DE

INVENIENDO MAXIMO VEL MINIMO VALORE FUNCTIONIS

$$y = Aa^x + Bb^x + Cc^x + \cdots + Mm^x + Nn^x + P,$$

DISSERTATIO;

QUAM

CONSENTIENTE AMPL. FAC. PHIL. REG. ACAD. ABOËNSIS,

PRÆSIDE

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PRO GRADU PHILOSOPHICO

PUBLICO EXAMINI OFFERT

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STIPEND. ARCKENHOLTZ. WIBURGENSIS.

In Auditorio Majori die VII Maji MDCCCII.

Horis a. m. consuetis.

ABOÆ, TYPIS FRENCKELLIANIS.

15.

In legibus naturæ eruendis id præcipue Phyſico eſſe agendum judicamus, ut primum experimenta, mutatis rebus illa ſpectantibus, plura & accurata inſtituat. Deinde autem hæc inter ſe conferendo ipſam inveniet legem phænomenorum omnium intra limites eosdem, qui facta experimenta comprehendebant, contentorum. Vulgo tum æquationem aliquam aut algebraicam aut etiam tranſcendentem ſibi fingunt, cujus quantitates incognitas conferendo cum valoribus ope experimentorum inventis determinant.

Inter plures tales interpolando inſervientes æquationes ſæpe magno calculi compendio adhiberi poteſt hæc: $y = Aa^x + Bb^x + Cc^x + \text{- - -} + Mm^x + Nn^x + P$, in qua pro cognitis quibusdam valoribus o, x', $2x'$, $3x'$, $4x'$, &c., quantitatis x per experimenta dati habebuntur valores reſpondentes functionis y, &, ſi modo poſitivæ ſint quantitates a^x, b^x, c^x, - - - - m^x, n^x, non difficulter determinantur conſtantes A, a, B, b, C, c - - - - M, m, N, n, & P.

Determinatis omnibus quantitatibus constantibus, sæpenumero utile est scire, num pro aliquo certo valore quantitatis x maxima vel minima fiat y, nec non quantæ in hoc casu sint x & y. Methodo vulgari utriusque membri æquationis fluxio quæritur, & evanescens ponitur *), ut, facto in præsenti casu Log. hyp. $a = \alpha$, Log. hyp. $b = \beta$, Log. hyp. $c = \gamma$, ------ Log. hyp. $m = \mu$, & Log. hyp. $n = \nu$, habeatur $dy = A\alpha a^x dx + B\beta b^x dx + C\gamma c^x dx + ------ + M\mu m^x dx + N\nu n^x dx$, adeoque $\frac{dy}{dx} = A\alpha a^x + B\beta b^x + C\gamma c^x + -----$ $+ M\mu m^x + N\nu n^x = 0$.

Ex hac ultima æquatione determinanda est quantitas x, quæ in æquatione $y = Aa^x + Bb^x + Cc^x + ----- + Mm^x + Nn^x + P$ substituta dat maximum vel minimum valorem quantitatis y. Cum autem, etiamsi reales sint omnes quantitates constantes, non semper inveniri queat aliquis valor quantitatis x, qua æquationi $0 = A\alpha a^x + B\beta b^x + ----- + M\mu m^x + N\nu n^x$ satisfit, nec semper iste valor functionis y determinari potest.

Ut

*) Vide: *Institutiones calculi differentialis*, *Auctore* LEONH. EULERO, *P. II. Cap. X. pag.* 578, *sqq.*

