ON THE THEORY OF BETA-RADIOACTIVITY III

THE INFLUENCE OF ELECTRIC AND MAGNETIC FIELDS ON POLARIZED ELECTRON BEAMS *)

by H. A. TOLHOEK and S. R. DE GROOT

Instituut voor theoretische natuurkunde, Universiteit, Utrecht, Nederland

Synopsis

The influence of electric and magnetic fields on the spin orientation (polarization) of electrons in a beam is calculated according to the P a u l i spin theory and the D i r a c theory. For the cases, where the field is perpendicular or parallel to a polarized electron beam, the following results are found.

Transverse electric field. In the non-relativistic approximation the spin orientation remains constant in space, even if the beam is deflected; the relativistic formula gives for the ratio of the rotation of the spin orientation and the angle of deflection of the beam: E_{kin}/E (ratio of kinetic energy and total energy, i.e., including the rest mass).

Transverse magnetic field. The spin orientation does not change in relation to the direction of propagation.

Longitudinal electric field. Though the beam is accelerated (or retarded) the spin orientation remains constant in space.

Longitudinal magnetic field. The spin orientation rotates about the direction of propagation.

It is shown that longitudinal polarization of electron beams (spins parallel or antiparallel to the direction of propagation) can be observed by means of an electric deflection of the beam and a scattering experiment in succession.

§ 1. *Introduction.* In this paper and the previous one of this series on beta-radioactivity [1] [2] [3] a number of problems concerning the polarization of electron beams are considered in view of application to the problem of polarization of the β-rays emitted by aligned nuclei [3]. We shall study the influence of transverse and longitudinal electric and magnetic fields on the orientation of the

*) Formulae from the first and second paper of this series [1] [2] will be quoted as, say, I (10) or II (12).

spin in electron beams (we call a field transverse or longitudinal according to its being perpendicular or parallel to the direction of propagation of the electron beam).

We have made the calculations by taking a plane wave as the starting-point and by calculating the deviation from the plane wave caused by the field in a first approximation. We consider only the case of homogeneous or slowly varying fields (i.e., the variation of the fields over distances of the order of the electron wave length is negligible). We shall start (§ 2) with a treatment of the deflection and acceleration of particles without spin, according to the relativistic wave-equation of Klein-Gordon as an introduction to the method of calculation. After that we treat the problem with the Pauli spin theory (§ 3) and the Dirac equation of the spin electron (§§ 4–6).

Generally the behaviour of electron beams in slowly varying fields is treated with the classical particle model. However, for the description of the spin of the electron we have to use wave equations from the beginning. By forming wave packets from our solutions, which are approximately plane waves, wave functions could be obtained that approximate the particles of the classical model.

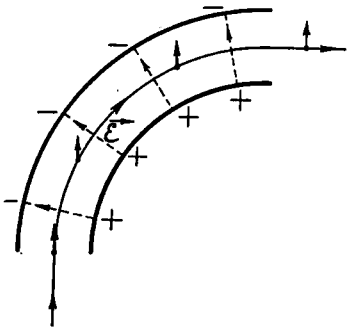

Fig. 1. Deflection of an electron beam in an electric field; non-relativistic approximation. Dotted lines: electric field lines, short arrows: spin orientation.

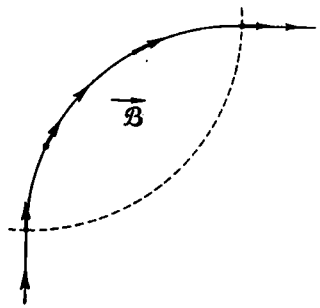

Fig. 2. Deflection of an electron beam in a magnetic field; magnetic field perpendicular to the plane of the figure. Short arrows: spin orientation.

In the preceding paper it was shown that in scattering experiments the transverse polarization is observed. Here we shall show that longitudinal polarization can be changed into transverse polarization by means of a transverse electric field (Fig. 1 and 2 show diagrams

for beams which are deflected over a right angle by an electric and by a magnetic field). So the complete determination of an arbitrary state of polarization is possible.

§ 2. *Deflection and acceleration of particles according to the Klein-Gordon equation.* We start with the Klein-Gordon equation

$$[(1/c^2)\,(E - \epsilon\,\Phi)^2 - \Sigma_{x,y,z}\,(p_x - (\epsilon/c)\,A_x)^2 - m^2c^2]\,\psi = 0, \qquad (1)$$

in which E and p_x are the operators: $E = -(\hbar/i)\partial/\partial t$ and $p_x = (\hbar/i)\partial/\partial x$. We take particles with charge ϵ; if necessary we can specialize to negatons by putting $\epsilon = -e$ and to positons by putting $\epsilon = e$. The values for Φ, the scalar potential and $\mathbf{A}$, the vector potential, will be inserted later for the cases to be studied (we suppose the fields constant in time). We look for solutions of (1) of the shape (cf. [4]))

$$\psi = \psi_0 \exp(i\,\epsilon\,f), \qquad (2)$$

in which ψ_0 satisfies (1) if we put $\epsilon = 0$, and for which we take a plane wave

$$\psi_0 = \exp[(i/\hbar)\,(\mathbf{p}\cdot\mathbf{x} - Et)] \quad \text{with} \quad E^2 = p^2c^2 + m^2c^4. \qquad (3)$$

f is a function of x, y, z restricted by an equation, which is obtained from (1) by the substitution of (2). If we take into account that ψ_0 satisfies (1) if we put $\epsilon = 0$, and if we neglect the terms with ϵ^2 (i.e., we calculate only small deviations from the plane wave), the following equation for f is found (in this deduction the relation $\operatorname{div}\mathbf{A} = 0$ is used)

$$2\hbar\mathbf{p}\cdot\operatorname{grad} f = -2E\Phi/c^2 + 2\mathbf{p}\cdot\mathbf{A}/c + i\hbar^2\,\Delta f. \qquad (4)$$

