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On the intersection of Padovan, Perrin sequences and Pell, Pell-Lucas sequences

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Abstract

In this paper, we find all the Padovan and Perrin numbers which are Pell or Pell-Lucas numbers.

Keywords: Padovan numbers, Perrin numbers, Pell numbers, Pell-Lucas numbers, Linear form in logarithms, reduction method.

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1. Introduction

Let (u_n) and (v_n) be two linear recurrent sequences. The problem of finding the common terms of (u_n) and (v_n) was treated in [3, 4, 6, 7, 9]. They proved, under some assumption, that the Diophantine equation

$$u_n = v_m$$

has only finitely many integer solutions (m, n). The aim of this paper is to study the common terms of Padovan, Perrin, Pell and Pell-Lucas sequences that we will recall below.

Let $\{P_n\}_{n\geq 0}$ be the Pell sequence given by

$$P_{m+2} = 2P_{m+1} + P_m,$$

for $m \geq 0$, where $P_0 = 0$ and $P_1 = 1$. This is the sequence A000129 in the OEIS and its first few terms are

$$0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, 13860, 33461, 80782, 195025, \ldots$$

We let $\{Q_m\}_{m\geq 0}$ be the companion Lucas sequence of the Pell sequence also called the sequence of Pell–Lucas numbers. It starts with $Q_0 = 2$, $Q_1 = 2$ and obeys the same recurrence relation

$$Q_{m+2} = 2Q_{m+1} + Q_m, \quad \text{for all} \quad m \geq 0$$

as the Pell sequence. This is the sequence A002203 in the OEIS and its first few terms are

$$2, 2, 6, 14, 34, 82, 198, 478, 1154, 2786, 6726, 16238, 39202, 94642, 228486, 551614, \ldots$$

The Padovan sequence $\{\mathcal{P}_n\}_{n\geq 0}$ is defined by

$$\mathcal{P}_{n+3} = \mathcal{P}_{n+1} + \mathcal{P}_n,$$

for $n \geq 0$, where $\mathcal{P}_0 = 0$ and $\mathcal{P}_1 = \mathcal{P}_2 = 1$. This is the sequence A000931 in the OEIS. A few terms of this sequence are

$$0, 1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, 65, 86, 114, 151, 200, \ldots$$

Let $\{E_n\}_{n\geq 0}$ be the Perrin sequence given by

$$E_{n+3} = E_{n+1} + E_n,$$

for $n \geq 0$, where $E_0 = 3$, $E_1 = 0$ and $E_2 = 2$. Its first few terms are

$$3, 0, 2, 3, 2, 5, 5, 7, 10, 12, 17, 22, 29, 39, 51, 68, 90, 119, 158, 209, 277, \ldots$$

It is the sequence A001608 in the OEIS.

Our proofs of our main theorems are mainly based on linear forms in logarithms of algebraic numbers and a reduction algorithm originally introduced by Baker and Davenport in [1]. Here, we use a version due to de Weger [2]. We organize this paper as follows. In Section 2, we recall the important results that will be used to prove our main results. Sections 4–6 are devoted to the statements and the proofs of our main results.

2. The tools

In this section, we recall all the tools that we will use to prove our main results.

2.1. Linear forms in logarithms

We need some results from the theory of lower bounds for nonzero linear forms in logarithms of algebraic numbers. We start by recalling Theorem 9.4 of [8], which is a modified version of a result of Matveev [5]. Let $\mathbb{L}$ be an algebraic number field of degree $d_{\mathbb{L}}$. Let $\eta_1, \eta_2, \ldots, \eta_l \in \mathbb{L}$ not 0 or 1 and $d_1, \ldots, d_l$ be nonzero integers. We put

$$D = \max\{|d_1|, \ldots, |d_l|\},$$

and

$$\Gamma = \prod_{i=1}^{l} \eta_i^{d_i} - 1.$$

Let $A_1, \ldots, A_l$ be positive integers such that

$$A_j \geq h'(\eta_j) := \max\{d_{\mathbb{L}} h(\eta_j), |\log \eta_j|, 0.16\}, \quad \text{for} \quad j = 1, \ldots l,$$

where for an algebraic number η of minimal polynomial

$$f(X) = a_0(X - \eta^{(1)}) \cdots (X - \eta^{(k)}) \in \mathbb{Z}[X]$$

over the integers with positive a_0, we write $h(\eta)$ for its Weil height given by

$$h(\eta) = \frac{1}{k}\left(\log a_0 + \sum_{j=1}^{k} \max\{0, \log|\eta^{(j)}|\}\right).$$

The following consequence of Matveev's theorem is Theorem 9.4 in [8].

