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Abstract

We survey a number of minor-monotone graph parameters and their relationship to the complexity of routing on graphs. In particular we compare the interval routing parameters $\kappa_{slir}(G)$ and $\kappa_{sir}(G)$ with Colin de Verdière's graph invariant $\mu(G)$ and its variants $\lambda(G)$ and $\kappa(G)$. We show that for all the known characterizations of $\theta(G)$ with $\theta(G)$ being $\mu(G)$, $\lambda(G)$ or $\kappa(G)$, that $\theta(G) \leq 2\kappa_{slir}(G) - 1$ and $\theta(G) \leq 2\kappa_{sir}(G)$ and conjecture that these inequalities always hold. We show that $\theta(G) \leq 4\kappa_{slir}(G) - 1$ and $\theta(G) \leq 4\kappa_{sir}(G) + 1$.

1 Introduction

Graphs are a common model for the interconnection structure of communication networks. The complexity of the underlying graph should be an indication of the complexity of routing information in the network. In this paper we survey a number of minor-monotone graph parameters and their relationship to the complexity of routing using a particular type of routing schemes. We focus on Colin de Verdière's graph parameter $\mu(G)$ and its variants $\lambda(G)$ and $\kappa(G)$ as a measure of complexity for the underlying graph, and on interval routing as the routing scheme of choice ([29, 30]).

Colin de Verdière's graph parameter $\mu(G)$ [9] is defined for any undirected graph G. It has generated quite some interest due to its rather nice graph-theoretical properties. For example, $\mu(G) \leq 3$ if and only if G is planar, and $\mu(G) \leq 4$ if and only if G is linklessly embeddable in $\mathbb{R}^3$ [25, 20]. Furthermore, $\mu(G)$ is minor-monotone, so by the graph minor theory of Robertson and Seymour (see e.g., [24]) it can be characterized by means of forbidden minors. Van der Holst, Laurent and Schrijver [16, 14] introduced related minor-monotone parameters $\lambda(G)$ and $\kappa(G)$ based on extensions of $\mu(G)$. Characterization results for small values of the above parameters are known and surveyed in e.g. [18, 27].

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Interval routing was first introduced by Santoro and Khatib [26], and subsequently generalized by van Leeuwen and Tan [29, 30]. In interval routing schemes, the nodes of a graph are numbered and the edges are labeled with intervals in order to allow the proper routing of messages to their set destinations. Interval routing schemes proved useful in the design of large processor networks and more recently, in routing issues for wireless ad-hoc networks. We are interested in the graph parameter $\kappa_{sir}(G)$ and its variant $\kappa_{slir}(G)$ that represent the complexity of a certain type of interval routing schemes.

The parameters $\kappa_{sir}(G)$ and $\kappa_{slir}(G)$ originate from the work by Frederickson and Janardan [12]. They introduced a variant of interval routing with 'dynamic link costs', where the cost of the edges in a graph can vary in such a way as to preserve the shortest path information. For the case of 'strict' interval routing schemes to be defined later, $\kappa_{sir}(G)$ is the maximum number of intervals minimally needed per edge to always be able to achieve shortest path routing by a suitable assignment of intervals. Frederickson and Janardan showed that $\kappa_{sir}(G) = 1$ if and only if G is outerplanar. The relevance for this survey stems from the result of Bodlaender, Tan, Thilikos and van Leeuwen [8] that $\kappa_{sir}(G)$ is minor-monotone and related to the treewidth of a graph, another well-known minor-monotone parameter due to Robertson and Seymour. The parameter $\kappa_{slir}(G)$ has similar properties and refers to a related, more restricted lass of interval routing schemes. For surveys on interval routing, see [13, 28].

In this paper we compare the Colin de Verdière parameter $\mu(G)$ and its variants $\lambda(G)$ and $\kappa(G)$, with the interval routing parameter $\kappa_{sir}(G)$ and its variant $\kappa_{slir}(G)$. We show that, with $\theta(G)$ denoting $\mu(G)$, $\lambda(G)$ or $\kappa(G)$:

i. $\kappa_{slir}(G) = \kappa_{sir}(G) = 0 \iff \theta(G) = 0$,

ii. $\kappa_{slir}(G) = 1 \Longrightarrow \theta(G) \leq 1$,

iii. $\kappa_{sir}(G) \leq 1 \Longrightarrow \theta(G) \leq 2$,

iv. $\kappa_{slir}(G) \leq 2 \Longrightarrow \theta(G) \leq 3$,

v. $\kappa_{slir}(G) \leq \lfloor \frac{n}{2} \rfloor \iff \theta(G) \leq n-1$, where n is the size of G,

vi. $\kappa_{sir}(G) \leq \lceil \frac{n}{2} \rceil \iff \theta(G) \leq n-1$.

For the invariants $\mu(G)$ and $\kappa(G)$, we also have

i. $\kappa_{sir}(G) \leq 2 \Longrightarrow \mu(G) \leq 4$,

ii. $0 < \kappa(G) \leq 2\kappa_{slir}(G) - 1$ for all graphs G,

iii. $\kappa(G) \leq 2\kappa_{sir}(G)$ for all graphs G.

Thus, we know for some values of $\theta(G)$ the following relations and we *conjecture* that these hold always:

i. $0 < \theta(G) \leq 2\kappa_{slir}(G) - 1$ and

ii. $\theta(G) \leq 2\kappa_{sir}(G)$.

For general graphs G we can show the following bounds:

i. $0 < \theta(G) \leq 4\kappa_{slir}(G) - 1$, and

ii. $\theta(G) \leq 4\kappa_{sir}(G) + 1$.

