On the abelianization of the Torelli group

De abelianisatie van de Torelligroep
(met een samenvatting in het Nederlands)

Proefschrift

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Contents

List of Notations

We use the following notations. The number refers to the page number where you can find the definition.

General	notations
$H_i(X)$	$H_i(X)$ is the i^{th} (group, singular, cellular) homology group of (a group, a topological space, a space with cell structure) X with integer coefficients,
$C_i(X)$	a chain complex to compute $H_i(X)$ is denoted by $C_i(X)$,
$\mathbb{Z}[G]$	$\mathbb{Z}[G]$ denotes the group ring of a group G,
$I[G]$	$I[G] := \operatorname{Ker}(\epsilon : \mathbb{Z}[G] \to \mathbb{Z})$ is the augmentation ideal,
$[a, b]$	$[a, b] := aba^{-1}b^{-1}$ is the commutator of a and b,
G_{ab}	we write G_{ab} for the abelianization $G/[G, G]$ of a group G,
$\mathbb{Z}^{(A)}$	If A is a set, then $\mathbb{Z}^{(A)}$ denotes the group of maps $A \to \mathbb{Z}$ with finite support.

Notations	introduced in Chapter 1	
$\operatorname{Rad}(H)$	$\operatorname{Rad}(H)$ is the radical of a quasi-unimodular symplectic module H,	p. 1
$\overline{H}$	$\overline{H} := H/\operatorname{Rad}(H)$,	p. 1
$g(H)$	$g(H) := \frac{1}{2}\operatorname{rk}(\overline{H})$ is the genus of H,	p. 1
$(\widetilde{H}, H, \Delta)$	$(\widetilde{H}, H, \Delta)$ denotes an extended surface module,	p. 2
(S, P)	(S, P) denotes a surface with boundary marking,	p. 2
$\operatorname{Sp}(\widetilde{H}, H)$	$\operatorname{Sp}(\widetilde{H}, H)$ is the group of automorphisms of an extended surface module $(\widetilde{H}, H, \Delta)$,	p. 3
$\operatorname{Sp}(H)$	when $\widetilde{H} = H$ the group $\operatorname{Sp}(\widetilde{H}, H)$ is denoted by $\operatorname{Sp}(H)$,	p. 3
δ_v	for every $v \in H$ we have $\delta_v(x) := x + (x \cdot v)v$, the symplectic transvection determined by v,	p. 3
$K(\widetilde{H}, H)$	$K(\widetilde{H}, H) := \operatorname{Ker}(\operatorname{Sp}(\widetilde{H}, H) \to \operatorname{Sp}(\overline{H}))$,	p. 3
S^2V	S^2V is the submodule of $V \otimes V$ of invariants under the involution determined by $a \otimes b \mapsto b \otimes a$,	p. 3

$V \circ V/U$	$V \circ V/U := S^2V/S^2U$ for a submodule U of V,	p. 3
Ω_H	if H is a symplectic module over $\mathbb{Z}/2$ then Ω_H denotes the affine space of associated quadratic forms on H,	p. 5
Ψ_H	Ψ_H is the set of quadratic forms of Arf-invariant zero,	p. 5
$B_r(\Omega_H)$	$B_r(\Omega_H)$ is the space of all polynomial functions on Ω_H of degree $\leq r$,	p. 5
$\lvert\Sigma\rvert$	$\lvert\Sigma\rvert$ is the topological realization of a simplicial complex Σ,	p. 5
$H_p(\Sigma, \mathcal{F})$	the p^{th} homology group of Σ with values in the system of coefficients $\mathcal{F}$ is denoted by $H_p(\Sigma, \mathcal{F})$,	p. 6
f/y	$f/y := \{x \in X : f(x) \leq y\}$,	p. 6
$f\backslash y$	$f\backslash y := \{x \in X : f(x) \geq y\}$,	p. 6
$\mathrm{Link}^-_X(y)$	$\mathrm{Link}^-_X(y) := X_{<y} = \{x \in X : x < y\}$,	p. 6
$\mathrm{Link}^+_X(y)$	$\mathrm{Link}^+_X(y) := X_{>y} = \{x \in X : x > y\}$,	p. 6
$\mathrm{Star}_X(y)$	$\mathrm{Star}_X(y) := \mathrm{Link}_X(y) \cup \{y\}$,	p. 6
CM_d	CM_d is an abbreviation for Cohen-Macaulay of dimension d,	p. 7
$\mathcal{O}(S)$	$\mathcal{O}(S)$ is the poset of nonempty finite subsets of S,	p. 8
$T(V, W)$	$T(V, W)$ is the poset of nonzero proper subspaces U of V such that $U \oplus W \to V$ is a primitive embedding,	p. 9
$\mathcal{P}(V, W)$	$\mathcal{P}(V, W)$ is the poset of partial bases E of V such that its span $\langle E \rangle$ is in $T(V, W)$	p. 9
$\mathcal{A}^o(H)$	$\mathcal{A}^o(H)$ is the poset of arc-sequences in H,	p. 9
$\mathcal{A}^o(H, \pi)$	if $\pi : H \to \mathbb{Z}$ is an epimorpism that factorizes of $\mathrm{Rad}(H)$, then $\mathcal{A}^o(H, \pi)$ is the poset of arc-sequences E^o in $\pi^{-1}(1)$ such that $E^o \in \mathcal{P}(H, \mathrm{Rad}(\pi^{-1}(0)))$,	p. 9
$I(H, I)$	$I(H, I)$ is the poset of $U \in T(H, I)$ such that $U + I$ is isotropic,	p. 10
$\mathcal{I}(H, I)$	$\mathcal{I}(H, I)$ is the poset of $E \in \mathcal{P}(H, I)$ such that its span $\langle E \rangle$ is in $I(H, I)$,	p. 10
$T(\pi)$	$T(\pi)$ is the poset of $U \in T(V)$ such that U is in general position relative to π,	p. 13
$T(\pi/\rho)$	$T(\pi/\rho)$ is the poset of $U \in T(\pi)$ such that U is primitive relative to ρ,	p. 13
$I(\pi)$	$I(\pi) := I(H) \cap T(\pi)$,	p. 13

Notations	introduced in Chapter 2	
$S^n_{g,r}$	$S^n_{g,r}$ denotes a compact, oriented, connected topological surface of genus g, with r boundary components and n distinct fixed points on the interior of $S^n_{g,r}$,	p. 35
$\mathfrak{F}S^n_{g,r}$	$\mathfrak{F}S^n_{g,r}$ denotes the group of orientation preserving homeomorphisms of $S^n_{g,r}$ that are the identity on the boundary of $S_{g,r}$ and fix the n distinct points pointwise,	p. 35

$\Gamma^n_{g,r}$	$\Gamma^n_{g,r}$ is the mapping class group of $S^n_{g,r}$, that means, the group of isotopy classes of $\mathfrak{F}S^n_{g,r}$,	p. 35
D_γ	if γ is an embedded circle on the interior of $S^n_{g,r}$ disjoint from the n fixed points, we denote by D_γ the left Dehn twist around γ,	p. 35
$D_\beta D_\alpha$	$D_\beta D_\alpha$ means first apply D_α then D_β,	
T_S	$T_S := \mathrm{Ker}(\Gamma_S \to \mathrm{Sp}(H_1(S,P), H_1(S)))$ is the Torelli group of S	p. 36
$\widetilde{T}_S$	$\widetilde{T}_S := \mathrm{Ker}(\Gamma_S \to \mathrm{Sp}(H_1(S)))$ is the big Torelli group of S	p. 36
SCC	simple closed curve,	p. 35
$BSCC$	bounding simple closed curve,	p. 38
BP	bounding pair,	p. 38
$\mathfrak{T}_k$	$\mathfrak{T}_k$ is the set of $BSCC$-maps that bound a subsurface of genus k,	p. 38
$\mathfrak{W}_k$	$\mathfrak{W}_k$ is the set of BP-maps that bound a subsurface of genus k,	p. 38
τ_m	for every $m \geq 1$ the Johnson homomorphism $\tau_m : \Gamma(m) \to \mathrm{Hom}(H, \pi_{[m]}/\pi_{[m+1]})$ is defined,	p. 40
t_α	for an oriented loop α without self intersection such that $\partial S \cap \alpha = \{p\}$ on a component ∂ of ∂S, we define $t_\alpha := D^{-1}_{\alpha_+} D_{\alpha_-}$. Here α_+, α_- are the boundary components of the regular neighborhood of $\alpha \cup \partial$ such that α_- is on the left of α and α_+ on the right,	p. 42
$BX(\Lambda, \Lambda^0)$	$BX(\Lambda, \Lambda^0)$ denotes the simplicial complex of (Λ, Λ^0)-arc systems defined by Harer,	p. 45
$\overline{BX}(p,q)$	$\overline{BX}(p,q) := T_S \backslash BX(p,q)$, sometimes also abbreviated by $\overline{BX}$,	p. 46

Notations	introduced in Chapter 3	
$F^n(S)$	if S is a surface then $F^n(S)$ is the configuration space of n pairwise distinct points of S,	p. 51
$P^n(S)$	$P^n(S) := \pi(F^n(S))$ the pure braid group of S on n strings,	p. 51
H_2	if H is a free module over $\mathbb{Z}$ then $H_2 := H \otimes_{\mathbb{Z}} \mathbb{Z}/2$,	p. 59
M_H	M_H is the set of unimodular symplectic subspaces of H of genus 1,	p. 59
N_H	N_H is the set of unimodular symplectic subspaces of H of genus 2,	p. 59
R_H	R_H is the set of elements $U \oplus U' - U - U'$, where $U, U' \in M_H$, $U \perp U'$ and $U \oplus U' \in N_H$,	p. 59
$\widetilde{G}_H$	$\widetilde{G}_H := \frac{\mathbb{Z}/2^{(N_H)} \oplus \mathbb{Z}^{(M_H)}}{R_H}$,	p. 59
G_H	$G_H := \frac{\mathbb{Z}/2^{(N_{H_2})} \oplus \mathbb{Z}/2^{(M_{H_2})}}{R_{H_2}}$.	p. 59

Introduction

Let S be a surface of genus g, possibly with boundary (in this thesis by surface is meant a connected, orientable and compact topological surface). The mapping class group Γ_S of S is the group of isotopy classes of the orientation preserving homeomorphisms of S that are the identity on the boundary ∂S. If we choose on each boundary component of S a point and denote this set of points by P, then Γ_S acts on $H_1(S, P)$ (relative homology with integer coefficients), leaving the submodule $H_1(S)$ invariant and preserving the intersection product $H_1(S, P) \times H_1(S) \to \mathbb{Z}$ that can be defined. The kernel of this action is by definition the Torelli group T_S of S. This means that if ∂S is empty or connected, then T_S is the subgroup of mapping classes that act trivially on the homology of S. If ∂S has more than one component we need this refined definition in order to make T_S functorial for inclusion of surfaces.

In this thesis we study the abelianization of the Torelli group of a surface S with an arbitrary (but finite) number of boundary components. This study takes up and continues the work of Johnson from around 1980. He computes, among other things, that for a surface of genus at least three and with one boundary component, $H_1(T_S) \cong \wedge^3 H_1(S)$ modulo 2-torsion, and the torsion is also completely described in terms of the homology of the surface, see [Johnson8]. Here we prove that this result holds for surfaces of $g \geq 3$ with an arbitrary number of boundary components. Our method of proof differs from his, is inductive in nature and may open the way to calculate the higher homology of these groups. We study how $H_1(T_S)$ changes compared to $H_1(T_{S'})$, where S is obtained from S' by gluing a pair of pants to it, by using the action of T_S on Harer's arc-complexes. Furthermore we study the Torelli group of a surface of low genus, the results of which are also needed to start the induction.

We finish this introduction with an overview of the content of all chapters in this thesis, but let us first be more explicit about the method of proof we use.

Let $p, q \in \partial S$. The basic tool in our study of T_S is its action on the highly connected arc-complexes $BX(p, q)$ that were introduced by Harer in [Harer]. The k-simplices of this simplicial complex are $(k+1)$-tuples of isotopy classes of arcs from p to q that can be represented by a $(k+1)$-tuple of embedded arcs which are disjoint away from p, q and whose complement in S is connected. They have the property that the stabilizers under the action of Γ_S, respectively T_S, are mapping class groups, respectively Torelli groups, of surfaces of lower genus or with fewer boundary components. Harer proves in that $BX(p,q)$ is spherical and, using the action of the mapping class group on $BX(p,q)$, he establishes the stability of the homology of the mapping class groups induced by the inclusion of surfaces. His approach is analogous to the proof of the stabilization of homology of various arithmetic groups. See also [Ivanov] on the stability of the homology of the mapping class group.

We use, just as Foisy does in [Foisy], the induced action of T_S on this arc-complex. Here the quotient space is a simplicial complex that is closely related to the complexes of (isotropic) partial bases of a (symplectic) lattice, studied by Maazen, Van der Kallen, Charney, Vogtmann and others to prove the homology stability for certain linear groups. See for example [Charney], [VdKallen], [Maazen], [Vogtmann]. Most of these complexes are known to be spherical and we show that this quotient space is connected up to a certain dimension. When the genus of the surface is ≥ 4 and we choose p, q on the same boundary component, this quotient is at least 2-connected and we get by a spectral sequence argument a description of $H_1(T_S)$ as an amalgam of the abelianization of the stabilizers of the vertices. When $g = 3$ the spectral sequence shows that $H_1(T_S)$ is a quotient of this amalgam, the Johnson and Birman-Craggs homomorphisms show that the kernel is trivial.

For low genera, ($g \leq 2$), we do not have a uniform description of the abelianized Torelli group. Mess shows that for a closed surface of genus two, T_S is infinitely freely generated by a set of Dehn twists around separating curves, see [Mess]. We give a description of $H_1(T_S)$ in some other cases.

The thesis consists of three chapters, culminating in the proof of Johnsons result for all surfaces of $g \geq 3$. The outline of this thesis is as follows.

The first chapter is about symplectic modules and simplicial complexes described in terms of such modules. We introduce the notion of an extended surface module which formalizes the situation of $H_1(S, P)$ and the structures it has. It will appear in the second chapter that the quotient of $BX(p,q)$ by the action of the Torelli group, $T_S \backslash BX(p,q)$, is again a simplicial complex and can be described in terms of symplectic modules. The main goal of this chapter is to prove that $T_S \backslash BX(p,q)$ is $(g-2)$-connected when p and q are on the same boundary component; when p and q are on different components we show that it is 1-connected if $g \geq 2$. We will do this in Sections 1.6 up to 1.10 and for this purpose we introduce some other complexes, some of them are known to be spherical, of others we prove connectedness properties here. We finish this chapter with a discussion about simply connected simplicial complexes with a group action. Using a spectral sequence we derive an exact sequence that relates the abelianization of the group to the low-dimensional homology of the quotient complex and the abelianization of the stabilizers.

The second chapter is about surfaces. We introduce the Torelli group and give an overview of the work of Johnson and others on the Torelli group. When S' is obtained from S by closing a hole of S, we have a surjection of T_S onto $T_{S'}$. We give a description of the kernel of this map. In the final section of this chapter we show that the quotient complexes of the arc-complexes of Harer by the action of the Torelli group are indeed isomorphic to the complexes introduced in Chapter 1.

In the third chapter we investigate the Torelli group of surfaces of low genus and establish an isomorphism between $H_1(T_S)$ and $\wedge^3 H_1(S) \oplus B_2(\Omega_S)$ when $g \geq 3$. Here $B_2(\Omega_S)$ is the $\mathbb{Z}/2$ linear space of polynomial functions of degree ≤ 2 on the space of symplectic quadratic forms on $H_1(S, \mathbb{Z}/2)$. If $g = 0$ then T_S is the commutator subgroup of a pure braid group P and therefore we can give a description of $H_1(T_S)$ as a module over $\mathbb{Z}[P_{\text{ab}}]$, the group ring of P_{ab}. When S is a torus then if S is closed, T_S is trivial; if ∂S is connected then T_S is infinitely cyclic and if S has two boundary components, we use the map from S to the torus with one boundary component to describe T_S. If S is a surface of genus two we know by the result of Mess that if S is closed, T_S is infinitely freely generated by Dehn twists around separating curves. When ∂S is connected, we use the map from S to the closed surface of genus two to compute $H_1(T_S)$. For a surface of genus two with two boundary components we were not able to compute its abelianized Torelli group, but we can describe it as a quotient of a group that is small enough to initiate the induction: it implies a surjection

$$\wedge^3 H_1(S) \oplus B_2(\Omega_S) \to H_1(T_S)$$

when S has genus three, and will establish the theorem for S using the Johnson homomorphism and the Birman-Cragg homomorphism. For a surface of higher genus the theorem follows by induction.

A list of notations used in this thesis is given on page v.

CHAPTER 1

Symplectic modules

1.1. Introduction

This chapter is about symplectic modules and simplicial complexes associated to symplectic modules. We start with recalling definitions concerning symplectic modules and introduce the notion of an extended surface module. We then assume that we have a symplectic module over $\mathbb{Z}/2$ and we recall some notions about quadratic forms. In the fourth section we give the definition of a simplicial complex and a poset and some related notions. We associate several simplicial complexes to a symplectic module, some of them are known to be spherical and we prove in the Sections 1.6 to 1.10 connectedness properties for the new ones. Finally, we discuss simply connected simplicial complexes with a group action and derive an exact sequence that relates the low dimensional homology groups of the quotient complex and the abelianization of the stabilizers with the abelianization of the group.

1.2. Surface modules

Let D denote $\mathbb{Z}$ or $\mathbb{Z}/2$ and let H be a *symplectic module over* D. With this we mean that H is a finitely generated free module over D with a pairing $H \times H \to D$ that is skew-symmetric, (so in fact symmetric if $D = \mathbb{Z}/2$). I f $v, w \in H$, we denote the pairing by $v \cdot w$. The *radical of* H is the submodule consisting of elements $v \in H$ such that $v \cdot w = 0$ for all $w \in H$. In other words, it is the kernel of the homomorphism $^* : H \to H^* = \mathrm{Hom}(H, D)$ defined by $v^*(x) = x \cdot v$. We denote the radical by $\mathrm{Rad}(H)$ and the quotient $H/\mathrm{Rad}(H)$ by $\overline{H}$. The symplectic module H is called *quasi-unimodular* if the induced map $\overline{H} \to \mathrm{Hom}(\overline{H}, D)$ is an isomorphism and is called *unimodular* if moreover $\mathrm{Rad}(H) = 0$. In that case we know that the rank of the module is even and we call the integer $g(H) := \frac{1}{2}\mathrm{rk}(\overline{H})$ the *genus of* H. Any quasi-unimodular symplectic module H has a basis

$$\{e_1, \ldots, e_g, e_{-g}, \ldots, e_{-1}, e_{g+1}, \ldots, e_{g+r}\},$$

where $\{e_{g+i}\}_{i=1}^r$ is a basis for $\mathrm{Rad}(H)$ and $\{\overline{e_{\pm i}}\}_{i=1}^g$ is a basis for $\overline{H}$, with intersection products $e_i \cdot e_{-i} = 1$ and all other combinations of the basis elements have intersection product zero. We refer to such a basis as a symplectic basis for H. A direct summand I of H is called *isotropic* if $v \cdot w = 0$ for all $v, w \in I$. If $a, b \in H$ and $a \cdot b = 1$, we call $\{a, b\}$ a *hyperbolic pair.*

We introduce the following notions to create a setting that is suitable for later formulations.

Definition 1.2.1. *A surface module is a pair* (H, Δ) *where* H *is a quasi-unimodular symplectic module and* $\Delta \subset \mathrm{Rad}(H)$ *is a finite generating set for* $\mathrm{Rad}(H)$ *such that the evident map* $\mathbb{Z}^{\Delta} \to \mathrm{Rad}(H)$ *has kernel spanned by* $\sum_{\partial \in \Delta} \partial$*. An extended surface module is a triple* $(\widetilde{H}, H, \Delta)$ *where* $\widetilde{H}$ *is a finitely generated free module over* D *and we have a filtration*

$$0 \subset \mathrm{Rad}(H) \subset H \subset \widetilde{H},$$

together with a pairing $\widetilde{H} \times H \to D$ *such that*

(i) (H, Δ) *with the restricted pairing is a surface module,*

(ii) *the induced map* $\widetilde{H}/H \to (\mathrm{Rad}(H))^* \cong \{\sum_{\partial \in \Delta} n_\partial \partial^* : \sum_{\partial} n_\partial = 0\}$ *is injective and the image is of the form* $\{\sum_{p \in P} n_p p^* : \sum_{p \in P} n_p = 0\}$ *for a subset* $P \subset \Delta$.

We call an extended surface module complete *if* $\widetilde{H}/H \to (\mathrm{Rad}(H))^*$ *is an isomorphism, or equivalently, if the induced pairing* $\widetilde{H}/\mathrm{Rad}(H) \times H \to D$ *is perfect. We call* P*, which is clearly unique, the marking of the extended surface module.*

If $(\widetilde{H}, H, \Delta)$ is an extended surface module, we define for every $v \in \widetilde{H}$ the *orthogonal complement in* H by $v^{\perp} := \mathrm{Ker}(v^* : H \to D)$. If $W \subset \widetilde{H}$ is a subset we define $W^{\perp} := \cap_{w \in W} w^{\perp}$.

The notion of an extended surface module is explained by the following example.

Example-Definition 1.2.2. *Let* S *be a compact orientable connected topological surfaces with boundary* ∂S*. A boundary marking of* S *is a subset* P *of* ∂S *such that the map induced by the inclusion*

$$\pi_0(P) \to \pi_0(\partial S)$$

is injective. We call it a complete boundary marking whenever this map is a bijection. The first homology group $H_1(S)$ *of* S *with integer coefficients has a symplectic form defined by the intersection of cycles, the radical is generated by the boundary cycles. If* Δ *is the image of the natural basis of* $H_1(\partial S)$ *in* $H_1(S)$ *then* $(H_1(S), \Delta)$ *has the structure of a surface module. If* P *is a boundary marking of* S *we have the commuting exact diagram*

$$\begin{array}{ccccccccccc}
 & 0 & & 0 & & & & H_0(P) & & \\
 & \downarrow & & \downarrow & & & & \downarrow & & \\
\cdots & H_1(\partial S) & \to & H_1(S) & \to & H_1(S, \partial S) & \to & H_0(\partial S) & \twoheadrightarrow & H_0(S) \\
 & \downarrow & & \downarrow & & \| & & \downarrow & & \downarrow \\
\cdots & H_1(\partial S, P) & \to & H_1(S, P) & \to & H_1(S, \partial S) & \to & H_0(\partial S, P) & \to & 0 \\
 & \downarrow & & \downarrow & & & & \downarrow & & \\
 & 0 & & H_0(P) & & & & 0 & &
\end{array}$$

Since we have the identification $H_1(S,\partial S) \cong H_1(S)^*$ *by the intersection product, we see from this diagram that we can extend the pairing on* $H_1(S)$ *to a pairing*

$$H_1(S,P) \times H_1(S) \to \mathbb{Z}$$

and that the map $H_1(S,P)/H_1(S) \to (\mathrm{Rad}(H_1(S)))^* \cong \{\sum_{\partial\in\Delta} n_\partial \partial^* | \sum_{\partial\in\Delta} n_\partial = 0\}$ *thus obtained is injective. The image consists of the elements* $\sum_{p\in\Delta_P} n_p p^*$ *such that* $\sum_{p\in\Delta_P} n_p = 0$*. Here* $\Delta_P \subset \Delta$ *is the set of cycles that are represented by a boundary component containing an element of* P*. Therefore* $(H_1(S,P), H_1(S), \Delta)$ *has the structure of an extended surface module and we remark that the marking is complete if and only if the module is complete. This finishes the example.*

Let $(\widetilde{H}, H, \Delta)$ be an extended surface module. We denote by $\mathrm{Sp}(\widetilde{H}, H)$ the group of automorphisms of $\widetilde{H}$ that preserve the filtration $0 \subset \mathrm{Rad}(H) \subset H \subset \widetilde{H}$, that also preserve the pairing and are the identity on $\mathrm{Rad}(H)$. This implies that such an automorphism induces the identity on $\widetilde{H}/H$. The group $\mathrm{Sp}(\widetilde{H}, H)$ is called the group of automorphisms of the extended surface module. As the notation suggests, it is independent of the choice of Δ. If $H = \widetilde{H}$ is just a surface module then we denote this group by $\mathrm{Sp}(H)$. To every element $v \in H$ we assign an element δ_v of $\mathrm{Sp}(\widetilde{H}, H)$ by defining

$$\delta_v(x) := x + (x \cdot v)v.$$

The element δ_v is called the *symplectic transvection determined by* v. An element of $\mathrm{Sp}(\widetilde{H}, H)$ maps H into itself and hence we have a group homomorphism $\mathrm{Sp}(\widetilde{H}, H) \to \mathrm{Sp}(H)$ defined which is surjective. Since an automorphism of (H, Δ) is the identity on Δ we have the induced surjection $\mathrm{Sp}(H) \to \mathrm{Sp}(\overline{H})$. The kernel can be identified with $\mathrm{Hom}(\overline{H}, \mathrm{Rad}(H))$ and is generated by the elements $\delta_{r+a}\delta_a^{-1}(x) = x + (x \cdot a)r$ for $r \in \mathrm{Rad}(H)$ and $a \in \overline{H}$. We give a description of the kernel $K(\widetilde{H}, H)$ of the composition map $\mathrm{Sp}(\widetilde{H}, H) \to \mathrm{Sp}(\overline{H})$. If V is a D-module, we denote by $S^2(V)$ the submodule of $V \otimes_D V$ that is invariant under the involution defined by $a \otimes b \mapsto b \otimes a$. If U is a submodule of V then S^2U is a submodule of S^2V and we write $V \circ V/U$ for the quotient S^2V/S^2U.

LEMMA 1.2.3. *The kernel* $K(\widetilde{H}, H)$ *of the surjection* $\mathrm{Sp}(\widetilde{H}, H) \to \mathrm{Sp}(\overline{H})$ *is a central abelian extension of* $\mathrm{Hom}(\overline{H}, \mathrm{Rad}(H))$ *by* $\mathrm{Rad}(H) \circ \mathrm{Rad}(H)/\langle \Delta - P\rangle$ *that splits.*

PROOF. We show that the kernel of $\mathrm{Sp}(\widetilde{H}, H) \to \mathrm{Sp}(H)$ can by identified with $\mathrm{Rad}(H) \circ \mathrm{Rad}(H)/\langle \Delta - P\rangle$ and is central in $\mathrm{Sp}(\widetilde{H}, H)$. The statement then follows from the commuting exact diagram

$$\begin{array}{ccccc}
0 & & 0 & & \\
\downarrow & & \downarrow & & \\
\mathrm{Rad}(H)\circ\mathrm{Rad}(H)/\langle\Delta-P\rangle & = & \mathrm{Rad}(H)\circ\mathrm{Rad}(H)/\langle\Delta-P\rangle & & \\
\downarrow & & \downarrow & & \\
K(\widetilde{H},H) & \hookrightarrow & \mathrm{Sp}(\widetilde{H},H) & \twoheadrightarrow & \mathrm{Sp}(\overline{H}) \\
\downarrow & & \downarrow & & \| \\
\mathrm{Hom}(\overline{H},\mathrm{Rad}(H)) & \hookrightarrow & \mathrm{Sp}(H) & \twoheadrightarrow & \mathrm{Sp}(\overline{H}) \\
\downarrow & & \downarrow & & \\
0 & & 1 & &
\end{array}$$

Let $a \in \mathrm{Rad}(H)$, we define a map from $S^2\mathrm{Rad}(H)$ to $\mathrm{Sp}(\widetilde{H}, H)$ by $a \otimes a \mapsto \delta_a$. Because a is in the radical of H, we see that the image of $a \otimes a$ is the identity on H and on $\widetilde{H}/\mathrm{Rad}(H)$, so the image is in $\mathrm{Ker}(\mathrm{Sp}(\widetilde{H}, H) \to \mathrm{Sp}(H))$. Since

$$\delta_a\delta_b(x) = x + (x \cdot b)b + (x \cdot a)a = \delta_b\delta_a(x)$$

if $a \cdot b = 0$ we see that the map is well-defined. The image is central in $\mathrm{Sp}(\widetilde{H}, H)$ because by definition every automorphism is the identity on $\mathrm{Rad}(H)$. From the definition it follows that $a\otimes b+b\otimes a$ maps to the automorphism $x \mapsto x+(x\cdot b)a+(x\cdot a)b$. The kernel is exactly $S^2\langle\Delta - P\rangle$ and if follows from the fact that any element of $\mathrm{Sp}(\widetilde{H}, H)$ induces the identity on $\widetilde{H}/H$ that $\mathrm{Rad}(H) \circ \mathrm{Rad}(H)/\langle\Delta - P\rangle$ maps surjectively onto $\mathrm{Ker}(\mathrm{Sp}(\widetilde{H}, H) \to \mathrm{Sp}(H))$. □

LEMMA 1.2.4. *The group* $\mathrm{Sp}(\widetilde{H}, H)$ *is generated by transvections.*

PROOF. It is well known that $\mathrm{Sp}(\overline{H})$ is generated by transvections (it is a quotient of the mapping class group, see [MKS] Theorem N13. The mapping class group is generated by Dehn twists and the image of the Dehn twists are the transvections, see Section 2.2). The subgroup $\mathrm{Hom}(\overline{H}, \mathrm{Rad}(H))$ is generated by the elements $\delta_{r+a}\delta_a^{-1}(x) = x + (x \cdot a)r$ for $r \in \mathrm{Rad}(H)$ and $a \in \overline{H}$, and the previous lemma shows that $\mathrm{Rad}(H) \circ \mathrm{Rad}(H)/\langle\Delta - P\rangle$ is generated by transvections as well. This proves the lemma. □

1.3. Quadratic forms

Let W be a vector space over a field k and U an affine space over W. We call a function $f : U \to k$ *affine-linear* if there exist a linear functional $g : W \to k$ such that $f(w+u) = g(w)+f(u)$ for all $w \in W$ and $u \in U$. The function f is determined by g and the value of f at some fixed point of U. Hence, the vector space of affine-linear functions on U is of dimension $\dim(W) + 1$. The *polynomial functions on* U are finite sums and products of linear functions. We denote by $B(U)$ the algebra of polynomial functions on U and by $B_r(U)$ the linear subspace of functions of degree $\leq r$.

Let H be a quasi-unimodular symplectic module over $\mathbb{Z}/2$ of genus g. An *associated quadratic form* is a function $\omega : H \to \mathbb{Z}/2$ such that $\omega(a+b) = \omega(a)+\omega(b)+a\cdot b$ for all $a, b \in H$. It is determined by its values on a basis, and these values can be arbitrarily chosen. The difference of two forms is a linear form on H, so the set of all associated-quadratic forms is an affine space over $\mathrm{Hom}(H, \mathbb{Z}/2)$. We denote this space by Ω_H. The group $\mathrm{Sp}(H)$ acts on Ω_H adjoint to the action of $\mathrm{Sp}(H)$ on H, that is, if $h \in \mathrm{Sp}(H)$ and $\omega \in \Omega_H$ then $(h\omega)(a) := \omega(h^{-1}(a))$. For every $v \in H$ we define a linear function $\overline{v} : \Omega_H \to \mathbb{Z}/2$ by $\overline{v}(\omega) := \omega(v)$. If $\{e_i, e_{-i}\}_{i=1}^{g} \cup \{e_{g+i}\}_{i=1}^{r}$ is a symplectic basis of H then 1 and $\{\overline{e_i}, \overline{e_{-i}}\}_{i=1}^{g} \cup \{\overline{e_{g+i}}\}_{i=1}^{r}$ is a basis of the space of linear functions on Ω_H. The group $\mathrm{Sp}(H)$ acts on the functions on Ω_H adjoint to its action on Ω_H. Because we work over the field $\mathbb{Z}/2$ every polynomial function on Ω_H can be written as a sum of square free monomials in the $\overline{e_i}$. They are called *Boolean polynomials.* If H is a symplectic module over $\mathbb{Z}$ then $B_r(\Omega_{H\otimes\mathbb{Z}/2})$ is also denoted by $B_r(\Omega_H)$. If $U \subset H$ is a subspace, then we have the restriction map $\Omega_H \to \Omega_U$, which induces an injection $B_r(\Omega_U) \to B_r(\Omega_H)$. Let H be unimodular, then the *Arf invariant* α is the quadratic Boolean function on Ω_H given by $\alpha := \sum_{i=1}^{g} \overline{e_i e_{-i}}$. It has the property that it is invariant under the action of $\mathrm{Sp}(H)$ and two forms ω, ω' are in the same orbit of $\mathrm{Sp}(H)$ if and only if $\alpha(\omega) = \alpha(\omega')$. The set of quadratic forms of Arf invariant zero is denoted by Ψ_H. See also [Arf] and [Johnson2] for more about this subject.

1.4. Simplicial complexes and posets

We first recall some basic definitions concerning simplicial complexes and posets and fix some notations. We finish this section with recalling some useful lemma's and stating a theorem of Quillen.

A *simplicial complex* Σ consists of a set V of *vertices* and a set of finite nonempty subsets of V, called the *simplices of* Σ, such that each vertex is a simplex and each nonempty subset of a simplex is again a simplex, called a *face* of the simplex. If a simplex contains exactly $k+1$ vertices we say that it is a *k-simplex* or that it is of *dimension* k, and we denote the set of all k-simplices of Σ by Σ_k. The *topological realization* of Σ is the set of all functions $x : V \to [0, 1]$ such that

(*i*) its support, $\mathrm{supp}(x)$, is a simplex of Σ,
(*ii*) $\sum_{v\in V} x(v) = 1$.

We denote this set by $|\Sigma|$. If σ is a simplex of Σ then $|\sigma| = \{x \in |\Sigma| : \mathrm{supp}(x) \subset \sigma\}$. Notice that if we number the vertices of σ, then $|\sigma|$ gets identified with the geometric simplex $\{x \in \mathbb{R}^{\dim(\sigma)+1} : 0 \le x_i \le 1,\ \sum x_i = 1\}$. This defines a topology on $|\sigma|$ and therefore on $|\Sigma|$, if we stipulate that $A \subset |\Sigma|$ is closed if and only if $A \cap |\sigma|$ is closed for all simplices σ of Σ.

We adopt the convention that a partially ordered set is abbreviated by *poset.* The set of all simplices of a simplicial complex is partially ordered by the face relation. Conversely, to a poset X we can associate a simplicial complex with X as vertex set and as k-simplices the set of sequences $x_0 < \ldots < x_k$, where $x_i \in X$. We denote it by $\Sigma(X)$. If we do this for a poset associated to a simplicial complex then the simplicial complex thus obtained is the barycentric subdivision of the original complex.

We can regard a simplicial complex or a poset as a category with morphisms given by the face relation or the partial ordering. We define a *system of coefficients on a simplicial complex* Σ as a contravariant functor from Σ to the category of abelian groups. Let $H_p(\Sigma, \mathcal{F})$ denote the p^{th} homology group of Σ with values in a system of coefficients $\mathcal{F}$. It is the homology of the complex

$$C_p(\Sigma, \mathcal{F}) := \bigoplus_{\sigma \in \Sigma_p} \mathcal{F}(\sigma)$$

with boundary maps the alternating sum of the restriction maps induced by the face relations.

We say that a poset X is of *dimension* n if $|\Sigma(X)|$ is of dimension n. A chain of elements $x_0 < x_1 < \ldots < x_n$ of length n is in that case called a *maximal chain* in this poset. If X and Y are posets and $f : X \to Y$ is a morphism between them, then there are two kinds of fibers over an element $y \in Y$; namely

$$f/y := \{x \in X : f(x) \leq y\} \text{ and } f\backslash y := \{x \in X : f(x) \geq y\}.$$

In case we have an inclusion $X \subset Y$ we also use the notation

$$X_{\leq y} := \{x \in X : x \leq y\},$$

where $y \in Y$ and similarly for $X_{\geq y}$. We define

$$\begin{aligned}
\mathrm{Link}^-_X(y) &:= X_{<y} = \{x \in X : x < y\}, \\
\mathrm{Link}^+_X(y) &:= X_{>y} = \{x \in X : x > y\}, \\
\mathrm{Link}_X(y) &:= \mathrm{Link}^-_X(y) \cup \mathrm{Link}^+_X(y), \\
\mathrm{Star}_X(y) &:= \mathrm{Link}_X(y) \cup \{y\}.
\end{aligned}$$

The *height* $h(x)$ of an element $x \in X$ is by definition the dimension of $X_{\leq x}$. We use the notation $X_{\leq k}$ for the subset of elements of height $\leq k$. The *join* $X * Y$ of two posets X and Y is the disjoint union of X and Y with the given partial ordering on X and on Y and $x < y$ for all $x \in X$, $y \in Y$. Remark that $\mathrm{Link}_X(y) = \mathrm{Link}^-_X(y) * \mathrm{Link}^+_X(y)$. Let x_i be elements of a poset X, indexed by a set I. If the supremum of $\{x_i\}_{i \in I}$ exists and is unique, we denote this element by $\sup_{i \in I}\{x_i\}$. A poset X is called *d-connected* if $|X|$ is d-connected, meaning that $\pi_i(|X|) = 0$ for $i \leq d$, and we say that it is *d-spherical* if $\dim(X) = d$ and $|X|$ is $(d-1)$-connected. We stipulate that X is (-1)-connected if X is nonempty. We see that a nonempty poset X is 0-connected if $|X|$ is connected and X is 1-connected if $|X|$ is simply connected. A useful definition related to this is the following definition.