Theorem 2.1. *If $\Gamma \neq 0$ and $\mathbb{L} \subseteq \mathbb{R}$, then*

$$\log|\Gamma| > -1.4 \cdot 30^{l+3} l^{4.5} d_{\mathbb{L}}^2 (1 + \log d_{\mathbb{L}})(1 + \log D) A_1 A_2 \cdots A_l.$$

2.2. The de Weger reduction

Here, we present a variant of the reduction method of Baker and Davenport due to de Weger [2]).

Let $\vartheta_1, \vartheta_2, \beta \in \mathbb{R}$ be given, and let $x_1, x_2 \in \mathbb{Z}$ be unknowns. Let

$$\Lambda = \beta + x_1\vartheta_1 + x_2\vartheta_2. \tag{2.1}$$

Let c, δ be positive constants. Set $X = \max\{|x_1|, |x_2|\}$. Let X_0, Y be positive. Assume that

$$|\Lambda| < c \cdot \exp(-\delta \cdot Y), \tag{2.2}$$

$$Y \leq X \leq X_0. \tag{2.3}$$

When $\beta = 0$ in (2.1), we get

$$\Lambda = x_1\vartheta_1 + x_2\vartheta_2.$$

Put $\vartheta = -\vartheta_1/\vartheta_2$. We assume that x_1 and x_2 are coprime. Let the continued fraction expansion of ϑ be given by

$$[a_0, a_1, a_2, \ldots],$$

and let the kth convergent of ϑ be p_k/q_k for $k = 0, 1, 2, \ldots$. We may assume without loss of generality that $|\vartheta_1| < |\vartheta_2|$ and that $x_1 > 0$. We have the following results.

Lemma 2.2 (See Lemma 3.2 in [2]). *Let*

$$A = \max_{0 \le k \le Y_0} a_{k+1},$$

where

$$Y_0 = -1 + \frac{\log(\sqrt{5}X_0 + 1)}{\log\left(\frac{1+\sqrt{5}}{2}\right)}.$$

If (2.2) *and* (2.3) *hold for* x_1, x_2 *and* $\beta = 0$, *then*

$$Y < \frac{1}{\delta} \log\left(\frac{c(A+2)X_0}{|\vartheta_2|}\right).$$

When $\beta \neq 0$ in (2.1), put $\vartheta = -\vartheta_1/\vartheta_2$ and $\psi = \beta/\vartheta_2$. Then, we have

$$\frac{\Lambda}{\vartheta_2} = \psi - x_1\vartheta + x_2.$$

Let p/q be a convergent of ϑ with $q > X_0$. For a real number x, we let $\|x\| = \min\{|x - n|, n \in \mathbb{Z}\}$ be the distance from x to the nearest integer. We have the following result.

Lemma 2.3 (See Lemma 3.3 in [2]). *Suppose that*

$$\| q\psi \| > \frac{2X_0}{q}.$$

Then, the solutions of (2.2) *and* (2.3) *satisfy*

$$Y < \frac{1}{\delta} \log\left(\frac{q^2 c}{|\vartheta_2| X_0}\right).$$

2.3. Properties of Padovan and Perrin sequences

In this subsection, we recall some facts and properties of the Padovan and the Perrin sequences which will be used later.

