Additively-separable and rank-discounted variable-population social welfare functions: A characterization

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Abstract

Economic policy evaluations require social welfare functions for variable-size populations. Two important candidates are critical-level generalized utilitarianism (CLGU) and rank-discounted critical-level generalized utilitarianism, which was recently characterized by Asheim and Zuber (2014) (AZ). AZ introduce a novel axiom, existence of egalitarian equivalence (EEE). First, we show that, under some uncontroversial criteria for a plausible social welfare relation, EEE suffices to rule out the Repugnant Conclusion of population ethics (without AZ's other novel axioms). Second, we provide a new characterization of CLGU: AZ's set of axioms is equivalent to CLGU when EEE is replaced by the axiom same-number independence.

Keywords: social welfare, population ethics, utilitarianism, rank-discounted utilitarianism, critical-level utilitarianism, separability

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1 Introduction

Population ethics studies the axiomatic properties of social welfare relations for different-sized populations. In one of the most important recent advances of this literature, Asheim and Zuber (2014) (hereafter AZ) have proposed and axiomatized rank-discounted generalized utilitarianism (RDGU), which has been further investigated by Pivato (2020). AZ show that RDGU escapes the "Repugnant Conclusion" (Parfit, 1984), a well-studied property of some approaches to population ethics.[1]

Here we document two further properties of RDGU, with special attention to AZ's novel "existence of egalitarian equivalence" axiom. First, we show that, under a set of minimal conditions for the normative plausibility of any approach to population ethics, this axiom is individually sufficient to avoid the Repugnant Conclusion. Then, we show that by replacing existence of egalitarian equivalence with an alternative axiom — same-number independence — but retaining AZ's six other axioms we obtain a novel characterization of critical-level generalized utilitarianism (CLGU, Blackorby and Donaldson, 1984).

2 Setting and basic axioms

Our notation follows AZ. $\mathbb{N}$ are the positive integers, $\mathbb{R}$ are the real numbers, $\mathbb{R}_{++}$, $\mathbb{R}_{+}$, $\mathbb{R}_{-}$ are positive, nonnegative, and nonpositive.

Populations $\mathbf{x}, \mathbf{y}$ are finite-length vectors of real numbers, where the ith position in the vector $\mathbf{x}$, denoted x_i, is the lifetime utility of person i. Utilities are normalized so that $x_i = 0$ is a neutral life for person i. An index enclosed in square brackets indicates rank from worst-off, so $x_{[2]} = 4$ means that the second-lowest utility in the population is 4. For any $\mathbf{x}$, $\mathbf{x}_{[]}$ is the nondecreasing reordering of $\mathbf{x}$ (note that we and AZ assume anonymity throughout). Write $\mathbf{x}_{[]} \geq \mathbf{y}_{[]}$ if $x_{[r]} \geq y_{[r]}$ for all ranks r and $\mathbf{x}_{[]} > \mathbf{y}_{[]}$ if $x_{[r]} \geq y_{[r]}$ for all ranks r and $x_{[r']} > y_{[r']}$ for some rank r'.

The size of $\mathbf{x}$ is $n(\mathbf{x}) \in \mathbb{N}$, so $\mathbf{x} \in \mathbb{R}^{n(\mathbf{x})}$. $(z)_n$ is a population of n equally well-off people with utility z each. Parentheses combine populations, so $n\left((\mathbf{x}, \mathbf{y})\right) = n(\mathbf{x}) + n(\mathbf{y})$. When

[1] Although the population ethics literature has long focused on avoiding the Repugnant Conclusion, a collaboration of 29 authors from economics and philosophy has recently proposed that "avoiding the Repugnant Conclusion is not a necessary condition for a minimally adequate candidate axiology, social ordering, or approach to population ethics" (Zuber et al., 2021). They present several arguments, including that the Repugnant Conclusion is not fundamentally dissimilar from related implications of many approaches to population ethics that are commonly understood to avoid the Repugnant Conclusion (e.g. Spears and Budolfson, 2021).

a population $\mathbf{x}$ is combined with a singleton population whose only member's lifetime utility is z, we write this as $(\mathbf{x}, z)$.

