;\mathcal{X},\mathcal{W})$ is *dissipative* if $V \geq 0$. The state/signal node is *energy preserving* if V is neutral: $[v,v]_{\mathfrak{K}} = 0$ for all $v \in V$.

A state/signal node is *passive* if V is a maximally nonnegative subspace of $\mathfrak{K}$. The state/signal node is *conservative* if V is a Lagrangian subspace of $\mathfrak{K}$.

According to Proposition 4.3 and Corollary 4.4 of [Kur10], dissipativity and energy preservation can be characterised in terms of the system trajectories. A state/signal node is *dissipative* if there exists a $T > 0$ such that

$$\forall t \in [0,T]: \quad \|x(t)\|_{\mathcal{X}}^2 - \|x(0)\|_{\mathcal{X}}^2 \leq \int_0^t [w(s),w(s)]_{\mathcal{W}}\,\mathrm{d}s \tag{3.10}$$

for all generalised trajectories $\begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{W}[0,T]$. A dissipative state/signal node $(V;\mathcal{X},\mathcal{W})$ is *passive* if its state/signal dual $(V^{[\perp]};\mathcal{X},\mathcal{W})$ is dissipative in the time-reflected sense that

$$\forall t \in [T,0]: \quad \|x^d(0)\|_{\mathcal{X}}^2 - \|x^d(t)\|_{\mathcal{X}}^2 \geq \int_t^0 [w^d(s),w^d(s)]_{\mathcal{W}}\,\mathrm{d}s \tag{3.11}$$

for some $T < 0$ and all $\begin{bmatrix} x^d \\ w^d \end{bmatrix} \in \mathfrak{W}^d[T,0]$.

The intuitive interpretation of (3.10) is that the energy $\|x(t)\|^2$ stored in the state at time $t \in [0,T]$ never exceeds the energy of the initial state $\|x(0)\|^2$ plus the total energy $\int_0^t [w(s),w(s)]_{\mathcal{W}}\,\mathrm{d}s$ absorbed from the environment.

The following theorem was proved as Theorem 4.5 of [Kur10]. The theorem is of fundamental importance for the theory of passive state/signal systems, because it establishes that every fundamental input/output pair is admissible for a passive state/signal node.

Theorem 3.7. *Assume that V is a maximally nonnegative subspace of $\mathfrak{K}$ and that $\left[\begin{smallmatrix} z \\ 0 \\ 0 \end{smallmatrix}\right] \in V \Longrightarrow z=0$. Then $(V;\mathcal{X},\mathcal{W})$ is a passive state/signal node for which every fundamental decomposition $\mathcal{W}=(\mathcal{W}_+,\mathcal{W}_-)$ is an admissible input/output pair.*

Let the operator node representation $\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix};\mathcal{X},\mathcal{W}_+,\mathcal{W}_-\right)$ correspond to a fundamental input/output pair. Then the operator $\begin{bmatrix} A\&B \\ C\&D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{W}_+ \end{bmatrix} \supset \operatorname{Dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}\right) \to \begin{bmatrix} \mathcal{X} \\ \mathcal{W}_- \end{bmatrix}$ is a system node that has a contraction semigroup $\mathfrak{A}$ on $\mathcal{X}$. The generator A of $\mathfrak{A}$ satisfies $\mathbb{C}^+ \subset \operatorname{Res}(A)$.

Thus, a triple $(V;\mathcal{X},\mathcal{W})$ is a passive (conservative) state/signal node if and only if $\left[\begin{smallmatrix} z \\ 0 \\ 0 \end{smallmatrix}\right] \in V \Longrightarrow z=0$ and V is a maximally nonnegative (Lagrangian) subspace of $\mathfrak{K}$. Compare this to the much more complicated Definitions 2.4 and 3.2 for input/state/output systems.

Definition 3.8. Let $V \subset \mathfrak{K}$. If the fundamental input/output pair $(\mathcal{W}_+,\mathcal{W}_-)$ is admissible for V, then we call the corresponding operator node representation in (2.21) a *scattering representation* of V. If the Lagrangian input/output pair $(\mathcal{U},\mathcal{Y})$ is admissible, then we call the corresponding operator node representation an *impedance representation* of V.

By (3.10) and (A.3), the classical trajectories (u,x,y) of a scattering representation of a passive state/signal node satisfy (for all $T>0$):

$$\|x(t)\|_{\mathcal{X}}^2 + \int_0^t \|y(s)\|_{|\mathcal{W}_-|}^2 \,\mathrm{d}s \le \|x(0)\|_{\mathcal{X}}^2 + \int_0^t \|u(s)\|_{\mathcal{W}_+}^2 \,\mathrm{d}s, \quad t \in [0,T]. \tag{3.12}$$

More generally, also the generalised trajectories $\begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{W}$ generated by a passive state/signal node satisfy (3.12) with $u := \mathcal{P}_{\mathcal{W}_+}^{\mathcal{W}_-} w$ and $y := \mathcal{P}_{\mathcal{W}_-}^{\mathcal{W}_+} w$. The validity of these claims follows directly from Definition 2.10 and (2.21).

The inequality (3.12) is associated with scattering-passive systems, see Definition 3.2, and it motivates the name "scattering representation". Combining (3.12) with Definition 2.11 and Theorem 3.7, we obtain that every passive state/signal node is well-posed.

Now assume that $\mathcal{W}=\mathcal{U}\dotplus\mathcal{Y}$ is a Lagrangian decomposition of $\mathcal{W}$, i.e., that $\mathcal{U}$ and $\mathcal{Y}$ are both Lagrangian subspaces of $\mathcal{W}$. By [AS07b, Lemma 2.3] there exist Hilbert-space inner products on $\mathcal{U}$ and $\mathcal{Y}$ and a unitary operator $\Psi:\mathcal{U}\to\mathcal{Y}$, such that the Kreĭn-space inner product on $\mathcal{W}$ is given by

$$\left[\begin{bmatrix} y^1 \\ u^1 \end{bmatrix}, \begin{bmatrix} y^2 \\ u^2 \end{bmatrix}\right]_{\mathcal{W}} = (y^1,\Psi u^2)_{\mathcal{Y}} + (\Psi u^1, y^2)_{\mathcal{Y}}. \tag{3.13}$$

Thus the following holds for any classical trajectory of an impedance representation of a passive state/signal node:

$$\begin{aligned}\left[\begin{bmatrix}\dot{x}(t)\\x(t)\\y(t)\\u(t)\end{bmatrix},\begin{bmatrix}\dot{x}(t)\\x(t)\\y(t)\\u(t)\end{bmatrix}\right]_{\mathfrak{K}} &= (y(t),\Psi u(t))_{\mathcal{Y}}+(\Psi u(t),y(t))_{\mathcal{Y}}-(\dot{x}(t),x(t))_{\mathcal{X}}-(x(t),\dot{x}(t))_{\mathcal{X}}\\ &= 2\mathrm{Re}\,(\Psi u(t),y(t))_{\mathcal{Y}}-2\mathrm{Re}\,(\dot{x}(t),x(t))_{\mathcal{X}}.\end{aligned} \tag{3.14}$$

Integrating (3.14) from 0 to $T>0$ we arrive at the following special case of (3.10), which holds for impedance representations of passive state/signal nodes:

$$\|x(t)\|^2_{\mathcal{X}} \le \|x(0)\|^2_{\mathcal{X}}+2\mathrm{Re}\int_0^t(\Psi u(s),y(s))_{\mathcal{Y}}\mathrm{d}s, \quad t\in[0,T], \tag{3.15}$$

for $\begin{bmatrix}x\\w\end{bmatrix}\in\mathfrak{W}[0,T]$ and $u=\mathcal{P}^{\mathcal{Y}}_{\mathcal{U}}w$, $y=\mathcal{P}^{\mathcal{U}}_{\mathcal{Y}}w$. This is the impedance-passivity inequality (3.2).

The energy inequalities (3.12) and (3.15) correspond to the fundamental and Lagrangian decompositions of $\mathcal{W}$, respectively, but the *property of passivity* is characterised by the maximal nonnegativity of V. Thus passivity is an *input/output invariant* property of the state/signal node.

3.4 Discrete-time state/signal nodes and the Cayley transformation

In this section we present the Cayley transformation which maps passive continuous-time state/signal nodes into passive discrete-time state/signal nodes. This transformation is a reinterpretation of the Cayley transformation of an input/state/output system node, which has been treated e.g. in [Sta05, Chap. 12].

We begin by giving some basic definitions for discrete systems. The theory for state/signal systems in discrete time has been studied by Arov and Staffans in [AS05, AS07a, AS07b, AS07c, AS09, Sta06], and these sources should be consulted for more information on discrete-time state/signal systems.

A discrete-time input/state/output system on the Banach-space triple $(\mathcal{U},\mathcal{X},\mathcal{Y})$ is usually given in the difference form

$$\begin{bmatrix}x(n+1)\\y(n)\end{bmatrix}=\begin{bmatrix}\mathbf{A}&\mathbf{B}\\\mathbf{C}&\mathbf{D}\end{bmatrix}\begin{bmatrix}x(n)\\u(n)\end{bmatrix}, \quad n\in\mathbb{Z}^+, \quad x(0)=x_0\in\mathcal{X} \text{ given}, \tag{3.16}$$

where $\begin{bmatrix}\mathbf{A}&\mathbf{B}\\\mathbf{C}&\mathbf{D}\end{bmatrix}$ is a bounded operator from $\begin{bmatrix}\mathcal{X}\\\mathcal{U}\end{bmatrix}$ to $\begin{bmatrix}\mathcal{X}\\\mathcal{Y}\end{bmatrix}$ and $\mathbb{Z}^+=\{0,1,2,\dots\}$.

In analogy to the construction in Section 2.2, the system (3.16) is turned into a state/signal system by a suitable identification $\mathcal{W}=\mathcal{U}\dot{+}\mathcal{Y}$ and by reinterpreting (3.16) as

$$\begin{bmatrix}x(n+1)\\x(n)\\w(n)\end{bmatrix}\in\mathbf{V},\ n\in\mathbb{Z}^+,\ x(0)=x_0\in\mathcal{X} \text{ given}, \quad \text{where} \quad \mathbf{V}=\begin{bmatrix}\mathbf{A}&\mathbf{B}\\1&0\\\mathbf{C}&\mathbf{D}+1\end{bmatrix}\begin{bmatrix}\mathcal{X}\\\mathcal{U}\end{bmatrix}. \tag{3.17}$$

A discrete-time input/state/output system is scattering passive, see [Sta02b, Sec. 5], if all its input/state/output trajectories (u,x,y) satisfy

$$\forall n \geq 0: \quad \|x(n+1)\|_{\mathcal{X}}^2 - \|x(n)\|_{\mathcal{X}}^2 \leq \|u(n)\|_{\mathcal{U}}^2 - \|y(n)\|_{\mathcal{Y}}^2.$$

Therefore the natural discrete counterpart of the continuous-time node space in Definition 2.7 is the Kreĭn space $\mathfrak{K}_d = \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{X}\\ \mathcal{W}\end{smallmatrix}\right]$ equipped with the power product

$$\left[\left[\begin{matrix}z^1\\ x^1\\ w^1\end{matrix}\right],\left[\begin{matrix}z^2\\ x^2\\ w^2\end{matrix}\right]\right]_{\mathfrak{K}_d} = \left[w^1,w^2\right]_{\mathcal{W}} - \left(z^1,z^2\right)_{\mathcal{X}} + \left(x^1,x^2\right)_{\mathcal{X}}. \tag{3.18}$$

This choice of power product in $\mathfrak{K}_d$ implies that the discrete-time system node $\left[\begin{matrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{matrix}\right]$ is scattering passive, i.e., $\left[\begin{matrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{matrix}\right]$ is a contraction, if and only if $\mathbf{V}$ in (3.17) is a maximally nonnegative subspace of $\mathfrak{K}_d$.

We now define a general discrete-time state/signal node, which is not a priori induced by a discrete-time input/state/output system node $\left[\begin{matrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{matrix}\right]$ through (3.17).

Definition 3.9. Let $\mathcal{X}$ be a Hilbert space and $\mathcal{W}$ a Kreĭn space and let $\mathbf{V} \subset \mathfrak{K}_d$.

The triple $(\mathbf{V};\mathcal{X},\mathcal{W})$ is a *discrete state/signal node* if it has the following properties:

(i) The space $\mathbf{V}$ is closed.

(ii) If $\left[\begin{smallmatrix}z\\ 0\\ 0\end{smallmatrix}\right] \in \mathbf{V}$ then $z=0$.

(iii) For every $x \in \mathcal{X}$, there exists some z and w, such that $\left[\begin{smallmatrix}z\\ x\\ w\end{smallmatrix}\right] \in \mathbf{V}$.

(iv) The space $\left\{\left[\begin{matrix}x\\ w\end{matrix}\right] \,\middle|\, \exists z: \left[\begin{smallmatrix}z\\ x\\ w\end{smallmatrix}\right] \in \mathbf{V}\right\}$ is a closed subspace of $\left[\begin{matrix}\mathcal{X}\\ \mathcal{W}\end{matrix}\right]$. ♦

Although the continuous-time and discrete-time generating subspaces V and $\mathbf{V}$ themselves are statical objects which do not depend on the choice of discrete or continuous time, already the choice of power product for the node space reflects the choice of continuous or discrete time.

We now aim at introducing the Cayley transformation for state/signal systems.

Definition 3.10. Let $\mathcal{W}_1$ and $\mathcal{W}_2$ be Kreĭn spaces with the inner products $[\cdot,\cdot]_1$ and $[\cdot,\cdot]_2$, respectively. We say that a continuous bijective linear operator $T:\mathcal{W}_1 \to \mathcal{W}_2$ is *Kreĭn unitary* if $[w,w']_1 = [Tw,Tw']_2$ for all $w,w' \in \mathcal{W}$. ♦

Any Kreĭn-unitary operator trivially maps (maximal) nonnegative subspaces one-to-one onto (maximal) nonnegative subspaces. Neutral subspaces are mapped one-to-one onto neutral subspaces, and Lagrangian subspaces are mapped one-to-one onto Lagrangian subspaces.