The paper is organized as follows. The next section gives some preliminaries on graph theory. In Section 3 we explain interval routing schemes and define the parameters κ_{sir} and κ_{slir} associated with these. In Section 4 we define treewidth and relate it to the interval routing parameters. Section 5 deals with the Colin de Verdière invariant $\mu(G)$ and its relationship with $\kappa_{sir}(G)$ and $\kappa_{slir}(G)$. Sections 6 and 7 contain definitions and results pertained to the parameters $\lambda(G)$ and $\kappa(G)$ respectively. The last section gives some conclusions and open problems.

2 Preliminaries

Let $G = (V, E)$ be a finite simple undirected graph. Let the set of vertices be $V = \{1, \cdots, n\}$ and write $ij \in E$ if the edge $\{i, j\} \in E$.

We need the following graph operations. The definitions illustrate some of the manners in which a new graph can be created from component graphs.

Definition 1 (Clique-sum) *A graph $G = (V, E)$ is a* clique-sum *of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ if $V = V_1 \cup V_2$, $E = E_1 \cup E_2$ and $V_1 \cap V_2$ is a clique in G_1 and G_2.*

Writing G as a clique-sum is a decomposition technique. Often a parameter for G can be deduced from the parameters of G_1 and G_2. For example, for a graph G which is a clique-sum of graphs G_1 and G_2, the chromatic number $\chi(G)$ =max$\{\chi(G_1), \chi(G_2)\}$.

Definition 2 (ΔY-transformation) *For a graph G,*

i. the ΔY-operation is as follows: choose a triangle in G and insert a new node v inside the triangle and connect it to all the three nodes of the triangle, and then delete the triangle.

ii. The $Y\Delta$-operation is the reverse operation: starting with a node v of degree three, make its three neighbors pairwise adjacent and then delete v with its three edges.

See Figure 3 for an example of ΔY-transformation.

Definition 3 (Product graphs) *For graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the* product *$G = G_1 \times G_2$ is the graph with $V = V_1 \times V_2$ and $E = \{\{(u, v_1), (u, v_2)\} : v_1v_2 \in E_2\}\} \cup \{\{(u_1, v), (u_2, v)\} : u_1u_2 \in E_1\}\}$, where $u \in V_1$ and $v \in V_2$.*

Definition 4 *A* subdivision *of a graph G is a graph obtained by inserting a new node along an edge of G.*

Edge contraction is the operation of replacing an edge $ij \in E$ by a new node v, making v adjacent to all nodes to which i and j are adjacent in G, and subsequently deleting nodes i and j and all their incident edges from G ([31]).

We now come to the important definition of a graph minor.

Definition 5 (Graph minors)

i. *A graph H is a* minor *of graph G if H can be obtained from G by a series of zero or more node deletions, edge deletions and edge contractions.*

ii. *A class of graphs $\mathcal{G}$ is* minor-closed *if for every $G \in \mathcal{G}$, every minor H of G belongs to $\mathcal{G}$.*

iii. *A* proper minor *of a graph G is a minor unequal to G.*

iv. *A graph H is a* forbidden minor *of the class of graphs $\mathcal{G}$, if $H \notin \mathcal{G}$ but every proper minor of H is.*

v. *A function $\theta : \mathcal{G} \to \mathbb{N}$ is* minor-monotone*, if $\theta(H) \leq \theta(G)$ whenever $G, H \in \mathcal{G}$ and H is a minor of G.*

Well-known classes of minor-closed graphs include the classes of: disjoint unions of paths, trees, outerplanar graphs, planar graphs, graphs that are knotlessly embeddable in $\mathbb{R}^3$, and graphs that are linklessly embeddable in $\mathbb{R}^3$ (see [25]).

Fact 6 (Robertson and Seymour [24]) *For every minor-closed class of graphs, the minimal set of forbidden minors is finite.*

It follows that every minor-closed class of graphs $\mathcal{G}$ has a finite characterisation in terms of a finite collection of forbidden minors $\mathcal{F}$: $G \in \mathcal{G}$ if and only if G does not contain any graph of $\mathcal{F}$ as a minor.

3 Interval Routing

Interval routing stems from compact routing methods in computer networks. Given a connected graph G with n nodes, an Interval Labeling Scheme (ILS) is a labeling of each node $v \in V$ with a unique name from $\{1, \cdots, n\}$ and of each incident edge (u, v) with a, possibly empty, interval $[a, b]$, where $a, b \in \{1, \cdots, n\}$. If $a > b$ then $[a, b] = \{a, a+1, \cdots, n, 1, \cdots, b\}$, i.e. wraparound is allowed. For each node u, with label $\ell(u)$, the set of labels assigned to all incident edges forms a partition of the set $\{1, \cdots, n\}$. In more general labelling schemes, more than one interval may be assigned to an incident edge.

Given an ILS, routing is done as follows. When a node receives a packet containing destination address w, it checks its set of edge intervals (routing table) for the interval that contains w and relays the packet via the edge marked with that interval; unless $w = \ell(v)$, in which case the packet was routed correctly and is not relayed any further. It can be shown that every graph admits an ILS such that messages are always routed correctly to their destination no matter where they depart, even when only one interval per incident edge is allowed and all intervals are required to be non-empty ([30]).

Normally each edge of G has a *length*, which is a non-negative integer. An ILS is *valid* if for all nodes v, w, all packets from v to w are routed correctly and eventually reach their destination via a path of minimum total length. An Interval Routing Scheme (IRS) is a valid ILS.