The characteristic equation

$$x^3 - x - 1 = 0,$$

has roots $\alpha, \beta, \gamma = \overline{\beta}$, where

$$\alpha = \frac{r_1 + r_2}{6}, \quad \beta = \frac{-r_1 - r_2 + i\sqrt{3}(r_1 - r_2)}{12},$$

and

$$r_1 = \sqrt[3]{108 + 12\sqrt{69}} \text{ and } r_2 = \sqrt[3]{108 - 12\sqrt{69}}.$$

Let

$$\begin{aligned} c_\alpha &= \frac{(1-\beta)(1-\gamma)}{(\alpha-\beta)(\alpha-\gamma)} = \frac{1+\alpha}{-\alpha^2+3\alpha+1}, \\ c_\beta &= \frac{(1-\alpha)(1-\gamma)}{(\beta-\alpha)(\beta-\gamma)} = \frac{1+\beta}{-\beta^2+3\beta+1}, \\ c_\gamma &= \frac{(1-\alpha)(1-\beta)}{(\gamma-\alpha)(\gamma-\beta)} = \frac{1+\gamma}{-\gamma^2+3\gamma+1} = \overline{c_\beta}. \end{aligned}$$

The Binet's formula of $\mathcal{P}_n$ is

$$\mathcal{P}_n = c_\alpha \alpha^n + c_\beta \beta^n + c_\gamma \gamma^n, \text{ for all } n \geq 0, \tag{2.4}$$

and that of E_n is

$$E_n = \alpha^n + \beta^n + \gamma^n, \text{ for all } n \geq 0. \tag{2.5}$$

Numerically, we have

$$\begin{aligned} &1.32 < \alpha < 1.33, \\ &0.86 < |\beta| = |\gamma| < 0.87, \\ &0.72 < c_\alpha < 0.73, \\ &0.24 < |c_\beta| = |c_\gamma| < 0.25. \end{aligned}$$

It is easy to check that

$$|\beta| = |\gamma| = \alpha^{-1/2}.$$

Further, using induction, we can prove that

$$\alpha^{n-2} \leq \mathcal{P}_n \leq \alpha^{n-1}, \quad \text{holds for all } n \geq 4 \tag{2.6}$$

and

$$\alpha^{n-2} \leq E_n \leq \alpha^{n+1}, \quad \text{holds for all } n \geq 2. \tag{2.7}$$

2.4. Pell and Pell-Lucas sequence

Let $\delta = 1+\sqrt{2}$ and $\overline{\delta} := 1-\sqrt{2}$ be the roots of the characteristic equation x^2-2x-1 of P_m and Q_m. The Binet formula of P_m is given by

$$P_m = \frac{\delta^m - \overline{\delta}^m}{2\sqrt{2}}, \quad \text{for all } m \geq 0, \tag{2.8}$$

$$Q_m = \delta^m + \overline{\delta}^m, \quad \text{for all } m \geq 0. \tag{2.9}$$

Moreover, we have

$$\delta^{m-2} < P_m < \delta^{m-1} \tag{2.10}$$

and

$$\delta^{m-1} < Q_m < \delta^{m+1}. \tag{2.11}$$

3. Padovan numbers which are Pell numbers

In this section, we will prove our first main result, which is the following.

Theorem 3.1. *The only solutions of the Diophantine equation*

$$\mathcal{P}_n = P_m \tag{3.1}$$

in positive integers m and n are

$$(n, m) \in \{(0,0), (1,1), (2,1), (3,1), (4,2), (5,2), (8,3), (11,4)\}.$$

Hence, $\mathcal{P} \cap P = \{0, 1, 2, 5, 12\}$.

Proof. A quick computation with Maple reveals that the solutions of the Diophantine equation (3.1) in the interval $[0, 60]$ are the solutions cited in Theorem 3.1.

From now, assuming that $n > 60$, then by (2.6) and (2.10), we have

$$\alpha^{n-2} < \delta^{m-1} \quad \text{and} \quad \delta^{m-2} < \alpha^{n-1}.$$

Thus, we get

$$(n-2)c_1 + 1 < m < (n-1)c_1 + 2, \quad \text{where } c_1 := \log\alpha/\log\delta.$$

Particularly, we have $n < 4m$. So to solve equation (3.1), it suffices to get a good upper bound on m.