Lemma 3.11. *For any $\alpha\in\mathbb{C}^+$, the operator* $\mathcal{C}_\alpha := \frac{1}{\sqrt{2\mathrm{Re}\,\alpha}}\begin{bmatrix} 1 & \overline{\alpha} & 0 \\ -1 & \alpha & 0 \\ 0 & 0 & \sqrt{2\mathrm{Re}\,\alpha}\end{bmatrix}$ *is Kreĭn unitary from $\mathfrak{K}$ given in* (2.19) *to $\mathfrak{K}_d$ given in* (3.18)*. Moreover, the bounded inverse of $\mathcal{C}_\alpha$ is* $\mathcal{C}_\alpha^{-1} = \frac{1}{\sqrt{2\mathrm{Re}\,\alpha}}\begin{bmatrix} \alpha & -\overline{\alpha} & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2\mathrm{Re}\,\alpha}\end{bmatrix}$.

The proof is straightforward and therefore omitted.

Definition 3.12. The operator $\mathcal{C}_\alpha$ in Lemma 3.11 is the *Cayley transformation* with parameter $\alpha\in\mathbb{C}^+$.

Given any continuous-time *state/signal node* $(V;\mathcal{X},\mathcal{W})$ and $\alpha\in\mathbb{C}^+$, we call the triple $(\mathcal{C}_\alpha V;\mathcal{X},\mathcal{W})$ the *Cayley transform* with parameter α of $(V;\mathcal{X},\mathcal{W})$. ♦

The Cayley transformation commutes with the operation of taking state/signal node adjoints, i.e.,

$$\mathcal{C}_\alpha(V^{[\perp]}) = (\mathcal{C}_\alpha V)^{[\perp]_d},$$

where $[\perp]$ denotes the orthogonal companion in $\mathfrak{K}$ and $[\perp]_d$ denotes the orthogonal companion in $\mathfrak{K}_d$. This is a direct consequence of the Kreĭn-unitarity of $\mathcal{C}_\alpha$.

Theorem 3.13. *Let* $\Sigma_{op} = \left(\begin{bmatrix} A\&B \\ C\&D\end{bmatrix};\mathcal{X},\mathcal{U},\mathcal{Y}\right)$ *be an operator node representation of a continuous-time state/signal node, and assume that $\alpha\in\mathrm{Res}(A)\cap\mathbb{C}^+$. Then*

$$\mathcal{C}_\alpha V = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ 1_{\mathcal{X}} & 0 \\ \mathbf{C} & \mathbf{D}+1_{\mathcal{U}}\end{bmatrix}\begin{bmatrix}\mathcal{X} \\ \mathcal{U}\end{bmatrix}, \tag{3.19}$$

where

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D}\end{bmatrix} = \begin{bmatrix} (\overline{\alpha}+A)(\alpha-A)^{-1} & \sqrt{2\mathrm{Re}\,\alpha}\,(\alpha-A|_{\mathcal{X}})^{-1}B \\ \sqrt{2\mathrm{Re}\,\alpha}\,C(\alpha-A)^{-1} & \widehat{\mathfrak{D}}(\alpha)\end{bmatrix}. \tag{3.20}$$

The operator $1+\mathbf{A}$ is injective with range dense in $\mathcal{X}$.

The triple $(\mathcal{C}_\alpha V;\mathcal{X},\mathcal{W})$ is a discrete state/signal node. Moreover, $(\mathcal{C}_\alpha V;\mathcal{X},\mathcal{W})$ is passive (conservative) if and only if $(V;\mathcal{X},\mathcal{W})$ is passive (conservative).

If $(\mathbf{V};\mathcal{X},\mathcal{W})$ is a passive discrete-time state/signal node with the property that $\left[\begin{smallmatrix}-x \\ x \\ 0\end{smallmatrix}\right]\in\mathbf{V}\Longrightarrow x=0$, *then $(\mathcal{C}_\alpha^{-1}\mathbf{V};\mathcal{X},\mathcal{W})$ is a passive continuous-time state/signal node for all $\alpha\in\mathbb{C}^+$.*

We omit the straightforward proof because it requires a significant amount of space.

Remark 3.14. The condition $\left[\begin{smallmatrix}-x \\ x \\ 0\end{smallmatrix}\right]\in\mathcal{C}_\alpha V\Longrightarrow x=0$ is equivalent to the condition $\left[\begin{smallmatrix}x \\ 0 \\ 0\end{smallmatrix}\right]\in V\Longrightarrow x=0$. If $(V;\mathcal{X},\mathcal{W})$ is passive and (3.19) holds, then the condition is also equivalent to the injectivity of $1+\mathbf{A}$.

Also note that $\mathbf{A}=(\overline{\alpha}+A)(\alpha-A)^{-1}$ in (3.20) is the standard operator Cayley transformation which maps a dissipative (skew-adjoint) operator A into a contractive (unitary) operator $\mathbf{A}$. ♦

Chapter 4

Port-Hamiltonian systems

In this chapter Dirac structures and port-Hamiltonian systems defined on Hilbert spaces are presented. We investigate how port-Hamiltonian systems relate to the previously-discussed state/signal systems. The notions and notation in this chapter are a bit different from the preceding chapters, because the historical background is different, as port-Hamiltonian systems originate from modelling of nonlinear physical systems using differential forms and geometrical methods; see [MvdS05].

One of the technical differences between this chapter and the previous chapters is that we consider Hilbert spaces over the real field in this chapter, whereas we usually assume that the field which underlies $\mathcal{X}$ and $\mathcal{W}$ is complex when we study state/signal systems. This difference, however, does not change anything conceptual and, in fact, many of the proofs in [Kur10] and [KZvdSB09] make no assumptions on the underlying field. Some proofs, however, might differ slightly between the real and the complex cases. The following simple lemma exposes another essential connection between the real and the complex cases.

Lemma 4.1. *Let $\mathcal{B}$ be a real Kreĭn space with inner product $[\cdot,\cdot]_{\mathcal{B}}$. Define $\mathfrak{K} := \mathcal{B} + i\mathcal{B}$ with inner product*

$$\left[b^1 + ib^2, b^3 + ib^4\right]_{\mathfrak{K}} := [b^1,b^3]_{\mathcal{B}} + [b^2,b^4]_{\mathcal{B}} + i[b^2,b^3]_{\mathcal{B}} - i[b^1,b^4]_{\mathcal{B}}, \quad b^k \in \mathcal{B}. \tag{4.1}$$

Then $\mathfrak{K}$ is a complex Kreĭn space and $[b^1,b^2]_{\mathfrak{K}} = [b^1,b^2]_{\mathcal{B}}$ for all $b^1, b^2 \in \mathcal{B}$.

Let $\mathcal{D} \subset \mathcal{B}$ and define $V := \mathcal{D} + i\mathcal{D}$. The orthogonal companion $V^{[\perp]}$ of V in $\mathfrak{K}$ is

$$V^{[\perp]} = \mathcal{D}^{[\perp]_{\mathcal{B}}} + i\mathcal{D}^{[\perp]_{\mathcal{B}}}, \tag{4.2}$$

where $\mathcal{D}^{[\perp]_{\mathcal{B}}}$ is the orthogonal companion of $\mathcal{D}$ in $\mathcal{B}$.

Moreover, $\mathcal{D}$ is a neutral subspace of $\mathcal{B}$ if and only if V is a neutral subspace of $\mathfrak{K}$. In particular, the space $\mathcal{D}$ is Lagrangian if and only if V is Lagrangian.

Proof. Let $\mathcal{B} = \mathcal{B}_+ \dot{+} \mathcal{B}_-$ be a fundamental decomposition of $\mathcal{B}$. We now show that $\mathfrak{K} = \mathfrak{K}_+ \dot{+} \mathfrak{K}_-$, where $\mathfrak{K}_\pm := \mathcal{B}_\pm + i\mathcal{B}_\pm$, is a fundamental decomposition of $\mathfrak{K}$. Obviously, $\mathfrak{K}_+ + \mathfrak{K}_- = (\mathcal{B}_+ + \mathcal{B}_-) + i(\mathcal{B}_+ + \mathcal{B}_-) = \mathfrak{K}$ and, moreover, $k \in \mathfrak{K}_+ \cap \mathfrak{K}_-$ implies $k = b^1 + ib^2$, where $b_k \in \mathcal{B}_+ \cap \mathcal{B}_- = \{0\}$. Thus $\mathfrak{K} = \mathfrak{K}_+ \dot{+} \mathfrak{K}_-$. From (4.1) we have that

$$[b^1 + ib^2, b^1 + ib^2]_{\mathfrak{K}} = [b^1,b^1]_{\mathcal{B}} + [b^2,b^2]_{\mathcal{B}} \tag{4.3}$$

and therefore $\mathfrak{K}_+$ is a Hilbert space whenever $\mathcal{B}_+$ is a Hilbert space. Analogously, $\mathfrak{K}_-$ is an anti-Hilbert space if $\mathcal{B}_-$ is an anti-Hilbert space. Moreover, (4.1) is zero if $b^1 + ib^2 \in \mathfrak{K}_+$ and $b^3 + ib^4 \in \mathfrak{K}_-$, because $\mathcal{B}_+ [\perp] \mathcal{B}_-$.

We now prove (4.2). It is clear from (4.1) that if $b^1,b^2 \in \mathcal{D}^{[\perp]}$ and $b^3,b^4 \in \mathcal{D}$, then $(b^1+ib^2)[\perp](b^3+ib^4)$ and therefore $\mathcal{D}^{[\perp]_\mathcal{B}}+i\mathcal{D}^{[\perp]_\mathcal{B}} \subset V^{[\perp]}$. Conversely, if $b^1+ib^2 \in V^{[\perp]}$, then by taking $b^4=0$ in (4.1), we in particular obtain that

$$\forall b^3 \in \mathcal{D}: \quad [b^1+ib^2,b^3]_\mathfrak{K} = [b^1,b^3]_\mathcal{B} + i[b^2,b^3]_\mathcal{B} = 0.$$

This means that $b^1,b^2 \in \mathcal{D}^{[\perp]}$, i.e., that $V^{[\perp]} \subset \mathcal{D}^{[\perp]_\mathcal{B}}+i\mathcal{D}^{[\perp]_\mathcal{B}}$.

If $\mathcal{D}$ is a neutral subspace of $\mathcal{B}$, i.e. $[b,b]_\mathcal{B}=0$ for all $b\in\mathcal{D}$, then V is a neutral subspace of $\mathfrak{K}$ by (4.3). If $V\subset V^{[\perp]}$ in $\mathfrak{K}$ then it follows by setting $b^2=0$ in (4.3) that $[b,b]_\mathcal{B}=0$ for all $b\in\mathcal{D}$. Thus $\mathcal{D}$ is neutral if and only if V is neutral.

Note that $V^{[\perp]}$ is of the same form as V with $\mathcal{D}$ is replaced by $\mathcal{D}^{[\perp]_\mathcal{B}}$. We thus conclude that $\mathcal{D}^{[\perp]_\mathcal{B}}$ is neutral if and only if $V^{[\perp]}$ is neutral, and in particular, $\mathcal{D}=\mathcal{D}^{[\perp]_\mathcal{B}}$ if and only if $V=V^{[\perp]}$. $\square$

Lemma 4.1 is not as limited as it might seem at first, due to the fact that the derivatives of the real and imaginary parts of a complex-valued function $f(z,t)=g(z,t)+ih(z,t)$, with real arguments, $t>0$ and $z\in\mathbb{R}^n$, are computed separately: $\frac{\partial f}{\partial t}(z,t)=\frac{\partial g}{\partial t}(z,t)+i\frac{\partial h}{\partial t}(z,t)$, $g(z,t)$ and $h(z,t)\in\mathbb{R}$. Therefore many complex Lagrangian subspaces V can be decomposed into a direct sum $V=\mathcal{D}\dotplus i\mathcal{D}$, where $\mathcal{D}$ is a real Lagrangian subspace.

4.1 The abstract Hamiltonian system

We need two ingredients in order to define a Hamiltonian system. The first one is a so-called *Dirac structure*, which describes how the system behaves under interconnection. The second ingredient is a *Hamiltonian*, which measures the total energy of the system at any given state; see Sections 4.2 and 4.4 of [MvdS05].

The Dirac structure was first introduced by Courant [Cou90] and Dorfman, see e.g. [Dor93], and they were adapted to the Hilbert-space context by Parsian and Shafei Deh Abad in [PSDA99]. Infinite-dimensional Dirac structures have later been studied in e.g. [PSDA99, GIZvdS04, LGZM05, ISG05, KZvdSB09].

In the set-up of Courant and Dorfman one starts with a linear space $\mathcal{E}$ and a duality pairing $\langle\cdot,\cdot\rangle_{\langle\mathcal{F},\mathcal{E}\rangle}$ between the so-called *space $\mathcal{E}$ of efforts* and its dual, the so-called *space $\mathcal{F}=\mathcal{E}'$ of flows.* These efforts and flows should be "power conjugated", so that $\langle e,f\rangle_{\langle\mathcal{E},\mathcal{F}\rangle}$ can be interpreted as power. One then defines the *bond space* as the product $\mathcal{B}:=\mathcal{F}\times\mathcal{E}=\begin{bmatrix}\mathcal{F}\\ \mathcal{E}\end{bmatrix}$ equipped with the bi-linear *power product*

$$\left\langle \begin{bmatrix} f^1 \\ e^1 \end{bmatrix}, \begin{bmatrix} f^2 \\ e^2 \end{bmatrix} \right\rangle := \langle e^1,f^2\rangle_{\langle\mathcal{E},\mathcal{F}\rangle} + \langle e^2,f^1\rangle_{\langle\mathcal{E},\mathcal{F}\rangle}. \tag{4.4}$$

We denote the orthogonal companion of $\mathcal{D}\subset\mathcal{B}$ with respect to the power product (4.4) by $\mathcal{D}^{\langle\perp\rangle}$, so that

$$\mathcal{D}^{\langle\perp\rangle} := \left\{ \begin{bmatrix} f' \\ e' \end{bmatrix} \in\mathcal{B} \,\middle|\, \forall \begin{bmatrix} f \\ e \end{bmatrix} \in\mathcal{D}: \ \left\langle \begin{bmatrix} f' \\ e' \end{bmatrix}, \begin{bmatrix} f \\ e \end{bmatrix} \right\rangle = 0 \right\}.$$

A *(constant) Dirac structure* is a subspace $\mathcal{D}\subset\mathcal{B}$ such that $\mathcal{D}^{\langle\perp\rangle}=\mathcal{D}$.

