Geometry of the roots of matrices with Onsager–Casimir symmetry

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The condition under which a matrix with Onsager–Casimir symmetry describing the coupling between m variables that are even (odd) under time reversal and 1 variable that is odd (even) has a pair of complex eigenvalues are analyzed, both graphically as well as algebraically.

I. INTRODUCTION

The problem of the maximum number of complex eigenvalues of matrices with Onsager–Casimir symmetry has been studied by Lekkerkerker and Laidlaw,[1,2] McLennan,[3] and Grmela and Iscoe.[4] Using the Onsager–Casimir symmetry relations it has been shown that the maximum number of complex roots of a hydrodynamic matrix describing the coupling between n variables that are even under time reversal and m variables that are odd is $2n$ or $2m$, whichever is smaller. In addition Lekkerkerker and Van Oost[5] have shown that in the dissipation-free limit there are $2n$ or $2m$ (whichever is smaller) purely imaginary roots and $|n-m|$ roots that are zero.

Further in the purely dissipative limit the hydrodynamic matrix is Hermitian and has only real roots. Knowing the limit situations (dissipation-free limit and purely dissipative limit) there remains the problem to determine the threshold conditions at which a pair of complex roots changes into real roots. In this paper we treat this problem for the simplest case, i.e., that of m even (odd) variables coupled to 1 odd (even) variable. In Sec. II we present a graphical analysis of this problem and in Sec. III we give an algebraic treatment.

II. GRAPHICAL ANALYSIS

The analysis presented in this section is closely related to the graphical determination of the changes of the eigenvalues caused by adding a single state to the Hamiltonian matrix in quantum mechanics.[6] It can be shown[1] that the hydrodynamic matrix describing the coupling of m variables that are even (odd) under time reversal and 1 variable that is odd (even) without loss of generality can be written in the form

$$\mathbf{M}=\begin{bmatrix} a_1 & 0 & \cdots & 0 & b_1 \\ 0 & a_2 & \cdots & 0 & b_2 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_m & b_m \\ -b_1^* & -b_2^* & \cdots & -b_m^* & c \end{bmatrix}, \tag{1}$$

where $a_1, a_2 \cdots a_m$ and c are real and if we assume the system to be stable they are in addition positive. The eigenvalue equation

$$\mathbf{M}\mathrm{V}=\lambda \mathrm{V} \tag{2}$$

can be written as

$$(a_j-\lambda)V_j+b_jV_{m+1}=0 \quad (j=1,2,\cdots,m), \tag{3}$$

$$-\sum_{j=1}^{m} b_j^* V_j+(c-\lambda)V_{m+1}=0. \tag{4}$$

Substituting the values of V_j that follow from (3) in (4) gives the equation

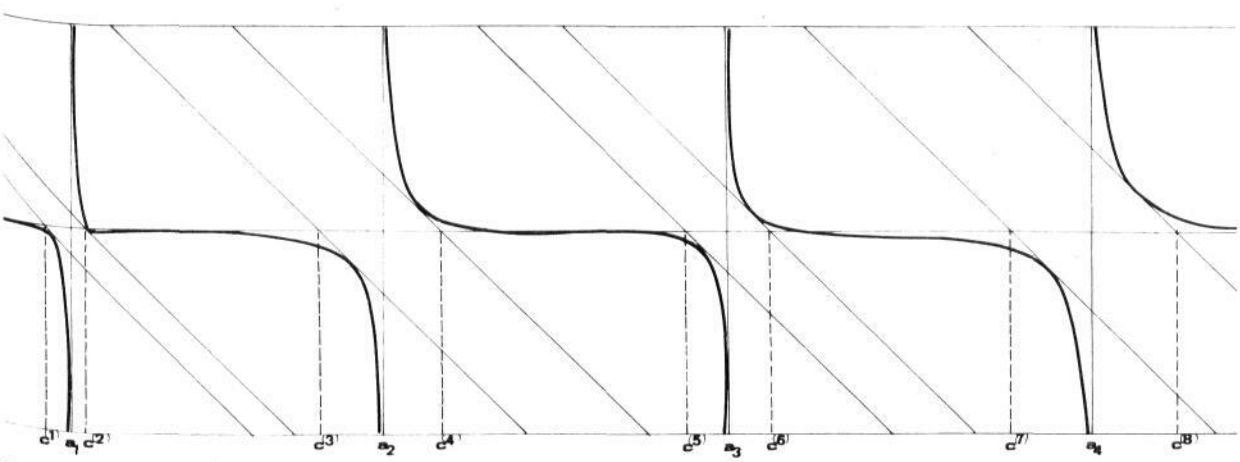

FIG. 1 Graphical determination of values of c for which $\mathbf{M}$ has complex eigenvalues. In this example $a_1=2$, $a_2=32$, $a_3=65$, $a_4=100$, $|b_1|=1$, $|b_2|=3$, $|b_3|=2$, $|b_4|=4$. One obtains $c^{(1)}=-.51$, $c^{(2)}=3.46$, $c^{(3)}=25.70$, $c^{(4)}=37.65$, $c^{(5)}=60.87$, $c^{(6)}=68.79$ $c^{(7)}=92.28$, $c^{(8)}=108.24$.