Equation (3.1) can be expressed as

$$c_\alpha \alpha^n - \frac{\delta^m}{2\sqrt{2}} = -c_\beta \beta^n - c_\gamma \gamma^n - \frac{\overline{\delta}^m}{2\sqrt{2}},$$

by using (2.4) and (2.8). Thus, we get

$$\left| c_\alpha \alpha^n - \frac{\delta^m}{2\sqrt{2}} \right| = \left| c_\beta \beta^n + c_\gamma \gamma^n + \frac{\overline{\delta}^m}{2\sqrt{2}} \right| < 0.85.$$

Multiplying through by $2\sqrt{2}\delta^{-m}$, we obtain

$$\left| (c_\alpha 2\sqrt{2})\alpha^n \delta^{-m} - 1 \right| < 2.41\delta^{-m}. \tag{3.2}$$

Now, we apply Matveev's theorem by choosing

$$\Lambda_1 = 2\sqrt{2}c_\alpha \alpha^n \delta^{-m} - 1$$

and

$$\eta_1 := 2\sqrt{2}c_\alpha, \quad \eta_2 := \alpha, \quad \eta_3 := \delta, \quad b_1 := 1, \quad b_2 := n, \quad b_3 := -m.$$

The algebraic numbers η_1, η_2 and η_3 belong to $\mathbb{K} := \mathbb{Q}(\alpha, \delta)$ for which $d_{\mathbb{K}} = 6$. Since $n < 4m$, therefore we can take $D := 4m = \max\{1, m, n\}$. Furthermore, we have

$$h(\eta_2) = \frac{\log \alpha}{3} \quad \text{and} \quad h(\eta_3) = \frac{\log \delta}{2},$$

thus, we can take

$$\max\{6h(\eta_2), |\log \eta_2|, 0.16\} < 0.58 := A_2$$

and

$$\max\{6h(\eta_3), |\log \eta_3|, 0.16\} = 2.65 := A_3.$$

On the other hand, the conjugates of η_1 are $\pm 2\sqrt{2}c_\alpha$, $\pm 2\sqrt{2}c_\beta$ and $\pm 2\sqrt{2}c_\gamma$, so the minimal polynomial of η_1 is

$$(x^2 - 8c_\alpha^2)(x^2 - 8c_\beta^2)(x^2 - 8c_\gamma^2) = \frac{529x^6 - 2024x^4 - 640x^2 - 512}{529}.$$

Since $2\sqrt{2}c_\alpha > 1$ and $\left|2\sqrt{2}c_\beta\right| = \left|2\sqrt{2}c_\gamma\right| < 1$, then we get

$$h(\eta_1) = \frac{\log 529 + 2\log(2\sqrt{2}c_\alpha)}{6}.$$

So, we can take

$$\max\{6h(\eta_1), |\log \eta_1|, 0.16\} < 7.8 := A_1.$$

To apply Matveev's theorem, we still need to prove that $\Lambda_1 \neq 0$. Assume the contrary, i.e. $\Lambda_1 = 0$. So, we get

$$\delta^m = 2\sqrt{2}c_\alpha \alpha^n.$$

Conjugating the above relation using the $\mathbb{Q}$-automorphism of Galois σ defined by $\sigma = (\alpha\beta)$ and taking the absolute value we obtain

$$1 < \delta^m = 2\sqrt{2}\,|c_\beta|\,|\beta|^n < 1,$$

which is a contradiction. Thus $\Lambda_1 \neq 0$.

Matveev's theorem tells us that

$$\begin{aligned}\log|\Lambda_1| &> -1.4 \times 30^6 \times 3^{4.5} \times 6^2(1 + \log 6)(1 + \log 4m) \times 7.8 \times 0.58 \times 2.65\\ &> -1.8 \times 10^{14} \times (1 + \log 4m).\end{aligned}$$

The last inequality together with (3.2) leads to

$$m < 1.99 \times 10^{14}(1 + \log 4m).$$

Thus, we obtain

$$m < 7.52 \times 10^{15}. \tag{3.3}$$

Now, we will use Lemma 2.3 to reduce the upper bound (3.3) on m.