Ut vero, quando fieri poteſt, ex hac æquatione eruatur x, omnes ejus termini in ſeriem ſunt evolvendi. Hoc modo invenitur:

$$A\alpha a^x = A\alpha \left(1 + \frac{\alpha x}{1} + \frac{\alpha^2 x^2}{1.2.} + \frac{\alpha^3 x^3}{1.2.3.} + \frac{\alpha^4 x^4}{1.2.3.4} + \&c.\right);$$

$$B\beta b^x = B\beta \left(1 + \frac{\beta x}{1} + \frac{\beta^2 x^2}{1.2.} + \frac{\beta^3 x^3}{1.2.3.} + \frac{\beta^4 x^4}{1.2.3.4.} + \&c.\right);$$

$$C\gamma c^x = C\gamma \left(1 + \frac{\gamma x}{1} + \frac{\gamma^2 x^2}{1.2.} + \frac{\gamma^3 x^3}{1.2.3.} + \frac{\gamma^4 x^4}{1.2.3.4.} + \&c.;\right.$$

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$$M\mu m^x = M\mu \left(1 + \frac{\mu x}{1} + \frac{\mu^2 x^2}{1.2.} + \frac{\mu^3 x^3}{1.2.3.} + \frac{\mu^4 x^4}{1.2.3.4.} + \&c.\right);$$

$$N\nu n^x = N\nu \left(1 + \frac{\nu x}{1} + \frac{\nu^2 x^2}{1.2.} + \frac{\nu^3 x^3}{1.2.3.} + \frac{\nu^4 x^4}{1.2.3.4.} + \&c.\right):$$

unde colligitur eſſe

$$\begin{array}{llllll}
0 = + A\alpha + A\alpha^2 & x + \frac{A\alpha^3}{1.2.} & x^2 + \frac{A\alpha^4}{1.2.3} & x^3 + \frac{A\alpha^5}{1.2.3.4.} & x^4 + \&c. \\
\quad + B\beta + B\beta^2 & + \frac{B\beta^3}{1.2.} & + \frac{B\beta^4}{1.2.3} & + \frac{B\beta^5}{1.2.3.4.} & \\
\quad + C\gamma + C\gamma^2 & + \frac{C\gamma^3}{1.2.} & + \frac{C\gamma^4}{1.2.3.} & + \frac{C\gamma^5}{1.2.3.4.} & \\
\quad - - - - & - - - & - - - & - - - - & \\
\quad + M\mu + M\mu^2 & + \frac{M\mu^3}{1.2.} & + \frac{M\mu^4}{1.2.3.} & + \frac{M\mu^5}{1.2.3.4.} & \\
\quad + N\nu + N\nu^2 & + \frac{N\nu^3}{1.2.} & + \frac{N\nu^4}{1.2.3.} & + \frac{N\nu^5}{1.2.3.4.} &
\end{array}$$

Fa-

Factis itaque

$$A^{0} = A\alpha + B\beta + C\gamma + \text{- - - - - -} + M\mu + N\nu,$$
$$A' = A\alpha^{2} + B\beta^{2} + C\gamma^{2} + \text{- - - - - -} + M\mu^{2} + N\nu^{2},$$
$$A'' = \frac{A\alpha^{3}}{1.2.} + \frac{B\beta^{3}}{1.2.} + \frac{C\gamma^{3}}{1.2.} + \text{- - - - - - -} + \frac{M\mu^{3}}{1.2.} + \frac{N\nu^{3}}{1.2.},$$
$$A''' = \frac{A\alpha^{4}}{1.2.3.} + \frac{B\beta^{4}}{1.2.3.} + \frac{C\gamma^{4}}{1.2.3} + \text{- - - - - - -} + \frac{M\mu^{4}}{1.2.3.} + \frac{N\nu^{4}}{1.2.3.},$$
$$A^{IV} = \frac{A\alpha^{5}}{1.2.3.4.} + \frac{B\beta^{5}}{1.2.3.4.} + \frac{C\gamma^{5}}{1.2.3.4.} + \text{- -} + \frac{M\mu^{5}}{1.2.3.4.} + \frac{N\nu^{5}}{1.2.3.4.},$$

&c. = &c.