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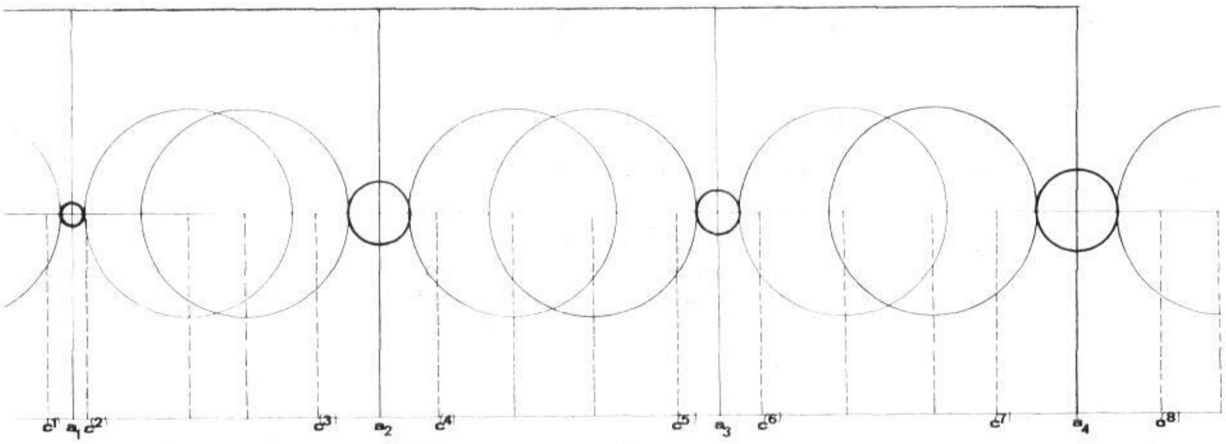

FIG. 2. Geršgorin's disks for the example treated in Sec. II (See caption Fig. 1).

$$\sum_{j=1}^{m} \frac{|b_j|^2}{\lambda - a_j} = (c - \lambda) \tag{5}$$

which is equivalent to the characteristic equation

$$|\lambda \mathbf{I} - \mathbf{M}| = 0.$$

In Fig. 1 the heavy curves represent the function

$$g(\lambda) = \sum_{j=1}^{m} \frac{|b_j|^2}{\lambda - a_j}$$

for the example noted and the light lines represent the lines $f(\lambda) = c - \lambda$ for values of c such that $f(\lambda)$ is tangent to $g(\lambda)$. It is clear that for values of c situated in the union of open intervals $]c^{(2k-1)}, c^{(2k)}[$ $(k = 1, \cdots, 4)$, $f(\lambda)$ intersects $g(\lambda)$ only three times [or in the general case $(m-1)$ times]. Thus for values of c situated in these intervals there are $(m-1)$ real roots and two complex roots whereas for all other c values there are $(m+1)$ real roots.

III. ALGEBRAIC ANALYSIS

In addition to a graphical analysis of the problem it is worthwhile to have an algebraic method at one's disposal to determine the values of the parameters for which the matrix **M** given by (1) has two complex eigenvalues. Using Geršgorin's theorem[7] it is possible to write down sufficient (but not necessary) conditions for which **M** has no complex eigenvalues. From the graphical analysis presented in the previous section it follows that the union of the m Geršgorin's disk

$|z - a_j| \leqslant |b_j|$ $(j = 1, 2, \cdots, m)$ contains at least $m-1$ real eigenvalues if the Geršgorin disk

$$|z - c| \leqslant \sum_{j=1}^{m} |b_j|$$

has no points in common with this union. Of the remaining two roots one is located in the isolated Geršgorin disk

$$|z - c| \leqslant \sum_{j=1}^{m} |b_j|$$

and the other in the union of the remaining disks. This means that these roots are real since complex roots would appear in the same Geršgorin disk. Thus a sufficient condition for the absence of complex eigenvalues is

$$|c - a_j| > |b_j| + \sum_{j=1}^{m} |b_j|, \quad j = 1, \cdots, m. \tag{6}$$

This condition is illustrated in Fig. 2 for the same example as was treated in Sec. II.

Since the characteristic polynomial of **M** is real and has at most two complex roots it appears logical to try to establish an extension of the relation $\Delta = A'^2_1 - 4A'_2 < 0$ which indicates that the real second degree polynomial $p_2(\lambda) = \lambda^2 + A'_1\lambda + A'_2$ has complex eigenvalues. Indeed it is possible to obtain such a generalization using determinant sequences.[8]

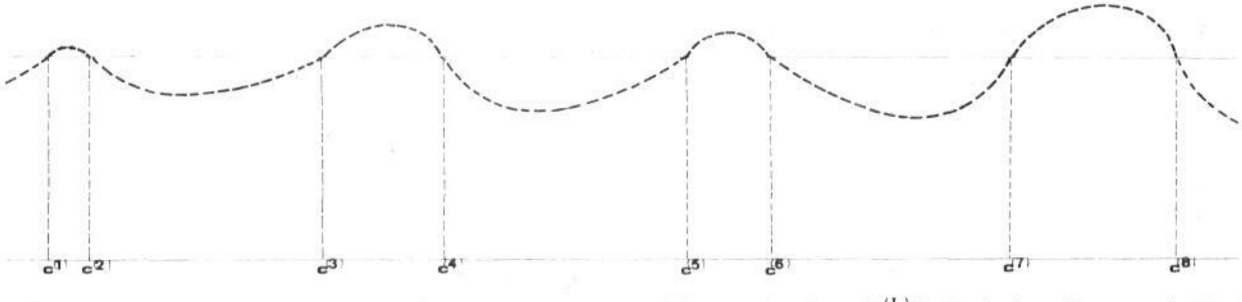

FIG. 3. Schematic illustration of the behavior of $\Delta(5,5)$ for the example of Sec. II. The values of $c^{(k)}$ for $k = 1, \ldots 8$ are the same as in Fig. 1.