See [MvdS00, MvdS01] for examples of nonlinear Dirac structures based on Stoke's theorem for differential forms arising in electrodynamics and fluid dynamical systems. We now specialise to the linear case.

In the Hilbert-space setting we let $\mathcal{E}$ and $\mathcal{F}$ be Hilbert spaces over the field of real numbers and assume that $\mathcal{E}$ and $\mathcal{F}$ have same cardinality, so that there exists a unitary map $r_{\mathcal{E},\mathcal{F}}$ from $\mathcal{E}$ to $\mathcal{F}$. The bond space $\mathcal{B}$ is $\mathcal{F}\times\mathcal{E}$ equipped with the power product

$$\left[\left[\begin{bmatrix} f^1 \\ e^1 \end{bmatrix}, \begin{bmatrix} f^2 \\ e^2 \end{bmatrix}\right]\right]_{\mathcal{B}} := (f^1, r_{\mathcal{E},\mathcal{F}} e^2)_{\mathcal{F}} + (r_{\mathcal{E},\mathcal{F}} e^1, f^2)_{\mathcal{E}}, \tag{4.5}$$

where $f^1,f^2\in\mathcal{F}$ and $e^1,e^2\in\mathcal{E}$. This bond space is a Kreĭn space with fundamental decomposition $\mathcal{B}=\mathcal{B}_+\dot{+}\mathcal{B}_-$, where

$$\mathcal{B}_\pm = \begin{bmatrix} \pm r_{\mathcal{E},\mathcal{F}} \\ I \end{bmatrix}\mathcal{E}, \quad \text{so that} \quad \mathcal{B}_+\geq 0,\ \mathcal{B}_-\leq 0, \text{ and } \mathcal{B}_+[\perp]\mathcal{B}_-.$$

Definition 4.2. A subspace $\mathcal{D}$ of the bond space $\mathcal{B}$ is a *(linear and constant) Dirac structure* on $\mathcal{B}$ if $\mathcal{D}=\mathcal{D}^{[\perp]}$ with respect to (4.5). ♦

In applications the Dirac structure is usually known, as it can be read out from the partial differential equations describing the system which is being studied. The Dirac structures in Sections 4 and 5 of [KZvdSB09] are for example of the type $\mathcal{D}=\begin{bmatrix} L \\ K \\ 1 \\ G \end{bmatrix}\mathrm{Dom}(L)$, where (L,K,G) is a boundary colligation, see Definition 2.13, but of course there exist more general Dirac structures. Most physical systems also have an identifiable set of elements which store energy and whose state x change as the system evolves with time.

We now proceed to discuss the Hamiltonian $\mathcal{H}:\mathcal{X}\to\mathbb{R}$, which measures the total energy $\mathcal{H}(x)$ of the system at state x. The Hamiltonian is assumed to have a variational derivative $\frac{\delta\mathcal{H}}{\delta x}$, which is given by:

$$\forall\xi\in\mathcal{E}: \quad \left(\frac{\delta\mathcal{H}}{\delta x}(x),\xi\right)_{\mathcal{E}} = \lim_{h\to 0}\frac{\mathcal{H}(x+h\xi)-\mathcal{H}(x)}{h}, \tag{4.6}$$

where it is essential that $h\in\mathbb{R}$.

In the linear setting, a natural choice of Hamiltonian is the quadratic form

$$\begin{gathered} \mathcal{H}(x)=\frac{1}{2}\|x\|_{\mathcal{E}}^2, \quad \text{that has} \quad \frac{\delta\mathcal{H}}{\delta x}(x)=x, \quad \text{because} \\ \forall\xi\in\mathcal{X}: \quad \lim_{h\to 0}\frac{\mathcal{H}(x+h\xi)-\mathcal{H}(x)}{h} = \lim_{h\to 0}\frac{(x+h\xi,x+h\xi)_{\mathcal{X}}-(x,x)_{\mathcal{X}}}{2h} = (x,\xi)_{\mathcal{E}}. \end{gathered} \tag{4.7}$$

In the distributed-parameter case, the Hamiltonian is usually of the form

$$\mathcal{H}(x)=\int_\Omega H(x,z)\,\mathrm{d}z,$$

where Ω is the domain on which the partial differential equation is considered; see [MvdS05, Sect. 4.4].

If we assume that x is a continuously differentiable function of time, then the continuity of the inner product $(\cdot,\cdot)_{\mathcal{E}}$ in both its arguments yields that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(x(t))=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(x(t),x(t))_{\mathcal{E}}=\frac{1}{2}(\dot{x}(t),x(t))_{\mathcal{E}}+\frac{1}{2}(x(t),\dot{x}(t))_{\mathcal{E}}=(x(t),\dot{x}(t))_{\mathcal{E}}. \quad (4.8)$$

Using the chain rule of the Gâteaux differential, one can show that (4.8) more generally has the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(x(t))=\left(\frac{\delta\mathcal{H}}{\delta x}(x(t)),\dot{x}(t)\right)_{\mathcal{E}} \quad (4.9)$$

for nonquadratic Hamiltonians. A system which preserves the total energy should satisfy $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(x(t))=0$ for all trajectories x and all $t\geq 0$. This agrees with the following definition of an abstract Hamiltonian system.

Definition 4.3. Let $\mathcal{D}\subset\mathcal{F}\times\mathcal{E}$ be a Dirac structure and assume that $\mathcal{H}:\mathcal{E}\to\mathbb{R}$ has a variational derivative. Let $x:t\to\mathcal{E}$, $t\geq 0$, be a differentiable trajectory taken by the energy storing elements of some physical system.

The *internal flows* at time $t\geq 0$ are given by $f_x(t)=r_{\mathcal{E},\mathcal{F}}\dot{x}(t)$ and the *internal efforts* are $e_x(t)=\frac{\delta\mathcal{H}}{\delta x}(x(t))$. The *Hamiltonian system* associated with the Dirac structure $\mathcal{D}$ and the Hamiltonian $\mathcal{H}$ is the set of internal flow/effort pairs (f_x,e_x) for which the inclusion

$$\begin{bmatrix} f_x(t) \\ e_x(t) \end{bmatrix}\in\mathcal{D}, \quad t\geq 0, \quad (4.10)$$

makes sense and is satisfied. ♦

In order to connect the Hamiltonian system in Definition 4.3 to other systems, we need to open up ports to the world outside of the system. This is the topic of the next section.

4.2 Hamiltonian systems with external ports

We now introduce the external efforts e_∂ and flows f_∂, which take values in the Hilbert spaces $\mathcal{E}_\partial$ and $\mathcal{F}_\partial$, respectively. These external signals are assumed to be power conjugated, so that the amount of energy flowing into the system through the external ports per time unit is given by $(r_\partial e_\partial, f_\partial)_{\mathcal{F}_\partial}$, where $r_\partial:\mathcal{E}_\partial\to\mathcal{F}_\partial$ is some given unitary operator. In particular we must demand that $\mathcal{E}_\partial$ and $\mathcal{F}_\partial$ are of the same cardinality. All energy exchange is assumed to take place through the external ports and we thus arrive at the condition

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(x(t))=(x(t),\dot{x}(t))_{\mathcal{E}}=(r_\partial e_\partial(t),f_\partial(t))_{\mathcal{F}_\partial}, \quad (4.11)$$

which should be satisfied for all system signals $\begin{bmatrix} f_x \\ e_x \\ f_\partial \\ e_\partial \end{bmatrix}$.

We remark that the notation f_∂, e_∂ originates from Dirac structures of boundary control type, where the external ports are given by the internal efforts evaluated at the boundary; see e.g. [MvdS05] or [LGZM05].

Let $\mathcal{E}$ and $\mathcal{F}$ be two Hilbert spaces. By $\mathcal{F}\times\mathcal{E}$ we denote the standard product of $\mathcal{F}$ and $\mathcal{E}$, i.e., the set of pairs $\begin{bmatrix} f \\ e \end{bmatrix}$, such that $f\in\mathcal{F}$ and $e\in\mathcal{E}$. By $\mathcal{F}\oplus\mathcal{E}$ we mean the Hilbert space obtained by equipping $\mathcal{F}\times\mathcal{E}$ with the inner product $\left(\begin{bmatrix} f^1 \\ e^1 \end{bmatrix}, \begin{bmatrix} f^2 \\ e^2 \end{bmatrix}\right)_{\mathcal{F}\oplus\mathcal{E}} = (f^1,f^2)_{\mathcal{F}}+(e^1,e^2)_{\mathcal{E}}$.

Definition 4.4. Let $\mathcal{E}_x,\mathcal{E}_\partial,\mathcal{F}_x,\mathcal{F}_\partial$ be Hilbert spaces and let $r_{\mathcal{E},\mathcal{F}}=\begin{bmatrix} r_x & 0 \\ 0 & -r_\partial \end{bmatrix}$ be a unitary operator from the space $\mathcal{E}:=\mathcal{E}_x\oplus\mathcal{E}_\partial$ of efforts to the space $\mathcal{F}:=\mathcal{F}_x\oplus\mathcal{F}_\partial$ of flows. Let $\mathcal{D}$ be a Dirac structure on the bond space $\mathcal{B}:=\mathcal{F}\times\mathcal{E}$ with power product (4.5).

The *linear port-Hamiltonian system*, which is induced by the Dirac structure $\mathcal{D}$ and the Hamiltonian $\mathcal{H}(x)=\frac{1}{2}\|x\|^2_{\mathcal{E}}$, is the set of all quadruples $\begin{bmatrix} r_x\dot{x} \\ x \\ f_\partial \\ e_\partial \end{bmatrix}$ of functions, such that $x\in C^1(\mathbb{R}^+;\mathcal{E}_x)$, $f_\partial\in C(\mathbb{R}^+;\mathcal{F}_\partial)$, $e_\partial\in C(\mathbb{R}^+;\mathcal{E}_\partial)$, for which the following inclusion holds:

$$\begin{bmatrix} r_x\dot{x}(t) \\ f_\partial(t) \\ x(t) \\ e_\partial(t) \end{bmatrix}\in\mathcal{D}, \quad t\geq 0. \quad \blacklozenge \tag{4.12}$$

In the case of an electrical circuit, the port effort e_∂ has the interpretation of voltage over the port, whereas the port flow f_∂ is the electrical current flowing into the system. An abstract port-Hamiltonian system is illustrated graphically in Figure 4.1. We will later expand this figure to illustrate the interconnection of two port-Hamiltonian systems in the next section.

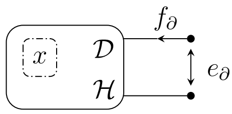

Figure 4.1: The abstract port-Hamiltonian system induced by the Dirac structure $\mathcal{D}$ and the Hamiltonian $\mathcal{H}$.

Remark 4.5. We sometimes need to consider systems which are of port-Hamiltonian type, i.e., a system described by a subspace $\mathcal{D}\subset\mathcal{B}$, a Hamiltonian $\mathcal{H}$ and the inclusion (4.12), but where $\mathcal{D}$ is not necessarily a Dirac structure. In this case we refer to $\mathcal{D}$ as the *interconnection structure* of Σ. $\blacklozenge$

Evaluating the power product $[\cdot,\cdot]_{\mathcal{B}}$ for a trajectory $\begin{bmatrix} r_x\dot{x} \\ f_\partial \\ x \\ e_\partial \end{bmatrix}$ at time t we obtain

$$\begin{aligned}\left[\begin{bmatrix} r_x\dot{x}(t) \\ f_\partial(t) \\ x(t) \\ e_\partial(t) \end{bmatrix}, \begin{bmatrix} r_x\dot{x}(t) \\ f_\partial(t) \\ x(t) \\ e_\partial(t) \end{bmatrix}\right]_{\mathcal{B}} &= 2\left(\begin{bmatrix} r_x\dot{x}(t) \\ f_\partial(t) \end{bmatrix}, \begin{bmatrix} r_x x(t) \\ -r_\partial e_\partial(t) \end{bmatrix}\right)_{\mathcal{F}_x\oplus\mathcal{F}_\partial} \\ &= 2\left(\dot{x}(t),x(t)\right)_{\mathcal{E}_x} - 2\left(f_\partial(t),r_\partial e_\partial(t)\right)_{\mathcal{F}_\partial}.\end{aligned} \tag{4.13}$$

Comparing this to (4.11), we see that the power product actually returns twice the rate at which energy, which does not enter through the external ports, accumulates in the state. If $\mathcal{D}$ is a Dirac structure then this difference equals zero.

In fact, taking $\mathcal{D}$ to be a *Tellegen structure*, i.e., assuming that $[d,d]_{\mathcal{B}}=0$ for all $d\in\mathcal{D}$, we would already obtain that every trajectory preserves energy when we take the energy flow through the external ports into consideration. The assumption that $\mathcal{D}$ is a Dirac structure essentially says that $\mathcal{D}$ is "large enough", in contrast to e.g. the Tellegen structure $\mathcal{D}=\{0\}$ that only allows the zero trajectory. The orthogonal companion $\{0\}^{[\perp]}=\mathcal{B}$, which corresponds to the dual port-Hamiltonian system, see Section 3.2, is then totally unstructured. It is easy to see that $\mathcal{D}$ is a Dirac structure if and only if $\mathcal{D}$ is closed, and $[d,d]_{\mathcal{B}}=0$ for all $d\in\mathcal{D}$ and all $d\in\mathcal{D}^{[\perp]}$.

Equation (4.13) motivates the choice of a diagonal operator $r_{\mathcal{E},\mathcal{F}}=\begin{bmatrix} r_x & 0 \\ 0 & -r_\partial \end{bmatrix}$ in Definition 4.4, because the diagonality represents the assumption that all state energy is exchanged through the external ports.

We now illustrate the most important concepts of the port-Hamiltonian theory using the lossless transfer line in Example 2.1.

