Element failure probability of soil slope under consideration of random groundwater level

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Abstract: The instability of soil slopes is directly related to both the shear parameters of the soil material and the groundwater, which usually causes some uncertainty. In this study, a novel method, the element failure probability method (EFP), is proposed to analyse the failure of soil slopes. Based on the upper bound theory, finite element discretization, and the stochastic programming theory, an upper bound stochastic programming model is established by simultaneously considering the randomness of shear parameters and groundwater level to analyse the reliability of slopes. The model is then solved by using the Monte-Carlo method based on the random shear parameters and groundwater levels. Finally, a formula is derived for the element failure probability (EFP) based on the safety factors and velocity fields of the upper bound method. The probability of a slope failure can be calculated by using the safety factor, and the distribution of failure regions in space can be determined by using the location information of the element. The proposed method is validated by using a classic example. This study has theoretical value for further research attempting to advance the application of plastic limit analysis to analyse slope reliability.

Keywords: soil slope; reliability; element failure probability; upper bound method; finite element discretization; stochastic programming

Introduction

Computation of the soil slopes stability is complex and uncertain as many factors may affect the stability of soil slopes. Among these factors, the soil shear parameters and groundwater are the most important ones and have significant effects on the reliability of soil slopes (Dai et al. 2002). Shear parameters are natural characteristics of the soil material of the slope and are directly related to the mechanical properties of the soil mass, such as the resistance to a landslide in the inner slope. The composition and random distribution of the soil material determines the variation and uncertain characteristics of the shear parameters of the soil mass. Variation in groundwater level often results in a landslide of the soil slope (Ching et al.2016; Cafaro and Cherubini 2002; Chen and Mayne 1996; Cho 2007). This effect is observed because the presence of groundwater can strongly reduce the shear resistance of soil, and variation of the seepage fields of a slope further changes the pore water pressure in soil (Ali and Lyamin 2014; Chiu et al. 2012; Lu and Griffiths 2004; Wang et al. 2019). Since both the soil shear parameters and the groundwater distribution in a slope have random characteristics, a random influence caused by these two factors should be considered in the stability analysis of the soil slope.

Extensive studies based on the rigid body limit equilibrium and the finite element methods have been conducted to investigate the slope reliability. The rigid limit equilibrium-based analysis could directly yield a mathematical distribution and the failure probability of the safety factor. This method is effective and widely employed, but the slip surface of a slope needs to be artificially determined (Malkawi and Hassan 2000; Ji et al. 2020; Lu and Griffiths 2004; Low et al. 2007). Finite element method (FEM) analysis is theoretically stricter than the former method. The stress-strain distribution in the inner of slope could be obtained by using FEM, but it is computationally costly (Griffiths and Fenton 2004; Griffiths et al. 2009; Dyson and Tolooiyan 2020). Recently, slope reliability analysis based on the plastic limit theory has been significantly developed. By using this method, a limit condition of the slope under instability could be obtained without considering the loading history of the slope or the constitutive relation of the soil material. The ultimate load (or safety factor) and the failure mechanism (stress field and velocity field) of the slope could also be obtained (Huang et al. 2013; Li and Liu. 2001; Zhang et al.2018; Li et al. 2019; Ali et al. 2017; Kasama and Whittle. 2011; Zhao et al. 2016). Since the analysis is greatly simplified, the plastic limit-based method has great potential applications in slope reliability analysis.

The integrated failure probability method is conventionally used to analyse slope failure, in which the overall failure of a slope is determined by a safety factor threshold of 1.0. The failure probability of the slope is calculated thereafter (Phoon and Kulhawy. 1999; Huang et al.2010). However, different failure modes of the soil slope would result in different failure consequences (sliding volume). For example, there is large difference in the failure consequence between deep sliding and near-surface sliding; however, the conventional method does not consider such differences and only considers whether the safety factor is less than 1.0. As a result, the distribution difference of the failure regions in the space of a slope is overlooked (Jiang et al. 2014; Low et al.2011; Li et al. 2019). To date, there have been great achievements on the relationship between the random parameters and failure modes of slope. However, the relationship between spatial distribution of failure probability and groundwater level has not been studied systematically. Novel theory and methods are required to solve this issue.

In this study, we apply the random distribution model of the shear parameters and the groundwater level of soil slopes together with their sampling. Based on the upper bound theory, the finite element discretization technique, the stochastic programming theory, and the Monte-Carlo method, an upper limit numerical method is developed to analyse the reliability of soil slopes by considering the shear parameters and the random groundwater level. A method is proposed to analyse the element failure probability (**EFP**) of soil slopes. This analysis provides novel theory and method for the failure analysis of soil slopes and the relationship between the groundwater levels of soil slopes and the failure probability. Additionally, this study investigates the evolution of the failure probability of soil slopes.

Random seepage field of soil slope

The soil body is a typical three-phase medium. The pore water flow in the soil mass forms seepage field. The existence condition of the soil slope is highly complicated. The seepage field in the soil slope is influenced by many random parameters with two important parameters. The first important parameter is the random distribution of the soil particles and pores in the soil mass, which results in random permeability of the soil material. The second important parameter is the random supply (i.e., rain, ground run-off, and irrigation) and drain (i.e., evaporation, ground pumping, and soil excavation) of the groundwater in the soil slope, which results in uncertain groundwater level in the soil slope. Consequently, the seepage flow in the soil slope has a random characteristic.

Extensive studies have been performed to investigate the influence of the random permeability of the soil material on the seepage field of the soil slope. These studies include the investigation of the effect of permeability variation on the stable seepage field, the random permeability-based analysis of slope seepage, and the random probability analysis of slope instability caused by permeation (Yang et al. 2004; Cho. 2012; Mouyeaux et al. 2019; Griffiths and Fenton.1998). However, few studies have been conducted to investigate the influence of the random groundwater on the random seepage field and reliability of soil slopes, which motivates this study. To this end, a plastic limit analysis numerical model of soil slopes is developed by considering the shear parameters and the random groundwater levels.

The seepage problem of the soil slope under the action of the random groundwater levels is shown in Figure 1. To simplify the analysis, this paper makes the following assumptions: (1) assuming that the soil slope groundwater level H_w^r is a random variable and varies randomly between the lower and upper bounds. Assuming that the underground water level at the slope angle is determined, the soil permeability coefficient is considered a determined parameter during the calculation of the random seepage field. (2) Only the saturated and stable seepage field of soil slope under the action of the random groundwater level is calculated, and the excess pore water pressure caused by sudden increase or decrease in water level is not considered. (3) The random variation of the groundwater level H_w^r will result in a random change of the saturation line location, assuming that the soil pore water pressure p^r at an arbitrary point $K^{'}$ above the saturation line is zero and is a random variable below the saturation line (as shown in Figure 1). The pore water pressure p^r and has a direct correlation with H_w^r.

Function of the reliability analysis of soil slopes

In calculating the stability of soil slopes, there are two general methods to make the slope reach the limit state of the instability. The first method is to find the overload coefficient by gradually increasing the external load, while the second approach is to find the safety factor by gradually reducing the shear parameters of the soil material. Because the instability of the soil slope is related to many random parameters, the overload coefficient and the safety factor are random variables.