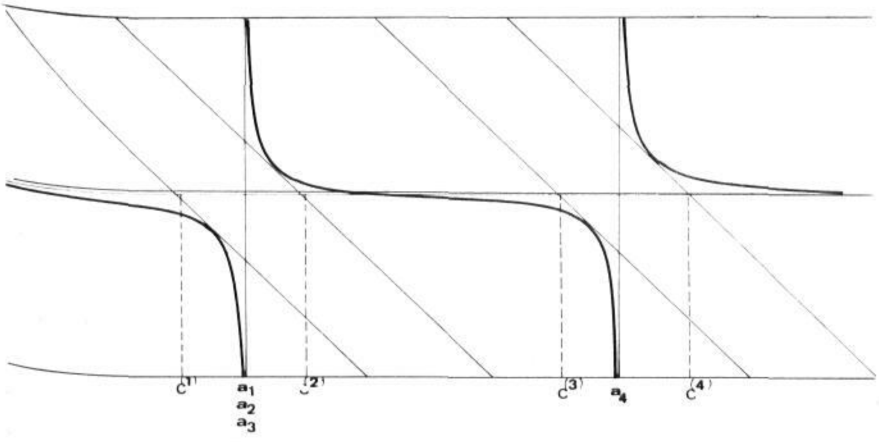

FIG. 4. Graphical determination of the values of c for which a hydrodynamic matrix with a twofold degenerate root has complex eigenvalues. In this example $a_1=a_2=a_3=4$, $a_4=40$, $|b_1|=1$, $|b_2|=|b_3|=2$, $|b_4|=3$. One obtains $c^{(1)}=-2.23$, $c^{(2)}=9.73$, $c^{(3)}=34.27$, and $c^{(4)}=46.23$.

Consider the nth degree polynomial

$$f(\lambda)=p_n(\lambda)-i\frac{dp_n(\lambda)}{d\lambda}$$
$$=\lambda^n+A_1\lambda^{n-1}+\cdots+A_n, \qquad (7)$$

where $p_n(\lambda)$ is the characteristic polynomial of **M**. Let the coefficients of $f(\lambda)$ be written in the form $A_k=A'_k+iA''_k$ where A'_k and A''_k are real. Let $\mathbf{A}(n,k)$ $(k=1,\cdots,n)$ denote the maxtrix formed from the first $(2k-1)$ elements in the first $(2k-1)$ rows of the matrix.

$$\begin{bmatrix} A''_1 & A''_2 & --- & A''_n & 0 & --- & 0 \\ 1 & A'_1 & --- & A'_{n-1} & A'_n & --- & 0 \\ 0 & A''_1 & --- & & A''_n & --- & 0 \\ | & | & & | & | & & | \\ | & | & & | & | & & | \\ | & | \quad 1 & & A'_1 & ------ & ------ & A'_n \\ 0 & ------ \quad 0 & & A''_1 & A''_2 & --- & A''_n \end{bmatrix}. \qquad (8)$$

Further let $\Delta(n,k)$ denote the determinant of the matrix $\mathbf{A}(n,k)$. It is possible to prove the following theorem.

Theorem I: Let $p_n(\lambda)$ be a real polynomial with at most two complex roots. Then $p_n(\lambda)$ has $(n-l)$ distinct roots iff $\Delta(n,n)=\cdots=\Delta(n,n-l+1)=0$ and $\Delta(n,n-l)\neq 0$. Further two of these are complex iff sign $\Delta(n,n-l)=$ sign $(-1)^{n-l+1}$. (The proof of this theorem is given in the Appendix.)

Let us apply Theorem I to the example analyzed in Sec. II. The behavior of $\Delta(5,5)$ is schematically illustrated in Fig. 3. As a further illustration of the application of Theorem I, we consider a hydrodynamic matrix with a twofold degenerate root. In Fig. 4 we first give a graphical analysis of such a case. For c values situated in the open intervals $]c^{(1)},c^{(2)}[$ and $]c^{(3)},c^{(4)}[$ there are complex eigenvalues.

The determinant $\Delta(5,5)$ for the example considered is zero for all real c and the behavior of $\Delta(5,4)$ is schematically illustrated in Fig. 5.

We hope that the simple examples treated here sufficiently demonstrate the usefulness of theorem I. An extension of theorem I to deal with the general case of n variables that are even under time reversal coupled to m variables that are odd would be desirable but so far we have not been able to find such an extension.

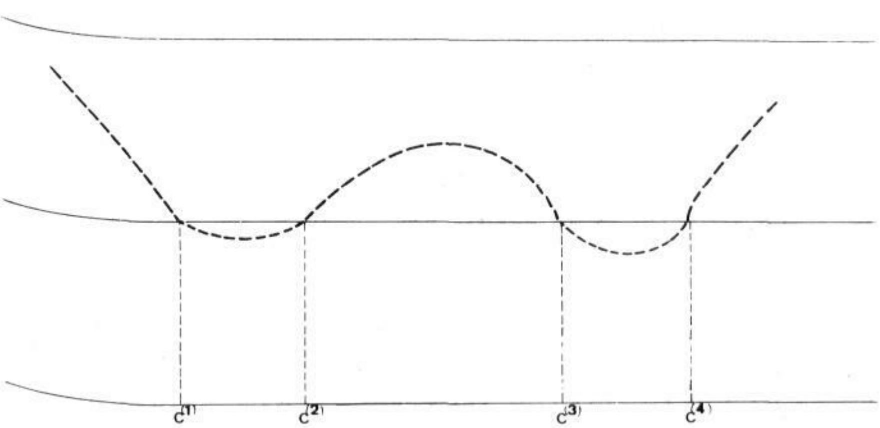

FIG. 5. Schematic illustration of the behavior of $\Delta(5,4)$ for the example given in Fig. 4. The values of $c^{(1)}$, $c^{(2)}$, $c^{(4)}$ are the same as in Fig. 4.

APPENDIX

Let the real polynomial $p_n(\lambda)=\lambda^n+A'_1\lambda^{n-1}\cdots+A'_n$ be written in the form

$$p_n(\lambda)=\prod_{i=1}^{r}(\lambda-\alpha_i)^{m_i} \tag{A1}$$

where the roots are ordered such that the first j roots have multiplicity higher than one.

Lemma 1: If $p_n(\lambda)$ has $n-1$ different roots then $\Delta(n,n)=\Delta(n,n-1)=\cdots=\Delta(n,n-s+1)=0$.