DEFINITION 1.4.1. *A poset is called Cohen-Macaulay of dimension d, abbreviated by CM_d, if*

(i) X is d-spherical,
(ii) for all $x \in X$ we have $X_{<x}$ is $(h(x)-1)$-spherical and $X_{>x}$ is $(d-h(x)-1)$-spherical,
(iii) for all $x, x' \in X$ such that $x > x'$ we have $X_{>x'} \cap X_{<x}$ is $(h(x)-h(x')-2)$-spherical.

We will often make use of the following three lemmas.

LEMMA 1.4.2. *Let X, Y be posets and $f : X \to Y$, $g : X \to Y$ morphisms of posets such that $f(x) \leq g(x)$ for all $x \in X$. Then $|f|$ and $|g|$ are homotopy equivalent.*

PROOF. The map $H : X \times \{0,1\} \to Y$ defined by $H(x,0) = f(x)$ and $H(x,1) = g(x)$ gives a homotopy between $|f|$ and $|g|$. □

LEMMA 1.4.3. *If X and Y are posets and if X is n-connected and Y is m-connected, then $X * Y$ is $(n+m+2)$-connected.*

PROOF. See [Maazen], Chapter I, Proposition (1.5) and Chapter II, Proposition (1.2). □

LEMMA 1.4.4. *Let Y be a poset, $X \subset Y$ a subposet. Assume that $Y \backslash X$ is discrete and X is n-spherical of the inclusion $X \subset Y$ is null-homotopic. If $\mathrm{Link}_X(y)$ is $(n-1)$-spherical for all $y \in Y \backslash X$, then Y is n-spherical.*

PROOF. See [Maazen], Chapter II, Theorem (3.2). □

We also recall a results of Quillen, see [Quillen2] (Theorem (9.1) and Corollary (9.7)).

THEOREM 1.4.5. *Let $f : X \to Y$ be a morphism of posets. Assume that Y is d-spherical, for every $y \in Y$ the poset $Y_{>y}$ is $(d-h(y)-1)$-spherical and f/y is $h(y)$-spherical. Then X is d-spherical. Moreover, there is a canonical filtration*

$$0 = F_{d+1} \subset F_d \subset \ldots \subset F_{-1} = \widetilde{H}_d(X)$$

and isomorphisms

$$\begin{aligned} F_{-1}/F_0 &\cong \widetilde{H}_d(Y) \\ F_q/F_{q+1} &\cong \bigoplus_{h(y)=q} \widetilde{H}_{d-q-1}(Y_{>y}) \otimes \widetilde{H}_q(f/y) \\ F_d &\cong \bigoplus_{h(y)=d} \widetilde{H}_d(f/y) \end{aligned}$$

for $0 \leq q \leq d-1$ and the sum is taken over the elements of height q in Y.

COROLLARY 1.4.6. *If $f : X \to Y$ is a strictly increasing map, Y is Cohen-Macaulay of dimension d and for all $y \in Y$ the poset f/y is Cohen-Macaulay of dimension $h(y)$, then X is Cohen-Macaulay of dimension d.*

For the proofs we refer to [Quillen2].

1.5. Definitions of simplicial complexes associated to a surface module.

We now restrict ourselves to special classes of posets. Let V be a finitely generated free abelian group. Since a direct summand of V is the same thing as the intersection of V with a linear subspace of $V \otimes \mathbb{Q}$ we shall use the term *subspace of V* as synonymous for a direct summand of V. We associate several posets to V; we use Roman letters for posets having subspaces of V as elements and calligraphic letters for posets consisting of finite subsets of V. When we assume that the finite subsets are totally ordered we use the superscript ^{o}. The subsets of V are partially ordered by inclusion, the ordered sequences of elements of V by refinement: we say that $(v_0, \ldots, v_k) \leq (w_0, \ldots, w_p)$ if and only if there exists a strictly increasing map $\varphi : \{0, \ldots, k\} \to \{0, \ldots, p\}$ such that $v_i = w_{\varphi(i)}$.

If S is a set, we define the *full simplex on S* by

$$\mathcal{O}(S) := \{E \subset S : E \text{ is finite and nonempty }\}.$$

We recall the following result of Maazen, see [Maazen], Chapter II, Corollary (5.5).

PROPOSITION 1.5.1. *Let S be a set, $z = (z_0, \ldots, z_n) \in \mathcal{O}^o(S)$ and let F be a poset contained in $S - \{z_0, \ldots, z_n\}$. Regard F as a subposet of $\mathcal{O}(S - \{z_0, \ldots, z_n\})$ and assume that its preimage in $\mathcal{O}^o(S - \{z_0, \ldots, z_n\})$ is CM_d. Then its preimage under the composite map*

$$\operatorname{Link}^+_{\mathcal{O}^o(S)}(z) \to \mathcal{O}^o(S - \{z_0, \ldots, z_n\}) \to \mathcal{O}(S - \{z_0, \ldots, z_n\})$$

(the first map suppresses the terms $z_0, \ldots, z_n$) is d-spherical.

According to Theorem (2.1), Chapter III in [Maazen] we have that for any integer $n \geq 1$, the poset $\mathcal{O}^o(\{1, \ldots, n\})$ is $(n-1)$-spherical. We show that it is also Cohen-Macaulay.

PROPOSITION 1.5.2. *For any integer $n \geq 1$ is $\mathcal{O}^o(\{1, \ldots, n\})$ Cohen-Macaulay of dimension $n-1$.*

PROOF. We know that it is $(n-1)$-spherical. Suppose $E^o \in \mathcal{O}^o(\{1, \ldots, n\})$ then $\mathcal{O}^o(\{1, \ldots, n\})_{<E^o}$ is spherical of dimension $h(E^o) - 1$, $\mathcal{O}^o(\{1, \ldots, n\})_{>E^o}$ is spherical of dimension $n - h(E_0) - 2$ by Propositon 1.5.1 and if $E^o < F^o$ then $\mathcal{O}^o(\{1, \ldots, n\})_{>E^o} \cap \mathcal{O}^o(\{1, \ldots, n\})_{<F^o}$ is spherical of dimension $h(F^o) - h(E^o) - 2$.
□

Let V be a free abelian group of finite rank. We call a nonempty subset E in $\mathcal{O}(V)$ a *partial basis* if it can be completed to a basis of V. We denote by $\langle E\rangle$ the span of E in V. Let $T(V)$ be the *Tits building* of the general linear group of V, that is, the poset of nonzero proper subspaces of V. For a subspace W of V we define

$$T(V,W) := \{U : U \in T(V), U \oplus W \to V \text{ is a primitive embedding}\},$$
$$\mathcal{P}(V,W) := \{E \subset V : \; E \text{ is a partial basis and } \langle E\rangle \in T(V,W)\}.$$

With *primitive embedding* we mean that $U \oplus W$ is a subspace of H, which means that $((U\oplus W)\otimes_{\mathbb{Z}}\mathbb{Q})\cap H = U \oplus W$. For a subset S of V we put

$$\mathcal{P}(S,W) := \mathcal{P}(V,W)\cap\mathcal{O}(S).$$

So the poset $\mathcal{P}^o(S,W)$ consists of totally ordered sequences $(v_0,\dots v_k)$ such that $\{v_0,\dots v_k\}\in\mathcal{P}(S,W)$. If $W=\emptyset$ we often omit W from the notation.

We now define the poset that interests us for the rest of this thesis.

DEFINITION 1.5.3. *Let H be a quasi-unimodular symplectic module. An ordered sequence $(v_0,\cdots,v_m)\in\mathcal{P}^o(H,\mathrm{Rad}(H))$ is called an arc-sequence if it satisfies the following three conditions:*

- *(i) if $0\le i<j\le m$ then $v_i\cdot v_j\in\{0,1\}$,*
- *(ii) if $0\le i<j<k\le m$ and $v_i\cdot v_k=1$ then $v_i\cdot v_j=1$ or $v_j\cdot v_k=1$,*
- *(iii) if $0\le i<j<k\le m$ and suppose that $v_i\cdot v_j=1$ and $v_j\cdot v_k=1$, then $v_i\cdot v_k=1$.*

Let $\mathcal{A}^o(H)$ be the poset of arc-sequences in H. For a subset S of H and a subspace $J\subset H$ we put $\mathcal{A}^o(S,J):=\mathcal{A}^o(H)\cap\mathcal{P}^o(S,J)$. Let $\pi: H\to\mathbb{Z}$ be an epimorphism of groups that factorizes over $\overline{\pi}:\overline{H}\to\mathbb{Z}$, so that π is given by taking the symplectic product with some $e\in H$. Then $\pi^{-1}(0)$ is a quasi-unimodular symplectic module and $\mathrm{Rad}(\pi^{-1}(0))\cong\mathrm{Rad}(H)\oplus\langle e\rangle$. *Define*

$$\mathcal{A}^o(H,\pi) := \mathcal{A}^o(\pi^{-1}(1),\mathrm{Rad}(\pi^{-1}(0))).$$

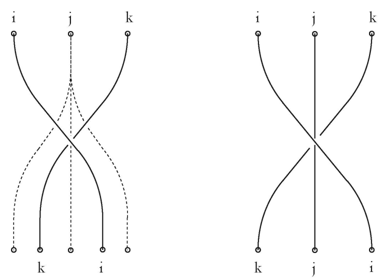

FIGURE 1.1. The conditions (ii) and (iii).

The goal of Sections 1.6 to 1.10 is to prove the following two theorems.

THEOREM 1.5.4. *Let H be a quasi-unimodular symplectic module of genus g over $\mathbb{Z}$, then $\mathcal{A}^o(H)$ is $(g-2)$-connected.*

THEOREM 1.5.5. *Let H be a quasi-unimodular symplectic module of genus g over $\mathbb{Z}$ and $\pi : H \to \mathbb{Z}$ an epimorphism that factorizes over $\overline{\pi} : \overline{H} \to \mathbb{Z}$. If $g \geq 3$ then $\mathcal{A}^o(H,\pi)$ is 1-connected.*

We remark that $\mathcal{A}^o(H)$ is of dimension $2g-1$ and $\mathcal{A}^o(H,\pi)$ is of dimension $2g-2$. To see this, we choose a symplectic basis

$$\{e_1, \dots, e_g, e_{-1}, \dots, e_{-g}, e_{g+1}, \dots, e_{g+r}\}$$

of H, then $(e_1, e_{-1}, e_2, e_{-2}, \dots, e_g, e_{-g}\}$ is an arc-sequence in H of maximal length. Assume that π is given by $\pi(x) = x \cdot e_{-1}$, then $\mathrm{Rad}(\pi^{-1}(0)) = \langle e_{-1}\rangle \oplus \mathrm{Rad}(H)$ and $E^o = (e_1, e_1 + e_2, e_1 + e_{-2}, \dots, e_1 + e_g, e_1 + e_{-g})$ is an arc-sequence in $\pi^{-1}(1)$ of maximal length such that $\langle E^o\rangle \oplus \mathrm{Rad}(\pi^{-1}(0)) \to H$ is a primitive embedding.

We give an overview of the proof of Theorem 1.5.4 and 1.5.5 now. For this, we need the following auxiliary posets. Let H be a quasi-unimodular symplectic module and $J \subset H$ an isotropic subspace that contains $\mathrm{Rad}(H)$. We define

$$I(H,J) := \{U \in T(H,J) : U + J \text{ is isotropic}\},$$
$$\mathcal{I}(H,J) := \{E \in \mathcal{P}(H,J) : \langle E\rangle \in I(H,J)\}.$$

So $E \in \mathcal{I}(H,J)$ means that its elements span together with J a primitive isotropic subspace of H of rank $|E| + \mathrm{rk}(J)$. For a subset S of H we put

$$\mathcal{I}(S,J) := \mathcal{I}(H,J) \cap \mathcal{O}(S).$$

If $J = \mathrm{Rad}(H)$ we usually omit J from the notation. Let

$$d(H,J) := g(H) - \mathrm{rk}(J/\mathrm{Rad}(H)) - 1.$$

In section 1.6 we prove the following proposition.

PROPOSITION 1.5.6. *Let H be a quasi-unimodular symplectic lattice, then $\mathcal{I}^o(H,J)$ is Cohen-Macaulay of dimension $d(H,J)$.*

In Section 1.8 we prove that

PROPOSITION 1.5.7. *Let H be a quasi-unimodular symplectic lattice of genus g and let $\pi : H \to \mathbb{Z}$ be an epimorphism such that it factorizes over $\overline{\pi} : \overline{H} \to \mathbb{Z}$. If $g = 1, 2, 3$ then $\mathcal{I}^o(\pi^{-1}(1))$ is Cohen-Macaulay of dimension $g-1$. If $g \geq 4$ then $\mathcal{I}^o(\pi^{-1}(1))_{\leq g-2}$ is Cohen-Macaulay of dimension $g-2$.*

We recall that $\mathcal{I}^o(\pi^{-1}(1))$ is by definition $\mathcal{I}^o(\pi^{-1}(1), \mathrm{Rad}(H))$, but we remark that it is a subposet of $\mathcal{A}^o(H,\pi)$. In Section 1.10 we show that $\mathcal{A}^o(H)$ arises from $\mathcal{I}^o(H)$ by attaching only $(g-1)$-cells and therefore the degree of connectedness does not change. In Section 1.10 we show that if $g \geq 2$ then any 1-cycle in $\mathcal{A}^o(H,\pi)$ is homotopic in $\mathcal{A}^o(H,\pi)$ to a 1-cycle in $\mathcal{I}^o(\pi^{-1}(1))$. This proves that for $g \geq 3$ the poset $\mathcal{A}^o(H,\pi)$ is 1-connected, if $\mathcal{I}^o(\pi^{-1}(1))$ is 1-connected.

To prove the connectedness properties of $\mathcal{I}^o(H)$ and $\mathcal{I}^o(\pi^{-1}(1))$, we use the restriction of the map

$$f : \mathcal{O}^o(S) \to \mathcal{O}(S)$$

that forgets the ordering. The fibers f/y are Cohen-Macaulay by Proposition 1.5.2 and so by Quillen's theorem it suffices to show that $\mathcal{I}(H)$ and $\mathcal{I}(\pi^{-1}(1))_{\leq g-2}$ are Cohen-Macaulay, and that if $g = 1, 2, 3$ then $\mathcal{I}(\pi^{-1}(1))$ is Cohen-Macaulay. For $\mathcal{I}(H)$ the proof we give in Section 1.6 is inspired by the proof of Maazen for $\mathcal{P}(H)$. Let $n = g - 1$ if $g = 1, 2, 3$ and $n = g - 2$ if $g \geq 4$. To prove the connectedness of $\mathcal{I}(\pi^{-1}(1))_{\leq n}$ we use the map $E \mapsto \langle E \rangle$ to the subposet of $I(H)_{\leq n}$ consisting of those subspaces X such that $\pi|_X$ is surjective. In Section 1.7 we prove that this poset is Cohen-Macaulay and by using Quillen's theorem once more, we conclude that $\mathcal{I}(\pi^{-1}(1))_{\leq n}$ is Cohen-Macaulay of the right dimension.

1.6. The Cohen-Macaulay property of $\mathcal{I}(H,J)$ and $\mathcal{I}^o(H,J)$

Let H be a symplectic quasi-unimodular module, J an isotropic submodule such that $\mathrm{Rad}(H) \subset J$ and put $d(H,J) := g(H) - \mathrm{rk}(J/\mathrm{Rad}(H)) - 1$. Then $\mathcal{I}(H,J)$ is of dimension $d(H,J)$. The proof of the following theorem is essentially the same proof of Maazen of the spherical property of $\mathcal{P}(H,W)$, see [Maazen], Chapter III, Theorem (4.2).

PROPOSITION 1.6.1. *The poset $\mathcal{I}(H,J)$ is Cohen-Macaulay of dimension $d(H,J)$.*

PROOF. It is clear that the proposition holds for $d(H,J) = 0$. We proceed with induction on $d := d(H,J)$ and assume that $d \geq 1$. In that case, we can choose a hyperbolic pair a, b, that means $a \cdot b = \pm 1$, such that $\{a,b\} \in \mathcal{P}(H,J)$ and $a, b \perp J$. For $k, l \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ we denote by $H_{(k,l)}$ the set of $e \in H$ such that $|e \cdot a| \leq k$ and $|e \cdot b| \leq l$ and by $\mathcal{I}_{(k,l)}(H,J)$ the set of $E \in \mathcal{I}(H,J)$ such that $E \cap H_{(k,l)} \neq \emptyset$. By Lemma 1.4.2 we see that the map $E \mapsto E \cap H_{(k,l)}$ gives a deformation retraction of $\mathcal{I}_{(k,l)}(H,J)$ onto $\mathcal{I}(H_{(k,l)},J)$, similarly we see that if $k \geq 1$ $\mathcal{I}(H_{(k,l)},J)$ is a deformation retract of $\mathcal{I}_{(k-1,l)}(H,J) \cup \mathcal{I}(H_{(k,l)},J)$ and the equivalent statement holds for $l \geq 1$.

Step 1. The inclusion $\mathcal{I}(H_{(0,0)},J) \subset \mathcal{I}(H_{(1,0)},J)$ is null-homotopic.

PROOF. For any $E \in \mathcal{I}(H_{(0,0)},J)$ we have $E \cup \{b\} \in \mathcal{I}(H_{(1,0)},J)$ and since $\{b\} \in \mathcal{I}(H_{(1,0)},J)$ this shows, by Lemma 1.4.2 again, that the inclusion is homotopic to the map that is constant equal to $\{b\}$.

Step 2. For $k \geq 1$ is the inclusion $\mathcal{I}(H_{(k-1,0)},J) \subset \mathcal{I}(H_{(k,0)},J)$ a d-cellular extension up to homotopy.

Proof. We shall prove the equivalent assertion that the inclusion

$$\mathcal{I}_{(k-1,0)}(H,J) \subset \mathcal{I}_{(k-1,0)}(H,J) \cup \mathcal{I}(H_{(k,0)},J)$$

is a d-cellular extension up to homotopy. Let $\mathcal{X}_q$ be the union of $\mathcal{I}_{(k-1,0)}(H,J)$ and the elements of $\mathcal{I}(H_{(k,0)},J)$ of height $\leq q$, so that we have the filtration

$$\mathcal{I}_{(k-1,0)}(H,J) = \mathcal{X}_{-1} \subset \mathcal{X}_0 \subset \ldots \subset \mathcal{X}_d = \mathcal{I}_{(k-1,0)}(H,J) \cup \mathcal{I}(H_{(k,0)},J).$$

We prove that for $q = 0, \ldots, d$ $\mathcal{X}_q \subset \mathcal{X}_{q-1}$ is a d-cellular extension up to homotopy. For this it is enough to show that for $E \in \mathcal{X}_q - \mathcal{X}_{q-1}$, $\mathrm{Link}_{\mathcal{X}_{q-1}}(E)$ is $(d-1)$-spherical, see Lemma 1.4.4. Notice that for $q = d$ and $k > 1$ $\mathcal{X}_q - \mathcal{X}_{q-1}$ is empty, in the other cases it means that $|E| = q$, $|e \cdot a| = k$ and $|e \cdot b| = 0$ for all $e \in E$. Now $\mathrm{Link}^-_{\mathcal{X}_{q-1}}(E)$ is just the poset of nonempty proper subsets of E and so is spherical of dimension $q-1$. If $q = d$ we are done, otherwise, by Lemma 1.4.3 it remains to show that $\mathrm{Link}^+_{\mathcal{X}_{q-1}}(E) = \mathrm{Link}_{\mathcal{I}_{(k-1,0)}(H,J)}(E)$ is $(d-q-1)$-spherical. We identify $\mathrm{Link}_{\mathcal{I}_{(k-1,0)}(H,J)}(E)$ with $\mathcal{I}_{(k-1,0)}(H, J + \langle E \rangle)$ via the map $E' \mapsto E' - E$, we know that the latter is homotopy equivalent to $\mathcal{I}(H_{(k-1,0)}, J + \langle E \rangle)$. We construct a poset retraction $R : \mathcal{I}(H, J + \langle E \rangle) \to \mathcal{I}(H_{(k-1,0)}, J + \langle E \rangle)$; this will finish the job, since we know by our induction hypothesis that $\mathcal{I}(H, J + \langle E \rangle)$ is spherical of dimension $d - q - 1$. Choose $e_k \in E$ and let $\{e\} \in \mathcal{I}(H, J + \langle E \rangle)$. If $|e \cdot a| \leq k-1$ then $R(\{e\}) = \{e\}$. If this is not the case, then divide $|e \cdot a|$ by k with remainder: we find that there exists an $n_e \in \mathbb{Z}$ such that $|(e + n_e e_k) \cdot a| \leq k-1$ and we put $R(\{e\}) = \{e + n_e e_k\}$; this is clearly an element of $\mathcal{I}(H_{(k-1,0)}, J + \langle E \rangle)$. This extends to a poset map as desired and thus completes the proof of step 2. We conclude that $\mathcal{I}(H_{(\infty,0)},J) = \varinjlim \mathcal{I}(H_{(k,0)},J)$ is spherical of dimension d.

Step 3. For $l \geq 1$ is the inclusion $\mathcal{I}(H_{(\infty,l-1)},J) \subset \mathcal{I}(H_{(\infty,l)},J)$ a d-cellular extension up to homotopy.

Proof. We prove the equivalent statement that the inclusion

$$\mathcal{I}_{(\infty,l-1)}(H,J) \subset \mathcal{I}_{(\infty,l-1)}(H,J) \cup \mathcal{I}(H_{(\infty,l)},J)$$

is a d-cellular extension up to homotopy, in the same way as we did in step 2. We have the filtration

$$\mathcal{I}_{(\infty,l-1)}(H,J) = \mathcal{Y}_{-1} \subset \mathcal{Y}_0 \subset \ldots \subset \mathcal{Y}_d = \mathcal{I}_{(\infty,l-1)}(H,J) \cup \mathcal{I}(H_{(\infty,l)},J),$$

where $\mathcal{Y}_q$ is the union of $\mathcal{I}_{(\infty,l-1)}(H,J)$ and the elements of $\mathcal{I}(H_{(\infty,l)},J)$ of height $\leq q$ and the argument goes exactly the same way as in step 2.

Step 4. Conclusion. As $\mathcal{I}(H,J) = \varinjlim \mathcal{I}(H_{(\infty,l)},J)$ the previous steps imply that $\mathcal{I}(H,J)$ is d-spherical. For the Cohen-Macaulay property, we note that for $E \in \mathcal{I}(H,J)$ we have that $\mathrm{Link}^-_{\mathcal{I}(H,J)}(E)$ is $(|E|-2)$-spherical and that if $E' \in \mathcal{I}(H,J)$ with $E < E'$ then $\mathrm{Link}^+_{\mathcal{I}(H,J)}(E) \cap \mathrm{Link}^-_{\mathcal{I}(H,J)}(E')$ is $(|E'-E|-2)$-spherical. Furthermore, $\mathrm{Link}^+_{\mathcal{I}(H,J)}(E) \cong \mathcal{I}(H, J + \langle E \rangle)$ is $(d - |E|)$-spherical. This proves the Cohen-Macaulay property. □

We show that this proposition implies Proposition 1.5.6 that states that $\mathcal{I}^o(H,J)$ is Cohen-Macaulay of dimension $d(H,J)$.

PROOF OF PROPOSITION 1.5.6. For $d(H,J)=0$ the proposition obviously holds. Assume $d:=d(H,J)>0$. Let $f:\mathcal{I}^0(H,J)\to\mathcal{I}(H,J)$ be the map that forgets the ordering. If $E\in\mathcal{I}(H,J)$ then $h(E)=|E|-1$ and $f/E\cong\mathcal{O}^0(\{1,\dots,|E|\})$. The latter is Cohen-Macaulay of dimension $|E|-1=h(E)$ by Proposition 1.5.2. By Theorem 1.4.5 of Quillen we conclude that $\mathcal{I}^0(H,J)$ is Cohen-Macaulay of dimension d, since by Proposition 1.6.1 we know that $\mathcal{I}(H,J)$ is Cohen-Macaulay of this dimension. □

1.7. The Cohen-Macaulay property $I(\pi)_{\leq g-2}$

In this section we define a poset $I(\pi)$ of dimension $g-1$ and prove that $I(\pi)_{\leq g-2}$ is Cohen-Macaulay. The proof is based on unpublished notes of Looijenga [1].

We introduce the following definitions. Let V be a free abelian group of finite rank r. By assigning to a subspace of V its nilspace in $V^*:=\mathrm{Hom}(V,\mathbb{Z})$, we may identify $T(V^*)$ with the poset opposite to $T(V)$. Given an epimorphism $\pi:V\to A$ of groups, we say that a subspace $X\subset V$ is *in general position relative to* π if $\pi|X$ is onto. Dually, if we are given an epimorphism $\rho:V^*\to B$, then we say that a subspace $X\subset V$ is *primitive relative to* ρ if $\rho|(V/X)^*$ is onto. (This terminology is explained by the fact that we can understand that property in terms of V as follows: if we put

$$V_\rho:=\{v\in V\otimes\mathbb{Q}\ :\ \phi(v)\in\mathbb{Z}\text{ for all }\phi\in\mathrm{Ker}(\rho)\},$$

then $V\subset V_\rho\subset V\otimes\mathbb{Q}$, V_ρ/V can be identified with $\mathrm{Hom}(B,\mathbb{Q}/\mathbb{Z})$ and the condition imposed on X amounts to: X is a direct summand of V_ρ. In case B is free abelian, then $\rho^*:B^*\to V$ is a primitive embedding, $V_\rho=V+(B^*\otimes\mathbb{Q})$ and the condition on X is equivalent to $X\oplus B^*\to V$ being a primitive embedding.) We denote by $T(\pi)$ the set of members of $T(V)$ that are in general position relative to π and by $T(\pi/\rho)$ those that have the additional property that they are primitive relative to ρ. Notice that $T(\pi/\rho)$ and $T(\rho/\pi)$ are opposite as posets. If H is a quasi-unimodular symplectic lattice and $\pi:H\to A$ is an epimorphism of groups, then we write

$$I(\pi):=I(H)\cap T(\pi).$$

If also is given a factorization of π over an epimorphism $\tilde{\pi}:H\to\tilde{A}$, then we write $I(\pi,\tilde{\pi})$ for the poset of $Y\in I(\pi)$ with $Y^\perp\in T(\tilde{\pi})$. It is clear that

$$I(\tilde{\pi})\subset I(\pi,\tilde{\pi})\subset I(\pi).$$

[1] In [Foisy], Lemma 5.1 it is claimed that $I(\pi)$ is spherical of dimension $g-1$, but Foisy and I agreed that the proof there is incomplete. We did not manage to solve this, but instead give a proof of the weaker result that $I(\pi)_{\leq g-2}$ is Cohen-Macaulay, Theorem 1.7.3.

PROPOSITION 1.7.1. *Let V be a lattice of rank r and $\pi : V \to A$ be a cyclic quotient of V, then $T(\pi)$ is $(r-2)$-spherical.*

PROOF. If $r \leq 1$ then $T(\pi) = \emptyset$ and the proposition holds. We proceed with induction on r and assume that $r \geq 2$. In that case we can choose a $\varphi \in V^*$ such that $\tilde{\pi} = (\pi, \varphi) : V \to A \oplus \mathbb{Z}$ is surjective.

Step 1.The inclusion $T(\tilde{\pi}) \subset T(\pi)$ in null-homotopic.

PROOF. If $X \in T(\tilde{\pi})$ then $X \cap \mathrm{Ker}(\varphi) \in T(\pi)$. So $X > X \cap \mathrm{Ker}(\varphi) < \mathrm{Ker}(\varphi)$ gives a path from X to $\mathrm{Ker}(\varphi)$. By applying Lemma 1.4.2 two times, this shows that the inclusion $T(\tilde{\pi}) \subset T(\pi)$ is null-homotopic.

Step 2. $T(\pi)$ is an $(r-2)$-cellular extension of $T(\tilde{\pi})$.

PROOF. For $\sigma \geq -1$ we define $T_\alpha := T(\tilde{\pi}) \cup \{X \in T(\pi) : \mathrm{rk}(X) \leq \alpha + 1\}$, then

$$T(\tilde{\pi}) = T_{-1} \subset T_0 \subset T_1 \subset \cdots \subset T_{r-2} = T(\pi)$$

gives a filtration. Let $X \in T_\alpha - T_{\alpha-1}$, then $\dim(X) = \alpha + 1$. We prove that $\mathrm{Link}_{T_{\alpha-1}}(X)$ is $(r-4)$-connected. We have that $\mathrm{Link}^-_{T_{\alpha-1}}(X) = T(\pi|_X)$ is $(\alpha-1)$-spherical with induction. We know that $\tilde{\pi}|_X$ is not onto, so that

$$\widetilde{A}_X := \mathrm{Coker}(\tilde{\pi}|_X : X \to A \oplus \mathbb{Z})$$

is nontrivial and cyclic, and the composition $V \xrightarrow{\tilde{\pi}} A \oplus \mathbb{Z} \to \widetilde{A}_X$ factorizes over a map $\tilde{\pi}_X : V/X \to \widetilde{A}_X$. We have $\mathrm{Link}^+_{T_{\alpha-1}}(X) = \{X \subsetneq Y \subsetneq V : \tilde{\pi}|_Y \text{ is surjective }\} \cong T(\tilde{\pi}_X)$ via the map $Y \mapsto Y/X$. The latter is spherical of dimension $(r-\alpha-3)$ with induction. Since the Link of X in $T_{\alpha-1}$ is the join of the lower and the upper Link of X in $T_{\alpha-1}$, we have by Lemma 1.4.3 that $\mathrm{Link}_{T_{\alpha-1}}(X)$ is $(\alpha-1)+(r-\alpha-3)+1 = (r-3)$-spherical. So T_α is an $(r-2)$-cellular extension of $T_{\alpha-1}$. This finishes the proof of Step 2. and therefore of the Proposition.

□

PROPOSITION 1.7.2. *Let V be a lattice of rank r, $\pi : V \to A$ and $\rho : V^* \to B$ cyclic quotients, then $T(\pi/\rho)$ is $(r-4)$-connected.*

PROOF. For $r \leq 2$ there is nothing to prove. Let $r \geq 3$, then $T(\pi/\rho)$ is nonempty. We proceed with induction on r and assume that $r \geq 3$.

Choose $\varphi \in V^*$ such that $\tilde{\pi} = (\pi, \varphi) : V \to A \oplus \mathbb{Z}$ is surjective and $\rho : (V/\ker(\varphi))^* \to B$ is surjective. This means that $\varphi \in \rho^{-1}(\overline{1})$, where $\overline{1}$ is a generator of B, when $r \geq 3$ such a φ exists. Then again $T(\tilde{\pi}/\rho) \subset T(\pi/\rho)$ is null-homotopic: if $X \in T(\tilde{\pi}/\rho)$, then $X > X \cap \mathrm{Ker}(\varphi) < \mathrm{Ker}(\varphi)$, so applying Lemma 1.4.2 two times we see that the inclusion is homotopic to the constant map. Let $T_\alpha := T(\tilde{\pi}/\rho) \cup \{X \in T(\pi/\rho) : \mathrm{rk}(X) \leq \alpha + 1\}$. Then

$$T(\tilde{\pi}/\rho) = T_{-1} \subset T_0 \subset T_1 \subset \cdots \subset T_{r-2} = T(\pi/\rho)$$

gives a filtration; we follow the procedure as usual.

Let $X \in T_\alpha - T_{\alpha-1}$, then $\mathrm{rk}(X) = \alpha + 1$ and $\tilde{\pi}|_X$ is not surjective. Let $\tilde{\pi}_X : V/X \to \widetilde{A}_X$ be defined as in the previous proof. Then $\mathrm{Link}^-_{T_{\alpha-1}}(X) = T(\pi|_X)$ is $(\alpha-1)$-spherical. We have

$$\mathrm{Link}^+_{T_{\alpha-1}}(X) = \{X \subsetneq Y \subsetneq V : \tilde{\pi}|_Y, \rho|_{(V/Y)^*} \text{ are surjective }\}.$$

Via the map $Y \mapsto Y/X$ this poset is isomorphic to $T(\tilde{\pi}_X/\rho|_{(V/X)^*})$, which is $(r-\alpha-5)$-connected by induction. So $\mathrm{Link}_{T_{\alpha-1}}(X)$ is $(r-5)$-connected by Lemma 1.4.3 and hence, T_α is $(r-4)$-connected by Lemma 1.4.4. □

THEOREM 1.7.3. *Let H be a unimodular symplectic lattice of genus $g \geq 1$ and let $\pi : H \to A$ be a cyclic quotient of H. Then $I(\pi)_{\leq g-2}$ is Cohen-Macaulay of dimension $g-2$.*

REMARK 1.7.4. *We are only interested in the special case when A is infinite cyclic, so that π is essentially given as the symplectic product with a primitive vector. But the inductive set-up of the proof requires that we prove this more general result. When A is trivial, $I(\pi) = I(H)$ is the Tits building associated to the symplectic group. By the Solomon-Tits Theorem it is spherical of dimension $(g-1)$; we shall use this special case in the proof below. A direct proof of the sphericalness property of $I(H)$ can be found in* [Vogtmann]*, Theorem 1.6, where she proves it for $O_{n,n}$ but the proof works for the symplectic group as well.*

PROOF OF THEOREM 1.7.3. It is clear that $I(\pi)$ is nonempty of dimension $g-1$. So for $g = 1$ there is nothing to prove. We continue with induction on g and assume that $g \geq 2$. We also assume that A is nontrivial.

Choose a line L that belongs to $I(\pi)$ and let

$$\tilde{\pi} : H \to A \oplus L^*, \quad v \mapsto (\pi(v), x \in L \mapsto x \cdot v \in \mathbb{Z}).$$

We have defined $I(\tilde{\pi})$ and $I(\pi, \tilde{\pi})$.

Step 1. The inclusion $I(\tilde{\pi}) \subset I(\pi)$ is null-homotopic.

PROOF. If $X \in I(\tilde{\pi})$, then $\tilde{\pi} : X \to A \oplus L^*$ is onto by definition, and so $\pi : X \cap L^\perp \to A$ onto. Since A is nontrivial, we cannot have that $X \cap L^\perp$ is trivial and so $X \cap L^\perp \in I(\pi)$. In other words, $X \cap L^\perp < X$ is a 1-simplex of $I(\pi)$. The submodule $(X \cap L^\perp) + L$ is primitive, because if $L = \langle v \rangle$, then, since $X \in I(\tilde{\pi})$, there exists a $w \in X$ such that $w \cdot v = 1$. This implies that $w^* : X \cap L^\perp + L \to \mathbb{Z}$ is surjective with kernel $X \cap L^\perp$. So $(X \cap L^\perp) + L \in I(\pi)$ and so $X \cap L^\perp < (X \cap L^\perp) + L > L$ are 1-simplices of $I(\pi)$. It follows that $I(\tilde{\pi}) \cup \mathrm{Star}_{I(\pi)}(L)$ contracts onto the singleton represented by L by Lemma 1.4.2.

Step 2. $I(\pi, \tilde{\pi})$ is a $(g-1)$-cellular extension of $I(\tilde{\pi})$.

Proof. Let I_α denote the union of $I(\tilde\pi)$ and the set of members of $I(\pi,\tilde\pi)$ of rank $\le 1+\alpha$ so that we have the filtration

$$I(\tilde\pi) = I_{-1} \subset I_0 \subset \cdots \subset I_{g-1} = I(\pi,\tilde\pi).$$

It suffices to show that for $\alpha = 0,\dots,g-1$, I_α is a $(g-1)$-cellular extension of $I_{\alpha-1}$. We prove this by showing that for every $X \in I_\alpha - I_{\alpha-1}$ ($\alpha = 0,\dots,g-1$), $\operatorname{Link}_{I_{\alpha-1}}(X)$ is $(g-2)$-spherical. Observe that for such an X, $\operatorname{rk}(X) = 1+\alpha$ (so that $X^\perp/X$ has genus $g-1-\alpha$), $\pi|X$ and $\tilde\pi|X^\perp$ are onto, but $\tilde\pi|X$ is not onto. In any case, the cokernel $\tilde A_X$ of $\tilde\pi|X$ is cyclic and we have an induced surjection $\tilde\pi_X : X^\perp/X \to \tilde A_X$.