Example 4.6. Equip the bond space $\mathcal{B}:=(L^2(\mathbb{R}^+;\mathbb{R}^2)\times\mathbb{R})\times(L^2(\mathbb{R}^+;\mathbb{R}^2)\times\mathbb{R})$ with the power product (4.5) with r_x and r_∂ the identity operators on $L^2(\mathbb{R}^+;\mathbb{R}^2)$ and $\mathbb{R}$, respectively. Define

$$\mathcal{D}:=\left\{\begin{bmatrix} f_U \\ f_I \\ f_\partial \\ e_U \\ e_I \\ e_\partial \end{bmatrix} \;\middle|\; \begin{bmatrix} f_U \\ f_I \\ f_\partial \\ e_\partial \end{bmatrix} = \begin{bmatrix} 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & 0 \\ 0 & -\varphi_0 \\ \varphi_0 & 0 \end{bmatrix} \begin{bmatrix} e_U \\ e_I \end{bmatrix},\; e_U,e_I\in H^1(\mathbb{R}^+;\mathbb{R})\right\}. \tag{4.14}$$

Then e_∂ represents the voltage at the left end of the transfer line and f_∂ is the current flowing *into* the transfer line at the left end, because $e_I(0,t)$ is the current flowing out from the transfer line, according to Figure 2.2. It is rather straightforward to prove that $\mathcal{D}$ is a Dirac structure, e.g. using [KZvdSB09, Thm 4.3].

In the approximation of the transfer line by discrete capacitors and inductors, which was presented in Figure 2.2, the energy-storing elements are the inductors and the capacitors. The energy stored in the inductor L is $\frac{1}{2}|I(z+l)|^2$ and the energy in the capacitor C is $\frac{1}{2}|U(z)|^2$. Therefore, letting $l\to 0^+$, we obtain that the inductance and capacitance distributed along the whole transfer line store the energy. The total energy of the transfer line is then given by

$$\mathcal{H}\left(\begin{bmatrix} U \\ I \end{bmatrix}\right)=\int_0^\infty \frac{1}{2}(|U(z)|^2+|I(z)|^2)\,\mathrm{d}z \tag{4.15}$$

and the Hamiltonian density can be read out from (4.15) as $H\left(\begin{bmatrix} u \\ i \end{bmatrix},z\right)=\frac{1}{2}(|u(z)|^2+|i(z)|^2)$, $z\in\Omega=(0,\infty)$.

The port-Hamiltonian system defined by the Dirac structure $\mathcal{D}$ in (4.14) and the Hamiltonian $\mathcal{H}$ in (4.15) is

$$\begin{cases} \frac{\partial}{\partial t}U(z,t) = \frac{\partial}{\partial z}I(z,t) \\ \frac{\partial}{\partial t}I(z,t) = \frac{\partial}{\partial z}U(z,t)\,, \quad t>0,\ z>0. \\ \qquad f_\partial = -I(0,t) \\ \qquad e_\partial = U(0,t) \end{cases} \tag{4.16}$$

Note that this is the same system as (2.3), with $f_\partial = u$ and $e_\partial = y$. In (2.3), however, u and y are regarded as the input and output, respectively, of the system, whereas in (4.16), there is neither input nor output. The signals e_∂ and f_∂ are rather considered to be general port variables and the input and output of (4.16) should be chosen as appropriate functions of these port variables; see [LGZM05, Sect. 4]. ♦

It seems to be common not to prove the existence of solutions of port-Hamiltonian systems mathematically. Often physical reasons are considered to imply existence of these solutions.

Remark 4.7. Assume that Ψ is a unitary operator from $\mathcal{E}_\partial$ to $\mathcal{F}_\partial$. Applying (4.1) to (4.13) in the case $\mathcal{E}_x = \mathcal{F}_x = \mathcal{X}, \mathcal{E}_\partial = \mathcal{U}, \mathcal{F}_\partial = \mathcal{Y}, r_x = 1, r_\partial = \Psi$, we obtain (3.14) but with a change of sign. Lemma 4.1 then yields that a subspace $\mathcal{D}$ of $\mathcal{B}$ is Lagrangian with respect to $[\cdot,\cdot]_\mathcal{B}$ if and only $\mathcal{D} + i\mathcal{D} \subset \mathfrak{K}$ is Lagrangian with respect to $[\cdot,\cdot]_\mathfrak{K}$. This further shows how closely Dirac structures are connected to impedance representations of conservative state/signal systems. ♦

We conclude that the terminology and notation of port-Hamiltonian systems differs from that of state/signal systems, but that the idea is essentially the same. However, neither approach can be considered a special case of the other, because port-Hamiltonian systems are usually allowed to be nonlinear, whereas state/signal systems allow more general forms of energy exchange through the external ports.

4.3 Interconnection of port-Hamiltonian systems and composition of Dirac structures

In order to be able to interconnect two port-Hamiltonian systems, we have to split the efforts and flows into two parts. One part is reserved for interconnection and the other part contains the rest of the variables. In the most general case we split both the port variables and the internal efforts and flows.

For the internal effort and flow spaces, the splitting is done by setting $\mathcal{E}_x = \mathcal{E}_{x,1} \oplus \mathcal{E}_{x,2}$ and $\mathcal{F}_x = \mathcal{F}_{x,1} \oplus \mathcal{F}_{x,2}$, where e.g. $\mathcal{E}_{x,2}$ is the part of the internal effort which is dedicated to interconnection and $\mathcal{E}_{x,1}$ contains the "remaining" internal efforts. Similarly we set $\mathcal{E}_\partial = \mathcal{E}_{\partial,1} \oplus \mathcal{E}_{\partial,2}$ and $\mathcal{F}_\partial = \mathcal{F}_{\partial,1} \oplus \mathcal{F}_{\partial,2}$ for the external efforts and flows. Then we group the interconnection and remaining signals together by setting $\mathcal{E}_1 := \mathcal{E}_{x,1} \oplus \mathcal{E}_{\partial,1}$, $\mathcal{F}_1 := \mathcal{F}_{x,1} \oplus \mathcal{F}_{\partial,1}$, $\mathcal{E}_2 := \mathcal{E}_{x,2} \oplus \mathcal{E}_{\partial,2}$, and $\mathcal{F}_2 := \mathcal{F}_{x,2} \oplus \mathcal{F}_{\partial,2}$. These splittings should be performed in such a way that there exist unitary operators $r_1 : \mathcal{E}_1 \to \mathcal{F}_1$ and $r_2 : \mathcal{E}_2 \to \mathcal{F}_2$.

Thus we have split the bond space into

$$\mathcal{B} = (\mathcal{F}_{x,1} \oplus \mathcal{F}_{\partial,1}) \times (\mathcal{F}_{x,2} \oplus \mathcal{F}_{\partial,2}) \times (\mathcal{E}_{x,1} \oplus \mathcal{E}_{\partial,1}) \times (\mathcal{E}_{x,2} \oplus \mathcal{E}_{\partial,2})$$

with power product given by (4.5), where $r_{\mathcal{E},\mathcal{F}} = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}$. The flows and efforts are split and recombined in the corresponding way, so that e.g. $f_x = \begin{bmatrix} f_{x,1} \\ f_{x,2} \end{bmatrix} \in \mathcal{F}_x$ and $f_\partial = \begin{bmatrix} f_{\partial,1} \\ f_{\partial,2} \end{bmatrix} \in \mathcal{F}_\partial$ are recombined into $f_1 = \begin{bmatrix} f_{x,1} \\ f_{\partial,1} \end{bmatrix} \in \mathcal{F}_1$ and $f_2 = \begin{bmatrix} f_{x,2} \\ f_{\partial,2} \end{bmatrix} \in \mathcal{F}_2$.

A Dirac structure which is used to define a split port-Hamiltonian system should be of the following kind.

Definition 4.8. Assume that the spaces of efforts and flows are decomposed as $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$ and $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2$, and that r_i are unitary operators from $\mathcal{E}_i$ onto $\mathcal{F}_i$, where $i = 1,2$.

A subspace $\mathcal{D} \subset \mathcal{B} = (\mathcal{F}_1 \oplus \mathcal{F}_2) \times (\mathcal{E}_1 \oplus \mathcal{E}_2)$ is called a *split Dirac structure* if it is a Dirac structure in the sense of Definition 4.2, with $r_{\mathcal{E},\mathcal{F}} = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}$. ♦

Now suppose that we have two port-Hamiltonian systems defined on the split Dirac structures $\mathcal{D}^A \subset (\mathcal{F}_1 \oplus \mathcal{F}_2) \times (\mathcal{E}_1 \oplus \mathcal{E}_2)$ and $\mathcal{D}^B \subset (\mathcal{F}_3 \oplus \mathcal{F}_2) \times (\mathcal{E}_3 \oplus \mathcal{E}_2)$. We wish to interconnect these two systems using the efforts and flows e_2^A, f_2^A, e_2^B and f_2^B of $\mathcal{D}^A$ and $\mathcal{D}^B$. We do this in such a manner that the efforts on the ports are the same and the flow out of system B goes into system A, i.e., $e_2^A = e_2^B$ and $f_2^A = -f_2^B$. This is an example of a so-called "energy-preserving interconnection". After we have done the interconnection we disregard the interconnection signals. Figure 4.2 illustrates interconnection in the simplified, but common, situation where the interconnection takes place only through the ports, i.e. where $\mathcal{E}_{x,2} = \{0\}$ and $\mathcal{F}_{x,2} = \{0\}$.

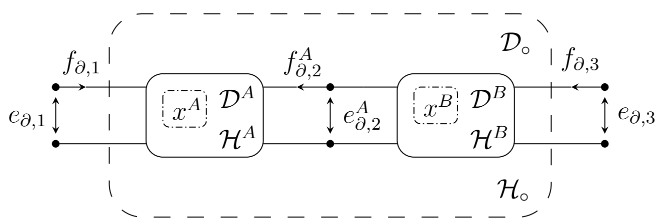

Figure 4.2: A graphical interpretation of interconnection. Here $\mathcal{D}_\circ$ and $\mathcal{H}_\circ$ are the interconnection structure and Hamiltonian, respectively, of the interconnected system.

We now proceed to study the interconnection structure $\mathcal{D}_\circ$ of the system obtained by the procedure described above.

From the inclusions

$$\begin{bmatrix} f_1^A \\ f_2^A \\ e_1^A \\ e_2^A \end{bmatrix} = \begin{bmatrix} f_{x,1}^A \\ f_{\partial,1}^A \\ \hline f_{x,2}^A \\ f_{\partial,2}^A \\ \hline e_{x,1}^A \\ e_{\partial,1}^A \\ \hline e_{x,2}^A \\ e_{\partial,2}^A \end{bmatrix} \in \mathcal{D}^A \quad \text{and} \quad \begin{bmatrix} f_3^B \\ f_2^B \\ e_3^B \\ e_2^B \end{bmatrix} = \begin{bmatrix} f_{x,3}^B \\ f_{\partial,3}^B \\ \hline f_{x,2}^B \\ f_{\partial,2}^B \\ \hline e_{x,3}^B \\ e_{\partial,3}^B \\ \hline e_{x,2}^B \\ e_{\partial,2}^B \end{bmatrix} \in \mathcal{D}^B$$

we immediately see that the flows and the efforts of the interconnected system live on the so-called *composition* $\mathcal{D}_\circ = \mathcal{D}^A \circ \mathcal{D}^B$ of $\mathcal{D}^A$ and $\mathcal{D}^B$, which we describe in the following definition.

Definition 4.9. Let $\mathcal{F}_i$ and $\mathcal{E}_i$, $i=1,2,3$, be Hilbert spaces and let

$$\mathcal{D}^A \subset (\mathcal{F}_1 \oplus \mathcal{F}_2) \times (\mathcal{E}_1 \oplus \mathcal{E}_2) \quad \text{and} \quad \mathcal{D}^B \subset (\mathcal{F}_3 \oplus \mathcal{F}_2) \times (\mathcal{E}_3 \oplus \mathcal{E}_2) \tag{4.17}$$

be split Dirac structures. Then the *composition* $\mathcal{D}^A \circ \mathcal{D}^B$ of $\mathcal{D}^A$ and $\mathcal{D}^B$ (through $\mathcal{F}_2 \times \mathcal{E}_2$) is defined as

$$\mathcal{D}^A \circ \mathcal{D}^B = \left\{ \begin{bmatrix} f_1 \\ f_3 \\ e_1 \\ e_3 \end{bmatrix} \middle| \exists f_2, e_2 : \begin{bmatrix} f_1 \\ f_2 \\ e_1 \\ e_2 \end{bmatrix} \in \mathcal{D}^A \text{ and } \begin{bmatrix} f_3 \\ -f_2 \\ e_3 \\ e_2 \end{bmatrix} \in \mathcal{D}^B \right\}.$$

The *bond space of the composition* is the space $\mathcal{B}_\circ := (\mathcal{F}_1 \oplus \mathcal{F}_3) \times (\mathcal{E}_1 \oplus \mathcal{E}_3)$ equipped with the power product

$$\left[\!\left[\begin{bmatrix} f_1^1 \\ f_3^1 \\ e_1^1 \\ e_3^1 \end{bmatrix}, \begin{bmatrix} f_1^2 \\ f_3^2 \\ e_1^2 \\ e_3^2 \end{bmatrix} \right]\!\right]_{\mathcal{B}_\circ} = \left(\begin{bmatrix} f_1^1 \\ f_3^1 \end{bmatrix}, \begin{bmatrix} r_1 & 0 \\ 0 & r_3 \end{bmatrix} \begin{bmatrix} e_1^2 \\ e_3^2 \end{bmatrix} \right)_{\mathcal{F}_1 \oplus \mathcal{F}_3} + \left(\begin{bmatrix} r_1 & 0 \\ 0 & r_3 \end{bmatrix} \begin{bmatrix} e_1^1 \\ e_3^1 \end{bmatrix}, \begin{bmatrix} f_1^2 \\ f_3^2 \end{bmatrix} \right)_{\mathcal{F}_1 \oplus \mathcal{F}_3}. \quad \blacklozenge$$

It is readily verified that the power product on $\mathcal{B}_\circ$ is obtained as the sum of the power products on $\mathcal{B}^A$ and $\mathcal{B}^B$ in the sense that $(f_1, f_2, e_1, e_2) \in \mathcal{D}^A$ and $(f_3, -f_2, e_3, e_2) \in \mathcal{D}^B$ imply that

$$\left[\!\left[\begin{bmatrix} f_1 \\ f_2 \\ e_1 \\ e_2 \end{bmatrix}, \begin{bmatrix} f_1 \\ f_2 \\ e_1 \\ e_2 \end{bmatrix} \right]\!\right]_{\mathcal{B}^A} + \left[\!\left[\begin{bmatrix} f_3 \\ -f_2 \\ e_3 \\ e_2 \end{bmatrix}, \begin{bmatrix} f_3 \\ -f_2 \\ e_3 \\ e_2 \end{bmatrix} \right]\!\right]_{\mathcal{B}^B} = \left[\!\left[\begin{bmatrix} f_1 \\ f_3 \\ e_1 \\ e_3 \end{bmatrix}, \begin{bmatrix} f_1 \\ f_3 \\ e_1 \\ e_3 \end{bmatrix} \right]\!\right]_{\mathcal{B}_\circ}. \tag{4.18}$$

Therefore the composition of two Dirac structures always satisfies $[d,d]_{\mathcal{B}_\circ} = 0$ for all $d \in \mathcal{D}^A \circ \mathcal{D}^B$. In the case where $\mathcal{E}_2$ and $\mathcal{F}_2$ have finite dimension, $\mathcal{D}^A \circ \mathcal{D}^B$ is always a Dirac structure, as can be seen from [KZvdSB09, Cor. 3.8]. However, in the Hilbert-space case it is not always true that $[d,d]_{\mathcal{B}_\circ} = 0$ for all $d \in (\mathcal{D}^A \circ \mathcal{D}^B)^{[\perp]}$, so that $\mathcal{D}$ is not a Dirac structure. For a counterexample see [Gol02, Ex. 5.2.23].