There are several variants of Interval Routing schemes, see [13, 28]. For our purpose, we need the following:

Definition 7 (Strict interval routing schemes)

i. *An IRS is* Strict *(SIRS) if for each node u, none of the intervals assigned to its incident edges contains its own label, $\ell(u)$.*

ii. *A k-SIRS is a SIRS where each incident edge is labeled with at most k intervals.*

iii. *$\kappa_{slir}(G)$ is the smallest k, such that there exists a labeling ℓ of the nodes of V with unique names from $\{1, \cdots, n\}$, with the property that for each assignment of non-negative lengths to the edges, there is a k-SIRS using labeling ℓ.*

iv. *A Strict Linear Interval Routing Scheme (SLIRS) is a SIRS where the interval labels are not allowed to wrap-around, i.e. for all non-empty interval labels $[a, b], a \leq b$.*

v. *$\kappa_{slir}(G)$ is the smallest k, such that there exists a labeling ℓ of the nodes of V with unique names from $\{1, \cdots, n\}$, with the property that for each assignment of non-negative lengths to the edges, there is a k-SLIRS using labeling ℓ.*

The parameters $\kappa_{sir}(G)$ and $\kappa_{slir}(G)$ represent the routing complexity of a graph.

Fact 8 (Frederickson, Janardan [12])

i. $\kappa_{sir}(G) = 1 \iff G$ *is outerplanar.*

ii. *$K_{2,2k+1}$ and K_{2k+2} are forbidden minors for the class of graphs $\{G \mid \kappa_{sir}(G) = k\}$.*

Fact 9 (Bodlaender, Tan, Thilikos, van Leeuwen [8]) *The graph parameters $\kappa_{slir}(G)$ and $\kappa_{sir}(G)$ are minor-monotone.*

The following bounds are obvious.

Fact 10 (Upper bounds)

i. $\kappa_{slir}(G) \leq \lfloor \frac{n}{2} \rfloor$.

ii. $\kappa_{sir}(G) \leq \lceil \frac{n}{2} \rceil$.

Fact 11 (Bakker, Bodlaender, Tan, van Leeuwen [5])

i. $K_{p,q}$ *is a forbidden minor for the class of graphs* $\{G \mid \kappa_{slir}(G) = k \geq 1\}$*, where* $2 \leq p \leq q$ *and* $p + q = 2k + 2$.

ii. K_{2k+1} *is a forbidden minor for the class of graphs* $\{G \mid \kappa_{slir}(G) = k \geq 1\}$.

iii. $K_{p,q}$ *is a forbidden minor for the class of graphs* $\{G \mid \kappa_{sir}(G) = k \geq 1\}$*, where* $2 \leq p \leq q$ *and* $p + q = 2k + 3$.

iv. $\kappa_{slir}(G) = 1 \iff G$ *is a path.*

4 Interval Routing and Treewidth

In this section we give the definition of treewidth of a graph and some results relating interval routing and treewidth.

Definition 12 *A* k-tree *is a graph defined recursively as follows: A clique with* $k + 1$ *nodes is a* k*-tree. A* k*-tree with* $n + 1$ *nodes can be formed from a* k*-tree with* n *nodes by adding a new node and making it adjacent to exactly all nodes of a* k*-clique in the original* k*-tree.*

Definition 13 *A graph is a* partial k*-tree if it is a subgraph of a* k*-tree.*

Definition 14 *The* treewidth $\tau\omega(G)$ *of a graph* G *is the minimum value* k *for which the graph is a subgraph of a* k*-tree.*

The definition above is due to Arnborg and Proskurowski, see e.g. [1]. The term treewidth and the commonly used definition in terms of tree decompositions was introduced by Robertson and Seymour [22]. There are other well-known equivalent definitions, see the surveys by Bodlaender [6, 7], for instance.

Fact 15 (Treewidth)

i. $\tau\omega(G) = 1 \iff G$ *is a tree.*

ii. $\tau\omega(G) \leq 2 \iff G$ *does not contain* K_4 *as a minor.*

iii. **(Arnborg, Proskurowski, Corneil [2])** $\tau\omega(G) \leq 3 \iff G$ *does not have a minor isomorphic to any of the graphs in Figure 1.*

iv. $\tau\omega(K_n) = n - 1$.

Colin de Verdière [11] defines a variant of the notion of treewidth which is simpler.

Definition 16 $\tau\omega_{CdV}(G)$ *(denoted* $la(G)$ *in [11]) is the smallest integer* n *such that* G *is a minor of* $T \times K_n$*, where* T *is a tree.*

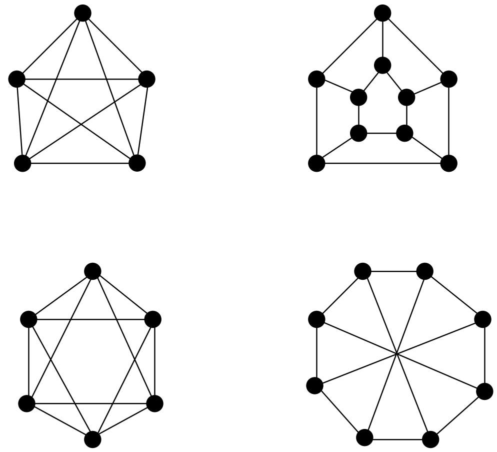

Figure 1: Forbidden Minors of $\tau\omega(G) \leq 3$

The two definitions of treewidth are very closely related and in fact their values differ by at most 1.