In this study, the volume weight overload is used to make the soil slope reach the limit state of the instability. The random quantity of the volume weight overload factor is defined as

$$\lambda_\gamma^r = \frac{\gamma_c(c^r, \varphi^r, H_w^r)}{\gamma_a} \tag{1}$$

where λ_γ^r is the random variable of the overload factor of volume weight and relates to c^r, φ^r, and H_w^r; $\gamma_c(c^r, \varphi^r, H_w^r)$ is the random variable of the ultimate value of volume weight when the soil reaches the limit state; c^r and φ^r are random variables of the cohesion and the internal friction angle of the soil materials; and γ_a is the real volume weight of the soil material.

The random quantity of the soil slope safety factor is defined as:

$$\lambda_m^r = \frac{c^r}{c'^r} = \frac{\varphi^r}{\varphi'^r} \tag{2}$$

where λ_m^r is the random variable of the safety factor that relates to c^r, φ^r, and H_w^r; c^r and c'^r are the random quantity of cohesion before and after the intensity reduction, respectively; φ^r and φ'^r are the random quantity of the internal friction angle before and after the intensity reduction, respectively.

In this study, the limit state equation for the reliability analysis of the soil slope is developed by considering the shear parameters of the soil material and the randomness of the ground water level. The limit state function of the soil slope reliability is defined as:

$$Z = (\lambda_m^r - 1.0)\begin{cases} > 0, & \text{Stable} \\ = 0, & \text{Critical state} \\ < 0, & \text{Failure} \end{cases} \tag{3}$$

Equation (3) shows that when Z > 0, that is, the safety factor $\lambda_m^r > 1.0$, the slope is in a stable state. When Z = 0, that is, the safety factor $\lambda_m^r = 1.0$, the slope is in a critical state. When Z<0, that is, the safety factor $\lambda_m^r < 1.0$, the slope is in a failure state.

Stochastic programming model of upper bound method

The upper bound theorem of plastic limit analysis is an efficient tool for solving the stability of soil slope. According to the upper bound theorem, among all the external loads corresponding to the kinematically admissible velocity fields, the minimum external load is the closest to the real load. This property means that the upper bound method is a mathematical programming problem for finding the minimum value of an external load. Extensive studies have been conducted by using the upper bound numerical method of plastic limit analysis to perform the deterministic analysis of soil slopes (Zhang et al. 2018; Sloan and Kleeman. 1995; Kim and Salgado. 1999, (Li et al. 2018)). Based on the previous work of Sloan and Kim, this study establishes an upper bound method stochastic programming model for the reliability analysis of soil slopes that simultaneously considers the shear parameters of soil and the randomness of groundwater level.

In this study, a non-common-node triangular element is used to discrete the soil slope (as shown in Figure 2 (a)), which were proposed by Sloan (Sloan and Kleeman. 1995). Each node has velocity along the direction *x* and *y*, as well as pore water pressure.

The velocity vector of finite element $\mathbf{u}^e$ can be expressed as:

$$\mathbf{u}^e = [u_{x1}^e \quad u_{y1}^e \quad u_{x2}^e \quad u_{y2}^e \quad u_{x3}^e \quad u_{y3}^e]^{\mathrm{T}} \tag{4}$$

where $e = (1, \cdots, n_e)$, n_e is the quantity of finite elements in the soil slope, and u_{xi}^e, u_{yi}^e are the velocity of nodes *i* (*i* = 1,…,3) in the finite element *e* along the *x* or *y* direction, respectively.

To construct the kinematically admissible velocity fields of the soil slope, there should be a velocity discontinuity between adjacent finite elements, as shown in Figure 2 (b). The velocity vector of the velocity discontinuity can be expressed as:

$$\mathbf{u}^d = [u_{x1}^d \quad u_{y1}^d \quad u_{x2}^d \quad u_{y2}^d \quad u_{x3}^d \quad u_{y3}^d \quad u_{x4}^d \quad u_{y4}^d]^T \tag{5}$$

where $d = (1,\cdots,n_d)$, n_d is the quantity of the velocity discontinuities in the soil slope, u_{xi}^d, u_{yi}^d are the velocity of the *i*th (*i* = (1...,4)) node on the velocity discontinuity *d* along the *x* or *y* direction, respectively. Nodes ① and ③ belong to the finite element *a*, and nodes ② and ④ belong to the finite element *b*.

The pore water pressure vector $\mathbf{p}_e^r$ of triangular element in the soil slope is defined as:

$$\mathbf{p}_e^r = [\, p_{e1}^r \quad p_{e2}^r \quad p_{e3}^r \,]^{\mathrm{T}} \tag{6}$$

where $e = (1,\cdots,n_e)$; $p_{e1}^r, p_{e2}^r, p_{e3}^r$ are the random variables of the pore water pressure at nodes ①, ②, and ③ in the finite element *e*, respectively, which are directly related to H_w^r of the soil slope. When considering the randomness of shear parameters and groundwater level, the velocity field is related to random parameters c^r , φ^r and $\mathbf{p}_e^r$. Therefore, a stochastic programming model of the upper bound method for soil slope reliability needs to be established based on plastic flow constraints of elements and discontinuities, velocity boundary conditions and objective function of the upper bound analysis. Based on plastic flow constraints of elements and discontinuities, velocity boundary conditions and objective function, a stochastic programming model of the upper bound method for soil slope reliability with a random seepage field is:

$$\begin{cases} Z = \lambda_m^r - 1.0 \\ Minimise: \ \lambda_\gamma^r = \mathbf{W}_{In1} + \mathbf{W}_{In2} - \mathbf{W}_{Ex2} - \mathbf{W}_{Ex3}^p - \mathbf{W}_{Ex4}^p \\ Subject\ to: \ \mathbf{A}_1^e\mathbf{u}^e - \mathbf{A}_2^e\boldsymbol{\lambda}^e = 0; \ \boldsymbol{\lambda}^e \geq 0; \ \ e = (1,\cdots,n_e) \\ \qquad\qquad\quad \mathbf{A}_1^d\mathbf{u}^d - \mathbf{A}_2^d\boldsymbol{\lambda}^d = 0; \ \boldsymbol{\lambda}^d \geq 0; \ d = (1,\cdots,n_d) \\ \qquad\qquad\quad \mathbf{A}^b\mathbf{u}^b = 0; \quad b = (1,\cdots,n_b) \\ \qquad\qquad\quad \mathbf{W}_{Ex1}^e = 1.0 \end{cases} \tag{7}$$

where $\mathbf{W}_{In1}$ is the internal power of elements; $\mathbf{W}_{In2}$ is the internal power of the velocity discontinuities; $\mathbf{W}_{Ex1}$ is the external work power exerted by the dead weight on the velocity of the element nodes; $\mathbf{W}_{Ex2}$ is the external power exerted by concentrated force and distributed load at the velocity of the element nodes; $\mathbf{W}_{Ex3}^p$ is the external work power of the pore water pressure in the element continuous body; $\mathbf{W}_{Ex4}^p$ is the external work power exerted by pore water pressure on the velocity discontinuities; $\mathbf{A}_1^e$ and $\mathbf{A}_2^e$ are the matrixes of plastic flow constraint conditions of element; $\mathbf{A}_1^d$ and $\mathbf{A}_2^d$ are the matrixes of plastic flow constraint conditions of velocity discontinuity; $\mathbf{A}^b$ is the coordinate transformation matrix of the boundary element and $\mathbf{u}^b$ is the velocity vector of the boundary element. The meaning of the symbols can be referred to literature (Zhang et al. 2018; Sloan and Kleeman. 1995).