Define

$$\Gamma_1 = n \log \alpha - m \log \delta + \log(2\sqrt{2}c_\alpha).$$

Clearly, we have $e^{\Gamma_1} - 1 = \Lambda_1$. Since $\Lambda_1 \neq 0$, then $\Gamma_1 \neq 0$. If $\Gamma_1 > 0$ the we get

$$0 < \Gamma_1 < e^{\Gamma_1} - 1 = \left|e^{\Gamma_1} - 1\right| = |\Lambda_1| < 2.41\delta^{-m}.$$

If $\Gamma_1 < 0$, so we have $1 - e^{\Gamma_1} = \left|e^{\Gamma_1} - 1\right| = |\Lambda_1| < 1/2$, because $n > 60$. Then $e^{|\Gamma_1|} < 2$. Thus, one can see that

$$0 < |\Gamma_1| < e^{|\Gamma_1|} - 1 = e^{|\Gamma_1|}\,|\Lambda_1| < 4.82\delta^{-m}.$$

From both cases, we deduce that

$$0 < \left|n(-\log \alpha) + m \log \delta - \log(2\sqrt{2}c_\alpha)\right| < 4.82 \exp(-0.88 \times m).$$

The inequality (3.3) implies that we can take $X_0 := 3.01 \times 10^{16}$. Furthermore, we can choose

$$c := 4.82, \quad \delta := 0.88, \quad \psi := -\frac{\log(2\sqrt{2}c_\alpha)}{\log \delta},$$

$$\vartheta := \frac{\log \alpha}{\log \delta}, \quad \vartheta_1 := -\log \alpha, \quad \vartheta_2 := \log \delta, \quad \beta := -\log(2\sqrt{2}c_\alpha).$$

With the help of Maple, we find that

$$q_{29} = 386003278073423723$$

satisfies the hypotheses of Lemma 2.3. Furthermore, Lemma 2.3 tells us

$$m < \frac{1}{0.88} \log\left(\frac{386003278073423723^2 \times 4.82}{\log \delta \times 3.01 \times 10^{16}}\right) \leq 57.$$

This contradicts the assumption that $n > 60$. Therefore, the theorem is proved. $\square$

4. Padovan numbers which are Pell-Lucas numbers

Our second result will be stated and proved in this section.

Theorem 4.1. *The only solutions of the Diophantine equation*

$$\mathcal{P}_n = Q_m \tag{4.1}$$

in positive integers m and n are

$$(n, m) \in \{(4, 0), (4, 1), (5, 0), (5, 1)\}.$$

Hence, we deduce that $\mathcal{P} \cap Q = \{2\}$.

Proof. A quick computation with Maple reveals that the solutions of the Diophantine equation (4.1) in the interval $[0, 60]$ are those cited in Theorem 4.1.

From now, we suppose that $n > 60$, then by (2.6) and (2.11), we have

$$\alpha^{n-2} < \delta^{m+1} \quad \text{and} \quad \delta^{m-1} < \alpha^{n-1}.$$

Thus, we get

$$(n-2)c_1 - 1 < m < (n-1)c_1 + 1, \quad \text{where } c_1 := \log \alpha / \log \delta.$$

Particularly, we have $n < 4m$. So, to solve equation (4.1), we will determine a good upper bound on m.

By using (2.4) and (2.9), equation (4.1) can be rewritten into the form

$$c_\alpha \alpha^n - \delta^m = -c_\beta \beta^n - c_\gamma \gamma^n - \overline{\delta}^m$$

So, we obtain

$$|c_\alpha \alpha^n - \delta^m| \leq 2\,|c_\beta \beta^n| + 1 < 1.5.$$

Multiplying both sides by δ^{-m}, we get

$$\left|c_\alpha \alpha^n \delta^{-m} - 1\right| < 1.5\delta^{-m}. \tag{4.2}$$

Now, we will apply Matveev's theorem to

$$\Lambda_2 = c_\alpha \alpha^n \delta^{-m} - 1$$

by taking

$$\eta_1 := c_\alpha, \quad \eta_2 := \alpha, \quad \eta_3 := \delta, \quad b_1 := 1, \quad b_2 := n, \quad b_3 := -m.$$