erit $0 = A^{0} + A'x + A''x^{2} + A'''x^{3} + A^{IV}x^{4} + \&c.$

Ut hanc ſeriem revertendo inveniamus aliam, quæ determinet quantitatem x, & cujus termini conſtantibus & datis quantitatibus progrediantur, fingatur esſe

$$x = M'A^{0} + N'A^{0\,2} + P'A^{0\,3} + Q'A^{0\,4} + \&c.,$$

ut habeantur

$$A'x = M'A'A^{0} + N'A'A^{0\,2} + P'A'A^{0\,3} + Q'A'A^{0\,4} + \&c.$$
$$A''x^{2} = M'^{2}A''A^{0\,2} + 2M'N'A''A^{0\,3} + \left.\begin{matrix}2M'P'A'' \\ + N'^{2}A''\end{matrix}\right\} A^{0\,4} + \&c.$$
$$A''x^{3} = M'^{3}A'''A^{0\,3} + 3M'^{2}N'A'''A^{0\,4} + \&c.$$
$$A^{IV}x^{4} = M'^{4}A^{IV}A^{0\,3} + \&c.;$$

quibus omnibus collectis & in ſerie revertenda ſubſtitutis, prodit

o =

$$0 = \left.\begin{matrix}+1\\+M'A'\end{matrix}\right] A^{\circ} \left.\begin{matrix}+N'A'\\+M'^2A''\end{matrix}\right] A^{\circ 2} \left.\begin{matrix}+P'A'\\+2M'N'A''\\+M'^3A'''\end{matrix}\right\} A^{\circ 3} \left.\begin{matrix}+Q'A'\\+2M'P'A''\\+N'^2A''\\+3M'^2N'A'''\\+M'^4A^{IV}\end{matrix}\right\} A^{\circ 4} + \&c.$$

In determinandis quantitatibus adhuc incognitis M', N', P', Q', &c., obfervandum eft, quum æquationis hujus membrum unum fit $= 0$, quascunque etiam coëfficientes quantitatis A° in altero membro evanefcentes esfe ponendas. Hac autem ratione obtinentur fequentes æquationes:

$1 + M'A' = 0;$
$N'A' + M'^2A'' = 0;$
$P'A' + 2M'N'A'' + M'^3A''' = 0;$
$Q'A' + 2M'P'A'' + N'^2A'' + 3M'^2N'A''' + M'^4A^{IV} = 0;$
$R'A' + 2M'Q'A'' + 2N'P'A'' + 3M'^2P'A''' +$
$3M'N'^2A''' + 4M'^3N'A^{IV} + M'^5A^{V} = 0;$
&c. = &c.

unde fequentes inveniuntur valores

$M' = -\frac{1}{A'};$

$N' = -\frac{A''}{A'^3};$

$P' = -\frac{2A''^2}{A'^5} + \frac{A'''}{A'^4};$

 $Q' =$

$$Q' = -\frac{5A''^3}{A'^7} + \frac{5A''A'''}{A'^6} - \frac{A^{IV}}{A'^5};$$

$$R' = -\frac{14A''^4}{A'^9} + \frac{21A''^2A'''}{A'^8} - \frac{6A''A^{IV}}{A'^7} - \frac{3A'''^2}{A'^7} + \frac{A^V}{A'^6};$$

&c. = &c.

Si hi jam hoc modo determinati valores ſubſtituuntur in æquatione antea asſumta, provenit æquatio, cujus ope computari poteſt ille valor quantitatis x, qui functionem y minimam vel maximam reddit. Habetur enim tum

$$x = -\frac{1}{A'}A^0 - \frac{A''}{A'^3}A^{02} - \left.\begin{matrix} \frac{2A''^2}{A'^5} \\ + \frac{A'''}{A'^4} \end{matrix}\right\} A^{03} - \left.\begin{matrix} \frac{5A''^3}{A'^7} \\ + \frac{5A''A'''}{A'^6} \\ - \frac{A^{IV}}{A'^5} \end{matrix}\right\} A^{04} + \&c.$$