$\mathbf{A}_2^e$, $\mathbf{A}_2^d$, $\mathbf{W}_{In1}$, $\mathbf{W}_{In2}$, $\mathbf{W}_{Ex3}^p$, and $\mathbf{W}_{Ex4}^p$ are all random matrices related to the random variables (e.g., soil cohesion c^r, internal friction angle φ^r, and groundwater level H_w^r). Therefore, Equation (7) is a stochastic programming problem with the safety factor as the objective function, the soil shear parameter c^r, φ^r, and the ground water level H_w^r as the random variables, and velocity $\mathbf{u}_e$, $\mathbf{u}^d$ and plasticity multiplier $\boldsymbol{\lambda}_e$, $\boldsymbol{\lambda}^d$ as the decision variables.

Solution of the stochastic programming model

Equation (7) is a large-scale stochastic programming model. To obtain its solution, we must solve for the upper bound solution of the safety factor according to the characteristics of the soil shear parameters and the random variables of groundwater level. For large-scale stochastic programming problems, there has been no direct solution to date. Therefore, an iterative method based on the Monte Carlo method is proposed to obtain the solution. The numerical iterative method of the reliability upper bound method for stochastic programming model with a random seepage field is as follows:

(1) Generating the random number of the ground water level of soil slope. Assuming that the variation of the groundwater level H_w^r conforms to the truncated normal distribution (Shadabfar et al. 2020). Using the Monte Carlo method to determine H_w^r, the random number of the groundwater level of the soil slope is then generated as following:

$$\begin{cases} H_w^r(t_w) = rand(Normal, \mu_w, \sigma_w, 1, n_w) \\ H_{lb} \le H_w^r(t_w) \le H_{ub} \end{cases} \tag{8}$$

where $t_w = (1, \cdots, n_w)$, n_w is the quantity of the Monte Carlo random numbers of the groundwater level of the soil slope, $H_w^r(t_w)$ is the t_w th random number of the groundwater level of the soil slope, μ_w is the mean groundwater level of the soil slope, σ_w is the standard deviation of the groundwater level of the soil slope, $rand$ is the normally distributed random number generation function, $Normal$ means that the random number conforms to the normal distribution, H_{lb} is the lower bound of the groundwater level of the soil slope, which takes the lowest water level, and H_{ub} is the upper bound of the groundwater level of the soil slope, which takes the highest water level.

(2) Generating the random numbers of the soil cohesion and the internal friction angle of the soil slope. It is assumed that the cohesion and friction angle of the soil material conform to a logarithmic normal distribution, and the random numbers of the material cohesion and the friction angle are generated as the following:

$$\begin{cases} c^r(t_m) = rand(lognormal, \mu_c, \sigma_c, 1, n_m) \\ \varphi^r(t_m) = rand(lognormal, \mu_\varphi, \sigma_\varphi, 1, n_m) \end{cases} \tag{9}$$

where $t_m = (1,\cdots,n_m)$, n_m is the quantity of material for the soil cohesion and the friction angle of the Monte Carlo random number, $c^r(t_m)$ is the t_mth random number on the materials of the soil cohesion, $\varphi^r(t_m)$ is the t_mth random number of the friction angle of the soil material, μ_c is the mean value of the material cohesion of the soil, μ_φ is the mean value of the friction angle of the soil material, σ_c is the standard deviation of the soil cohesion, σ_φ is the standard deviation of the friction angle of the soil materials, and *lognormal* means the random number that has a logarithmic normal distribution.

Using the Monte Carlo Method to determine the random variables, the volume weight overload factor of soil slope is defined as:

$$\lambda_\gamma(t_w,t_m) = \frac{\gamma_c(c^r(t_m),\varphi^r(t_m),H_w^r(t_w))}{\gamma_a} \tag{10}$$

where $t_w = (1,\cdots,n_w)$, $t_m = (1,\cdots,n_m)$, $\lambda_\gamma(t_w,t_m)$ is the volume weight overload factor corresponding to the random number of the t_mth random shear parameter for the t_wth groundwater level, and $\gamma_c(c^r(t_m),\varphi^r(t_m),H_w^r(t_w))$ is the ultimate volume weight of the soil slope in the limit state when it reaches the instability due to the t_wth groundwater level, which is related to the t_mth random shear parameters.

Using the Monte Carlo Method to determine the random variables while considering both the shear parameters and the randomness of the ground water level, the safety factor of the soil slope due to the groundwater level is then defined as:

$$\lambda_m(t_w,t_m) = \frac{c^r(t_m)}{c'^r(t_m)} = \frac{\varphi^r(t_m)}{\varphi'^r(t_m)} \tag{11}$$

where $t_w = (1,\cdots,n_w)$, $t_m = (1,\cdots,n_m)$, $\lambda_m(t_w,t_m)$ is the safety factor of the random number corresponding to the t_mth random shear parameter under the action of the t_wth groundwater level, $c'^r(t_m)$ is the t_mth random number of the soil cohesion after strength reduction, and $\varphi'^r(t_m)$ is the t_mth random number of the internal friction angle of soil after strength reduction.

(3) Taking $H_w^r(t_w)$ as the water head boundary condition for the calculation of the stable seepage field, the n_w stable seepage fields of the soil slope are calculated from $t_w = 1$ to $t_w = n_w$ (Lu and Griffiths. 2004), and the random pore water pressure at each finite element node in the soil slope is then obtained: $p_{e1}^r(t_w), p_{e2}^r(t_w), p_{e3}^r(t_w)$, where $t_w = (1,\cdots,n_w)$, $e = (1,\cdots,n_e)$.

(4) Repeating $p_{e1}^r(t_w), p_{e2}^r(t_w), p_{e3}^r(t_w)$ from $t_w = 1$ to $t_w = n_w$, the random pore water pressure at all nodes in the n_w seepage fields is successively substituted into the stochastic programming model of the soil slope reliability (Equation (7)). $c^r(t_m)$ and $\varphi^r(t_m)$ are nested from $t_m = 1$ to

$t_m = n_m$ in each iterative loop from $t_w = 1$ to $t_w = n_w$. Substituting the soil material cohesion and the random number of the friction angle in group n_m into Equation (7), Equation (7) then becomes a linear programming problem in which all constraint matrices are fixed values. The upper bound method linear programming model of the soil slope reliability is solved by using the dual simplex optimization algorithm in IBM's CPLEX software (IBM.2016). The random number $[\lambda_\gamma(t_w,t_m),\ t_w = 1,\cdots,n_w, t_m = 1,\cdots,n_m]$ and the corresponding velocity fields of $n_w \times n_m$ volume weight overload factors are obtained through iterative calculation.