If $\alpha = 0$ then $\operatorname{Link}^-_{I_{\alpha-1}}(X) = \emptyset$. Assume $\alpha > 0$. Let $Y \in I(H)$ be such that $Y < X$. Then $Y \in I_{\alpha-1}$ iff $\pi|Y$ is onto. This shows that $\operatorname{Link}^-_{I_{\alpha-1}}(X) = T(\pi|X)$ and this is spherical of dimension $\operatorname{rk}(X) - 2 = \alpha - 1$ by Proposition 1.7.1.

Next consider the case when $Y \in I(H)$ is such that $Y > X$. Then $Y \subset X^\perp$ and Y/X is a member of $I(X^\perp/X)$. We have $Y \in I_{\alpha-1}$ iff $Y \in I(\tilde\pi)$. This means that $\tilde\pi|Y$ is onto, or equivalently, that a $\tilde\pi_X|(Y/X)$ is onto. So $\operatorname{Link}^+_{I_{\alpha-1}}(X) \cong I(\tilde\pi_X)$. The latter is spherical of dimension $(g-1-\alpha)-1 = g-\alpha-2$ by induction.

It follows by Lemma 1.4.3 that $\operatorname{Link}_{I_{\alpha-1}}(X)$ is spherical of dimension $(\alpha-1)+1+(g-\alpha-2) = g-2$.

Step 3. $I(\pi)$ is a $(g-2)$-cellular extension of $I(\pi,\tilde\pi)$ up to homotopy.

Proof. Let J_α denote the union of $I(\pi,\tilde\pi)$ and the set of members of $I(\pi)$ of rank $\ge g-\alpha$ so that

$$I(\pi,\tilde\pi) = J_{-1} \subset J_0 \subset \cdots \subset J_{g-1} = I(\pi).$$

We prove that for $\alpha = 0,\dots,g-1$, J_α is a $(g-2)$-cellular extension of $J_{\alpha-1}$ by showing that for every $X \in J_\alpha - J_{\alpha-1}$, $\operatorname{Link}_{J_{\alpha-1}}(X)$ is $(g-4)$-connected. For such an X, $\operatorname{rk}(X) = g-\alpha$ (so that $X^\perp/X$ has genus α), $\pi|X$ is onto, whereas $\tilde\pi|X^\perp$ isn't. So the cokernel $\tilde A^X$ of $\tilde\pi|X^\perp$ is cyclic and we have an induced surjection $\tilde\pi^X : X^* \cong H/X^\perp \to \tilde A^X$.

Let $Y \in I(H)$ be such that $Y < X$. Then $Y \in I_{\alpha-1}$ iff $Y \in I(\pi,\tilde\pi)$, that is, iff $\pi|Y$ and $\tilde\pi|Y^\perp$ are onto. The latter property is equivalent to $Y^\perp/X^\perp \to \tilde A^X$ is onto. Since we can identify $Y^\perp/X^\perp$ with the dual of X/Y, this amounts to: $\tilde\pi^X|(X/Y)^*$ is onto. This shows that $\operatorname{Link}^-_{J_{\alpha-1}}(X) = T(\pi|X \,/\, \tilde\pi^X)$. According to Proposition 1.7.2 that poset is $(\operatorname{rk}(X)-4) = (g-\alpha-4)$-connected.

Now consider the case when $Y \in I(H)$ is such that $Y > X$. Then $Y \in I_{\alpha-1}$ automatically (there is no such Y when $\alpha = 0$) and so $\operatorname{Link}^+_{J_{\alpha-1}}(X) \cong I(X^\perp/X)$. The latter is $(\alpha-1)$-spherical by the Solomon-Tits Theorem.

It follows that $\operatorname{Link}_{J_{\alpha-1}}(X)$ is $(g-\alpha-3)+(\alpha-1) = (g-4)$-connected.

Step 4. Conclusion.

The previous steps imply that $I(\pi)$ is $(g-3)$-connected, so $I(\pi)_{\leq g-2}$ is spherical. For the Cohen-Macaulay property of $I(\pi)_{\leq g-2}$, we note that for $X \in I(\pi)_{\leq g-2}$, $\mathrm{Link}^-_{I(\pi)_{\leq g-2}}(X) = T(\pi|X)$ is $(\mathrm{rk}(X)-2)$-spherical,

$$\mathrm{Link}^+_{I(\pi)_{\leq g-2}}(X) \cong I(X^\perp/X)_{\leq g-\mathrm{rk}(X)-2}$$

is $(g-\mathrm{rk}(X)-2)$-spherical and that if we are also given $X' \in I(\pi)_{\leq g-2}$ with $X < X'$, then $\mathrm{Link}^-_{I(\pi)_{\leq g-2}}(X') \cap \mathrm{Link}^+_{I(\pi)_{\leq g-2}}(X) = T(X'/X)$ is $(\mathrm{rk}(X')-\mathrm{rk}(X)-2)$-spherical by the Solomon-Tits Theorem for the general group, see for example [Quillen1]. This proves the Cohen-Macaulay property of $I(\pi)_{\leq 2}$. ☐

QUESTION 1.7.5. *Is $I(\pi)$ Cohen-Macaulay of dimension $g-1$?*

1.8. The Cohen-Macaulay property of $\mathcal{I}(\pi^{-1}(1))_{\leq g-2}$ and $\mathcal{I}^o(\pi^{-1}(1))_{\leq g-2}$

PROPOSITION 1.8.1. *Let H be a unimodular symplectic lattice of genus g and $\pi : H \to \mathbb{Z}$ an epimorphism. Then $\mathcal{I}(\pi^{-1}(1))_{\leq g-2}$ is Cohen-Macaulay of dimension $g-2$.*

PROOF. Let $\varphi : \mathcal{I}(\pi^{-1}(1))_{\leq g-2} \to I(\pi)_{\leq g-2}$ be defined by $E \mapsto \langle E \rangle$. Then φ is a strictly increasing map. If $X \in I(\pi)$ then $\varphi/X \cong \mathcal{P}((\pi|_X)^{-1}(1))$ is Cohen-Macaulay of dimension $\dim(X)-1$ by [Maazen], Chapter III, Corollary (5.7). By Corollary 1.4.6 we conclude that $\mathcal{I}(\pi^{-1}(1))$ is Cohen-Macaulay of dimension $g-1$, since by Theorem 1.7.3 $I(\pi)_{\leq g-2}$ is Cohen-Macaulay of dimension $g-2$. ☐

We are now in the position to prove Proposition 1.5.7 for $g \geq 4$, which states that for any quasi-unimodular symplectic lattice H of genus g and epimorphism $\pi : H \to \mathbb{Z}$ that factorizes over $\overline{\pi} : \overline{H} \to \mathbb{Z}$ the poset $\mathcal{I}^o(\pi^{-1}(1))_{\leq g-2}$ is Cohen-Macaulay of dimension $g-2$.

PROOF OF PROPOSITION 1.5.7 FOR $g \geq 4$. We first show that we may assume that H is unimodular. We choose a section of the projection $H \to \overline{H}$ so that we have a decomposition $H = \overline{H} \oplus \mathrm{Rad}(H)$. Let $E^o = ((v_0, r_0), \dots, (v_k, r_k)) \subset \overline{H} \oplus \mathrm{Rad}(H)$ be written according to this decomposition. Then E^o is in $\mathcal{I}^o(\pi^{-1}(1))_{\leq g-2}$ if and only if $(v_0, \dots, v_k)$ is in $\mathcal{I}^o(\overline{\pi}^{-1}(1))_{\leq g-2}$. By [Maazen], Corollary (6.3), Chapter II, it follows that if $\mathcal{I}^o(\overline{\pi}^{-1}(1))_{\leq g-2}$ is Cohen-Macaulay, then $\mathcal{I}^o(\pi^{-1}(1))_{\leq g-2}$ is Cohen-Macaulay of the same dimension. The proof of the CM-property of $\mathcal{I}^o(\overline{\pi}^{-1}(1))_{\leq g-2}$ is completely analogous to the proof of Proposition 1.5.6, see Section 1.6, where we use the map $f : \mathcal{I}^o(\overline{\pi}^{-1}(1))_{\leq g-2} \to \mathcal{I}(\overline{\pi}^{-1}(1))_{\leq g-2}$ that forgets the ordering. ☐

1.9. The Cohen-Macaulay property of $\mathcal{I}(\pi^{-1}(1))$ and $\mathcal{I}^o(\pi^{-1}(1))$ when $g = 1, 2, 3$

Let H be a unimodular symplectic lattice of genus g and $\pi : H \to \mathbb{Z}$ an epimorphism. In this section we prove that $\mathcal{I}(\pi^{-1}(1))$ is spherical when $g = 1, 2, 3$, that means, the geometric realization is nonempty when $g = 1$, connected when $g = 2$ and simply connected when $g = 3$[1]. As before, using the map $f : \mathcal{I}^o(\pi^{-1}(1)) \to \mathcal{I}(\pi^{-1}(1))$ that forgets the ordering, this implies the same result for $\mathcal{I}^o(\pi^{-1}(1))$. the For $g = 1$ the statement is trivially true. Let $e_1, e_{-1}, \ldots, e_g, e_{-g}$ be an ordered symplectic basis of H and assume that π is given by $\pi(x) = x \cdot e_{-1}$. In this section we often write a vector with respect to this basis as a column vector. We sometimes abbreviate $\mathcal{I}(\pi^{-1}(1))$ with $\mathcal{I}$. We use the following terminology when we are dealing with a poset P that is a subposet of a poset that is associated to a simplicial complex Σ. An element of height k will, in that case, be called a k-simplex; in particular, an element of height zero is called a vertex and an element of height one an edge. Although in the geometric realization of P an element of height k is represented by a vertex, as it is the subspace of the geometric realization of the barycentric subdivision of Σ, we hope that this does not cause any confusion.

PROPOSITION 1.9.1. *If $g = 2$ then $\mathcal{I}(\pi^{-1}(1))$ is connected.*

PROOF. We show that every vertex of $\mathcal{I}$ can be connected to the vertex e_1. Let $\begin{pmatrix}1\\a\\b\\0\end{pmatrix} \in \mathcal{I}$, assume $a \geq 0, b > 0$, then

$$\begin{pmatrix}1\\a\\b\\0\end{pmatrix}, \begin{pmatrix}1\\a\\b-1\\0\end{pmatrix}, \ldots, \begin{pmatrix}1\\a\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\a\end{pmatrix}, \begin{pmatrix}1\\0\\0\\a-1\end{pmatrix}, \ldots, \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

is a path in $\mathcal{I}$. If $b = 0$ then $\begin{pmatrix}1\\a\\0\\0\end{pmatrix}, \begin{pmatrix}1\\a\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\a\end{pmatrix}, \ldots, \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ is a path in $\mathcal{I}$. The cases with $a < 0$ or $b < 0$ are similar. Let $\begin{pmatrix}1\\a\\b\\c\end{pmatrix} \in \mathcal{I}$, then we can apply a symplectic

[1] Since we could not answer Question 1.7.5 positively, we prove the low-dimensional case "by hand ". With many thanks to Wilberd van der Kallen.

transformation φ that stabilizes e_1 and maps $\begin{pmatrix} 1 \\ a \\ b \\ c \end{pmatrix}$ to an element $\begin{pmatrix} 1 \\ a \\ b' \\ 0 \end{pmatrix}$, so φ applied to the path above gives a path between this element and e_1 again. □

PROPOSITION 1.9.2. *If H is a unimodular symplectic lattice of genus 3 and $\pi : H \to \mathbb{Z}$ is an epimorphism, then $\mathcal{I}(\pi^{-1}(1))$ is simply connected.*

PROOF. The proof consists of four lemmas. First we define the bigger simplicial complex $\mathcal{I}_{\mathbb{Q}}(\pi^{-1}(1))$ by stating that an isotropic subset $\{v_0, \dots, v_p\}$ of $\pi^{-1}(1)$ is in $\mathcal{I}_{\mathbb{Q}}(\pi^{-1}(1))$ if and only if they are independent vectors over $\mathbb{Q}$, so we drop the assumption that they span a *primitive* subspace. We abbreviate this poset by $\mathcal{I}_{\mathbb{Q}}$. In Lemma 1.9.5 we prove, using Lemma 1.9.3 and 1.9.4, that $\mathcal{I}$ is a retract of $\mathcal{I}_{\mathbb{Q}}$. It suffices then to prove that $\mathcal{I}_{\mathbb{Q}}$ is simply connected, which we do in Lemma 1.9.6. To prove that $\mathcal{I}$ is a retract of $\mathcal{I}_{\mathbb{Q}}$ we use the following definition and two lemmas. Let $\sigma \in \mathcal{I}_{\mathbb{Q}}$ then we define $\mathcal{I}_\sigma := \mathcal{I}(\pi^{-1}(1) \cap \langle \sigma \rangle^\perp)$.

LEMMA 1.9.3. *If $\sigma \in \mathcal{I}_{\mathbb{Q}}$ is an edge, then $\mathcal{I}_\sigma$ is simply connected.*

PROOF. After applying a suitable symplectic transformation we may assume that $\sigma = \{e_1, e_1 + ke_2\}$, for some $k \neq 0$. Then $\mathcal{I}_\sigma = \mathcal{I}(\pi^{-1}(1) \cap \langle e_1, e_2 \rangle^\perp)$. The subposet $\mathcal{I}(\pi^{-1}(1) \cap \langle e_1, e_2, e_3 \rangle^\perp)$ is simply connected by a Theorem of Maazen, see [Maazen], Chapter III, Theorem (5.5). We define an infinite filtration $Y_0 \subset Y_1 \subset Y_2 \subset \cdots$ such that $Y_0 := \mathcal{I}(\pi^{-1}(1) \cap \langle e_1, e_2, e_3 \rangle^\perp)$ and $\lim_{n\to\infty} Y_n = \mathcal{I}_\sigma$. If $\{v\} \in \mathcal{I}_\sigma - Y_0$ then we may write $v = \begin{pmatrix} 1 \\ 0 \\ a \\ 0 \\ bp \\ bq \end{pmatrix}$, with $b > 0$ and $\gcd(p, q) = 1$. Let Y_n be the subposet of $\mathcal{I}_\sigma$ consisting of $E \in \mathcal{I}_\sigma$ such that an elements of E is a vertex of Y_0 or a vertex such that $b \leq n$. For each $n \geq 1$, let $\widetilde{Y}_{n-1}$ be the poset of $E \in Y_n$ such that E contains a vertex of Y_{n-1}. Then deleting the vertices not in Y_{n-1} gives, by Lemma 1.4.2 a homotopy equivalence between $\widetilde{Y}_{n-1}$ and Y_{n-1}. We define the subfiltration $(X_{n-1})_m := \widetilde{Y}_{n-1} \cup (Y_n)_{\leq m}$ for $m = -1, 0, 1, 2$. We show in the following steps that if $i = -1, 0, 1$ and $\tau \in (X_{n-1})_{i+1} - (X_{n-1})_i$ then $\mathrm{Link}_{(X_{n-1})_i}(\tau)$ is connected. This implies that if Y_{n-1} is simply connected then so is Y_n, since Y_0 is simply connected we can conclude that $\mathcal{I}_\sigma = \lim_{n\to\infty} Y_n$ is simply connected.

Step 1. Suppose $\tau \in (X_{n-1})_0 - (X_{n-1})_{-1}$. Then $\tau = e_1 + ae_2 + npe_3 + nqe_{-3}$ with $q \neq 0$. We claim that $\mathrm{Link}_{(X_{n-1})_{-1}}(\tau)$ is connected. If a vertex in $(X_{n-1})_{-1}$ forms an edge with τ in $(X_{n-1})_{-1}$ then it is of the form $\begin{pmatrix}1\\0\\x\\0\\yp\\yq\end{pmatrix}$ with $|y| < n$. After applying a suitable symplectic transformation of $\langle e_3, e_{-3}\rangle$ we may assume that $\begin{pmatrix}p\\q\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$. Apply now the symplectic transformation $\delta^a_{e_2}\delta^a_{e_{-1}}\delta^{-a}_{e_2+e_{-1}}(x) = x - a(x\cdot e_2)e_{-1} - a(x\cdot e_{-1})e_2$; it maps $e_1+ae_2+ne_{-3}$ to e_1+ne_{-3} and leaves the set of elements in $(X_{n-1})_{-1}$ invariant. Then $\{\begin{pmatrix}1\\0\\0\\0\\0\\n\end{pmatrix}, \begin{pmatrix}1\\0\\x\\0\\0\\y\end{pmatrix}\}$ is an edge in $\mathrm{Link}_{(X_{n-1})_{-1}}(\tau)$ if and only if $|y| < n$ and $\gcd(x, n-y) = 1$. Suppose that $y \neq n-1$. This means that there are integers α, β such that $(\alpha - n)x - \beta(y-n) = 1$, $\alpha - n < 0$ and $|\alpha - n| < n - y$, so $y < \alpha < n$. Then $\{\begin{pmatrix}1\\0\\0\\0\\0\\n\end{pmatrix}, \begin{pmatrix}1\\0\\x\\0\\0\\y\end{pmatrix}, \begin{pmatrix}1\\0\\\beta\\0\\0\\\alpha\end{pmatrix}\} \in \mathrm{Link}^+_{(X_{n-1})_{-1}}(\tau)$ and with induction we see that we may assume that $y = n-1$. Then $\{\begin{pmatrix}1\\0\\0\\0\\0\\n\end{pmatrix}, \begin{pmatrix}1\\0\\x\\0\\0\\n-1\end{pmatrix}, \begin{pmatrix}1\\0\\x\pm 1\\0\\0\\n-1\end{pmatrix}\} \in \mathrm{Link}^+_{(X_{n-1})_{-1}}(\tau)$, so with induction we see that it is connected to $\{\begin{pmatrix}1\\0\\0\\0\\0\\n\end{pmatrix}, \begin{pmatrix}1\\0\\1\\0\\0\\n-1\end{pmatrix}\}$. This proves that $\mathrm{Link}^+_{(X_{n-1})_{-1}}(\tau)$ is connected and hence $(X_{n-1})_0$ is simply connected if X_{n-1} is.

Step 2. Let $\tau \in (X_{n-1})_1 - (X_{n-1})_0$. Again, we may assume that $\tau = \{v, w\}$ with $v = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ n \end{pmatrix}$, then w must be $\begin{pmatrix} 1 \\ 0 \\ \pm 1 \\ 0 \\ 0 \\ n \end{pmatrix}$ or $\begin{pmatrix} 1 \\ 0 \\ x \\ 0 \\ 0 \\ -n \end{pmatrix}$. In the first case $\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ n \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ \pm 1 \\ 0 \\ 0 \\ n \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ n-1 \end{pmatrix}\}$ is a 2-simplex, so $\mathrm{Link}^+_{(X_{n-1})_0}(\tau)$ is nonempty in that case. If $w = \begin{pmatrix} 1 \\ 0 \\ x \\ 0 \\ 0 \\ -n \end{pmatrix}$ then $\gcd(x, 2n) = 1$, because $\{v, w\}$ is an edge, so there exist α, β with $-2n < \alpha < 0$ such that $\alpha x + \beta 2n = 1$. Write $y = n + \alpha$, then $-n < y < n$, and with the conditions on α and β we see that $\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ n \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ x \\ 0 \\ 0 \\ -n \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ \beta \\ 0 \\ 0 \\ y \end{pmatrix}\}$ is a 2-simplex in $\mathrm{Link}^+_{(X_{n-1})_0}(\tau)$.

Step 3. Let $\tau \in (X_{n-1})_2 - (X_{n-1})_1$, then trivially $\mathrm{Link}_{(X_{n-1})_1}(\tau)$ is connected. This finishes the proof of Lemma 1.9.3. □

LEMMA 1.9.4. *If $\sigma \in \mathcal{I}_{\mathbb{Q}}$ is a 2-simplex, then $\mathcal{I}_\sigma$ is simply connected.*

PROOF. Let $\sigma = \{v, w, u\} \in \mathcal{I}_{\mathbb{Q}}$. After applying a symplectic transformation we may and will assume that $v = e_1$, $w = e_1 + ae_2$ and $u = e_1 + be_2 + ce_3$ for integers a, b, c. So $\mathcal{I}_\sigma = \mathcal{I}(\pi^{-1}(1) \cap \langle e_1, e_2, e_3\rangle^\perp)$. This poset is spherical, that means, simply connected, by [Maazen], Chapter III, Theorem (5.5). □

Using these two lemmas we prove the following lemma.

LEMMA 1.9.5. *The simplicial complex $\mathcal{I}$ is a retract of $\mathcal{I}_{\mathbb{Q}}$.*

PROOF. We define a retract $\mathcal{I}_{\mathbb{Q}} \to \mathcal{I}$. Let $v \in \mathcal{I}_{\mathbb{Q}}$ be a vertex, then v is a vertex of $\mathcal{I}$. Let $\{v, w\}$ be an edge of $\mathcal{I}_{\mathbb{Q}}$. Then $\mathcal{I}_{\{v,w\}}$ is connected, so we can choose a path $e_{(v,w)}$ in $\mathcal{I}_{\{v,w\}}$ between v, w; let $e_{(v,w)}$ be the image of $\{v, w\}$. Let $\{v, w, u\}$ be a 2-simplex of $\mathcal{I}_{\mathbb{Q}}$. Then we choose a point x in $\mathcal{I}_{\{v,w,u\}}$ and paths $e_{(x,v)}$, $e_{(x,w)}$, $e_{(x,u)}$ in $\mathcal{I}_{(v,w,u)}$ between x and v, x and w, and x and u respectively. Because $e_{(x,v)}$, $e_{(v,w)}$, $e_{(x,w)}$ gives a closed path in $\mathcal{I}_{\{v,w\}}$ it can be filled in this space by Lemma 1.9.3 with a 2-cell. We do the same for $\{v, u, x\}$ and $\{w, u, x\}$ respectively and this defines the image of $\{v, w, u\}$. □

This lemma implies that it suffices to prove that $\mathcal{I}_{\mathbb{Q}}$ is simply connected, which we prove in Lemma 1.9.6.

LEMMA 1.9.6. *The simplicial complex $\mathcal{I}_{\mathbb{Q}}$ is simply connected.*

PROOF. The proof of Lemma 1.9.6 consists of the following steps. We have the filtration

$$\mathcal{I}_{\mathbb{Q}}(\pi^{-1}(1)\cap\langle e_1, e_2, e_3\rangle^{\perp}) \subset \mathcal{I}_{\mathbb{Q}}(\pi^{-1}(1)\cap\langle e_1, e_2\rangle^{\perp}) \subset \mathcal{I}_{\mathbb{Q}}(\pi^{-1}(1)\cap\langle e_1\rangle^{\perp}) \subset \mathcal{I}_{\mathbb{Q}}(\pi^{-1}(1)).$$

In the first step we show that $\mathcal{I}_{\mathbb{Q}}(\pi^{-1}(1) \cap \langle e_1, e_2, e_3\rangle^{\perp})$ is simply connected. Let $U \subset V \subset H$ be subspaces, then $\mathcal{I}_{\mathbb{Q}}(\pi^{-1}(1)\cap U) \subset \mathcal{I}_{\mathbb{Q}}(\pi^{-1}(1)\cap V)$. We introduce the following notation. Let $\widetilde{\mathcal{I}}_{\mathbb{Q}}(\pi^{-1}(1)\cap U)$ be the poset consisting of $E \in \mathcal{I}_{\mathbb{Q}}(\pi^{-1}(1)\cap V)$ such that $E\cap U \neq \emptyset$, then $\widetilde{\mathcal{I}}_{\mathbb{Q}}(\pi^{-1}(1)\cap U)$ is homotopy equivalent to $\mathcal{I}_{\mathbb{Q}}(\pi^{-1}(1)\cap U)$. Define further

$$X_m := \widetilde{\mathcal{I}}_{\mathbb{Q}}(\pi^{-1}(1) \cap U) \cup \mathcal{I}_{\mathbb{Q}}(\pi^{-1}(1) \cap V)_{\leq m},$$

then we have the subfiltration

$$\widetilde{\mathcal{I}}_{\mathbb{Q}}(\pi^{-1}(1) \cap U) = X_{-1} \subset X_0 \subset X_1 \subset X_2 = \mathcal{I}_{\mathbb{Q}}(\pi^{-1}(1) \cap V).$$

In Steps 2,3 and 4 we apply this to $(U, V) = (\langle e_1, e_2, e_3\rangle^{\perp}, \langle e_1, e_2\rangle^{\perp})$, $(U, V) = (\langle e_1, e_2\rangle^{\perp}, \langle e_1\rangle^{\perp})$ and $(U, V) = (\langle e_1\rangle^{\perp}, H)$ respectively. We show in each step that if $\sigma \in X_{i+1} - X_i$ then $\mathrm{Link}_{X_i}(\sigma)$ is connected, or for some edges σ where this is not the case, we show that σ is homotopic relative the endpoints in X_2 to a path consisting of edges with a connected link in X_i. This will show that if X_i is simply connected then so is X_{i+1} and will finish the proof.

Step 1. The poset $\mathcal{I}_{\mathbb{Q}}(\pi^{-1}(1) \cap \langle e_1, e_2, e_3\rangle^{\perp})$ is simply connected.

PROOF. Any path γ in $\mathcal{I}_{\mathbb{Q}}(\pi^{-1}(1) \cap \langle e_1, e_2, e_3\rangle^{\perp})$ is homotopic to a path in the 1-skeleton of this poset, so assume that this is the case. Let $e_1 + p_0e_2 + q_0e_3, e_1 + p_1e_2 + q_1e_3, \ldots, e_1 + p_ne_2 + q_ne_3$ be the vertices of γ. A vertex $e_1 + ae_2 + be_3$ with $a, b \in \mathbb{Z}$ spans a 2-simplex with the edges $e_1 + p_ie_2 + q_ie_3, e_1 + p_{i+1}e_2 + q_{i+1}e_3$ if and only if $\begin{pmatrix} a \\ b \end{pmatrix}$ is not on the line through $\begin{pmatrix} p_i \\ q_i \end{pmatrix}$ and $\begin{pmatrix} p_{i+1} \\ q_{i+1} \end{pmatrix}$, where $i \in \{0, \ldots, n\}$. Because there are only finitely many of such lines, there exists a point $e_1 + ae_2 + be_3$ such that γ is in the star of this point, which shows that the path γ is contractible. □

Step 2. The poset $\mathcal{I}_{\mathbb{Q}}(\pi^{-1}(1) \cap \langle e_1, e_2\rangle^{\perp})$ is simply connected.

PROOF. Let $U = \langle e_1, e_2, e_3\rangle^{\perp}$, $V = \langle e_1, e_2\rangle^{\perp}$ and X_i for $-1 \le i \le 2$ as in the notation of the beginning of the proof of the lemma.

Let $\sigma \in X_0 - X_{-1}$, then σ is of the form $\sigma = e_1 + ae_2 + be_3 + ce_{-3}$ with $c \neq 0$. Then $\mathrm{Link}_{X_{-1}}(\sigma) \cong \mathcal{I}_{\mathbb{Q}}(\pi^{-1}(1) \cap \langle e_1, e_2, e_3, e_{-3}\rangle^{\perp})$ via the map $E \mapsto E - \{\sigma\}$. The latter is connected since every vertex of this poset is connected to e_1.

Let $\sigma \in X_1 - X_0$, then $\mathrm{Link}^{-}_{X_0}(\sigma)$ is clearly nonempty. We show that $\mathrm{Link}^{+}_{X_0}(\sigma)$ is nonempty as well. Let $\sigma = \{e_1 + a_0e_2 + b_0e_3 + c_0e_{-3}, e_1 + a_1e_2 + b_1e_3 + c_1e_{-3}\}$ with $c_i \neq 0$ for $i = 0, 1$. Then $\sigma \cup \{e_1\}$ is a 2-simplex in X_0 if $\begin{pmatrix} a_0 \\ b_0 \\ c_0 \end{pmatrix} \neq \lambda \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}$, but in case of equality, $\sigma \cup \{e_1 + e_2\}$ is a 2-simplex in X_0.

Let $\sigma \in X_2 - X_1$, then trivially $\mathrm{Link}_{X_1}(\sigma) = \mathrm{Link}^{-}_{X_1}(\sigma)$ is connected.

□

Step 3. The poset $\mathcal{I}_{\mathbb{Q}}(\pi^{-1}(1) \cap \langle e_1\rangle^{\perp})$ is simply connected.

PROOF. Let $U = \langle e_1, e_2\rangle^{\perp}$, $V = \langle e_1\rangle^{\perp}$ and X_i for $-1 \le i \le 2$ as in the notation of the beginning of the proof of the lemma.

Let $\sigma \in X_0 - X_{-1}$, then $\sigma = e_1 + ae_2 + be_{-2} + ce_3 + de_{-3}$ with $b \neq 0$. The vertices of X_{-1} that form an edge in X_{-1} together with σ, are of the form $v = e_1 + xe_2 + ye_3 + ze_{-3}$ and orthogonal to σ. Since $\{\sigma, v, e_1\} \in \mathrm{Link}_{X_{-1}}(\sigma)$, every element in $\mathrm{Link}_{X_{-1}}(\sigma)$ is connected to $\{\sigma, e_1\}$.

Let $\sigma \in X_1 - X_0$, then after applying a suitable symplectic transformation of $\langle e_3, e_{-3}\rangle$ we may assume that $\sigma = \{\begin{pmatrix} 1 \\ 0 \\ a_0 \\ b_0 \\ c_0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ a_1 \\ b_1 \\ c_1 \\ d_1 \end{pmatrix}\}$, with $b_0, b_1 \neq 0$. Then $\sigma \cup \{e_1\}$ is a 2-simplex in X_0, except for the special case that $\begin{pmatrix} a_0 \\ b_0 \\ c_0 \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{pmatrix}$ with $\lambda \neq 1$, but in that case $e_1 + e_3$ will do the job.

Let $\sigma \in X_2 - X_1$, then trivially $\mathrm{Link}_{X_1}(\sigma) = \mathrm{Link}^{-}_{X_1}(\sigma)$ is connected. □

Step 4. The poset $\mathcal{I}_{\mathbb{Q}}(\pi^{-1}(1))$ is simply connected.

PROOF. Let $U = \langle e_1\rangle^{\perp}$, $V = H$ and X_i for $-1 \le i \le 2$ as in the notation of the beginning of the proof of the lemma. In this step the procedure is more subtle, a reason for this is that $\mathrm{Link}_{X_{-1}}(e_1 + ae_{-1}) = \emptyset$ if $a \neq 0$. For every pair $a \le b$ of integers we define $Y_{[a,b]}$ to be the subposet of $\mathcal{I}_{\mathbb{Q}}$ consisting of $E \in \mathcal{I}_{\mathbb{Q}}$ such that an elements of E is a vertex of $\mathcal{I}_{\mathbb{Q}}(\pi^{-1}(1) \cap \langle e_1 \rangle^{\perp})$ or is an element $e_1 + xe_{-1} + ye_2 + ze_{-2} + te_3 + ue_{-3}$ such that $a \leq x \leq b$ and $\begin{pmatrix} y \\ z \\ t \\ u \end{pmatrix}$ is unimodular over $\mathbb{Z}$. By the previous step we know that $Y_{[0,0]} = X_{-1}$ is simply connected. Assume that $Y_{[a,b]}$ is simply connected, we prove that this implies that $Y_{[a,b+1]}$ is simply connected and by the same way of arguing that $Y_{[a-1,b]}$ is simply connected. We use the same procedure as before.

Let $\widetilde{Y}_{[a,b]}$ be the subposet consisting of $E \in Y_{[a,b+1]}$ such that E contains a vertex of $Y_{[a,b]}$. Then $\widetilde{Y}_{[a,b]}$ is homotopy equivalent to $Y_{[a,b]}$. We have the filtration

$$\widetilde{Y}_{[a,b]} = X_{-1} \subset X_0 \subset X_1 \subset X_2 = Y_{[a,b+1]},$$

where $X_m := \widetilde{Y}_{[a,b]} \cup (Y_{[a,b+1]})_{\leq m}$.

Let $\sigma \in X_0 - X_{-1}$, then, after applying a suitable symplectic transformation that stabilizes $\langle e_1, e_{-1} \rangle$, we may assume that $\sigma = e_1 + ce_{-1} + e_2$ with $c = b+1$. If a vertex v_0 in X_{-1} forms an edge in X_{-1} together with σ, then $v_0 = \begin{pmatrix} 1 \\ x \\ y \\ c-x \\ z \\ t \end{pmatrix}$ with $a \leq x \leq b$. We show that $\{\sigma, v_0\}$ can be connected to $\{\sigma, e_1 + be_{-1} + e_{-2}\}$ via a path in $\mathrm{Link}_{X_{-1}}(\sigma)$. By applying a suitable symplectic transformation of $\langle e_3, e_{-3} \rangle$ we may assume that $t = 0$. Let $v_1 = \begin{pmatrix} 1 \\ x \\ y \\ c-x \\ 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ b \\ 0 \\ 1 \\ 1 \\ s \end{pmatrix}$, $v_3 = \begin{pmatrix} 1 \\ b \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ where $s = -b + x - y$. Then $\{\sigma, v_0, v_1\}$, $\{\sigma, v_1, v_2\}$, $\{\sigma, v_2, v_3\} \in \mathrm{Link}_{X_{-1}}(\sigma)$ connect them.

Let $\sigma \in X_1 - X_0$. For investigating $\mathrm{Link}_{X_0}(\sigma)$ we may assume that $\sigma = \{v = \begin{pmatrix} 1 \\ c \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, w_0 = \begin{pmatrix} 1 \\ c \\ d_0 \\ 0 \\ f_0 \\ 0 \end{pmatrix}\}$ with $\gcd(d_0, f_0) = 1$ and $f_0 \geq 0$. Then $\mathrm{Link}^{+}_{X_0}(\sigma)$ may be empty because if a vertex $u = e_1 + xe_{-1} + ye_2 + (c-x)e_{-2} + ze_3 + te_{-3}$ in X_0 forms a 2-simplex with σ then $a \leq x \leq b$ and $(c-x)(d_0 - 1) + f_0 t = 0$. This means that if $f_0 = \pm 1$, then the link is nonempty. We show that otherwise we can find vertices $w_1, \ldots, w_k$ in X_0 such that $\{v, w_i, w_{i+1}\}$ is a 2-simplex in $Y_{[a,c]}$ for all $i \in \{0, \ldots, k-1\}$ and $\mathrm{Link}^+_{X_0}(\{w_i, w_{i+1}\})$, $\mathrm{Link}^+_{X_0}(\{w_k, v\})$ are nonempty. This will imply the following. Any closed path in $Y_{[a,c]}$ is homotopic to a path in the 1-skeleton of $Y_{[a,c]}$ and, by what we show now, any edge of the path is homotopic to a path with edges such that their upper link in X_0 is nonempty. So this path is homotopic to a path in X_0 and hence is contractible. We show now that we can find the vertices with the given property.

Since $\gcd(d_0, f_0) = 1$ and $f_0 \geq 0$ we can find with induction pairs of integers (d_i, f_i) such that $\gcd(d_i, f_i) = 1$, $d_i f_{i+1} - f_i d_{i+1} = 1$, $0 < f_{i+1} < f_i$ and $f_k = 1$. Let $w_i = \begin{pmatrix} 1 \\ c \\ d_i \\ 0 \\ f_i \\ 0 \end{pmatrix}$, we have that $\{\begin{pmatrix} 1 \\ c \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ c \\ d_i \\ 0 \\ f_i \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ c \\ d_{i+1} \\ 0 \\ f_{i+1} \\ 0 \end{pmatrix}\}$ is a 2-simplex in $Y_{[a,c]}$, since with our choices $\det \begin{pmatrix} 1 & 1 & 1 \\ 1 & d_i & d_{i+1} \\ 0 & f_i & f_{i+1} \end{pmatrix} \neq 0$. Also $\mathrm{Link}^+_{X_0}(\{w_i, w_{i+1}\})$ is nonempty, since one can compute that $\{\begin{pmatrix} 1 \\ c \\ d_i \\ 0 \\ f_i \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ c \\ d_{i+1} \\ 0 \\ f_{i+1} \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ x \\ 0 \\ y \end{pmatrix}\}$ is a 2-simplex if $x = c(f_i - f_{i+1})$ and $y = c(d_{i+1} - d_i)$.

Let $\sigma \in X_2 - X_1$, then trivially $\mathrm{Link}_{X_1}(\sigma) = \mathrm{Link}^-_{X_1}(\sigma)$ is connected.

This proves that $Y_{(-\infty,\infty)} := \lim_{a \to -\infty} \lim_{b \to \infty} Y_{[a,b]}$ is simply connected. Let $\widetilde{Y}_{(-\infty,\infty)}$ be the poset of $E \in \mathcal{I}_{\mathbb{Q}}$ such that E contains vertices of $Y_{(-\infty,\infty)}$. Then $\widetilde{Y}_{(-\infty,\infty)}$ is homotopy equivalent to $Y_{(-\infty,\infty)}$, we define as usual $X_m := \widetilde{Y}_{(-\infty,\infty)} \cup (\mathcal{I}_{\mathbb{Q}})_{\leq m}$.