Proof: Consider the real functions γ and κ of the integer i_j for $j\geqslant 1$

$$\gamma(i_j)=\sum_{i_{j-1}=0}^{i_j}\cdots\sum_{i_0=0}^{i_1}\left[\prod_{k=1}^{j}\alpha_k^{i_k-i_{k-1}}\binom{m_{k-2}+i_k-i_{k-1}}{i_k-i_{k-1}}\right]A'_{i_0}$$

$$\kappa(i_j)=\sum_{i_{j-1}=0}^{i_j}\cdots\sum_{i_0=0}^{i_1}\left[\prod_{k=1}^{j}\alpha_k^{i_k-i_{k-1}}\binom{m_{k-2}+i_k-i_{k-1}}{i_k-i_{k-1}}\right]$$

$$(n-i_0)A'_{i_0}$$

where $A'_{i_0}=0$ for $i_0>n$ and $A'_0=1$.

After some algebra we find that

$$\frac{p_n(\lambda)}{\Pi_{i=1}^{r}(\lambda-\alpha_i)^{m_i-1}}=\sum_{i_j=0}^{r}\gamma(i_j)\lambda^{r-i_j}$$

and

$$\frac{dp_n(\lambda)/d\lambda}{\Pi_{i=1}^{r}(\lambda-\alpha_i)^{m_i-1}}=\sum_{i_j=0}^{r-1}\kappa(i_j)\lambda^{r-1-i_j}.$$

Further one can show that for any $k\geqslant 0$

$$\gamma(r+k+1)=\frac{1}{(m_j-2)!}\frac{d^{(m_j-2)}}{d\lambda^{(m_j-2)}}\times\left(\lambda^k\cdot\frac{p_n(\lambda)}{\Pi_{i=1}^{j-1}(\lambda-\alpha_i)^{m_i-1}}\right)\bigg|_{\lambda=\alpha_j}$$

and

$$\kappa(r+k)=\frac{1}{(m_j-2)!}\frac{d^{(m_j-2)}}{d\lambda^{(m_j-2)}}\times\left(\lambda^k\cdot\frac{dp_n(\lambda)/d\lambda}{\Pi_{i=1}^{j-1}(\lambda-\alpha_i)^{m_i-1}}\right)\bigg|_{\lambda=\alpha_j}.$$

From the last two relations it is clear that

$$\gamma(r+k+1)=0,\quad\text{for } k\geqslant 0, \tag{A2}$$

$$\kappa(r+k)=0,\quad\text{for } k\geqslant 0. \tag{A.3}$$

If $p_n(\lambda)$ has $n-s$ different roots, we use relation (A.2) and (A.3) to construct a linear combination of the columns of the matrices $\mathbf{A}(n,n),\cdots,\mathbf{A}(n,n-s+1)$, which is zero.

Lemma 2: Let α_i be a root of $p_n(\lambda)$ with multiplicity m_i and let $\Delta(n-g,n-g)$ denote the determinant associated with the polynomial

$$p_n(\lambda)/(\lambda-\alpha_i)^g,\ \text{for } g\leqslant m_i$$

then

$$\Delta(n,n-g+1)=-g\left(\frac{1}{g!}\frac{d^{(g)}}{d\lambda^{(g)}}(p_n(\lambda))\bigg|_{\lambda=\alpha_i}\right)^2\cdot\Delta(n-g,n-g). \tag{A.4}$$

Proof: Let $\mathbf{S}$ and $\mathbf{T}$ be $2(n-g+1)-1$ dimensional upper triangular square matrices

$$\mathbf{S}=\begin{bmatrix}1 & g & -\alpha_i & 0 & \cdots & \cdots & \cdots & 0\\ 0 & 1 & 0 & -\alpha_i & 0 & \cdots & \cdots & 0\\ 0 & 0 & 1 & g & -\alpha_i & 0 & \cdots & 0\\ & & & \ddots & \ddots & \ddots & \ddots & \vdots\\ & & & & 1 & g & & -\alpha_i\\ & & & & & 1 & & 0\\ & & & & & & & 1\end{bmatrix} \tag{A5}$$

and $\mathbf{T}=(t_{h,k})$ with $t_{h,k}=0$ if $h>k$.

For $h\leqslant k$

$$t_{n,k}=\binom{g-h+k}{k-h}\cdot\alpha_i^{k-h},\quad 1\leqslant k\leqslant 2n-2g$$

and

$$t_{n,k}=\binom{2n-g-h}{2n-2g+1-h}\cdot\alpha_i^{2n-2g+1-h},\ \text{if } k=2n-2g+1. \tag{A6}$$

The matrix $\mathbf{S}\cdot\mathbf{A}(n,n-g+1)$. $\mathbf{T}$ has the form

$$\mathbf{S}\cdot\mathbf{A}(n,n-g+1)\cdot\mathbf{T}=\left[\begin{array}{c|c}\mathbf{A}(n-g,n-g) & \mathbf{0}\\ \hline \mathbf{X} & \mathbf{B}\end{array}\right],$$

where $\mathbf{0}$ is a $(2n-2g-1)\times 2$ dimensional zeroblock

$$\mathbf{B}=\begin{bmatrix}\dfrac{1}{g!}\dfrac{d^{(g)}p_n(\lambda)}{d\lambda^{(g)}}\bigg|_{\lambda=\alpha_i} & 0\\ x & -\dfrac{g}{g!}\dfrac{d^{(g)}p_n(\lambda)}{d\lambda^{(g)}}\bigg|_{\lambda=\alpha_i}\end{bmatrix}.$$

Taking into account that $\det(\mathbf{S})=\det(\mathbf{T})=1$ it follows that

$$\det(\mathbf{S}\cdot\mathbf{A}(n,n-g+1)\cdot\mathbf{T})=\Delta(n,n-g+1)=\Delta(n-g,n-g)\cdot\det\mathbf{B}$$

$$=-g\left(\frac{1}{g!}\frac{d^{(g)}p_n(\lambda)}{d\lambda^{(g)}}\bigg|_{\lambda=\alpha_i}\right)^2\cdot\Delta(n-g,n-g)$$

which was to be proved. □

Lemma 3: If $p_n(\lambda)$ has $n-s$ different roots and we denote by $\mathbf{A}(n-m_k,n-m_k)$ the matrix of the polynomial