(5) In each iteration from $t_m = 1$ to $t_m = n_m$, "dichotomy" is used to iteratively solve $n_w \times n_m$ safety factors $[\lambda_m(t_w,t_m),\ t_w = 1,\cdots,n_w, t_m = 1,\cdots,n_m]$ and the corresponding velocity field of the soil slope. For each safety factor, the "dichotomy" iteration is used to calculate the volume weight overload coefficient about 10 to 12 times. The specific iteration process is shown in Figure 3.

(6) The result of the safety factor is substituted into the limit state equation to calculate the reliability index of the slope. Based on the calculation results, the safety factor histogram, the probability density curve and the cumulative probability density curve of the soil slope, as well as the mean value and standard deviation of the safety factor, are plotted. The relation diagram of the change of the water level under the failure probability of the slope and the velocity field of the slope are plotted.

In this study, *Python* is used to program the upper bound method for the slope reliability with the random seepage field. The calculation program consists of three parts: the pre-processing module, the computational module, and the post-processing module. Due to the large scale of the computational samples, in order to improve the computational efficiency, the Parallel Computing Toolbox in *Python* (John V Guttag. 2013) is used to develop a Parallel optimization solution program. The optimization solution is solved by calling the dual simplex method in *CPlex* 12.71. The program is able to run stably on a workstation (Processor: AMD ThreadRipper 3970X with 32 Cores, Physical Memory: 128GB) with high efficiency.

Element failure probability of soil slope

Integrated failure probability (IFP)

Traditional slope failure analysis mainly applies the integrated failure probability method, which solves the integrated failure probability of the slope according to the safety factor of the slope (Griffiths and Fenton.2004; Phoon and Kulhawy.1999). Many commercial software products, such as GEO-Slope, SLIDE, use this method to compute the slope failure probability. The calculation principle states that when the safety factor of the slope is greater than or equal to 1.0, the slope is stable (overall safety), and when the safety factor of the slope is smaller than 1.0, the slope is unstable (overall failure). The failure function of the slope used in the integrated failure probability method is:

$$I_z(t_w,t_m)=\begin{cases}0,\ \text{if}\ \ \lambda_m(t_w,t_m)\geq 1.0\\ 1,\ \text{if}\ \ \lambda_m(t_w,t_m)<1.0\end{cases} \tag{12}$$

where $t_w=(1,\cdots,n_w)$, $t_m=(1,\cdots,n_m)$, and $I_z(t_w,t_m)$ is the failure function of the soil slope corresponding to the random number of the t_m th random shear parameter for the t_w th groundwater level.

The integrated failure probability of the slope for groundwater level t_w is:

$$P_f^z(t_w)=\frac{1}{n_m}\sum_{t_m=1}^{n_m} I_z(t_w,t_m)\times 100\% \tag{13}$$

where $t_w=(1,\cdots,n_w)$ and $P_f^z(t_w)$ is the integrated failure probability of the slope for the t_w th groundwater level.

The integrated failure probability P_F^z of the soil slope for all groundwater levels is:

$$P_F^z=\frac{1}{n_w\times n_m}\sum_{t_w=1}^{n_w}\sum_{t_m=1}^{n_m} I_z(t_w,t_m)\times 100\%=\frac{1}{n_w}\sum_{t_w=1}^{n_w} P_f^z(t_w) \tag{14}$$

Element failure probability (EFP)

The integrated failure probability Equation (13) is widely used in the calculation of the slope failure probability. However, there are some shortcomings in this method: (1) Equation (13) only considers the size of the safety factor and does not consider the failure range of the soil corresponding to each safety factor. Therefore, the failure probability does not correspond to the failure modes; (2) Equation (13) implicitly assumes that the slope has only a single failure mode, which is inconsistent with the existence of the multiple failure modes in the slope (Huang et al. 2013; Zhang et al. 2018). To overcome these shortcomings, the authors proposed a new method for the failure analysis of the rock slope – the element failure probability method (Li et al. 2019). According to the theory of the upper bound method, when the mass element of the rock slope has plastic flow, it will have a relative velocity based on the fixed element on the boundary. Therefore, in the velocity field obtained by the upper bound method, when the element velocity is greater than 0, the element has plastic flow (element failure). When the element velocity is equal to 0, the element does not have plastic flow (element safety). In this method, the probability of the failure of the rock slope is calculated by using the safety factor, and the location information of the failure element is used to calculate the spatial distribution of the failure area of the rock slope. The failure probability of the rock slope with multiple failure modes can be accurately calculated. Based on (Li et al. 2019), this study takes the soil slope as the research object, applies the finite element to discrete the soil slope and considers the effect of the random groundwater level on the slope stability. Both the shear parameters and the random groundwater level parameter samples are considered in defining the element failure probability of the soil slope. The failure function of the finite element *e* of the soil slope is defined as:

$$I_e(t_w,t_m)=\begin{cases}0 \text{ if } \lambda_m(t_w,t_m) \geq 1.0 \\ 0 \text{ if } \lambda_m(t_w,t_m) < 1.0 \text{ and } u_c^e(t_w,t_m)=0 \\ 1 \text{ if } \lambda_m(t_w,t_m) < 1.0 \text{ and } u_c^e(t_w,t_m)>0\end{cases} \tag{15}$$

where $t_w=(1,\cdots,n_w)$, $t_m=(1,\cdots,n_m)$, and $e=(1,\cdots,n_e)$; $I_e(t_w,t_m)$ is the failure function of the finite element *e* corresponding to the random number of the t_mth shear parameter under the action of the t_wth groundwater level, $\lambda_m(t_w,t_m)$ is the random number of the safety factor related to the random number $c^r(t_m),\varphi^r(t_m)$ of the t_mth shear parameter for the t_wth groundwater level, $u_c^e(t_w,t_m)$ is the resultant velocity at the centre of the finite element *e* in the velocity field calculated by using the random number $c^r(t_m),\varphi^r(t_m)$ of the t_mth shear parameter under the action of the t_wth groundwater level.

The resultant velocity $u_c^e(t_w,t_m)$ at the centroid of the finite element *e* is calculated as:

$$u_c^e(t_w,t_m)=\sqrt{\left(\frac{1}{3}\sum_{i=1}^{3}u_{xi}^e(t_w,t_m)\right)^2+\left(\frac{1}{3}\sum_{i=1}^{3}u_{yi}^e(t_w,t_m)\right)^2} \tag{16}$$

where $t_w=(1,\cdots,n_w)$, $t_m=(1,\cdots,n_m)$, $e=(1,\cdots,n_e)$, $u_{xi}^e(t_w,t_m)$, and $u_{yi}^e(t_w,t_m)$ are the velocities of node *i* (*i* = (1,2,3)) in the finite element *e* along the *x* and *y* direction calculated by using the random number $c^r(t_m),\varphi^r(t_m)$ of the t_mth shear parameter for the t_wth groundwater level.

The specific meaning of Equation (15) is the following:

(1) When the safety factor of the slope is greater than or equal to 1, i.e. $\lambda_m(t_w,t_m)\geq 1.0$, the slope remains stable and all elements in the slope do not fail. Therefore, the failure function of the finite element *e* is $I_e(t_w,t_m)$ = 0.