Sic quidem generaliter iſte valor quantitatis x determinatur, qui functionem y maximam vel minimam, ſi fieri posſit, reddit. In caſibus autem quibusdam ſpecialioribus prolixo hoc calculo uti opus non eſt. Quando nimirum in æquatione $0 = A\alpha a^x + B\beta b^x + - - - - + M\mu m^x + N\nu n^x$ omnia producta $A\alpha a^x$, $B\beta b^x$, &c., ſunt vel poſitiva vel negativa, videndum eſt, an omnes quantitates $a, b, c, - - - m$ & n ſint

sint vel majores vel minores unitate, adeoque earum Logarithmi α, β, γ, - - - - μ & ν vel positivi vel negativi. Si hi omnes sunt positivi, summa productorum $+ (A\alpha a^x + B\beta b^x + \text{- - - -} M\mu m^x + N\nu n^x)$ evanescere non potest, nisi sumatur $x = -\infty$, qui valor reddit minimam $y = \frac{1}{\infty} + P = P$. Si autem omnes Logarithmi α, β, γ, - - - - μ & ν sunt negativi, summa productorum $- (A\alpha a^x + B\beta b^x + \text{- - -} + M\mu m^x + N\nu n^x)$ non evanescit nisi posita $x = \infty$, qui valor reddit functionem $y = \infty$, unde mox apparet, maximam vel minimam y secundum receptam loquendi rationem in hoc casu non dari.

Hujus rei exemplum præbet æquatio $y = Aa^x + P$, ubi primo sumatur $a > 1$. Pro casu maximæ vel minimæ y habetur æquatio $\frac{dy}{dx} = A\alpha a^x = 0$, quæ, si realis est, dat $x = -\infty$, adeoque, facta substitutione, $y = Aa^{-\infty} + P = P$. Si itaque est A quantitas positiva, invenitur minima $y = P$. Sed pro A negativa evadit maxima $y = P$. Si autem est $a < 1$, æquatio $A\alpha a^x = 0$ dat $x = \infty$, quæ reddit $y = Aa^\infty + P = \pm \infty$. Idem quoque ope seriei allatæ invenitur. Est enim in hoc exemplo $A^0 = A\alpha$; $A' = A\alpha^2$;

$A'' =$

$A'' = \frac{A\alpha^3}{1.2.}$; $A''' = \frac{A\alpha^4}{1.2.3.}$; $A^{IV} = \frac{A\alpha^5}{1.2.3.4.}$; $A^{V} = \frac{A\alpha^6}{1.2.3.4.5.}$; &c., atque $M' = -\frac{1}{A\alpha^2}$; $N' = -\frac{1}{2A^2\alpha^3}$; $P' = -\frac{1}{3A^3\alpha^4}$; $Q' = -\frac{1}{4A^4\alpha^5}$; $R' = -\frac{1}{5A^5\alpha^6}$, &c.; unde pro casu minimæ y invenitur $x = -\frac{1}{\alpha}(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} +$ &c.). Hujus vero seriei summa est $= \infty$ *), adeoque si α est positivus, pro maxima vel minima y erit $x = -\frac{\infty}{\alpha}$, & $y = P$. Si vero est α negativus, habetur $x = \frac{\infty}{\alpha}$, & $y = \pm\infty$, ut antea.

Præterea quoque interdum, quando sunt $C = 0$, --- $M = 0$, $N = 0$, ut sit $0 = A\alpha a^x + B\beta b^x$, sine adhibita serie inveniri potest maxima vel minima y. Est enim $\left(\frac{a}{b}\right)^x = -\frac{B\beta}{A\alpha}$, adeoque si quantitas $-\frac{B\beta}{A\alpha}$ est positiva, invenitur x (Log. a. — Log. b) = Log. ($-B\beta$) — Log. $A\alpha$, seu = Log. $B\beta$ — Log. ($-A\alpha$), & $x =$