If $\sigma \in X_0 - X_{-1}$ then we may assume that $\sigma = e_1 + ae_{-1} + be_2$ with $|b| \neq 1$. Let $v = \begin{pmatrix} 1 \\ a - by \\ x \\ y \\ z \\ t \end{pmatrix} \in X_{-1}$ and assume that it forms an edge in X_{-1} with σ. We claim that $\{\sigma, v\}$ is connected to $\{\sigma, \tau\}$, where $\tau = \begin{pmatrix} 1 \\ a-b \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$. Assume first that $t = 0$, then $\{\sigma, v, w\} \in \mathrm{Link}^{+}_{X_{-1}}(\sigma)$ with $w = \begin{pmatrix} 1 \\ a \\ b \\ 0 \\ 1 \\ 0 \end{pmatrix}$, except when $x = b$ and $y = 0$. In that case , let $u = \begin{pmatrix} 1 \\ a-b \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ then $\{\sigma, v, u\}$ and $\{\sigma, u, w\}$ are 2-simplices in $\mathrm{Link}^{+}_{X_{-1}}(\sigma)$. This shows that $\{\sigma, v\}$ is connected to $\{\sigma, w\}$, but $\{\sigma, w\}$ is connected to $\{\sigma, \tau\}$ since $\{\begin{pmatrix} 1 \\ a \\ b \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ a \\ b \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ a-b \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\}$ is a 2-simplex in $\mathrm{Link}_{X_{-1}}(\sigma)$. If $t \neq 0$ we apply a suitable symplectic transformation of $\langle e_3, e_{-3} \rangle$ so that it becomes zero, this leaves τ and σ invariant.

If $\sigma \in X_1 - X_0$ then we may assume that $\sigma = \{\begin{pmatrix} 1 \\ a \\ b \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ a-bs \\ r \\ s \\ t \\ 0 \end{pmatrix}\}$ with $|b| \neq 1$ and we can choose $z \in \mathbb{Z}$ such that $\sigma \cup \begin{pmatrix} 1 \\ a-bs \\ r-t \\ s \\ z \\ -s \end{pmatrix} \in \mathrm{Link}^{+}_{X_0}(\sigma)$. This shows that $\mathrm{Link}^{+}_{X_0}(\sigma)$ is nonempty.

Let $\sigma \in X_2 - X_1$, then trivially $\mathrm{Link}_{X_1}(\sigma) = \mathrm{Link}^{-}_{X_1}(\sigma)$ is connected.

The steps $1, 2, 3$ and 4 together prove that $\mathcal{I}_{\mathbb{Q}}$ is simply connected. □

COROLLARY 1.9.7. *If H is a unimodular symplectic lattice of genus ≤ 3 and $\pi : H \to \mathbb{Z}$ an epimorphism, then $\mathcal{I}(\pi^{-1}(1))$ is Cohen-Macaulay.*

PROOF. We check the three conditions of the definition of CM on p. 7. The only nontrivial condition we have to check is if $g(H) = 3$ and $\{v\} \in \mathcal{I}(\pi^{-1}(1))$ then $\mathcal{I}(\pi^{-1}(1))_{>\{v\}}$ is connected. We may assume that $v = e_1$, then $\mathcal{I}(\pi^{-1}(1))_{>\{v\}} \cong \mathcal{I}(\langle e_2, e_{-2}, e_3, e_{-3}\rangle)$, which is spherical by Proposition 1.6.1. □

We are now able to finish the proof of Proposition 1.5.7 for $g \leq 3$ which states that $\mathcal{I}^o(\pi^{-1}(1))$ is Cohen-Macaulay of dimension $g-1$. Again, we use the map $f : \mathcal{I}^o(\pi^{-1}(1)) \to \mathcal{I}(\pi^{-1}(1))$ that forgets the ordering and the proof is exactly the same as for $g \geq 4$, see p. 17.

1.10. The connectedness of $\mathcal{A}^o(H)$ and $\mathcal{A}^o(H,\pi)$

We are now in the position to prove Theorem 1.5.4 that says that if H is a quasi-unimodular symplectic lattice of genus g, then $\mathcal{A}^o(H)$ is $(g-2)$-connected.

PROOF OF THEOREM 1.5.4. Every $\sigma = (v_0, \ldots, v_m) \in \mathcal{A}^o(H)$ determines a subgraph $\mathcal{G}_\sigma$ on its set of vertices, where two different vertices v_i, v_j are connected via an edge if and only if $v_i \cdot v_j \neq 0$. We say that a vertex is *isolated* if there are no edges emerging from this vertex. Let k_σ be the number of non-isolated vertices in this graph. Notice that k_σ is never 1 and is $2g$ at most, so if we define $\mathcal{A}^o(H)_k := \{\sigma \in \mathcal{A}^o(H) : k_\sigma \leq k+1\}$ then we have a filtration

$$\mathcal{I}^o(H) = \mathcal{A}^o(H)_0 \subset \mathcal{A}^o(H)_1 \subset \cdots \subset \mathcal{A}^o(H)_{2g-1} = \mathcal{A}^o(H).$$

Since $\mathcal{I}^o(H)$ is spherical of dimension $g-1$ we want to see that for $k \geq 1$ $\mathcal{A}^o(H)_k$ is a $(g-1)$-cellular extension of $\mathcal{A}^o(H)_{k-1}$. Let Σ_k denote the set of $\sigma = (v_0, \ldots, v_k) \in \mathcal{A}^o(H)$ with $k_\sigma = k+1$. Then every member of $\mathcal{A}^o(H)_k - \mathcal{A}^o(H)_{k-1}$ is in the link of a unique member of Σ_k. We claim that $\mathcal{A}^o(H)_{k-1}$ is a deformation retract of $\mathcal{A}^o(H)_k - \Sigma_k$. Let

$$X_m = \mathcal{A}^o(H)_{k-1} \cup \{\tau \in \mathcal{A}^o(H)_k - \Sigma_k : h(\tau) \leq m\}.$$

If $m \leq k$ then $X_m - X_{m-1} = \emptyset$. If $m > k$ and $\tau \in X_m - X_{m-1}$ then there is a unique $\sigma \in \Sigma_k$ such that $\sigma < \tau$. Then $\mathrm{Link}^-_{X_{m-1}}(\tau)$ is contractible since it consists of all the faces of τ unequal to σ. This implies that $\mathrm{Link}_{X_{m-1}}(\tau)$ is contractible as well. So X_{m-1} is a deformation retract of X_m and hence, $\mathcal{A}^o(H)_{k-1}$ of $\mathcal{A}^o(H)_k - \Sigma_k$. Therefore it suffices to prove that $(\mathcal{A}^o(H)_k, \mathcal{A}^o(H)_k - \Sigma_k)$ is $(g-2)$-connected. Let $\sigma \in \Sigma_k$. Then $\mathcal{G}_\sigma$ has no isolated points and therefore $\dim(\sigma) = k$,

$\mathrm{Link}^-_{\mathcal{A}^o(H)_k}(\sigma) \subset \mathcal{A}^o(H)_{k-1} \subset \mathcal{A}^o(H)_k - \Sigma_k$ and is a sphere of dimension $k-1$. Furthermore,

$\mathrm{Link}^+_{\mathcal{A}^o(H)_k-\Sigma_k}(\sigma) = \{\tau \in \mathcal{A}^o(H) : \tau > \sigma$ and all vertices of τ not in σ are isolated$\}$,

By the definition of an arc-sequence it is possible to write $\sigma = (\sigma_1, \ldots, \sigma_m)$ such that $\mathcal{G}_{\sigma_1}, \ldots, \mathcal{G}_{\sigma_m}$ are the connected components of $\mathcal{G}_\sigma$. If $\tau \in \mathrm{Link}^+_{\mathcal{A}^o(H)_k-\Sigma_k}(\sigma)$ then deleting σ gives a map to $\mathcal{I}^o(\langle\sigma\rangle^\perp)$; recall that $\mathcal{I}^o(\langle\sigma\rangle^\perp) = \mathcal{I}^o(\langle\sigma\rangle^\perp, \mathrm{Rad}(\langle\sigma\rangle^\perp))$. We show that $\mathrm{Link}^+_{\mathcal{A}^o(H)_k-\Sigma_k}(\sigma)$ can be identified with the preimage of $\mathcal{I}^o(\langle\sigma\rangle^\perp)$ under the map

$$\mathrm{Link}^+_{\mathcal{O}^o(\langle\sigma\rangle^\perp \cup \{\sigma_1,\ldots,\sigma_m\})}((\sigma_1, \ldots, \sigma_m)) \to \mathcal{O}^o(\langle\sigma\rangle^\perp)$$

that deletes $\sigma_1, \ldots, \sigma_m$. This means, we have to show that for every $\rho \in \mathcal{I}^o(\langle\sigma\rangle^\perp)$ the map $\langle\rho\rangle + \langle\sigma\rangle + \mathrm{Rad}(H) \to H$ is a primitive embedding.

We have $\langle\sigma\rangle \cong \overline{\langle\sigma\rangle} \oplus \langle\sigma\rangle \cap \langle\sigma\rangle^\perp$ and $\langle\sigma\rangle^\perp \cong \overline{\langle\sigma\rangle^\perp} \oplus (\langle\sigma\rangle \cap \langle\sigma\rangle^\perp + \mathrm{Rad}(H))$. So if $\rho \in \mathcal{I}^o(\langle\sigma\rangle^\perp)$ then

$$\langle\rho\rangle + \langle\sigma\rangle \cap \langle\sigma\rangle^\perp + \mathrm{Rad}(H) \to H$$

is a primitive embedding, and because $\langle\rho\rangle \cap \overline{\langle\sigma\rangle} = \emptyset$ we have

$$\langle\rho\rangle + \langle\sigma\rangle + \mathrm{Rad}(H) = \langle\rho\rangle + \overline{\langle\sigma\rangle} \oplus \langle\sigma\rangle \cap \langle\sigma\rangle^\perp + \mathrm{Rad}(H) = \overline{\langle\sigma\rangle} \oplus (\langle\rho\rangle + \langle\sigma\rangle \cap \langle\sigma\rangle^\perp) + \mathrm{Rad}(H)$$

maps injectively to a primitive subspace of H. Since σ contains a hyperbolic pair we have that $g(\langle\sigma\rangle^\perp) \geq g-k$, and hence $\mathcal{I}^o(\langle\sigma\rangle^\perp)$ is $(g-k-2)$-connected. It follows by Proposition 1.5.1 that $\mathrm{Link}^+_{\mathcal{A}^o(H)_k}(\sigma)$ is $(g-k-2)$-connected. We conclude that $\mathrm{Link}_{\mathcal{A}^o(H)_k-\Sigma_k}(\sigma) = \mathrm{Link}^-_{\mathcal{A}^o(H)_k-\Sigma_k}(\sigma) * \mathrm{Link}^+_{\mathcal{A}^o(H)_k-\Sigma_k}(\sigma)$ is $(k-1)+(g-k-2)+1 = g-2$ connected, so if $\mathcal{A}^o(H)_k - \Sigma_k$ is $g-2$ connected, then so is $\mathcal{A}^o(H)_k$. □

□

We now give the proof of Theorem 1.5.5, which states that if H is a quasi-unimodular symplectic lattice of genus g and $\pi : H \to \mathbb{Z}$ an epimorphism that factorizes over $\overline{\pi} : \overline{H} \to \mathbb{Z}$, then $\mathcal{A}^o(H, \pi)$ is 1-connected if $g \geq 3$.

Proof of Theorem 1.5.5. We show that any vertex $(v, w) \in \mathcal{A}^o(H, \pi)$ is homotopic, relative the endpoints, to a path in $\mathcal{I}^o(\pi^{-1}(1))$. Since $\mathcal{I}^o(\pi^{-1}(1))$ is 1-connected by Theorem 1.5.7, this proves the Theorem. We choose a symplectic basis

$$\{e_1, e_{-1}, \ldots, e_g, e_{-g}, e_{g+1}, \ldots, e_{g+r}\}$$

of H and assume that $\pi : H \to \mathbb{Z}$ is given by $\pi(x) = x \cdot e_{-1}$. Assume that (v, w) is not an edge of $\mathcal{I}^o(\pi^{-1}(1))$, otherwise we are done. After applying a suitable symplectic transformation we may assume that $v = e_1, w = e_1 + e_{-1} + ae_2$ for some $a \in \mathbb{Z}$. If $a = \pm 1$ then $(e_1, e_1 + e_{-1} + ae_2, e_1 + ae_{-2})$ will do the job. If $a \neq \pm$ then

$(e_1, e_1 + e_{-1} + ae_2, e_1 + e_{-1} + (a \pm 1)e_2)$ is a 2-simplex in $\mathcal{A}^o(H, \pi)$, so with induction we can reduce this to the case $a = \pm 1$. □

□

1.11. Simplicial complexes with a group action

Let Σ be a simplicial complex and G a group that acts simplicially on Σ. Assume that there are no edges $\{v, w\}$ such that v and w are in the same orbit of the action of G. If this is not the case, we can pass to the barycentric subdivision to achieve this situation. The orbit space $\overline{\Sigma} := G \backslash \Sigma$ can in that case be viewed as a set of vertices of a simplicial complex, such that the projection map $\pi : \Sigma \to \overline{\Sigma}$ is a morphism of simplicial complexes (k-simplices are mapped to k-simplices), by stating that $\{v_0, \ldots, v_k\} \subset \overline{\Sigma}$ forms a k-simplex if and only if $v_i = \overline{w_i}$ for $w_i \in \Sigma$ such that $\{w_0, \ldots, w_k\}$ is a k-simplex of Σ.

The action of G on Σ defines for every $q \geq 0$ a system of coefficients on Σ, as follows. If σ is a simplex, we denote by G_σ the stabilizer of σ in G and we assign to σ the group $H_q(G_\sigma)$. If $\tau \leq \sigma$ then $G_\sigma \subset G_\tau$ and this induces the restriction map $\rho_{\sigma,\tau} : H_q(G_\sigma) \to H_q(G_\tau)$. We get an induced system of coefficients $\mathcal{H}_q$ on $\overline{\Sigma}$ defined by

$$\mathcal{H}_q(\overline{\sigma}) := (\bigoplus_{\tau \in \pi^{-1}(\overline{\sigma})} H_q(G_\tau))_G,$$

the group of co-invariants under the conjugation action of G.

Let F_* be a projective resolution of $\mathbb{Z}$ as $\mathbb{Z}G$-module and let C_* denote the simplicial chain complex of $|\Sigma|$. The double complex $F_* \otimes_G C_*$ gives rise to two spectral sequences converging to the homology of the total complex of $F_* \otimes_{\mathbb{Z}G} C_*$, denoted by $H_*(G, C_*)$. The first spectral sequence (where we take the differential of C_* first) has E^1-term

$$E^1_{p,q}(I) = H_q(C_* \otimes_G F_p) \cong H_q(\Sigma) \otimes_G F_p$$

and thus $E^2_{p,q}(I) = H_p(G, H_q(\Sigma))$. The other spectral sequence has

$$E^1_{p,q}(II) = H_q(G, C_p) \cong \bigoplus_{\overline{\sigma} \in \overline{\Sigma}_p} H_q(G, \bigoplus_{\tau \in \pi^{-1}(\overline{\sigma})} \mathbb{Z}).$$

Since this is isomorphic to

$$\bigoplus_{\overline{\sigma} \in \overline{\Sigma}_p} (\bigoplus_{\tau \in \pi^{-1}(\overline{\sigma})} H_q(G, \mathbb{Z}G \otimes_{\mathbb{Z}G_\tau} \mathbb{Z}))_G,$$

we conclude by Shapiro's Lemma, see [Brown2], that this is isomorphic to

$$\bigoplus_{\overline{\sigma} \in \overline{\Sigma}_p} (\bigoplus_{\tau \in \pi^{-1}(\overline{\sigma})} H_q(G_\tau))_G = \bigoplus_{\overline{\sigma} \in \overline{\Sigma}_p} \mathcal{H}_q(\overline{\sigma}) =: C_p(\overline{\Sigma}, \mathcal{H}_q).$$

Hence $E^2_{p,q}(II) = H_p(\overline{\Sigma}, \mathcal{H}_q)$ for all $p \geq 0, q \geq 0$. In the next lemma we apply these spectral sequences to the special case when Σ is simply connected.

LEMMA 1.11.1. *Let Σ be a simply connected simplicial complex and G a group which acts on Σ in such a manner that there are no edges $\{v, w\}$ such that v and w are in the same orbit. Then there is an exact sequence*

$$H_2(\overline{\Sigma}) \to H_0(\overline{\Sigma}, \mathcal{H}_1) \to H_1(G) \to H_1(\overline{\Sigma}) \to 0.$$

PROOF. The assumption that Σ is simply connected implies that $E^\infty_{p,1}(I) = E^2_{p,1}(I) = 0$ for all $p \geq 0$. Also, we have that $E^\infty_{1,0}(I) = E^2_{1,0}(I) = H_1(G, H_0(\Sigma)) = H_1(G)$. Using the exact sequence

$$0 \to E^\infty_{0,1} \to H_1(G, C_*) \to E^\infty_{1,0} \to 0,$$

that holds for both spectral sequences, we see that $H_1(G, C_*) \cong H_1(G)$. Since $E^\infty_{0,1}(II) = E^3_{0,1}(II) = \operatorname{Coker}(H_2(\overline{\Sigma}) \to H_0(\overline{\Sigma}, \mathcal{H}_1))$ and $E^\infty_{1,0}(II) = E^2_{1,0}(II) = H_2(\overline{\Sigma})$, the lemma follows using the exact sequence again. $\square$

1.12. Computation of $H_0(\Sigma, \mathcal{F})$

In this section we compute $H_0(\Sigma, \mathcal{F})$ for $\mathcal{A}^o(H)$ and $\mathcal{A}^o(H, \pi)$ and certain systems of coefficients $\mathcal{F}$. Let H be a symplectic quasi-unimodular module and $\pi : H \to \mathbb{Z}$ an epimorphism. We define the systems of coefficients $\mathcal{F}_f$, $\mathcal{F}_t$ on $\mathcal{A}^o(H)$ by

$$\begin{aligned} \mathcal{F}_f(\sigma) &:= \wedge^3 \sigma^\perp, \\ \mathcal{F}_t(\sigma) &:= B_2(\Omega_{\sigma^\perp}). \end{aligned}$$

Because $\mathcal{F}_f$ has image in the category of free abelian groups, we use the subscript f here and since $\mathcal{F}_t$ maps into the category of 2-torsion groups, we use the subscript t there.

On $\mathcal{A}^o(H, \pi)$ we define the following systems of coefficients, by abuse of notation denoted by the same letters,

$$\begin{aligned} \mathcal{F}_f(\sigma) &:= \wedge^3(\sigma^\perp \cap \pi^{-1}(0)), \\ \mathcal{F}_t(\sigma) &:= B_2(\Omega_{\sigma^\perp \cap \pi^{-1}(0)}). \end{aligned}$$

The restriction maps are in each case the inclusion maps.

REMARK 1.12.1. If $U, V \subset H$ are subspaces then

$$(\wedge^k U) \cap (\wedge^k V) = \wedge^k (U \cap V),$$

for if $K, L_1, L_2 \subset H$ such that $U \cap V = (K \oplus L_1) \cap (K \oplus L_2) = K$ then $\wedge^k(K \oplus L_1) \cap \wedge^k(K \oplus L_2) = \bigoplus_{i=0}^{k} (\wedge^i K \otimes \wedge^{k-i} L_1) \cap \bigoplus_{i=0}^{k} (\wedge^i K \otimes \wedge^{k-i} L_2) = \wedge^k K$.

We state a lemma that is helpful in the computations of $H_0(\Sigma, \mathcal{F})$.

LEMMA 1.12.2. *Let Σ be a simplicial complex and $\mathcal{F}$ a system of coefficients on Σ. We assume the following three conditions on Σ and $\mathcal{F}$:*

(i) *the functor $\mathcal{F}$ takes values in the category of subgroups of some abelian group A,*
(ii) *the group A is spanned by $\cup_{\sigma\in\Sigma_0}\mathcal{F}(\sigma)$ for all σ,*
(iii) *if σ_0, σ_1 are vertices of Σ and $x \in \mathcal{F}(\sigma_0) \cap \mathcal{F}(\sigma_1)$ then σ_0, σ_1 are connected in $\Sigma_x := \{\sigma \in \Sigma : x \in \mathcal{F}(\sigma)\}$.*

Then $H_0(\Sigma, \mathcal{F}) \cong A$.

PROOF. By the first two assumptions we know that there is a surjective map $\epsilon : C_0(\Sigma, \mathcal{F}) \to A$ which factors through $H_0(\Sigma, \mathcal{F})$. A section of ϵ can be defined on an element in $\mathcal{F}(\sigma)$ by the inclusion in the $\mathcal{F}(\sigma)$-summand. Because of assumption (iii) this map is well-defined and it is an inverse of $\bar{\epsilon} : H_0(\Sigma, \mathcal{F}) \to A$. ☐

PROPOSITION 1.12.3. *Let H be a quasi-unimodular symplectic module over $\mathbb{Z}$ of genus g. If $g \geq 2$ then $H_0(\mathcal{A}^o(H), \mathcal{F}_f)$ surjects onto $\wedge^3 H$ and $H_0(\mathcal{A}^o(H), \mathcal{F}_t)$ surjects onto $B_2(\Omega_H)$. If $g \geq 4$ then $H_0(\mathcal{A}^o(H), \mathcal{F}_f) \cong \wedge^3 H$ and $H_0(\mathcal{A}^o(H), \mathcal{F}_t) \cong B_2(\Omega_H)$.*

PROOF. We only prove the statement for $\mathcal{F}_f$ as the proof of the statement for $\mathcal{F}_t$ is similar. We check the conditions of Lemma 1.12.2. First we see that after choosing a symplectic basis of H, every element of $\wedge^3 H$ is a sum of basis elements and these are obviously in a summand $\mathcal{F}_f(\sigma)$ as long as $g \geq 2$. So $H_0(\mathcal{A}^o(H), \mathcal{F}_f)$ surjects onto $\wedge^3 H$. Suppose that v and w are vertices of $\mathcal{A}^o(H)$, $x \in \mathcal{F}(v) \cap \mathcal{F}(w)$ and that $g \geq 4$. If we look at the proof of Lemma 1.12.2, we see that we may assume that $x = x_0 \wedge x_1 \wedge x_2$ for some $x_0, x_1, x_2 \in H$. If (v, w) (or (w, v)) is an edge of Σ_x we are done. The other cases are proved in the following steps.

Case 1. Suppose now that (v, w) is an edge of $\mathcal{P}(H, \mathrm{Rad}(H))$ but not of $\mathcal{A}^o(H)$. Then we can choose a symplectic basis for H such that $v = e_1$ and $w = \alpha e_1 + \beta e_{-1} + \gamma e_2$ for integers α, β, γ with $\gcd(\beta, \gamma) = 1$. Then $v^\perp \cap w^\perp$ has basis

$$\beta = \{e_2, \ldots, e_g, \gamma e_1 + \beta e_{-2}, e_{-3}, \ldots, e_{-g}, e_{g+1}, \ldots, e_{g+r}\}.$$

If $g \geq 4$ we can find for every triple $x_0, x_1, x_2 \in \beta$ an element $u_0 \in \beta$ such that $x_0 \wedge x_1 \wedge x_2 \in \wedge^3 u_0^\perp$ and v, u_0, w forms a path in Σ_x.

Case 2. Suppose that $\langle v, w\rangle \oplus \mathrm{Rad}(H) \to H$ is an embedding but the image is not primitive. We may assume that $v = e_1$, then $w = \alpha_0 e_1 + \beta_0 u \bmod \mathrm{Rad}(H)$ for some integers α_0, β_0 with $\gcd(\alpha_0, \beta_0) = 1$ and primitive element u of H. With induction we can find pairs $(\alpha_1, \beta_1), \cdots, (\alpha_n, \beta_n)$ such that $\alpha_i\beta_{i+1} - \alpha_{i+1}\beta_i = 1$ and $|\alpha_{i+1}| < |\alpha_i|$, $|\beta_{i+1}| < |\beta_i|$ and $|\alpha_n| \leq 1$, $|\beta_n| \leq 1$. Then $v, \alpha_n v + \beta_n u, \cdots, \alpha_0 v + \beta_0 u$ are vertices in the primitive hull of $\langle v, w\rangle$ of a path in $\mathcal{P}(H, \mathrm{Rad}(H))$ and if $x_0 \wedge x_1 \wedge x_2 \in \mathcal{F}_f(v) \cap \mathcal{F}_f(w)$ then it is in the summand of each of the vertices of this path. The previous case shows how we can deal with each of the edges of this path.

Case 3. Remains the case where v, w are vertices of Σ_x but $\langle v, w\rangle \oplus \mathrm{Rad}(H) \to H$ is not an embedding. In that case we may assume that $v = e_1$ and $w = e_1 + \rho$ for some $\rho \in \mathrm{Rad}(H)$. Then $v^{\perp} \cap w^{\perp}$ has basis

$$\beta = \{e_1, \dots, e_g, e_{-2}, \dots, e_{-g}, e_{g+1}, \dots, e_{g+r}\}$$

and it is easy to see that that for each triple $x_0, x_1, x_2 \in \beta$ there is an element $u_0 \in \beta$ such that $x_0, x_1, x_2 \perp u_0$ and v, u_0, w are vertices of a path in $\mathcal{A}^o(H)$.

This proves all cases and therefore the proposition. □

We now state the affine version of Proposition 1.12.3.

PROPOSITION 1.12.4. *Let H be a quasi-unimodular symplectic module and $\pi : H \to \mathbb{Z}$ an epimorphism. If $g \geq 3$ then $H_0(\mathcal{A}^o(H,\pi), \mathcal{F}_f)$ surjects onto $\wedge^3\pi^{-1}(0)$ and $H_0(\mathcal{A}^o(H,\pi)), \mathcal{F}_t)$ surjects onto $B_2(\Omega_{\pi^{-1}(0)})$. If $g \geq 4$ then*

$$H_0(\mathcal{A}^o(H,\pi)), \mathcal{F}_f) \cong \wedge^3\pi^{-1}(0)$$

and

$$H_0(\mathcal{A}^o(H,\pi), \mathcal{F}_t) \cong B_2(\Omega_{\pi^{-1}(0)}).$$

PROOF. Again we only prove the statement for $\mathcal{F}_f$. Let $e_{-1} \in H$ be primitive such that π is given by e^*_{-1} and complete e_{-1} to a symplectic basis of H. Then

$$\beta = \{e_2, \dots, e_g, e_{-1}, \dots e_{-g}, e_{g+1}, \dots e_{g+r}\}$$

is a basis for $\pi^{-1}(0)$. If $x, y, z \in \beta - \{e_{-1}\}$ then $x \wedge y \wedge z \in \wedge^3 e_1^{\perp} \cap \pi^{-1}(0)$. For distinct elements $e_{-1}, y, z \in \beta$, we can choose a hyperbolic pair $\{e_i, e_{-i}\} \subset \beta$ such that $y, z \perp e_{-i}$ (this is possible if $g \geq 3$), then

$$(e_{-1} + e_i) \wedge y \wedge z \in \wedge^3(e_1 - e_i)^{\perp} \cap \pi^{-1}(0).$$

This shows that $\wedge^3\pi^{-1}(0)$ has a basis for which each element is in a summand of $C_0(\mathcal{A}^o(H,\pi), \mathcal{F}_f)$. This shows that condition (ii) holds. Suppose that v and w are vertices of Σ_x, $x \in \mathcal{F}(v) \cap \mathcal{F}(w)$ and $g \geq 4$.

If (v, w) (or (w, v)) be an edge of Σ_x we are done. Again, we may assume that $x = x_0 \wedge x_1 \wedge x_2$.

Case 1. If (v, w) is an edge of $\mathcal{P}(\pi^{-1}(1), \mathrm{Rad}(H))$ but not of Σ_x, we may assume, after applying a suitable element of $\mathrm{Sp}(H, \pi^{-1}(0))$, that $v = e_1$ and $w = e_1 + \lambda e_{-1} + \mu e_2$ for some integers λ, μ such that $\gcd(\lambda, \mu) = 1$. Then $v^{\perp} \cap w^{\perp} \cap \pi^{-1}(0)$ has basis $\beta = \{e_2, \dots, e_g, e_{-3}, \dots, e_{-g}, e_{g+1}, \dots, e_{g+r}\}$ and for every $x_0, x_1, x_2 \in \beta$ the path $e_1, e_1 + \frac{\lambda}{|\lambda|}e_{-1} + e_2, \dots, e_1 + (|\lambda| - 1)\frac{\lambda}{|\lambda|}e_{-1} + e_2, e_1 + \lambda e_{-1} + \mu e_2$ is in Σ_x and $x_0 \wedge x_1 \wedge x_2$ is in every summand of $\mathcal{F}_f$ evaluated on each of the vertices of this path.

Case 2. Suppose now that $\langle v, w\rangle \oplus \mathrm{Rad}(H) \to H$ is an embedding but the image is not primitive, then we may assume that $v = e_1$ and $w = e_1 + \lambda u \bmod \mathrm{Rad}(H)$ for some integers λ and primitive element u of H such that $\langle v, u\rangle$ is primitive modulo $\mathrm{Rad}(H)$. Then $e_1, e_1 + \frac{\lambda}{|\lambda|}u, e_1 + 2\frac{\lambda}{|\lambda|}u, \cdots, e_1 + \lambda u$ is a path in $\mathcal{P}(\pi^{-1}(1), \mathrm{Rad}(H))$ and for each of these edges we showed in the previous case what to do.

Case 3. As in the proof of Proposition 1.12.3 the case that v, w are vertices of $\mathcal{A}^o(H)$ but $\langle v, w\rangle \oplus \mathrm{Rad}(H) \to H$ is not an embedding is clear once we have chosen a symplectic basis and remember that $g \geq 4$.

This proves all cases and therefore the proposition.

□

CHAPTER 2

Surfaces

2.1. Introduction

This chapter is about surfaces. We start with introducing the Torelli group, a subgroup of the mapping class group, and explain about the work of Johnson on the Torelli group that he published around 1980. In Section 2.4 we describe the kernel of the map of Torelli groups induced by an closing a hole of the surface. We finish this chapter with recalling the definitions of the arc-complexes associated to a surface, that were introduced by Harer. The mapping class group acts on these complexes and therefore we also have an action of the Torelli group on it. We show that the orbit space under the action of the Torelli group on the arc-complexes is isomorphic to a poset that we defined in Chapter 1.

2.2. The Torelli group

Let $S^n_{g,r}$ denote a compact, connected, oriented surface of genus g with r boundary components and n distinct fixed points chosen on the interior of the surface. We shall associate several groups or spaces to $S^n_{g,r}$, they are indexed by ${}^n_{g,r}$ or ${}_S$. The indices will often be omitted when this can not cause any confusion. We usually omit the index n or r when it is zero. We shall denote the boundary of S by ∂S.

Let $\mathfrak{F}S^n_{g,r}$ be the topological group of orientation preserving homeomorphisms of S that fix each of the n points and are the identity on the boundary of the surface, where the topology is the compact-open topology. The *mapping class group* or *modular group* $\Gamma^n_{g,r}$ of $S^n_{g,r}$ is the group of path components of $\mathfrak{F}S^n_{g,r}$. Up to isomorphism, a surface can be given a unique C^∞-structure and we will sometimes assume such a structure on S. It is well known that every homeomorphism in $\mathfrak{F}S^n_{g,r}$ is isotopic to a diffeomorphism of S.

When γ is a (non-oriented) loop on S without self intersections, also called a *simple closed curve,* and abbreviated by SCC, we associate to it a mapping class D_γ, the so called *left Dehn twist around* γ. The word left in this definition will often be suppressed in the rest of this thesis. It is represented by the map defined by choosing a regular neighborhood of γ, twist this cylinder around clockwise over 2π and extend the map by the identity on the rest of the surface, see Figure 2.1. The Dehn twist D_γ only depends on the isotopy class of γ. Let $p, q \in \partial S$ be on different components and α an arc between them without self intersections. By abuse of

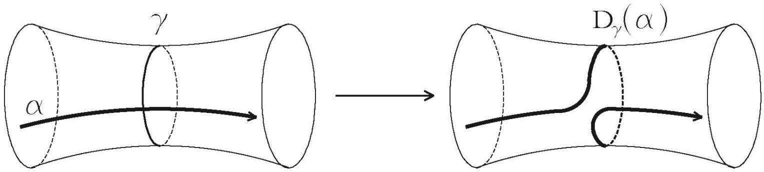

FIGURE 2.1. The effect of the left Dehn twist around γ on α.

notation we write D_α for the Dehn twist around the curve that is the boundary of a regular neighborhood of α and the boundary components containing p and q. The mapping class group is finitely generated by Dehn twists; Wajnryb shows that it is generated by $2g+1$ Dehn twists, with finitely many relations between them, see [Wajnryb]. When $r \geq 1$, we can define a map $\Gamma^n_{g,r} \to \Gamma^{n+1}_{g,r-1}$ by closing a hole ∂_r with a disc and fix a point on the disc. The kernel of this map is the infinitely cyclic group generated by the Dehntwist around a curve isotopic to the boundary curve that is closed. This is because if f is a mapping class in the kernel of this map, we have an isotopy on $S^{n+1}_{g,r-1}$ between f and the identity; by composing this isotopy with a suitable power of D_{∂_r}, we can assume that this isotopy is the identity on a neighborhood of ∂_r and therefore this composition with f is the identity in $\Gamma^n_{g,r}$.

Let (S,P) be a surface with boundary marking (see Definition 1.2.2. The mapping class group acts on the relative homology group $H_1(S,P)$ and because the mapping classes are orientation preserving and the identity on the boundary of the surface, this defines a map

$$\Gamma_S \to \mathrm{Sp}(H_1(S,P), H_1(S)).$$

The image of a Dehn twist around a *SCC* γ is, if we orient γ in the right way, precisely the symplectic transvection $\delta_{[\gamma]}$, determined by the class $[\gamma] \in H_1(S)$. So according to Lemma 1.2.4 the image of Γ_S is exactly $\mathrm{Sp}(H_1(S,P), H_1(S))$.

PROPOSITION-DEFINITION 2.2.1. *Let (S,P) be a surface with a complete boundary marking. The Torelli group T_S of S is the kernel of the representation of Γ_S on $\mathrm{Sp}(H_1(S,P), H_1(S))$. It is independent of the choice of P.*

PROOF. If P' is another complete boundary marking of S then we can choose a subset A of ∂S such that A intersected with each boundary component is an interval meeting a point of P and of P'. We see that $H_1(S,P) \cong H_1(S,A) \cong H_1(S,P')$ as representations of Γ_S. □

We should remark that there exist an ambiguity in the literature about the definition of the Torelli group. In some references one works with the bigger group that is the kernel of the action of Γ_S on $H_1(S)$. We refer to the latter as the *big Torelli group*, $\widetilde{T_S}$. A problem of this definition (or more generally, when one drops the assumption that the boundary marking is complete) is that it not functorial;

when $S \subset S'$ is an inclusion of surfaces, this induces a homomorphism $\Gamma_S \to \Gamma_{S'}$ by extending the mapping class on S by the identity on the rest of the surface of S', but in general this will not restrict to a map on the big Torelli group whenever S has more than one boundary component. A counter example can be the Dehn twist around a boundary curve. With the more refined definition, where we do assume the boundary marking to be complete, we do not encounter this problem, as the next lemma shows.

LEMMA 2.2.2. *An inclusion $S \subset S'$ of surfaces induces a homomorphism $T_S \to T_{S'}$ of Torelli groups. If $H_1(S) \to H_1(S')$ is injective then T_S is the preimage of $T_{S'}$ in Γ_S.*

PROOF. We choose complete boundary markings P of S, P' of S' and R of $\overline{S'-S}$ such that on intersections they agree, see Figure 2.2 The Mayer-Vietoris

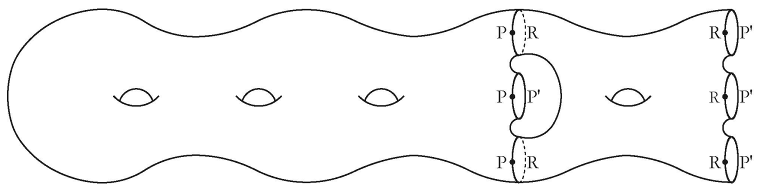

FIGURE 2.2. The surfaces S, S' and $\overline{S-S'}$ with boundary markings.

sequence of a regular neighborhood of the pairs $(\overline{S'-S}, R)$ and (S, P) shows that

$$H_1(\overline{S'-S}, R) \oplus H_1(S, P) \to H_1(S', P' \cup P)$$

is surjective because $H_0(S \cap (\overline{S'-S}), P \cap R) = 0$. The map commutes with the action of Γ_S and since T_S acts trivial on both summands of the domain, the action on $H_1(S', P' \cup P)$ is trivial. This last module contains $H_1(S', P')$ and we see that an element of T_S extends to an element of $T_{S'}$.