If we want the interconnected system to be a port-Hamiltonian system, then its interconnection structure $\mathcal{D}^A \circ \mathcal{D}^B$ by definition must be a Dirac structure. This shows how important it is to know under which circumstances the composition of two Dirac structures is a Dirac structure.

At least in the case, where the interconnection takes place only through the external ports, the natural Hamiltonian of the interconnected system is $\mathcal{H}_\circ(x^A, x^B) = \mathcal{H}^A(x^A) + \mathcal{H}^B(x^B)$, where $\mathcal{H}^A$ and $\mathcal{H}^B$ are the Hamiltonians of the original systems A and B, respectively. This corresponds to the fact that the total energy of the interconnected system should equal the sum of the energies stored in the two subsystems.

In the next chapter we interconnect the complex version of the system (4.16) with another conservative system via the whole spatial domain, through an infinite-dimensional channel.

Chapter 5

A motivating example

We now continue Example 4.6 and at the same time we further connect the state/signal framework to that of the Dirac structures by treating Example 3.9 from [KZvdSB09] within the state/signal framework. The function spaces which appear in this chapter have complex scalar fields.

We will use Theorem [Kur10, Thm 4.11] and for the convenience of the reader we include the relevant parts of that theorem here. Recall from Corollary 3.5 that if the input/output pair $(\mathcal{U},\mathcal{Y})$ is admissible for the subspace $V\subset\mathfrak{K}$, then $(\mathcal{U}^{[\perp]},\mathcal{Y}^{[\perp]})$ is admissible for the orthogonal companion $V^{[\perp]}$.

Theorem 5.1. *Assume that $V\subset\mathfrak{K}$ has the property that $\left[\begin{smallmatrix} z\\0\\0\end{smallmatrix}\right]\in V \Longrightarrow z=0$.*

Then $(V;\mathcal{X},\mathcal{W})$ is a conservative state/signal node if and only if $V=V^{[\perp]}$. This holds if and only if V satisfies the following three conditions:

(i) The space V is neutral: $[v,v]_{\mathfrak{K}}=0$ for all $v\in V$.

(ii) There exists an admissible input/output pair $(\mathcal{U},\mathcal{Y})$ for V such that also the dual pair $(\mathcal{U}^{[\perp]},\mathcal{Y}^{[\perp]})$ is an admissible input/output pair for V.

(iii) The main operators of the operator node representations $A^\times$ and A^d of V and $V^{[\perp]}$, see Definitions 2.4 and 2.11, corresponding to the dual input/output pair $(\mathcal{U}^{[\perp]},\mathcal{Y}^{[\perp]})$ have non-disjoint resolvent sets:

$$\operatorname{Res}\left(A^\times\right)\cap\operatorname{Res}\left(A^d\right)\neq\emptyset. \tag{5.1}$$

Let $\mathcal{E}_{x,1}=\mathcal{F}_{x,1}=\mathcal{E}_2=\mathcal{F}_2=L^2(\mathbb{R}^+;\mathbb{C})$ and $\mathcal{E}_{\partial,1}=\mathcal{F}_{\partial,1}=\mathbb{C}$, and let the bond space be

$$\mathcal{B}=\mathcal{F}\times\mathcal{E}=(\mathcal{F}_{x,1}\oplus\mathcal{F}_{\partial,1}\oplus\mathcal{F}_2)\times(\mathcal{E}_{x,1}\oplus\mathcal{E}_{\partial,1}\oplus\mathcal{E}_2)$$

with power product (4.5), where $r_{\mathcal{E},\mathcal{F}}$ is the identity operator on $\mathcal{E}=\mathcal{F}$.

By Lemma 4.1, (4.14) is a real Dirac structure if and only if

$$\mathcal{D}^A:=\left\{\begin{bmatrix} f_{x,1}\\ f_{\partial,1}\\ f_2\\ e_{x,1}\\ e_{\partial,1}\\ e_2\end{bmatrix}\;\middle|\;\begin{bmatrix} f_{x,1}\\ f_2\\ f_{\partial,1}\\ e_{\partial,1}\end{bmatrix}=\begin{bmatrix} 0 & \frac{\partial}{\partial z}\\ \frac{\partial}{\partial z} & 0\\ 0 & -\varphi_0\\ \varphi_0 & 0\end{bmatrix}\begin{bmatrix} e_{x,1}\\ e_2\end{bmatrix},\ e_{x,1},e_2\in H^1(\mathbb{R}^+;\mathbb{C})\right\} \tag{5.2}$$

is a complex Dirac structure. We want to compose the complex transmission line Dirac structure (5.2) with the Dirac structure

$$\mathcal{D}^B = \left\{ \begin{bmatrix} f_2 \\ e_2 \end{bmatrix} \middle| f_2 = ie_2,\ e_2 \in L^2(\mathbb{R}^+;\mathbb{C}) \right\} \subset \begin{bmatrix} \mathcal{F}_2 \\ \mathcal{E}_2 \end{bmatrix}.$$

Recall that the energy-preserving composition of two Dirac structures through a finite-dimensional space $\begin{bmatrix} \mathcal{F}_2 \\ \mathcal{E}_2 \end{bmatrix}$ always is a Dirac structure. At this time it is not clear if the composition $\mathcal{D}^A \circ \mathcal{D}^B$ is a Dirac structure or not, because the composition is done through the infinite-dimensional state. This is the extreme opposite of the interconnection depicted in Figure 4.2, since now no port signals are used for interconnection.

The composition of $\mathcal{D}^A$ and $\mathcal{D}^B$, as given in Definition 4.9, is

$$\begin{aligned} \mathcal{D}^A \circ \mathcal{D}^B &= \left\{ \begin{bmatrix} f_{x,1} \\ f_{\partial,1} \\ e_{x,1} \\ e_{\partial,1} \end{bmatrix} \middle| \begin{bmatrix} f_{x,1} \\ 0 \\ f_{\partial,1} \\ e_{\partial,1} \end{bmatrix} = \begin{bmatrix} 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & i \\ 0 & -\varphi_0 \\ \varphi_0 & 0 \end{bmatrix} \begin{bmatrix} e_{x,1} \\ e_2 \end{bmatrix},\ e_{x,1}, e_2 \in H^1(\mathbb{R}^+;\mathbb{C}) \right\} \\ &= \begin{bmatrix} 0 & \frac{\partial}{\partial z} \\ 0 & -\varphi_0 \\ 1 & 0 \\ \varphi_0 & 0 \end{bmatrix} \left\{ \begin{bmatrix} e_{x,1} \\ e_2 \end{bmatrix} \middle| e_2 = i\frac{\partial}{\partial z} e_{x,1},\ e_{x,1}, e_2 \in H^1(\mathbb{R}^+;\mathbb{C}) \right\} \\ &= \begin{bmatrix} i\frac{\partial^2}{\partial z^2} \\ -i\varphi_0 \frac{\partial}{\partial z} \\ 1 \\ \varphi_0 \end{bmatrix} H^2(\mathbb{R}^+;\mathbb{C}). \end{aligned} \tag{5.3}$$

Denote $L := i\frac{\partial^2}{\partial z^2}$, $K := -i\varphi_0 \frac{\partial}{\partial z}$ and $G := \varphi_0$, all defined on $\mathrm{Dom}(L) := H^2(\mathbb{R}^+;\mathbb{C})$. Let $\mathcal{X} := L^2(\mathbb{R}^+;\mathbb{C})$ and let $\mathcal{W} := \mathbb{C}^2$ with

$$\left[\begin{bmatrix} f_\partial^1 \\ e_\partial^1 \end{bmatrix}, \begin{bmatrix} f_\partial^2 \\ e_\partial^2 \end{bmatrix} \right]_{\mathcal{W}} = f_\partial^1 \overline{e_\partial^2} + e_\partial^1 \overline{f_\partial^2}.$$

Note that $\mathcal{U} = \begin{bmatrix} \{0\} \\ \mathcal{E}_{\partial,1} \end{bmatrix}$ and $\mathcal{Y} = \begin{bmatrix} \mathcal{F}_{\partial,1} \\ \{0\} \end{bmatrix}$ satisfy $\mathcal{U}^{[\perp]} = \mathcal{U}$ and $\mathcal{Y}^{[\perp]} = \mathcal{Y}$ in $\mathcal{W}$, i.e., that the input/output pair $(\mathcal{U},\mathcal{Y})$ is Lagrangian. If this input/output pair is admissible for $V := \mathcal{D}^A \circ \mathcal{D}^B$, then the corresponding main operators A and $A^\times$ in Theorem 5.1 coincide.

Moreover, the composition $\mathcal{D}^A \circ \mathcal{D}^B$ is a neutral or Lagrangian subspace of the complex bond space $\mathcal{B}$ if and only if it is neutral or Lagrangian, respectively, in the node space $\mathfrak{K}$, because $[\cdot,\cdot]_{\mathfrak{K}} = -[\cdot,\cdot]_{\mathcal{B}}$, cf. Remark 4.7. Obviously $V := \begin{bmatrix} L \\ 1 \\ G+K \end{bmatrix} \mathrm{Dom}(L)$ satisfies the condition $\begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix} \in V \Longrightarrow z = 0$. Theorem 5.1 therefore yields that $\mathcal{D}^A \circ \mathcal{D}^B$ is a Dirac structure if and only if $(\mathcal{D}^A \circ \mathcal{D}^B; \mathcal{X}, \mathcal{W})$ is a conservative state/signal node.

The composition $\mathcal{D}^A \circ \mathcal{D}^B$ satisfies condition (i) of Theorem 5.1 due to (4.18). We now show that $(\mathcal{U},\mathcal{Y})$ is an admissible input/output pair for $\mathcal{D}^A \circ \mathcal{D}^B$ by showing that (L,K,G) is a boundary node on $(\mathcal{U},\mathcal{X},\mathcal{Y}) = (\mathbb{C}, L^2(\mathbb{R}^+;\mathbb{C}), \mathbb{C})$ and applying [MS06, Thm 2.3]; see Section 2.3.

We need to verify the conditions in Definition 2.13. By construction, the operators K, L and G all have domain $H^2(\mathbb{R}^+;\mathbb{C})$, which is a Hilbert space, and they are continuous maps from their domain to their respective co-domains. Therefore, the operator triple $\left[\begin{smallmatrix} L \\ K \\ G \end{smallmatrix}\right]$ is necessarily a closed operator. Moreover, the operator $G = \varphi_0\big|_{H^2(\mathbb{R}^+;\mathbb{C})}$ is surjective, because for every $a \in \mathbb{C}$, the function $f(z) = \frac{a}{1+z}$ lies in $H^2(\mathbb{R}^+;\mathbb{C})$ and $Gf = a$. We are done proving that (L,K,G) is a boundary node if we manage to prove that

$$A := L\big|_{\mathcal{N}(\varphi_0)} = i\frac{\partial^2}{\partial z^2}\bigg|_{\{x\in H^2(\mathbb{R}^+;\mathbb{C}) \mid x(0)=0\}} \tag{5.4}$$

generates a C_0 semigroup on $\mathcal{X}$. According to Stone's Theorem [Paz83, Thm 1.10.8], every skew-adjoint operator A, i.e. every operator which satisfies $A^* = -A$, generates a C_0 semigroup of unitary operators.

The spectrum of a skew adjoint operator lies on the imaginary axis. Indeed, if $A^* = -A$ then $(iA)^* = iA$, and it is well-known that the spectrum of a self-adjoint operator is real. This means that $\mathbb{C}^+ \cup \mathbb{C}^- \subset \mathrm{Res}(A)$, which obviously implies (5.1) when we take into account that $A^\times = A$. We now proceed to prove that A in (5.4) is skew-adjoint on $\mathcal{X} = L^2(\mathbb{R}^+;\mathbb{C})$.

Firstly, A is skew-symmetric, because for all $x \in \mathrm{Dom}(A) = \mathcal{N}(G)$:

$$\left[\begin{bmatrix} Lx \\ x \\ Kx \end{bmatrix}, \begin{bmatrix} Lx \\ x \\ Kx \end{bmatrix}\right]_{\mathfrak{K}} = -(x,Ax)_{\mathcal{X}} - (Ax,x)_{\mathcal{X}} = 0.$$

By Definition A.4, this implies that $Ax = -A^*x$ for all $x \in \mathrm{Dom}(A)$, and we still need to show that $\mathrm{Dom}(A^*) \subset \mathrm{Dom}(A)$.