(2) When the safety factor of the slope is smaller than 1, i.e. $\lambda_m(t_w,t_m)<1.0$, the slope is unstable. At the same time, if the centroid velocity of a finite element *e* in the slope is equal to 0, i.e., $u_c^e(t_w,t_m)=0$, no plastic flow will occur in this element, and no failure will occur in this element. At this time, the failure function of the finite element *e* is $I_e(t_w,t_m)$ = 0.

(3) When the safety factor of the slope is smaller than 1.0, i.e., $\lambda_m(t_w,t_m)<1.0$, the slope is unstable. At the same time, if the element velocity of finite element *e* in the slope is larger than 0, i.e. $u_c^e(t_w,t_m)>0$, plastic flow occurs in this element, and the element fails. At this point, $I_e(t_w,t_m)$ = 1.0.

The failure probability of the soil slope for ground water level t_w is calculated as:

$$P_f^e(t_w) = \frac{1}{n_m}\sum_{t_m=1}^{n_m} I_e(t_w, t_m) \times 100\% \quad (17)$$

where $t_w = (1,\cdots,n_w)$, $e = (1,\cdots,n_e)$, and $P_f^e(t_w)$ is the failure probability of the finite element *e* in the soil slope for the t_w th groundwater level.

The element failure probability of the slope for all possible groundwater levels is calculated as:

$$P_F^e = \frac{1}{n_w \times n_m}\sum_{t_w=1}^{n_w}\sum_{t_m=1}^{n_m} I_e(t_w, t_m) \times 100\% = \frac{1}{n_w}\sum_{t_w=1}^{n_w} P_f^e(t_w) \quad (18)$$

where $e = (1,\cdots,n_e)$ and P_F^e is the failure probability of the finite element *e* in the slope for all possible groundwater levels.

Validation and application

To verify the rationale and correctness of the proposed reliability upper bound analysis method of the soil slope for random seepage fields, a classical calculation example of the heterogeneous soil slope is selected. The stability of the slope is calculated and analysed with the program written by the authors.

Basic information regarding heterogeneous soil slopes

The example selected in this study is a heterogeneous slope with two layers of soil (as shown in Figure 4) (Ji et al. 2017). The top width of the heterogeneous slope is 40.0 m, the total height is 28.0 m, the slope height is 24.0 m, and the slope ratio of the slope surface is 0.75:1.0. The thickness of soil layer 1 is 18.0 m and is 10.0 m for the soil layer 2. The groundwater level on the right side of the slope is H_w^r, and the groundwater level at the left slope angle is flush with the surface. In this study, the volume weight and the permeability coefficient of the slope soil material are set as the determined values, and the cohesion, the internal friction angle, and the right groundwater level H_w^r of the two layers of the soil material are set as random variables. The finite element grid of the slope is shown in Figure 4. The slope is divided into 624 finite elements, 765 velocity discontinuities, and 1,404 finite element nodes. The purpose of this example is to: (1) calculate the mathematical distribution of the safety factor for the random groundwater level; (2) calculate the probability density curve of the safety factor and the cumulative probability density curve based on the random groundwater level; and (3) calculate the relationship between the integrated failure probability, the element failure probability, and the groundwater level.

Random dispersion of shear parameters of slope soil

The heterogeneous slope is composed of two soil layers. The statistical values of the physical and mechanical parameters of the two soil materials are shown in Table 1. The shear parameter of the lower layer is 1.5 times that of the upper layer. In this study, it is assumed that both the cohesion and the internal friction angle of soil layer conform to a logarithmic normal distribution. Because the sample is very large, this study does not consider the correlation of the soil shear parameters in the two layers. The quantity of random numbers n_m for the soil cohesion and the internal friction is 4,000. Based on the shear parameters of the mean value and coefficient of variation, the random number of the shear parameter is obtained by discretizing Equation (9). The random distribution of the soil cohesion and the internal friction angle is shown in Figure 5 and Figure 6.

Random groundwater level and random seepage field of slope

In this study, it is assumed that the groundwater level conforms to the truncated normal distribution. The mean μ_w of the groundwater level H_w^r is set as 16.0 m, the coefficient of variation δ_w is set as 0.25, and the standard deviation σ_w is set as 4.0. The lower bound H_{lb} of the groundwater level H_w^r is set as 4.01 m, and the upper bound H_{ub} is set as 27.99 m. The number of random numbers for the groundwater level n_w is set as 50. The groundwater level at the edge slope angle is set as 10.0 m. According to Equation (8), 50 random numbers for the groundwater level of H_w^r can be obtained discretely (as shown in Table 2). The histograms of 50 random groundwater level numbers are shown in Figure 6. Among the 50 random numbers, the frequency of the random numbers near the mean value is relatively high, with 21 random numbers appearing between 15.0 m and 18.0 m. The frequency of both the high water level and the low water level is relatively low. The groundwater level sample below 5.0 m appears twice, and the groundwater level sample above 27.0 m appears once.

According to the random number $H_w^r(t_w)$ of 50 groundwater levels, 50 stable seepage fields of the heterogeneous soil slopes are calculated. Pore-water pressure at all nodes in each seepage field is obtained. Figure 7 depicts the contour map of the pore water pressure of the slope for $H_w^r(10)$ = 10.9536 m, $H_w^r(30)$ = 16.7785 m, $H_w^r(40)$ = 19.3883 m, and $H_w^r(50)$ = 27.1907 m. When the local lower water level is low, the variation of the pore water pressure contour line is relatively gradual; when the local lower water level is gradually increasing, the infiltration line is gradually increasing, and the pore water pressure contour line also becomes steep.

Figure 8 shows the variation of the pore water pressure with the water level at three key points in the slope (see Figure 4 for details of the location of the key points). The coordinates of these three key points are P1 (20.00, 4.00), P2 (31.98, 4.00), and P3 (44.00, 4.00), respectively. It is seen that the pore water pressure at three locations increases gradually with the increase of the groundwater level. Figure 9 shows the histogram of 50 random pore water pressures at these three locations. It is seen that the frequency distribution of the pore water pressure is high in the middle and low on both sides. The mean pore water pressure at locations P1, P2, and P3 is -37.28, -59.88, and -77.45 kPa, respectively, and the standard deviations of the pore water pressure are 18.05, 29.02, and 36.93, respectively.

Analysis of the distribution of slope safety factor

According to the stochastic mathematical model Equation (7) of the upper bound method and the calculation process (Figure 3), the reliability of the soil slope is calculated by taking the soil shear parameters and the randomness of the ground water level into account. For each group of groundwater levels, 4,000 groups of the shear parameter samples are investigated. In total, 50×4,000 = 200,000 samples are calculated. The upper bound solutions of 200,000 safety factors and the corresponding velocity fields are obtained. Based on the parallel computational program developed in this study, the calculation on a small workstation (Processor: AMD ThreadRipper 3970X with 32 Cores, Physical Memory: 128GB) takes approximately 74 hours for 200,000 samples with an average time of 1.33 s for each sample.