Suppose that $H_1(S) \to H_1(S') \subset H_1(S', P')$ is injective. This means that we do not close a hole of S with a disc. Any arc α in S with endpoint in $P \cap R$ can be extended in $S' - S$ to a loop $\beta\alpha$ representing an element in $H_1(S', P')$. If $g \in \mathrm{Im}(\Gamma_S \to \Gamma_{S'}) \cap T_{S'}$ then $[\beta\alpha] = [g(\beta\alpha)] = [\beta g(\alpha)]$ and thus $g|_S \in T_S$ because by assumption $g|_S$ act trivial on $H_1(S)$. So T_S is the preimage of $T_{S'}$ in Γ_S. $\square$

We further remark that if S has at most one boundary component then both definitions of the Torelli group agree. More precisely, the Torelli group and the big Torelli group are related as follows.

PROPOSITION 2.2.3. *If S is a surface, T_S the Torelli group and $\widetilde{T_S}$ the big Torelli group of S, then we have a short exact sequence*

$$1 \to T_S \to \widetilde{T_S} \to S^2\mathrm{Rad}(H_1(S)) \to 0.$$

Proof. This follows from the exact sequence

$$0 \to S^2\mathrm{Rad}(H_1(S)) \to \mathrm{Sp}(H_1(S,P), H_1(S)) \to \mathrm{Sp}(H_1(S)) \to 1$$

and the snake lemma. □

We describe two types of typical elements in T_S. If γ is a simple closed curve on S that is the boundary of a subsurface $S_{k,1}$ of S, then $[\gamma] = 0$ in $H_1(S)$, so D_γ is in T_S. We abbreviate such a curve with $BSCC$, *bounding simple closed curve*, and we refer to D_γ as a $BSCC$-map. The set of $BSCC$-maps such that γ bounds a subsurface of genus k is denoted by $\mathfrak{T}_k$. When S is a closed surface of genus g we see that $\mathfrak{T}_k = \mathfrak{T}_{g-k}$. Let $\mathfrak{T} = \cup_{k=1}^{g}\mathfrak{T}_k$. The second type of element is described as follows. If $\{\gamma_1, \gamma_2\}$ is a pair of oriented SCC's that together bound a subsurface $S_{k,2}$ of S and are oriented such that $[\gamma_1]+[\gamma_2] = 0$ in $H_1(S)$, then the mapping class $D_{\gamma_1}D_{\gamma_2}^{-1}$ is another element in T_S. We abbreviate such a pair with BP, *bounding pair*, and we refer to $D_{\gamma_1}D_{\gamma_2}^{-1}$ by BP-map. The set of BP-maps such that $\{\gamma_1, \gamma_2\}$ bounds a subsurface of genus k is denoted by $\mathfrak{W}_k$. Each of the sets $\mathfrak{T}_k$ and $\mathfrak{W}_k$ is a full conjugate class of the mapping class group. The importance of these maps is explained in the next section.

2.3. The work of Johnson and others on the Torelli group

We give a short overview of some results on the Torelli group and focus only on the work of Johnson from around 1980, the work of Mess on T_2 and of the work of Foisy, because he uses methods similar to those here. We do not at all pretend to be complete.

In this section we assume that the genus of S is ≥ 3 unless otherwise stated. The first article which I found on the Torelli group is that of Birman [Birman3] on Siegel's Modular Group in 1971. She gives here a finite presentation of $\mathrm{Sp}(H)$ where H is a unimodular symplectic $\mathbb{Z}$-module of rank $2g$ with $g \geq 3$. The liftings of the words in $\mathrm{Sp}(H)$ that represent the relations to the mapping class group, are elements of the Torelli group that normally generate it. Powell computes in [Powell] that these lifted relations are of type $\mathfrak{T}_1$, $\mathfrak{T}_2$ and $\mathfrak{W}_1$. Since two such elements are conjugated in Γ if and only if they bound homeomorphic subsurfaces, this implies that $\mathfrak{T}_1$, $\mathfrak{T}_2$ and $\mathfrak{W}_1$ together are a set of generators for $T_{g,0}$. With use of the so called lantern relation, see Figure 2.3, Johnson shows in [Johnson1] that $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are already in $\mathfrak{W}_1$. In [Johnson6] he proves that the Torelli group of a closed surface or a surface with one boundary component is finitely generated. The generators he gives are all BP-maps but not necessarily in $\mathfrak{W}_1$. The proof requires a lot of computations and the remark that a normal subgroup of Γ which is contained in T and contains an element of $\mathfrak{W}_1$ must be T itself. Whether the Torelli group is also finitely presented for $g >> 0$ is not known.

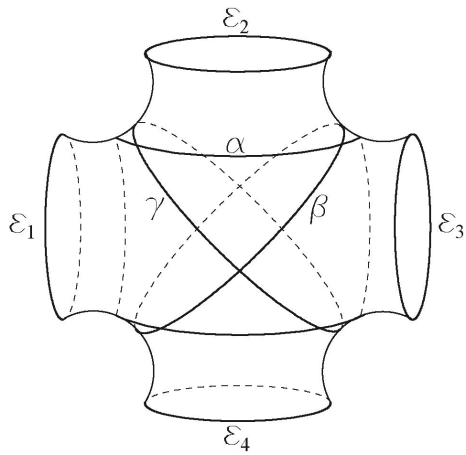

FIGURE 2.3. The lantern relation $D_\gamma D_\beta D_\alpha = D_{\epsilon_1} D_{\epsilon_2} D_{\epsilon_3} D_{\epsilon_4}$.

It was until this last article that the "Torelli group" did not carry a name. Johnson tells us here that an analyst had told him that this group was classically known as the Torelli group. From then on this name has become standard, also for topologists.

In the meantime Johnson had done an extensive study of abelian quotients of T and published a sequence of papers that led to a characterization of $H_1(T)$. We review his results on this now.

Birman and Craggs have produced a number of maps $T_g \to \mathbb{Z}/2$ using the Rochlin invariant for 3-manifolds, see [Birman-Craggs]. In [Johnson2] we find a refined version of these homomorphisms, to describe them we need the following definitions and facts that we recall from that article.

Let W be an oriented $\mathbb{Z}/2$-homology 3-sphere and $h : S_g \to W$ a *Heegaard embedding* of a closed surface. This means that $h(S_g)$ splits W into two handlebodies A and B. We agree that A is on the positive side of $h(S_g)$. The surface $h(S_g)$ is called a *Heegaard surface*. The *Seifert linking form* λ is defined on $H_1(h(S_g), \mathbb{Z}/2)$ by stating that $\lambda(a, b)$ is the modulo 2 linking number of a and b^+, where b^+ is obtained from b by moving it in the positive normal direction. The map $\omega(a) = \lambda(a, a)$ is a quadratic form on $H_1(h(S_g), \mathbb{Z}/2)$ and via h we get a quadratic form ω_h on $H_1(S_g, \mathbb{Z}/2)$. This form has Arf invariant zero, so by definition $\omega_h \in \Psi$ (recall that Ψ denotes the set of quadratic forms of Arf invariant zero). Conversely, every $\omega \in \Psi$ can be realized as the quadratic form of some Heegaard embedding in a $\mathbb{Z}/2$-homology 3-sphere. When $h : S_g \to S^3$ is a Heegaard embedding into the 3-sphere and $k \in T$, we produce a new 3-manifold $M(h, k)$ by splitting S^3 along $h(S_g)$ and glue the boundary by the identification $x \sim hkh^{-1}(x)$ for all $x \in h(S_g)$. The fact that $k \in T$ implies that $M(h, k)$ is a homology 3-sphere again. For any

$\mathbb{Z}/2$-homology 3-sphere we define the *Rochlin-invariant* $\mu \in \mathbb{Z}/2$, see [HNK] for definitions; in case of $M(h,k)$ it is denoted by $\mu(h,k)$. Johnson shows by computing $\mu(h,k)$ on generators of T that $\mu(h,-)$ only depends on ω_h and therefore we have a function $\rho_\omega : T \to \mathbb{Z}/2$ for every $\omega \in \Psi$. Let $\mathfrak{C} = \cap_{\omega\in\Psi} \ker(\rho_\omega)$. The map

$$\sigma : T/\mathfrak{C} \to \{\text{ functions on } \Psi \text{ }\}$$

defined by $\sigma(k)(\omega) = \rho_\omega(k)$ is injective onto $B_3(\Psi)$ and the image of $\mathfrak{T}$ is exactly $B_2(\Psi)$. On generators the map σ is defined as follows.

Let γ be a *BSCC* that bounds a subsurface $S_{k,1}$ and $\{e_i, e_{-i}\}_{i=1}^k$ a symplectic basis of $H_1(S_{k,1}, \mathbb{Z}/2)$, then $\sigma(D_\gamma) = \sum_{i=1}^{k} \overline{e_i e_{-i}}$. Let $\{\gamma_1, \gamma_2\}$ be a *BP* that bounds a subsurface $S_{1,2}$ and $\{e_1, e_{-1}, e_2\}_{i=1}^k$ a symplectic basis of $H_1(S_{1,2}, \mathbb{Z}/2)$ such that $[\gamma_1] = e_2$, then $\sigma(D_{\gamma_1} D_{\gamma_2}^{-1}) = \overline{e_1 e_{-1}}(\overline{e_2} + 1)$.

All this also extends to surfaces with one boundary component and Johnson proves that in that case all quadratic forms occur as self linking forms. We get the isomorphism

$$\sigma : T/\mathfrak{C} \to B_3(\Omega_S)$$

and the image of $\mathfrak{T}$ is $B_2(\Omega_S)$. We will call the homomorphism σ the *Birman-Craggs homomorphism.*

Johnson produced another abelian quotient of $T_{g,1}$ and $T_{g,0}$ in [Johnson3], via what are now called *Johnson homomorphisms.* To define them,let $\pi := \pi_1(S_{g,1})$, $\pi_{[0]} := \pi$ and $\pi_{[m]} := [\pi, \pi_{[m-1]}]$ $(m \geq 1)$, the m^{th} term of the lower central series of π. Let $\pi/\pi_{[m]}$ be the m^{th} nilpotent quotient of π; we will denote $\pi/\pi_{[1]}$ by H. The mapping class group $\Gamma_{g,1}$ acts on $\pi/\pi_{[m]}$, let $\Gamma(m)$ be the kernel of this action. This means that $\Gamma(0) = \Gamma$, $\Gamma(1) = T$ and Johnson proves in [Johnson7] that $\Gamma(2) = \mathfrak{T}$. The *Johnson homomorphisms*

$$\tau_m : \Gamma(m) \to \operatorname{Hom}(H, \pi_{[m]}/\pi_{[m+1]}),$$

are defined by

$$\tau_m(k)(c) = [k(\gamma)\gamma^{-1}],$$

where $c \in H$, γ is a lift of c in π and $[k(\gamma)\gamma^{-1}]$ is the image in $\pi_{[m]}/\pi_{[m+1]}$ of $k(\gamma)\gamma^{-1} \in \pi_{[m]}$. See also [Johnson5], [Hain] and [Morita] for equivalent definitions and more about these maps. We denote τ_1 just by τ. We have the identification $\operatorname{Hom}(H, \pi_{[1]}/\pi_{[2]}) \cong H^* \otimes \wedge^2 H \cong H \otimes \wedge^2 H$ and $\wedge^3 H$, can be embedded via

$$a \wedge b \wedge c \mapsto a \otimes b \wedge c + b \otimes c \wedge a + c \otimes a \wedge b.$$

Johnson shows that the image of T under τ is precisely $\wedge^3 H$ via this identification and τ is $\operatorname{Sp}(H)$-equivariant. The image of a *BP*-map $D_{\gamma_1} D_{\gamma_2}^{-1} \in \mathfrak{W}_k$ induced by an oriented bounding pair $\{\gamma_1, \gamma_2\}$ that bounds a subsurface $S_{k,2}$, is $(\sum_{i=1}^{k} e_i \wedge e_{-i}) \wedge f$, where $\{e_i, e_{-i}\}_{i=1}^k$ is a symplectic basis of a maximal unimodular subspace of $H_1(S')$

and $f = [\gamma_1]$. The $BSCC$-maps generate precisely the kernel of τ. In the final paper [Johnson8] of Johnson in this series he proves that $\overline{\sigma} : T/T^2 \to B_3(\Omega_H)$ is an isomorphism, the elements of $\mathfrak{T}/[T,T]$ are of order 2 and we had seen that the image of this group under $\overline{\sigma}$ is $B_2(\Omega_H)$. Hence we have the short exact sequence

$$0 \to B_2(\Omega_H) \to H_1(T) \to \wedge^3 H \to 0.$$

If S is closed, he proves an analogous result. Let in that case $\theta = \sum_{i=1}^{g} e_i \wedge e_{-i} \in \wedge^2 H_1(S)$ be the fundamental class of S, where $\{e_i, e_{-i}\}_{i=1}^{g}$ is a symplectic basis of $H_1(S)$ (but θ is independent of the choice of this basis). For a closed surface we have the short exact sequence

$$0 \to B_2(\Psi_{H_1(S)}) \to H_1(T) \to \wedge^3 H_1(S)/\theta \wedge H_1(S) \to 0.$$

If $g = 2$ the situation is completely different. Mess shows that T_2 is freely generated by a set of $BSCC$-maps that corresponds one-to-one with the set of splittings of $H_1(S_2)$ into two unimodular symplectic and mutually orthogonal subspaces of rank 2, see [Mess]. To prove this, he uses the period map from the Torelli space, the quotient of the Teichmüller space by the action of the Torelli group, to the Siegel space, which is the space of all symmetric $g \times g$ complex matrices with positive definite imaginary part. When $g = 2$ this is an injection with image the period matrices of abelian varieties of dimension 2 with smooth theta divisor. The complement is the set of period matrices of abelian varieties that are products of two elliptic curves with singular theta divisor. With this description the image can be studied and using Morse theory we can give the Torelli space a cell decomposition with a single 0-cell and a 1-cell for every homology splitting. Because the action of the Torelli group on the contractible Teichmüller space is free, this detects the free generators of T_2.

We also want to mention the work of Foisy, see [Foisy]. A key step on his way of proving his main result that $H_2(\Gamma[2], \mathbb{Q}) \cong \mathbb{Q}$, where $\Gamma[2]$ is the level 2 mapping class group, is that $H_2(ET, \mathbb{Q})$ is finitely generated when $g \geq 3$ and the surface has one boundary component. Here ET is the *extended Torelli group*, the extension of $\mathbb{Z}/2$ by T. The method he uses in proving this is very similar to the method we use in this thesis. He lets ET act on the arc complexes defined by Harer and Ivanov, that we will discuss in Section 2.5. The importance of working with the extended Torelli group is that $H_1(ET, \mathbb{Q}) \cong H_1(T, \mathbb{Q})_{\mathbb{Z}/2} = 0$ by Johnsons result. This will imply that in the associated spectral sequence explained in Section 1.11, $E^2_{1,1}(II) = 0$. This, together with the connectedness of both the arc-system and the quotient by ET, will induce the result that $H_2(ET, \mathbb{Q})$ is finitely generated. The final result that $H_2(\Gamma[2], \mathbb{Q}) \cong \mathbb{Q}$ then follows using representation theory.

Other overviews of work on the Torelli group include [Hain-Looijenga] and [Johnson5].

2.4. Closing a hole of a surface

Let (S', P') be a surface with complete boundary marking and p a point on the interior of S'. We can regard the real oriented blow up of S' in p as a surface $S \xrightarrow{\varphi} S'$ with p replaced by one extra boundary component, because in the blow up process p is replaced by all directions through p. A point q on the boundary component $\varphi^{-1}(p)$ determines a unit tangent vector $v_q \in T_pS'$ and vice versa, a nonzero tangent vector $v \in T_pS'$ determines a point q_v on $\varphi^{-1}(p)$. Let P be a complete boundary marking of S that extends P' and $\partial = \varphi^{-1}(p)$ a connected component of ∂S. We assume that the Euler characteristic of S' is negative. The map φ induces an epimorphism on mapping class groups and restricts to an epimorphism $T_S \to T_{S'}$ of Torelli groups. If S' has at least one boundary component then the map splits by gluing a pair of pants to a boundary component of S'. The kernel of the map $\Gamma_S \to \Gamma_{S'}$ is described in [Johnson6]. We recall this result in this section and we determine the image and kernel of the action of $\mathrm{Ker}(\Gamma_S \to \Gamma_{S'})$ on $H_1(S, P)$.

We introduce some notations. If $\partial \cap P = \{q\}$ and α is an oriented SCC that meets ∂S only in q, we define a mapping class t_α; it has the effect of sliding the hole ∂ along the path α. We choose a regular neighborhood of $\alpha \cup \partial$ and label the boundary components of this neighborhood unequal to ∂ with α_+ and α_-, in such a way that α_+ is on the left of α. Then $t_\alpha = D_{\alpha_+} D_{\alpha_-}^{-1}$, see Figure 2.4.

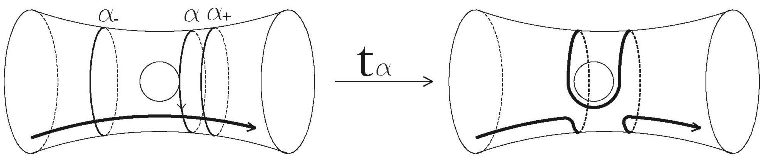

FIGURE 2.4. The effect of the map t_α.

Let US' be the unit tangent bundle of S' and $\widetilde{\pi} := \pi_1(US', v_q)$, $\pi := \pi_1(S', p)$. The loop α determines an element $\tilde{\alpha} \in \widetilde{\pi}$ by translating the vector v_q parallel along the image of α in S'. The mapping class t_α only depends on $\tilde{\alpha}$, so we denote it by $t_{\tilde{\alpha}}$. The image of a loop homotopic to ∂ gives the central element in $\widetilde{\pi}$ that generates the kernel of the projection $\widetilde{\pi} \to \pi$; it is a clockwise rotation of v_q and we denote it again by ∂. The group $\widetilde{\pi}$ is generated by the images $\tilde{\alpha}$ of SCC's α on S through q and the claim is that the correspondence $\tilde{\alpha} \mapsto t_{\tilde{\alpha}}$ gives a well defined isomorphism of $\widetilde{\pi}$ onto $\mathrm{Ker}(\Gamma_S \to \Gamma_{S'})$. A proof of this can be found in [Johnson6], where he uses the inverse map: if we have a mapping class on S that is isotopic to the identity on S' then the orbit of v_p under this isotopy determines an element of $\widetilde{\pi}$. This element is independent of the chosen isotopy because $\pi_1(\mathrm{Diff}^+(S'), id) = \{1\}$ whenever the Euler characteristic of S' is negative, see [Gramain].

If the orientation of the SCC α near q is the same as ∂ and (α_+, α_-) are oriented such that $[\alpha_+] = [\alpha] = a$ and $[\alpha_-] = [\alpha] - [\partial]$ in $H_1(S)$, then the action of

$t_\alpha = D_{\alpha_+} D_{\alpha_-}^{-1}$ on $H_1(S,P)$ is given by

$$(*) \qquad x \mapsto x + (x \cdot a)[\partial] + (x \cdot [\partial])a - (x \cdot [\partial])[\partial].$$

Therefore we can write t_a for the image of $t_{\tilde{\alpha}}$ in $\mathrm{Sp}(H_1(S,P), H_1(S))$. The elements t_a generate the image of $\widetilde{\pi}$ in $\mathrm{Sp}(H_1(S,P), H_1(S))$. We see that when S' is a closed surface then the action is trivial because in that case $[\partial] = 0$. We further remark that the correspondence $a \in H_1(S) \mapsto t_a \in \mathrm{Sp}(H_1(S,P), H_1(S))$ is not a homomorphism:

$$t_a t_b(x) = x + (x \cdot (a+b))\partial + (x \cdot \partial)(a+b) - 2(x \cdot \partial)\partial + (x \cdot \partial)(b \cdot a)\partial = t_{a+b} D_\partial^{-1-(a \cdot b)}(x)$$

so that $[t_a, t_b] = D_\partial^{-2(a \cdot b)}$. We see that if α and β are both SCC's with the agreed orientation then the composition $\alpha\beta$ is homotopic to a SCC with the agreed orientation if and only if $a \cdot b = -1$.

Proposition 2.4.1. *Let $\pi : S \to S'$ be the real oriented blow up of a surface S' in a point $p \in S'$ and let $q \in \varphi^{-1}(p)$. Assume that the Euler characteristic of S' is negative. Then* $\ker(\Gamma_S \to \Gamma_{S'}) \cong \pi_1(US', v_q)$*. If the surface S' is closed, then* $\ker(T_S \to T_{S'}) \cong \pi_1(US', v_q)$*. If the boundary of S' is nonempty then* $\ker(T_S \to T_{S'})$ *can be identified with* $[\pi_1(S', p), \pi_1(S', p)]$ *via*

$$[\alpha, \beta] \mapsto [t_\alpha, t_\beta] D_\partial^{2(a \cdot b)}$$

whenever α, β are represented by SCC's.

Proof. We remarked that if S' is closed then the action is trivial. So assume that S' has at least one boundary component. Define $\tilde{\kappa} := \ker(T_S \to T_{S'})$. We first compute the kernel κ of the action of $\widetilde{\pi}$ on $H_1(S', P' \cup \{p\}) \cong H_1(S,P)/\langle \partial \rangle$. Let $\tilde{\alpha} \in \tilde{\pi}$. From $(*)$ we see that $t_{\tilde{\alpha}}$ acts on this group via $t_a(x) = x + (x \cdot p)a$ where a is the image of $\tilde{\alpha}$ in $H_1(S')$ and that the map $a \mapsto t_a$ is a homomorphism, $t_a t_b = t_{a+b}$. We conclude that $H_1(S')$ acts faithfully on $H_1(S', P')$. So κ contains $[\tilde{\pi}, \tilde{\pi}] = [\pi, \pi]$ and the central element ∂. In fact $\kappa = [\pi, \pi] \times \langle \partial \rangle$ since

$$H_1(S') \cong \frac{\pi}{[\pi, \pi]} \cong \frac{\widetilde{\pi}}{[\pi, \pi] \times \langle \partial \rangle}.$$

It contains $\tilde{\kappa}$ and the inclusion of the central element gives an isomorphism $\langle \partial \rangle \cong ([\pi, \pi] \times \langle \partial \rangle)/\tilde{\kappa}$. This means that we have a map $[\pi, \pi] \times \langle \partial \rangle \to \langle \partial \rangle$ given by $(g, h) \mapsto f(g)h$ which is constant e on $\tilde{\kappa}$. So $\tilde{\kappa}$ is the graph of a homomorphism $[\pi, \pi] \to \langle \partial \rangle$, that factorizes over a skew-symmetric form $H_1(S) \times H_1(S) \to \mathbb{Z}$ and we saw that it is defined by $(a, b) \mapsto 2(a \cdot b)$. $\square$

Notice that the map $[\pi, \pi] \to T_{\widehat{S}}$ commutes with the action of $\Gamma_{\widehat{S}}$. We show that the short exact sequence

$$1 \to [\pi, \pi] \to T_{\widehat{S}} \to T_S \to 1$$

implies that $T_{\widehat{S}}$ is generated by the elements of type $\mathfrak{W}_1$, $\mathfrak{T}_1$ if $g \geq 3$ and by $\mathfrak{W}_1$, $\mathfrak{T}_1$ and $\mathfrak{T}_2$ if $g = 2$. Johnson proved that $T_{g,1}$ $(g \geq 3)$ is finitely generated. For $T_{g,r}$

with $r \geq 2$ we do not know whether this is the case. In the following proposition we only use that $T_{g,1}$ is generated by $\mathfrak{W}_1$ (see overview Section 2.3).

PROPOSITION 2.4.2. *Let S be a surface with $g(S) \geq 1$. If $g(S) = 2$ then T_S is generated by the elements of type $\mathfrak{T}_1$, $\mathfrak{T}_2$ and $\mathfrak{W}_1$. Otherwise, T_S is generated by the elements of type $\mathfrak{T}_1$ and $\mathfrak{W}_1$.*

PROOF. If S is a closed torus, T_S is trivial and if S is a torus with one boundary component, T_S is infinitely free generated by the Dehn twist around the boundary curve (see Section 3.3). So in that case the proposition holds. If S is closed and $g(S) = 2$, the proposition is true by Mess. If S has one boundary component and $g(S) = 2$, we have the short exact sequence in the notation of Proposition 2.4.1

$$1 \to \tilde{\pi} \to T_S \to T_{S'} \to 1.$$

The Mess-generators are of type $\mathfrak{T}_1$ and lift to elements of the same type in T_S. For $\tilde{\pi}$ we can choose generators such that they map to the Dehn twist around the boundary, which is in $\mathfrak{T}_2$, or to elements in $\mathfrak{W}_1$. If $g \geq 3$ and S is closed or has connected boundary, the proposition is true by Johnson, see [Johnson1].

We proceed with induction on the number of boundary components r and assume that $r \geq 2$. In that case S is the real oriented blow up in a point of a surface S' as before, so we have by Proposition 2.4.1 the exact sequence

$$1 \to [\pi, \pi] \to T_S \to T_{S'} \to 1.$$

The generators of $T_{S'}$ of type $\mathfrak{W}_1$, $\mathfrak{T}_1$ (and $\mathfrak{T}_2$ if $g(S) = 2$) lift to elements in T_S of the same type.

Let $\alpha_1, \alpha_{-1}, \ldots, \alpha_g, \alpha_{-g}, \alpha_{g+1}, \ldots, \alpha_{g+r-2}$ be a set of generators of π that are represented by $SCC's$ on S', disjoint outside the basepoint p and such that their images in $H_1(S')$ form a symplectic basis of $H_1(S')$.Then $[\alpha_i, \alpha_{-i}]$ maps to $t_\gamma D_\partial^{-1} = D_{\gamma_-}^{-1} D_{\gamma_+} D_\partial^{-1}$, where γ is a SCC homotopic to $[\alpha_i, \alpha_{-i}]$ and ∂ is the boundary of the hole that is closed in S', see Figure 3.4 in Chapter 3. Then $D_{\gamma_-}^{-1} \in \mathfrak{W}_1$ and $D_{\gamma_+} D_\partial^{-1} \in \mathfrak{T}_1$. In general, if $j \in \{1, \ldots, g+r-2, -1, \ldots, -g\}$, we have $[\alpha_i, \alpha_j] = [\alpha_i, \alpha_j \alpha_{-i}] \alpha_j [\alpha_{-i}, \alpha_i] \alpha_j^{-1}$. Then $\alpha_j [\alpha_{-i}, \alpha_i] \alpha_j^{-1}$ maps to a conjugate of the image of $[\alpha_{-i}, \alpha_i]$ so is in the group generated by $\mathfrak{W}_1$ and $\mathfrak{T}_1$. If $\alpha_j \alpha_{-i}$ is homotopic to a SCC (if not, replace α_{-i} by α_{-i}^{-1}), then $[\alpha_i, \alpha_j \alpha_{-i}]$ is in the same orbit as $[\alpha_i, \alpha_{-i}]$ under the action of $\Gamma_{\widehat{S}}$, so maps to an element in $\mathfrak{W}_1 \mathfrak{T}_1$. Since $[\pi, \pi]$ is normally generated by these elements, this proves the proposition. □

We show that the maps $\tau : T_{g,1} \to \wedge^3 H_1(S_{g,1})$ and $\sigma : T_{g,1} \to B_3(\Omega_{g,1})$ defined by Johnson and Birman and Craggs (see the overview in Section 2.3), can be extended to maps defined on Torelli groups of surfaces S with an arbitrary number of boundary components. The images are $\wedge^3 H_1(S)$ and $B_3(\Omega_S)$ respectively. For τ this is also done in [Johnson7].

Let $S = S_{g,r}$ be a surface and $S'' = S_{g+r-1,1}$ with $g \geq 3$ or we have $g \geq 2$ and $r \geq 2$. We embed $S \subset S''$ by gluing a surface $S_{0,r+1}$ to S. This inclusion induces a map on Torelli groups $T_S \to T_{S''}$, composition with $\tau_{S''}$, $\sigma_{S''}$ give homomorphisms

$$\tau_S : T_S \to T_{S''} \to \wedge^3 H_1(S''),$$
$$\sigma_S : T_S \to T_{S''} \to B_3(\Omega_{S''}).$$

PROPOSITION 2.4.3. *The image of T_S under τ_S is $\wedge^3 H_1(S)$, the image of T_S under σ_S is $B_3(\Omega_S)$.*

PROOF. By Proposition 2.4.2 we know that T_S is generated by the elements of type $\mathfrak{W}_1$, $\mathfrak{T}_1$, and $\mathfrak{T}_2$ if $g = 2$. The images of these elements are obviously in $\wedge^3 H_1(S)$ respectively in $B_3(\Omega_S)$ (see section 2.3 for a formula of these images), and $\wedge^3 H_1(S)$, $B_3(\Omega_S)$ are both generated by the images of type $\mathfrak{W}_1$ and $\mathfrak{T}_1$. □

2.5. The arc-complexes of Harer

Let $\Lambda \subset \partial S$ be a finite set (different points do not necessarily lie on distinct components) and let $\Lambda^0 \subsetneq \Lambda$ be a proper subset. A Λ-*arc* α is an isotopy class of a C^∞-embedded path in S with endpoints in Λ. We say that it is *nontrivial* if it is not homotopic (relative the endpoints) to an arc in $(\partial S - \Lambda) \cup \partial\alpha$. A (Λ, Λ^0)-*arc* is a Λ-arc with one endpoint in Λ^0 and the other in $\Lambda - \Lambda^0$. A family $\{\alpha_0, \dots, \alpha_k\}$ of nontrivial (Λ, Λ^0)-arcs that have the property that they can be represented by disjoint arcs, except that they may intersect at their endpoints, is called *a* (Λ, Λ^0)-*arc system of height* k. We define the *arc-complex* $BX(\Lambda, \Lambda^0)$ defined by Harer.

DEFINITION 2.5.1. *Let $BX(\Lambda, \Lambda^0)$ be the simplicial complex with k-simplices the (Λ, Λ^0)-arc systems $\{\alpha_0, \dots, \alpha_k\}$ of height k on S that can be represented by a $(k+1)$-tuple of embedded arcs, which are disjoint away from the endpoints and whose complement in S is connected. When $\Lambda = \{p, q\}$ and $\Lambda^0 = \{p\}$ we write $BX(p, q)$ instead and in this case we orient the arcs from p to q. If p and q are on the same boundary component we refer to it as the 1-component case, when they are on different components we say that we are in the 2-component case.*

Notice that if α is an arc with both endpoints on the same boundary component, then the surface obtained from S by removing α has genus $g(S) - 1$ and the number of boundary components has increased by one. If on the other hand, α is an arc with endpoints on different components, then the genus remains unchanged if we remove α from S and the number of boundary components has decreased by one. We deduce with induction that the dimension of $BX(\Lambda, \Lambda^0)$ is $2g - 2 + r$, where r is the number of boundary components containing a point of Λ. Harer proved in [Harer], Theorem 1.4, the following important theorem.

THEOREM 2.5.2. *The complex $BX(\Lambda, \Lambda^0)$ is spherical of dimension $2g - 2 + r$.*

We orient the boundary of S such that S is on the left of ∂S. The vertices of a k-simplex of $BX(p,q)$ inherit an ordering via this orientation, as they all depart at p; their order of arrival at q determines a permutation of $\{0,\dots,k\}$. We will denote an ordered k-simplex by $\alpha = (\alpha_0,\dots,\alpha_k)$ and the induced permutation by π_α. We remark that if $k \leq g-1$ then every permutation occurs, if $k \geq g$ this is not the case. The mapping class group acts on the simplicial complex $BX(\Lambda,\Lambda^0)$. By the classification of surfaces we have that two arc-systems of $BX(p,q)$ of the same height are in the same orbit if and only if they determine the same permutation, see [Harer], Lemma 3.2. The induced action of the Torelli group preserves the homology classes in $H_1(S,\{p,q\})$ determined by the arc-system. Conversely, if two arc-systems represent the same ordered sequence of elements in $H_1(S,\{p_0,p_1\})$ and determine the same permutation, then they are in the same orbit of the action of T_S, see [Foisy], Lemma 3.2.

If σ is a simplex of $BX(p,q)$, we denote by T_σ the stabilizer of σ in T and by S_σ the surface obtained by cutting S along arcs that represent the vertices of σ and are disjoint away from the endpoints. In the 1-component case, we denote by γ_1 that part of the boundary going from p to q. In the 2-component case, let $\widetilde{S} = S_{g+1,r-1}$ be a surface obtained from S by gluing a pair of pants $S_{0,3}$ to the two boundary components that contain the points p and q. We fix an arc γ_2 on this pair of pants that connects q with p. Let $[\partial_0] \in H_1(\widetilde{S})$ be the homology class of the boundary component containing p and let $\pi : H_1(\widetilde{S}) \to \mathbb{Z}$ be defined by $v \mapsto [\partial_0]\cdot v$. We denote by $\overline{BX}(p,q)$ the quotient space of the action of the Torelli group on $BX(p,q)$. Define two maps,

$$\varphi_1 : BX(p,q) \to \mathcal{O}^o(H_1(S))$$

in the 1-component case and

$$\varphi_2 : BX(p,q) \to \mathcal{O}^o(\pi^{-1}(1))$$

in the 2-component case, by

$$\varphi_i((\alpha_0,\dots,\alpha_k)) := ([\gamma_i\alpha_0],\dots,[\gamma_i\alpha_k])$$

for $i=1,2$. Then φ_1 and φ_2 factorize over $\overline{BX}(p,q)$. The next proposition identifies $\overline{BX}(p,q)$ with a subposet of the codomain.

PROPOSITION 2.5.3. *The map φ_1 factorizes over an isomorphism*

$$\overline{\varphi_1} : \overline{BX}(p,q) \to \mathcal{A}^o(H_1(S))$$

in the 1-component case and in the 2-component case, φ_2 factorizes over an isomorphism

$$\varphi_2 : \overline{BX}(p,q) \to \mathcal{A}^o(H_1(\widetilde{S}),\pi).$$

PROOF. Let $(\alpha_0, \dots, \alpha_k)$ be a k-simplex of $BX(p,q)$. We first show that the arc-system maps to an element of $\mathcal{P}^o(H_1(S), \mathrm{Rad}(H_1(S)))$ in the 1-component case and of $\mathcal{P}^o(\pi^{-1}(1), \mathrm{Rad}(\pi^{-1}(0)))$ in the 2-component case; here we have

$$\mathrm{Rad}(\pi^{-1}(0)) \cong \mathrm{Rad}(H_1(S)) \cong \langle \partial_0 \rangle \oplus \mathrm{Rad}(H_1(\widetilde{S})).$$

The conditions of an arc-sequence are then easily checked.

When $k = 0$ this is clearly the case. We proceed with induction on k. Let $i = 1, 2$. Since the complement of the arc-sequence in S is connected, we can find for α_0 an element $(\beta_0) \in BX(p,q)$ such that $[\gamma_i\beta_0] \cdot [\gamma_i\alpha_0] = 1$ and $[\gamma_i\beta_0] \cdot [\gamma_i\alpha_j] = 1$ if $[\gamma_i\alpha_0] \cdot [\gamma_i\alpha_j] = 1$ but $[\gamma_i\beta_0] \cdot [\gamma_i\alpha_j] = 0$ otherwise, as follows. We take an arc disjoint away from the endpoints with the other arcs, starting from the left of α_0 and arriving on the right of α_0, between α_0 and the next one to arrive. Because S remains connected after cutting it along the arcs, such an arc exists. Let

$$\lambda_0[\gamma_i\alpha_0] + \cdots + \lambda_k[\gamma_i\alpha_k] + r = \lambda v$$

for some $\lambda_0, \dots, \lambda_k, \lambda \in \mathbb{Z}$, $r \in \mathrm{Rad}(H_1(S))$ and and in the 1-component case $v \in H_1(S)$; in the 2-component case $v \in H_1(\widetilde{S})$. Taking the product with $[\gamma_i\beta_0] - [\gamma_i\alpha_0]$ shows that $\lambda|\lambda_0$, so with induction we find that $\lambda|\lambda_j$ and $\lambda|r$. This proves that the maps are well-defined.