*) Cfr. *Introduct. in Analysin infinitorum*, *Auct.* LEONH. EULERO, *Lausannæ* 1748, *Tom. I. pag.* 229.

$$x = \frac{\text{Log.}(-B\beta) - \text{Log.}\,A\alpha}{\text{Log.}\,a - \text{Log.}\,b}, \text{ seu } x = \frac{\text{Log.}\,B\beta - \text{Log.}(-A\alpha)}{\text{Log.}\,a - \text{Log.}\,b}.$$

Hujus casus exemplum præbet æquatio $y = 2^x + 0{,}3^x + 1$. Est enim

$A = 1$;	$B = 1$;
$a = 2$;	$b = 0{,}3$;
$\alpha = 0{,}6931472$;	$\beta = -1{,}2039728$.

Invenitur itaque $A\alpha = 0{,}6931472$, & $-B\beta = 1{,}2039728$, adeoque $x = \frac{\text{Log. } 1{,}2039728 - \text{Log. } 0{,}6931472}{\text{Log. } 2 - \text{Log. } 0{,}3} = \frac{0{,}2397912}{0{,}8239087}$; quare erit Log. $x = 0{,}4639542 - 1$, & $x = 0{,}2910410$. Hoc autem valore substituto eruitur minima $y = 2{,}9279253$.

Si ulterius omnia producta $A\alpha a^x$, $B\beta b^x$, &c., ita sunt vel positiva vel negativa, ut unus vel plures, non autem omnes, Logarithmi α, β, γ, &c., sint positivi, æquatio $0 = \pm (A\alpha a^x + B\beta b^x + \text{- - - -} + M\mu m^x + N\nu n^x)$ est imaginaria. Factis ex. gr. α & β negativis atque μ & ν positivis, pro aliquo finito valore quantitatis x non evanescit membrum posterius æquationis; pro $x = \infty$ autem fit $0 = \pm (\text{- - - -} M\mu + N\nu)\,\infty$, nec non pro $x = -\infty$, $0 = \pm (A\alpha + B\beta + \text{- - - -})\,\infty$, quarum æquationum utraque est impossibilis. In talibus itaque casibus nullus valor quantitati x da-

x dari poteſt, qui functionem y maximam vel minimam redderet.

Generaliter quidem, quando inventus eſt aliquis maximus vel minimus valor functionis y, judicare licet, an maximus vel minimus ille ſit, ſi majores & minores valores quantitatis x ſucceſſive ſubſtituuntur. Directe autem illud e ſignis quantitatum in æquatione datarum facilius cognoſcitur, ſi facto $ddx = 0$ valor functionis $\frac{ddy}{dx^2}$ eruitur. Nam pro $\frac{ddy}{dx^2} = v$ invenitur minima y, ſed pro $\frac{ddy}{dx^2} = - v$ maxima, ſi v eſt quantitas quædam affirmativa *). Habetur autem ex æquatione noſtra generali, $\frac{dy}{dx} = A\alpha a^x + B\beta b^x + C\gamma c^x + - - - - + M\mu m^x + N\nu n^x = 0$, unde erit $N\nu n^x = - A\alpha a^x - B\beta b^x - C\gamma c^x - M\mu m^x$. Hac facta ſubſtitutione in æquatione $\frac{ddy}{dx^2} = A\alpha^2 a^x + B\beta^2 b^x + C\gamma^2 c^x + - - - + M\mu^2 m^x + N\nu^2 n^x$, invenitur $\frac{ddy}{dx^2} = A\alpha(\alpha - \nu) a^x + B\beta(\beta - \nu) b^x + C\gamma(\gamma - \nu) c^x + - - - - + M\mu(\mu - \nu) m^x$, ubi quantitates conſtantes determinant an poſitivus vel negativus ſit valor $\frac{ddy}{dx^2}$, adeoque an vel minima vel maxima fiat functio y.

*) Cfr. L. Euleri *Inſtitut. Calculi differentialis*, pag. 583.