If we take for ψ_0 a plane wave in the x-direction, we have $p_x = p$, $p_y = 0$, $p_z = 0$ and (4) reduces to

$$\partial f/\partial x = -E\Phi/\hbar pc^2 + A_x/\hbar c + i(\hbar/2p)\,\Delta f. \qquad (5)$$

We shall study the influence of the following fields ($\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the unit vectors in the x, y and z-direction)

a) transverse electric field:

$$\mathcal{E} = \mathcal{E}\mathbf{j},\ \Phi = -\mathcal{E}y,\ \mathbf{A} = 0, \qquad (6)$$

b) transverse magnetic field:

$$\mathcal{B} = \mathcal{B}\mathbf{k},\ \Phi = 0,\ A_x = -\mathcal{B}y,\ A_y = 0,\ A_z = 0, \qquad (7)$$

c) longitudinal electric field:

$$\mathcal{E} = \mathcal{E}\mathbf{i},\ \Phi = -\mathcal{E}x,\ \mathbf{A} = 0, \tag{8}$$

d) longitudinal magnetic field:

$$\mathcal{B} = \mathcal{B}\mathbf{i},\ \Phi = 0,\ A_x = 0,\ A_y = -\mathcal{B}z,\ A_z = 0. \tag{9}$$

It is easily verified that the following expressions for f satisfy (5) for these cases

a) $$f = (E\mathcal{E}/\hbar pc^2)\, xy, \tag{10}$$

b) $$f = -(\mathcal{B}/\hbar c)\, xy, \tag{11}$$

c) $$f = (E\mathcal{E}/2\hbar pc^2)\, x^2 + i(E\mathcal{E}/2p^2c^2)\, x, \tag{12}$$

d) $$f = 0. \tag{13}$$

In this way we find the solutions

a) $\psi = \exp\{(i/\hbar)\,[(px - Et) + (\mathrm{e}\,E\mathcal{E}/pc^2)\, xy]\}$, (14)

b) $\psi = \exp\{(i/\hbar)\,[(px - Et) - (\mathrm{e}\,\mathcal{B}/c)\, xy]\}$, (15)

c) $\psi = \exp\{(i/\hbar)\,[(px - Et) + (\mathrm{e}\,E\mathcal{E}/2pc^2)\, x^2]\} \exp[-(\mathrm{e}E\mathcal{E}/2p^2c^2)x]$, (16)

d) $\psi = \exp\{(i/\hbar)\,(px - Et)\}$. (17)

The influence of the fields exists in a certain change of momentum if the beam has traversed a distance x; we calculate the changes in (kinetic) momentum $\Delta\pi_x$, $\Delta\pi_y$, $\Delta\pi_z$ for this case. (In the results π_x, $\Delta\pi_x$, etc. are meant as mean values of the operators $\pi_x = (\hbar/i)\partial/\partial x - (\mathrm{e}/c_x)\, A_x$, ... for the solutions (14)–(17) in a sense that is explaned in the appendix). For the transverse fields we define the angle $\Delta\gamma = \Delta\pi_y/\pi_x$, which determines the angle of deflection of the beam. For $x = 0$: $\pi_x = p$, $\pi_y = 0$, $\pi_z = 0$. We obtain the results

a) $\Delta\pi_x = 0,\ \Delta\pi_z = 0,\ \Delta\pi_y = (\mathrm{e}\,E\mathcal{E}/pc^2)\, x,\ \Delta\gamma_{\mathcal{E}} = (\mathrm{e}\,E\mathcal{E}/p^2c^2)\, x,$ (18)

b) $\Delta\pi_x = 0,\ \Delta\pi_z = 0,\ \Delta\pi_y = -(\mathrm{e}\,\mathcal{B}/c)\, x,\ \Delta\gamma_{\mathcal{B}} = -(\mathrm{e}\,\mathcal{B}/pc)\, x,$ (19)

c) $\Delta\pi_y = 0,\ \Delta\pi_z = 0,\ \Delta\pi_x = (\mathrm{e}\,E\mathcal{E}/pc^2)\, x.$ (20)

d) $\Delta\pi_x = 0,\ \Delta\pi_y = 0,\ \Delta\pi_z = 0.$ (21)

A longitudinal magnetic field has no influence on the beam. In case of the magnetic fields, the calculation takes a rather different shape if we take other forms for $\mathbf{A}$, but the final result is of course the same, because of the gauge invariance in quantum mechanics.

§ 3. *Deflection and acceleration of electrons according to the Pauli spin theory.* In this section, we study the change of the spin orienta-

tion of electrons in a beam in electric and magnetic fields, according to the Pauli spin theory.

We start with the wave equation

$$[(E - e\Phi) - (p^2/2m) + (e/mc)(\mathbf{p}\cdot\mathbf{A}) + \\ + (e\hbar/2mc)\boldsymbol{\sigma}\cdot(\mathcal{B} + \mathcal{E}\curlywedge\mathbf{p}/2mc)]\psi = 0, \tag{22}$$

in which $\boldsymbol{\sigma}$ is the "vector" with the Pauli spin matrices as components. We shall investigate solutions of (22) in the same way as in § 2: we again use the substitution

$$\psi = [\exp(ief)]\psi_0. \tag{23}$$

The wave functions have now two components; ψ_0 satisfies (22) if we put $e = 0$; we take for ψ_0

$$\psi_0 = \begin{pmatrix} A \\ B \end{pmatrix} \exp[(i/\hbar)(\mathbf{p}\cdot\mathbf{x} - Et)] \quad \text{with} \quad E = p^2/2m. \tag{24}$$

The function f now becomes a 2×2 matrix, which can be expressed, e.g., with the aid of the Pauli spin matrices. Substituting (23) in (22) we get

$$[-2\hbar\mathbf{p}\cdot\operatorname{grad} f - 2m\Phi + 2\mathbf{p}\cdot\mathbf{A}/c + i\hbar^2\Delta f + \\ + (\hbar/c)\boldsymbol{\sigma}\cdot(\mathcal{B} + \mathcal{E}\curlywedge\mathbf{p}/2mc)][\exp(ief)]\psi_0 = 0. \tag{25}$$