For every arc-sequence $a = (a_0, \dots, a_k)$ we define a permutation π_a of $\{0, \dots k\}$. It will have the property that for all $i < j$ we have that $a_i \cdot a_j = 0$ if and only if $\pi_a(i) < \pi_a(j)$, and if $a = \varphi_i(\alpha)$ then $\pi_a = \pi_\alpha$. If $k = 0$ then $\pi_a(0) = 0$. Assume that $k \geq 1$. Let $\tilde{a} := (a_0, \dots, a_{k-1})$ and suppose with induction that $\pi_{\tilde{a}}$ is defined with the above property.

CLAIM 2.5.4. *If* $a_{\pi_{\tilde{a}}^{-1}(i)} \cdot a_k = 1$ *then* $a_{\pi_{\tilde{a}}^{-1}(j)} \cdot a_k = 1$ *for all* $0 \leq i < j \leq k-1$.

PROOF. If $\pi_{\tilde{a}}^{-1}(i) < \pi_{\tilde{a}}^{-1}(j)$ then $a_{\pi_{\tilde{a}}^{-1}(i)} \cdot a_{\pi_{\tilde{a}}^{-1}(j)} = 0$ so by property (ii) of the definition of an arc-sequence (see p. 9) we know that $a_{\pi_{\tilde{a}}^{-1}(i)} \cdot a_k = 1$. If on the other hand $\pi_{\tilde{a}}^{-1}(i) > \pi_{\tilde{a}}^{-1}(j)$ then $a_{\pi_{\tilde{a}}^{-1}(j)} \cdot a_{\pi_{\tilde{a}}^{-1}(i)} = 1$ and by property (iii) of the definition of an arc-sequence we know that $a_{\pi_{\tilde{a}}^{-1}(j)} \cdot a_k = 1$. This proves the claim. □

From the claim it follows that there is a unique $i_a \in \{0, \dots, k\}$ such that for all $i < k$ we have $a_{\tilde{\pi}^{-1}(i)} \cdot a_k = 1$ if and only if $i \geq i_a$. We define

$$\begin{aligned} \pi_a(i) &:= \pi_{\tilde{a}}(i) && \text{if} \quad \pi_{\tilde{a}}(i) < i_a, \\ \pi_a(k) &:= i_a, \\ \pi_a(i) &:= \pi_{\tilde{a}}(i) + 1 && \text{if} \quad \pi_{\tilde{a}}(i) \geq i_a. \end{aligned}$$

We check that π_a satisfies the condition that for $i < j$ we have that $a_i \cdot a_j = 0$ if and only if $\pi_a(i) < \pi_a(j)$. If $j < k$ then $a_i \cdot a_j = 0$ if and only if $\pi_{\tilde{a}}(i) < \pi_{\tilde{a}}(j)$ and this is equivalent to $\pi_a(i) < \pi_a(j)$ by the above definition. If $j = k$ then $a_i \cdot a_k = 0$ if and only if $\tilde{\pi}(i) < i_a$ and this is equivalent to $\pi_a(i) < \pi_a(k)$. By construction we see that if a is the image of an arc-system then π_a is the same as the permutation determined by the arc-system. This implies that the maps φ_1 and φ_2 are injective. We prove that they are also surjective. For this it is enough to see that the permutations associated to the arc-sequences also occur as permutations coming from the arc-systems. This is because if a, b are arc-sequences inducing the same permutation, then $\langle a\rangle \cong \langle b\rangle$ and therefore in the 1-component case we can find an element in $\mathrm{Sp}(H_1(S))$ mapping a onto b. In the 2-component case we know that $(a, [\partial_0])$ and $(b, [\partial_0])$ are arc-sequences in $\mathcal{A}^o(H_1(\widetilde{S}))$ inducing the same permutation, so we can find an element f in the stabilizer of $[\partial_0]$ in $\mathrm{Sp}(H_1(\widetilde{S}), H_1(S))$ mapping a onto b. This shows that if a and b are arc-sequences inducing the same permutation, then they are in the same orbit under the induced action of Γ_S.

For $k = 0$ we can clearly lift (a_0) to an arc in $BX(p, q)$. Suppose that we know that this is the case for all permutations of $\{0, \dots, i\}$ with $i \leq k-1$. Let π_a be the permutation induced by $a \in \mathcal{A}^o(H)_k$ or $\mathcal{A}^o(H_1(\widetilde{S}), \pi)_k$. Then we know that if $\tilde{a} = (a_0, \dots, a_{k-1})$, then $\pi_{\tilde{a}}$ is induced by an arc-system $\tilde{\alpha} = (\alpha_0, \dots, \alpha_{k-1})$. Let $S_{\tilde{\alpha}}$ be the closure of the surface obtained by removing $\alpha_0, \dots, \alpha_{k-1}$. The integer i_a determines points p', q' on $\partial S_{\tilde{\alpha}}$ such that there is an embedded arc α_k from p' to q' on $S_{\tilde{\alpha}}$ with the property that the complement remains connected, since otherwise $\langle a_0, \dots a_{k-1}\rangle + \mathrm{Rad}(H_1(S)) \cong H_1(S)$ or $\langle a_0, \dots, a_{k-1}\rangle + \mathrm{Rad}(H_1(S)) \cong H_1(\widetilde{S})$ in the 2-component case. The arc system $(\alpha_0, \dots, \alpha_k)$ has permutation π_a.

$\square$

CHAPTER 3

The abelianization of the Torelli group

3.1. Introduction

In this chapter we study the abelianization of the Torelli group. For a surface $S_{g,1}$ with $g \geq 3$, Johnson has computed that

$$H_1(T_S) \cong \wedge^3 H_1(S) \oplus B_2(\Omega_S),$$

using the Johnson homomorphism $\tau : T_S \to \wedge^3 H_1(S)$ and the Birman-Craggs homomorphism $\sigma : T_S \to B_3(\Omega_S)$. For lower genera, the only known nontrivial result is that of Mess which says that $T_{2,0}$ is infinitely free generated by Dehn twists around separating curves that are in one-to-one correspondence with the homology splittings of $H_1(S)$.

We show that the result of Johnson holds for surfaces of genus $g \geq 3$ having an arbitrary number of boundary components. The method we use is different from the one Johnson uses and this gives an alternative proof of his result. The outline of the proof is as follows.

We choose two points p, q on ∂S, if ∂S is not connected we may choose them on different components. The Torelli group acts on the arc-complex $BX := BX(p, q)$ and the stabilizer of a vertex is a Torelli group of a surface of lower genus in the 1-component case, or with fewer boundary components in the 2-component case. In any case S is obtained from a subsurface by gluing a pair of pants to it. Harer shows that the arc-complex BX is spherical, of dimension $2g-1$ in the 1-component case and of dimension $2g$ in the 2-component case. The quotient by the action of the Torelli group, that we denote by $\overline{BX}$, is a poset isomorphic to the poset of arc-sequences introduced in Section 1.5. We proved in Section 1.10 that they are $(g-2)$-connected when p, q are on the same component, and 1-connected otherwise. This means that when $g \geq 4$ we get in the 1-component case by a spectral sequence argument as discussed in Section 1.11 an isomorphism

$$H_1(T_S) \cong H_0(\overline{BX}, \mathcal{H}_1).$$

Hence with induction we can compute $H_1(T_S)$, since $H_0(\overline{BX}, \mathcal{H}_1)$ is completely described in terms of the stabilizers of vertices and edges. When $g = 3$ we get by this spectral sequence an epimorphism

$$H_0(\overline{BX}, \mathcal{H}_1) \to H_1(T_S)$$

which also allows us to compute $H_1(T_S)$. In order to let the induction start we need to know more about the abelianization of Torelli groups of surfaces of low genera. We did not succeed in computing them all but we learned enough about them to let the induction begin.

When $g = 0$ we describe T_S in terms of a colored braid group P; it turns out that $T \cong [P, P]$. Because there is a presentation of P having all the relations in the commutator subgroup, we can give a finite presentation of $H_1(T)$ as a module over the group ring $\mathbb{Z}[P_{\text{ab}}]$.

When $g = 1$ it is easily seen that $T_{1,0} = \{1\}$ and $T_{1,1}$ is infinitely cyclic, generated by the Dehn twist around the boundary curve. We give a presentation of $H_1(T_{1,2})$ using the exact sequence

$$1 \to [\pi, \pi] \to T_{1,2} \to T_{1,1} \to 1$$

of Section 1.11. For $T_{1,r}$ with $r \geq 3$ the computation becomes more complicated this way.

When $g = 2$ we know $H_1(T_{2,0})$ by the result of Mess. Using the short exact sequence

$$1 \to \tilde{\pi} \to T_{2,1} \to T_{2,0} \to 1$$

we compute $H_1(T_{2,1})$. For $H_1(T_{2,2})$ we use the complex $BX(p,q)$ where p, q are on different boundary components. We know that the quotient $\overline{BX}$ is 1-connected and therefore we have by Lemma 1.11.1 the exact sequence

$$H_2(\overline{BX}) \to H_0(\overline{BX}, \mathcal{H}_1) \to H_1(T_S) \to 0,$$

which enables us to describe $H_1(T_S)$ as a quotient of a group that we can compute. Since we do not know the image of $H_2(\overline{BX})$ in $H_0(\overline{BX}, \mathcal{H}_1)$ we cannot compute $H_1(T_S)$.

When $S = S_{3,1}$ we use the spherical arc-complex with both points on one boundary component of S; then $\overline{BX}$ is 1-connected, so we get again the above exact sequence. We cannot compute $H_0(\overline{BX}, \mathcal{H}_1)$ since the stabilizer of a vertex is $T_{2,2}$ but we can bound this group from above and hence $H_1(T_S)$. This bound turns out to be the same as the lower bound for $H_1(T_S)$ that one gets via the Johnson homomorphism and the Birman-Craggs homomorphism.

When $g = 3$ and $r \geq 2$ we choose p, q on different components; then $\overline{BX}$ is 1-connected and therefore we have the exact sequence

$$H_2(\overline{BX}) \to H_0(\overline{BX}, \mathcal{H}_1) \to H_1(T_S) \to 0.$$

We can compute $H_0(\overline{BX}, \mathcal{H}_1)$ since the stabilizer of a vertex is $T_{3,1}$ and of an edge is $T_{2,2}$ and we know enough of the latter to see which identifications we get in $H_0(\overline{BX}, \mathcal{H}_1)$. We use the Johnson homomorphism and the Birman-Craggs homomorphism to show that $H_1(T_S) \cong H_0(\overline{BX}, \mathcal{H}_1)$.

When $g \geq 4$ and $r \geq 1$ we choose the points p, q on the same component, then $\overline{BX}$ is 2-connected, so we get an isomorphism

$$H_0(\overline{BX}, \mathcal{H}_1) \cong H_1(T_S)$$

and we have enough information to compute $H_0(\overline{BX}, \mathcal{H}_1)$.

3.2. Genus zero

If the genus of the surface is zero, we will see that the mapping class group $\Gamma^n_{0,1}$ is isomorphic to $P^n(D^2)$, the colored braid group on n strings of the disc. Of the latter we have a presentation. This isomorphism makes it possible to express the Torelli group in terms of this braid group and in Theorem 3.2.2 we show that $T_{0,r}$ is isomorphic to $[P^{r-1}(D^2), P^{r-1}(D^2)]$. For $r \geq 4$ this group is not finitely generated, but since the relations in the presentation of $P^{r-1}(D^2)$ are all in the commutator subgroup, we can give a finite presentation of $H_1(T)$ as a module over $\mathbb{Z}[P^{r-1}(D^2)_{ab}]$, as we do in Corollary 3.2.8.

We start with the relation between the mapping class group and the pure braid group. A reference is [Birman4].

Let S be a surface and $F^n(S)$ the configuration space of pairwise distinct points $(z_1, \ldots, z_n) \in \Pi^n_{i=1} S$. Let $p \in F^n(S)$, the *pure braid group* $P^n(S)$ *of* S *on* n *strings* is defined by

$$P^n(S) := \pi_1(F^n(S), p).$$

Because $F^n(S)$ is connected this does not depend on the choice of the basepoint p. Recall that $\mathfrak{F}S^n_{g,r}$ is the set of orientation preserving homeomorphisms of $S^n_{g,r}$ that are the identity on ∂S and fix the n points. The evaluation map $\epsilon : \mathfrak{F}S_{g,r} \to F^n S$, defined by

$$\epsilon(f) := (f(p_1), \ldots, f(p_n)),$$

relates the mapping class group of S with the pure braid group of this surface. It is a locally trivial fibration with fiber $\mathfrak{F}S^n_{g,r}$ and the tail of the long exact sequence of homotopy groups is

$$\ldots \to \pi_1(\mathfrak{F}S_{g,r}) \to P^n(S_{g,r}) \to \pi_0(\mathfrak{F}S^n_{g,r}) \to \pi_0(\mathfrak{F}S_{g,r}) \to 1.$$

When S is the disc D^2 this comes down to

$$\pi_1(\mathfrak{F}D^2) \to P^n(D^2) \to \Gamma^n_{0,1} \to \Gamma_{0,1} = 1.$$

We show that $\pi_1(\mathfrak{F}D^2) = 1$ using a homotopy given in [Birman4]. Suppose that $\{f_t\}_{t\in[0,1]}$ is a continuous family of homeomorphisms $f_t : D^2 \to D^2$ such that $f_t|_{\partial D^2}$ is the identity and $f_0 = f_1$ is the identity on D^2. We write $f_t(r,\theta) = (R_t(r,\theta), \Theta_t(r,\theta))$ in polar coordinates and extend f_t to $\mathbb{R}^2$ by the identity outside D^2. We show that the loop f_t is homotopic to the identity by shrinking the disc on which f_t is not the identity to the origin. For $s \in (0,1]$ we define

$H(s,t)(r,\theta) := (sR_t(r/s,\theta), \Theta_t(r/s,\theta))$. As for $(s,t) \to (0,\tilde{t})$ for any $\tilde{t} \in [0,1]$, the limit of $H(s,t)$ is the identity on D^2, we can extend H continuously to $s = 0$ by $H(0,t)(r,\theta) := (r,\theta)$. This shows that $\{f_t\}$ is homotopic to the constant loop, hence we have proved the following lemma.

LEMMA 3.2.1. *For every $n \geq 0$ is $\Gamma^n_{0,1} \cong P^n(D^2)$.*

Originally, Artin gave in [Artin] a geometrical definition of the group of braids of n-strings embedded in $\mathbb{R}^3$ up to isotopy. They correspond as follows. If $[\gamma] \in P^n(D^2)$ is represented by the loop $\gamma = (\gamma_1, \ldots, \gamma_n) : [0,1] \to \prod_{i=1}^n D^2$ then the graphs of $\gamma_1, \ldots, \gamma_n$ are disjoint in $[0,1] \times D^2$ and their union represents a braid in $\mathbb{R}^3$. We give a well known presentation of $P^n(D^2)$ in Theorem 3.2.3 and in Corollary 3.2.4 we rewrite this presentation in a somewhat simpler way.

For every pair $i, j \in \{1, \ldots, n\}$ let γ^{ij} be the arc $[0,1] \to F^n(D^2)$ with coordinates $\gamma^{ij}_1, \ldots, \gamma^{ij}_n$, where γ^{ij}_k is constant for $k \neq i, j$, γ^{ij}_i is an arc from p_i to p_j and γ^{ij}_j an arc from p_j to p_i such that the composition $\gamma^{ij}_j \gamma^{ij}_i$ of arcs is a loop starting at p_i, that bounds a disc counterclockwise and does not enclose any point p_k. Let D_{ij} be the element in $P^n(D^2)$ represented by the loop $(\gamma^{ij})^2$, so $D_{ij} = D_{ji}$, see Figure 3.1.

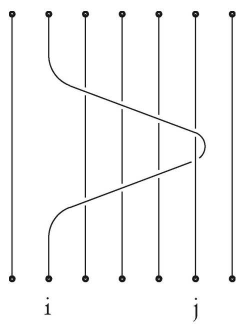

FIGURE 3.1. The element D_{ij} as geometric braid.

PROPOSITION 3.2.2. *The group $T_{0,0} = 1$ and for all $r \geq 1$ is*

$$T_{0,r} \cong [P^{r-1}(D^2), P^{r-1}(D^2)].$$

PROOF. Since $\Gamma_{0,0} = 1$ we have that $T_{0,0} = 1$.

Let $S = S_{0,r}$ and P be a complete boundary marking of S then

$$\mathrm{Sp}(H_1(S,P), H_1(S)) \cong S^2 H_1(S).$$

Let $H_1(S)$ be generated by $\epsilon_1, \ldots, \epsilon_{r-1}$, then

$$(P^{r-1}(D^2))_{ab} \cong H_1(F^{r-1}(D^2)) \cong S^2 H_1(S)/\langle \epsilon_i \otimes \epsilon_i : i = 1, \ldots, r-1 \rangle,$$

where each D_{ij} maps to the cycle in $F^{r-1}(D^2)$ around the hyperplane $z_i = z_j$ that corresponds to $\epsilon_i \otimes \epsilon_j + \epsilon_j \otimes \epsilon_i$ in $S^2 H_1(S)/\langle \epsilon_i \otimes \epsilon_i : i = 1, \ldots, r-1\rangle$. We number the r boundary components of $S_{0,r}$; we close the first $r-1$ holes with a disc and choose a point on each of the discs. By the previous lemma we get a map

$$\Gamma_{0,r} \to \Gamma_{0,1}^{r-1} \overset{\cong}{\to} P^{r-1}(D^2).$$

As we have remarked in Section 2.2, the kernel of this map is the free abelian group generated by the Dehn twist around the boundary curves. The following commuting exact diagram shows that $T_{0,r}$ maps isomorphicly onto the commutator subgroup of $P^{r-1}(D^2)$

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 1 & \to & T_{0,r} & \to & [P^{r-1}(D^2), P^{r-1}(D^2)] & \to & 1 \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \mathbb{Z}^{r-1} & \to & \Gamma_{0,r} & \longrightarrow & P^{r-1}(D^2) & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & \mathbb{Z}^{r-1} & \to & S^2H_1(S) & \longrightarrow & (P^{r-1}(D^2))_{ab} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

This proves the proposition. □

The following presentation of $P^n(D^2)$ appeared already in the original paper [Artin] of Artin.

Proposition 3.2.3. *The group $P^n(D^2)$ is generated by the elements D_{ij} for $1 \le i < j \le n$. The relations are generated by*

$$D_{kl} D_{ij} D_{kl}^{-1} = \begin{cases} D_{ij} & \text{if } i < k < l < j \\ & \text{or } k < l < i < j, \\ D_{kj}^{-1} D_{ij} D_{kj} & \text{if } k < i = l < j, \\ D_{kj}^{-1} D_{lj}^{-1} D_{ij} D_{lj} D_{kj} & \text{if } i = k < l < j, \\ D_{kj}^{-1} D_{lj}^{-1} D_{kj} D_{lj} D_{ij} D_{lj}^{-1} D_{kj}^{-1} D_{lj} D_{kj} & \text{if } k < i < l < j. \end{cases}$$

We now choose the n fixed points $p_1, \ldots, p_n$ on a circle in the interior of D^2 and label them in a counterclockwise manner with the elements of $\mathbb{Z}/n$. For every pair $i, j \in \mathbb{Z}/n$ of distinct elements we have defined the element D_{ij}. We denote the line segment between p_i and p_j by $\overline{p_i p_j}$. In the following corollary we rewrite the presentation of Proposition 3.2.3 in a way more convenient for us.

Corollary 3.2.4. *The group $P^n(D^2)$ is generated by the elements $D_{ij} = D_{ji}$ where $i \neq j \in \mathbb{Z}/n$. The relations are generated by* (*i*) *for* $i, j, k \in \mathbb{Z}/n$ *pairwise distinct and ordered counter clockwise is* $D_{ij}D_{jk}D_{ki}$ *cyclic invariant, or equivalently*

$$[D_{ij}, D_{jk}]D_{jk}[D_{ij}, D_{ik}]D_{jk}^{-1} = 1,$$

(*ii*) *for* $i, j, k, l \in \mathbb{Z}/n$ *pairwise distinct and* $\overline{p_ip_j} \cap \overline{p_kp_l} = \varnothing$ *is* $[D_{ij}, D_{kl}] = 1$ *and*

(*iii*) *for* $i, j, k, l \in \mathbb{Z}/n$ *pairwise distinct and* $(p_j - p_i, p_l - p_k)$ *clockwise oriented is*

$$[D_{ij}, D_{kl}] = [D_{ij}, [D_{kl}, D_{jk}]].$$

See Figure 3.2 for the three cases.

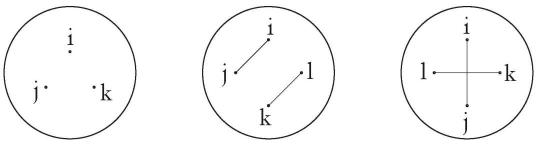

FIGURE 3.2. The positions of p_i, p_j, p_k, p_l in the three cases of the corollary.

PROOF. The first case of the relations in Proposition 3.2.3, $D_{kl}D_{ij}D_{kl}^{-1}D_{ij}^{-1} = 1$, is equivalent to relation (*ii*). In the second case we have $D_{ki}D_{ij}D_{ki}^{-1} = D_{jk}^{-1}D_{ij}D_{jk} \Leftrightarrow D_{jk}D_{ki}D_{ij} = D_{ij}D_{jk}D_{ki}$; therefore it is equivalent to relation (*i*) because the condition $k < i = l < j$ is equivalent to $i, j, k \in \mathbb{Z}/n$ are ordered counterclockwise . The equivalence in relation (*i*) of the corollary holds because if x, y, z are group elements then

$$xyz = yzx \Leftrightarrow xyx^{-1}y^{-1} = yzxz^{-1}x^{-1}y^{-1} \Leftrightarrow [x, y] = y[z, x]y^{-1}.$$

In the third case of Proposition 3.2.3,

$$\begin{aligned} &D_{il}D_{ji}D_{il}^{-1} = D_{ji}^{-1}D_{lj}^{-1}D_{ji}D_{lj}D_{ji} \Leftrightarrow \\ &D_{lj}D_{ji}D_{il}D_{ji} = D_{ji}D_{lj}D_{ji}D_{il} \overset{(i)}{\Longleftrightarrow} \\ &D_{ji}D_{il}D_{lj}D_{ji} = D_{ji}D_{lj}D_{ji}D_{il} \Leftrightarrow \\ &D_{il}D_{lj}D_{ji} = D_{lj}D_{ji}D_{il} \end{aligned}$$

which is again equivalent to (*i*). The relation (*iii*) in the corollary is equivalent to the last equation in Proposition 3.2.3 because

$$\begin{aligned} &D_{kl}D_{ij}D_{kl}^{-1} = D_{jk}^{-1}D_{lj}^{-1}D_{jk}D_{lj}D_{ij}D_{lj}^{-1}D_{jk}^{-1}D_{lj}D_{jk} \overset{(*)}{\Longleftrightarrow} \\ &D_{kl}D_{ij}D_{kl}^{-1}D_{ij}^{-1} = D_{kl}D_{jk}D_{kl}^{-1}D_{jk}^{-1}D_{ij}D_{jk}D_{kl}D_{jk}^{-1}D_{kl}^{-1}D_{ij}^{-1} \Leftrightarrow \\ &[D_{ij}, D_{kl}] = [D_{ij}, [D_{kl}, D_{jk}]], \end{aligned}$$

where $(*)$ holds because if k, l, j are ordered counterclockwise, we know from (i) that $D_{kl}D_{lj}D_{jk}$ is cyclic invariant and we get

$$D_{jk}D_{kl}D_{lj} = D_{lj}D_{jk}D_{kl} \Leftrightarrow$$
$$D_{jk}^{-1}D_{lj}^{-1}D_{jk} = D_{kl}D_{lj}^{-1}D_{kl}^{-1} = D_{kl}D_{jk}D_{jk}^{-1}D_{lj}^{-1}D_{kl}^{-1} = D_{kl}D_{jk}D_{kl}^{-1}D_{jk}^{-1}D_{lj}^{-1} \Leftrightarrow$$
$$D_{jk}^{-1}D_{lj}^{-1}D_{jk}D_{lj} = D_{kl}D_{jk}D_{kl}^{-1}D_{jk}^{-1}.$$

□

Notice that the presentation implies that $[P^n(D^2), P^n(D^2)]$ is normally generated by the commutators $[D_{ij}, D_{jk}]$ for $i, jk, \in \mathbb{Z}/n$ ordered counter clockwise.

A lift of the braid D_{ij} to $\Gamma_{0,r}$ is the mapping class $D_{\overline{p_i p_j}}$ that is again denoted by D_{ij}. Let $i, j, k \in \mathbb{Z}/r-1$ be pairwise different and ordered counterclockwise and let γ be an oriented arc from p_j to p_k. We define the mapping class $B_{\gamma,i} := D^{-1}_{D_{\overline{p_i p_j}}(\gamma)} D_\gamma$, see Figure 3.3.

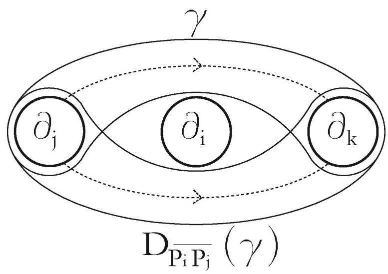

FIGURE 3.3. The mapping class $B_{\gamma,i}$ corresponding to an oriented arc γ from p_j to p_k.

COROLLARY 3.2.5. *The groups $T_{0,0}$, $T_{0,1}$, $T_{0,2}$, $T_{0,3}$ are trivial and for $r \geq 4$ we have that*

$$T_{0,r} \cong [P^{r-1}(D^2), P^{r-1}(D^2)]$$

is not finitely generated. Generators are in that case $B_{\gamma,i}$, where γ is an embedded arc from p_j to p_k and $i, j, k \in \mathbb{Z}/(r-1)$ are pairwise different and ordered counter-clockwise.

PROOF. The property that $T_{0,r}$ is trivial for $r \leq 3$ follows immediately from Theorem 3.2.2 because $P^0(D^2)$ and $P^1(D^2)$ are trivial and $P^2(D^2)$ is infinitely cyclic.

Let F_n denote the free group on n generators. We have the short exact sequence

$$1 \to F_{r-2} \to P^{r-1}(D^2) \to P^{r-2}(D^2) \to 1$$

obtained by forgetting the r^{th} string, here F_{r-2} is generated by the elements $D_{i,r-1}$ with $1 \leq i \leq r-2$. Because all the relations of the presentation of $P^{r-1}(D^2)$ of

Corollary 3.2.4 are in the commutator subgroup, we have that $P^{r-1}(D^2)_{\text{ab}}$ is the free abelian group generated by the images of D_{ij}. We see that $(F_{r-2})_{\text{ab}}$ maps injectively into $P^{r-1}(D^2)_{\text{ab}}$. Hence, the sequence

$$1 \to [F_{r-2}, F_{r-2}] \to [P^{r-1}(D^2), P^{r-1}(D^2)] \to [P^{r-2}(D^2), P^{r-2}(D^2)] \to 1$$

is exact. It follows by induction that $[P^{r-1}(D^2), P^{r-1}(D^2)]$ is not finitely generated when $r \geq 4$, because $P^2(D^2)$ is isomorphic to $\mathbb{Z}$ and hence $[P^3(D^2), P^3(D^2)] \cong [F_2, F_2]$, which is not finitely generated. Generators are of the form $w[D_{ij}, D_{jk}]w^{-1}$, where $w \in P^{r-1}(D^2)$ and $i, j, k \in \mathbb{Z}/r-1$ are ordered counterclockwise. This follows immediately from the relations in the generators D_{ij} of $P^{r-1}(D^2)$ and the fact that any commutator subgroup $[G, G]$ is normally generated in G by the commutators of the generators of G. The element $D_{ij}D_{jk}D_{ij}^{-1}D_{jk}^{-1} = D_{D_{ij}(\overline{p_j p_k})}D_{jk}^{-1} = B_{(\overline{p_j p_k}),i}$ and conjugation by all $w \in P^{r-1}(D^2)$ give all the generators of the corollary.

□

Although $T_{0,r}$ is not finitely generated when $r \geq 4$, we do have a finite presentation of $H_1(T_{0,r})$ as a module over $\mathbb{Z}[P^{r-1}(D^2)_{\text{ab}}]$, as the following general discussion shows.

Let G be a group. Then G acts on $[G, G]$ by conjugation and this induces an action of the group ring $\mathbb{Z}[G_{\text{ab}}]$ on $H_1([G, G])$. Suppose that G is of rank n generated by $a_1, \ldots, a_n$. We denote the images of a_i in G_{ab} by α_i and that of $[a_i, a_j]$ in $H_1([G, G])$ by e_{ij}, where $i, j \in \{1, \ldots, n\}$. The group ring $\mathbb{Z}[G_{\text{ab}}]$ is isomorphic to $\mathbb{Z}[\alpha^{\pm 1}, \ldots, \alpha^{\pm n}]$, the ring of Laurent polynomials. Let us first assume that the group in question is a free group F on n letters. The following lemma gives in that case a presentation of $H_1([F, F])$.

Lemma 3.2.6. *The module $H_1([F, F])$ over $\mathbb{Z}[F_{\text{ab}}]$ admits the following presentation. It is generated by e_{ij}, and the relations are generated by*

(*i*) $e_{ij} + e_{ji}$ *and*
(*ii*) $\alpha_i e_{jk} + \alpha_j e_{ki} + \alpha_k e_{ij}$.

where $i, j, k \in \{1, \ldots, n\}$ are pairwise distinct.

Proof. We construct a graph with an action of F_{ab} such that

$$H_1(X) \cong H_1([F, F])$$

as $\mathbb{Z}[F_{\text{ab}}]$-modules. Let $\vee^n S^1$ be a bouquet of n-circles. We view $\vee^n S^1$ as a graph with one vertex p and n oriented edges and we make the identification $\pi_1(\vee^n S^1, p) = F$. We embed $\vee^n S^1$ in the n-dimensional torus $\mathbb{R}^n/\mathbb{Z}^n$ in an obvious way and define the graph X to be the preimage of $\vee^n S^1$ in $\mathbb{R}^n$ of the projection map $\mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$. Using the choice of a basepoint we can identify $\pi_0(\mathbb{Z}^n)$ with the fiber $\mathbb{Z}^n$ of the fibration $X \to \vee^n S^1$. The long exact sequence of a fibration gives us the short exact sequence

$$1 \to \pi_1(X) \to F \xrightarrow{\text{ab}} F_{\text{ab}} \to 0.$$

We conclude that $\pi_1(X) \cong [F, F]$. The group F_{ab} acts on X, so we have an induced action of $\mathbb{Z}[F_{\text{ab}}]$ on $H_1(X)$ and the isomorphism with $H_1([F, F])$ is F_{ab}-equivariant.

The cellular chain complex of X induces the exact sequence

$$0 \to H_1(X) \to \mathbb{Z}[F_{\text{ab}}]^n \to \mathbb{Z}[F_{\text{ab}}] \to \mathbb{Z} \to 0,$$

where the i^{th} generating directed edge that generate the i^{th} component of $\mathbb{Z}[F_{\text{ab}}]^n$ maps to $\alpha_i - 1$ in $I[F_{\text{ab}}]$, the augmentation ideal of $\mathbb{Z}[F_{\text{ab}}]$. We recognize the tail of the Koszul complex coming from the regular sequence $(\alpha_1 - 1, \ldots, \alpha_n - 1)$ in $\mathbb{Z}[F_{\text{ab}}]$ and thus

$$\wedge^3 \mathbb{Z}[F_{\text{ab}}]^n \to \wedge^2 \mathbb{Z}[F_{\text{ab}}]^n \to H_1(X) \to 0$$

is a presentation of $H_1(X)$, which is the presentation of the lemma. $\square$

Geometrically we see that the generators correspond to the $\binom{n}{2}$ oriented squares in X in the first quadrant and the relations correspond to the $\binom{n}{3}$ unit cubes in each first octant.

Suppose now that G is finitely presented, with generators $a_1, \ldots, a_n$ and relations $r_1, \ldots, r_N$. We assume that the relations are all in the commutator subgroup, that is, for every $1 \leq j \leq N$, we can write

$$r_j = \prod_{i=1}^{k_j} w_{ji} [a_{p_{ji}}, a_{q_{ji}}]^{l_{ji}} w_{ji}^{-1}$$

where w_{ji} is a word in $a_1, \ldots, a_n$ and $k_j, p_{ji}, q_{ji}, l_{ji} \geq 1$. The next proposition gives a presentation of $H_1([G, G])$, as module over $\mathbb{Z}[G_{ab}]$.

PROPOSITION 3.2.7. *Let G be a finitely presented group with relations in the commutator subgroup. Then, in the notation given above, $H_1([G, G])$ admits the following presentation as $\mathbb{Z}[G_{ab}]$-module. It is generated by the elements e_{ij} and the relations are generated by*

(i) $e_{ij} + e_{ji}$,
(ii) $\alpha_i e_{jk} + \alpha_j e_{ki} + \alpha_k e_{ij}$ *and*
(iii) $\sum_{i=1}^{k_j} l_{ji} \overline{w_{ji}} e_{p_{ji}, q_{ji}}$ *for* $1 \leq j \leq N$,

where $\overline{w}$ denotes the image in G_{ab} of a word w in G. Here $i, j, k \in \{1, \ldots, n\}$ are pairwise distinct.

PROOF. Since $[G, G]$ is generated by $w[g_i, g_j]w^{-1}$, where $w \in G$ and $1 \leq i, j \leq n$, the module $H_1([G, G])$ is generated by the images e_{ij} of $[g_i, g_j]$ in $H_1([G, G])$. Let

$$1 \to R_G \to F \xrightarrow{p} G \to 1$$

be a presentation of G such that the relations are in the commutator subgroup. Because $R_G \subseteq [F,F]$, the group $[G,G]$ has presentation

$$1 \to R_G \to [F,F] \stackrel{[p,p]}{\to} [G,G] \to 1$$

and thus the kernel of the induced map $[p,p]_* : H_1([F,F]) \to H_1([G,G])$ is generated by the images in $H_1([F,F])$ of the generators of R_G, that means, by the elements $\sum_{i=1}^{k_j} l_{ji}\overline{w_{ji}}e_{p_{ji},q_{ji}}$. Let

$$0 \to R_{H_1([F,F])} \to \bigoplus_{i,j}\mathbb{Z}[F_{ab}] \to H_1([F,F]) \to 0$$

be the presentation of $H_1([F,F])$ given in Proposition 3.2.6. So $R_{H_1([F,F])}$ is generated by $e_{ij}+e_{ji}$ and $\alpha_i e_{jk} + \alpha_j e_{ki} + \alpha_k e_{ij}$. Applying the snake lemma to the diagram

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & \bigoplus_{i,j}\mathbb{Z}[F_{ab}] & \stackrel{\cong}{\to} & \bigoplus_{i,j}\mathbb{Z}[G_{ab}] & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathrm{Ker}([p,p]_*) & \to & H_1([F,F]) & \stackrel{[p,p]_*}{\longrightarrow} & H_1([G,G]) & \to & 0, \end{array}$$

shows that the kernel of $\bigoplus_{i,j}\mathbb{Z}[G_{ab}] \to H_1([G,G])$ is generated by the elements (i), (ii) and (iii) of the proposition. □

In the presentation of $P^{r-1}(D^2)$ in Corollary 3.2.4 we see that all the relations are in the commutator subgroup , therefore we can apply Proposition 3.2.7 to obtain a finite presentation of $H_1([P^{r-1}(D^2), P^{r-1}(D^2)])$, and therefore of $H_1(T_{0,r})$, as a module over $\mathbb{Z}[P^{r-1}(D^2)_{ab}]$. We write $[ij,kl]$ for the images of $[D_{ij}, D_{kl}]$ in $H_1(T_{0,r})$ and α_{ij} for the image of D_{ij} in $\mathbb{Z}[P^{r-1}(D^2)_{ab}]$.

COROLLARY 3.2.8. *For every $0 \le r \le 3$ we have $H_1(T_{0,r}) = 0$. For $r \ge 4$ $H_1(T_{0,r})$ is a module over $\mathbb{Z}[P^{r-1}(D^2)_{ab}]$ via the isomorphism of Theorem 3.2.2, with the following presentation. It is generated by the elements $[ij,kl]$ with $i,j,k,l \in \mathbb{Z}/r-1$, $i \neq j$, $k \neq l$, and the relations are generated by*

(i) *$[ij,kl]$ is symmetric in (i,j) and in (k,l),*
(ii) *$[ij,kl] = -[kl,ij]$,*
(iii) *$(\alpha_{ij}-1)[kl,mn] + (\alpha_{kl}-1)[mn,ij] + (\alpha_{mn}-1)[ij,kl] = 0$,*
(iv) *$[ij,jk] = \alpha_{jk}[ki,ij]$, if $i,j,k \in \mathbb{Z}/r-1$ are ordered counter clockwise,*
(v) *$[ij,kl] = 0$ if $\overline{p_ip_j} \cap \overline{p_kp_l} = \varnothing$ and*
(vi) *$[ij,kl] = (\alpha_{ij}-1)[lk,kj]$ if (p_l-p_k, p_j-p_i) is an oriented basis.*

From the relations we see that the elements $[ij,jk]$ already generate the module $H_1(T_{0,r})$ where $i,j,k \in \mathbb{Z}/r-1$ are ordered counterclockwise.