Fact 17 (Colin de Verdière [11], van der Holst [14])
$\tau\omega(G) \leq \tau\omega_{CdV}(G) \leq \tau\omega(G) + 1$.

Fact 18 (Bodlaender,Tan,Thilikos,van Leeuwen [8])

i. $\tau\omega(G) \leq 4\kappa_{slir}(G) - 2$.

ii. $\tau\omega(G) \leq 4\kappa_{sir}(G)$.

Corollary 19

i. $\tau\omega_{CdV}(G) \leq 4\kappa_{slir}(G) - 1$.

ii. $\tau\omega_{CdV}(G) \leq 4\kappa_{sir}(G) + 1$.

For known values of treewidth, the bounds can be improved.

Theorem 20 *Let G be a connected graph.*

i. $\kappa_{slir}(G) = 1 \Longrightarrow \tau\omega(G) = 1$.

ii. $\kappa_{sir}(G) = 1 \Longrightarrow \tau\omega(G) \leq 2$.

iii. $\kappa_{slir}(G) \leq 2 \Longrightarrow \tau\omega(G) \leq 3$.

iv. $\kappa_{sir}(G) \leq \lceil \frac{n}{2} \rceil \iff \tau\omega(G) \leq n - 1$.

v. $\kappa_{slir}(G) \leq \lfloor \frac{n}{2} \rfloor \iff \tau\omega(G) \leq n - 1$.

Proof. (i) $\kappa_{slir}(G) = 1 \iff G$ is a path, so the equality holds.
(ii) $\kappa_{sir}(G) = 1 \iff G$ is an outerplanar graph, and outerplanar graphs cannot contain K_4 or $K_{2,3}$ as a minor, so have treewidth at most 2.
(iii) See Figure 1. Each graph from the set of forbidden minors for the class of graphs with $\tau\omega(G) \leq 3$ contains K_5 (the first graph), $K_{2,4}$ (the next two graphs) or $K_{3,3}$ (the last graph); all are also forbidden minors for the class of graphs with $\kappa_{slir}(G) = 2$. So when $\kappa_{slir}(G) \leq 2$, $\tau\omega(G)$ cannot be larger than 3. The bound is tight as $\kappa_{slir}(K_4) = 2$ and $\tau\omega(K_4) = 3$.
(iv)(v) Follows from the upper bounds when $G = K_n$. □

Thus we can sharpen the bounds to $\tau\omega(G) \leq 2\kappa_{slir}(G) - 1$ and $\tau\omega(G) \leq 2\kappa_{sir}(G)$ for known characterizations of $\tau\omega(G)$.

5 Interval Routing and $\mu(G)$

In this section we give the relations between the interval routing parameters $\kappa_{slir}(G)$ and $\kappa_{sir}(G)$ with the Colin de Verdière's invariant $\mu(G)$ introduced in [9]. The motivation of the definition stems from Schrödinger operators in differential geometry and is rather technical in nature. We introduce three conditions on matrices, related to graphs.

Condition (M1)(Discrete Schrödinger Operator)
$\mathcal{O}_G$ is the set of real symmetric $V \times V$ matrices $M = (M_{ij})$ such that

i. $M_{ij} < 0$ *if* $ij \in E$, *and*

ii. $M_{ij} = 0$ *if* $i \neq j$ *and* $ij \notin E$.

There is no restriction on the diagonal entries of M.

As $M \in \mathcal{O}_G$ is real symmetric, M has n real *eigenvalues* (counting multiplicities) $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, called the *spectrum* of M. If G is connected, then by Perron-Frobenius Theorem the first eigenvalue λ_1 must be of multiplicity 1. There is no loss of generality in restricting attention to connected graphs, since the spectrum of a disconnected graph is the union of the spectra of its connected components.

Condition (M2)(Normalization)
$M \in \mathcal{O}_G$ has exactly one negative eigenvalue, of multiplicity 1.

As there is no restriction on the diagonal entries, we can replace M by $M - \lambda_2 I$. Then the new λ_1 is the only negative eigenvalue and the new λ_2 now is zero. Thus by shifting the spectrum of M this condition can always be achieved.

Condition (M3)(Strong Arnol'd Hypothesis - SAH)
There is no nonzero real symmetric matrix X such that $MX = 0$ with $X_{ij} = 0$ whenever $i = j$ or $ij \in E$.

This is a non-degeneracy condition on M. It is equivalent to the concept of *transversal intersection* of differential manifolds. Transversal intersections have nice properties because near them the manifolds behave like affine subspaces. (See [18], pp. 8-9, for other equivalent conditions of SAH.)

Definition 21 *A matrix M satisfying conditions (M1), (M2) and (M3) is called a* Colin de Verdière matrix *for G.*

We are now ready to define the Colin de Verdière's invariant.

Definition 22 *$\mu(G)$ is the maximum corank of any Colin de Verdière matrix of G.*

As noted above, the second eigenvalue λ_2 can be shifted such that it is zero, with the same multiplicity. Thus $Mx = \lambda_2 x = 0x = 0$ and the dimension of the kernel (null space) of M, is the corank of M. This coincides with the multiplicity of λ_2. So $\mu(G)$ can also be defined as the *maximum multiplicity* of the *second smallest eigenvalue* of M satisfying conditions (M1) and (M3), i.e. $M \in \mathcal{O}_G$ satisfying SAH. For more detailed explanations of $\mu(G)$, see Colin de Verdière's original paper [9] ([10] for an English translation) or [18] for a more leisurely paced survey.