The set of all possible populations is $\mathbf{X} = \bigcup_{n\in\mathbb{N}} \mathbb{R}^n$. This paper describes $\succsim$, a social welfare relation on $\mathbf{X}$, which is a binary relation with the interpretation of "at least as good as." The asymmetric and symmetric parts of $\succsim$ are $\succ$ and $\sim$, respectively.

Three of AZ's axioms are uncontroversial in population economics:

Axiom 1 (Order). *The relation $\succsim$ is complete, transitive, and reflexive on $\mathbf{X}$.*

Axiom 2 (Continuity). *For all $n \in \mathbb{N}$ and all $\mathbf{x} \in \mathbb{R}^n$, the sets $\{\mathbf{y} \in \mathbb{R}^n : \mathbf{y} \succsim \mathbf{x}\}$ and $\{\mathbf{y} \in \mathbb{R}^n : \mathbf{x} \succsim \mathbf{y}\}$ are closed.*

Axiom 3 (Suppes-Sen). *For all $n \in \mathbb{N}$ and all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, if $\mathbf{x}_{[]} > \mathbf{y}_{[]}$, then $\mathbf{x} \succ \mathbf{y}$.*

AZ characterize RDGU, which is defined as:

$$\mathbf{x} \succsim \mathbf{y} \Leftrightarrow V^{RDGU}(\mathbf{x}) = \sum_{r=1}^{n(\mathbf{x})} \beta^r \left(g\left(x_{[r]}\right) - g(c)\right) \geq V^{RDGU}(\mathbf{y}) = \sum_{r=1}^{n(\mathbf{y})} \beta^r \left(g\left(y_{[r]}\right) - g(c)\right) \tag{RDGU}$$

where $\beta \in (0, 1)$ and the constant $c \geq 0$ is the critical-level parameter. We characterize CLGU, which is defined (without reference to ranks, as these are irrelevant to CLGU) as:

$$\mathbf{x} \succsim \mathbf{y} \Leftrightarrow V^{CLGU}(\mathbf{x}) = \sum_{i=1}^{n(\mathbf{x})} \left(g\left(x_i\right) - g(c)\right) \geq V^{CLGU}(\mathbf{y}) = \sum_{i=1}^{n(\mathbf{y})} \left(g\left(y_i\right) - g(c)\right) \tag{CLGU}$$

3 Existence of egalitarian equivalence and the Repugnant Conclusion

Some variable-population social welfare relations entail the Repugnant Conclusion (hereafter RC), formalized by AZ as:

The Repugnant Conclusion. *For all $y, x \in \mathbb{R}$ with $y > x > 0$ and all $k \in \mathbb{N}$, there is $n > k$ such that $(x)_n \succ (y)_k$.*

RDGU avoids the RC through a novel axiom:

Axiom 7 (Existence of egalitarian equivalence). *For all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, if $\mathbf{x} \succ \mathbf{y}$, then there exists $z \in \mathbb{R}$ such that, for all $n \in \mathbb{N}$, $\mathbf{x} \succ (z)_m \succ \mathbf{y}$ for some $m \geq n$.*

AZ note that existence of egalitarian equivalence (hereafter EEE) "is weaker than directly requiring avoidance of the [Repugnant Conclusion]" by providing a normatively implausible example in which the value of increasing population size depends on whether population size is odd or even. But no approach to population ethics defended in the literature accepts EEE and the RC. This is because no normatively plausible social welfare relation can do so, if the following two properties are minimal requirements for normative plausibility:

- Consistent egalitarian expansion: For all $x \in \mathbb{R}$ and all $k, m, n \in \mathbb{N}$, $k \geq m > n$, if $(x)_m \succ (x)_n$ then $(x)_k \succsim (x)_m$.
- Weak egalitarian negative expansion: For all $x \in \mathbb{R}_-$ and all $m, n \in \mathbb{N}$, $m \geq n$, $(x)_n \succsim (x)_m$.