$[p_n(\lambda)/(\lambda-\alpha_k)]m_k$,

where α_k is a m_k — fold degenerate root of $p_n(\lambda)$ then,

$$\Delta(n,n-s)$$
$$=-m_k\Delta(n-m_k,n-s-1)$$
$$\times\left(\frac{d}{d\lambda}\left(\frac{p_n(\lambda)}{\Pi_{i=1}^r(\lambda-\alpha_i)^{m_i-1}}\right)\Big|_{\lambda=\alpha_k}\right)^2.$$

Proof: The proof follows the same lines as that of Lemma 2. We introduce the matrices $\mathbf{S}_{n-s}$ and $\mathbf{T}_{n-s}$ formed from the last $2(n-s)-1$ elements from the last $2(n-s)-1$ rows of $\mathbf{S}$ and $\mathbf{T}$ as defined by (A5) and (A6).

Then

$$\mathbf{S}_{n-s}\cdot\mathbf{A}(n,n-s)\cdot\mathbf{T}_{n-s}=\begin{pmatrix}\mathbf{A}(n-m_k,n-s-1) & \mathbf{C}\\ \mathbf{X} & \mathbf{B}\end{pmatrix},$$

where $\mathbf{C}$ consists of the last two elements of the first $2(n-s)-3$ rows of $\mathbf{A}(n-m_k,n-s)$.

From Lemma 1 it follows that $\Delta(n-m_k,n-s)=0$. Using the structure of the matrices $\mathbf{A}(n,k)$ for $1\leqslant k\leqslant n$, this relation implies that there exists linear combinations of the columns of $\mathbf{S}_{n-s}\mathbf{A}(n,n-s)\,\mathbf{T}_{n-s}$ which transform $\mathbf{C}$ into a zeroblock. Let $\mathbf{B}'$ be the result of this transformation applied to $\mathbf{B}$, then

$$\det(B')=-m_k\left(\frac{d}{d\lambda}\left(\frac{p_n(\lambda)}{\Pi_{i=1}^r(\lambda-\alpha_i)^{m_i-1}}\right)\Big|_{\lambda=\alpha_k}\right)^2.$$

which leads to the relation that we had to prove. □

Lemma 4: If $p_n(\lambda)$ has $n-s$ different roots and $\mathbf{A}(n-s,n-s)$ denotes the matrices associated to the polynomial

$$p_n(\lambda)\Big/\prod_{i=1}^r(\lambda-\alpha_i)^{m_i-1}$$

then

$$\Delta(n,n-s)=\prod_{i=1}^r m_i\cdot\Delta(n-s,n-s). \qquad \text{(A7)}$$

Proof: Consider first the case that $p_n(\lambda)$ has one degenerate root α_1 with multiplicity m_1 and thus $s=m_1-1$. Using Lemma 2 and the relation

$$\frac{p_n(\lambda)}{(\lambda-\alpha_i)^k}\Big|_{\lambda=\alpha_i}=\frac{1}{k!}\,\frac{d^{(k)}p_n(\lambda)}{d\lambda^k}\Big|_{\lambda=\alpha_i},$$

for $0\leqslant k\leqslant m_i$, where m_i is the multiplicity of the root α_i of $p_n(\lambda)$, we obtain (A7) for $s=m_1-1$. Let us now assume that (A7) is valid for any polynomial with at most $j-1$ degenerate roots

This implies that

$$\Delta(n-m_1,n-s-1)=m_2\cdots m_r\,\Delta(n-s-1,n-s-1), \qquad \text{(A8)}$$

where $\Delta(n-m_1,n-s-1)$ is the determinant associated with the polynomial

$$p_n(\lambda)/(\lambda-\alpha_i)^{m_1}$$

and $\Delta(n-s-1,n-s-1)$ is associated with

$$p_n(\lambda)/(\lambda-\alpha_1)^{m_1}\,\Pi_{i=2}^r(\lambda-\alpha_i)^{m_i-1}.$$

Then

$$\Delta(n,n-s)$$
$$=-m_1\left(\frac{d}{d\lambda}\left(\frac{p_n(\lambda)}{\Pi_{i=1}^r(\lambda-\alpha_i)^{m_i-1}}\right)\Big|_{\lambda=\alpha_1}\right)^2$$
$$\cdot\Delta(n-m_1,n-s-1)\ \text{(Lemma 3)}$$
$$=-m_1\left(\frac{d}{d\lambda}\left(\frac{p_n(\lambda)}{\Pi_{i=1}^r(\lambda-\alpha_i)^{m_i-1}}\right)\Big|_{\lambda=\alpha_1}\right)^2$$
$$\cdot m_2\cdots m_r\Delta(n-s-1,n-s-1) \qquad \text{(A9)}$$
$$=\prod_{i=1}^r m_i\Delta(n-s,n-s)\quad \text{(Lemma 2)}. \qquad □$$

Proof of Theorem I: We first prove the theorem for the case that $l=0$ by induction on n, the degree of the polynomial $p_n(\lambda)$. For $n=2$, the proof is trivial since $\Delta(2,2)=A_1'^2-4A_2'$. Let us assume that the theorem is valid for any real $(n-1)$th degree polynomial with a maximum of two complex roots. From Lemma 1 it follows that

$$\Delta(n,n)=-\left(\frac{dp_n(\lambda)}{d\lambda}\Big|_{\lambda=x}\right)^2\ \Delta(n-1,n-1), \qquad \text{(A10)}$$

where x is any root of $p_n(\lambda)$ and $\Delta(n-1,n-1)$ the determinant associated to $p_n(\lambda)/(\lambda-x)$. From (A7) it follows that $\Delta(n,n)\neq 0$ iff

$$\frac{dp_n(\lambda)}{d\lambda}\Big|_{\lambda=x}\neq 0$$

and $\Delta(n-1,n-1)\neq 0$, which is equivalent to say that $p_n(\lambda)$ has n different roots.