We now state some known results of $\mu(G)$.

Fact 23 (Colin de Verdière [9, 11])

i. $\mu(G)$ *is minor-monotone.*

ii. $\mu(G) = 0 \iff G$ *is a single node.*

iii. $\mu(G) = 1 \iff G$ *is a disjoint union of paths.*

iv. $\mu(G) \leq 2 \iff G$ *is outerplanar.*

v. $\mu(G) \leq 3 \iff G$ *is planar.*

vi. $\mu(G) \leq n - 1$.

vii. $\mu(K_n) = n - 1$.

viii. $\mu(G) \leq \tau\omega_{CdV}(G) \leq \tau\omega(G) + 1$.

A graph G is *linklessly embeddable* if it can be embedded in $\mathbb{R}^3$ so that any two disjoint circuits in G form unlinked closed curves in $\mathbb{R}^3$. (See [25] for a more detailed explanation.) The following results are due to Robertson, Seymour and Thomas[25] and Lovász and Schrijver [20].

Fact 24 (Linkless embedding)

i. **(Robertson, Seymour and Thomas [25])** $\mu(G) \leq 4 \Longrightarrow G$ *is linklessly embeddable.*

ii. **(Lovász and Schrijver [20])** *G is linklessly embeddable* $\Longrightarrow \mu(G) \leq 4$.

We need the following characterization of a linklessly embeddable graph.

Definition 25 ((Petersen Family)) *The* Petersen Family *consists of the seven graphs in Figure 2. The first one is K_6 and the others are its ΔY-transformations, one of which is the Petersen graph.*

Fact 26 (Robertson, Seymour and Thomas[25]) *A graph G is linklessly embeddable $\iff$ G has no minor isomorphic to a member of the Petersen family.*

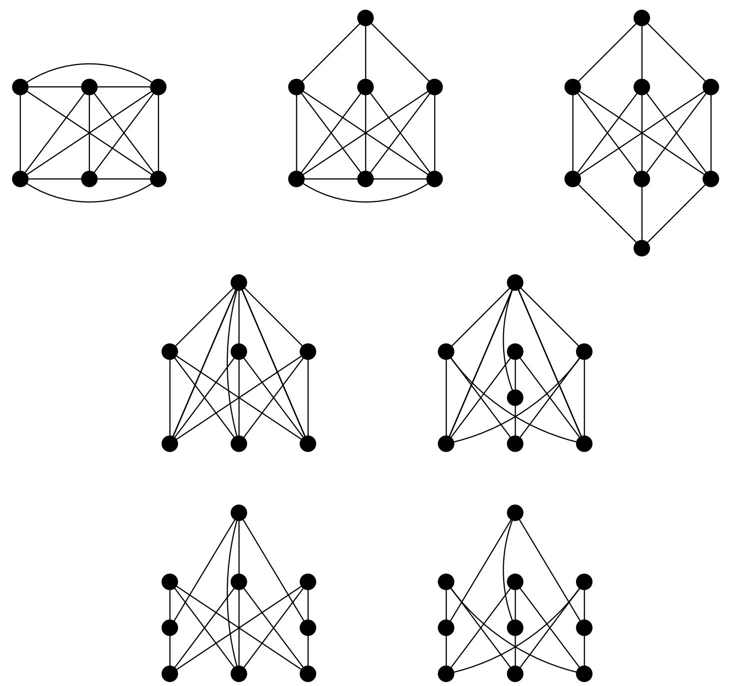

Figure 2: The Petersen Family

Theorem 27 *Let G be a connected graph.*

i. $\kappa_{slir}(G) = \kappa_{sir}(G) = 0 \iff \mu(G) = 0.$

ii. $\kappa_{slir}(G) = 1 \iff \mu(G) = 1.$

iii. $\kappa_{sir}(G) = 1 \iff \mu(G) \leq 2.$

iv. $\kappa_{slir}(G) = 2 \Longrightarrow \mu(G) \leq 3.$

v. $\kappa_{sir}(G) = 2 \Longrightarrow \mu(G) \leq 4.$

vi. $\kappa_{slir}(G) \leq \lfloor \frac{n}{2} \rfloor \iff \mu(G) \leq n-1.$

vii. $\kappa_{sir}(G) \leq \lceil \frac{n}{2} \rceil \iff \mu(G) \leq n-1.$

Proof.

i. This is trivially true.

ii. $\kappa_{slir}(G) = 1$ implies that G is a path, so $\mu(G) = 1$.

iii. This follows from the fact that $\kappa_{sir}(G) = 1$ if and only if G is outerplanar if and only if $\mu(G) \leq 2$.

iv. If $\kappa_{slir}(G) = 2$, then G does not contain K_5 or $K_{3,3}$ as a minor, so G is planar. Combining this with Colin de Verdière's result of $\mu(G) \leq 3$ yields the inequality. The bound is tight as $\kappa_{slir}(K_4) = 3$.

v. Suppose there is a G with $\kappa_{sir}(G) = 2$ but $\mu(G) > 4$. Then G must contain a minor in the Petersen family. Now, K_6 and $K_{3,4}$ are both forbidden minors for G. By inspection, each graph from the Petersen family contains both of these as minors: a contradiction. The bound is tight as $\kappa_{sir}(K_5) = 2$ and $\mu(K_5) = 4$.

vi. Follows from the upper bounds when $G = K_n$.

vii. Again follows from the case $G = K_n$.