(25) is a matrix equation, in which $\mathbf{p}$ is no longer an operator, but has the value from (24). The two constants A and B of ψ_0 in (25) can have an arbitrary ratio as we study waves with arbitrary spin orientation. From this it can be derived that the expression between brackets in (25) can be put equal to 0 separately. If we again take a plane wave in the x-direction for ψ_0, we get

$$\partial f/\partial x = -m\Phi/\hbar p + A_x/\hbar c + i(\hbar/2p)\Delta f + \\ + (1/2pc)\boldsymbol{\sigma}\cdot(\mathcal{B} + \mathcal{E}\curlywedge\mathbf{p}/2mc). \tag{26}$$

If we again take the cases (6)–(9) for the electromagnetic field it is easily verified that the following expressions for f satisfy (26)

$$a)\quad f = (m\mathcal{E}/\hbar p)xy - (\mathcal{E}/4mc^2)\sigma_z x, \tag{27}$$

$$b)\quad f = -(\mathcal{B}/\hbar c)xy + (\mathcal{B}/2pc)\sigma_z x, \tag{28}$$

$$c)\quad f = (m\mathcal{E}/2\hbar p)x^2 + i(m\mathcal{E}/2p^2)x, \tag{29}$$

$$d)\quad f = (\mathcal{B}/2pc)\sigma_x x. \tag{30}$$

Hence we find for ψ

$$a)\quad \psi = \exp\left[-i(\mathrm{e}\mathcal{E}/4mc^2)\,\sigma_z x\right]\begin{pmatrix}A\\B\end{pmatrix}\exp\left\{(i/\hbar)\left[(px - Et) + (\mathrm{e}\,m\mathcal{E}/p)\,xy\right]\right\}, \tag{31}$$

$$b)\quad \psi = \exp\left[i(\mathrm{e}\,\mathcal{B}/2pc)\,\sigma_z x\right]\begin{pmatrix}A\\B\end{pmatrix}\exp\left\{(i/\hbar)\left[(px - Et) - (\mathrm{e}\,\mathcal{B}/c)\,xy\right]\right\}, \tag{32}$$

$$c)\quad \psi = \begin{pmatrix}A\\B\end{pmatrix}\exp\left\{(i/\hbar)\left[(px - Et) + (\mathrm{e}\,m\mathcal{E}/2p)\,x^2\right]\right\} \exp\left[-(\mathrm{e}\,m\mathcal{E}/2p^2)\,x\right], \tag{33}$$

$$d)\quad \psi = \exp\left[i\,(\mathrm{e}\,\mathcal{B}/2pc)\,\sigma_x x\right]\begin{pmatrix}A\\B\end{pmatrix}\exp\left\{(i/\hbar)\,(px - Et)\right\}. \tag{34}$$

We use the same symbols as in § 2 for the change of momentum and the deflection. As the Pauli spin matrices with the factor $-(i/2)$ are the operators for the infinitesimal rotations (cf. II (13)), it is an immediate consequence of (31)–(34) that we can describe the change of the spin orientation as a rotation the absolute value of which we shall denote as $\Delta\alpha$. The results in the four cases are

$$a)\quad \Delta\pi_x = 0,\ \Delta\pi_z = 0,\ \Delta\pi_y = (\mathrm{e}\,m\mathcal{E}/p)\,x,\ \Delta\gamma_{\mathcal{E}} = (\mathrm{e}\,m\mathcal{E}/p^2)\,x, \tag{35}$$

$$\Delta\alpha_{\mathcal{E}\perp} = (\mathrm{e}\,\mathcal{E}/2mc^2)\,x,\ \text{rotation about the } z\text{-axis}, \tag{36}$$

$$\Delta\alpha_{\mathcal{E}\perp}/\Delta\gamma_{\mathcal{E}} = p^2/2m^2c^2 = \tfrac{1}{2}(v/c)^2; \tag{37}$$

$$b)\quad \Delta\pi_x = 0,\ \Delta\pi_z = 0,\ \Delta\pi_y = -(\mathrm{e}\,\mathcal{B}/c)\,x,\ \Delta\gamma_{\mathcal{B}} = -(\mathrm{e}\,\mathcal{B}/pc)\,x, \tag{38}$$

$$\Delta\alpha_{\mathcal{B}\perp} = -(\mathrm{e}\,\mathcal{B}/pc)\,x,\ \text{rotation about the } z\text{-axis}, \tag{39}$$

$$\Delta\alpha_{\mathcal{B}\perp}/\Delta\gamma_{\mathcal{B}} = 1; \tag{40}$$

$$c)\quad \Delta\pi_y = 0,\ \Delta\pi_z = 0,\ \Delta\pi_x = (\mathrm{e}\,m\mathcal{E}/p)\,x,\ \Delta\alpha_{\mathcal{E}\parallel} = 0; \tag{41}$$

$$d)\quad \Delta\pi_x = 0,\ \Delta\pi_y = 0,\ \Delta\pi_z = 0, \tag{42}$$

$$\Delta\alpha_{\mathcal{B}\parallel} = -(\mathrm{e}\,\mathcal{B}/pc)\,x,\ \text{rotation about the } x\text{-axis}. \tag{43}$$

For the cases of a deflection of the beam by transverse fields we have calculated the quotients $\Delta\alpha/\Delta\gamma$ to compare the rotation of the spin orientation with the deflection of the beam.

In the formulae (35)–(43) $p = mv$; we have used the non-relativistic approximation for the translatory motion of the electron; in this approximation these formulae agree with (18)–(21). For a transverse magnetic field we see that the orientation of the spin remains the same in relation to the direction of the beam (cf. Fig. 2). For a

transverse electric field the rotation of the spin orientation is of the relativistic order $\frac{1}{2}(v/c)^2$, so that the spin orientation remains nearly constant in space for small kinetic energies, which result could be expected (cf. Fig. 1).