We know of no approach defended in the literature that rejects either of these properties. If they are minimal requirements, then EEE entails avoiding the Repugnant Conclusion.

Proposition 1. *No transitive social welfare relation that satisfies Suppes-Sen, consistent egalitarian expansion, and weak egalitarian negative expansion can both satisfy existence of egalitarian equivalence and entail the Repugnant Conclusion.*

Proof. Assume that $\succsim$ entails the RC; we will show that no z can satisfy EEE. Fix any y, x, k, and n from the RC and use $\mathbf{y} = (y)_k$ and $\mathbf{x} = (x)_n$ for EEE. First, z cannot be non-positive because then $(y)_k \succ (z)_k$ by Suppes-Sen and, for any $\ell \geq k$, $(z)_k \succsim (z)_\ell$ by weak egalitarian negative expansion. Hence, by transitivity, $(y)_k \succ (z)_\ell$. Next consider $0 < z < x$. By the RC there exists $n' > n$ such that $(z)_{n'} \succ (x)_n$; by Suppes-Sen, $(x)_n \succ (z)_n$; hence, by transitivity, $(z)_{n'} \succ (z)_n$. So, by consistent egalitarian expansion, $(z)_\ell \succsim (z)_{n'}$, for any $\ell \geq n'$; and, by transitivity, $(z)_\ell \succ (x)_n$. Finally, for $z \geq x$, let us now re-label z, which we used in the last step, z'. Then, by Suppes-Sen, we have $(z)_\ell \succ (z')_\ell$, so, by the result in the last step and transitivity, $(z)_\ell \succ (x)_n$. This excludes all possible z. □

Proposition 1 implies that the RC is ruled out even if AZ's Axioms 1-6 are replaced by transitivity, Suppes-Sen, consistent egalitarian expansion, and weak egalitarian negative expansion, while keeping EEE. Three examples that satisfy these axioms and avoid the RC are average utilitarianism $\left(\frac{\sum_i x_i}{n(\mathbf{x})}\right)$, number-dampened generalized utilitarianism (NDGU) with a bounded transformation—i.e., $f(n(\mathbf{x}))\frac{\sum_i g(x_i)}{n(\mathbf{x})}$, where f is bounded, in addition to being increasing and concave—and RDGU but ranked in descending, rather than ascending, order. Each of these violates AZ's Axiom 6, which is presented below.

Is EEE an attractive axiom? As one consideration, we highlight that EEE implies that some unboundedly large populations of negative lives will not be worse than some finite populations of negative lives. Consider, for example, $\mathbf{x} = (-1)_1$ and $\mathbf{y} = (-2)_1$. Then z in the statement of EEE will be a negative utility level (negative because $\mathbf{x}$ and $\mathbf{y}$ are both negative) such that, for unboundedly large n, $(z)_n \succ (-2)_1$. This is true of AU, NDGU, and RDGU.

4 Same-number independence and CLGU

In their characterization of RDGU, AZ use three further axioms which we will employ:

Axiom 4 (Existence independence of the best-off)**.** *For all* $n \in \mathbb{N}$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, *and* $z \in \mathbb{R}$ *satisfying* $z \geq \max\left\{x_{[n]}, y_{[n]}\right\}$, $(\mathbf{x}, z) \succsim (\mathbf{y}, z) \Leftrightarrow \mathbf{x} \succsim \mathbf{y}$.

Axiom 5 (Existence independence of the worst-off)**.** *For all* $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ *and* $z \in \mathbb{R}$ *satisfying* $z \leq \min\left\{x_{[1]}, y_{[1]}\right\}$, $(\mathbf{x}, z) \succsim (\mathbf{y}, z) \Leftrightarrow \mathbf{x} \succsim \mathbf{y}$.

Axiom 6 (Existence of a critical level)**.** *There exist* $c \in \mathbb{R}_+$ *and* $n \in \mathbb{N}$ *such that, for all* $\mathbf{x} \in \mathbb{R}^n$ *satisfying* $x_{[n]} \leq c$, $(\mathbf{x}, c) \sim \mathbf{x}$.