□

The above theorem shows that for several possible values of $\kappa_{slir}(G)$ and $\kappa_{sir}(G)$, we have $\mu(G) \leq 2\kappa_{slir}(G) - 1$ and $\mu(G) \leq 2\kappa_{sir}(G)$. It is tempting to generalize this to any G. We formulate it as a conjecture based on the evidence that results from the known characterizations to date.

Conjecture 28 *For any G and $\mu(G) > 0$,*

i. $\mu(G) \leq 2\kappa_{slir}(G) - 1$.

ii. $\mu(G) \leq 2\kappa_{sir}(G)$.

We can prove the following, weaker result.

Theorem 29 *For $\kappa_{slir}(G)$,$\kappa_{sir}(G) \geq 3$,*

i. $\mu(G) \leq 4\kappa_{slir}(G) - 1$.

ii. $\mu(G) \leq 4\kappa_{sir}(G) + 1$.

Proof. Since $\mu(G) \leq \tau\omega(G) + 1$ and $\tau\omega(G) \leq 4\kappa_{slir}(G) - 2$, $\tau\omega(G) \leq 4\kappa_{sir}(G)$, the inequalities follow. □

We now consider some of the effects on $\mu(G)$, $\kappa_{slir}(G)$ and $\kappa_{sir}(G)$ of some common graph operations.

Fact 30 (Colin de Verdière [9]) $\mu(G) \leq \mu(G - v) + 1$

In essence, the result states that when a node v and its associated edges are deleted from G, $\mu(G)$ and $\mu(G-v)$ differs by at most one. We show that this is not the case for $\kappa_{slir}(G)$ and $\kappa_{sir}(G)$, except for the simple cases when $\kappa_{slir}(G) \leq 1$. In fact, $\kappa_{slir}(G)$ and $\kappa_{slir}(G-v)$ can differ by as much as we want (up to $\kappa_{slir}(G)$).

Theorem 31 *For any constant $s \geq 0$,*

i. there exists a graph G and a node v such that $\kappa_{slir}(G) \geq \kappa_{slir}(G-v) + s$,

ii. there exists a graph G and a node v such that $\kappa_{sir}(G) \geq \kappa_{sir}(G-v) + s$.

Proof. (i) When $\kappa_{slir}(G) = 1$, let $G = K_{1,2}$. If $s = 1$, let node v be the interior node, then $\kappa_{slir}(G) = 1$ and $\kappa_{slir}(G-v) = 0$. If $s = 0$, let node v to be one of the leaf nodes, then $\kappa_{slir}(G-v) = 1$. Thus the inequality holds.

Suppose now that $\kappa_{slir}(G) \geq 2$. If $s = 0$, then $\kappa_{slir}(G) = \kappa_{slir}(G+v)$, where $G+v$ is G with an extra node attached to any node in G. Deleting this v will satisfy the inequality. Let $s \geq 1$. Consider the graph $G = K_{2,2s+2}$ and let v be one of the $2-$nodes. Then $G - v = K_{1,2s+2}$. Now, $\kappa_{slir}(G) = s+2$ but $\kappa_{slir}(G-v) \leq 2$, so $\kappa_{slir}(G) \geq \kappa_{slir}(G-v) + s$.

(ii) The proof is similar as above, except that we take $G = K_{2,2s+1}$. □

We now investigate the clique-sum of graphs. In [17], it is shown that under the right conditions, $\mu(G) = \max\{\mu(G_1), \mu(G_2)\}$.

Theorem 32 *For any constant $s \geq 0$,*

i. there is a graph G with clique-sum G_1 and G_2 such that $\kappa_{slir}(G) \geq \max\{\kappa_{slir}(G_1), \kappa_{slir}(G_2)\} + s$;

ii. there is a graph G with clique-sum G_1 and G_2 such that $\kappa_{sir}(G) \geq max\{\kappa_{sir}(G_1), \kappa_{sir}(G_2)\} + s$.

Proof. When $s = 0$, we can just choose $G = K_{1,2}$ with $G_1 = K_2 = G_2$ intersecting at a single node. Then $\kappa_{slir}(G) = \kappa_{slir}(G_1) = \kappa_{slir}(G_2) = 1$. When $s = 1$, we let $G_1 = K_{1,2}$ and $G_2 = K_2$ with $G = K_{1,3}$ so that G_1 and G_2 intersect at the interior node of G_1. Then $\kappa_{slir}(G_1) = \kappa_{slir}(G_2) = 1$ but $\kappa_{slir}(G) = 2$.

Assume now $s \geq 2$. Let $K^+_{2,r}$ denote the graph $K_{2,r}$ with an extra edge connecting the 2-nodes. Let $G_1 = K^+_{2,2s-1} = G_2$, $G = G_1 \cup G_2 = K^+_{2,4s-2}$ and $G_1 \cap G_2 = K_2$. Then $\kappa_{slir}(G_1) = \kappa_{slir}(G_2) = s$, but $\kappa_{slir}(G) = \kappa_{slir}(K^+_{2,4s-2}) = 2s$. Thus $\kappa_{slir}(G) \geq \max\{\mu(G_1), \mu(G_2)\} + s$.

(ii) Similar to above. Choose $G_1 = K_{2,2s} = G_2$. □

Subdivision of G does not seem to affect $\mu(G)$ very much.

Fact 33 (Colin de Verdière [9]) *Let G' be a* subdivision *of G. Then*

i. $\mu(G) \leq \mu(G')$.

ii. $\mu(G) = \mu(G')$ *if* $\mu(G) \geq 3$.