The algebraic numbers η_1, η_2 and η_3 belong to $\mathbb{K} := \mathbb{Q}(\alpha, \delta)$ with $d_{\mathbb{K}} = 6$. As above, we take

$$D = 4m, \;\; A_2 = 0.58, \;\; A_3 = 2.65.$$

On the other hand, the minimal polynomial of c_α is

$$23x^3 - 23x^2 - 6x - 1,$$

which has roots c_α, c_β and c_γ. Since $c_\alpha < 1$ and $|c_\beta| = |c_\gamma| < 1$, then we get

$$h(\eta_1) = \frac{\log 23}{3}.$$

So, we can take

$$\max\{6h(\eta_1), |\log \eta_1|, 0.16\} < 6.28 := A_1.$$

To apply Matveev's theorem, we will prove that $\Lambda_2 \neq 0$. Suppose the contrary, i.e $\Lambda_2 = 0$. Thus, we get

$$\delta^m = c_\alpha \alpha^n.$$

Conjugating the above relation using the $\mathbb{Q}$-automorphism of Galois σ defined by $\sigma = (\alpha\beta)$ and taking the absolute value, we obtain

$$1 < \delta^m = |c_\beta| \, |\beta|^n < 1,$$

which is a contradiction. Thus, we deduce that $\Lambda_2 \neq 0$.

We use Matveev's theorem to obtain

$$\begin{aligned}\log|\Lambda_2| &> -1.4 \times 30^6 \times 3^{4.5} \times 6^2(1+\log 6)(1+\log 4m) \times 6.28 \times 0.58 \times 2.65\\ &> -1.39 \times 10^{14}(1+\log 4m).\end{aligned}$$

The last inequality together with (4.2) leads to

$$m < 1.58 \times 10^{14}(1+\log 4m).$$

Thus, we obtain

$$m < 6.05 \times 10^{15}. \tag{4.3}$$

Now, we will use Lemma 2.3 to reduce the upper bound (4.3) on m.

Putting

$$\Gamma_2 = n\log\alpha - m\log\delta + \log(c_\alpha),$$

we proceed like in Section 3 to obtain

$$0 < |n(-\log\alpha) + m\log\delta - \log(c_\alpha)| < 3\exp(-0.88 \times m).$$

Using inequality (4.3), we take $X_0 := 2.42 \times 10^{16}$. Moreover, we choose

$$c := 3, \quad \delta := 0.88, \quad \psi := -\frac{\log(c_\alpha)}{\log\delta},$$

$$\vartheta := \frac{\log\alpha}{\log\delta}, \quad \vartheta_1 := -\log\alpha, \quad \vartheta_2 := \log\delta, \quad \beta := -\log(c_\alpha).$$

We use Maple to find that

$$q_{29} = 38600327807734237233$$

satisfies the hypotheses of Lemma 2.3. Therefore, we get

$$m < \frac{1}{0.88}\log\left(\frac{38600327807734237233^2 \times 3}{\log\delta \times 2.42 \times 10^{16}}\right) \leq 56.$$

This contradicts the assumption that $n > 60$. Therefore, the proof of Theorem 4.1 is complete. $\square$

5. Perrin numbers which are Pell numbers

In this section, we will state and prove our third main result.

Theorem 5.1. *The only solutions of the Diophantine equation*

$$E_n = P_m \tag{5.1}$$

in positive integers m and n are

$$(n, m) \in \{(0,1), (2,2), (4,2), (5,3), (6,3), (9,4), (8,3), (12,5)\}.$$

Hence, this implies that $E \cap P = \{0, 2, 5, 12, 29\}$.

Proof. A quick computation with Maple gives the solutions of the Diophantine equation (5.1) in the interval $[0, 55]$, cited in Theorem 5.1.

From now, assuming that $n > 55$, then by (2.7) and (2.10), we have

$$\alpha^{n-2} < \delta^{m-1} \quad \text{and} \quad \delta^{m-2} < \alpha^{n+1}.$$

Thus, we get

$$(n-2)c_1 + 1 < m < (n+1)c_1 + 2, \quad \text{where } c_1 := \log\alpha / \log\delta.$$

Particularly, we have $n < 4m$. So to solve equation (5.1), we will determine a good upper bound on m.