§ 4. *Generalities on the treatment with the Dirac equation.* In §§ 5 and 6 we shall treat the same problem as in § 3, but starting from the Dirac equation. For this treatment we shall use some formulae given in this section. The Dirac equation consists of four simultaneous first order equations for the four components of the wave functions. We shall show that we can split it up for special problems into two pairs of equations with each only two functions. Further we give the reduction to one second order differential equation for a number of cases (cf. also [5]) [6]) [7]).

The Dirac equation for an electron in an electromagnetic field will be used in the notation

$$\left.\begin{aligned}
(\varepsilon/c + mc)\,\psi_1 + (\pi_x - i\pi_y)\,\psi_4 + \pi_z\psi_3 &= 0,\\
(\varepsilon/c + mc)\,\psi_2 + (\pi_x + i\pi_y)\,\psi_3 - \pi_z\psi_4 &= 0,\\
(\varepsilon/c - mc)\,\psi_3 + (\pi_x - i\pi_y)\,\psi_2 + \pi_z\psi_1 &= 0,\\
(\varepsilon/c - mc)\,\psi_4 + (\pi_x + i\pi_y)\,\psi_1 - \pi_z\psi_2 &= 0,
\end{aligned}\right\} \tag{44}$$

with

$$\varepsilon = -(\hbar/i)\,\partial/\partial t - e\,\Phi,\quad \pi_x = (\hbar/i)\,\partial/\partial x - (e/c)\,A_x, \text{ etc.} \tag{45}$$

If we treat problems, which are independent of the z-coordinate and for which $A_z = 0$, (44) is reduced to two pairs of equations

$$\left.\begin{aligned}
(\varepsilon/c + mc)\,\psi_1 + (\pi_x - i\pi_y)\,\psi_4 &= 0,\\
(\varepsilon/c - mc)\,\psi_4 + (\pi_x + i\pi_y)\,\psi_1 &= 0,
\end{aligned}\right\} \tag{46}$$

$$\left.\begin{aligned}
(\varepsilon/c + mc)\,\psi_2 + (\pi_x + i\pi_y)\,\psi_3 &= 0,\\
(\varepsilon/c - mc)\,\psi_3 + (\pi_x - i\pi_y)\,\psi_2 &= 0.
\end{aligned}\right\} \tag{47}$$

It is seen that (47) is obtained from (46) by the substitution

$$\psi_1 \to \psi_2,\ \psi_4 \to \psi_3,\ \pi_y \to -\pi_y. \tag{48}$$

We give a further reduction of the problem for four cases

α) ε commutes with π_x and π_y,
β) π_y commutes with ε and π_x,
γ) π_x commutes with ε and π_y,
δ) No electric field exists: $\mathfrak{E} = 0$.

It is not necessary that the last two operators of the cases α, β, γ commute. We now give the ways in which the reductions can be carried out.

α) We obtain from (46)

$$(\varepsilon/c + mc)\,(\varepsilon/c - mc)\,\psi_4 = -(\varepsilon/c + mc)\,(\pi_x + i\pi_y)\,\psi_1 = \\ = -(\pi_x + i\pi_y)\,(\varepsilon/c + mc)\,\psi_1.$$

Hence

$$(\varepsilon^2/c^2 - m^2c^2)\,\psi_4 - (\pi_x + i\pi_y)\,(\pi_x - i\pi_y)\,\psi_4 = 0. \tag{49}$$

β) Instead of ψ_1 and ψ_4 we can introduce the function φ_1 and φ_4

$$\left.\begin{aligned} \varphi_1 &= \tfrac{1}{2}(\psi_1 + \psi_4) \\ \varphi_4 &= \tfrac{1}{2}(\psi_1 - \psi_4) \end{aligned}\right\} \qquad \left.\begin{aligned} \psi_1 &= \varphi_1 + \varphi_4 \\ \psi_4 &= \varphi_1 - \varphi_4 \end{aligned}\right\} \tag{50}$$

From (46) we get by addition and subtraction

$$\left.\begin{aligned} (\varepsilon/c + \pi_x)\,\varphi_1 + (mc + i\pi_y)\,\varphi_4 &= 0, \\ (\varepsilon/c - \pi_x)\,\varphi_4 + (mc - i\pi_y)\,\varphi_1 &= 0. \end{aligned}\right\} \tag{51}$$

From (51) a second order equation for φ_4 can be deduced

$$(\varepsilon/c + \pi_x)\,(\varepsilon/c - \pi_x)\,\varphi_4 - (m^2c^2 + \pi_y^2)\,\varphi_4 = 0. \tag{52}$$

γ) We introduce the functions ω_1 and ω_4 according to

$$\left.\begin{aligned} \omega_1 &= \tfrac{1}{2}(\psi_1 + i\psi_4) \\ \omega_4 &= \tfrac{1}{2}(\psi_1 - i\psi_4) \end{aligned}\right\} \qquad \left.\begin{aligned} \psi_1 &= \omega_1 + \omega_4 \\ i\psi_4 &= \omega_1 - \omega_4 \end{aligned}\right\} \tag{53}$$

From (46) the following equations for ω_1 and ω_4 are obtained

$$\left.\begin{aligned} (\varepsilon/c - \pi_y)\,\omega_1 + (mc + i\pi_x)\,\omega_4 &= 0, \\ (\varepsilon/c + \pi_y)\,\omega_4 + (mc - i\pi_x)\,\omega_1 &= 0. \end{aligned}\right\} \tag{54}$$

From (54) we get as a second order equation

$$(\varepsilon/c - \pi_y)\,(\varepsilon/c + \pi_y)\,\omega_4 - (m^2c^2 + \pi_x^2)\,\omega_4 = 0. \tag{55}$$

If the 4-components have been obtained from the second order equation, the 1-components must be calculated from a first order differential expression according to (47), (51) or (54). The 2- and 3-components can be treated analogously to the 1- and 4-components, using the substitution (48).