Therefore assume that $y \in \mathrm{Dom}(A^*)$. Integrating twice by parts, we get for all $x \in \mathrm{Dom}(A)$ that:

$$\begin{aligned} (Ax,y)_{\mathcal{X}} &= \int_0^\infty i\frac{\partial^2 x}{\partial z^2}(z)\overline{y(z)}\,\mathrm{d}z = i\left[\frac{\partial x}{\partial z}(z)\overline{y(z)}\right]_0^\infty - i\int_0^\infty \frac{\partial x}{\partial z}(z)\overline{\frac{\partial y}{\partial z}(z)}\,\mathrm{d}z \\ &= -i\frac{\partial x}{\partial z}(0)\overline{y(0)} - i\left[x(z)\overline{\frac{\partial y}{\partial z}(z)}\right]_0^\infty + i\int_0^\infty x(z)\overline{\frac{\partial^2 y}{\partial z^2}(z)}\,\mathrm{d}z \\ &= -i\frac{\partial x}{\partial z}(0)\overline{y(0)} - \left(x, i\frac{\partial^2}{\partial z^2}y\right)_{\mathcal{X}} = (x,A^*y)_{\mathcal{X}}. \end{aligned} \tag{5.5}$$

The space of all $x \in H^2(\mathbb{R}^+;\mathbb{C})$, such that $x(0) = 0$ and $\frac{\partial x}{\partial z}(0) = i$, is well-known to be dense in $L^2(\mathbb{R}^+;\mathbb{C})$. We can therefore find a sequence $x_n \in H^2(\mathbb{R}^+;\mathbb{C})$ which tends to zero in $L^2(\mathbb{R}^+;\mathbb{C})$ and satisfies $x_n(0) = 0$ and $\frac{\partial x_n}{\partial z}(0) = i$. Then the last line of (5.5) yields that

$$\left(x_n, i\frac{\partial^2}{\partial z^2}y\right)_{\mathcal{X}} + (x_n, A^*y)_{\mathcal{X}} = \overline{y(0)},$$

where the left-hand side tends to zero by the Cauchy-Schwarz inequality. This proves that $y(0) = 0$ for all $y \in \mathrm{Dom}(A^*)$, which means that $\mathrm{Dom}(A^*) \subset \mathrm{Dom}(A)$ and we are finished proving that A is skew-adjoint. Thus $(\mathcal{D}^A \circ \mathcal{D}^B)^{[\perp]} = \mathcal{D}^A \circ \mathcal{D}^B$. ♦

The preceding computation does at the first glance not seem much simpler than [KZvdSB09, Ex. 3.9], but the approach given here has a few advantages over that taken in [KZvdSB09]. For instance, we always have $A \subset -A^*$, because $\mathcal{D}^A \circ \mathcal{D}^B$ is automatically neutral when $\mathcal{D}^A$ and $\mathcal{D}^B$ are Dirac structures and the composition is energy preserving. There is no need to check this separately for every example. In contrast to [KZvdSB09, Ex. 3.9], we did not need to compute the trajectories of $\mathcal{D}^A \circ \mathcal{D}^B$ in (5.3) explicitly. One can often check if a given operator A is skew adjoint on a complex Hilbert space by looking it up in the literature, using the fact that A is skew adjoint if and only if iA is self adjoint.

Chapter 6

Summaries of the included articles and their contributions

This chapter contains short summaries of the three articles included in this dissertation. Accounts of their relevance for the research field of infinite-dimensional linear systems are also provided.

6.1 Article I: Well-posed state/signal systems in continuous time

We introduce the class of L^p-well-posed state/signal systems for $1 \leq p < \infty$ and present their basic properties. This paper is mostly a technical exposition of how to represent these systems and how to work with their trajectories, but we also give Examples 6.8 and 6.9, which indicate how systems that behave badly in the input/state/output setting can be modelled within the state/signal framework.

In Article I we characterise the well-posed input/output pairs of a given well-posed state/signal system in various ways and show how to obtain the corresponding input/state/output representations; see Definition 2.7, Theorem 4.13 and Theorem 6.6 of [KS09]. A comparison of classical and generalised trajectories of the state/signal node and how they relate to the classical and generalised trajectories of an input/state/output representation is also an essential part of this article; also see Section 5 of the article.

It is clear that every state/signal node determines a state/signal system uniquely through Definitions 2.9 and 2.10, but the converse still remains an open question. We prove in [KS09, Section 6] that there always exists a unique *maximal* state/signal node V_{max}, which generates a well-posed state/signal system. Here maximality means that if V_{max} is maximal and V' generates the same space of generalised trajectories as V_{max}, then $V' \subset V_{max}$. The maximal generating state/signal node of an L^p-well-posed state/signal node is L^p-well-posed. The converse of the above question, however, is still open; I do not know if a well-posed state/signal system $(\mathfrak{W}; \mathcal{X}, \mathcal{W})$ determines a generating well-posed state/signal node $(V; \mathcal{X}, \mathcal{W})$ uniquely if the maximality condition is dropped.

6.2 Article II: On passive and conservative state/signal systems in continuous time

In Article II we specialise the theory in Article I to the case $p=2$ and more structured generating subspaces V. The main focus now lies on passive state/signal systems. In fact passive and conservative systems are the main reason for introducing and studying state/signal systems at all, because of their very useful extra structure. The additional assumption of passivity is quite reasonable, because it essentially means that the system has no internal sources of energy, and many physical systems have this property.

Section 2 of [Kur10] mostly concerns operator nodes and system nodes, and a method for using these to represent generating subspaces is explored. In section 3 we introduce the dual state/signal node, which is a prerequisite for understanding passivity. Theorem 3.6 of [Kur10] is a quite general result that yields an operator node representation of the state/signal dual in terms of an arbitrary operator node representation of the original state/signal node.

A significant amount of background is necessary before we are able to define a passive state/signal system in an intuitively understandable way, but once this background is there, we have a very simple characterisation of passive and conservative state/signal nodes. Indeed, in section 4 we prove that for an arbitrary $V\subset\mathfrak{K}$, the triple $(V;\mathcal{X},\mathcal{W})$ is a passive state/signal node if and only if

$$\begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix} \in V \quad \Longrightarrow \quad z=0 \tag{6.1}$$

and V is a maximally nonnegative subspace of the node space $\mathfrak{K}$ with respect to the power product (2.19). The triple $(V;\mathcal{X},\mathcal{W})$ is a conservative state/signal node if and only if (6.1) holds and V is a Lagrangian subspace of $\mathfrak{K}$: $V=V^{[\perp]}$.

Comparing these characterisations to Definitions 2.4 and 3.2, we see that it is easier to discuss passivity and conservativity in the state/signal framework than in the input/state/output counterpart. And, moreover, state/signal passivity covers several different types of input/state/output passivity, cf. Section 3.3. The condition (6.1) is not critical, because we can apply [Kur10, Prop. 4.7] to any maximally nonnegative (Lagrangian) subspace of $\mathfrak{K}$ in order to turn it into a passive (conservative) state/signal node. In [Kur10, Thm 4.11] we give a list of useful characterisations of conservative state/signal nodes.

We do not formally introduce the scattering representation until section 5 of Article II, but in fact Theorem 4.5 of that article is fundamental to almost all of the theory that we give for passive state/signal systems. Many of the results in section 4 of [Kur10] are based on this theorem, even if there is no explicit reference to fundamental input/output pairs. I therefore consider Theorem 4.5 to be the main result of the article.

The usefulness of Theorem 4.5 mostly lies in the fact that it yields that every fundamental input/output pair is admissible for a passive state/signal system. Even though it is relatively easy to characterise all well-posed input/output pairs given one such pair, it seems difficult to characterise the admissible but ill-posed input/output pairs in a simple and useful way, because the system node in Definition 2.4 is a rather complicated object. Section five contains two characterisations of passive state/signal nodes that are related to scattering representations.

Although Article II is intended to be a continuation of Article I, some differences do exist. Indeed, when one analyses general well-posed state/signal systems, the generalised trajectories are most useful, because the local continuity condition (2.22) ensures that these behave reasonably well. Passivity, however, implies that the generating subspace V of the system is maximally nonnegative and this additional structure can be used quite extensively. Therefore a passive state/signal node $(V;\mathcal{X},\mathcal{W})$ is more convenient to study than its set of trajectories. An implication of this fact is that admissibility of an input/output pair carries a different meaning in Article II than in Article I, as we have already noted. Indeed, all well-posed input/output pairs, i.e., those that we call admissible in Article I, are also admissible in the sense of Article II, because of [KS09, Thm 4.9 and Lem. 5.2].

6.3 Article III: Dirac structures and their composition on Hilbert spaces

Article III seems quite different from Articles I and II on first sight. However, a linear Dirac structure is essentially the same thing as a conservative state/signal node with a Lagrangian decomposition of the external signal space, as we have already noted in Remark 4.7.

In Section 2 of Article III we define the Dirac structure and show how Dirac structures fit into the theory of Kreĭn spaces. We derive the scattering representation of a Dirac structure using an extension of the operator Cayley transformation, which was mentioned in Remark 3.14, to skew-adjoint linear relations. We now describe a slightly simplified version of the scattering representation of a Dirac structure.

Theorem 6.1. *If $\mathcal{D}$ is a Dirac structure on the bond space $\mathcal{B}=\mathcal{F}\times\mathcal{E}$ with power product induced by $r_{\mathcal{E},\mathcal{F}}$, then there exists a unique unitary operator $\mathcal{O}$ on $\mathcal{E}$ such that*

$$\begin{bmatrix} f \\ e \end{bmatrix} \in \mathcal{D} \Longleftrightarrow (e+r^*_{\mathcal{E},\mathcal{F}}f)=\mathcal{O}(e-r^*_{\mathcal{E},\mathcal{F}}f). \tag{6.2}$$

On the other hand, if $\mathcal{O}$ is a unitary operator on $\mathcal{E}$, then

$$\mathcal{D}:=\left\{\begin{bmatrix} r_{\mathcal{E},\mathcal{F}}(\mathcal{O}g-g) \\ (\mathcal{O}g+g) \end{bmatrix} \mid g\in\mathcal{E}\right\} \tag{6.3}$$

defines a Dirac structure on $\mathcal{B}$ for which (6.2) *holds.*

Note that the terms scattering representation and Cayley transformation carry a different meaning in the context of Article III than in that of state/signal systems.

In Section 3 of Article III we introduce so-called "split Dirac structures", which are used to define port-Hamiltonian systems as described in Chapter 4 of this summary. We then proceed to give necessary and sufficient conditions in terms of scattering representations for the composition of two split Dirac structures to be a split Dirac structure.

We also give the scattering representation of the composed Dirac structures. The first part of [KZvdSB09, Cor. 3.8] gives a simple sufficient condition for the composition to be a Dirac structure and the second part of the corollary confirms that an energy-preserving composition of two Dirac structures through a finite-dimensional channel is always a Dirac structure. This is a nice and simple generalisation of the well-known result that the energy-preserving composition of two finite-dimensional Dirac structures is always a Dirac structure.

In Sections 4 and 5 Dirac structures associated to boundary control problems are studied. We first give necessary and sufficient conditions for the graph $\begin{bmatrix} L \\ K \\ 1 \\ G \end{bmatrix} \mathrm{Dom}(L)$ of a boundary colligation (L,K,G) to be a Dirac structure; see Definition 2.13. Theorem 4.6 of [KZvdSB09] clarifies the connection between such Dirac structures and the so-called boundary triplets studied in [GG91]. We mainly consider strong boundary colligations in Article III, but [KZvdSB09, Thm 4.7] gives some sufficient conditions for the graph of a non-strong boundary colligation to be a Dirac structure as well. We end the paper by showing that the class of Dirac structures, which in the above sense are graphs of strong boundary colligations, is invariant under energy-preserving composition through the external port variables.

6.4 Contributions made to the research field

This section summarises the advances made in the three appended articles.

Article I

To my knowledge, the concepts of equal treatment of inputs and outputs, which is the main feature of state/signal systems, and that of well-posedness have not previously been combined in the case of continuous-time systems. It should be pointed out that the discrete-time state/signal systems studied by Arov and Staffans are well-posed in the appropriate discrete-time sense.

Even though most of the results presented in Article I turn out as one would expect, it takes a significant effort to sort out the technical details. Theorem 5.8 of Article I should be known to most researchers in the field of infinite-dimensional linear systems, but it seems not to have been written down in this form earlier.

Article II

Passive and conservative systems have been studied quite extensively in e.g. [Aro95, Aro99, AN96, MS06, MS07, MSW06, Sta02a, Sta02b, TW03, WST01]. The Cayley transformation described in Section 3.4 would allow us to establish most of the results of this article by reinterpreting the corresponding discrete-time results in [AS07a] in a way similar to what was done in [AN96]. However, that approach also requires a significant amount of work, and a fair amount of explanation is necessary in order to understand how the theorems should be interpreted.

Indeed, we could immediately have defined $(V;\mathcal{X},\mathcal{W})$ to be a passive state/signal node if (6.1) holds and V is maximally nonnegative, but this definition would have been completely unintuitive and therefore difficult to understand. Moreover, it is interesting in its own right to build the continuous-time theory independently of the discrete-time counterpart. To my best knowledge, Article II contains the first development of an input/output-free theory for infinite-dimensional passive systems, and the development is done throughout in continuous time.

Recall from the introduction that modular modelling, and therefore interconnection considerations, was one of the main motivations for introducing state/signal systems. Moreover, in analogy to the behavioural theory, control of state/signal systems is done by interconnection. The interconnection theory is a very important aspect of systems theory, which still remains to be developed for state/signal systems, and in my opinion, the steps taken towards that interconnection theory is the most important contribution of Article II; also see Chapter 5 of this summary.

Article III

Dirac structures on Hilbert spaces were introduced by Parsian and Shafei Deh Abad in [PSDA99], where also their scattering representations were established. Some further development of these Dirac structures was done in [GIZvdS04, ISG05] and other publications by these authors. However, the theory of interconnection of infinite-dimensional port-Hamiltonian systems is still only in its infancy and to the knowledge of the author no comprehensive study of the composition of two Dirac structures on Hilbert spaces has been carried out.

The proof of [KZvdSB09, Thm 3.4] extends the ideas in [Gol02, Sec. 5.2.3] but the rest of the results in sections 3 to 5 of [KZvdSB09] should be new. Note that this article also contributes towards nonlinear port-Hamiltonian systems, because in some nonlinear port-Hamiltonian systems the nonlinearities can be incorporated into the Hamiltonian while the Dirac structure is linear; see [Vil07, Ex. 1.14].

Chapter 7

A few ideas for the future

The study of state/signal systems and irregular interconnection of infinite-dimensional systems has merely been started and much remains to be done. Many questions remain to be answered in connection with stability, the properties of the state/signal dual and the properties of impedance representations. The articles by Arov and Staffans on discrete-time state/signal systems contain a large number of results, which should be considered in continuous time as well. In particular the frequency-domain behaviour of state/signal systems should be worked out.