If moreover we assumed that x is real (which is always possible for $n\geqslant 3$) it follows from (A7) that $\Delta(n,n)$ and $\Delta(n-1,n-1)$ have opposite signs, and since p_n has complex roots iff $p_n(\lambda)/(\lambda-x)$ has complex roots, the theorem is proved for $l=0$.

We now consider the case that $l\neq 0$.

Let us first assume that $p_n(\lambda)$ has $(n-l)$ distinct roots. From Lemma 1 it follows then that

$\Delta(n,n)=\cdots\Delta(n,n-l+1)=0$. Further the $(n-l)$th degree polynomial

$$p_n(\lambda)/\prod_{i=1}^r(\lambda-\alpha_i)^{m_i-1}$$

has $(n-l)$ distinct roots and thus statisfies the conditions of the first part of the proof. This means that $\Delta(n-l,n-l)\neq 0$ and sign $\Delta(n-l,n-l)=\text{sign}\,(-1)^{n-l+1}$ iff

$$p_n(\lambda)\Big/\prod_{i=1}^r(\lambda-\alpha_i)^{m_i-1}$$

has complex roots. Using Lemma 4 the above means that $\Delta(n,n-l)\neq 0$ and sign $\Delta(n,n-l)=\text{sign}(-1)^{n-l+1}$ iff $p_n(\lambda)$

has complex roots. Finally we still have to show that if $\Delta(n,n)=\cdots\Delta(n,n-l+1)=0$ and $\Delta(n,n-l)\neq 0$ that $p_n(\lambda)$ has $(n\text{-}l)$ distinct roots. Indeed if the number of distinct roots of $p_n(\lambda)$ is different from $(n-l)$, this leads immediately to contradictions. □

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Fibre bundle analysis of topological charges in spontaneously broken gauge theories

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We study the topological properties of spontaneously broken gauge theories in the context of fibre bundle theory. In particular, we discuss the conditions under which the topological charges of gauge and Higgs fields are the same.

I. INTRODUCTION

The aim of the present paper is to study topological properties of spontaneously broken gauge theories, giving an explicit geometrical description in terms of connections and cross sections in a principal and associated fibre bundles.

As it was noted by Popov,[1] this approach not only enables one to "calculate" but renders calculations more transparent.

In Sec. II we study the simple case of an Abelian Higgs theory in $d=2$ (Euclidean) dimensions and show how the geometrical interpretation leads naturally to the identification of the topological charges of both gauge fields and Higgs field.

The discussion of the general case is done in Sec. III: for a compact Lie group G, conditions under which a single K-uple of integers labels the topological charges of gauge and Higgs fields are obtained. These conditions, already obtained by Woo[2] for the case $d=4$ (Euclidean) dimensions, using rather different techniques, arises in a transparent manner in the context of the fibre bundle theory. Several examples are discussed at the end of this section.

II. THE ABELIAN CASE

We will consider in this section the Abelian Higgs model: an Abelian gauge field A_μ coupled to a complex scalar field ϕ, with dynamics determined by the Lagrangian density

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\partial_\mu + iA_\mu)\phi^*(\partial_\mu - iA_\mu)\phi - \mathcal{U}(\phi), \quad (2.1)$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \qquad \phi = \phi_1 + i\phi_2, \quad (2.2)$$

and

$$\mathcal{U}(\phi) = \tfrac{1}{2}\lambda(\phi^*\phi - \mu^2/2\lambda)^2. \quad (2.3)$$

This Lagrangian density is invariant under the local gauge transformations generated by the group $G = \mathcal{U}(1)$; we are interested in the case $\mu^2 > 0$, that is, when the symmetry is spontaneously broken. It was noted by Nielsen and Olesen[3] that this model allows for vortex solutions-static, cylindrically symmetric field configurations of finite energy per unit length. The Nielsen–Olesen solution depends only on two variables. Then, the vortex field can be considered as a pseudoparticle configuration (with finite action) in the Euclidean version of the two-dimensional theory. If the action is to be finite, then, at Euclidean infinity (that is, on the sphere S^1_∞)

$$\lim_{r\to\infty} F_{\mu\nu} = 0, \quad (2.4)$$

$$\lim_{r\to\infty} (\partial_\mu - iA_\mu)\phi = 0, \quad (2.5)$$

$$\lim_{r\to\infty} \mathcal{U}(\phi) = 0, \quad (2.6)$$

where $r^2 = x_1^2 + x_2^2$.

These conditions can be interpreted geometrically in terms of the theory of fibre bundles. To this end, we consider S^1_∞ as the base space of a principal fibre bundle $P(S^1_\infty, \mathcal{U}(1))$ with structure group $\mathcal{U}(1)$. If we call $\mathcal{J}$ the associated vector bundle with fibre $\mathbb{C}$—the group $\mathcal{U}(1)$ acts on the left on $\mathbb{C}$ by usual multiplication—then the scalar field ϕ can be considered as a global cross section on $\mathcal{J}$[4].

Because $\phi \neq 0$ on S^1_∞ [from Eq. (2.3) and condition (2.6)] $\mathcal{J}$ is trivial and so is P.

The one form $A_\mu\, dx^\mu$ defines a connection on P. We can then consider on $\mathcal{J}$ the covariant derivative D induced by the connection [its explicit expression is $(\partial_\mu - iA_\mu)$].