δ) An analogous reduction of the problem is possible in the case that there is *no electric field*: $\Phi = 0$, $\mathbf{A}$ constant, $\varepsilon = E$. Then the iterated Dirac equation contains only two wave functions; it follows directly from (44) (here the problem may contain the z-coordinate)

$$[E^2/c^2 - (\pi_z^2 + \pi_y^2 + \pi_2^2) - m^2c^2]\begin{pmatrix}\psi_3\\ \psi_4\end{pmatrix} + (e\,\hbar/c)\,\mathcal{B}\cdot\boldsymbol{\sigma}\begin{pmatrix}\psi_3\\ \psi_4\end{pmatrix} = 0. \tag{56}$$

After determination of ψ_3 and ψ_4, ψ_1 and ψ_2 can be derived from the first two equations of (44).

§ 5. *The influence of electric fields on electron beams according to the Dirac theory.* In this section we study the influence of transverse and longitudinal electric fields on the spin electron with the aid of the Dirac equation. We take a plane wave moving in the x-direction in case there would be no field. We use such fields that the z-coordinate does not enter. Then we can apply the considerations of § 4.

$\mathcal{E}_\perp$ *Transverse electric field*

$$\boldsymbol{\mathcal{E}} = \mathcal{E}\mathbf{j}, \quad \Phi = -\mathcal{E}y, \quad \mathbf{A} = 0. \tag{57}$$

$$\varepsilon = -(\hbar/i)\,\partial/\partial t + \mathfrak{e}\,\mathcal{E}y = E + \mathfrak{e}\,\mathcal{E}y, \quad \boldsymbol{\pi} = \mathbf{p}. \tag{58}$$

We have the case γ of § 4; using (58) equation (55) becomes

$$(E^2/c^2 - m^2c^2)\,\omega_4 + \hbar^2\,(\partial^2/\partial x^2 + \partial^2/\partial y^2)\omega_4 - \mathfrak{e}\,\mathcal{E}\,(\hbar/ic - 2Ey/c^2)\,\omega_4 = 0 \tag{59}$$

(terms of the order $\mathfrak{e}^2$ are neglected). We put

$$\omega_4 = \omega_{40} \exp(i\mathfrak{e}f) \tag{60}$$

with

$$\omega_{40} = \bar{B} \exp[(i/\hbar)\,(px - Et)]. \tag{61}$$

From (59), (60) and (61) an equation for f is obtained

$$\partial f/\partial x - i(\hbar/2p)\,(\partial^2 f/\partial x^2 + \partial^2 f/\partial y^2) - (\mathcal{E}/2pc)\,i - (\mathcal{E}E/\hbar pc^2)\,y = 0. \tag{62}$$

It is easily verified that we have the solution

$$f = (\mathcal{E}E/\hbar pc^2)\,xy + i(\mathcal{E}/2pc)\,x. \tag{63}$$

We get as solution for ω_4

$$\omega_4 = \bar{B} \exp\{(i/\hbar)\,[(px - Et) + (\mathfrak{e}\,\mathcal{E}E/pc^2)\,xy]\} \exp[-(\mathfrak{e}\,\mathcal{E}/2pc)\,x]. \tag{64}$$

ω_1 is calculated according to (54)

$$\omega_1 = -[1/(mc - ip)]\,[(E/c) + (\mathfrak{e}\,\mathcal{E}\,E/pc^2)\,x + \mathfrak{e}\,uy + \mathfrak{e}s]\,\omega_4 \tag{65}$$

with

$$\left.\begin{aligned} s &= \hbar\mathcal{E}(mc + ip)/2pE, \\ u &= im\mathcal{E}/p. \end{aligned}\right\} \tag{66}$$

(The values of s and u have no influence on our result (74)).

Analogously it is calculated for ω_3 and ω_2

$$\omega_3 = \bar{A} \exp\{(i/\hbar)\,[(px - Et) + (\mathfrak{e}\,\mathcal{E}\,E/pc^2)\,xy]\} \exp[(\mathfrak{e}\,\mathcal{E}/2pc)\,x], \tag{67}$$

$$(\omega_2 = -[1/(mc - ip)]\,[(E/c) - (\mathfrak{e}\,\mathcal{E}\,E/pc^2)\,x + \mathfrak{e}\,uy - \mathfrak{e}s)\,\omega_3. \tag{68}$$

According to (48) and (53) we have

$$i\begin{pmatrix}\psi_3\\ \psi_4\end{pmatrix} = \begin{pmatrix}\omega_2\\ \omega_1\end{pmatrix} - \begin{pmatrix}\omega_3\\ \omega_4\end{pmatrix}. \tag{69}$$

From (69), (64), (65), (67) and (68) we obtain the following expression by a straightforward though rather lengthy calculation (in which terms of order e^2 are neglected)

$$\begin{pmatrix}\psi_3\\ \psi_4\end{pmatrix}_{x,t} = (1 + Mx)\begin{pmatrix}\psi_3\\ \psi_4\end{pmatrix}_{\substack{x=0\\ t=0}} K(x, y, t) \tag{70}$$

with

$$\begin{pmatrix}\psi_3\\ \psi_4\end{pmatrix}_{\substack{x=0\\ t=0}} = \begin{pmatrix}A\\ B\end{pmatrix} = \{[-pc + i(E + mc^2)]/E\}\,(1 - e\,C s\sigma_z)\begin{pmatrix}\bar{A}\\ \bar{B}\end{pmatrix}, \tag{71}$$

$$K(x, y, t) = (1 + e\,Cuy)\exp\{(i/\hbar)\,[(px - Et) + (e\,E\,\mathcal{E}/pc^2)\,xy]\}, \tag{72}$$

$$C = [(E/c) + mc - ip]^{-1} = c(E + mc^2 + ipc)\,[2E(E + mc^2)]^{-1}, \tag{73}$$

while we find for the matrix M

$$M = -(i/2)\,\sigma_z[e\,\mathcal{E}/(E + mc^2)]. \tag{74}$$

These formulae contain the results for the influence of the electric field on the beam. For the deflection of the beam we again find the relativistic formula (18). From (74) we obtain the formula for the rotation of the spin orientation

$$\Delta a_{\mathcal{E}\perp} = [e\,\mathcal{E}/(E + mc^2)]\,x, \text{ rotation about the } z\text{-axis.} \tag{75}$$

This is the relativistic generalization of (36). From (75) and (18) it follows immediately that

$$\Delta a_{\mathcal{E}\perp}/\Delta\gamma_{\mathcal{E}} = p^2c^2/E(E + mc^2)$$

or

$$\Delta a_{\mathcal{E}\perp}/\Delta\gamma_{\mathcal{E}} = E_{kin}/E \quad \text{with} \quad E_{kin} = E - mc^2 \tag{76}$$

It is clear that (37) is a first approximation of (76).