Instead of EEE, we add a separability axiom that only applies to comparisons of same-sized populations:

Axiom 8 (Same-number independence)**.** *For all* $n, m \in \mathbb{N}$, *for all* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, *and for all* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^m$, $(\mathbf{x}, \mathbf{u}) \succsim (\mathbf{y}, \mathbf{u}) \Leftrightarrow (\mathbf{x}, \mathbf{v}) \succsim (\mathbf{y}, \mathbf{v})$.

Same-number independence was previously studied by Blackorby, Bossert and Donaldson (2005). It enjoys wide implicit normative acceptance in the many same-number policy evaluations that do not incorporate estimates of the number and welfare of unaffected (e.g., past) lives.

Theorem 1. *The following two statements are equivalent:*

(i) $\succsim$ *satisfies Axioms 1-6 and Axiom 8.*

(ii) $\succsim$ *is CLGU.*

Proof. We modify AZ's method. By AZ's Lemma 1, axioms 1-5 are sufficient for there to be a $\beta \in \mathbb{R}_{++}$ and a continuous, increasing g such that $\forall n \in \mathbb{N}, \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

$$\mathbf{x} \succsim \mathbf{y} \Leftrightarrow \sum_{r=1}^{n(x)} \beta^r g\left(x_{[r]}\right) \geq \sum_{r=1}^{n(y)} \beta^r g\left(y_{[r]}\right).$$

By Theorem 4.7 of Blackorby, Bossert, and Donaldson (2005), same-number independence implies same-number generalized utilitarianism given the biconditional above, so $\beta = 1$. Note that any fixed critical level would cancel.

By AZ's Lemma 2, axioms 1-6 (with $c \in \mathbb{R}_+$ from axiom 6) are sufficient for there to be a $\beta \in \mathbb{R}_{++}$ and a continuous, increasing g such that

$$\mathbf{x} \succsim \mathbf{y} \Leftrightarrow \sum_{r=1}^{n(x)} \beta^r \left(g\left(x_{[r]}\right) - g(c)\right) \geq \sum_{r=1}^{n(y)} \beta^r \left(g\left(y_{[r]}\right) - g(c)\right),$$

if $\mathbf{x}, \mathbf{y} \in \mathbf{X}_c$, where $\mathbf{X}_c = \left\{\mathbf{x} \in \mathbf{X} : x_{[n(\mathbf{x})]} \leq c\right\}$. Because this includes same-number cases, $\beta = 1$. By same-number generalized utilitarianism and transitivity, Lemma 2 extends to all $\mathbf{x}$ such that there exists a $\mathbf{y} \in \mathbf{X}_c$ where $\mathbf{x} \sim \mathbf{y}$ and $n(\mathbf{x}) = n(\mathbf{y})$.

Extend Lemma 2 to all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ by choosing $z < \min\{x_{[1]}, y_{[1]}, c\}$. For all $\mathbf{x} \in \mathbf{X}$, define $e(\mathbf{x})$ as $e \in \mathbb{R}$ such that $(e)_{n(\mathbf{x})} \sim \mathbf{x}$; by same-number generalized utilitarianism, $e(\mathbf{x}) = g^{-1}\left(\frac{1}{n(\mathbf{x})} \sum_i^{n(\mathbf{x})} g(x_i)\right)$. Then by same-number generalized utilitarianism, $\exists k \in \mathbb{N}$ such that $e\left((\mathbf{x}, (z)_k)\right) < c$ and $e\left((\mathbf{y}, (z)_k)\right) < c$. Then by Lemma 2 and transitivity

$$(\mathbf{x}, (z)_k) \succsim (\mathbf{y}, (z)_k) \Leftrightarrow \sum_i^{n(\mathbf{x})} \left(g\left(x_i\right) - g(c)\right) \geq \sum_i^{n(\mathbf{y})} \left(g\left(y_i\right) - g(c)\right),$$

because the k z-terms cancel. By Axiom 5 and because z is worst-off, $(\mathbf{x}, (z)_k) \succsim (\mathbf{y}, (z)_k) \Leftrightarrow \mathbf{x} \succsim \mathbf{y}$, which completes the proof.