To verify the proposed model, we compare the calculated results from the upper bound method with the results from the rigid body limit equilibrium method. Two different groundwater levels are considered for comparative analysis, namely, $H_w^r(1)$ = 4.1234 m (the first sample) and $H_w^r(47)$ = 24.8945 m (the 47th sample). To execute the rigid body limit equilibrium method, the Probabilistic Analysis module in GEO-Slope, a widely used commercial software product, is used to calculate the reliability of the slope stability, while the Bishop method is used to calculate the slope stability. The safety factor and the integrated failure probability calculated by the two methods are shown in Table 3. The failure mode of the slope is shown in Figure 10. The comparison of the cumulative probability density curve of the slope safety factor is shown in Figure 11. The analysis and calculation results show that:

(1) The upper bound solutions of the mean value and the standard deviation of the slope safety factor are smaller than those calculated by the Bishop method. The difference of the slope safety calculated using two methods decreases with the increase of the groundwater level. At a low groundwater level ($H_w^r(1)$ = 4.1234 m), the mean difference in the safety factors is 1.5%, and the difference in the standard deviations is 35.1%. At a high groundwater level ($H_w^r(47)$ = 24.8945 m), the mean difference between the safety factors is 0.53%, while the difference in standard deviations is 6.4%.

(2) In terms of slope failure mode, only a shallow slope landslide occurs for the low water level ($H_w^r(1)$ = 4.1234 m) when the Bishop method is used (as shown in Fig. 10a); for the high water level ($H_w^r(47)$ = 24.8945 m), only a deep slope landslide occurs, and the upper and lower soil masses are unstable simultaneously (as shown in Fig. 10b). However, when the upper bound method is used to calculate the slope, a shallow landslide and a deep landslide may occur regardless of the

groundwater level. The velocity fields of a shallow landslide and a deep landslide are shown in Figure 10.

(3) The integrated failure probability $P_f^z(t_w)$ of the slope, as shown in Table 3, demonstrates that there is a large difference between the calculated integrated failure probabilities produced by the upper bound method and the Bishop method. At a low groundwater level (e.g. $H_w^r(1)$ = 4.1234 m), the upper bound solution of the integrated failure probability $P_f^z(1)$ is 0.4%, which is less than that in the high groundwater level (e.g. 0.80% at $H_w^r(47)$ = 24.8945 m). This is consistent with the actual law. However, when the Bishop method is used, the integrated failure probability is 1.88% at the low groundwater level ($H_w^r(1)$ = 4.1234 m) and is 1.01% at the high groundwater level ($H_w^r(47)$ = 24.8945 m), which is inconsistent with the actual law. The main reason for this abnormal phenomenon is that the critical slip surface in the Bishop method is calculated according to the mean value of shear parameters. When the safety factor is calculated using other shear parameter samples, the possibility of deep sliding is ignored at the low groundwater levels, and the possibility of shallow sliding is ignored at the high groundwater levels (as shown in Fig. 10).

(4) Figure 11 shows that the cumulative probability density curve of the Bishop solution is higher than that of the upper bound solution. The difference of the solutions by two methods increases with the decrease of the groundwater level. It is mainly due to the fact that the Bishop method does not fully consider all failure modes of the slope. Actually, the slope is dominated by shallow landslide at the low groundwater level; on the contrary, the slope is dominated by deep landslide at the high groundwater level. However, The Bishop method only considers shallow instability at the low groundwater level and deep instability at the high groundwater level.

According to Fig. 3, the upper bound method is used to calculate the safety factor of the slope stability for 50 groundwater levels. The distribution histogram of the upper bound solution of the safety factor at t_w = 10, 30, 40, and 50 is shown in Figure 12. The 50 probability density curves and cumulative probability density curves of the slope safety factor are shown in Figure 13 and Figure 14. The mean value and standard deviation of the slope safety factor are shown in Figure 15 and Figure 16. Analysis of these figures shows the following: (1) The safety factor of the slope generally fits with the normal distribution. With the rise of the groundwater level, the mean value of the safety factor gradually decreases, and the probability density curve and the cumulative probability density curve gradually shift to the left. This finding indicates that the higher the groundwater level is, the lower the safety of the slope is. (2) As the groundwater level rises, the standard deviation of the safety factor gradually decreases, the distribution range of the probability density curve gradually becomes narrow, and the cumulative probability density curve gradually steepens. (3) According to the polynomial fitting of the data of the mean value and standard deviation of the safety factor, the relationship between the standard deviation of the safety factor and the groundwater level can be obtained as:

$$Std = 0.000003H_w^3 - 0.000148H_w^2 + 0.001101H_w + 0.265096 \quad (19)$$

The mean value and standard deviation of the safety factor are negatively correlated with the groundwater level.

(4) Both the histograms of the mean value and standard deviation of the safety factor are low on both sides and high in the middle, which is similar to the distribution of the groundwater head. The mean maximum frequency of the safety factor is between 1.56 and 1.57, and the maximum frequency was 14 times. The highest frequency of the standard deviation of the safety factor occurs between 0.255 and 0.258, and the highest frequency is 11 times.

(5) The 200,000 safety factors calculated by the upper bound method are statistically analysed to obtain the distribution characteristics of the random slope safety factors, which take into account both the groundwater level and the shear parameters, as shown in Figure 17. The probability density curve and the cumulative probability density curve of the 200,000 safety factors are given in Figure 18. The mean value of the upper bound solution of 200,000 safety factors is 1.569 and the standard deviation is 0.260.

Failure probability analysis of slope

In this study, two methods are used to analyse the failure of the slope, which are the traditional integrated failure probability method and the element failure probability method developed in this study.

According to Equations (13), the integrated failure probability of homogeneous soil slopes under the action of 50 random groundwater levels is calculated, as shown in Table 4. The variation of the integrated failure probability of the slope for groundwater level t_w is shown in Figure 19. When the local underground water level gradually increases, the integrated failure probability of the slope gradually increases from 0.40% to 1.425%, and the safety of the slope gradually decreases. According to the data of discrete point $P_f^z(t_w)$ of integrated failure probability, the relationship between the integrated failure probability and the groundwater level can be obtained by best fitting:

$$P_f^z = -0.00044H_w^5 + 0.00689H_w^4 - 0.06596H_w^3 + 0.37341H_w^2 - 1.13746H_w + 1.82329 \quad (20)$$

The above equation shows that the relationship between the integrated failure probability of the slope and the groundwater level is a 5-power polynomial. The integrated failure probability of the slope changes only slightly when the local water level is within the range of 4.0 m to 20.0 m. When the local lower water level is greater than 20.0 m, the integrated failure probability of the slope increases rapidly, indicating that the higher the groundwater level is, the greater the integrated failure probability of the slope is.

According to Equation (14), the integrated failure probability of the slope under the action of all possible groundwater levels is calculated to be P_F^z = 0.487%. According to the calculation of the integrated failure probability, the relationship between P_F^z and $P_f^z(t_w)$ is $P_F^z = \sum_{t_w=1}^{n_w} P_f^z(t_w) / n_w$, that is, P_F^z is the average value of $\left(P_f^z(t_w),\ t_w = (1, \cdots, n_w)\right)$.