3.3. Genus one

Let $\Gamma_{1,1} \to \Gamma_{1,0}$ be the map induced by closing the hole of the surface with a disc. The kernel of this map is the infinite cyclic group generated by D_∂, where ∂ denotes the boundary of the surface. It is well known that $\Gamma_{1,0} \cong \mathrm{Sl}(2,\mathbb{Z}) \cong \mathrm{Sp}(H_1(S_{1,0}))$ and so $T_{1,0} = \{1\}$ and $T_{1,1} = \langle D_\partial \rangle$. In this section we give a description of $T_{1,2}$ and compute $H_k(T_{1,2})$ for $k \geq 0$.

We start with introducing notations. Let H be a symplectic quasi-unimodular module over $\mathbb{Z}$, we write H_2 for $H \otimes_{\mathbb{Z}} \mathbb{Z}/2$. If A is a set, then $\mathbb{Z}^{(A)}$ denotes the group of maps $A \to \mathbb{Z}$ with finite support.

Definition 3.3.1. *We define the following sets and groups*

$$\begin{aligned} M_H &:= \{U : U \textit{ is a unimodular symplectic subspace of } H \textit{ of genus } 1\}, \\ N_H &:= \{U : U \textit{ is a unimodular symplectic subspace of } H \textit{ of genus } 2\}, \\ R_H &:= \langle U - U' - U \oplus U' : U, U' \in M_H,\ U \perp U' \textit{ and } U \oplus U' \in N_H\}, \\ \widetilde{G}_H &:= \frac{\mathbb{Z}/2^{(N_H)} \oplus \mathbb{Z}^{(M_H)}}{R_H} \textit{ and} \\ G_H &:= \frac{\mathbb{Z}/2^{(N_{H_2})} \oplus \mathbb{Z}/2^{(M_{H_2})}}{R_{H_2}}. \end{aligned}$$

If $H = H_1(S)$ we write M_S, N_S, $\widetilde{G}_S$, G_S instead. We denote the class of a subspace U in $\widetilde{G}_H$ and G_H respectively, by $[U]$.

Let $S = S_{1,2}$ and label the boundary components with ∂_0 and ∂_1. Let $S' = S_{1,1}$ be the surface obtained from S by closing ∂_1 with a disc and fix a point p on the boundary of this disc. As we have seen in Section 2.4, we have the following exact sequence

$$1 \to [\pi,\pi] \to T_S \to \langle D_\partial \rangle \to 1,$$

where $\pi = \pi_1(S', p)$, so π is a free group on two generators. When α, β are represented by $SCC's$ and $a, b \in H_1(S')$ the classes represented by α, β respectively, then the map $[\pi,\pi] \to T_S$ is given by (see Section 2.4)

$$[\alpha,\beta] \mapsto [t_\alpha, t_\beta] D_{\partial_0}^{2(a\cdot b)}.$$

Let α and β be two generators of π such that $[\alpha,\beta]$ is represented by a SCC γ that is isotopic to ∂_0. We compute the image of $[\alpha,\beta]$. In Γ_S we have that

$$[t_\alpha, t_\beta] = t_\alpha t_\beta t_\alpha^{-1} t_\beta^{-1} = t_\gamma D_{\partial_1}^{-1}.$$

So the element $[\alpha,\beta]$ maps to $t_\gamma D_{\partial_1}^{-1} D_{\partial_1}^{2(a\cdot b)} = D_{\gamma^+} D_{\partial_0}^{-1} D_{\partial_1}$ in T_S. See Figure 3.4 where we compute it on a covering of $S_{1,1}$. Notice that $D_{\partial_0}^{-1} D_{\partial_1}$ is central in Γ_S. The group $[\pi,\pi]$ is normally generated by the elements $[\alpha,\beta]$ and any conjugate $g[\alpha,\beta]g^{-1}$ maps to $D_\omega D_{\partial_0}^{-1} D_{\partial_1}$, where $\omega = \tilde{g}(\gamma^+)$ and $\tilde{g} \in \mathrm{Ker}(\Gamma_S \to \Gamma_{S'})$ is the

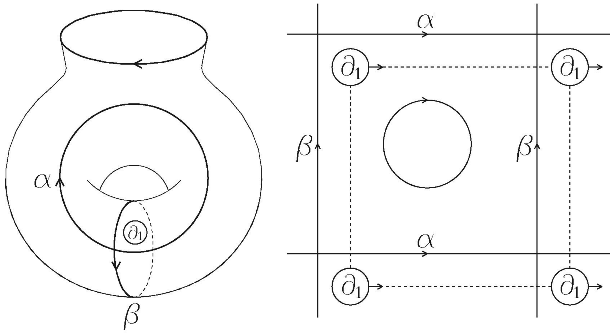

FIGURE 3.4. Computation of $[t_\alpha, t_\beta] = t_\gamma D_{\partial_1}^{-1}$ on a covering of $S_{1,1}$ and $\gamma^+, \partial_0, \partial_1$ on $S_{1,2}$.

map associated to a lift of g in $\tilde{\pi} = \pi_1(US', v_p)$. In this way we get $SCC's$ ω that separate $S_{1,2}$ into a surface $S_{1,1}$ and a pair of pants $S_{0,3}$.

We can split the above exact sequence by lifting D_∂ to the central element $D_{\partial_0}^{-1} D_{\partial_1}$ and therefore $T_S \cong [\pi, \pi] \times \langle D_\partial \rangle$. We construct a set of free generators for $[\pi, \pi]$.

We close the hole of S' to a point q and get the two-pointed torus S_1^2. Assume that the base point p is the unit element $0 \in S_1^2$ and that q is not on a circle subgroup of S_1^2. Let $g : \widehat{S_1^2} \to S_1^2$ be the universal cover of S_1^2 and fix $\hat{0} \in g^{-1}(0)$. Then $[\pi, \pi]$ is the fundamental group of the graph that is the preimage in $\widehat{S_1^2}$ of the embedded circles α and β based at 0. So a set of free generators of $[\pi, \pi]$ corresponds one to one with the set of unit squares in this graph. It is an affine set with group of translations $H_1(S_1^2)$. The unit squares correspond one-to-one with the elements of $g^{-1}(q)$. Since we have chosen q general enough, the images under g of the line segments $[\hat{0}, \hat{q}]$, with $\hat{q} \in g^{-1}(q)$, correspond with the set of geodesics on S_1^2 from 0 to q without self intersection. The set of classes of such arcs in $H_1(S_1^2, \{0, q\}) \cong H_1(S, \partial S)$ corresponds one-to-one with the set of linear forms on $H_1(S)$ that split $H_1(S)$ in the radical and a complementary symplectic summand of rank two, that is, with the set M_S. When γ is a geodesic from p to q and ω is the boundary of a regular neighborhood of γ, then $[\gamma]$ corresponds to the mapping class $D_\omega D_{\partial_0}^{-1} D_{\partial_1}$ on $S_{1,2}$.

We compute the homology of T_S using its product structure. It shows that

$$H_1(T_S) \cong \mathbb{Z} \oplus \mathbb{Z}^{(M_S)}$$

and $H_k(T_S) = 0$ if $k \geq 3$. Using the Künneth formula we see that $H_2(T_S) \cong \mathbb{Z}^{(M_S)}$.

The previous discussion proves the following theorem.

THEOREM 3.3.2. *Let $S = S_{1,2}$ then*

$$\begin{aligned} H_1(T_S) &= \mathbb{Z} \oplus \mathbb{Z}^{(M_S)} \cong \wedge^3 H_1(S) \oplus \widetilde{G}_S, \\ H_2(T_S) &= \mathbb{Z}^{(M_S)} \text{ and} \\ H_k(T_S) &= 0 \text{ if } k \geq 3. \end{aligned}$$

REMARK 3.3.3. *Let p, q be two points on different boundary components of $S_{1,2}$ and $BX(p,q)$ the associated arc-complex, which is of dimension two. The stabilizer of a vertex α is the Torelli group $T_{1,1} \cong \langle D_\alpha \rangle$. The stabilizer of an edge is trivial. The vertices of $\overline{BX} := T_S \backslash BX(p,q)$ correspond one-to-one with the set M_S and hence $H_0(\overline{BX}, \mathcal{H}_1) \cong \mathbb{Z}^{(M_S)}$. Therefore $H_1(\overline{BX}) \cong \mathbb{Z}$ is not trivial, a generator corresponds with the central element $D_{\partial_0}^{-1} D_{\partial_1}$. This implies that $\mathcal{A}^o(H,\pi)$ is not 1-connected in this case.*

When the number of boundary components of S is at least three, then the action of $T_{S'}$ on $[\pi_{S'}, \pi_{S'}]$ is not trivial. In that case, T_S is isomorphic to a successive semi-direct product via the split exact sequence

$$1 \to [\pi_{S'}, \pi_{S'}] \to T_S \to T_{S'} \to 1.$$

We do not compute its homology.

REMARK 3.3.4. *As in the case of genus zero, we can relate the Torelli group of a surface of genus one to the associated braid group. Let $S = S_{1,r}$, $\mathbb{T}$ be the torus, $F^r(U\mathbb{T})$ the configuration space of pairwise distinct points $(v_1, \ldots, v_r) \in \Pi_{i=1}^r U\mathbb{T}$, where $U\mathbb{T}$ is the unit tangent bundle and $P_r(\mathbb{T}) = \pi_1(F^r(U\mathbb{T}))$. The kernel of the map $\Gamma_S \to \mathrm{Sl}(2,\mathbb{Z})$ defined by filling the r holes with a disc is isomorphic to $P_r(\mathbb{T})/Z(P_r(\mathbb{T}))$, where $Z(P_r(\mathbb{T}))$ is the center of $P_r(\mathbb{T})$ (see* [Birman4]*). The center $Z(P_r(\mathbb{T}))$ is the free abelian group generated by the two loops that move the r holes together around α and β respectively. We denote $P_r(\mathbb{T})/Z(P_r(\mathbb{T}))$ by $\overline{P}$.*

The Torelli group T_S is in $\overline{P}$ with quotient $K(H_1(S,P), H_1(S))$ (see Section 1.2 for notations). The group $\overline{P}$ has an explicit presentation (see [Birman1]*); using this presentation one can derive the following commuting exact diagram*

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & [\overline{P},[\overline{P},\overline{P}]] & \to & [\overline{P},\overline{P}] & \to & R \circ R & \to & R/2R & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & T & \to & \overline{P} & \to & K(\widetilde{H}, H) & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H_0(\partial S) & \to & \overline{P}_{\mathrm{ab}} & \to & \mathrm{Hom}(H,R) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & R/2R & & 0 & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

that relates T_S *with* $[\overline{P},[\overline{P},\overline{P}]]$. *Here* $R := \mathrm{Rad}(H_1(S))$, $H := H_1(S)$ *and* $\widetilde{H} := H_1(S,P)$. *The computation of* $H_1([\overline{P},[\overline{P},\overline{P}]])$ *becomes complicated when we follow the method of Section 3.2. We did not succeed in computing it.*

3.4. Genus two

When $S = S_{2,0}$ it is proved in [Mess] that the Torelli group of S is freely generated by an infinite set of $BSCC$-maps that corresponds one-to-one with the set of homology splittings of $H_1(S)$ into two mutually orthogonal unimodular symplectic subspaces of rank two. Using this result we compute in Theorem 3.4.1 the group $H_1(T_{2,1})$. For a surface of genus two and more boundary components the computation becomes difficult, as we explain in Remark 3.4.3, but for $S_{2,2}$ we can describe $H_1(T_S)$ as a quotient of a certain group. This will be useful in the computations of $H_1(T_S)$ for surfaces of genus three and four.

In our notation, introduced in Section 3.3, we know by the result of Mess that if $S = S_{2,0}$ then

$$H_1(T_S) \cong \frac{\mathbb{Z}^{(M_S)}}{\langle U - U' : U \oplus U' = H_1(S), U \perp U' \rangle} = \frac{\widetilde{G}_S}{\langle H_1(S) \rangle}.$$

When the surface has one boundary component, we can regard it as the blow up $\varphi : S \to S'$ of a closed surface S' such that $\varphi^{-1}(p) = \partial S$ for some $p \in S'$. By Proposition 2.4.1 we have the exact sequence

$$1 \to \pi_1(US', v_q) \to T_S \to T_{S'} \to 1,$$

where $q \in \partial S$. Using the Gysin sequence one finds that $H_1(US') \cong H_1(S') \oplus \mathbb{Z}/2 \cong H_1(S) \oplus \mathbb{Z}/2$. The action of $T_{S'}$ on $H_1(US')$ is trivial, hence the short exact sequence reduces to the exact sequence

$$0 \to H_1(S) \oplus \mathbb{Z}/2 \to H_1(T_S) \to H_1(T_{S'}) \to 0$$

on homology. The generator of $\mathbb{Z}/2$ is represented by the Dehn twist D_∂ around the boundary component of S'. There are two natural lifts of a free Mess-generator of T_2 to $T_{2,1}$; on homology they differ by the order two element D_∂. Notice that $H_1(S)$ has rank four and therefore $N_S = \{H_1(S)\}$. We have a one-to-one correspondence between the set M_S and the set of the two liftings of all the Mess-generators. With this correspondence their relations are precisely expressed in R_S. Hence, we have the following proposition.

PROPOSITION 3.4.1. *Let* $S = S_{2,1}$, *then*

$$H_1(T_S) \cong H_1(S) \oplus \widetilde{G}_S \cong \wedge^3 H_1(S) \oplus \widetilde{G}_S.$$

An element $[\gamma] \in H_1(S)$ represented by an oriented SCC γ based at $q \in \partial S$ corresponds to the class of t_γ in $H_1(T_S)$.

When $S = S_{2,2}$ we do not know how to compute $H_1(T_S)$ but we show that it is a quotient of $\wedge^3 H_1(S) \oplus \widetilde{G}_S$. Let ∂_0, ∂_1 be the two boundary components of S; we write $\widetilde{S} = S_{3,1}$ for the surface obtained from S by gluing a pair of pants to ∂_0, ∂_1. Let $\widetilde{H} := H_1(\widetilde{S})$ and $\pi : \widetilde{H} \to \mathbb{Z}$ be the map defined by $v \mapsto [\partial_0]\cdot v$, then $\pi^{-1}(0) \cong H_1(S)$. If $v \in \widetilde{H}$ let $\breve{v} := v^\perp \cap \pi^{-1}(0)$. We choose two points p, q on the two boundary components of S and associated to them we have the arc-complex $BX(p,q)$, see Definition 2.5.1. By Proposition 2.5.3 we know that $T \backslash BX(p,q) \cong \mathcal{A}^o(\widetilde{H}, \pi)$. On this complex we have defined the system of coefficients $\mathcal{H}_1$, see Section 1.11 for the definition. This system can be identified with

$$\mathcal{H}_1(v) \cong \wedge^3 \breve{v} \oplus \widetilde{G}_{\breve{v}},$$

with the inclusions as boundary maps . This is because the stabilizer of a vertex is the Torelli group $T_{2,1}$ and the stabilizer of an edge is either $T_{1,2}$ or $T_{0,3} = \{1\}$. The stabilizer of a 3-simplex is trivial in all cases.

The poset $BX(p,q)$ is spherical of dimension four and $\mathcal{A}^o(\widetilde{H}, \pi)$ is simply connected by Theorem 1.5.5. Hence, by Lemma 1.11.1 we have the exact sequence

$$H_2(\mathcal{A}^o(\widetilde{H}, \pi)) \to H_0(\mathcal{A}^o(\widetilde{H}, \pi), \mathcal{H}_1) \to H_1(T_S) \to 0.$$

Proposition 3.4.2. *If $S = S_{2,2}$ then $H_1(T_S)$ is a quotient of $\wedge^3 H_1(S) \oplus \widetilde{G}_S$.*

Proof. The system $\mathcal{H}_1$ decomposes into a direct sum of $\mathcal{F}_f(\breve{v}) = \wedge^3 \breve{v}$ and $\mathcal{F}_G(\breve{v}) = \widetilde{G}_{\breve{v}}$. According to Proposition 1.12.4 we know that $H_0(\mathcal{A}^o(\widetilde{H}, \pi), \mathcal{H}_f)$ surjects onto $\wedge^3 H_1(S)$ and by the same argument we have that

$$H_0(\mathcal{A}^o(\widetilde{H}, \pi), \mathcal{F}_G)$$

surjects onto $\tilde{G}_S$. We prove that

$$H_0(\mathcal{A}^o(\widetilde{H}, \pi), \mathcal{H}_1) \to H_1(T_S)$$

factorizes over

$$\wedge^3 H_1(S) \oplus \widetilde{G}_S \to H_1(T_S).$$

Let $x = e_1 \wedge e_{-1} \wedge e_2 \in \wedge^3 H_1(S)$ such that $\{e_1, e_{-1}, e_2\}$ is a symplectic basis of $H_1(S_x)$, where $S_x \cong S_{1,2}$ is a subsurface of S, and that can be extended to a symplectic basis

$$\{e_1, e_{-1}, e_2, e_{-2}, e_3, e_{-3}\}$$

of $H_1(\widetilde{S})$ with $e_3 = [\partial_0]$. Any primitive element $v \in \pi^{-1}(1)$ such that $x \in \wedge^3 \breve{v}$, is in $e_{-3} + \langle e_2, e_3 \rangle$. This implies that it can be represented by an embedded arc on the subsurface of $\widetilde{S}$ that is the closure of $\widetilde{S} - S_x$. Because the image of $x \in \wedge^3 \breve{v}$ in $H_1(T_S)$ is the mapping class $D_{\gamma_1} D_{\gamma_2}^{-1}$, where γ_1, γ_2 are the two boundary curves of

S_x, oriented such that $[\gamma_1]+[\gamma_2]=0$, we see that every lift of x to $H_0(\mathcal{A}^o(\widetilde{H},\pi),\mathcal{H}_1)$ has the same image in $H_1(T_S)$.

Let $U=\langle e_1,e_{-1}\rangle\in M_S$ such that $\{e_1,e_{-1}\}$ is a symplectic basis of $H_1(S_U)$, where $S_U\cong S_{1,1}$ is a subsurface of S and that can be extended to a symplectic basis

$$\{e_1,e_{-1},e_2,e_{-2},e_3,e_{-3}\}$$

of $H_1(\widetilde{S})$ with $e_3=[\partial_0]$. Any primitive element $v\in\pi^{-1}(1)$ such that $U\in\widetilde{G}_{\check{v}}$ is in $e_{-3}+\langle e_2,e_{-2},e_3\rangle$. Therefore, it can be represented by an embedded arc on the subsurface of $\widetilde{S}$ that is the closure of $\widetilde{S}-S_U$. Because the image of $[U]$ in $H_1(T_S)$ is the Dehn twist D_γ, where γ is homotopic to the boundary of S_U, we see that every lift of U to $H_0(\mathcal{A}^o(\widetilde{H},\pi),\mathcal{H}_1)$ has the same image in $H_1(T_S)$.

Because the elements x and $[U]$ generate $\wedge^3H_1(S)\oplus\widetilde{G}_S$ we have the above factorization. □

REMARK 3.4.3. *One difficulty we encounter in the computation of $H_1(T_{2,2})$ is that we do not know generators of $H_2(\mathcal{A}^o(\widetilde{H},\pi))$.*

REMARK 3.4.4. *If ∂_0,∂_1 denote the two boundary components, then $D_{\partial_0}^{-1}D_{\partial_1}$ is a central element in T_S. Its class is represented by $e_1\wedge e_{-1}\wedge e_3+e_2\wedge e_{-2}\wedge e_3$ when $[\partial_0]=e_3$ is a generator of* $\mathrm{Rad}(H_1(S))$ *and $\{e_1,e_2,e_{-1},e_{-2}\}$ is a symplectic basis for $\overline{H_1(S)}$. We have that $e_1\wedge e_{-1}\wedge e_3=e_{-1}\wedge e_1\wedge(e_3+e_2)-e_1\wedge e_{-1}\wedge e_2$ and $e_1\wedge e_{-1}\wedge(e_3+e_2)\in(e_{-3}-e_{-2})$-summand and $e_1\wedge e_{-1}\wedge e_2\in e_{-3}$-summand of $C_0(\overline{BX},\mathcal{H}_1)$ is represented by a BP-map. This shows how $e_1\wedge e_{-1}\wedge e_3$ is represented by Dehn twists. Analogous, we see how $e_2\wedge e_{-2}\wedge e_3$ is represented. This shows that the central element is indeed represented in $H_0(\overline{BX},\mathcal{H}_1)$, where this was not the case for $S_{1,2}$, see Remark 3.3.3.*

3.5. Genus three or more

This section is devoted to the computations of $H_1(T_S)$ when S is a surface of genus three or more and having an nonempty boundary. We prove in this case that

$$H_1(T_S)\cong\wedge^3H_1(S)\oplus B_2(\Omega_S).$$

Johnson has proved this result for $S=S_{g,1}$ and $g\geq 3$ using a different method than is used here, see [Johnson8] and the review in Section 2.3. The method we apply is as follows. We use the arc-complex $BX(p,q)$ defined by Harer that enables us to compute $H_1(T_S)$ by inductive means. We let T_S act on the spherical simplicial complex $BX(p,q)$. The quotient $\overline{BX}$ is in the 1-component case isomorphic to $\mathcal{A}^o(H)$ case and hence $(g-2)$-connected. In the 2-component case it is isomorphic to $\mathcal{A}(\widetilde{H},\pi)$ for a certain $\widetilde{H}$ and π and therefore 1-connected. By Lemma 1.11.1, we have the short exact sequence

$$H_2(\overline{BX})\to H_0(\overline{BX},\mathcal{H}_1)\to H_1(T_S)\to 0,$$

which tells us how $H_1(T_S)$ is related to the abelianization of Torelli groups of lower genera or less boundary components. If $g \geq 4$ then $H_2(\overline{BX}) = 0$ when p, q are on the same component, so in that case, we have an isomorphism

$$H_0(\overline{BX}, \mathcal{H}_1) \cong H_1(T_S).$$

If $g = 3$ we get by the exact sequence an upper bound for $H_1(T_S)$. We use the Johnson epimorphism $\tau : T_S \to \wedge^3 H_1(T_S)$ and the Birman-Craggs epimorphism $\sigma : T_S \to B_3(\Omega_{H_1(S)})$ to find a lower bound for $H_1(T_S)$ which turns out to be equal to our upper bound.

We start with computing $H_1(T_S)$ when $S = S_{3,1}$. Fix two distinct points p, q on ∂S and let $BX := BX(p,q)$ be the associated complex in the 1-component case. We denote the quotient of BX by the action of T_S by $\overline{BX}$ and denote $H_1(S)$ by H. By Proposition 2.5.3 and Theorem 1.5.4 we know that $\overline{BX} \cong \mathcal{A}^o(H)$ is 1-connected and therefore by Lemma 1.11.1 we have the exact sequence

$$H_2(\overline{BX}) \to H_0(\overline{BX}, \mathcal{H}_1) \to H_1(T_S) \to 0.$$

If $\alpha \in BX$ is a vertex we denote by $S_\alpha \cong S_{2,2}$ the closure of $S - \{\alpha\}$. Let γ be the part of ∂S going from p to q. If $v \in H$ and $v = [\gamma\alpha]$ for some embedded $\alpha \in BX$ then $v^\perp \cong H_1(S_\alpha)$. The group $H_0(\overline{BX}, \mathcal{H}_1)$ is by definition a quotient of $\bigoplus_{v \in \mathcal{A}^o(H)_0} \mathcal{H}_1(v)$. The stabilizer of a lift of $v = [\gamma\alpha]$ is exactly the Torelli group of S_α, so by Proposition 3.4.2 we have that $\bigoplus_{v \in \mathcal{A}^o(H)_0} \mathcal{H}_1(v)$ is a quotient of

$$\bigoplus_{v \in \mathcal{A}^o(H)_0} \wedge^3 v^\perp \oplus \widetilde{G}_{v^\perp}.$$

Let $f_v : \widetilde{G}_{v^\perp} \to G_{v^\perp}$ be the projection map that reduces modulo two.

Lemma 3.5.1. *The group* $\bigoplus_{v \in \mathcal{A}(H)_o} \mathrm{Ker}(f_v)$ *is in the kernel of the surjection*

$$\bigoplus_{v \in \mathcal{A}^0(H)_0} \wedge^3 v^\perp \oplus \widetilde{G}_{v^\perp} \to H_1(T_S).$$

Proof. The group $\mathrm{Sp}(2g, \mathbb{Z})[2] := \ker(\mathrm{Sp}(2g, \mathbb{Z}) \to \mathrm{Sp}(2g, \mathbb{Z}/2))$ is generated by all squares δ_v^2, see the appendix in [Johnson8]. This implies that in each summand of $v \in \mathcal{A}^o(H)_0$, the kernel of f_v is generated by

(*i*) $2[U]$ for $U \in M_{v^\perp}$,

(*ii*) $[U] - [\delta_a^2(U)]$ with $U \in M_{v^\perp}$, $a \in v^\perp$ and

(*iii*) $[W] - [\delta_a^2(W)]$ with $W \in N_{v^\perp}$, $a \in v^\perp$.

We will prove that these relations also hold in $H_1(T_S)$.

The image of an element $U \in M_{v^\perp}$ in $H_1(T_S)$ is represented by the Dehn twist around the boundary curve of a separating subsurface $S_U \subset S$ such that $H_1(S_U) = U$. It follows from the lantern relation that $2[U] = 0$ in $H_1(T_S)$, when $g(S) \geq 3$, see Figure 2.3 and [Johnson8] Lemma 2, for an explanation.

Let $b \in H$ and $a \in b^\perp$, we define a homomorphism $\varphi_{a,b} : b^\perp \to b^\perp$ by

$$\varphi_{a,b}(x) := x + (a \cdot x)b.$$

Then $\varphi_{a,b}$ is a symplectic automorphism, $\varphi_{a+a',b} = \varphi_{a,b}\varphi_{a',b}$ and $\varphi_{a,b+b'} = \varphi_{a,b}\varphi_{a,b'}$. The proof that the relations (ii) and (iii) hold in $H_1(T_S)$ follows from the following lemmas.

Lemma 3.5.2. *Let $U \in M_H \cup N_H$ and a in H. We decompose $H = U \oplus U'$ where U' is a symplectic subspace such that $U \perp U'$ and write $a = u + u'$ according to this decomposition. Then $\delta_a^2(U) = \varphi_{2u,u'}(U)$.*

Proof. For any $k \in \mathbb{Z}$ and $x \in U$ is

$$\begin{aligned}\delta_a^k(x) &= x + k(a \cdot x)a = x + k(u \cdot x)(u + u') = x + k(u \cdot x)u + k(u \cdot x)u' \\ &= \varphi_{ku,u'}(x + k(u \cdot x)u) = \varphi_{ku,u'}\delta_u^k(x).\end{aligned}$$

If $u \in U$ then $x \in U$ if and only if $\delta_u^k(x) \in U$, so $\delta_a^k(U) = \varphi_{ku,u'}(U)$. □

We see that it suffices to prove the relations for $\varphi_{2u,u'}$. We further deduce that it is enough to prove that relation (iii) holds in $H_1(T_S)$. This is because if H is quasi-unimodular of rank five with cyclic radical, $U \in M_H$, $H = U \oplus U'$ a decomposition as above and $u' \in U'$, then we can choose a hyperbolic pair $\{a, b\}$ in U' such that $u' \in \langle a, b\rangle^\perp$. Then

$$\begin{aligned}[U] - [\varphi_{2u,u'}(U)] &= [U] - [\langle a, b\rangle] - [\varphi_{2u,u'}(U)] + [\langle a, b\rangle] \\ &= [U \oplus \langle a, b\rangle] - [\varphi_{2u,u'}(U \oplus \langle a, b\rangle)],\end{aligned}$$

in $H_1(T_S)$.

Lemma 3.5.3. *Suppose $W \in N_H$, $v_1, v_2 \in W$ such that $u_1 \cdot u_2 = 0$ and let $w' \in W^\perp$. Then*

$$[\varphi_{u_1+u_2,w'}(W)] + [\varphi_{u_1,w'}(W)] + [\varphi_{u_2,w'}(W)] + [W] = 0$$

in $H_1(T_S)$.

PROOF. Assume first that $u_1 \neq \pm u_2$. Then there exists a unimodular decomposition $W = U_1 \oplus U_2$ such that $u_i \in U_i$ for $i = 1, 2$ and $U_1 \perp U_2$. Then

$$\begin{aligned}
&[\varphi_{u_1+u_2,w'}(W)] = [\varphi_{u_1,w'}(U_1) \oplus \varphi_{u_2,w'}(U_2)] = [\varphi_{u_1,w'}(U_1)] + [\varphi_{u_2,w'}(U_2)],\\
&[\varphi_{u_1,w'}(W)] = [\varphi_{u_1,w'}(U_1) \oplus U_2] = [\varphi_{u_1,w'}(U_1)] + [U_2],\\
&[\varphi_{u_2,w'}(W)] = [U_1 \oplus \varphi_{u_2,w'}(U_2)] = [U_1] + [\varphi_{u_2,w'}(U_2)],\\
&[W] = [U_1 \oplus U_2] = [U_1] + [U_2],
\end{aligned}$$

in $H_1(T_S)$, because this relation holds already in each summand of $\bigoplus_{v \in \mathcal{A}^o(H)_0} \widetilde{G}_{v^\perp}$. We see that their sum is zero in $H_1(T_S)$.

If $u_1 = u_2$ we can choose $u_3 \in W$ such that $u_1 \cdot u_3 = 0$, $u_1 \neq u_3$, then

$$\begin{aligned}
&[\varphi_{2u_1,w'}(W)] + [\varphi_{u_3,w'}(W)] + [W] = [\varphi_{2u_1+u_3,w'}(W)] = [\varphi_{u_1+(u_1+u_3),w'}(W)] =\\
&[\varphi_{u_1,w'}(W)] + [\varphi_{u_1+u_3,w'}(W)] + [W] = [\varphi_{u_3,w'}(W)]
\end{aligned}$$

in $H_1(T_S)$, hence $[\varphi_{2u_1,w'}(W)] = [W]$ in $H_1(T_S)$. If $u_1 = -u_2$, the lemma is trivially true. □

If we choose $u_1 = u_2$ this finishes the proof of Lemma 3.5.1 □

Lemma 3.5.1 implies that we have the surjection

$$(*) \qquad \bigoplus_{v \in \mathcal{A}^o(H)_0} \wedge^3 v^\perp \oplus G_{v^\perp} \to H_1(T_S).$$

LEMMA 3.5.4. *If $v \in H$ then $G_{v^\perp} \cong B_2(\Omega_{v^\perp})$.*

PROOF. If $U \in M_H \cup N_H$, let $\operatorname{arf}(U) := \sum_{i=1}^{g(U)} \overline{e_i e_{-i}}$ be the Arf-invariant of U where $\{e_{\pm i}\}_{i=1}^{g(U)}$ is a symplectic basis of U. We have an epimorphism $G_{v^\perp} \to B_2(\Omega_{v^\perp})$ defined by $U \mapsto \operatorname{arf}(U)$. Since $\dim(B_2(\Omega_{v^\perp})) = 16$ we prove that this map is injective by showing that $\dim(G_{v^\perp}) \leq 16$.

For each $v \in H$ we have that $v^\perp$ is quasi-unimodular of genus two and with radical $\langle v \rangle$. Let V be a module over $\mathbb{Z}/2$ of this type, that means, quasi-unimodular, $g(V) = 2$ and $\operatorname{Rad}(V)$ has rank one.

The projection $V \to \overline{V}$ induces $G_V \to G_{\overline{V}}$. Now $\overline{V}$ is unimodular of rank four so $[\overline{V}]$ is a special element in it and a direct computation shows that $G_{\overline{V}}/[\overline{V}]$ has dimension ten, so $\dim(G_{\overline{V}}) = 11$. So it suffices to show that the kernel K of $G_V \to G_{\overline{V}}$ is at most of dimension five. A priori K is generated by the elements $[U] + [U']$ with $U, U' \in M_V$ and $[W] + [W']$ with $W, W' \in N_V$, that have the same image in $G_{\overline{V}}$. But if $U, U' \in M_V$ such that $U = U' \bmod \operatorname{Rad}(V)$, then we can choose $U_1 \in M_V$ such that $U \perp U_1$ and $U' \perp U_1$ and so

$$[U] + [U'] = [U \oplus U_1] + [U' \oplus U_1] \text{ in } G_V.$$

Therefore K is generated by the second type of elements only. Let v_0 be a generator of $\operatorname{Rad}(V)$. The set N_V is an affine space over $\overline{V}$; if $v \in \overline{V}$ and $W \in N_V$ then the translation of W over v is defined by $W + v := \varphi_{v,v_0}(W)$. We fix an element $W_0 \in M_V$. Let $\{e_1, e_2, e_{-1}, e_{-2}\}$ be a symplectic basis of $\overline{V}$. It follows by Lemma 3.5.3 that K is generated by the six elements $[W_0 + e_{\pm i}] + [W_0]$ and $[W_0 + e_i + e_{-i}] + [W_0]$, for $i = 1, 2$. In K we have the relation

$$\begin{aligned}&[W_0 + e_1 + e_{-1}] + [W_0 + e_2 + e_{-2}] =\\&[W_0 + e_1 + e_{-1} + e_2 + e_{-2}] + [W_0] =\\&[W_0 + e_1 + e_2] + [W_0 + e_{-1} + e_{-2}] =\\&[W_0 + e_1] + [W_0 + e_2] + [W_0 + e_{-1}] + [W_0 + e_{-2}].\end{aligned}$$

Hence $\dim(K) \leq 5$ and thus $\dim(G_V) \leq 16$. □

If v is a vertex of $\mathcal{A}^o(H)$ and $U \in M_{v^\perp}$ is a generating element, then the image of $[U]$ in $H_1(T_S)$ is represented by the Dehn twist around the boundary of a genus one subsurface $S_U \subset S$ such that $H_1(S_U) \otimes \mathbb{Z}/2 \cong U$ and $H_1(S_U) \subset v^\perp$. If v, w are two vertices of $\mathcal{A}^o(H)$ and $U \in M_{v^\perp} \cap M_{w^\perp}$ then we may choose S_U such that v and w can be represented by arcs on $S - S_U$. This implies that

$$\bigoplus_{v \in \mathcal{A}^o(H)_0} G_{v^\perp} \cong \bigoplus_{v \in \mathcal{A}^o(H)_0} B_2(\Omega_{v^\perp}) \to H_1(T_S)$$

factorizes over $G_S \cong B_2(\Omega_S) \to H_1(T_S)$.

If v is a vertex of $\mathcal{A}^o(H)$ then $\wedge^3 v^\perp$ is generated by the elements $x = a \wedge b \wedge c$ such that $\{a, b\}$ is a hyperbolic pair and $c \in \langle a, b\rangle^\perp$ is primitive. Then $\langle a, b, c\rangle = H_1(S_x) \subset v^\perp$ for a subsurface $S_x \subset S$ with oriented boundary curves γ_1, γ_2 such that $[\gamma_1] = c$ and the image of x in $H_1(T_S)$ is $D_{\gamma_2}^{-1} D_{\gamma_1}$. If v, w are two vertices of $\mathcal{A}^o(H)$ and $x \in \wedge^3(v^\perp \cap w^\perp)$ then this surface S_x is such that v and w can be represented by arcs on $S - S_x$. This implies that the map $(*)$ factorizes over an epimorphism

$$\wedge^3 H_1(S) \oplus B_2(\Omega_S) \to H_1(T_S).$$

In the overview of the work of Johnson, Section 2.3, we have recalled the Johnson epimorphism $\tau : T_S \to \wedge^3 H_1(T_S)$ and the Birman-Craggs epimorphism $\sigma : T_S \to B_3(\Omega_S)$. The composition

$$\wedge^3 H_1(S) \oplus B_2(\Omega_S) \to H_1(T_S) \to \wedge^3 H_1(S) \oplus B_3(\Omega_S)$$

is the map

$$(a \wedge b \wedge c, \beta) \mapsto (a \wedge b \wedge c, \beta + \overline{a}\overline{b}(\overline{c} + 1)),$$

when $\{a, b, c\}$ is a symplectic basis for a subsurface $S_{1,2}$ such that $a \cdot b = 1$. So this composition is injective and hence

$$H_1(T_S) \cong \wedge^3 H_1(S) \oplus B_2(\Omega_S).$$

Assume now that $S = S_{3,r}$ with $r \geq 2$. We choose two points p, q on different boundary components of S and let BX be the associated arc-complex. The quotient,

$\overline{BX}$, by the action of the Torelli group is 1-connected by Theorem 1.5.5 and Theorem 2.5.3. By Lemma 1.11.1 we have the exact sequence

$$H_2(\overline{BX}) \to H_0(\overline{BX}, \mathcal{H}_1) \to H_1(T_S) \to 1.$$

The stabilizer of a vertex of BX is the Torelli group of a surface $S_{3,r-1}$ and that of an edge is the Torelli group of a surface $S_{2,r}$.