$\mathcal{E}_{\parallel}$ *Longitudinal electric field.*

$$\boldsymbol{\mathcal{E}} = \mathcal{E}\mathbf{i},\ \Phi = -\mathcal{E}x,\ \mathbf{A} = 0. \tag{77}$$

$$\varepsilon = -(\hbar/i)\,\partial/\partial t + e\,\mathcal{E}x = E + e\,\mathcal{E}\,x,\ \boldsymbol{\pi} = \mathbf{p}, \tag{78}$$

We have the case β of § 4; using (78) equation (52) becomes

$$(E^2/c^2 - m^2c^2)\,\varphi_4 + \hbar^2(\partial^2/\partial x^2 + \partial^2/\partial y^2)\,\varphi_4 + e\,\mathcal{E}\,(\hbar/ic + 2Ex/c^2)\,\varphi_4 = 0. \tag{79}$$

We put

$$\varphi_4 = \varphi_{40}\exp(i\,e\,f) \tag{80}$$

with

$$\varphi_{40} = \bar{B}\exp[(i/\hbar)\,(px - Et)]. \tag{81}$$

From (79), (80) and (81) we obtain as equation for f

$$\partial f/\partial x - i(\hbar/2p)\,(\partial^2 f/\partial x^2 + \partial^2 f/\partial y^2) + (\mathcal{E}/2pc)\,i - (\mathcal{E}E/\hbar pc^2)x = 0. \quad (82)$$

From this equation we find a solution

$$f = (\mathcal{E}E/2\hbar pc^2)\,x^2 + i[E\mathcal{E}/2p^2c^2 - \mathcal{E}/2pc]\,x. \quad (83)$$

This gives for φ_4

$$\varphi_4 = \bar{B}\exp\{(i/\hbar)\,[(px - Et) + (\mathfrak{e}\,E\mathcal{E}/2p^2c^2)\,x^2]\}$$
$$\exp[-(E/pc - 1)\,(\mathfrak{e}\,\mathcal{E}/2pc)\,x]. \quad (84)$$

φ_1 is calculated according to (51)

$$\varphi_1 = (1/mc)\,[E/c - p + (\mathfrak{e}\,\mathcal{E}/c)\,x - (\mathfrak{e}\,E\,\mathcal{E}/p^2c^2)\,x +$$
$$+ (\hbar/i)\,(E/pc - 1)\,(\mathfrak{e}\,\mathcal{E}/2pc)]\,\varphi_4. \quad (85)$$

φ_2 and φ_3 can be calculated in an analogous way; from the general equations (51) and (52) it is already clear that their values are exactly the same as for φ_1 and φ_4 (cf. (48)) except for a constant factor. Hence, according to (48) and (50) ψ_3/ψ_4 is independent of x, in other words: *The spin orientation is not changed by a longitudinal electric field, by which the beam is accelerated.*

This is the same result that was expressed in (41) by $\Delta a_{\mathcal{E}\|} = 0$, which thus remains valid for the relativistic theory. This result can already be deduced from the general form of the equations, without calculating explicitly the solution, as we have done. From this solution it is easy to establish again the relativistic formula (20) for the change of momentum.

§ 6. *The influence of magnetic fields on electron beams according to the Dirac theory.* For the treatment of the influence of magnetic fields on a beam we again take a plane wave in the x-direction in case there is no field. We choose the magnetic field in such a way that we can apply § 4.

$\mathcal{B}_\perp$ *Transverse magnetic field.*

$$\mathcal{B} = \mathcal{B}\mathbf{k},\ \Phi = 0,\ A_x = -\mathcal{B}y,\ A_y = 0,\ A_z = 0. \quad (86)$$

$$\varepsilon = -(\hbar/i)\,\partial/\partial t,\ \pi_x = p_x - (\mathfrak{e}/c)\,A_x = (\hbar/i)\,\partial/\partial x + \mathfrak{e}\,\mathcal{B}y/c,\ \pi_y = p_y,\ \pi_z = p_z. \quad (87)$$

We can proceed according to the cases α or δ of § 4. From the equations (49) or (56) we obtain the same equation for ψ_4, if we take a situation that no dependence on the z-coordinate exists ((87) is used).

$$(E^2/c^2 - m^2c^2)\,\psi_4 + \hbar^2\,(\partial^2/\partial x^2 + \partial^2/\partial y^2)\,\psi_4 - (\mathfrak{e}\,\hbar\mathcal{B}/c)\,\psi_4 -$$
$$- (2\,\mathfrak{e}\,\hbar\mathcal{B}y/ic)\,\partial\psi_4/\partial x = 0. \quad (88)$$

We put

$$\psi_4 = \psi_{40} \exp(i\,\mathrm{e}\,f) \tag{89}$$

with

$$\psi_{40} = B \exp[(i/\hbar)(px - Et)]. \tag{90}$$

From (88), (89) and (90) we get an equation for f

$$\partial f/\partial x - i(\hbar/2p)(\partial^2 f/\partial x^2 + \partial^2 f/\partial y^2) + (\mathcal{B}/2pc) + (\mathcal{B}/\hbar c)y = 0. \tag{91}$$

From this equation a solution for f can be obtained

$$f = -(\mathcal{B}/\hbar c)\,xy - (\mathcal{B}/2pc)\,x. \tag{92}$$

Hence we have for ψ_4

$$\psi_4 = B \exp\{(i/\hbar)[(px - Et) - (\mathrm{e}\,\mathcal{B}/c)\,xy]\} \exp[-i(\mathrm{e}\,\mathcal{B}/2pc)\,x]. \tag{93}$$

Analogously a solution for ψ_3 can be calculated

$$\psi_3 = A \exp\{(i/\hbar)[(px - Et) - (\mathrm{e}\,\mathcal{B}/c)\,xy]\} \exp[i(\mathrm{e}\,\mathcal{B}/2pc)\,x]. \tag{94}$$

w_+ and w_- can be calculated according to (46) and (47).