Fifty random seepage fields of inhomogeneous soil slope for element failure probability are calculated based on Equation (17). Figure 20 shows the slope element failure probability contour for t_w = 5, 30, 40, 42, 45 and 50, while Figure 21 shows the failure mode of the slope velocity field. The relationship between element failure probability and groundwater level of a characteristic element is shown in Figure 22. Figures 20 - 22 show:

(1) When the local lower groundwater level is less than 20.0 m ($t_w \le 40$) (as shown in Figure 20 (a, b, c)), the failure element is mainly located slope in the upper soil mass. At this time, only a shallow landslide occurs in the slope, and the failure area of the slope is consistent with that calculated by the Bishop method (see Figure 11 (a)). When groundwater level is between 4.0 m and 20.0 m, the slope failure is mainly in the range of a height of 10.0 m to 28.0 m in the upper layer soil. The maximum element failure probability in the upper soil changes between 0.400 and 0.475%. The main reason is that the saturation line of pore water pressure on the slope of the upper soil has an insignificant effect and the upper soil pore water pressure is zero. The stability of the slope is not sensitive to the change of the groundwater level.

(2) When the local lower groundwater level is greater than 20.0 m and gradually increases, the failure probability of the element gradually increases, and the failure element of the slope gradually moves towards the lower soil. The higher the groundwater level is, the more elements of deep failure occur in the lower soil, as shown in Figure 20 (d) - (f). It can be seen from the figure that the failure probability of the element of shallow soil is greater than that of the element of deep soil, which indicates that the slope has both shallow and deep landslides when the groundwater level rises. The probability of a shallow landslide is greater than that of a deep landslide.

(3) When the local lower groundwater level is greater than 20.0 m (t_w > 40), the distribution area of failures of the slope gradually expands, which indicates that there are multiple failure modes in the slope as the groundwater level increases. According to the statistical analysis of all the velocity fields, there are four typical velocity fields when the slope fails, as shown in Figure 21. Among these velocity fields, failure mode 1 and failure mode 2 belong to shallow failures, while failure mode 3 and failure mode 4 belong to deep failures. Theoretically, if there is only one failure mode of the slope, the failure probability of each element in the sliding body area of the slope should be equal (as

shown in Figure 21 (a)). In contrast, if there are multiple failure modes in the slope, each failure mode occurs at different times, and the failure area of each failure mode is different in size, then the failure probability of each part of the slope is different. This property means that the failure probability of elements in different positions of the soil is different (as shown in Figure 21 (e ~ f)). When the groundwater level $H_w^r(45)$ = 22.5527 m (as shown in Figure 21 (e)), the failure probability of the soil area of the shallow landslide is between 0.258% and 0.650%, while the failure probability of the soil area of the deep landslide is between 0.071% and 0.258%. When the local water level rises to $H_w^r(50)$ =27.1907 m (as shown in Figure 21 (f)), the failure probability of the soil area of the shallow landslide is between 0.552% and 1.425%, while the failure probability of the soil area of the deep landslide is between 0.149% and 0.552%. In the multiple failure modes, the overlapping soil area has the highest failure probability, as shown in the red area.

(4) The relationship between the failure probability of the five characteristic elements in the slope and the groundwater level is shown in Figure 22, and the location of the five elements is shown in Figure 4. As seen from the figure: (i) Under the action of the same groundwater level, the failure probability of each part of the slope is different. (ii) The failure probability of the same element in the slope is also different for different groundwater levels. The failure probability of the element is positively correlated with the groundwater level. With the increase in groundwater level, the failure probability of all elements increases gradually. (iii) The failure probability of the upper soil is higher than that of the lower soil. Element E1 is located at the top of slope, and element E3 is located at the bottom of the upper soil. Failure occurs in both the shallow and the deep landslides. Element E2 is located at the slope foot, while element E4 and element E5 are located at the upper and middle part of the subsoil, respectively. These three elements, E2, E4, and E5, only fail when the slope has a deep landslide. When the local lower groundwater level is greater than 20.0 m (t_w > 40), both the shallow and the deep landslides will occur in the slope at the same time. The failure probability of the E1 and E3 elements of the upper soil increases faster than that of the E2, E4 and E5 elements. When t_w > 43, the failure probability of the 5 elements adheres to the following rules: E1 > E3 > E2 and E4 > E5.

(5) According to the calculation results of the traditional Bishop method, only shallow landslides occur when the groundwater level is low, and only deep landslides occur when the groundwater level is high. These limitations are caused by the calculation principle. The Bishop method calculates the critical slip surface of slope according to the average shear parameters. According to this critical slip surface with a random numbers of shear parameters, the reliability is then calculated using the Monte Carlo calculation. Therefore, for each set of shear parameters from the random sample, the critical slip surface obtained by the Bishop method is not realistic. Therefore, the failure

mode calculated by the Bishop method is not complete. The limitations of the Bishop method lead to the ignorance of some samples of the failure mode. Based on the upper bound method, the proposed element failure probability method can completely determine all the slope failure modes.

According to Equation (18), the element failure probability of the slope for all possible groundwater levels is calculated. The contour map of the element failure probability of the slope is shown in Figure 23. The maximum element failure probability of the slope is 0.4865%. The shallow landslide failure occurs in the soil area surrounded by the 0.43% isoline. The element failure probability of the deep landslide is 0.008 to 0.43%. Shallow failure is more likely than deep failure.

Through the example analysis in this study, the differences between IFP and EFP can be summarised as follows:

(1) The IFP only determines whether the slope fails according to whether the safety factor of the slope stability is less than 1.0 and IFP only reflects the degree of the failure probability of slope. While the EFP proposed in this paper investigates the failure possibility of each element in the slope, it simultaneously considers the two attributes of the slope failure, namely: the slope failure probability is calculated by the safety factors, and the difference in the spatial distribution of the slope failure area is calculated by the location information of the element. The safety degree of each part of slope is accurately described by the EFP.

(2) The IFP is specific to the overall stability of the slope, while the EFP calculates the failure of specific elements in the slope. EFP firstly judges the slope instability by the safety factor being less than 1.0, under the condition of slope instability, the element failure can be judged by whether the element velocity is greater than 0. Each element in the slope is in a different position, which produces a different failure probability for each element. If the same samples of safety factors is used by the IFP and the EFP, element failure probability must be equal to the integral failure probability.

(3) For slopes with only one failure mode, the two methods have the same results. However, for slopes with multiple failure modes, the EFP method is more accurate and efficient than the IFP.

Conclusion

This study provides a new approach for the reliability analysis of the soil slope stability, which is based on the upper bound method of the plastic limit analysis theory, as it considers the shear parameters of soil and the randomness of the groundwater level. The distribution of the safety factor and the element failure probability of the soil slope by considering the randomness of the shear parameters and the groundwater level is obtained.

The traditional slope failure analysis (IFP) has a large error when it is used to analysis the failure probability of slope with multiple failure modes. This paper studies the possibility of each element in the soil slope failure by considering the double attribute of the slope failure, that is, the application of the safety factor probability of the slope failure and the application of the location information of the element statistical differences in the spatial distribution of the slope failure area. The results calculated using the element failure probability (EFP) method proposed in this study and the traditional integrated failure probability method are similar under the single failure mode of the slope. However, in the calculation of the slope failure probability with multiple failure modes, the element failure probability method shows its advantage. The element failure probability method of soil slopes for each element has the comprehensive safety assessment of the slope has reference significance. The slope is transformed from "*integrated failure probability analysis*" to "*element failure probability analysis*", which provides a new method for slope failure analysis.