Contrary to $\mu(G)$, $\kappa_{slir}(G')$ can be greater than $\kappa_{slir}(G)$ for $\kappa_{slir}(G) \geq 2$.

Theorem 34 (Subdivision)

i. For $k \geq 2$, there exists a graph G and a subdivision G' with $k = \kappa_{slir}(G) < \kappa_{slir}(G')$.

ii. For $k \geq 1$, there exists a graph G and a subdivision G' with $k = \kappa_{sir}(G) < \kappa_{sir}(G')$.

Proof. (i) Let $G = K^+_{2,2k-1}$. Then $\kappa_{slir}(G) = k$, but a subdivision along the edge connecting the 2-nodes yields $G' = K_{2,2k}$, a forbidden minor.

(ii) Similar for $\kappa_{sir}(G)$. Let $G = K^+_{2,2k}$. □

However the values of $\kappa_{slir}(G)$ and $\kappa_{sir}(G')$ can only differ by at most one.

Theorem 35 *Let G' be a subdivision of a graph G. Then*

i. $\kappa_{slir}(G') \leq \kappa_{slir}(G) + 1$,

ii. $\kappa_{sir}(G') \leq \kappa_{sir}(G) + 1$.

Proof. Assume the edge where the subdivision occurs is adjacent to nodes u and u' and the new node is v. Then label node v with $n+1$. Any path to v must either go through node u or through u', so in the worst case at each edge, we add the extra interval $[v, v]$, i.e., we have an increase with at most one extra interval. The edge (v, u) is labeled with the original intervals of edge (u', u) and likewise edge (v, u') with original intervals of edge (u, u'), with no increase in the number of intervals. □

Bacher and Colin de Verdière [3] showed that for $\mu(G) \geq 4$, $\mu(G)$ is invariant under the ΔY- and $Y\Delta$-operations. We show that this is not the case for $\kappa_{slir}(G)$ and $\kappa_{sir}(G)$.

Theorem 36 (ΔY-transformations)

i. For $k \geq 2$, there exists a graph G and a graph G' obtained from G by a ΔY-transformation, such that $\kappa_{slir}(G) = k < \kappa_{slir}(G')$.

ii. For $k \geq 1$, there exists a graph G and a graph G' obtained from G by a ΔY-transformation, such that $\kappa_{sir}(G) = k < \kappa_{sir}(G')$.

Proof. (i) Assume $k \geq 2$. In the top of Figure 3 the graph Δ which is outerplanar and contains $K_{2,2}$ is transformed to the graph Y which contains $K_{2,3}$. By adding $2k-3$ extra nodes and then connecting them to the graphs Δ and Y, we obtain the graphs G and G'; each of the extra nodes is made adjacent to two vertices, as is shown in Figure 3. Now $\kappa_{slir}(G) = k$ and $\kappa_{slir}(G') = k+1$ as it contains the forbidden minor $K_{2,2k}$.

(ii) Similar to the above construction, except that G is the Δ-graph with $2k-2$ extra nodes connected to Δ. □

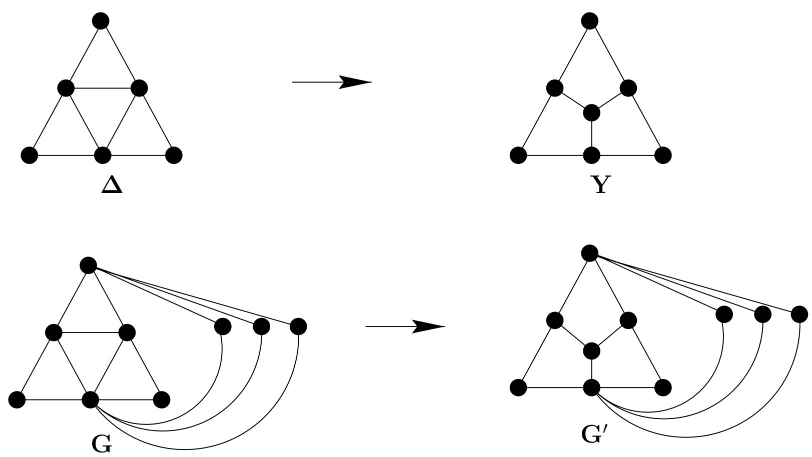

Figure 3: The ΔY Transformation

It is an interesting open question what the maximum difference can be between $\kappa_{sir}(G)$ and $\kappa_{sir}(G')$ and between $\kappa_{sir}(G)$ and $\kappa_{sir}(G')$ when G' is obtained from G by a ΔY-transformation.

6 Interval Routing and $\lambda(G)$

In [16] van der Holst, Laurent and Schrijver introduced an interesting graph parameter based on Colin de Verdière's $\mu(G)$.

Definition 37 ($\lambda(G)$)

i. *For any vector* $x \in \mathbb{R}^n$*, the positive support is* $supp_+(x) = \{i | x_i > 0\}$.

ii. *A linear subspace* $L \subset \mathbb{R}^n$ *is a* valid representation *of* $G = (V, E)$ *with* $V = \{1, 2, \cdots, n\}$*, if for each nonzero* $x \in L$, $supp_+(x) \neq \emptyset$ *and the subgraph induced by the set of nodes* $supp_+(x)$ *in* G *is connected.*

iii. $\lambda(G) = \max\{dim(L) | L$ *is a valid representation of* $G\}$.

The motivation from $\mu(G)$ is that, when x is in the null space of a Colin de Verdière matrix M (with minimal support), the subspace induced by $supp_+(x)$ is nonempty and connected.