Let γ be a given curve in S^1_∞ from x_0 to x. If $(x_0, g) \in P_{x_0}$ and $(x_0, v) \in \mathcal{J}_{x_0}$ [$\mathcal{J}_{x_0}(P_{x_0})$ is the fibre of $\mathcal{J}(P)$ over x_0], we will call $(x, T_\gamma g) \in P_x$ the parallel displacement of (x_0, g) along γ, given by the connection $A_\mu\, dx^\mu$, and $(x, \tilde{T}_\gamma v) \in \mathcal{J}_x$ the parallel displacement of (x, v) along γ given by D. Since D is induced by the connection,

$$\tilde{T}_\gamma v = (T_\gamma g) v/g \quad (2.7)$$

and

Lemma: for any curve γ' in S^1_∞ from x_0 to x, $T'_\gamma g = T_\gamma g\ \forall\ g \in \mathcal{U}(1)$ and $\tilde{T}'_\gamma v = \tilde{T}_\gamma v\ \forall\ v \in \mathbb{C}$.

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Proof: From Eq. (2.5), $\phi(x)=\widetilde{T}_\gamma\phi(x_0)$; then for any $g\in U(1)$

$$\phi(x)=(T_\gamma g)\phi(x_0)/g. \tag{2.8}$$

But Eq. (2.5) also implies that $\phi(x)=\widetilde{T}'_\gamma\phi(x_0)$. Hence, $\phi(x)=(T'_\gamma g)\,\phi(x_0)/g$. Since $\phi(x_0)\neq 0$, it follows $T_\gamma g=T'_\gamma g$. Now, from (2.7) it is evident that $\widetilde{T}_\gamma v=\widetilde{T}'_\gamma v$. That is, the parallel displacement from x_0 to x is independent of the particular choice of γ in S^1_∞. ∎

Hence, if we fix a point $x_0\in S^1_\infty$, the given connection $A_\mu dx^\mu$ can be identified with a well-defined mapping

$$\widetilde{A}:S^1_\infty\to U(1),\qquad \widetilde{A}(x)=T_\gamma(I), \tag{2.9}$$

where I is the identity of $U(1)$ and γ is any curve from x_0 to x in S^1_∞. We note that the explicit form of $\widetilde{A}(x)$ reads

$$\widetilde{A}(x)=\exp(i\int_{x_0}^{x}A_\mu\,dx^\mu).$$

Taking $g=I$ in Eq. (2.7), we have

$$\phi(x)=\widetilde{A}(x)\phi(x_0). \tag{2.10}$$

Now, condition (2.6) shows that ϕ defines a mapping $\phi:S^1_\infty\to S^1$. Because $U(1)\equiv S^1$, $\widetilde{A}$ can be also considered as a mapping $\widetilde{A}:S^1_\infty\to S^1$. Then, relation (2.10) implies that ϕ and $\widetilde{A}$ *belong to the same class in* $\Pi(S^1)\equiv Z$. Hence, *the same integer characterizes topologically the gauge field* A_μ *and the scalar field* ϕ.

This integer, the topological charge "n" is related to the quantization of the magnetic flux of the vortex. In terms of the gauge field A_μ it is given by the expression

$$n=\frac{1}{2\pi}\oint_{S^1_\infty}A_\mu\,dx^\mu$$

that, according to Stokes theorem, can be written as

$$n=\frac{1}{4\pi}\int_{\mathbb{R}^2}F_{\mu\nu}dx^\mu\wedge dx^\nu.$$

Our derivation of the topological equivalence between Higgs and gauge fields can be understood intuitively as follows: The principal fibre bundle P can be visualized as a torus (A) (see Fig. 1). Then the topological charge, associated with the gauge field counts the number of times that the curve $\bar{\gamma}$ joining the points $(x,\widetilde{A}(x))$ in P winds around the torus. On the other hand, we can take on each fibre $\mathbb{C}$ of the vector bundle $\mathcal{J}$ the circle $|z|=a_0(a_0=(\mu^2/2\lambda)^{1/2})$, that is, all the possible values of ϕ on S^1_∞. We thus obtain another torus (B) and we can take on it the curve $\widetilde{\tau}$ joining the points $(x,\phi(x))$. The topological charge associated with the Higgs field corresponds to the number of times the curve $\widetilde{\tau}$ winds around this torus. Relation (2.10) states the equality between both numbers.

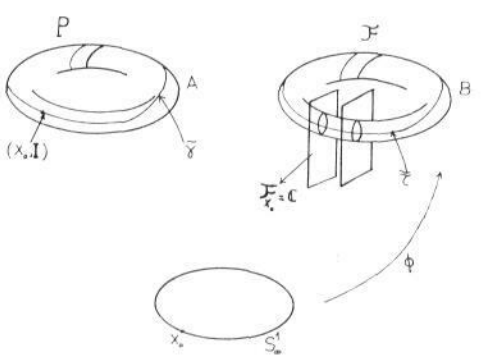

FIG. 1. We represent at the bottom the base space S^1_∞ of the principal fiber bundle P (torus A). Taking on each fibre C of the vector bundle $\mathcal{J}$ the circles $|Z|=a_0$, we obtain another torus (B).

III. HIGHER-DIMENSIONAL CASES

In this section, we consider the general case of a spontaneously broken gauge theory in a Euclidean d-dimensional space ($d>2$). Let G be a compact Lie group and

$$L=-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}-\tfrac{1}{2}D_\mu\phi D^\mu\phi-U(\phi) \tag{3.1}$$

a Yang-Mills Lagrangian density for G. The field ϕ transforms in accordance with an N-dimensional unitary irreducible representation of G, and the fields $A_\mu=(A^1_\mu,\ldots,A^n_\mu)$ assume values in the algebra of the Lie Group G. $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu-i[A_\mu,A_\nu]$, D_μ is the covariant derivative, and $U(\phi)$ is a G-invariant function which has a minimum (this minimum is assumed to be equal to zero). As in the Abelian case, finiteness of the action imposes the following conditions on S^{d-1}_∞:

$$\lim_{r\to\infty}F_{\mu\nu}=0, \tag{3.2}$$

$$\lim_{r\to\infty}D_\mu\phi=0, \tag{3.3}$$

$$\lim_{r\to\infty}U(\phi)=0, \tag{3.4}$$

where $r^2=x_1^2+\ldots+x_d^2$. This situation corresponds to the following geometrical description: a flat connection $A_\mu dx^\mu$ [condition (3.2)] on a principal fibre bundle $P(S^{d-1}_\infty,G)$ with base manifold S^{d-1}_∞ and structure group G.