PROPOSITION 3.5.5. *If $S = S_{3,r}$ with $r \geq 1$ then*

$$H_1(T_S) \cong \wedge^3 H_1(S) \oplus B_2(\Omega_S).$$

PROOF. For $r = 1$ we know that the proposition holds. We proceed with induction on r and assume that $r \geq 2$. We compute $H_0(\overline{BX}, \mathcal{H}_1)$. Let $S_{4,r-1}$ be the surface obtained from S by gluing a pair of pants to the two boundary components that contain a point of p, q. Let $\pi : H_1(S_{4,r-1}) \to \mathbb{Z}$ be the epimorphism defined by $v \mapsto [\partial_0] \cdot v$ where $[\partial_0]$ is the class determined by the boundary component containing p. We know by Proposition 2.5.3 that $\overline{BX} \cong \mathcal{A}^o(H_1(S_{4,r-1}), \pi)$. If (v, w) is an edge of BX, then the stabilizer of (v, w) maps via the boundary map onto $\wedge^3(v^\perp \cap w^\perp) \oplus B_2(\Omega_{v^\perp \cap w^\perp})$ in the summand of a and b respectively of $C_0(\overline{BX}, \mathcal{H}_1)$. Hence by Proposition 1.12.3 and induction we have $H_0(\overline{BX}, \mathcal{H}_1) \cong \wedge^3 H_1(S) \oplus B_2(\Omega_S)$. This implies that we have an epimorphism

$$\wedge^3 H_1(S) \oplus B_2(\Omega_S) \to H_1(T_S).$$

In Proposition 2.4.3, we have extended the Johnson epimorphism $\tau : T_S \to \wedge^3 H_1(T_S)$ and the Birman-Craggs epimorphism $\sigma : T_S \to B_3(\Omega_S)$ to Torelli groups of surfaces with arbitrarily many boundary components. The composition

$$\wedge^3 H_1(S) \oplus B_2(\Omega_S) \to H_1(T_S) \to \wedge^3 H_1(S) \oplus B_3(\Omega_S)$$

is the map

$$(a \wedge b \wedge c, \beta) \mapsto (a \wedge b \wedge c, \beta + \overline{a}\overline{b}(\overline{c} + 1)),$$

when $\{a, b, c\}$ is a symplectic basis for a subsurface $S_{1,2}$ such that $a \cdot b = 1$. So this composition is injective and hence

$$H_1(T_S) \cong \wedge^3 H_1(S) \oplus B_2(\Omega_S).$$

□

Let $S = S_{4,r}$. Choose two points p, q on the same boundary component of S and let $BX(p, q)$ be the associated arc-complex. Then $BX(p, q)$ and $T_S \backslash BX(p, q)$ are both 2-connected, thus $H_1(T_S) \cong H_0(\mathcal{A}^o(H), \mathcal{H}_1)$, by Lemma 1.11.1. The stabilizer of a vertex of $BX(p, q)$ is the Torelli group of a surface $S_{3,r+1}$ and the stabilizer of an edge is the Torelli group of either a surface $S_{3,r}$ or $S_{2,r+2}$. In both cases we have that if (v, w) is the edge, then $H_1(T_{(v,w)})$ maps by Proposition 2.4.3 onto $\wedge^3(v^\perp \cap w^\perp) \oplus B_2(\Omega_{v^\perp \cap w^\perp})$ in the summand of v and w in $C_0(\mathcal{A}^o(H), \mathcal{H}_1)$. By

Proposition 1.12.3 we know that $H_0(\mathcal{A}^o(H),\mathcal{H}_1) \cong \wedge^3 H_1(S)\oplus B_2(\Omega_S)$. We conclude that

THEOREM 3.5.6. *If $S = S_{g,r}$ with $g \geq 3$ and $r \geq 1$ then*

$$H_1(T_S) \cong \wedge^3 H_1(S) \oplus B_2(\Omega_S).$$

Summarizing we get (Corollary 3.2.5, Theorem 3.3.2, Mess, Proposition 3.4.1, Proposition 3.4.2 and Theorem 3.5.6)

$$\begin{array}{lcl} H_1(T_{0,r}) & = & 0 \ \text{ if } r \leq 3, \\ H_1(T_{0,r}) & \cong & H_1([P^{r-1}(D^2), P^{r-1}(D^2)]) \ \text{ if } r \geq 4, \\ H_1(T_{1,0}) & = & 0, \\ H_1(T_{1,1}) & \cong & \mathbb{Z} \cong \wedge^3 H_1(S_{1,1}) \oplus \widetilde{G}_{S_{1,1}}, \\ H_1(T_{1,2}) & \cong & \wedge^3 H_1(S_{1,2}) \oplus \widetilde{G}_{S_{1,2}}, \\ H_1(T_{2,0}) & \cong & \dfrac{\widetilde{G}_{S_{2,0}}}{\langle H_1(S_{2,0})\rangle}, \\ H_1(T_{2,1}) & \cong & \wedge^3 H_1(S_{2,1}) \oplus \widetilde{G}_{S_{2,1}}, \\ H_1(T_{2,2}) & & \text{is a quotient of } \wedge^3 H_1(S_{2,2}) \oplus \widetilde{G}_{S_{2,2}} \text{ and} \\ H_1(T_{g,r}) & \cong & \wedge^3 H_1(S_{g,r}) \oplus B_2(\Omega_{S_{g,r}}) \cong \wedge^3 H_1(S_{g,r}) \oplus G_{S_{1,2}} \ \text{ if } g \geq 3,\ r \geq 1. \end{array}$$

Recall that the "big Torelli group", denoted by $\widetilde{T_S}$, is the subgroup of Γ_S of mapping classes that act trivially on $H_1(S)$. We can compute $H_1(\widetilde{T_S})$ when $g \geq 3$, using Theorem 3.5.6.

COROLLARY 3.5.7. *If $g \geq 3$ then we have a short exact sequence*

$$0 \to \wedge^3 H_1(S) \oplus B_2(\Omega_S) \to H_1(\widetilde{T_S}) \to S^2\mathrm{Rad}(H_1(S)) \to 0.$$

If $\mathrm{Rad}(H_1(S)) = \langle [\partial_1], \dots, [\partial_r] : [\partial_1] + \cdots + [\partial_r] = 0\rangle$ where ∂_i is the i^{th} boundary component, we have for $i, j \in \{1, \dots, r\}$ that $[\partial_i]\otimes[\partial_i]$ lifts to D_{∂_i} and $([\partial_i]+[\partial_j])\otimes([\partial_i]+[\partial_j])$ lifts to $D_{\gamma_{ij}}$, where γ_{ij} is a SCC such that $[\gamma_{ij}] = [\partial_i] + [\partial_j]$.

PROOF. Let $U := S^2\mathrm{Rad}(H_1(S))$, we have by Proposition 2.2.3 the short exact sequence

$$1 \to T_S \to \widetilde{T_S} \to U \to 0.$$

It induces the exact sequence

$$\cdots \to \wedge^2 U \to H_1(T_S)_U \to H_1(\widetilde{T_S}) \to U \to 0$$

on homology. By Theorem 3.5.6 we have $H_1(T_S) \cong \wedge^3 H_1(S) \oplus B_2(\Omega_S)$. The lifts D_{∂_i} and $D_{\gamma_{ij}}$, where γ_{ij} is such that $[\gamma_{ij}] = [\partial_i] + [\partial_j]$, act via this isomorphism on $\wedge^3 H_1(S) \oplus B_2(\Omega_S)$ by the transvections determined by $[\partial_i]$ and $[\gamma_{ij}]$ respectively, hence trivially on this module. Let $x, y \in U$ and $\tilde{x}, \tilde{y}$ liftings in $\widetilde{T_S}$ of x, y respectively, then $[\tilde{x}, \tilde{y}]$ is in T_S. We show that the image of $\wedge^2 U \to \wedge^3 H_1(S) \oplus B_2(\Omega_S)$ is trivial, where $x \wedge y$ maps to the image of $\overline{[\tilde{x}, \tilde{y}]}$ in $H_1(T_S) \cong H_1(S) \oplus B_2(\Omega_S)$.

The elements D_{∂_i} are central. We have for $i, j, k, l \in \{1, \dots, r\}$ that

$$([\partial_i] + [\partial_j] \otimes [\partial_i] + [\partial_j]) \wedge ([\partial_k] + [\partial_l] \otimes [\partial_k] + [\partial_l]) \mapsto [D_{\gamma_{ij}}, D_{\gamma_{kl}}] = D_{\gamma_{ij}} D^{-1}_{D_{\gamma_{kl}}(\gamma_{ij})}.$$

If $\{i, j\} \cap \{k, l\} = \emptyset$ we can choose γ_{ij} and γ_{kl} such that they are disjunct on S, so the image is 0 in that case. Otherwise, we can assume without loss of generality that $l = j$. Suppose $S = S_{g,r}$. If $r \leq 2$ there is nothing to prove. If $r \geq 3$ let $S^* = S_{g+r-3,3}$ be the surface obtained from S by gluing to each boundary component not equal to $\partial_i, \partial_j, \partial_k$ a torus $S_{1,1}$, see Figure 3.5. So if $r = 3$ then $S = S^*$. The map $H_1(S) \to H_1(S^*)$ is injective and hence by Theorem 3.5.6 the

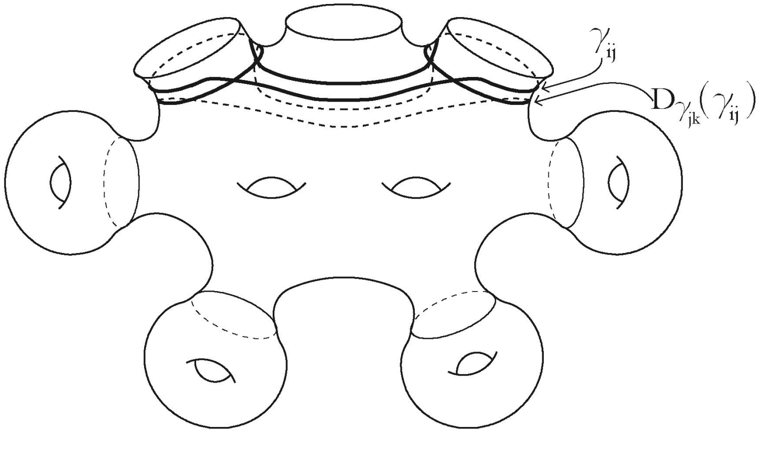

FIGURE 3.5. The surface S^* with γ_{ij} and $D_{\gamma_{jk}}(\gamma_{ij})$.

induced map $H_1(T_S) \to H_1(T_{S^*})$ is injective. In $H_1(T_{S^*})$ we have $D_{\gamma_{ij}} D^{-1}_{D_{\gamma_{jk}}(\gamma_{ij})} = D_{\gamma_{ij}} D^{-1}_{\partial_k} D_{\partial_k} D^{-1}_{D_{\gamma_{jk}(\gamma_{ij})}}$. We remark that $D_{\gamma_{ij}} D^{-1}_{\partial_k}$ and $D_{\partial_k} D^{-1}_{D_{\gamma_{jk}(\gamma_{ij})}}$ are both elements in T_{S^*} representing opposite elements in $\wedge^3 H_1(S^*) \oplus B_2(\Omega_S)$. This is because they are BP-maps representing $-(\sum_{i=1}^{g+r-3} e_i \wedge e_{-i}) \wedge [\partial_k]$ and $(\sum_{i=1}^{g+r-3} e_i \wedge e_{-i}) \wedge [\partial_k]$ respectively, where $\{e_i, e_{-i}, [\partial_i], [\partial_j], [\partial_k]\}_{i=1}^{g+r-3}$ is a symplectic basis of $H_1(S^*)$. So in $H_1(T_{S^*})$ their sum is zero and hence in $H_1(T_S)$ it is too. This finishes the proof of the Corollary. $\square$

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Samenvatting

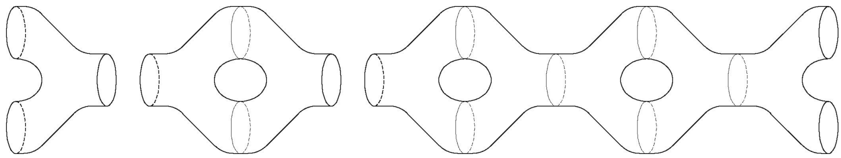
Deze samenvatting probeert aan lezers die niet bekend zijn met wiskunde een indruk te geven van waar dit proefschrift over gaat. Soms zullen er ook technische termen gebruikt worden (vaak tussen haakjes) om ook de niet-leken duidelijkheid te geven. Het is de bedoeling dat er ook zonder die termen een begrijpelijk verhaal overblijft.

We beginnen met het definiëren van de oppervlakken die een hoofdrol spelen in dit verhaal, daarna beschrijven we een constructie die we toe kunnen passen op deze oppervlakken, de zogenoemde Dehnse verwringing. Met behulp hiervan kunnen we de Torelligroep definiëren, het onderwerp van dit proefschrift. Vervolgens leggen we uit wat het belangrijkste resultaat is van deze studie en schetsen we de methode die we hiervoor gebruikt hebben.

Oppervlakken

De objecten waar we ons in dit proefschrift mee bezighouden zijn oppervlakken; hiermee bedoelen we compacte, samenhangende en oriënteerbare topologische oppervlakken. Voor de lezer die niet weet wat dit betekent, is de volgende stelling (over de classificatie van oppervlakken) van belang: de oppervlakken die hier beschreven worden zijn precies de oppervlakken die men verkrijgt als men willekeurig veel *stretchbroeken* (een boloppervlak met alleen drie randen: de onderkant van de twee broekspijpen en de taille) met de randen aan elkaar naait, en eventueel daarna nog randen dichtmaakt met een schijfje. Met de keuze voor *stretch*broeken drukken we uit dat we het hebben over *topologische* oppervlakken: twee oppervlakken beschouwen we als gelijk als de één na uitrekken en/of inkrimpen gelijk is aan de ander. Zie Figuur 3.6 voor voorbeelden: links één stretchbroek, midden twee, en rechts vijf aan elkaar genaaid. De notatie in het onderschrift wordt nu uitgelegd.

Ieder samenhangend deel van de rand, zoals de onderkant van de broekspijp of de hals van een hemd, noemen we een *randcomponent*. Wanneer we iedere randcomponent dichtmaken met een schijfje, ontstaat er een *gesloten* oppervlak dat de buitenkant is van een driedimensionaal object. Het aantal gaten dat dit object heeft

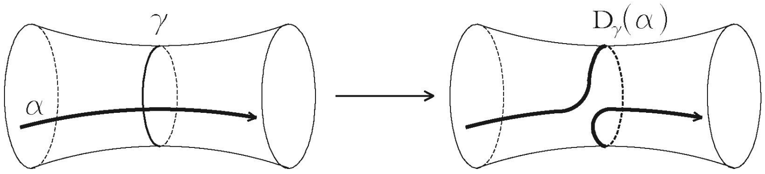


FIGUUR 3.6. Van links naar rechts: oppervlakken $S_{0,3}$, $S_{1,2}$ en $S_{2,3}$

heet het *geslacht* van het oppervlak. Een oppervlak van geslacht g met r randcomponenten wordt genoteerd met $S_{g,r}$, of, als dit niet tot verwarring leidt, met S. Een *sfeer*, oftewel een voetbal, is een gesloten oppervlak $S_{0,0}$ van geslacht nul, een *broek* is dus een oppervlak $S_{0,3}$, een *hemd* een oppervlak $S_{0,4}$ en is dus topologisch hetzelfde als een coltrui. Een gesloten oppervlak van geslacht één, $S_{1,0}$, wordt een *torus* genoemd en ziet er uit als een zwemband. In Figuur 3.6 zijn oppervlakken $S_{0,3}$, $S_{1,2}$ en $S_{2,3}$ getekend.

Dehnse verwringingen

Laat $S_{g,r}$ een oppervlak zijn en γ een gesloten lus op $S_{g,r}$ die zichzelf niet doorsnijdt; in het vervolg noemen we dit een *ingebedde cirkel op S*. Gegeven γ definiëren we nu een constructie die we uit kunnen voeren op S: de *(linkse) Dehnse verwringing langs* γ genoemd en die we noteren met D_γ. Kies een cylinderomgeving C_γ van γ en knip dit uit het oppervlak. Houd één randcomponent vast en draai de andere kant van C_γ met de wijzers van de klok mee, rond om de cylinderas over 360° en plaats de getwiste cylinder terug in het oppervlak waarbij de randen van de cylinder weer op dezelfde plek worden teruggeplaatst, zie Figuur 3.7.



FIGUUR 3.7. Het effect van de Dehnse verwringing langs γ op een boog α.

De Dehnse verwringing langs γ definiëren we op continue vervormingen van γ na, en, wanneer we de rand van het oppervlak vastlaten, ook op vervormingen van het oppervlak na (we bekijken isotopieklassen van homeomorfismen waarbij de rand puntsgewijs vastgelaten wordt). Wanneer γ bijvoorbeeld de rand is van een schijfje op S, is D_γ triviaal omdat na het uitvoeren van D_γ het oppervlak weer terug te vervormen is in zijn oude toestand. Wanneer γ is zoals in Figuur 3.8 is dit niet het geval. Er zijn oneindig veel verschillende ingebedde cirkels op S te bedenken zodat er oneindig veel verschillende Dehnse verwringingen op S uit te voeren zijn. De *inverse Dehnse verwringing* is de afbeelding die we krijgen wanneer we de cylinder C_γ tegen de wijzers van de klok in hadden gedraaid, deze wordt met D_γ^{-1} genoteerd.

Afbeeldingsklassegroep

We kunnen (inverse) Dehnse verwringingen achter elkaar uitvoeren en zo ontstaat de *afbeeldingsklassegroep van* S, genoteerd met Γ_S, bestaande uit alle samenstellingen van Dehnse verwringingen en inverse verwringingen uitgevoerd op S. Bijvoorbeeld, als γ_1, γ_2 ingebedde cirkels op S zijn, dan bedoelen we met $D_{\gamma_2} D_{\gamma_1}^{-1}$ eerst $D_{\gamma_1}^{-1}$ uitvoeren en vervolgens D_{γ_2}. Elementen van Γ_S worden afbeeldingsklassen van S genoemd. We zeggen dat de Dehnse verwringingen de afbeeldingsklassegroep *voortbrengen* (hier per definitie, maar de afbeeldingsklassegroep wordt ook gedefinieerd als de groep van isotopieklassen van de orientatiebehoudende homeomorfismen van S die de rand puntsgewijs vastlaten en dan is dit een stelling). De Dehnse verwringingen zijn dus de basiselementen, voortbrengers van Γ_S genoemd; de afbeeldingsklassen zijn alle mogelijke recepten waarin verteld wordt in welke volgorde bepaalde Dehnse verwringingen en inverse verwringingen uitgevoerd moeten worden. Hatcher and Thurston bewijzen in [Hatcher-Thurston] een sterker resultaat: de groep Γ_S heeft een *eindige presentatie*, in dit geval betekent dit dat hij door eindig veel Dehnse verwringingen wordt voortgebracht en er slechts eindig veel *relaties* gelden tussen deze voortbrengers. Een relatie is een vergelijking die geldt tussen voortbrengers, hiermee kun je soms inzien dat je met minder voortbrengers af kunt.

Abelianisatie

Een voorbeeld van een relatie in Γ_S is het volgende. Laat γ_1, γ_2 twee ingebedde cirkels op S zijn die elkaar niet snijden. Je kunt dan eenvoudig inzien dat het dan niet uitmaakt in welke volgorde we de Dehnse verwringingen om γ_1 en γ_2 uitvoeren. Met andere woorden, in Γ_S geldt voor zulke lussen de *relatie*

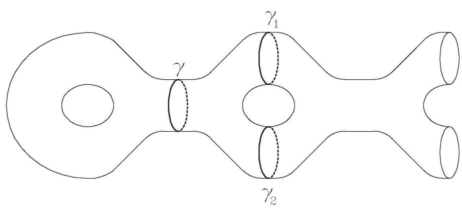
$$D_{\gamma_2} D_{\gamma_1} = D_{\gamma_1} D_{\gamma_2}.$$

Wanneer γ_1 en γ_2 elkaar precies één keer snijden, dan geldt juist $D_{\gamma_2} D_{\gamma_1} \neq D_{\gamma_1} D_{\gamma_2}$. We zeggen dat de groep Γ_S niet abels is, want een groep G is namelijk per definitie *abels* als $fh = hf$ voor álle elementen f, h in de groep. Een voorbeeld van een abelse groep is de verzameling van alle gehele getallen $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ waarin we optellen; hier geldt $x + y = y + x$ voor alle gehele getallen x, y. In het algemeen zijn groepen die niet abels zijn veel moeilijker te bestuderen dan abelse groepen. Om die reden voert men vaak een constructie uit die de groep *abels maakt*; we vervangen

G door een nieuwe groep genoteerd als $H_1(G)$. Deze groep $H_1(G)$ lijkt in alles op G, behalve dat we voor alle f en h uit G nu wél laten gelden dat $fh = hf$; deze groep wordt de *abelianisatie van* G genoemd. De notatie suggereert dat er ook groepen $H_k(G)$ bestaan, dit is inderdaad het geval voor $k \geq 0$, het zijn objecten die men bestudeert om de groep G te leren kennen, maar hier zullen we niet verder op ingaan.

Torelligroep

De Torelligroep T_S van S is een deelverzameling van Γ_S (een normale ondergroep), die we nu zullen beschrijven door te vertellen door welke elementen het wordt voortgebracht. We doen dit hier voor $g \geq 1$, waarbij g het geslacht is van S. In dat geval wordt het door twee typen elementen voortgebracht. Het eerste type voortbrenger is als volgt. Laat γ een ingebedde cirkel zijn op S zodat γ precies de rand is van een deeloppervlak $S_{g',1}$ van S, met $g' \leq g$, zie bijvoorbeeld Figuur 3.8.



FIGUUR 3.8. De lus γ is precies de rand van het deeloppervlak $S_{1,1}$ links van γ. De lussen γ_1 en γ_2 vormen samen precies de rand van het deeloppervlak $S_{1,2}$ links van hen.

Dan is D_γ een element van de Torelligroep van S. We noemen zo'n cirkel een *BSCC, bounding simple closed curve*, en D_γ een *BSCC-afbeelding*. Voor de beschrijving van het tweede type voortbrenger, laat γ_1 en γ_2 twee ingebedde cirkels op S zijn zodat γ_1 en γ_2 samen precies de rand van een deeloppervlak $S_{g',2}$ van S zijn, met $g' \leq g$, zie opnieuw Figuur 3.8. Dan is ook $D_{\gamma_2}^{-1}D_{\gamma_1}$ een element van de Torelligroep van S. We noemen zo'n paar een *BP, bounding pair* en de afbeeldingsklasse $D_{\gamma_2}^{-1}D_{\gamma_1}$ een *BP-afbeelding*. Als $g \geq 1$ dan brengen alle $BSCC$-afbeeldingen en BP-afbeeldingen de Torelligroep voort, dat wil dus zeggen dat de Torelligroep in dat geval precies bestaat uit alle samenstellingen (recepten) van $BSCC$ en BP-afbeelingen en inverses daarvan. Dit is een stelling wanneer we de definitie van de Torelligroep geven die deze afbeeldingen karakteriseert; het begrijpen van deze definitie vergt wel enige wiskundige kennis. Kies hiervoor op iedere randcomponent een punt en laat P de verzameling van deze punten zijn. Wiskundig gezegd is de Torelligroep de ondergroep van Γ_S bestaande uit die afbeeldingsklassen die triviaal werken op de relatieve homologiegroep $H_1(S, P; \mathbb{Z})$. Voor $S_{2,0}$ is dit door Mess bewezen, zie

[Mess], voor $S_{g,1}$ met $g \geq 3$ is dit door [Powell] bewezen. Zij bewijzen eigenlijk een veel sterker resultaat. Voor willekeurige oppervlakken $S_{g,r}$ van geslacht minstens één is dit nu af te leiden uit deze resultaten, zie Hoofdstuk 2 van dit proefschrift. Voor $g = 0$ kan men inzien dat er geen niet triviale $BSCC$ en BP-afbeeldingen zijn, maar kan men andere voortbrengers aanwijzen (zie Paragraaf 3.2, Hoofdstuk 3).

Resultaten

In dit proefschrift bestuderen we de abels gemaakte Torelligroep $H_1(T_S)$. Het is een vervolg op het werk van Johnson van rond 1980, waarin hij de Torelligroep bestudeert. Zie [Johnson1] tot en met [Johnson8]. Hij bewijst hier onder meer dat de Torelligroep T_S van een oppervlak van geslacht minstens drie en hoogstens één randcomponent, eindig is voortgebracht. Of er ook eindig veel voortbrengers te vinden zijn waartussen ook maar eindig veel relaties gelden is niet bekend. Vervolgens berekent hij de abels gemaakte Torelligroep, opnieuw alleen voor oppervlakken $S_{g,1}$ waarbij $g \geq 3$. Hij bewijst dat voor zulke oppervlakken S

$$H_1(T_S) \cong \wedge^3 H_1(S) \oplus B_2(\Omega_S).$$

Het teken $\cong$ betekent dat we elementen uit het linkerlid kunnen identificeren met elementen uit het rechterlid en dat na deze identificaties, $H_1(T_S)$ gelijk is aan het rechterlid. Voor de niet wiskundige lezer vergt het te veel voorkennis om kort uit te kunnen leggen wat het rechterlid is. Van belang is dat $\wedge^3 H_1(S) \oplus B_2(\Omega_S)$ volledig bekend is, en hiermee dus ook $H_1(T_S)$. Voor wiskundigen zal de notatie $B_2(\Omega_S)$ meestal niet bekend zijn, het betekent het volgende. Laat Ω_S de verzameling van alle kwadratische vormen op $H_1(S, \mathbb{Z}/2)$ zijn, die het intersectieproduct modulo $\mathbb{Z}/2$ bepalen, dat wil zeggen, functies $\omega : H_1(S, \mathbb{Z}/2) \to \mathbb{Z}/2$ met de eigenschap dat $\omega(a + b) = \omega(a) + \omega(b) + a \cdot b$. Dit is een affiene ruimte over $\text{Hom}(H_1(S, \mathbb{Z}/2), \mathbb{Z}/2)$ en laat $B_2(\Omega_S)$ de $\mathbb{Z}/2$-lineaire ruimte van polynomiale functies van graad ≤ 2 zijn op Ω_S. De groep $B_2(\Omega_S)$ is het 2-torsiegedeelte van $H_1(T_S)$ van rang $\sum_{i=0}^{2} \binom{2g}{i}$, de groep $\wedge^3 H_1(S)$ is het vrije gedeelte van $H_1(T_S)$ van rang $\binom{2g}{3}$.

We bewijzen in dit proefschrift dat dit resultaat van Johnson over $H_1(T_S)$ geldig is voor álle oppervlakken met rand en $g \geq 3$, maar nu dus met een willekeurig aantal randcomponenten. Verder berekenen we $H_1(T_S)$ voor $g = 0$, voor $g = 1$ en $r = 0, 1, 2$ en voor $g = 2$ en $r = 1$ (voor $r = 0$ is dit gedaan door Mess, zie [Mess]). Voor $g = 1, 2$ en meer randcomponenten waren we niet in staat de Torelligroep te berekenen. De methode die we gebruiken voor de berekening van $H_1(T_S)$ als $g \geq 3$ is anders dan de methode gebruikt door Johnson en zal nu worden toegelicht. Het geeft in het bijzonder een alternatief bewijs voor het resultaat van Johnson over $H_1(T_S)$, en geeft mogelijk een manier om ook $H_k(T_S)$ te bepalen voor $k \geq 2$.

Bewijsmethode

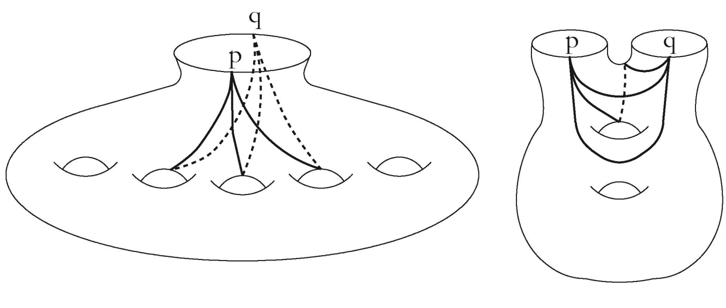
Laat $S_{g,r}$ een oppervlak zijn, dat we kortweg noteren met S, waarbij we veronderstellen dat $r \geq 1$, oftewel het oppervlak is niet gesloten, maar heeft één of meerdere randcomponenten. Kies twee punten p, q op de rand van S, ze mogen op dezelfde randcomponent liggen, en laat α een pad van p naar q zijn op S dat zichzelf niet doorsnijdt. We leggen de eis op aan α dat wanneer we S langs α openknippen, het nieuwe oppervlak S_α dat zo ontstaat nog uit één stuk bestaat. Het is dan van het volgende type: als p en q op dezelfde randcomponent liggen dan is $S_\alpha = S_{g-1,r+1}$; als p en q op verschillende randen liggen dan is $S_\alpha = S_{g,r-1}$. In beide gevallen krijg je S weer terug uit S_α door het aanhechten van een broek. De bewijsmethode die we gebruiken om $H_1(T_S)$ te bepalen is een *bewijs met inductie*: van oppervlakken van geslacht 1 en 2 proberen we genoeg te weten te komen over de Torelligroep (de *inductiestart*) en vervolgens willen we weten hoe $H_1(T_{S_\alpha})$ verandert als we S_α een broek aannaaien (de *inductiestap*). Omdat ieder oppervlak met rand en $g \geq 3$ onstaat uit een oppervlak $S_{2,1}$ door hier genoeg broeken aan te naaien, kunnen we zo $H_1(T_S)$ berekenen voor willekeurige oppervlakken $S_{g,r}$ met $g \geq 3$ en $r \geq 1$. Voor oppervlakken van geslacht ≤ 2 kunnen we inderdaad genoeg berekenen om de inductie te laten starten (Paragraaf 3.2 t/m 3.4 in het proefschrift), hier zullen we in deze samenvatting niet verder op ingaan. We schetsen nu hoe we de inductiestap kunnen nemen.

Boogsystemen

Als γ een ingebedde cirkel is op S dan kunnen we D_γ toepassen op de boog α. Als γ en α elkaar niet snijden gebeurt er niets met α, als ze elkaar wel snijden ontstaat er na het twisten een nieuwe boog, die we noteren met $D_\gamma(\alpha)$. Zie opnieuw Figuur 3.7. Dit kunnen we voor willekeurige afbeeldingsklassen $f \in \Gamma_S$ doen door de verschillende (inverse) Dehnse verwringingen achter elkaar uit te voeren volgens het recept van f en we noteren het resultaat met $f(\alpha)$. De afbeeldingsklassen die α onveranderd laten zijn precies de afbeeldingsklassen die leven op het oppervlak S_α en zoals we al opmerkten is S_α of van lager geslacht, of heeft deze minder randcomponenten dan S. De verzameling van zulke afbeeldingsklassen heet de *stabilisator van α in Γ_S*. Deze wordt genoteerd met $(\Gamma_S)_\alpha$ en we hebben net opgemerkt dat

$$(\Gamma_S)_\alpha \cong \Gamma_{S_\alpha}.$$

Wanneer we ons beperken tot de afbeeldingsklassen die in de Torelligroep zitten dan hebben we opnieuw dat $(T_S)_\alpha \cong T_{S_\alpha}$. We kunnen het bovenstaande verhaal veralgemeniseren door *boogsystemen* $(\alpha_0, \ldots, \alpha_k)$ te bekijken waarbij iedere α een boog is van p naar q die zichzelf niet doorsnijdt en die behalve in p en q, geen punten gemeenschappelijk heeft met de andere bogen. Opnieuw bekijken we de bogen op vervormingen waarbij de eindpunten vastgelaten worden, na. De verzameling van alle boogsystemen op S met de eigenschap dat als we S langs de bogen $\alpha_0, \ldots, \alpha_k$ tegelijk openknippen, er een oppervlak ontstaat dat nog steeds uit één stuk bestaat, wordt genoteerd met $BX(p,q)$. Zie Figuur 3.9 voor een voorbeeld als $k = 2$.



FIGUUR 3.9

Deze verzameling $BX(p,q)$ is geïntroduceerd door Harer en is door hem en Ivanov gebruikt voor de bestudering van de afbeeldingsklassegroep, zie [Harer], [Ivanov] (zij bewijzen met behulp van de werking van Γ_S op $BX(p,q)$ de stabiliteit van de homologie van de afbeeldingsklassegroep).

Partiëel geordende verzamelingen

De verzameling $BX(p,q)$ heeft de stuctuur van een *partiëel geordende verzameling*: we zeggen dat

$$(\alpha_0, \ldots, \alpha_k) \leq (\beta_0, \ldots, \beta_m)$$

precies dan als $\{\alpha_0, \ldots, \alpha_k\}$ een deelverzameling is van $\{\beta_0, \ldots, \beta_m\}$. Bijvoorbeeld, als (α_0, α_1) en $(\alpha_0, \beta, \alpha_1)$ boogsystemen in $BX(p,q)$ zijn dan $(\alpha_0, \alpha_1) \leq (\alpha_0, \beta, \alpha_1)$. De verzameling van natuurlijke getallen, $\{0, 1, 2, 3, \ldots\}$, heeft zoals bekend een natuurlijke ordening ($4 < 7$ etcetera). In tegenstelling tot deze geordende verzameling kun je van twee boogrijtjes niet altijd zeggen of de één kleiner is dan de ander of andersom, vandaar de naam *partiëel* geordend. Partiëel geordende verzamelingen hebben een meetkundige structuur in zich (de meetkundige realisatie van het bijbehorende simpliciaalcomplex) en Harer bewijst dat de meetkundige structuur van $BX(p,q)$ een hele bijzondere is. (Namelijk $BX(p,q)$ bestaat uit een boeket van sferen van dimensie $2g - 1$ als p, q op dezelfde rand liggen, en van dimensie $2g$ in het andere geval).

Met behulp van $BX(p,q)$ maken we een nieuwe partiëel geordende verzameling, genoteerd als $T_S \backslash BX(p,q)$, door te stellen dat twee boogrijtjes $(\alpha_0, \ldots, \alpha_k)$ en $(\beta_0, \ldots, \beta_k)$ uit $BX(p,q)$ in $T_S \backslash BX(p,q)$ met elkaar geïdentificeerd worden als er een element t uit de Torelligroep is zodat $t(\alpha_i) = \beta_i$ voor iedere i met $0 \leq i \leq k$. We laten in dit proefschrift zien dat $T_S \backslash BX(p,q)$ ook een bijzondere meetkundige structuur heeft (namelijk dat deze sferisch is als p, q op dezelfde rand liggen en enkelvoudig samenhangend als p, q op verschillende randen liggen).

Inductiestap

Met behulp van een techniek uit de algebraïsche topologie, de techniek van de *spectraalrijen*, die we hier niet uit zullen leggen maar die zeer krachtig is, kunnen we nu een verband leggen tussen $H_k(T_S)$, $H_k((T_S)_\alpha)$ en de meetkunde van $BX(p,q)$ en $T_S \backslash BX(p,q)$. Dit verband stelt ons in staat om $H_1(T_S)$ te berekenen, gegeven dat we $H_1(T_{S_\alpha})$ kennen, en voltooit dus de inductiestap.

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Curriculum vitae

Barbara van den Berg werd op 22 februari 1971 in Groningen geboren. Na daar de Vrije School te hebben doorlopen behaalde ze in 1990 op het Noordelijk Avondcollege haar VWO-diploma. In 1991 studeerde ze wijsbegeerte aan de Universiteit Utrecht en haalde haar propedeuse daarin. Het volgende jaar begon zij met haar studie wiskunde aan de Universiteit Utrecht en in 1997 behaalde zij haar doctoraaldiploma. Van 1997 tot 2002 was ze Onderzoeker in Opleiding aan dezelfde faculteit binnen het NWO-project "Algebraïsche krommen en Riemannoppervlakken" met als promotor professor E.J.N. Looijenga. In dit kader bezocht ze in de lente van 2000 professor R.M. Hain aan Duke University in Durham NC (VS). Vanaf september 2002 was zij junior-docent aan de Wiskunde & Informatica faculteit van de Universiteit Utrecht en voltooide zij dit proefschrift, dat op 6 oktober 2003 verdedigd wordt.