Fact 38 (van der Holst, Laurent, Schrijver [16])

i. $\lambda(G) = 0 \iff$ *G has no* K_2 *as a minor.*

ii. $\lambda(G) = 1 \iff$ *G has no* K_3 *as a minor.*

iii. $\lambda(G) \leq 2 \iff$ *G has no* K_4 *as a minor.*

iv. $\lambda(G) \leq 3 \iff$ *G has no* K_5 *or* V_8 *as a minor (*V_8 *is the last graph in Figure 1).*

v. $\lambda(K_n) = n - 1$.

vi. $\lambda(G)$ *is minor-monotone.*

There is an interesting connection between $\mu(G)$ and $\lambda(G)$.

Fact 39 (Pendavingh [21]) $\mu(G) \leq \lambda(G) + 2$.

It is conjectured that $\mu(G) \leq \lambda(G) + 1$.

We now consider the relationship between the interval routing parameters and $\lambda(G)$.

Theorem 40 *Let G be a connected graph.*

i. $\kappa_{slir}(G) = \kappa_{sir}(G) = 0 \iff \lambda(G) = 0$.

ii. $\kappa_{slir}(G) = 1 \Longrightarrow \lambda(G) = 1$.

iii. $\kappa_{sir}(G) = 1 \Longrightarrow \lambda(G) \leq 2$.

iv. $\kappa_{slir}(G) = 2 \Longrightarrow \lambda(G) \leq 3$.

v. $\kappa_{slir}(G) \leq \lfloor \frac{n}{2} \rfloor \iff \lambda(G) \leq n - 1$.

vi. $\kappa_{sir}(G) \leq \lceil \frac{n}{2} \rceil \iff \lambda(G) \leq n - 1$.

Proof. The results follow from the forbidden minors of the parameters. For the inequality (iii), we note that the graph V_8 contains $K_{3,3}$ as a minor, which is forbidden for G with $\kappa_{slir}(G) = 2$. Again, the last two inequalities follow from the upper bounds of $G = K_n$. □

The above theorem shows again that for several values of $\kappa_{slir}(G)$ and $\kappa_{sir}(G)$, $\lambda(G) \leq 2\kappa_{slir}(G) - 1$ and $\lambda(G) \leq 2\kappa_{sir}(G)$. The following conjecture may also hold.

Conjecture 41 *For any* $\lambda(G) > 0$,

i. $\lambda(G) \leq 2\kappa_{slir}(G) - 1$.

ii. $\lambda(G) \leq 2\kappa_{sir}(G)$.

7 Interval Routing and $\kappa(G)$

In [16], van der Holst, Laurent and Schrijver introduced yet another graph parameter related to $\lambda(G)$, which they called $\kappa(G)$. The definition is rather technical but similar to $\lambda(G)$.

Definition 42 ($\kappa(G)$) *Let* $G = (V, E)$ *be a* connected *graph and* $\phi : V \to \mathbb{R}^d$ *be a function such that:*

i. $\phi(V)$ *affinely spans a d-dimensional affine space,*

ii. for each affine halfspace $H \subseteq \mathbb{R}^d$, $\phi^{-1}(H)$ *induces a connected subgraph of* G *(possibly empty).*

$\kappa(G)$ *is the largest* d *for which the above conditions hold, for a suitable* ϕ.

It turns out that $\kappa(G)$ has a complete forbidden minor characterization.

Fact 43 (van der Holst, Laurent, Schrijver [16])

i. $\kappa(G)$ *is minor-monotone.*

ii. $\kappa(G) \leq k \iff G$ *does not contain* K_{k+2} *as a minor.*

iii. $\kappa(G) \leq \lambda(G)$.

It turns out that Conjecture 41 actually can be shown for the case of $\kappa(G)$.

Theorem 44 *For every kappa* ≥ 1,

i. $\kappa(G) \leq 2\kappa_{slir}(G) - 1$.

ii. $\kappa(G) \leq 2\kappa_{sir}(G)$.

Proof. (i) This follows from the fact that K_{2k+1} is a forbidden minor of the class of graphs G with $\kappa_{slir}(G) = k$ and $\kappa_{slir}(K_{2k}) = k$, $\kappa(K_{2k}) = 2k - 1$.

(ii) This also follows from the fact that K_{2k+2} is a forbidden minor of the class of graphs G with $\kappa_{sir}(G) = k$ with $\kappa_{slir}(K_{2k+1}) = k$ and $\kappa(K_{2k+1}) = 2k$. □

8 Conclusion

In this paper, we have surveyed and explored several graph parameters that are indicative for the complexity of a graph and for routing information in a graph. In particular we have made comparisons between Colin de Verdière's graph parameter $\mu(G)$ and its variants $\lambda(G)$ and $\kappa(G)$, and the parameters $\kappa_{slir}(G)$ and $\kappa_{sir}(G)$ related to certain types of interval routing schemes. Interesting relationships could be observed based on the known characterisations of the parameters, which are all minor-monotone. With $\theta(G)$ denoting $\mu(G)$, $\lambda(G)$ or $\kappa(G)$, it appears that $0 < \theta(G) \leq 2\kappa_{slir}(G) - 1$ and $\theta(G) \leq 2\kappa_{sir}(G)$, for the known characterizations of the respective parameters (and the case $\kappa(G)$).

It may be that the above inequalities also hold for any graph G but we leave this as a conjecture. Also, all the results derived are based on the known characterizations by forbidden minors. It would be interesting to see if there are any direct proofs by using the definitions.

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