By using (2.5) and (2.8), equation (5.1) can be expressed as

$$\alpha^n - \frac{\delta^m}{2\sqrt{2}} = -\beta^n - \gamma^n - \frac{\overline{\delta}^m}{2\sqrt{2}}.$$

Thus, we get

$$\left|\alpha^n - \frac{\delta^m}{2\sqrt{2}}\right| = \left|\beta^n + \gamma^n + \frac{\overline{\delta}^m}{2\sqrt{2}}\right| < 2.36.$$

Dividing through by $\delta^m/(2\sqrt{2})$, we obtain

$$\left|2\sqrt{2}\alpha^n\delta^{-m} - 1\right| < 6.68\delta^{-m}. \tag{5.2}$$

Now, we apply Matveev's theorem to

$$\Lambda_3 = 2\sqrt{2}\alpha^n\delta^{-m} - 1$$

and take

$$\eta_1 := 2\sqrt{2}, \quad \eta_2 := \alpha, \quad \eta_3 := \delta, \quad b_1 := 1, \quad b_2 := n, \quad b_3 := -m.$$

The algebraic numbers η_1, η_2 and η_3 belong to $\mathbb{K} := \mathbb{Q}(\alpha, \delta)$, with $d_{\mathbb{K}} = 6$. As before we can take

$$D = 4m, \quad A_2 = 0.58 \quad \text{and} \quad A_3 = 2.65$$

Furthermore, since $h(\eta_1) = \log(2\sqrt{2})$, we choose

$$\max\{6h(\eta_1), |\log \eta_1|, 0.16\} < 6.24 := A_1.$$

Similarly to what was done above, one can check that $\Lambda_3 \neq 0$. We deduce from Matveev's theorem that

$$\begin{aligned}\log|\Lambda_3| &> -1.4 \times 30^6 \times 3^{4.5} \times 6^2(1+\log 6)(1+\log 4m) \times 6.24 \times 0.58 \times 2.65\\ &> -1.39 \times 10^{14} \times (1+\log 4m).\end{aligned}$$

The last inequality together with (5.2) leads to

$$m < 1.57 \times 10^{14}(1+\log 4m).$$

Thus, we solve the above inequality to obtain

$$m < 6.1 \times 10^{15}. \tag{5.3}$$

Now, we will use Lemma 2.3 to reduce the upper bound (5.3) on m.

Define

$$\Gamma_3 = n\log\alpha - m\log\delta + \log(2\sqrt{2}).$$

Like above, we use Γ_3 to obtain

$$0 < \left|n(-\log\alpha) + m\log\delta - \log(2\sqrt{2})\right| < 13.36\exp(-0.88 \times m)$$

Inequality (5.3) implies $X_0 := 2.44 \times 10^{16}$. Now, we take

$$c := 13.36, \quad \delta := 0.88, \quad \psi := -\frac{\log(2\sqrt{2})}{\log\delta},$$

$$\vartheta := \frac{\log\alpha}{\log\delta}, \quad \vartheta_1 := -\log\alpha, \quad \vartheta_2 := \log\delta, \quad \beta := -\log(2\sqrt{2}).$$

We use Maple to see that

$$q_{28} = 153529568750401532$$

satisfies the hypotheses of Lemma 2.3. Applying Lemma 2.3, we get

$$m < \frac{1}{0.88}\log\left(\frac{153529568750401532^2 \times 13.36}{\log\delta \times 2.44 \times 10^{16}}\right) \leq 51.$$

This contradicts the assumption that $n > 55$. Therefore, This completes the proof of Theorem 5.1. $\square$

6. Perrin numbers which are Pell-Lucas numbers

In this section, we will state and prove our last main result.

Theorem 6.1. *The only solutions of the Diophantine equation*

$$E_n = Q_m \tag{6.1}$$

in positive integers m and n are

$$(n, m) \in \{(2, 0), (2, 1), (4, 0), (4, 1)\}.$$

Hence, we see that $E \cap Q = \{2\}$.

Proof. A quick computation with Maple in the interval $[0, 50]$ gives the solutions of Diophantine equation (6.1) cited in Theorem 6.1.

We suppose that $n > 50$, then by (2.7) and (2.11), we have

$$\alpha^{n-2} < \delta^{m+1} \quad \text{and} \quad \delta^{m-1} < \alpha^{n+1}.$$

Thus, we get

$$(n-2)c_1 - 1 < m < (n+1)c_1 + 1, \quad \text{where } c_1 := \log \alpha / \log \delta.$$

Particularly, we have $n < 4m$. So to solve equation (6.1), We will find a good upper bound on m.