Data Availability Statement

All data generated or used during the study are available from the corresponding author by request.

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Appendix I. Notation

The following symbols are used in this paper:

$\mathbf{A}^b$	=	coordinate transformation matrix of the finite element b on the boundary
$\mathbf{A}_1^e$	=	matrix of plastic flow constraint conditions of the finite element e
$\mathbf{A}_2^e$	=	matrix of plastic flow constraint conditions of the finite element e
$\mathbf{A}_1^d$	=	matrix of plastic flow constraint conditions of the velocity discontinuity d
$\mathbf{A}_2^d$	=	matrix of plastic flow constraint conditions of the velocity discontinuity d
c^r	=	random variables of the cohesion of the soil materials
c'^r	=	random quantity of cohesion after the intensity reduction
$c^r(t_m)$	=	the t_m th random number on the materials of the soil cohesion
$c'^r(t_m)$	=	the t_m th random number of the soil cohesion after strength reduction
H_w^r	=	random variable of the groundwater level of the soil slope
$H_w^r(t_w)$	=	the t_w th random number of the groundwater level of the soil slope
H_{lb}	=	the lower bound of the groundwater level of the soil slope
H_{ub}	=	the upper bound of the groundwater level of the soil slope
$I_z(t_w, t_m)$	=	failure function of the soil slope corresponding to the random number of the t_m the random shear parameter under the action of the t_w th groundwater level
$I_e(t_w, t_m)$	=	the failure function of the finite element e corresponding to the random number of the shear t_m th parameter under the action of the t_w th underground level
n_b	=	the quantity of the finite elements on the boundary of the soil slope
n_d	=	the quantity of the velocity discontinuities in the soil slope
n_e	=	the quantity of finite elements in the soil slope
n_m	=	the quantity of material for the soil cohesion and the friction angle of the Monte Carlo random number
n_w	=	the quantity of the Monte Carlo random numbers of the groundwater level of the soil slope
$\mathbf{p}_e^r$	=	pore water pressure vector of finite element e
$p_{ei}^r(t_w)$	=	the pore water pressure at nodes i (i=1,2,3) in finite element e under the action of t_w th groundwater level

p_{ei}^{r}	=	random variable of the pore water pressure at nodes i in finite element e
P_F^z	=	the integrated failure probability of the slope under the action of all possible groundwater levels
P_F^e	=	the failure probability of the finite element e in the slope under the action of all possible groundwater levels
$P_f^e(t_w)$	=	the failure probability of the finite element e in the soil slope under the action of the t_w th groundwater level
$P_f^z(t_w)$	=	the integrated failure probability of the slope under the action of the t_w th groundwater level
$\mathbf{T}^b$	=	transformation matrix of the finite element b on the boundary
$\mathbf{T}^d$	=	transformation matrix of the velocity discontinuity d
$\mathbf{u}^b$	=	velocity vector of the boundary finite element b
$\mathbf{u}^d$	=	velocity vector of the velocity discontinuity d
$\mathbf{u}^e$	=	velocity vector of finite element e
$u_c^e(t_w,t_m)$	=	resultant velocity at the centroid of the finite element e corresponding to the random number of the shear t_m th parameter under the action of the t_w th underground water level
u_{xi}^d	=	the velocity of the ith (i=(1,...,4)) node on the velocity discontinuity plane d along the direction x
u_{yi}^d	=	the velocity of the ith (i=(1,...,4)) node on the velocity discontinuity plane d along the direction y
u_{xi}^e	=	velocity of nodes i (i=1,…,3) in the finite element e along the direction x
$u_{xi}^e(t_w,t_m)$	=	the velocity of node i (i =(1,2,3)) in the finite element e along the x direction calculated by using the random number $c^r(t_m),\varphi^r(t_m)$ of the t_m th shear parameter under the action of the t_w th groundwater level.
u_{yi}^e	=	velocity of nodes i (i=1,…,3) in the finite element e along the direction y
$u_{yi}^e(t_w,t_m)$	=	the velocity of node i (i =(1,2,3)) in the finite element e along the y direction calculated by using the random number $c^r(t_m),\varphi^r(t_m)$ of the t_m th shear parameter under the action of the t_w th groundwater level.
$\mathbf{W}_{Ex1}$	=	external work power done by the dead weight on the velocity of the finite element nodes
$\mathbf{W}_{Ex2}$	=	external power done by concentrated force and distributed load at the

		velocity of the finite element nodes
$\mathbf{W}_{Ex3}^{p}$	=	external work power of the pore water pressure in the finite element continuous body
$\mathbf{W}_{Ex4}^{p}$	=	external work power done by pore water pressure on the finite element velocity discontinuities
$\mathbf{W}_{In1}$	=	internal power of finite elements
$\mathbf{W}_{In2}$	=	internal power of the velocity discontinuities
Z	=	limit state function of the soil slope reliability
γ_a	=	the real volume weight of the soil material
$\gamma_c(c^r, \varphi^r, H^r)$	=	random variable of the ultimate value of volume weight that relates to c^r, φ^r and H^r when the soil reaches the limit state
$\gamma_c(c^r(t_m), \varphi^r(t_m), H_w^r(t_w))$	=	the ultimate volume weight of the soil slope in the limit state when it reaches the instability related to the t_m th random shear parameter under the action of t_w th groundwater level
γ_e	=	volume weight of finite element e .
θ_d	=	inclination angle of the velocity discontinuity *d*,
θ_b	=	dip angle of the boundary
$\boldsymbol{\lambda}^d$	=	vector of non-negative plastic multiplier of the velocity discontinuity *d*
$\boldsymbol{\lambda}^e$	=	vector of nonnegative plastic multiplier of finite element e
λ_m^r	=	the random variable of the safety factor that relates to c^r, φ^r and H^r
$\lambda_m(t_w, t_m)$	=	the safety factor of the random number corresponding to the t_m th random shear parameter under the action of the t_w th ground water level
λ_γ^r	=	random variable of the overload factor of volume weight that relates to c^r, φ^r and H^r
$\lambda_\gamma(t_w, t_m)$	=	volume weight overload factor corresponding to the random number of the random shear parameter t_m under the action of the t_w th ground water level
μ_c	=	the mean value of the material cohesion of the soil
μ_w	=	the mean groundwater level of the soil slope
μ_φ	=	the mean value of the friction angle of the soil material
σ_c	=	the standard deviation of the soil cohesion
σ_w	=	the standard deviation of the groundwater level of the soil slope

σ_{φ}	=	the standard deviation of the friction angle of the soil materials
φ^{r}	=	random variables of the internal friction angle of the soil materials
φ_{d}^{r}	=	random quantity of the internal friction angle of the velocity discontinuity plane *d*
φ_{e}^{r}	=	random quantity of friction angle of the finite element *e* of soil slope.
$\varphi^{r}(t_{m})$	=	the t_{m} th random number of the friction angle of the soil material
φ'^{r}	=	random quantity of the internal friction angle after the intensity reduction
$\varphi'^{r}(t_{m})$	=	the t_{m} th random number of the internal friction angle of soil after strength reduction

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List of Tables

List of Figures