Also regarding interconnection there is much work undone. The results presented in [KZvdSB09] should be generalised to more general types of interconnection. One may also ask when an interconnection of two state/signal systems is a regular feedback interconnection of two input/state/output systems and which system behaviours can be achieved through interconnection with a controller. The finite-dimensional formulations of these problems can be found in [JWBT05]. A study in which cases well-posedness is preserved by interconnection also remains to be done.

Finally, the state/signal framework needs to be tested on more complicated physical examples. This will surely open up many new interesting questions, which will give directions for future research.

Appendix A

Brief background on Kreĭn spaces

In this appendix we collect some standard terminology and results from the theory of Kreĭn spaces. More background can be found e.g. in [AS07a] and [Bog74]. The claims we make here were proved in the appendix of [Kur10].

Definition A.1. The vector space $(\mathcal{W};[\cdot,\cdot]_{\mathcal{W}})$, where $[\cdot,\cdot]_{\mathcal{W}}$ is an indefinite sesquilinear product, is an *anti-Hilbert space* if $-\mathcal{W}:=(\mathcal{W};-[\cdot,\cdot]_{\mathcal{W}})$ is a Hilbert space. In this case we for clarity denote the Hilbert space $-\mathcal{W}$ by $|\mathcal{W}|$.

The space $(\mathcal{W};[\cdot,\cdot]_{\mathcal{W}})$ is a *Kreĭn space* if it admits a direct-sum decomposition $\mathcal{W}=\mathcal{W}_+\dot{+}\mathcal{W}_-$, such that:

(i) the spaces $\mathcal{W}_+$ and $\mathcal{W}_-$ are $[\cdot,\cdot]_{\mathcal{W}}$-orthogonal, i.e., $[w_+,w_-]_{\mathcal{W}}=0$ for all $w_+\in\mathcal{W}_+$ and $w_-\in\mathcal{W}_-$, and

(ii) the space $\mathcal{W}_+$ is a Hilbert space and $\mathcal{W}_-$ is an anti-Hilbert space.

In this case we call the decomposition $\mathcal{W}=\mathcal{W}_+\dot{+}\mathcal{W}_-$ a *fundamental decomposition* of $\mathcal{W}$ and we always denote it by $\mathcal{W}=(\mathcal{W}_+,\mathcal{W}_-)$, so that the second space in the pair is the anti-Hilbert space. ♦

Let $\mathcal{U}$ and $\mathcal{Y}$ be subspaces of the Kreĭn space $\mathcal{W}$. By writing $\mathcal{U}[\perp]\mathcal{Y}$ we mean that $\mathcal{U}$ and $\mathcal{Y}$ are orthogonal to each other with respect to $[\cdot,\cdot]_{\mathcal{W}}$. The *orthogonal companion* of $\mathcal{U}$ is the space

$$\mathcal{U}^{[\perp]}:=\{w\in\mathcal{W}\,|\,\forall u\in\mathcal{U}:\quad [u,w]_{\mathcal{W}}=0\}. \tag{A.1}$$

Proposition A.2. *Let $\alpha\in\mathbb{C}^+$ and let $\mathcal{W}$ be a Kreĭn space with fundamental decomposition $\mathcal{W}=(\mathcal{W}_+,\mathcal{W}_-)$. Then the node space $\mathfrak{K}$ in Definition 2.7 is a Kreĭn space with fundamental decomposition $\mathfrak{K}=(\mathfrak{K}_+,\mathfrak{K}_-)$, where*

$$\mathfrak{K}_+=\begin{bmatrix}\begin{bmatrix}-\overline{\alpha}\\1\end{bmatrix}\mathcal{X}\\ \mathcal{W}_+\end{bmatrix}\quad \textit{and}\quad \mathfrak{K}_-=\begin{bmatrix}\begin{bmatrix}\alpha\\1\end{bmatrix}\mathcal{X}\\ \mathcal{W}_-\end{bmatrix}. \tag{A.2}$$

The fundamental decomposition (A.2) is closely connected to the Cayley transformation in Lemma 3.11.

Let $\mathcal{W}=(\mathcal{W}_+,\mathcal{W}_-)$ be a fundamental decomposition. Then it follows from Definition A.1 that all $w_+^1+w_-^1,w_+^2+w_-^2\in\mathcal{W}$, where $w_\pm^1,w_\pm^2\in\mathcal{W}_\pm$, satisfy

$$[w_+^1+w_-^1,w_+^2+w_-^2]_{\mathcal{W}}=[w_+^1,w_+^2]_{\mathcal{W}_+}+[w_-^1,w_-^2]_{\mathcal{W}_-}=(w_+^1,w_+^2)_{\mathcal{W}_+}-(w_-^1,w_-^2)_{|\mathcal{W}_-|}. \tag{A.3}$$

Therefore we can turn $\mathcal{W}$ into a Hilbert space by changing the sign on the restriction of $[\cdot,\cdot]_{\mathcal{W}}$ to $\mathcal{W}_-$, as described in the following definition.

Definition A.3. We call the Hilbert-space inner products on $\mathcal{W}$ that arise from fundamental decompositions $\mathcal{W}=(\mathcal{W}_+,\mathcal{W}_-)$ through

$$(w_+^1+w_-^1,w_+^2+w_-^2)_{\mathcal{W}}=(w_+^1,w_+^2)_{\mathcal{W}_+}+(w_-^1,w_-^2)_{|\mathcal{W}_-|}$$

admissible inner products. A norm induced by an admissible inner product is called an *admissible norm.* ♦

Only Hilbert and anti-Hilbert spaces have unique fundamental decompositions, but all admissible norms are equivalent. Every admissible inner product turns a closed subspace of a Kreĭn space into a Hilbert space, and thus in particular, every closed subspace of a Kreĭn space is a reflexive Banach space.

In contrast to Hilbert spaces, not every closed subspace $\mathcal{U}$ of a Kreĭn space $\mathcal{W}$ is itself a Kreĭn space. More precisely, a closed subspace $\mathcal{U}$ is a Kreĭn space if and only if it is ortho-complemented: $\mathcal{U}\dotplus\mathcal{U}^{[\perp]}=\mathcal{W}$; see [Bog74, Thm V.3.4]. In the state/signal theory we often encounter Lagrangian subspaces, which are closed non-Kreĭn subspaces of Kreĭn spaces.

The orthogonal companion (A.1) of any subspace $\mathcal{U}$ of $\mathcal{W}$ is a closed subspace of $\mathcal{W}$ with respect to the admissible norms. Denoting the closure of a subspace $\mathcal{U}\subset\mathcal{W}$ with respect to any admissible norm by $\overline{\mathcal{U}}$, we have that $(\mathcal{U}^{[\perp]})^{[\perp]}=\overline{\mathcal{U}}$.

The following definition makes use of the continuous dual $\mathcal{U}'$ of a Banach space $\mathcal{U}$. Recall that this continuous dual is the space of all continuous linear functionals on $\mathcal{U}$.

Definition A.4. Let $\mathcal{W}=\mathcal{U}\dotplus\mathcal{Y}$ be a direct-sum decomposition of a Kreĭn space. According to [AS07c, Lemma 2.3], we can identify the continuous duals of $\mathcal{U}$ and $\mathcal{Y}$ with $\mathcal{Y}^{[\perp]}$ and $\mathcal{U}^{[\perp]}$, respectively, using the following restrictions of $[\cdot,\cdot]_{\mathcal{W}}$ as duality pairings:

$$\begin{aligned}\langle u,u'\rangle_{\langle\mathcal{U},\mathcal{U}'\rangle}&=[u,u']_{\mathcal{W}},\quad u\in\mathcal{U},\ u'\in\mathcal{Y}^{[\perp]}\quad\text{and}\\ \langle y,y'\rangle_{\langle\mathcal{Y},\mathcal{Y}'\rangle}&=[y,y']_{\mathcal{W}},\quad y\in\mathcal{Y},\ y'\in\mathcal{U}^{[\perp]}.\end{aligned}$$

Let T map a dense subspace of $\mathcal{U}$ linearly into $\mathcal{Y}$. By $T^\dagger$ we denote the (possibly unbounded) *adjoint* of T computed with respect to these duality pairings, so that $T^\dagger:\mathcal{Y}'\to\mathcal{U}'$ is the maximally defined operator that satisfies

$$\forall u\in\mathrm{Dom}\,(T)\,,y'\in\mathrm{Dom}\left(T^\dagger\right):\quad\langle Tu,y'\rangle_{\langle\mathcal{Y},\mathcal{Y}'\rangle}=\langle u,T^\dagger y'\rangle_{\langle\mathcal{U},\mathcal{U}'\rangle}.\tag{A.4}$$

Here $\mathrm{Dom}\left(T^\dagger\right)$ is the subspace consisting of those $y'\in\mathcal{Y}'$, for which there exists some $u'\in\mathcal{U}'$, such that $\langle Tu,y'\rangle_{\langle\mathcal{Y},\mathcal{Y}'\rangle}=\langle u,u'\rangle_{\langle\mathcal{U},\mathcal{U}'\rangle}$ for all $u\in\mathrm{Dom}\,(T)$. ♦

The condition (A.4) can also be written

$$\forall u\in\mathrm{Dom}\,(T)\,,y'\in\mathrm{Dom}\left(T^\dagger\right):\quad[Tu,y']_{\mathcal{W}}=[u,T^\dagger y']_{\mathcal{W}},\tag{A.5}$$

but note that T is not densely defined on $\mathcal{W}$ in general, and therefore (A.5) does not determine $T^\dagger$ as an operator on $\mathcal{W}$ uniquely. However, if $\mathcal{U}=\mathcal{Y}=\mathcal{W}$ and this is a Hilbert space with inner product $(\cdot,\cdot)_{\mathcal{W}}$, then the construction in Definition A.4 leads to an identification $\mathcal{W}'=\mathcal{W}$, using the standard Hilbert-space duality pairing $\langle w,w'\rangle_{\langle\mathcal{W},\mathcal{W}'\rangle}=(w,w')_{\mathcal{W}}$. In this case we denote the adjoint $T^\dagger$ of T by T^* in order to emphasise that the adjoint is computed with respect to a Hilbert-space inner product.

Definition A.5. The subspace $V \subset \mathcal{W}$ is *nonnegative* if $[v,v]_{\mathcal{W}} \geq 0$ for all $v \in V$ and we denote this by $V \geq 0$. The subspace V is nonpositive, which we denote by $V \leq 0$, if $[v,v]_{\mathcal{W}} \leq 0$ for all $v \in V$. In both of these cases V is said to be *semidefinite* and V is *maximally semidefinite* if V has no proper extension to a semidefinite subspace of $\mathcal{W}$.

A *vector* $v \in \mathcal{W}$ is *neutral* if $[v,v]_{\mathcal{W}} = 0$. The *space* V is *neutral* if all $v \in V$ are neutral and V is *Lagrangian* if $V = V^{[\perp]}$. ♦

One can use polarisation to prove that V is a neutral subspace of the Kreĭn space $\mathcal{W}$ if and only if $[v^1, v^2]_{\mathcal{W}} = 0$ for all $v^1, v^2 \in V$. This means that V is neutral if and only if $V \subset V^{[\perp]}$. The closure of a semidefinite subspace is semidefinite and, therefore, every maximally semidefinite subspace is closed.

Appendix B

Useful function spaces

Here we define the spaces of functions which we need in this dissertation. We also introduce some operators for manipulating these functions.

Definition B.1. Let I and I' be subsets of $\mathbb{R}$ and let $\mathcal{U}$ be a Banach space.

(i) The vector space of functions defined everywhere on I with values in $\mathcal{U}$ is denoted by $\mathcal{U}^I$.

(ii) For $f \in \mathcal{U}^I$ and $a \in I$ we define the *point-evaluation operator* φ_a through $\varphi_a f := f(a)$.

(iii) The reflection operator $\boldsymbol{Я}$ about zero is defined as

$$(\boldsymbol{Я} f)(v) = f(-v), \quad f \in \mathcal{U}^I, \ -v \in I.$$

(iv) For all $t \in \mathbb{R}$ we define the *shift operator* τ^t, which maps functions in $\mathcal{U}^I$ into functions in $\mathcal{U}^{I-t}$, by $(\tau^t f)(v) = f(v+t)$ for $f \in \mathcal{U}^I$ and $v+t \in I$. If $t > 0$ then τ^t is a left shift by the amount t.

(v) The operator $\pi_I : \mathcal{U}^I \to \mathcal{U}^{\mathbb{R}}$ is defined by

$$(\pi_I f)(v) := \begin{cases} f(v), & v \in I \\ 0, & v \in \mathbb{R} \setminus I \end{cases}.$$

(vi) For $I' \supset I$, the *restriction operator* $\rho_I : \mathcal{U}^{I'} \to \mathcal{U}^I$ is given by

$$(\rho_I f)(v) = f(v), \ v \in I, \quad \text{i.e.} \quad \rho_I f = f|_I, \quad f \in \mathcal{U}^{I'}.$$

We briefly write $\pi_+ := \pi_{[0,\infty)}$ and $\rho_+ := \rho_{[0,\infty)}$. ♦

We note that $\tau^0 = 1$ and that for all $s,t \in \mathbb{R}$ we have $\tau^s \tau^t = \tau^{s+t}$. Thus, the shift operators $t \to \tau^t$ form a group on $\mathcal{U}^{\mathbb{R}}$. If $s,t \geq 0$ then $\rho_+ \tau^s \rho_+ \tau^t = \rho_+ \tau^{s+t}$, i.e. $\rho_+ \tau$ is a semigroup on $\mathcal{U}^{\mathbb{R}^+}$.

Definition B.2. Let $\mathcal{U}$ be a Banach space and let $-\infty < a < b < \infty$.