We can write (93) and (94) together as

$$\begin{pmatrix}\psi_3\\ \psi_4\end{pmatrix}_{x,t} = (1 + Mx)\begin{pmatrix}\psi_3\\ \psi_4\end{pmatrix}_{\substack{x=0\\ t=0}} K(x, y, t), \tag{95}$$

with

$$\begin{pmatrix}\psi_3\\ \psi_4\end{pmatrix}_{\substack{x=0\\ t=0}} = \begin{pmatrix}A\\ B\end{pmatrix}, \tag{96}$$

$$K(x, y, t) = \exp\{(i/\hbar)[(px - Et) - (\mathrm{e}\,\mathcal{B}/c)\,xy]\}, \tag{97}$$

while we find for the matrix M

$$M = (i/2)\,\sigma_z\,(\mathrm{e}\,\mathcal{B}/pc). \tag{98}$$

From these formulae we find again the relativistic formula (19) for the deflection of the beam. From (98) we obtain the relativistic formula for the rotation of the spin orientation.

$$\Delta\alpha_{\mathcal{B}\perp} = -(\mathrm{e}\,\mathcal{B}/pc)\,x, \text{ rotation about the } z\text{-axis.} \tag{99}$$

(19) and (99) have as their quotient

$$\Delta\alpha_{\mathcal{B}\perp}/\Delta\gamma_{\mathcal{B}} = 1. \tag{100}$$

Hence the non-relativistic formula (40) remains vali

$\mathcal{B}_{\|}$ *Longitudinal magnetic field.*

$$\boldsymbol{\mathcal{B}} = \mathcal{B}\mathbf{i},\ \Phi = 0,\ A_x = 0,\ A_y = -\mathcal{B}z,\ A_z = 0. \tag{101}$$

$$\varepsilon = -(\hbar/i)\,\partial/\partial t = \mathrm{E},\ \pi_x = p_x,\ \pi_z = p_z,\ \pi_y = p_y - (\mathrm{e}/c)\,A_y = (\hbar/i)\,\partial/\partial y + \mathrm{e}\,\mathcal{B}z/c. \tag{102}$$

We proceed according to δ § 4; from (56) and (102) we obtain

$$[(E^2/c^2 - m^2c^2) + \hbar^2 \Delta] \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix} + (e \hbar \mathcal{B}/c)\, \sigma_x \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix} - \\ - (2 e \hbar \mathcal{B}z/ic)\, (\partial/\partial y) \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix} = 0. \tag{103}$$

We proceed in a way analogous to § 3 by putting

$$\begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix} = [\exp (i e f)] \begin{pmatrix} \psi_{30} \\ \psi_{40} \end{pmatrix}, \tag{104}$$

with

$$\begin{pmatrix} \psi_{30} \\ \psi_{40} \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} \exp [(i/\hbar)\, (px - Et)]. \tag{105}$$

From (103), (104) and (105) we get by an analogous argument as in § 3 an equation for f

$$\partial f/\partial x - i(\hbar/2p)\, (\partial^2 f/\partial x^2 + \partial^2 f/\partial y^2 + \partial^2 f/\partial z^2) - (\mathcal{B}/2pc)\, \sigma_x = 0. \tag{106}$$

From this equation we find a solution

$$f = (\mathcal{B}/2pc)\, \sigma_x x. \tag{107}$$

Hence we can write for the wave functions

$$\begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}_{x,t} = (1 + Mx) \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}_{\substack{x=0 \\ t=0}} K(x, t), \tag{108}$$

with

$$\begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}_{\substack{x=0 \\ t=0}} = \begin{pmatrix} A \\ B \end{pmatrix}, \tag{109}$$

$$K(x, t) = \exp [(i/\hbar)\, (px - Et)], \tag{110}$$

while we have for the matrix M

$$M = (i/2)\, \sigma_x\, (e \mathcal{B}/pc). \tag{111}$$

Hence we find that the beam is not deflected or accelerated, as is described by (21) and that the relativistic formula for the rotation of the spin orientation becomes

$$\Delta a_{\mathcal{B}||} = - (e \mathcal{B}/pc)\, x, \text{ rotation about the } x\text{-axis.} \tag{112}$$

The results on the influence of magnetic fields on the electron spin are essentially already contained in [5]) and [7]).

§ 7. *Discussion.* From the investigations in §§ 2–6 on the influence of electric and magnetic fields on the polarization of electron beams, we can draw the following conclusions: A transverse magnetic field does not change the spin orientation relative to the direction of

propagation but a transverse electric field performs this change. It follows from this result that longitudinal polarization can be changed into transverse polarization with the aid of a transverse electric field (not with a transverse magnetic field). After this change by an electric deflection the transverse polarization can be observed with the aid of a scattering experiment according to II, § 3. If a transverse electric field is used according to Fig. 1 to make a circular beam, the beam must traverse an angle

$$(\pi/2)/(1 - E_{kin}/E) \tag{113}$$

to change longitudinal polarization into transverse polarization. Hence the angle is $\pi/2$ for low energies, but might be rather big for energies which are high compared with the rest mass.

It thus appears that the use of electric fields in combination with scattering experiments [2]) offers experimental methods which can be realized and are sufficient to make a complete investigation of the polarization of electron beams.

It follows from II § 4 that the asymmetries in scattering experiments become smaller for low (< 30 keV) and high (> 1 MeV) kinetic energies of the electrons. According to §§ 3 and 5 the polarization does not change if a beam is accelerated (or retarded) by a longitudinal electric field. Hence it will be useful to accelerate (retard) a beam with too small (too big) energy before measuring the polarization by means of scattering.