By using (2.5) and (2.9), one can see that equation (6.1) can be rewritten as

$$\alpha^n - \delta^m = -\beta^n - \gamma^n - \overline{\delta}^m.$$

We deduce that

$$|\alpha^n - \delta^m| \leq 2|\beta^n| + 1 < 3.$$

Dividing both sides by δ^m, we get

$$\left|\alpha^n \delta^{-m} - 1\right| < 3\delta^{-m}. \tag{6.2}$$

To apply Matveev's theorem to

$$\Lambda_4 = \alpha^n \delta^{-m} - 1$$

we take

$$\eta_1 := \alpha, \quad \eta_2 := \delta, \quad b_1 := n, \quad b_2 := -m, \quad D = 4m, \quad A_1 = 0.58 \quad \text{and} \quad A_2 = 2.65.$$

Moreover, one can show that $\Lambda_4 \neq 0$. Thus, we apply Matveev's theorem to obtain

$$\begin{aligned}\log|\Lambda_4| &> -1.4 \times 30^5 \times 2^{4.5} \times 6^2(1 + \log 6)(1 + \log 4m) \times 0.58 \times 2.65 \\ &> -1.19 \times 10^{11}(1 + \log 4m).\end{aligned}$$

The last inequality together with (6.2) implies

$$m < 1.35 \times 10^{11}(1 + \log 4m).$$

Thus, we obtain

$$m < 4.19 \times 10^{12}. \tag{6.3}$$

Now, we will use Lemma 2.2 to reduce the upper bound (6.3) on m.

Put

$$\Gamma_4 = n \log \alpha - m \log \delta.$$

We proceed as above and use Γ_4 to obtain

$$0 < |n(-\log \alpha) + m \log \delta| < 6 \exp(-0.88 \times m).$$

From inequality (6.3), we take $X_0 := 1.68 \times 10^{13}$. So, we have $Y := 63.95005\ldots$. Moreover, we choose

$$c := 6, \quad \delta := 0.88, \quad \vartheta := \frac{\log \alpha}{\log \delta}, \quad \vartheta_1 := -\log \alpha, \quad \vartheta_2 := \log \delta.$$

With the help of Maple, we find that

$$\max_{0 \le k \le 64} a_{k+1} = 1029.$$

So, Lemma 2.2 gives

$$m < \frac{1}{0.88} \log \left(\frac{6 \times 1031 \times 1.68 \times 10^{13}}{\log \delta} \right) \le 45.$$

This contradicts the assumption that $n > 50$. Therefore, Theorem 6.1 is completely proved. □

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References

[1] A. Baker, H. Davenport: *The equations $3x^2 - 2 = y^2$ and $8x^2 - 7 = z^2$*, Quart. J. Math. Oxford Ser. (2) 20 (1969), pp. 129–137, DOI: https://doi.org/10.1093/qmath/20.1.129.

[2] B. M. De Weger: *Algorithms for Diophantine equations*, CWI tracts 65 (1989).

[3] P. Kiss: *On common terms of linear recurrences*, Acta Mathematica Academiae Scientiarum Hungarica 40.1-2 (1982), pp. 119–123.

[4] M. Laurent: *Équations exponentielles polynômes et suites récurrentes linéaires*, Astérisque 147.148 (1987), pp. 121–139.

[5] E. M. Matveev: *An explicit lower bound for a homogeneous rational linear form in the logarithms of algebraic numbers. II*, Izvestiya: Mathematics 64.6 (2000), p. 1217.

[6] M. Mignotte: *Une extension du théoreme de Skolem–Mahler*, CR Acad. Sci. Paris 288 (1979), pp. 233–235.

[7] M. Mignotte: *Intersection des images de certaines suites récurrentes linéaires*, Theoretical Computer Science 7.1 (1978), pp. 117–121.

[8] M. Mignotte, Y. Bugeaud, S. Siksek: *Classical and modular approaches to exponential Diophantine equations I. Fibonacci and Lucas perfect powers*, Annals of mathematics 163.3 (2006), pp. 969–1018.

[9] H. P. Schlickewei, W. M. Schmidt: *The intersection of recurrence sequences*, Acta Arithmetica 72.1 (1995), pp. 1–44.