Since S^{d-1}_∞ is simply connected ($d>2$) and the connection is flat, then it follows that P is trivial and also that the parallel displacement along a curve γ in S^{d-1}_∞ from x_0 to x, induced by the connection is independent of the particular choice of the curve γ. (These results follow from the general theory of flat connections, see, for example, Ref. 4).

As we did in Sec. II, let us fix a point $x_0\in S^{d-1}_\infty$. The connection $A_\mu dx^\mu$ determines a well defined mapping

$$\widetilde{A}:S^{d-1}_\infty\to G, \tag{3.5}$$

$$\widetilde{A}(x)=T_\gamma I, \tag{3.6}$$

where I is the identity of G and $(x,T_\gamma I)\in P_x$ is the parallel displacement of $(x_0,I)\in P_{x_0}$ along any curve γ in S^{d-1}_∞ from x_0 to x.

Let $\mathcal{J}$ be the associated bundle with fibre K, where K is a vector space-tensor space—according to the choice of ϕ. (The group G acts on the left on K in the usual way).

Let us call D the covariant derivative induced on $\mathcal{J}$ by the connection given on P. From Eq. (3.3) the Higgs field ϕ can be considered as a global cross section of the bundle $\mathcal{J}$ with

$$\phi(x)=\tilde{T}_\gamma\phi(x_0), \tag{3.7}$$

where $(x, \tilde{T}_\gamma\phi(x_0))\in\mathcal{J}_x$ is the parallel displacement of $(x_0, \phi(x_0))\in\mathcal{J}_{x_0}$ along any curve γ from x_0 to x, induced by D.

Now Eq. (3.7) implies that

$$\phi(x)=\tilde{A}(x)\phi(x_0). \tag{3.8}$$

Let us call $a_0\in K$ a fixed element of K satisfying $\mathcal{U}(a_0)=0$ on S_∞^{d-1}. Because $\mathcal{U}(\phi)$ is G-invariant, ga_0 also satisfies condition (3.4) for every $g\in G$. We will assume that all the zeros of $\mathcal{U}$ on S_∞^{d-1} are of this form.[5] Hence, the set of zeros of $\mathcal{U}$ can be identified with the left coset space G/H where $H=\{h\in G/ha_0=a_0\}$, the isotropy group of a_0, is called the unbroken group. Then ϕ can be regarded as a mapping $\phi=S_\infty^{d-1}\to G/H$.

From this and relation (3.8) the mapping ϕ can be identified with the mapping $\Pr\cdot\tilde{A}$ where Pr is the projection

$$\Pr: G\to G/H.$$

The element of $\Pi_{d-1}(G)$ represented by ϕ is then $\Pr*[\tilde{A}]$ where

$$\Pr*:\Pi_{d-1}(G)\to\Pi_{d-1}(G/H) \tag{3.9}$$

is the mapping induced by the projection Pr and $[\tilde{A}]$ is the class of $\tilde{A}$ in $\Pi_{d-1}(G)$.

Now, $\Pi_{d-1}(G)$ is an Abelian group, then $[\tilde{A}]$ is represented by a k-uple of integers $(n_1,\ldots,n_k)$ (the signs and ordering depending on the choice of the generators; note that if $\Pi_{d-1}(G)$ is not free, some of these integers n_i are elements of Z_{p_i}). Then, the equivalence between topological charges of Higgs and gauge fields can be characterized in the following way: *One can choose the generators of $\Pi_{d-1}(G/H)$ in order to have the class of ϕ represented by the same k-uple $(n_1,\ldots,n_k)$ if and only if $Pr*$ is an isomorphism.*

We wish to stress at this point that the interpretation of Yang–Mills fields in terms of connection in the principal fibre bundle and the Higgs field as a global cross section in the associated fibre bundle leads naturally to a condition for the equivalence between Higgs fields and gauge fields topological charges. [in the Abelian case (with $d=2$) studied in Sec. II, $H=\{I\}$, and the precedent discussion was not necessary.] From the exact sequence of homotopy

$$\cdots\Pi_{d-1}(H)\xrightarrow{i*}\Pi_{d-1}(G)\xrightarrow{\Pr*}\Pi_{d-1}(G/H)\to\Pi_{d-2}(H)\to\cdots$$

it can be deduced that Pr* is an isomorphism if

$$\Pi_{d-1}(H)=0, \tag{3.10a}$$

$$\Pi_{d-2}(H)=0. \tag{3.10b}$$

For the case $d=4$, Eq. (3.10b) is always verified since H is compact (H is closed in G), and Eq. (3.10a) reduces to the condition derived by Woo[2]: $\Pi_3(H)=0$. This is the case when the gauge group is broken (i) entirely, (ii) down to a subgroup of Abelian factors. Then, the topological charges of the gauge and Higgs fields are the same.

We consider lastly the example of the group $G=SO(3)$ and a scalar field forming an isovector $\boldsymbol{\phi}$, with the usual potential $\mathcal{U}(\phi)=\frac{1}{2}\lambda(|\boldsymbol{\phi}|^2-1)^2$, that is, the Gerogi–Glashow model without fermions. The group G can be identified in this case with P^3 (the three-dimensional projective space) and $H=\mathcal{U}(1)$.

If $d=3$, then $\Pi_2(G)=0$. On the other hand, the Higgs field topological charge is characterized by $\Pi_2(G/H)=Z$. What happens in this example is that, although condition (3.10a) is satisfied, (3.10b) is not, since $\Pi_1(H)=Z$. Then, our analysis does not apply. If $d=4$, then $\Pi_3(G)=Z$, and $\Pi_3(G/H)=Z$, that is, a single integer labels the Higgs field topological charge. Note that $A:S_\infty^3\to P^3$ while $\phi:S_\infty^3\to G/H\equiv S^2$.

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