The influence of a longitudinal magnetic field consists in a rotation of the spin direction about the direction of propagation, which is analogous to the influence of a quartz-plate on polarized light. Although an experimental confirmation of this effect would of course be interesting, it seems not especially useful for the observation of polarization.

Our results were calculated for homogeneous fields. After this it is easy to obtain the effect of arbitrary fields. Locally the fields can be taken as homogeneous, as long as the electron wave length is small in comparison with the distances over which the fields vary appreciably. For the influence of an arbitrary non-homogeneous field on electron beams it is important that we can calculate electron *trajectories* without taking an influence of the electron magnetic moment into account (cf. the considerations of Bohr on this subject [8])). This is different from the influence of inhomogeneous magnetic fields on beams of neutral particles with a magnetic moment

in the Stern-Gerlach experiment where the beam is separated into different parts. After calculating the trajectory the change of the spin orientation along the trajectory can be obtained by using the results of §§ 5 and 6 as differential equations determining the change of the spin direction.

We wish to point out a possible way of measuring directly the magnetic moment of the electron μ_e (or the g-factor of the electron). In the double scattering experiment for electrons apply a homogeneous magnetic field $\mathcal{B}$, with direction perpendicular to the plane of the electron beams beyond the first scattering (cf. II Fig. 2). We assume that the electrons are scattered at right angles at the two places even if the beam is deflected by the magnetic field. We know that the directions of the spin, which are perpendicular to the plane of the beams are not changed by this magnetic field. Let $\Delta E = 2\mu_e \mathcal{B}$ be the energy difference for electrons at rest for the two afore-mentioned spin directions in the magnetic field $\mathcal{B}$. The electron beam in the magnetic field could be depolarized by applying a radiofrequency field with a frequency ν, which is determined by $\Delta E = h\nu$ (in a way analogous to the resonance method of Gorter and Rabi for the determination of atomic magnetic moments with atomic beams). In this way the asymmetry after the second scattering would disappear. Hence one could obtain a direct measurement of μ_e by measuring the frequency for which this asymmetry disappears.

In §§ 2–6 we exclusively discussed phenomena on a macroscopic scale in slowly varying fields; they are essentially different from, e.g., scattering at a nucleus (Mott [8], [9]).

Appendix. The use of wave packets and the calculation of $\Delta\pi_x$ and $\Delta\pi_y$. In the foregoing sections we have calculated solutions for charged particles in electric and magnetic fields, which are approximately plane waves. From these solutions we obtained the changes in kinetic momentum $\Delta\pi_x$ and $\Delta\pi_y$ by calculating the result of the operators $\pi_x = (\hbar/c)\,\partial/\partial x - (e/c)\,A_x$, $\pi_y = \ldots$. acting on the wave function ψ. In this connection we will call attention to the following point that otherwise might give rise to confusion. In the foregoing sections the calculations give often results of the shape (π_k, k^{th} component of $\boldsymbol{\pi}$)

$$\pi_k \psi = (a + bx)\,\psi, \tag{114}$$

where a and b do not depend on x, y, z or t (cf., e.g., the results for $\pi_x\psi$ according to the formulae (16), (31), (33), (84), (85) and for $\pi_y\psi$ in

(70) with (72)). a may be a (spin) matrix, b is a number. The result for the changes in kinetic momentum $\Delta\pi_k$ has then been given as

$$\Delta\pi_k = bx. \tag{115}$$

The expression for the momentum density for particles without spin is

$$\tfrac{1}{2}(\psi^* \pi\psi + c.c.). \tag{116}$$

For particles with spin there is still an additional term while in (116) the sum over the spin variables must be taken. The expression for the total momentum of a system, which is obtained by integration of the expression for the momentum density, is for particles with or without spin (cf. e.g. [10]))

$$\int \psi^* \pi_k \psi \, d\tau. \tag{117}$$

We need not use the expression for the momentum density and we can start directly from (117). However, we cannot form this integral for (approximately) plane wave solutions, for it does not converge. But we can form solutions $\tilde{\psi}(x, y, z, t)$ of the wave equation representing wave packets, that correspond to (approximately) plane wave solutions $\psi(x, y, z, t)$. If we form the mean value $\bar{\pi}_k = \int \tilde{\psi}^* \pi_k \tilde{\psi} d\tau$ (117) for the solutions $\tilde{\psi}$ and if we use (114), it can be shown that only the term bx (but not the term a) in (114) gives rise to a change in $\bar{\pi}_k$ if the wave packet changes its place. If x gives approximately the place of the wave packet the change $\Delta\pi_k$ of $\bar{\pi}_k$ is found to be determined by (115). Wave packets have mainly been considered for particles without spin; for particles with spin some calculations are given in [11]) and [12]).

Received 22-12-50.

REFERENCES

1) Groot, S. R. de and Tolhoek, H. A., On beta-radioactivity I, Physica **16** (1950) 456.
2) Tolhoek, H. A. and Groot, S. R. de, On beta-radioactivity II, Physica **17** (1951) 1.
3) Tolhoek, H. A. and Groot, S. R. de, On beta-radioactivity IV, Physica **17** (1951) 81.
4) Beth, E. W., Natuurphilosophie, Gorinchem, 1948 p. 83 (in Dutch).
5) Fues, E. and Hellman, H., Phys. Z. **31** (1930) 465.
6) Hellmann, H., Z. Phys. **96** (1935) 247.
7) Alexandrow, W., Z. Phys. **56** (1929) 818 and Ann. Physik **2** (1929) 477.
8) Mott, N. F., Proc. roy. Soc. London A **124** (1929) 425.
9) Mott, N. F., Proc. roy. Soc. London A **135** (1932) 429.
10) Pauli, W., Handbuch der Physik (2. Aufl.), Band 24/1, Berlin, 1933, p. 236.
11) Darwin, C. G., Proc. roy. Soc. London A **117** (1927) 258.
12) Broglie, L. de, L'électron magnétique, Paris, 1934, p. 167 ff.