(i) The space of continuous $\mathcal{U}$-valued functions with domain $[a,b]$ is denoted by $C([a,b];\mathcal{U})$. This space is equipped with the supremum norm

$$\|f\|_{C([a,b];\mathcal{U})} := \sup_{t \in [a,b]} \|f(t)\|_{\mathcal{U}}.$$

(ii) The space of all $\mathcal{U}$-valued functions defined on $[a,b]$ with $n \in \mathbb{Z}^+$ continuous derivatives is denoted by $C^n([a,b];\mathcal{U})$ and equipped with the norm

$$\|f\|_{C^n([a,b];\mathcal{U})} := \sum_{k=0}^{n} \|f^{(k)}\|_{C([a,b];\mathcal{U})}. \tag{B.1}$$

(iii) The space of $\mathcal{U}$-valued functions defined on $[a,\infty)$ with $n \in \mathbb{Z}^+$ continuous derivatives is denoted by $C^n([a,\infty);\mathcal{U})$. This space is equipped with the compact-open topology induced by the family

$$\|f\|_n := \|\rho_{[a,a+n]} f\|_{C^n([a,a+n];\mathcal{U})}, \quad n \in \mathbb{Z}^+,$$

of seminorms. By writing $C([a,\infty);\mathcal{U})$ we mean $C^0([a,\infty);\mathcal{U})$. ♦

The space $C^n([a,b];\mathcal{U})$ is a Banach space and $C^n([a,\infty);\mathcal{U})$ is a Fréchet space for all $n \in \mathbb{Z}^+$. Convergence to zero of a sequence f_m in a Fréchet space means that $\|f_m\|_n \to 0$ for all $n \in \mathbb{Z}^+$.

Definition B.3. Let $\mathcal{U}$ be a Banach space and let $I = [a,b]$ or $I = [a,\infty)$.

(i) By $L^p(I;\mathcal{U})$ we denote the space of all $\mathcal{U}$-valued Lebesgue-measurable functions f defined on I, such that

$$\|f\|_{L^p(I;\mathcal{U})} := \left(\int_I \|f(v)\|_{\mathcal{U}}^p \,\mathrm{d}v\right)^{1/p} < \infty. \tag{B.2}$$

(ii) The space $L^p_{loc}(I;\mathcal{U})$ consists of all Lebesgue-measurable functions, which map I into $\mathcal{U}$, such that $\rho_{[a,b]} f \in L^p([a,b];\mathcal{U})$ for all bounded subintervals $[a,b]$ of I. A family of seminorms on $L^p_{loc}([a,\infty);\mathcal{U})$ is given by

$$\|f\|_n := \|\rho_{[a,a+n]} f\|_{L^p([a,a+n];\mathcal{U})}, \quad n \in \mathbb{Z}^+.$$

(iii) The space of functions $f \in L^2(I;\mathcal{U})$ with a distribution derivative g in $L^2(I;\mathcal{U})$ is denoted by $H^1(I;\mathcal{U})$. By this we mean that $f \in L^2(I;\mathcal{U})$ lies in $H^1(I;\mathcal{U})$ if and only if there exists a $g \in L^2(I;\mathcal{U})$ such that

$$\forall t \geq a: \quad f(t) = \int_a^t g(v)\,\mathrm{d}v. \tag{B.3}$$

We denote the space of $f \in L^2(I;\mathcal{U})$ that possess $n \in \mathbb{Z}^+$ distribution derivatives in $L^2(I;\mathcal{U})$ by $H^n(I;\mathcal{U})$. The standard norm on $H^n(I;\mathcal{U})$ is $\|\cdot\|_{H^n(I;\mathcal{U})}$, where

$$\|f\|^2_{H^n(I;\mathcal{U})} := \|f\|^2_{L^2(I;\mathcal{U})} + \|f^{(1)}\|^2_{L^2(I;\mathcal{U})} + \dots + \|f^{(n)}\|^2_{L^2(I;\mathcal{U})}$$

and $f^{(k)}$ denotes the k:th distribution derivative of f.

The space of functions $f \in H^1(\mathbb{R}^+;\mathcal{U})$ that have the property $f(0) = 0$ is denoted by $H^1_0(\mathbb{R}^+;\mathcal{U})$. ♦

The space $L^p(I;\mathcal{U})$ is a Banach space for $p \geq 1$ whenever $\mathcal{U}$ is a Banach space, whereas $L^p_{loc}([a,\infty);\mathcal{U})$ is only a Fréchet space. For finite intervals $[a,b]$, the spaces $L^p_{loc}([a,b];\mathcal{U})$ and $L^p([a,b];\mathcal{U})$ coincide. The point-evaluation operator φ_0 is continuous from $H^1(\mathbb{R}^+;\mathbb{C})$ to $\mathbb{C}$.

If $\mathcal{W}$ is a Kreĭn space, then $L^2(I;\mathcal{W})$ is a Kreĭn space with the inner product $[w,w'] := \int_I [w(v),w'(v)]_{\mathcal{W}}\, \mathrm{d}v$, because every fundamental decomposition $\mathcal{W} = (\mathcal{W}_+,\mathcal{W}_-)$ induces the fundamental decomposition

$$L^2(I;\mathcal{W}) = \big(L^2(I;\mathcal{W}_+), L^2(I;\mathcal{W}_-)\big).$$

The operators τ, $\boldsymbol{Я}$, π and ρ of Definition B.1 have obvious extensions to the L^p-type spaces in Definition B.3. We can also apply the pointwise-projection operator $\mathcal{P}^{\mathcal{Y}}_{\mathcal{U}}$ to a function which belongs to an L^p-type space by defining that $\mathcal{P}^{\mathcal{Y}}_{\mathcal{U}} w = u$ if $\mathcal{P}^{\mathcal{Y}}_{\mathcal{U}} w(t) = u(t)$ almost everywhere.

Bibliography

[AN96] Damir Z. Arov and Mark A. Nudelman, *Passive linear stationary dynamical scattering systems with continuous time*, Integral Equations Operator Theory **24** (1996), 1–45.

[Aro95] Damir Z. Arov, *A survey on passive networks and scattering sytstems which are lossless or have minimal losses*, Archiv für Elektronik und Übertragungstechnik **49** (1995), 252–265.

[Aro99] ———, *Passive linear systems and scattering theory*, Dynamical Systems, Control Coding, Computer Vision, Progress in Systems and Control Theory, vol. 25, Birkhäuser Verlag, 1999, pp. 27–44.

[AS05] Damir Z. Arov and Olof J. Staffans, *State/signal linear time-invariant systems theory. Part I: Discrete time systems*, The State Space Method, Generalizations and Applications, Operator Theory: Advances and Applications, vol. 161, Birkhäuser-Verlag, 2005, pp. 115–177.

[AS07a] ———, *State/signal linear time-invariant systems theory, Passive discrete time systems*, International Journal of Robust and Nonlinear Control **17** (2007), 497–548.

[AS07b] ———, *State/signal linear time-invariant systems theory. Part III: Transmission and impedance representations of discrete time systems*, Operator Theory, Structured Matrices, and Dilations. Tiberiu Constantinescu Memorial Volume (Bucharest, Romania), Theta Foundation, 2007, pp. 104–140.

[AS07c] ———, *State/signal linear time-invariant systems theory, Part IV: Affine representations of discrete time systems*, Complex Analysis and Operator Theory **1** (2007), 457–521.

[AS09] ———, *Two canonical passive state/signal shift realizations of passive discrete time behaviors*, Journal of Functional Analysis **257** (2009), 2573–2634.

[Bel68] V. Belevitch, *Classical network theory*, Holden-Day, San Francisco, Calif., 1968.

[Bog74] János Bognár, *Indefinite inner product spaces*, Springer-Verlag, New York, 1974, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 78.

[BT02] Madhu N. Belur and H. L. Trentelman, *Stabilization, pole placement, and regular implementability*, IEEE Trans. Automat. Control **47** (2002), no. 5, 735–744.

[Cou90] Theodore James Courant, *Dirac manifolds*, Trans. Amer. Math. Soc. **319** (1990), no. 2, 631–661.

[CZ95] Ruth F. Curtain and Hans Zwart, *An introduction to infinite-dimensional linear systems theory*, Springer-Verlag, New York, 1995.

[Dor93] Irene Dorfman, *Dirac structures and integrability of nonlinear evolution equations*, Nonlinear Science: Theory and Applications, John Wiley & Sons Ltd., Chichester, 1993.

[Fat68] H. O. Fattorini, *Boundary control systems*, SIAM J. Control **6** (1968), 349–385.

[GG91] Valentivna I. Gorbachuk and M. L. Gorbachuk, *Boundary value problems for operator differential equations*, Mathematics and its Applications (Soviet Series), vol. 48, Kluwer Academic Publishers Group, Dordrecht, 1991, translated and revised from the 1984 Russian original.

[GIZvdS04] G. Golo, O.V. Iftime, H. Zwart, and A.J. van der Schaft, *Tools for analysis of Dirac structures on Hilbert spaces*, Memorandum Faculteit TW 1729, Univesiteit Twente, Enschede (2004).

[Gol02] G. Golo, *Interconnection structures in port-based modelling: Tools for analysis and simulation*, Twente University Press, 2002, Ph.D. Thesis. Available at `http://doc.utwente.nl/`.

[ISG05] O.V. Iftime, A. Sandovici, and G. Golo, *Tools for analysis of dirac structures on banach spaces*, Decision and Control, 2005 and 2005 European Control Conference. CDC-ECC '05. 44th IEEE Conference on, Dec. 2005, pp. 3856–3861.

[JWBT05] A. A. Julius, J. C. Willems, M. N. Belur, and H. L. Trentelman, *The canonical controllers and regular interconnection*, Systems Control Lett. **54** (2005), no. 8, 787–797.

[KS07] Mikael Kurula and Olof Staffans, *A complete model of a finite-dimensional impedance-passive system*, Math. Control Signals Systems **19** (2007), no. 1, 23–63.

[KS08] ———, *Well-posed state/signal systems in continuous time*, Proceedings of the Mathematical Theory of Networks and Systems, 2008.

[KS09] Mikael Kurula and Olof J. Staffans, *Well-posed state/signal systems in continuous time*, To appear in Complex Analysis and Operator Theory, SpringerLink Online First version available with DOI 10.1007/s11785-009-0021-5, 2009.

[Kur10] Mikael Kurula, *On passive and conservative state/signal systems in continuous time*, To appear in Integral Equations and Operator Theory, SpringerLink Online First version available with DOI 10.1007/s00020-010-1787-6, 2010.

[KZvdS06] Mikael Kurula, Hans Zwart, and Arjan van der Schaft, *Composition of infinite-dimensional linear dirac-type structures*, Proceedings of the Mathematical Theory of Networks and Systems, 2006.

[KZvdSB09] Mikael Kurula, Hans Zwart, Arjan van der Schaft, and Jussi Behrndt, *Dirac structures and their composition on Hilbert spaces*, submitted, draft available at `http://users.abo.fi/mkurula/`, 2009.

[LGZM05] Y. Le Gorrec, H. Zwart, and B. Maschke, *Dirac structures and boundary control systems associated with skew-symmetric differential operators*, SIAM J. Control Optim. **44** (2005), no. 5, 1864–1892 (electronic).

[MM05] Alessandro Macchelli and Claudio Melchiorri, *Control by interconnection of mixed port Hamiltonian systems*, IEEE Trans. Automat. Control **50** (2005), no. 11, 1839–1844.

[MS06] Jarmo Malinen and Olof J. Staffans, *Conservative boundary control systems*, J. Differential Equations **231** (2006), no. 1, 290–312.

[MS07] ______, *Impedance passive and conservative boundary control systems*, Complex Anal. Oper. Theory **1** (2007), no. 2, 279–300.

[MSW06] Jarmo Malinen, Olof J. Staffans, and George Weiss, *When is a linear system conservative?*, Quart. Appl. Math. **64** (2006), no. 1, 61–91.

[MvdS00] Bernhard Maschke and Arjan van der Schaft, *Portcontrolled hamiltonian representation of distributed parameter systems*, Memorandum 1510, Faculty of Mathematical Sciences, University of Twente, The Netherlands, 2000.

[MvdS01] ______, *Fluid dynamical systems as hamiltonian boundary control systems*, Memorandum 1575, Faculty of Mathematical Sciences, University of Twente, The Netherlands, 2001.

[MvdS05] Bernhard Maschke and Arjan van der Schaft, *Compositional modelling of distributed-parameter systems*, Advanced topics in control systems theory, Lecture Notes in Control and Inform. Sci., vol. 311, Springer, London, 2005, pp. 115–154.

[Paz83] Amnon Pazy, *Semi-groups of linear operators and applications to partial differential equations*, Springer-Verlag, Berlin, 1983.

[PSDA99] A. Parsian and A. Shafei Deh Abad, *Dirac structures on Hilbert spaces*, Int. J. Math. Math. Sci. **22** (1999), no. 1, 97–108.

[PW98] Jan Willem Polderman and Jan C. Willems, *Introduction to mathematical systems theory: A behavioral approach*, Springer-Verlag, New York, 1998.

[Sta02a] Olof J. Staffans, *Passive and conservative continuous-time impedance and scattering systems. Part I: Well-posed systems*, Math. Control Signals Systems **15** (2002), 291–315.

[Sta02b] ______, *Passive and conservative infinite-dimensional impedance and scattering systems (from a personal point of view)*, Mathematical Systems Theory in Biology, Communication, Computation, and Finance (New York), IMA Volumes in Mathematics and its Applications, vol. 134, Springer-Verlag, 2002, pp. 375–414.

[Sta05] ______, *Well-posed linear systems*, Cambridge University Press, Cambridge and New York, 2005.

[Sta06] Olof J. Staffans, *Passive linear discrete time-invariant systems*, International Congress of Mathematicians. Vol. III, Eur. Math. Soc., Zürich, 2006, pp. 1367–1388.

[TW03] Marius Tucsnak and George Weiss, *How to get a conservative well-posed linear system out of thin air. Part II. Controllability and stability*, SIAM J. Control Optim. **42** (2003), 907–935.

[vdS00] A. J. van der Schaft, *L_2-gain and passivity techniques in nonlinear control*, Springer Communications and Control Engineering series, vol. 218, Springer-Verlag, London, 2000, 2nd revised and enlarged edition.

[Vil07] J. A. Villegas, *A port-hamiltonian approach to distributed parameter systems*, Twente University Press, 2007, Ph.D. Thesis. Available online at `http://doc.utwente.nl/`.

[Wil07] Jan C. Willems, *The behavioral approach to open and interconnected systems: modeling by tearing, zooming, and linking*, IEEE Control Syst. Mag. **27** (2007), no. 6, 46–99.

[WST01] George Weiss, Olof J. Staffans, and Marius Tucsnak, *Well-posed linear systems – a survey with emphasis on conservative systems*, Internat. J. Appl. Math. Comput. Sci. **11** (2001), 7–34.

ISBN 978-952-12-2410-2 (Hard copy)
ISBN 978-952-12-2418-8 (Electronic)
Painosalama Oy
Åbo, Finland, 2010