Alternatively, we can establish constant critical levels by showing that $\forall \mathbf{x}, (\mathbf{x}, c) \sim \mathbf{x}$. By same-number generalized utilitarianism, choose $k \in \mathbb{N}$ such that $\xi = e\left((\mathbf{x}, (z)_k)\right) < c$. By transitivity and then by Axiom 6, $(\mathbf{x}, (z)_k) \sim (\xi)_{n(\mathbf{x})+k} \sim ((\xi)_{n(\mathbf{x})+k}, c)$. By same-number generalized utilitarianism and transitivity, $((\xi)_{n(\mathbf{x})+k}, c) \sim (\mathbf{x}, (z)_k, c) \sim (\mathbf{x}, (z)_k)$. So, by repeated use of Axiom 5 $(\mathbf{x}, c) \sim \mathbf{x}$.

$\square$

5 Discussion

AZ highlight that RDGU escapes the RC. Proposition 1 shows that it does so because of EEE, since any minimally plausible social welfare relation (including all defended in the population economics literature) avoids the RC if it satisfies EEE. Of course, in avoiding the RC through one key axiom, RDGU is not unusual: Blackorby and Donaldson's (1984) assumption of a fixed, positive critical level is similarly sufficient to avoid the RC for any transitive relation that satisfies same-number Pareto.[2]

Theorem 1 shows that the difference between RDGU and CLGU is the substitution of same-number independence for EEE. A reader may find Axioms 1-6 apparently weaker or more attractive than the existence independence or fixed critical level axioms that have been used to characterized CLGU in the literature. As AZ explain: "While ordinary critical-level generalized utilitarianism allows for unrestricted independence to adding an individual (see Blackorby *et al*. 2005), our axioms impose such independence only if the added individual is best off (relative to two allocations with the same population size) or worst off" (p. 634). Those who are attracted to Axioms 1-6 but also want to preserve the ordinary policy-evaluation practice of same-number independence may therefore adopt CLGU — which includes, as special cases, priority for the worse-off (for concave g) and utilitarianism (for linear g).

References

Asheim, Geir B and Stéphane Zuber. 2014. "Escaping the repugnant conclusion: Rank-discounted utilitarianism with variable population." *Theoretical Economics*, 9(3): 629–650.

Blackorby, Charles and David Donaldson. 1984. "Social criteria for evaluating population change." *Journal of Public Economics*, 25(1-2): 13–33.

Blackorby, Charles, Walter Bossert, and David J Donaldson. 2005. *Population issues in social choice theory, welfare economics, and ethics*: Cambridge University Press.

Parfit, Derek. 1984. *Reasons and Persons*: Oxford.

Pivato, Marcus. 2020. "Rank-additive population ethics." *Economic Theory*, 69(4): 861–918.

[2]To see this, take any utilities $y > c$ and any sizes $n > k$; then $(y)_k \succ (c)_k \sim (c)_n$.

Spears, Dean and Mark Budolfson. 2021. "Repugnant conclusions." *Social Choice and Welfare*: 1–22.

Zuber, Stéphane, Nikhil Venkatesh, Torbjörn Tännsjö, Christian Tarsney, H. Orri Stefánsson, Katie Steele, Dean Spears, Jeff Sebo, Marcus Pivato, Toby Ord, Yew-Kwang Ng, Michal Masny, William MacAskill, Nicholas Lawson, Kevin Kuruc, Michelle Hutchinson, Johan E. Gustafsson, Hilary Greaves, Lisa Forsberg, Marc Fleurbaey, Diane Coffey, Susumu Cato, Clinton Castro, Tim Campbell, Mark Budolfson, John Broome, Alexander Berger, Nick Beckstead, and Geir B. Asheim. 2021. "What should we agree on about the Repugnant Conclusion?" *Utilitas*.